Chapter 1 – Reviewing algebra

Solutions to Exercise 1A

1 **a** Add indices:
$x^3 \times x^4 = x^{3+4} = x^7$

b Add indices:
$a^5 \times a^{-3} = a^{5+-3} = a^2$

c Add indices:
$x^2 \times x^{-1} \times x^2 = x^{2+-1+2} = x^3$

d Subtract indices:
$\dfrac{y^3}{y^7} = y^{3-7} = y^{-4}$

e Subtract indices:
$\dfrac{x^8}{x^{-4}} = x^{8-(-4)} = x^{12}$

f Subtract indices:
$\dfrac{p^{-5}}{p^2} = p^{-5-2} = p^{-7}$

g Subtract indices:
$a^{\frac{1}{2}} \div a^{\frac{2}{3}} = a^{\frac{3}{6}-\frac{4}{6}} = a^{-\frac{1}{6}}$

h Multiply indices:
$(a^{-2})^4 = a^{-2\times 4} = a^{-8}$

i Multiply indices:
$(y^{-2})^{-7} = y^{-2\times(-7)} = y^{14}$

j Multiply indices:
$(x^5)^3 = x^{5\times 3} = x^{15}$

k Multiply indices:
$(a^{-20})^{\frac{3}{5}} = a^{-20\times\frac{3}{5}} = a^{-12}$

l Multiply indices:
$\left(x^{-\frac{1}{2}}\right)^{-4} = x^{-\frac{1}{2}\times -4} = x^2$

m Multiply indices:
$(n^{10})^{\frac{1}{5}} = n^{10\times\frac{1}{5}} = n^2$

n Multiply the coefficients and add the indices:
$2x^{\frac{1}{2}} \times 4x^3 = (2 \times 4)x^{\frac{1}{2}+3} = 8x^{\frac{7}{2}}$

o Multiply the first two indices and add the third:

$$\begin{aligned}(a^2)^{\frac{5}{2}} \times a^{-4} &= a^{2\times\frac{5}{2}} \times a^{-4}\\ &= a^{5+(-4)}\\ &= a^1 = a\end{aligned}$$

p $\dfrac{1}{x^{-4}} = x^{1\div\frac{1}{4}} = x^4$

q

$$\begin{aligned}\left(2n^{-\frac{2}{5}}\right)^5 \div (4^3n^4) &= 2^5n^{-\frac{2}{5}\times 5} \div ((2^2)^3n^4)\\ &= 2^5n^{-2} \div (2^6n^4)\\ &= 2^{5-6}n^{-2-4}\\ &= 2^{-1}n^{-6} = \frac{1}{2n^6}\end{aligned}$$

r Multiply the coefficients and add the indices.

$$\begin{aligned}&x^3 \times 2x^{\frac{1}{2}} \times -4x^{-\frac{3}{2}}\\ &= (1 \times 2 \times -4)x^{3+\frac{1}{2}+\left(-\frac{3}{2}\right)}\\ &= -8x^2\end{aligned}$$

s

$$\begin{aligned}&(ab^3)^2 \times a^{-2}b^{-4} \times \frac{1}{a^2b^{-3}}\\ &= a^2b^6 \times a^{-2}b^{-4} \times a^{-2}b^3\\ &= a^{2+-2+-2}b^{6+-4+3}\\ &= a^{-2}b^5\end{aligned}$$

t $(2^2p^{-3} \times 4^3p^5 \div ((6p^{-3}))^0 = 1$
Anything to the power zero is 1.

2 **a** $25^{\frac{1}{2}} = \sqrt{25} = 5$

b $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

c $\left(\frac{16}{9}\right)^{\frac{1}{2}} = \frac{16^{\frac{1}{2}}}{9^{\frac{1}{2}}}$

$= \frac{\sqrt{16}}{\sqrt{9}} = \frac{4}{3}$

d $16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}}$

$= \frac{1}{\sqrt{16}} = \frac{1}{4}$

e $\left(\frac{49}{36}\right)^{-\frac{1}{2}} = \frac{1}{\left(\frac{49}{36}\right)^{\frac{1}{2}}}$

$= \frac{1}{\frac{\sqrt{49}}{\sqrt{36}}}$

$= \frac{\sqrt{36}}{\sqrt{49}} = \frac{6}{7}$

f $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

g $144^{\frac{1}{2}} = \sqrt{144} = 12$

h $64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2 = 4^2 = 16$

i $9^{\frac{3}{2}} = \left(9^{\frac{1}{2}}\right)^3$

$= 3^3 = 27$

j $\left(\frac{81}{16}\right)^{\frac{1}{4}} = \frac{81^{\frac{1}{4}}}{16^{\frac{1}{4}}}$

$= \frac{3}{2}$

k $\left(\frac{23}{5}\right)^0 = 1$

l $128^{\frac{3}{7}} = \left(128^{\frac{1}{7}}\right)^3$

$= 2^3 = 8$

3 **a** $4.35^2 = 18.9225 \approx 18.92$

b $2.4^5 = 79.62624 \approx 79.63$

c $\sqrt{34.6921} = 5.89$

d $0.02^{-3} = 125\,000$

e $\sqrt[3]{0.729} = 0.9$

f $\sqrt[4]{2.3045} = 1.23209\ldots \approx 1.23$

g $(345.64)^{-\frac{1}{3}} = 0.14249\ldots \approx 0.14$

h $(4.558)^{\frac{2}{5}} = 1.83607\ldots \approx 1.84$

i $\frac{1}{(0.064)^{-\frac{1}{3}}} = (0.064)^{\frac{1}{3}} = 0.4$

4 **a** $\frac{a^2b^3}{a^{-2}b^{-4}} = a^{2--2}b^{3--4}$

$= a^4b^7$

b $\frac{2a^2(2b)^3}{(2a)^{-2}b^{-4}} = \frac{2a^2 \times 2^3b^3}{2^{-2}a^{-2}b^{-4}}$

$= \frac{2^4a^2b^3}{2^{-2}a^{-2}b^{-4}}$

$= 2^{4--2}\,a^{2--2}b^{3--4}$

$= 2^6a^4b^7 = 64a^4b^7$

c $\frac{a^{-2}b^{-3}}{a^{-2}b^{-4}} = a^{-2--2}b^{-3--4}$

$= a^0b^1 = b$

d $\dfrac{a^2b^3}{a^{-2}b^{-4}} \times \dfrac{ab}{a^{-1}b^{-1}}$

$$= \frac{a^{2+1}b^{3+1}}{a^{-2+-1}b^{-4+-1}}$$

$$= \frac{a^3b^4}{a^{-3}b^{-5}}$$

$$= a^{3--3}b^{4--5} = a^6b^9$$

e $\dfrac{(2a)^2 \times 8b^3}{16a^{-2}b^{-4}} = \dfrac{4a^2 \times 8b^3}{16a^{-2}b^{-4}}$

$$= \frac{32a^2b^3}{16a^{-2}b^{-4}}$$

$$= \frac{32}{16}a^{2--2}b^{3--4}$$

$$= 2a^4b^7$$

f $\dfrac{2a^2b^3}{8a^{-2}b^{-4}} \div \dfrac{16ab}{(2a)^{-1}b^{-1}}$

$$= \frac{2a^2b^3}{8a^{-2}b^{-4}} \times \frac{(2a)^{-1}b^{-1}}{16ab}$$

$$= \frac{2a^2b^3}{8a^{-2}b^{-4}} \times \frac{2^{-1}a^{-1}b^{-1}}{16ab}$$

$$= \frac{2^{1+-1}a^{2+-1}b^{3+-1}}{8 \times 16 \times a^{-2+1}b^{-4+1}}$$

$$= \frac{2^0a^1b^2}{128a^{-1}b^{-3}}$$

$$= \frac{1}{128}a^{1--1}b^{2--3} = \frac{a^2b^5}{128}$$

5 $\dfrac{2^n \times 8^n}{2^{2n} \times 16} = \dfrac{2^n \times (2^3)^n}{2^{2n} \times 2^4}$

$$= \frac{2^n \times 2^{3n}}{2^{2n} \times 2^4}$$

$$= \frac{2^{n+3n-2n}}{2^4}$$

$$= 2^{2n} \times 2^{-4}$$

$$= 2^{2n-4}$$

6 $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$

$$= (2 \times 3)^{-x} \times 6^{2x} \times (2 \times 3)^{2x}$$

$$= 6^{-x} \times 6^{2x} \times 6^{2x}$$

$$= 6^{-x+2x+2x}$$

$$= 6^{3x}$$

7 In each case, add the fractional indices.

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}} = 2^{\frac{2}{6}+\frac{1}{6}+-\frac{4}{6}}$

$$= 2^{-\frac{1}{6}} = \left(\frac{1}{2}\right)^{\frac{1}{6}}$$

b $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}} = a^{\frac{5}{20}+\frac{8}{20}+-\frac{2}{20}}$

$$= a^{\frac{11}{20}}$$

c $2^{\frac{2}{3}} \times 2^{\frac{5}{6}} \times 2^{-\frac{2}{3}} = 2^{\frac{4}{6}+\frac{5}{6}+-\frac{4}{6}}$

$$= 2^{\frac{5}{6}}$$

d $\left(2^{\frac{1}{3}}\right)^2 \times \left(2^{\frac{1}{2}}\right)^5 = 2^{\frac{2}{3}} \times 2^{\frac{5}{2}}$

$$= 2^{\frac{4}{6}+\frac{15}{6}} = 2^{\frac{19}{6}}$$

e $\left(2^{\frac{1}{3}}\right)^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}} = 2^{\frac{2}{3}} \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$

$$= 2^{\frac{2}{3}+\frac{1}{3}+-\frac{2}{5}} = 2^{\frac{3}{5}}$$

8 **a**

$$\sqrt[3]{a^3b^2} \div \sqrt[3]{a^2b^{-1}} = (a^3b^2)^{\frac{1}{3}} \div (a^2b^{-1})^{\frac{1}{3}}$$

$$= a^1b^{\frac{2}{3}} \div a^{\frac{2}{3}}b^{-\frac{1}{3}}$$

$$= a^{1-\frac{2}{3}}b^{\frac{2}{3}--\frac{1}{3}} = a^{\frac{1}{3}}b$$

b $\sqrt{a^3b^2} \times \sqrt{a^2b^{-1}}$

$$= (a^3b^2)^{\frac{1}{2}} \times (a^2b^{-1})^{\frac{1}{2}}$$

$$= a^{\frac{3}{2}}b^1 \times a^1b^{-\frac{1}{2}}$$

$$= a^{\frac{3}{2}+1}b^{1+-\frac{1}{2}} = a^{\frac{5}{2}}b^{\frac{1}{2}}$$

c

$$\sqrt[5]{a^3b^2} \times \sqrt[5]{a^2b^{-1}} = (a^3b^2)^{\frac{1}{5}} \times (a^2b^{-1})^{\frac{1}{5}}$$

$$= a^{\frac{3}{5}}b^{\frac{2}{5}} \times a^{\frac{2}{5}}b^{-\frac{1}{5}}$$

$$= a^{\frac{3}{5}+\frac{2}{5}}b^{\frac{2}{5}+-\frac{1}{5}} = ab^{\frac{1}{5}}$$

d $\sqrt{a^{-4}b^2} \times \sqrt{a^3b^{-1}}$

$$= (a^{-4}b^2)^{\frac{1}{2}} \times (a^3b^{-1})^{\frac{1}{2}}$$

$$= a^{-2}b^1 \times a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{-2+\frac{3}{2}}b^{1+-\frac{1}{2}}$$

$$= a^{-\frac{1}{2}}b^{\frac{1}{2}}$$

$$= \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}} = \left(\frac{b}{a}\right)^{\frac{1}{2}}$$

e $\sqrt{a^3b^2c^{-3}} \times \sqrt{a^2b^{-1}c^{-5}}$

$$= (a^3b^2c^{-3})^{\frac{1}{2}} \times (a^2b^{-1}c^{-5})^{\frac{1}{2}}$$

$$= a^{\frac{3}{2}}b^1c^{-\frac{3}{2}} \times a^1b^{-\frac{1}{2}}\ c^{-\frac{5}{2}}$$

$$= a^{\frac{3}{2}+1}b^{1+-\frac{1}{2}}c^{-\frac{3}{2}+-\frac{5}{2}}$$

$$= a^{\frac{5}{2}}b^{\frac{1}{2}}c^{-4}$$

f $\sqrt[5]{a^3b^2} \div \sqrt[5]{a^2b^{-1}}$

$$= (a^3b^2)^{\frac{1}{5}} \div (a^2b^{-1})^{\frac{1}{5}}$$

$$= a^{\frac{3}{5}}b^{\frac{2}{5}} \div a^{\frac{2}{5}}b^{-\frac{1}{5}}$$

$$= a^{\frac{3}{5}-\frac{2}{5}}b^{\frac{2}{5}--\frac{1}{5}} = a^{\frac{1}{5}}b^{\frac{3}{5}}$$

g $\dfrac{\sqrt{a^3b^2}}{a^2b^{-1}c^{-5}} \times \dfrac{\sqrt{a^{-4}b^2}}{a^3b^{-1}} \times \sqrt{a^3b^{-1}}$

$$= \frac{(a^3b^2)^{\frac{1}{2}}}{a^2b^{-1}c^{-5}} \times \frac{(a^{-4}b^2)^{\frac{1}{2}}}{a^3b^{-1}} \times (a^3b^{-1})^{\frac{1}{2}}$$

$$= \frac{a^{\frac{3}{2}}b^1}{a^2b^{-1}c^{-5}} \times \frac{a^{-2}b^1}{a^3b^{-1}} \times a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{\frac{3}{2}-2}b^{1--1}c^{0--5} \times a^{-2-3}b^{1--1}$$

$$\times a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{-\frac{1}{2}}b^2c^5 \times a^{-5}b^2 \times a^{\frac{3}{2}}b^{-\frac{1}{2}}$$

$$= a^{-\frac{1}{2}+-5+\frac{3}{2}}b^{2+2+-\frac{1}{2}}c^5$$

$$= a^{-4}b^{\frac{7}{2}}c^5$$

Solutions to Exercise 1B

1 **a** $47.8 = 4.78 \times 10^1 = 4.78 \times 10$

b $6728 = 6.728 \times 10^3$

c $79.23 = 7.923 \times 10^1 = 7.923 \times 10$

d $43\,580 = 4.358 \times 10^4$

e $0.0023 = 2.3 \times 10^{-3}$

f $0.000\,000\,56 = 5.6 \times 10^{-7}$

g $12.000\,34 = 1.2000\,34 \times 10^1$
$= 1.2000\,34 \times 10$

h Fifty million $= 50\,000\,000$
$= 5.0 \times 10^7$

i $23\,000\,000\,000 = 2.3 \times 10^{10}$

j $0.000\,000\,0013 = 1.3 \times 10^{-9}$

k 165 thousand $= 165\,000$
$= 1.65 \times 10^5$

l $0.000\,014\,567 = 1.4567 \times 10^{-5}$

2 **a** 2.99×10^{-23}

b The decimal point moves 8 places to the right $= 1.0 \times 10^{-8}$

c 3.432×10^2

d 3.1536×10^7

e 6.09×10^9

f 3.057×10^{21}

3 **a** $1\,390\,000\,000$

b $0.000\,0075$

c $0.000\,000\,000\,000\,0056$

4 **a** $456.89 \approx 4.569 \times 10^2$
(4 significant figures)

b $34567.23 \approx 3.5 \times 10^4$
(2 significant figures)

c $5679.087 \approx 5.6791 \times 10^3$
(5 significant figures)

d $0.04536 \approx 4.5 \times 10^{-2}$
(2 significant figures)

e $0.09045 \approx 9.0 \times 10^{-2}$
(2 significant figures)

f $4568.234 \approx 4.5682 \times 10^3$
(5 significant figures)

5 **a**
$$\begin{aligned}\frac{324\,000 \times 0.000\,000\,7}{4000} &= \frac{3.24 \times 10^5 \times 7 \times 10^{-7}}{4 \times 10^3}\\ &= \frac{3.24 \times 7}{4} \times 10^{5+-7-3}\\ &= 5.67 \times 10^{-5}\\ &= 0.0000567\end{aligned}$$

b $\dfrac{5\,240\,000 \times 0.8}{42\,000\,000}$

$$= \frac{5.24 \times 10^6 \times 8 \times 10^{-1}}{4.2 \times 10^7}$$

$$= \frac{41.92 \ \times 10^5}{4.2 \times 10^7}$$

$$= \frac{4192 \ \times 10^3}{42\,000 \times 10^3}$$

$$= \frac{4192}{42\,000} = \frac{262}{2625}$$

6 a $\dfrac{\sqrt[3]{a}}{b^4} = \dfrac{\sqrt[3]{2 \times 10^9}}{3.215^4}$

$$= \frac{\sqrt[3]{2} \times \sqrt[3]{10^9}}{106.8375\ldots}$$

$$= \frac{1.2599\ldots \times 10^3}{106.8375\ldots}$$

$$= 0.011\,792 \ \ldots \ \times 10^3 \approx 11.8$$

b

$$\frac{\sqrt[4]{a}}{4b^4} = \frac{\sqrt[4]{2 \times 10^{12}}}{4 \times 0.05^4}$$

$$= \frac{\sqrt[4]{2} \times \sqrt[4]{10^{12}}}{4 \times 0.000\,006\,25}$$

$$= \frac{1.189\,2\ldots \ \times 10^3}{4 \times 6.25 \times 10^{-6}}$$

$$= 0.047\,568\ldots \times 10^9 \approx 4.76 \times 10^7$$

Solutions to Exercise 1C

1 **a**
$$\begin{aligned} 3x + 7 &= 15 \\ 3x &= 15 - 7 \\ &= 8 \\ x &= \frac{8}{3} \end{aligned}$$

b
$$\begin{aligned} 8 - \frac{x}{2} &= -16 \\ -\frac{x}{2} &= -16 - 8 \\ &= -24 \\ -\frac{x}{2} \times -2 &= -24 \times -2 \\ x &= 48 \end{aligned}$$

c
$$\begin{aligned} 42 + 3x &= 22 \\ 3x &= 22 - 42 \\ &= -20 \\ x &= -\frac{20}{3} \end{aligned}$$

d
$$\begin{aligned} \frac{2x}{3} - 15 &= 27 \\ \frac{2x}{3} &= 27 + 15 \\ &= 42 \\ \frac{2x}{3} \times \frac{3}{2} &= 42 \times \frac{3}{2} \\ x &= 63 \end{aligned}$$

e
$$\begin{aligned} 5(2x + 4) &= 13 \\ 10x + 20 &= 13 \\ 10x &= 13 - 20 \\ &= -7 \\ x &= -\frac{7}{10} = -0.7 \end{aligned}$$

f
$$\begin{aligned} -3(4 - 5x) &= 24 \\ -12 + 15x &= 24 \\ 15x &= 24 + 12 \\ &= 36 \\ x &= \frac{36}{15} \\ &= \frac{12}{5} = 2.4 \end{aligned}$$

g
$$\begin{aligned} 3x + 5 &= 8 - 7x \\ 3x + 7x &= 8 - 5 \\ 10x &= 3 \\ x &= \frac{3}{10} = 0.3 \end{aligned}$$

h
$$\begin{aligned} 2 + 3(x - 4) &= 4(2x + 5) \\ 2 + 3x - 12 &= 8x + 20 \\ 3x - 10 &= 8x + 20 \\ 3x - 8x &= 20 + 10 \\ -5x &= 30 \\ x &= \frac{30}{-5} = -6 \end{aligned}$$

i
$$\begin{aligned} \frac{2x}{5} - \frac{3}{4} &= 5x \\ \frac{2x}{5} \times 20 - \frac{3}{4} \times 20 &= 5x \times 20 \\ 8x - 15 &= 100x \\ 8x - 100x &= 15 \\ -92x &= 15 \\ x &= -\frac{15}{92} \end{aligned}$$

j
$$\begin{aligned}6x + 4 &= \frac{x}{3} - 3\\ 6x \times 3 + 4 \times 3 &= \frac{x}{3} \times 3 - 3 \times 3\\ 18\ x + 12 &= x - 9\\ 18\ x - x &= -9 - 12\\ 17x &= -21\\ x &= -\frac{21}{17}\end{aligned}$$

d
$$\begin{aligned}\frac{5x}{4} - \frac{4}{3} &= \frac{2x}{5}\\ \frac{5x}{4} \times 60 - \frac{4}{3} \times 60 &= \frac{2x}{5} \times 60\\ 75x - 80 &= 24x\\ 75x - 24x &= 80\\ 51x &= 80\\ x &= \frac{80}{51}\end{aligned}$$

2 a
$$\begin{aligned}\frac{x}{2} + \frac{2x}{5} &= 16\\ \frac{x}{2} \times 10 + \frac{2x}{5} \times 10 &= 16 \times 10\\ 5x + 4x &= 160\\ 9x &= 160\\ x &= \frac{160}{9}\end{aligned}$$

b
$$\begin{aligned}\frac{3x}{4} - \frac{x}{3} &= 8\\ \frac{3x}{4} \times 12 - \frac{x}{3} \times 12 &= 8 \times 12\\ 9x - 4x &= 96\\ 5x &= 96\\ x &= \frac{96}{5} = 19.2\end{aligned}$$

c
$$\begin{aligned}\frac{3x - 2}{2} + \frac{x}{4} &= -8\\ \frac{3x - 2}{2} \times 4 + \frac{x}{4} \times 4 &= -8 \times 4\\ 2(3x - 2) + x &= -32\\ 6x - 4 + x &= -32\\ 7x &= -32 + 4\\ &= -28\\ x &= -4\end{aligned}$$

e
$$\begin{aligned}\frac{x - 4}{2} + \frac{2x + 5}{4} &= 6\\ \frac{x - 4}{2} \times 4 + \frac{2x + 5}{4} \times 4 &= 6 \times 4\\ 2(x - 4) + (2x + 5) &= 24\\ 2x - 8 + 2x + 5 &= 24\\ 4x &= 24 + 8 - 5\\ &= 27\\ x &= \frac{27}{4} = 6.75\end{aligned}$$

f
$$\begin{aligned}\frac{3 - 3x}{10} - \frac{2(x + 5)}{6} &= \frac{1}{20}\\ \frac{3 - 3x}{10} \times 60 - \frac{2(x + 5)}{6} \times 60 &= \frac{1}{20} \times 60\\ 6(3 - 3x) - 20(x + 5) &= 3\\ 18 - 18x - 20x - 100 &= 3\\ -38x &= 3 - 18 + 100\\ &= 85\\ x &= -\frac{85}{38}\end{aligned}$$

g $$\frac{3-x}{4} - \frac{2(x+1)}{5} = -24$$

$$\frac{3-x}{4} \times 20 - \frac{2(x+1)}{5} \times 20 = -24 \times 20$$
$$5(3-x) - 8(x+1) = -480$$
$$15 - 5x - 8x - 8 = -480$$
$$-13x = -480 - 15 + 8$$
$$= -487$$
$$x = \frac{487}{13}$$

h $$\frac{-2(5-x)}{8} + \frac{6}{7} = \frac{4(x-2)}{3}$$

$$\frac{-2(5-x)}{8} \times 168 + \frac{6}{7} \times 168 = \frac{4(x-2)}{3} \times 168$$
$$-42(5-x) + 144 = 224(x-2)$$
$$-210 + 42x + 144 = 224x - 448$$
$$42x - 224x = -448 + 210 - 144$$
$$-182x = -382$$
$$x = \frac{382}{182} = \frac{191}{91}$$

3 a $3x + 2y = 2;\ 2x - 3y = 6$

Use elimination. Multiply the first equation by 3 and the second equation by 2.

$9x + 6y = 6$ ①

$4x - 6y = 12$ ②

① + ②:

$$13x = 18$$
$$x = \frac{18}{13}$$

Substitute into the first equation:

$$3 \times \frac{18}{13} + 2y = 2$$
$$\frac{54}{13} + 2y = 2$$
$$2y = 2 - \frac{54}{13}$$
$$= -\frac{28}{13}$$
$$y = -\frac{14}{13}$$

b $5x + 2y = 4;\ 3x - y = 6$

Use elimination. Multiply the second equation by 2.

$5x + 2y = 4$ ①

$6x - 2y = 12$ ②

① + ②:

$$11x = 16$$
$$x = \frac{16}{11}$$

Substitute into the second, simpler equation:

$$3 \times \frac{16}{11} - y = 6$$
$$\frac{48}{11} - y = 6$$
$$-y = 6 - \frac{48}{11}$$
$$y = -\frac{18}{11}$$

c $2x - y = 7;\ 3x - 2y = 2$

Use substitution. Make y the subject of the first equation.

$y = 2x - 7$

Substitute into the second equation:

$$3x - 2(2x - 7) = 2$$
$$3x - 4x + 14 = 2$$
$$-x = 2 - 14$$
$$x = 12$$

Substitute into the equation in which y is the subject:

$$y = 2 \times 12 - 7$$
$$= 17$$

d $x + 2y = 12;\ x - 3y = 2$

Use substitution. Make x the subject of the first equation.

$x = 12 - 2y$

Substitute into the second equation:

$12 - 2y - 3y = 2$

$$-5y = 2 - 12$$
$$= -10$$
$$y = 2$$

Substitute into the first equation:

$x + 2 \times 2 = 12$

$$x + 4 = 12$$
$$x = 8$$

e $7x - 3y = -6; x + 5y = 10$

Use substitution. Make x the subject of the second equation.

$x = 10 - 5y$

Substitute into the first equation:

$7(10 - 5y) - 3y = -6$

$$70 - 35y - 3y = -6$$
$$-38y = -6 - 70$$
$$= -76$$
$$y = \frac{-76}{-38} = 2$$

Substitute into the second equation:

$x + 5 \times 2 = 10$

$$x + 10 = 10$$
$$x = 0$$

f $15x + 2y = 27; 3x + 7y = 45$

Use elimination. Multiply the second equation by 5.

$$15x + 2y = 27 \qquad ①$$
$$15x + 35y = 225 \qquad ②$$

① − ②:

$$-33y = -198$$
$$y = \frac{-198}{-33} = 6$$

Substitute into the second equation:

$3x + 7 \times 6 = 45$

$$3x + 42 = 45$$
$$3x = 45 - 42$$
$$= 3$$
$$x = 1$$

Solutions to Exercise 1D

1 **a** $4(x-2) = 60$

$$\begin{aligned} 4x - 8 &= 60 \\ 4x &= 60 + 8 \\ &= 68 \\ x &= 17 \end{aligned}$$

b The length of the square is $\dfrac{2x+7}{4}$.

$$\begin{aligned} \left(\frac{2x+7}{4}\right)^2 &= 49 \\ \frac{2x+7}{4} &= 7 \\ 2x + 7 &= 7 \times 4 = 28 \\ 2x &= 28 - 7 = 21 \\ x &= 10.5 \end{aligned}$$

c The equation is length = twice width.

$$\begin{aligned} x - 5 &= 2(12 - x) \\ x - 5 &= 24 - 2x \\ x + 2x &= 24 + 5 \\ 3x &= 29 \\ x &= \frac{29}{3} \end{aligned}$$

d $y = 2((2x+1) + (x-3))$

$$\begin{aligned} &= 2(2x + 1 + x - 3) \\ &= 2(3x - 2) \\ &= 6x - 4 \end{aligned}$$

e $Q = np$

f If a 10% service charge is added, the total price will be multiplied by 110%, or 1.1.
$R = 1.1pS$

g Using the fact that there are 12 lots of 5 min in an hour ($60 \div 12 = 5$),
$\dfrac{60n}{5} = 2400$

h $a = \text{circumference} \times \dfrac{60}{360}$

$$\begin{aligned} &= 2\pi(x+3) \times \frac{60}{360} \\ &= 2\pi(x+3) \times \frac{1}{6} \\ &= \frac{\pi}{3}(x+3) \end{aligned}$$

2 Let the value of Bronwyn's sales in the first week be $\$s$.

$$\begin{aligned} s + (s + 500) + (s + 1000) & \\ + (s + 1500) + (s + 2000) & \\ = 17500 & \\ 5s + 5000 &= 17500 \\ 5s &= 12500 \\ s &= 2500 \end{aligned}$$

The value of her first week's sales is \$2500.

3 Let d be the number of dresses bought and h the number of handbags bought.
$65d + 26h = 598$

$d + h = 11$

Multiply the second equation by 26 (the smaller number).

$65d + 26h = 598$ ①

$26d + 26h = 286$ ②

① − ②:

$$39d = 312$$

$$d = \frac{312}{39} = 8$$

$$h + 8 = 11$$

$$h = 3$$

Eight dresses and three handbags.

4 Let the courtyard's width be w metres.

$$3w + w + 3w + w = 67$$

$$8w = 67$$

$$w = 8.375$$

The width is 8.375 m.

The length = $3 \times 8.375 = 25.125$ m.

5 Let p be the full price of a case of wine. The merchant will pay 60% (0.6) on the 25 discounted cases.

$$25p + 25 \times 0.6p = 2260$$

$$25p + 15p = 2260$$

$$40p = 2260$$

$$p = 56.5$$

The full price of a case is \$56.50.

6 Let x be the number of houses with an \$11 500 commission and y be the number of houses with a \$13 000 commission.

We only need to find x.

$$x + y = 22$$

$$11\,500x + 13\,000y = 272\,500$$

To simplify the second equation, divide both sides by 500.

$$23x + 26y = 545$$

Using the substitution method:

$$23x + 26y = 545$$

$$y = 22 - x$$

$$23x + 26(22 - x) = 545$$

$$23x + 572 - 26x = 545$$

$$-3x = 545 - 572$$

$$= -27$$

$$x = 9$$

He sells nine houses with an \$11 500 commission.

7 It is easiest to let the third boy have m marbles, in which case the second boy will have $2m$ marbles and the first boy will have $2m - 14$.

$$(2m - 14) + 2m + m = 71$$

$$5m - 14 = 71$$

$$5m = 85$$

$$m = 17$$

The first boy has 20 marbles, the second boy has 34 and the third boy has 17 marbles, for a total of 71.

8 Let Belinda's score be b.

Annie's score will be 110% of Belinda's or $1.1b$.

Cassie's will be 60% of their combined scores:

$$0.6(1.1b + b) = 0.6 \times 2.1b$$

$$= 1.26b$$

$$1.1b + b + 1.26b = 504$$

$$3.36b = 504$$

$$b = \frac{5.04}{3.36}$$

$$= 150$$

Belinda scores 150
Annie scores $1.1 \times 150 = 165$
Cassie scores $0.6 \times (150 + 165) = 189$

9 Let r km/h be the speed Kim can run. Her cycling speed will be $(r + 30)$ km/h. Her time cycling will be $48 + 48 \div 3 = 64$ min. Converting the times to hours ($\div 60$) and using distance = speed × time gives the following equation:

$$r \times \frac{48}{60} + (r + 30) \times \frac{64}{60} = 60$$
$$48r + 64(r + 30) = 60 \times 60$$
$$48r + 64r + 1920 = 3600$$
$$112r + 1920 = 3600$$
$$112r = 1680$$
$$r = \frac{1680}{112} = 15$$

She can run at 15 km/h

10 Let c g be the mass of a carbon atom and x g be the mass of an oxygen atom.
(o is too confusing a symbol to use)
$2c + 6x = 2.45 \times 10^{-22}$
$$x = \frac{c}{3}$$
Use substitution.
$$2c + 6 \times \frac{c}{3} = 2.45 \times 10^{-22}$$
$$2c + 2c = 2.45 \times 10^{-22}$$
$$4c = 2.45 \times 10^{-22}$$
$$c = \frac{2.45 \times 10^{-22}}{4}$$
$$= 6.125 \times 10^{-23}$$
$$x = \frac{c}{3}$$
$$= \frac{6.125 \times 10^{-23}}{3}$$
$$\approx 2.04 \times 10^{-23}$$
The mass of an oxygen atom is 2.04×10^{-23} g.

11 Let x be the number of pearls.
$$\frac{x}{6} + \frac{x}{3} + \frac{x}{5} + 9 = x$$
$$\frac{5x + 10x + 6x}{30} + 9 = x$$
$$21x + 270 = 30x$$
$$7x + 90 = 10x$$
$$3x = 90$$
$$x = 30$$
There are 30 pearls.

12 Let the oldest receive $\$x$.
The middle child receives $\$(x - 12)$.
The youngest child receives $\$\left(\frac{x-12}{3}\right)$
$$x + x - 12 + \frac{x - 12}{3} = 96$$
$$2x - 12 + \frac{x - 12}{3} = 96$$
$$2x - 12 + \frac{x}{3} = 100$$
$$6x - 36 + x = 300$$
$$7x = 336$$
$$x = 48$$

Oldest \$48, Middle \$35, Youngest \$12

13 Let S be the sum of her marks on the first four tests.

Then $\frac{S}{4} = 88$

$\therefore S = 352$

Let x be her mark on the fifth test.

$\frac{S + x}{5} = 90$

$352 + x = 450$

$x = 98$

Her mark on the fifth test has to be 98%

14 Let N be the number of students in the class.

$0.72N$ students have black hair

After 5 leave the class there are $0.72N - 5$ students with black hair.

There are now $N - 5$ students in the class.

Hence $\frac{0.72N - 5}{N - 5} = 0.65$

$$\therefore 0.72N - 5 = 0.65(N - 5)$$
$$\therefore 0.72N = 0.65N + 1.75$$
$$\therefore 0.07N = 1.75$$
$$7N = 175$$
$$N = 25$$

There were originally 25 students

15 Amount of water in tank A at time t minutes $= 100 - 2t$

Amount of water in tank B at time t minutes $= 120 - 3t$

$100 - 2t = 120 - 3t$

$t = 20$

After 20 minutes the amount of water in the tanks will be the same.

16 Height of candle A at t minutes $= 10 - 5t$

Height of candle B at t minutes $= 8 - 2t$

a $10 - 5t = 8 - 2t$

$3t = 2$

$t = \frac{2}{3}$

$\therefore$ equal after 40 minutes.

b $10 - 5t = \frac{1}{2}(8 - 2t)$

$10 - 5t = 4 - t$

$4t = 6$

$t = \frac{3}{2}$

$\therefore$ half the height after 90 minutes.

c $10 - 5t = 8 - 2t + 1$

$10 - 5t = 9 - 2t$

$3t = 1$

$t = \frac{1}{3}$

$\therefore$ one centimetre more after 20 minutes.

17 Let t be the time the motorist drove at 100 km/h

$100t + 90(\frac{10}{3} - t) = 320$

$100t + 300 - 90t = 320$

$10t = 20$

$t = 2$

Therefore the motorist travelled 200 km at 100 km/h

18 Let v km/h be Jarmila's usual speed

Therefore distance travelled= $\frac{14v}{3}$ km

$v + 3$ is the new speed and it takes $\frac{13}{3}$ hours.

$$\therefore \frac{13}{3}(v+3) = \frac{14v}{3}$$

$$13(v+3) = 14v$$

$$v = 39$$

Her usual speed is 39 km/h

Solutions to Exercise 1E

1 Let k be the number of kilometres travelled in a day. The unlimited kilometre alternative will become more attractive when $0.32k + 63 > 108$.
Solve for $0.32k + 63 = 108$:

$$\begin{aligned}0.32k &= 108 - 63\\ &= 45\\ k &= \frac{45}{0.32} = 140.625\end{aligned}$$

The unlimited kilometre alternative will become more attractive when you travel more than 140.625 km.

2 Let g be the number of guests. Solve for the equality.

$$\begin{aligned}300 + 43g &= 450 + 40g\\ 43g - 40g &= 450 - 300\\ 3g &= 150\\ g &= 50\end{aligned}$$

Company A is cheaper when there are more than 50 guests.

3 Let a be the number of adults and c the number of children.

$$\begin{aligned}45a + 15c &= 525\,000\\ a + c &= 15\,000\end{aligned}$$

Multiply the second equation by 15.

$$\begin{aligned}45a + 15c &= 525\,000 \quad ①\\ 15a + 15c &= 225\,000 \quad ②\end{aligned}$$

① − ②:

$$\begin{aligned}30a &= 300\,000\\ a &= 10\,000\end{aligned}$$

10 000 adults and 5000 children bought tickets.

4 Let $\$m$ be the amount the contractor paid a man and $\$b$ the amount he paid a boy.

$$\begin{aligned}8m + 3b &= 2240\\ 6m + 18b &= 4200\end{aligned}$$

Multiply the first equation by 6.

$$\begin{aligned}48m + 18b &= 13\,440 \quad ①\\ 6m + 18b &= 4200 \quad ②\end{aligned}$$

① − ②:

$$\begin{aligned}42m &= 9240\\ m &= 220\end{aligned}$$

Substitute into the first equation:

$$\begin{aligned}8 \times 220 + 3b &= 2240\\ 1760 + 3b &= 2240\\ 3b &= 2240 - 1760\\ &= 480\\ b &= 160\end{aligned}$$

He paid the men \$220 each and the boys \$160.

5 Let the numbers be x and y.

$$\begin{aligned}x + y &= 212 \quad ①\\ x - y &= 42 \quad ②\end{aligned}$$

① + ②:

$$\begin{aligned}2x &= 254\\ x &= 127\\ 127 + y &= 212\\ y &= 85\end{aligned}$$

The numbers are 127 and 85.

6 Let x L be the amount of 40% solution and y L be the amount of 15% solution. Equate the actual substance.

$$0.4x + 0.15y = 0.24 \times 700$$
$$= 168$$
$$x + y = 700$$

Multiply the second equation by 0.15.

$$0.4x + 0.15y = 168 \quad ①$$
$$0.15x + 0.15y = 105 \quad ②$$

① − ②:

$$0.25x = 63$$
$$x = 63 \times 4$$
$$= 252$$
$$252 + y = 700$$
$$y = 448$$

Use 252 L of 40% solution and 448 L of 15% solution.

7 Form two simultaneous equations.

$$x + y = 220 \quad ①$$
$$x - \frac{x}{2} = y - 40$$
$$\frac{x}{2} - y = -40 \quad ②$$

① + ②:

$$\frac{3x}{2} = 180$$
$$x = 120$$
$$120 + y = 220$$
$$y = 100$$

They started with 120 and 100 marbles and ended with 60 each.

8 Let $\$x$ be the amount initially invested at 10% and $\$y$ the amount initially invested at 7%. This earns \$31 000.
$0.1x + 0.07y = 31\,000$
When the amounts are interchanged, she earns \$1000 more, i.e. \$32 000.
$0.07x + 0.1y = 32\,000$
Multiply the first equation by 100 and the second equation by 70.

$$10x + 7y = 3\,100\,000 \quad ①$$
$$4.9x + 7y = 2\,240\,000 \quad ②$$

① − ②:

$$5.1x = 860\,000$$
$$x = \frac{860\,000}{5.1} \approx 168\,627.451$$

$$10 \times 168\,627.451 + 7y = 3\,100\,000$$
$$1\,686\,274.51 + 7y = 3\,100\,000$$
$$7y = 1\,413\,725.49$$
$$y = 201\,960.78$$

The total amount invested is
$x + y = 168\,627.45 + 201\,960.78$
$= \$370\,588.23$
$= \$370\,588$
correct to the nearest dollar.

9 Let a be the number of adults and s the number of students who attended.

$$30a + 20s = 37\,000 \quad ①$$
$$a + s = 1600$$
$$20a + 20s = 1600 \times 20$$
$$= 32\,000 \quad ②$$

① − ②:

$$10a = 5000$$
$$a = 500$$
$$500 + s = 1600$$
$$s = 1100$$

500 adults and 1100 students attended the concert.

10 There were $\frac{12 \times 11}{2} = 66$ matches. Let x be the number of wins and y the number of draws.

$x + y = 66 \ldots (1)$ and

$3x + 2y = 180 \ldots (2)$

Multiply (1) by 2.

$2x + 2y = 132 \ldots (1')$

Subtract (1') from (2)

$x = 48$

Therefore $y = 18$

Solutions to Exercise 1F

1 a $v = u + at$

$= 15 + 2 \times 5$

$= 25$

b $I = \dfrac{PrT}{100}$

$= \dfrac{600 \times 5.5 \times 10}{100}$

$= 330$

c $V = \pi r^2 h$

$= \pi \times 4.25^2 \times 6$

≈ 340.47

d $S = 2\pi r(r + h)$

$= 2\pi \times 10.2 \times (10.2 + 15.6)$

≈ 1653.48

e $V = \dfrac{4}{3}\pi r^2 h$

$= \dfrac{4\pi \times 3.58^2 \times 11.4}{3}$

≈ 612.01

f $s = ut + \dfrac{1}{2}at^2$

$= 25.6 \times 3.3 + \dfrac{1}{2} \times -1.2 \times 3.3^2$

≈ 77.95

g $T = 2\pi\sqrt{\dfrac{l}{g}}$

$= 2\pi \times \sqrt{\dfrac{1.45}{9.8}}$

$= 2\pi \times 0.3846\ldots$

≈ 2.42

h $\dfrac{1}{f} = \dfrac{1}{v} + \dfrac{1}{u}$

$= \dfrac{1}{3} + \dfrac{1}{7} = \dfrac{10}{21}$

$f = \dfrac{21}{10}$

$= 2.1$

i $c^2 = a^2 + b^2$

$= 8.8^2 + 3.4^2$

$= 89$

$c = \sqrt{89}$

≈ 9.43

j $v^2 = u^2 + 2as$

$= 4.8^2 + 2 \times 2.25 \times 13.6$

$= 91.04$

$v = \sqrt{91.04}$

≈ 9.54

2 a $v = u + at$

$v - u = at$

$\therefore \quad a = \dfrac{v - u}{t}$

b $S = \dfrac{n}{2}(a + l)$

$2S = n(a + l)$

$a + l = \dfrac{2S}{n}$

$\therefore \quad l = \dfrac{2S}{n} - a$

c $$A = \frac{1}{2}bh$$
$$2A = bh$$
$$\therefore b = \frac{2A}{h}$$

d $$P = I^2R$$
$$\frac{P}{R} = I^2$$
$$\therefore I = \pm\sqrt{\frac{P}{R}}$$

e $$s = ut + \frac{1}{2}at^2$$
$$s - ut = \frac{1}{2}at^2$$
$$2(s - ut) = at^2$$
$$\therefore \quad a = \frac{2(s - ut)}{t^2}$$

f $$E = \frac{1}{2}mv^2$$
$$2E = mv^2$$
$$v^2 = \frac{2E}{m}$$
$$\therefore v = \pm\sqrt{\frac{2E}{m}}$$

g $$Q = \sqrt{2gh}$$
$$Q^2 = 2gh$$
$$\therefore h = \frac{Q^2}{2g}$$

h $$-xy - z = xy + z$$
$$-xy - xy = z + z$$
$$-2xy = 2z$$
$$\therefore \quad x = \frac{2z}{-2y}$$
$$= -\frac{z}{y}$$

i $$\frac{ax + by}{c} = x - b$$
$$ax + by = c(x - b)$$
$$ax + by = cx - bc$$
$$ax - cx = -bc - by$$
$$x(a - c) = -b(c + y)$$
$$\therefore \quad x = \frac{-b(c + y)}{a - c}$$
$$= \frac{b(c + y)}{c - a}$$

j $$\frac{mx + b}{x - b} = c$$
$$mx + b = c(x - b)$$
$$mx + b = cx - bc$$
$$mx - cx = -bc - b$$
$$x(m - c) = -b(c + 1)$$
$$\therefore \quad x = \frac{-b(c + 1)}{m - c}$$

3 a $$F = \frac{9C}{5} + 32$$
$$= \frac{9 \times 28}{5} + 32$$
$$= 82.4^\circ$$

b $$F = \frac{9C}{5} + 32$$
$$F - 32 = \frac{9C}{5}$$
$$9C = 5(F - 32)$$
$$\therefore \quad C = \frac{5(F - 32)}{9}$$

Substitute $F = 135$.
$$C = \frac{5(135 - 32)}{9}$$
$$= \frac{515}{9}$$
$$\approx 57.22^\circ$$

4 a $S = 180(n-2)$

$= 180(8-2)$

$= 1080°$

b $S = 180(n-2)$

$\frac{S}{180} = n - 2$

$\therefore \quad n = \frac{S}{180} + 2$

$= \frac{1260}{180} + 2$

$= 7 + 2 = 9$

Polygon has 9 sides (a nonagon).

5 a $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 3.5^2 \times 9$

$\approx 115.45 \text{ cm}^3$

b $V = \frac{1}{3}\pi^2 h$

$3V = \pi r^2 h$

$\therefore h = \frac{3V}{\pi r^2}$

$= \frac{3 \times 210}{\pi 4^2}$

≈ 12.53 cm

c $V = \frac{1}{3}\pi r^2 h$

$3V = \pi r^2 h$

$r^2 = \frac{3V}{\pi h}$

$\therefore \quad r = \sqrt{\frac{3V}{\pi h}}$

$= \sqrt{\frac{3 \times 262}{\pi \times 10}}$

≈ 5.00 cm

6 a $S = \frac{n}{2}(a + l)$

$= \frac{7}{2}(-3 + 22)$

$= 66.5$

b $S = \frac{n}{2}(a + l)$

$2S = n(a + l)$

$\frac{2S}{n} = a + l$

$\therefore a = \frac{2S}{n} - l$

$= \frac{2 \times 1040}{13} - 156$

$= 4$

c $S = \frac{n}{2}(a + l)$

$2S = n(a + l)$

$\therefore n = \frac{2S}{a + l}$

$= \frac{2 \times 110}{25 + -5}$

$= 11$

There are 11 terms.

Solutions to Exercise 1G

1 a $\dfrac{2x}{3} + \dfrac{3x}{2} = \dfrac{4x + 9x}{6}$
$= \dfrac{13x}{6}$

b $\dfrac{3a}{2} - \dfrac{a}{4} = \dfrac{6a - a}{4}$
$= \dfrac{5a}{4}$

c $\dfrac{3h}{4} + \dfrac{5h}{8} - \dfrac{3h}{2} = \dfrac{6h + 5h - 12h}{8}$
$= -\dfrac{h}{8}$

d $\dfrac{3x}{4} - \dfrac{y}{6} - \dfrac{x}{3} = \dfrac{9x - 2y - 4x}{12}$
$= \dfrac{5x - 2y}{12}$

e $\dfrac{3}{x} + \dfrac{2}{y} = \dfrac{3y + 2x}{xy}$

f $\dfrac{5}{x - 1} + \dfrac{2}{x} = \dfrac{5x + 2(x - 1)}{x(x - 1)}$
$= \dfrac{5x + 2x - 2}{x(x - 1)}$
$= \dfrac{7x - 2}{x(x - 1)}$

g $\dfrac{3}{x - 2} + \dfrac{2}{x + 1} = \dfrac{3(x + 1) + 2(x - 2)}{(x - 2)(x + 1)}$
$= \dfrac{3x + 3 + 2x - 4}{(x - 2)(x + 1)}$
$= \dfrac{5x - 1}{(x - 2)(x + 1)}$

h $\dfrac{2x}{x + 3} - \dfrac{4x}{x - 3} - \dfrac{3}{2}$
$= \dfrac{4x(x - 3) - 8x(x + 3) - 3(x + 3)(x - 3)}{2(x + 3)(x - 3)}$
$= \dfrac{4x^2 - 12x - 8x^2 - 24x - 3(x^2 - 9)}{2(x + 3)(x - 3)}$
$= \dfrac{4x^2 - 12x - 8x^2 - 24x - 3x^2 + 27}{2(x + 3)(x - 3)}$
$= \dfrac{-7x^2 - 36x + 27}{2(x + 3)(x - 3)}$

i $\dfrac{4}{x + 1} + \dfrac{3}{(x + 1)^2} = \dfrac{4(x + 1) + 3}{(x + 1)^2}$
$= \dfrac{4x + 4 + 3}{(x + 1)^2}$
$= \dfrac{4x + 7}{(x + 1)^2}$

j $\dfrac{a - 2}{a} + \dfrac{a}{4} + \dfrac{3a}{8}$
$= \dfrac{8(a - 2) + 2a^2 + 3a^2}{8a}$
$= \dfrac{5a^2 + 8a - 16}{8a}$

k $2x - \dfrac{6x^2 - 4}{5x} = \dfrac{10x^2 - (6x^2 - 4)}{5x}$
$= \dfrac{10x^2 - 6x^2 + 4}{5x}$
$= \dfrac{4x^2 + 4}{5x}$
$= \dfrac{4(x^2 + 1)}{5x}$

l $\dfrac{2}{x+4} - \dfrac{3}{x^2+8x+16}$

$= \dfrac{2}{x+4} - \dfrac{3}{(x+4)^2}$

$= \dfrac{2(x+4)-3}{(x+4)^2}$

$= \dfrac{2x+8-3}{(x+4)^2}$

$= \dfrac{2x+5}{(x+4)^2}$

m $\dfrac{3}{x-1} + \dfrac{2}{(x-1)(x+4)}$

$= \dfrac{3(x+4)+2}{(x-1)(x+4)}$

$= \dfrac{3x+12+2}{(x-1)(x+4)}$

$= \dfrac{3x+14}{(x-1)(x+4)}$

n $\dfrac{3}{x-2} - \dfrac{2}{x+2} + \dfrac{4}{x^2-4}$

$= \dfrac{3}{x-2} - \dfrac{2}{x+2} + \dfrac{4}{(x-2)(x+2)}$

$= \dfrac{3(x+2)-2(x-2)+4}{(x-2)(x+2)}$

$= \dfrac{3x+6-2x+4+4}{(x-2)(x+2)}$

$= \dfrac{x+14}{(x-2)(x+2)}$

o

$\dfrac{5}{x-2} + \dfrac{3}{x^2+5x+6} + \dfrac{2}{x+3}$

$= \dfrac{5}{x-2} + \dfrac{3}{(x+2)(x+3)} + \dfrac{2}{x+3}$

$= \dfrac{5(x+3)(x+2)+3(x-2)+2(x-2)(x+2)}{(x-2)(x+2)(x+3)}$

$= \dfrac{5(x^2+5x+6)+3x-6+2(x^2-4)}{(x-2)(x+2)(x+3)}$

$= \dfrac{5x^2+25x+30+3x-6+2x^2-8}{(x-2)(x+2)(x+3)}$

$= \dfrac{7x^2+28x+16}{(x-2)(x+2)(x+3)}$

p $x-y-\dfrac{1}{x-y} = \dfrac{(x-y)(x-y)-1}{x-y}$

$= \dfrac{(x-y)^2-1}{x-y}$

q $\dfrac{3}{x-1} - \dfrac{4x}{1-x} = \dfrac{3}{x-1} + \dfrac{4x}{x-1}$

$= \dfrac{4x+3}{x-1}$

r $\dfrac{3}{x-2} + \dfrac{2}{2-x} = \dfrac{3}{x-2} - \dfrac{2x}{x-2}$

$= \dfrac{3-2x}{x-2}$

2 **a** $\dfrac{x^2}{2y} \times \dfrac{4y^3}{x} = \dfrac{4y^3x^2}{2yx}$

$= 2xy^2$

b $\dfrac{3x^2}{4y} \times \dfrac{y^2}{6x} = \dfrac{3x^2y^2}{24yx}$

$= \dfrac{xy}{8}$

c $\dfrac{4x^3}{3} \times \dfrac{12}{8x^4} = \dfrac{48x^3}{24x^4}$

$= \dfrac{2}{x}$

d $\dfrac{x^2}{2y} \div \dfrac{3xy}{6} = \dfrac{x^2}{2y} \times \dfrac{6}{3xy}$

$= \dfrac{6x^2}{6xy^2}$

$= \dfrac{x}{y^2}$

e $\dfrac{4-x}{3a} \times \dfrac{a^2}{4-x} = \dfrac{a^2(4-x)}{3a(4-x)}$

$= \dfrac{a}{3}$

f $\dfrac{2x+5}{4x^2+10x} = \dfrac{2x+5}{2x(2x+5)}$

$= \dfrac{1}{2x}$

g $\dfrac{(x-1)^2}{x^2+3x-4} = \dfrac{(x-1)^2}{(x-1)(x+4)}$

$= \dfrac{x-1}{x+4}$

h $\dfrac{x^2-x-6}{x-3} = \dfrac{(x-3)(x+2)}{x-3}$

$= x+2$

i $\dfrac{x^2-5x+4}{x^2-4x} = \dfrac{(x-1)(x-4)}{x(x-4)}$

$= \dfrac{x-1}{x}$

j $\dfrac{5a^2}{12b^2} \div \dfrac{10a}{6b} = \dfrac{5a^2}{12b^2} \times \dfrac{6b}{10a}$

$= \dfrac{30a^2b}{120ab^2}$

$= \dfrac{a}{4b}$

k $\dfrac{x-2}{x} \div \dfrac{x^2-4}{2x^2}$

$= \dfrac{x-2}{x} \times \dfrac{2x^2}{x^2-4}$

$= \dfrac{x-2}{x} \times \dfrac{2x^2}{(x-2)(x+2)}$

$= \dfrac{2x^2}{x(x+2)}$

$= \dfrac{2x}{x+2}$

l $\dfrac{x+2}{x(x-3)} \div \dfrac{4x+8}{x^2-4x+3}$

$= \dfrac{x+2}{x(x-3)} \div \dfrac{4(x+2)}{(x-1)(x-3)}$

$= \dfrac{x+2}{x(x-3)} \times \dfrac{(x-1)(x-3)}{4(x+2)}$

$= \dfrac{1}{x} \times \dfrac{x-1}{4}$

$= \dfrac{x-1}{4x}$

m

$\dfrac{2x}{x-1} \div \dfrac{4x^2}{x^2-1} = \dfrac{2x}{x-1} \times \dfrac{x^2-1}{4x^2}$

$= \dfrac{2x}{x-1} \times \dfrac{(x-1)(x+1)}{4x^2}$

$= \dfrac{2x(x+1)}{4x^2}$

$= \dfrac{x+1}{2x}$

n $\dfrac{x^2-9}{x+2} \times \dfrac{3x+6}{x-3} \div \dfrac{9}{x}$

$= \dfrac{(x-3)(x+3)}{x+2} \times \dfrac{3(x+2)}{x-3} \times \dfrac{x}{9}$

$= \dfrac{3x(x-3)(x+3)(x+2)}{9(x+2)(x-3)}$

$= \dfrac{x(x+3)}{3}$

o $\dfrac{3x}{9x-6} \div \dfrac{6x^2}{x-2} \times \dfrac{2}{x+5}$

$= \dfrac{3x}{3(3x-2)} \times \dfrac{x-2}{6x^2} \times \dfrac{2}{x+5}$

$= \dfrac{2x(x-2)}{6x^2(3x-2)(x+5)}$

$= \dfrac{x-2}{3x(3x-2)(x+5)}$

3 a $\dfrac{1}{x-3} + \dfrac{2}{x-3} = \dfrac{3}{x-3}$

b $\dfrac{2}{x-4}+\dfrac{2}{x-3}=\dfrac{2(x-3)+2(x-4)}{(x-4)(x-3)}$

$=\dfrac{2x-6+2x-8}{x^2-7x+12}$

$=\dfrac{4x-14}{x^2-7x+12}$

c $\dfrac{3}{x+4}+\dfrac{2}{x-3}=\dfrac{3(x-3)+2(x+4)}{(x+4)(x-3)}$

$=\dfrac{3x-9+2x+8}{x^2+x-12}$

$=\dfrac{5x-1}{x^2+x-12}$

d $\dfrac{2x}{x-3}+\dfrac{2}{x+4}=\dfrac{2x(x+4)+2(x-3)}{(x-3)(x+4)}$

$=\dfrac{2x^2+8x+2x-6}{x^2+x-12}$

$=\dfrac{2x^2+10x-6}{x^2+x-12}$

e $\dfrac{1}{(x-5)^2}+\dfrac{2}{x-5}=\dfrac{1+2(x-5)}{(x-5)^2}$

$=\dfrac{1+2x-10}{x^2-10x+25}$

$=\dfrac{2x-9}{x^2-10x+25}$

f $\dfrac{3x}{(x-4)^2}+\dfrac{2}{x-4}=\dfrac{3x+2(x-4)}{(x-4)^2}$

$=\dfrac{3x+2x-8}{x^2-8x+16}$

$=\dfrac{5x-8}{x^2-8x+16}$

g $\dfrac{1}{x-3}-\dfrac{2}{x-3}=\dfrac{-1}{x-3}$

$=\dfrac{1}{3-x}$

h $\dfrac{2}{x-3}-\dfrac{5}{x+4}=\dfrac{2(x+4)-5(x-3)}{(x-3)(x+4)}$

$=\dfrac{2x+8-5x+15}{x^2+x-12}$

$=\dfrac{23-3x}{x^2+x-12}$

i

$\dfrac{2x}{x-3}+\dfrac{3x}{x+3}=\dfrac{2x(x+3)+3x(x-3)}{(x-3)(x+3)}$

$=\dfrac{2x^2+6x+3x^2-9x}{x^2-9}$

$=\dfrac{5x^2-3x}{x^2-9}$

j $\dfrac{1}{(x-5)^2}-\dfrac{2}{x-5}=\dfrac{1-2(x-5)}{(x-5)^2}$

$=\dfrac{1-2x+10}{x^2-10x+25}$

$=\dfrac{11-2x}{x^2-10x+25}$

k $\dfrac{2x}{(x-6)^3}-\dfrac{2}{(x-6)^2}=\dfrac{2x-2(x-6)}{(x-6)^3}$

$=\dfrac{2x-2x+12}{(x-6)^3}$

$=\dfrac{12}{(x-6)^3}$

l

$\dfrac{2x+3}{x-4}-\dfrac{2x-4}{x-3}$

$=\dfrac{(2x+3)(x-3)-(2x-4)(x-4)}{(x-4)(x-3)}$

$=\dfrac{(2x^2-3x-9)-(2x^2-12x+16)}{x^2-7x+12}$

$=\dfrac{2x^2-3x-9-2x^2+12x-16}{x^2-7x+12}$

$=\dfrac{9x-25}{x^2-7x+12}$

4 a $\sqrt{1-x}+\dfrac{2}{\sqrt{1-x}}$

$=\dfrac{\sqrt{1-x}\sqrt{1-x}+2}{\sqrt{1-x}}$

$=\dfrac{1-x+2}{\sqrt{1-x}}$

$=\dfrac{3-x}{\sqrt{1-x}}$

b $\dfrac{2}{\sqrt{x-4}}+\dfrac{2}{3}=\dfrac{2\sqrt{x-4}+6}{3\sqrt{x-4}}$

c $\dfrac{3}{\sqrt{x+4}}+\dfrac{2}{\sqrt{x+4}}=\dfrac{5}{\sqrt{x+4}}$

d $\dfrac{3}{\sqrt{x+4}}+\sqrt{x+4}$

$=\dfrac{3+\sqrt{x+4}\sqrt{x+4}}{\sqrt{x+4}}$

$=\dfrac{3+x+4}{\sqrt{x+4}}$

$=\dfrac{x+7}{\sqrt{x+4}}$

e $\dfrac{3x^3}{\sqrt{x+4}}-3x^2\sqrt{x+4}$

$=\dfrac{3x^3-3x^2\sqrt{x+4}\sqrt{x+4}}{\sqrt{x+4}}$

$=\dfrac{3x^3-3x^2(x+4)}{\sqrt{x+4}}$

$=\dfrac{3x^3-3x^3-12x^2}{\sqrt{x+4}}$

$=-\dfrac{12x^2}{\sqrt{x+4}}$

f $\dfrac{3x^3}{2\sqrt{x+3}}+3x^2\sqrt{x+3}$

$=\dfrac{3x^3+6x^2\sqrt{x+3}\sqrt{x+3}}{2\sqrt{x+3}}$

$=\dfrac{3x^3+6x^2(x+3)}{2\sqrt{x+3}}$

$=\dfrac{3x^3+6x^3+18x^2}{2\sqrt{x+3}}$

$=\dfrac{9x^3+18x^2}{2\sqrt{x+3}}$

$=\dfrac{9x^2(x+2)}{2\sqrt{x+3}}$

5 a $(6x-3)^{\frac{1}{3}}-(6x-3)^{-\frac{2}{3}}$

$=(6x-3)^{\frac{1}{3}}-\dfrac{1}{(6x-3)^{\frac{2}{3}}}$

$=\dfrac{(6x-3)^{\frac{1}{3}}(6x-3)^{\frac{2}{3}}-1}{(6x-3)^{\frac{2}{3}}}$

$=\dfrac{6x-3-1}{(6x-3)^{\frac{2}{3}}}$

$=\dfrac{6x-4}{(6x-3)^{\frac{2}{3}}}$

b $(2x+3)^{\frac{1}{3}}-2x(2x+3)^{-\frac{2}{3}}$

$=(2x+3)^{\frac{1}{3}}-\dfrac{2x}{(2x+3)^{\frac{2}{3}}}$

$=\dfrac{(2x+3)^{\frac{1}{3}}(2x+3)^{\frac{2}{3}}-2x}{(2x+3)^{\frac{2}{3}}}$

$=\dfrac{2x+3-2x}{(2x+3)^{\frac{2}{3}}}$

$=\dfrac{3}{(2x+3)^{\frac{2}{3}}}$

c $(3-x)^{\frac{1}{3}} - 2x(3-x)^{-\frac{2}{3}}$

$$= (3-x)^{\frac{1}{3}} - \frac{2x}{(3-x)^{\frac{2}{3}}}$$

$$= \frac{(3-x)^{\frac{1}{3}}(3-x)^{\frac{2}{3}} - 2x}{(3-x)^{\frac{2}{3}}}$$

$$= \frac{3-x-2x}{(3-x)^{\frac{2}{3}}}$$

$$= \frac{3-3x}{(3-x)^{\frac{2}{3}}}$$

Since $(3-x)^2 = (x-3)^2$, the answer is equivalent to $\dfrac{3-3x}{(x-3)^{\frac{2}{3}}}$.

Solutions to Exercise 1H

1 a $ax + n = m$

$$ax = m - n$$
$$x = \frac{m-n}{a}$$

b $ax + b = bx$

$$ax - bx = -b$$
$$x(a-b) = -b$$
$$x = \frac{-b}{a-b}$$

This answer is correct, but to avoid a negative sign, multiply numerator and denominator by -1.

$$x = \frac{-b}{a-b} \times \frac{-1}{-1}$$
$$= \frac{b}{b-a}$$

c $\frac{ax}{b} + c = 0$

$$\frac{ax}{b} = -c$$
$$ax = -bc$$
$$x = -\frac{bc}{a}$$

d $px = qx + 5$

$$px - qx = 5$$
$$x(p-q) = 5$$
$$x = \frac{5}{p-q}$$

e $mx + n = nx - m$

$$mx - nx = -m - n$$
$$x(m-n) = -m - n$$
$$x = \frac{-m-n}{m-n}$$
$$= \frac{m+n}{n-m}$$

f $\frac{1}{x+a} = \frac{b}{x}$

Take reciprocals of both sides:

$$x + a = \frac{x}{b}$$
$$x - \frac{x}{b} = -a$$
$$\frac{x}{b} - x = a$$
$$\frac{x - xb}{b} = a$$
$$\frac{x - xb}{b} \times b = ab$$
$$x - xb = ab$$
$$x(1-b) = ab$$
$$x = \frac{ab}{1-b}$$

g $\frac{b}{x-a} = \frac{2b}{x+a}$

Take reciprocals of both sides:

$$\frac{x-a}{b} = \frac{x+a}{2b}$$
$$\frac{x-a}{b} \times 2b = \frac{x+a}{2b} \times 2b$$
$$2(x-a) = x + a$$
$$2x - 2a = x + a$$
$$2x - x = a + 2a$$
$$x = 3a$$

h

$$\frac{x}{m} + n = \frac{x}{n} + m$$
$$\frac{x}{m} \times mn + n \times mn = \frac{x}{n} \times mn + m \times mn$$
$$nx + mn^2 = mx + m^2n$$
$$nx - mx = m^2n - mn^2$$
$$x(n-m) = mn(m-n)$$
$$x = \frac{mn(m-n)}{n-m}$$

Note that $n - m = -m + n$

$$= -1(m - n)$$

$$\therefore \quad x = \frac{-mn(n - m)}{n - m}$$

$$= -mn$$

i $-b(ax + b) = a(bx - a)$

$$-abx - b^2 = abx - a^2$$

$$-abx - abx = -a^2 + b^2$$

$$-2abx = -a^2 + b^2$$

$$x = -\frac{(-a^2 + b^2)}{2ab}$$

$$= \frac{a^2 - b^2}{2ab}$$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

$$p^2 - p^2x - 2pqx = q^2 + q^2x$$

$$-p^2x - 2pqx - q^2x = q^2 - p^2$$

$$-x(p^2 + 2pq + q^2) = q^2 - p^2$$

$$x = \frac{-(q^2 - p^2)}{p^2 + 2pq + q^2}$$

$$= \frac{p^2 - q^2}{(p + q)^2}$$

$$= \frac{(p - q)(p + q)}{(p + q)^2}$$

$$= \frac{p - q}{p + q}$$

k $\frac{x}{a} - 1 = \frac{x}{b} + 2$

$$\frac{x}{a} \times ab - ab = \frac{x}{b} \times ab + 2ab$$

$$bx - ab = ax + 2ab$$

$$bx - ax = 2ab + ab$$

$$x(b - a) = 3ab$$

$$x = \frac{3ab}{b - a}$$

l

$$\frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}$$

$$\frac{x(a - b)(a + b)}{a - b} + \frac{2x(a + b)(a - b)}{a + b} = \frac{(a + b)(a - b)}{a^2 - b^2}$$

$$x(a + b) + 2x(a - b) = 1$$

$$ax + bx + 2ax - 2bx = 1$$

$$3ax - bx = 1$$

$$x(3a - b) = 1$$

$$x = \frac{1}{3a - b}$$

m

$$\frac{p - qx}{t} + p = \frac{qx - t}{p}$$

$$\frac{pt(p - qx)}{t} + p \times pt = \frac{pt(qx - t)}{p}$$

$$p(p - qx) + p^2t = t(qx - t)$$

$$p^2 - pqx + p^2t = qtx - t^2$$

$$-pqx - qtx = -t^2 - p^2 - p^2t$$

$$-qx(p + t) = -(t^2 + p^2 + p^2t)$$

$$x = \frac{t^2 + p^2 + p^2t}{q(p + t)} \text{ or}$$

$$\frac{p^2 + p^2t + t^2}{q(p + t)}$$

n $\frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}$

Multiply each term by$(x + a)(x + 2a)(x + 3a)$.

$$(x + 2a)(x + 3a) + (x + a)(x + 3a) = 2(x + a)(x + 2a)$$

$$x^2 + 5ax + 6a^2 + x^2 + 4ax + 3a^2 = 2x^2 + 6ax + 4a^2$$

$$2x^2 + 9ax + 9a^2 = 2x^2 + 6ax + 4a^2$$

$$2x^2 - 9ax - 2x^2 - 6ax = 4a^2 - 9a^2$$

$$3ax = -5a^2$$

$$x = \frac{-5a^2}{3a}$$

$$= -\frac{5a}{3}$$

2 $ax + by = p; bx - ay = q$

Multiply the first equation by a and the second equation by b.

$$a^2x + aby = ap \quad ①$$

$$b^2x - aby = bp \quad ②$$

①s + ②:

$$x(a^2 + b^2) = ap + bq$$

$$x = \frac{ap + bq}{a^2 + b^2}$$

Substitute into $ax + by = p$:

$$a \times \frac{ap + bq}{a^2 + b^2} + by = p$$

$$a(ap + bq) + by(a^2 + b^2) = p(a^2 + b^2)$$

$$a^2p + abq + by(a^2 + b^2) = a^2p + b^2p$$

$$by(a^2 + b^2) = a^2p + b^2p - a^2p - abq$$

$$by(a^2 + b^2) = b^2p - abq$$

$$y = \frac{b(bp - aq)}{b(a^2 + b^2)}$$

$$= \frac{bp - aq}{a^2 + b^2}$$

3 $\frac{x}{a} + \frac{y}{b} = 1; \frac{x}{b} + \frac{y}{a} = 1$

First, multiply both equations by ab, giving the following:

$bx + ay = ab$

$ax + by = ab$

Multiply the first equation by b and the second equation by a:

$$b^2x + aby = ab^2 \quad ①$$

$$a^2x + aby = a^2b \quad ②$$

① − ②:

$$x(b^2 - a^2) = ab^2 - a^2b$$

$$x = \frac{ab^2 - a^2b}{b^2 - a^2}$$

$$= \frac{ab(b - a)}{(b - a)(b + a)}$$

$$= \frac{ab}{a + b}$$

Substitute into $bx + ay = ab$:

$$b \times \frac{ab}{a + b} + ay = ab$$

$$\frac{ab^2(a + b)}{a + b} + ay(a + b) = ab(a + b)$$

$$ab^2 + ay(a + b) = a^2b + ab^2$$

$$ay(a + b) = a^2b + ab^2 - ab^2$$

$$ay(a + b) = a^2b$$

$$y = \frac{a^2b}{a(a + b)}$$

$$= \frac{ab}{a + b}$$

4 **a** Multiply the first equation by b.

$$abx + by = bc \quad ①$$

$$x + by = d \quad ②$$

① − ②:

$$x(ab - 1) = bc - d$$

$$x = \frac{bc - d}{ab - 1}$$

$$= \frac{d - bc}{1 - ab}$$

It is easier to substitute in the first equation for x:

$$a \times \frac{bc-d}{ab-1} + y = c$$

$$\frac{a(bc-d)(ab-1)}{ab-1} + y(ab-1) = c(ab-1)$$

$$abc - ad + y(ab-1) = abc - c$$

$$y(ab-1) = abc - c$$

$$- abc + ad$$

$$y(ab-1) = -c + ad$$

$$y = \frac{ad-c}{ab-1}$$

$$= \frac{c-ad}{1-ab}$$

b Multiply the first equation by a and the second equation by b.

$$a^2x - aby = a^3 \quad ①$$

$$b^2x - aby = b^3 \quad ②$$

① − ②:

$$x(a^2-b^2) = a^3 - b^3$$

$$x = \frac{a^3-b^3}{a^2-b^2}$$

$$= \frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}$$

$$= \frac{a^2+ab+b^2}{a+b}$$

In this case it is easier to start again, but eliminate x.
Multiply the first equation by b and the second equation by a.

$$abx - b^2y = a^2b \quad ③$$

$$abx - a^2y = ab^2 \quad ④$$

③ − ④:

$$y(-b^2+a^2) = a^2b - ab^2$$

$$y(a^2-b^2) = ab(a-b)$$

$$y = \frac{ab(a-b)}{a^2-b^2}$$

$$= \frac{ab(a-b)}{(a-b)(a+b)}$$

$$= \frac{ab}{a+b}$$

c Add the starting equations:

$$ax + by + ax - by = t + s$$

$$2ax = t + s$$

$$x = \frac{t+s}{2a}$$

Subtract the starting equations:

$$ax + by - (ax - by) = t - s$$

$$2by = t - s$$

$$y = \frac{t-s}{2b}$$

d Multiply the first equation by a and the second equation by b.

$$a^2x + aby = a^3 + 2a^2b - ab^2 \quad ①$$

$$b^2x + aby = a^2b + b^3 \quad ②$$

① − ②:

$$x(a^2-b^2) = a^3 + a^2b - ab^2 - b^3$$

$$x = \frac{a^3+a^2b-ab^2-b^3}{a^2-b^2}$$

$$= \frac{a^2(a+b) - b^2(a+b)}{a^2-b^2}$$

$$= \frac{(a^2-b^2)(a+b)}{a^2-b^2}$$

$$= a + b$$

Substitute into the second, simpler equation.

$$b(a+b) + ay = a^2 + b^2$$

$$ab + b^2 + ay = a^2 + b^2$$

$$ay = a^2 + b^2 - ab - b^2$$

$$ay = a^2 - ab$$

$$y = \frac{a^2-ab}{a}$$

$$= a - b$$

e Rewrite the second equation, then multiply the first equation by $b + c$

and the second equation by c.

$$(a+b)(b+c)x + c(c+c)y$$
$$= bc(b+c) \quad ①$$
$$acx + c(b+c)y$$
$$= -abc \quad ②$$

① − ②:

$$x((a+b)(b+c) - ac)$$
$$= bc(b+c) + abc$$
$$x(ab + ac + b^2 + bc - ac)$$
$$= bc(b+c+a)$$
$$x(ab + b^2 + bc) = bc(a+b+c)$$
$$xb(a+b+c) = bc(a+b+c)$$
$$x = \frac{bc(a+b+c)}{b(a+b+c)}$$
$$= c$$

Substitute into the first equation. (It has the simpler y term.)

$$c(a+b) + cy = bc$$
$$ac + bc + cy = bc$$
$$cy = bc - ac - bc$$
$$cy = -ac$$
$$y = \frac{-ac}{c}$$
$$= -a$$

f First simplify the equations.

$$3x - 3a - 2y - 2a = 5 - 4a$$
$$3x - 2y = 5 - 4a$$
$$+ 3a + 2a$$
$$3x - 2y = a + 5 \quad ①$$
$$2x + 2a + 3y - 3a = 4a - 1$$
$$2x + 3y = 4a - 1$$
$$- 2a + 3a$$
$$2x + 3y = 5a - 1 \quad ②$$

Multiply① by 3 and ② by 2.

$$9x - 6y = 3a + 15 \quad ③$$
$$4x + 6y = 10a - 2 \quad ④$$

③ + ④:

$$13x = 13a + 13$$
$$x = a + 1$$

Substitute into ②:

$$2(a+1) + 3y = 5a - 1$$
$$2a + 2 + 3y = 5a - 1$$
$$3y = 5a - 1 - 2a - 2$$
$$3y = 3a - 3$$
$$y = a - 1$$

5 **a** $s = ah$

$$= a(2a+1)$$

b Make h the subject of the second equation.

$$h = a(2+h)$$
$$= 2a + ah$$
$$h - ah = 2a$$
$$h(1-a) = 2a$$
$$h = \frac{2a}{1-a}$$

Substitute into the first equation.

$$s = ah$$
$$= a \times \frac{2a}{1-a}$$
$$= \frac{2a^2}{1-a}$$

c $h + ah = 1$

$$h(1 + a) = 1$$

$$h = \frac{1}{(1 + a)} = \frac{1}{a + 1}$$

$$as = a + h$$

$$= a + \frac{1}{a + 1}$$

$$= \frac{a(a + 1) + 1}{a + 1}$$

$$= \frac{a^2 + a + 1}{a + 1}$$

$$s = \frac{a^2 + a + 1}{a(a + 1)}$$

d Make h the subject of the second equation.

$$ah = a + h$$

$$ah - h = a$$

$$h(a - 1) = a$$

$$h = \frac{1}{a - 1}$$

Substitute into the first equation.

$$as = s + h$$

$$as = s + \frac{a}{a - 1}$$

$$as - s = \frac{a}{a - 1}$$

$$s(a - 1) = \frac{a}{a - 1}$$

$$s(a - 1)(a - 1) = \frac{a(a - 1)}{a - 1}$$

$$s(a - 1)^2 = a$$

$$s = \frac{a}{(a - 1)^2}$$

e $s = h^2 + ah$

$$= (3a^2)^2 + a(3a^2)$$

$$= 9a^4 + 3a^3$$

$$= 3a^3(3a + 1)$$

f $as = a + 2h$

$$= a + 2(a - s)$$

$$= a + 2a - 2s$$

$$as + 2s = 3a$$

$$s(a + 2) = 3a$$

$$s = \frac{3a}{a + 2}$$

g $s = 2 + ah + h^2$

$$= 2 + a\left(a - \frac{1}{a}\right) + \left(a - \frac{1}{a}\right)^2$$

$$= 2 + a^2 - 1 + a^2 - 2 + \frac{1}{a^2}$$

$$= 2a^2 - 1 + \frac{1}{a^2}$$

h Make h the subject of the second equation.

$$as + 2h = 3a$$

$$2h = 3a - as$$

$$h = \frac{3a - as}{2}$$

Substitute into the first equation.

$$3s - ah = a^2$$

$$3s - \frac{a(3a - as)}{2} = a^2$$

$$6s - a(3a - as) = 2a^2$$

$$6s - 3a^2 + a^2 s = 2a^2$$

$$a^2 s + 6s = 2a^2 + 3a^2$$

$$s(a^2 + 6) = 5a^2$$

$$s = \frac{5a^2}{a^2 + 6}$$

Solutions to Exercise 1I

Use your CAS calculator to find the solutions to these problems. The exact method will vary depending on the calculator used.

1 **a** $x = a - b$

b $x = 7$

c $x = -\dfrac{a \pm \sqrt{a^2 + 4ab - 4b^2}}{2}$

d $x = \dfrac{a + c}{2}$

2 **a** $(x - 1)(x + 1)(y - 1)(y + 1)$

b $(x - 1)(x + 1)(x + 2)$

c $(a^2 - 12b)(a^2 + 4b)$

d $(a - c)(a - 2b + c)$

3 **a** $axy + b = (a + c)y$

$bxy + a = (b + c)y$

Dividing by y yields:

$ax + \dfrac{b}{y} = a + c$

$bx + \dfrac{a}{y} = b + c$

let $n = \dfrac{1}{y}$ and the equations become:

$ax + bn = a + c$

$bx + an = b + c$

$$\therefore \quad x = \frac{a + b + c}{a + b}$$

$$y = \frac{a + b}{c}$$

b

$$x(b - c) + by - c = 0$$

$$y(c - a) - ax + c = 0$$

$$(b - c)x + by = c$$

$$-ax + (c - a)y = -c$$

$$\therefore \quad x = \frac{-(a - b - c)}{a + b - c}$$

$$y = \frac{a - b + c}{a + b - c}$$

Solutions to short-answer questions

1 a $(x^3)^4 = x^{3\times4}$
$= x^{12}$

b $(y^{-12})^{\frac{3}{4}} = y^{-12\times\frac{3}{4}}$
$= y^{-9}$

c $3x^{\frac{3}{2}} \times -5x^4 = (3\times-5)x^{\frac{3}{2}+4}$
$= -15x^{\frac{11}{2}}$

d $(x^3)^{\frac{4}{3}} \times x^{-5} = x^{3\times\frac{4}{3}} \times x^{-5}$
$= x^{4-5}$
$= x^{-1}$

2 $23\times10^{-6}\times14\times10^{15}$
$= (14\times23)\times10^{15-6}$
$= 322\times10^{9}$
$= 3.22\times10^{11}$

3 a $\dfrac{3x}{5}+\dfrac{y}{10}-\dfrac{2x}{5}=\dfrac{6x+y-4x}{10}$
$=\dfrac{2x+y}{10}$

b $\dfrac{4}{x}-\dfrac{7}{y}=\dfrac{4y-7x}{xy}$

c $\dfrac{5}{x+2}+\dfrac{2}{x-1}=\dfrac{5(x-1)+2(x+2)}{(x+2)(x-1)}$
$=\dfrac{5x-5+2x+4}{(x+2)(x-1)}$
$=\dfrac{7x-1}{(x+2)(x-1)}$

d $\dfrac{3}{x+2}+\dfrac{4}{x+4}=\dfrac{3(x+4)+4(x+2)}{(x+2)(x+4)}$
$=\dfrac{3x+12+4x+8}{(x+2)(x+4)}$
$=\dfrac{7x+20}{(x+2)(x+4)}$

e $\dfrac{5x}{x+4}+\dfrac{4x}{x-2}-\dfrac{5}{2}$
$=\dfrac{10x(x-2)+8x(x+4)-5(x+4)(x-2)}{2(x+4)(x-2)}$
$=\dfrac{10x^2-20x+8x^2+32x-5(x^2+2x-8)}{2(x+4)(x-2)}$
$=\dfrac{10x^2-20x+8x^2+32x-5x^2-10x+40}{2(x+4)(x-2)}$
$=\dfrac{13x^2+2x+40}{2(x+4)(x-2)}$

f $\dfrac{3}{x-2}-\dfrac{6}{(x-2)^2}=\dfrac{3(x-2)-6}{(x-2)^2}$
$=\dfrac{3x-6-6}{(x-2)^2}$
$=\dfrac{3x-12}{(x-2)^2}$
$=\dfrac{3(x-4)}{(x-2)^2}$

4 a $\dfrac{x+5}{2x-6}\div\dfrac{x^2+5x}{4x-12}$
$=\dfrac{x+5}{2x-6}\times\dfrac{4x-12}{x^2+5x}$
$=\dfrac{x+5}{2(x-3)}\times\dfrac{4(x-3)}{x(x+5)}$
$=\dfrac{4}{2x}=\dfrac{2}{x}$

b $\dfrac{3x}{x+4} \div \dfrac{12x^2}{x^2-16}$

$= \dfrac{3x}{x+4} \times \dfrac{x^2-16}{12x^2}$

$= \dfrac{3x}{x+4} \times \dfrac{(x-4)(x+4)}{12x^2}$

$= \dfrac{3x(x-4)}{12x^2}$

$= \dfrac{x-4}{4x}$

c $\dfrac{x^2-4}{x-3} \times \dfrac{3x-9}{x+2} \div \dfrac{9}{x+2}$

$= \dfrac{x^2-4}{x-3} \times \dfrac{3x-9}{x+2} \times \dfrac{x+2}{9}$

$= \dfrac{(x-2)(x+2)}{x-3} \times \dfrac{3(x-3)}{x+2}$

$\times \dfrac{x+2}{9}$

$= \dfrac{(x+2)(x-2)}{3} = \dfrac{x^2-4}{3}$

d $\dfrac{4x+20}{9x-6} \times \dfrac{6x^2}{x+5} \div \dfrac{2}{3x-2}$

$= \dfrac{4(x+5)}{3(3x-2)} \times \dfrac{6x^2}{x+5} \times \dfrac{3x-2}{2}$

$= \dfrac{4 \times 6x^2}{3 \times 2} = 4x^2$

5 500 ml of blood contains 2.5×10^{12} red blood cells.
$(2.5 \times 10^{12}) \div (2.5 \times 10^6) = 10^6$
It will take 10^6 seconds

6 $(1.5 \times 10^8) \div (3 \times 10^6) = 0.5 \times 10^2$
$= 50$

7 Let g be the number of games the team lost. They won $2g$ games and drew one third of 54 games, i.e. 18 games.

$g + 2g + 18 = 54$

$3g = 54 - 18$

$= 36$

$g = 12$

They have lost 12 games.

8 Let s be the number of science fiction sold. The store sold $1.1s$ crime and $1.5(s + 1.1s)$ romance, totalling 420 books.

$s + 1.1\,s + 1.5 \times 2.1s = 420$

$5.25s = 420$

$s = \dfrac{420}{5.25}$

$= 80$

$1.1s = 1.1 \times 80 = 88$

$1.5 \times 2.1s = 1.5 \times 2.1 \times 80$

$= 252$

80 science fiction, 88 crime and 252 romance (totalling 420)

9 **a** $V = \pi r^2 h$

$= \pi \times 5^2 \times 12$

$= 300\pi \approx 942\text{ cm}^3$

b $h = \dfrac{V}{\pi r^2}$

$= \dfrac{585}{\pi \times 5^2}$

$= \dfrac{117}{5\pi} \approx 7.4\text{ cm}$

c $r^2 = \dfrac{V}{\pi h}$

$r = \sqrt{\dfrac{V}{\pi h}}$ (use positive root)

$= \sqrt{\dfrac{786}{\pi \times 6}}$

$= \sqrt{\dfrac{128}{\pi}} \approx 40.7$ cm

10 a $xy + ax = b$

$x(y + a) = b$

$x = \dfrac{b}{a + y}$

b $\dfrac{a}{x} + \dfrac{b}{x} = c$

$\dfrac{ax}{x} + \dfrac{bx}{x} = cx$

$a + b = cx$

$x = \dfrac{a + b}{c}$

c $\dfrac{x}{a} = \dfrac{x}{b} + 2$

$\dfrac{xab}{a} = \dfrac{xab}{b} + 2ab$

$bx = ax + 2ab$

$bx - ax = 2ab$

$x(b - a) = 2ab$

$x = \dfrac{2ab}{b - a}$

d

$$\frac{a - dx}{d} + b = \frac{ax + d}{b}$$

$$\frac{bd(a - dx)}{d} + bd \times b = \frac{bd(ax + d)}{b}$$

$$b(a - dx) + b^2 d = d(ax + d)$$

$$ab - bdx + b^2 d = adx + d^2$$

$$-bdx - adx = d^2 - ab - b^2 d$$

$$-x(bd + ad) = -(ab + b^2 d - d^2)$$

$$x = \frac{-(ab + b^2 d - d^2)}{-(bd + ad)}$$

$$= \frac{ab + b^2 d - d^2}{bd + ad}$$

11 a $\dfrac{p}{p + q} + \dfrac{q}{p - q} = \dfrac{p(p - q) + q(p + q)}{(p + q)(p - q)}$

$= \dfrac{p^2 - qp + qp + q^2}{p^2 - pq + pq - q^2}$

$= \dfrac{p^2 + q^2}{p^2 - q^2}$

b $\dfrac{1}{x} - \dfrac{2y}{xy - y^2} = \dfrac{(xy - y^2) - 2xy}{x(xy - y^2)}$

$= \dfrac{-xy - y^2}{x^2 y - xy^2}$

$= \dfrac{y(-x - y)}{xy(x - y)}$

$= \dfrac{-x - y}{x(x - y)}$

$= \dfrac{x + y}{x(y - x)}$

c $\dfrac{x^2 + x - 6}{x + 1} \times \dfrac{2x^2 + x - 1}{x + 3}$

$= \dfrac{(x - 2)(x + 3)}{x + 1} \times \dfrac{(x + 1)(2x - 1)}{x + 3}$

$= (x - 2)(2x - 1)$

d $\dfrac{2a}{2a+b} \times \dfrac{2ab+b^2}{ba^2}$

$= \dfrac{2a}{2a+b} \times \dfrac{b(2a+b)}{ba^2}$

$= \dfrac{2ab}{ba^2}$

$= \dfrac{2}{a}$

12 Let A's age be a, B's age be b and C's age be c.

$$a = 3b$$

$$b + 3 = 3(c + 3)$$

$$a + 15 = 3(c + 15)$$

Substitute for a and simplify:

$$b + 3 = 3(c + 3)$$

$$b + 3 = 3c + 9$$

$$b = 3c + 6 \qquad ①$$

$$3b + 15 = 3(c + 15)$$

$$3b + 15 = 3c + 45$$

$$3b = 3c + 30$$

$$b = c + 10 \qquad ②$$

① = ②:

$$3c + 6 = c + 10$$

$$3c - c = 10 - 6$$

$$2c = 4$$

$$c = 2$$

$$b = 3 \times 2 + 6$$

$$= 12$$

$$a = 3 \times 12$$

$$= 36$$

A, B and C are 36, 12 and 2 years old respectively.

13 **a** Simplify the first equation:

$$a - 5 = \frac{1}{7}(b + 3)$$

$$7(a - 5) = b + 3$$

$$7a - 35 = b + 3$$

$$7a - b = 38$$

Simplify the second equation:

$$b - 12 = \frac{1}{5}(4a - 2)$$

$$5(b - 12) = 4a - 2$$

$$5b - 60 = 4a - 2$$

$$-4a + 5b = 58$$

Multiply the first equation by 5, and add the second equation.

$$35a - 5b = 190 \qquad ①$$

$$-4a + 5b = 58 \qquad ②$$

① + ②:

$$31a = 248$$

$$a = 8$$

Substitute into the first equation:

$$7 \times 8 - b = 38$$

$$56 - b = 38$$

$$b = 56 - 38 = 18$$

b Multiply the first equation by p.

$$(p - q)x + (p + q)y = (p + q^2)$$

$$p(p - q)x + p(p + q)y = p(p + q^2) \qquad ①$$

Multiply the second by $(p + q)$.

$$qx - py = q^2 - pq$$

$$q(p + q)x - p(p + q)y = (p + q)(q^2 - pq) \qquad ②$$

① + ②:

$$(p(p - q) + q(p + q))x = p(p + q)^2 + (p + q)(q^2 - pq)$$

$(p^2 - pq + pq + q^2)x$

$= p(p^2 + 2pq + q^2)$

$\quad + pq^2 - p^2q + q^3 - pq^2$

$(p^2 + q^2)x$

$= p^3 + 2p^2q + pq^2 - p^2q + q^3$

$= p^3 + p^2q + pq^2 + q^3$

$= p^2(p + q) + q^2(p + q)$

$= (p + q)(p^2 + q^2)$

$x = p + q$

Substitute into the second equation, factorising the right side.

$q(p + q) - py = q^2 - pq$

$pq + q^2 - py = q^2 - pq$

$-py = q^2 - pq - pq - q^2$

$-py = -2pq$

$y = \dfrac{-2pq}{-p}$

$= 2q$

14 Time $= \dfrac{\text{distance}}{\text{speed}}$

Remainder = 50 − 7 − 7 = 36 km

$\dfrac{7}{x} + \dfrac{7}{4x} + \dfrac{36}{6x + 3} = 4$

$\dfrac{7}{x} + \dfrac{7}{4x} + \dfrac{12}{2x + 1} = 4$

$(4x(2x + 1)) \times \left(\dfrac{7}{x} + \dfrac{7}{4x} + \dfrac{12}{2x + 1}\right)$

$= 4 \times 4x(2x + 1)$

$28(2x + 1) + 7(2x + 1) + 48x$

$= 16x(2x + 1)$

$56x + 28 + 14x + 7 + 48x$

$= 32x^2 + 16x$

$56x + 28 + 14x + 7 + 48x$

$- 32x^2 - 16x = 0$

$-32x^2 + 102x + 35 = 0$

$32x^2 - 102x - 35 = 0$

$(2x - 7)(16x + 5) = 0$

$2x - 7 = 0$ or $16x + 5 = 0$

$x > 0$, so $2x - 7 = 0$

$x = 3.5$

15 **a** $2n^2 \times 6nk^2 \div 3n = \dfrac{2n^2 \times 6nk^2}{3n}$

$= \dfrac{12n^3k^2}{3n}$

$= 4n^2k^2$

b $\dfrac{8c^2x^3y}{6a^2b^3c^3} \div \dfrac{\frac{1}{2}xy}{15abc^2}$

$= \dfrac{8c^2x^3y}{6a^2b^3c^3} \div \dfrac{xy}{30abc^2}$

$= \dfrac{8c^2x^3y}{6a^2b^3c^3} \times \dfrac{30abc^2}{xy}$

$= \dfrac{240abc^4x^3y}{6a^2b^3c^3xy}$

$= \dfrac{40cx^2}{ab^2}$

16 $\dfrac{x + 5}{15} - \dfrac{x - 5}{10} = 1 + \dfrac{2x}{15}$

$\dfrac{30(x + 5)}{15} - \dfrac{30(x - 5)}{10} = 30 \times \left(1 + \dfrac{2x}{15}\right)$

$2(x + 5) - 3(x - 5) = 30 + 4x$

$2x + 10 - 3x + 15 = 30 + 4x$

$2x - 3x - 4x = 30 - 10 - 15$

$-5x = 5$

$x = -1$

Solutions to multiple-choice questions

1 A

$$5x + 2y = 0$$
$$2y = -5x$$
$$\frac{y}{x} = -\frac{5}{2}$$

2 A Multiply both sides of the second equation by 2.

$$3x + 2y = 36 \quad ①$$
$$6x - 2y = 24 \quad ②$$

① + ②:

$$9x = 60$$
$$x = \frac{20}{3}$$
$$3 \times \frac{20}{3} - y = 12$$
$$20 - y = 12$$
$$y = 8$$

3 C

$$t - 9 = 3t - 17$$
$$t - 3t = 9 - 17$$
$$-2t = -8$$
$$t = 4$$

4 A

$$m = \frac{n - p}{n + p}$$
$$m(n + p) = n - p$$
$$mn + mp = n - p$$
$$mp + p = n - mn$$
$$p(m + 1) = n(1 - m)$$
$$p = \frac{n(1 - m)}{1 + m}$$

5 B

$$\frac{3}{x - 3} - \frac{2}{x + 3} = \frac{3(x + 3) - 2(x - 3)}{(x - 3)(x + 3)}$$
$$= \frac{3x + 9 - 2x + 6}{x^2 - 9}$$
$$= \frac{x + 15}{x^2 - 9}$$

6 E

$$9x^2y^3 \div 15(xy)^3 = \frac{9x^2y^3}{15(xy)^3}$$
$$= \frac{9x^2y^3}{15x^3y^3}$$
$$= \frac{9}{15x}$$
$$= \frac{3}{5x}$$

7 B

$$V = \frac{1}{3}h(l + w)$$
$$3V = h(l + w)$$
$$3V = hl + hw$$
$$hl = 3V - hw$$
$$l = \frac{3V - hw}{h}$$
$$= \frac{3V}{h} - w$$

8 B

$$\frac{(3x^2y^3)^2}{2x^2y} = \frac{9x^4y^6}{2x^2y}$$
$$= \frac{9x^2y^5}{2}$$
$$= \frac{9}{2}x^2y^5$$

9 B $Y = 80\% \times Z = \frac{4}{5}Z$

$$X = 150\% \times Y = \frac{3}{2}Y$$
$$= \frac{3}{2} \times \frac{4Z}{5}$$
$$= \frac{12Z}{10}$$
$$= 1.2Z$$

= 20% greater than Z

10 B Let the other number be n.

$$\frac{x+n}{2} = 5x+4$$
$$x+n = 2(5x+4)$$
$$= 10x+8$$
$$n = 10x+8-x$$
$$= 9x+8$$

11 E

$$\frac{4}{(x+3)^2} + \frac{2x}{x+1} = \frac{4(x+1)+2x(x+3)^2}{(x+3)^2(x+1)}$$
$$= \frac{4x+4+2x(x^2+6x+9)}{(x+3)^2(x+1)}$$
$$= \frac{2x^3+12x^2+22x+4}{(x+3)^2(x+1)}$$
$$= \frac{2(x^3+6x^2+11x+2)}{(x+3)^2(x+1)}$$

Solutions to extended-response questions

1 Jack cycles $10x$ km.
Benny drives $40x$ km.

a
$$\text{Distance} = \text{speed} \times \text{time}$$
$$\therefore \quad \text{time} = \frac{\text{distance}}{\text{speed}}$$
$$\therefore \quad \text{time taken by Jack} = \frac{10x}{8}$$
$$= \frac{5x}{4} \text{ hours}$$

b
$$\text{Time taken by Benny} = \frac{40x}{70}$$
$$= \frac{4x}{7} \text{ hours}$$

c
$$\text{Jack's time–Benny's time} = \frac{5x}{4} - \frac{4x}{7}$$
$$= \frac{(35 - 16)x}{7}$$
$$= \frac{19x}{28} \text{ hours}$$

d **i** If the difference is 30 mins $= \frac{1}{2}$ hour
$$\text{then } \frac{19x}{28} = \frac{1}{2}$$
$$\therefore \quad x = \frac{14}{19}$$
$$= 0.737 \text{ (correct to three decimal places)}$$

ii
$$\text{Distance for Jack} = 10 \times \frac{14}{19}$$
$$= \frac{140}{19}$$
$$= 7 \text{ km (correct to the nearest km)}$$
$$\text{Distance for Benny} = 40 \times \frac{14}{19}$$
$$= \frac{560}{19}$$
$$= 29 \text{ km (correct to the nearest km)}$$

2 **a** Dinghy is filling with water at a rate of
$27\,000 - 9\,000 = 18\,000$ cm^3 per minute.

b After t minutes there are $18\,000t$ cm^3 water in the dinghy,
i.e. $V = 18\,000t$

c $V = \pi r^2 h$ is the formula for the volume of a cylinder

$$\therefore \quad h = \frac{V}{\pi r^2} = \frac{18\,000t}{\pi r^2}$$

The radius of this cylinder is 40 cm

$$\therefore \quad h = \frac{18\,000t}{1600\pi} = \frac{45t}{4\pi}$$

i.e. the height h cm water at time t is given by $h = \dfrac{45t}{4\pi}$

d When $t = 9$, $h = \dfrac{45 \times 9}{4\pi}$
$\approx 32.228\ldots$
The dinghy has filled with water, before $t = 9$, i.e. Sam is rescued after the dinghy completely filled with water.

3 **a** Let Thomas have a cards. Therefore Henry has $\dfrac{5a}{6}$ cards, George has $\dfrac{3a}{2}$ cards, Sally has $(a - 18)$ cards and Zeb has $\dfrac{a}{3}$ cards.

b $\dfrac{3a}{2} + a - 18 + \dfrac{a}{3} = a + \dfrac{5a}{6} + 6$

c $\therefore \quad 9a + 6a - 108 + 2a = 6a + 5a + 36$

$$\therefore \quad 6a = 144$$

$$\therefore \quad a = 24$$

Thomas has 24 cards, Henry has 20 cards, George has 36 cards, Sally has 6 cards and Zeb has 8 cards.

4 **a** $F = \dfrac{6.67 \times 10^{-11} \times 200 \times 200}{12^2}$

$= 1.852\ldots \times 10^{-8}$

$= 1.9 \times 10^{-8}$ N (correct to two significant figures)

b $m_1 = \dfrac{Fr^2}{m_2 \times 6.67 \times 10^{-11}}$

$= \dfrac{Fr^2 \times 10^{11}}{6.67 m_2}$

c If $F = 2.4 \times 10^4$

$r = 6.4 \times 10^6$

and $m_2 = 1500$

$m_1 = \dfrac{2.4 \times 10^4 \times (6.4 \times 10^6)^2 \times 10^{11}}{6.67 \times 1500}$

$= 9.8254 \ldots \times 10^{24}$

The mass of the planet is 9.8×10^{24} kg (correct to two significant figures).

5 **a** $V = 3 \times 10^3 \times 6 \times 10^3 \times d$

$= 18 \times 10^6 d$

b When $d = 30$, $V = 18 \times 10^6 \times 30$

$= 540\,000\,000$

$= 5.4 \times 10^8$

The volume of the reservoir is 5.4×10^8 m^3.

c $E = kVh$

$1.06 \times 10^{15} = k \times 200 \times 5.4 \times 10^8$

$k = \dfrac{1.06 \times 10^{15}}{200 \times 5.4 \times 10^8}$

$= 9.81 \ldots \times 10^3$

$k = 9.81 \times 10^3$ correct to three significant figures.

d $E = (9.81 \times 10^3) \times 5.4 \times 10^8 \times 250$

$= 1.325 \times 10^{15}$ correct to four significant figures.

The amount of energy produced is 1.325×10^{15} J.

e Let t be the time in seconds.

$5.2 \times t = 5.4 \times 10^8$

$t = 103.846\,153\,8$

$\therefore$ number of days $= 103.846\,153\,8 \div (24 \times 60 \times 60)$

$= 1201.92 \ldots$

The station could operate for approximately 1202 days.

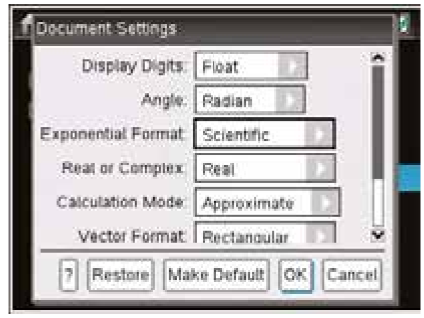
CAS calculator techniques for Question 5

5 b Calculations involving scientific notation and significant figures can be accomplished with the aid of a graphics calculator.

When $d = 30, \; V = 18 \times 10^6 \times 30$

$$= 540\,000\,000$$

This calculation can be completed as shown here.



T1: Press c → **5**: **Settings** → **2**: **Document Settings** and change the Exponential Format to Scientific. Click on Make Default.

Return to the Calculator application.

Type **18** × **10**^**6** × **30** or **18i6** × **30**

CP: In the Main application tap ◯ → **Basic Format**

Change the Number Format to Sci2

Type **18** × **10**^**6** × **30**

c **T1**: Press c→ **5**: **Settings** → **2**: **Document Settings** and change the Display Digits to Float 3. Click on Make Default.

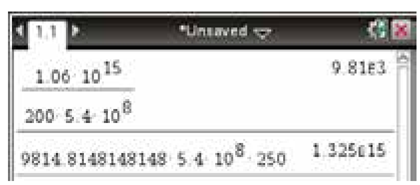
Return to the home screen and press and complete as shown.

CP: tap ◯ → **Basic Format**

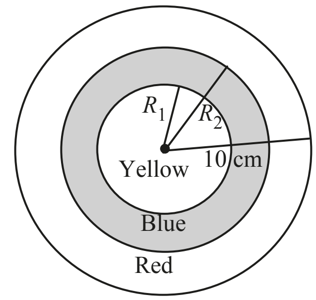
Change the Number Format to Sci3 Complete calculation as shown

d The calculation is as shown. **T1**: Display Digits is Float 4 **CP**: Number Format is Sci4

Simply type ×**5.4** × **10**^**8** × **25**



6 Let R_1 cm and R_2 cm be the radii of the inner circles.



$$\therefore \quad \text{Yellow area} = \pi R_1^2$$

$$\text{Blue area} = \pi R_2^2 - \pi R_1^2$$

$$\text{Red area} = 100\pi - \pi R_2^2$$

$$\therefore \quad 100\pi - \pi R_2^2 = \pi R_2^2 - \pi R_1^2 = \pi R_1^2$$

Firstly, $\pi R_2^2 - \pi R_1^2 = \pi R_1^2$

implies $R_2^2 = 2R_1^2$ ①

and $100\pi - \pi R_2^2 = \pi R_2^2 - \pi R_1^2$

implies $100 = 2R_2^2 - R_1^2$ ②

Substitute from ①in ②

$$\therefore \quad 100 = 4R_1^2 - R_1^2$$

$$100 = 3R_1^2$$

and $R_1 = \dfrac{10}{\sqrt{3}}$

$$= \frac{10\sqrt{3}}{3} \qquad \left(\text{Note : } R_2^2 = \frac{200}{3}\right)$$

The radius of the innermost circle is $\dfrac{10\sqrt{3}}{3}$ cm.

7 If $C = F$,

$$F = \frac{5}{9}(F - 32)$$

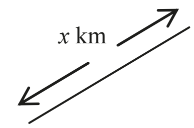
$$9F = 5F - 160$$

$$\therefore 4F = -160$$

$$\therefore F = -40$$

Therefore $-40°\text{F} = -40°\text{C}$.

8 Let x km be the length of the slope.



$$\therefore \quad \text{time to go up} = \frac{x}{15}$$

$$\therefore \quad \text{time to go down} = \frac{x}{40}$$

$$\begin{aligned}
\therefore \quad \text{total time} &= \frac{x}{15} + \frac{x}{40} \\
&= \frac{11x}{120}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \text{average speed} &= \frac{\text{total distance}}{\text{total time}} \\
&= 2x \div \frac{11x}{120} \\
&= 2x \times \frac{120}{11x} \\
&= \frac{240}{11} \\
&\approx 21.82 \text{ km/h}
\end{aligned}$$

9 1 litre = 1000 cm^3

a

$$\begin{aligned}
\text{Volume} &= \text{Volume of cylinder} + \text{Volume of hemisphere} \\
&= \pi r^2 h + \frac{2}{3}\pi r^3
\end{aligned}$$

It is known that $r + h = 20$

$$\therefore \quad h = 20 - r$$

b **i**

$$\begin{aligned}
\text{Volume} &= \pi r^2(20 - r) + \frac{2}{3}\pi r^3 \\
&= 20\pi r^2 - \pi r^3 + \frac{2}{3}\pi r^3 \\
&= 20\pi r^2 - \frac{\pi}{3} r^3
\end{aligned}$$

ii If Volume = 2000 cm^3

then $20\pi r^2 - \frac{\pi}{3} r^3 = 2000$

Use a CAS calculator to solve this equation for r, given that $0 < r < 20$. This gives $r = 5.943999\cdots$

Therefore $h = 20 - r$

$$\begin{aligned}
&= 20 - 5.943\,99\ldots \\
&= 14.056\,001\ldots
\end{aligned}$$

The volume is two litres when $r = 5.94$ and $h = 14.06$, correct to two decimal places.

10 **a** Let x and y be the amount of liquid (in cm^3) taken from bottles A and B respectively. Since the third bottle has a capacity of $1000\ cm^3$,

$$x + y = 1000 \qquad ①$$

Now $$x = \frac{2}{3}x \text{ wine} + \frac{1}{3}x \text{ water}$$

and $$y = \frac{1}{6}y \text{ wine} + \frac{5}{6}y \text{ water}$$

$\therefore$ $\frac{2}{3}x + \frac{1}{6}y = \frac{1}{3}x + \frac{5}{6}y$ since the proportion of wine and water must be the same.

$$\therefore \quad 4x + y = 2x + 5y$$

$$\therefore \quad 2x = 4y$$

$$\therefore \quad x = 2y$$

From ② $$2y + y = 1000$$

$$\therefore \quad y = \frac{1000}{3} \text{ and } x = \frac{2000}{3}$$

Therefore, $\frac{2000}{3}\ cm^3$ and $\frac{1000}{3}\ cm^3$ must be taken from bottles A and B respectively so that the third bottle will have equal amounts of wine and water, i.e. $\frac{2}{3}L$ from A and $\frac{1}{3}L$ from B

b $$x + y = 1000 \qquad ①$$

$$\frac{1}{3}x + \frac{3}{4}y = \frac{2}{3}x + \frac{1}{4}y$$

$$\therefore \quad 4x + 9y = 8x + 3y$$

$$\therefore \quad 6y = 4x$$

$$\therefore \quad x = \frac{3}{2}y \qquad ②$$

From ① $$\frac{3}{2}y + y = 1000$$

$$\therefore \quad y = \frac{2}{5} \times 1000$$

$$= 400$$

$$\therefore \quad x = 600$$

Therefore, 600 cm^3 and 400 cm^3 must be taken from bottles A and B respectively so that the third bottle will have equal amounts of wine and water, i.e. 600 mL from A and 400 mL from B

c

$$x + y = 1000 \qquad ①$$

$$\frac{m}{m+n}x + \frac{p}{p+q}y = \frac{n}{m+n}x + \frac{q}{p+q}y$$

$$\therefore \quad m(p+q)x + p(m+n)y = n(p+q)x + q(m+n)y$$

$$\therefore \quad (m(p\ +q) - n(p+q))x = (q(m+n) - p(m+n))y$$

$$\therefore \quad (m-n)(p+q)x = (q-p)(m+n)y$$

$$\therefore \quad x = \frac{(m+n)(q-p)}{(m-n)(p+q)}y,\ m \neq n, p \neq q \qquad ②$$

From ① $$\frac{(m+n)(q-p)}{(m-n)(p+q)}y + y = 1000$$

$$\therefore \quad \frac{(m+n)(q-p) + (m-n)(p+q)}{(m-n)(p+q)}y = 1000$$

$$\therefore \quad \frac{mq - mp + nq - np + mp + mq - np - nq}{(m-n)(p+q)}y = 1000$$

$$\therefore \quad \frac{2(mq-np)}{(m-n)(p+q)}y = 1000$$

$$\therefore \quad y = \frac{500(m-n)(p+q)}{mq-np},$$

$$mq \neq np$$

From ① $x = \dfrac{(m+n)(q-p)}{(m-n)(p+q)} \times \dfrac{500(m-n)(p+q)}{mq-np}$

$$= \frac{500(m+n)(q-p)}{mq-np},\ \frac{n}{q} \neq \frac{q}{p}$$

Therefore, $\dfrac{500(m+n)(q-p)}{mq-np}$ cm^3 and $\dfrac{500(m-n)(p+q)}{mq-np}$ cm^3 must be taken from bottles A and B respectively so that the third bottle will have equal amounts of wine and water. In litres this is $\dfrac{(m+n)(q-p)}{2(mq-np)}$ litres from A and $\dfrac{(m-n)(p+q)}{2(mq-np)}$ litres from B. Also note that $\dfrac{n}{m} \geq 1$ and $\dfrac{q}{p} \leq 1$ or $\dfrac{n}{m} \leq 1$ and $\dfrac{q}{p} \geq 1$.

11 **a** $$\frac{20-h}{20}=\frac{r}{10}$$

$$\therefore \quad 10(20-h)=20r$$

$$\therefore \quad 200-10h=20r$$

$$\therefore \quad 20-h=2r$$

$$\therefore \quad h=20-2r$$

$$=2(10-r)$$

b $V=\pi r^2 h$

$=2\pi r^2(10-r)$

c Use CAS calculator to solve the equation $2\pi x^2(10-r)=500$, given that $0<r<10$.
This gives $r=3.49857\ldots$ or $r=9.02244\ldots$
When $r=3.498\,57\ldots,\ h=2(10-3.498\,57\ldots)$

$=13.002\,85\ldots$

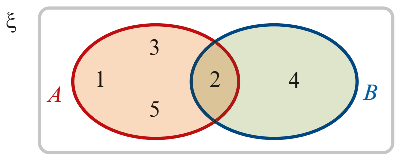
When $r=9.022\,44\ldots,\ h=2(10-9.022\,44\ldots)$

$=1.955\,11\ldots$

Therefore the volume of the cylinder is 500 cm^3 when $r=3.50$ and $h=13.00$ or when $r=9.02$ and $h=1.96$, correct to two decimal places.

Chapter 2 – Number systems and sets

Solutions to Exercise 2A

1 

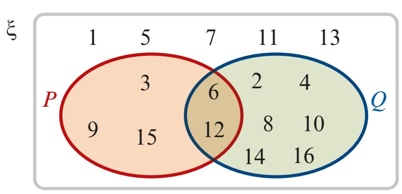
a $A' = \{4\}$

b $B' = \{1, 3, 5\}$

c $A \cup B = \{1, 2, 3, 4, 5\}$, or ξ

d $(A \cup B)' = \varnothing$

e $A' \cap B' = \varnothing$

2 

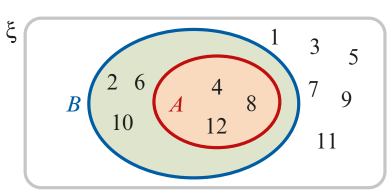
a $P' = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16\}$

b $Q' = \{1, 3, 5, 7, 9, 11, 13, 15\}$

c $P \cup Q =$ $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}$

d $(P \cup Q)' = \{1, 5, 7, 11, 13\}$

e $P' \cap Q' = \{1, 5, 7, 11, 13\}$

3 

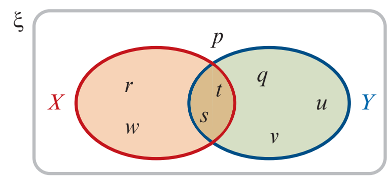
a $A' = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$

b $B' = \{1, 3, 5, 7, 9, 11\}$

c $A \cup B = \{2, 4, 6, 8, 10, 12\}$

d $(A \cup B)' = \{1, 3, 5, 7, 9, 11\}$

e $A' \cap B' = \{1, 3, 5, 7, 9, 11\}$

4 

a $X' = \{p, q, u, v\}$

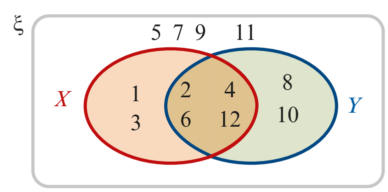
b $Y' = \{p, r, w\}$

c $X' \cap Y' = \{p\}$

d $X' \cup Y' = \{p, q, r, u, v, w\}$

e $X \cup Y = \{q, r, s, t, u, v, w\}$

f $(X \cup Y)' = \{p\}$ **c** and **f** are equal.

5 

a $X' = \{5, 7, 8, 9, 10, 11\}$

b $Y' = \{1, 3, 5, 7, 9, 11\}$

c $X' \cup Y' = \{1, 3, 5, 7, 8, 9, 10, 11\}$

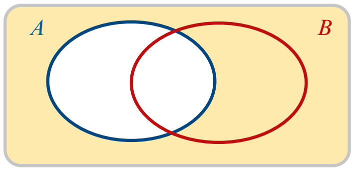
d $X' \cap Y' = \{1, 3, 5, 7, 8, 9, 10, 11\}$

e $X \cup Y = \{1, 2, 3, 4, 6, 8, 10, 12\}$

f $(X \cup Y)' = \{5, 7, 9, 11\}$ **d** and **f** are equal.

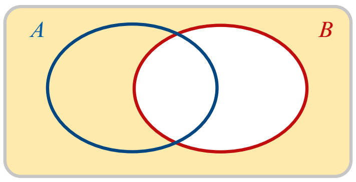
6 a ξ

A'



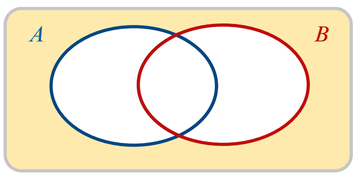
b ξ

B'



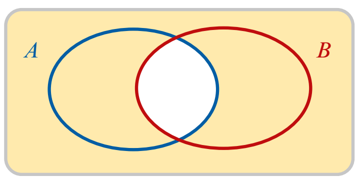
c ξ

$A' \cap B'$



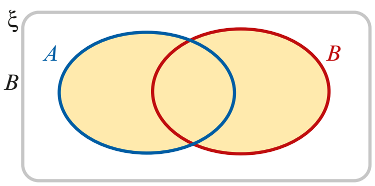
d ξ

$A' \cup B'$



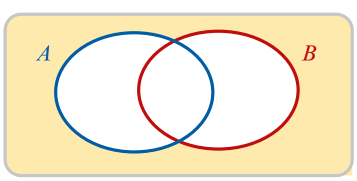
e ξ

$A \cup B$

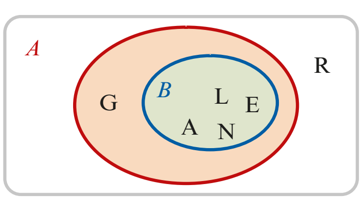


f ξ

$(A \cup B)'$



7 ξ



a $A' = \{R\}$

b $B' = \{G, R\}$

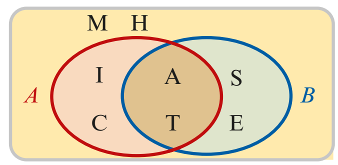
c $A \cap B = \{L, E, A, N\}$

d $A \cup B = \{A, N, G, E, L\}$

e $(A \cup B)' = \{R\}$

f $A' \cup B' = \{G, R\}$

8 ξ



a $A' = \{E, H, M, S\}$

b $B' = \{C, H, I, M\}$

c $A \cap B = \{A, T\}$

d $(A \cup B)' = \{H, M\}$

e $A' \cup B' = \{C, E, H, I, M, S\}$

f $A' \cap B' = \{H, M\}$

9 a There are 10 subsets with 2 elements.
$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\},$
$\{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}$

b There are 10 subsets with 3 elements.
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}$
$\{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\},$

c When you take out 3 a second set of 2 is formed and vica versa

Solutions to Exercise 2B

1 **a** Yes

b Yes

c Yes

2 **a** No. The sum may be rational or irrational, for instance, $\sqrt{2} + \sqrt{3}$ is irrational; $\sqrt{2} + (3 - \sqrt{2}) = 3$ is rational.

b No.The product may be rational or irrational. For instance, $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ is irrational; $\sqrt{2} \times 3\sqrt{2} = 6$ is rational.

c No. The quotient may be rational or irrational. For instance $\dfrac{\sqrt{2}}{\sqrt{3}}$ is irrational; $\dfrac{3\sqrt{2}}{\sqrt{2}} = 3$ is rational.

3 **a** $0.45 = \dfrac{45}{100} = \dfrac{9}{20}$

b
$$\begin{aligned} 0.\dot{2} &= 0.22222\ldots \\ 0.\dot{2} \times 10 &= 2.2222\ldots \\ 0.\dot{2} \times 9 &= 2 \\ \therefore 0.\dot{2} &= \frac{2}{9} \end{aligned}$$

c
$$\begin{aligned} 0.\dot{2}\dot{7} &= 0.272727\ldots \\ 0.\dot{2}\dot{7} \times 100 &= 27.272727\ldots \\ 0.\dot{2}\dot{7} \times 99 &= 27 \\ \therefore 0.\dot{2}\dot{7} &= \frac{27}{99} = \frac{3}{11} \end{aligned}$$

d $0.12 = \dfrac{12}{100} = \dfrac{3}{25}$

e
$$\begin{aligned} 0.\dot{3}\dot{6} &= 0.363636\ldots \\ 0.\dot{3}\dot{6} \times 100 &= 36.3636\ldots \\ 0.\dot{3}\dot{6} \times 99 &= 36 \\ \therefore \quad 0.\dot{3}\dot{6} &= \frac{36}{99} = \frac{4}{11} \end{aligned}$$

f
$$\begin{aligned} 0.\dot{2}8571\dot{4} &= 0.285714285714\ldots \\ 0.\dot{2}8571\dot{4} \times 10^6 &= 285714.285714\ldots \\ 0.\dot{2}8571\dot{4} \times (10^6 - 1) &= 285714 \\ \therefore \quad 0.\dot{2}8571\dot{4} &= \frac{285714}{999999} = \frac{2}{7} \end{aligned}$$

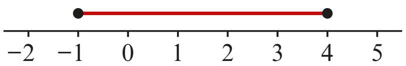
4 **a**
$$\begin{aligned} \frac{2}{7} &= 7\overline{)2.000000}\ldots \\ &= 0.2857142857\ldots \\ &= 0.\dot{2}8571\dot{4} \end{aligned}$$

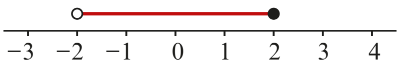
b
$$\begin{aligned} \frac{5}{11} &= 11\overline{)5.000000}\ldots \\ &= 0.454545\ldots \\ &= 0.\dot{4}\dot{5} \end{aligned}$$

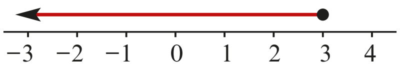
c
$$\begin{aligned} \frac{7}{20} &= 20\overline{)7.00} \\ &= 0.35 \end{aligned}$$

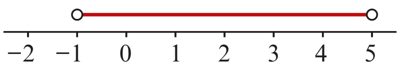
d
$$\begin{aligned} \frac{4}{13} &= 13\overline{)4.000000}\ldots \\ &= 0.30769230\ldots \\ &= 0.\dot{3}0769\dot{2} \end{aligned}$$

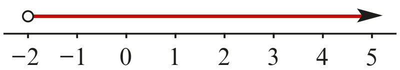
e
$$\begin{aligned} \frac{1}{17} &= 17\overline{)1.00000000000000000}\ldots \\ &= 0.0588235294117647058\ldots \\ &= 0.\dot{0}58823529411764\dot{7} \end{aligned}$$

5 a 

b 

c 

d 

e 

6 a $(-\infty, 3)$

b $[-3, \infty)$

c $(-\infty, -3]$

d $(5, \infty)$

e $[-2, 3)$

f $[-2, 3]$

g $(-2, 3]$

h $(-5, 3)$

Solutions to Exercise 2C

1 a $\sqrt{8} = \sqrt{4} \times \sqrt{2}$

$= 2\sqrt{2}$

b $\sqrt{12} = \sqrt{4} \times \sqrt{3}$

$= 2\sqrt{3}$

c $\sqrt{27} = \sqrt{9} \times \sqrt{3}$

$= 3\sqrt{3}$

d $\sqrt{50} = \sqrt{25} \times \sqrt{2}$

$= 5\sqrt{2}$

e $\sqrt{45} = \sqrt{9} \times \sqrt{5}$

$= 3\sqrt{5}$

f $\sqrt{1210} = \sqrt{121} \times \sqrt{10}$

$= 11\sqrt{10}$

g $\sqrt{98} = \sqrt{49} \times \sqrt{2}$

$= 7\sqrt{2}$

h $\sqrt{108} = \sqrt{36} \times \sqrt{3}$

$= 6\sqrt{3}$

i $\sqrt{25} = 5$

j $\sqrt{75} = \sqrt{25} \times \sqrt{3}$

$= 5\sqrt{3}$

k $\sqrt{512} = \sqrt{256} \times \sqrt{2}$

$= 16\sqrt{2}$

2 a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$

$= \sqrt{4 \times 2} + \sqrt{9 \times 2} - 2\sqrt{2}$

$= 2\sqrt{2} + 3\sqrt{2} - 2\sqrt{2}$

$= 3\sqrt{2}$

b

$\sqrt{75} + 2\sqrt{12} - \sqrt{27}$

$= \sqrt{25 \times 3} + 2\sqrt{4 \times 3} - \sqrt{9 \times 3}$

$= 5\sqrt{3} + 4\sqrt{3} - 3\sqrt{3}$

$= 6\sqrt{3}$

c $\sqrt{28} + \sqrt{175} - \sqrt{63}$

$= \sqrt{4 \times 7} + \sqrt{25 \times 7} - \sqrt{9 \times 7}$

$= 2\sqrt{7} + 5\sqrt{7} - 3\sqrt{7}$

$= 4\sqrt{7}$

d

$\sqrt{1000} - \sqrt{40} - \sqrt{90}$

$= \sqrt{100 \times 10} - \sqrt{4 \times 10} - \sqrt{9 \times 10}$

$= 10\sqrt{10} - 2\sqrt{10} - 3\sqrt{10}$

$= 5\sqrt{10}$

e

$\sqrt{512} + \sqrt{128} + \sqrt{32}$

$= \sqrt{256 \times 2} + \sqrt{64 \times 2} + \sqrt{16 \times 2}$

$= 16\sqrt{2} + 8\sqrt{2} + 4\sqrt{2}$

$= 28\sqrt{2}$

f

$\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

$= \sqrt{4 \times 6} - 3\sqrt{6} - \sqrt{36 \times 6} + \sqrt{49 \times 6}$

$= 2\sqrt{6} - 3\sqrt{6} - 6\sqrt{6} + 7\sqrt{6}$

$= 0$

3 a $\sqrt{75}+\sqrt{108}+\sqrt{14}$

$= \sqrt{25\times 3}+\sqrt{36\times 3}+\sqrt{14}$

$= 5\sqrt{3}+6\sqrt{3}+\sqrt{14}$

$= 11\sqrt{3}+\sqrt{14}$

b $\sqrt{847}-\sqrt{567}+\sqrt{63}$

$= \sqrt{121\times 7}-\sqrt{81\times 7}$

$+\sqrt{9\times 7}$

$= 11\sqrt{7}-9\sqrt{7}+3\sqrt{7}$

$= 5\sqrt{7}$

c $\sqrt{720}-\sqrt{245}-\sqrt{125}$

$= \sqrt{144\times 5}-\sqrt{49\times 5}$

$-\sqrt{25\times 5}$

$= 12\sqrt{5}-7\sqrt{5}-5\sqrt{5}$

$= 0$

d $\sqrt{338}-\sqrt{288}+\sqrt{363}-\sqrt{300}$

$= \sqrt{169\times 2}-\sqrt{144\times 2}$

$+\sqrt{121\times 3}-\sqrt{100\times 3}$

$= 13\sqrt{2}-12\sqrt{2}+11\sqrt{3}$

$-10\sqrt{3}$

$= \sqrt{2}+\sqrt{3}$

e $\sqrt{12}+\sqrt{8}+\sqrt{18}+\sqrt{27}+\sqrt{300}$

$= \sqrt{4\times 3}+\sqrt{4\times 2}+\sqrt{9\times 2}$

$+\sqrt{9\times 3}+\sqrt{100\times 3}$

$= 2\sqrt{3}+2\sqrt{2}+3\sqrt{2}$

$+3\sqrt{3}+10\sqrt{3}$

$= 5\sqrt{2}+15\sqrt{3}$

f

$2\sqrt{18}+3\sqrt{5}-\sqrt{50}+\sqrt{20}-\sqrt{80}$

$= 2\sqrt{9\times 2}+3\sqrt{5}-\sqrt{25\times 2}$

$+\sqrt{4\times 5}-\sqrt{16\times 5}$

$= 6\sqrt{2}+3\sqrt{5}-5\sqrt{2}+2\sqrt{5}-4\sqrt{5}$

$= \sqrt{2}+\sqrt{5}$

4 a $\dfrac{1}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{\sqrt{5}}{5}$

b $\dfrac{1}{\sqrt{7}}\times\dfrac{\sqrt{7}}{\sqrt{7}}=\dfrac{\sqrt{7}}{7}$

c $-\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=-\dfrac{\sqrt{2}}{2}$

d $\dfrac{2}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{2\sqrt{3}}{3}$

e $\dfrac{3}{\sqrt{6}}\times\dfrac{\sqrt{6}}{\sqrt{6}}=\dfrac{3\sqrt{6}}{6}=\dfrac{\sqrt{6}}{2}$

f $\dfrac{1}{2\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{2}}{4}$

g $\dfrac{1}{\sqrt{2}+1}\times\dfrac{\sqrt{2}-1}{\sqrt{2}-1}=\dfrac{\sqrt{2}-1}{2-1}$

$=\dfrac{\sqrt{2}-1}{1}$

$=\sqrt{2}-1$

h $\dfrac{1}{2-\sqrt{3}}\times\dfrac{2+\sqrt{3}}{2+\sqrt{3}}=\dfrac{2+\sqrt{3}}{4-3}$

$=2+\sqrt{3}$

i $\dfrac{1}{4-\sqrt{10}}\times\dfrac{4+\sqrt{10}}{4+\sqrt{10}}=\dfrac{4+\sqrt{10}}{16-10}$

$=\dfrac{4+\sqrt{10}}{6}$

j $\dfrac{2}{\sqrt{6}+2} \times \dfrac{\sqrt{6}-2}{\sqrt{6}-2} = \dfrac{2\sqrt{6}-4}{6-4}$

$= \dfrac{2\sqrt{6}-4}{2}$

$= \sqrt{6}-2$

k $\dfrac{1}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \dfrac{\sqrt{5}+\sqrt{3}}{5-3}$

$= \dfrac{\sqrt{5}+\sqrt{3}}{2}$

l

$\dfrac{3}{\sqrt{6}-\sqrt{5}} \times \dfrac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}} = \dfrac{3(\sqrt{6}+\sqrt{5})}{6-5}$

$= 3(\sqrt{6}+\sqrt{5})$

m $\dfrac{1}{3-2\sqrt{2}} \times \dfrac{3+2\sqrt{2}}{3+2\sqrt{2}} = \dfrac{3+2\sqrt{2}}{9-8}$

$= 3+2\sqrt{2}$

5 a $\dfrac{2}{3-2\sqrt{2}} \times \dfrac{3+2\sqrt{2}}{3+2\sqrt{2}} = \dfrac{6+4\sqrt{2}}{9-8}$

$= 6+4\sqrt{2}$

b $(\sqrt{5}+2)^2 = (\sqrt{5})^2 + 4\sqrt{5} + 4$

$= 5 + 4\sqrt{5} + 4$

$= 9 + 4\sqrt{5}$

c $(1+\sqrt{2})(3-2\sqrt{2})$

$= 3 - 2\sqrt{2} + 3\sqrt{2} - 4$

$= -1 + \sqrt{2}$

d $(\sqrt{3}-1)^2 = 3 - 2\sqrt{3} + 1$

$= 4 - 2\sqrt{3}$

e

$\dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{27}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{27}}{\sqrt{27}} - \dfrac{1}{\sqrt{27}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{3\sqrt{3}-\sqrt{3}}{9}$

$= \dfrac{2\sqrt{3}}{9}$

f $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1} = \dfrac{\sqrt{3}+2}{2\sqrt{3}-1} \times \dfrac{2\sqrt{3}+1}{2\sqrt{3}+1}$

$= \dfrac{6+\sqrt{3}+4\sqrt{3}+2}{12-1}$

$= \dfrac{8+5\sqrt{3}}{11}$

g $\dfrac{\sqrt{5}+1}{\sqrt{5}-1} = \dfrac{\sqrt{5}+1}{\sqrt{5}-1} \times \dfrac{\sqrt{5}+1}{\sqrt{5}+1}$

$= \dfrac{5+2\sqrt{5}+1}{5-1}$

$= \dfrac{6+2\sqrt{5}}{4}$

$= \dfrac{3+\sqrt{5}}{2}$

h $\dfrac{\sqrt{8}+3}{\sqrt{18}+2} = \dfrac{2\sqrt{2}+3}{3\sqrt{2}+2}$

$= \dfrac{2\sqrt{2}+3}{3\sqrt{2}+2} \times \dfrac{3\sqrt{2}-2}{3\sqrt{2}-2}$

$= \dfrac{12-4\sqrt{2}+9\sqrt{2}-6}{18-4}$

$= \dfrac{6+5\sqrt{2}}{14}$

6 a $(2\sqrt{a}-1)^2 = (2\sqrt{a}-1)(2\sqrt{a}-1)$

$= 4a - 2\sqrt{a} - 2\sqrt{a} + 1$

$= 4a - 4\sqrt{a} + 1$

b $(\sqrt{x+1} + \sqrt{x+2})^2$

$= (\sqrt{x+1} + \sqrt{x+2})$

$\times (\sqrt{x+1} + \sqrt{x+2})$

$= x + 1 + 2\sqrt{(x+1)(x+2)}$

$+ x + 2$

$= 2x + 3 + 2\sqrt{(x+1)(x+2)}$

7 $7, 3\sqrt{5}, 5\sqrt{2}, 4\sqrt{3}$
Squaring these: 49, 45, 50, 48
Hence $3\sqrt{5} < 4\sqrt{3} < 7 < 5\sqrt{2}$

8 a $(5 - 3\sqrt{2}) - (6\sqrt{2} - 8)$

$= 5 - 3\sqrt{2} - 6\sqrt{2} + 8$

$= 13 - 9\sqrt{2}$

$= \sqrt{169} - \sqrt{162}$

> 0

$5 - 3\sqrt{2}$ is larger.

b $(2\sqrt{6} - 3) - (7 - 2\sqrt{6})$

$= 2\sqrt{6} - 3 - 7 + 2\sqrt{6}$

$= 4\sqrt{6} - 10$

$= \sqrt{96} - \sqrt{100}$

< 0

$7 - 2\sqrt{6}$ is larger.

9 a $\frac{4}{3} < \frac{9}{2} \Rightarrow \frac{2}{\sqrt{3}} < \frac{3}{\sqrt{2}}$

b $\frac{7}{9} < \frac{5}{4} \Rightarrow \frac{\sqrt{7}}{3} < \frac{\sqrt{5}}{2}$

c $\frac{3}{49} < \frac{1}{5} \Rightarrow \frac{\sqrt{3}}{7} < \frac{\sqrt{5}}{5}$

d $\frac{10}{4} < \frac{64}{3} \Rightarrow \frac{\sqrt{10}}{2} < \frac{8}{\sqrt{3}}$

10 a $(x - \sqrt{3})(x + \sqrt{3}) = x^2 - 3$
Therefore $b = 0$ and $c = -3$

b $(x - 2\sqrt{3})(x + 2\sqrt{3}) = x^2 - 12$
Therefore $b = 0$ and $c = -12$

c $(x - (1 - \sqrt{2})(x - (1 + \sqrt{2}) =$
$x^2 - 2x - 1$
Therefore $b = -2$ and $c = -1$

d $(x - (2 - \sqrt{3})(x - (2 + \sqrt{1}) =$
$x^2 - 4x + 1$
Therefore $b = -4$ and $c = 1$

e $(x - (3 - 2\sqrt{2})(x - (3 + 2\sqrt{2}) =$
$x^2 - 6x + 1$
Therefore $b = -6$ and $c = 1$

f $(x - (4 - 7\sqrt{5})(x - (3 + 2\sqrt{5}) =$
$x^2 - (= -7 + 5\sqrt{5})x - 58 - 13\sqrt{5}$
Therefore $b = -7 + 5\sqrt{5}$ and
$c = -58 - 13\sqrt{5}$

11 $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}} \times \dfrac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$

$= \dfrac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{(\sqrt{2}+\sqrt{3})^2-5}$

$= \dfrac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{(2+3+2\sqrt{6}-5}$

$= \dfrac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}}$

$= \dfrac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}} \times \dfrac{\sqrt{6}}{\sqrt{6}}$

$= \dfrac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{12}$

$= \dfrac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$

12 **a** Note $a - b = \left(a^{\frac{1}{3}}\right)^3 - \left(b^{\frac{1}{3}}\right)^3$

b $\dfrac{1}{1-2^{\frac{1}{3}}} \times \dfrac{1+2^{\frac{1}{3}}+2^{\frac{2}{3}}}{1+2^{\frac{1}{3}}+2^{\frac{2}{3}}}$

$= -(1+2^{\frac{1}{3}}+2^{\frac{2}{3}})$

13 $\dfrac{1}{\sqrt{4}+\sqrt{5}} + \dfrac{1}{\sqrt{5}+\sqrt{6}} + \dfrac{1}{\sqrt{5}+\sqrt{6}} +$

$\cdots + \dfrac{1}{\sqrt{23}+\sqrt{24}} + \dfrac{1}{\sqrt{24}+\sqrt{25}}$

Rationalising each term:

$\dfrac{\sqrt{4}-\sqrt{5}}{4-5} + \dfrac{\sqrt{5}-\sqrt{6}}{5-6} + \cdots +$

$\dfrac{\sqrt{23}-\sqrt{24}}{23-24} + \dfrac{\sqrt{24}-\sqrt{25}}{24-25}$

$= \sqrt{5}-\sqrt{4}+\sqrt{6}-\sqrt{5}+\sqrt{7}-\sqrt{6}+$

$\cdots + \sqrt{24}-\sqrt{23}+\sqrt{25}-\sqrt{24}$

$= 5 - 2$

$= 3$

Solutions to Exercise 2D

1 **a** $2^2 \times 3 \times 5$

b $2^2 \times 13^2$

c $2^2 \times 3 \times 19$

d $2^2 \times 3^2 \times 5^2$

e $2^2 \times 3^2 \times 7$

f $2^2 \times 3^2 \times 5^2 \times 7$

g

$$\begin{array}{r|r}
2 & 68\,640 \\
2 & 34\,320 \\
2 & 17\,160 \\
2 & 8580 \\
2 & 4290 \\
3 & 2145 \\
5 & 715 \\
11 & 143 \\
13 & 13 \\
\hline
 & 1
\end{array}$$

Prime decomposition

$= 2^5 \times 3 \times 5 \times 11 \times 13$

h

$$\begin{array}{r|r}
2 & 96\,096 \\
2 & 48\,048 \\
2 & 24\,024 \\
2 & 12\,012 \\
2 & 6006 \\
3 & 3003 \\
7 & 1001 \\
11 & 143 \\
13 & 13 \\
\hline
 & 1
\end{array}$$

Prime decomposition

$= 2^5 \times 3 \times 7 \times 11 \times 13$

i

$$\begin{array}{r|r}
2 & 32\,032 \\
2 & 16\,016 \\
2 & 8008 \\
2 & 4004 \\
2 & 2002 \\
7 & 1001 \\
11 & 143 \\
13 & 13 \\
\hline
 & 1
\end{array}$$

Prime decomposition

$= 2^5 \times 7 \times 11 \times 13$

j $2\overline{)544\,544}$

$2\overline{)272\,272}$

$2\overline{)136\,136}$

$2\overline{)\ 68\,068}$

$2\overline{)\ 34\,034}$

$7\overline{)\ 17\,017}$

$11\overline{)\ 2431}$

$13\overline{)\ \ 221}$

$17\overline{)\ \ \ 17}$

$\overline{\ \ \ \ \ \ \ 1}$

Prime decomposition
$= 2^5 \times 7 \times 11 \times 13 \times 17$

2 For each part, first find the prime decomposition of each number.

a $4361 = 7^2 \times 89$
Neither 7 nor 89 are factors of 9281.
HCF = 1

b $999 = 3^3 \times 37$

$2160 = 2^4 \times 3^3 \times 5$

$\text{HCF} = 3^3 = 27$

c $5255 = 5 \times 1051$
716 845 is divisible by 5 but not 1051.
HCF = 5

d $1271 = 31 \times 41$

$3875 = 5^3 \times 31$

HCF = 31

e $804 = 2^2 \times 3 \times 67$

$2358 = 2 \times 3^2 \times 131$

$\text{HCF} = 2 \times 3 = 6$

3 **a** $18 = 3^2 \times 2$
Factors: 1, 2, 3,6, 9,18.
$36 = 3^2 \times 2^2$
Factors: 1, 2, 4, 3, 6, 12, 9, 18, 36

b 36 is a perfect square

c $121 = 11^2$. It has to be a perfect square to have an odd number of factors. To have 3 it must be the perfect square of a prime.

4 $1050 = 7 \times 5^2 \times 3 \times 2$
Children are teenagers: Ages:
$7 \times 2 = 14$
$5 \times 3 = 15$
5

5 $60 = 2^2 \times 3 \times 5$
Therefore $60 \times 3 \times 5 = 2^2 \times 3^2 \times 5^2$
Hence $n = 15$ is the smallest natural number.

6 $22^2 \times 55^2 = 10^2 \times n^2$
$(11 \times 2)^2 \times (11 \times 5^2) = 10^2 \times n^2$
$\therefore 11^2 \times 11^2 \times (5 \times 2)^2 = 10^2 \times n^2$
$\therefore n = 121$

7 $5 \times 3 \times 7 \times 3 = 7 \times 5 \times 3^2$.
This has 12 factors Therefore the starting number is $7 \times 5 \times 3 = 105$. It has 8 factors.

8 $720 = 5 \times 3^2 \times 2^4$
$720 = 2^3 \times 2 \times 3^2 \times 5$
$720 = 8 \times 9 \times 10$. $n = 8$

9 $30 = 2 \times 3 \times 5$
Factors are: $1, 3, 5, 2, 2 \times 3, 2 \times 5, 3 \times 5, 2 \times 3 \times 5$
Product $= 2^4 \times 3^4 \times 5^4 = 30^4$

10 LCM is 252 which is 4 hours and 12 minutes. That is 1:12 pm.

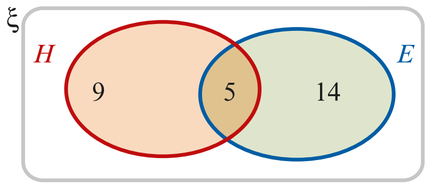
11 The LCM is formed by taking the highest power of each of the prime factors and the HCF formed by taking the lowest power.
So the two numbers are each of the form $2^\alpha 3^\beta 5^\gamma$.

- One number could be $2^3 \times 3 \times 5^2 = 600$ and the other will be $2^5 \times 3^3 \times 5^3 = 108000$
- One number could be $2^5 \times 3 \times 5^2 = 2400$ and the other will be $2^3 \times 3^3 \times 5^3 = 27\,000$
- One number could be $2^3 \times 3^3 \times 5^2 = 5400$ and the other will be $2^5 \times 3 \times 5^3 = 12\,000$
- One number could be $2^3 \times 3 \times 5^3 = 3000$ and the other will be $2^5 \times 3^3 \times 5^2 = 21600$

600 and 108 000
2400 and 27 000
3000 and 21 600
5400 and 12 000

Solutions to Exercise 2E

1 **a**



Since all students do at least one of these subjects, $9 + 5 + x = 28$

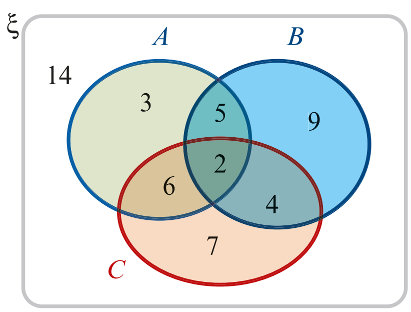
$$x = 14$$

b **i** $5 + 14 = 19$

ii 9

iii $9 + 14 = 23$ or $28 - 5 = 23$

2 **a**



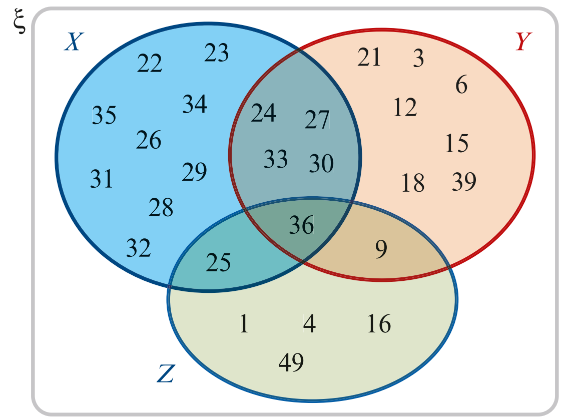
b **i** $n(A' \cap C') = 9 + 14 = 23$

ii

$$n(A \cup B') = 3 + 6 + 5 + 2 + 7 + 14$$
$$= 37$$

iii $n(A' \cap B \cap C') = 9$

3



Since 40% don't speak Greek,

$y + 20\% = 40\%$

$y = 20\%$

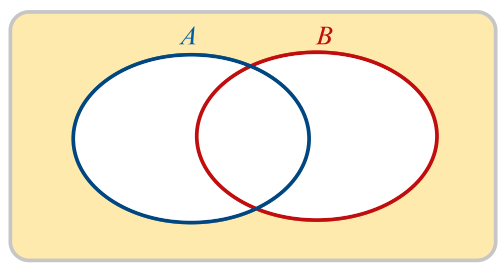
Since 40% speak Greek,

$x + 20\% = 40\%$

$x = 20\%$

20% speak both languages.

4 **a**



$(A \cup B)' = A' \cap B'$ is shaded

We must assume every delegate spoke at least one of these languages. If 70 spoke English, and 25 spoke English and French, 45 spoke English but not French.

$\therefore$ $45 + 50 = 95$ spoke either English or French or both.

$\therefore$ $105 - 95 = 10$ spoke only Japanese.

If 50 spoke French, and 15 spoke French and Japanese, 35 spoke

French but not Japanese.

$\therefore$ 35 + 50 = 85 spoke either French or Japanese or both.

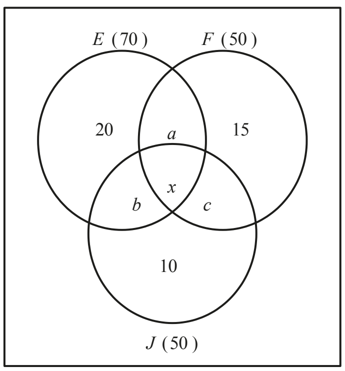
$\therefore$ 105 − 85 = 20 spoke only English.

If 50 spoke Japanese, and 30 spoke Japanese and English, 20 spoke Japanese but not English.

$\therefore$ 20 + 70 = 90 spoke either Japanese or English or both.

$\therefore$ 105 − 90 = 15 spoke only French.

We can now fill in more of the Venn diagram.



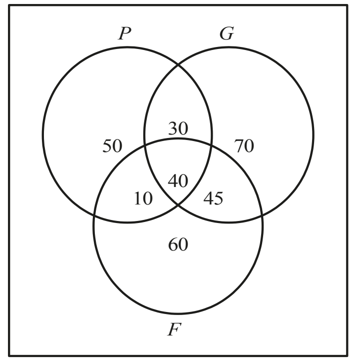
c is the number who don't speak English.

$$105 - 70 = 10 + c + 15$$
$$c + 25 = 35$$
$$c = 10$$
$$x + c = 15$$
$$x = 5$$

5 delegates speak all five languages.

b We have already found that 10 spoke only Japanese.

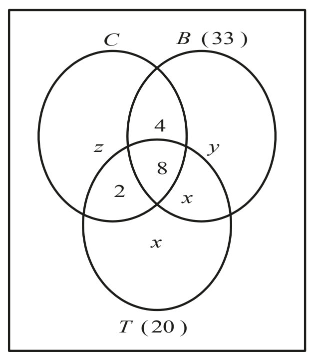
5 Enter the information into a Venn diagram.



Number having no dessert

$$= 350 - 50 - 30 - 70 - 10$$
$$- 40 - 45 - 60$$
$$= 45$$

6 Insert the given information on a Venn diagram. Place y as the number taking a bus only, and z as the number taking a car only.



a Using $n(T) = 20$, $2x + 10 = 20$

$$x = 5$$

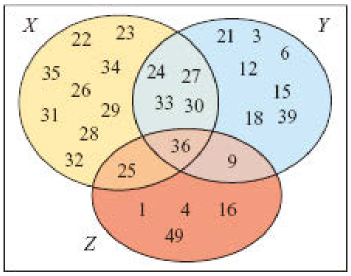
b Using $n(B) = 33$ and $x = 5$,

$$12 + 5 + y = 33$$
$$y = 16$$

c Assume they all used at least one of these forms of transport.

$$z + 4 + 8 + 16 + 2 + 5 + 5 = 40$$
$$z = 0$$

7 **a**



b **i**

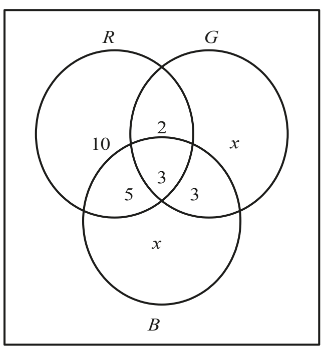
$(X \cap Y \cap \mathbb{Z})$ = intersection of all sets

= 36 (from diagram)

ii $|X \cap Y|$ = number of elements in both X and Y

= 5 (from diagram)

8 The following information can be placed on a Venn diagram.



The additional information gives $5 > x$ and $x > 3$.

$\therefore \quad x = 4$

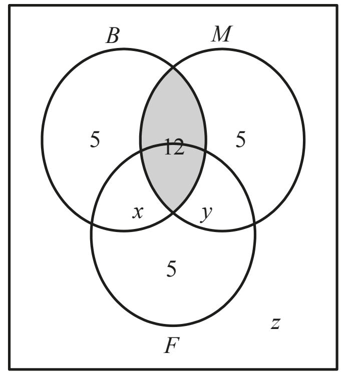
Number of students

$= 10 + 2 + 4 + 5 + 3 + 3 + 4$

$= 31$

20 bought red pens, 12 bought green pens and 15 bought black pens.

9 Enter the given information as below. $B \cap M$ is shaded.



$5 + 12 + 5 + 5 + x + y + z = 28$

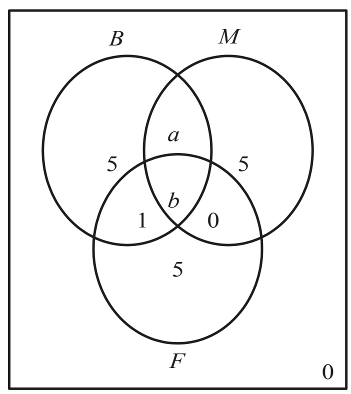
$27 + x + y + z = 28$

$x + y + z = 1$

This means that exactly one of x, y and z must equal 1, and the other two will equal zero.

Since $|F \cap B| > |M \cap F|$, the Venn diagram shows that this means $x > y$.

$\therefore \quad x = 1, y = z = 0$



$a + b = 12$

$|M \cap F \cap B| = |F'|$

$\therefore \quad b = a + 10$

Substitute in $a + b = 12$:

$a + (a + 10) = 12$

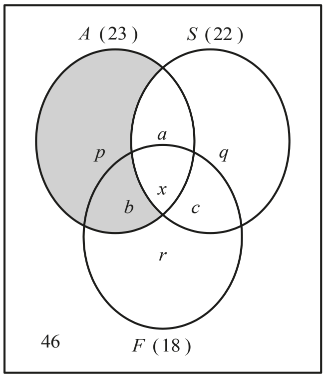
$2a = 2$

$a = 1$

$b = a + 10 = 11$

$|M \cap F| = b + 0 = 11$

10 Enter the given information as below.



$a + x = |A \cap S| = 10$

The number of elements in the shaded region is given by

$$|A \cap S'| = |A| - (a + x)$$
$$= 23 - 10$$
$$= 13$$
$$|A \cup S| = 10 + 22$$
$$= 32$$
$$\therefore r + 46 = 80 - 32 = 48$$
$$r = 2$$

Use similar reasoning to show

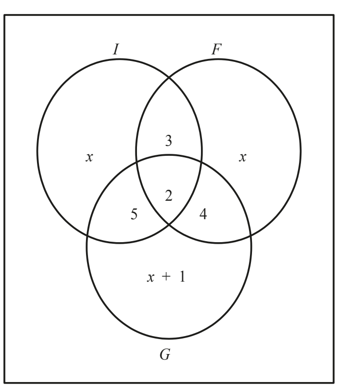
$$c + r = 18 - (b + x)$$
$$= 18 - 11 = 7$$

Since $r = 2$, $c = 5$

Since $x + c = |S \cup F| = 6$ and $c = 5$, $x = 1$

One person plays all three sports.

11 Enter the information into a Venn diagram.



Since they are all proficient in at least one language,

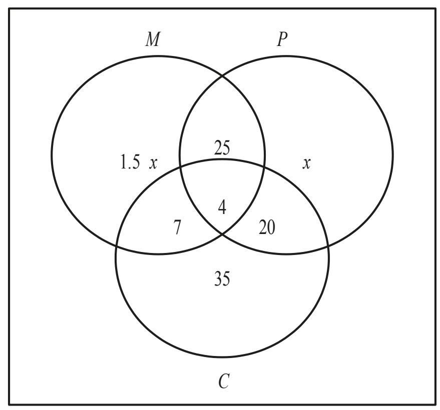
$$x + 3 + x + 5 + 2 + 4 + x + 1 = 33$$
$$3x + 15 = 33$$
$$3x = 18$$
$$x = 6$$

The number proficient in Italian

$= 6 + 3 + 2 + 5$

$= 16$

12 Enter the given information into a Venn diagram.



$$1.5x + 25 + x + 7 + 4 + 20 + 35 = 201$$
$$2.5x + 91 = 201$$
$$2.5x = 110$$
$$x = \frac{110}{2.5}$$
$$= 44$$

The number studying Mathematics

$= 1.5x + 25 + 7 + 4$

$= 66 + 25 + 7 + 4$

$= 102$

Solutions to short-answer questions

1 a

$$\begin{aligned} 0.0\dot{7} &= 0.07777\ldots \\ 0.0\dot{7} \times 10 &= 0.7777\ldots \\ 0.0\dot{7} \times 9 &= 0.7 = \frac{7}{10} \\ 0.0\dot{7} &= \frac{7}{90} \end{aligned}$$

b

$$\begin{aligned} 0.\dot{4}\dot{5} &= 0.454545\ldots \\ 0.\dot{4}\dot{5} \times 100 &= 45.4545\ldots \\ 0.\dot{4}\dot{5} \times 99 &= 45 \\ 0.\dot{4}\dot{5} &= \frac{45}{99} = \frac{5}{11} \end{aligned}$$

c $0.005 = \dfrac{5}{1000} = \dfrac{1}{200}$

d $0.405 = \dfrac{405}{1000} = \dfrac{81}{200}$

e

$$\begin{aligned} 0.2\dot{6} &= 0.26666\ldots \\ 0.2\dot{6} \times 10 &= 2.6666\ldots \\ 0.2\dot{6} \times 9 &= 2.4 = \frac{24}{10} \\ 0.2\dot{6} &= \frac{24}{90} = \frac{4}{15} \end{aligned}$$

f

$$\begin{aligned} &0.1\dot{7}1428\dot{5} \\ &\quad = 0.1714825714\ldots \\ &0.1\dot{7}1428\dot{5} \times 10^6 \\ &\quad = 171\,428.5714285\ldots \\ &0.1\dot{7}1428\dot{5} \times (10^6 - 1) \\ &\quad = 171\,428.4 \\ &\quad = \frac{1\,714\,284}{10} \\ &0.1\dot{7}1428\dot{5} \\ &\quad = \frac{1\,714\,284}{9\,999\,990} \\ &\quad = \frac{6}{35} \end{aligned}$$

2

$$\begin{array}{r|r} 2 & 504 \\ 2 & 252 \\ 2 & 126 \\ 3 & 63 \\ 3 & 21 \\ 7 & 7 \\ \hline & 1 \end{array}$$

$504 = 2^3 \times 3^2 \times 7$

3 a

$$\begin{aligned} \frac{2\sqrt{3} - 1}{\sqrt{2}} &= \frac{2\sqrt{3} - 1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{6} - \sqrt{2}}{2} \end{aligned}$$

b

$$\begin{aligned} \frac{\sqrt{5} + 2}{\sqrt{5} - 2} &= \frac{\sqrt{5} + 2}{\sqrt{5} - 2} \times \frac{\sqrt{5} + 2}{\sqrt{5} + 2} \\ &= \frac{5 + 4\sqrt{5} + 4}{5 - 4} \\ &= 4\sqrt{5} + 9 \end{aligned}$$

c $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \times \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

$= \dfrac{3+2\sqrt{6}+2}{3-2}$

$= 2\sqrt{6}+5$

4 $\dfrac{3+2\sqrt{75}}{3-\sqrt{12}} = \dfrac{3+2\sqrt{25\times 3}}{3-\sqrt{4\times 3}}$

$= \dfrac{3+2\times 5\sqrt{3}}{3-2\sqrt{3}}$

$= \dfrac{3+10\sqrt{3}}{3-2\sqrt{3}} \times \dfrac{3+2\sqrt{3}}{3+2\sqrt{3}}$

$= \dfrac{9+6\sqrt{3}+30\sqrt{3}+60}{9-12}$

$= \dfrac{69+36\sqrt{3}}{-3}$

$= -23-12\sqrt{3}$

5 **a** $\dfrac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}} = \dfrac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$

$\times \dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$

$= \dfrac{36+12\sqrt{6}}{18-12}$

$= \dfrac{36+12\sqrt{6}}{6}$

$= 6+2\sqrt{6}$

b $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$

$= \dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$

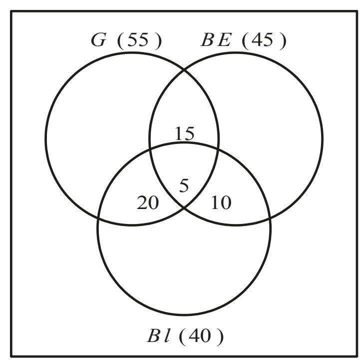
$\times \dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$

$= \dfrac{a+b-2\sqrt{(a+b)(a-b)}+a-b}{(a+b)-(a-b)}$

$= \dfrac{2a-2\sqrt{a^2-b^2}}{2b}$

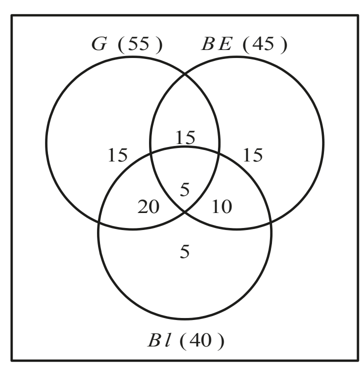
$= \dfrac{a-\sqrt{a^2-b^2}}{b}$

6 First enter the information on a Venn diagram.



a It is obvious to make up the 40 blonds that 5 must be blond only, so the number of boys (not girls) who are blond is 5 + 10 = 15.

b The rest of the Venn diagram can be filled in the same way:

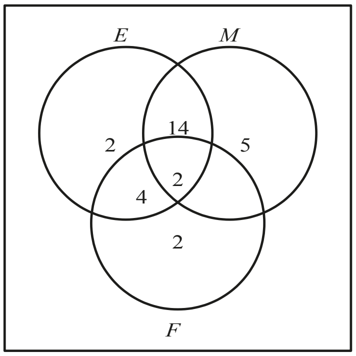


Boys not blond or blue-eyed

$= 100 - 15 - 15 - 15 - 20 - 5 - 10 - 5$

$= 15$

7 Fill in a Venn diagram.

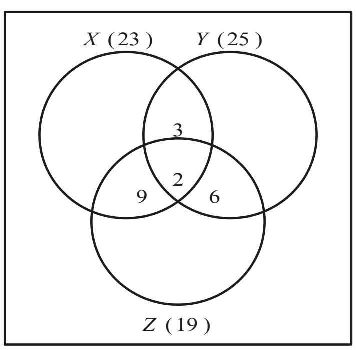


a $30 - 2 - 14 - 5 - 4 - 2 - 2 = 1$ (since all received at least one prize.)

b $14 + 5 + 2 + 1 = 22$

c $2 + 14 + 4 + 2 = 22$

8 Enter the given information on a Venn diagram as below.

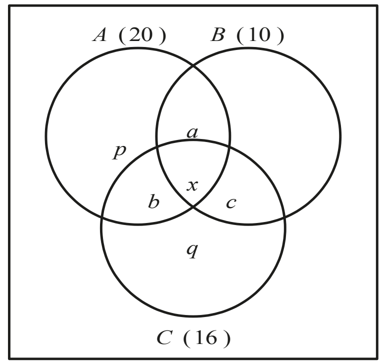


The numbers liking X only, Y only and Z only are 9, 14 and 2 respectively. The number who like none of them

$= 50 - 9 - 3 - 14 - 9 - 2 - 6 - 2$

$= 5$

9 The rectangles can be represented by circles for clarity. Enter the data:



Note: $a + x = 3$, $b + x = 6$ and $c + x = 4$

$$p + b + a + x = 20$$
$$p + b + 3 = 20$$
$$p + b = 17$$
$$q + (p + b) + n(B) = 35$$
$$q + 17 + 10 = 35$$
$$\therefore q = 8$$
$$q + (b + x) + c = n(C) = 16$$
$$8 + 6 + c = 16$$
$$\therefore c = 2$$
$$c + x = 4$$
$$\therefore x = 2$$

There is 2 cm^2 in common.

10 $\sqrt{112} - \sqrt{63} - \dfrac{224}{\sqrt{28}}$

$$= \sqrt{16 \times 7} - \sqrt{9 \times 7} - \frac{224}{\sqrt{4 \times 7}}$$
$$= 4\sqrt{7} - 3\sqrt{7} - \frac{224}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$
$$= 4\sqrt{7} - 3\sqrt{7} - \frac{224\sqrt{7}}{14}$$
$$= 4\sqrt{7} - 3\sqrt{7} - 16\sqrt{7}$$
$$= -15\sqrt{7}$$

11 Cross multiply:

$$(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3}) = x^2$$
$$7 - 3 = x^2$$
$$4 = x^2$$
$$x = \pm 2$$

12 $$\frac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} + \frac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}}$$

$$= \frac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} \times \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}}$$
$$+ \frac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}}$$
$$= \frac{\sqrt{5} - \sqrt{5} + \sqrt{10} - \sqrt{6}}{5 - 3}$$
$$+ \frac{\sqrt{5} + \sqrt{5} - \sqrt{10} - \sqrt{6}}{5 - 3}$$
$$= \frac{2\sqrt{5} - 2\sqrt{6}}{2}$$
$$= \sqrt{5} - \sqrt{6}$$

13 $$\sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\frac{48}{25}}$$

$$= \sqrt{9 \times 3} - \sqrt{4 \times 3}$$
$$+ 2\sqrt{25 \times 3} - \frac{\sqrt{16 \times 3}}{\sqrt{25}}$$
$$= 3\sqrt{3} - 2\sqrt{3} + 10\sqrt{3} - \frac{4\sqrt{3}}{5}$$
$$= \frac{15\sqrt{3} - 10\sqrt{3} + 50\sqrt{3} - 4\sqrt{3}}{5}$$
$$= \frac{51\sqrt{3}}{5}$$

14 $17 + 6\sqrt{8} = 17 + 2 \times \sqrt{9} \times \sqrt{8}$

$$= 17 + 2\sqrt{72}$$

$a + b = 17$; $ab = 72$

$a = 8$, $b = 9$ (or $a = 9$, $b = 8$, giving the same answer.)

$(\sqrt{8} + \sqrt{9})^2 = 17 + 6\sqrt{8}$

So the square root of

$17 + 6\sqrt{8} = \sqrt{8} + \sqrt{9}$

$$= 2\sqrt{2} + 3$$

15 **a** $|A \cup B| = 32 + 7 + 15 + 3 = 57$

b $C = 3$

c $B' \cap A = 32$

Solutions to multiple-choice questions

1 A $$\frac{4}{3+2\sqrt{2}} = \frac{4}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$
$$= \frac{12-8\sqrt{2}}{9-8}$$
$$= 12-8\sqrt{2}$$

2 D
$$\begin{array}{r|r} 2 & 86\,400 \\ 2 & 43\,200 \\ 2 & 21\,600 \\ 2 & 10\,800 \\ 2 & 5400 \\ 2 & 2700 \\ 2 & 1350 \\ 3 & 675 \\ 3 & 225 \\ 3 & 75 \\ 5 & 25 \\ 5 & 5 \\ \hline & 1 \end{array}$$
Prime decomposition
$$= 2^7 \times 3^3 \times 5^2$$

3 D $(\sqrt{6}+3)(\sqrt{6}-3)$
$$= (\sqrt{6})^2 + 3\sqrt{6} - 3\sqrt{6} - 9$$
$$= 6 - 9$$
$$= -3$$

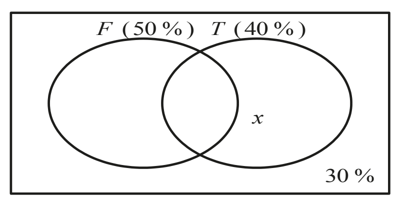
4 D $B' \cap A$ = numbers in set A that are not also in set B
$$= \{1, 2, 4, 5, 7, 8\}$$

5 C $(3, \infty) \cap (-\infty, 5]$
$$= \{x \in R : x > 3\} \cap \{x \in R : x \le 5\}$$
$$= \{x \in R : 3 < x \le 5\}$$
$$= (3, 5]$$

6 D The next time will be both a multiple of 6 and a multiple of 14.
$$\text{LCM} = \frac{6 \times 14}{3}$$
$$= 42$$
The next time is in 42 minutes.

7 B
$X \cap Y \cap \mathbb{Z}$ = set of numbers that are multiples of 2, 5 and 7
$$\text{LCM} = 2 \times 5 \times 7$$
$$= 35$$

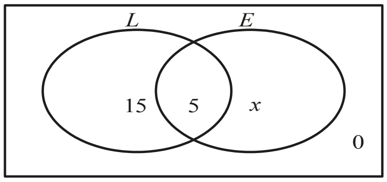
8 B Draw a Venn diagram.



Since 50% don't play football,
$$x + 30\% = 50\%$$
$$x = 20\%$$
Since 40% play tennis, it can be seen that 20% play both sports.

9 C $$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} = \frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} \times \frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}-\sqrt{6}}$$
$$= \frac{7-2\sqrt{42}+6}{7-6}$$
$$= 13-2\sqrt{42}$$

10 A Draw a Venn diagram.



$15 + 5 + x = 40$

$x = 20$

20 students take only Economics.

11 D You can choose any number of 2s from 0 to p in $(p + 1)$ ways. For each of these, you can choose any number of 3s from 0 to q in $(q + 1)$ ways, and for each of these combinations you can choose any number of 5s from 0 to r in $(r + 1)$ ways.
The total number of ways = $(p + 1)(q + 1)(r + 1)$

12 B $m + n = mn$

$$n = mn - m$$
$$= m(n - 1)$$
$$m = \frac{n}{n-1}$$

This will only be an integer if $n = 2, m = 2$ or $n = 0, m = 0$.
There are two solutions.

Solutions to extended-response questions

1 **a** $(\sqrt{x}+\sqrt{y})^2 = (\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y})$

$$= \sqrt{x}(\sqrt{x}+\sqrt{y}) + \sqrt{y}(\sqrt{x}+\sqrt{y})$$
$$= x + \sqrt{x}\sqrt{y} + \sqrt{y}\sqrt{x} + y$$
$$= x + y + 2\sqrt{x}\sqrt{y}$$
$$= x + y + 2\sqrt{xy}$$

b From **a**, $(\sqrt{3}+\sqrt{5})^2 = 3 + 5 + 2\sqrt{3}\sqrt{5}$

$$= 8 + 2\sqrt{15}$$

$$\therefore \quad \sqrt{3}+\sqrt{5} = \sqrt{8+2\sqrt{15}}$$

c **i** $(\sqrt{11}+\sqrt{3})^2 = 11 + 3 + 2\sqrt{11}\sqrt{3}$

$$= 14 + 2\sqrt{33}$$

$$\therefore \sqrt{14+2\sqrt{33}} = \sqrt{11}+\sqrt{3}$$

ii $(\sqrt{8}-\sqrt{7})^2 = 8 + 7 - 2\sqrt{8}\sqrt{7}$ (also consider $-\sqrt{8}+\sqrt{7}$)

$$= 15 - 2\sqrt{56}$$

$$\therefore \sqrt{15-2\sqrt{56}} = \sqrt{8}-\sqrt{7}$$
$$= 2\sqrt{2}-\sqrt{7}$$

iii $(\sqrt{27}-\sqrt{24})^2 = 27 + 24 - 2\sqrt{27}\sqrt{24}$

$$= 51 - 2\times 3\sqrt{3}\times 2\sqrt{3}\sqrt{2}$$
$$= 51 - 36\sqrt{2}$$

$$\therefore \sqrt{51-36\sqrt{2}} = \sqrt{27}-\sqrt{24}$$
$$= 3\sqrt{3}-2\sqrt{6}$$

2 **a** $(2+3\sqrt{3}) + (4+2\sqrt{3}) = 2 + 4 + 3\sqrt{3} + 2\sqrt{3}$

$$= 6 + 5\sqrt{3}$$

Hence $a = 6$ and $b = 5$.

b $(2 + 3\sqrt{3})(4 + 2\sqrt{3}) = 2(4 + 2\sqrt{3}) + 3\sqrt{3}(4 + 2\sqrt{3})$

$$= 8 + 4\sqrt{3} + 12\sqrt{3} + 18$$

$$= 26 + 16\sqrt{3}$$

Hence $p = 26$ and $q = 16$.

c $\dfrac{1}{3 + 2\sqrt{3}} = \dfrac{1}{3 + 2\sqrt{3}} \times \dfrac{3 - 2\sqrt{3}}{3 - 2\sqrt{3}}$

$$= \frac{3 - 2\sqrt{3}}{9 - 12}$$

$$= \frac{3 - 2\sqrt{3}}{-3}$$

$$= -1 + \frac{2}{3}\sqrt{3}$$

Hence $a = -1$ and $b = \dfrac{2}{3}$.

d **i** $(2 + 5\sqrt{3})x = 2 - \sqrt{3}$

$$\therefore x = \frac{2 - \sqrt{3}}{2 + 5\sqrt{3}}$$

$$= \frac{2 - \sqrt{3}}{2 + 5\sqrt{3}} \times \frac{2 - 5\sqrt{3}}{2 - 5\sqrt{3}}$$

$$= \frac{(2 - \sqrt{3})(2 - 5\sqrt{3})}{4 - 75}$$

$$= \frac{2(2 - 5\sqrt{3}) - \sqrt{3}(2 - 5\sqrt{3})}{-71}$$

$$= \frac{4 - 10\sqrt{3} - 2\sqrt{3} + 15}{-71}$$

$$= \frac{19 - 12\sqrt{3}}{-71}$$

$$= \frac{12\sqrt{3} - 19}{71}$$

ii $(x - 3)^2 - 3 = 0$

$$\therefore \quad (x - 3)^2 = 3$$

$$\therefore \quad x - 3 = \pm\sqrt{3}$$

$$\therefore \quad x = 3 \pm \sqrt{3}$$

iii $(2x - 1)^2 - 3 = 0$

$\therefore \quad (2x - 1)^2 = 3$

$\therefore \quad 2x - 1 = \pm\sqrt{3}$

$\therefore \quad 2x = 1 \pm \sqrt{3}$

$\therefore \quad x = \dfrac{1 \pm \sqrt{3}}{2}$

e If $b = 0$, $a + b\sqrt{3} = a$. Hence every rational number, a, is a member of $\{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$.

3 a $x = 2mn$

$= 2 \times 5 \times 2$

$= 20$

$y = m^2 - n^2$

$= 5^2 - 2^2$

$= 25 - 4$

$= 21$

$z = m^2 + n^2$

$= 5^2 + 2^2$

$= 25 + 4$

$= 29$

b

$$\begin{aligned} x^2 + y^2 &= (2mn)^2 + (m^2 - n^2)^2 \\ &= 4m^2n^2 + m^4 - 2m^2n^2 + n^4 \\ &= 2m^2n^2 + m^4 + n^4 \\ z^2 &= (m^2 + n^2)^2 \\ &= m^4 + 2m^2n^2 + n^4 \end{aligned}$$

$\therefore x^2 + y^2 = z^2$

4 a i $2^3 = 8$. Factors of 8 are 1, 2, 4 and 8. Hence 2^3 has four factors.

ii $3^7 = 2187$. Factors of 2187 are 1, 3, 9, 27, 81, 243, 729 and 2187. Hence 3^7 has eight factors.

$2^1 = 2$ Factors are 1, 2. Hence 2^1 has two factors.
$2^2 = 4$ Factors are 1, 2, 4. Hence 2^2 has three factors.

b $2^3 = 8$ Factors are 1, 2, 4, 8. Hence 2^3 has four factors.
$2^4 = 16$ Factors are 1, 2, 4, 8, 16. Hence 2^4 has five factors.
2^n has $n + 1$ factors.

c **i** $2^1.3^1 = 6$. Factors are 1, 2, 3, 6. There are four factors.
$2^1.3^2 = 18$. Factors are 1, 2, 3, 6, 9, 18. There are six factors.
$2^2.3^2 = 36$. Factors are 1, 2, 3, 4, 6, 9, 12, 18, 36. There are nine factors.
$2^2.3^3 = 108$. Factors are 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108. There are twelve factors.
$2^3.3^7$ has $(3 + 1)(7 + 1) = 32$ factors.

ii $2^n.3^m$ has $(n + 1)(m + 1)$ factors.

d The following table investigates the relationship between the number of factors of x and its prime factorisation.

x	Factors	Number of factors	Prime factorisation	Number of factors
1	1	1		0 + 1
2	1, 2	2	2^1	1 + 1
3	1, 3	2	3^1	1 + 1
4	1, 2, 4	3	2^2	2 + 1
5	1, 5	2	5^1	1 + 1
6	1, 2, 3, 6	4	$2^1.3^1$	(1 + 1)(1 + 1)

e For any number x, there are $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_n + 1)$ factors.

$$\begin{aligned} 8 &= 4 \times 2 \\ &= (3 + 1)(1 + 1) \end{aligned}$$

Now $2^3.3^1 = 24$
The smallest number which has eight factors is 24.

5 **a** $1080 = 2^3 \times 3^3 \times 5$ $25\,200 = 2^4 \times 3^2 \times 5^2 \times 7$

b Least common multiple of 1080 and 25 200 is $2^4 \times 3^3 \times 5^2 \times 7 = 75\,600$

c HCF of m and n = $p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \cdots p_n^{\min(\alpha_n,\beta_n)}$
$\therefore$ the product of the HCF and LCM

$$\begin{aligned} &= p_1^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)}\, p_2^{\min(\alpha_2,\beta_2)+\max(\alpha_2,\beta_2)} \cdots Pn^{\min(\alpha_n,\beta_n)+\max(\alpha_n,\beta_n)} \\ &= p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2}\; p_n^{\alpha_n+\beta_n} \\ &= mn \end{aligned}$$

d **i** The lowest common multiple of 5, 7, 9 and 11 is 3465.
Now 3465 + 11 is divisible by 11, 3465 + 9 is divisible by 9, 3465 + 7 is divisible by 7, 3465 + 5 is divisible by 5.
Therefore choose numbers 3476, 3474, 3472 and 3470.

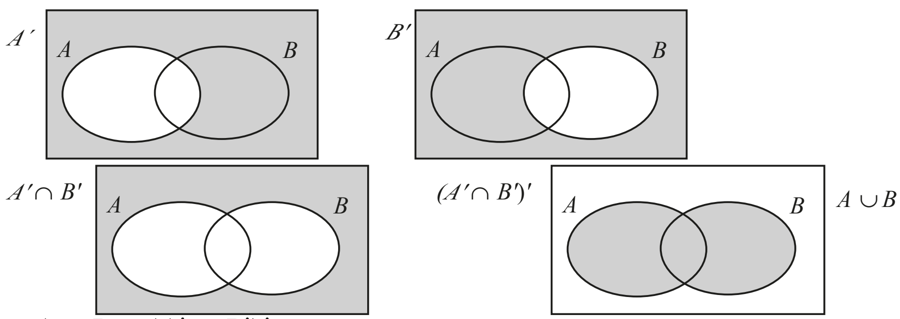
ii Divide by 2 to obtain 4 consecutive natural numbers, i.e. 1738, 1737, 1736, 1735.

6 **a** **i** B' denotes the set of students at Sounion Secondary College 180 cm or shorter.

ii $A \cup B$ denotes the set of students at Sounion Secondary College either female or taller than 180 cm or both.

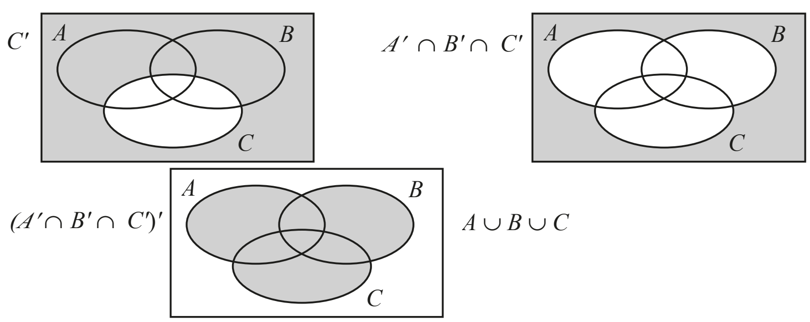
iii $A' \cap B'$ denotes the set of students at Sounion Secondary College who are males 180 cm or shorter.

b



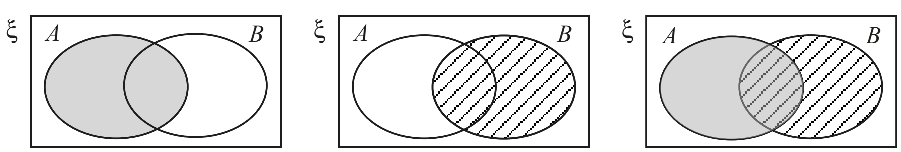
$\therefore A \cup B = (A' \cap B')'$

c

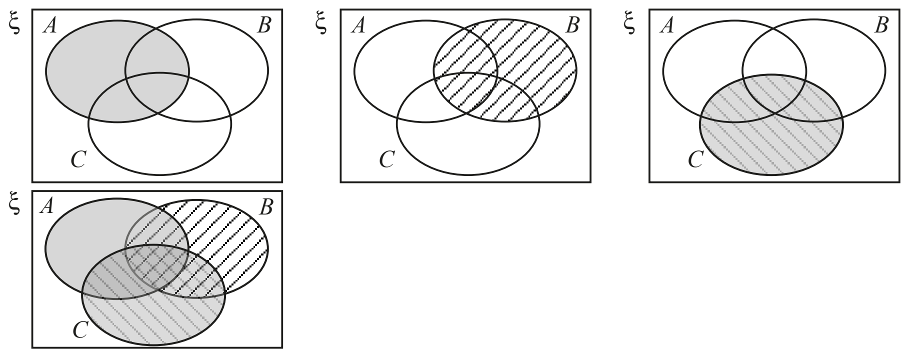


$\therefore A \cup B \cup C = (A' \cap B' \cap C')'$

7 **a**



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

b 

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

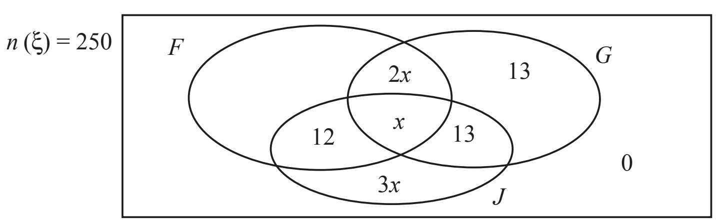
8 **a** **i** Region 8, $B' \cap F' \cap R'$

ii Region 1, $B \cap F' \cap R$ represents red haired, blue eyed males.

iii Region 2, $B \cap F' \cap R'$ represents blue eyed males who do not have red hair.

b Let ξ be the set of all students at Argos Secondary College studying French, Greek or Japanese.

$n(\xi) = n(F \cup G \cup J) = 250$
$n(F' \cap G' \cap J') = 0$
$n((G \cup J) \cap F') = 41$
$n(F \cap J \cap G') = 12$
$n(J \cap G \cap F') = 13$
$n(G \cap J' \cap F') = 13$
$n(F \cap G \cap J') = 2 \times n(F \cap G \cap J)$
$n(J \cap G' \cap F') = n(F \cap G)$



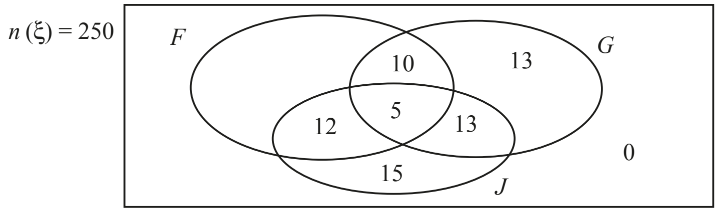
Now $n((G \cup J) \cap F') = 13 + 13 + 3x$

$= 26 + 3x$

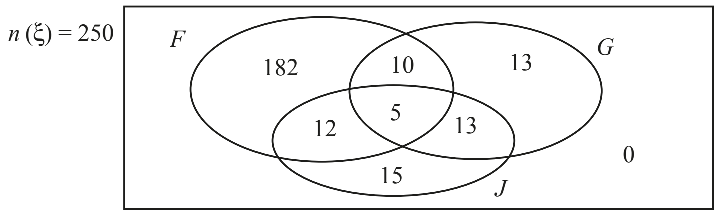
$\therefore \quad 26 + 3x = 41$

$\therefore \quad 3x = 15$

$\therefore \quad x = 5$



$$n(F \cap G' \cap J') = 250 - (10 + 12 + 5 + 13 + 13 + 15 + 0)$$
$$= 250 - 68$$
$$= 182$$

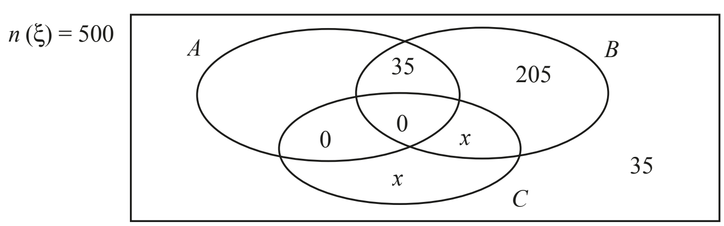


i $n(F \cap G \cap J) = 5$, the number studying all three languages.

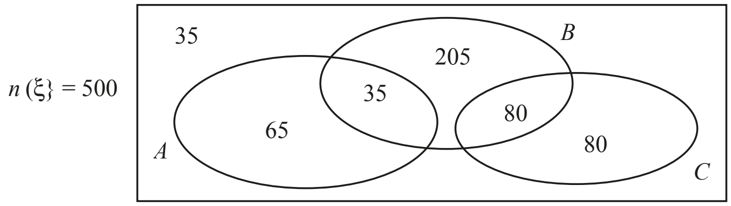
ii $n(F \cap G' \cap J') = 182$, the number studying only French.

9
$$n(\xi) = 500$$
$$n(A \cap C) = 0$$
$$n(A) = 100$$
$$n(B \cap A' \cap C') = 205$$
$$n(C) = 2 \times n(B \cap C)$$
$$n(A \cap B \cap C') = 35$$
$$n(A' \cap B' \cap C') = 35$$

a



$$n(A \cap B' \cap C') = 100 - 35$$
$$= 65$$
$$2x + 35 + 65 + 205 + 35 = 500$$
$$\therefore \quad 2x + 340 = 500$$
$$\therefore \quad 2x = 160$$
$$\therefore \quad x = 80$$



b $n(C) = 160$, regular readers of C.

c $n(A \cap B' \cap C') = 65$, regular readers of A only.

d $n(A \cap B \cap C) = 0$, regular readers of A, B and C.

Chapter 3 – Sequences and series

Solutions to Exercise 3A

1 a $t_1 = 3$

$t_2 = 3 + 4 = 7$

$t_3 = 7 + 4 = 11$

$t_4 = 11 + 4 = 15$

$t_5 = 15 + 4 = 19$

b $t_1 = 5$

$t_2 = 3 \times 5 + 4 = 19$

$t_3 = 3 \times 19 + 4 = 61$

$t_4 = 3 \times 61 + 4 = 187$

$t_5 = 3 \times 187 + 4 = 565$

c $t_1 = 1$

$t_2 = 5 \times 1 = 5$

$t_3 = 5 \times 5 = 25$

$t_4 = 5 \times 25 = 125$

$t_5 = 5 \times 125 = 625$

d $t_1 = -1$

$t_2 = -1 + 2 = 1$

$t_3 = 1 + 2 = 3$

$t_4 = 3 + 2 = 5$

$t_5 = 5 + 2 = 7$

e $t_1 = 1$

$t_2 = 3$

$t_3 = 2 \times 3 + 1 = 7$

$t_4 = 2 \times 7 + 3 = 17$

$t_5 = 2 \times 17 + 7 = 41$

2 a $t_2 = t_1 + 3$

$t_3 = t_2 + 3$

$\therefore t_n = t_{n-1} + 3, \ t_1 = 3$

b $t_2 = 2t_1$

$t_3 = 2t_2$

$\therefore t_n = 2t_{n-1}, \ t_1 = 1$

c $t_2 = -2 \times t_1$

$t_3 = -2 \times t_2$

$\therefore t_n = -2t_{n-1}, \ t_1 = 3$

d $t_2 = t_1 + 3$

$t_3 = t_2 + 3$

$\therefore t_n = t_{n-1} + 3, \ t_1 = 4$

e $t_2 = t_1 + 5$

$t_3 = t_2 + 5$

$\therefore t_n = t_{n-1} + 5, \ t_1 = 4$

3 a $t_n = \dfrac{1}{n}$

$t_1 = \dfrac{1}{1} = 1$

$t_2 = \dfrac{1}{2}$

$t_3 = \dfrac{1}{3}$

$t_4 = \dfrac{1}{4}$

b $t_n = n^2 + 1$

$t_1 = 1^2 + 1 = 2$

$t_2 = 2^2 + 1 = 5$

$t_3 = 3^2 + 1 = 10$

$t_4 = 4^2 + 1 = 17$

c $t_n = 2n$

$t_1 = 2 \times 1 = 2$

$t_2 = 2 \times 2 = 4$

$t_3 = 2 \times 3 = 6$

$t_4 = 2 \times 4 = 8$

d $t_n = 2^n$

$t_1 = 2^1 = 2$

$t_2 = 2^2 = 4$

$t_3 = 2^3 = 8$

$t_4 = 2^4 = 16$

e $t_n = 3n + 2$

$t_1 = 3 \times 1 + 2 = 5$

$t_2 = 3 \times 2 + 2 = 8$

$t_3 = 3 \times 3 + 2 = 11$

$t_4 = 3 \times 4 + 2 = 14$

f $t_n = (-1)^n n^3$

$t_1 = (-1)^1 \times 1^3 = -1$

$t_2 = (-1)^2 \times 2^3 = 8$

$t_3 = (-1)^3 \times 3^3 = -27$

$t_4 = (-1)^4 \times 4^3 = 64$

g $t_n = 2n + 1$

$t_1 = 2 \times 1 + 1 = 3$

$t_2 = 2 \times 2 + 1 = 5$

$t_3 = 2 \times 3 + 1 = 7$

$t_4 = 2 \times 4 + 1 = 9$

h $t_n = 2 \times 3^{n-1}$

$t_1 = 2 \times 3^0 = 2$

$t_2 = 2 \times 3^1 = 6$

$t_3 = 2 \times 3^2 = 18$

$t_4 = 2 \times 3^3 = 54$

4 a $t_n = 3n$

b $t_n = 2^{n-1}$

c $t_n = \dfrac{1}{n^2}$

d $t_n = 3(-2)^{n-1}$

e $t_n = 3n + 1$

f $t_n = 5n - 1$

5
$$\begin{aligned} t_n &= 3n + 1 \\ t_{n+1} &= 3(n + 1) + 1 \\ &= 3n + 4 \\ t_{2n} &= 3(2n) + 1 \\ &= 6n + 1 \end{aligned}$$

6 a $t_n = t_{n-1} + 3,\ t_1 = 15$

b $t_1 = 15$

$t_2 = 15 + 3$

$t_3 = (15 + 3) + 3$

$= 15 + 2 \times 3$

$\therefore t_n = 15 + (n - 1) \times 3$

$= 3n + 12$

c $t_{13} = 3 \times 13 + 12$

$= 51$

7 **a** 4% reduction is equivalent to 96% of the original.
$t_n = 0.96t_{n-1}$
$t_1 = 94.3$

b $t_1 = 94.3$

$t_2 = 0.96 \times 94.3$

$t_3 = 0.96 \times (0.96 \times 94.3)$

$= 0.96^2 \times 94.3$

$\therefore t_n = 94.3 \times 0.96^{n-1}$

c $t_9 = 94.3 \times 0.96^8$

≈ 68.03 seconds

8 **a** $t_n = 1.8t_{n-1} + 20$
$t_0 = 100$

b $t_1 = 1.8 \times 100 + 20 = 200$

$t_2 = 1.8 \times 200 + 20 = 380$

$t_3 = 1.8 \times 380 + 20 = 704$

$t_4 = 1.8 \times 704 + 20 = 1287$

$t_5 = 1.8 \times 1287 + 20 = 2336$

9 **a** $t_1 = 2000 \times 1.06$

$= \$2120$

$t_2 = (2120 + 400) \times 1.06$

$= \$2671.20$

$t_3 = (2671.2 + 400) \times 1.06$

$= \$3255.47$

b $t_n = (t_{n-1} + 400) \times 1.06$

$= 1.06(t_{n-1} + 400),\ t_1 = 2120$

c Method will depend on the calculator or spreadsheet used.
$t_{10} = \$8454.02$

10 **a** 1, 4, 7, 10, 13, 16

b 3, 1, −1, −3, −5, −7

c $\frac{1}{2}$, 1, 2, 4, 8, 16

d 32, 16, 8, 4, 2, 1

11 **a** 1.1, 1.21, 1.4641, 2.144, 4.595, 21.114

b 27, 18, 12, 8, $\frac{16}{3}$, $\frac{32}{9}$

c −1, 3, 11, 27, 59, 123

d −3, 7, −3, 7, −3, 7

12 **a** $t_n = 2^{n-1}$

$t_1 = 2^0 = 1$

$t_2 = 2^1 = 2$

$t_3 = 2^2 = 4$

b $u_n = \frac{1}{2}(n^2 - n) + 1$

$u_1 = \frac{1}{2}(1^2 - 1) + 1 = 1$

$u_2 = \frac{1}{2}(2^2 - 2) + 1 = 2$

$u_3 = \frac{1}{2}(3^2 - 3) + 1 = 4$

c The sequences are the same for the first three terms.

$t_1 = u_1$

$t_2 = u_2$

$t_3 = u_3$

d $t_4 = 2^3 = 8$

$u_4 = \frac{1}{2}(4^2 - 4) + 1 = 7$

The sequences are not the same after the first three terms.

13 $S_1 = a \times 1^2 + b \times 1 = a + b$

$S_2 = a \times 2^2 + b \times 2 = 4a + 2b$

$S_3 = a \times 3^2 + b \times 3 = 9a + 3b$

$S_{n+1} - S_n$

$= a(n + 1)^2 + b(n + 1) - an^2 - bn$

$= a(n^2 + 2n + 1) + bn + b - an^2 - bn$

$= an^2 + 2an + a + b - an^2$

$= 2an + a + b$

14 $t_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$

$t_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \approx 1.4168$

$t_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{577}{408} \approx 1.4142$

Comparing the terms to real numbers between 1 and 1.5, it can be seen that the sequence gives an approximation of $\sqrt{2} = 1.4142\ldots$

15

$$\begin{aligned} F_3 &= F_2 + F_1 \\ &= 1 + 1 = 2 \\ F_4 &= F_3 + F_2 \\ &= 2 + 1 = 3 \\ F_5 &= F_4 + F_3 \\ &= 3 + 2 = 5 \\ F_{n+2} &= F_{n+1} + F_n \\ \therefore F_{n+1} &= F_n + F_{n-1} \\ \therefore F_{n+2} &= (F_n + F_{n-1}) + F_n \\ &= 2F_n + F_{n-1} \end{aligned}$$

Solutions to Exercise 3B

1 $t_n = a + (n-1)d$

a $t_1 = 0 + (1-1) \times 2 = 0$

$t_2 = 0 + (2-1) \times 2 = 2$

$t_3 = 0 + (3-1) \times 2 = 4$

$t_4 = 0 + (4-1) \times 2 = 6$

b $t_1 = -3 + (1-1) \times 5 = -3$

$t_2 = -3 + (2-1) \times 5 = 2$

$t_3 = -3 + (3-1) \times 5 = 7$

$t_4 = -3 + (4-1) \times 5 = 12$

c $t_1 = -\sqrt{5} + (1-1) \times -\sqrt{5} = -\sqrt{5}$

$t_2 = -\sqrt{5} + (2-1) \times -\sqrt{5} = -2\sqrt{5}$

$t_3 = -\sqrt{5} + (3-1) \times -\sqrt{5} = -3\sqrt{5}$

$t_4 = -\sqrt{5} + (4-1) \times -\sqrt{5} = -4\sqrt{5}$

d $t_1 = 11 + (1-1) \times -2 = 11$

$t_2 = 11 + (2-1) \times -2 = 9$

$t_3 = 11 + (3-1) \times -2 = 7$

$t_4 = 11 + (4-1) \times -2 = 5$

2 **a** $t_{13} = a + 12d$

$= 5 + 12 \times -3 = -31$

b $t_{10} = a + 9d$

$= -12 + 9 \times 4 = 24$

c $t_9 = a + 8d$

$= 25 + 8 \times -2.5 = 5$

d $t_5 = a + 4d$

$= 2\sqrt{3} + 4 \times \sqrt{3}$

$= 6\sqrt{3}$

3 **a** $a + (1-1)d = 3$

$a = 3$

$3 + (2-1)d = 7$

$d = 7 - 3 = 4$

$\therefore t_n = 3 + 4(n-1)$

$= 4n - 1$

b $a + (1-1)d = 3$

$a = 3$

$3 + (2-1)d = -1$

$d = -1 - 3 = -4$

$\therefore t_n = 3 + -4(n-1)$

$= 7 - 4n$

c $a + (1-1)d = -\frac{1}{2}$

$a = -\frac{1}{2}$

$-\frac{1}{2} + (2-1)d = \frac{3}{2}$

$d = \frac{3}{2} - \frac{1}{2} = 2$

$t_n = -\frac{1}{2} + 2(n-1)$

$= 2n - \frac{5}{2}$

d

$$a + (1 - 1)d = 5 - \sqrt{5}$$
$$a = 5 - \sqrt{5}$$
$$(5 - \sqrt{5}) + (2 - 1)d = 5$$
$$d = 5 - (5 - \sqrt{5})$$
$$= \sqrt{5}$$
$$t_n = (5 - \sqrt{5}) + \sqrt{5}(n - 1)$$
$$= n\sqrt{5} + 5 - 2\sqrt{5}$$

4 **a** $a = 6$ and $d = 4$

$$6 + 4(n - 1) = 54$$
$$4(n - 1) = 48$$
$$n - 1 = 12$$
$$n = 13$$

b $a = 5$ and $d = -3$

$$5 - 3(n - 1) = -16$$
$$-3(n - 1) = -21$$
$$n - 1 = 7$$
$$n = 8$$

c $a = 16$ and $d = 16 - 13 = 3$

$$16 + 3(n - 1) = -41$$
$$-3(n - 1) = -57$$
$$n - 1 = 19$$
$$n = 20$$

d $a = 7$ and $d = 11 - 7 = 4$

$$7 + 4(n - 1) = 227$$
$$4(n - 1) = 220$$
$$n - 1 = 55$$
$$n = 56$$

5

$$t_4 = 7$$
$$t_{30} = 85$$
$$a + 3d = 7 \ldots (1)$$
$$a + 29d = 85 \ldots (2)$$

Equation (2) − Equation (1)

$$26d = 78$$
$$d = 3$$
$$\therefore a = -2$$
$$t_7 = -2 + 6 \times 3$$
$$= 16$$

6

$$a + 2d = 18 \qquad \ldots (1)$$
$$a + 5d = 486 \qquad \ldots (2)$$

Equation (2) − Equation (1)

$$3d = 468$$
$$d = 156$$
$$a + 2 \times 156 = 18$$
$$a + 312 = 18$$
$$a = -294$$
$$\therefore t_n = -294 + 156(n - 1)$$
$$= 156n - 450$$

7
$$a + 6d = 0.6 \ldots (1)$$
$$a + 11d = -0.4 \ldots (1)$$
Equation (2) − Equation (1)
$$5d = -1.0$$
$$d = -0.2$$
$$a + 6 \times -0.2 = 0.6$$
$$a - 1.2 = 0.6$$
$$a = 1.8$$
$$\therefore t_{20} = 1.8 + 19 \times -0.2$$
$$= -2$$

8
$$a + 4d = 24 \ldots (1)$$
$$a + 9d = 39 \ldots (2)$$
Equation (2) − Equation (1)
$$5d = 15$$
$$d = 3$$
$$a + 4 \times 3 = 24$$
$$a + 12 = 24$$
$$a = 12$$
$$\therefore t_{15} = 12 + 14 \times 3$$
$$= 54$$

9
$$a + 14d = 3 + 9\sqrt{3} \ldots (1)$$
$$a + 19d = 38 - \sqrt{3} \ldots (2)$$
Equation (2) − Equation (1)
$$5d = 35 - 10\sqrt{3}$$
$$d = 7 - 2\sqrt{3}$$
$$a + 14 \times (7 - 2\sqrt{3}) = 3 + 9\sqrt{3}$$
$$a + 98 - 28\sqrt{3} = 3 + 9\sqrt{3}$$
$$a = 37\sqrt{3} - 95$$
$$t_6 = 37\sqrt{3} - 95 + 5 \times (7 - 2\sqrt{3})$$
$$= 37\sqrt{3} - 95 + 35 - 10\sqrt{3}$$
$$= 27\sqrt{3} - 60$$

10 a 672

b 91st week

11 a P is the 16th row. $a = 25,\ d = 3$
$$t_{16} = a + 15d$$
$$= 25 + 15 \times 3$$
$$= 70 \text{ seats}$$

b X is the 24th row. $a = 25,\ d = 3$
$$t_{24} = a + 23d$$
$$= 25 + 23 \times 3$$
$$= 94 \text{ seats}$$

c
$$t_n = 25 + 3(n - 1) = 40$$
$$3(n - 1) = 15$$
$$n - 1 = 5$$
$$n = 6$$
Row F

12 $t_6 = 3 + 5d = 98$

$5d = 95$

$d = 19$

$t_7 = t_6 + 19$

$= 117$

13 $4 + 9d = 30$

$9d = 26$

$d = \frac{29}{9}$

$t_2 = 4 + 1 \times \frac{26}{9} = \frac{62}{9}$

$t_3 = 4 + 2 \times \frac{26}{9} = \frac{88}{9}$

$t_4 = 4 + 3 \times \frac{26}{9} = \frac{38}{3}$

$t_5 = 4 + 4 \times \frac{26}{9} = \frac{140}{9}$

$t_6 = 4 + 5 \times \frac{26}{9} = \frac{166}{9}$

$t_7 = 4 + 6 \times \frac{26}{9} = \frac{64}{3}$

$t_8 = 4 + 7 \times \frac{26}{9} = \frac{218}{9}$

$t_9 = 4 + 8 \times \frac{26}{9} = \frac{244}{9}$

14 $5 + 5d = 15$

$5d = 10$

$d = 2$

$t_2 = 5 + 1 \times 2 = 7$

$t_3 = 5 + 2 \times 2 = 9$

$t_4 = 5 + 3 \times 2 = 11$

$t_5 = 5 + 4 \times 2 = 13$

15 $a + (m - 1)d = 0$

$(m - 1)d = -a$

$d = -\frac{a}{m - 1}$

$t_n = a - \frac{a(n - 1)}{m - 1}$

This could be simplified as follows:

$$t_n = \frac{a(m - 1) - a(n - 1)}{m - 1}$$

$$= \frac{a(m - 1 + n + 1)}{m - 1}$$

$$= \frac{a(m - n)}{m - 1}$$

16 **a** $c = \frac{a + b}{2}$

$= \frac{8 + 15}{2} = 11.5$

b $c = \frac{a + b}{2}$

$$= \frac{1}{2}\left(\frac{1}{2\sqrt{2} - 1} + \frac{1}{2\sqrt{2} + 1}\right)$$

$$= \frac{2\sqrt{2} + 1 + 2\sqrt{2} - 1}{2(2\sqrt{2} - 1)(2\sqrt{2} + 1)}$$

$$= \frac{4\sqrt{2}}{2 \times (8 - 1)}$$

$$= \frac{2\sqrt{2}}{7}$$

17 $3x - 2 = \frac{5x + 1 + 11}{2}$

$6x - 4 = 5x + 12$

$x = 16$

18 Use the fact that the difference is constant.

$$(8a - 13) - (4a - 4) = (4a - 4) - a$$
$$8a - 13 - 4a + 4 = 4a - 4 - a$$
$$4a - 9 = 3a - 4$$
$$a = 5$$

19 $t_m = a + (m - a)d = n$

$t_n = a + (n - a)d = m$

Subtract:

$$\begin{aligned}(m - n)d &= n - m \\ &= -1(m - n) \\ d &= \frac{-1(m - n)}{m - n} \\ &= -1\end{aligned}$$

Substitute:

$$a + (m - a) \times -1 = n$$
$$a = m + n - 1$$
$$\begin{aligned}t_{m+n} &= a + (m + n - 1)d \\ &= n + m - 1 + (m + n - 1) \times -1 \\ &= n + m - 1 - m - n + 1 \\ &= 0\end{aligned}$$

20 Use the fact that the difference is constant.

$$a^2 - 2a = 2a - a$$
$$a^2 - 3a = 0$$
$$a(a - 3) = 0$$
$$a = 3 \text{ (since } a \neq 0)$$

21 If a is a prime number, then the nth term is $a + (n - 1)d$ Since a is a natural number there is an n such that $n - 1 = a$. The term $t_{a+1} = a + ad = a(d + 1)$ which is divisible by a. It is composite since $d + 1 \geq 2$ and $a \geq 2$). Hence no infinite arithmetic sequence of primes exists.

Solutions to Exercise 3C

1 a $a = 8,\ d = 5,\ n = 12$

$$t_{12} = 8 + 11 \times 5 = 63$$

$$S_{12} = \frac{12}{2}(8 + 63)$$

$$= 6 \times 71$$

$$= 426$$

b $a = -3.5,\ d = 2,\ n = 10$

$$t_{10} = -3.5 + 9 \times 2 = 14.5$$

$$S_{10} = \frac{10}{2}(-3.5 + 14.5)$$

$$= 5 \times 11$$

$$= 55$$

c $a = \dfrac{1}{\sqrt{2}},\ d = \dfrac{1}{\sqrt{2}},\ n = 15$

$$t_{15} = \frac{1}{\sqrt{2}} + 14 \times \frac{1}{\sqrt{2}}$$

$$= \frac{15}{\sqrt{2}}$$

$$S_{15} = \frac{15}{2}\left(\frac{1}{\sqrt{2}} + \frac{15}{\sqrt{2}}\right)$$

$$= 60\sqrt{2}$$

d $a = -4,\ d = 5,\ n = 8$

$$t_8 = -4 + 7 \times 5 = 31$$

$$S_8 = \frac{8}{2}(-4 + 31)$$

$$= 108$$

2 $a = 7,\ d = 3,\ n = 7$

$$S_7 = \frac{7}{2}(14 + 6 \times 3)$$

$$= 112$$

3 $a = 5,\ d = 5,\ n = 16$

$$S_{16} = \frac{16}{2}(10 + 15 \times 5)$$

$$= 680$$

4 There will be half of 98 = 49 numbers:

$a = 2,\ d = 2,\ n = 49$

$$S_{49} = \frac{49}{2}(4 + 48 \times 2)$$

$$= 2450$$

5 a 14

b 322

6 a 20

b −280

7 a 12

b 105

8 a 180

b $S_n = \dfrac{n}{2}(8 + (n - 1) \times 4)$

$$= 180$$

$$n(8 + 4n - 4) = 360$$

$$4n^2 + 4n - 360 = 0$$

$$n^2 + n - 90 = 0$$

$$(n - 9)(n + 10) = 0$$

$$n = 9 \text{ as } n > 0.$$

9 $S_n = \dfrac{n}{2}(30 + (n - 1) \times -1) = 110$

$$n(30 - n + 1) = 220$$
$$-n^2 + 31n - 220 = 0$$
$$n^2 - 31n + 220 = 0$$
$$(n - 11)(n - 20) = 0$$
$$n = 11 \text{ or } n = 20$$

Reject any value of $n > 15$, as this would involve a negative number of logs in a row. There will be 11 layers.

10 $a = -5,\ d = 4$

$$S_m = \frac{m}{2}(-10 + (m-1) \times 4)$$
$$= 660$$
$$m(-10 + 4m - 4) = 1320$$
$$4m^2 - 14m - 1320 = 0$$
$$(m - 20)(4m + 66) = 0$$
$$m = 20 \text{ as } m > 0$$

11 $S_n = \frac{n}{2}(a + \ell) \therefore S_n = 0$

12 **a** $a = 6$

$$t_{15} = 6 + 14d = 27$$
$$14d = 21$$
$$d = 1.5$$
$$t_8 = 6 + 7 \times 1.5$$
$$= 16.5 \text{ km}$$

b $S_5 = \frac{5}{2}(12 + 4 \times 1.5)$

$$= 45 \text{ km}$$

c 7 walks

d Total distance:

$$S_{15} = \frac{15}{2}(12 + 14 \times 1.5)$$
$$= 247.5$$

Distance missed $= 18 + 19.5 + 21$

$= 58.5$ km

(8th day = 16.5 km)

Distance Dora walks $= 247.5 - 58.5$

$= 189$ km

13 **a** $a = 30,\ d = 5$

$$S_n = \frac{n}{2}(60 + (n-1) \times 5)$$
$$= 500$$
$$n(60 + 5n - 10) = 1000$$
$$5n^2 + 50n - 1000 = 0$$
$$n^2 + 10n - 200 = 0$$
$$(n - 10)(n + 20) = 0$$
$$n = 10,\ \text{as } n > 0$$

10 days

b $a = 50,\ n = 5$

$$S_5 = \frac{5}{2}(100 + 4d)$$
$$= 500$$
$$100 + 4d = 200$$
$$d = \frac{200 - 100}{4}$$

$= 25$ pages per day

14 **a** Row J $= t_{10}$

$= 50 + 9 \times 4 = 86$

b $S_{26} = \frac{26}{2}(100 + 25 \times 4)$

$$= 2600$$

c $50 + 54 + 58 + 62 = 224$

d $2600 - 224 = 2376$

e $$\begin{aligned} S_n &= \frac{n}{2}(100 + (n-1) \times 4) \\ &= 3410 \\ n(100 + 4n - 4) &= 6820 \\ 4n^2 + 96n - 6820 &= 0 \\ n^2 + 24n - 1705 &= 0 \\ (n-31)(n+55) &= 0 \\ n &= 31 \text{ as } n > 0 \end{aligned}$$

There are 5 extra rows (from 26 to 31).

15 Total members

$$\begin{aligned} S_{12} &= \frac{12}{2}(80 + 11 \times 15) \\ &= 1470 \end{aligned}$$

$$\begin{aligned} \text{Total fees} &= 1470 \times \$120 \\ &= \$176\,400 \end{aligned}$$

16 $$\begin{aligned} a + d &= -12 \\ 6(2a + 11d) &= 18 \\ 2a + 11d &= 3 \end{aligned}$$

Substitute $a = -12 - d$:

$$\begin{aligned} -24 - 2d + 11d &= 3 \\ 9d - 24 &= 3 \\ d &= 3 \\ a + 3 &= -12 \\ a &= -15 \\ t_6 &= -15 + 5 \times 3 \\ &= 0 \\ S_6 &= \frac{6}{2}(-30 + 5 \times 3) \\ &= -45 \end{aligned}$$

17 $$\begin{aligned} 5(2a + 9d) &= 120 \\ 2a + 9d &= 24 \ldots (1) \\ 10(2a + 19d) &= 840 \\ 2a + 19d &= 84 \ldots (2) \end{aligned}$$

Equation (2) − Equation (1)

$$\begin{aligned} 10d &= 60 \\ d &= 6 \\ 2a + 9 \times 6 &= 24 \\ a &= -15 \\ S_{30} &= \frac{30}{2}(-30 + 29 \times 6) \\ &= 2160 \end{aligned}$$

18 $$\begin{aligned} a + 5d &= 16 \ldots (1) \\ a + 11d &= 28 \ldots (2) \end{aligned}$$

Equation (2) − Equation (1)

$$\begin{aligned} 6d &= 12 \\ d &= 2 \\ a + 10 &= 16 \\ a &= 6 \\ S_{14} &= \frac{14}{2}(12 + 13 \times 2) \\ &= 266 \end{aligned}$$

19 a

$$a + 2d = 6.5 \ldots (1)$$

$$4(2a + 7d) = 67$$

$$a + 3.5d = \frac{67}{8} = 8.375 \ldots (2)$$

Equation (2) − Equation (1)

$$1.5d = 1.875$$

$$d = 1.25$$

$$a + 1.25 \times 2 = 6.5$$

$$a = 4$$

$$t_n = 4 + 1.25(n - 1)$$

$$= 2.75 + 1.25n$$

$$= \frac{5}{4}n + \frac{11}{4}$$

b

$$a + 3d = \frac{6}{\sqrt{5}}$$

$$= \frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{6\sqrt{5}}{5} \ldots (1)$$

$$\frac{5}{2}(2a + 4d) = 16\sqrt{5}$$

$$5(a + 2d) = 16\sqrt{5}$$

$$a + 2d = \frac{16\sqrt{5}}{5} \ldots (2)$$

Equation (2) − Equation (1)

$$d = \frac{6\sqrt{5}}{5} - \frac{16\sqrt{5}}{5}$$

$$= -\frac{10\sqrt{5}}{5}$$

$$a + 2 \times \frac{-10\sqrt{5}}{5} = \frac{16\sqrt{5}}{5}$$

$$a = \frac{16\sqrt{5}}{5} + \frac{20\sqrt{5}}{5}$$

$$= 36\sqrt{5}$$

$$t_n = \frac{36\sqrt{5}}{5} - \frac{10\sqrt{5}}{5}$$

$$(n - 1)$$

$$= \frac{46\sqrt{5}}{5} - \frac{10\sqrt{5}}{5}n$$

$$= \frac{46\sqrt{5}}{5} - 2\sqrt{5}n$$

20 a $t_{n+1} - t_n = b(n + 1) - bn$

$$= b$$

b $S_n = \frac{n}{2}(2b + (n - 1)b)$

$$= \frac{n}{2}(2b + nb - b)$$

$$= \frac{n}{2}(nb + b)$$

This can be factorised to $\dfrac{nb(n+1)}{2}$.

21 $a = 10,\ d = -5$

$$\begin{aligned} t_5 &= 10 + 4 \times -5 \\ &= -10 \\ S_{25} &= \frac{25}{2}(20 + 24 \times -5) \\ &= -1250 \end{aligned}$$

22
$$\begin{aligned} S_{20} &= 10(2a + 19d) \\ &= 25a \\ 20a + 190d &= 25a \\ 190d &= 5a \\ a &= 38d \\ S_{30} &= 15(76d + 29d) \\ &= 1575d \end{aligned}$$

23 **a** $S_{n-1} = 17(n-1) - 3(n-1)^2$
$$\begin{aligned} &= 17n - 17 - 3(n^2 - 2n + 1) \\ &= 17n - 17 - 3n^2 + 6n - 3 \\ &= 23n - 3n^2 - 20 \end{aligned}$$

b $t_n = S_n - S_{n-1}$
$$\begin{aligned} &= 17n - 3n^2 - 23n + 3n^2 + 20 \\ &= 20 - 6n \end{aligned}$$

c $t_{n+1} - t_n = 20 - 6(n+1) - (20 - 6n)$
$$\begin{aligned} &= 20 - 6n - 6 - 20 + 6n \\ &= -6 \end{aligned}$$
The sequence has a constant difference of −6 and so is arithmetic.

$$\begin{aligned} a &= t_1 \\ &= 20 - 6 \times 1 = 14 \\ a &= 14 \end{aligned}$$

24 Let the terms be $a,\ a + d,\ a + 2d$.

Sum $= 3a + 3d = 36$

$a + d = 12$

Product $= a(a + d)(a + 2d)$

$= 1428$

Substitute $d = 12 - a$.
$$\begin{aligned} a(a + 12 - a)(a + 24 - 2a) &= 1428 \\ 12a(24 - a) &= 1428 \\ a(24 - a) &= 119 \\ 24a - a^2 &= 119 \\ a^2 - 24a + 119 &= 0 \\ (a - 7)(a - 17) &= 0 \\ a = 7 \text{ or } a &= 17 \\ \therefore d &= 12 - 7 = 5 \\ \text{or } d &= 12 - 17 = -5 \end{aligned}$$
The three terms are either 7, 12, 17 or 17, 12, 7.

Note: in cases like this, it is sometimes easier to call the terms $a - d,\ a,\ a + d$.

25 **a** There are n terms in the sequnce and $a = 1$ and $l = 2n - 1$. Therefore

$$\begin{aligned} S_n &= \frac{n}{2}(a + l) \\ &= \frac{n}{2}(1 + 2n - 1) \\ &= \frac{n}{2}(2n) \\ &= n^2 \end{aligned}$$

as required.

b **i** Since $S_n = n^2$ we find that

$$\begin{aligned} S_{2n} - S_n &= (2n)^2 - n^2 \\ &= 4n^2 - n^2 \\ &= 3n^2 \\ &= 3S_n, \end{aligned}$$

as required.

ii Each term in the sequence is of the form $\dfrac{S_n}{S_{2n} - S_n}$. Therefore,

$$\begin{aligned} \frac{S_n}{S_{2n} - S_n} &= \frac{S_n}{S_{2n} - S_n} \\ &= \frac{S_n}{3S_n} \\ &= \frac{1}{3}, \end{aligned}$$

as required

26 The middle terms will be t_n and t_{n+1}.

$$\begin{aligned} t_n &= a + (n-1)d \\ t_{n+1} &= a + nd \\ t_n + t_{n+1} &= 2a + (2n-1)d \\ n(t_n + t_{n+1}) &= n(2a + (2n-1)d) \\ S_{2n} &= \frac{2n}{2}(2a + (2n-1)d) \\ &= n(2a + (2n-1)d) \\ &= n(t_n + t_{n+1}) \end{aligned}$$

27 There are 60 numbers divisible by 2.
$S_{60} = 30(4 + 59 \times 2) = 3660$
There are 40 numbers divisible by 3.
$S_{40} = 20(6 + 39 \times 3) = 2460$
There are 20 numbers divisible by 6
$S_{60} = 10(12 + 19 \times 6) = 1260$
The sum of the numbers divisible by 2 or 3 $= 3660 + 2460 - 1260 = 4860$

28 Let the numbers be $a - d, a, a + d, a + 2d$.
The sum is $4a + 2d = 100$ which simplifies to $2a + d = 50$.
One solution is $a = 25$ and $d = 0$.
The others are $(24, 2), (23, 4), \ldots (17, 16)$
The sequence for the first solution is 25,25,25,25.
One other sequence is 22,24,26,28.
There are 9 sequences in total.

29 Let the angles be $a - d, a$ and $a + d$.
Then $3a = 180$. Hence $a = 60$. The angles are, $60 - d, 60$ and $60 + d$.
There are 60 such triangles: Listing:
$(1, 60, 119), (2, 60, 118), \ldots, (60, 60, 60)$

Solutions to Exercise 3D

1 $t_n = ar^{n-1}$

a $t_1 = 3 \times 2^{1-1} = 3$

$t_2 = 3 \times 2^{2-1} = 6$

$t_3 = 3 \times 2^{3-1} = 12$

$t_4 = 3 \times 2^{4-1} = 24$

b $t_1 = 3 \times -2^{1-1} = 3$

$t_2 = 3 \times -2^{2-1} = -6$

$t_3 = 3 \times -2^{3-1} = 12$

$t_4 = 3 \times -2^{4-1} = -24$

c $t_1 = 10\,000 \times 0.1^{1-1} = 10\,000$

$t_2 = 10\,000 \times 0.1^{2-1} = 1000$

$t_3 = 10\,000 \times 0.1^{3-1} = 100$

$t_4 = 10\,000 \times 0.1^{4-1} = 10$

d $t_1 = 3 \times 3^{1-1} = 3$

$t_2 = 3 \times 3^{2-1} = 9$

$t_3 = 3 \times 3^{3-1} = 27$

$t_4 = 3 \times 3^{4-1} = 81$

2 a $a = \dfrac{15}{7}$

$r = \dfrac{1}{3}$

$t_6 = \dfrac{15}{7} \times \left(\dfrac{1}{3}\right)^5 = \dfrac{5}{567}$

b $a = 1$

$r = -\dfrac{1}{4}$

$t_5 = 1 \times \left(-\dfrac{1}{4}\right)^4 = \dfrac{1}{256}$

c $a = \sqrt{2}$

$r = \sqrt{2}$

$t_{10} = \sqrt{2} \times (\sqrt{2})^9 = 32$

d $a = a^x$

$r = a$

$t_6 = a^x \times a^5 = a^{x+5}$

3 a $a = 3$

$r = \dfrac{2}{3}$

$t_n = 3 \times \left(\dfrac{2}{3}\right)^{n-1}$

b $a = 2$

$r = \dfrac{-4}{2} = -2$

$t_n = 2 \times (-2)^{n-1}$

c $a = 2$

$r = \dfrac{2\sqrt{5}}{2} = \sqrt{5}$

$t_n = 2 \times (\sqrt{5})^{n-1}$

4 a $a = 2$ and $t_6 = 486$

Let r be the common ratio

$\therefore 2 \times r^5 = 486$

$\therefore r^5 = 243$

$\therefore r = 3$

b $a = 25$ and $t_5 = \dfrac{16}{25}$

Let r be the common ratio

$\therefore 25 \times r^4 = \dfrac{16}{25}$

$\therefore r^4 = \dfrac{16}{625}$

$$\therefore r = \pm\frac{2}{5}$$

5 $$\frac{1}{4}2^{n-1} = 64$$
$$2^{n-1} = 64 \times 4$$
$$= 2^8$$
$$n = 9$$
Thus t_9, the ninth term.

6 a $$a = 2,\ r = 3$$
$$2 \times 3^{n-1} = 486$$
$$3^{n-1} = 243$$
$$= 3^5$$
$$n = 6$$

b $$a = 5,\ r = 2$$
$$5 \times 2^{n-1} = 1280$$
$$2^{n-1} = 256$$
$$= 2^8$$
$$n = 9$$

c $$a = 768,\ r = \frac{1}{2}$$
$$768 \times \left(\frac{1}{2}\right)^{n-1} = 3$$
$$\frac{1}{2^{n-1}} = \frac{3}{768}$$
$$= \frac{1}{256} = \frac{1}{2^8}$$
$$n = 9$$

d $$a = \frac{8}{9},\ r = \frac{3}{2}$$
$$\frac{8}{9} \times \frac{3^{n-1}}{2^{n-1}} = \frac{27}{4}$$
$$\frac{3^{n-1}}{2^{n-1}} = \frac{27}{4} \times \frac{9}{8}$$
$$= \frac{3^5}{2^5}$$
$$n = 6$$

e $$a = -\frac{4}{3},\ r = -\frac{1}{2}$$
$$-\frac{4}{3} \times \left(-\frac{1}{2}\right)^{n-1} = \frac{1}{96}$$
$$\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{96} \times -\frac{3}{4}$$
$$= -\frac{1}{32 \times 4}$$
$$= -\frac{1}{2^7} = \left(-\frac{1}{2}\right)^7$$
$$n = 8$$

7 $$ar^{14} = 54$$
$$ar^{11} = 2$$
$$r^3 = \frac{54}{2} = 27$$
$$r = 3$$
$$a \times 3^{11} = 2$$
$$a = \frac{2}{3^{11}}$$
$$t_7 = \frac{2}{3^{11}} \times 3^6$$
$$= \frac{2}{3^5}$$

8 $ar^1 = \dfrac{1}{2\sqrt{2}}$

$ar^3 = \sqrt{2}$

$r^2 = \sqrt{2} \div \dfrac{1}{2\sqrt{2}}$

$= 4$

$r = 2$

$a \times 2 = \dfrac{1}{2\sqrt{2}}$

$a = \dfrac{1}{4\sqrt{2}}$

$t_8 = \dfrac{1}{4\sqrt{2}} \times 2^7$

$= \dfrac{32}{\sqrt{2}}$

Rationalise the denominator:

$t_8 = \dfrac{32}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{32\sqrt{2}}{2} = 16\sqrt{2}$

9 a $ar^5 = 768$

$ar^2 = 96$

$r^3 = \dfrac{768}{96} = 8$

$r = 2$

$a \times 2^2 = 96$

$a = 24$ fish

b $24 \times 2^9 = 12\,888$ fish

10 a At the end of 7 days, it will have increased 7 times.

$10 \times 3^7 = 21870$ m^2

b $10 \times 3^n \geq 200\,000$

$3^n \geq 20\,000$

$n \log_{10} 3 \geq \log_{10} 20\,000$

$n \geq 9.014\ldots$

It will cover the lake early in the tenth day.

11 $r = \dfrac{3}{4}$.

First bounce: $\dfrac{3}{2}$ m

Second bounce: $\dfrac{9}{8}$ m

Third bounce: $\dfrac{27}{32}$ m

Fourth bounce: $\dfrac{81}{128}$ m

Fifth bounce: $\dfrac{243}{512}$ m

12 a At the end of 10 years, it will have increased 10 times.

$2500 \times 1.08^9 = \$5397.31$

b $2500 \times 1.08^n \geq 100\,000$

$1.08^n \geq \dfrac{100\,000}{2500} = 40$

$n \log_{10} 1.08 \geq \log_{10} 40$

$n \geq 47.93\ldots$

It will take 48 years until the value exceeds \$100 000. Alternatively, use the solve command of a CAS calculator to solve $2500 \times 1.08^n \geq 100\,000$.

This gives $n > 47.93\ldots$ directly.

13 **a** $120 \times 0.9^7 \approx 57.4$ km

b
$$\begin{aligned} 120 \times 0.9^{n-1} &= 30.5 \\ 0.9^{n-1} &= \frac{30.5}{120} \\ &= 0.251\ldots \\ (n-1)\log_{10} 0.9 &= \log_{10} 0.251\ldots \\ n-1 &= 13.0007\ldots \\ n &= 14 \end{aligned}$$

The 14th day.

14 $a = 1$ and $r = 2$
$t_{30} = 2^{29} = 5\,368\,709.12$
She would receive \$ 5 368 709.12. To the nearest thousand dollars this is \$5 369 000.

15
$$\begin{aligned} a &= 4,\ r = 2 \\ 4 \times 2^{n-1} &> 2000 \\ 2^{n-1} &> 500 \\ 2^9 &= 512 \end{aligned}$$

The tenth term is the required term since $t_{10} = 4 \times 2^9 = 2048$.

16
$$\begin{aligned} a &= 3,\ r = 3 \\ 3 \times 3^{n-1} &> 500 \\ 3^n &> 500 \\ 3^5 &= 243 \text{ and } 3^6 = 729 \end{aligned}$$

The sixth term is the first to exceed 500.

17 Solve for n:

$$\begin{aligned} 40\,960 \times \left(\frac{1}{2}\right)^{n-1} &= 40 \times 2^{n-1} \\ \frac{40\,960}{40} &= 2^{n-1} \times 2^{n-1} \\ 1024 = 2^{2n-2} &= 2^{10} \\ 2n - 2 &= 10 \\ n &= 6 \end{aligned}$$

But $n = 1$ corresponds to the initial numbers present, so they are equal after 5 weeks.

18 **a** $\sqrt{5 \times 720} = \sqrt{3600} = 60$

b $\sqrt{1 \times 6.25} = \sqrt{6.25} = 2.5$

c $\sqrt{\frac{1}{\sqrt{3}} \times \sqrt{3}} = \sqrt{1} = 1$

d
$$\begin{aligned} \sqrt{x^2y^3 \times x^6y^{11}} &= \sqrt{x^8y^{14}} \\ &= x^4y^7 \end{aligned}$$

19
$$\begin{aligned} r &= \frac{t_7}{t_4} = \frac{t_{16}}{t_7} \\ \frac{a+6d}{a+3d} &= \frac{a+15d}{a+6d} \\ (a+6d)^2 &= (a+15d)(a+3d) \\ a^2 + 12ad + 36d^2 &= a^2 + 18ad + 45d^2 \\ 9d^2 + 6ad &= 0 \\ 3d(3d+2a) &= 0 \\ 3d + 2a &= 0 \text{ (see below)} \\ d &= -\frac{2}{3}a \\ r &= \frac{a+6d}{a+3d} \\ &= \frac{a-4a}{a-2a} \\ &= \frac{-3a}{-a} = 3 \end{aligned}$$

Note: $d = 0$ gives the trivial case $r = \frac{a}{a} = 1$.
(All the terms are the same.)

20 $a^{n-1} + a^n = a^{n+1}$

$\therefore a^{n-1}(1 + a - a^2) = 0$

$\therefore a = \dfrac{1 \pm \sqrt{5}}{2}$ or $a = 0$

21 **a** When the first 300 mL is withdrawn there is 700 mL of ethanol left.
When the second 300 mL withdrawn there is $0.7^2 \times 1000$ mL of ethanol left
After 5 such withdrawals there is $0.7^5 \times 1000 \approx 168.07$ mL left.

b Solve the inequality $1000 \times 0.7^n < 1$ for a positive integer n to find $n = 20$.

22 **a** The perimeter of the rectangle is $2a + 2b$. Each side of the corresponding square will be $\dfrac{a+b}{2}$, the arithmetic mean of a and b.

b The area of the rectangle is ab. The side length of the corresponding square is $\sqrt{ab}$, the geometric mean of a and b.

Solutions to Exercise 3E

1 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

a $a = 5$

$r = \dfrac{10}{5} = 2$

$S_{10} = \dfrac{5(2^{10} - 1)}{2 - 1}$

$= 5115$

b $a = 1$

$r = \dfrac{-3}{1} = -3$

$S_6 = \dfrac{1(-3^6 - 1)}{-3 - 1}$

$= -182$

c $a = -\dfrac{4}{3}$

$r = \dfrac{2}{3} \div -\dfrac{4}{3} = -\dfrac{1}{2}$

$$S_9 = \frac{-\frac{4}{3}\left(\left(-\frac{1}{2}\right)^9 - 1\right)}{-\frac{1}{2} - 1}$$

$= -\dfrac{57}{64}$

2 a $a = 2$

$r = \dfrac{-6}{2} = -3$

$t_n = 1458 = 2 \times -3^{n-1}$

$-3^{n-1} = 729$

$n = 7$

$S_7 = \dfrac{2 \times (-3^7 - 1)}{-3 - 1}$

$= 1094$

b $a = -4$

$r = \dfrac{8}{-4} = -2$

$t_n = -1024 = -4 \times -2^{n-1}$

$-2^{n-1} = 256$

$n = 9$

$S_9 = \dfrac{-4 \times (-2^9 - 1)}{-2 - 1}$

$= -684$

c $a = 6250$

$r = \dfrac{1250}{6250} = 0.2$

$t_n = 2 = 6250 \times (0.2)^{n-1}$

$(0.2)^{n-1} = \dfrac{2}{6250} = \dfrac{1}{3125}$

$n = 6$

$S_6 = \dfrac{6250 \times ((0.2)^6 - 1)}{0.2 - 1}$

$= 7812$

3 $a = 3$ and $r = 2$

$\therefore S_n = \dfrac{3(2^n - 1)}{2 - 1}$

If $S_n = 3069$ then

$3(2^n - 1) = 3069$

$2^n - 1 = 1023$

$2^n = 1024$

$n = 10$

4 $a = 24$ and $r = -\dfrac{1}{2}$

$$\therefore S_n = \frac{24(1 + (\frac{1}{2})^n)}{1 + \frac{1}{2}}$$

$S_n = 16(1 - (\frac{1}{2})^n)$

If $S_n = \dfrac{129}{8}$ then

$$16(1 + (\tfrac{1}{2})^n) = \frac{129}{8}$$

$$1 + (\tfrac{1}{2})^n = \frac{129}{128}$$

$$(\tfrac{1}{2})^n = \frac{1}{128}$$

$$n = 7$$

5 $a = 600,\ r = 1.1$

a $t_7 = 600 \times 1.1^6$

$= 1062.9366$

About 1062.9 mL

b $S_7 = \dfrac{600 \times (1.1^7 - 1)}{1.1 - 1}$

$= 5692.3026$

About 5692.3 mL

c 11 days

6 $a = 20,\ r = \dfrac{25}{20} = 2.5$

a $t_5 = 20 \times 1.25^4$

$= 48.828125$

49 minutes (to the nearest minute)

b $S_5 = \dfrac{20 \times (1.25^5 - 1)}{1.25 - 1}$

$= 164.140625$

164 minutes, or 2 hours and 44 minutes

c

$$S_n > 15 \times 60 = 900$$

$$\frac{20 \times (1.25^n - 1)}{0.25} > 900$$

$$1.25^n - 1 > 900 \times \frac{0.25}{20}$$

$$= 11.25$$

$$1.25^n > 12.25$$

$$n \log_{10} 1.25 > \log_{10} 12.25$$

$$n > 11.228$$

$12 - 7 = 5,$ so Friday.

7 $a = 15,\ r = \dfrac{2}{3}$

$$S_{10} = \frac{15 \times \left(1 - \left(\frac{2}{3}\right)^{10}\right)}{1 - \frac{2}{3}}$$

$$= 3 \times 15 \times \frac{3^{10} - 2^{10}}{3^{10}}$$

$$= 5 \times \frac{3^{10} - 2^{10}}{3^8}$$

$$= \frac{5 \times 58\,025}{6561}$$

$$= \frac{290\,125}{6561}$$

The bounces will all be doubled (up and down) except for the first (down only).

$$\text{Distance} = 2 \times \frac{290\,125}{6561} - 15$$

$$= \frac{481\,835}{6561}$$

$$= 73\frac{2882}{6561} \text{ m}$$

8 $a = \$15\,000,\ r = 1.05$

a $t_5 = 15\,000 \times 1.05^4$

$= 18\,232.593\ldots$

\$18 232.59

b $S_5 = \dfrac{15\,000 \times (1.05^5 - 1)}{1.05 - 1}$

$= 82\,844.4686$

\$82 884.47

9 **a**

$$
\begin{aligned}
ar^2 &= 20 \\
ar^5 &= 160 \\
r^3 &= \frac{160}{20} = 8 \\
r &= 2 \\
a \times 2^2 &= 20 \\
a &= 5 \\
S_5 &= \frac{5 \times (2^5 - 1)}{2 - 1} \\
&= 155
\end{aligned}
$$

b

$$
\begin{aligned}
ar^2 &= \sqrt{2} \\
ar^7 &= 8 \\
r^5 &= \frac{8}{\sqrt{2}} \\
&= \frac{\sqrt{64}}{\sqrt{2}} \\
&= \sqrt{32} = (\sqrt{2})^5 \\
r &= \sqrt{2} \\
a \times (\sqrt{2})^2 &= \sqrt{2} \\
a &= \frac{1}{\sqrt{2}} \\
S_8 &= \frac{\frac{1}{\sqrt{2}} \times ((\sqrt{2})^8 - 1)}{\sqrt{2} - 1} \\
&= \frac{\frac{1}{\sqrt{2}} \times 15}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \\
&= \frac{\frac{15}{\sqrt{2}} \times (\sqrt{2} + 1)}{2 - 1} \\
&= 15 + \frac{15\sqrt{2}}{2}
\end{aligned}
$$

10 $a = 1,\ r = 2$

a

$$
\begin{aligned}
S_n &= 255 \\
\frac{1 \times (2^n - 1)}{2 - 1} &= 255 \\
2^n - 1 &= 255 \\
2^n &= 256 \\
n &= 8
\end{aligned}
$$

b $$S_n > 1\,000\,000$$
$$\frac{1 \times (2^n - 1)}{2 - 1} > 1\,000\,000$$
$$2^n - 1 > 1\,000\,000$$
$$2^n > 1\,000\,001$$
$$n \log_{10} 2 > \log_{10} 1\,000\,001$$
$$n > 19.931\ldots$$
$\{n : n > 19\}$ or $\{n : n \geq 20\}$, since n is a positive integer.

11 $a = 1,\ r = -x^2$

Note that there are $(m + 1)$ terms.

$$S_{m+1} = \frac{1 \times (-x^2)^{m+1} - 1}{-x^2 - 1}$$
$$= \frac{-x^{2(m+1)} - 1}{-x^2 - 1}$$
$$= \frac{x^{2m+2} + 1}{x^2 + 1}$$

12 **a** The thickness of each piece is 0.05 mm.

There are $1 + 2 + 4 + \cdots 2^{40}$ pieces of paper of this thickness.

That is,$1 + 2 + 4 + \cdots 2^{40} = \dfrac{2^{40} - 1}{2 - 1}$.

The thickness is $\dfrac{2^{40} - 1}{2 - 1} \times 0.05 \approx$ 54976 km

b Solve the inequality

$0.05 \times 2^n \geq 384400 \times 10^6$ for n an integer to find $n = 43$.

13 Option 1: \$52 million;

Option 2: \$45 040 000 million

14 $S_{50} = 60$ and $S_{100} = 80$

$$\frac{a(r^{50} - 1)}{r - 1} = 60 \ldots (1) \text{ But } r = 1$$
$$\frac{a(r^{100} - 1)}{r - 1} = 80 \ldots (2)$$
$$(2) \div (1)$$
$$\frac{r^{100} - 1}{r^{50} - 1} = \frac{4}{3}$$

Let $a = r^{50}$

$$3(a^2 - 1) = 4(a - 1)$$
$$3a^2 - 4a + 1 = 0$$
$$(3a - 1)(a - 1) = 0$$
$$\therefore a = \frac{1}{3} \text{ or } a = 1$$
$$\therefore r = 1 \text{ or } r = \left(\frac{1}{3}\right)^{\frac{1}{50}}$$

is not acceptable in our calulation and cleary $r = 1$ does not give the correct result. Therefore

$$r = \left(\frac{1}{3}\right)^{\frac{1}{50}}$$

Solutions to Exercise 3F

1 $A_n = Pr^n$ where $r = 1 + \dfrac{R}{100}$

$A = 5000, R = 6$

a $r = 1.06$ Value of the

$\therefore A_6 = 5000 \times 1.06^6$

$= 7092.5955...$

investment is \$7092.60 after 6 years.

b $10\,000 = 5000 \times 1.06^n$

$2 = 1.06^n$

$\log_{10} 2 = n \log_{10} 1.06$

$n = 11.89566\ldots$

It will take 12 years to double the money.

2 $A = ?, R = 8.5, A_{12} = 8000$

$A_{12} = A \times 1.085^{12}$

$8000 = A \times 1.085^{12}$

$A = 8000 \div 1.085^{12}$

$= 3005.6134...$

You would need to invest \$3005.61.

3 **a** $P = 60\,000(1.15)^{n-1}$

b $60\,000(1.15)^{n-1} > 1\,200\,000$ In the

$(1.15)^{n-1} > 20$

$n > 22.4345...$

23rd year.

c Use $S_n = \dfrac{a(r^{n-1} - 1)}{r - 1}$

$S_n = \dfrac{60\,000(1.15^{n-1} - 1)}{1.15 - 1}$

$= 400\,000(1.15^n - 1)$

4 **a** $D_3 = 65\,000 \times 0.85^3$

$= 39918.125$

It will be worth \$39 918.13

b $65\,000 \times 0.85^n < 32\,500$

$0.85^n < 0.5$

$n > \dfrac{\log_{10} 0.5}{\log_{10} 0.85}$

$n > 4.265...$

During the 5th year

5 $3A = A \times r^{10}$

$3 = \times r^{10}$

$r = 1.116...$

The required interest rate is 11.6% p.a.

6 $D_n = 40\,000 \times 0.85^n$

$40\,000 \times 0.85^n < 10\,000$

$0.85^n < 0.25$

$n > \dfrac{\log_{10} 0.25}{\log_{10} 0.85}$

$n > 8.53...$

During the 9th year

7 Use $A_n = \dfrac{Pr(r^n - 1)}{r - 1}$

$A_{10} = \dfrac{25\,000 \times 1.05 \times (1.05^{10} - 1)}{0.05}$

$= 330169.679\ldots$

Total is \$330 169.68

8 Use $A_n = \dfrac{Pr(r^n - 1)}{r - 1}$

$100\,000 = \dfrac{P \times 1.1 \times (1.1^{20} - 1)}{0.1}$

$P = 1587.2386\ldots\ldots$

Annual payments should be $1587.24.

9 Use $A_n = \dfrac{Pr(r^n - 1)}{r - 1}$

a $A_{20} = \dfrac{20\,000 \times 1.06 \times (1.06^{10} - 1)}{0.06}$

$A_{20} = 279432.852\ldots$

The investment is worth $279 432.85 after 10 years.

b

$$200\,000 = \frac{20\,000 \times 1.06 \times (1.06^n - 1)}{0.06}$$

$$0.56603\ldots = 1.06^n - 1$$

$$1.056603\ldots = 1.06^n$$

$$n = \frac{\log_{10} 1.056603\ldots}{\log_{10} 1.06}$$

$$n \approx 7.697\ldots$$

After 8 years

10 We can use $D_n = Pr^n - \dfrac{Q(r^n - 1)}{r - 1}$

$P = 100\,000$, $Q = 10\,000$, $r = 1.05$

a

$$D_{10} = 100000 \times 1.05^{10} - \frac{10000(1.05^{10} - 1)}{0.05}$$

$$= 37110.537\ldots$$

He owes $37 110.54 after 10 years

b

$$100000 \times 1.05^n - \frac{10000(1.05^n - 1)}{0.05} = 0$$

$$100000 \times 1.05^n = \frac{10000(1.05^n - 1)}{0.05}$$

$$5000 \times 1.05^n = 10000 \times (1.05^n - 1)$$

$$2 = 1.05^n$$

$$n = 14.206\ldots$$

After 14 years he owes $ 2006.84

At the end of the 15th year, the final repayment is $2107.18

11 We can use $D_n = Pr^n - \dfrac{Q(r^n - 1)}{r - 1}$

$$D_n = Pr^n - \frac{Q(r^n - 1)}{r - 1}$$

$$0 = 50000(1.04)^{15} - \frac{Q(1.04^{15} - 1)}{0.04}$$

$$Q = 4497.055\ldots$$

The equal installments are $4497.06

12 Andrew invested with simple interest at 20% for 10 years.

Total interest = $1000 \times 0.2 \times 10$

The total amount at the end of 10 years = $3000.

Bianca invested $1000 at 12.5% for 10 with compound interest.

Total amount = $1000 \times 1.125^10 = 3247.32\ldots$

Bianca $3247.32; Andrew $3000

13 **a** **i** $20\,000 \times 1.15 - 2000 = 21\,000$

ii $21\,000 \times 1.15 - 2000 = 22\,150$

iii $22\,150 \times 1.15 - 2000 = 23\,473$

b $P_n = 1.15P_{n-1} - 2000$

c $P_n = 20\,000 \times 1.15^n - \dfrac{40\,000}{3}(1.15^n - 1)$

d 67 580

14 **a** **i** \$290 000

ii \$279 000

iii \$266 900

b $A_n = 1.1A_{n-1} - 40\,000$

c $A_n = 300\,000 \times 1.1^n - 400\,000(1.1^n - 1)$

d At the end of the 15th year, the final payment is \$22 275.18

Solutions to Exercise 3G

1 **a** $t_n = 3t_{n-1} + 4. \quad t_1 = 6$

$t_2 = 3 \times 6 + 4 = 22$

$t_3 = 3 \times 22 + 4 = 70$

$t_4 = 3 \times 70 + 4 = 214$

b $s_n = 6s_{n-1} + 2. \quad s_1 = 1$

$t_2 = 6 \times 1 + 2 = 8$

$t_3 = 6 \times 8 + 2 = 50$

$t_4 = 6 \times 50 + 2 = 302$

c $t_{n+1} = 3t_n - 4. \quad t_1 = 6$

$t_2 = 3 \times 6 - 4 = 14$

$t_3 = 3 \times 14 - 4 = 38$

$t_4 = 3 \times 38 - 4 = 110$

d $u_{n+1} = 4u_n + 1. \quad u_1 = 2$

$u_2 = 4 \times 2 + 1 = 9$

$u_3 = 4 \times 9 + 1 = 37$

$u_4 = 4 \times 37 + 1 = 149$

2 **a** $2, 6, 26, \ldots$

$t_1 = 2$

$t_2 = 6$

$$t_2 = 5 \times t_1 + d$$

$$6 = 5 \times 2 + d$$

$\therefore d = -4$

b $2, 6, 26, \ldots$

$t_1 = 500$

$t_2 = 650$

$$t_2 = r \times t_1 - 100$$

$$650 = r \times 500 - 100$$

$$750 = 500r$$

$$\therefore r = \frac{3}{2}$$

c $1000, 100, -80, \ldots$

$T_1 = 1000$

$T_2 = 100$

$$T_2 = 0.2 \times T_1 + d$$

$$100 = 0.2 \times 1000 + d$$

$$\therefore d = -100$$

d $a, 22, 90, \ldots$

$s_1 = a$

$s_2 = 22$

$$s_2 = 4 \times s_1 + 2$$

$$22 = 4 \times a + 2$$

$$\therefore a = 5$$

3 **a** $2, 5, 11, \ldots$

$t_1 = 2, t_2 = 5, t_3 = 11$

$$5 = 2r + d \qquad (1)$$

$$11 = 5r + d \qquad (2)$$

$$(2) - (1)$$

$$6 = 3r$$

$$r = 2$$

$$d = 1 \qquad \text{From (1))}$$

b $512, 192, 32, \ldots$

$v_1 = 512, v_2 = 192, v_3 = 32$

$$192 = 512r + d \qquad (1)$$

$$32 = 192r + d \qquad (2)$$

$$(1) - (2)$$

$$160 = 320r$$

$$r = \frac{1}{2}$$

$$d = -64 \qquad \text{From (1))}$$

c $a, 10, 55, \ldots$

$t_1 = a, t_2 = 10, t_3 = 55$

$10 = 5a + d \qquad (1)$

$55 = 5 \times 10 + d \qquad (2)$

From (2) :

$d = 5$

Substitute in(1)

$10 = 5a + 5$

$a = 1$

d 200, 500, 1400, . . .

$t_1 = 200, t_2 = 500, t_3 = 1400$

$500 = 200r + d \qquad (1)$

$1400 = 500r + d \qquad (2)$

(2) − (1)

$900 = 300r$

$r = 3$

$d = -100 \qquad$ From (1))

4 a $a_1 = k, a_n = 5a_{n-1} + 3$

$a_1 = k$

$a_2 = 5k + 3$

$a_3 = 5(5k + 3) + 3$

$= 25k + 18$

b $a_4 = 5a_3 + 5$

$= 5(25k + 18) + 3$

$= 125k + 93$

Sum of the first 4 terms

$= k + 5k + 3 + 25k + 18 + 125k + 93$

$= 156k + 114$

5 We use

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

a $r = 2, d = -6, t_1 = 7$

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

$$= 2^{n-1} \times 7 + \frac{(-6)(2^{n-1} - 1)}{2 - 1}$$

$$= 7 \times 2^{n-1} - 6(2^{n-1} - 1)$$

$$= 2^{n-1} + 6$$

b $r = 2, d = -2, t_1 = 1$

$$t_n = 2^{n-1} \times 1 + \frac{(-2)(2^{n-1} - 1)}{2 - 1}$$

$$= 2^{n-1} - 2(2^{n-1} - 1)$$

$$= 2 - 2^{n-1}$$

c $r = \frac{1}{2}, d = 10, t_1 = 20$

$$t_n = \frac{1}{2}^{n-1} \times 20 + \frac{10 \times (\frac{1}{2}^{n-1} - 1)}{\frac{1}{2} - 1}$$

$$= 20 \times \frac{1}{2}^{n-1} - 2(10 \times \frac{1}{2}^{n-1} - 1)$$

$$= 20$$

d $r = \frac{1}{2}, d = 14, t_1 = 20$

$$t_n = \frac{1}{2}^{n-1} \times 20 + \frac{14 \times (\frac{1}{2}^{n-1} - 1)}{\frac{1}{2} - 1}$$

$$= 20 \times \frac{1}{2}^{n-1} - 2(14 \times \frac{1}{2}^{n-1} - 1)$$

$$= 28 - 8 \times \frac{1}{2}^{n-1}$$

$$= 28 - \frac{8}{2^{n-1}}$$

e $r = \frac{1}{2}, d = -10, t_1 = 20$

$$t_n = \frac{1}{2}^{n-1} \times 20 + \frac{(-10) \times (\frac{1}{2}^{n-1} - 1)}{\frac{1}{2} - 1}$$

$$= 20 \times \frac{1}{2}^{n-1} - 2((-10) \times \frac{1}{2}^{n-1} - 1)$$

$$= 40 \times \frac{1}{2}^{n-1} - 20$$

$$= \frac{40}{2^{n-1}} - 20$$

f $r = \frac{1}{2}, d = \frac{1}{2}, t_1 = 1$

$$t_n = \frac{1}{2}^{n-1} \times 1 + \frac{(\frac{1}{2}) \times (\frac{1}{2}^{n-1} - 1)}{\frac{1}{2} - 1}$$

$$= 1 \times \frac{1}{2}^{n-1} - 2((\frac{1}{2}) \times (\frac{1}{2}^{n-1} - 1)$$

$$= 1$$

6 **a** $r = \frac{1}{2}, d = 5, t_1 = 6$

$$t_n = r^{n-1} t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

$$= \left(\frac{1}{2}\right)^{n-1} \times 6 + \frac{5\left(\left(\frac{1}{2}\right)^{n-1}\right)}{\frac{1}{2} - 1}$$

$$= 6 \times \left(\frac{1}{2}\right)^{n-1} - 10\left(\left(\frac{1}{2}\right)^{n-1} - 1\right)$$

$$= -4 \times \left(\frac{1}{2}\right)^{n-1} + 10$$

b $t_1 = 6$

$t_2 = -4 \times \frac{1}{2} + 10 = 8$

$t_3 = -4 \times \frac{1}{4} + 10 = 9$

$t_4 = -4 \times \frac{1}{8} + 10 = \frac{19}{2}$

c $t_n = -4 \times \left(\frac{1}{2}\right)^{n-1} + 10$

As $n \to \infty, \left(\frac{1}{2}\right)^{n-1} \to 0.$

$t_n < 10$ for all n

Hence $t_n \to 10$ from below.

7 **a** $r = -\frac{1}{2}, d = 5, t_1 = 6$

$$t_n = r^{n-1} t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

$$= \left(-\frac{1}{2}\right)^{n-1} \times 6 + \frac{5\left(\left(-\frac{1}{2}\right)^{n-1}\right)}{-\frac{1}{2} - 1}$$

$$= 6 \times \left(-\frac{1}{2}\right)^{n-1} - \frac{10}{3}\left(\left(-\frac{1}{2}\right)^{n-1} - 1\right)$$

$$= \frac{8}{3} \times \left(-\frac{1}{2}\right)^{n-1} + \frac{3}{2}$$

$$= \frac{1}{3}\left(8 \times \left(-\frac{1}{2}\right)^{n-1} + 10\right)$$

b $t_1 = 6$

$t_2 = -8 \times \frac{1}{6} + \frac{10}{3} = 2$

$t_3 = \frac{2}{3} + \frac{10}{3} = 4$

$t_4 = \frac{1}{3} + \frac{10}{3} = 3$

c $\frac{1}{3}\left(8 \times \left(-\frac{1}{2}\right)^{n-1} + 10\right)$ As $n \to \infty, \left(-\frac{1}{2}\right)^{n-1} \to 0.$

Hence $t_n \to \frac{10}{3}.$

8 From the first three terms we have

$$7 = A + B \ldots (1)$$

$$31 = Ar + B \ldots (2)$$

$$103 = Ar^2 + B \ldots (3)$$

Subtract (1) from (2) and (3).

$24 = A(r - 1) \dots (4)$

$96 = A(r^2 - 1) \dots (5)$

Divide (5) by (4)

$4 = r + 1$

$r = 3$ Substitute in (4)

$A = 12$

Substitute in (1)

$B = -5$

Therefore $t_n = 12 \times 3^{n-1} - 5$

9 From the first three terms we have

$16 = A + B \dots (1)$

$5 = Ar + B \dots (2)$

$-\frac{1}{2} = Ar^2 + B \dots (3)$

Subtract (1) from (2) and (3).

$-11 = A(r - 1) \dots (4)$

$-\frac{33}{2} = A(r^2 - 1) \dots (5)$

Divide (5) by (4)

$\frac{3}{2} = r + 1$

$r = \frac{1}{2}$ Substitute in (4)

$A = 22$

Substitute in (1)

$B = -6$

Therefore $t_n = 22 \times \frac{1}{2}^{n-1} - 6$

10 a $t_n = 3 \times 2^n - 4$

$t_1 = 2, t_2 = 8, t_3 = 20$

$\therefore 8 = 2r + d \dots (1)$ and

$20 = 8r + d \dots (2)$

Subtract (1) from (2).

$12 = 6r$

$r = 2$

$\therefore d = 4$

$t_n = 2t_{n-1} + 4$

b $3 \times 2^n - 4 > 1000$

$$2^n > \frac{1004}{3}$$

$$n > 8.386\ldots$$

$$n \geq 9$$

Smallest value is 9.

c $$\begin{aligned} t_{n+1} - t_n &= 3 \times 2^{n+1} - 4 - (3 \times 2^n - 4) \\ &= 3 \times 2^{n+1} - 3 \times 2^n \\ &= 3 \times 2^n(2 - 1) \\ &= 3 \times 2^n > 0 \qquad \text{(for all } n \in \mathbb{N}\text{)} \end{aligned}$$

11 $t_n = t_{n-1} + 2n$

$t_1 = 5$

$t_2 = 5 + 2 \times 2$

$t_3 = (5 + 2 \times 2) + 2 \times 3 = 5 + 2 \times (2 + 3)$

$t_4 = (5 + 2 \times 2 + 2 \times 3) + 2 \times 4 = 5 + 2 \times (2 + 3 + 4)$

$\vdots$

$\vdots$

$$\begin{aligned} t_n &= 5 + 2 \times (2 + 3 + 4 + \cdots + n) \\ &= 5 + 2 \times \frac{n-1}{2}(4 + (n - 2)) \\ &= 5 + (n - 1)(n + 2) \\ &= 5 + n^n + n - 2 \\ &= n^2 + n + 3 \end{aligned}$$

12 $t_n = t_{n-1} + 2n + 1$

$t_1 = 5$

$t_2 = 5 + 2 \times 2 + 1$

$t_3 = 5 + 2 \times 2 + 1 + 2 \times 1$

The result is the same as in the previous question but adding $(n - 1) \times 1$

Therefore:

$t_n = n^2 + n + 3 + n - 1 = n^2 + 2n + 2$

13 a $a_n = 4a_{n-1} - 1$ and $a_2 = 43$

$43 = 4a_1 - 1$

$a_1 = 11$

$a_3 = 4 \times 43 - 1 = 171$

b We use

$$a_n = r^{n-1}a_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

$a_1 = 11, r = 4, d = -1$

$$a_n = 4^{n-1} \times 11 + \frac{(-1)(4^{n-1} - 1)}{3}$$

$$= 4^{n-1} \times 11 - \frac{1}{3}(4^{n-1} - 1)$$

$$= \frac{1}{3}(32 \times 4^{n-1} + 1)$$

14 a $s_1 = 1$ and $s_n = -\frac{1}{2}s_{n-1} + 3$

We use

$$s_n = r^{n-1}s_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

$s_1 = 1, r = -\frac{1}{2}, d = 3$

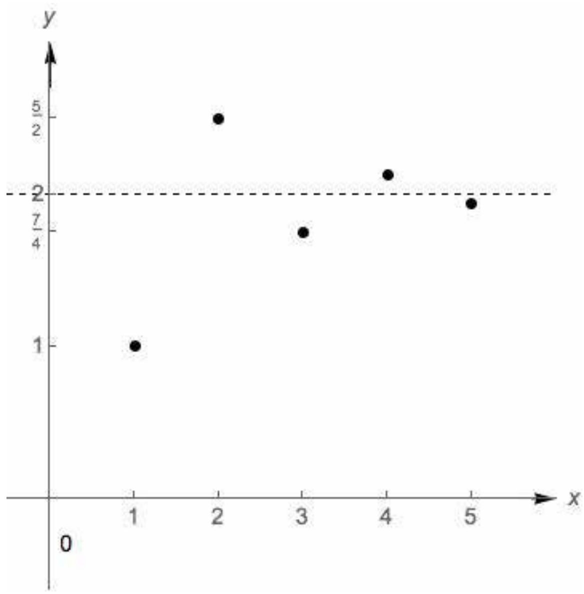
$$s_n = \left(-\frac{1}{2}\right)^{n-1} \times 1 + \frac{3(\left(-\frac{1}{2}\right)^{n-1} - 1)}{-\frac{1}{2} - 1}$$

$$= \left(-\frac{1}{2}\right)^{n-1} - 2((\left(-\frac{1}{2}\right)^{n-1} - 1)$$

$$= 2 - \left(-\frac{1}{2}\right)^{n-1}$$

b $1, \frac{5}{2}, \frac{7}{4}, \frac{17}{8}, \frac{31}{16}$

c



d As n becomes large s_n becomes close to 2 and oscillates from side to side.

15 **a** $N_n = 1.22N_{n-1} - 250, N_1 = 1356$

b The deer population continues to increase.

16 **a** t_n is the population at the start of the nth year. $t_n = 1.085t_{n-1} + 250, t_1 = 3000$

b $r = 1.085, d = 250, t_1 = 3000$

$$t_n = 1.085^{n-1} \times 3000 + \frac{250(1.085^{n-1} - 1)}{1.085 - 1}$$

$$\therefore t_n = 1.085^{n-1} \times 3000 + \frac{50\,000}{17}(1.085^{n-1} - 1)$$

Simplifying:

$$\therefore t_n = \frac{1}{17}(101\,000 \times 1.085^{n-1} - 50\,000)$$

c $t_{11} = \frac{1}{17}\left(101\,000 \times 1.085^{10} - 50\,000\right) \approx 10492$

d Using CAS, 2013

17 **a** $t_n = t_{n-1} - 0.03t_{n-1} + 0.2t_{n-1} + 2 \times 0.05t_{n-1}$

$t_n = 1.27t_{n-1} - 120 \quad t_1 = 2000$

b $t_n = 1.27^{n-1} \times 200 - \frac{20(1.27^{n-1} - 1)}{1.27 - 1}$

$= \frac{3400}{27} \times 1.27^{n-1} + \frac{2000}{27}$

$= \frac{200}{27}\left(17 \times 1.27^{n-1} + 10\right)$

c With CAS, 1156

18 **a** $A_{n+1} = 1.007A_n - 400, A_1 = 15\,000$

b $A_n = 1.007^{n-1} \times 15\,000 - \frac{400(1.007^{n-1} - 1)}{1.007 - 1}$

$A_n = -\frac{295\,000}{7} \times 1.007^{n-1} + \frac{400\,000}{7}$

$A_n = \frac{1}{7}\left(-295\,000 \times 1.007^{n-1} + 400\,000\right)$

c CAS calc. After 44.6 . . . . Paid off by the start of the 45th month.

19 **a** $t_n = 0.6t_{n-1} + 60 \quad t_1 = 32 \; t_n = 0.6^{n-1} \times 32 + \frac{60(0.6^{n-1} - 1)}{0.6 - 1}$

$t_n = -118 \times 0.6^{n-1} + 150$

b $t_{n+1} - t_n = (-118 \times 0.6^n + 150) - (-118 \times 0.6^{n-1} + 150)$

$= 118 \times 0.6^{n-1} - 118 \times 0.6^n$

$= 118 \times 0.6^{n-1}(1 - 0.6)$

$= \frac{236 \times 0.6^{n-1}}{5} > 0$

for all $n \in \mathbb{N}$.

The sequence is increasing

c $\frac{236 \times 0.6^{n-1}}{5} < 0.001$

$n > 22.068\ldots$

$n \geq 23$

d $150 - 118 \times 0.6^{n-1}$

$118 \times 0.6^{n-1} > 0$ for all $n \in \mathbb{N}$

Hence $150 - 118 \times 0.6^{n-1} \leq 150$ for all $n \in \mathbb{N}$

e As $n \to \infty, 0.6^{n-1} \to 0$

Hence $t_n \to 150$

f $s_n = 0,6s_{n-1} + d, s_1 = 32$

$s_n = 0.6^{n-1} \times 32 + \frac{d(0.6^{n-1} - 1)}{0.6 - 1}$

$$s_n = 0.6^{n-1} \times 32 - \frac{5d}{2}(0.6^{n-1} - 1)$$

$$s_n = 0.6^{n-1} \times 32 - \frac{5d}{2}(0.6^{n-1} - 1)$$

$\therefore$ if $n \to \infty$ implies $s_n \to 200$

$$\frac{5d}{2} = 200$$

$$d = 80$$

Solutions to Exercise 3H

1 $S_\infty = \dfrac{a}{1-r}$

a $a = 1$

$r = \dfrac{1}{5} \div 1 = \dfrac{1}{5}$

$S_\infty = \dfrac{1}{1 - \dfrac{1}{5}}$

$= \dfrac{5}{4}$

b $a = 1$

$r = -\dfrac{2}{3} \div 1 = -\dfrac{2}{3}$

$S_\infty = \dfrac{1}{1 - -\dfrac{2}{3}}$

$= \dfrac{3}{5}$

2 Each side, and hence each perimeter, will be half the larger side.

$r = \dfrac{1}{2},\ a = p$

Perimeter of nth triangle $= p \times \left(\dfrac{1}{2}\right)^{n-1}$

$= \dfrac{p}{2^{n-1}}$

$S_\infty = \dfrac{p}{1 - \dfrac{1}{2}}$

$= 2p$

Area $= \dfrac{p^2\sqrt{3}}{9 \times 4^n}$

Sum of the areas $= \dfrac{p^2\sqrt{3}}{27}$

3 $a = 200,\ r = 0.94$

$S_\infty = \dfrac{200}{1 - 0.94}$

$= 3333\dfrac{1}{3}$ m

4 $a = 3,\ r = 0.5$

$S_\infty = \dfrac{3}{1 - 0.5} = 6$

It's not perfectly clear when the problem ceases to be realistic, however he can only make the joumey if he walks for an infinite time. Clearly, this is not possible.

5 $a = 2,\ r = \dfrac{3}{4}$

$S_\infty = \dfrac{2}{1 - 0.75} = 8$

The frog only travels 8 metres (in the limit).

6 $r = 70\% = 0.7$

$S_\infty = \dfrac{a}{1 - 0.7} = 40$

$a = 0.3 \times 40$

$= 12$ m

7 Note: all distances will be double (up and down) except the first (down only).

Use $a = 30,\ r = \dfrac{2}{3}$ and subtract 15 m from the answer.

$S_\infty = \dfrac{30}{1 - \dfrac{2}{3}} = 90$

Distance $= 90 - 15 = 75$ m

8 **a** $a = 0.4,\ r = 0.1$

$S_\infty = \dfrac{0.4}{1 - 0.1} = \dfrac{4}{9}$

b $a = 0.03,\ r = 0.1$

$S_\infty = \dfrac{0.03}{1 - 0.1}$

$= \dfrac{3}{90} = \dfrac{1}{30}$

c $a = 0.3,\ r = 0.1$

$$S_\infty = \frac{0.3}{1 - 0.1} = \frac{3}{9} = \frac{1}{3}$$

Decimal $= 10\frac{1}{3} = \frac{31}{3}$

d $a = 0.035,\ r = 0.01$

$$S_\infty = \frac{0.035}{1 - 0.01} = \frac{35}{990} = \frac{7}{198}$$

e $a = 0.9,\ r = 0.1$

$$S_\infty = \frac{0.9}{1 - 0.1} = \frac{9}{9} = 1$$

f $a = 0.1,\ r = 0.1$

$$S_\infty = \frac{0.1}{1 - 0.1} = \frac{1}{9}$$

Decimal $= 4\frac{1}{9} = \frac{37}{9}$

9 a The series converges if and only if

$$|\tfrac{a-b}{a}| < 1$$

$$|a - b| < |a| = a$$

$$-a < a - b < a$$

$$-2a < -b < 0$$

$$0 < b < 2a$$

b
$$S_\infty = \frac{a}{1 - r} = \frac{a}{1 - \frac{a-b}{a}} = \frac{a^2}{a - (a - b)} = \frac{a^2}{b)}.$$

10 a The series converges if and only if $|x| < 1$. That is, $-1 < x < 1$. As $a = 1, r = x$ we find that $S_\infty = \frac{1}{1 - x}$.

b The series converges if and only if

$$|2a - 1| < 1$$

$$-1 < 2a - 1 < 10 < a < 1$$

As the first term is a and $r = 2a - 1$ we find that $S_\infty = \frac{a}{1 - (2a - 1)} = \frac{a}{2(1 - a)}$.

c The series converges if and only if

$$\left|\frac{x}{3x - 1}\right| < 1$$

$$x < \frac{1}{4} \text{ or } x > \frac{1}{2}.$$

As $a = \frac{3x - 1}{x}, r = \frac{x}{3x - 1}$ we find that

$$S_\infty = \frac{\frac{3x - 1}{x}}{1 - \frac{x}{3x - 1}} = \frac{(3x - 1)^2}{x(2x - 1)}$$

d The series converges if and only if

$$\left|\frac{x^2}{3x^2 - 1}\right| < 1$$

Therefore

$$x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

We can replace x with x^2 in the previous answer to find that

$$S_\infty = \frac{(3x^2 - 1)^2}{x^2(2x^2 - 1)}.$$

11
$$S_4 = \frac{a(1 - r^4)}{1 - r} = 30$$

$$S_\infty = \frac{a}{1 - r} = 32$$

$a = 32(1 - r)$

Substitute for a:

$$\frac{32(1-r)(1-r^4)}{1-r} = 30$$

$$32(1-r^4) = 30$$

$$1 - r^4 = \frac{30}{32}$$

$$r^4 = 1 - \frac{30}{32}$$

$$= \frac{2}{32} = \frac{1}{16}$$

$$r = \frac{1}{2} \text{ or } r = -\frac{1}{2}$$

If $r = \frac{1}{2}$: $a = 32\left(1 - \frac{1}{2}\right)$

$= 16$

If $r = -\frac{1}{2}$: $a = 32\left(1 - \left(-\frac{1}{2}\right)\right)$

$= 48$

The first two terms are 16 and 8, or 48 and −24. In both cases, the sum is 24.

12 $S_\infty = \dfrac{a}{1 + \frac{1}{4}}$

$$= \frac{4a}{5} = 8$$

$$a = 10$$

$$t_3 = 10 \times \left(-\frac{1}{4}\right)^2$$

$$= \frac{5}{8}$$

13 $\dfrac{5}{1-r} = 15$

$$5 = 15(1 - r)$$

$$1 - r = \frac{1}{3}$$

$$r = 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

14 $\dfrac{2}{1-r} = x$

Solve for r

$$\frac{2}{x} = 1 - r$$

$$r = 1 - \frac{2}{x}$$

Since $x > 2$, $\frac{2}{x} < 1$ and so $r = 1 - \frac{2}{x} < 1$

Solutions to short-answer questions (technology-free)

1 **a** $t_1 = 3$

$t_2 = 3 - 4 = -1$

$t_3 = -1 - 4 = -5$

$t_4 = -5 - 4 = -9$

$t_5 = -9 - 4 = -13$

$t_6 = -13 - 4 = -17$

b $t_1 = 5$

$t_2 = 2 \times 5 + 2 = 12$

$t_3 = 2 \times 12 + 2 = 26$

$t_4 = 2 \times 26 + 2 = 54$

$t_5 = 2 \times 54 + 2 = 110$

$t_6 = 2 \times 110 + 2 = 222$

2 **a** $t_1 = 2 \times 1 = 2$

$t_2 = 2 \times 2 = 4$

$t_3 = 2 \times 3 = 6$

$t_4 = 2 \times 4 = 8$

$t_5 = 2 \times 5 = 10$

$t_6 = 2 \times 6 = 12$

b $t_1 = -3 \times 1 + 2 = -1$

$t_2 = -3 \times 2 + 2 = -4$

$t_3 = -3 \times 3 + 2 = -7$

$t_4 = -3 \times 4 + 2 = -10$

$t_5 = -3 \times 5 + 2 = -13$

$t_6 = -3 \times 6 + 2 = -16$

3 **a** End of first year:
$\$5000 \times 1.05 = \$5250$
Start of second year:
$\$5250 + \$500 = \$5750$
End of second year:
$\$5750 \times 1.05 = \$6037.50$

b $t_n = 1.05(t_{n-1} + 500),\ t_1 = 5250$

4

$$a + 3d = 19 \ldots (1)$$
$$a + 6d = 43 \ldots (2)$$

Equation (2) − Equation (1)

$$3d = 24$$
$$d = 8$$
$$a + 3 \times 8 = 19$$
$$a = -5$$
$$t_{20} = -5 + 19 \times 8$$
$$= 147$$

5

$$a + 4d = 0.35 \ldots (1)$$
$$a + 8d = 0.15 \ldots (2)$$

Equation (2) − Equation (1)

$$4d = -0.2$$
$$d = -0.05$$
$$a + 4 \times -0.05 = 0.35$$
$$a = 0.35 + 0.2$$
$$= 0.55$$
$$t_{14} = 0.55 + 13 \times -0.55$$
$$= -0.1$$

6 $a + 5d = -24 \ldots (1)$

$a + 13d = 6 \ldots (2)$

Equation (2) − Equation (1)

$8d = 30$

$d = 3.75$

$a + 5 \times 3.75 = -24$

$a = -24 - 18.75$

$= -42.75$

$S_{10} = 5 \times (-85.5$

$+ 9 \times 3.75)$

$= -258.75$

7 $a = -5,\ d = 7$

$S_n = \frac{n}{2}(-10 + 7(n-1))$

$= 402$

$n(-10 + 7(n-1)) = 804$

$7n^2 - 10n - 7n = 804$

$7n^2 - 17n - 804 = 0$

$(7n + 67)(n - 12) = 0$

$n = 12$ (since $n > 0$)

$\{n : S_n = 402\} = \{n : n = 12\}$

8 $ar^5 = 9$

$ar^9 = 729$

$r^4 = 81$

$r = 3$ or $r = -3$

$r = 3 : a \times 3^5 = 9$

$a = \frac{9}{243} = \frac{1}{27}$

$t_4 = \frac{1}{27} \times 3^3 = 1$

$r = -3 : a \times (-3)^5 = 9$

$a = -\frac{1}{27}$

$t_4 = -\frac{1}{27} \times (-3)^3 = 1$

So for either case, $t_4 = 1$

9 $a = 1000$

$r = 1.035$

$t_n = ar^n$

$= 1000 \times 1.035^n$

10 Solution:

$t_n = 2^{n-1} \times 4 - 3(2^{n-1} - 1)$

Therefore, $t_n = 2^{n-1} + 3$

11 $9r^2 = 4$

$r^2 = \frac{4}{9}$

$r = \pm\frac{2}{3}$

$t_2 = ar = \pm 6$

$t_4 = ar^3 = \pm\frac{8}{3}$

Terms $= 6,\ \frac{8}{3}$ or $-6, -\frac{8}{3}$

12 $a + ar + ar^2 = 24$

$ar^3 + ar^4 + ar^5 = 24$

$r^3(a + ar + ar^2) = 24$

$r^3 = 1$

$r = 1$

All terms will be the same: $t_n = \frac{24}{3} = 8$

$S_{12} = 12 \times 8 = 96$

13 $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$S_8 = \dfrac{6 \times (-3^8 - 1)}{-3 - 1}$

$= -9840$

14 $a = 1,\ r = -\dfrac{1}{3}$

$S_\infty = \dfrac{1}{1 - -\frac{1}{3}}$

$= \dfrac{3}{4}$

15

$$\frac{x+4}{x} = \frac{2x+2}{x+4}$$

$$(x+4)^2 = x(2x+2)$$

$$x^2 + 8x + 16 = 2x^2 + 2x$$

$$2x^2 + 2x - x^2 - 8x - 16 = 0$$

$$x^2 - 6x - 16 = 0$$

$$(x-8)(x+2) = 0$$

$$x = 8 \text{ or } x = -2$$

Solutions to multiple-choice questions

1 D $t_1 = 3 \times 1 + 2 = 5$

$t_2 = 3 \times 2 + 2 = 8$

$t_3 = 3 \times 3 + 2 = 11$

2 B $t_2 = 3 + 3 = 6$

$t_3 = 6 + 3 = 9$

$t_4 = 9 + 3 = 12$

3 A $a = 10$

$d = 8 - 10 = -2$

$t_{10} = 10 + (9 \times -2)$

$= -8$

4 A $a = 10,\ d = -2$

$S_{10} = \dfrac{10}{2}(10 + -8)$

$= 10$

5 B $a = 8$

$d = 13 - 8 = 5$

$t_n = 8 + 5(n - 1) = 58$

$5(n - 1) = 50$

$n - 1 = 10$

$n = 11$

6 D $a = 12$

$r = \dfrac{8}{12} = \dfrac{2}{3}$

$t_6 = 12 \times \left(\dfrac{2}{3}\right)^5$

$= \dfrac{128}{81}$

7 E $a = 8$

$r = \dfrac{4}{8} = \dfrac{1}{2}$

$S_6 = \dfrac{8 \times \left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \frac{1}{2}}$

$= 15\dfrac{3}{4}$

8 C $a = 8$

$r = \dfrac{4}{8} = \dfrac{1}{2}$

$S_\infty = \dfrac{8}{1 - \frac{1}{2}}$

$= 16$

9 E Value $= 2000 \times 1.055^6$

$= \$2757.69$

10 D $\dfrac{a}{1 - \frac{1}{3}} = 37.5$

$a = 37.5 \times \dfrac{2}{3}$

$= 25$

11 B

$t_3 = 4 \times 19 - 5 = 71$

12 E $t_3 = \frac{1}{2}t_2 + 2$

$12 = \frac{1}{2}t_2 + 2$

$t_2 = 20$

$t_2 = \frac{1}{2}t_1 + 2$

$20 = \frac{1}{2}t_1 + 2$

$t_1 = 36$

13 B $t_0 = 6$

$t_1 = 2 \times 6 - 6 = 6$ $t_2 = 2 \times 6 - 6 = 6$

$t_3 = 2 \times 6 - 6 = 6$ $t_4 = 2 \times 6 - 6 = 6$

Solutions to extended-response questions

1 **a** $0.8, 1.5, 2.2, \ldots$

b $d = 0.7$ and so the sequence is conjectured to be arithmetic.

c
$$\begin{aligned} t_n &= 0.8 + (n-1) \times 0.7 \\ \therefore t_{12} &= 0.8 + (12-1) \times 0.7 \\ &= 8.5 \end{aligned}$$
The length of moulding in the kit size 12 is 8.5 metres.

2 **a** $d = 25$ and so the sequence is arithmetic.

b
$$\begin{aligned} t_n &= a + (n-1)d \\ &= 50 + (n-1) \times 25 \\ &= 50 + 25n - 25 \\ &= 25n + 25 \end{aligned}$$

c
$$\begin{aligned} t_{25} &= 25 \times 25 + 25 \\ &= 650 \end{aligned}$$
There are 650 seeds in the 25th size packet.

3 The distances $5, 5-d, \ 5-2d, \ldots, 5-6d$ form an arithmetic sequence of seven terms with common difference $-d$.

$$\begin{aligned} \text{Now } S_n &= \frac{n}{2}(a + \ell) \\ \therefore S_7 &= \frac{7}{2}(5 + 5 - 6d) \\ \text{Since } S_7 = 32 - 3 = 29, \ 29 &= \frac{7}{2}(10 - 6d) \\ \therefore \frac{58}{7} &= 10 - 6d \\ \therefore 6d &= \frac{12}{7} \\ \therefore d &= \frac{2}{7} \end{aligned}$$

The distance of the fifth pole from town A is given by S_5.

$$\begin{aligned} S_5 &= \frac{5}{2}\left(5 + 5 - 4 \times \frac{2}{7}\right) \\ &= \frac{155}{7} \\ &= 22\frac{1}{7} \text{ and } 32 - 22\frac{1}{7} = 9\frac{6}{7} \end{aligned}$$

The fifth pole is $22\frac{1}{7}$ km from town A and $9\frac{6}{7}$ km from town B.

4 **a** $D_n = a + (n - 1)d$

$= 2 + (n - 1) \times 7$

$= 2 + 7n - 7$

$= 7n - 5$

b $D_{n+1} = 191$

$\therefore 7(n + 1) - 5 = 191$

$\therefore 7(n + 1) = 196$

$\therefore n + 1 = 28$

$\therefore n = 27$

The firm made 27 different thicknesses.

5 $t_1 = 4,\ t_2 = 16,\ t_3 = 28 \qquad \therefore d = 12$

$t_{40} = a + (40 - 1)d$

$= 4 + 39 \times 12$

$= 472$

The house will slip 472 mm in the 40th year.

6 $t_1 = 16,\ t_2 = 24,\ t_3 = 32 \qquad \therefore d = 8$

$S_{10} = \frac{10}{2}(2 \times 16 + (10 - 1) \times 8)$

$= 5\,(32 + 72)$

$= 520$

She will have sent 520 cards altogether in 10 years.

7 **a** $a = 90,\ r = \frac{1}{10},$

$$\therefore S_6 = \frac{90\left(1 - \left(\frac{1}{10}\right)^6\right)}{1 - \frac{1}{10}}$$

$= 99.9999$

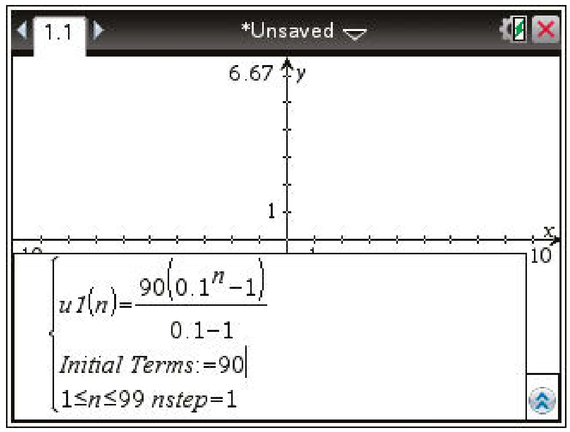
After six rinses, Joan will have washed out 99.9999 mg of shampoo.

b
$$S_\infty = \frac{a}{1-r} = \frac{90}{1-\frac{1}{10}} = 100$$

There were 100 mg present at the beginning.

CAS calculator techniques for Question 8

TI: Open a Graphs page. Press **Menu** → **3** : **Graph Entry/Edit** → **6** : **Sequence** → **1** : **Sequence** and input the equation and initial term as shown. Press ENTER then press /T to view the sequence.

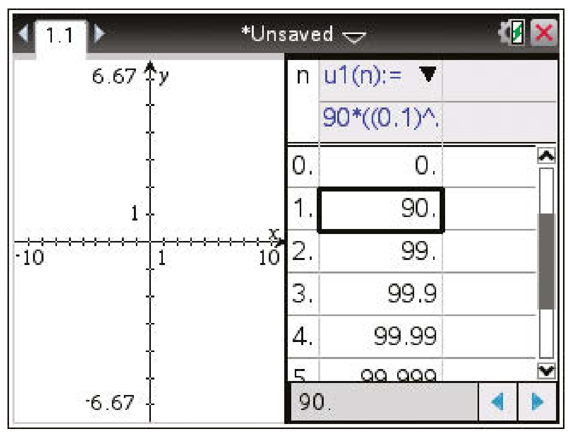


CP: Open the Sequence application. Input the following:

$$a_{n+1} = \frac{\mathbf{90(0.1^n - 1)}}{\mathbf{0.1 - 1}}$$

$$a_0 = \mathbf{90}$$

Tap # to view the sequence.



8 **a** $t_1 = \frac{1}{3}$, $t_2 = \left(\frac{1}{3}\right)^2$, $t_6 = \left(\frac{1}{3}\right)^6 = \frac{1}{729}$

The water level will rise by $\frac{1}{729}$ metres at the end of the sixth hour.

b $S_6 = \dfrac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^6\right)}{1-\frac{1}{3}}$

$= \dfrac{364}{729}$

$= 0.499314\ldots$

The total height of the water level after six hours will be 1.499 m, correct to three decimal places.

$S_\infty = \dfrac{\frac{1}{3}}{1-\frac{1}{3}}$

$= 0.5$

The maximum height the water will reach is 1.5 metres. If the prisoner is able to keep his head above this level, he will not drown.

9 a $\dfrac{400}{500} = \dfrac{320}{400} = 0.8$

$a = 500,\ r = 0.8,$

$\therefore t_n = 500(0.8)^{n-1}$

$t_{14} = 500(0.8)^{14-1}$

$= 27.487\,790\ldots$

On the 14th day they were subjected to 27.49 curie hours, correct to two decimal places.

b $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_5 = \dfrac{500(1-0.8^5)}{1-0.8}$

$= 1680.8$

During the first five days, they were subjected to 1680.8 curie hours.

10 a $t_1 = \dfrac{2}{3} \times 81$

$t_2 = \left(\dfrac{2}{3}\right)^2 \times 81$

$t_6 = \left(\dfrac{2}{3}\right)^6 \times 81$

$= 7\dfrac{1}{9}$

After the sixth bounce, the ball reaches a height of $7\frac{1}{9}$ metres.

b Total distance $= 81 + \frac{2}{3} \times 81 + \frac{2}{3} \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \ldots$

$$= 81 + 2\left(\frac{2}{3} \times 81 + \left(\frac{2}{3}\right)^2 \times 81 + \ldots\right)$$

$$= 81 + 2 \times \frac{\frac{2}{3} \times 81}{1 - \frac{2}{3}}$$

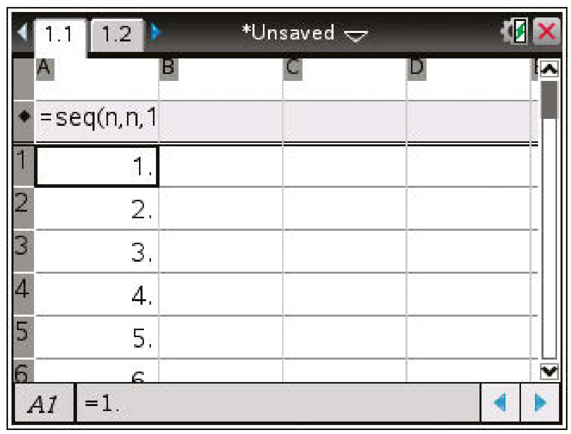
$$= 81 + 324$$

$$= 405$$

The total distance travelled by the ball is 405 metres.

CAS calculator techniques for Question 11

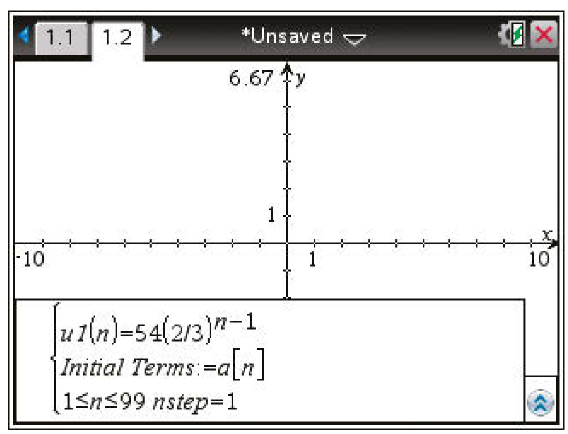
TI: Open a Lists & Spreadsheet application. Type **seq**(**n**, **n**, **1**, **30**, **1**) into the formula cell for column A. This will place the number 1–30 into column A.



Open a Graphs application and input the following sequence.

Navigate back to the Lists & Spreadsheet page. Type **seq**(**ul**(**n**), **n**, **1**, **30**, **1**) into the formula cell for columns B.

Type **2** × **b**[] into the formula cell for column C.

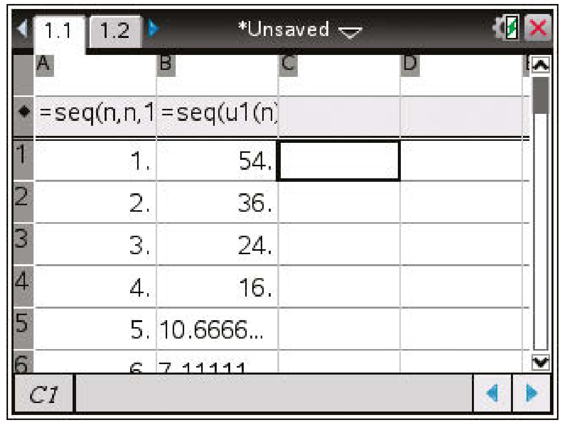


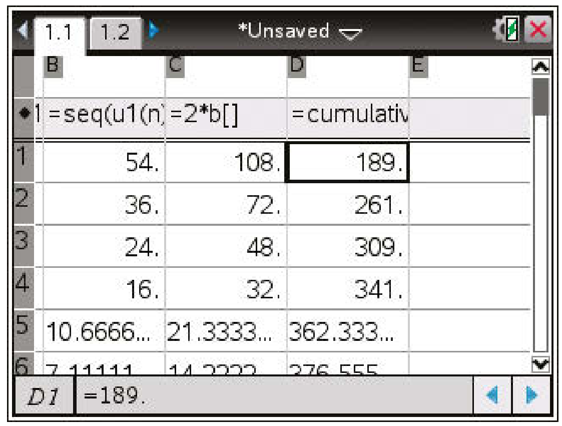
Type **cumulativeSum**(**c**[]) + **81** into the formula cell for column D.

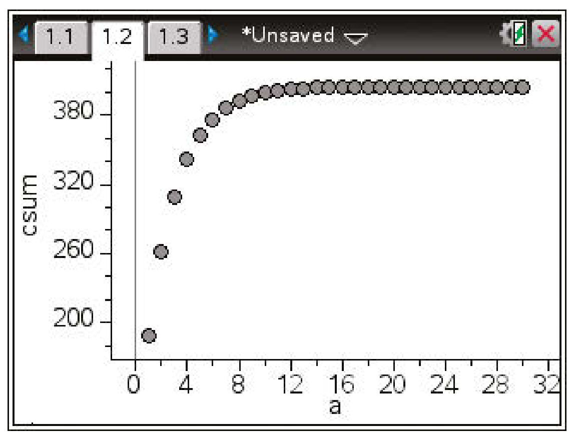
Give column D the name **csum** and column A the name **a**

The graphs of these relations can now be considered. In a Data & Statistics application sketch the graph of **csum** against **a** as shown. This is the total distance travelled against the number of bounces.

The limiting behaviour is demonstrated by this graph.







11 $t_1 = 1 = 2^0$

$t_2 = 2 = 2^1$

$t_3 = 4 = 2^2$

$\therefore t_n = 2^{n-1}$

$S_n = \dfrac{a(1-r^n)}{1-r}$ where $a = 1,\ r = 2$

$\therefore S_{64} = \dfrac{1(1-2^{64})}{1-2}$

$= 2^{64} - 1$

The king had to pay $2^{64} - 1 = 1.845 \times 10^{19}$ grains of rice.

12 a i The amount of cement produced is an arithmetic sequence.

Let C_n be the amount of cement produced (in tonnes) in the nth month.

$C_n = a + (n-1)d$ where $a = 4000,\ d = 250$

$= 4000 + (n-1) \times 250$

$= 4000 + 250n - 250$

$\therefore C_n = 250n + 3750$

ii Let S_n be the amount of cement (in tonnes) produced in the first n months.

$S_n = \dfrac{n}{2}(a + 1)$ where $a = 4000,\ l = 250n + 3750$

$= \dfrac{n}{2}(4000 + 250n + 3750)$

$= \dfrac{n}{2}(250n + 7750)$

$= n(125n + 3875)$

$\therefore S_n = 125n(n + 31)$

$= 3875n + 125n^2$

iii When $C_n = 9250$,

$250n + 3750 = 9250$

$\therefore 250n = 5500$

$\therefore n = 22$

The amount of cement produced is 9250 tonnes in the 22nd month.

iv $C_n = 250n + 3750$

$\therefore T = 250m + 3750$

$\therefore m = \dfrac{1}{250}T - 15$

v $S_p = 522\,750$ and $S_p = \dfrac{p}{2}(a + l)$

$$\therefore 522\,750 = \frac{p}{2}(4000 + 250p + 3750)$$

$$\therefore 1045\,500 = p(250p + 7750)$$

$$\therefore 4182 = p(p + 31)$$

$\therefore p^2 + 31p - 4182 = 0$

Using the general quadratic formula,

$$p = \frac{-31 \pm \sqrt{31^2 - 4 \times 1 \times (-4182)}}{2}$$

$$= \frac{-31 \pm 131}{2}$$

$= -82$ or 51

$= 51$ as $p > 0$

b **i** The total amount of cement produced is a geometric series. Total amount of cement produced after n months is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ where } a = 3000,\ r = 1.08$$

$$= \frac{3000(1.08^n - 1)}{0.08}$$

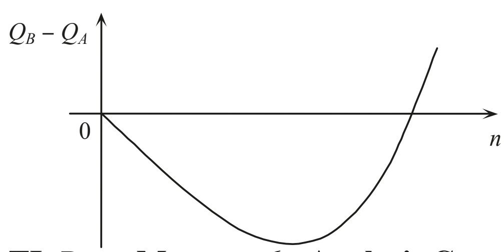
$\therefore S_n = 37500(1.08^n - 1)$

ii $Q_A = 125n(n + 31)$ and $Q_B = 37\,500(1.08^n - 1)$

$\therefore Q_B - Q_A = 37\,500(1.08^n - 1) - 125n(n + 31)$

Using a CAS calculator,

sketch $\boldsymbol{f1} = \mathbf{37\,500(1.08^x} - 1) - \mathbf{125x(x + 31)}$



TI: Press **Menu** → **6** : **Analysis Graph** → **1** : **Zero**

CP: Tap **Analysis** → **G - Solve** → **Root** to yield a horizontal axis intercept at $(17.28, 0)$, correct to two decimal places. Hence, the smallest value of n for which $Q_B - Q_A \geq 0$ is 18.

13 Let P_n be the population of birds at the old swamp at the end of n years and Q_n be the population of birds at the new swamp at the end of n years. (just after 30 birds have been moved)

a $Q_1 = 1.15 \times 30 + 30 = 64.5$ and
$P_1 = 1.15 \times 320 \times 1.5 - 30 = 338$

b $Q_n = 1.15Q_{n-1} + 30$ and $Q_0 = 30$

c We use

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

and use Q_1 and P_1

Hence:

$$\begin{aligned} P_n &= 1.15^{n-1} \times 338 + \frac{(-30)(1.15^{n-1} - 1)}{1.15 - 1} \\ &= 338 \times 1.15^{n-1} - 200(1.15^{n-1} - 1) \\ &= 138 \times 1.15^{n-1} + 200 \end{aligned}$$

$$\begin{aligned} P_n &= 1.15^{n-1} \times 338 + \frac{(-30)(1.15^{n-1} - 1)}{1.15 - 1} \\ &= 338 \times 1.15^{n-1} - 200(1.15^{n-1} - 1) \\ &= 138 \times 1.15^{n-1} + 200 \end{aligned}$$

$$\begin{aligned} Q_n &= 1.15^{n-1} \times 64.5 + \frac{(30)(1.15^{n-1} - 1)}{1.15 - 1} \\ &= 64.5 \times 1.15^{n-1} + 200(1.15^{n-1} - 1) \\ &= 264.5 \times 1.15^{n-1} - 200 \end{aligned}$$

d **i** $P_5 = 138 \times 1.15^4 + 200 = 442.363$
$Q_5 = 264.5 \times 1.15^4 - 200 = 262.612$

ii Solve $264.5 \times 1.15^{n-1} - 200 = 264.5 \times 1.15^{n-1} - 200$ for n.
The solution is 9.237.

14 **a** Geometric sequence with $a = 1$ and $r = 3$:
Number of white triangles after step n is 3^{n-1}

b Geometric sequence with $a = 1$ and $r = \frac{1}{2}$

Side length of white triangle in diagram n is $\left(\frac{1}{2}\right)^{n-1}$

c Geometric sequence with $a = 1$ and $r = \frac{3}{4}$:

Fraction that is white $= \left(\frac{3}{4}\right)^{n-1}$

d As $n \to \infty$ the fraction that is white approaches 0.

15 **a** Geometric sequence with $a = 1$ and $r = 8$:
Number of white squares after step n is 8^{n-1}

b Geometric sequence with $a = 1$ and $r = \frac{1}{3}$:
Side length of white square in diagram n is $\left(\frac{1}{3}\right)^{n-1}$

c Geometric sequence with $a = 1$ and $r = \frac{8}{9}$:
Fraction that is white $= \left(\frac{8}{9}\right)^{n-1}$

d As $n \to \infty$ the fraction that is white approaches 0.

Chapter 4 – Additional algebra

Solutions to Exercise 4A

1 $ax^2 + bx + c = 10x^2 - 7$

$= 10x^2 + 0x - 7$

$a = 10,\ b = 0,\ c = -7$

2
$$2a - b = 4 \quad ①$$
$$a + 2b = -3 \quad ②$$
$$4a - 2b = 8 \quad ③$$

② + ③:

$$5a = 5$$
$$a = 1$$
$$a \times 1 - b = 4$$
$$b = -2$$

3
$$2a - 3b = 7 \quad ①$$
$$3a + b = 5 \quad ②$$

① + 3 × ②:

$$11a = 22$$
$$a = 2$$
$$3 \times 2 + b = 5$$
$$b = -1$$
$$c = 7$$

4 $a(x + b)^2 + c = ax^2 + 2abx + ab^2 + c$

$$a = 2$$
$$2ab = 4$$
$$b = 1$$
$$ab^2 + c = 5$$
$$2 + c = 5$$
$$c = 3$$

5 $c(x + 2)^2 + a(x + 2) + 2$

$= cx^2 + 4cx + 4c + ax + 2a + d$

$$c = 1$$
$$4c + a = 0$$
$$a = -4$$
$$4c + 2a + d = 0$$
$$4 - 8 + d = 0$$
$$d = 4$$
$$\therefore \quad x^2 = (x + 2)^2 - 4(x + 2) + 4$$

6 $(x + 1)^3 + a(x + 1)^2 + b(x + 1) + c$

$= x^3 + 3x^2 + 3x + 1 + ax$

$+ a + bx + b + c$

$$3 + a = 0$$
$$a = -3$$
$$3 + 2a + b = 0$$
$$3 - 6 + b = 0$$
$$b = 3$$
$$1 + a + b + c = 0$$
$$c = -1$$
$$\therefore \quad x^3 = (x + 1)^3 - 3(x + 1)^2 + 3(x + 1) - 1$$

7 $ax^2 + 2ax + a + bx + c = x^2$

$$a = 1$$
$$2a + b = 0$$
$$b = -2$$
$$a + c = 0$$
$$c = -1$$

8 **a** $a(x + b)^3 + c = ax^3 + 3abx^2 + 3ab^2x + ab^3 + c = 3x^3 - 9x^2 + 8x + 12$

$$a = 3$$
$$3ab = -9$$
$$3 \times 3 \times b = -9$$
$$b = -1$$

Equating x terms:

$3ab^2 = 8$

$3ab^2 = 3 \times 3 \times (-1)^2 = 9$

The equality is impossible.

b Clearly this expression can be expressed in this form, if $a = 3$, $b = -1$ and

$$ab^3 + c = 2$$
$$-3 + c = 2$$
$$c = 5$$

9 Expanding gives the following:

$$n^3 = an^3 + 6an^2 + 11an + 6a + bn^2 + 3bn + 2b + cn + c + d$$
$$a = 1$$
$$6a + b = 0$$
$$b = -6$$
$$11a + 3b + c = 0$$
$$11 - 18 + c = 0$$
$$c = 7$$
$$6a + 2b + c + d = 0$$
$$6 - 12 + 7 + d = 0$$
$$d = -1$$

10 **a** Expanding gives the following:

$$n^2 = an^2 + 3an + 2a + bn^2 + 5bn + 6b$$
$$a + b = 1 \quad ①$$
$$3a + 5b = 0 \quad ②$$
$$2a + 6b = 0$$
$$a + 3b = 0 \quad ③$$

③ − ①:

$$2b = -1$$
$$b = -\frac{1}{2}$$
$$a + -\frac{1}{2} = 1$$
$$a = 1\frac{1}{2}$$

These do not satisfy the second equation, as $3 \times 1\frac{1}{2} + 5 \times -\frac{1}{2} = 2$.

b
$$\begin{aligned} n^2 &= an^2 + 3an + 2a \\ &\quad + bn + b + c \\ a &= 1 \\ 3a + b &= 0 \\ b &= -3 \\ 2a + b + c &= 0 \\ 2 - 3 + c &= 0 \\ c &= 1 \\ \therefore \quad n^2 &= (n+1)(n+2) \\ &\quad - 3(n+1) + 1 \end{aligned}$$

11 **a** $a(x^2 + 2bx + b^2) + c =$
$ax^2 + 2abx + ab^2 + c$

b
$$\begin{aligned} ax^2 + bx + c &= A(x+B)^2 + C \\ &= Ax^2 + 2ABx \\ &\quad + AB^2 + C \\ A &= a \\ 2AB &= b \\ B &= \frac{b}{2a} \\ AB^2 + C &= c \\ a \times \frac{b^2}{4a^2} + C &= c \\ C &= c - \frac{b^2}{4a} \\ \therefore \quad ax^2 + bx + x &= a\left(x + \frac{b}{2a}\right)^2 \\ &\quad + \frac{4ac - b^2}{4a} \end{aligned}$$

12 $(x-1)^2(px+q) = (x^2 - 2x + 1)(px + q)$
$$\begin{aligned} &= px^3 + (q - 2p)x^2 \\ &\quad + (p - 2q)x + q \end{aligned}$$

Equating x^3 and x^2 terms:
$$\begin{aligned} p &= a \\ q - 2p &= b \\ q - 2a &= b \\ q &= 2a + b \end{aligned}$$
Equating x and constant terms:
$$\begin{aligned} q &= d \\ p - 2q &= c \\ p &= c + 2d \end{aligned}$$
Equating the two different expressions for p and q gives:
$d = 2a + b \qquad (q)$
$\therefore \quad b = d - 2a$
$a = c + 2d \qquad (p)$
$\therefore \quad c = a - 2d$

13
$$\begin{aligned} c(x-a)(x-b) &= cx^2 - acx \\ &\quad - bcx + abc \\ &= 3 \\ c &= 3 \\ -ac - bc &= 10 \\ -3a - 3b &= 10 \\ abc &= 3 \\ 3ab &= 3 \\ ab &= 1 \\ b &= \frac{1}{a} \\ -3a - \frac{3}{a} &= 10 \end{aligned}$$

$$3a^2 + 3 = -10a$$
$$3a^2 + 10a + 3 = 0$$
$$(3a + 1)(a + 3) = 0$$
$$a = -\frac{1}{3},\ b = -3,\ c = 3$$
$$\text{or } a = -3,\ b = -\frac{1}{3},\ c = 3$$

14 $n^2 = a(n-1)^2 + b(n-2)^2 + c(n-3)^2$

$= an^2 - 2an + a + bn^2 - 4bn + 4b + cn^2 + 9c$

$a + b + c = 1$ ①

$-2a - 4b - 6c = 0$

$a + 2b + 3c = 0$ ②

$a + 4b + 9c = 0$ ③

② − ①:

$b + 2c = -1$ ④

③ − ②:

$2b + 6c = 0$

$b + 3c = 0$ ⑤

⑤ − ④:

$c = 1$

$b + 3 \times 1 = 0$

$b = -3$

$a + b + c = 1$

$a - 3 + 1 = 1$

$a = 3$

$\therefore\ n^2 = 3(n-1)^2 - 3(n-2)^2 + (n-3)^2$

15 $(x-a)^2(x-b) = (x^2 - 2ax + a^2)(x-b)$

$= x^3 - 2ax^2 - bx^2 + a^2x + 2abx - a^2b$

$-2a - b = 3$

$a^2 + 2ab = -9$

Substitute $b = -2a - 3$:

$a^2 + 2a(-2a - 3) = -9$

$a^2 - 4a^2 - 6a = -9$

$-3a^2 - 6a + 9 = 0$

$a^2 + 2a - 3 = 0$

$(a + 3)(a - 1) = 0$

$a = -3$ or $a = 1$

$b = -2a - 3$

$b = 3$ or $b = -5$

Comparing the constant terms:

$c = -a^2b$

$c = (-3)^2 \times 3 = -27$

or $c = (-1)^2 \times -5 = 5$

So $a = 1,\ b = -5, c = 5$

or $a = -3,\ b = 3, c = -27$

16 **a** If $P(x)$ is even, then

$ax^4 + bx^3 + cx^2 + dx + e$

$= a(-x)^4 + b(-x)^3 + c(-x)^2 + d(-x) + e$

$ax^4 + bx^3 + cx^2 + dx + e$

$= ax^4 - bx^3 + cx^2 - dx + e$

$bx^3 + dx = -bx^3 - dx$

$2bx^3 + 2dx = 0$

$b = d = 0,$

b If $P(x)$ is odd, then $-P(x) = P(-x)$ so that

$$- ax^5 - bx^4 - cx^3 - dx^2 - ex - f$$
$$= a(-x)^5 + b(-x)^4 + c(-x)^3 + d(-x)^2 + e(-x) + f$$
$$- ax^5 - bx^4 - cx^3 - dx^2 - ex - f$$
$$= -ax^5 + bx^4 - cx^3 + dx^2 - ex + f$$
$$- bx^4 - dx^2 - f = bx^4 + dx^2 + f$$
$$2bx^4 + 2dx^2 + 2fx = 0$$
$$b = d = f = 0$$

Solutions to Exercise 4B

1 a
$$
\begin{aligned}
x^2 - 2x &= -1 \\
x^2 - 2x + 1 &= 0 \\
(x-1)^2 &= 0 \\
x &= 1
\end{aligned}
$$

b
$$
\begin{aligned}
x^2 - 6x + 9 &= 0 \\
(x-3)^2 &= 0 \\
x &= 3
\end{aligned}
$$

c Divide both sides by 5:
$$
\begin{aligned}
x^2 - 2x &= \frac{1}{5} \\
x^2 - 2x + 1 &= \frac{6}{5} \\
(x-1)^2 &= \frac{6}{5} = \frac{30}{25} \\
x - 1 &= \pm\frac{\sqrt{30}}{5} \\
x &= 1 \pm \frac{\sqrt{30}}{5}
\end{aligned}
$$

d Divide both sides by −2:
$$
\begin{aligned}
x^2 - 2x &= -\frac{1}{2} \\
x^2 - 2x + 1 &= \frac{1}{2} \\
(x-1)^2 &= \frac{1}{2} = \frac{2}{4} \\
x - 1 &= \pm\frac{\sqrt{2}}{2} \\
x &= 1 \pm \frac{\sqrt{2}}{2}
\end{aligned}
$$

e Divide both sides by 2:
$$
\begin{aligned}
x^2 + 2x &= \frac{7}{2} \\
x^2 + 2x + 1 &= \frac{9}{2} \\
(x+1)^2 &= \frac{9}{2} = \frac{9 \times 2}{4} \\
x + 1 &= \pm\frac{3\sqrt{2}}{2} \\
x &= -1 \pm \frac{3\sqrt{2}}{2}
\end{aligned}
$$

f $6x^2 + 13x + 1$
$$
\begin{aligned}
&= 0 \\
&x \\
&= \frac{-13 \pm \sqrt{169 - 4 \times 6 \times 1}}{12} \\
&= \frac{-13 \pm \sqrt{145}}{12}
\end{aligned}
$$

2 a
$$\Delta = 9 - 4m$$
No solutions:$\Delta < 0$
$$
\begin{aligned}
9 - 4m &< 0 \\
m &> \frac{9}{4}
\end{aligned}
$$

b
$$\Delta = 25 - 4m$$
Two solutions:$\Delta > 0$
$$
\begin{aligned}
25 - 4m &> 0 \\
m &< \frac{25}{4}
\end{aligned}
$$

c
$$\Delta = 25 + 32m$$
One solution:$\Delta = 0$
$$
\begin{aligned}
25 + 32m &= 0 \\
m &= -\frac{25}{32}
\end{aligned}
$$

d $\Delta = m^2 - 36$

Two solutions:$\Delta > 0$

$m^2 - 36 > 0$

$m > 6$ or $m < -6$

e $\Delta = m^2 - 16$

Nosolutions:$\Delta < 0$

$m^2 - 16 < 0$

$-4 < m < 4$

f $\Delta = m^2 + 16m$

One solution:$\Delta = 0$

$m^2 + 16m = 0$

$m = -16$ or $m = 0$

3 a $2x^2 - x - 4t = 0$

$$x = \frac{1 \pm \sqrt{1 - 4 \times 2 \times -4t}}{4}$$

$$= \frac{1 \pm \sqrt{32t + 1}}{4}$$

$32t + 1 \geq 0$

$32t \geq -1$

$t \geq -\frac{1}{32}$

b $4x^2 + 4x - t - 2 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 4 \times -(t + 2)}}{8}$$

$$= \frac{-4 \pm \sqrt{16 + 32 + 16t}}{8}$$

$$= \frac{-4 \pm \sqrt{16t + 48}}{8}$$

$$= \frac{-4 \pm 4\sqrt{t + 3}}{8}$$

$$= \frac{-1 \pm \sqrt{t + 3}}{2}$$

$t + 3 \geq 0$

$t \geq -3$

c $5x^2 + 4x - t + 10 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 5 \times (-t + 10)}}{10}$$

$$= \frac{-4 \pm \sqrt{16 + 20t - 200}}{10}$$

$$= \frac{-4 \pm \sqrt{20t - 184}}{10}$$

$$= \frac{-4 \pm \sqrt{4(5t - 46)}}{10}$$

$$= \frac{-4 \pm 2\sqrt{5t - 46}}{10}$$

$$= \frac{-2 \pm \sqrt{5t - 46}}{5}$$

$5t - 46 \geq 0$

$5t \geq 46$

$t \geq \frac{46}{5}$

d $tx^2 + 4tx - t + 10 = 0$

$$x = \frac{-4t \pm \sqrt{16t^2 - 4 \times t \times (-t + 10)}}{2t}$$

$$= \frac{-4t \pm \sqrt{16t^2 + 4t^2 - 40t}}{2t}$$

$$= \frac{-4t \pm \sqrt{20t^2 - 40t}}{2t}$$

$$= \frac{-4t \pm 2\sqrt{5t^2 - 10t}}{2t}$$

$$= \frac{-2t \pm \sqrt{5t(t-2)}}{t}$$

$5t(t - 2) \geq 0$

This is a quadratic with a minimum and solutions $t = 0$, $t = 5$.

$\therefore \quad t < 0, \; t \geq 2$

Note: $t = 0$ gives denominator zero, so it must be checked by substituting $t = 0$ in the original equation. In this case it gives $10 = 0$, and so is not a solution, but it should be checked.

(e.g. $tx^2 + 5x + 4 = t$ gives a solution with t on the denominator, but substituting $t = 0$ gives $5x + 4 = 0$, which has a solution.)

4 a $x = \dfrac{-p \pm \sqrt{p^2 - 4 \times 1(-16)}}{2}$

$= \dfrac{-p \pm \sqrt{p^2 + 64}}{2}$

b $p = 0$ gives $x = \dfrac{0 + \sqrt{64}}{2} = 4$

$p = 6$ gives $x = \dfrac{-6 + \sqrt{100}}{2} = 2$

5 a $2x^2 - 3px + (3p - 2) = 0$

$$\Delta = 9p^2 - 8(3p - 2)$$

$$= 9p^2 - 24p + 16$$

$$= (3p - 4)^2$$

$\Delta$is a perfect square

b $\Delta = 0 \Rightarrow p = \dfrac{4}{3}$

c Solution is $x = \dfrac{3p \pm (3p - 4)}{4}$

i When $p = 1$, $x = \dfrac{3 \pm 1}{4}$

$\therefore x = 1$ or $x = \dfrac{1}{2}$

ii When $p = 2$, $x = \dfrac{6 \pm 2}{4}$

$\therefore x = 2$ or $x = 1$

iii When $p = -1$, $x = \dfrac{-3 \pm 7}{4}$

$\therefore x = 1$ or $x = -\dfrac{5}{2}$

6 a $4(4p - 3)x^2 - 8px + 3 = 0$

$\Delta = 64p^2 - 48(4p - 3)$

$= 64p^2 - 192p + 144$

$= 16(4p^2 - 12p + 9)$

$= 16(2p - 3)^2$

$\Delta$is a perfect square

b $\Delta = 0 \Rightarrow p = \dfrac{3}{2}$

c Solution is $x = \dfrac{8p \pm 4(2p - 3)}{8(4p - 3)}$

That is $x = \dfrac{1}{2}$ or $x = \dfrac{3}{2(4p - 3)}$

i When $p = 1$, $x = \dfrac{8 \pm 4}{8}$

$\therefore x = \dfrac{1}{2}$ or $x = \dfrac{3}{2}$

ii When $p = 2$, $x = \dfrac{16 \pm 20}{40}$

$\therefore x = \dfrac{1}{2}$ or $x = \dfrac{3}{10}$

iii When $p = -1$, $x = \dfrac{-8 \pm 20}{-56}$

$\therefore x = \dfrac{1}{2}$ or $x = -\dfrac{3}{14}$

7 Use Pythagoras' Theorem:

$$(8 - x)^2 + (6 + x)^2 = 100$$
$$64 - 16x + x^2 + 36 + 12x + x^2 = 100$$
$$2x^2 - 4x = 0$$
$$2x(x - 4) = 0$$
$x = 2$ since

$x \neq 0$

8 Let x be the length of one part.
The other part has length $100 - x$
Let the second one be the larger.

$$\left(\frac{200 - x}{4}\right)^2 = 9\frac{x^2}{16}$$
$$(200 - x)^2 = 9x^2$$
$$200 - x = 3x$$
$$x = 50$$

$\therefore 200 - x = 150$
The length of the sides of the larger square is 37.5 cm

9 **a** Let $a = \sqrt{x}$
$$a^2 - 8a + 12 = 0$$
$$(a - 6)(a - 2) = 0$$
$$a = 6 \text{ or } a = 2$$
$$\therefore x = 36 \text{ or } x = 4$$

b Let $a = \sqrt{x}$
$$a^2 - 2a - 8 = 0$$
$$(a - 4)(a + 2) = 0$$
$$a = 4 \text{ or } a = -2$$
$$\therefore x = 16$$

c Let $a = \sqrt{x}$
$$a^2 - 5a - 14 = 0$$
$$(a - 7)(a + 2) = 0$$
$$a = 7 \text{ or } a = -2$$
$$\therefore x = 49$$

d Let $a = \sqrt[3]{x}$
$$a^2 - 9a + 8 = 0$$
$$(a - 8)(a - 1) = 0$$
$$a = 8 \text{ or } a = 1$$
$$\therefore x = 512 \text{ or } x = 1$$

e Let $a = \sqrt[3]{x}$
$$a^2 - a - 6 = 0$$
$$(a - 3)(a + 2) = 0$$
$$a = 3 \text{ or } a = -2$$
$$\therefore x = 27 \text{ or } x = -8$$

f Let $a = \sqrt{x}$
$$a^2 - 29a + 100 = 0$$
$$(a - 25)(a - 4) = 0$$
$$a = 25 \text{ or } a = 4$$
$$\therefore x = 625 \text{ or } x = 16$$

10 $3x^2 - 5x + 1 = a(x^2 + 2bx + b^2) + c$
Equating coefficients:
x^2 : $3 = a$
x : $-5 = 2ba \Rightarrow b = -\frac{5}{6}$
constant: $1 = b^2a + c \Rightarrow c = -\frac{13}{12}$
Minimum value is $-\frac{13}{12}$

11
$$2 - 4x - x^2 = 24 + 8x + x^2$$
$$2x^2 + 12x + 22 = 0$$
$$x^2 + 6x + 11 = 0$$
$$\Delta = 36 - 4 \times 11 < 0$$

Therefore no intersection

12 $(b-c)x^2+(c-a)x+(a-b)=0$

$$((b-c)x-(a-b))(x-1))=0$$

$$x=\frac{a-b}{b-c} \text{ or } x=1$$

13 $2x^2-6x-m=0$

$$x=\frac{6\pm\sqrt{36+8m}}{4}$$

The difference of the two solutions

$$=\frac{\sqrt{36+8m}}{2}$$

$$\frac{\sqrt{36+8m}}{2}=5$$

$$36+8m=100$$

$$8m=64$$

$$m=8$$

14 a $(b^2-2ac)x^2+4(a+c)x-8=0$

$$\begin{aligned}\Delta &= 16(a+c)^2+32(b^2-2ac)\\ &= 16(a^2+2ac+c^2)+32b^2-64ac\\ &= 16a^2-32ac+16c^2+32b^2\\ &= 16(a^2-2ac+c^2+2b^2)\\ &= 16((a-c)^2+2b^2)>0\end{aligned}$$

b One solution if $a=c$ and $b=0$

15 $\frac{1}{2}+\frac{1}{x+k}=\frac{1}{x}$

$$x(x+k)+2x=2(x+k)$$

$$x^2+xk+2x=2x+2k$$

$$x^2+kx-2k=0$$

$$\Delta=k^2+8k$$

$$\Delta<0\Rightarrow k^2+8k<0$$

$$k^2+8k<0$$

$$k(k+8)<0$$

$$-8<k<0$$

16 $3x^2+px+7=0$ $\Delta=p^2-84$

$p^2>84$

The smallest such integer is 10.

Solutions to Exercise 4C

1 **a** $\dfrac{6(x+3)-6x}{x(x+3)} = \dfrac{18}{x(x+3)}$

b
$$\frac{18}{x(x+3)} = 1$$
$$\frac{18 - x(x+3)}{x(x+3)} = 0$$
$$18 - x(x+3) = 0$$
$$18 - x - 3x = 0$$
Re-arrange and divide by -1:
$$x^2 + 3x - 18 = 0$$
$$(x-3)(x+6) = 0$$
$$x = 3 \text{ or } x = -6$$

2
$$\frac{300}{x+5} = \frac{300}{x} - 2$$
$$300x = 300(x+5) - 2x(x+5)$$
$$300x = 300x + 1500 - 2x^2 + 10x$$
$$2x^2 - 10x - 1500 = 0$$
$$x^2 - 5x - 750 = 0$$
$$(x+25)(x-30) = 0$$
$$x = 25 \text{ or } x = -30$$

3 Let the numbers be n and $n+2$.
$$\frac{1}{n} + \frac{1}{n+2} = \frac{36}{323}$$
$$\frac{1}{n} + \frac{1}{n+2} - \frac{36}{323} = 0$$
$$\frac{323(n+2) + 323n - 36n(n+2)}{323n(n+2)} = 0$$
$$323n + 646 + 323n - 36n^2 - 72n = 0$$
Re-arrange and divide by -1:
$$36n^2 - 574n - 646 = 0$$
$$18n^2 - 287 - 323 = 0$$
$$(n-17)(18n+19) = 0$$
$$n = 17$$
The numbers are 17 and 19.

4 **a** $\dfrac{40}{x}$

b $\dfrac{40}{x-2}$

c
$$\frac{40}{x-2} - \frac{40}{x} = 1$$
$$40x - 40(x-2) = x(x-2)$$
$$80 = x^2 - 2x$$
$$x^2 - 2x - 80 = 0$$
$$(x-10)(x+8) = 0$$
$$\therefore x = 10$$

5 **a** Car $= \dfrac{600}{x}$ km/h; Plane $= \dfrac{600}{x} + 220$ km/h

b Since the plane takes $x - 5.5$ hours to cover 600 km its average speed is also given by $\dfrac{600}{x-5.5}$. Hence:

$$\frac{600}{x} + 220 = \frac{600}{x - 5.5}$$

$$600(x - 5.5) + 220x(x - 5.5) = 600x$$

$$600x - 3300 + 220x^2 - 1210x = 600x$$

$$220x^2 - 1210x - 3300 = 0$$

$$2x^2 - 11x - 30 = 0$$

$$(2x - 15)(x + 2) = 0$$

$$x = 7.5$$

$$(x > 0)$$

Average speed of car $= \frac{600}{7.5} =$ 80 km/h

Average speed of plane $= 80 + 220$

$= 300$ km/h

6 Time taken by car $= \frac{200}{x}$ h

Time taken by train $= \frac{200}{x + 5}$ h $= \frac{200}{x} - 2$ h

$$\frac{200}{x + 5} = \frac{200}{x} - 2$$

$$\frac{200}{x + 5} \times x(x + 5) = \frac{200}{x} \times x(x + 5) - 2 \times x(x + 5)$$

$$200x = 200(x + 5) - 2x(x + 5)$$

$$= 200x + 1000 - 2x^2 - 10x$$

$$2x^2 + 10x - 1000 = 0$$

$$x^2 + 5x - 500 = 0$$

$$(x - 20)(x + 25) = 0$$

$x = 20$ since $x > 0$

7 Let his average speed be x km/h.

His time for the journey is $\frac{108}{x}$ h.

$$\frac{108}{x} - 4\frac{1}{2} = \frac{108}{x + 2}$$

$$108 \times 2(x + 2) - 4\frac{1}{2} \times 2x(x + 2) = 108 \times 2x$$

$$216x + 432 - 9x^2 - 18x = 216x$$

$$-9x^2 - 18x + 432 = 0$$

$$x^2 + 2x - 48 = 0$$

$$(x - 6)(x + 8) = 0$$

$$x = 6$$

since $x > 0$

His average speed is 6 km/h.

8 **a** Usual time $= \frac{75}{x}$ h.

$$\frac{75}{x} - \frac{18}{60} = \frac{75}{x + 12.5}$$

$$\frac{75}{x} - \frac{3}{10} = \frac{75}{x + 12.5}$$

$$75(x + 12.5) - 0.3x(x + 12.5) = 75x$$

$$75x + 937.5 - 0.3x^2 - 3.75x = 75x$$

$$-0.3x^2 - 3.75x + 937.5 = 0$$

Divide by 0.15:

$$2x^2 + 25x - 6250 = 0$$

$$(x - 50)(2x + 125) = 0$$

$$x = 50$$

b Average speed $= x + 12.5 = 62.5$

Time $= \frac{75}{62.5} = 1.2$ h,
or 1 hour 12 minutes, or 72 minutes.

9 Let the speed of the slow train be

x km/h. The slow train takes

$$3\frac{1}{2} - \frac{10}{60} = \frac{7}{2} - \frac{1}{6} = \frac{20}{6} = \frac{10}{3}$$

hours longer. Now compare the times:

$$\frac{250}{x+20} + \frac{10}{3} = \frac{250}{x}$$
$$750x + 10x(x+20) = 750(x+20)$$
$$750x + 10x^2 + 200x = 750x + 15\,000$$
$$10x^2 + 200x - 15\,000 = 0$$
$$x^2 + 20x - 1500 = 0$$
$$(x-30)(x+50) = 0$$
$$x = 30$$

Slow train: 30 km/h
Fast train: 50 km/h

10 Let the original speed of the car be x km/h. Compare the times:

$$\frac{105}{x+10} = \frac{105}{x} - \frac{1}{4}$$
$$420x = 420(x+10) - x(x+10)$$
$$420x = 420x + 4200 - x - 10x$$
$$x^2 + 10x - 4200 = 0$$
$$(x-60)(x+70) = 0$$
$$x = 60 \text{ km/h}$$

11 Let x min be the time the larger pipe takes, and C the capacity of the tank. Form an equation using the rates:

$$\frac{C}{x} + \frac{C}{x+5} = \frac{C}{11\frac{1}{9}}$$
$$\frac{C}{x} + \frac{C}{x+5} = \frac{9C}{100}$$
$$\frac{1}{x} + \frac{1}{x+5} = \frac{9}{100}$$
$$100(x+5) + 100x = 9x(x+5)$$
$$100x + 500 + 100x = 9x^2 + 45x$$
$$200x + 500 = 9x^2 + 45x$$
$$9x^2 - 155x - 500 = 0$$
$$(x-20)(9x+25) = 0$$
$$x = 20 \text{ since } x > 0$$

The larger pipe takes 20 min and the smaller pipe takes 25 min.

12 Let x min be the original time the first pipe takes, and y min be the original time the second pipe takes.
Let C be the capacity of the tank.
The original rates are $\frac{C}{x}$ and $\frac{C}{y}$.
The combined rate is $\frac{C}{x} + \frac{C}{y}$.
Total time taken = capacity ÷ rate

$$C \div \left(\frac{C}{x} + \frac{C}{y}\right) = C \div \frac{Cy + Cx}{xy} = C \times \frac{xy}{Cx + Cy} = \frac{xy}{x+y} = \frac{20}{3}$$

New rates are $\frac{C}{x-1}$ and $\frac{C}{y+2}$.

The combined rate is $\frac{C}{x-1} + \frac{C}{y+2}$.

$$C \div \left(\frac{C}{x-1} + \frac{C}{y+2}\right)$$

$$= C \div \frac{C(y+2) + C(x-1)}{(x-1)(y+2)}$$

$$= C \times \frac{(x-1)(y+2)}{Cx + Cy + C}$$

$$= \frac{(x-1)(y+2)}{x+y+1} = 7$$

Solve the simultaneous equations:

$$\frac{xy}{x+y} = \frac{20}{3}$$

$$\frac{(x-1)(y+2)}{x+y+1} = 7$$

Multiply both sides of the first equation by $3(x + y)$:

$$3xy = 20x + 20y$$

$$3xy - 20y = 20x$$

$$y(3x - 20) = 20x$$

$$y = \frac{20x}{3x-20}$$

Substitute into the second equation, after multiplying both sides by $x + y + 1$:

$$(x-1)(y+2)$$

$$= 7x + 7y + 7$$

$$(x-1)\left(\frac{20x}{3x-20} + 2\right)$$

$$= 7x + \frac{140x}{3x-20} + 7$$

$$(x-1)\frac{20x + 2(3x-20)}{3x-20}$$

$$= 7x + \frac{140x}{3x-20} + 7$$

$$(x-1)\frac{26x-40}{3x-20}$$

$$= 7x + \frac{140x}{3x-20} + 7$$

$$(x-1)(26x-40)$$

$$= 7x(3x-20)$$

$$+ 140x + 7(3x-20)$$

$$26x^2 - 66x + 40$$

$$= 21x^2 - 140x$$

$$+ 140x + 21x - 140$$

$$5x^2 - 87x + 180$$

$$= 0$$

$$(5x-12)(x-15)$$

$$= 0$$

$$x = 2.4 \text{ or } x = 15$$

$$y = \frac{20x}{3x-20} < 0 \text{ if } x = 2.4$$

$$\therefore \quad x = 15$$

$$y = \frac{20 \times 15}{3 \times 15 - 20} = 12$$

The first pipe now takes one minute less, i.e. $15 - 1 = 14$ minutes.
The second pipe now takes two minutes more, i.e. $12 + 2 = 14$ minutes.

13 Let the average speed for rail and sea be $x + 25$ km/h and x km/h respectively.

Time for first route = $\frac{233}{x+25} + \frac{126}{x}$ hours.

Time for second route = $\frac{405}{x+25} + \frac{39}{x}$ hours.

$$\frac{233}{x+25} + \frac{126}{x}$$

$$= \frac{405}{x+25} + \frac{39}{x} + \frac{5}{6}$$

$$233 \times 6x + 126 \times 6(x+25)$$

$$= 405 \times 6x + 39 \times 6(x+25) + 5x(x+25)$$

$$1398x + 756x + 18\,900$$

$$= 2430x + 234x + 5850 + 5x^2 + 125x$$

$$-5x^2 - 635x + 13\,050 = 0$$

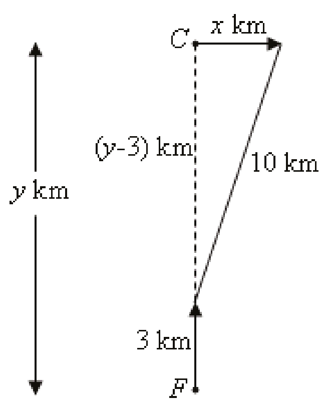
$$x^2 + 127x - 2625 = 0$$

$$x = \frac{-127 + \sqrt{127^2 - 4 \times 1 \times 2625}}{2}$$

$$\approx 18.09$$

(Ignore negative square root as $x > 0$.)
Speed by rail is 18 + 25 = 43 km/h and by sea is 18 km/h.

14 After 15 min, the freighter has travelled 3 km, bringing it to 12 km from where the cruiser was.
Let x km be the distance the cruiser has travelled in 15 minutes and y km the original distance apart of the ships.
The distance the cruiser has travelled can be calculated using Pythagoras' theorem.



$$x^2 + (y - 3)^2 = 10^2 = 100$$

After a further 15 minutes, the distances will be $2x$ km and $(y - 6)$ km.

$$(2x)^2 + (y - 6)^2 = 13^2$$

$$4x^2 + (y - 6)^2 = 169$$

Multiply the first equation by 4 and subtract:

$$4(y - 3)^2 - (y - 6)^2 = 400 - 169$$

$$4y^2 - 24y + 36 - y^2 + 12y - 36 = 231$$

$$3y^2 - 12y - 231 = 0$$

$$y^2 - 4y - 77 = 0$$

$$(y - 11)(y + 7) = 0$$

$$y = 11$$

$$x^2 + 8^2 = 10^2$$

$$x = 6$$

The speed of the cruiser is $6 \div 0.25 = 24$ km/h. The cruiser will be due east of the freighter when the freighter has travelled 11 km.
This will take $\frac{11}{12}$ hours. During that time the cruiser will have travelled $24 \times \frac{11}{12} = 22$ km.
They will be 22 km apart.

15 Let x be the amount of wine first taken out of cask A.
After water is added, the concentration of wine in cask B is $\frac{x}{20}$.
If cask A is filled, it will receive x litres at concentration $\frac{x}{20}$.
The amount of wine in cask A will be $(20 - x) + x \times \frac{x}{20} = 20 - x + \frac{x^2}{20}$.

The concentration of wine in cask A will be $\dfrac{20 - x + \dfrac{x^2}{20}}{20} = 1 - \dfrac{x}{20} + \dfrac{x^2}{400}$.

The amount of wine in cask B will be

$(20 - x) \times \dfrac{x}{20} = x - \dfrac{x^2}{20}$.

Mixture is transferred again.

The amount of wine transferred is

$$\left(1 - \frac{x}{20} + \frac{x^2}{400}\right) \times \frac{20}{3} = \frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}.$$

Amount of wine in A =

$$\left(20 - x + \frac{x^2}{20}\right) - \left(\frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}\right).$$

Amount of wine in B =

$$\left(x - \frac{x^2}{20}\right) + \left(\frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}\right)$$

$$\left(20 - x + \frac{x^2}{20}\right) - \left(\frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}\right)$$

$$= \left(x - \frac{x^2}{20}\right) + \left(\frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}\right)$$

$$20 - x + \frac{x^2}{20} - \frac{20}{3} + \frac{x}{3} - \frac{x^2}{60}$$

$$= x - \frac{x}{20} + \frac{20}{3} - \frac{x}{3} + \frac{x^2}{60}$$

$$-\frac{4x^2}{60} - \frac{4x}{3} + \frac{20}{3} = 0$$

$$\frac{x^2}{15} + \frac{4x}{3} - \frac{20}{3} = 0$$

$$x^2 + 20x - 100 = 0$$

$$(x - 10)^2 = 0$$

10 litres was first taken out of cask A.

16 Let v km/h be the speed of train B

The speed of train A is $v + 5$ km/h.

Time for train A = $\dfrac{80}{v + 5}$

Time for train B = $\dfrac{80}{v}$

$$\frac{80}{v} - \frac{80}{v + 5} = \frac{1}{3}$$

$$80(v + 5) - 80v = \frac{1}{3}(v(v + 5)$$

$$80v + 400 - 80v = \frac{1}{3}(v(v + 5))$$

$$1200 = v^2 + 5v$$

$$v^2 + 5v - 1200 = 0$$

$$v = \frac{5(\sqrt{193} - 1)}{2}$$

or $v = -\dfrac{5(\sqrt{193} + 1)}{2}$.

The speed of train B is ≈ 32.23 km/h and the speed of train A is ≈ 37.23km/h

17 **a** $a + \sqrt{a^2 - 24a}$ minutes, $a - 24 + \sqrt{a^2 - 24a}$ minutes

b **i** 84 minutes, 60 minutes

ii 48 minutes, 24 minutes

iii 36 minutes, 12 minutes

iv 30 minutes, 6 minutes

18 **a** 120 km

b 20 km/h, 30 km/h

Solutions to Exercise 4D

1 a

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$
$$= \frac{Ax+Bx+2A-B}{(x-1)(x+2)}$$

$$A + B = 5 \quad ①$$
$$2A - B = 1 \quad ②$$

① + ②:

$$3A = 6$$
$$A = 2$$
$$2 + B = 5$$
$$B = 3$$

$$\therefore \quad \frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$$

b

$$\frac{-1}{(x+1)(2x+1)}$$
$$= \frac{A}{x+1} + \frac{B}{2x+1}$$
$$= \frac{A(2x+1)+B(x+1)}{(x+1)(2x+1)}$$
$$= \frac{2Ax+Bx+A+B}{(x+1)(2x+1)}$$

$$2A + B = 0 \quad ①$$
$$A + B = -1 \quad ②$$

① − ②:

$$A = 1$$
$$1 + B = -1$$
$$B = -2$$

$$\therefore \quad \frac{-1}{(x+1)(2x+1)} = \frac{1}{x+1} - \frac{2}{2x+1}$$

c

$$\frac{3x-2}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$
$$= \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$
$$= \frac{Ax+Bx-2A+2B}{(x+2)(x-2)}$$

$$A + B = 3$$
$$2A + 2B = 6 \quad ①$$
$$-2A + 2B = -2 \quad ②$$

① + ②:

$$4B = 4$$
$$B = 1$$
$$A + 1 = 3$$
$$A = 2$$

$$\therefore \quad \frac{3x-2}{(x+2)(x-2)} = \frac{2}{x+2} + \frac{1}{x-2}$$

d

$$\frac{4x+7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$
$$= \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}$$
$$= \frac{Ax+Bx-2A+3B}{(x+3)(x-2)}$$

$$A + B = 4$$
$$2A + 2B = 8 \quad ①$$
$$-2A + 3B = 7 \quad ②$$

① + ②:

$$5B = 15$$
$$B = 3$$
$$A + 3 = 4$$
$$A = 1$$

$$\therefore \quad \frac{4x+7}{(x+3)(x-2)} = \frac{1}{x+3} + \frac{3}{x-2}$$

e

$$\frac{7-x}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$$
$$= \frac{Ax+Bx+A-4B}{(x-4)(x+1)}$$

$$A + B = -1 \quad ①$$
$$A - 4B = 7 \quad ②$$

① − ②:

$$5B = -8$$
$$B = -\frac{8}{5}$$
$$A - \frac{8}{5} = -1$$
$$A = \frac{3}{5}$$
$$\therefore \quad \frac{7-x}{(x-4)(x+1)} = \frac{3}{5(x-4)} - \frac{8}{5(x+1)}$$

2 a

$$\frac{2x+3}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$
$$= \frac{A(x-3)+B}{(x-3)^2}$$
$$= \frac{Ax-3A+B}{(x-3)^2}$$

$$A = 2$$
$$-3A + B = 3$$
$$-6 + B = 3$$
$$B = 9$$
$$\therefore \quad \frac{2x+3}{(x-3)^2} = \frac{2}{x-3} + \frac{9}{(x-3)^2}$$

b

$$\frac{9}{(1+2x)(1-x)^2}$$
$$= \frac{A}{1+2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$
$$= \frac{A(1-x)^2 + B(1+2x)(1-x) + C(1+2x)}{(1+2x)(1-x)^2}$$
$$= \frac{A - 2Ax + Ax^2 + B + Bx - 2Bx^2 + C + 2Cx}{(1+2x)(1-x)^2}$$

$$A - 2B = 0 \quad ①$$
$$-2A + B + 2C = 0 \quad ②$$
$$A + B + C = 9 \quad ③$$
$$2A + 2B + 2C = 18 \quad ④$$

④ − ②:

$$4A + B = 18$$

① × ④: $4A - 8B = 0$

$$9B = 18$$
$$B = 2$$
$$4A + 2 = 18$$
$$A = 4$$
$$4 + 2 + C = 9$$
$$C = 3$$
$$\therefore \quad \frac{9}{(1+2x)(1-x)^2} = \frac{4}{1+2x} + \frac{2}{1-x} + \frac{3}{(1-x)^2}$$

c

$$\frac{2x-2}{(x+1)(x-2)^2}$$
$$= \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$
$$= \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}$$
$$= \frac{Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C}{(x+1)(x-2)^2}$$

$$A + B = 0 \quad ①$$

$$-4A - B + C = 2 \quad ②$$

$$4A - 2B + C = -2 \quad ③$$

$$③ - ②: 8A - B = -4 \quad ④$$

$$④ + ①: 9A = -4$$

$$A = -\frac{4}{9}$$

$$A + B = 0$$

$$B = \frac{4}{9}$$

$$4A - 2B + C = -2$$

$$-\frac{16}{9} - \frac{8}{9} + C = -2$$

$$C = -2 + \frac{24}{9} = \frac{2}{3}$$

$$\therefore \quad \frac{2x-2}{(x+1)(x-2)^2} = -\frac{4}{9(x+1)} + \frac{4}{9(x-2)} + \frac{2}{3(x-2)^2}$$

3 a $\frac{3x+1}{(x+1)(x^2+x+1)}$

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{A(x^2+x+1) + (Bx+C)(x+1)}{(x+1)(x^2+x+1)}$$

$$= \frac{Ax^2 + Ax + A + Bx^2 + Bx + Cx + C}{(x+1)(x^2+x+1)}$$

$$A + B = 0 \quad ①$$

$$A + B + C = 3 \quad ②$$

$$A + C = 1 \quad ③$$

$$② - ①: C = 3$$

$$A + 3 = 1$$

$$A = -2$$

$$A + B + C = 3$$

$$-2 + B + 3 = 3$$

$$B = 2$$

$$\therefore \quad \frac{3x+1}{(x+1)(x^2+x+1)} = -\frac{2}{x+1} + \frac{2x+3}{x^2+x+1}$$

b $\frac{3x^2+2x+5}{(x^2+2)(x+1)}$

$$= \frac{Ax+B}{x^2+2} + \frac{C}{x+1}$$

$$= \frac{(Ax+B)(x+1) + C(x^2+2)}{(x^2+2)(x+1)}$$

$$= \frac{Ax^2 + Ax + Bx + B + Cx^2 + 2C}{(x^2+2)(x+1)}$$

$$A + C = 3 \quad ①$$

$$A + B = 2 \quad ②$$

$$B + 2C = 5 \quad ③$$

$① - ②$:

$$C - B = 1 \quad ④$$

$③ + ④$:

$$3C = 6$$

$$C = 2$$

$$A + 2 = 3$$

$$A = 1$$
$$1 + B = 2$$
$$B = 1$$
$$\therefore \quad \frac{3x^2 + 2x + 5}{(x^2 + 2)(x + 1)} = \frac{x + 1}{x^2 + 2} + \frac{2}{x + 1}$$

c Factorise the denominator:

$$2x^3 + 6x^2 + 2x + 6$$
$$= 2x^2(x + 3) + 2(x + 3)$$
$$= 2(x^2 + 1)(x + 3)$$

The 2 factor can be put with either fraction.

$$\frac{x^2 + 2x - 13}{2(x^2 + 1)(x + 3)}$$
$$= \frac{Ax + B}{x^2 + 1} + \frac{C}{2(x + 3)}$$
$$= \frac{2(Ax + B)(x + 3) + C(x^2 + 1)}{2(x^2 + 1)(x + 3)}$$
$$= \frac{2Ax^2 + 6Ax + 2Bx + 6B + Cx^2 + C}{2(x^2 + 1)(x + 3)}$$

$2A + C = 1$ ①

$6A + 2B = 2$

$9A + 3B = 3$ ②

$6B + C = -13$ ③

① − ③:

$2A - 6B = 14$

$A - 3B = 7$ ④

② + ④:

$10A = 10$

$A = 1$

$2 + C = 1$

$C = -1$

$3A + B = 1$

$A + B = 1$

$B = -2$

$$\therefore \quad \frac{x^2 + 2x - 13}{2(x^2 + 1)(x + 3)} = \frac{x - 2}{x^2 + 1} - \frac{1}{2(x + 3)}$$

4 $(x - 1)(x - 2) = x^2 - 3x + 2$

First divide:

$$3x^2 - 4x - 2 = 3(x^2 - 3x + 2) + 5x - 8$$
$$\frac{3x^2 - 4x - 2}{(x - 1)(x - 2)} = \frac{5x - 8}{(x - 1)(x - 2)} + 3$$
$$\frac{5x - 8}{(x - 1)(x - 2)} = \frac{A}{x - 1} + \frac{B}{x - 2}$$
$$= \frac{A(x - 2) + B(x - 1)}{(x - 1)(x - 2)}$$
$$= \frac{Ax + Bx - 2A - B}{(x - 1)(x - 2)}$$

$A + B = 5$ ①

$-2A - B = -8$ ②

① + ②:

$-A = -3$

$A = 3$

$3 + B = 5$

$B = 2$

$$\therefore \quad \frac{5x - 8}{(x - 1)(x - 2)} = \frac{3}{x - 1} + \frac{2}{x - 2}$$

Use the previous working:

$$\frac{3x^2 - 4x - 2}{(x - 1)(x - 2)} = 3 + \frac{3}{x - 1} + \frac{2}{x - 2}$$

5 $\dfrac{2x+10}{(x+1)(x-1)^2}$

$$= \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + C(x+1)}{(x+1)(x-1)^2}$$

$$= \frac{Ax^2 - 2Ax + A + Cx + C}{(x+1)(x-1)^2}$$

$$A = 0$$

$$-2A + C = 2$$

$$C = 2$$

$$A + C = 10$$

$$0 + 2 \neq 10$$

It is impossible to find A and C to satisfy this equation.

6 a

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

$$= \frac{Ax + Bx + A - B}{(x-1)(x+1)}$$

$$A + B = 0 \quad ①$$

$$A - B = 1 \quad ②$$

① + ②:

$$2A = 1$$

$$A = \frac{1}{2}$$

$$\frac{1}{2} + B = 0$$

$$B = -\frac{1}{2}$$

$$\therefore \quad \frac{1}{(x-1)(x+1)} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

b

$$\frac{x}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$= \frac{A(x+3) + B(x-2)}{(x-2)(x+3)}$$

$$= \frac{Ax + Bx + 3A - 2B}{(x-2)(x+3)}$$

$$A + B = 1$$

$$2A + 2B = 2 \quad ①$$

$$3A - 2B = 0 \quad ②$$

① + ②:

$$5A = 2$$

$$A = \frac{2}{5}$$

$$\frac{2}{5} + B = 1$$

$$B = \frac{3}{5}$$

$$\therefore \quad \frac{x}{(x-2)(x+3)} = \frac{2}{5(x-2)} + \frac{3}{5(x+3)}$$

c

$$\frac{3x+1}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$= \frac{A(x+5) + B(x-2)}{(x-2)(x+5)}$$

$$= \frac{Ax + Bx + 5A - 2B}{(x-2)(x+5)}$$

$$A + B = 3$$

$$2A + 2B = 6 \quad ①$$

$$5A - 2B = 1 \quad ②$$

① + ②:

$$7A = 7$$

$$A = 1$$

$$1 + B = 3$$

$$B = 2$$

$$\therefore \quad \frac{3x+1}{(x-2)(x+5)} = \frac{1}{x-2} + \frac{2}{x+5}$$

d $\dfrac{1}{(2x-1)(x+2)}$

$$= \frac{A}{2x-1} + \frac{B}{x+2}$$

$$= \frac{A(x+2) + B(2x-1)}{(2x-1)(x+2)}$$

$$= \frac{Ax + 2Bx + 2A - B}{(2x-1)(x+2)}$$

$$A + 2B = 0$$

$$2A + 4B = 0 \quad ①$$

$$2A - B = 1 \quad ②$$

① + ②:

$$5B = -1$$

$$B = -\frac{1}{5}$$

$$A + 2B = 0$$

$$A = \frac{2}{5}$$

$$\therefore \quad \frac{1}{(2x-1)(x+2)} = \frac{2}{5(2x-1)} - \frac{1}{5(x+2)}$$

e

$$\frac{3x+5}{(3x-2)(2x+1)}$$

$$= \frac{A}{3x-2} + \frac{B}{2x+1}$$

$$= \frac{A(2x+1) + B(3x-2)}{(3x-2)(2x+1)}$$

$$= \frac{2Ax + 3Bx + A - 2B}{(3x-2)(2x+1)}$$

$$2A + 3B = 3 \quad ①$$

$$A - 2B = 5$$

$$2A - 4B = 10 \quad ②$$

① − ②:

$$7B = -7$$

$$B = -1$$

$$A - 2 \times -1 = 5$$

$$A = 3$$

$$\therefore \quad \frac{3x+5}{(3x-2)(2x+1)} = \frac{3}{3x-2} - \frac{1}{2x+1}$$

f $\dfrac{2}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$$= \frac{A(x-1) + Bx}{x(x-1)}$$

$$= \frac{Ax + Bx - A}{x(x-1)}$$

$$A + B = 0$$

$$-A = 2$$

$$A = -2$$

$$-2 + B = 0$$

$$B = 2$$

$$\therefore \quad \frac{2}{x(x-1)} = \frac{2}{x-1} - \frac{2}{x}$$

g $\dfrac{3x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$$= \frac{A(x^2+1) + x(Bx+C)}{x(x^2+1)}$$

$$= \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2+1)}$$

$$A + B = 0$$

$$C = 3$$

$$A = 1$$

$$1 + B = 0$$

$$B = -1$$

$$\therefore \quad \frac{3x+1}{x(x^2+1)} = \frac{1}{x} + \frac{3-x}{x^2+1}$$

h $\dfrac{3x^2+8}{x(x^2+4)}$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4)+x(Bx+C)}{x(x^2+4)}$$

$$= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}$$

$$A+B=3$$

$$C=0$$

$$4A=8$$

$$A=2$$

$$2+B=3$$

$$B=1$$

$$\therefore \quad \frac{3x^2+8}{x(x^2+4)} = \frac{2}{x} + \frac{x}{x^2+4}$$

i $\dfrac{1}{x(x-4)} = \dfrac{A}{x} + \dfrac{B}{x-4}$

$$= \frac{A(x-4)+Bx}{x(x-4)}$$

$$= \frac{Ax+Bx-4A}{x(x-4)}$$

$$A+B=0$$

$$-4A=1$$

$$A=-\frac{1}{4}$$

$$-\frac{1}{4}+B=0$$

$$B=\frac{1}{4}$$

$$\therefore \quad \frac{1}{x(x-4)} = \frac{1}{4(x-4)} - \frac{1}{4x}$$

j $\dfrac{x+3}{x(x-4)} = \dfrac{A}{x} + \dfrac{B}{x-4}$

$$= \frac{A(x-4)+Bx}{x(x-4)}$$

$$= \frac{Ax+Bx-4A}{x(x-4)}$$

$$A+B=1$$

$$-4A=3$$

$$A=-\frac{3}{4}$$

$$-\frac{3}{4}+B=1$$

$$B=\frac{7}{4}$$

$$\therefore \quad \frac{x+3}{x(x-4)} = \frac{7}{4(x-4)} - \frac{3}{4x}$$

k First divide $x^2 - x^2 - 1$ by $x^2 - x$. You might observe a pattern in the question.

$$\frac{x^3-x^2-1}{x^2-x} = \frac{x(x^2-x)-1}{x^2-x} = x - \frac{1}{x^2-x}$$

Express $-\dfrac{1}{x^2-x}$ in partial fractions.

$$-\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$= \frac{A(x-1)+Bx}{x(x-1)}$$

$$= \frac{Ax+Bx-A}{x(x-1)}$$

$$A+B=0$$

$$-A=-1$$

$$A=1$$

$$1 + B = 0$$
$$B = -1$$
$$\therefore \quad \frac{-1}{x(x-1)} = \frac{1}{x} - \frac{1}{x-1}$$
$$\frac{x^3 - x^2 - 1}{x^2 - x} = x + \frac{1}{x} - \frac{1}{x-1}$$

l First divide $(x^2 - x^2 - 6)$ by $(-x^2 + 2x)$.

$$\begin{array}{r}
-x - 1 \\
-x^2 + 2x \overline{) x^3 - x^2 - 6} \\
\underline{x^3 - 2x^2 } \\
x^2 - 6 \\
\underline{x^2 - 2x} \\
2x - 6
\end{array}$$

$\therefore$ $(x^3 - x^2 - 6) \div (-x^2 + 2x) = -x - 1 + \dfrac{2x-6}{x(2-x)}$

Separate $\dfrac{2x-6}{x(2-x)}$ into partial fractions.

$$\frac{2x-6}{x(2-x)} = \frac{A}{x} + \frac{B}{2-x}$$
$$= \frac{A(2-x) + Bx}{x(2-x)}$$
$$= \frac{-Ax + Bx + 2A}{x(2-x)}$$

$$-A + B = 2$$
$$2A = -6$$
$$A = -3$$
$$3 + B = 2$$
$$B = -1$$
$$\therefore \quad \frac{2x-6}{x(2-x)} = -\frac{3}{x} - \frac{1}{2-x}$$
$$\frac{x^3 - x^2 - 6}{2x - x^2} = -x - 1 - \frac{3}{x} - \frac{1}{2-x}$$

m $\dfrac{x^2 - x}{(x+1)(x^2+2)}$

$$= \frac{A}{x+1} + \frac{Bx + C}{x^2 + 2}$$
$$= \frac{A(x^2+2) + (Bx+C)(x+1)}{(x+1)(x^2+2)}$$
$$= \frac{Ax^2 + 2A + Bx^2 + Bx + Cx + C}{(x+1)(x^2+2)}$$

$$A + B = 1 \quad ①$$
$$B + C = -1 \quad ②$$
$$2A + C = 0 \quad ③$$
$$① - ②: A - C = 2 \quad ④$$
$$③ + ④: 3A = 2$$
$$A = \frac{2}{3}$$
$$\frac{2}{3} + B = 1$$
$$B = \frac{1}{3}$$
$$\frac{1}{3} + C = -1$$
$$C = -\frac{4}{3}$$
$$\therefore \quad \frac{x^2 - x}{(x+1)(x^2+2)} = \frac{2}{3(x+1)} + \frac{x-4}{3(x^2+2)}$$

n $x^3 - 3x - 2$ can be factorised into $(x-2)(x+1)^2$.

$$\frac{x^2 + 2}{(x-2)(x+1)^2}$$
$$= \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$= \frac{A(x+1)^2 + B(x+1)(x-2) + C(x-2)}{(x-2)(x+1)^2}$$
$$= \frac{Ax^2 + 2Ax + A + Bx^2 - Bx - 2B + Cx - 2C}{(x-2)(x+1)^2}$$

$$A + B = 1 \quad ①$$
$$2A - B + C = 0$$
$$4A - 2B + 2C = 0 \quad ②$$
$$A - 2B - 2C = 2 \quad ③$$
②+③:
$$5A - 4B = 2 \quad ④$$
④−4×①:
$$9A = 6$$
$$A = \frac{2}{3}$$
$$A + B = 1$$
$$B = \frac{1}{3}$$
$$\frac{4}{3} - \frac{1}{3} + C = 0$$
$$C = -1$$
$$\therefore \quad \frac{x^2+2}{(x-2)(x+1)^2} = \frac{2}{3(x-2)} + \frac{1}{3(x+1)} - \frac{1}{(x+1)^2}$$

o $\dfrac{2x^2+x+8}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$
$$= \frac{A(x^2+4)+x(Bx+C)}{x(x^2+4)}$$
$$= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}$$
$$A + B = 2$$
$$C = 1$$
$$4A = 8$$
$$A = 2$$
$$2 + B = 2$$
$$B = 0$$
$$\therefore \quad \frac{2x^2+x+8}{x(x^2+4)} = \frac{2}{x} + \frac{1}{x^2+4}$$

p $2x^2 + 7x + 6 = (2x+3)(x+2)$
$$\frac{1-2x}{(2x+3)(x+2)} = \frac{A}{2x+3} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(2x+3)}{(2x+3)(x+2)}$$
$$= \frac{Ax+2Bx+2A+3B}{(2x+3)(x+2)}$$
$$A + 2B = -2$$
$$2A + 4B = -4 \quad ①$$
$$2A + 3B = 1 \quad ②$$
①−②:
$$B = -5$$
$$A + 2 \times -5 = -2$$
$$A = 8$$
$$\therefore \quad \frac{1-2x}{(2x+3)(x+2)} = \frac{8}{2x+3} - \frac{5}{x+2}$$

q $\dfrac{3x^2-6x+2}{(x-1)^2(x+2)}$
$$= \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{(x-1)^2(x+2)}$$
$$= \frac{Ax^2-2Ax+A+Bx^2+Bx-2B+Cx+2C}{(x-1)^2(x+2)}$$

$$A + B = 3$$
$$4A + 4B = 12 \quad ①$$
$$-2A + B + C = -6 \quad ②$$
$$A - 2B + 2C = 2 \quad ③$$
③ − ②:
$$5A - 4B = 14 \quad ④$$
①+④:
$$9A = 26$$
$$A = \frac{26}{9}$$
$$\frac{26}{9} + B = 3$$
$$B = \frac{1}{9}$$
$$-\frac{52}{9} + \frac{1}{9} + C = -6$$
$$C = -\frac{1}{3}$$
$$\therefore \quad \frac{3x^2 - 6x + 2}{(x-1)^2(x+2)} = \frac{26}{9(x+2)} + \frac{1}{9(x-1)} - \frac{1}{3(x-1)^2}$$

r $$\frac{4}{(x-1)^2(2x+1)}$$
$$= \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2 + B(2x+1)(x-1) + C(2x+1)}{(x-1)^2(2x+1)}$$
$$= \frac{Ax^2 - 2Ax + A + 2Bx^2 - Bx - B + 2Cx + C}{(x-1)^2(2x+1)}$$

$$A + 2B = 0 \quad ①$$
$$-2A - B + 2C = 0 \quad ②$$
$$A - B + C = 4$$
$$2A - 2B + 2C = 8 \quad ③$$
③ − ②:
$$4A - B = 8$$
$$8A - 2B = 16 \quad ④$$
① + ④:
$$9A = 16$$
$$A = \frac{16}{9}$$
$$\frac{16}{9} + 2B = 0$$
$$B = -\frac{8}{9}$$
$$\frac{16}{9} + \frac{8}{9} + C = 4$$
$$C = \frac{4}{3}$$
$$\therefore \quad \frac{4}{(x-1)^2(2x+1)} = \frac{16}{9(2x+1)} - \frac{8}{9(x-1)} + \frac{4}{3(x-1)^2}$$

s Divide:

$$\begin{array}{r|l} & x - 2 \\ \hline x^2 - 4 & x^3 - 2x^2 - 3x + 9 \\ & \underline{x^3 - 0x^2 - 4x} \\ & \quad -2x^2 + x \\ & \quad \underline{-2x^2 + 8} \\ & \qquad\quad x + 1 \end{array}$$

$$\frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4} = x - 2 + \frac{x+1}{x^2 - 4}$$

$$\frac{x+1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2) + B(x+2)}{(x+2)(x-2)} = \frac{Ax + Bx - 2A + 2B}{(x+2)(x-2)}$$

$$A + B = 1$$

$$2A + 2B = 2 \quad ①$$

$$-2A + 2B = 1 \quad ②$$

① + ②:

$$4B = 3$$

$$B = \frac{3}{4}$$

$$A + \frac{3}{4} = 1$$

$$A = \frac{1}{4}$$

$$\therefore \quad \frac{x+1}{(x+2)(x-2)} = \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$$

$$\frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4} = x - 2 + \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$$

t Divide:

$$\begin{array}{r} x \\ x^2 - 1\overline{\smash{)}x^3 + 3} \\ \underline{x^3 - x} \\ x + 3 \end{array}$$

$$\frac{x^3 + 3}{(x+1)(x-1)} = x + \frac{x+3}{(x+1)(x-1)}$$

$$\frac{x+3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)} = \frac{Ax + Bx - A + B}{(x+1)(x-1)}$$

$$A + B = 1 \quad ①$$

$$-A + B = 3 \quad ②$$

① + ②:

$$2B = 4$$

$$B = 2$$

$$A + 2 = 1$$

$$A = -1$$

$$\therefore \quad \frac{x+3}{(x+1)(x-1)} = -\frac{1}{x+1} + \frac{2}{x-1}$$

$$\frac{x^3 + 3}{(x+1)(x-1)} = x - \frac{1}{x+1} + \frac{2}{x-1}$$

u $\dfrac{2x-1}{(x+1)(3x+2)}$

$$= \frac{A}{x+1} + \frac{B}{3x+2} = \frac{A(3x+2) + B(x+1)}{(x+1)(3x+2)} = \frac{3Ax + Bx + 2A + B}{(x+1)(3x+2)}$$

$$3A + B = 2 \quad ①$$

$$2A + B = -1 \quad ②$$

① − ②: $A = 3$

$$9 + B = 2$$

$$B = -7$$

$$\therefore \quad \frac{2x-1}{(x+1)(3x+2)} = \frac{3}{x+1} - \frac{7}{3x+2}$$

Solutions to Exercise 4E

1 a A simple start is often to subtract the equations.

$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$
$$\text{If } x = 0,\ y = 0$$
$$\text{If } x = 1,\ y = 1$$

The points of intersection are $(0, 0)$ and $(1, 1)$.

b Subtract the equations:

$$2x^2 - x = 0$$
$$x(2x-1) = 0$$
$$x = 0 \text{ or } x = \frac{1}{2}$$

If $x = 0,\ y = 0$

If $x = \frac{1}{2},\ y = \frac{1}{2}$

The points of intersection are $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$.

c Subtract the equations:

$x^2 - 3x - 1 = 0$

$$x = \frac{3 \pm \sqrt{9 - 4 \times 1 \times -1}}{2}$$
$$= \frac{3 \pm \sqrt{13}}{2}$$
$$= \frac{3 + \sqrt{13}}{2} \text{ or } \frac{3 - \sqrt{13}}{2}$$

$$\text{If } x = \frac{3 + \sqrt{13}}{2},\ y = 2 \times \frac{3 + \sqrt{13}}{2} + 1$$
$$= 4 + \sqrt{13}$$

$$\text{If } x = \frac{3 - \sqrt{13}}{2},\ y = 2 \times \frac{3 - \sqrt{13}}{2} + 1$$
$$= 4 - \sqrt{13}$$

The points of intersection are

$$\left(\frac{3 + \sqrt{13}}{2}, 4 + \sqrt{13}\right) \text{ and}$$
$$\left(\frac{3 - \sqrt{13}}{2}, 4 - \sqrt{13}\right).$$

2 a Substitute $y = 16 - x$ into $x^2 + y^2 = 178$

$$x^2 + (16 - x)^2 = 178$$
$$x^2 + 256 - 32x + x^2 = 178$$
$$2x^2 - 32x + 78 = 0$$
$$x^2 - 16x + 39 = 0$$
$$(x - 3)(x - 13) = 0$$
$$x = 3 \text{ or } x = 13$$

If $x = 3, y = 16 - x = 13$

If $x = 13, y = 16 - x = 3$

The points of intersection are $(3, 13)$ and $(13, 3)$.

b Substitute $y = 15 - x$ into $x^2 + y^2 = 125$.

$$x^2 + (15 - x)^2 = 125$$
$$x^2 + 225 - 30x + x^2 = 125$$
$$2x^2 - 30x + 100 = 0$$
$$x^2 - 15x + 50 = 0$$
$$(x - 5)(x - 10) = 0$$
$$x = 5 \text{ or } x = 10$$

If $x = 5,\ y = 15 - x = 10$

If $x = 10,\ y = 15 - x = 5$

The points of intersection are $(5, 10)$ and $(10, 5)$.

c Substitute $y = x - 3$ into $x^2 + y^2 = 185$.

$$x^2 + (x-3)^2 = 185$$
$$x^2 + x^2 - 6x + 9 = 185$$
$$2x^2 - 6x - 176 = 0$$
$$x^2 - 3x - 88 = 0$$
$$(x-11)(x+8) = x = 0$$
$$x = 11 \text{ or } x = -8$$
If $x = 11, y = x - 3 = 8$

If $x = -8, y = x - 3 = -11$

The points of intersection are $(11, 8)$ and $(-8, -11)$.

d Substitute $y = 13 - x$ into $x^2 + y^2 = 97$.
$$x^2 + (13-x)^2 = 97$$
$$x^2 + 169 - 26x + x^2 = 97$$
$$2x^2 - 26x + 72 = 0$$
$$x^2 - 13x + 36 = 0$$
$$(x-4)(x-9) = 0$$
$$x = 4 \text{ or } x = 9$$
If $x = 4,\ y = 13 - x = 9$

If $x = 9,\ y = 13 - x = 4$

The points of intersection are $(4, 9)$ and $(9, 4)$.

e Substitute $y = x - 4$ into $x^2 + y^2 = 106$.
$$x^2 + (x-4)^2 = 106$$
$$x^2 + x^2 - 8x + 16 = 106$$
$$2x^2 - 8x - 90 = 0$$
$$x^2 - 4x - 45 = 0$$
$$(x-9)(x+5) = 0$$
$$x = 9 \text{ or } x = -5$$
If $x = 9,\ y = x - 4 = 5$

If $x = -5,\ y = x - 4 = -9$

The points of intersection are $(9, 5)$ and $(-5, -9)$.

3 a Substitute $y = 28 - x$ into $xy = 187$.
$$x(28-x) = 187$$
$$28x - x^2 = 187$$
$$x^2 - 28x + 187 = 0$$
$$(x-11)(x-17) = 0$$
$$x = 11 \text{ or } x = 17$$
If $x = 11,\ y = 28 - x = 17$

If $x = 17,\ y = 28 - x = 11$

The points of intersection are $(11, 17)$ and $(17, 11)$.

b Substitute $y = 51 - x$ into $xy = 518$.
$$x(51-x) = 518$$
$$51x - x^2 = 518$$
$$x^2 - 51x + 518 = 0$$
$$(x-14)(x-37) = 0$$
$$x = 14 \text{ or } x = 37$$
If $x = 14,\ y = 51 - x = 37$

If $x = 37,\ y = 51 - x = 14$

The points of intersection are $(14, 37)$ and $(37, 14)$.

c Substitute $y = x - 5$ into $xy = 126$.
$$x(x-5) = 126$$
$$x^2 - 5x = 126$$
$$x^2 - 5x - 126 = 0$$
$$(x-14)(x+9) = 0$$
$$x = 14 \text{ or } x = -9$$
If $x = 14,\ y = x - 5 = 9$
If $x = -9,\ y = x - 5 = -14$
The points of intersection are $(14, 9)$ and $(-9, -14)$.

4 Substitute $y = 2x$ into the equation of the circle.

$$(x - 5)^2 + (2x)^2 = 25$$
$$x^2 - 10x + 25 + 4x^2 = 25$$
$$5x^2 - 10x = 0$$
$$5x(x - 2) = 0$$
$$x = 0 \text{ or } x = 2$$

If $x = 0, y = 2x = 0$
If $x = 2, y = 2x = 4$

The points of intersection are $(0, 0)$ and $(2, 4)$.

5 Substitute $y = x$ into the equation of the second curve.

$$x = \frac{1}{x - 2} + 3$$
$$x(x - 2) = 1 + 3(x - 2)$$
$$x^2 - 2x = 1 + 3x - 6$$
$$x^2 - 5x + 5 = 0$$
$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 5}}{2}$$
$$= \frac{5 \pm \sqrt{5}}{2}$$
$$= \frac{5 + \sqrt{5}}{2} \text{ or } \frac{5 - \sqrt{5}}{2}$$

Since $y = x$, the points of intersection are

$\left(\frac{5 + \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right)$ and
$\left(\frac{5 - \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2}\right)$.

6 Substitute $x = 3y$ into the equation of the circle.

$$9y^2 + y^2 - 30y - 5y + 25 = 0$$
$$10y^2 - 35y + 25 = 0$$
$$2y^2 - 7y + 5 = 0$$
$$(2y - 5)(y - 1) = 0$$
$$y = \frac{5}{2} \text{ or } y = 1$$

If $y = \frac{5}{2}$, $x = 3y = \frac{15}{2}$
If $y = 1$, $x = 3y = 3$

The points of intersection are $\left(\frac{15}{2}, \frac{5}{2}\right)$ and $(3, 1)$.

7 Make y the subject in $\frac{y}{4} - \frac{x}{5} = 1$.

$$\frac{y}{4} = \frac{x}{5} + 1$$
$$y = \frac{4x}{5} + 4$$

Substitute into $x^2 + 4x + y^2 = 12$.

$$x^2 + 4x + \left(\frac{4x}{5} + 4\right)^2 = 12$$
$$x^2 + 4x + \frac{16x^2}{25} + \frac{32x}{5} + 16 = 12$$
$$25x^2 + 100x + 16x^2 + 160x + 400 = 300$$
$$41x^2 + 260x + 100 = 0$$
$$x = \frac{-260 \pm \sqrt{67\,600 - 4 \times 41 \times 100}}{82}$$
$$= \frac{-260 \pm \sqrt{51\,200}}{82}$$
$$= \frac{-260 \pm \sqrt{25\,600 \times 2}}{82}$$
$$= \frac{-260 \pm 160\sqrt{2}}{82}$$
$$= \frac{-130 \pm 80\sqrt{2}}{41}$$

If $x = \dfrac{-130 + 80\sqrt{2}}{41}$,

$$y = \frac{4 \times (-130 + 80\sqrt{2})}{5 \times 41} + 4$$
$$= \frac{4 \times (-26 + 16\sqrt{2})}{41} + \frac{4 \times 41}{41}$$
$$= \frac{-104 + 64\sqrt{2} + 164}{41}$$
$$= \frac{60 + 64\sqrt{2}}{41}$$

Likewise, if $x = \dfrac{-130 - 80\sqrt{2}}{41}$,

$$y = \frac{60 - 64\sqrt{2}}{41}$$

The points of intersection are

$\left(\dfrac{-130 + 80\sqrt{2}}{41}, \dfrac{60 + 64\sqrt{2}}{41}\right)$ and $\left(\dfrac{-130 - 80\sqrt{2}}{41}, \dfrac{60 - 64\sqrt{2}}{41}\right)$.

8 Subtract the second equation from the first.

$$\frac{1}{x+2} - 3 + x = 0$$
$$1 - 3(x + 2) + x(x + 2) = 0$$
$$1 - 3x - 6 + x^2 + 2x = 0$$
$$x^2 - x - 5 = 0$$
$$x = \frac{1 \pm \sqrt{1 - 4 \times 1 \times -5}}{2}$$
$$= \frac{1 \pm \sqrt{21}}{2}$$

If $x = \dfrac{1 + \sqrt{21}}{2}$, $y = -x = \dfrac{-1 - \sqrt{21}}{2}$

If $x = \dfrac{1 - \sqrt{21}}{2}$, $y = -x = \dfrac{-1 + \sqrt{21}}{2}$

The points of intersection are

$\left(\dfrac{1 + \sqrt{21}}{2}, \dfrac{-1 - \sqrt{21}}{2}\right)$ and $\left(\dfrac{1 - \sqrt{21}}{2}, \dfrac{-1 + \sqrt{21}}{2}\right)$

9 Substitute $y = \dfrac{9x + 4}{4}$ into the equation of the parabola.

$$\left(\frac{9x + 4}{4}\right)^2 = 9x$$
$$\frac{(9x + 4)^2}{16} = 9x$$
$$(9x + 4)^2 = 9x \times 16$$
$$81x^2 + 72x + 16 = 144x$$
$$81x^2 - 72x + 16 = 0$$
$$(9x - 4)^2 = 0$$
$$x = \frac{4}{9}$$
$$y = \frac{9x + 4}{4}$$
$$= \frac{4 + 4}{4} = 2 \left(\frac{4}{9}, 2\right)$$

Note: Substitute into the linear equation, as substituting into the quadratic introduces a second answer that is not actually a solution.

10 Substitute $y = 2x + 3\sqrt{5}$ into the equation of the circle.

$$x^2 + (2x + 3\sqrt{5})^2 = 9$$
$$x^2 + 4x^2 + 12\sqrt{5}x + 45 = 9$$
$$5x^2 + 12\sqrt{5}x + 36 = 0$$
$$x^2 + \frac{12\sqrt{5}}{5}x + \frac{36}{5} = 0$$
$$x^2 + \frac{2 \times 6\sqrt{5}}{5}x + \frac{(6\sqrt{5})^2}{25} = 0$$
$$\left(x + \frac{6\sqrt{5}}{5}\right)^2 = 0$$

$$x = -\frac{6\sqrt{5}}{5}$$

$$y = 2x + 3\sqrt{5}$$

$$= -\frac{12\sqrt{5}}{5} + \frac{15\sqrt{5}}{5}$$

$$= \frac{3\sqrt{5}}{5}\left(-\frac{6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$$

11 Substitute $y = \frac{1}{4}x + 1$ into $y = -\frac{1}{x}$.

$$\frac{1}{4}x + 1 = -\frac{1}{x}$$

$$\frac{x+4}{4} = -\frac{1}{x}$$

$$x(x+4) = -4$$

$$x^2 + 4x + 4 = 0$$

$$(x+2)^2 = 0$$

$$x = -2$$

$$y = -\frac{1}{x}$$

$$= \frac{1}{2}\left(-2, \frac{1}{2}\right)$$

12 Substitute $y = x - 1$ into $y = \frac{2}{x-2}$.

$$x - 1 = \frac{2}{x-2}$$

$$(x-1)(x-2) = 2$$

$$x^2 - 3x + 2 = 2$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0 \text{ or } x = 3$$

If $x = 0,\ y = x - 1 = -1$
If $x = 3,\ y = x - 1 = 2$
The points of intersection are $(0, -1)$ and $(3, 2)$.

13 a $2x^2 - 4x + 1 = 2x^2 - x - 1$

$$-3x = -2$$

$$x = \frac{2}{3}$$

$$y = -\frac{7}{9}$$

b

$$-2x^2 + x + 1 = 2x^2 - x - 1$$

$$4x^2 - 2x - 2 = 0$$

$$2x^2 - x - 1 = 0$$

$$(2x+1)(x-1) = 0$$

$x = -\frac{1}{2}$ or $x = 1$

Solutions: $\left(\frac{-1}{2}, 0\right), (1, 0)$

c $x^2 + x + 1 = x^2 - x - 2$

$$2x + 3 = 0$$

$$x = -\frac{3}{2}$$

$$y = \frac{7}{4}$$

d

$$3x^2 + x + 2 = x^2 - x + 2$$

$$2x^2 + 2x- = 0$$

$$2x(x+1) = 0$$

$x = 0$ or $x = -1$
Solutions: $(-1, 4), (0, 2)$

e If $x = 1$ and $y = b$ is a solution to this pair of equations then:

$$5 + 4b = 11$$

$$2 + ab + 4b^2 = 24.$$

From the first equation $b = \frac{6}{4} = \frac{3}{2}$.
Substituting into the second equation

gives

$$2 + \frac{3a}{2} + 4\left(\frac{3}{2}\right)^2 = 24$$

$$2 + \frac{3a}{2} + 9 = 24$$

$$\frac{3a}{2} + 11 = 24$$

$$\frac{3a}{2} = 13$$

$$a = \frac{26}{3}.$$

The two equations then become

$$5x + 4y = 11$$

$$2x^2 + \frac{26}{3}xy + 4y^2 = 24.$$

From the first equation we find that $y = \frac{11-5x}{4}$. Substituting this into the second equation gives

$$2x^2 + \frac{26}{3}x\left(\frac{11-5x}{4}\right) + 4\left(\frac{11-5x}{4}\right)^2 = 24.$$

Expanding this and rearranging gives

$$-31x^2 - 44x + 75 = 0.$$

Solving for x gives $x = 1$ and $x = -\frac{75}{31}$. The corresponding values of y are $y = \frac{3}{2}$ and $y = \frac{179}{31}$.

Solutions to technology-free questions

1
$$3a + b = 11$$
$$6a + 2b = 22 \quad ①$$
$$a - 2b = -1 \quad ②$$
① + ②:
$$7a = 21$$
$$a = 3$$
$$3 \times 3 + b = 11$$
$$b = 2$$
$$2 + 2c = 4$$
$$c = 1$$

2
$$x^3 = (x-1)^3 + a(x-1)^2 + b(x-1) + c$$
$$= x^3 - 3x^2 + 3x - 1 + ax^2 - 2ax + a + bx - b + c$$
$$-3 + a = 0$$
$$a = 3$$
$$3 - 2 \times 3 + b = 0$$
$$b = 3$$
$$-1 + 3 - 3 + c = 0$$
$$c = 1$$
$$\therefore \quad x^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$$

3
$$(x+1)^2(px+q)$$
$$= (x^2 + 2x + 1)(px + q)$$
$$= px^3 + (q + 2p)x^2 + (p + 2q)x + q$$
$$a = p$$
$$b = q + 2p$$
$$c = p + 2q$$
$$d = q$$
$$2a + d = 2p + q = b$$
$$a + 2d = p + 2q = c$$

4
$$(x-2)^2(px+q) = (x^2 - 4x + 4)(px + q)$$
$$= px^3 + (q - 4p)x^2 + (4p - 4q)x + 4q$$
$$a = p$$
$$b = q - 4p$$
$$c = 4p - 4q$$
$$d = 4q$$
$$-4a + \frac{1}{4}d = -4p + q = b$$
$$4a - d = 4p - 4q = c$$

5 **a**
$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x = -4 \text{ or } x = 3$$

b
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1 \text{ or } 2$$

c
$$x^2 - 3x - 11 = -1$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
$$x = 5 \text{ or } x = -2$$

d $2x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4 \times 2 \times 1}}{4}$$

$$= \frac{4 \pm \sqrt{8}}{4}$$

$$= \frac{2 \pm \sqrt{2}}{2}$$

e $3x^2 - 2x + 5 - t = 0$

$$x = \frac{2 \pm \sqrt{4 - 4 \times 3 \times (5 - t)}}{6}$$

$$= \frac{2 \pm \sqrt{4 - 60 + 12t}}{6}$$

$$= \frac{2 \pm \sqrt{12t - 56}}{6}$$

$$= \frac{2 \pm \sqrt{4(3t - 14)}}{6}$$

$$= \frac{2 \pm 2\sqrt{3t - 14}}{6}$$

$$= \frac{1 \pm \sqrt{3t - 14}}{3}$$

f $tx^2 - tx + 4 = 0$

$$x = \frac{t \pm \sqrt{t^2 - 4 \times t \times 4}}{2t}$$

$$= \frac{t \pm \sqrt{t^2 - 16t}}{2t}$$

6 $\dfrac{2(x + 2) - 3(x - 1)}{(x - 1)(x + 2)} = \dfrac{1}{2}$

$$2(2x + 4 - 3x + 3) = (x - 1)(x + 2)$$

$$2(-x + 7) = x^2 + x - 2$$

$$-2x + 14 = x^2 + x - 2$$

$$x^2 + 3x - 16 = 0$$

$a = 1,\ b = 3,\ c = -16$

$$x = \frac{-3 \pm \sqrt{9 - 4 \times 1 \times -16}}{2}$$

$$= \frac{-3 \pm \sqrt{73}}{2}$$

7 **a** $\dfrac{-3x + 4}{(x - 3)(x + 2)} = \dfrac{A}{x - 3} + \dfrac{B}{x + 2}$

$$= \frac{A(x + 2) + B(x - 3)}{(x - 3)(x + 2)}$$

$$= \frac{Ax + Bx + 2A - 3B}{(x - 3)(x + 2)}$$

$$A + B = -3$$

$$3A + 3B = -9 \quad ①$$

$$2A - 3B = 4 \quad ②$$

① + ②:

$$5A = -5$$

$$A = -1$$

$$-1 + B = -3$$

$$B = -2$$

$$\therefore \quad \frac{-3x + 4}{(x - 3)(x + 2)} = -\frac{1}{x - 3} - \frac{2}{x + 2}$$

b

$$\frac{7x + 2}{(x + 2)(x - 2)} = \frac{A}{x + 2} + \frac{B}{x - 2}$$

$$= \frac{A(x - 2) + B(x + 2)}{(x + 2)(x - 2)}$$

$$= \frac{Ax + Bx - 2A + 2B}{(x + 2)(x - 2)}$$

$$A + B = 7$$

$$2A + 2B = 14 \quad ①$$

$$-2A + 2B = 2 \quad ②$$

① + ②:

$$4B = 16$$

$$B = 4$$

$$A + 4 = 7$$

$$A = 3$$

$$\therefore \quad \frac{7x + 2}{(x + 2)(x - 2)} = \frac{3}{x + 2} + \frac{4}{x - 2}$$

c

$$\frac{7-x}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$
$$= \frac{A(x+5)+B(x-3)}{(x-3)(x+5)}$$
$$= \frac{Ax+Bx+5A-3B}{(x-3)(x+5)}$$

$$A+B=-1$$
$$3A+3B=-3 \quad ①$$
$$5A-3B=7 \quad ②$$

① + ②:

$$8A=4$$
$$A=\frac{1}{2}$$
$$\frac{1}{2}+B=-1$$
$$B=-\frac{3}{2}$$

$$\therefore \quad \frac{7-x}{(x-3)(x+5)} = \frac{1}{2(x-3)} - \frac{3}{2(x+5)}$$

d

$$\frac{3x-9}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-5)}{(x-5)(x+1)}$$
$$= \frac{Ax+Bx+A-5B}{(x-5)(x+1)}$$

$$A+B=3$$
$$5A+5B=15 \quad ①$$
$$A-5B=-9 \quad ②$$

① + ②:

$$6A=6$$
$$A=1$$
$$1+B=3$$
$$B=2$$

$$\therefore \quad \frac{3x-9}{(x-5)(x+1)} = \frac{1}{x-5} + \frac{2}{x+1}$$

e

$$\frac{3x-4}{(x+3)(x+2)^2}$$
$$= \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$
$$= \frac{A(x+2)^2+B(x+3)(x+2)+C(x+3)}{(x+3)(x+2)^2}$$
$$= \frac{Ax^2+4Ax+4A+Bx^2+5Bx+6B+Cx+3C}{(x+3)(x+2)^2}$$

$$A+B=0$$
$$8A+8B=0 \quad ①$$
$$4A+5B+C=3$$
$$12A+15B+3C=9 \quad ②$$
$$4A+6B+3C=-4 \quad ③$$

② − ③:

$$8A+9B=13 \quad ④$$

④ − ①:

$$B=13$$
$$A+13=0$$
$$A=-13$$
$$4\times -13+5\times 13+C=3$$
$$C=-10$$

$$\therefore \quad \frac{3x-4}{(x+3)(x+2)^2} = -\frac{13}{x+3} + \frac{13}{x+2} - \frac{10}{(x+2)^2}$$

f

$$\frac{6x^2-5x-16}{(x-1)^2(x+4)}$$
$$= \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2+B(x+4)(x-1)+C(x+4)}{(x-1)^2(x+4)}$$
$$= \frac{Ax^2-2Ax+A+Bx^2+3Bx-4B+Cx+4C}{(x-1)^2(x+4)}$$

$$A + B = 6$$
$$16A + 16B = 96 \quad ①$$
$$-2A + 3B + C = -5$$
$$-8A + 12B + 4C = -20 \quad ②$$
$$A - 4B + 4C = -16 \quad ③$$
③ − ②:
$$9A - 16B = 4 \quad ④$$
① + ④:
$$25A = 100$$
$$A = 4$$
$$4 + B = 6$$
$$B = 2$$
$$-2 \times 4 + 3 \times 2 + C = -5$$
$$C = -3$$
$$\therefore \quad \frac{6x^2 - 5x - 16}{(x-1)^2(x+4)} = \frac{4}{x+4} + \frac{2}{x-1} - \frac{3}{(x-1)^2}$$

g $$\frac{x^2 - 6x - 4}{(x^2+2)(x+1)}$$
$$= \frac{Ax + B}{x^2 + 2} + \frac{C}{x+1}$$
$$= \frac{(Ax+B)(x+1) + C(x^2+2)}{(x^2+2)(x+1)}$$
$$= \frac{Ax^2 + Ax + Bx + B + Cx^2 + 2C}{(x^2+2)(x+1)}$$
$$A + C = 1 \quad ①$$
$$A + B = -6 \quad ②$$
$$B + 2C = -4 \quad ③$$
① − ②:
$$C - B = 7 \quad ④$$
③ + ④:
$$3C = 3$$
$$C = 1$$
$$A + 1 = 1$$
$$A = 0$$
$$0 + B = -6$$
$$B = -6$$
$$\therefore \quad \frac{x^2 - 6x - 4}{(x^2+2)(x+1)} = \frac{1}{x+1} - \frac{6}{x^2+2}$$

h
$$\frac{-x + 4}{(x-1)(x^2+x+1)}$$
$$= \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}$$
$$= \frac{A(x^2+x+1) + (Bx+C)(x-1)}{(x-1)(x^2+x+1)}$$
$$= \frac{Ax^2 + Ax + A + Bx^2 - Bx + Cx - C}{(x-1)(x^2+x+1)}$$
$$A + B = 0 \quad ①$$
$$A - B + C = -1 \quad ②$$
$$A - C = 4 \quad ③$$
② + ③:
$$2A - B = 3 \quad ④$$
① + ④:

$$3A = 3$$
$$A = 1$$
$$B = -1$$
$$1 - C = 4$$
$$C = -3$$
$$\therefore \quad \frac{-x+4}{(x-1)(x^2+x+1)} = \frac{1}{x-1} - \frac{x+3}{x^2+x+1}$$

i $$\frac{-4x+5}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$
$$= \frac{A(x-3)+B(x+4)}{(x+4)(x-3)}$$
$$= \frac{Ax+Bx-3A+4B}{(x+4)(x-3)}$$
$$A + B = -4$$
$$3A + 3B = -12 \quad ①$$
$$-3A + 4B = 5 \quad ②$$
$$① + ②: 7B = 7$$
$$B = -1$$
$$A - 1 = -4$$
$$A = -3$$
$$\therefore \quad \frac{-4x+5}{(x+4)(x-3)} = -\frac{3}{x+4} - \frac{1}{x-3}$$
$$= \frac{1}{3-x} - \frac{3}{x+4}$$

j
$$\frac{-2x+8}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$
$$= \frac{A(x-3)+B(x+4)}{(x+4)(x-3)}$$
$$= \frac{Ax+Bx-3A+4B}{(x+4)(x-3)}$$
$$A + B = -2$$
$$3A + 3B = -6 \quad ①$$
$$-3A + 4B = 8 \quad ②$$
$$① + ②: 7B = 2$$
$$B = \frac{2}{7}$$
$$A + \frac{2}{7} = -2$$
$$A = -\frac{16}{7}$$
$$\therefore \quad \frac{-2x+8}{(x+4)(x-3)} = \frac{2}{7(x-3)} - \frac{16}{7(x+4)}$$

8 a
$$\frac{14x-28}{(x-3)(x^2+x+2)}$$
$$= \frac{A}{x-3} + \frac{Bx+C}{x^2+x+2}$$
$$= \frac{A(x^2+x+2)+(Bx+C)(x-3)}{(x-3)(x^2+x+2)}$$
$$= \frac{Ax^2+Ax+2A+Bx^2-3Bx+Cx-3C}{(x-3)(x^2+x+2)}$$

$$A + B = 0$$
$$9A + 9B = 0 \quad ①$$
$$A - 3B + C = 14$$
$$3A - 9B + 3C = 42 \quad ②$$
$$2A - 3C = -28 \quad ③$$
$$②+③: 5A - 9B = 14 \quad ④$$
$$①+④: 14A = 14$$
$$A = 1$$
$$1 + B = 0$$
$$B = -1$$
$$1 - 3 \times -3 + C = 14$$
$$C = 10$$

$$\therefore \frac{14x - 28}{(x-3)(x^2+x+2)} = \frac{1}{x-3} + \frac{-x+10}{x^2+x+2} = \frac{1}{x-3} - \frac{x-10}{x^2+x+2}$$

b $$\frac{1}{(x+1)(x^2-x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+2} = \frac{A(x^2-x+2)+(Bx+C)(x+1)}{(x+1)(x^2-x+2)} = \frac{Ax^2 - Ax + 2A + Bx^2 + Bx + Cx + C}{(x+1)(x^2-x+2)}$$

$$A + B = 0 \quad ①$$
$$-A + B + C = 0 \quad ②$$
$$2A + C = 1 \quad ③$$
$$③-②: 3A - B = 1 \quad ④$$
$$①+④: 4A = 1$$
$$A = \frac{1}{4}$$
$$\frac{1}{4} + B = 0$$
$$B = -\frac{1}{4}$$
$$-\frac{1}{4} - \frac{1}{4} + C = 0$$
$$C = \frac{1}{2}$$

$$\therefore \frac{1}{(x+1)(x^2-x+2)} = \frac{1}{4(x+1)} + \frac{-x+2}{4(x^2-x+2)} = \frac{1}{4(x+1)} - \frac{x-2}{4(x^2-x+2)}$$

c First divide $3x^3$ by $x^2 - 5x + 4$.

$$\begin{array}{r} 3x + 15 \\ x^2 - 5x + 4 \enclose{longdiv}{3x^3} \\ 3x^3 - 15x^2 + 12x \\ \hline 15x^2 - 12x \\ 15x^2 - 75x + 60 \\ \hline 63x - 60 \end{array}$$

$\frac{3x^3}{x^2 - 5x + 4} = 3x + 15 + \frac{63x - 60}{(x-4)(x-1)}$ (factorising the denominator)

$$\frac{63x - 60}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} = \frac{A(x-1) + B(x-4)}{(x-4)(x-1)} = \frac{Ax + Bx - A - 4B}{(x-4)(x-1)}$$

$$A + B = 63 \quad ①$$

$$-A - 4B = -60 \quad ②$$

$$① + ②: -3B = 3$$

$$B = -1$$

$$A - 1 = 63$$

$$A = 64$$

$$\therefore \quad \frac{63x - 60}{(x-4)(x-1)} = \frac{64}{x-4} - \frac{1}{x-1}$$

$$\frac{3x^3}{x^2 - 5x + 4} = 3x + 15 + \frac{64}{x-4} - \frac{1}{x-1}$$

9 a

$$x^2 = -x$$

$$x^2 + x = 0$$

$$x(x+1) = 0$$

$$x = 0 \text{ or } x = -1$$

If $x = 0,\ y = 0$
If $x = -1,\ y = 1$
The points of intersection are $(0, 0)$ and $(-1, 1)$.

b Substitute $y = 4 - x$ into $x^2 + y^2 = 16$.

$$x^2 + (4 - x)^2 = 16$$

$$x^2 + 16 - 8x + x^2 = 16$$

$$2x^2 - 8x = 0$$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0 \text{ or } x = 4$$

If $x = 0,\ y = 4$
If $x = 4,\ y = 0$
The points of intersection are $(0, 4)$ and $(4,\ 0)$.

c Substitute $y = 5 - x$ into $xy = 4$.

$$x(5 - x) = 4$$

$$5x - x^2 - 4 = 0$$

$$x^2 - 5x + 4 = 0$$

$$(x - 4)(x - 1) = 0$$

$x = 4$ or $x = 1$
If $x = 4,\ y = 1$
If $x = 1,\ y = 4$
The points of intersection are $(4, 1)$ and $(1, 4)$.

10 Substitute $x = 3y - 1$ into the circle.

$$(3y - 1)^2 + 2(3y - 1) + y^2 = 9$$

$$9y^2 - 6y + 1 + 6y - 2 + y^2 = 9$$

$$10y^2 - 10 = 0$$

$$y^2 - 1 = 0$$

$$(y + 1)(y - 1) = 0$$

$$y = 1 \text{ or } y = -1$$

If $y = -1,\ x = -4$
If $y = 1,\ x = 2$
The points of intersection are $(2, 1)$ and $(-4, -1)$.

11 a $t = \dfrac{135}{x}$

b $t = \dfrac{135}{x - 15}$

c $x = 60$

d 60 km/h, 45 km/h

Solutions to multiple-choice questions

1 C $x^2 = (x+1)^2 + b(x+1) + c$

$= x^2 + 2x + 1 + bx + b + c$

$b + 2 = 0$

$b = -2$

$b + c + 1 = 0$

$c = 1$

2 D $x^3 = a(x+2)^3 + b(x+2)^2 + c(x+2) + d$

$= ax^3 + 6ax^2 + 12ax + 8a + bx^2 + 4bx + 4b + cx + 2c + d$

$a = 1$

$b + 6a = 0$

$b = -6$

$12a + 4b + c = 0$

$c = 12$

$8a + 4b + 2c + d = 0$

$d = -8$

3 D $a = 3,\ b = -6,\ c = 3$

$$x = \frac{6 \pm \sqrt{36 - 4 \times 3 \times 3}}{2 \times 3}$$

$$= \frac{6 \pm \sqrt{0}}{6}$$

$= 1$

4 C $(x-4)(x+6) = 0$

$x^2 + 2x - 24 = 0$

$x^2 + 2x = 24$

$2x^2 + 4x = 48$

5 C $$\frac{7x^2 + 13}{(x-1)(x^2 + x + 2)}$$

$$= \frac{a}{x-1} + \frac{bx + c}{x^2 + x + 2}$$

$$= \frac{a(x^2 + x + 2) + (bx + c)(x-1)}{(x-1)(x^2 + x + 2)}$$

$$= \frac{ax^2 + ax + 2a + bx^2 - bx + cx - c}{(x-1)(x^2 + x + 2)}$$

$a + b = 7$ ①

$a - b + c = 0$ ②

$2a - c = 13$ ③

② + ③:

$3a - b = 13$ ④

① + ④:

$4a = 20$

$a = 5$

$5 + b = 7$

$b = 2$

$a - b + c = 0$

$C = -3$

6 D $$\frac{4x - 3}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

$$= \frac{A(x-3) + B}{(x-3)^2}$$

$$= \frac{Ax - 3A + B}{(x-3)^2}$$

$A = 4$

$-3 \times 4 + B = -3$

$B = 9$

$$\therefore \frac{4x - 3}{(x-3)^2} = \frac{4}{x-3} + \frac{9}{(x-3)^2}$$

7 B $2x^2 + 5x + 2 = (2x+1)(x+2)$

$$\frac{8x+7}{(2x+1)(x+2)}$$

$$= \frac{A}{2x+1} + \frac{B}{x+2}$$

$$= \frac{A(x+2)+B(2x+1)}{(2x+1)(x+2)}$$

$$= \frac{Ax+2Bx+2A+B}{(2x+1)(x+2)}$$

$A + 2B = 8$

$2A + 4B = 16$ ①

$2A + B = 7$ ②

① − ②:

$3B = 9$

$B = 3$

$A + 2B = 8$

$A = 2$

$$\therefore \quad \frac{8x+7}{(2x+1)(x+2)} = \frac{2}{2x+1} + \frac{3}{x+2}$$

8 B $\dfrac{-3x^2+2x-1}{(x^2+1)(x+1)}$

$$= \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$$= \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$$

$$= \frac{Ax^2+Ax+Bx+B+Cx^2+C}{(x^2+1)(x+1)}$$

$A + C = -3$ ①

$A + B = 2$ ②

$B + C = -1$ ③

① − ②:

$C - B = -5$

$2C = -6$

$C = -3$

$A + -3 = -3$

$A = 0$

$0 + B = 2$

$B = 2$

$$\therefore \quad \frac{-3x+2x+5}{(x^2+1)(x+1)} = \frac{2}{x^2+1} - \frac{3}{x+1}$$

9 C Let $y = 2k - x$ in $x^2 + y^2 = k$ to give

$$x^2 + (2k-x)^2 = k$$

$$x^2 + 4k^2 - 4kx + x^2 = k$$

$2x^2 - 4kx + (4k^2 - k) = 0.$

If the line touches the circle then this equation has one solution. This occurs if and only if $\Delta = 0$. That is,

$\Delta = b^2 - 4ac$

$= (-4k)^2 - 4(2)(4k^2 - k)$

$= 8k(1 - 2k)$

If $k > 0$ and $8k(1 - 2k) = 0$, then $k = \frac{1}{2}$.

10 B Let $y = bx - 1$ in $y = x^2 + x$ to give

$$x^2 + x = bx - 1$$

$x^2 + (1 - b)x + 1 = 0$

This has one solution if and only if $\Delta = 0$. That is,

$\Delta = b^2 - 4ac$

$= (1 - b)^2 - 4(1)(1)$

$= b^2 - 2b - 3$

$= (b - 3)(b + 1)$

If $b > 0$ and $(b - 3)(b + 1) = 0$, then $b = 3$.

11 B If this is true for all values of x then set $x = 0$. This gives

$$5c = -10.$$

Therefore $c = 2$. Now expand the left-hand side to give

$$(bx + 2)(2x - 5) = 12x^2 + kx - 10$$

$$2bx^2 + (4 - 5b)x - 10 = 12x^2 + kx - 10$$

Therefore $2b = 12$, so $b = 6$. Also,

$$k = 4 - 5b = 4 - 5(6) = -26.$$

Solutions to extended-response questions

1 **a** If $x^2 + bx + c = 0$ and $x = 2 - \sqrt{3}$

then $(2 - \sqrt{3})^2 + b(2 - \sqrt{3}) + c = 0$

$\therefore \quad 4 - 4\sqrt{3} + 3 + 2b - \sqrt{3}b + c = 0$

$\therefore \quad (7 + 2b + c) + (-4 - b)\sqrt{3} = 0$

$\therefore \quad 7 + 2b + c = 0 \quad \text{and} \quad -4 - b = 0$

$\therefore \quad 7 + 2(-4) + c = 0 \quad b = -4$

$\therefore \quad 7 - 8 + c = 0$

$\therefore \quad -1 + c = 0$

$\therefore \quad c = 1$

b $x^2 - 4x + 1 = 0$

Using the same procedure as in **3 c**, $x = 2 \pm \sqrt{3}$.

Hence $2 + \sqrt{3}$ is the other solution.

c **i** If $x^2 + bx + c = 0$ and $x = m - n\sqrt{q}$

then $(m - n\sqrt{q})^2 + b(m - n\sqrt{q}) + c = 0$

$\therefore \quad m^2 - 2mn\sqrt{q} + n^2q + bm - bn\sqrt{q} + c = 0$

$\therefore \quad (m^2 + n^2q + bm + c) + (-2mn - bn)\sqrt{q} = 0$

$\therefore \quad m^2 + n^2q + bm + c = 0 \text{ and } -2mn - bn = 0$

$-2mn = bn$

$-2m = b$

ii $m^2 + n^2q + (-2m)m + c = 0$

$\therefore \quad m^2 + n^2q + -2m^2 + c = 0$

$\therefore \quad n^2q - m^2 + c = 0$

$\therefore \quad c = m^2 - n^2q$

iii If $x^2 + bx + c = 0$, the general quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \text{ (as } a = 1)$$

Given $b = -2m$ and $c = m^2 - n^2q$

$$x = \frac{2m \pm \sqrt{4m^2 - 4(m^2 - n^2q)}}{2}$$

$$= \frac{2m \pm \sqrt{4m^2 - 4m^2 + 4n^2q}}{2}$$

$$= \frac{2m \pm 2n\sqrt{q}}{2}$$

$$= m \pm n\sqrt{q}$$

$$\therefore \quad x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))$$

or, by completing the square,

$$x^2 - 2mx + m^2 - n^2q = x^2 - 2mx + m^2 + m^2 - n^2q - m^2$$

$$= (x - m)^2 - (n\sqrt{q})^2$$

$$= (x - m - n\sqrt{q})(x - m + n\sqrt{q})$$

2 a Let V km/h be the initial speed.

$V - 4$ is the new speed.

It takes 2 more hours to travel at the new speed,

$$\therefore \quad \frac{240}{V} + 2 = \frac{240}{V - 4} \qquad \ldots \boxed{1}$$

$$\therefore \quad 240(V - 4) + 2V(V - 4) = 240V$$

$$\therefore \quad 240V - 960 + 2V^2 - 8V = 240V$$

$$\therefore \quad 2V^2 - 8V - 960 = 0$$

$$\therefore \quad V^2 - 4V - 480 = 0$$

$$\therefore \quad (V - 24)(V + 20) = 0$$

$$\therefore \quad V = 24 \text{ or } V = -20$$

Actual speed is 24 km/h.

b If it travels at $V - a$ km/h and takes 2 more hours, equation $\boxed{1}$ from **a** becomes

$$\frac{240}{V} + 2 = \frac{240}{V - a}$$

$$\therefore \quad 240(V - a) + 2V(V - a) = 240V$$

$$\therefore \quad 240V - 240a + 2V^2 - 2Va = 240V$$

$$\therefore \quad 2V^2 - 2aV - 240a = 0$$

$$\therefore \quad V^2 - aV - 120a = 0$$

Using the general quadratic formula,

$$V = \frac{a + \sqrt{a^2 + 480a}}{2}$$

When $a = 60$, $V = 120$, i.e. the speed is 120 km/h, a fairly fast speed. So if speed is less than this, practical values are $0 < a < 60$ and then $0 < V < 120$.

c If it travels at $V - a$ km/h and takes a more hours, equation $\boxed{1}$ from **a** becomes

$$\frac{240}{V} + a = \frac{240}{V - a}$$

$$\begin{aligned} \therefore \quad 240(V - a) + aV(V - a) &= 240V \\ \therefore \quad 240V - 240a + aV^2 - a^2V &= 240V \\ \therefore \quad aV^2 - a^2V - 240a &= 0 \\ \therefore \quad V^2 - aV - 240 &= 0 \end{aligned}$$

Using the general quadratic formula,

$$V = \frac{a + \sqrt{a^2 + 960}}{2}$$

The only pairs of integers for a and V are found in the table below.

a	1	8	14	22	34	43	56	77	118
V	16	20	24	30	40	48	60	80	120

3 A table is a useful way to display the speed, time taken and distance covered for each train.

	distance (km)	time (h)	speed (km/h)
Faster train	b	$\frac{b}{v}$	v
Slower train	b	$\frac{b}{v} + a$	$b \div \left(\frac{b}{v} + a\right) = \frac{bv}{b + av}$

a In c hours, the faster train travels a distance of cv km.

In c hours, the slower train travels a distance of $\frac{bcv}{b + av}$ km.

Since the slower train travels 1 km less than the faster one in c hours,

$$cv - 1 = \frac{bcv}{b + av}$$

$$\begin{aligned} \therefore (cv - 1)(b + av) &= bcv \\ \therefore bcv + acv^2 - b - av &= bcv \\ \therefore acv^2 - av - b &= 0 \end{aligned}$$

Using the general quadratic formula,

$$\begin{aligned} v &= \frac{a \pm \sqrt{a^2 + 4abc}}{2ac} \\ &= \frac{a + \sqrt{a^2 + 4abc}}{2ac} \quad \text{since } v > 0 \end{aligned}$$

Therefore the speed of the faster train is $\frac{a + \sqrt{a^2 + 4abc}}{2ac}$ km/h.

b If the speed of the faster train is a rational number, then $a^2 + 4abc$ must be a square number.

Set 1

If $a = 1$,

then $a^2 + 4abc = 1 + 4bc$

e.g. $a = 1,\ b = 1,\ c = 2$

in which case $v = \dfrac{a + \sqrt{a^2 + 4abc}}{2ac}$

becomes $v = \dfrac{1 + \sqrt{1^2 + 4 \times 1 \times 1 \times 2}}{2 \times 1 \times 2}$

$= \dfrac{1 + \sqrt{9}}{4}$

$= 1$ km/h

Set 2

If $a = 1$ and $b = 100$,

then $a^2 + 4abc = 1 + 400c$

Choose $c = \dfrac{11}{10}$

then $a^2 + 4ac = 1 + 400 \times \dfrac{11}{10}$

$= 441$

$= 21^2$

When $a = 1,\ b = 100$ and $c = \dfrac{11}{10}$,

$$v = \frac{a + \sqrt{a^2 + 4abc}}{2ac}$$

becomes $v = \dfrac{1 + 21}{2 \times 1 \times \frac{11}{10}}$

$= \dfrac{22 \times 10}{22}$

$= 10$ km/h

Set 3

If $a = \dfrac{1}{2},\ b = 15,\ c = 1$

$$\text{then } v = \frac{a + \sqrt{a^2 + 4abc}}{2ac}$$

$$\text{becomes } v = \frac{\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 4 \times \frac{1}{2} \times 15 \times 1}}{2 \times \frac{1}{2} \times 1}$$

$$= \frac{\frac{1}{2} + \sqrt{\frac{121}{4}}}{1}$$

$$= \frac{1}{2} + \frac{11}{2}$$

$$= 6 \text{ km/h}$$

Set 4

If $a = \frac{1}{4}$,

then $a^2 + 4abc = \frac{1}{16} + bc$

e.g. $a = 1,\ b = 5,\ c = 1$

$$\text{in which case } v = \frac{a + \sqrt{a^2 + 4abc}}{2ac}$$

$$\text{becomes } v = \frac{\frac{1}{4} + \sqrt{\left(\frac{1}{4}\right)^2 + 4 \times \frac{1}{4} \times 5 \times 1}}{2 \times 1 \times 1}$$

$$= \frac{\frac{1}{4} + \sqrt{\frac{81}{16}}}{2}$$

$$= \frac{5}{4} \text{ km/h}$$

Set 5

If $a = 1$ and $b = 1$,

then $a^2 + 4abc = 1 + 4c$

Choose $c = 6$

then $a^2 + 4ac = 1 + 4 \times 6$

$= 25$

$= 5^2$

When $a = 1,\ b = 1$ and $c = 6$,

$$v = \frac{a + \sqrt{a^2 + 4abc}}{2ac}$$

becomes $$v = \frac{1 + \sqrt{1^2 + 4 \times 1 \times 1 \times 6}}{2 \times 1 \times 6}$$

$$= \frac{1+5}{12}$$

$$= \frac{1}{2} \text{ km/h}$$

4 a

	Volume	Time	Rate
Large pipe	1	T_L	r_L
Small pipe	1	T_S	r_S
Both pipes	1	T_B	$r_L + r_S$

T_L is the time for the large pipe to fill the tank
T_S is the time for the small pipe to fill the tank
T_B is the time for both pipes to fill the tank
where it is assumed without loss of generality that the volume of the tank is 1 unit.
Given

$$T_S = T_L + a \qquad \ldots \boxed{1}$$

$$T_S = T_B + b \qquad \ldots \boxed{2}$$

Note that $r_B = r_S + r_L$.

$$\begin{aligned} T_B &= \frac{1}{r_B} \\ &= \frac{1}{r_S + r_L} \\ &= \frac{1}{\frac{1}{T_S} + \frac{1}{T_L}} \\ &= \frac{T_S T_L}{T_S + T_L} \end{aligned}$$

From $\boxed{1}$ and $\boxed{2}$

$$\begin{aligned} T_L + a &= T_B + b \\ &= \frac{T_S T_L}{T_S + T_L} + b \end{aligned}$$

$\therefore T_L(T_L + T_S) + a(T_L + T_S) = T_S T_L + b(T_L + T_S)$

$\therefore T_L(2T_L + a) + a(2T_L + a) = T_L(T_L + a) + b(2T_L + a)$

$\therefore 2T_L^2 + aT_L + 2aT_L + a^2 = T_L^2 + aT_L + 2bT_L + ba$

$\therefore T_L^2 + 2(a - b)T_L + a^2 - ba = 0$

$$\therefore \quad T_L = \frac{2(b-a) + \sqrt{4(a^2 - 2ab + b^2) - 4(a^2 - ba)}}{2} \text{ since } T_L > 0$$

$$= \frac{2(b-a) + \sqrt{4a^2 - 8ab + 4b^2 - 4a^2 + 4ba}}{2}$$

$$= b - a + \sqrt{-ab + b^2}$$

Also from $\boxed{1}$ $T_S = T_L + a$

$$= b - a + \sqrt{b^2 - ab} + a$$

$$= b + \sqrt{b^2 - ab}$$

b If $a = 24$ and $b = 32$,

$T_S = 32 + \sqrt{32^2 - 32 \times 24}$

$= 48$

$T_L = T_S - a$

$= 48 - 24$

$= 24$

c $b^2 - ab$ is a perfect square, and $T_S = b + \sqrt{b^2 - ab}$.

Let $b = a + 1$. Then $T_S = a + 1 + \sqrt{(a+1)^2 - a(a+1)}$

$$= a + 1 + \sqrt{a^2 + 2a + 1 - a^2 - a}$$

$$= a + 1 + \sqrt{a + 1}$$

Note: This means b must be a perfect square.

a	3	8	15	24	35
b	4	9	16	25	36
T_S	8	18	32	50	72
T_L	5	10	17	26	37

5 **a** $k(1 - 2x) = x^2 + 2$

$k - 2kx = x^2 + 2$

$x^2 + 2kx + 2 - k = 0$

Consider discriminant

$\Delta = 0 \Rightarrow 4k^2 - 4(2 - k) = 0$

$4k^2 + 4k - 8 = 0$

$k^2 + k - 2 = 0$

$(k + 2)(k - 1) = 0$
Therefore, $k = -2$ or $k = 1$

b $x^2 + (2x + c)^2 = 20$

$x^2 + 4x^2 + 4xc + c^2 - 20 = 0$

$5x^2 + 4xc + c^2 - 20 = 0$

Consider discriminant

$\Delta > 0 \Rightarrow 16c^2 - 20(c^2 - 20) > 0$

$4c^2 - 5(c^2 - 20) > 0$

$-c^2 + 100 > 0$

$(10 - c)(10 + c) > 0$
Therefore, $-10 < c < 10$

c $x^2 + (1 - p)x + 2p = 6$

$x^2 + (1 - p)x + 2p - 6 = 0$

Consider discriminant

$\Delta = 0 \Rightarrow (1 - p)^2 - 4(2p - 6) = 0$

$1 - 2p + p^2 - 8p + 24 = 0$

$p^2 - 10p + 25 = 0$

$(p - 5)^2 = 0$
Therefore, $p = 5$

6 We first find that

$$\begin{aligned} (x - \alpha)(x - \beta) &= x^2 - px + 3 \\ x^2 - (\alpha + \beta)x + \alpha\beta &= x^2 - px + 3 \\ \therefore \alpha + \beta &= p \\ \alpha\beta &= 3. \end{aligned}$$

a **i** $(\alpha - 2p) + (\beta - 2p) = (\alpha + \beta) - 4p$

$$= p - 4p$$

$$= -3p$$

ii $(\alpha - 2p)(\beta - 2p) = (\alpha\beta - 2p(\alpha + \beta) + 4p^2$

$$= 3 - 2p^2 + 4p^2$$

$$= 3 + 2p^2$$

b $(x - (\alpha - 2p))(x - (\beta - 2p)) = x^2 - x(\alpha - 2p + \beta - 2p) + (\alpha - 2p)(\beta - 2p)$

$$= x^2 + 3px + 3 + 2p^2$$

Consider

$$x^2 + mx + n = x^2 + 3px + 3 + 2p^2$$

$$\Rightarrow m = -3p, n = 3 + 2p^2$$

7 $y = \dfrac{p}{q}x - \dfrac{1}{q}$

a
$$\frac{p}{q}x - \frac{1}{q} = ax^2$$

$$ax^2 - \frac{p}{q}x - \frac{1}{q} + \frac{1}{q} = 0 \ldots (1)$$

Consider discriminant

$$\left(\frac{p}{q}\right)^2 - 4a \times \frac{1}{q} = 0$$

$$4a \times \frac{1}{q} = \left(\frac{p}{q}\right)^2$$

$$a = \frac{p^2}{4q}$$

b When $a = \dfrac{p^2}{4q}$ equation (1) becomes

$\dfrac{p^2}{4q}x^2 - \dfrac{p}{q}x - \dfrac{1}{q} + \dfrac{1}{q} = 0)$

$p^2x^2 - 4pv + 4 = 0$

$(px - 2)^2 = 0$

$x = \dfrac{2}{p}$

And $y = \dfrac{p}{q} \times \dfrac{2}{p} - \dfrac{1}{q} = \dfrac{1}{q}$

c $X\left(\frac{1}{p}, 0\right)$ and $Y\left(0, -\frac{1}{q}\right)$

$$PX^2 = \left(\frac{1}{p} - \frac{2}{p}\right)^2 + \left(\frac{1}{q} + \frac{1}{q}\right)^2$$

$$= \left(\frac{1}{p}\right)^2 + \left(\frac{1}{q}\right)^2$$

$$= \frac{p^2 + q^2}{p^2q^2}$$

Also $XY^2 = \left(\frac{1}{p}\right)^2 + \left(\frac{1}{q}\right)^2 = \frac{p^2 + q^2}{p^2q^2}$

Hence $PX^2 = XY^2 = \frac{p^2 + q^2}{p^2q^2}$

d **i** $x - y = 1 \Leftrightarrow p = 1$ and $q = 1$

Therefore $a = \frac{1}{4}$ and the equation of the parabola is:

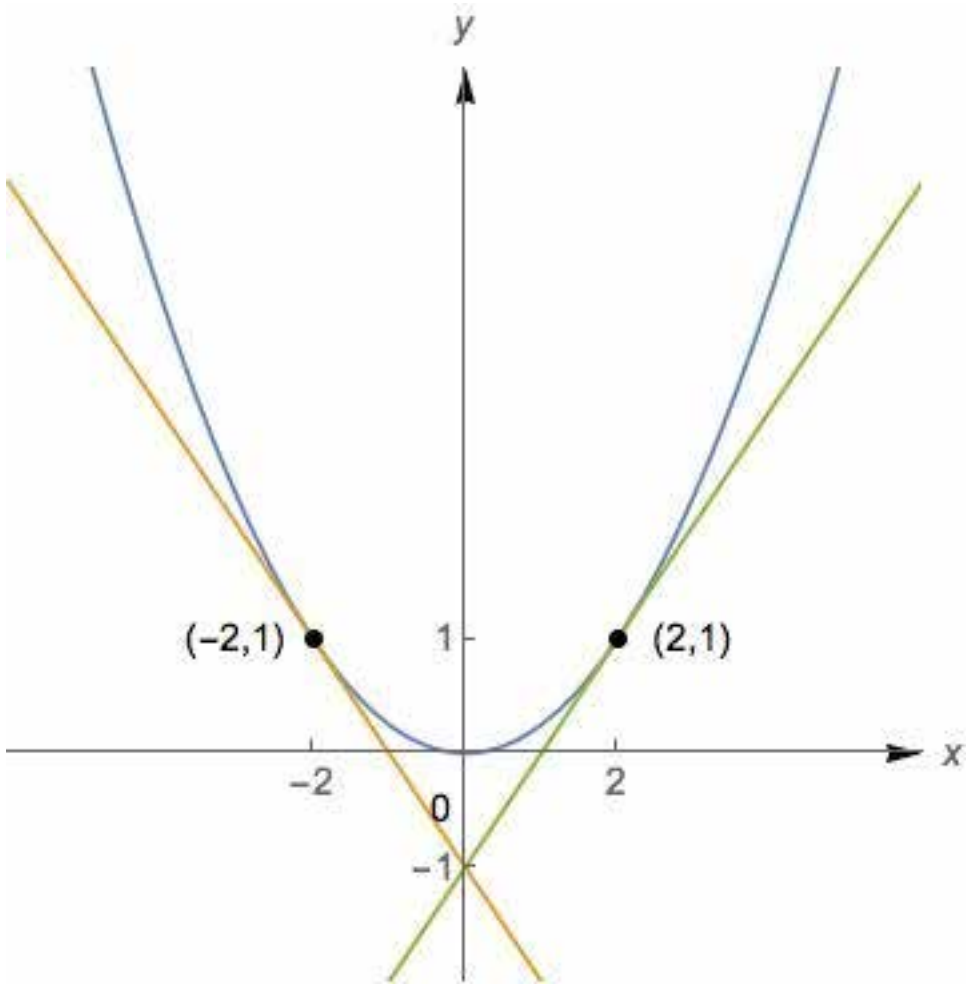
$$y = \frac{1}{4}x^2$$

ii Touches the parabola at $\left(\frac{2}{p}, \frac{1}{q}\right) = (2, 1)$

iii $PX^2 = \frac{p^2 + q^2}{p^2q^2} = \frac{1 + 1}{1} = 2$

Therefore $PX = \sqrt{2}$

iv In this case $p = -1, q = 1$. Therefore touches at $(-2, 1)$



8 $y = -\dfrac{p}{q}x + \dfrac{1}{q}$

a
$$x^2 + \left(\frac{p}{q}x - \frac{1}{q}\right)^2 = a^2$$
$$x^2 + \frac{1}{q^2}(p^2x^2 - 2px + 1) = a^2$$
$$q^2x^2 + p^2x^2 - 2px + 1 = a^2q^2$$
$$(p^2 + q^2)x^2 - 2px + 1 - a^2q^2 = 0$$

Consider discriminant
$$\Delta = 4p^2 - 4(p^2 + q^2)(1 - a^2q^2) = 0$$
$$4p^2 = 4(p^2 + q^2)(1 - a^2q^2)$$
$$\frac{4p^2}{4(p^2 + q^2)} = 1 - a^2q^2$$
$$\frac{p^2}{p^2 + q^2} = 1 - a^2q^2$$
$$a^2q^2 = 1 - \frac{p^2}{p^2 + q^2}$$
$$a^2q^2 = \frac{q^2}{p^2 + q^2}$$
$$a^2 = \frac{1}{p^2 + q^2}$$

b Consider the equation:
$$q^2x^2 + p^2x^2 - 2px + 1 = \frac{q^2}{p^2 + q^2}$$
$$(p^2 + q^2)x^2 - 2px + 1 - \frac{q^2}{p^2 + q^2} = 0$$
$$(p^2 + q^2)x^2 - 2px + \frac{p^2}{p^2 + q^2} = 0$$
$$(p^2 + q^2)^2x^2 - 2p(p^2 + q^2)x + p^2 = 0$$
$$((p^2 + q^2)x - p)^2 = 0$$

Therefore $x = \dfrac{p}{p^2 + q^2}$

and since $y = -\dfrac{p}{q}x + \dfrac{1}{q}$

$\therefore y = \dfrac{p}{q} \times \dfrac{-p}{p^2 + q^2} + \dfrac{1}{q}$

$= \dfrac{1}{q} \times \dfrac{-p^2 + (p^2 + q^2)}{p^2 + q^2}$

$= \dfrac{q}{p^2 + q^2}$

Cooordinates are $\left(\frac{p}{p^2+q^2}, \frac{q}{p^2+q^2}\right)$

c **i** Since $p = q = 1$ we find that $a^2 = \frac{1}{p^2+q^2} = \frac{1}{2}$.

ii The cooordinates are

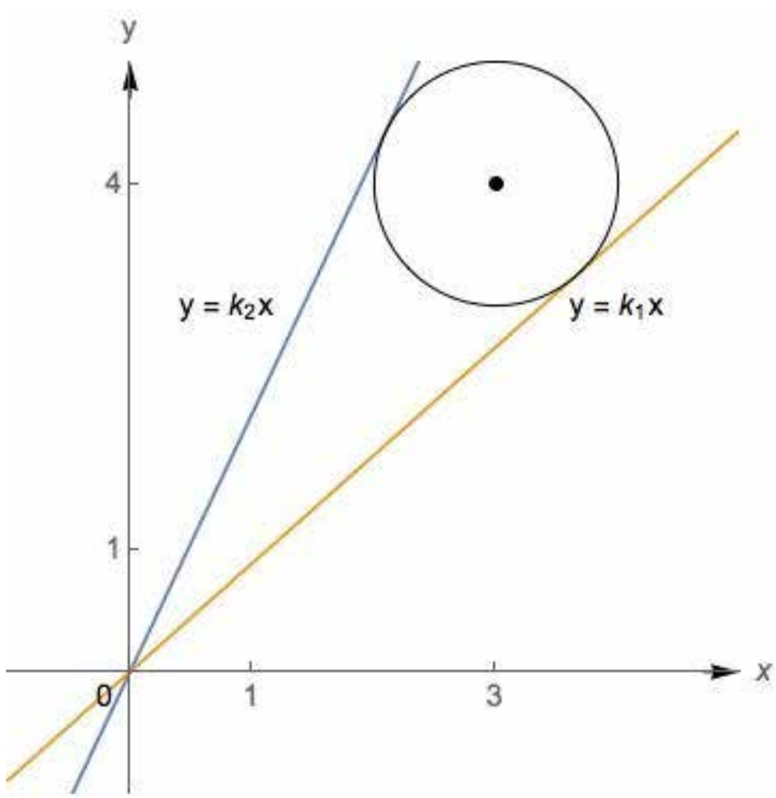
$$\left(\frac{p}{p^2+q^2}, \frac{q}{p^2+q^2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

iii By symmetry the other point must be $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

9 **a** **i** $x^2 - 6x + y^2 - 8y + 24 = 0$ Centre $(3, 4)$ and radius 1.

$$x^2 - 6x + 9 + y^2 - 8y + 16 = 1$$
$$(x-3)^2 + (y-4)^2 = 1$$

ii



iii
$$x^2 - 6x + (mx)^2 - 8mx + 24 = 0$$
$$(m^2 + 1)x^2 - (6 + 8m)x + 24 = 0$$
Consider discriminant
$$\Delta = (6 + 8m)^2 - 4 \times 24 \times (m^2 + 1) = 0$$
$$36 + 96m + 64m^2 - 96(m^2 + 1) = 0$$
$$36 + 96m + 64m^2 - 96m^2 - 96 = 0$$
$$-32m^2 + 96m - 60 = 0$$
$$8m^2 - 24m + 15 = 0$$
$$m = \frac{24 \pm \sqrt{576 - 480}}{16}$$
$$= \frac{24 \pm \sqrt{96}}{16}$$
$$= \frac{24 \pm 4\sqrt{6}}{16}$$
$$= \frac{6 \pm \sqrt{6}}{4}$$
$$m_1 = \frac{6 - \sqrt{6}}{4}, m_2 = \frac{6 + \sqrt{6}}{4}$$

iv Use CAS (a little messy by hand).
$$\left(\frac{72 - 8\sqrt{6}}{25}, \frac{96 + 6\sqrt{6}}{25}\right),$$
$$\left(\frac{72 + 8\sqrt{6}}{25}, \frac{96 - 6\sqrt{6}}{25}\right)$$

b Equation of circle $(x - 3)^2 + (y - 4)^2 = a^2$

i
$$(x - 3)^2 + (2x - 4)^2 = a^2$$
$$x^2 - 6x + 9 + 4x^2 - 16x + 16 - a^2 = 0$$
$$5x^2 - 22x + 25 - a^2 = 0 \ldots (1)$$
Consider discriminant
$$\Delta = 484 - 20(25 - a^2) = 0$$
$$484 - 500 + 20a^2 = 0$$
$$20a^2 = 16$$
$$a^2 = \frac{4}{5}$$

The equation of the circle is $(x-3)^2+(y-4)^2=\frac{4}{5}$

ii Use equation (1) with $a^2=\frac{4}{5}$

$$5x^2-22x+25-\frac{4}{5}=0$$
$$25x^2-110x+125-4=0$$
$$25x^2-110x+121=0$$
$$(5x-11)^2=0$$
$$x=\frac{11}{5}$$

Therefore coordinates are $\left(\frac{11}{5},\frac{22}{5}\right)$

iii $(x-3)^2+(y-4)^2=\frac{4}{5}$

$y=m_3x$

Consider,

$$(x-3)^2+(m_3x-4)^2=\frac{4}{5}$$
$$x^2-6x+9+m_3^2x^2-8m_3x+16=\frac{4}{5}$$
$$(1+m_3^2)x^2-(6+8m_3)x+25=\frac{4}{5}$$
$$(1+m_3^2)x^2-(6+8m_3)x+\frac{121}{5}=0$$

Discriminant = 0

$$(6+8m_3)^2-4\times\frac{121}{5}\times(1+m_3^2)=0$$
$$(3+4m_3)^2-\frac{121}{5}\times(1+m_3^2)=0$$
$$9+24m_3+16m_3^2-\frac{121}{5}\times(1+m_3^2)=0$$
$$45+120m_3+80m_3^2-121-121m_3^2=0$$
$$-76+120m_3-41m_3^2=0$$
$$41m_3^2-120m+76=0$$
$$(41m_3-38)(m_3-2)=0$$
$$m=2 \text{ or } m=\frac{38}{41}$$

c **i** Circle has equation $(x-h)^2+(y-k)^2=1$

Consider each of the two lines touching the circle to obtain two equations to determine h and k. First consider intersection with $y=2x$

$$(x-h)^2 + (2x-k)^2 = 1 \ldots (1)$$

$$x^2 - 2xh + h^2 + 4x^2 - 4kx + k^2 - 1 = 0$$

$$5x^2 - (2h+4k)x + (h^2+k^2-1) = 0$$

Consider discriminant

$$\Delta = (2h+4k)^2 - 20(h^2+k^2-1) = 0$$

$$(h+2k)^2 - 5h^2 - 5k^2 + 5 = 0$$

$$h^2 + 4hk + 4k^2 - 5h^2 - 5k^2 + 5 = 0$$

$$-4h^2 - k^2 + 4hk + 5 = 0 \ldots (1')$$

Next consider intersection with $y = \frac{1}{2}x$

$$(x-h)^2 + (\frac{1}{2}x - k)^2 = 1 \ldots (2)$$

$$x^2 - 2xh + h^2 + \frac{1}{4}x^2 - kx + k^2 - 1 = 0$$

$$\frac{5x^2}{4} - (2h+k)x + (h^2+k^2-1) = 0$$

Consider discriminant

$$\Delta = (2h+k)^2 - 5(h^2+k^2-1) = 0$$

$$4h^2 + 4hk + k^2 - 5h^2 - 5k^2 + 5 = 0$$

$$-h^2 - 4k^2 + 4hk + 5 = 0 \ldots (2')$$

Subtract $(2')$ from $(1')$

$-3h^2 + 3k^2 = 0$

$h = k$and $h = -k$ is not possible since the straight lines pass through the first and third quadrants

Substitute in $(1')$

$-5k^2 + 4k^2 + 5 = 0$

$-k^2 + 5 = 0$ $k = \pm\sqrt{5}$

That is, $h = \sqrt{5}$ and $k = \sqrt{5}$ or

$h = -\sqrt{5}$ and $k = -\sqrt{5}$

ii Assume $h = k = \sqrt{5}$

For the touch point of $y = 2x$

The equation $5x^2 - (2h+4k)x + (h^2+k^2-1) = 0$ becomes

$5x^2 - 6\sqrt{5}x + 9 = 0$

$(\sqrt{5}x - 3)^2 = 0$

$x = \frac{3\sqrt{5}}{5}$ and $y = \frac{6\sqrt{5}}{5}$

For the touch point of $y = \frac{1}{2}x$

$$\frac{5x^2}{4} - (2h + k)x + (h^2 + k^2 - 1) = 0 \text{ becomes } \frac{5x^2}{4} - 3\sqrt{5}x + 9 = 0$$

$$\left(\frac{\sqrt{5}x}{2} - 3\right)^2 = 0$$

$$x = \frac{6\sqrt{5}}{5} \text{ and } y = \frac{3\sqrt{5}}{5}$$

The solution with $h = -\sqrt{5}$ and $k = -\sqrt{5}$ follows the same path. The coefficient of x in each of the equations is positive and so you get the symmetric results.

Chapter 5 – Revision of Chapters 1–4

Solutions to technology-free questions

1 **a** $2002 = 2 \times 1001$
$= 2 \times 7 \times 143$
$= 2 \times 7 \times 11 \times 13$

b $555 = 5 \times 111$
$= 5 \times 3 \times 37$

c $7007 = 7 \times 1001$
$= 7 \times 7 \times 143$
$= 7 \times 7 \times 11 \times 13$
$= 7^2 \times 11 \times 13$

d $10\,000 = 10^4$
$= 2^4 \times 5^4$

2 $$\frac{5m - 2p}{4m^2 + mp - 3p^2} - \frac{1}{4m - 3p}$$
$$= \frac{5m - 2p}{(m + p)(4m - 3p)} - \frac{1}{4m - 3p}$$
$$= \frac{1}{4m - 3p}\left(\frac{5m - 2p}{m + p} - 1\right)$$
$$= \frac{1}{4m - 3p}\left(\frac{5m - 2p - (m + p)}{m + p}\right)$$
$$= \frac{1}{4m - 3p}\left(\frac{4m - 3p}{m + p}\right)$$
$$= \frac{1}{m + p}$$

3 **a** $(\sqrt{3} + \sqrt{2})(\sqrt{3} - 1)$
$= \sqrt{3}(\sqrt{3} - 1)) + \sqrt{2}(\sqrt{3} - 1)$
$= 3 - \sqrt{3} + \sqrt{6} - \sqrt{2}$
$= \sqrt{6} - \sqrt{3} - \sqrt{2} + 3$

b
$(5\sqrt{3} - \sqrt{6})(2\sqrt{6} + 3\sqrt{3})$
$= 5\sqrt{3}(2\sqrt{6} + 3\sqrt{3}) - \sqrt{6}(2\sqrt{6} + 3\sqrt{3})$
$= 30\sqrt{2} + 45 - 12 - 9\sqrt{2}$
$= 21\sqrt{2} + 33$

c $(2\sqrt{x} - 3)^2 = 4x - 12\sqrt{x} + 9$

d $(\sqrt{x - 2} - 3)^2 = x - 2 - 6\sqrt{x - 2} + 9$
$= x + 7 - 6\sqrt{x - 2}$

4 **a** $$\frac{1}{\sqrt{2} - 3} = \frac{1}{\sqrt{2} - 3} \times \frac{\sqrt{2} + 3}{\sqrt{2} + 3}$$
$$= -\frac{\sqrt{2} + 3}{7}$$

b $$\frac{3}{\sqrt{5} - 1} = \frac{3}{\sqrt{5} - 1} \times \frac{\sqrt{5} + 1}{\sqrt{5} + 1}$$
$$= \frac{3(\sqrt{5} + 1)}{4}$$

c $$\frac{2}{2\sqrt{2} - 1} = \frac{2}{2\sqrt{2} - 1} \times \frac{2\sqrt{2} + 1}{2\sqrt{2} + 1}$$
$$= \frac{4\sqrt{2} + 2}{7}$$

d $$\frac{3}{\sqrt{5} - \sqrt{3}} = \frac{3}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}}$$
$$= \frac{3(\sqrt{5} + \sqrt{3})}{2}$$

e $$\frac{1}{\sqrt{7} - \sqrt{2}} = \frac{1}{\sqrt{7} - \sqrt{2}} \times \frac{\sqrt{7} + \sqrt{2}}{\sqrt{7} + \sqrt{2}}$$
$$= \frac{\sqrt{7} + \sqrt{2}}{5}$$

f

$$\frac{1}{2\sqrt{5}-\sqrt{3}} = \frac{1}{2\sqrt{5}-\sqrt{3}} \times \frac{2\sqrt{5}+\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$$
$$= \frac{2\sqrt{5}+\sqrt{3}}{17}$$

5 **a** $-1 < -\frac{1}{a} < 1$ and $a \neq 0$

$\Leftrightarrow -1 < \frac{1}{a} < 1$and $a \neq 0$

$\Leftrightarrow |a| > 1$

$$S_\infty = \frac{a^4}{1+\frac{1}{a}}$$
$$= \frac{a^5}{a+1}$$

b $-1 < -\frac{b}{a} < 1$ and $a \neq 0$

$\Leftrightarrow -1 < \frac{b}{a} < 1$

$$S_\infty = \frac{\frac{1}{a}}{1+\frac{b}{a}}$$
$$= \frac{1}{a+b}$$

c $-1 < -\frac{x}{2x+1} < 1$ and $x \neq -\frac{1}{2}$

$\Leftrightarrow -1 < \frac{x}{2x+1} < 1$and $a \neq -\frac{1}{2}$

Case 1 : $x > 0$

$x < 2x+1$

$\Leftrightarrow x > -1$

Case 2 $-\frac{1}{2} < x < 0$

$-x < 2x+1$

$\Leftrightarrow x > -\frac{1}{3}$

Case 3 $x < -\frac{1}{2}$

$-x < -2x-1$

$\Leftrightarrow x < -1$

Therefore $x < -1$ or $x > -\frac{1}{3}$

$$S_\infty = \frac{\frac{2x+1}{x}}{1+\frac{x}{2x+1}}$$
$$= \frac{(2x+1)^2}{x(3x+1)}$$

d $-1 < -\frac{1}{4x-2} < 1$ and $x \neq \frac{1}{2}$

$\Leftrightarrow x > \frac{3}{4}$ or $x < \frac{1}{4}$

$$S_\infty = \frac{1}{1+\frac{1}{4x-2}}$$
$$= \frac{4x-2}{4x-1}$$

6 **a** $x^2 + x - 1 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1+4}}{2}$

$x^2 + bx + 1 = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2-4}}{2}$

Case1

$$-b + \sqrt{b^2-4} = -1 + \sqrt{5}$$
$$1 - b = \sqrt{5} - \sqrt{b^2-4}$$
$$1 - 2b + b^2 = 5 - 2\sqrt{5(b^2-4)} + b^2 - 4$$
$$-2b = -2\sqrt{5(b^2-4)}$$
$$b^2 = 5(b^2-4)$$
$$20 = 4b^2$$
$$b = \pm\sqrt{5}$$

Case2

$$-b + \sqrt{b^2-4} = -1 - \sqrt{5}$$
$$1 - b = -\sqrt{5} - \sqrt{b^2-4}$$
$$1 - 2b + b^2 = 5 + 2\sqrt{5(b^2-4)} + b^2 - 4$$
$$-2b = +2\sqrt{5(b^2-4)}$$
$$b^2 = 5(b^2-4)$$
$$20 = 4b^2$$
$$b = \pm\sqrt{5}$$

Comment: Because of squaring

solutions should be checked

b **i** If $b = \sqrt{5}$, solutions of $x^2 + bx + 1 = 0$ are:

$$x = \frac{-\sqrt{5} \pm \sqrt{5-4}}{2}$$

$$x = \frac{-\sqrt{5} \pm 1}{2}$$

The common solution is $\dfrac{-1-\sqrt{5}}{2}$

ii If $b = -\sqrt{5}$, solutions of $x^2 + bx + 1 = 0$ are:

$$x = \frac{\sqrt{5} \pm \sqrt{5-4}}{2}$$

$$x = \frac{\sqrt{5} \pm 1}{2}$$

The common solution is $\dfrac{-1+\sqrt{5}}{2}$

7 $n^2 - 6n - 7 = a + bn + cn^2 - cn$

Equating coefficients

$c = 1, a = -7, b - c = -6$

$\therefore a = -7, b = -5, c = 1$

8 $a = k_1 n$ and $b = k_2 n$

$\therefore a - b = k_1 n - k_2 n = (k_1 - k_2)n$

9 **a** $576 = 2^6 \times 3^2$,

$\sqrt{576} = 2^3 \times 3 = 24$

b $1225 = 5^2 \times 7^2$,

$\sqrt{1225} = 5 \times 7 = 35$

c $1936 = 4^2 \times 11^2$,

$\sqrt{1936} = 4 \times 11 = 44$

d $1296 = 6^4$,

$\sqrt{1296} = 6^2 = 36$

10 $\dfrac{x+b}{x-c} = 1 - \dfrac{x}{x-c}$

$x + b = x - c - x$

$x = -b - c$

11

$$\frac{1}{x-a} + \frac{1}{x-b} = \frac{2}{x}$$

$$x(x-b) + x(x-a) = 2(x-a)(x-b)$$

$$x^2 - bx + x^2 - ax = 2x^2 - 2(a+b)x + 2ab$$

$$-(a+b)x = -2(a+b)x + 2ab$$

$$x = \frac{2ab}{a+b}$$

12 The sum of the first n natural numbers

$= \dfrac{n(n+1)}{2}$

The sum of the first 1000 natural numbers

$= 500 \times 1001 = 500\ 500$

a There are 333 numbers divisible by 3. The sum of these numbers is

$= \dfrac{333(6 + 332 \times 3)}{2}$

$= 166\ 833$

$\therefore$ sum of numbers not divisible by 3

$= 333\ 667$

b The sum of the numbers divisible by 2

$= \dfrac{500(4 + 499 \times 2)}{2}$

$= 250\ 500$

The sum of the numbers divisible by 6

$= \dfrac{166(12 + 165 \times 6)}{2}$

$= 83\ 166$

$\therefore$ the sum of the numbers divisible by

2 or 3
= 166 833 + 250 500 − 83 116 = 334 217.
$\therefore$ the sum of the numbers not divisible by 2 or 3
= 166 283

13 $$x^2 - 4x - 8 - \lambda(x^2 - 2x - 5) = a(x^2 - 2bx + b^2)$$

Equating coefficients

$$x^2 : 1 + \lambda = a \ldots (1)$$

$$x : -4 - 2\lambda = -2ab \ldots (2)$$

$$\text{Constant} : -8 - 5\lambda = ab^2 \ldots (3)$$

Substitute from (1) in (2) and (3)

$$-4 - 2\lambda = -2b(1 + \lambda) \ldots (4)$$

$$-8 - 5\lambda = (1 + \lambda)b^2 \ldots (5)$$

From (4), $b = \dfrac{2 + \lambda}{1 + \lambda}$

Substitute in (5)

$$-8 - 5\lambda = \frac{(2 + \lambda)^2}{1 + \lambda}$$

$$(-8 - 5\lambda)(1 + \lambda) = (2 + \lambda)^2$$

$$\lambda = -\frac{3}{2} \text{ or } \lambda = -\frac{4}{3}$$

Find $a = -\frac{1}{3},\ b = -2,\ \lambda = -\frac{4}{3};$

$a = -\frac{1}{2},\ b = -1,\ \lambda = -\frac{3}{2}$

14 $S_k = \dfrac{3(2^k - 1)}{2 - 1} = 3(2^k - 1)$

Hence $3(2^k - 1) = 189$

$$(2^k - 1) = 63$$

$$2^k = 64$$

$$k = 6$$

15 **a** $t_n = \frac{1}{2}t_{n-1}, \quad t_1 = 2$

Geometric with first term 2 and common d $\frac{1}{2}$

$$t_n = 2 \times \left(\frac{1}{2}\right)^{n-1}$$

b $t_n = t_{n-1} - \frac{5}{2}, \quad t_1 = 2$

Arithmetc with first term 2 and common difference $-\frac{5}{2}$

$$t_n = 2 - \frac{5(n-1)}{2}$$

c $t_n = \frac{1}{2}t_{n-1} - \frac{5}{2}, \quad t_1 = 2$

$r = \frac{1}{2}, d = -\frac{5}{2}, t_1 = 2$

$$t_n = 2 \times \left(\frac{1}{2}\right)^{n-1} - \frac{5}{2} \times \frac{\left(\frac{1}{2}\right)^{n-1} - 1}{\frac{1}{2} - 1}$$

$$\therefore t_n = 2 \times \left(\frac{1}{2}\right)^{n-1} + 5\left(\frac{1}{2}\right)^{n-1} - 5$$

$$\therefore t_n = 7 \times \left(\frac{1}{2}\right)^{n-1} - 5$$

16 $4 + 2 + 1 + \frac{1}{2} \ldots$

$$S = \frac{4}{1 - \frac{1}{2}} = 8$$

The frog will jump 8 m.

17 The three sides are a, ar and $ar^2 = 36$ assuming the sequence is increasing.
Also $a + ar + ar^2 = 76$
Therefore $a(1 + r) = 40$
Also $\dfrac{ar^2}{a(1 + r)} = \dfrac{9}{10}$

$$\therefore 10r^2 = 9 + 9r$$

$$10r^2 - 9r - 9 = 0$$

$$r = \frac{3}{2} \text{ or } r = -\frac{3}{5}$$

$$\text{If } r = \frac{3}{2}, a = 16$$

The other value does not give side lengths of a triangle.

18 Let three terms be $a - d, a, a + d$ and the sum of th three terms is 36.
Hence $a = 12$
The new terms are:
$13 - d$, 16 and $55 + d$
They are in geometric sequence,
Hence

$$\frac{16}{13 - d} = \frac{55 + d}{16}$$

$$256 = (55 + d)(13 - d)$$

$$256 = -d^2 - 42d + 715$$

$$d^2 + 42d - 459 = 0$$

$$(d - 9)(d + 51) = 0$$

$$d = 9 \text{ or } d = -51$$

19 $2x^2 - 4x - 2 = -2x^2 - 4x + 2$

$$4x^2 - 4 = 0$$

$$x^2 = 1$$

$$x = 1 \text{ or } x = -1$$

Graphs intersect at $(1, -4)$ and$(-1, 4)$

20

$$\frac{4}{x^2 - x - 2} + \frac{3}{x^2 - 4} = \frac{2}{x^2 + 3x + 2}$$

$$\frac{4}{(x - 2)(x + 1)} + \frac{3}{(x - 2)(x + 2)} = \frac{2}{(x + 2)(x + 1)}$$

$$4(x + 2) + 3(x + 1) = 2(x - 2)$$

$$4x + 8 + 3x + 3 = 2x - 4$$

$$5x = -15$$

$$x = -3$$

21 Train travels $55 \times 2 + 70 \times 3 = 320$ km in 5 hours.
Therefore average speed
$= \frac{320}{5} = 64$ km/h

22 $\frac{\sqrt{2}}{\sqrt{3} + \sqrt{2}} = \frac{\sqrt{2}}{\sqrt{3} + \sqrt{2}} \times \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

$$= \sqrt{6} - 2$$

23 **a**

$$\frac{2x}{3(x - 2)(x + 2)} = \frac{1}{3}\left(\frac{a}{x - 2} + \frac{b}{x + 2}\right)$$

$$2x = a(x + 2) + b(x - 2)$$

When $x = 2, 4 = 4a \Rightarrow a = 1$

When $x = -2, -4 = -4b \Rightarrow b = -1$

$$\therefore \frac{2x}{3(x - 2)(x + 2)} = \frac{1}{3}\left(\frac{1}{x - 2} - \frac{1}{x + 2}\right)$$

b $\frac{2x + 5}{(x + 2)(x + 3)} = \frac{a}{x + 2} + \frac{b}{x + 3}$

$$2x + 5 = a(x + 3) + b(x + 2)$$

When $x = -3, -1 = -b \Rightarrow b = 1$

When $x = -2, 1 = a \Rightarrow a = 1$

$$\frac{2x + 5}{(x + 2)(x + 3)} = \frac{1}{x + 2} + \frac{1}{x + 3}$$

c

$$\frac{5x^2+4x+4}{(x+2)(x^2+4} = \frac{a}{x+2} + \frac{bx+c}{x^2+4}$$

$$5x^2+4x+4 = a(x^2+4)+(bx+c)(x+2)$$

When $x=-2, 16=8a \Rightarrow a=2$

Equate coefficients:constant

$$4 = 4a+2c \Rightarrow c=-2$$

Equate coefficients:x^2

$$5 = a+b \Rightarrow b=3$$

$$\therefore \frac{5x^2+4x+4}{(x+2)(x^2+4} = \frac{2}{x+2} + \frac{3x-2}{x^2+4}$$

d

$$\frac{2(x^2-2x-1)}{(x+1)(x-1)^2} = \frac{a}{x+1} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$$

$$2x^2-4x-2 = a(x-1)^2+b(x-1)(x+1)+c(x+1)$$

When $x=1, -4=2c \Rightarrow c=-2$

Equate coefficients:x

$$-4 = -2a+c \Rightarrow a=1$$

Equate coefficients:x^2

$$2 = a+b \Rightarrow b=1$$

$$\therefore \frac{2(x^2-2x-1)}{(x+1)(x-1)^2} = \frac{1}{x+1} + \frac{1}{x-1} - \frac{2}{(x-1)^2}$$

e

$$\frac{2x^2-3x+1}{(x^2+1)(x-3)} = \frac{a}{x-3} + \frac{bx+c}{x^2+1}$$

$$2x^2-3x+1 = a(x^2+1)+(bx+c)(x-3)$$

When $x=3, 10=10a \Rightarrow a=1$

Equate coefficients:x^2

$$2 = a+b \Rightarrow b=1$$

Equate coefficients:x

$$-3 = -3b+c \Rightarrow c=0$$

$$\therefore \frac{2x^2-3x+1}{(x^2+1)(x-3)} = \frac{1}{x-3} + \frac{x}{x^2+1}$$

f

$$\frac{3x^2-x+6}{(x^2+4)(x-2)} = \frac{a}{x-2} + \frac{bx+c}{x^2+4}$$

$$3x^2-x+6 = a(x^2+4)+(bx+c)(x-2)$$

When $x=2, 16=8a \Rightarrow a=2$

Equate coefficients:x^2

$$3 = a+b \Rightarrow b=1$$

Equate coefficients:x

$$-1 = -2b+c \Rightarrow c=1$$

$$\therefore \frac{3x^2-x+6}{(x^2+4)(x-2)} = \frac{2}{x-2} + \frac{x+1}{x^2+4}$$

Solutions to multiple-choice questions

1 E 5 is seven less than 3 times $(x+1)$

$$5 = 3 \times (x+1) - 7$$
$$5 = 3x + 3 - 7$$
$$5 = 3x - 4$$

2 B $$\frac{3}{x-3} - \frac{2}{x+3} = \frac{3(x+3) - 2(x-3)}{(x-3)(x+3)}$$
$$= \frac{3x + 9 - 2x + 6}{x^2 - 9}$$
$$= \frac{x + 15}{x^2 - 9}$$

3 D This is an arithmetic progression with $a = 1$ and $d = 2$.

$$S_n = \frac{n}{2}(2 + 2(n-1))$$
$$= 100$$
$$n(2 + 2n - 2) = 200$$
$$n^2 = 100$$
$$n = 10 \text{ terms}$$
$$t_n = 1 + 2 \times (10 - 1)$$
$$= 19$$

4 B $$a = S_1 = 2^2 - 2 = 2$$
$$S_2 = 2^3 - 2 = 6$$
$$t_2 = S_2 - S_1 = 4$$
$$r = \frac{t_2}{t_1} = 2$$
$$t_n = ar^{n-1}$$
$$= 2 \times 2^{n-1}$$
$$= 2^n$$

5 C $$A \cap (B \cup C) = A \cap \{1, 2, 3, 4, 5, 6, 7\}$$
$$= \{2, 3, 4\}$$

6 C $$0.\dot{7}\dot{2} = 0.727272\ldots$$
$$0.\dot{7}\dot{2} \times 100 = 72.7272\ldots$$
$$0.\dot{7}\dot{2} \times 99 = 72$$
$$0.\dot{7}\dot{2} = \frac{72}{99}$$

7 A $$\frac{-4}{x-1} - \frac{3}{1-x} + \frac{x}{x-1}$$
$$= \frac{-4}{x-1} + \frac{3}{x-1} + \frac{x}{x-1}$$
$$= \frac{x-1}{x-1}$$
$$= 1$$

8 C $$\frac{x+2}{3} - \frac{5}{6} = \frac{2x+4}{6} - \frac{5}{6}$$
$$= \frac{2x-1}{6}$$

9 C $$a - 1 = \frac{1}{1+b}$$
$$\frac{1}{a-1} = 1 + b$$
$$\frac{1}{a-1} - 1 = b$$
$$b = \frac{1}{a-1} - 1$$

10 A $$0.\dot{3}\dot{6} = 0.363636\ldots$$
$$0.\dot{3}\dot{6} \times 100 = 36.3636\ldots$$
$$0.\dot{3}\dot{6} \times 99 = 36$$
$$0.\dot{3}\dot{6} = \frac{36}{99} = \frac{4}{11}$$

Numerator + denominator = 4 + 11

= 15

11 B Multiply both sides by $4(2x + y)$.

$$\begin{aligned} 4(2x - y) &= 3(2x + y) \\ 8x - 4y &= 6x + 3y \\ 8x - 6x &= 3y + 4y \\ 2x &= 7y \\ \frac{2x}{2y} &= \frac{7y}{2y} \\ \frac{x}{y} &= \frac{7}{2} \end{aligned}$$

12 D $a = \frac{1}{2},\ r = -\frac{1}{2}$

$$\begin{aligned} S_\infty &= \frac{a}{1 - r} \\ &= \frac{\frac{1}{2}}{1 - -\frac{1}{2}} \\ &= \frac{\frac{1}{2}}{\frac{3}{2}} \\ &= \frac{1}{3} \end{aligned}$$

13 B Multiply both sides by $(3 + y)$.

$$\begin{aligned} 3 &= 4(3 + y) \\ 3 &= 12 + 4y \\ -9 &= 4y \\ y &= -\frac{9}{4} \end{aligned}$$

14 B Multiply the first equation by 5, then subtract. $15x + 5y = -35$ ①

$2x + 5y = 4$ ②

① − ②:

$$\begin{aligned} 13x &= -39 \\ x &= -3 \\ 3 \times -3 + y &= -7 \\ y &= 2 \end{aligned}$$

$(-3, 2)$

15 A Multiply both sides by 4.

$$\begin{aligned} (m + 2) - (2 - m) &= 2 \\ m + 2 - 2 + m &= 2 \\ 2m &= 2 \\ m &= 1 \end{aligned}$$

16 D

$$\begin{array}{r|r} 2 & 46\,200 \\ 2 & 23\,100 \\ 2 & 11\,550 \\ 3 & 5575 \\ 5 & 1925 \\ 5 & 385 \\ 7 & 77 \\ 11 & 11 \\ & 1 \end{array}$$

$= 2^3 \times 3 \times 5^2 \times 7 \times 11$

17 A The difference between terms is constant.

$$\begin{aligned} (y - 1) - y &= (2y - 1) - (y - 1) \\ y - 1 - y &= 2y - 1 - y + 1 \\ -1 &= y \\ y &= -1 \end{aligned}$$

18 B Order is $n - 6,\ n - 5,\ n - 1,\ n + 1,\ n + 4$. Middle number is $n - 1$.

19 A $t_4 = a + 3d$

$= 3 + 3d = 9$

$3d = 6$

$d = 2$

$t_{11} = a + (n - 1)d$

$= 3 + 10 \times 2$

$= 23$

20 B $\dfrac{4}{n+1} + \dfrac{3}{n-1} = \dfrac{4(n-1) + 3(n+1)}{(n+1)(n-1)}$

$= \dfrac{4n - 4 + 3n + 3}{n^2 - 1}$

$= \dfrac{7n - 1}{n^2 - 1}$

$= \dfrac{7n - 1}{n^2 - 1} \times \dfrac{-1}{-1}$

$= \dfrac{1 - 7n}{1 - n^2}$

21 A $(\sqrt{7} + 3)(\sqrt{7} - 3) = 7 - 9$

$= -2$

22 C $2x^2 - 9x + 4$

$= (x - 4)(2x - 1)$

$\dfrac{13x - 10}{(x-4)(2x-1)}$

$= \dfrac{P}{x-4} + \dfrac{Q}{2x-1}$

$= \dfrac{P(2x-1) + Q(x-4)}{(x-4)(2x-1)}$

$= \dfrac{2Px + Qx - P - 4Q}{(x-4)(2x-1)}$

$2P + Q = 13$ ①

$-P - 4Q = -10$

$-2P - 8Q = -20$ ②

① + ②:

$-7Q = -7$

$Q = 1$

$2P + 1 = 13$

$2P = 12$

$P = 6$

23 C $\dfrac{a}{1-r} = 4a$

Multiply both sides by $\dfrac{1-r}{a}$.

$1 = 4(1 - r)$

$1 = 4 - 4r$

$4r = 4 - 1$

$r = \dfrac{3}{4}$

24 A $\dfrac{5x}{(x+2)(x-3)}$

$= \dfrac{P}{x+2} + \dfrac{Q}{x-3}$

$= \dfrac{P(x-3) + Q(x+2)}{(x+2)(x-3)}$

$= \dfrac{Px + Qx - 3P + 2Q}{(x+2)(x-3)}$

$P + Q = 5$

$3P + 3Q = 15$ ①

$-3P + 2Q = 0$ ②

① + ②:

$5Q = 15$

$Q = 3$

$P + 3 = 5$

$P = 2$

25 E Assuming n is an integer, and $n = m^2$, then the next largest perfect square is $(m + 1)^2$

$(m + 1)^2 = m^2 + 2m + 1$

Since $n = m^2$, $m = \sqrt{n}$

$(m + 1)^2 = n + 2\sqrt{n} + 1$

The next largest perfect square is $n + 2\sqrt{n} + 1$.

26 C 0.4 and 4.125 are terminating decimals.

$\frac{3}{8} = 0.125$

$\sqrt{16} = 4$

Only $\sqrt{5}$ cannot be expressed as a rational number.

27 C $x = \frac{b}{a}$ and $y = \frac{1}{a-b}$

$$x + y = \frac{b}{a} + \frac{1}{a-b}$$
$$= \frac{b(a-b)+a}{a(a-b)}$$
$$= \frac{ba - b^2 + a}{a(a-b)}$$

28 E The perfect square could be $(3x-2)^2$ or $(3x+2)^2$

The middle term of the expansion would be $-12x$ or $12x$ respectively. This means m would be 3 or -3, i.e. ± 3.

29 D $x = (n+1)(n+2)(n+3),\ n > 0$

When $n = 1$,

$x = (1+1) \times (1+2) \times (1+3)$

$= 2 \times 3 \times 4 = 12$

When $n = 2$,

$x = (2+1) \times (2+2) \times (2+3)$

$= 3 \times 4 \times 5 = 60$

1, 2, 3 and 6 are factors in both equations, but not 5.

30 C There are 8 terms, $a = -4$ and $t_8 = 10$.

$$a + 7d = 10$$
$$-4 + 7d = 10$$
$$7d = 14$$
$$d = 2$$

The required sum is $S_7 - a$.

$$S_7 - a = \frac{7}{2}(-8 + 6 \times 2) - -4$$
$$= 14 + 4$$
$$= 18$$

Note. A faster solution can be found by noting that

$a + f = b + e = c + d = -4 + 10 = 6.$

Therefore

$$a+b+c+d+e+f = (a+f)+(b+e)+(c+d) = 18.$$

31 A An odd number plus an odd number is always an even number, so $n + p$. (The other options all produce odd numbers for all n and p.)

32 C Arithmetic. $S_{10} = 5(8 + 5 \times 9) = 5 \times 53 = 265$

33 C $t_{20} = 3 \times 4^{19}$

34 D $a = 9$ and $t_{16} = 9 + 15d = 144$

Hence $d = 9$

35 A The sequence oscillates and $t_2 = 13$. Only A satisfies this. Check the remaining values.

36 A $r = 1 + 0.15 - 0.11 = 1.04$

37 D LCM $= 5 \times 3 \times 2$ and HCF $= 3 \times 2$

38 C $x^2 = (x^2 - 4x + 4) + b(x-2) + c$.

Therefore $b = 4$ and

$0 = 4 - 8 + c \Rightarrow c = 4$

Solutions to extended-response questions

1 **a** When $h = 10$, $d = \dfrac{10}{5} + 6$

$= 8$

b When $h = 8.5$, $d = \dfrac{8.5}{5} + 6$

$= 7.7$

c The diameter of the bottom of the glass can be calculated when $h = 0$.

$\therefore d = \dfrac{0}{5} + 6$

$= 6$

The diameter of the bottom of the glass is 6 cm.

d When $d = 9$, $9 = \dfrac{h}{5} + 6$

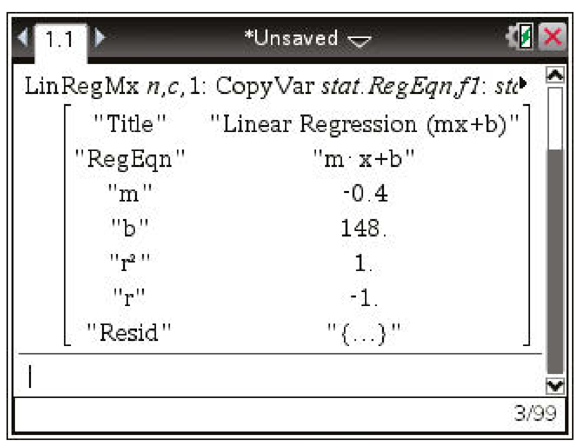
$\therefore \quad 3 = \dfrac{h}{5}$

$\therefore \quad h = 15$

The height of the glass is 15 cm.

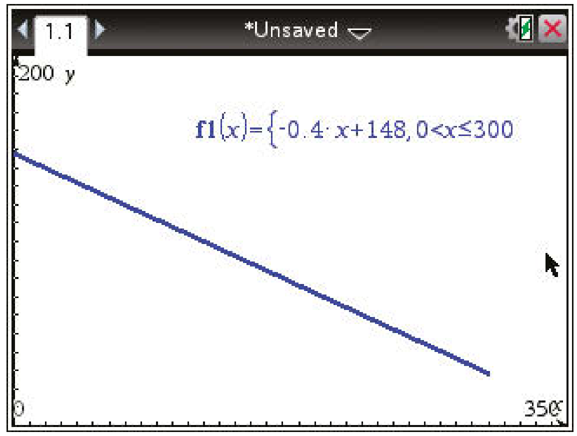
CAS calculator techniques for Question 4

TI: In the Calculator page type {**100**, **120**}**→n** then ENTER followed by {**108**, **100**}**→c** then ENTER.
Press **Menu** → **6**: **Statistics** → **1**: **Stat Calculations→**3**Linear Regression** (**mx**+ **b**).
Set X List to **n** and Y List to **c**.
The equation of the line is
$C = -0.4n + 148$



In a Graphs page input **−0.4*x*** + **148|0<*x*≤300** into ***f*1** then ENTER. In a Calculator page type ***f*1(200)** to yield a value of 68.
To answer part **d** sketch ***f*2** = **48.8**.
Press **Menu→6: Analyze Graph→4**: **Intersection**

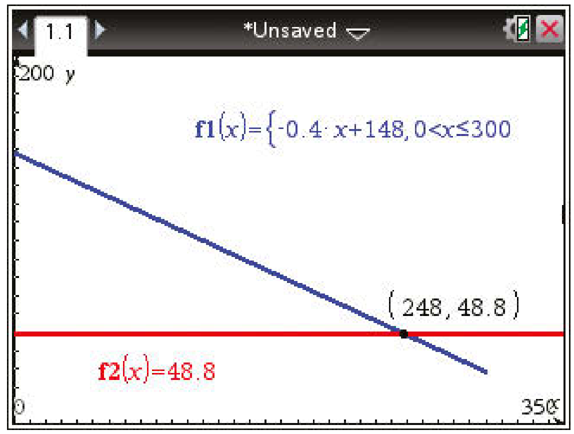
CP: In the Main application type {**100**, **120**}**Bn** then EXE followed by {**108**, **100**}**Bc** then EXE. In the (tab of the Keyboard select **LinearReg** and complete the command as **LinearReg n,c** followed by EXE. Tap **Action→Command→DispStat**



The equation of the line is
$C = -0.4n + 148$

In a Graph&Table application input

−0.4*x*+148|0<*x*≤300 into ***y*1** then EXE. Tap $ then **Analysis→G−Solve→y − Cal** and input 200 as the x-value to yield a value of 68.



To answer part **d** sketch ***y*2** = **48.8**. Tap **Analysis→G-Solve→Intersect**

2 **a** When $n = 1$, $P = -9000$,

$\therefore \quad -9000 = a + b \quad \ldots \boxed{1}$

When $n = 5$, $P = 15\,000$,

$\therefore \quad 15\,000 = 5a + b \quad \ldots \boxed{2}$

Subtract $\boxed{1}$ from $\boxed{2}$

$\therefore \quad 24\,000 = 4a$

$\therefore \quad 6000 = a$

Substitute $a = 6000$ in $\boxed{1}$

$\therefore \quad -9000 = 6000 + b$

$\therefore \quad -15\,000 = b$

b $P = 6000n - 15\,000$

When $n = 12$, $P = 6000 \times 12 - 15\,000$

$= 57\,000$

The profit is $57 000.

c When $P = 45\,000$, $45\,000 = 6000n - 15\,000$

$$\therefore \quad 60\,000 = 6000n$$

$$\therefore \quad 10 = n$$

The profit will be \$45 000 at the end of 2016, after 10 years of operation.

3 a i When $n = 180$, $A = 180 - \dfrac{360}{180}$

$$= 178$$

ii When $n = 360$, $A = 180 - \dfrac{360}{360}$

$$= 179$$

iii When $n = 720$, $A = 180 - \dfrac{360}{720}$

$$= 179.5$$

iv When $n = 7200$, $A = 180 - \dfrac{360}{7200}$

$$= 179.95$$

b i As n becomes very large, A approaches 180.

ii As n becomes very large, the shape of the polygon approaches that of a circle.

c When $A = 162$, $162 = 180 - \dfrac{360}{n}$

$$\therefore \quad \frac{360}{n} = 18$$

$$\therefore \quad n = \frac{360}{18}$$

$$= 20$$

d $A = 180 - \dfrac{360}{n}$

$$\therefore \frac{360}{n} = 180 - A$$

$$\therefore n = \frac{360}{180 - A}$$

e For an octagon, $n = 8$

$$\therefore \quad A = 180 - \frac{360}{8}$$

$$= 135$$

At the point where the two octagons and the third regular polygon meet, the three angles sum to 360°,

$\therefore 135 + 135 + x = 360$

where $x°$ is the size of the interior angle of the third regular polygon.

$\therefore 270 + x = 360$

$\therefore x = 90$

Thus the third regular polygon is a square.

4 a Volume of hemisphere, $V_H = \frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi t^3$

Volume of cylinder, $V_{CY} = \pi r^2 h = \pi t^2 s$

Volume of cone, $V_{co} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi t^2 w$

b **i** $If V_H = V_{CY} = V_{CO}$

then $\frac{2}{3}\pi t^3 = \pi t^2 s \qquad \ldots \boxed{1}$

and $\pi t^2 s = \frac{1}{3}\pi t^2 w \qquad \ldots \boxed{2}$

From $\boxed{1}$ $\frac{t^3}{t^2} = \frac{3}{2}s$

From $\boxed{2}$ $w = \frac{\pi t^2 s}{\frac{1}{3}\pi t^2}$

$= 3s \qquad \therefore \quad w : s : t = 3s : s : \frac{3}{2}s$

$= 3 : 1 : \frac{3}{2}$

$= 6 : 2 : 3$

ii If $w + s + t = 11$

then $3s + s + \frac{3}{2}s = 11$

$\therefore \quad \frac{11}{2}s = 11$

$\therefore \quad s = 2$

$$\begin{aligned} w &= 3 \times 2 \\ &= 6 \\ t &= \frac{3}{2} \times 2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{Total volume} &= V_H + V_{CY} + V_{CO} \\ &= \frac{2}{3}\pi t^3 + \pi t^2 s + \frac{1}{3}\pi t^2 w \\ &= \frac{2}{3}\pi \times 3^3 + \pi \times 3^2 \times 2 + \frac{1}{3}\pi \times 3^2 \times 6 \\ &= 18\pi + 18\pi + 18\pi \\ &= 54\pi \end{aligned}$$

The total volume is 54π cubic units.

5 Needs Venn diagram

a $|A \cup B| = 140$

b $A \cup B \cup C| = 180$

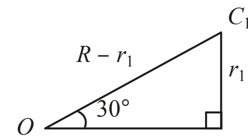
c $A \cup B \cup C| = 180$

d $A' \cap B \cap C| = 20$

e $A \cap B' \cap C'| = 10$

6 **a** **i** $OC_1 = R - r_1$

ii $\sin 30^\circ = \dfrac{1}{2}$



$$\text{and } \sin 30^\circ = \frac{r_1}{R - r_1} = \frac{r_1}{OC_1}$$

$$\therefore \quad \frac{r_1}{OC_1} = \frac{1}{2}$$

$$\therefore \quad \frac{r_1}{R - r_1} = \frac{1}{2}$$

$$\therefore \quad 2r_1 = R - r_1$$

$$\therefore \quad 3r_1 = R$$

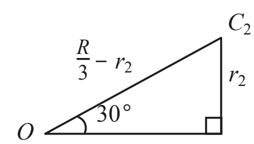
$$\therefore \quad r_1 = \frac{R}{3}$$

b **i** $OC_2 = (R - 2r_1) - r_2$

$$= R - 2 \times \frac{R}{3} - r_2$$

$$= \frac{R}{3} - r_2$$

ii $\sin 30^\circ = \dfrac{1}{2}$



$$\text{and } \sin 30^\circ = \frac{r_2}{\frac{R}{3} - r_2}$$

$$\therefore \quad \frac{r_2}{\frac{R}{3} - r_2} = \frac{1}{2}$$

$$\therefore \quad 2r_2 = \frac{R}{3} - r_2$$

$$\therefore \quad 3r_2 = \frac{R}{3}$$

$$\therefore \quad r_2 = \frac{R}{9}$$

c **i** The common ratio is $r = \dfrac{r_2}{r_1}$

$$= \frac{R}{9} \div \frac{R}{3}$$

$$= \frac{R}{9} \times \frac{3}{R} = \frac{1}{3}$$

ii $r_1 = \frac{R}{3}$

and $r_2 = \frac{R}{9} = \frac{R}{3^2}$

$\therefore \quad r_n = \frac{R}{3^n}$

iii $S_\infty = \frac{a}{1-r}$

$$= \frac{\frac{R}{3}}{1-\frac{1}{3}}$$

$$= \frac{R}{3} \div \frac{2}{3}$$

$$= \frac{R}{3} \times \frac{3}{2} = \frac{R}{2}$$

The sum to infinity is $\frac{R}{2}$.

iv Let A_n be the area of the circle with radius r_n.

$\therefore \quad A_n = \pi r_n^2$

$\therefore \quad A_1 = \pi r_1^2$

$$= \pi\left(\frac{R}{3}\right)^2$$

$$= \frac{\pi R^2}{9}$$

and $\quad A_2 = \pi r_2^2$

$$= \pi\left(\frac{R}{9}\right)^2$$

$$= \frac{\pi R^2}{81}$$

The common ratio is $\quad r = \frac{A_2}{A_1}$

$$= \frac{\pi R^2}{81} \div \frac{\pi R^2}{9}$$

$$= \frac{\pi R^2}{81} \times \frac{9}{\pi R^2} = \frac{1}{9}$$

$$
\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
&= \frac{\pi R^2}{9} \div \left(1 - \frac{1}{9}\right) \\
&= \frac{\pi R^2}{9} \times \frac{9}{8} \\
&= \frac{\pi R^2}{8}
\end{aligned}
$$

The sum to infinity of the area of the circles with radii $r_1,\ r_2,\ r_3,\ \ldots$ is $\dfrac{\pi R^2}{8}$ square units.

7 a i Production of Company A in nth month $= 1000 + 80(n-1)$

$$
\begin{aligned}
&= 1000 + 80n - 80 \\
&= 80n + 920
\end{aligned}
$$

ii Production of Company A in 24th month $= 920 + 80 \times 24$

$$= 2840$$

Production of Company B in 24th month $= 1000 \times 1.04^{23}$

$$= 2464.71554\ldots$$

Company A and Company B produced 2840 and 2465 tonnes respectively, to the nearest tonne, in December 2019.

iii For Company A, the total production over n months is

$$
\begin{aligned}
S_n &= \frac{n}{2}(2a + (n-1)d) \text{ where } a = 1000 \text{ and } d = 80 \\
&= \frac{n}{2}(2000 + 80(n-1)) \\
&= \frac{n}{2}(2000 + 80n - 80) \\
&= \frac{n}{2}(80n + 1920) \\
&= 40n^2 + 960n \\
&= 40n(n + 24)
\end{aligned}
$$

iv For Company A, $S_{24} = (40 \times 24)(24 + 24) = 46\,080$

For Company B, $S_n = \dfrac{a(r^n - 1)}{r - 1}$ where $a = 1000$ and $r = 1.04$

$$\therefore \quad S_{24} = \frac{1000(1.04^{24} - 1)}{1.04 - 1} = 39\,082.604\,12$$

The total production for the period January 2018 to December 2019 inclusive, of Company A and Company B, is 46 080 and 39 083 tonnes respectively, to the nearest tonne.

b Find n for which $S_n > 100\,000$ for Company A,

$$\therefore \quad 40n(n+24) > 100\,000$$

$$\therefore \quad 40n^2 + 960n - 100\,000 > 0$$

$$\therefore \quad n^2 + 24n - 2500 > 0$$

When $n = 39$,

$39^2 + 24 \times 39 - 2500 = -43 < 0$

When $n = 40$,

$40^2 + 24 \times 40 - 2500 = 60 > 0$

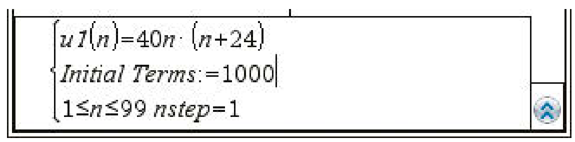
The 40th month represents Apri1 2021.

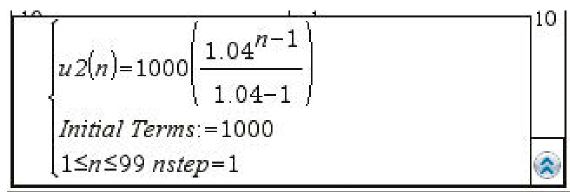
The total production of Company A passes 100 000 tonnes in Apri1 2021.

CAS calculator techniques for Question 7

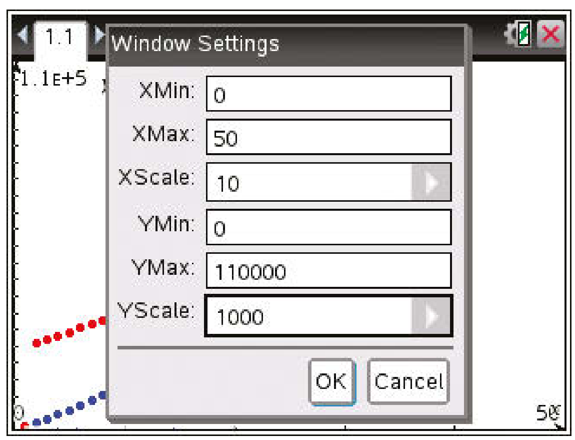
TI: In the graphs page input the following sequences.

It is important to set the window correctly to obtain informative graphs.





Press /T to view a table of values. Scroll down to $n = 24$ to see the value of $u1(24)$ and $u2(24)$. Scroll through the table to find when the production is 100 000 for $u1$.



CP: In the Sequence application input the following:

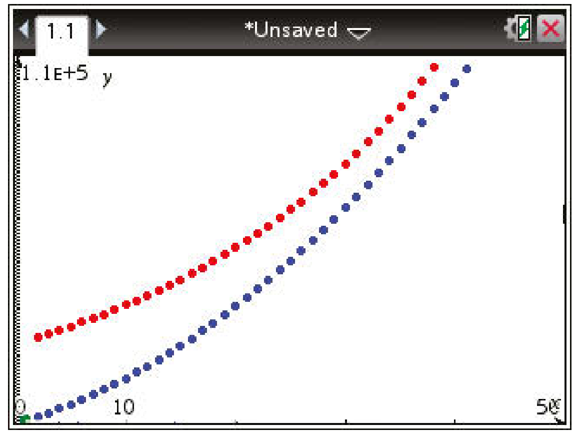
$$\boldsymbol{a_{n+1} = 40n \times (n+24)}$$

$$\boldsymbol{a_0 = 1000}$$

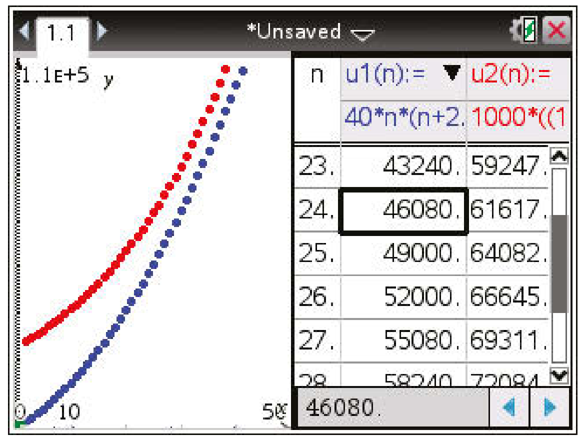
$$\boldsymbol{b_{n+1} = 1000\left(\frac{1.04^{n-1}}{1.04-1}\right)}$$

$$\boldsymbol{b_0 = 1000}$$

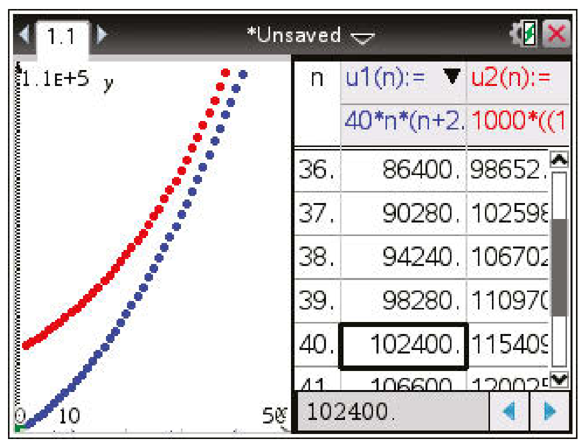
Tap 8 and set the table Start value to 1 and the End value to 50. Now tap # to see the table of values.



Scroll down to n = 25 to see the value of $a_n(24)$ and $b_n(24)$.



Scroll through the table to find when the production is 100 000 for a_n.



8 **a** $P_1 = 4 \times 1 = 4$

b $P_2 = 3 \times 1 + 6 \times \dfrac{1}{2}$

$= 3 + 3$

$= 6$

c $P_3 = 3 \times 1 + 5 \times \frac{1}{2} + 10 \times \frac{1}{4}$

$= 3 + \frac{5}{2} + \frac{5}{2}$

$= 8$

d The common difference is 2 as $8 - 6 = 2$ and $6 - 4 = 2$.

e **i** $P_4 = P_3 + 2$

$= 8 + 2$

$= 10$

ii $P_n = P_{n-1} + 2$

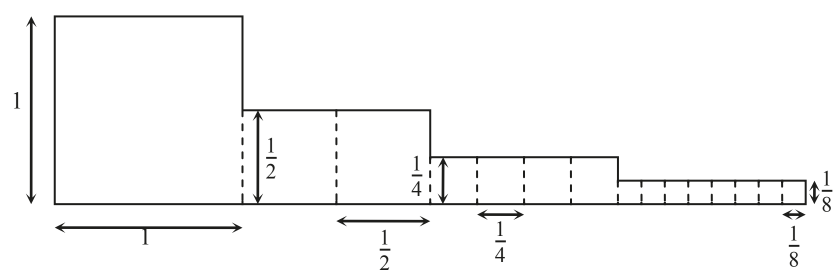
iii $P_n = P_1 + (n - 1) \times 2$

$= 4 + 2(n - 1)$

$= 4 + 2n - 2$

$= 2n + 2$

iv



9 **a** a and n are integers. $\frac{n}{2}(2a + (n - 1) = 10$

$2an + n^2 - n = 20$

$n^2 + (2a - 1)n - 20 = 0$

Possible factors of 20 are −20 and 1, −1 and 20, −4 and 5, −5 and 4, −10 and 2, 10 and −2.

Consider the possiblities: $(n - 20)(n + 1) = n^2 - 19n - 20$. If this is a solution $2a - 1 = -19$ (not positive value of a.)

$(n + 20)(n - 1) = n^2 + 19n - 20$. If this is a solution $2a - 1 = 19$. $a = 10$. $10 = 10$✓

$(n - 4)(n + 5) = n^2 + n - 20$. If this is a solution $2a - 1 = 1$. $a = 1$ and $1 + 2 + 3 + 4 = 10$✓

$(n + 4)(n - 5) = n^2 - n - 20$. If this is a solution $2a - 1 = -1$ (not positive value of a.)

$(n + 10)(n - 2) = n^2 + 8n - 20$. If this is a solution $2a - 1 = 8$ (not integer value of value of a.)

$(n - 10)(n + 2) = n^2 - 8n - 20$. If this is a solution $2a - 1 = -8$ (not integer value of value of a.)
There are only 2.

b a and n are integers. $\frac{n}{2}(2a + (n - 1) = 100$
$2an + n^2 - n = 200$
$n^2 + (2a - 1)n - 200 = 0$
Possible factors of -200 which give a result. These are $-5, 40, 25, -8$ Consider the possiblities: $(n - 5)(n + 40) = n^2 + 35n - 200$. If this is a solution $2a - 1 = 35$ and $a = 18$.)
Therefore $18 + 19 + 20 + 21 + 22$ $(n + 25)(n - 8) = n^2 + 17n - 200$. If this is a solution $2a - 1 = 17$ and $a = 9$.)
Therefore $9 + 10 + 11 + 12 + 13 + 14 + 15 + 16$

c 8 ways.

10 **a** Perimeter of rectangle $= 2(3x + x)$

$= 8x$

The perimeter of the rectangle is $8x$ cm.

b Perimeter of square = length of wire − perimeter of rectangle

$= 28 - 8x$

The perimeter of the square is $(28 - 8x)$ cm.

c Side length of square $= \dfrac{28 - 8x}{4}$

$= 7 - 2x$

The length of each side of the square is $(7 - 2x)$ cm.

d A = area of rectangle + area of square

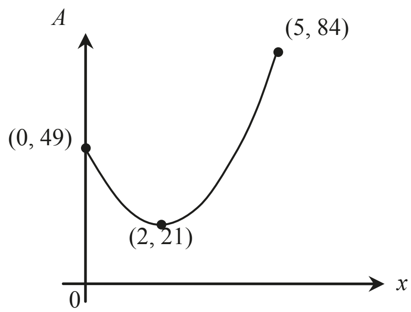
$= 3x \times x + (7 - 2x)^2$

$= 3x^2 + 49 - 28x + 4x^2$

$= 7x^2 - 28x + 49$

$= 7(x^2 - 4x + 7)$ as required .

e



f $A = 7x^2 - 28x + 49$

Minimum value occurs at $x = \frac{-b}{2a}$, where $a = 7$ and $b = -28$

$$= \frac{28}{14}$$

$$= 2$$

When $x = 2$, $A = 7(2^2 - 4 \times 2 + 7)$

$$= 21$$

A has a minimum value of 21 when $x = 2$.

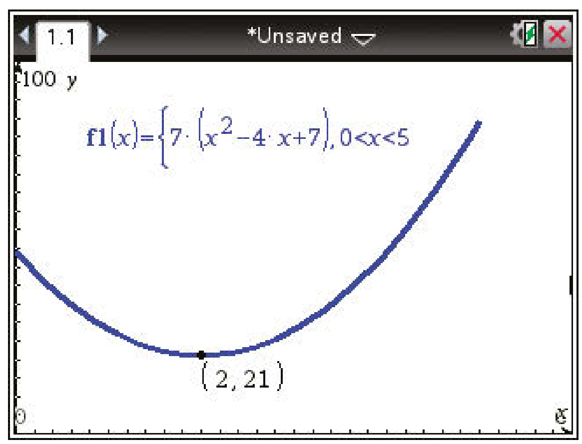
CAS calculator techniques for Question 9

TI: Sketch the graph of ***f*1(*x*)=7(*x*^2 – 4*x* + 7)|0<*x*<5**

Press **Menu→ 6: AnalyzeGraph→2: Minimum** to yield the minimum value.

CP: Sketch the graph of ***y*1(*x*)=7(*x*^2 – 4*x* + 7)|0 < *x* < 5**

Press **Analysis→G–Solve→Min** to yield the minimum value.



11 a i

$$\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$$

$$\Rightarrow \quad (\sqrt{7x-5} - \sqrt{2x})^2 = (\sqrt{15-7x})^2$$

$$\Rightarrow \quad (\sqrt{7x-5})^2 - 2(\sqrt{7x-5})(\sqrt{2x}) + (\sqrt{2x})^2 = 15-7x$$

$$\Rightarrow \quad 7x-5-2\sqrt{(7x-5)(2x)} + 2x = 15-7x$$

$$\Rightarrow \quad 9x-5-2\sqrt{14x^2-10x} = 15-7x$$

$$\Rightarrow \quad 9x-5-15+7x = 2\sqrt{14x^2-10x}$$

$$\Rightarrow \quad 16x-20 = 2\sqrt{14x^2-10x}$$

$$\Rightarrow \quad 8x-10 = \sqrt{14x^2-10x}, \text{ as required.}$$

ii

$$(8x-10)^2 = (\sqrt{14x^2-10x})^2$$

$$\therefore \quad 64x^2 - 160x + 100 = 14x^2 - 10x$$

$$\therefore \quad 64x^2 - 160x + 100 - 14x^2 + 10x = 0$$

$$\therefore \quad 50x^2 - 150x + 100 = 0$$

$$\therefore \quad x^2 - 3x + 2 = 0, \text{ as required.}$$

iii Consider $\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$

When $x = 1$, LHS $= \sqrt{7 \times 1 - 5} - \sqrt{2 \times 1}$

$= \sqrt{2} - \sqrt{2} = 0$

RHS $= \sqrt{15 - 7 \times 1}$

$= \sqrt{8} \neq 0$

Hence LHS $\neq$ RHS and $x = 1$ is not a solution.

When $x = 2$, LHS $= \sqrt{7 \times 2 - 5} - \sqrt{2 \times 2}$

$= \sqrt{9} - \sqrt{4}$

$= 3 - 2 = 1$

RHS $= \sqrt{15 - 7 \times 2}$

$= \sqrt{1} = 1$

Hence LHS = RHS and $x = 2$ is a solution.

b **i** $$\sqrt{x+2} - 2\sqrt{x} = \sqrt{x+1}$$

$$\Rightarrow \quad (\sqrt{x+2} - 2\sqrt{x})^2 = (\sqrt{x+1})^2$$

$$\Rightarrow \quad x + 2 - 4\sqrt{x+2}\sqrt{x} + 4x = x + 1$$

$$\Rightarrow \quad 5x + 2 - 4\sqrt{(x+2)x} = x + 1$$

$$\Rightarrow \quad 5x + 2 - x - 1 = 4\sqrt{x^2 + 2x}$$

$$\Rightarrow \quad 4x + 1 = 4\sqrt{x^2 + 2x}$$

$$\Rightarrow \quad (4x+1)^2 = (4\sqrt{x^2+2x})^2$$

$$\Rightarrow \quad 16x^2 + 8x + 1 = 16(x^2 + 2x)$$

$$\Rightarrow \quad 16x^2 + 8x + 1 = 16x^2 + 32x$$

$$\Rightarrow \quad 1 = 24x$$

$$\Rightarrow \quad x = \frac{1}{24}$$

Consider $\sqrt{x+2} - 2\sqrt{x} = \sqrt{x+1}$

When $x = \frac{1}{24}$, LHS $= \sqrt{\frac{1}{24} + 2} - 2\sqrt{\frac{1}{24}}$

$$= \sqrt{\frac{49}{24}} - \frac{2}{2\sqrt{6}}$$

$$= \frac{7}{2\sqrt{6}} - \frac{2}{2\sqrt{6}} = \frac{5}{2\sqrt{6}}$$

and RHS $= \sqrt{\frac{1}{24} + 1}$

$$= \sqrt{\frac{25}{24}}$$

$$= \frac{5}{2\sqrt{6}}$$

Hence LHS = RHS and $x = \frac{1}{24}$ is a solution.

ii

$$2\sqrt{x+1} + \sqrt{x-1} = 3\sqrt{x}$$

$$\Rightarrow \quad (2\sqrt{x+1} + \sqrt{x-1})^2 = (3\sqrt{x})^2$$

$$\Rightarrow \quad 4(x+1) + 4\sqrt{x+1}\sqrt{x-1} + x - 1 = 9x$$

$$\Rightarrow \quad 4x + 4 + 4\sqrt{(x+1)(x-1)} + x - 1 = 9x$$

$$\Rightarrow \quad 5x + 3 + 4\sqrt{x^2-1} = 9x$$

$$\Rightarrow \quad 4\sqrt{x^2-1} = 4x - 3$$

$$\Rightarrow \quad (4\sqrt{x^2-1})^2 = (4x-3)^2$$

$$\Rightarrow \quad 16(x^2-1) = 16x^2 - 24x + 9$$

$$\Rightarrow \quad 16x^2 - 16 = 16x^2 - 24x + 9$$

$$\Rightarrow \quad 24x = 25$$

$$\Rightarrow \quad x = \frac{25}{24}$$

Consider $2\sqrt{x+1} + \sqrt{x-1} = 3\sqrt{x}$

When $x = \frac{25}{24}$, LHS $= 2\sqrt{\frac{25}{24}+1} + \sqrt{\frac{25}{24}-1}$

$$= 2\sqrt{\frac{49}{24}} + \sqrt{\frac{1}{24}}$$

$$= \frac{2 \times 7}{2\sqrt{6}} + \frac{1}{2\sqrt{6}}$$

$$= \frac{15}{2\sqrt{6}}$$

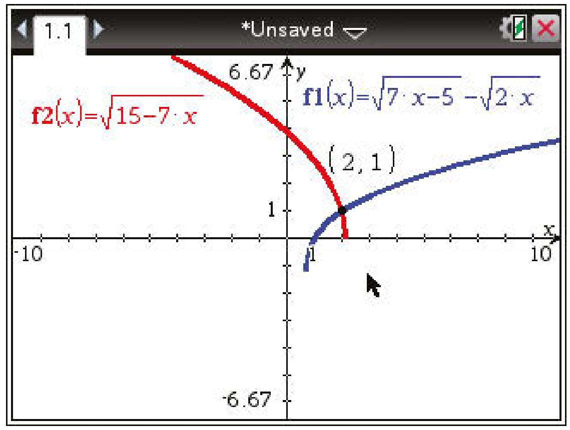
and RHS $= 3\sqrt{\frac{25}{24}}$

$$= \frac{3 \times 5}{2\sqrt{6}}$$

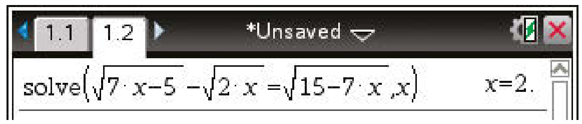
$$= \frac{15}{2\sqrt{6}}$$

Hence LHS = RHS and $x = \frac{25}{24}$ is a solution.

CAS calculator techniques for Question 12



TI: Sketch the graphs of $\boldsymbol{f1} = \sqrt{\boldsymbol{7x - 5}} - \sqrt{\boldsymbol{2x}}$ and $\boldsymbol{f2} = \sqrt{\boldsymbol{15 - 7x}}$
Press **Menu→6: Analyze Graph→4: Intersection**
CP: Sketch the graphs of $\boldsymbol{y1} = \sqrt{\boldsymbol{7x - 5}} - \sqrt{\boldsymbol{2x}}$ and $\boldsymbol{y2} = \sqrt{\boldsymbol{15 - 7x}}$ Tap **Analysis →G-Solve→Intersect**



Alternatively, type
solve ($\sqrt{\boldsymbol{7x - 5}} - \sqrt{\boldsymbol{2x}} = \sqrt{\boldsymbol{15 - 7x}}, \boldsymbol{x}$)
in a Calculator/Main page.

12 a $n + 25$ is a perfect square implies

$$n + 25 = b^2$$

$$\therefore \quad n = b^2 - 25$$

$$= (b - 5)(b + 5)$$

Let $a = b - 5$

then $b + 5 = a + 10$

$$\therefore \quad n = a(a + 10)$$

b Note: $0 < a(a + 10) < 50$

$\therefore \quad a(a + 10) - 50 < 0 \quad \ldots \boxed{1}$

and $\quad a(a + 10) > 0 \quad \ldots \boxed{2}$

From $\boxed{1}$ $\quad a^2 + 10a + 25 - 75 < 0$

$\therefore (a + 5)^2 - (5\sqrt{3})^2 < 0$

$\therefore (a + 5 - 5\sqrt{3})(a + 5 + 5\sqrt{3}) < 0$

$\therefore a < -5 + 5\sqrt{3}$ and $a > -5 - 5\sqrt{3}$

From $\boxed{2}$, $\; a < -10$ or $a > 0$

$\therefore \quad a = 3$ or 2 or 1 or 13 or -12 or -11

Consider $10p + q = a^2 + 10a$.

$a = 1,\ p = 1,\ q = 1$

$a = 2,\ p = 2,\ q = 4$

$a = 3,\ p = 3,\ q = 9$

$a = -11,\ p = 1,\ q = 1$

$a = -12,\ p = 2,\ q = 4$

$a = -13,\ p = 3,\ q = 9$

Hence $q = p^2$.

c From the above, $n = 11$ or 24 or 39.

13 a
$$\begin{aligned}\text{Distance} &= 0.5 + 9 \times 1.5\\ &= 0.5 + 13.5\\ &= 14\end{aligned}$$
The distance between the fence and the tenth row of carrots is 14 metres.

b
$$\begin{aligned}t_n &= 0.5 + (n-1) \times 1.5\\ &= 0.5 + 1.5n - 1.5\\ &= 1.5n - 1\end{aligned}$$

c
$$\begin{aligned}1.5n - 1 &< 80\\ \therefore\quad 1.5n &< 81\\ \therefore\quad n &< \frac{81}{1.5}\\ \therefore\quad n &< 54\end{aligned}$$
The largest number of rows possible is 53.

d
$$\begin{aligned}\text{Distance run by rabbit} &= 2 \times 0.5 + 2 \times (0.5 + 1.5) + 2 \times (0.5 + 2 \times 1.5) +\\ &\quad \cdots + 2 \times (0.5 + 14 \times 1.5)\\ &= 2(0.5 + (0.5 + 1.5) + (0.5 + 2 \times 1.5) +\\ &\quad \cdots + (0.5 + 14 \times 1.5))\\ &= 2\left(\frac{15}{2}(2 \times 0.5 + (15-1) \times 1.5)\right)\\ &= 15(1 + 21)\\ &= 330\end{aligned}$$
The shortest distance the rabbit has to run is 330 metres.

14 a i $a = 50\,000,\ d = 5000$

ii When $t_n = 2t_1$,

$50\,000 + (n-1) \times 5000 = 2 \times 50\,000$

$\therefore \quad 50\,000 + 5000n - 5000 = 100\,000$

$\therefore \quad 5000n + 45\,000 = 100\,000$

$\therefore \quad n = \dfrac{55\,000}{5000}$

$= 11$

The original production will double in the 11th month.

iii $S_n = \dfrac{n}{2}(2a + (n-1)d)$

$\therefore \quad S_{36} = \dfrac{36}{2}(2 \times 50\,000 + (36-1) \times 5000)$

$= 18(100\,000 + 35 \times 5000)$

$= 4\,950\,000$

In the first 36 months, 4 950 000 litres in total will be produced.

b **i** A geometric sequence applies in this case.

$q_n = ar^{n-1}$ where $a = 12\,000$ and $r = 1.1$

$= 12\,000(1.1)^{n-1}$

ii $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$\therefore \quad S_{12} = \dfrac{12\,000(1.1^{12} - 1)}{11 - 1}$

$= 256\,611.4052$

The total amount produced in the first 12 months is 256 611 litres, to the nearest litre.

c

$$q_n > t_n$$

$$\therefore \quad 12\,000(1.1)^{n-1} > 5000n + 45\,000$$

$$\therefore \quad (1.1)^{n-1} > \frac{5000(n+9)}{12\,000}$$

$$\therefore \quad (1.1)^{n-1} > \frac{5}{12}(n+9)$$

When $n = 30,\ (1.1)^{30-1} = 15.863\,09\ldots$ and $\frac{5}{12}(30+9) = 16.25$

$$\therefore \quad (1.1)^{30-1} < \frac{5}{12}(30+9)$$

When $n = 31,\ (1.1)^{31-1} = 17.449\,40\ldots$ and $\frac{5}{12}(31+9) = 16.666\,66\ldots$

$$\therefore \quad (1.1)^{30-1} > \frac{5}{12}(31+9)$$

Production of the second factory will exceed that of the first factory in the 31st month.

CAS calculator techniques for Question 14

TI: Input the following sequences into the graphs page.

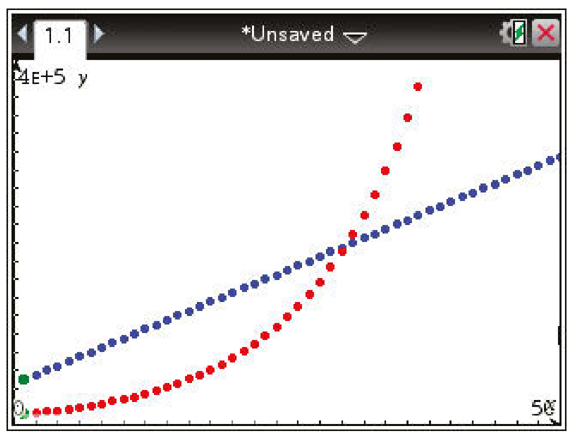
***u*1(*n*) = 50000+(*n* − 1)×5000**

***Initial Terms*:=50000** and

***u*2(*n*)=12000(1.1)$^{n-1}$**

***Initial Terms*:=50000**

Press /T to view the sequence.



CP: Input the following sequences into the Sequence application.

$\boldsymbol{a_{n+1} = 50000 + (n-1) \times 5000}$

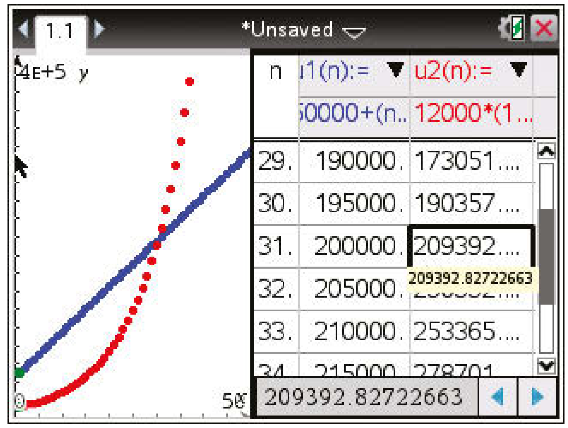
$\boldsymbol{a_0 = 50000}$

$\boldsymbol{b_{n+1} = 12000\,(1.1)^{n-1}}$

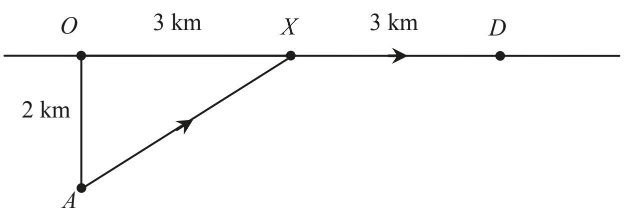
$\boldsymbol{b_0 = 12000}$

Tap 8 and change the Table Start value to 1 and the End value to 50. Tap # to view the sequence.

It can be seen that for $n = 31$ the geometric sequence exceeds the arithmetic sequence for the first time.



15 a



From the diagram,

$$AX = \sqrt{2^2 + 3^2}$$
$$= \sqrt{4 + 9}$$
$$= \sqrt{13}$$

Distance travelled = speed × time

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$\therefore \quad \text{Time taken for } AX = \frac{\sqrt{13}}{3}$$

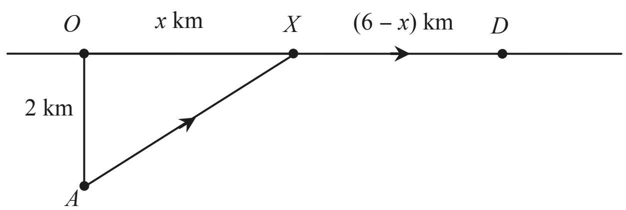
$$\text{Time taken for } XD = \frac{3}{8}$$

$$\text{Total time taken} = \frac{\sqrt{13}}{3} + \frac{3}{8}$$
$$= 1.57685\ldots$$

$$1.576\,85\ldots \text{ hours} = 1 \text{ hour and } 0.576\,85\ldots \times 60 \text{ minutes}$$
$$= 1 \text{ hour } 34.611\,02\ldots \text{ minutes}$$

The time taken was 1 hour 35 minutes, correct to the nearest minute.

b



From the diagram, $AX = \sqrt{2^2 + x^2}$

$= \sqrt{x^2 + 4}$

Off–road he walks at 3 km/h

$\therefore$ Time taken for $AX = \dfrac{\sqrt{x^2 + 4}}{3}$

On–road he walks at 8Km/h for a distance of $(6 - x)$ km

$\therefore$ Time taken for $XD = \dfrac{6 - x}{8}$

Total time taken $= \dfrac{\sqrt{x^2 + 4}}{3} + \dfrac{6 - x}{8} = \dfrac{3}{2}$

$\therefore \quad 8\sqrt{x^2 + 4} + 3(6 - x) = 36$

$\therefore \quad 8\sqrt{x^2 + 4} + 18 - 3x = 36$

$\therefore \quad 8\sqrt{x^2 + 4} = 3x + 18$

$\therefore \quad (8\sqrt{x^2 + 4})^2 = (3x + 18)^2$

$\therefore \quad 64(x^2 + 4) = 9x^2 + 108x + 324$

$\therefore \quad 64x^2 + 256 = 9x^2 + 108x + 324$

$\therefore \quad 55x^2 - 108x - 68 = 0$

$$x = \frac{+108 \pm \sqrt{(-108)^2 - 4 \times 55 \times (-68)}}{2 \times 55}$$

$= -0.501\,53\ldots, 2.465\,16\ldots$

but $x > 0$, $\therefore \quad x = 2.465\,16\ldots$

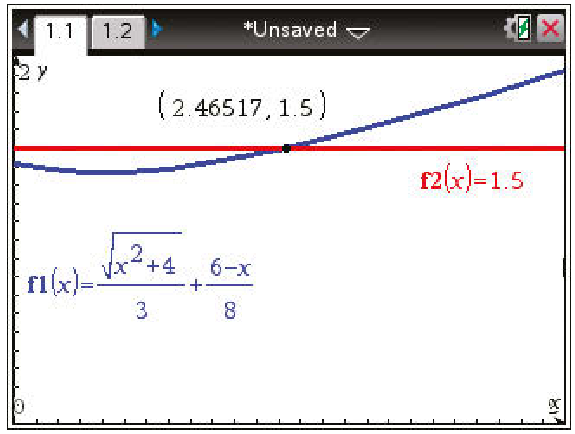
If the total time taken was $1\frac{1}{2}$ hours, OX is 2.5 km correct to one decimal place.

CAS calculator techniques for Question 19

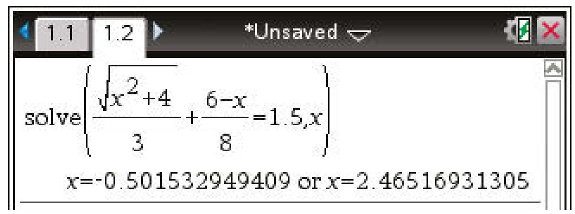
Sketch the graphs of $\boldsymbol{f1(x)} = \dfrac{\sqrt{x^2+4}}{3} + \dfrac{6-x}{8}$ and $\boldsymbol{f2(x)} = \mathbf{1.5}$

T1: Press **Menu→6**: **Analyze Graph→4**: **Intersection**

CP: Tap **Analysis→G-Solve→Intersect**



Alternatively, type $\mathbf{solve}\left(\dfrac{\sqrt{x^2+4}}{3} + \dfrac{6-x}{8} = \mathbf{1.5}, x\right)$ and interpret answers recalling $x > 0$.



16 Let x km/h be the average speed for the whole journey. Then $x - 2$ km/h is the speed for the last hour.

$$\frac{80 \times 0.25 + (x-2)}{1.25} = x$$
$$18 + x = 1.25x$$
$$18 = 0.25x$$
$$x = 72$$

a Distance= $72 \times 1.25 = 90$ km

b $x - 2 = 70$ km/h

17 **a** Let the original two digit number be $10a + b$ where a and b are digits. The new number is $1000 + 100a + 10b + 1$.
We require:
$1000 + 100a + 10b + 1 = 21(10a + b)$ There is only one possible solution. The

$$1001 = 110a + 11b$$
$$91 = 10a + b$$

number is 91.

b Let the number be $10\,000a + 1000b + 100c + 10d + e$ Adding a '1' at the back gives:
$100\,000a + 10\,000b + 1000a + 100b + 10c + 1$

Adding a 1 at the front gives $100\,000 + 10\,000a + 1000b + 100c + 10d + e$
We require:
$100\,000a + 10\,000b + 1000a + 100b + 10c + 1$

$= 3(100\,000 + 10\,000a + 1000b + 100c + 10d + e)$

$70\,000a + 7000b + 700c + 70d + 7e + 1 = 300\,000$

$7(10\,000a + 1000b + 100c + 10d + e) = 299999$

$10\,000a + 1000b + 100c + 10d + e = 42587$
This number satisfies the required property.

18 **a** The discriminant is $k^2 + 64 > 0$ for all integers k

The solutions are $x = \dfrac{-k \pm \sqrt{k^2 + 64}}{2}$. We need the discriminant to be a perfect square and $-k \pm \sqrt{k^2 + 64}$ to be even.
If $k^2 + 64$ is a perfct square there exists a positive integer n such that

$$k^2 + 64 = n^2$$

that is

$$n^2 - k^2 = 64$$

Factorising

$$(n - k)(n + k) = 64$$

We know $64 = 2^6$ and so all factors are powers of 2 Consider the simultaneous equations $n - k = 2^m$ and $k + n = 2^{6-m}$ for some positive integer m.

- We start with $n - k = 2$ and $k + n = 32$. The solutions are $n = 17$ and $k = 15$
  The quadratic $x^2 + 15x - 16 = 0$ has solutions $x = 1, x = -16$.

- Next $n - k = 4$ and $k + n = 16$. The solutions are $n = 10$ and $k = 6$
  The quadratic $x^2 + 6x - 16 = 0$ has solutions $x = 2, x = -8$.

- Next $n - k = 8$ and $k + n = 8$. The solutions are $n = 8$ and $k = 0$
  The quadratic $x^2 - 16 = 0$ has solutions $x = 4, x = -4$.

- Next $n - k = 16$ and $k + n = 4$. The solutions are $n = 10$ and $k = -6$
  The quadratic $x^2 - 6x - 16 = 0$ has solutions $x = 8, x = -2$.

- Next $n - k = 32$ and $k + n = 2$. The solutions are $n = 17$ and $k = -15$
  The quadratic $x^2 - 15k - 16 = 0$ has solutions $x = -1, x = 16$.

The possible values are $0, \pm 6, \pm 15$

b The discriminant is $k^2 - 80 > 0$ for all integers k

The solutions are $x = \dfrac{-k \pm \sqrt{k^2 - 80}}{2}$. We need the discriminant to be a perfect square and $-k \pm \sqrt{k^2 - 80}$ to be even.

If $k^2 - 80$ is a perfct square there exists a positive integer n such that

$$k^2 - 80 = n^2$$

that is

$$k^2 - n^2 = 80$$

Factorising

$$(k - n)(k + n) = 80$$

We know $80 = 2^4 \times 5$ As in **a** work through the factors. There are 10 factors.

- $k - n = 2$ and $k + n = 40, k = 21$.
  The quadratic is $x^2 + 21x + 20 = 0$. The solutions are $x = -20$ and $x = -1$

- $k - n = 8$ and $k + n = 10, k = 9$.
  The quadratic is $x^2 + 9x + 20 = 0$. The solutions are $x = -4$ and $x = -5$

- $k - n = 4$ and $k + n = 20, k = 12$.
  The quadratic is $x^2 + 12x + 20 = 0$. The solutions are $x = -2$ and $x = -10$

- and similarly there are another 3 values

c The discriminant is $144 - 4k > 0$ for all integers k

The solutions are $x = \dfrac{-12 \pm \sqrt{144 - 4k}}{2}$. We need the discriminant to be a perfect square and $-k \pm \sqrt{k^2 - 80}$ to be even.

We use trial and error with numbers 0 to 36 The numbers are: 11, 20, 27, 32, 35, 36

19 a
$$\begin{aligned} x^2 + (1 - x)^2 &= x^2 + 1 - 2x + x^2 \\ &= 2x^2 - 2x + 1 \\ &= 2[x^2 - \frac{x}{2} + \frac{1}{2}] \\ &= 2[x^2 - \frac{x}{2} + \frac{1}{4} + \frac{1}{4}] \\ &= 2[(x - \frac{1}{4})^2 + \frac{1}{4}] \end{aligned}$$

b Hence if $0 \le x \le 1$ maximum value of $x^2 + (1 - x)^2 = 2[(x - \frac{1}{4})^2 + \frac{1}{4}]$ occurs when $x = 0$ or $x = 1$. Hence maximum value is 1.

Also from the 'completed the square' form $x^2 + (1 - x)^2 = 2[(x - \frac{1}{4})^2 + \frac{1}{4}] \geq \frac{1}{2}$

Hence, $\frac{1}{2} \leq x^2 + (1 - x)^2 \leq 1$

c Let $ABCD$ be the unit square and the quadrilateral $PQRS$ with vertices on the square such that

- P is on side AB,
- Q is on side BC,
- R is on side CD,
- S is on side DA.

Let $AP = x, BQ = y, CR = z$ and $PS = w$
Let $PQ = a, QR = b, RS = c, SP = d$
Then
$a^2 = (1 - x)^2 + y^2$

$b^2 = (1 - y)^2 + z^2$

$c^2 = (1 - z)^2 + w^2$

$d^2 = (1 - w)^2 + x^2$

$$\begin{aligned}&a^2 + b^2 + c^2 + d^2\\&= (1 - x)^2 + x^2 + (1 - y)^2 + y^2\\&+ (1 - z)^2 + z^2 + (1 - w)^2 + w^2\end{aligned}$$

Hence

$$4 \times \frac{1}{2} \leq a^2 + b^2 + c^2 + d^2 \leq 4 \times 1$$

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$$

20 a If $x^2 + bx + c + 1 = 0$ has only one solution

Discriminant $= b^2 - 4(c + 1) = 0$

Therefore $b^2 = 4(c + 1)$

$$\begin{aligned}\frac{b^2}{4} &= c + 1\\ c &= \frac{b^2}{4} - 1\\ &= \frac{b^2 - 4}{4}\end{aligned}$$

b $x^2 + bx + c - 3 = (x - k)(x - 2k)$ $x^2 + bx + c - 3 = x^2 - 3k + 2k^2$

Hence

$b = -3k$ and $c - 3 = 2k^2$

$$\therefore c = \frac{2b^2}{9} + 3$$

c $$\frac{b^2 - 4}{4} = \frac{2b^2 + 27}{9}$$

$$9b^2 - 36 = 8b^2 + 108$$

$$b^2 = 144$$

$$b = \pm 12$$

$$\therefore c = 35$$

21 a $$\sqrt{9 - 4\sqrt{5}} = \sqrt{m} - n$$

$$9 - 4\sqrt{5} = m - 2n\sqrt{m} + n^2$$

Choose $m = 5$

Then $m + n^2 = 9$ and $2n = 4$

$\therefore m = 5$ and $n = 2$

b The other solution must be $-\sqrt{5} - 2$. $(x - (\sqrt{5} - 2))(x - (-\sqrt{5} - 2)) = x^2 + 4x - 1$

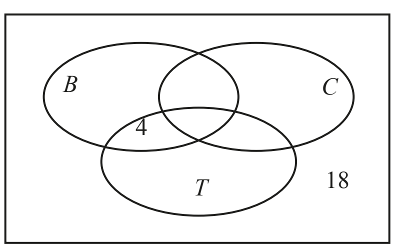
$\therefore b = 4, c = -1$

22 a $|B' \cap C' \cap T| = |C \cap T|$

$|B \cap C' \cap T'| = 3|B' \cap C \cap T'|$

$|B \cap C' \cap T| = 4$

b $n(\xi) = 76$



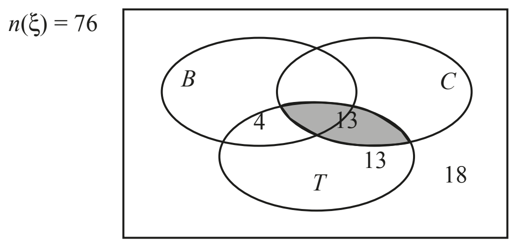
$|C \cap T| + |B' \cap C' \cap T| + |B \cap C' \cap T| = |T|$

$$\therefore \quad 2|C \cap T| + 4 = 30 \text{ as } |C \cap T| = |B' \cap C' \cap T|$$

$$\therefore \quad |C \cap T| = \frac{30 - 4}{2}$$

$$= 13$$

$$\therefore \quad |B' \cap C' \cap T| = 13$$



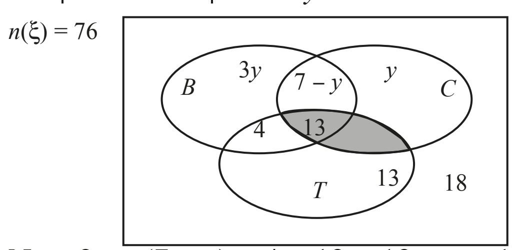
Let $|B' \cap C \cap T| = y$

$|B \cap C' \cap T'| = 3y$

$$|C| = |B' \cap C \cap T'| + |C \cap T| + |B \cap C \cap T'|$$

$$\therefore \quad 20 = y + 13 + |B \cap C \cap T'|$$

$\therefore \quad |B \cap C \cap T'| = 7 - y$



Now $3y + (7 - y) + 4 + 13 + 13 + y + 18 = 76$

$\therefore \quad 3y + 55 = 76$

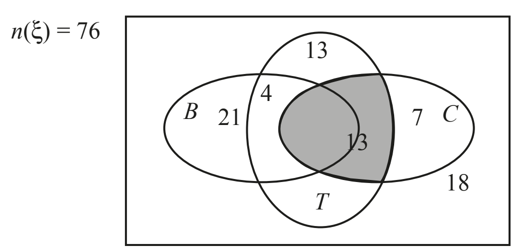
$\therefore \quad 3y = 21$

$\therefore \quad y = 7$

$|B' \cap C \cap T'| = 7$

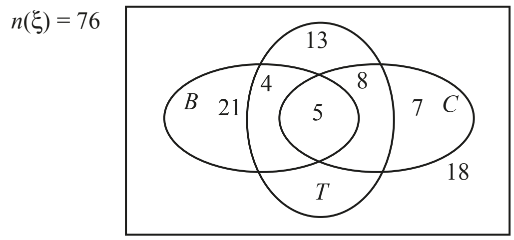
$|B \cap C' \cap T'| = 21$

$|B \cap C \cap T'| = 0$



$$|B \cap C \cap T| = |B| - |B \cap C' \cap T'| - |B \cap C' \cap T|$$
$$= 30 - 21 - 4$$
$$= 5$$

$\therefore \quad |B' \cap C \cap T| = 13 - 5 = 8$



c **i** $|B \cap C \cap T| = 5$

ii $|B \cap C \cap T'| = 0$

23 **a** **i** $600 \times 1.05 - 24 = 606$

ii $(600 \times 1.05 - 24) \times 1.05 - 24 \approx 612$

iii $612 \times 1.05 - 24 \approx 619$

b $t_n = 1.05t_{n-1} - 24, t_0 = 600$

c $t_n = 600 \times 1.05^n - \dfrac{24(1.05^n - 1)}{0.05}$
$t_n = 600 \times 1.05^n - 480((1.05)^n - 1))$

d $t_{12} = 600 \times 1.05^{12} - 480(1.05^{12} - 1)$
≈ 696

e $t_n = 0.85 \times t_{n-1} + 24, t_0 = 600 \ t_n = 600 \times 0.85^n - \dfrac{24(0.85^n - 1)}{0.15}$

i 534

ii 478

iii 223

f If stable: $t_n = t_{n-1}$
$\Rightarrow t_n = 0.85t_n + 24$
$\Rightarrow 0.15t_n = 24$
$\Rightarrow t_n = 160$

Use a spreadsheet or repeated use of formula. Solving the inequality with your calculator does give you a good start. You know what you are looking for.

$$600 \times 0.85^n - \frac{24(0.85^n - 1)}{0.15} < 160.5$$

$$n > 41.717...$$

This is an approximation and should be further tested with a spreadsheet.

24 a $1.35 \div (1.023)^{10} \approx 1.075$

The population 10 years ago was approximately 1.075 milllion

b $1.35 \times (1.028)^{10} \approx 1.779$

The population in ten years time will be approximately 1.779 million.

c Population of Beta now $= 1.25 \times 1.019^5 \approx 1.373$ million

Population of Beta is greater.

d In ten years time Alpha will have a population 0f 1.779 million. In ten years time population of Beta will be 1.658 million. The population of Alpha will be greater.

e

$$1.25 \times (1.019)^5 \times 1.019^n = 1.35 \times 1.028^n$$

$$n \approx 1.95$$

The populations will be approximately equal in two years time.

25 a $(a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$

$= a^2(a^2 + 2b^2 - 2ab) + 2b^2(a^2 + 2b^2 - 2ab) + 2ab(a^2 + 2b^2 - 2ab)$

$= a^4 + 2a^2b^2 - 2a^3b + 2a^2b^2 + 4b^4 - 4ab^3 + 2a^3b + 4ab^3 - 4a^2b^2$

$= a^4 + (2a^2b^2 + 2a^2b^2 - 4a^2b^2) + (-2a^3b + 2a^3b) + (-4ab^3 + 4ab^3) + 4b^4$

$= a^4 + 4b^4$

b From the identity If n is odd, $n = 2k + 1$ for some positive integer k.

$n^4 + 4^n = n^4 + 4^{2k+1} = n^4 + 4 \times (2^k)^4$ which is of the form of Sophie Germain's identity. Hence with $n = a$ and $b = 2^k$,

$n^4 + 4^n = (n^2 + 2 \times (2^k)^2 + 2 \times (2^k)n)(n^2 + 2 \times (2^k)^2 - 2 \times (2^k)n)$

c For $4^{545} + 545^4$ take $n = 545$ and $k = 272$

$4^{545} + 545^4 = (545^2 + 2 \times 2^{544} + 545 \times 2^{273})(545^2 + 2 \times 2^{544} - 545 \times 2^{273})$

26 a $a = 1, r = 4$

i $t_{10} = 4^9 = 262\,144$

ii Sum of n terms$= \frac{4^n - 1}{3}$. $\frac{4^n - 1}{3} = 349\,525$

$$4^n = 1\,048\,576$$

$$n = 10$$

b $a = 1, r = \frac{1}{4}$

i $t_{10} = \left(\frac{1}{4}\right)^9 = \frac{1}{262\,144}$

ii $S_{10} = \frac{1 - (\frac{1}{4})^{10}}{1 - \frac{1}{4}}$

$= \frac{4}{3}(1 - (\frac{1}{4})^{10}) \quad \approx 1.33$

c **i** We split the use of the 2: $2 = 1 + 1$

Consider the whole numbers first:

$1 + 4 + 16 + 64 + 256 + \cdots$

$S_{10} = \frac{4^{10} - 1}{4 - 1}$

$= \frac{1}{3}(4^{10} - 1)$

Now the fractional part.

$1 + \frac{1}{4} + \frac{1}{16} + \cdots$

$S_{10} = 1.33$ (from above)

Therefore total sum of 10 terms = 349 526.333

ii Using the two parts:

$S_n = \frac{4}{3}(1 - (\frac{1}{4})^n) + \frac{4^n - 1}{3}$

$\therefore S_n = \frac{1}{3}(4 - 4 \times \frac{1}{4})^n + 4^n - 1) \therefore S_n = \frac{1}{3}(-4^{1-n} + 4^n) + 1$

Solutions for Investigations

1 **a** Let ℓ be the length and w the width.

Let P be the constant perimeter.

Then $\ell + w = \frac{P}{2}$

Area, A, of the rectangle$= \ell \times w$ and

$\sqrt{\ell w} \leq \frac{1}{2}(\ell + w) = \frac{P}{4}$by the AM-GM inequality

Hence $\sqrt{A} = \sqrt{\ell w} \leq \frac{P}{4}$

Hence $A \le \left(\frac{P}{4}\right)^2$

Also if $\ell = w, A = \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$

Hence the maximum area is $\frac{P^2}{16}$ and this occurs when $\ell = w$

b Let ℓ be the length and w the width.

Let A be the constant area.

Then $A = \ell \times w$.

Hence $\sqrt{A} = \sqrt{\ell \times w} \le \frac{1}{2}(\ell + w)$ by the AM-GM inequality.

$\therefore \frac{P}{4} \ge \sqrt{A}$

$\therefore P \ge 4\sqrt{A}$

When $w = \ell = \sqrt{A}, P = 2(\sqrt{A} + \sqrt{A}) = 4\sqrt{A}$

Hence the minimum perimeter is $4\sqrt{A}$ and this occurs when $w = \ell = \sqrt{A}$

c $\frac{x^3 + y^3 + z^3}{3} \ge xyz$

$\iff x^3 + y^3 + z^3 - 3xyz \ge 0$

$\iff (x + y + z)((x - y)^2 + (y - z)^2 + (z - a)^2) \ge 0$

Hence $\frac{x^3 + y^3 + z^3}{3} \ge abc$

Let $a = \sqrt[3]{x}, b = \sqrt[3]{y}$ and $c = \sqrt[3]{z}$

Then $\frac{a + b + c}{3} \ge \sqrt[3]{abc}$

d Let a, b and c be the length, width and height of the rectangular prism.

Let S be the surface area of the prism which we assume to be constant.

Let V be the volume.

$S = 2(ab + ac + bc)$ and $(ab + bc + ca) = \frac{S}{2}$ and $V = abc$

Using the AM-GM inequality

$\sqrt[3]{a^2b^2c^2} \le \frac{1}{3}(ab + bc + ca)$

$\Rightarrow \sqrt[3]{V^2} \le \frac{S}{6}$

$\Rightarrow V \le \sqrt{\left(\frac{S}{6}\right)^3}$

Furthermore when $a = b = c, V = \sqrt{\left(\frac{S}{6}\right)^3}$.

2 **a** The sequences a_n, b_n are shown with $a_1 = 2$ and $b_1 = 3$

A	B	C
n	a	b
1	2	3
2	2.5	2.44948974
3	2.47474487	2.474616
4	2.47468044	2.47468044
5	2.47468044	2.47468044
6	2.47468044	2.47468044
7	2.47468044	2.47468044
8	2.47468044	2.47468044
9	2.47468044	2.47468044

n	a	b
1	2	3
=A2+1	=0.5*(B2+C2)	=SQRT(B2*C2)
=A3+1	=0.5*(B3+C3)	=SQRT(B3*C3)
=A4+1	=0.5*(B4+C4)	=SQRT(B4*C4)
=A5+1	=0.5*(B5+C5)	=SQRT(B5*C5)
=A6+1	=0.5*(B6+C6)	=SQRT(B6*C6)
=A7+1	=0.5*(B7+C7)	=SQRT(B7*C7)
=A8+1	=0.5*(B8+C8)	=SQRT(B8*C8)
=A9+1	=0.5*(B9+C9)	=SQRT(B9*C9)

It converges very quickly.

b
$$\begin{aligned} a_{n+1} - b_{n+1} &= \frac{1}{2}(a_n + b_n) - \sqrt{a_n b_n} \\ &= \frac{1}{2}(a_n + b_n - 2\sqrt{a_n b_n}) \\ &= \frac{1}{2}(\sqrt{a_n} - \sqrt{b_n})^2 \end{aligned}$$

Also $a_{n+1} - b_{n+1} \leq \frac{1}{2}(a_n - b_n)$

c Observe that $a_n <$ arithmetic-geometric mean $< b_n$ if $a_1 \neq b_1$

We know $b_n \leq a_n$ by the AM-GM inequality. Therefore $b_{n+1} = \sqrt{b_n a_n} \geq \sqrt{b_n^2} = b_n$.

It is also true that $x_1 <$ arithmetic-geometric mean $< b_1$

It is bounded above and below and is increasing.

There must be a limit b of the sequence $\{b_n\}$. Furthermore $a_n = \dfrac{b_{n+1}^2}{g_n}$ and we can see from this that the sequence $\{a_n\}$ must also approach b.

This number is arithmetic-geometric mean.

d We find that

$$\begin{aligned} a_{n+1} - b_{n+1} &= \frac{a_n + b_n}{2} - b_{n+1} \\ &\leq \frac{a_n + b_n}{2} - b_n \quad (\text{since } b_{n+1} \geq b_n) \\ &= \frac{1}{2}(a_n - b_n) \end{aligned}$$

as required.

3 **a**

A	B	C
n	x	y
1	30000	5000
2	27000	6000
3	26400	6900
4	25860	7080
5	25752	7242
6	25655	7274.4
7	25635	7303.56
8	25618	7309.392
9	25614	7314.6408
10	25611	7315.69056
11	25611	7316.63534
12	25610	7316.8243
13	25610	7316.99436
14	25610	7317.02837
15	25610	7317.05899
16	25610	7317.06511
17	25610	7317.07062
18	25610	7317.07172
19	25610	7317.07271

A	B	C
n	x	y
1	30000	5000
=A2+1	=30000-0.6*C2	=15000-0.3*B2
=A3+1	=30000-0.6*C3	=15000-0.3*B3
=A4+1	=30000-0.6*C4	=15000-0.3*B4
=A5+1	=30000-0.6*C5	=15000-0.3*B5
=A6+1	=30000-0.6*C6	=15000-0.3*B6
=A7+1	=30000-0.6*C7	=15000-0.3*B7
=A8+1	=30000-0.6*C8	=15000-0.3*B8
=A9+1	=30000-0.6*C9	=15000-0.3*B9
=A10+1	=30000-0.6*C10	=15000-0.3*B10
=A11+1	=30000-0.6*C11	=15000-0.3*B11
=A12+1	=30000-0.6*C12	=15000-0.3*B12
=A13+1	=30000-0.6*C13	=15000-0.3*B13
=A14+1	=30000-0.6*C14	=15000-0.3*B14
=A15+1	=30000-0.6*C15	=15000-0.3*B15
=A16+1	=30000-0.6*C16	=15000-0.3*B16
=A17+1	=30000-0.6*C17	=15000-0.3*B17
=A18+1	=30000-0.6*C18	=15000-0.3*B18
=A19+1	=30000-0.6*C19	=15000-0.3*B19

Consider

$$
\begin{aligned}
x_{n+1} &= x_n \\
30\,000 - 0.6y_n &= x_n \\
30\,000 - 0.6(15\,000 - 0.3x_{n-1}) &= x_n \\
30\,000 - 0.6(15\,000 - 0.3x_n) &= x_n \\
x_n &= 25609.756\ldots
\end{aligned}
$$

Therefore $y_n = 7317.07\ldots$ Equilibrium occurs when $x = 25\,610$ and $y = 7317$. That is 25 606 units from distributor X and 7317 from distributor Y

b

n	x	y	z
1	30000	5000	1000
2	27500	8800	27500
3	25600	4000	25600
4	28000	4760	28000
5	27620	3800	27620
6	28100	3952	28100
7	28024	3760	28024
8	28120	3790.4	28120
9	28105	3752	28104.8
10	28124	3758.08	28124
11	28121	3750.4	28120.96
12	28125	3751.616	28124.8
13	28124	3750.08	28124.192
14	28125	3750.3232	28124.96
15	28125	3750.016	28124.8384
16	28125	3750.06464	28124.992
17	28125	3750.0032	28124.9677
18	28125	3750.01293	28124.9984
19	28125	3750.00064	28124.9935
20	28125	3750.00259	28124.9997
21	28125	3750.00013	28124.9987
22	28125	3750.00052	28124.9999
23	28125	3750.00003	28124.9997

Equilibrium occurs when $x = 28\,125$, $y = 3750$ and $z = 28\,124$. That is 28125 units from distributor X , 3750 from distributor Y and 28 124 units from distributor Z.

4 a $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{3}$

$$\frac{x+y}{xy} = 3$$

$$3x + 3y = xy$$

$$3x = y(x-3)$$

$$y = \frac{3x}{x-3}$$

$$= 3 + \frac{9}{x-3}$$

Since x and y are positive integers, $x - 3 \le 9$ and $x - 3$ divides 9.

Possible values for $x - 3$ are 1, 3 and 9. Therefore the possible values of x are 4, 6 and 12.

The required pairs are (4, 12), (6, 6) and (12, 4).

b $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{11}$

$$\frac{x+y}{xy} = 11$$

$$11x + 11y = xy$$

$$11x = y(x-11)$$

$$y = \frac{11x}{x-11}$$

$$= 11 + \frac{121}{x-11}$$

As x and y are positive integers, $x - 11 \le 11$ and $x - 11$ divides 121.

Possible values for $x - 11$ are 1, 11 and 121. Therefore the possible values of x are 12, 22 and 132.

The required pairs are (12, 132), (22, 22) and (132, 12).

c As in the above cases $y = p + \dfrac{p^2}{x-p}$ Hence $x \le p^2$ and $x - p$ divides p^2. Since p is prime the factors of p^2 are 1, p and p^2. Hence $x = p + 1, 2p$ or $p^2 + p$.

The ordered pairs are $(p + 1, p^2 + p), (2p, 2p)$ and $(p^2 + p, p + 1)$

d We will provide another approach that allows us to quickly see that there will always be a solution and will also allow us to identify the number of solutions:

$$\begin{aligned}
\frac{1}{x}+\frac{1}{y} &= \frac{1}{n}\\
\frac{x+y}{xy} &= \frac{1}{n}\\
nx+ny &= xy\\
xy-nx-ny &= 0\\
x(y-n)-ny &= 0\\
x(y-n)-n(y-n)-n^2 &= 0\\
(x-n)(y-n) &= n^2
\end{aligned}$$

For each factor $a > 0$ of n^2 we can write $n^2 = ab$, giving the solution

$$x-n=a \text{ and } y-n=b \Longrightarrow x=a+n \text{ and } y=b+n \Longrightarrow (x,y)=(a+n,b+n).$$

Therefore the number of solutions is equal to the number of factors of n^2.

5 a i $\text{RHS} = \frac{1}{n}-\frac{1}{n+1}$

$$\begin{aligned}
&= \frac{n+1-n}{n(n+1)}\\
&= \frac{1}{n(n+1)}\\
&= \text{LHS}
\end{aligned}$$

ii $\text{LHS} = \frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{99\times 100}$

Using part **ai**

$$\begin{aligned}
&= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{98}-\frac{1}{99}\right)+\left(\frac{1}{99}-\frac{1}{100}\right)\\
&= 1-\frac{1}{100}\\
&= \frac{99}{100}\\
&= \text{RHS}
\end{aligned}$$

iii First note that we have partial sums of the arithmetic sequence $1, 2, 3, \ldots$ in the denominator of the fractions.

- $1+2 = \frac{2}{2}(2+(2-1)) = \frac{2\times 3}{2}$

- $1 + 2 + 3 = \frac{3}{2}(2 + (3 - 1) = \frac{3 \times 4}{2}$

$\vdots$

$\vdots$

- $1 + 2 + 3 + \cdots + 99 = \frac{99}{2}(2 + (99 - 1) = \frac{99 \times 100}{2}$

Hence,

$$\frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+99}$$

$$= \frac{2}{2 \times 3} + \frac{2}{3 \times 4} + \cdots \frac{2}{99 \times 100}$$

$$= \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) + \cdots + \left(\frac{2}{99} - \frac{2}{100}\right)$$

$$= 1 - \frac{1}{50}$$

$$= \frac{49}{50}$$

b **i**

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

$$n = -2 \Rightarrow -2B = 1 \Rightarrow B = -\frac{1}{2}$$

$$n = 0 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

$$\therefore \frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$$

ii $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{99 \times 101}$

Using part **bi**

$$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots + \left(\frac{1}{194} - \frac{1}{198}\right) + \left(\frac{1}{198} - \frac{1}{202}\right)$$

$$= \frac{1}{2} - \frac{1}{202}$$

$$= \frac{50}{101}$$

iii $$\frac{1}{n(n+5)} = \frac{A}{n} + \frac{B}{n+5}$$

$$1 = A(n+5) + Bn$$

$$n = -5 \Rightarrow -5B = 1 \Rightarrow B = -\frac{1}{5}$$

$$n = 0 \Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5}$$

$$\therefore \frac{1}{n(n+5)} = \frac{1}{5n} - \frac{1}{5(n+5)}$$

We use this result in the following:

$$\frac{1}{1\times 6} + \frac{1}{6\times 11} + \frac{1}{11\times 16} + \cdots + \frac{1}{96\times 101}$$

Using part **bi**

$$= \left(\frac{1}{5} - \frac{1}{30}\right) + \left(\frac{1}{30} - \frac{1}{55}\right) + \left(\frac{1}{55} - \frac{1}{80}\right) + \cdots + \left(\frac{1}{455} - \frac{1}{480}\right) + \left(\frac{1}{480} - \frac{1}{505}\right)$$

$$= \frac{1}{5} - \frac{1}{505}$$

$$= \frac{20}{101}$$

c **i** $$\text{RHS} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{n(n+1)}\right) - \frac{1}{2}\left(\frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{1}{2(n+1)}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \frac{1}{n(n+1)(n+2)}$$

$$= \text{LHS}$$

ii $$\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \cdots + \frac{1}{98\times 99\times 100}$$

Using part **ci**

$$= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + + \cdots + \left(\frac{1}{98} - \frac{1}{99}\right) - \left(\frac{1}{99} - \frac{1}{100}\right)\right]$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) - \left(\frac{1}{99} - \frac{1}{100}\right)\right]$$

$$= \frac{4949}{19800}$$

d **i** $\dfrac{1}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \times \dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$

$= \sqrt{n+1}-\sqrt{n}$

ii $\dfrac{1}{\sqrt{2}+\sqrt{1}} + \dfrac{1}{\sqrt{3}+\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{100}+\sqrt{99}}$

Using part **di**

$= \sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\cdots+\sqrt{100}-\sqrt{99}$

$= \sqrt{100}-1$

$= 9$

iii $\dfrac{1}{\sqrt{n+2}+\sqrt{n}} = \dfrac{1}{\sqrt{n+2}+\sqrt{n}} \times \dfrac{\sqrt{n+2}-\sqrt{n}}{\sqrt{n+2}-\sqrt{n}}$

$= \dfrac{1}{2}\left(\sqrt{n+2}-\sqrt{n}\right)$

Using this result, $\dfrac{1}{\sqrt{3}+\sqrt{1}} + \dfrac{1}{\sqrt{5}+\sqrt{4}} + \cdots + \dfrac{1}{\sqrt{121}+\sqrt{119}}$

$= \dfrac{1}{2}(\sqrt{3}-\sqrt{1}+\sqrt{5}-\sqrt{3}+\cdots+\sqrt{121}-\sqrt{119})$

$= \sqrt{121}-1$

$= 10$

Chapter 6 – Number and Proof

Solutions to Exercise 6A

1 **a** As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= 2p + 2q \\ &= 2(p + q), \end{aligned}$$

is an even number.

b As m and n are even, $m = 2p$ and $n = 2q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (2p)(2q) \\ &= 4pq \\ &= 2(2pq), \end{aligned}$$

is an even number.

2 As m and n are odd, $m = 2p + 1$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m + n &= (2p + 1) + (2q + 1) \\ &= 2p + 2q + 2 \\ &= 2(p + q + 1), \end{aligned}$$

is an even number.

3 As m is even and n is odd, $m = 2p$ and $n = 2q + 1$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= 2p(2q + 1) \\ &= 2(2pq + p), \end{aligned}$$

is an even number.

4 **a** If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3p)(7q) \\ &= 21pq, \end{aligned}$$

is divisible by 21.

b If m is divisible by 3 and n is divisible by 7, then $m = 3p$ and $n = 7q$ where $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m^2 n &= (3p)^2(7q) \\ &= 9p^2(7q) \\ &= 63p^2 q \end{aligned}$$

is divisible by 63.

5 If m and n are perfect squares then $m = a^2$ and $n = b^2$ for some $a, b \in \mathbb{Z}$. Therefore,

$$mn = (a^2)(b^2) = (ab)^2,$$

is also a perfect square.

6 Expanding both brackets gives,

$$\begin{aligned} &(m + n)^2 - (m - n)^2 \\ =&m^2 + 2mn + n^2 - (m^2 - 2mn + n^2) \\ =&m^2 + 2mn + n^2 - m^2 + 2mn - n^2 \\ =&4mn, \end{aligned}$$

which is divisible by 4.

7 (Method 1) If n is even then n^2 is even and $6n$ is even. Therefore the expression is of the form

$$\text{even} - \text{even} + \text{odd} = \text{odd}.$$

(Method 2) If n is even then $n = 2k$

where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 - 6n + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 4k^2 - 12k + 4 + 1 \\ &= 2(2k^2 - 6k + 2) + 1, \quad = \end{aligned}$$

is odd.

8 (Method 1) If n is odd then n^2 is odd and $8n$ is even. Therefore the expression is of the form

odd + even + odd = even.

(Method 2) If n is odd then $n = 2k + 1$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^2 + 8n + 5 &= (2k+1)^2 + 8(2k+1) + 3 \\ &= 4k^2 + 4k + 1 + 16k + 8 + 3 \\ &= 4k^2 + 20k + 12 \\ &= 2(2k^2 + 10k + 6), \end{aligned}$$

is even.

9 First suppose n is even. Then $5n^2$ and $3n$ are both even. Therefore the expression is of the form

even + even + odd = odd.

Now suppose n is odd. Then $5n^2$ and $3n$ are both odd. Therefore the expression is of the form

odd + odd + odd = odd.

10 Firstly, if $x > y$ then $x - y > 0$. Secondly, since x and y are positive, $x + y > 0$.

Therefore,

$$\begin{aligned} & x^4 - y^4 \\ =& (x^2 - y^2)(x^2 + y^2) \\ =& (x - y)(x + y)(x^2 + y^2) \\ =& \overbrace{(x - y)}^{\text{positive}}\overbrace{(x + y)}^{\text{positive}}\overbrace{(x^2 + y^2)}^{\text{positive}} \\ >& 0. \end{aligned}$$

Therefore, $x^4 > y^4$.

11 We have,

$$\begin{aligned} & x^2 + y^2 - 2xy \\ =& x^2 - 2xy + y^2 \\ =& (x - y)^2 \\ \geq& 2xy. \end{aligned}$$

Therefore, $x^2 + y^2 \geq 2xy$.

12 **a** We prove that Alice is a knave, and Bob is a knight.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice and Bob are both knaves
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are not both knaves
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

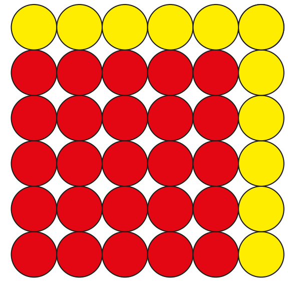
b We prove that Alice is a knave, and Bob is a knight.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ They are both of the same kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is lying
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is a knight and a knave.
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice and Bob are of a different kind
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Alice is a knave, and Bob is a knight

c We will prove that Alice is a knight, and Bob is a knave.

Suppose Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Bob is a knight
$\Rightarrow$ Bob is telling the truth
$\Rightarrow$ Neither of them are knaves
$\Rightarrow$ Both of them are knights
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Bob is lying
$\Rightarrow$ At least one of them is a knave
$\Rightarrow$ Bob is a knave
$\Rightarrow$ Alice is a knight, and Bob is a knave.

13 a In the diagram below, there are 11 yellow tiles. We can also count the yellow tiles by subtracting the number of red tiles, 5^2, from the total number of tiles, 6^2. Therefore $11 = 6^2 - 5^2$.



b Every odd number is of the form $2k + 1$ for some $k \in \mathbb{Z}$. Moreover,

$$\begin{aligned}(k+1)^2 - k^2 &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1,\end{aligned}$$

so that every odd number can be written as the difference of two squares.

c Since $101 = 2 \times 50 + 1$, we have,

$$51^2 - 50^2 = 101.$$

14 a Since

$$\frac{9}{10} = \frac{99}{110} \text{ and } \frac{10}{11} = \frac{100}{110},$$

it is clear that

$$\frac{10}{11} > \frac{9}{10}.$$

b We have,

$$\begin{aligned}&\frac{n}{n+1} - \frac{n-1}{n} \\ &= \frac{n^2}{n(n+1)} - \frac{n(n-1)}{n(n+1)} \\ &= \frac{n^2 - n(n-1)}{n(n+1)} \\ &= \frac{n^2 - n^2 + n}{n(n+1)} \\ &= \frac{1}{n(n+1)} \\ &> 0\end{aligned}$$

since $n(n+1) > 0$. Therefore,

$$\frac{n}{n+1} > \frac{n-1}{n}.$$

15 a We have,

$$\begin{aligned}&\frac{1}{10}-\frac{1}{11}\\=&\frac{11}{110}-\frac{10}{110}\\=&\frac{1}{110}\\<&\frac{1}{100},\end{aligned}$$

since $110 > 100$.

b We have,

$$\begin{aligned}\frac{1}{n}-\frac{1}{n+1}&=\frac{n+1}{n(n+1)}-\frac{n}{n(n+1)}\\&=\frac{n+1-n}{n(n+1)}\\&=\frac{1}{n(n+1)},\\&=\frac{1}{n^2+n},\\&<\frac{1}{n^2},\end{aligned}$$

since $n^2 + n > n^2$.

16 We have,

$$\begin{aligned}&\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\\=&\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}\\=&\frac{2a^2+2b^2}{4}-\frac{a^2+2ab+b^2}{4}\\=&\frac{2a^2+2b^2-a^2-2ab-b^2}{4}\\=&\frac{a^2-2ab+b^2}{4}\\=&\frac{(a-b)^2}{4}\\\geq&0.\end{aligned}$$

17 a Expanding gives,

$$\begin{aligned}&(x-y)(x^2+xy+y^2)\\=&x^3+x^2y+xy^2-x^2y-xy^2-y^3\\=&x^3-y^3,\end{aligned}$$

which is the difference of two cubes.

b Completing the square by treating y as a constant gives,

$$\begin{aligned}&x^2+yx+y^2\\=&x^2+yx+\frac{y^2}{4}-\frac{y^2}{4}+y^2\\=&\left(x^2+yx+\frac{y^2}{4}\right)+\frac{3y^2}{4}\\=&\left(x+\frac{y}{2}\right)^2+\frac{3y^2}{4}\\\geq&0\end{aligned}$$

c Firstly, if $x \geq y$ then $x - y \geq 0$. Therefore,

$$\begin{aligned}&x^3-y^3\\=&\overbrace{(x-y)}^{\geq 0}\overbrace{(x^2+xy+y^2)}^{\geq 0}\\\geq&0.\end{aligned}$$

Therefore, $x^3 > y^3$.

18 a Let D be the distance to and from work. The time taken to get to work is $D/12$ and the time taken to get home from work is $D/24$. The total distance is $2D$ and the total time is

$$\frac{D}{12}+\frac{D}{24}$$
$$=\frac{2D}{24}+\frac{D}{24}$$
$$=\frac{3D}{24}$$
$$=\frac{D}{8}$$

The average speed will then be

$$\text{distance} \div \text{time}$$
$$=2D \div \frac{D}{8}$$
$$=2D \times \frac{8}{D}$$
$$=16 \text{ km/hour.}$$

b Let D be the distance to and from work. The time taken to get to work is D/a and the time taken to get home from work is D/b. The total distance is $2D$ and the total time is

$$\frac{D}{a}+\frac{D}{b}$$
$$=\frac{bD}{ab}+\frac{aD}{ab}$$
$$=\frac{aD+bD}{ab}$$
$$=\frac{(a+b)D}{ab}$$

The average speed will then be

$$\text{distance} \div \text{time}$$
$$=2D \div \frac{(a+b)D}{ab}$$
$$=2D \times \frac{ab}{(a+b)D}$$
$$=\frac{2ab}{a+b} \text{ km/hour.}$$

c We first note that $a+b>0$. Secondly,

$$\frac{a+b}{2}-\frac{2ab}{a+b}$$
$$=\frac{(a+b)^2}{2(a+b)}-\frac{4ab}{2(a+b)}$$
$$=\frac{(a+b)^2-4ab}{2(a+b)}$$
$$=\frac{a^2+2ab+b^2-4ab}{2(a+b)}$$
$$=\frac{a^2-2ab+b^2}{2(a+b)}$$
$$=\frac{(a-b)^2}{2(a+b)}$$
$$\geq 0$$

since $(a-b)\geq 0$ and $a+b>0$. Therefore,

$$\frac{a+b}{2}\geq\frac{2ab}{a+b}.$$

Solutions to Exercise 6B

1 **a** $P : 1 > 0$ (true)
not $P : 1 \leq 0$ (false)

b P : 4 is divisible by 8 (false)
not P : 4 is not divisible by 8 (true)

c P : Each pair of primes has an even sum (false)
not P : Some pair of primes does not have an even sum (true)

d P : Some rectangle has 4 sides of equal length (true)
not P : No rectangle has 4 sides of equal length (false)

2 **a** P : 14 is divisible by 7 and 2 (true)
not P : 14 is not divisible by 7 or 14 is not divisible by 2 (false)

b P : 12 is divisible by 3 or 4 (true)
not P : 12 is not divisible by 4 and 12 is not divisible by 3 (false)

c P : 15 is divisible by 3 and 6 (false)
not P : 15 is not divisible by 3 or 15 is not divisible by 6 (true)

d P : 10 is divisible by 2 or 3 (false)
not P : 10 is not divisible by 2 and 10 is not divisible by 3 (true)

3 We will prove that Alice is a knave, and Bob is a knave.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice is a knight OR Bob is a knave
$\Rightarrow$ Bob is a knave, as Alice is not a knight
$\Rightarrow$ Alice and Bob are both knaves.

4 **a** If there are no clouds in the sky, then it is not raining.

b If you are not happy, then you are not smiling.

c If $2x \neq 2$, then $x \neq 1$.

d If $x^5 \leq y^5$, then $x \leq y$.

e Option 1: If n is not odd, then n^2 is not odd.
Option 2: If n is even, then n^2 is even.

f Option 1: If mn is not odd, then n is not odd or m is not odd.
Option 2: If If mn is even, then n is even or m is even.

g Option 1: If m and n are not both even or both odd, then $m + n$ is not even.
Option 2: If n and n are not both even or both odd, then $m + n$ is odd.

5 **a** Contrapositive: If n is even then $3n + 5$ is odd.
Proof: Suppose n is even. Then

$n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 3n + 5 &= 3(2k) + 5 \\ &= 6k + 5 \\ &= 6k + 4 + 1 \\ &= 2(3k + 2) + 1 \end{aligned}$$

is odd.

b Contrapositive: If n is even, then n^2 is even.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

is even.

c Contrapositive: If n is even, then $n^2 - 8n + 3$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 - 8n + 3 &= (2k)^2 - 8(2k) + 3 \\ &= 4k^2 - 16k + 3 \\ &= 4k^2 - 16k + 2 + 1 \\ &= 2(2k^2 - 8k + 1) + 1 \end{aligned}$$

is odd.

d Contrapositive: If n is divisible by 3, then n^2 is divisible by 3.
Proof: Suppose n is divisible by 3. Then $n = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (3k)^2 \\ &= 9k^2 \\ &= 3(3k^2) \end{aligned}$$

is divisible by 3.

e Contrapositive: If n is even, then $n^3 + 1$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 + 1 &= (2k)^3 + 1 \\ &= 8k^3 + 1 \\ &= 2(4k^3) + 1 \end{aligned}$$

is odd.

f Contrapositive: If m or n are divisible by 3, then mn is divisible by 3.
Proof: If m or n is divisible by 3 then we can assume that m is divisible by 3. Then, $m = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3k)n \\ &= 3(kn) \end{aligned}$$

is divisible by 3.

g Contrapositive: If $m = n$, then $m + n$ is even.
Proof: Suppose that $m = n$. Then

$$\begin{aligned} m + n &= n + n \\ &= 2n \end{aligned}$$

is even.

6 a Contrapositive: If $x \geq 0$, then $x^2 + 3x \geq 0$.
Proof: Suppose that $x \geq 0$. Then,

$$x^2 + 3x = x(x + 3) \geq 0,$$

since $x \geq 0$ and $x + 3 \geq 0$.

b Contrapositive: If $x \leq -1$, then $x^3 - x \leq 0$.
Proof: Suppose that $x \leq -1$. Then,

$$x^3 - x = x(x^2 - 1) \leq 0,$$

since $x^2 - 1 \geq 0$ and $x \leq 0$.

c Contrapositive: If $x < 1$ and $y < 1$, then $x + y < 2$.
Proof: If $x < 1$ and $y < 1$ then,

$$x + y < 1 + 1 = 2,$$

as required.

d Contrapositive: If $x < 3$ and $y < 2$, then $2x + 3y < 12$.
Proof: If $x < 3$ and $y < 2$ then,

$$2x + 3y < 2 \times 3 + 3 \times 2 = 6 + 6 = 12,$$

as required.

7 **a** Contrapositive: If m is odd or n is odd, then mn is odd or $m + n$ is odd.

b Proof:
(Case 1) Suppose m is odd and n is odd. Then clearly mn is odd.
(Case 2) Suppose m is odd and n is even. Then clearly $m + n$ will be odd.
It is likewise, if m is even and n is odd.

8 **a** We rationalise the right hand side to give,

$$\begin{aligned}
&\frac{x-y}{\sqrt{x}+\sqrt{y}} \\
&= \frac{x-y}{\sqrt{x}+\sqrt{y}}\,\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}} \\
&= \frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)} \\
&= \frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{(x-y)} \\
&= \sqrt{x}-\sqrt{y}.
\end{aligned}$$

b If $x > y$ then $x - y > 0$. Then, using the above equality, we see that,

$$\sqrt{x}-\sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}} > 0,$$

since the numerator and denominator are both positive. Therefore, $\sqrt{x} > \sqrt{y}$.

c Contrapositive: If $\sqrt{x} \le \sqrt{y}$, then $x \le y$.
Proof: If $\sqrt{x} \le \sqrt{y}$ then, since both sides are positive, we can square both sides to give $x \le y$.

Solutions to Exercise 6C

1 If all three angles are less than 60°, then the sum of interior angles of the triangle would be less than 180°. This is a contradiction as the sum of interior angles is exactly 180°.

2 Suppose there is some least positive rational number $\frac{p}{q}$. Then since,

$$\frac{p}{2q} < \frac{p}{q},$$

there is some lesser positive rational number, which is a contradiction. Therefore, there is no least positive rational number.

3 Suppose that $\sqrt{p}$ is an integer. Then

$$\sqrt{p} = n,$$

for some $n \in \mathbb{Z}$. Squaring both sides gives

$$p = n^2.$$

Since $n \neq 1$, this means that p has three factors: $1, n$ and n^2. This is a contradiction since every prime number has exactly two factors.

4 Suppose that x is rational so that $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 3^x = 2 \\ \Rightarrow\ & 3^{\frac{p}{q}} = 2 \\ \Rightarrow\ & \left(3^{\frac{p}{q}}\right)^q = 2^q \\ \Rightarrow\ & 3^p = 2^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

5 Suppose that $\log_2 5$ is rational so that $\log_2 5 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 2^{\frac{p}{q}} = 5 \\ \Rightarrow\ & 2^{\frac{p}{q}} = 5 \\ \Rightarrow\ & \left(2^{\frac{p}{q}}\right)^q = 5^q \\ \Rightarrow\ & 2^p = 5^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

6 Suppose the contrary, so that $\sqrt{x}$ is rational. Then

$$\sqrt{x} = \frac{p}{q},$$

where $p, q \in \mathbb{Z}$. Then, squaring both sides of the equation gives,

$$x = \frac{p^2}{q^2},$$

where $p^2, q^2 \in \mathbb{Z}$. Therefore, x is rational, which is a contradiction.

7 Suppose, on the contrary that $a + b$ is rational. Then

$$b = \overbrace{(a+b)}^{\text{rational}} - \overbrace{a}^{\text{rational}}.$$

Therefore, b is the difference of two rational numbers, which is rational. This is a contradiction.

8 Suppose b and c are both natural numbers. Then

$$c^2 - b^2 = 4$$
$$(c-b)(c+b) = 4.$$

The only factors of 4 are 1, 2 and 4. And since $c + b > c - b$,

$$c - b = 1 \text{ and } c + b = 4.$$

Adding these two equations gives $2c = 5$ so that $c = \frac{2}{5}$, which is not a whole number.

9 Suppose that there are two different solutions, x_1 and x_2. Then,

$$ax_1 + b = c \text{ and } ax_2 + b = c.$$

Equating these two equations gives,

$$ax_1 + b = ax_2 + b$$
$$ax_1 = ax_2$$
$$x_1 = x_2, \qquad (\text{since } a \neq 0)$$

which is a contradiction since the two solutions were assumed to be different.

10 **a** Every prime $p > 2$ is odd since if it were even then p would be divisible by 2.

b Suppose there are two primes p and p such that $p + q = 1001$. Then since the sum of two odd numbers is even, one of the primes must be 2. Assume $p = 2$ so that $q = 999$. Since 999 is not prime, this gives a contradiction.

11 **a** Suppose that

$$42a + 7b = 1.$$

Then

$$7(6a + b) = 1.$$

This implies that 1 is divisible by 7, which is a contradiction since the only factor of 1 is 1.

b Suppose that

$$15a + 21b = 2.$$

Then

$$3(5a + 7b) = 2.$$

This implies that 2 is divisible by 3, which is a contradiction since the only factors of 2 are 1 and 2.

12 **a** Contrapositive: If n is not divisible by 3, then n^2 is not divisible by 3.
Proof: If n is not divisible by 3 then either $n = 3k + 1$ or $n = 3k + 2$.
(Case 1) If $n = 3k + 1$ then,

$$\begin{aligned} n^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

is not divisible by 3.
(Case 2) If $n = 3k + 2$ then,

$$\begin{aligned} n^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

is not divisible by 3.

b This will be a proof by contradiction. Suppose $\sqrt{3}$ is rational so that $\sqrt{3} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^2 = 3q^2 \qquad (1)$$

$\Rightarrow$ p^2 is divisible by 3

$\Rightarrow$ p is divisible by 3

$\Rightarrow$ $p = 3k$ for some $k \in \mathbb{N}$

$\Rightarrow$ $(3k)^2 = 3q^2$(substitituting into (1))

$\Rightarrow$ $3q^2 = 9k^2$

$\Rightarrow$ $q^2 = 3k^2$

$\Rightarrow$ q^2 is divisible by 3

$\Rightarrow$ q is divisible by 3.

So p and q are both divisible by 3, which contradicts the fact that they have no factors in common.

13 a Contrapositive: If n is odd, then n^3 is odd.
Proof: If n is odd then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

is odd. Otherwise, we can simply quote the fact that the product of 3 odd numbers will be odd.

b This will be a proof by contradiction. Suppose $\sqrt[3]{2}$ is rational so that $\sqrt[3]{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^3 = 2q^3 \qquad (1)$$

$\Rightarrow$ p^3 is divisible by 2

$\Rightarrow$ p is divisible by 2

$\Rightarrow$ $p = 2k$ for some $k \in \mathbb{N}$

$\Rightarrow$ $(2k)^3 = 2q^3$(substitituting into (1))

$\Rightarrow$ $2q^3 = 8k^3$

$\Rightarrow$ $q^3 = 4k^3$

$\Rightarrow$ q^3 is divisible by 2

$\Rightarrow$ q is divisible by 2.

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

14 This will be a proof by contradiction, so we suppose there is some $a, b \in \mathbb{Z}$ such that

$$a^2 - 4b - 2 = 0$$

$$\Rightarrow a^2 = 4b + 2$$

$$\Rightarrow a^2 = 2(2b+1) \qquad (1)$$

which means that a^2 is even. However, this implies that a is even, so that $a = 2k$, for some $k \in \mathbb{Z}$. Substituting this into equation (1) gives,

$$\begin{aligned} (2k)^2 &= 2(2b+1) \\ 4k^2 &= 2(2b+1) \\ 2k^2 &= 2b+1 \\ 2k^2 - 2b &= 1 \\ 2(k^2 - b) &= 1. \end{aligned}$$

This implies that 1 is divisible by 2, which is a contradiction since the only factor of 1 is 1.

15 a Suppose on the contrary, that $a > \sqrt{n}$

and $b > \sqrt{n}$. Then

$$ab > \sqrt{n}\sqrt{n} = n,$$

which is a contradiction since $ab = n$.

b If 97 were not prime then we could write $97 = ab$ where $1 < a < b < n$. By the previous question, we know that

$$a \leq \sqrt{97} < \sqrt{100} = 10.$$

Therefore a is one of

$$\{2, 3, 4, 5, 6, 7, 8, 9\}.$$

However 97 is not divisible by any of these numbers, which is a contradiction. Therefore, 97 is a prime number.

16 a Let $m = 4n + r$ where $r = 0, 1, 2, 3$.
($r = 0$) We have,

$$\begin{aligned} m^2 &= (4n)^2 \\ &= 16n^2 \\ &= 4(4n^2) \end{aligned}$$

is divisible by 4.
($r = 1$) We have,

$$\begin{aligned} m^2 &= (4n + 1)^2 \\ &= 16n^2 + 8n + 1 \\ &= 4(4n^2 + 2n) + 1 \end{aligned}$$

has a remainder of 1.
($r = 2$) We have,

$$\begin{aligned} m^2 &= (4n + 2)^2 \\ &= 16n^2 + 16n + 4 \\ &= 4(4n^2 + 4n + 1) \end{aligned}$$

is divisible by 4.

($r = 3$) We have,

$$\begin{aligned} m^2 &= (4n + 3)^2 \\ &= 16n^2 + 24n + 9 \\ &= 16n^2 + 24n + 8 + 1 \\ &= 4(4n^2 + 6n + 2) + 1 \end{aligned}$$

has a remainder of 1.

Therefore, the square of every integer is divisible by 4 or leaves a remainder of 1.

b Suppose the contrary, so that both a and b are odd. Then $a = 2k + 1$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k + 1)^2 + (2m + 1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 = 4m + 1 \\ &= 4(k^2 + m^2 + k + m) + 2. \end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 4, which is a contradiction.

17 a Suppose by way of contradiction either $a \neq c$ or $b \neq d$. Then clearly both $a \neq c$ and $b \neq d$. Therefore,

$$\begin{aligned} a + b\sqrt{2} &= c + d\sqrt{2} \\ (b - d)\sqrt{2} &= c - a \\ \sqrt{2} &= \frac{c - a}{b - d} \end{aligned}$$

Since $\frac{c - a}{b - d} \in \mathbb{Q}$, this contradicts the irrationality of $\sqrt{2}$.

b Squaring both sides gives,

$$3 + 2\sqrt{2} = \left(c + d\sqrt{2}\right)^2$$
$$3 + 2\sqrt{2} = c^2 + 2cd\sqrt{2} + 2d^2$$
$$3 + 2\sqrt{2} = c^2 + 2d^2 + 2cd\sqrt{2}$$

Therefore

$$c^2 + 2d^2 = 3 \qquad (1)$$
$$cd = 1 \qquad (2)$$

Since c and d are integers, this implies that $c = d = 1$.

18 There are many ways to prove this result. We will take the most elementary approach (but not the most elegant). Suppose that

$$ax^2 + bx + c = 0 \qquad (1)$$

has a rational solution, $x = \frac{p}{q}$. We can assume that p and q have no factors in common (or else we could cancel).

Equation (1) then becomes

$$ax^2 + bx + c = 0$$
$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$$
$$ap^2 + bpq + cq^2 = 0 \qquad (2)$$

Since p and q cannot both be even, we need only consider three cases.
(Case 1) If p is odd and q is odd then equation (2) is of the form

$$\text{odd} + \text{odd} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.
(Case 2) If p is odd and q is even then equation (2) is of the form

$$\text{odd} + \text{even} + \text{even} = \text{odd} = 0.$$

This is not possible since 0 is even.
(Case 3) If p is even and q is odd then equation (2) is of the form

$$\text{even} + \text{even} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.

Solutions to Exercise 6D

1 **a** Converse: If $x = 1$, then $2x + 3 = 5$.
Proof: If $x = 1$ then

$$2x + 3 = 2 \times 1 + 3 = 5.$$

b Converse: If $n - 3$ is even, then n is odd.
Proof: If $n - 3$ is even then $n - 3 = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$n = 2k + 3 = 2k + 2 + 1 = 2(k + 1) + 1$$

is odd.

c Converse: If m is odd, then $m^2 + 2m + 1$ is even.
Proof 1: If m is odd then the expression $m^2 + 2m + 1$ is of the form,

$$\text{odd} + \text{even} + \text{odd} = \text{even}.$$

Proof 2: If m is odd then $m = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} &m^2 + 2m + 1 \\ &= (2k + 1)^2 + 2(2k + 1) + 1 \\ &= 4k^2 + 4k + 1 + 4k + 2 + 1 \\ &= 4k^2 + 8k + 4 \\ &= 2(2k^2 + 4k + 2), \end{aligned}$$

is clearly even.

d Converse: If n is divisible by 5, then n^2 is divisible by 5.
Proof: If n is divisible by 5 then $n = 5k$ for some $k \in \mathbb{Z}$. Therefore,

$$n^2 = (5k)^2 = 25k^2 = 5(5k^2),$$

which is divisible by 5.

2 **a** Converse: If mn is a multiple of 4, then m and n are even.

b This statement is not true. For instance, 4×1 is a multiple of 4, and yet 1 is clearly not even.

3 **a** These statements are not equivalent.
($P \Rightarrow Q$) If Vivian is in China then she is in Asia, since Asia is a country in China.
($Q \not\Rightarrow P$) If Vivian is in Asia, she is not necessarily in China. For example, she could be in Japan.

b These statements are equivalent.
($P \Rightarrow Q$) If $2x = 4$, then dividing both sides by 2 gives $x = 2$.
($Q \Rightarrow P$) If $x = 2$, then multiplying both sides by 2 gives $2x = 4$.

c These statements are not equivalent.
($P \Rightarrow Q$) If $x > 0$ and $y > 0$ then $xy > 0$ since the product of two positive numbers is positive.
($Q \not\Rightarrow P$) If $xy > 0$, then it may not be true that $x > 0$ and $y > 0$. For example, $(-1) \times (-1) > 0$, however $-1 < 0$.

d These statements are equivalent.
($P \Rightarrow Q$) If m or n are even then mn will be even.
($Q \Rightarrow P$) If mn is even then either m or n are even since otherwise the product of two odds numbers would give an odd number.

4 ($\Rightarrow$) If $n + 1$ is odd then, $n + 1 = 2k + 1$, where $k \in \mathbb{Z}$. Therefore,

$$n + 2 = 2k + 2$$
$$= 2(k + 1),$$

so that $n + 2$ is even.
($\Leftarrow$) If $n + 2$ is even then, $n + 2 = 2k$, where $k \in \mathbb{Z}$. Therefore,

$$n + 1 = 2k - 1$$
$$= 2k - 2 + 1$$
$$= 2(k - 1) + 1$$

so that $n + 1$ is odd.

5 ($\Rightarrow$) Suppose that $n^2 - 4$ is prime. Since

$$n^2 - 4 = (n - 2)(n + 2)$$

expresses $n^2 - 4$ as the product of two numbers, either $n - 2 = 1$ or $n + 2 = 1$. Therefore, $n = 3$ or $n = -1$. However, n must be positive, so $n = 3$.
($\Leftarrow$) If $n = 3$ then

$$n^2 - 4 = 3^2 - 4 = 5$$

is prime.

6 ($\Rightarrow$) We prove this statement in the contrapositive. Suppose n is not even. Then $n = 2k + 1$ where $k \in \mathbb{Z}$. Therefore,

$$n^3 = (2k + 1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^4 + 6k^2 + 3k) + 1$$

is odd.

($\Leftarrow$) If n is even then $n = 2k$. Therefore,

$$n^3 = (2k)^3$$
$$= 8k^3$$
$$= 2(4k^3)$$

is even.

7 ($\Rightarrow$) Suppose that n is odd. Then $n = 2m + 1$, for some $m \in \mathbb{Z}$. Now either m is even or m is odd. If m is even, then $m = 2k$ so that

$$n = 2m + 1$$
$$= 2(2k) + 1$$
$$= 4k + 1.$$

as required. If m is odd then $m = 2q + 1$ so that

$$n = 2m + 1$$
$$= 2(2q + 1) + 1$$
$$= 4q + 3$$
$$= 4q + 4 - 1$$
$$= 4(q + 1) - 1$$
$$= 4k - 1, \text{ where } k = q + 1,$$

as required.
($\Leftarrow$) If $n = 4k \pm 1$ then either $n = 4k + 1$ or $n = 4k - 1$. If $n = 4k + 1$, then

$$n = 4k + 1$$
$$= 2(2k) + 1$$
$$= 2m + 1, \text{ where } m = 2k,$$

is odd, as required. Likewise, if

$n = 4k - 1$, then

$$\begin{aligned} n &= 4k - 1 \\ &= 4k - 2 + 1 \\ &= 2(2k - 1) + 1 \\ &= 2m + 1, \text{ where } m = 2k - 1, \end{aligned}$$

is odd, as required.

8 ($\Rightarrow$) Suppose that,

$$\begin{aligned} (x + y)^2 &= x^2 + y^2 \\ x^2 + 2xy + y^2 &= x^2 + y^2 \\ 2xy &= 0 \\ xy &= 0 \end{aligned}$$

Therefore, $x = 0$ or $y = 0$.
($\Leftarrow$) Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\begin{aligned} (x + y)^2 &= (0 + y)^2 \\ &= y^2 \\ &= 0^2 + y^2 \\ &= x^2 + y^2, \end{aligned}$$

as required.

9 **a** Expanding gives

$$\begin{aligned} &(m - n)(m^2 + mn + n^2) \\ =&m^3 + m^2n + mn^2 - m^2n - mn^2 - n^3 \\ =&m^3 - n^3. \end{aligned}$$

b ($\Leftarrow$) We will prove this in the contrapositive. Suppose that $m - n$ were odd. Then either m is odd and n is even or visa versa.
Case 1 - If m is odd and n is even
The expression $m^2 + mn + n^2$ is of the form,

$$\text{odd} + \text{even} + \text{even} = \text{odd}.$$

Case 2 - m is even and n is odd
The expression $m^2 + mn + n^2$ is of the form,

$$\text{even} + \text{even} + \text{odd} = \text{odd}.$$

In both instances, the expression $m^2 + mn + n^2$ is odd. Therefore,

$$m^3 - n^3 = (m - n)(m^2 + mn + n^2)$$

is the product of two odd numbers, and will therefore be odd.

10 We first note that any integer n can be written in the form $n = 100x + y$ where $x, y \in \mathbb{Z}$ and y is the number formed by the last two digits. For example, $1234 = 100 \times 12 + 34$. Then

$$\begin{aligned} &n \text{ is divisible by } 4 \\ \Leftrightarrow &n = 100x + y = 4k, \text{for some } k \in \mathbb{Z} \\ \Leftrightarrow &y = 4k - 100x \\ \Leftrightarrow &y = 4(k - 25x) \\ \Leftrightarrow &y \text{ is divisible by } 4. \end{aligned}$$

Solutions to Exercise 6E

1 **a** For all

b There exists

c For all

d For all

e There exists

f There exists

g For all

h For all

2 **a** True

b False

c True

d False

e False

3 **a** There exists a natural number $n \in \mathbb{N}$ for which $2n^2 - 4n + 31$ is not prime.

b There exists $x \in \mathbb{R}$ for which $x^2 \leq x$.

c For all $x \in \mathbb{R}$, $2 + x^2 \neq 1 - x^2$.

d There exists $x, y \in \mathbb{R}$ for which $(x + y)^2 \neq x^2 + y^2$.

e For all $x, y \in \mathbb{R}$, $x \geq y$ or $x^2 \leq y^2$.

4 **a** If we let $n = 31$ it is clear that

$$2n^2 - 4n + 31 = 2 \times 31^2 - 4 \times 31 + 31$$

is divisible by 31 and so cannot be prime.

b Let $x = 1$ and $y = -1$ so that

$$(x + y)^2 = (1 + (-1))^2 = 0,$$

while,

$$x^2 + y^2 = 1^2 + (-1)^2 = 1 + 1 = 2,$$

c If $x = \frac{1}{2}$, then,

$$x^2 = \frac{1}{4} < \frac{1}{2} = x.$$

d If $n = 3$ then,

$$n^3 - n = 27 - 3 = 24$$

is even, although 3 is not.

e If $m = n = 1$ then $m + n = 2$ while $mn = 1$.

f Since 6 divides $2 \times 3 = 6$ but 6 does not divide 2 or 3, the statement is false.

5 **a** Negation: For all $n \in \mathbb{N}$, the number $9n^2 - 1$ is not a prime number.
Proof: Since

$$9n^2 - 1 = (3n - 1)(3n + 1),$$

and since each factor is greater than 1, the number $9n^2 - 1$ is not a prime number.

b Negation: For all $n \in \mathbb{N}$, the number $n^2 + 5n + 6$ is not a prime number.
Since

$$n^2 + 5n + 6 = (n + 2)(n + 3),$$

and since each factor is greater than 1, the number $9n^2 + 5n + 6$ is not a prime number.

c Negation: For all $x \in \mathbb{R}$, we have $2 + x^2 \neq 1 - x^2$

Proof: Suppose that $2 + x^2 = 1 - x^2$. Rearranging the equation gives,

$$\begin{aligned} 2 + x^2 &= 1 - x^2 \\ 2x^2 &= -1 \\ x^2 &= -\frac{1}{2}, \end{aligned}$$

which is impossible since $x^2 \geq 0$.

6 a Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of a and b are irrational, although $ab = 2$ is not.

b Let $a = \sqrt{2}$ and $b = -\sqrt{2}$. Then clearly each of a and b are irrational, although $a + b = 0$ is not.

c Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then clearly each of a and b are irrational, although $\frac{a}{b} = 1$ is not.

7 a If a is divisible by 4 then $a = 4k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 = (4k)^2 = 16k^2 = 4(4k^2)$$

is divisible by 4.

b Converse: If a^2 is divisible by 4 then a is divisible by 4.

This is clearly not true, since $2^2 = 4$ is divisible by 4, although 2 is not.

8 a If $a - b$ is divisible by 3 then $a - b = 3k$ for some $k \in \mathbb{Z}$. Therefore,

$$a^2 - b^2 = (a - b)(a + b) = 3k(a + b)$$

is divisible by 3.

b Converse: If $a^2 - b^2$ is divisible by 3 then $a - b$ is divisible by 3.

The converse is not true, since $2^2 - 1^2 = 3$ is divisible by 3, although $2 - 1 = 1$ is not.

9 a This statement is not true since for all $a, b \in \mathbb{R}$,

$$a^2 - 2ab + b^2 = (a - b)^2 \geq 0 > -1.$$

b This statement is not true since for all $x \in \mathbb{R}$, we have,

$$\begin{aligned} &x^2 - 4x + 5 \\ =&x^2 - 4x + 4 - 4 + 5 \\ =&(x - 2)^2 + 1 \\ \geq&1 \\ >&\frac{3}{4}. \end{aligned}$$

10 **a** The numbers can be paired as follows:

$$16+9=25,\quad 15+10=25$$
$$14+11=25,\quad 13+12=25$$
$$1+8=9,\quad 2+7=9,$$
$$4+5=9,\quad 3+6=9.$$

b We now list each number, in descending order, with each of its potential pairs.

12	4
11	5
10	6
9	7
8	1
7	2, 9
6	3, 10
5	4
4	5
3	1, 6
2	7
1	3, 8

Notice that the numbers 2 and 9 must be paired with 7. Therefore, one cannot pair all numbers in the required fashion.

11 If we let $x = c$, then

$$f(c) = ac^2 + bc + c = c(ac + b + 1)$$

is divisible by $c \geq 2$.

Solutions to Exercise 6F

1 a $\boxed{P(n)}$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1$$

and

$$\text{RHS} = \frac{1(1+1)}{2} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \qquad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1)\\
&= 1 + 2 + \cdots + k + (k+1)\\
&= \frac{k(k+1)}{2} + (k+1) \quad \text{(by (1))}\\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}\\
&= \frac{k(k+1) + 2(k+1)}{2}\\
&= \frac{(k+1)(k+2)}{2}\\
&= \frac{(k+1)((k+1)+1)}{2}\\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 + x$$

and

$$\text{RHS} = \frac{(1 - x^2)}{1 - x} = \frac{(1-x)(1+x)}{1-x} = 1 + x.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}. \qquad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1)\\
&= 1 + x + x^2 + \cdots + x^k + x^{k+1}\\
&= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \quad \text{(by (1))}\\
&= \frac{1 - x^{k+1}}{1 - x} + \frac{x^{k+1}(1-x)}{1-x}\\
&= \frac{1 - x^{k+1} + x^{k+1}(1-x)}{1-x}\\
&= \frac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1-x}\\
&= \frac{1 - x^{k+2}}{1-x}\\
&= \frac{1 - x^{(k+1)+1}}{1-x}\\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^2 - 1$$

and

$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \quad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$$\begin{aligned}
&= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by (1))} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$$1 \cdot 2 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 \times 2 = 2$$

and

$$\text{RHS} = \frac{1 \times 2 \times 3}{3} = 2.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 \cdot 2 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}. \quad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$$\begin{aligned}
&= 1 \cdot 2 + \cdots + k \cdot (k+1) + (k+1) \cdot (k+2) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(by (1))} \\
&= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\
&= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\
&= \frac{(k+1)(k+2)(k+3)}{3} \\
&= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}
\end{aligned}$$

=RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

e $\boxed{P(n)}$

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = \frac{1}{1 \times 3} = \frac{1}{3}$$

and

$$\text{RHS} = \frac{1}{2 \times 1 + 1} = \frac{1}{3}.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}. \quad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$$=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \text{(by (1))}$$

$$=\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{k+1}{(2k+3)}$$

$$=\frac{k+1}{(2(k+1)+1)}$$

$=$RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f $\boxed{P(n)}$

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

$\boxed{P(2)}$

If $n=2$ then

$$\text{LHS}=1-\frac{1}{2^2}=\frac{3}{4}$$

and

$$\text{RHS}=\frac{2+1}{2\times 2}=\frac{3}{4}.$$

Therefore $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$$=\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \quad \text{(by (1))}$$

$$=\frac{k+1}{2k}\left(\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\frac{(k+1)(k^2+2k)}{2k(k+1)^2}$$

$$=\frac{k(k+1)(k+2)}{2k(k+1)^2}$$

$$=\frac{(k+2)}{2(k+1)}$$

$$=\frac{(k+1)+1}{2(k+1)}$$

$=$RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 a $\boxed{P(n)}$

11^n-1 is divisible by 10

$\boxed{P(1)}$

If $n=1$ then

$$11^1-1=11-1=10$$

is divisible by 10. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$11^k - 1 = 10m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}11^{k+1} - 1 &= 11 \times 11^k - 1\\ &= 11 \times (10m + 1) - 1 \quad \text{(by (1))}\\ &= 110m + 11 - 1\\ &= 110m + 10\\ &= 10(11m + 1)\end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$3^{2n} + 7$ is divisible by 8

$\boxed{P(1)}$

If $n = 1$ then

$$3^{2\times 1} + 7 = 9 + 7 = 16 = 2 \times 8$$

is divisible by 8. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^{2k} + 7 = 8m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}3^{2(k+1)} + 7 &= 3^{2k+2} + 7\\ &= 3^{2k} \times 3^2 + 7\\ &= (8m - 7) \times 9 + 7 \quad \text{(by (1))}\\ &= 72m - 63 + 7\\ &= 72m - 56\\ &= 8(9m - 7)\end{aligned}$$

is divisible by 8. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$7^n - 3^n$ is divisible by 4

$\boxed{P(1)}$

If $n = 1$ then

$$7^1 - 3^1 = 7 - 3 = 4$$

is divisible by 4. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$7^k - 3^k = 4m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}&7^{k+1} - 3^{k+1}\\ &= 7 \times 7^k - 3^{k+1}\\ &= 7 \times (4m + 3^k) - 3 \times 3^k \quad \text{(by (1))}\\ &= 28m + 7 \times 3^k - 3 \times 3^k\\ &= 28m + 4 \times 3^k\\ &= 4(7m + 3^k)\end{aligned}$$

is divisible by 4. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$5^n + 6 \times 7^n + 1$ is divisible by 4

$\boxed{P(1)}$

If $n = 1$ then

$$5^1 + 6 \times 7^1 + 1 = 48 = 4 \times 12$$

is divisible by 4. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$5^k + 6 \times 7^k + 1 = 4m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}&5^{k+1} + 6 \times 7^{k+1} + 1\\ &= 5 \times 5^k + 6 \times 7 \times 7^k + 1\\ &= 5 \times (4m - 6 \times 7^k - 1) + 42 \times 7^{k+1}\\ &= 20m - 30 \times 7^k - 5 + 42 \times 7^k + 1\\ &= 20m + 12 \times 7^k - 4\\ &= 4(5m + 3 \times 7^k - 1)\end{aligned}$$

is divisible by 4. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

3 a $\boxed{P(n)}$

$4^n > 10 \times 2^n$ where $n \geq 4$

$\boxed{P(4)}$

If $n = 4$ then

LHS $= 4^4 = 256$ and RHS $= 10 \times 2^4 = 160$.

Since LHS > RHS, $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^k > 10 \times 2^k \text{ where } k \geq 4. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$4^{k+1} > 10 \times 2^{k+1}.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= 4^{k+1}\\ &= 4 \times 4^k\\ &> 4 \times 10 \times 2^k \quad \text{(by (1))}\\ &= 40 \times 2^k \quad \text{(as } 10 > 2)\\ &= 20 \times 2^{k+1}\\ &> 10 \times 2^{k+1}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Since $P(5)$ is true and $P(k+1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$3^n > 5 \times 2^n$ where $n \geq 5$

$\boxed{P(5)}$

If $n = 5$ then

LHS $= 3^5 = 243$ and RHS $= 5 \times 2^5 = 160$.

Since LHS > RHS, $P(5)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^k > 5 \times 2^k \text{ where } k \geq 5. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$3^{k+1} > 5 \times 2^{k+1}.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= 3^{k+1}\\ &= 3 \times 3^k\\ &> 3 \times 5 \times 2^k \quad \text{(by (1))}\\ &= 15 \times 2^k \quad \text{(as } 10 > 2)\\ &> 10 \times 2^k\\ &= 5 \times 2^{k+1}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Since $P(5)$ is true and $P(k+1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 5$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$2^n > 2n$ where $n \geq 3$

$\boxed{P(3)}$

If $n = 3$ then

$$\text{LHS} = 2^3 = 8 \text{ and RHS} = 2 \times 3 = 6.$$

Since LHS > RHS, $P(3)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$2^k > 2k \text{ where } k \geq 3. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$2^{k+1} > 2(k+1).$$

$$\begin{aligned}\text{LHS of } P(k+1) &= 2^{k+1}\\ &= 2 \times 2^k\\ &> 2 \times 2k \quad \text{(by (1))}\\ &= 4k\\ &= 2k + 2k\\ &\geq 2k + 2 \quad \text{(as } 2k \geq 2\text{)}\\ &= 2(k+1)\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all integers $n \geq 3$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$n! > 2^n$ where $n \geq 4$

$\boxed{P(4)}$

If $n = 4$ then

$$\text{LHS} = 4! = 24 \text{ and RHS} = 2^4 = 16.$$

Since LHS > RHS, $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k! > 2^k \text{ where } k \geq 4. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$(k+1)! > 2^{k+1}.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= (k+1)!\\ &= (k+1)k!\\ &> (k+1) \times 2^k \quad \text{(by (1))}\\ &> 2 \times 2^k \quad \text{(as } k+1 > 2\text{)}\\ &= 2^{k+1}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

4 a $\boxed{P(n)}$

$a_n = 2^n + 1$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = a_1 = 3 \text{ and RHS} = 2^1 + 1 = 3.$$

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 2^k + 1. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 2^{k+1} + 1.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= a_{k+1}\\ &= 2a_k - 1 \quad \text{(by definition)}\\ &= 2(2^k+1)-1 \quad \text{(by (1))}\\ &= 2^{k+1}+2-1\\ &= 2^{k+1}+1\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$a_n = 5^n - 1$

$\boxed{P(1)}$

If $n = 1$ then

LHS $= a_1 = 4$ and RHS $= 5^1 - 1 = 4$.

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 5^k - 4. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 5^{k+1} - 4.$$

$$\begin{aligned}\text{LHS} &= a_{k+1}\\ &= 5a_k + 4 \quad \text{(by definition)}\\ &= 5(5^k-1)+4 \quad \text{(by (1))}\\ &= 5^{k+1}-5+4\\ &= 5^{k+1}-1\\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$a_n = 2^n + n$

$\boxed{P(1)}$

If $n = 1$ then

LHS $= a_1 = 3$ and RHS $= 2^1 + 1 = 3$.

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 2^k + k. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 2^{k+1} + k + 1.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= a_{k+1}\\ &= 2a_k - k + 1 \quad \text{(by definition)}\\ &= 2(2^k+k)-k+1 \quad \text{(by (1))}\\ &= 2^{k+1}+2k-k+1\\ &= 2^{k+1}+k+1\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

5 $\boxed{P(n)}$

3^n is odd where $n \in \mathbb{N}$

$\boxed{P(1)}$

If $n = 1$ then clearly

$$3^1 = 3$$

is odd. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^k = 2m + 1 \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$3^{k+1} = 3 \times 3^k$$
$$= 3 \times (2m + 1) \quad \text{(by (1))}$$
$$= 6m + 3$$
$$= 6m + 2 + 1$$
$$= 2(3m + 1) + 1$$

is odd, so that $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 a $\boxed{P(n)}$

$n^2 - n$ is even, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^2 - \times 1 = 0$$

is even. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^2 - k$ is even. Therefore,

$$k^2 - k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k + 1)}$

$$(k + 1)^2 - (k + 1)$$
$$= k^2 + 2k + 1 - k - 1$$
$$= k^2 + k$$
$$= (k^2 - k) + 2k$$
$$= 2m + 2k \quad \text{(by (1))}$$
$$= 2(m + k)$$

Since this is even, $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^2 - n = n(n - 1).$$

As this is the product of two consecutive numbers, one of them must be even, so that the product will also be even.

7 a $\boxed{P(n)}$

$n^3 - n$ is divisible by 3, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^3 - 1 = 0$$

is divisible by 3. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^3 - k$ is divisible by 3. Therefore,

$$k^3 - k = 3m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k + 1)}$

We have to show that $(k+1)^3 - (k+1)$ is divisible by 3.

$$(k + 1)^3 - (k + 1)$$
$$= k^3 + 3k^2 + 3k + 1 - k - 1$$
$$= k^3 - k + 3k^2 + 3k$$
$$= (k^3 - k) + 3k^2 + 3k$$
$$= 3m + 3k^2 + 3k \quad \text{(by (1))}$$
$$= 3(m + k^2 + k)$$

Since this is divisible by 3, $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^3 + n = n(n^2 - 1) = n(n - 1)(n + 1).$$

As this is the product of three consecutive numbers, one of them must be divisible by 3, so that the product will also be divisible by 3.

8 a

n	1	2	3	4	5
a_n	9	99	999	9999	99999

b We claim that $a_n = 10^n - 1$.

c $\boxed{P(n)}$

$a_n = 10^n - 1$

$\boxed{P(1)}$

If $n = 1$, then

LHS $= a_1 = 9$ and RHS $= 10^1 - 1 = 9$.

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 10^k - 1. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 10^{k+1} - 1.$$

$$\begin{aligned}\text{LHS} &= a_{k+1}\\ &= 10a_k + 9 \quad \text{(by definition)}\\ &= 10(10^k - 1) + 9 \quad \text{(by (1))}\\ &= 10^{k+1} - 10 + 9\\ &= 10^{k+1} - 1\\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

9 a

n	1	2	3	4	5	6	7	8	9	10
f_n	1	1	2	3	5	8	13	21	34	55

b $\boxed{P(n)}$

$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_3 - 1 = 2 - 1 = 1.$$

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$f_1 + f_2 + \cdots + f_k = f_{k+2} - 1. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}\text{LHS of } P(k+1) &= f_1 + f_2 + \cdots + f_k + f_{k+1}\\ &= f_{k+2} - 1 + f_{k+1} \quad \text{(by (1))}\\ &= f_{k+1} + f_{k+2} - 1\\ &= f_{k+3} - 1 \quad \text{(by definition)}\\ &= f_{(k+1)+2} - 1\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $f_1 = 1$

$f_1 + f_3 = 1 + 2 = 3$

$f_1 + f_3 + f_5 = 3 + 5 = 8$

$f_1 + f_3 + f_5 + f_7 = 8 + 13 = 21$

d From the pattern observed above, we claim that

$$f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}.$$

e $\boxed{P(n)}$

$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_2 = 1.$$

Since LHS = RHS, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}\text{LHS} &= f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1}\\ &= f_{2k} + f_{2k+1} \quad \text{(by (1))}\\ &= f_{2k+2} \quad \text{(by definition)}\\ &= f_{2(k+1)}\\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f $\boxed{P(n)}$

The Fibonacci number f_{3n} is even.

$\boxed{P(1)}$

If $n = 1$ then

$$f_3 = 2$$

is even, therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that f_{3k} is even. That is,

$$f_{3k} = 2m \quad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}f_{3(k+1)} &= f_{3k+3}\\ &= f_{3k+2} + f_{3k+1} \quad \text{(by definition)}\\ &= f_{3k+1} + f_{3k} + f_{3k+1}\\ &= 2f_{3k+1} + f_{3k}\\ &= 2f_{3k+1} + 2m \quad \text{(by (1))}\\ &= 2(f_{3k+1} + m)\end{aligned}$$

Since this is even, $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

10 $\boxed{P(n)}$

Since we're only interested in odd numbers our proposition is:
$4^{2n-1} + 5^{2n-1}$ is divisible by 9, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$4^1 + 5^1 = 9$$

is divisible by 9. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^{2k-1} + 5^{2k-1} = 9m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

The next odd number will be $2k + 1$. Therefore, we have to prove that

$$4^{2k+1} + 5^{2k+1}$$

is divisible by 9.

$$4^{2k+1} + 5^{2k+1}$$
$$= 4^2 \times 4^{2k-1} + 5^2 \times 5^{2k-1}$$
$$= 16 \times (9m - 5^{2k-1}) + 25 \times 5^{2k-1} \quad \text{(by (1))}$$
$$= 144m - 16 \times 5^{2k-1} + 25 \times 5^{2k-1}$$
$$= 144m + 9 \times 5^{2k-1}$$
$$= 9(16 + 5^{2k-1})$$

Since this is divisible by 9, we've shown that $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

11 $\boxed{P(n)}$

A set of numbers S with n numbers has a largest element.

$\boxed{P(1)}$

If $n = 1$, then set S has just one element. This single element is clearly the largest element in the set.

$\boxed{P(k)}$

Assume that $P(k)$ is true. This means that a set of numbers S with k numbers has a largest element.

$\boxed{P(k + 1)}$

Suppose set S has $k + 1$ numbers. Remove one of the elements, say x, so that we now have a set with k numbers. The reduced set has a largest element, y. Put x back in set S, so that its largest element will be the larger of x and y. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

12 $\boxed{P(n)}$

It is possible to walk around a circle whose circumference includes n friends and n enemies (in any order) without going into debt.

$\boxed{P(1)}$

If $n = 1$, there is one friend and one enemy on the circumference of a circle. Start your journey at the friend, receive \$1, then walk around to the enemy and lose \$1. At no point will you be in debt, so $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true. This means that it is possible to walk around a circle with k friends and k enemies (in any order) without going into debt, provided you start at the correct point.

$\boxed{P(k + 1)}$

Suppose there are $k + 1$ friends and $k + 1$ enemies located on the circumference of the circle, in any order. Select a friend whose next neighbour is an enemy (going clockwise), and remove these two people. As there are now k friends and k enemies, it is possible to walk around the circle without going into debt, provided you start at the correct point. Now reintroduce the two people, and start walking from the same point. For every part of the journey you'll have the same amount of money as before except when you meet the added friend, who gives you \$1, which is immediately lost to the added enemy.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

13 $\boxed{P(n)}$

Every integer j such that $2 \le j \le n$ is divisible by some prime.

$\boxed{P(2)}$

If $n = 2$, then $j = 2$ is clearly divisible by a prime, namely itself. Therefore $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true. Therefore, every integer j such that $2 \leq j \leq k$ is divisible by some prime.

$\boxed{P(k+1)}$

We need to show that integer j such that $2 \leq j \leq k + 1$ is divisible by some prime. By the induction assumption, we already know that every j with $2 \leq j \leq k$ is divisible by some prime. We need only prove that $k + 1$ is divisible by a prime. If $k + 1$ is a prime number, then we are finished. Otherwise we can find integers a and b such that $k + 1 = ab$ and $2 \leq a \leq k$ and $2 \leq b \leq k$. By the induction assumption, the number a will be divisible by some prime number. Therefore $k + 1$ is divisible by some prime number.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

14 If such a colouring of the regions is possible we will call it a **satisfactory colouring**.

$\boxed{P(n)}$

If n lines are drawn then the resulting regions have a satisfactory colouring.

$\boxed{P(1)}$

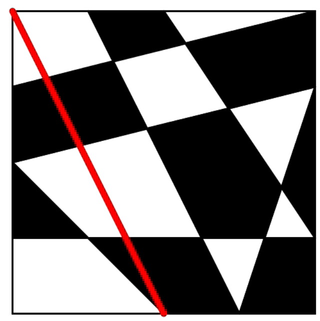
If $n = 1$, then there is just one line. We colour one side black and one side white. This is a satisfactory colouring. Therefore $P(1)$ is true.

$\boxed{P(k)}$

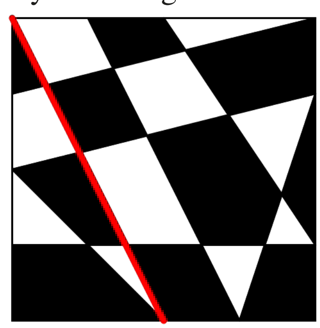
Assume that $P(k)$ is true. This means that we can obtain a satisfactory colouring if there are k lines drawn.

$\boxed{P(k+1)}$

Now suppose that there are $k + 1$ lines drawn. Select one of the lines, and remove it. There are now k lines, and the resulting regions have a satisfactory colouring since we assumed $P(k)$ is true. Now add the removed line. This will divide some regions into into two new regions with the same colour, so this is not a satisfactory colouring.



However, if we switch each colour on **one** side of the line we obtain a satisfactory colouring.



This is because inverting a satisfactory colouring will always give a satisfactory colouring, and regions separated the new line will not have the same colour.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to short-answer questions

1 a Let the 3 consecutive integers be $n, n+1$ and $n+2$. Then,

$$n + (n+1) + (n+2) = 3n + 3$$
$$= 3(n+1)$$

is divisible by 3.

b This statement is not true. For example, $1 + 2 + 3 + 4 = 10$ is not divisible by 4

2 (Method 1) If n is even then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$n^2 - 3n + 1 = (2k)^2 - 2(2k) + 1$$
$$= 4k^2 - 4k + 1$$
$$= 2(2k^2 - 2k) + 1$$

is odd.

(Method 2) If n is even then $n^2 - 3n + 1$ is of the form

$$\text{even} - \text{even} + \text{odd} = \text{odd}.$$

3 a (Contrapositive) If n is not even, then n^3 is not even. (Alternative) If n is odd, then n^3 is odd.

b If n is odd then $n = 2k + 1$, for some $k \in \mathbb{Z}$. Therefore,

$$n^3 = (2k+1)^3$$
$$= 8k^3 + 12k^2 + 6k + 1$$
$$= 2(4k^3 + 6k^2 + 3k) + 1$$

is odd.

c This will be a proof by contradiction. Suppose $\sqrt[3]{6}$ is rational so that $\sqrt[3]{6} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$p^3 = 6q^3 \qquad (1)$$
$$\Rightarrow \quad p^3 \text{ is divisible by } 2$$
$$\Rightarrow \quad p \text{ is divisible by } 2$$
$$\Rightarrow \quad p = 2k \text{ for some } k \in \mathbb{N}$$
$$\Rightarrow \quad (2k)^3 = 6q^3 (\text{substitituting into (1)})$$
$$\Rightarrow \quad 8k^3 = 6q^3$$
$$\Rightarrow \quad 4k^2 = 3q^2$$
$$\Rightarrow \quad q^2 \text{ is divisible by } 2$$
$$\Rightarrow \quad q \text{ is divisible by } 2.$$

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

4 a Suppose n is the first of three consecutive numbers. If n is divisible by 3 then there is nothing to prove. Otherwise, it is of the form $n = 3k + 1$ or $n = 3k + 2$. In the first case,

$$n = 3k + 1$$
$$n + 1 = 3k + 2$$
$$n + 2 = 3k + 3 = 3(k+1)$$

so that the third number is divisible by 3. In the second case,

$$n = 3k + 2$$
$$n + 1 = 3k + 3 = 3(k+1)$$
$$n + 2 = 3k + 4$$

so that the second number is divisible by 3.

b The expression can be readily factorised so that

$$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2)$$
$$= n(n + 1)(n + 2)$$

is the product of 3 consecutive integers. As one of these integers must be divisible by 3, the product must also be divisible by 3.

5 a if m and n are divisible by d then $m = pd$ and $n = qd$ for some $p, q \in \mathbb{Z}$. Therefore,

$$m - n = pd - qd$$
$$= d(p - q)$$

is divisible by d.

b Take any two consecutive numbers n and $n + 1$. If d divides n and $n + 1$ then d must divide $(n + 1) - n = 1$. As the only integer that divides 1 is 1, the highest common factor must be 1, as required.

c We know that any factor of 1002 and 999 must also divide $1002 - 999 = 3$. As the only factors of 3 are 1 and 3, the highest common factor must be 3.

6 a If $x = 9$ and $y = 16$ then the left hand side equals

$$\sqrt{9 + 16} = \sqrt{25} = 5$$

while the right hand side equals

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

b ($\Rightarrow$)

$$[t]\ \sqrt{x + y} = \sqrt{x} + \sqrt{y}$$
$$\Rightarrow x + y = \left(\sqrt{x} + \sqrt{y}\right)^2$$
$$\Rightarrow x + y = x + \sqrt{xy} + y$$
$$\Rightarrow 0 = sqrtxy$$
$$\Rightarrow xy = 0$$
$$\Rightarrow x = 0 \text{ or } y = 0$$

($\Leftarrow$) Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\sqrt{x + y} = \sqrt{y + 0}$$
$$= \sqrt{y}$$
$$= \sqrt{y} + \sqrt{0}$$
$$= \sqrt{y} + \sqrt{x},$$

as required.

7 (Case 1) If n is even then the expression is of the form

even + even + even = even.

(Case 1) If n is odd then the expression is of the form

odd + odd + even = even.

8 a If $a = b = c = d = 1$ then the left hand side equals

$$\frac{1}{1} + \frac{1}{1} = 2$$

while the right hand side equals

$$\frac{1+1}{1+1} = 1.$$

b first note that if $\dfrac{c}{d} > \dfrac{a}{b}$ then $bc > ad$. Therefore,

$$\begin{aligned}
&\frac{a+c}{b+d} - \frac{a}{b} \\
&= \frac{b(a+c)}{b(b+d)} - \frac{a(b+d)}{b(b+d)} \\
&= \frac{b(a+c) - a(b+d)}{b(b+d)} \\
&= \frac{ab + bc - ab - ad}{b(b+d)} \\
&= \frac{bc - ad}{b(b+d)} \\
&> 0
\end{aligned}$$

since $bc > ad$. This implies that

$$\frac{a+c}{b+d} > \frac{a}{b}.$$

Similarly, we can show that

$$\frac{a+c}{b+d} < \frac{c}{d}.$$

9 a $\boxed{P(n)}$

$6^n + 4$ is divisible by 10

$\boxed{P(1)}$

If $n = 1$ then

$$6^1 + 4 = 10$$

is divisible by 10. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$6^k + 4 = 10m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}
6^{k+1} + 4 &= 6 \times 6^k + 4 \\
&= 6 \times (10m - 4) + 4 \quad \text{(by (1))} \\
&= 60m - 24 + 4 \\
&= 60m - 20 \times 3^k \\
&= 10(6m - 2)
\end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

$\boxed{P(1)}$

If $n = 1$ then LHS$= 1^2 = 1$and

$$\text{RHS} = \frac{1(2 \times 1 - 1)(2 \times 1 + 1)}{3} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^2 + 3^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$$
\begin{aligned}
&= 1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad \text{(by (1))} \\
&= \frac{k(2k-1)(2k+1)}{3} + \frac{3(2k+1)^2}{3} \\
&= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} \\
&= \frac{(2k+1)(k(2k-1) + 3(2k+1))}{3} \\
&= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3} \\
&= \frac{(2k+1)(2k+3)(k+1)}{3} \\
&= \frac{(k+1)(2k+1)(2k+3)}{3} \\
&= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}
\end{aligned}
$$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to multiple-choice questions

1 E The expression $m - 3n$ is of the form

$$\text{even} - \text{odd} = \text{odd}.$$

2 E If m is divisible by 6 and n is divisible by 15 then $m = 6p$ and $n = 15q$ for $p, q \in \mathbb{Z}$. Therefore,

$$m \times n = 90pq$$

$$m + n = 6p + 15q = 3(2p + 5q)$$

From these two expressions, it should be clear that A,B,C and D are true, while E might be false. For example, if $m = 6$ and $n = 15$ then $m + n = 21$ is not divisible by 15.

3 C We obtain the contrapositive by switching P and Q and negating both. Therefore, the contrapositive will be

$$\text{not } Q \Rightarrow \text{not } P$$

.

4 B We obtain the converse by switching P and Q. Therefore, the converse will be

$$Q \Rightarrow P$$

.

5 C If $m + n = mn$ then

$$n = mn - m$$

$$n = m(n - 1)$$

This means that n is divisible by $n - 1$, which is only possible if $n = 2$ or $n = 0$. If $n = 0$, then $m = 0$. If $n = 2$, then $m = 2$. Therefore there are only two solutions, $(0, 0)$ and $(2, 2)$.

6 D The only statement that is true for all real numbers a, b and c is D. Counterexamples can be found for each of the other expressions, as shown below.

A $\quad \dfrac{1}{3} < \dfrac{1}{2}$

B $\quad \dfrac{1}{2} > \dfrac{1}{-1}$

C $\quad 3 \times -1 < 2 \times -1$

E $\quad 1^2 < (-2)^2$

7 D As n is the product of 3 consecutive integers, one of which will be divisible by 3 and one of which will be divisible by 2. The product will be then be divisible by 1, 2, 3 and 6. On the other hand, it won't necessarily be divisible by 5 since $2 \times 3 \times 4$ is not divisible by 5.

8 C Each of the statements is true except the third. In this instance, $1 + 3$ is even, although 1 and 3 are not even.

Solutions to extended-response questions

1 **a** The number of dots can be calculated two ways, either by addition,

$$(1 + 2 + 3 + 4) + (1 + 2 + 3 + 4)$$

or by multiplication,

$$4 \times 5.$$

Equating these two expressions gives,

$$\begin{aligned}(1 + 2 + 3 + 4) + (1 + 2 + 3 + 4) &= 4 \times 5\\ 2(1 + 2 + 3 + 4) &= 4 \times 5\\ 1 + 2 + 3 + 4 &= \frac{4 \times 5}{2}\end{aligned}$$

The argument obviously generalises to more dots, giving equation (1).

b We have,

$$\begin{aligned}1 + 2 \cdots + 99 &= \frac{99 \times 100}{2}\\ &= 99 \times 50,\end{aligned}$$

which is divisible by 99.

c Suppose that m is the first number, so that the n connective numbers are

$$m, m + 1, \ldots, m + n - 1.$$

Then,

$$\begin{aligned}&m + (m + 1) + (m + 2) + \cdots + (m + n - 1)\\ &= n \times m + (1 + 2 + \cdots (n - 1))\\ &= nm + \frac{(n-1)n}{2}\\ &= n\left(m + \frac{n-1}{2}\right)\end{aligned}$$

Since n is odd, $n - 1$ is even. This means that $\frac{n-1}{2}$ is an integer. Therefore, the term in brackets is an integer, which means the expression is divisible by n.

d Since

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

we need to prove the following statement:

$\boxed{P(n)}$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^3 = 1$$

and

$$\text{RHS} = \frac{1^2(1+1)^2}{4} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1)\\
=&1^3 + 2^3 + \cdots + k^3 + (k+1)^3\\
=&\frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{(by (1))}\\
=&\frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}\\
=&\frac{k^2(k+1)^2 + 4(k+1)^3}{4}\\
=&\frac{(k+1)^2(k^2 + 4(k+1))}{4}\\
=&\frac{(k+1)^2(k^2 + 4k + 4)}{4}\\
=&\frac{(k+1)^2(k+2)^2}{4}\\
=&\frac{(k+1)^2((k+1)+1)^2}{4}\\
=&\text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 a The first number is divisible by 2, the second by 3, the third by 4 and so on. As each number has a factor greater than 1, each is a composite number. Therefore this is a sequence of 9 consecutive composite numbers.

b We consider the this sequence of 10 consecutive numbers,

$$11! + 2, 11! + 3, \ldots, 11! + 11.$$

The first number is divisible by 2, the second by 3 and so on. Therefore as each number has a factor greater than 1, each is a composite number.

3 a Since (a, b, c) is a Pythagorean triple, we know that $a^2 + b^2 = c^2$. Then (na, nb, nc) is also a Pythagorean triple since,

$$\begin{aligned}(na)^2 + (nb)^2 &= n^2a^2 + n^2b^2\\ &= n^2(a^2 + b^2)\\ &= n^2(c^2)\\ &= (nc)^2,\end{aligned}$$

as required.

b Suppose that $(n, n + 1, n + 2)$ is a Pythagorean triple. Then

$$\begin{aligned}n^2 + (n + 1)^2 &= (n + 2)^2\\ n^2 + n^2 + 2n + 1 &= n^2 + 4n + 4\\ n^2 - 2n - 3 &= 0\\ (n - 3)(n + 1) &= 0\\ n &= 3, -1.\end{aligned}$$

However, since $n > 0$, we obtain only one solution, $n = 3$, which corresponds to the famous $(3, 4, 5)$ triangle.

c Suppose some triple (a, b, c) contained the number 1. Then clearly, 1 will be the smallest number. Therefore, we can suppose that

$$\begin{aligned}1^2 + b^2 &= c^2\\ c^2 - b^2 &= 1\\ (c - b)(c + b) &= 1\end{aligned}$$

Since the only divisor of 1 is 1, we must have

$$\begin{aligned}&c + b = 1\\ &c - b = 1\\ &\Rightarrow b = 0 \text{ and } c = 1.\end{aligned}$$

This is a contradiction, since b must be a positive integer. Now suppose some triple (a, b, c) contained the number 2. Then 2 will be smallest number. Therefore, we can suppose that

$$\begin{aligned}2^2 + b^2 &= c^2\\ c^2 - b^2 &= 4\\ (c-b)(c+b) &= 4\end{aligned}$$

Since the only divisors of 4 are 1, 2 and 4, we must have

$$c + b = 4$$

$$c - b = 1$$

$$\Rightarrow b = \frac{3}{2}, c = \frac{5}{2}$$

or

$$c + b = 2$$

$$c - b = 2$$

$$\Rightarrow b = 0, c = 2$$

In both instances, we have a contradiction since b must be a positive integer.

4 **a** (Case 1) If $a = 3k + 1$ then

$$\begin{aligned}a^2 &= (3k+1)^2\\ &= 9k^2 + 6k + 1\\ &= 3(3k^2 + 2k) + 1\end{aligned}$$

leaves a remainder of 1 when divided by 3.
(Case 2) If $a = 3k + 2$ then

$$\begin{aligned}a^2 &= (3k+2)^2\\ &= 9k^2 + 12k + 4\\ &= 9k^2 + 12k + 3 + 1\\ &= 3(3k^2 + 4k + 1) + 1\end{aligned}$$

also leaves a remainder of 1 when divided by 3.

b Suppose by way of contradiction that neither a nor b are divisible by 3. Then using the previous question, each of a^2 and b^2 leave a remainder of 1 when divided by 3. Therefore $a^2 = 3k + 1$ and $b^2 = 3m + 1$, for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned}c^2 &= a^2 + b^2\\ &= 3k + 1 + 3m + 1\\ &= 3(k+m) + 2.\end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 3, which is not possible.

5 **a** $\boxed{P(n)}$

$n^2 + n$ is divisible by 2, where $n \in \mathbb{Z}$.

$\boxed{P(1)}$

If $n = 1$ then $1^2 + 1 = 2$ is divisible by 2. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k^2 + k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

Letting $n = k + 1$ we have,

$$\begin{aligned}
&(k+1)^2 + (k+1) \\
=&k^2 + 2k + 1 + k + 1 \\
=&k^2 + 3k + 2 \\
=&(k^2 + k) + (2k + 2) \\
=&2m + 2(k+1) \qquad \text{(by (1))} \\
=&2(m + k + 1)
\end{aligned}$$

is divisible by 2. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Since

$$n^2 + n = n(n+1)$$

is the product of two consecutive integers, one of them must be even. Therefore the product will also be even.

c If n is odd, then $n = 2k + 1$. Therefore

$$\begin{aligned}
n^2 - 1 &= (2k+1)^2 - 1 \\
&= 4k^2 + 4k + 1 - 1 \\
&= 4k^2 + 4k \\
&= 4k(k+1) \\
&= 4 \times 2k \qquad \text{(since the product of consecutive integers is even)} \\
&= 8k
\end{aligned}$$

as required.

6 **a** If n is divisible by 8, then $n = 8k$ for some $k \in \mathbb{Z}$. Therefore

$$n^2 = (8k)^2 = 64k^2 = 8(8k^2)$$

is divisible by 8.

b (Converse) If n^2 is divisible by 8, then n is divisible by 8.

c The converse is not true. For example, $4^2 = 16$ is divisible by 8 however 4 is not divisible by 8.

7 a There are many possibilities. For example $3 + 97 = 100$ and $5 + 97 = 102$.

b Suppose 101 could be written as the sum of two prime numbers. Then one of these primes must be 2, since all other pairs of primes have an even sum. Therefore $101 = 2 + 99$, however 99 is not prime.

c There are many possibilities. For example, $7 + 11 + 83 = 101$.

d Consider any odd integer n greater than 5. Then $n - 3$ will be an even number greater than 2. If the Goldbach Conjecture is true, then $n - 3$ is the sum of two primes, say p and q. Then $n = 3 + p + q$, as required.

8 a We have,

$$\begin{aligned}\frac{1}{n-1} - \frac{1}{n} &= \frac{n}{n(n-1)} - \frac{n-1}{n(n-1)}\\ &= \frac{n-(n-1)}{n(n-1)}\\ &= \frac{n-n+1}{n(n-1)}\\ &= \frac{1}{n(n-1)}.\end{aligned}$$

b Using the identity developed in the previous question, we have,

$$\begin{aligned}&\frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{n(n+1)}\\ =&\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n}\\ =&\frac{1}{1} - \frac{1}{n}\\ =&1 - \frac{1}{n}\end{aligned}$$

as required.

c True when $n = 2$ since $\dfrac{1}{2\times 1} = 1 - \dfrac{1}{2}$

Assume true for $n = k$

$$\frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{k(k+1)} = 1 - \frac{1}{k}$$

For $n = k + 1$

$$\frac{1}{2\times 1}+\frac{1}{3\times 2}+\cdots+\frac{1}{k(k-1)}+\frac{1}{(k+1)(k)} = 1-\frac{1}{k}+\frac{1}{(k+1)(k)}$$
$$= 1-\frac{1}{k+1}$$

d Since $k^2 > k(k-1)$ for all $k \in \mathbb{N}$,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots+\frac{1}{n^2}$$
$$=\frac{1}{1^2}+\left(\frac{1}{2^2}+\frac{1}{3^2}\cdots+\frac{1}{n^2}\right)$$
$$<\frac{1}{1^2}+\left(\frac{1}{2\times 1}+\frac{1}{3\times 2}\cdots+\frac{1}{n(n-1)}\right)$$
$$=\frac{1}{1^2}+1-\frac{1}{n}$$
$$=2-\frac{1}{n}$$
$$<2,$$

as required.

9 a We have,

$$\frac{x+y}{2}-\sqrt{xy} = \frac{a^2+b^2}{2}-\sqrt{a^2b^2}$$
$$= \frac{a^2+b^2}{2}-ab$$
$$= \frac{a^2+b^2}{2}-\frac{2ab}{2}$$
$$= \frac{a^2-2ab+b^2}{2}$$
$$= \frac{(a-b)^2}{2}$$
$$\geq 0.$$

It is also worth noting that we get equality if and only if $x = y$.

b i Using the above inequality, we obtain,

$$a+\frac{1}{a} \geq 2\sqrt{a\cdot\frac{1}{a}}$$
$$= 2\sqrt{1}$$
$$= 2.$$

as required.

ii Using the above inequality three times, we obtain,

$$\begin{aligned}(a+b)(b+c)(c+a) &\geq 2\sqrt{ab} \times 2\sqrt{bc} \times 2\sqrt{ca}\\ &= 8(\sqrt{a})^2(\sqrt{b})^2(\sqrt{c})^2\\ &= 8abc,\end{aligned}$$

as required.

iii This inequality is a little trickier. We have,

$$\begin{aligned}a^2+b^2+c^2 &= \left(\frac{a^2}{2}+\frac{b^2}{2}\right)+\left(\frac{b^2}{2}+\frac{c^2}{2}\right)+\left(\frac{a^2}{2}+\frac{c^2}{2}\right)\\ &= \frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{a^2+c^2}{2}\\ &\geq \sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{a^2c^2}\\ &= ab+bc+ac,\end{aligned}$$

as required.

c If a rectangle has length x and width y then its perimeter will be $2x + 2y$. A square with the same perimeter will have side length,

$$\frac{2x+2y}{4} = \frac{x+y}{2}.$$

Therefore,

$$A(\text{square}) = \left(\frac{x+y}{2}\right)^2 \geq xy = A(\text{rectangle}).$$

10 We show that it is only possible for Kaye to be the liar.

case 1

Suppose Jaye is lying

$\Rightarrow$ Kaye is not lying

$\Rightarrow$ Elle is lying

$\Rightarrow$ There are two liars

$\Rightarrow$ This is impossible.

case 2

Suppose Kaye is lying

$\Rightarrow$ Jaye is not lying and Elle is not lying

$\Rightarrow$ Kaye is the only liar

case 3

Suppose Elle is lying

$\Rightarrow$ Mina is not lying

$\Rightarrow$ Karl is lying

$\Rightarrow$ There are two liars

$\Rightarrow$ This is impossible.

11 First note that the four sentences can be recast as:

- Exactly three of these statements are true.
- Exactly two of these statements are true.
- Exactly one of these statements are true.
- None of these statements are true.

At most one of these statements can be true, or else we obtain a contradiction. If none of the statements is true, then the last statement is true. This means that at least one of the statements is true. This also gives a contradiction. Therefore, only one of the statements is true, that is, the third statement.

12 **a** There is only one possibility,

$\boxed{1, 2, 4, 8}\ \boxed{3, 5, 6, 7}$

b We know that we can split the numbers $1, 2, \ldots, 8$,

$\boxed{1, 2, 4, 8}\ \boxed{3, 5, 6, 7}$

Deleting the largest number, 8, will give a splitting of $1, 2, \ldots, 7$.

$\boxed{1, 2, 4}\ \boxed{3, 5, 6, 7}$

Continuing this process, deleting the 7, will be a splitting of the numbers $1, 2, \ldots, 6$, and so on.

c We first note that if a set can be split then two numbers can't appear in the same group as their difference. To see this, if x and y and $x - y$ all belong to the same group then $(x - y) + y = x$. Let's now try to split the numbers $1, 2, \ldots, 9$. Call the two groups X and Y. We can assume that $1 \in X$. We now consider four cases for the groups containing elements 2 and 9.

(case 1) Suppose $2 \in X$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
(assumed)	9		
		3	$(1, 2 \in X)$
		7	$(2, 9 \in X)$
$(3, 7 \in Y)$	4		
		5	$(1, 4 \in X)$
		6	$(2, 4 \in X)$
$(5, 6 \in Y)$	8		

This doesn't work, since X is forced to contain the numbers $1, 8$ and 9.

(case 2) Suppose $2 \in X$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
		9	(assumed)
		3	$(1, 2 \in X)$
$(3, 9 \in Y)$	6		
		4	$(2, 6 \in X)$
		5	$(1, 6 \in X)$

This doesn't work, since Y is forced to contain the numbers $4, 5$ and 9.

(case 3) Suppose $2 \in Y$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
(assumed)	9		
		8	$(1, 9 \in X)$
$(2, 8 \in Y)$	6		
		3	$(6, 8 \in X)$
$(2, 8 \in Y)$	5		$(3, 8 \in X)$

This doesn't work, since X is forced to contain the numbers $1, 5$ and 6.

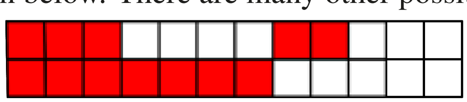
(case 4) Suppose $2 \in Y$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
		9	(assumed)
$(2, 9 \in Y)$	7		
		6	$(1, 7 \in X)$
$(2, 8 \in Y)$	4		
		3	$(4, 7 \in X)$

This doesn't work, since Y is forced to contain the numbers $3, 6$ and 9.

d If the numbers $1, 2, \ldots, n$ could be split, where $n \geq 9$, then we could successively eliminate the largest term to obtain a splitting of the numbers $1, 2, \ldots, 9$. However, we already know that this is impossible.

13 **a** A suitable tiling is shown below. There are many other possibilities.



b Tile E must go into a corner. This is because there are only two other tiles (A and

B) that it can go next to. Tile F must also go into a corner. This is because there are only two other tiles (B and C) that it can go next to.

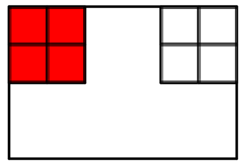
(Case 1) Tile E and tile F are in different rows

Since tile B must go next to both tiles E and F, this is impossible.

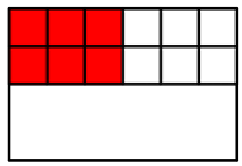
(Case 2) Tile E and tile F are in the same row

Assume tile F is in the top left position.

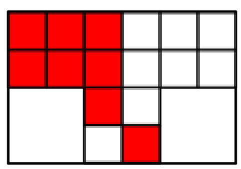
Then tile E goes in the top right position.



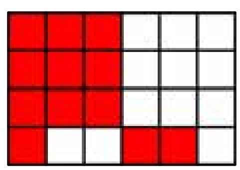
Therefore tile B must go between them.



Tile C must then go beneath tile F and tile A must go beneath tile E. Consequently, tile D must go beneath tile B. Therefore, there is only one valid orientation of tile D.



This fixes the orientation of tiles A and C.

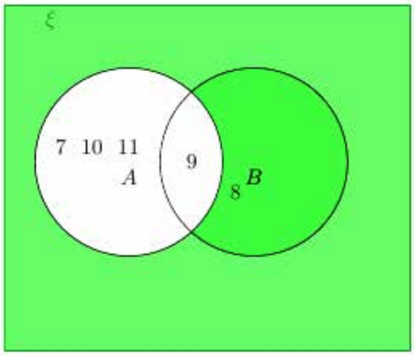


Since tile F could have gone into any one of the four corners, there are only four ways to tile the grid.

Chapter 7 – Logic

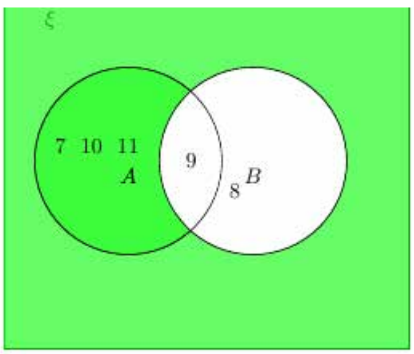
Solutions to Exercise 7A

1 a



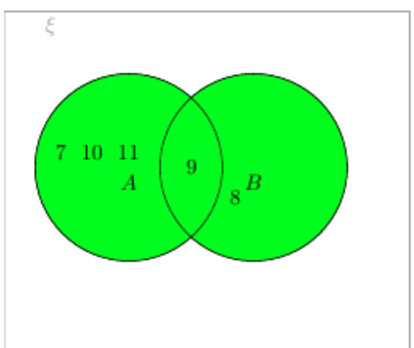
$A' = \{8\}$

b



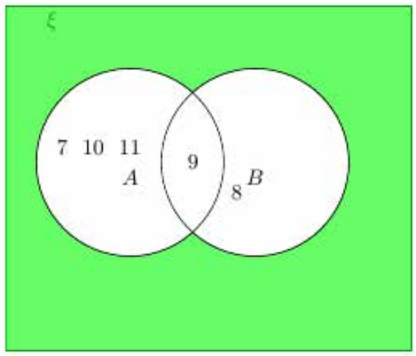
$B' = \{7, 10, 11\}$

c



$A \cup B = \{7, 8, 9, 10, 11\}$

d, e



$(A \cup B)' = A' \cap B' = \varnothing$

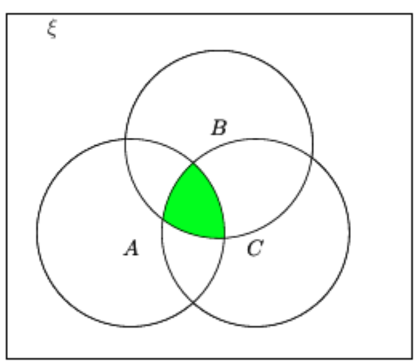
2 $X = \{1, 2, 3, 4, 5\}$ and $Y = \{2, 3, 5, 7, \dots\}$

a $\{2, 3, 5\}$
$= \{1, 2, 3, 4, 5\} \cap \{2, 3, 5, 7, \dots\}$
$= X \cap Y$

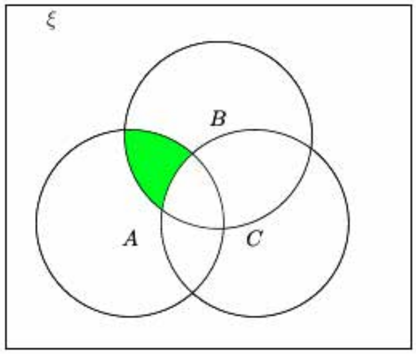
b One and 4 are the non-primes less than or equal to 5. Therefore $\{1, 4\} = X \cap Y'$

c Numbers greater than 5 are given by X' and composites are given by Y'. Therefore required set is $Y' \cap X'$

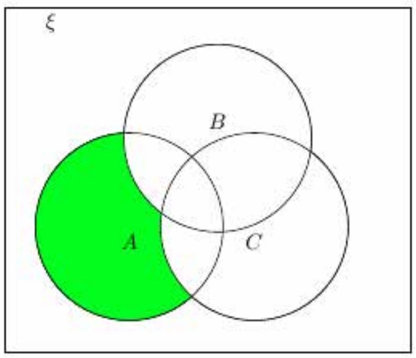
3 a



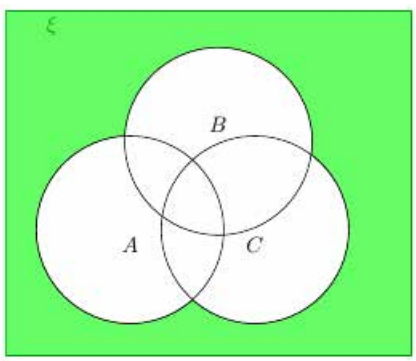
b



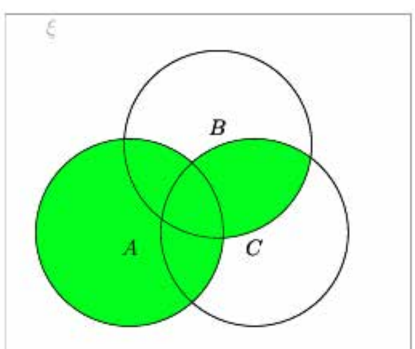
c

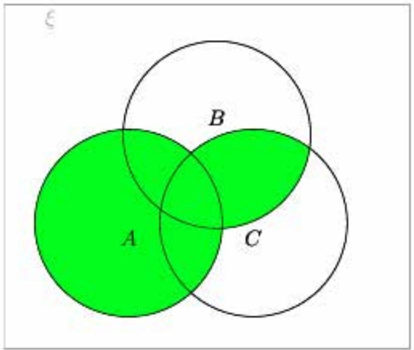


d



e



f 

Note that $(A \cup B) \cap (A \cup C) = A \cap (B \cup C)$

4 $A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6, 8, 10, 12 \ldots\}, C = \{3, 6, 9, 12 \ldots\}$

a $[3, 6] = A \cap C$

b $[1, 3, 5] = B' \cap A$

c $\{6, 12, 18, 24 \ldots\} = B \cap C$

d Natural numbers greater than 6 $= A'$
Therefore required set is $B \cap A'$

e $C \cup B'$

5 For a given statement about sets, the dual statement is obtained by interchanging:
$\cup$ with $\cap$
$\varnothing$ with ξ
$\subseteq$ with $\supseteq$

a $(A \cap \varnothing) \cup (A \cup \xi) = \xi$

b If $A \cup B = \xi$, then $A' \cap B = A'$.

c $A \cup B \supseteq A \cap B$

6 **a** To prove $A \cup B = B \cup A$
Let $x \in A \cup B$.
Then $x \in A$ or $x \in B$
Thus $x \in B$ or $x \in A$
Hence $x \in A \cup B$
$A \cup B \subseteq B \cup A$.
In the same way: $B \cup A \subseteq A \cup B$.

Hence $A \cup B = B \cup A$

b To prove $A \cap B = B \cap A$
Let $x \in A \cap B$.
Then $x \in A$ and $x \in B$
Thus $x \in B$ and $x \in A$
Hence $x \in A \cap B$
$A \cap B \subseteq B \cap A$.
In the same way: $B \cap A \subseteq A \cap B$.
Hence $A \cap B = B \cap A$

c To prove $(A \cap B)' = A' \cup B'$
Let $x \in (A \cap B)'$.
Then $x \notin (A \cap B)$.
Therefore $x \notin A$ or $x \notin B$.
That is $x \in A' \cup B'$
Hence $(A \cap B)' \subseteq A' \cup B'$
Now let $x \in A' \cup B'$.
Then $x \in A'$ or $x \in B'$
Hence $x \notin A$ or $x \notin B$
Hence $x \notin A \cap B$
That is $x \in (A \cap B)'$
Hence $A' \cup B' \subseteq (A \cap B)'$
Therefore,
$(A \cap B)' = A' \cup B'$

d To prove $(A \cup B) \cap (A \cup B') = A$
Let $x \in (A \cup B) \cap (A \cup B')$
The $x \in A \cup B$ and $x \in A \cup B'$
We proceed by contradiction
Assume $x \notin A$. Then $x \in B$ and $x \in B'$ but this is impossible. Hence $x \in A$ We have,
$(A \cup B) \cap (A \cup B') \subseteq A$
Conversely if $x \in A$ then $x \in A \cup B$ and $x \in A \cup B'$.
Hence $A \subseteq (A \cup B) \cap (A \cup B')$ and we have the result:
$(A \cup B) \cap (A \cup B') = A$

e To prove: $A = (A \cap B) \cup (A \cap B')$
Let $x \in A$.
First assume $x \in B$ then $x \in A \cap B$.
If $x \notin B$ then $x \in B'$ and hence $x \in (A \cap B')$.
Hence $x \in (A \cap B) \cup (A \cap B')$.

We have $A \subseteq (A \cap B) \cup (A \cap B')$
Conversely assume:
$x \in (A \cap B) \cup (A \cap B')$
Proceed by contradiction: Assume $x \notin A$.
Then $x \notin A \cap B$ and $x \notin A \cap B'$
Hence a contradiction and $x \in A$.
Thus $(A \cap B) \cup (A \cap B') \subseteq A$
and we have:
$A = (A \cap B) \cup (A \cap B')$

f To prove:
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$
If $x \in A$ then $x \in (A \cup B)$ and $x \in (A \cup C)$. Hence:
$x \in (A \cup B) \cap (A \cup C)$.
If $x \in B \cap C$ then $x \in B$ and $x \in C$.
Hence $x \in (A \cup B) \cap (A \cup C)$ and we have:
$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
Conversely:
Let $x \in (A \cup B) \cap (A \cup C)$
Then $x \in A \cup B$ and $x \in A \cup C$.
If $x \in A$ then $x \in A \cup (B \cap C)$.
If $x \notin A$ then $x \in (B \cap C)$and we thus have the result:
$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
and
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

g has a similar proof.

7 a $X \cup (Y \cup X) = (X \cup X) \cup Y$ (Commutative and associative)
$= X \cup Y$ (Primary $A \cup A = A$)

b $(Y \cup Y') \cap Y = \xi \cap Y$ $(A \cup A' = \xi)$
$= Y$ $(A \cap \xi = A)$

c $X \cap (X' \cap Y) = (X \cap X') \cap Y$ associative
$= \varnothing \cap Y$ $(A \cap A' = \varnothing)$
$= \varnothing$ $(\varnothing \cap A = \varnothing)$

d $X \cap (Y \cup X) = X \cup \varnothing \cap (Y \cup X)$

$= (X \cup \varnothing) \cap (X \cup Y)$ commutative

$= X \cup (\varnothing \cap Y)$ distributive

$= X \cup \varnothing$ $\varnothing \cap A = \varnothing$

$= X$ $\varnothing \cup A = A$

e $X \cup (Y' \cap X) = (X \cap \xi) \cup (Y' \cap X)$ $(A \cap \xi = A)$

$= (X \cap \xi) \cup (X \cap Y')$ commutative

$= X \cap (\xi \cup Y')$ distributive

$= X \cap \xi$ $(A \cup \xi = \xi)$

$= X$ $(A \cap \xi = A)$

f $[X' \cup (Y \cap Z)]' = X \cap (Y \cap Z)'$

$= X \cap Y' \cup Z'$

g $(X' \cup Y')' = X'' \cap Y''$

$= X \cap Y$

h $(X' \cap Y')' = X'' \cup Y''$

$= X \cup Y$

i $(X \cap Y') \cap (X' \cap Y') = X \cap X' \cap Y' \cap Y'$

$= \varnothing \cap Y'$

$= \varnothing$

j $(X \cap Y) \cup (X \cap Y') = X \cap (Y \cup Y')$

$= X \cap \xi$

$= X$

k $(X \cup Y) \cap (X \cup Y')]' = (X \cup Y)' \cup (X \cup Y')'$

$= (X' \cap Y') \cup (X' \cap Y'')$ Here we use the result that $(A \cup B)' = A' \cap B'$

$= X' \cap (Y' \cup Y)$

$= X' \cap \xi$

$= X'$

l $(X \cup Y') \cap [(X \cap Z) \cup (X \cap Z')]'$

$= (X \cup Y') \cap (X \cap Z)' \cap (X \cap Z')'$

$= (X \cup Y') \cap (X' \cup Z') \cap (X' \cup Z)$

$= (X \cup Y') \cap (X' \cup (Z \cap Z')$

$= (X \cup Y') \cap (X' \cup \varnothing)$

$= (X \cup Y') \cap X'$

$= X \cap X' \cup (Y' \cap X')$

$= \varnothing \cup (Y' \cap X')$

$= Y' \cap X'$

8 **a** Given that $A \subseteq B$ and $B \subseteq C$.
Let $x \in A \Rightarrow x \in B$ since $A \subseteq B$.
Hence $x \in C$ since $B \subseteq C$.
Thus $A \subseteq C$.

b Given that $A \subseteq B$ and $A \subseteq C$.
Let $x \in A$. Then $x \in B$ and $x \in C$.
Hence $x \in B \cap C$.
Thus $A \subseteq B \cup C$.

c Given that $A \subseteq B$.
Let $x \in B' \Rightarrow x \notin B$.
Since $A \subseteq B, x \notin A$
This implies $x \in A'$.
Hence $B' \subseteq A'$
Conversely assume $B' \subseteq A'$
Let $x \in A$.
We establish $x \in B$ by establishing a contradiction.
Assume $x \in B'$ then by the assumption $B' \subseteq A', x \in A'$ which is a contradiction.
Hence $x \in B$ and $A \subseteq B$

9 (We use the properties of sets discussed in this section and $A \backslash B = A \cap B'$)

a $P \backslash (Q \backslash R) = P \cap (Q \backslash R)'$
$= P \cap (Q \cap R')'$
$= P \cap (Q' \cup R)$ $\quad ((A \cap B)' = A' \cup B')$
$= (P \cap Q') \cup (P \cap R)$ $\quad$ (Distributive)
$= (P \backslash Q') \cup (P \cap R)$

b $P \cap (Q \backslash R)$

$= P \cap (Q \cap R')$

$= P \cap Q \cap R'$ (Associative)

$= \varnothing \cup (P \cap Q \cap R')$ $\varnothing \cup A = A$

$= ((P \cap Q) \cap P') \cup ((P \cap Q) \cap R')$ $A \cap A' = \varnothing$ and then $\varnothing \cap B = \varnothing$

$= (P \cap Q) \cap (P' \cup R')$ (Distributive)

$= (P \cap Q) \cap (P \cap R)'$

$= (P \cap Q) \backslash (P \cap R)$

Note Probably easier to work from the right side - work backwards through this proof.

Solutions to Exercise 7B

1 **a** $1 \vee 0' = 1 \vee 1 = 1$

b $1' \wedge 0 = 0 \wedge 0 = 0$

c $1' \vee 0' = 0 \vee 1 = 1$

d $(1 \vee 0) \vee 1' = 1 \vee 0 = 1$

e $(1 \vee 0) \vee 1' = 1 \vee 0 = 1$

f $0 \wedge (1' \vee 0) = 0 \wedge (0 \vee 0) = 0$

g $(1' \vee 1) \wedge (1 \vee 0) = 1 \wedge 1 = 1$

h $(1 \vee 0) \wedge (1' \vee 0) = 1 \wedge 0 = 0$

2 **a**

x	y	y'	$x \vee y'$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

b

x	y	y'	$x \wedge y'$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

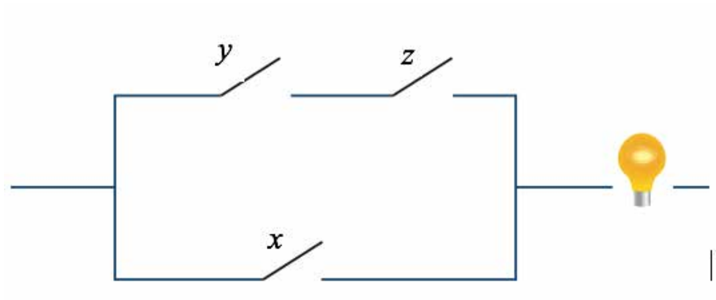
c

x	y	x'	y'	$x' \wedge y'$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

d

x	y	x'	y'	$x' \vee y'$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

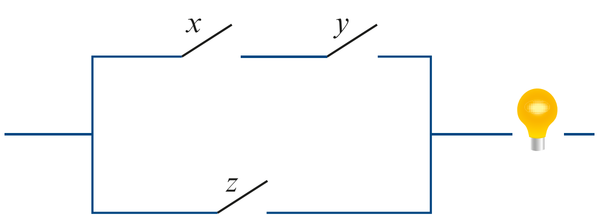
3 a i



ii

x	y	z	$y \wedge z$	$x \vee (y \wedge z)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

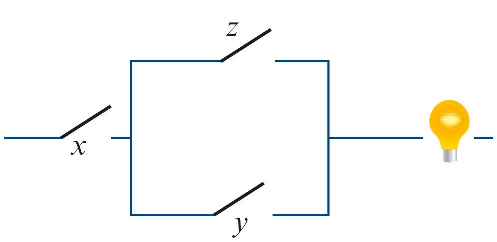
b i



ii

x	y	z	$x \wedge y$	$(x \wedge y) \vee z$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

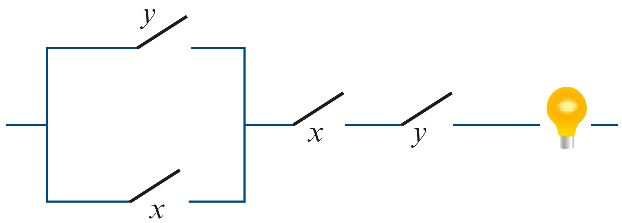
c i



ii

x	y	z	$y \vee z$	$x \wedge (y \vee z)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

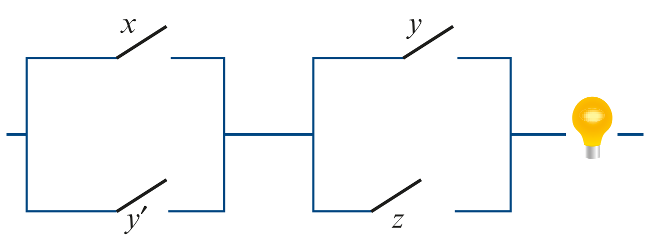
d **i**



ii

x	y	$x \vee y$	$x \wedge y$	$(x \vee y) \wedge (x \wedge y)$
0	0	0	0	0
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

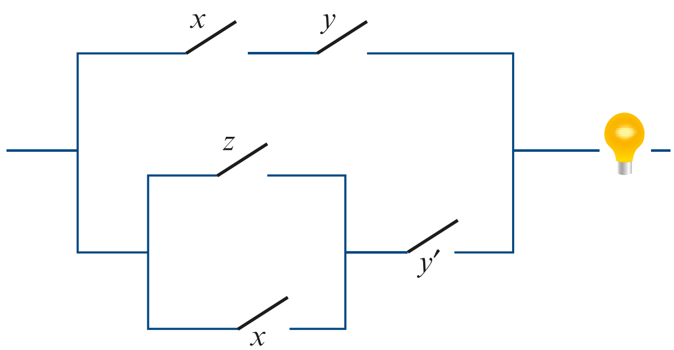
e t



ii

x	y	z	$a = x \vee y'$	$b = y \vee z$	$a \wedge b$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

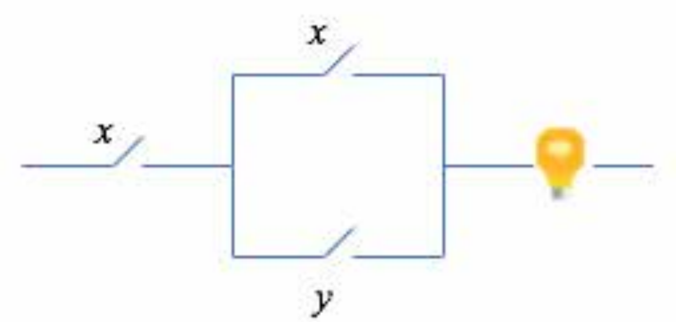
f **i**

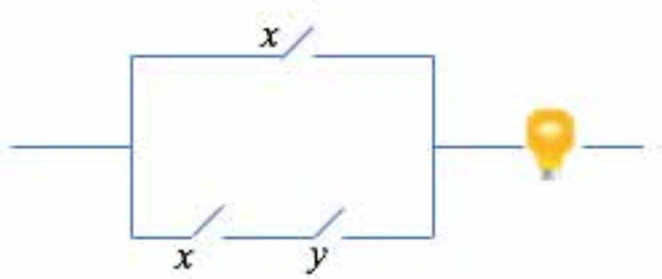


ii

x	y	z	$a = x \wedge y$	$b = (z \vee x) \wedge y'$	$a \vee b$
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	0	1

4





x	y	$x \vee y$	$x \wedge y$	$x \wedge (x \vee y)$	$x \vee (x \wedge y)$
0	0	0	0	0	0
0	1	1	0	0	0
1	0	1	0	1	1
1	1	1	1	1	1

Both $x \wedge (x \vee y)$ and $x \vee (x \wedge y)$ take the same values as x.

5 **a**

x	y	z	$y \vee z$	$x \wedge (y \vee z)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

x	y	z	$a = x \wedge y$	$b = x \wedge z$	$a \vee b$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

Hence $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

b

x	y	z	$y \wedge z$	$x \vee (y \vee z)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

x	y	z	$a = x \vee y$	$b = x \vee z$	$a \wedge b$
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

Hence $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

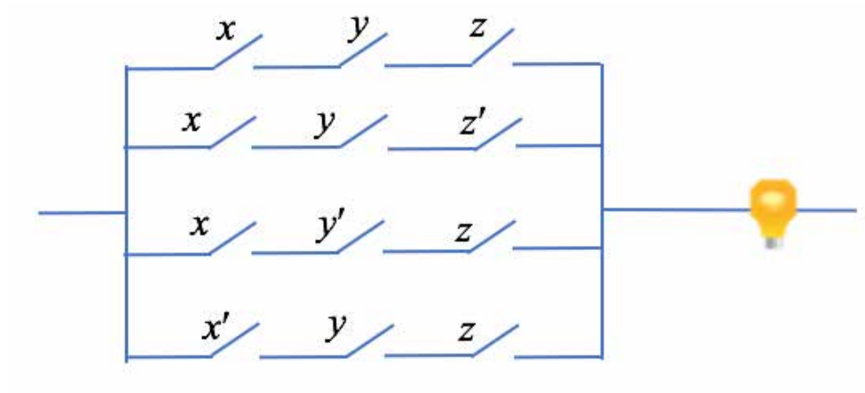
c

x	y	z	$x \wedge y$	$(x \wedge y) \vee z$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

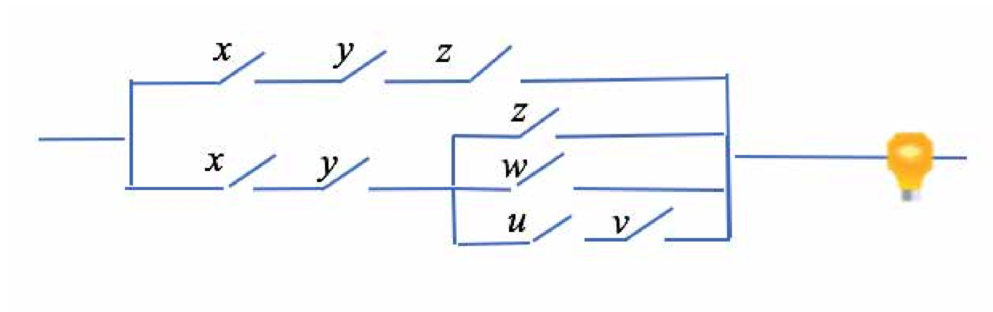
x	y	z	$x \vee y$	$(x \vee y) \wedge z$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

Therefore not equivalent.

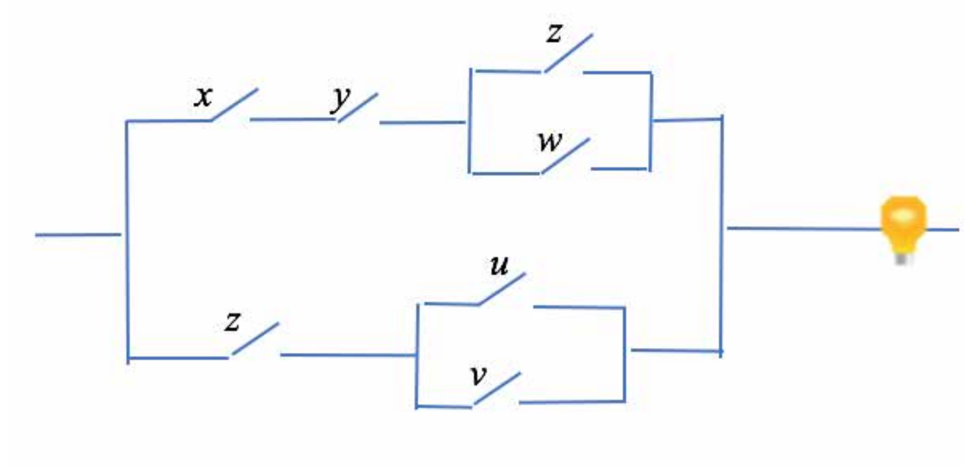
6 a



b



c



Solutions to Exercise 7C

Properties of Boolean algebras

Primary	$x \vee x = x$	$x \wedge x = x$
	$x \vee 0 = x$ (A4)	$x \wedge 1 = x$ (A4)
	$x \vee 1 = 1$	$x \wedge 0 = 0$
Associativity (A2)	$(x \vee y) \vee z = x \vee (y \vee z)$	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
Commutativity (A1)	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
Distributivity (A3)	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
Absorption	$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$
Complements	$x \vee x' = 1$ (A5)	$x \wedge x' = 0$ (A5)
	$0' = 1$	$1' = 0$
	$(x \vee y)' = x' \wedge y'$	$(x \wedge y)' = x' \vee y'$
	$(x')' = x$	

1 In this question any of the properties can be used to simplify

a $a \wedge (b \wedge a') = a \wedge (a' \wedge b)$ (Commutative)

$= (a \wedge a') \wedge b$ (Associative)

$= 0 \wedge b$ (Complementation)

$= 0$ ($0 \wedge x = 0$. See comment below)

$0 \wedge b = 0$ is not an axiom. We prove it here in two steps.

First we prove $x^{x} = x$ from the axioms.

PROOF 1

$x \wedge x = (x \wedge x) \vee 0$ (Axiom A4)

$= (x \wedge x) \vee (x \wedge x')$ (Axiom A5)

$= x \wedge (x \vee x')$ (Axiom A3)

$= x \wedge 1$ (Axiom A5)

$= x$ (Axiom A4)

We next prove that $0 \wedge x = 0$.

PROOF 2

$$0 \wedge x = (x \wedge x') \wedge x \qquad \text{(Axiom5)}$$
$$= (x \wedge x) \wedge x' \qquad \text{(Axioms A1 and A2)}$$
$$= x \wedge x' \qquad \text{(Above result)}$$
$$= 0 \qquad \text{(Axiom5)}$$

b $(a \wedge b') \wedge a' = (a \wedge a') \wedge b'$ (Commutative and associative)

$= 0 \wedge b'$ (Complementation)

$= 0$ (Proved in **a**)

c $a \vee (b \vee a') = (a \vee a') \vee b$ (Commutative and associative)

$= 1 \vee b$ (Complementation)

$= 1$

d $(a \vee b') \vee a' = (a \vee a') \vee b'$(Commutative and associative)

$= 1 \vee b'$(Complementation)

$= 1$

e $(a \vee b) \wedge a' = (a \wedge a') \vee (b \wedge a')$ (Commutative and associative)

$= 0 \vee (b \wedge a')$(Complementation)

$= b \wedge a'$

f $a \vee (b \wedge a') = (a \vee b) \wedge (a \vee a')$ (distributive)

$= (a \vee b) \wedge 1$(Complementation)

$= a \vee b$

g $a \wedge (b \vee a') = (a \wedge b) \vee (a \wedge a')$ (distributive)

$= (a \wedge b) \vee 0$(Complementation)

$= a \wedge b$

h $(a \wedge b) \vee (a' \wedge b) = (a \vee a') \wedge b$(Commutative and associative)

$= 1 \wedge b$ (Complementation)

$= b$

i $(a \vee b) \vee (a' \vee b) = (a \vee a') \vee (b \vee b)$ (Commutative and associative)

$= 1 \vee b$ (Complementation)

$= 1$

2 a LHS $= x \vee x$

$= (x \vee x) \wedge 1$ (Axiom 4)

$= (x \vee x) \wedge (x \vee x')$ (Axiom 5)

$= x \vee (x \wedge x')$ (Axiom 3)

$= x \vee 0$ (Axiom 5)

$= x$ (Axiom 4)

$=$ RHS

b LHS $= x \wedge x$

$= (x \wedge x) \vee 0$ (Axiom 4)

$= (x \wedge x) \vee (x \wedge x')$ (Axiom 5)

$= x \wedge (x \vee x')$ (Axiom 3)

$= x \wedge 1$ (Axiom 5)

$= x$ (Axiom 4)

$=$ RHS

c LHS $= x \vee (x \wedge y)$

$= (x \wedge 1) \vee (x \wedge y)$ (Axiom 4)

$= x \wedge (1 \vee y)$ (Axiom 3)

$= x \wedge (y \vee 1)$ (Axiom 1)

$= x \wedge 1$ (Example 6a)

$= x$ (Axiom 4)

$=$ RHS

d LHS $= x \wedge (x \vee y)$

$= (x \vee 0) \wedge (x \vee y)$ (Axiom 4)

$= x \vee (0 \wedge y)$ (Axiom 3)

$= x \vee (y \wedge 0)$ (Axiom 1)

$= x \vee 1$ (Example 6b)

$= x$ (Axiom 4)

$=$ RHS

e Refers to example 6c
If $a \vee b = 1$ and $a \wedge b = 0$, then $a' = b$

$a = 0$ and $b = 1$

$a \vee b = 0 \vee 1 = 1$ (Axioms 1 and 4)

$a \wedge b = 0 \wedge 1 = 0$ (Axiom 4)
Therefore $0' = 1$

f Refers to example 6c
If $a \vee b = 1$ and $a \wedge b = 0$, then $a' = b$
$a = 1$ and $b = 0$

$a \vee b = 1 \vee 0 = 1$ (Axioms 1 and 4)

$a \wedge b = 1 \wedge 0 = 0$ (Axioms 1 and 4)
Therefore $0' = 1$

g Refers to example 6c
If $a \vee b = 1$ and $a \wedge b = 0$, then $a' = b$
$a = x'$ and $b = x$

$a \vee b = x' \vee x = 1$ (Axioms 1 and 5)

$a \wedge b = x' \wedge x = 0$ (Axioms 1 and 4)
Therefore $(x')' = x$

3 $(b \wedge c') \wedge (d \wedge b') = (b \wedge b') \wedge (c' \wedge d)$

$= 0 \wedge (c' \wedge d)$

$= 0$

It is now easy to show
$a \vee (b \wedge c') \wedge (d \wedge b') = a \vee 0 = a$

4 a

x	y	x'	$x \vee y$	$f(x, y)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

b

x	y	x'	y'	$x \vee y'$	$x' \vee y'$	$f(x, y)$
0	0	1	1	1	1	1
0	1	1	0	0	1	0
1	0	0	1	1	1	1
1	1	0	0	1	0	0

Note that it is the same table of values as for y'.

c

x	y	x'	y'	$x \wedge y'$	$x' \wedge y'$	$f(x, y)$
0	0	1	1	0	1	0
0	1	1	0	0	0	0
1	0	0	1	1	0	0
1	1	0	0	0	0	0

d

x	y	z	y'	$x \wedge y'$	$f(x, y, z)$
0	0	0	1	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	0	1

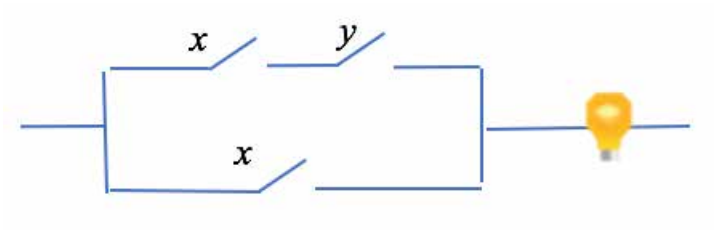
e

x	y	z	$x \vee y$	$f(x, y, z)$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

f

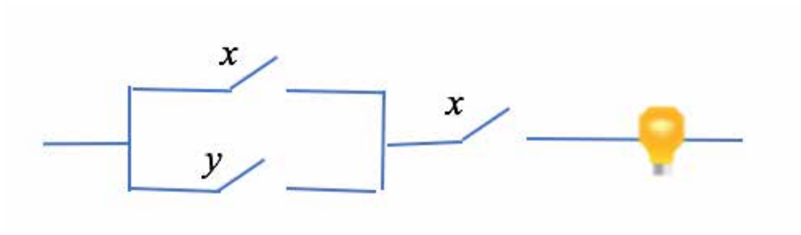
x	y	z	$x \vee y$	$y \vee z$	$z \vee x$	$f(x, y, z)$
0	0	0	0	0	0	0
0	0	1	0	1	1	0
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	1	0	1	0
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

5 a



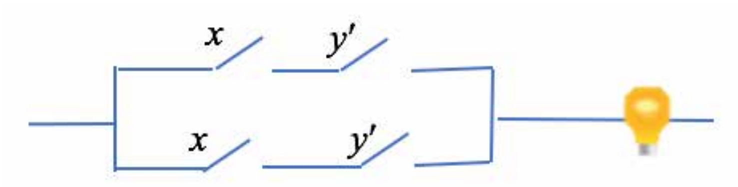
$(x \wedge y) \vee x = x$ the circuit can be simplified by an x switch.

b



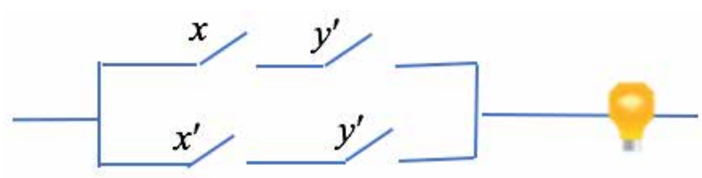
$(x \vee y) \wedge x = x$ the circuit can be simplified by an x switch.

c



$(x \wedge y') \vee (x \wedge y') = x \wedge y'$. Circuit can be simplified by x switch and y' switch in series.

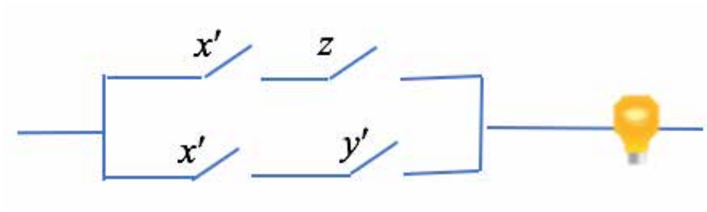
d



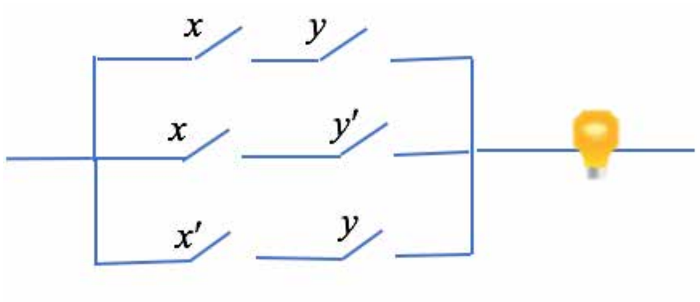
$$
\begin{aligned}
(x \wedge y') \vee (x' \wedge y') &= y' \wedge (x \vee x') \\
&= y' \wedge 1 \\
&= y'
\end{aligned}
$$

The circuit can be simplified to a y' switch

e



$(x' \wedge y') \vee (x' \wedge z) = x' \wedge (y' \vee z)$

f 

$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y)$

$= x \wedge (y \vee y') \vee (x' \wedge y)$

$= x \wedge 1 \vee (x' \wedge y)$

$= x \vee (x' \wedge y)$

$= (x \vee x') \wedge (x \vee y)$

$= x \vee y$

The circuit can be simplified to an x switch and a y switch in parallel.

6 a $(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$

b $(x' \wedge y') \vee (x' \wedge y)$

c $(x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y \wedge z) = (x' \wedge y' \wedge z') \vee (z \wedge y)$

d $(x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y' \wedge z)$

7 a

x	y	x'	y'	$x' \vee y$	$x \vee y'$	$(x' \vee y) \wedge (x \vee y')$
0	0	1	1	1	1	1
0	0	1	1	1	1	1
0	1	1	0	1	0	0
0	1	1	0	1	0	0
1	0	0	1	0	1	0
1	0	0	1	0	1	0
1	1	0	0	1	1	1
1	1	0	0	1	1	1

x	y	x'	y'	$x \wedge y$	$x' \wedge y'$	$(x \wedge y) \vee (x' \wedge y')$
0	0	1	1	0	1	1
0	0	1	1	0	1	1
0	1	1	0	0	0	0
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1
1	1	0	0	1	0	1

They represent the same Boolean function.

b
$$\begin{aligned}(x' \vee y) \wedge (x \vee y') &= ((x' \vee y) \wedge x) \vee ((x' \vee y) \wedge y') \\ &= ((x' \wedge x) \vee (y \wedge x)) \vee ((x' \wedge y') \vee (y \wedge y')) \\ &= (0 \vee (y \wedge x)) \vee ((x' \wedge y') \vee 0) \\ &= (y \wedge x) \vee (x' \wedge y')\end{aligned}$$

Solutions to Exercise 7D

1 **a** Your eyes are not blue.

b The sky is not grey.

c This integer is even.

d I do not live in Switzerland.

e $x \leq 2$

f This number is greater than or equal to 100.

2 **a** It is dark or it is cold.

b It is dark and cold.

c It is light and cold.

d It is light or hot.

e It is good or light.

f It is light and hard.

g It is dark or hard.

3 **a** $B \wedge A$

b $D \vee C$

c $\neg C \wedge D$

d $\neg A \wedge \neg B$

e $\neg D \wedge \neg C$

f $B \vee A$

4 **a** It is wet or rough.

b It is wet and rough.

c It is dry and rough.

d It is dry or smooth.

e It is difficult or dry.

f It is dry and inexpensive.

g It is wet or inexpensive.

5 **a** n is a prime number or an even number.

b n is divisible by 6.

c n is 2.

d n is an even number greater than 2.

e n is not 2.

f n is not prime.

g n is neither prime nor divisible by 6.

h n is not divisible by 6.

6

A	B	$A \wedge B$	$\neg(A \wedge B)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	T

7

A	B	$A \vee B$	$\neg B$	$(A \vee B) \wedge (\neg B)$
T	T	T	F	F
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

8 a

A	B	$\neg(A \vee B)$	$\neg A \wedge \neg B$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

b

A	$\neg(\neg A)$
T	T
F	F

c

A	$A \vee A$
T	T
F	F

d

A	B	$A \vee B$	$\neg(\neg A \wedge \neg B)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

e

A	B	$A \wedge B$	$\neg(\neg A \vee \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

f

A	B	$A \wedge \neg B$	$\neg(\neg A \vee B)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

9

A	B	$\neg A$	$\neg B$	$A \vee B$	$\neg A \wedge \neg B$	$(\neg A \wedge \neg B) \vee (A \vee B)$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	T

Hence a tautology.

10

A	B	$\neg B$	$A \wedge B$	$(A \wedge B) \wedge \neg B$
T	T	F	T	F
T	F	T	F	F
F	T	F	F	F
F	F	T	F	F

Hence a contradiction

11

A	B	$\neg A$	$\neg A \wedge B$	$(\neg A \wedge B) \wedge A$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Hence a contradiction

12 **a**

A	B	$A \wedge B$	$(A \wedge B) \Rightarrow A$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

b

A	B	$A \vee B$	$(A \vee B) \Rightarrow A$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

c

A	B	$\neg A$	$\neg B$	C: $\neg B \vee \neg A$	$C \Rightarrow A$
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	F	T	F
F	F	T	T	T	F

d

A	B	$\neg B$	$\neg B \wedge A$	$(\neg B \wedge A) \Rightarrow A$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

e

A	B	$\neg A$	$B \vee \neg A$	$(B \vee \neg A) \Rightarrow \neg A$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

f

A	B	C: $\neg B \vee \neg A$	D: $\neg B \wedge A$	$C \Rightarrow D$
T	T	F	F	T
T	F	T	T	T
F	T	T	F	F
F	F	T	F	F

g

A	B	C: $\neg B \vee A$	D: $\neg(B \wedge A)$	$C \Rightarrow D$
T	T	T	F	F
T	F	T	T	T
F	T	F	T	T
F	F	T	T	T

h

A	B	$\neg B$	$\neg B \Rightarrow A$	$\neg B \wedge (\neg B \Rightarrow A)$
T	T	F	T	F
T	F	T	T	T
F	T	F	T	F
F	F	T	F	F

13 **a** Truth table for $A \wedge B$

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for $\neg(A \Rightarrow \neg B)$

A	B	$\neg B$	$A \Rightarrow \neg B$	$\neg(A \Rightarrow \neg B)$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

They are equivalent.

b Truth table for $A \vee B$

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Truth table for $\neg A \Rightarrow B$

A	B	$\neg A$	$\neg A \Rightarrow B$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

They are equivalent.

c Truth table for $A \Leftrightarrow B$

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Truth table for $\neg[(A \Rightarrow B) \Rightarrow \neg(B \Rightarrow A)]$

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$\neg(B \Rightarrow A)$	$\neg[(A \Rightarrow B) \Rightarrow \neg(B \Rightarrow A)]$
T	T	T	T	F	T
T	F	F	T	F	F
F	T	T	F	T	F
F	F	T	T	F	T

14 **a**

A	B	$A \wedge B$	$A \vee B$	$(A \wedge B) \Rightarrow (A \vee B)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

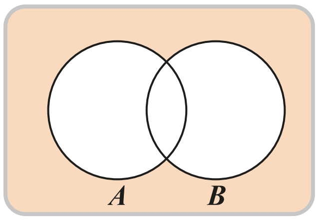
b

A	B	$A \Rightarrow B$	$A \wedge (A \Rightarrow B)$	$[A \wedge (A \Rightarrow B)] \Rightarrow B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

c

A	B	$A \vee B$	$\neg A$	$[(A \vee B) \wedge (\neg A)]$	$[(A \vee B) \wedge (\neg A)] \Rightarrow B$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

15



$(A \cup B)' = A' \cap B'$

16 a

A	B	$A \downarrow B$	$B \downarrow A$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

b

A	$A \downarrow A$	$\neg A$
T	F	F
F	T	T

c **Note:** $A \downarrow A$ is equivalent to $\neg A$ by part b

A	B	$\neg A \downarrow \neg B$	$A \wedge B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

d

A	B	$\neg(A \downarrow B)$	$A \vee B$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

17

A	B	$\neg A$	$\neg B$	$A \Rightarrow B$	$\neg B \Rightarrow \neg A$	$B \Rightarrow A$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

18

A	B	$\neg B$	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$A \wedge \neg B$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

19 **a** **i** If x is an even integer, then $x = 6$.

ii If x is not an even integer, then $x \neq 6$.

iii x is not an even inteer and $x = 6$

b **i** If public transport improves, then I was elected.

ii If public transport does not improve, then I was not elected.

iii Public transport does not improve and I was elected

c **i** If I qualify as an actuary, then I passed the exam.

ii If I do not qualify as an actuary, then I failed the exam.

iii I did not qualify as an actuary and I passed the exam

Solutions to Exercise 7E

1

A	B	$A \vee B$	$\neg A$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

Argument is valid.

2

A	B	$A \vee B$	$\neg A$	$\neg B$
T	T	T	F	F
T	F	T	F	T
F	T	T	T	F
F	F	F	T	T

Argument is not valid.

3

A	B	C	$A \Rightarrow B$	$B \Rightarrow C$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

Argument is valid.

4 A: You eat lots of garlic; B: You don't have many friends.

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Argument is valid.

5 B: Four is even; A: Five is odd.

A	B	$B \Rightarrow A$
T	T	T
T	F	T
F	T	F
F	F	T

Argument is not valid.

6 C: I will buy a car; M: I will buy a motorcycle; L I will need a loan

C	M	L	$C \vee M$	$C \wedge M$	$C \wedge M \Rightarrow L$	$M \wedge \neg L$	$\neg C$
T	T	T	T	T	T	F	F
T	T	F	T	T	F	T	F
T	F	T	T	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	T	F	T	T	T
F	T	F	T	F	T	T	T
F	F	T	F	F	T	T	T
F	F	F	F	F	F	T	T

Argument is valid.

7 **a**

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Argument is valid.

b

A	B	$A \Rightarrow B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

Argument is valid.

c

A	$\neg A$	B	$\neg B$	$A \wedge B$	$\neg A \Rightarrow B$
T	F	T	F	T	T
T	F	F	T	F	T
F	T	T	F	F	T
F	T	F	T	F	F

Argument is not valid.

d

A	B	$\neg B$	$A \Rightarrow \neg B$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

Argument is not valid

8 **a** Argument is valid

b Argument is not valid

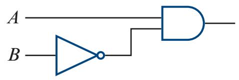
9 **a** $((J \Rightarrow W) \wedge W) \Rightarrow J$. Not a tautology

b $((\neg\text{Sunny} \Rightarrow \neg\text{Running}) \wedge \text{Running}) \Rightarrow \text{Sunny}$. Tautology

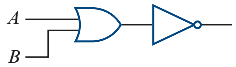
c $((K \Rightarrow J) \wedge (J \Rightarrow S)) \Rightarrow (K \Rightarrow S)$.Tautology

Solutions to Exercise 7F

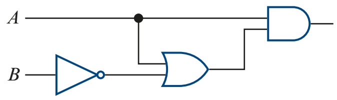
1 a



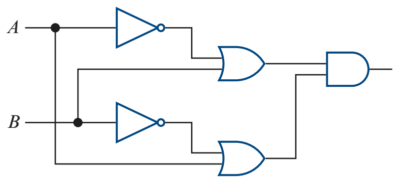
b



c



d



2 a $\neg(A \wedge B)$

A	B	$A \wedge B$	$\neg(A \wedge B)$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

b $\neg A \wedge \neg B$

A	B	$\neg A$	$\neg B$	$\neg A \wedge \neg B$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

c $\neg X \vee (X \wedge Y) \equiv \neg X \vee Y$

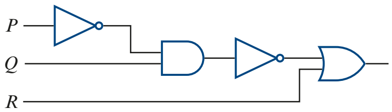
X	Y	$\neg X$	$X \wedge Y$	$\neg X \vee (X \wedge Y)$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	0
1	1	0	1	1

d $\neg A \wedge (A \vee B) \equiv \neg A \wedge B$

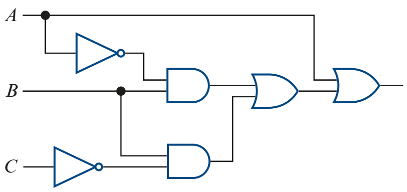
A	B	$\neg A$	$A \vee B$	$\neg A \wedge (A \vee B)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

3 $((X \wedge \neg Y) \vee Y) \vee Z$

4 **a**

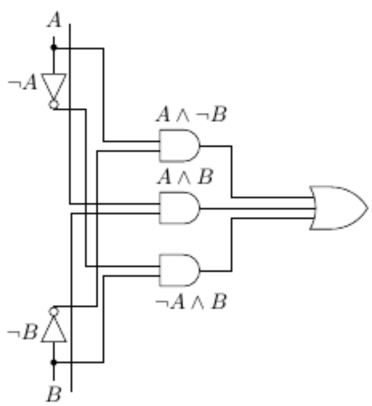


b

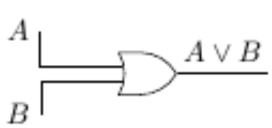


5 **a** $(\neg A \wedge B) \vee (A \wedge \neg B) \vee (A \wedge B)$

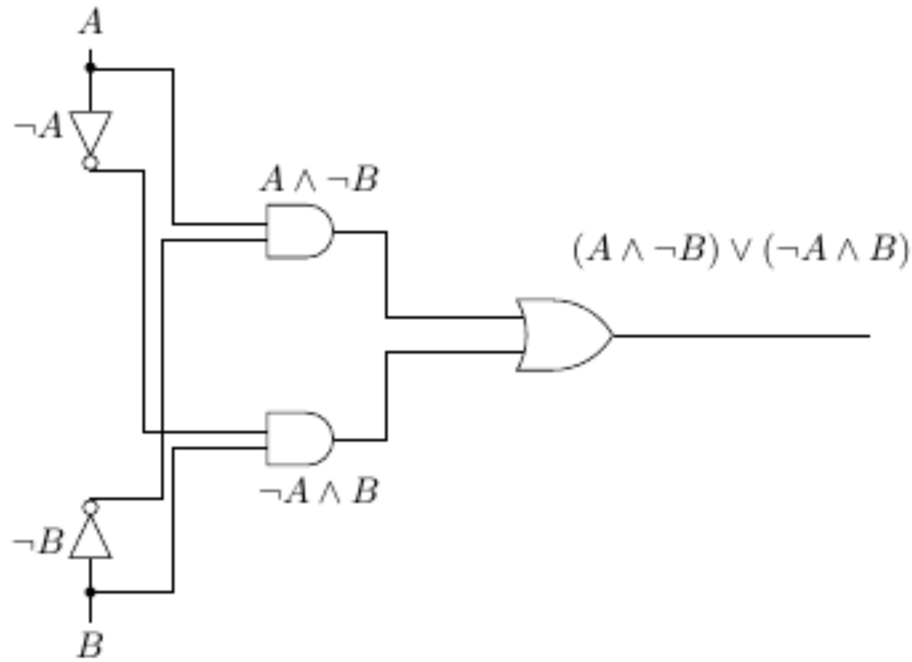
b



c
$$\begin{aligned}(\neg A \wedge B) \vee (A \wedge \neg B) \vee (A \wedge B) &= (\neg A \wedge B) \vee (A \wedge (B \vee \neg B))\\ &= (\neg A \wedge B) \vee (A \wedge 1)\\ &= (\neg A \wedge B) \vee A\\ &= (\neg A \vee A) \wedge (B \vee A)\\ &= 1 \wedge (B \vee A)\\ &= B \vee A\end{aligned}$$



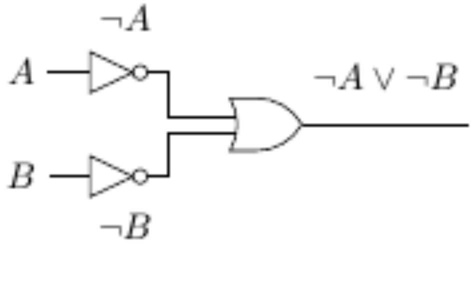
6 $(\neg A \wedge B) \vee (A \wedge \neg B)$



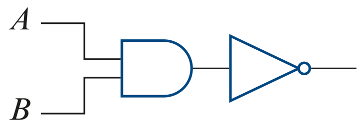
7 **a** $(A \wedge \neg B) \vee (\neg A \wedge \neg B) \vee (B \wedge \neg A)$

b
$$
\begin{aligned}
(A \wedge \neg B) \vee (\neg A \wedge \neg B) \vee (B \wedge \neg A) &= \neg B \wedge (A \vee \neg A) \vee (B \wedge \neg A) \\
&= (\neg B \wedge 1) \vee (B \wedge \neg A) \\
&= \neg B \vee (B \wedge \neg A) \\
&= \neg B \vee B \wedge \neg B \vee \neg A \\
&= 1 \wedge (\neg B \vee \neg A) \\
&= \neg B \vee \neg A
\end{aligned}
$$

c



d $(\neg A \vee \neg B) = \neg(A \wedge B)$


A
B

Solutions to Exercise 7G

1 **a**

	y	y'
x	1	
x'	1	

$(x \wedge y) \vee (x' \wedge y) = y$

b

	y	y'
x		1
x'	1	1

$(x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = x' \vee y'$

c

	y	y'
x	1	1
x'		1

$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y') = x \vee y'$

2 **a** y

b $y' \vee (y \wedge z') = y' \vee z'$

c $(x \wedge y') \vee (x' \wedge z')$

3 **a**

	yz	$y'z$	$y'z'$	yz'
x	1	1	1	
x'		1		1

$(x \wedge y') \vee (x \wedge z) \vee (x' \wedge y \wedge z')$

b

	yz	$y'z$	$y'z'$	yz'
x	1			1
x'			1	1

$(x \wedge y) \vee (x' \wedge z')$

c

	yz	$y'z$	$y'z'$	yz'
x	1	1	1	
x'	1		1	1

	yz	$y'z$	$y'z'$	yz'
x	1	1	1	
x'	1		1	1

$(x \wedge y') \vee (x' \wedge z') \vee (y \wedge z)$ or $(x \wedge z) \vee (x' \wedge y) \vee (y' \wedge z')$

4 a $(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y') = x' \vee y'$

b $(x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z) = x \vee (y \wedge z')$

5 a

X	Y	$\neg X$	$\neg Y$	$X \wedge \neg Y$	$\neg X \wedge \neg Y$	$(X \wedge \neg Y) \vee (\neg X \wedge \neg Y)$
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	1	0	1
1	1	0	0	0	0	0

b $\neg Y$

c circuit with $\neg Y$

6 a

X	Y	Z	$Y \wedge Z$	$(X \wedge \neg Y) \vee (\neg X \wedge \neg Y) \vee (Y \wedge Z)$
0	1	1	1	1
0	1	0	0	0
0	0	1	0	1
0	0	0	0	1
1	1	1	1	1
1	1	0	0	0
1	0	1	0	1
1	0	0	0	1

b $\neg Y \vee Z$

c circuit with $\neg Y \vee Z$

Solutions to technology-free questions

1 True: a,b, d, e,f
False c

2 **a** It is not raining.

b It is raining.

c $x \neq 5$ or $y \neq 5$

d $x \neq 3$ and $x \neq 5$ (i.e. $x \notin \{3, 5\}$)

e It is raining or it is windy.

f It is snowing and it is not cold

3 **a**

A	B	$A \oplus B$	$A \oplus (A \oplus B)$
T	T	F	T
T	F	T	F
F	T	T	T
F	F	F	F

Note: $A \oplus (A \oplus B) \equiv B$

b

A	B	$A \vee B$	$A \oplus (A \vee B)$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	F

4

A	B	$\neg A$	$A \Rightarrow B$	$\neg A \Rightarrow (A \Rightarrow B)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

5 **a** **i**

x	y	x’	$x' \wedge y$	$x \vee (x' \wedge y)$	$x \vee y$
1	1	0	0	1	1
1	0	0	0	1	1
0	1	1	1	1	1
0	0	1	0	0	0

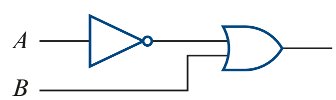
ii
$$\begin{aligned} x \vee (x' \wedge y) &= (x \vee x') \wedge (x \vee y) \\ &= 1 \wedge (x \vee y) \\ &= x \vee y \end{aligned}$$

b **i**

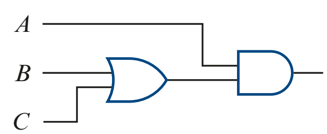
x	y	x’	$x \vee y$	$x' \vee y$	$(x \vee y) \wedge (x' \vee y)$
1	1	0	1	1	1
1	0	0	1	0	0
0	1	1	1	1	1
0	0	1	0	1	0

ii
$$\begin{aligned} (x \vee y) \wedge (x' \vee y &= y \vee (x \wedge x') \\ &= y \vee 0 \\ &= y \end{aligned}$$

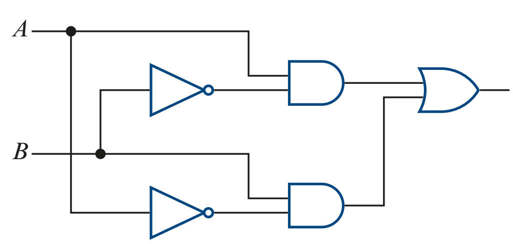
6 **a**



b



c



Solutions to multiple-choice questions

1 B

2 C The dual of $A \cap (A \cup B)' = \varnothing$ is $A \cup (A \cap B)' = \xi$

3 C $(x \wedge y)' = x' \vee y'$

4 D D is the contrapositive of the statement. Note the negation of (2 does not divide n **and** 3 does not divide n) is (2 does not divivide n **or** 3 does not divide n)

5 B Let A be the statement: Tom is Jane's father. Let B be the statement Jane is Bill's niece. Then P is the statement $A \Rightarrow B$.

We consider the contrapositive of the statement $(A \Rightarrow B) \Rightarrow Q)$. That is $\neg Q \Rightarrow \neg(A \Rightarrow B)$, We know that$\neg(A \Rightarrow B) = A \wedge \neg B$
So the required contrapositive is:
$\neg Q \Rightarrow A \wedge \neg B$

6 D

7 A Check the rows with 1 as the far right entry.

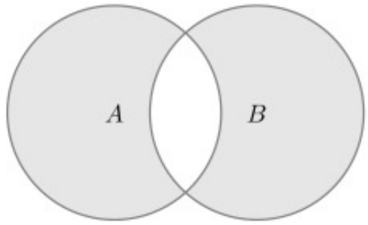
8 B

9 D

10 E

Solutions to extended-response questions

1 a



b We work from the right-hand side.

$$\begin{aligned}(A \cup B)\backslash(A \cap B) &= (A \cup B) \cap (A \cap B)' \\ &= (A \cup B) \cap (A' \cup B') \\ &= [(A \cup B) \cap A'] \cup [(A \cup B) \cap B'] \\ &= [(A \cap A') \cup (B \cap A')] \cup [(A \cap B') \cup (B \cap B')] \\ &= [\varnothing \cup (B \cap A')] \cup [(A \cap B') \cup \varnothing] \\ &= (B \cap A') \cup (A \cap B') \\ &= (B\backslash A) \cup (A\backslash B) \\ &= A \oplus B\end{aligned}$$

c We use the result that $P \cap (Q\backslash R) = (P \cap Q)\backslash(P \cap R)$ proved in 7A.

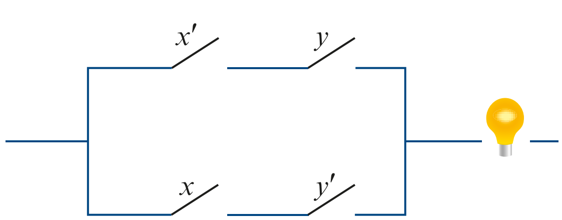
$$\begin{aligned}(A \cap B) \oplus (A \cap C) &= [A \cap B) \cup (A \cap C)]\ (A \cap B \cap A \cap C) \\ &= [A \cap (B \cup C)]\backslash(A \cap (B \cap C)] \\ &= A \cap [(B \cup C)\backslash(B \cap C)] \\ &= A \cap (B \oplus C)\end{aligned}$$

2 a

x	y	Light
0	0	0
0	1	1
1	0	1
1	1	0

b $(x' \wedge y) \vee (x \wedge y')$

c



3 a

x	y	z	Light
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

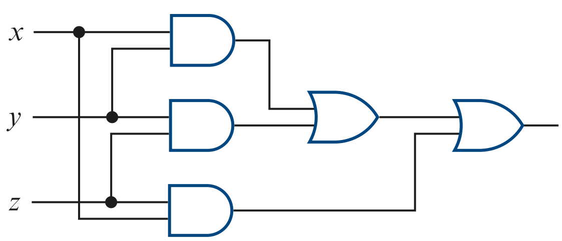
b $(x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$

c

	yz	$y'z$	$y'z'$	yz'
x	1	1		1
x'	1			

$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

d



4 a **i** $\ell = 1$

ii $h = 30$

b LCM$(x, x') = 30 = h$, for all $x \in B$;
HCF$(x, x') = 1 = \ell$, for all $x \in B$

5 a **i** d

ii 1

iii 0

b $d \vee d' = d \neq 1$ and $d \wedge d' = d \neq 0$

c

x	y	x'	y'	$x \vee y$	$x' \wedge y'$	$(x \vee y)'$
0	0	1	1	0	1	1
d	0	d	1	d	d	d
1	0	0	1	1	0	0
0	d	1	d	d	d	d
d	d	d	d	d	d	d
1	d	0	d	1	0	0
0	1	1	0	1	0	0
d	1	d	0	1	0	0
1	1	0	0	1	0	0

Chapter 8 – Algorithms

Solutions to Exercise 8A

1 **a** $F = 27, G = 14, H = 110, K = 69;$

$92 \times 37 = 3404$

b $F = 8, G = 18, H = 56, K = 30;$

$43 \times 26 = 1118$

c $F = 2, G = 63, H = 90, K = 25;$

$27 \times 19 = 513$

d $F = 10, G = 21, H = 60, K = 29;$

$57 \times 23 = 1311$

2 **a**

n	q	r
342	171	0
171	85	1
85	42	1
42	21	0
21	10	1
10	5	0
5	2	1
2	1	0
1	0	0

The binary number is 101010110

b

n	q	r
127	63	1
63	31	1
31	15	1
15	7	1
7	3	1
3	1	1
1	0	1

The binary number is 1111111

c 11011110001

d 100110100100

3 a
- **Step 1** Input *number*
- **Step 2** Let q be the quotient when *number* is divided by 8
- **Step 3** Let r be the remainder when *number* is divided by 8
- **Step 4** Record r
- **Step 5** Let n have the value of q.
- **Step 6** If *number* > 0, then return to Step 2.
- **Step 7** Write the recordeof r in reverse order.

b **i** 526

ii 13056

iii 705

iv 22657

4 a $9284 = 2 \times 4361 + 562$

$4361 = 7 \times 562 + 427$

$562 = 1 \times 427 + 135$

$427 = 3 \times 135 + 22$

$135 = 6 \times 22 + 3$

$22 = 7 \times 3 + 1$

$3 = 3 \times 1 + 0$

$\text{HCF}(9284, 562) = 1$

b $2160 = 2 \times 999 + 162$

$999 = 6 \times 162 + 27$

$162 = 6 \times 27 + 0$

$\text{HCF}(2160, 999) = 27$

c $762 = 2 \times 372 + 18$

$372 = 20 \times 18 + 12$

$18 = 1 \times 12 + 6$

$12 = 2 \times 6 + 0$

$\text{HCF}(762, 372) = 6$

d $716\,485 = 136 \times 5255 + 1805$

$5255 = 2 \times 1805 + 1645$

$1805 = 1 \times 1645 + 160$

$1645 = 10 \times 160 + 45$

$160 = 3 \times 45 + 25$

$45 = 1 \times 25 + 20$

$25 = 1 \times 20 + 5$

$20 = 4 \times 5 + 0$

$\text{HCF}(716\,485, 5255) = 5$

5 **a** $2x^2 + 3x + 4 = x(2x + 3) + 4$

b $x^3 + 3x^2 - 4x + 5 = (x^2 + 3x - 4)x + 5$

$= (x + 3)x - 4)x + 5$

c $4x^3 + 6x^2 - 5x - 4 = ((4x + 6)x - 5)x - 4$

6 **Theorem** Let a and b be two integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then $\text{HCF}(a, b) = \text{HCF}(a, r)$.

Proof

If d is a common factor of a and r, then d divides the right-hand side of the equation $b = aq + r$, and so d divides b.

This proves that all common factors of a and r are also common factors of a and b.

But $\text{HCF}(a, r)$ is a common factor of a and r, and therefore $\text{HCF}(a, r)$ must divide a and b. It follows that $\text{HCF}(a, r)$ must divide $\text{HCF}(a, b)$. That is,

$$\text{HCF}(a, b) = m \cdot \text{HCF}(a, r) \qquad \text{for some integer } m \qquad (1)$$

Now rewrite the equation $b = aq + r$ as $r = b - aq$.

If d is a common factor of a and b, then d divides the right-hand side of the equation $r = b - aq$, and so d divides r.

This proves that all common factors of a and b are also common factors of a and r.

It follows that HCF(a, b) must divide HCF(a, r). That is,

$$\text{HCF}(a, r) = n \cdot \text{HCF}(a, b) \qquad \text{for some integer } n \qquad (2)$$

From equations (1) and (2), we obtain

$$\text{HCF}(a, r) = mn \cdot \text{HCF}(a, r)$$

$$\therefore \qquad 1 = mn$$

This equation in integers m, n is possible only if $m = n = 1$ or $m = n = -1$.
Hence HCF(a, b) = HCF(a, r), since both must be positive

7 a ■ **Step 1** Choose an initial estimate x for $\sqrt{N}$

■ **Step 2** Let $x_{\text{new}} = \frac{1}{2}\left(x + \frac{N}{x}\right)$

■ **Step 3** Let x have the value of x_{new}

■ **Step 4** Repeat fom Step 2 unless $-0.01 < x^2 - N < 0.01$

■ **Step 5** The required number is x.

b **i** $\sqrt{5} \approx 2.2$

ii $\sqrt{345} \approx 18.6$

iii $\sqrt{1563} \approx 39.5$

iv $\sqrt{7856} \approx 88.6$

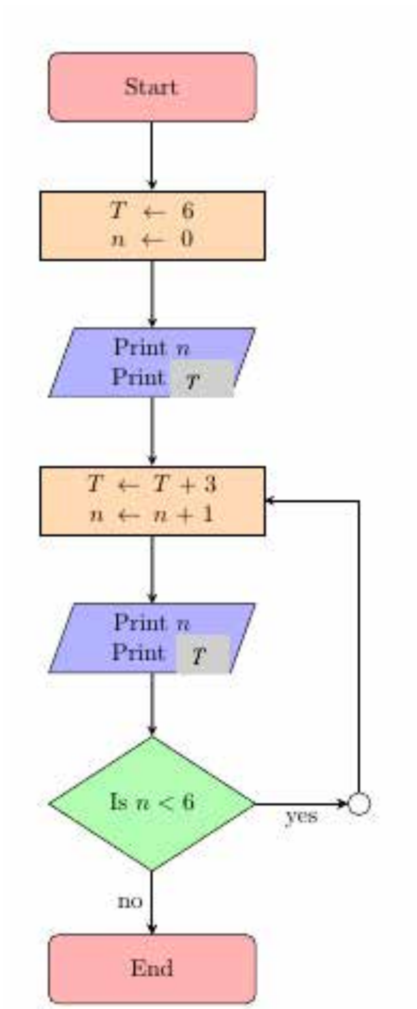
8

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Solutions to Exercise 8B

1 **a**
- **Step 1** $T \leftarrow 6$ and $n \leftarrow 1$
- **Step 2** Print n and print T
- **Step 3** $T \leftarrow T + 3$ and $n \leftarrow n + 1$
- **Step 4** print T and print n
- **Step 5** Repeat from **Step 3** while $n < 6$

b



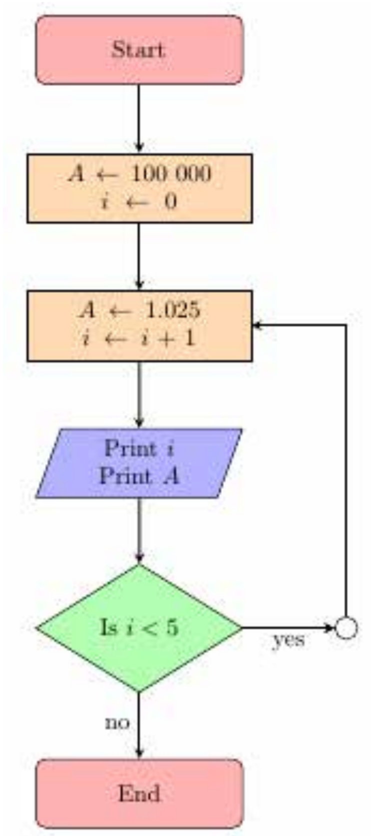
c

i	1	2	3	4	5	6
T	6	9	12	15	18	21

2 **a**
- **Step 1** $A \leftarrow 100\,000$ and $i \leftarrow 0$

- **Step 2** $A \leftarrow 1.025A$ and $i \leftarrow i + 1$
- **Step 3** print A and print i
- **Step 4** Repeat from **Step 2** while $i < 5$

b



c In the following table values are fiven to the nearest dollar.

i	0	1	2	3	4	5
A	100 000	102 500	105 063	107 689	110 381	113 141

3 a

n	1	2	3	4	5	6
A	10	15	20	25	30	35

b

n	1	2	3	4	5
A	2	6	18	54	162

4 a
- **Step 1** Let $sum \leftarrow 0$ and $n \leftarrow 1$
- **Step 2** $sum \leftarrow sum + \dfrac{1}{n^2}$

- **Step 3** $n \leftarrow n + 1$
- **Step 4** Repeat from **Step 2** while $n \le N$

b
- **Step 1** Let $sum \leftarrow 0$ and $n \leftarrow 1$
- **Step 2** $sum \leftarrow sum + \frac{1}{n}$
- **Step 3**$n \leftarrow n + 1$
- **Step 4** Repeat from **Step 2** while $n \le N$

5
- **Step 1** $n \leftarrow 1$
- **Step 2** If n is even $T \leftarrow 5 - 2n$.

  Otherwise $T \leftarrow n^2 + 1$
- **Step 3** Print T
- **Step 4** $n \leftarrow n + 1$
- **Step 5** Repeat from **Step 2** while $n \le N$

n	1	2	3	4	5	6	7
T	2	1	10	−3	26	−7	-

6 a

Step	p	i
1	0	3
2	1	2
3	5	1
4	12	0
5	37	−1

$P(3) = 37$

b

Step	p	i
1	0	3
2	2	2
3	5	1
4	19	0
5	55	−1

$P(3) = 55$

c

Step	p	i
1	0	3
2	-4	2
3	-10	1
4	-31	0
5	-94	−1

$P(3) = -94$

7 **b**
- **Step 1** $n \leftarrow 1$
- **Step 2** Draw forward for 3 cm
- **Step 3** turn through 90° anticlockwise
- **Step 4** $n \leftarrow n + 1$
- **Step 5** Repeat from Step 2 while $n \leq 4$

c
- **Step 1** $n \leftarrow 1$
- **Step 2** Draw forward for 3 cm
- **Step 3** turn through 60° anticlockwise
- **Step 4** $n \leftarrow n + 1$
- **Step 5** Repeat from Step 2 while $n \leq 6$

d

8 **a**
- **Step 1** Input n
- **Step 2** Print n
- **Step 3** If $n = 1$, then stop
- **Step 4** If n is even $n \leftarrow n \div 2$

Otherwise $n \leftarrow 3n + 1$

- **Step 5** Repeat from Step 2

b **i** $5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

ii $7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

iii $8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

Solutions to Exercise 8C

1 input a, b
if $a \le b$ then
    print a
else
    print b
end if

2 input *score*
if *mark* ≥ 95 then
    print 'A'
else if *mark* ≥ 85 then
    print 'B'
else if *mark* ≥ 65 then
    print 'C'
else if *mark* ≥ 55 then
    print 'D'
else
    print 'E'
end if

3 a 15

b 16

c 20

4 a 0

b 5

c 25

5 a 5

b 6

c 10

6 a $a = 6, b = 15$

b $a = 8, b = 29$

c $a = 7, b = 18$

7 a input n
$sum \leftarrow 0$
for i from 1 to n
    $sum \leftarrow sum + 2^i$
end for
print *sum*

b input n
$product \leftarrow 1$
for i from 1 to n
    $product \leftarrow product \times 2^i$
end for
print *sum*

8 input n
$sum \leftarrow 0$
for i from 1 to n
    $sum \leftarrow sum + i^3$
end for
print *sum*

9

a	b
8	6
2	18
2	6
−22	−18
410	366

10 $n \leftarrow 1$
$x \leftarrow 4$
while $x < 1000$

$n \leftarrow n + 1$

$x \leftarrow 3x + 2$

end while

print n, x

Desk check:

n	x
1	4
2	14
3	44
4	134
5	404
6	1214

11 $sum \leftarrow 0$

$n \leftarrow 0$

while $sum < 1\,000\,000$:

$n \leftarrow n + 1$

$sum \leftarrow sum + n^n$

print (n, sum)

end while

n	sum
1	1
2	5
3	32
4	288
5	3413
6	50 069
7	873 612
8	17 650 828

12 $n \leftarrow 1$

while $2^n \leq 10n^2$

$n \leftarrow n + 1$

print $(n, 2^n, 10n^2)$

end while

$n = 10$

13 $n \leftarrow 0$

$x \leftarrow 3$

$y \leftarrow 1000$

while $x \leq y$

$n \leftarrow n + 1$

$x \leftarrow 2 \times n + 3$

$y \leftarrow 0.9^n \times 1000$

print (n, x, y)

end while

$n = 28$

14 **a** **i** 8

ii 20

iii 16

b It employs the Euclidean algorithm to find the highest common factor of two numbers.

Solutions to Exercise 8D

1 Replacement line of code given.

a $sum \leftarrow sum + i^3$

b $sum \leftarrow sum + 2^i$

c $sum \leftarrow sum + i \times (i + 1)$

2 **a** It anounces the creation of a list.

b The 'filling' of the list statement is:

append 2^i to A

c
```
define power(n)
A ← []
for i from 0 to n
    append 2^(n−i) to A
end for
return A
```

3
```
define min(A)
min ← A[1]
for i from 1 to length(A)
    if A[i] < min then
        min ← A[i]
    end if
end for
return min
```

4 **a**
```
define sum(n)
sum ← 0
for i from 1 to n
    sum ← sum + factorial(i)
end for
return sum
```

b
```
1 ← n
while factorial(n) ≤ 10^n
    n ← n + 1
    print n
end while
```

5 **a**

a	b	c
1	1	1
1	2	2
1	3	3
2	1	4
2	2	5
2	3	6
3	1	7
3	2	8
3	3	9

b

a	b	c
2	3	6
2	4	14
3	3	23
3	4	35

6 **a**

i	$A[i]^2$	$tally$
1	1	1
2	9	10
3	25	35
4	64	99

b It calculates the sum of the squares of the elements of the list.

7 **a**

i	$A[i]$	$A[i+1]$	$A[]$
1	1	1	[1,1,2]
2	1	2	[1,1,2,3]
3	2	3	[1,1,2,3,5]
4	3	5	[1,1,2,3,5,8]
5	5	8	[1,1,2,3,5,8,13]

b
```
A ← [1, 1]
i ← 1
while A[i] ≤ 1000
    append A[i] + A[i + 1]) to A
    i ← i + 1
end while
    print i, A[i]
```

c
```
define Fib(n)
    A ← [1, 1]
    for i from 1 to n − 2
        append A[i] + A[i + 1] to A
return A[n]
```

d
```
A ← [0, 1, 1]
for i from 1 to 7
    append A[i] + A[i + 1] + A[i + 2]
to A
end for
print A
```

8 **a** $A[5] = 25$

b
```
A ← []
n ← 1
while n^3 ≤ 100 000
    append n^3 to A
    n ← n + 1
end while
print A
```

c 46

9
```
for x from 1 to 10
    for y from 1 to 6
        for z from 1 to 4
            if 3x + 5y + 7z = 30 then
                print (x, y, z)
            end if
        end for
    end for
end for
```
Solutions: $(1, 4, 1), (2, 2, 2), (6, 1, 1)$

10 **a**
```
for x from 1 to 5
    for y from 1 to 5
        for z from 1 to 3
            if x^2 + y^2 + 10z = 30
then
                print (x, y, z)
            end if
        end for
    end for
end for
```

b $(1, 3, 2), (2, 4, 1), (3, 1, 2), (4, 2, 1)$

c
```
for x from 1 to 7
    for y from 1 to 7
        for z from 1 to 7
            if x^2 + y^2 + 10z = 30
and
            x + y + z = 7 then
                print (x, y, z)
            end if
        end for
    end for
end for
```

$(2, 4, 1), (4, 2, 1)$

11 define $f(n)$

```
    A ← []
    for x from 1 to n
        for y from 1 to n
            if x^2 + y^2 = n then
                append [x, y] to A
            end if
    end for
end for
```

12 It gives the number and proportion of cases when there is no real solutions of the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are integers between -10 and 10 inclusively.

For questions 13 and 14 the approacch given in the answers is different.

13 a define $primecheck(n)$

```
    if length(factors(n)) = 2 then
        print 'prime'
    else
        print 'not prime'
```

b For numbers up to 1000. This bound can be changed.

define $prime(n)$

```
    B=[]
        for i from 1 to 1000
    if length(factors(i)) = 2 then
        append i to B
    print B[n]
```

14 a define $power(n)$

```
    i ← 0
    while quotient(n, 2) > 0
        n ← quotient(n, 2)
        i ← i + 1
    end while
    return i
```

b First function to return the highest common factor of two natural numbers:

define $hcf(x, y)$

```
    if x > y then
        smaller ← y
    else
        smaller ← x
    end if
    i ← 1
    while i ≤ (smaller + 1)
        if rem(x, i) = 0 and
rem(y, i) = 0
            h ← i
        end if
        i ← i + 1
    end while
    return h
```

Now the function to give the required smallest number:

define $number(n)$:

```
    ans ← 1
    for i from 1 to n
        ans ← (ans × i)/hcf(ans, i)
    end for
    return ans
```

15 a define $pell(n)$

```
    A ← [1, 2]
    for i from 1 to n − 2
      append A[i] + A[i + 1]) to A
    end for
    return A[n]
```

b

```
sum ← 0
for i from 1 to n
    sum ← sum + pell(n)
end for
```

print(sum)

c $A \leftarrow [1, 2]$

$i \leftarrow 1$

while $A[i] < 10^{999}$

  append $A[i] + 2 \times A[i+1]$ to A

  $i \leftarrow i + 1$

end while

print $A[i]$

16 **a** input a, b

print (a, b)

$i \leftarrow 0$

while $a \neq b$ and $i < 100$

  if $a < b$ then

    $b \leftarrow b - a$

    $a \leftarrow 2 * a$

  else if $b < a$

    $a \leftarrow a - b$

    $b \leftarrow 2 * b$

  end if

  print (a, b)

  $i \leftarrow i + 1$

end while

Note: The condition $i < 100$ is only there to ensure that the program stops.

b **i** It cycles indefinitely : $(21, 28), (42, 7), (25, 14), (21, 28), \ldots$

ii It cycles indefinitely : $(21, 49), (42, 28), (14, 56), (28, 42), (56, 14), (42, 28), \ldots$

iii Goes to $(35, 105), (70, 70)$.

iv Goes to $(19, 133), (38, 114), (76, 76)$

v Goes to $(148, 148)$ after 2 moves.

c For example: $(5, 15), (5, 27), (5, 35), (5, 59, (11, 165)$

17 See answer to 14b

Solutions to technology-free questions

1 **a** 8

b 18

c 93

d 9,75

2 **a**
$sum \leftarrow 0$
for n from 1 to 6
  $sum \leftarrow sum + n^n$
end for
print sum

b
$sum \leftarrow 0$
for n from 1to 6
  $sum \leftarrow sum + (-1)^{(n+1)} n(7-n)$
end for
print sum

3

n	a	b	c
1	2	4	4
2	4	12	12
3	12	44	44
4	44	200	200
5	200	1088	1088

4 **a** $a_1 = 2, a_2 = 8, a_3 = 26$

b
$a \leftarrow 0$
for i from 1 to 50
  $a \leftarrow 3a + 2$
end for
print a

c
$a \leftarrow 0$
$sum \leftarrow 0$
for n from 1 to 50
  $a \leftarrow 3a + 2$
  $sum \leftarrow sum + a$
end for
print sum

5 **a**
Input N
for n from 1 to N
  if $remainder(n, 2) = 0$ then
    $T \leftarrow 6 - 2n$
  else
    $T \leftarrow 3n + 1$
  end if
  print T
end for

n	T
1	4
2	2
3	10
4	−2
5	16

b
Input N
$sum \leftarrow 0$
for n from 1 to N
  if $remainder(n, 2) = 0$ then
    $sum \leftarrow sum + 6 - 2n$
  else
    $sum \leftarrow sum + 3n + 1$
  end if
  $sum \leftarrow sum + T$
end for
  print sum

6
for a from -6 to 6
  for b from-6 to 6
    if $9 \leq a^2 + b^2 \leq 36$ then
      print (a, b)

end if
end for
end for

7 **a**

a	m	b	$f(a)$	$f(m)$	$f(b)$
0	1	2	-2	-1	2
1	1.5	2	-1	0.25	2
1.	1.25	1.5	-1	-0.4375	0.25
1.25	1.375	1.5	-0.4375	-0.109...	0.25
1.375	1.4375	1.5			

b define $f(x) = x^2 - 3$
$a \leftarrow 0$
$b \leftarrow 3$
$m \leftarrow 1.5$
while $b - a \geq 2 \times 0.01$
if $f(a) \times f(m) < 0$ then
$b \leftarrow m$
else
$a \leftarrow m$
end if
$m \leftarrow \dfrac{a+b}{2}$
print (a, m, b)
end while
print m

Solutions to multiple-choice questions

1 E A desk check give the following sequence of values of a:
$1, 2, 4, 8, 16$

2 D

i	sum
1	1
2	3
3	6
4	10

3 C

It is simply which index gives the same element for both lists.

4 E

i	sum
1	−1
2	1
3	-2
4	2

5 A

i	B
1	[2]
2	[2,5]
3	[2,5,8]

6 C

i	j	sum
1	1	1
1	2	3
2	1	5
2	2	9

7 C

$F(2, 3) = f(3, 2) = 8$

8 E

n	$count$
10	1
5	2
4	3
2	4
1	5

9 E

n	i	$f(n)$
16	1	[1]
16	2	[1, 2]
16	4	[1, 2, 4]
16	8	[1, 2, 4, 8]
16	16	[1, 2, 4, 8, 16]

10 C

$sum = 0 + 1 \times 4$
$sum = 4 + 2 \times 3$
$sum = 10 + 3 \times 2$
$sum = 16 + 4 \times 1 = 20$

Solutions to extended-response questions

1 **a** **i** $[1, 0, 0, 0, 0, 0, 1]$

ii $[1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1]$

iii $[1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0]$

b define $baseb(b, n)$

$A \leftarrow []$

while $n > 0$

  $r \leftarrow remainder(n, b)$

  append r to A

  $n \leftarrow quotient(n, b)$

end while

$A \leftarrow reverse(A)$

return A

i $[1, 0, 1]$

ii $[1, 0, 7, 2, 7]$

iii $[1, 5, 3, 0, 0, 2]$

c **i**

i	B
0	$[10]$
1	$[10, 8]$
2	$[10, 8, 6]$
3	$[10, 8, 6, 4]$
4	$[10, 8, 6, 4, 2]$

ii There would be new entries in the list being used for calculations rather than from the old list.

Output $A = [10, 8, 6, 8, 10]$

2 a

i	j	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$
	0	1	9	3	2	7	6
1	1	1	9	3	2	7	6
1	2	1	3	9	2	7	6
1	3	1	3	2	9	7	6
1	4	1	3	2	7	9	6
1	5	1	2	3	7	6	9
2	1	1	3	2	7	6	9
2	2	1	2	3	7	6	9
2	3	1	2	3	7	6	9
2	4	1	2	3	6	7	9
2	5	1	2	3	6	7	9

b You are losing one of the swapped values and replacing with the other.

c The changes: i from 1 to $length(A)$ and j from 1 to $length(A) - i$.

3 a

n	a	$reverse$
567	8	8
56	7	87
5	6	876
0	5	8765

b The algorithm reverses the digits of a number by considering remainders and quotients when dividing by 10.

c
```
for n from 1 to 1000
    m = n^2
    if R(m) = m then
        print(m)
    end if
end for
```

4 a **i** 4321

ii 5555

iii 8765

iv 14443

b The sum of each pair of such digits is less than 10.

c

n	$R(n)$	$n + R(n)$
1756	6571	15565
15565	56551	72116
72116	61127	133243
133243	342331	475574

d 82 (tested up to 1000 iterations for each number)

5 **a** 8,15

b 12,35

c $4(m+1)^2 + m^2(m+2)^2 = (m^2 + 2m + 2)^2$

6 For example consider the list $A = [100, 25, 13, 32, 17, 34]$

```
for i from 1 to 6
    minin ← i
    for j from i + 1 to 5
        if A[minin] > A[j] then
            minind ← j
        end if
    end for
    x ← A[i]
    A[i] ← A[minind]
    A[minind] ← x
end for
print(A)
```

7 Here is a basic prime facorisation program to return a list with the prime factors

```
define primefactors(n)
    A = []
    c = 2
    while n > 1
        if (n is divisible by c) then
            Append c to A
            n ← n/c
        else:
            c ← c + 1
        end if
    end while
    return (A)
```

You can now work withhe and devise a count method to give the multiplicity.

Chapter 8 – Algorithms

Solutions to Exercise 8A

1 **a** $F = 27, G = 14, H = 110, K = 69;$

$92 \times 37 = 3404$

b $F = 8, G = 18, H = 56, K = 30;$

$43 \times 26 = 1118$

c $F = 2, G = 63, H = 90, K = 25;$

$27 \times 19 = 513$

d $F = 10, G = 21, H = 60, K = 29;$

$57 \times 23 = 1311$

2 **a**

n	q	r
342	171	0
171	85	1
85	42	1
42	21	0
21	10	1
10	5	0
5	2	1
2	1	0
1	0	0

The binary number is 101010110

b

n	q	r
127	63	1
63	31	1
31	15	1
15	7	1
7	3	1
3	1	1
1	0	1

The binary number is 1111111

c 11011110001

d 100110100100

3 **a**
- **Step 1** Input *number*
- **Step 2** Let q be the quotient when *number* is divided by 8
- **Step 3** Let r be the remainder when *number* is divided by 8
- **Step 4** Record r
- **Step 5** Let n have the value of q.
- **Step 6** If *number* > 0, then return to Step 2.
- **Step 7** Write the recordeof r in reverse order.

b **i** 526

ii 13056

iii 705

iv 22657

4 **a** $9284 = 2 \times 4361 + 562$

$4361 = 7 \times 562 + 427$

$562 = 1 \times 427 + 135$

$427 = 3 \times 135 + 22$

$135 = 6 \times 22 + 3$

$22 = 7 \times 3 + 1$

$3 = 3 \times 1 + 0$

$\text{HCF}(9284, 562) = 1$

b $2160 = 2 \times 999 + 162$

$999 = 6 \times 162 + 27$

$162 = 6 \times 27 + 0$

$\text{HCF}(2160, 999) = 27$

c $762 = 2 \times 372 + 18$

$372 = 20 \times 18 + 12$

$18 = 1 \times 12 + 6$

$12 = 2 \times 6 + 0$

$\text{HCF}(762, 372) = 6$

d $716\,485 = 136 \times 5255 + 1805$

$5255 = 2 \times 1805 + 1645$

$1805 = 1 \times 1645 + 160$

$1645 = 10 \times 160 + 45$

$160 = 3 \times 45 + 25$

$45 = 1 \times 25 + 20$

$25 = 1 \times 20 + 5$

$20 = 4 \times 5 + 0$

$\text{HCF}(716\,485, 5255) = 5$

5 **a** $2x^2 + 3x + 4 = x(2x + 3) + 4$

b $x^3 + 3x^2 - 4x + 5 = (x^2 + 3x - 4)x + 5$

$= (x + 3)x - 4)x + 5$

c $4x^3 + 6x^2 - 5x - 4 = ((4x + 6)x - 5)x - 4$

6 **Theorem** Let a and b be two integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then $\text{HCF}(a, b) = \text{HCF}(a, r)$.

Proof

If d is a common factor of a and r, then d divides the right-hand side of the equation $b = aq + r$, and so d divides b.

This proves that all common factors of a and r are also common factors of a and b.

But $\text{HCF}(a, r)$ is a common factor of a and r, and therefore $\text{HCF}(a, r)$ must divide a and b. It follows that $\text{HCF}(a, r)$ must divide $\text{HCF}(a, b)$. That is,

$$\text{HCF}(a, b) = m \cdot \text{HCF}(a, r) \qquad \text{for some integer } m \qquad (1)$$

Now rewrite the equation $b = aq + r$ as $r = b - aq$.

If d is a common factor of a and b, then d divides the right-hand side of the equation $r = b - aq$, and so d divides r.

This proves that all common factors of a and b are also common factors of a and r.

It follows that HCF(a, b) must divide HCF(a, r). That is,

$$\text{HCF}(a, r) = n \cdot \text{HCF}(a, b) \qquad \text{for some integer } n \qquad (2)$$

From equations (1) and (2), we obtain

$$\text{HCF}(a, r) = mn \cdot \text{HCF}(a, r)$$

$$\therefore \qquad 1 = mn$$

This equation in integers m, n is possible only if $m = n = 1$ or $m = n = -1$.
Hence HCF(a, b) = HCF(a, r), since both must be positive

7 **a**
- **Step 1** Choose an initial estimate x for $\sqrt{N}$
- **Step 2** Let $x_{\text{new}} = \frac{1}{2}\left(x + \frac{N}{x}\right)$
- **Step 3** Let x have the value of x_{new}
- **Step 4** Repeat fom Step 2 unless $-0.01 < x^2 - N < 0.01$
- **Step 5** The required number is x.

b **i** $\sqrt{5} \approx 2.2$

ii $\sqrt{345} \approx 18.6$

iii $\sqrt{1563} \approx 39.5$

iv $\sqrt{7856} \approx 88.6$

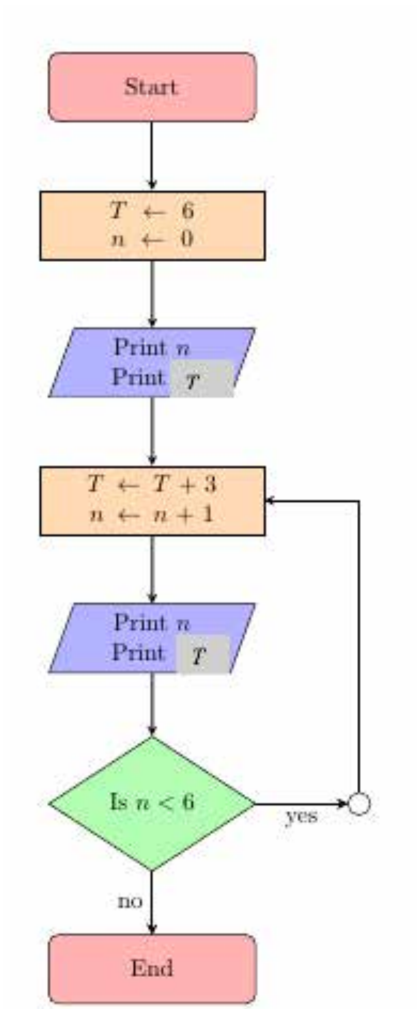
8

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Solutions to Exercise 8B

1 a
- **Step 1** $T \leftarrow 6$ and $n \leftarrow 1$
- **Step 2** Print n and print T
- **Step 3** $T \leftarrow T + 3$ and $n \leftarrow n + 1$
- **Step 4** print T and print n
- **Step 5** Repeat from **Step 3** while $n < 6$

b



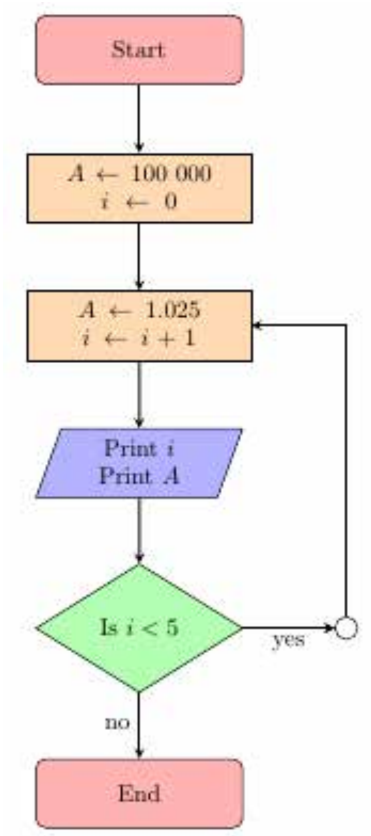
c

i	1	2	3	4	5	6
T	6	9	12	15	18	21

2 a
- **Step 1** $A \leftarrow 100\,000$ and $i \leftarrow 0$

- **Step 2** $A \leftarrow 1.025A$ and $i \leftarrow i + 1$
- **Step 3** print A and print i
- **Step 4** Repeat from **Step 2** while $i < 5$

b



c In the following table values are fiven to the nearest dollar.

i	0	1	2	3	4	5
A	100 000	102 500	105 063	107 689	110 381	113 141

3 a

n	1	2	3	4	5	6
A	10	15	20	25	30	35

b

n	1	2	3	4	5
A	2	6	18	54	162

4 a
- **Step 1** Let $sum \leftarrow 0$ and $n \leftarrow 1$
- **Step 2** $sum \leftarrow sum + \dfrac{1}{n^2}$

- **Step 3** $n \leftarrow n + 1$
- **Step 4** Repeat from **Step 2** while $n \leq N$

b
- **Step 1** Let $sum \leftarrow 0$ and $n \leftarrow 1$
- **Step 2** $sum \leftarrow sum + \frac{1}{n}$
- **Step 3**$n \leftarrow n + 1$
- **Step 4** Repeat from **Step 2** while $n \leq N$

5
- **Step 1** $n \leftarrow 1$
- **Step 2** If n is even $T \leftarrow 5 - 2n$.

  Otherwise $T \leftarrow n^2 + 1$
- **Step 3** Print T
- **Step 4** $n \leftarrow n + 1$
- **Step 5** Repeat from **Step 2** while $n \leq N$

n	1	2	3	4	5	6	7
T	2	1	10	−3	26	−7	-

6 a

Step	p	i
1	0	3
2	1	2
3	5	1
4	12	0
5	37	−1

$P(3) = 37$

b

Step	p	i
1	0	3
2	2	2
3	5	1
4	19	0
5	55	−1

$P(3) = 55$

c

Step	p	i
1	0	3
2	-4	2
3	-10	1
4	-31	0
5	-94	−1

$P(3) = -94$

7 b
- **Step 1** $n \leftarrow 1$
- **Step 2** Draw forward for 3 cm
- **Step 3** turn through 90° anticlockwise
- **Step 4** $n \leftarrow n + 1$
- **Step 5** Repeat from Step 2 while $n \leq 4$

c
- **Step 1** $n \leftarrow 1$
- **Step 2** Draw forward for 3 cm
- **Step 3** turn through 60° anticlockwise
- **Step 4** $n \leftarrow n + 1$
- **Step 5** Repeat from Step 2 while $n \leq 6$

d

8 a
- **Step 1** Input n
- **Step 2** Print n
- **Step 3** If $n = 1$, then stop
- **Step 4** If n is even $n \leftarrow n \div 2$

Otherwise $n \leftarrow 3n + 1$

- **Step 5** Repeat from Step 2

b **i** $5 \to 16 \to 8 \to 4 \to 2 \to 1$

ii $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

iii $8 \to 4 \to 2 \to 1$

Solutions to Exercise 8C

1 input a, b
if $a \le b$ then
    print a
else
    print b
end if

2 input *score*
if *mark* ≥ 95 then
    print 'A'
else if *mark* ≥ 85 then
    print 'B'
else if *mark* ≥ 65 then
    print 'C'
else if *mark* ≥ 55 then
    print 'D'
else
    print 'E'
end if

3 **a** 15

b 16

c 20

4 **a** 0

b 5

c 25

5 **a** 5

b 6

c 10

6 **a** $a = 6, b = 15$

b $a = 8, b = 29$

c $a = 7, b = 18$

7 **a** input n
$sum \leftarrow 0$
for i from 1 to n
    $sum \leftarrow sum + 2^i$
end for
print *sum*

b input n
$product \leftarrow 1$
for i from 1 to n
    $product \leftarrow product \times 2^i$
end for
print *sum*

8 input n
$sum \leftarrow 0$
for i from 1 to n
    $sum \leftarrow sum + i^3$
end for
print *sum*

9

a	b
8	6
2	18
2	6
−22	−18
410	366

10 $n \leftarrow 1$
$x \leftarrow 4$
while $x < 1000$

$n \leftarrow n + 1$

$x \leftarrow 3x + 2$

end while

print n, x

Desk check:

n	x
1	4
2	14
3	44
4	134
5	404
6	1214

11 $sum \leftarrow 0$

$n \leftarrow 0$

while $sum < 1\,000\,000$:

$n \leftarrow n + 1$

$sum \leftarrow sum + n^n$

print (n, sum)

end while

n	sum
1	1
2	5
3	32
4	288
5	3413
6	50 069
7	873 612
8	17 650 828

12 $n \leftarrow 1$

while $2^n \leq 10n^2$

$n \leftarrow n + 1$

print $(n, 2^n, 10n^2)$

end while

$n = 10$

13 $n \leftarrow 0$

$x \leftarrow 3$

$y \leftarrow 1000$

while $x \leq y$

$n \leftarrow n + 1$

$x \leftarrow 2 \times n + 3$

$y \leftarrow 0.9^n \times 1000$

print (n, x, y)

end while

$n = 28$

14 **a** **i** 8

ii 20

iii 16

b It employs the Euclidean algorithm to find the highest common factor of two numbers.

Solutions to Exercise 8D

1 Replacement line of code given.

a $sum \leftarrow sum + i^3$

b $sum \leftarrow sum + 2^i$

c $sum \leftarrow sum + i \times (i + 1)$

2 **a** It anounces the creation of a list.

b The 'filling' of the list statement is:

append 2^i to A

c
```
define power(n)
A ← []
for i from 0 to n
    append 2^(n−i) to A
end for
return A
```

3
```
define min(A)
min ← A[1]
for i from 1 to length(A)
    if A[i] < min then
        min ← A[i]
    end if
end for
return min
```

4 **a**
```
define sum(n)
sum ← 0
for i from 1 to n
    sum ← sum + factorial(i)
end for
return sum
```

b
```
1 ← n
while factorial(n) ≤ 10^n
    n ← n + 1
    print n
end while
```

5 **a**

a	b	c
1	1	1
1	2	2
1	3	3
2	1	4
2	2	5
2	3	6
3	1	7
3	2	8
3	3	9

b

a	b	c
2	3	6
2	4	14
3	3	23
3	4	35

6 **a**

i	$A[i]^2$	$tally$
1	1	1
2	9	10
3	25	35
4	64	99

b It calculates the sum of the squares of the elements of the list.

7 **a**

i	$A[i]$	$A[i+1]$	$A[]$
1	1	1	[1,1,2]
2	1	2	[1,1,2,3]
3	2	3	[1,1,2,3,5]
4	3	5	[1,1,2,3,5,8]
5	5	8	[1,1,2,3,5,8,13]

b
```
A ← [1, 1]
i ← 1
while A[i] ≤ 1000
    append A[i] + A[i + 1]) to A
    i ← i + 1
end while
    print i, A[i]
```

c
```
define Fib(n)
    A ← [1, 1]
    for i from 1 to n − 2
        append A[i] + A[i + 1] to A
return A[n]
```

d
```
A ← [0, 1, 1]
for i from 1 to 7
    append A[i] + A[i + 1] + A[i + 2]
to A
end for
print A
```

8 a $A[5] = 25$

b
```
A ← []
n ← 1
while n³ ≤ 100 000
    append n³ to A
    n ← n + 1
end while
print A
```

c 46

9
```
for x from 1 to 10
    for y from 1 to 6
        for z from 1 to 4
            if 3x + 5y + 7z = 30 then
                print (x, y, z)
            end if
        end for
    end for
end for
```
Solutions: $(1, 4, 1), (2, 2, 2), (6, 1, 1)$

10 a
```
for x from 1 to 5
    for y from 1 to 5
        for z from 1 to 3
            if x² + y² + 10z = 30
then
                print (x, y, z)
            end if
        end for
    end for
end for
```

b $(1, 3, 2), (2, 4, 1), (3, 1, 2), (4, 2, 1)$

c
```
for x from 1 to 7
    for y from 1 to 7
        for z from 1 to 7
            if x² + y² + 10z = 30
and
            x + y + z = 7 then
                print (x, y, z)
            end if
        end for
    end for
end for
```

$(2, 4, 1), (4, 2, 1)$

11 define $f(n)$

$A \leftarrow []$

for x from 1 to n

for y from 1 to n

if $x^2 + y^2 = n$ then

append $[x, y]$ to A

end if

end for

end for

12 It gives the number and proportion of cases when there is no real solutions of the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are integers between -10 and 10 inclusively.

For questions 13 and 14 the approacch given in the answers is different.

13 a define $primecheck(n)$

if $length(factors(n)) = 2$ then

print 'prime'

else

print 'not prime'

b For numbers up to 1000. This bound can be changed.

define $prime(n)$

B=[]

for i from 1 to 1000

if $length(factors(i)) = 2$ then

append i to B

print $B[n]$

14 a define $power(n)$

$i \leftarrow 0$

while $quotient(n, 2) > 0$

$n \leftarrow quotient(n, 2)$

$i \leftarrow i + 1$

end while

return i

b First function to return the highest common factor of two natural numbers:

define $hcf(x, y)$

if $x > y$ then

$smaller \leftarrow y$

else

$smaller \leftarrow x$

end if

$i \leftarrow 1$

while $i \leq (smaller + 1)$

if $rem(x, i) = 0$ and $rem(y, i) = 0$

$h \leftarrow i$

end if

$i \leftarrow i + 1$

end while

return h

Now the function to give the required smallest number:

define $number(n)$:

$ans \leftarrow 1$

for i from 1 to n

$ans \leftarrow (ans \times i)/hcf(ans, i)$

end for

return ans

15 a define $pell(n)$

$A \leftarrow [1, 2]$

for i from 1 to $n - 2$

append $A[i] + A[i + 1])$ to A

end for

return $A[n]$

b $sum \leftarrow 0$

for i from 1 to n

$sum \leftarrow sum + pell(n)$

end for

print(sum)

c $A \leftarrow [1, 2]$

$i \leftarrow 1$

while $A[i] < 10^{999}$

  append $A[i] + 2 \times A[i+1]$ to A

  $i \leftarrow i + 1$

end while

print $A[i]$

16 **a** input a, b

print (a, b)

$i \leftarrow 0$

while $a \neq b$ and $i < 100$

  if $a < b$ then

    $b \leftarrow b - a$

    $a \leftarrow 2 * a$

  else if $b < a$

    $a \leftarrow a - b$

    $b \leftarrow 2 * b$

  end if

  print (a, b)

  $i \leftarrow i + 1$

end while

Note: The condition $i < 100$ is only there to ensure that the program stops.

b **i** It cycles indefinitely : $(21, 28), (42, 7), (25, 14), (21, 28), \ldots$

ii It cycles indefinitely : $(21, 49), (42, 28), (14, 56), (28, 42), (56, 14), (42, 28), \ldots$

iii Goes to $(35, 105), (70, 70)$.

iv Goes to $(19, 133), (38, 114), (76, 76)$

v Goes to $(148, 148)$ after 2 moves.

c For example: $(5, 15), (5, 27), (5, 35), (5, 59, (11, 165)$

17 See answer to 14b

Solutions to technology-free questions

1 **a** 8

b 18

c 93

d 9,75

2 **a**

```
sum ← 0
for n from 1 to 6
    sum ← sum + n^n
end for
print sum
```

b

```
sum ← 0
for n from 1to 6
    sum ← sum + (−1)^(n+1) n(7 − n)
end for
print sum
```

3

n	a	b	c
1	2	4	4
2	4	12	12
3	12	44	44
4	44	200	200
5	200	1088	1088

4 **a** $a_1 = 2, a_2 = 8, a_3 = 26$

b

```
a ← 0
for i from 1 to 50
    a ← 3a + 2
end for
print a
```

c

```
a ← 0
sum ← 0
for n from 1 to 50
    a ← 3a + 2
    sum ← sum + a
end for
print sum
```

5 **a**

```
Input N
for n from 1 to N
    if remainder(n, 2) = 0 then
        T ← 6 − 2n
    else
        T ← 3n + 1
    end if
    print T
end for
```

n	T
1	4
2	2
3	10
4	−2
5	16

b

```
Input N
sum ← 0
for n from 1 to N
    if remainder(n, 2) = 0 then
        sum ← sum + 6 − 2n
    else
        sum ← sum + 3n + 1
    end if
    sum ← sum + T
end for
    print sum
```

6

```
for a from −6 to 6
    for b from−6 to 6
        if 9 ≤ a^2 + b^2 ≤ 36 then
            print (a, b)
```

```
        end if
    end for
end for
```

7 **a**

a	m	b	$f(a)$	$f(m)$	$f(b)$
0	1	2	−2	−1	2
1	1.5	2	−1	0.25	2
1.	1.25	1.5	−1	−0.4375	0.25
1.25	1.375	1.5	−0.4375	-0.109...	0.25
1.375	1.4375	1.5			

b define $f(x) = x^2 - 3$
$a \leftarrow 0$
$b \leftarrow 3$
$m \leftarrow 1.5$
while $b - a \geq 2 \times 0.01$
    if $f(a) \times f(m) < 0$ then
        $b \leftarrow m$
    else
        $a \leftarrow m$
    end if
    $m \leftarrow \dfrac{a + b}{2}$
    print (a, m, b)
end while
print m

Solutions to multiple-choice questions

1 E A desk check give the following sequence of values of a:
$1, 2, 4, 8, 16$

2 D

i	sum
1	1
2	3
3	6
4	10

3 C

It is simply which index gives the same element for both lists.

4 E

i	sum
1	−1
2	1
3	-2
4	2

5 A

i	B
1	[2]
2	[2,5]
3	[2,5,8]

6 C

i	j	sum
1	1	1
1	2	3
2	1	5
2	2	9

7 C

$F(2, 3) = f(3, 2) = 8$

8 E

n	$count$
10	1
5	2
4	3
2	4
1	5

9 E

n	i	$f(n)$
16	1	[1]
16	2	[1, 2]
16	4	[1, 2, 4]
16	8	[1, 2, 4, 8]
16	16	[1, 2, 4, 8, 16]

10 C

$sum = 0 + 1 \times 4$
$sum = 4 + 2 \times 3$
$sum = 10 + 3 \times 2$
$sum = 16 + 4 \times 1 = 20$

Solutions to extended-response questions

1 **a** **i** $[1, 0, 0, 0, 0, 0, 1]$

ii $[1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1]$

iii $[1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0]$

b define $baseb(b, n)$
$A \leftarrow []$
while $n > 0$
    $r \leftarrow remainder(n, b)$
    append r to A
    $n \leftarrow quotient(n, b)$
end while
$A \leftarrow reverse(A)$
return A

i $[1, 0, 1]$

ii $[1, 0, 7, 2, 7]$

iii $[1, 5, 3, 0, 0, 2]$

c **i**

i	B
0	$[10]$
1	$[10, 8]$
2	$[10, 8, 6]$
3	$[10, 8, 6, 4]$
4	$[10, 8, 6, 4, 2]$

ii There would be new entries in the list being used for calculations rather than from the old list.
Output $A = [10, 8, 6, 8, 10]$

2 **a**

i	j	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$
	0	1	9	3	2	7	6
1	1	1	9	3	2	7	6
1	2	1	3	9	2	7	6
1	3	1	3	2	9	7	6
1	4	1	3	2	7	9	6
1	5	1	2	3	7	6	9
2	1	1	3	2	7	6	9
2	2	1	2	3	7	6	9
2	3	1	2	3	7	6	9
2	4	1	2	3	6	7	9
2	5	1	2	3	6	7	9

b You are losing one of the swapped values and replacing with the other.

c The changes: i from 1 to $length(A)$ and j from 1 to $length(A) - i$.

3 **a**

n	a	$reverse$
567	8	8
56	7	87
5	6	876
0	5	8765

b The algorithm reverses the digits of a number by considering remainders and quotients when dividing by 10.

c

for n from 1 to 1000
    $m = n^2$
    if $R(m) = m$ then
        print(m)
    end if
end for

4 **a** **i** 4321

ii 5555

iii 8765

iv 14443

b The sum of each pair of such digits is less than 10.

c

n	$R(n)$	$n + R(n)$
1756	6571	15565
15565	56551	72116
72116	61127	133243
133243	342331	475574

d 82 (tested up to 1000 iterations for each number)

5 **a** 8,15

b 12,35

c $4(m+1)^2 + m^2(m+2)^2 = (m^2 + 2m + 2)^2$

6 For example consider the list $A = [100, 25, 13, 32, 17, 34]$

```
for i from 1 to 6
    minin ← i
    for j from i + 1 to 5
        if A[minin] > A[j] then
            minind ← j
        end if
    end for
    x ← A[i]
    A[i] ← A[minind]
    A[minind] ← x
end for
print(A)
```

7 Here is a basic prime facorisation program to return a list with the prime factors

```
define primefactors(n)
    A = []
    c = 2
    while n > 1
        if (n is divisible by c) then
            Append c to A
            n ← n/c
        else:
            c ← c + 1
        end if
    end while
    return (A)
```

You can now work withhe and devise a count method to give the multiplicity.

Chapter 10 – Revision of chapters 6-9

Solutions to Short answer questions

1 If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n^2 + n &= (2k + 1)^2 + (2k + 1) \\ &= 4k^2 + 4k + 1 + 2k + 1 \\ &= 4k^2 + 6k + 2 \\ &= 2(2k^2 + 3k + 1) \end{aligned}$$

is even.

2 Since m and n are consecutive integers, we know that $n - m = 1$. Therefore,

$$\begin{aligned} n^2 - m^2 &= (n - m)(n + m) \\ &= 1 \times (n + m) \\ &= n + m. \end{aligned}$$

3 **a** Converse: If n is odd, then $5n + 3$ is even.

b If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 5n + 3 &= 5(2k + 1) + 3 \\ &= 10k + 5 + 3 \\ &= 10k + 8 \\ &= 2(5k + 4) \end{aligned}$$

is even.

c Contrapositive: If n is even, then $5n + 3$ is odd.

d If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 5n + 3 &= 5(2k) + 3 \\ &= 10k + 3 \\ &= 10k + 2 + 1 \\ &= 2(5k + 1) + 1 \end{aligned}$$

is odd.

4 **Method 1:** Suppose that $x+1$ were rational. Then there would be $p, q \in \mathbb{Z}$ such that

$$x + 1 = \frac{p}{q}.$$

It follows that,

$$\begin{aligned} x &= \frac{p}{q} - 1 \\ &= \frac{p}{q} - \frac{q}{q} \\ &= \frac{p-q}{q}. \end{aligned}$$

Since $p - q \in \mathbb{Z}$ and $q \in \mathbb{Z}$ this implies that x is rational. This is a contradiction.
Method 2: Suppose that $x+1$ were rational. Then

$$x = \overbrace{(x+1)}^{\text{rational}} - \overbrace{\ \ 1}^{\text{rational}}.$$

Therefore, x is the difference of two rational numbers, which is rational. This is a contradiction.

5 Suppose on the contrary that 6 can be written as the difference of two perfect squares m and n.Then

$$6 = n^2 - m^2$$
$$6 = (n-m)(n+m)$$

The only factors of 6 are $1, 2, 3$ and 6. And since $n + m > n - m$ we need only consider two cases.
Case 1: If $n - m = 2$ and $n + m = 3$ then we add these two equations together to give $2n = 5$. This means that $n = 5/2$, which is not a whole number.
Case 2: If $n - m = 1$ and $n + m = 6$ then we add these two equations together to give $2n = 7$. This means that $n = 7/2$, which is not a whole number.

6 ($\Rightarrow$) Suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 3n + 1 &= 3(2k+1) + 1 \\ &= 6k + 3 + 1 \\ &= 6k + 4 \\ &= 2(3k+2) \end{aligned}$$

is even.
($\Leftarrow$) We will prove the equivalent contrapositive statement.
Contrapositive: If n is even, then $3n + 1$ is odd.

Proof. Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} 3n + 1 &= 3(2k) + 1 \\ &= 6k + 1 \\ &= 2(3k) + 1 \end{aligned}$$

is odd.

7 **a** This is false, for each of 2 and 5 are prime numbers and so too is $2 + 5 = 7$.

b Any number $x \leq 1$ will provide a counter-example. For example, let $x = 1/2$. Then,

$$x^3 = 1/8 < 1/4 = x^2.$$

Alternatively, you could let $x = -1$. Then,

$$x^3 = -1 < 1 = x^2.$$

8 We need to show that the opposite is true. That is, for all $n \in \mathbb{N}$, the number $25n^2 - 9$ is a composite number. To see this, note that

$$25n^2 - 9 = (5n - 3)(5n + 3)$$

And since $5n - 3 \geq 2$ and $5n + 3 > 2$, we have expressed $25n^2 - 9$ as the product of two natural numbers greater than 1.

9 **a** $\boxed{P(n)}$

$2 + 4 + \cdots + 2n = n(n + 1)$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 2$$

and

$$\text{RHS} = 1 \times 2 = 2.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$2 + 4 + \cdots + 2k = k(k + 1). \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$=2+4+\cdots+2k+2(k+1)$

$=k(k+1)+2(k+1) \quad$ (by (1))

$=(k+1)(k+2)$

$=(k+1)((k+1)+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$11^n - 6$ is divisible by 5

$\boxed{P(1)}$

If $n = 1$ then $11^1 - 6 = 5$ is divisible by 5. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$11^n - 6 = 5m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned} 11^{k+1} - 6 &= 11 \times 11^k - 6 \\ &= 11 \times (5m+6) - 6 \quad \text{(by (1))} \\ &= 55m + 66 - 6 \\ &= 55m + 60 \\ &= 5(11m + 12) \end{aligned}$$

is divisible by 5. Therefore $P(k+1)$ is true.

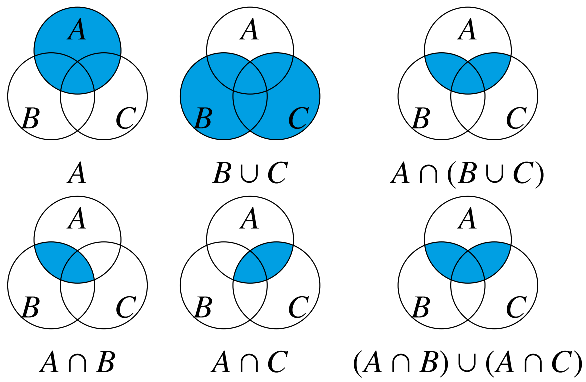
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

10 a We note that

$$\begin{aligned} & x \in A \cap (B \cup C) \\ \Leftrightarrow\ & x \in A \text{ and } x \in B \cup C \\ \Leftrightarrow\ & x \in A \text{ and } (x \in B \text{ or } x \in C) \\ \Leftrightarrow\ & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ \Leftrightarrow\ & (x \in A \cap B) \text{ or } (x \in A \cap C) \\ \Leftrightarrow\ & x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

It is important to note that each of the above steps is reversible. Therefore

$x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$, in which case $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

b 

11 **a** $A \cap \varnothing = \varnothing$

b $A \cup \xi = \xi$

c By reordering terms, we find that

$$\begin{aligned}(A \cup B) \cup A &= (A \cup A) \cup B \\ &= A \cup B.\end{aligned}$$

d For any set C, we know that $C \cup \varnothing = C$. Therefore,

$$(A \cap B) \cup \varnothing = A \cap B.$$

e Since A and its complements A' do not intersect, $A \cap A' = \varnothing$.

f Since every element in ξ is either in A or its compliment, we see that $A \cup A' = \xi$.

g By reordering terms we find that

$$\begin{aligned}(A \cap B) \cap B' &= A \cap (B \cap B') \\ &= A \cap \varnothing \\ &= \varnothing\end{aligned}$$

h By reordering terms we find that

$$\begin{aligned}(A \cup B') \cup B &= A \cup (B' \cup B) \\ &= A \cup \xi \\ &= \xi\end{aligned}$$

i Using distributivity, we find that

$$\begin{aligned} A \cup (B \cap A) &= (A \cup B) \cap (A \cup A) \\ &= (A \cup B) \cap A \\ &= A \end{aligned}$$

where the line line follows from the fact that $A \subset A \cup B$.

j Using distributivity, we find that

$$\begin{aligned} A \cap (A' \cup B) &= (A \cap A') \cup (A \cap B) \\ &= \varnothing \cup (A \cap B) \\ &= A \cap B \end{aligned}$$

k Using De Morgan's Laws and then reordering, we find that

$$\begin{aligned} B \cap (A \cup B)' &= B \cap (A' \cap B') \\ &= (B \cap B') \cap A' \\ &= \varnothing \cap A' \\ &= \varnothing \end{aligned}$$

l Using De Morgan's Laws and then distributing, we find that

$$\begin{aligned} A \cap (A \cap B)' &= A \cap (A' \cup B') \\ &= (A \cap A') \cup (A \cap B') \\ &= \varnothing \cup (A \cap B') \\ &= A \cap B' \end{aligned}$$

12 **a** As 1 is the identity for $\wedge$, we know that $x \wedge 1 = x$.

b As 0 is the identity for $\vee$, we know that $x \vee 0 = x$.

c By the properties of a Boolean algebra, $x' \wedge x = 0$

d By the properties of a Boolean algebra, $x' \wedge x = 1x' \vee x$

e We find that

$$\begin{aligned} (x \vee x') \vee x &= 1 \vee x \\ &= 1. \end{aligned}$$

f By first reordering terms, we find that

$$\begin{aligned}(x \wedge y) \wedge x &= x \wedge (y \wedge x)\\ &= x \wedge (x \wedge y)\\ &= (x \wedge x) \wedge y\\ &= x \wedge y\end{aligned}$$

g By the properties of a Boolean algebra,

$$\begin{aligned}(x \wedge x') \wedge y &= 0 \wedge y\\ &= 0.\end{aligned}$$

h By first reordering terms, we find that

$$\begin{aligned}x \vee (x' \vee y) &= (x \vee x') \vee y\\ &= 1 \vee y\\ &= 1\end{aligned}$$

i By the properties of a Boolean algebra,

$$\begin{aligned}(x \wedge 0) \wedge y &= 0 \wedge y\\ &= 0\end{aligned}$$

j By the properties of a Boolean algebra,

$$\begin{aligned}(x \vee 1) \wedge x' &= 1 \wedge x'\\ &= x'\end{aligned}$$

k By distributing, we find that

$$\begin{aligned}y \wedge (x \vee y') &= (y \wedge x) \vee (y \wedge y')\\ &= (y \wedge x) \vee 0\\ &= y \wedge x.\end{aligned}$$

l By using De Morgan's Laws and then reordering we find that

$$\begin{aligned}x \wedge (x \vee y)' &= x \wedge (x' \wedge y')\\ &= (x \wedge x') \wedge y'\\ &= 0 \wedge y'\\ &= 0\end{aligned}$$

13 Rows 1 and 3 have a 1 in the right most column.

$$\text{Row 1:} \quad x' \wedge y' = 1$$

$$\text{Row 3:} \quad x \wedge y' = 1$$

Piecing these together gives

$$f(x, y) = (x' \wedge y') \vee (x \wedge y').$$

This can be simplified using distributivity,

$$\begin{aligned} f(x, y) &= (x' \wedge y') \vee (x \wedge y') \\ &= (x' \vee x) \wedge y' \\ &= 1 \wedge y' \\ &= y'. \end{aligned}$$

14 **a** $\neg A$

b $A \wedge B$

c $A \Rightarrow \neg B$

d $A \vee (\neg A \Rightarrow B)$

e $(A \wedge B) \vee (\neg A \wedge (\neg B))$

15 **a** **i** $P \wedge Q$

ii $P \Rightarrow Q$

b Yasmin is not a member of the school orchestra if and only if Yasmin does not play violin.

16 **a** We need to show that the two statements have the same truth values.

A	B	$\neg A$	$\neg B$	$\neg A \vee B$	$A \wedge \neg B$	$\neg(A \wedge \neg B)$
T	T	F	F	T	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	F	T

The highlighted columns have the same values. Therefore $\neg(A \wedge B)$ is equivalent to $\neg A \vee \neg B$.

b We need to show that the statement is false in all circumstances.

A	B	$\neg A$	$\neg B$	$A \vee B$	$\neg A \wedge \neg B$	$(A \vee B) \wedge (\neg A \wedge \neg B)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

As the final column is false in all circumstances, the statement is a contradiction.

c We need to show that the statement is true in all circumstances.

A	B	$A \wedge B$	$A \vee B$	$(A \wedge B) \Rightarrow (A \vee B)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

As the final column is true in all circumstances, the statement $(A \wedge B) \Rightarrow (A \vee B)$ is a tautology.

d We need to show that the two statements have the same truth values.

A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$B \wedge C$	$(A \wedge B) \vee (A \wedge C)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

The highlighted columns have the same values. Therefore $A \wedge (B \vee C)$ is equivalent to $(A \wedge B) \vee (A \wedge C)$.

17 **a** I was not paid.

b $Q \Rightarrow P$

c We need to show that the statement is true in all circumstances.

P	Q	$\neg Q$	$P \vee \neg Q$	$(P \vee \neg Q) \wedge Q$	$(P \vee \neg Q) \wedge Q \Rightarrow P$
T	T	F	T	T	T
T	F	T	T	F	T
F	T	F	F	F	T
F	F	T	T	F	T

As the final column is true in all circumstances, the statement is a tautology.

18 We need to show that the two statements have the same truth values.

P	Q	$\neg P$	$\neg Q$	$P \vee \neg Q$	$\neg(P \vee \neg Q)$	$\neg P \wedge \neg Q$	$\neg(P \vee \neg Q) \vee (\neg P \wedge \neg Q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

The highlighted columns have the same values, so these two statements are logically equivalent.

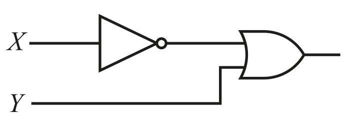
19 **a** The Boolean expression is $\neg X \vee (X \wedge Y)$.

b Using the distributive law we find that

$$\begin{aligned}\neg X \vee (X \wedge Y) &= (\neg X \vee X) \wedge (\neg X \vee Y)\\ &= 1 \wedge (\neg X \vee Y)\\ &= \neg X \vee Y.\end{aligned}$$

Note that 1 is the identity.

c We have to first negate X and then feed this, along with Y, into an OR gate.



20 **a** To show that this is a valid argument, we need to show that if each of the premises is true, then the conclusion is also true. The truth table is shown below.

A	B	C	$A \wedge B$	$A \wedge B \Rightarrow C$	$\neg B$	$\neg C$
T	T	T	T	T	F	F
T	T	F	T	F	F	T
T	F	T	F	T	T	F
T	F	F	F	T	T	T
F	T	T	F	T	F	F
F	T	F	F	T	F	T
F	F	T	F	T	T	F
F	F	F	F	T	T	T

Consider the highlighted row shown above. In this row, each of the premises is true, however the conclusion is false. Therefore, this **is not** a valid argument.

b To show that this is a valid argument, we need to show that if each of the premises is true, then the conclusion is also true. The truth table is shown below.

A	B	C	$A \vee B$	$A \Rightarrow C$	$B \Rightarrow C$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	F	T	T
F	F	F	F	T	T

In the table above, there are three rows for which every premise is true. In these same rows, the conclusion is also true. Therefore this **is** a valid argument.

21 Let A be the statement "I am Sam's father". Let B be the statement "Sam is Will's brother". We need to determine if the following argument is valid:

$$\begin{array}{rl} \text{Premise 1} & A \Rightarrow B \\ \text{Premise 2} & B \\ \hline \text{Conclusion} & \neg A \end{array}$$

To show that this is a valid argument, we need to show that if each of the premises is true, then the conclusion is also true. The truth table is shown below.

A	B	$A \Rightarrow B$	$\neg A$
T	T	T	F
T	F	F	F
F	T	T	T
F	F	T	T

In the table above, there are two rows for which every premise is true. In the top row, we see that the conclusion is not true. Therefore this **is not** a valid argument.

22 **a** Initially A is empty. The value of i ranges from 1 to 16. Whenever i is not the sum of two elements in A, we place it in A. The deck check that keeps track of the value of i and A at each step is given below.

i	A
	[]
1	[1]
2	$[1, 2]$
3	$[1, 2]$
4	$[1, 2, 4]$
5	$[1, 2, 4]$
6	$[1, 2, 4]$
7	$[1, 2, 4, 7]$
8	$[1, 2, 4, 7]$
9	$[1, 2, 4, 7]$
10	$[1, 2, 4, 7, 10]$
11	$[1, 2, 4, 7, 10]$
12	$[1, 2, 4, 7, 10]$
13	$[1, 2, 4, 7, 10, 13]$
14	$[1, 2, 4, 7, 10, 13]$
15	$[1, 2, 4, 7, 10, 13]$
16	$[1, 2, 4, 7, 10, 13, 16]$

Therefore the final list is $A = [1, 2, 4, 7, 10, 13, 16]$.

b Initially A is empty. The value of i ranges from 1 to 16. Whenever i is not the sum of two or more elements in A, we place it in A. The deck check that keeps track of the value of i and A at each step is given below.

i	A
	[]
1	[1]
2	$[1,2]$
3	$[1,2]$
4	$[1,2,4]$
5	$[1,2,4]$
6	$[1,2,4]$
7	$[1,2,4]$
8	$[1,2,4,8]$
9	$[1,2,4,8]$
10	$[1,2,4,8]$
11	$[1,2,4,8]$
12	$[1,2,4,8]$
13	$[1,2,4,8]$
14	$[1,2,4,8]$
15	$[1,2,4,8]$
16	$[1,2,4,8,16]$

Therefore the final list is $A = [1,2,4,8,16]$

c The elements in A are simply the powers of 2. (Note: every number can expressed as the sum of powers of 2 less than that number)

23 **a** This function given the sum of the squares from 1 to n.

b If $n = 4$, then

$$function(4) = 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

c From the previous question, $function(4) = 30$. Since $5^2 = 25$, we see that

$$\begin{aligned} function(5) &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 55. \end{aligned}$$

d Now the initial value has to be $prod = 1$. The code can be rewritten as follows.

define $function(n)$
$prod \leftarrow 1$
for i **from** 1 **to** n
    $prod \leftarrow prod \times i^3$
end for
return $prod$

24 a The desk check is shown below.

n	120	60	30	15
total	0	1	2	3

b The function repeatedly divides the integer by 2 until it has no further factors of 2. It returns a count of how many times it does this. Therefore, the function return the highest power of 2 that is a factor of n.

c The numbers that return an output of 8 are of the form $8n$, where n is not divisible by 2.

25 Four different objects can be arranged in $4! = 24$ different ways.

26 A teacher must occupy the first position. There are 3 choices for this position. There are five more people to be arranged in 5! ways. Therefore, using the multiplication principle there are a total of

$$3 \times 5! = 360$$

arrangements.

27 a There are five digits to choose from, and each can be used as many times as you like. Therefore, using the multiplication principle, there are

$$5 \times 5 \times 5 = 125$$

possibilities.

b There are 5 choices for the first digit, 4 for the second and 3 for the third. Therefore, using the multiplication principle, there are

$$5 \times 4 \times 3 = 60$$

possibilities.

28 a $1 + 2 \times 4 = 9$.

b $1 + (1 \times 3 \times 2 \times 2) + (3 \times 2 \times 2 \times 1)$

$=1 + 12 + 12$

$=25$

29 a $4! = 4 \times 3 \times 2 \times 1 = 24$

b $\frac{6!}{4!} = \frac{6 \times 5 \times 4!}{4!}$

$= 6 \times 5$

$= 30$

c $\frac{8!}{6!2!} = \frac{8 \times 7 \times 6!}{6! \times 2!}$

$= \frac{8 \times 7}{2}$

$= 28$

d ${}^{10}C_2 = \frac{10!}{8!2!}$

$= \frac{10 \times 9 \times 8!}{8! \times 2!}$

$= \frac{10 \times 9}{2}$

$= 45$

30 **a** There are fives choices for the first position, four for the second, three for the third and two for the fourth. This gives a total of

$$5 \times 4 \times 3 \times 2 = 120$$

arrangements.

b Five children can be arranged in five spaces in $5! = 120$ ways.

31 **a** There are a total of 5 items and these can be arranged in $5! = 120$ different ways.

b We group the three mathematics books together so that we now have just three items: $\{M_1, M_2, M_3\}, P_1, P_2$. These three items can be arranged in $3! = 6$ different ways. However, the three mathematics books can be arranged within the group in $3! = 6$ different ways. This gives a total of $6 \times 6 = 36$ different arrangements.

32 **a** Although the question states that there are no restrictions, we still can't have the zero in the first position or else the number wouldn't be a five-digit number. Therefore there are on 4 possibilities for the first digit. The remaining four digits can be arranged in $4!$ ways. This gives a total of $4 \times 4! = 96$ numbers.

b If the number is divisible by 10 then the last digit must be zero. The remaining four digits can be arranged without restriction in $4! = 24$ different ways.

c If the number is greater than 20000 then the first digit can one of three options: 2, 3 or 4. The remaining four digits can be arranged in 4! ways. This gives a total of $3 \times 4! = 72$ different numbers.

d Obviously the last digit must be either 0, 2 or 4. We need to consider two cases. **Case 1:** If the last digit is 0 then the remaining four digits can be arranged without further restriction in $4! = 24$ ways. **Case 2:** If the last digit is 2 or 4 then there are two possibilities for the final digit. As the first digit cannot be 0 there remains just 3 possibilities. The remaining three digits can be arranged in $3! = 6$ different ways. This gives a total of $3 \times 3! \times 2 = 36$ numbers.
Using the addition principle, there are a total of $24 + 36 = 60$ different numbers.

33 There are five items in total, of which a group of three are alike and a group of two are alike. These can be arranged in

$$\frac{5!}{2! \times 3!} = 10$$

different ways.

34 **a** Three children from six can be selected in ${}^6C_3$ ways. This gives,

$$\begin{aligned} {}^6C_3 &= \frac{6!}{3!3!} \\ &= \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} \\ &= \frac{6 \times 5 \times 4}{6} \\ &= 20. \end{aligned}$$

b Two letters from twenty-six can be selected in ${}^{26}C_2$ ways. This gives,

$$\begin{aligned} {}^{26}C_2 &= \frac{26!}{2!24!} \\ &= \frac{26 \times 25 \times 24!}{2! \times 24!} \\ &= \frac{26 \times 25}{2} \\ &= 325. \end{aligned}$$

c Four numbers out of ten can be selected in ${}^{10}C_4$ ways. This gives,

$$\begin{aligned}
{}^{10}C_4 &= \frac{10!}{4!6!} \\
&= \frac{10 \times 9 \times 8 \times 7 \times 6!}{4! \times 6!} \\
&= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \\
&= 210.
\end{aligned}$$

d Three sides out of eight can be selected in ${}^{8}C_3$ ways. This gives,

$$\begin{aligned}
{}^{8}C_3 &= \frac{8!}{3!5!} \\
&= \frac{8 \times 7 \times 6 \times 5!}{3! \times 5!} \\
&= \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \\
&= 56.
\end{aligned}$$

35 **a** Two elements from eight can be chosen in ${}^{8}C_2$ ways. This gives,

$$\begin{aligned}
{}^{8}C_2 &= \frac{8!}{2!6!} \\
&= \frac{8 \times 7 \times 6!}{2! \times 6!} \\
&= \frac{8 \times 7}{2 \times 1} \\
&= 28.
\end{aligned}$$

b Each set must contain the number 8. We are still to choose two more numbers from the set $\{1, 2, \ldots, 7\}$. These can be chosen in ${}^{7}C_2$ ways. This gives,

$$\begin{aligned}
{}^{7}C_2 &= \frac{7!}{2!5!} \\
&= \frac{7 \times 6 \times 5!}{2! \times 5!} \\
&= \frac{7 \times 6}{2 \times 1} \\
&= 21.
\end{aligned}$$

c A set with eight elements will have $2^8 = 256$ subsets (including the empty set, and the entire set).

36 Three boys can be selected from five in ${}^{5}C_3$ ways. Two girls can be selected from four in ${}^4C_2$ ways. Using the multiplication principle, we can make both selections in

$${}^5C_3 \times {}^4C_2 = 60$$

ways.

37 There are three cases to consider.

Labor	Liberal	Selections
1	3	${}^4C_1 \times {}^5C_3$
2	2	${}^4C_2 \times {}^5C_2$
3	1	${}^4C_3 \times {}^5C_1$

This gives a total of 120 selections.

38 Label three holes with the colours blue, green and red.

$$\boxed{\text{B}}\ \boxed{\text{G}}\ \boxed{\text{R}}$$

Clearly, selecting six balls is not sufficient as you might pick two balls of each colour. Now select seven balls and place each sock in the hole whose label corresponds to the colour of the sock. As there are seven balls and three holes, the Pigeonhole Principle guarantees that some hole contains at least three balls. Therefore the answer is seven.

39 Label fifty boxes with the numbers

$$\boxed{\text{1 or 99}}\ \boxed{\text{2 or 98}}\ \cdots \boxed{\text{49 or 51}}\ \boxed{\text{50}}$$

Selecting 50 different numbers is not sufficient as you might pick one number belonging to each box. Now select 51 numbers, and place each one in its corresponding hole. As there are 51 numbers and 50 holes, some hole contains 2 numbers. The two numbers in this hole are different, and so their sum is 100.

40 Let A and B be the sets comprising of multiples 2 and 3 respectively. Clearly $A \cap B$ consists of the multiples of 2 and 3, that is, multiples of 6. Therefore, $|A| = 60, |B| = 40$ and $|A \cap B| = 20$. We then use the Inclusion Exclusion Principle to find that,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= 60 + 40 - 20 \\ &= 80.\end{aligned}$$

Solutions to multiple-choice questions

1 **E** All are true except the last option. For if m and n are odd then so is mn. Therefore $mn + 1$ will be even.

2 **E**

- Item A is true. Since n is divisible by 12, it will be divisible by 3. Therefore, $m \times n$ will be divisible by 3.
- Item B is true. Since $m = 4j$ and $n = 12k$ we know that $m \times n = 48jk$, which is divisible by 48.
- Item C is true. Since $m = 4j$ and $n = 12k$ we know that $m + n = 4j + 12k = 4(j + 3k)$, which is divisible by 4.
- Item D is true. Since $m \times n$ is divisible by 48, it follows that m^2n will also be divisible by 48.
- Item E may be false. For example, $n = 12$ is divisible by 12 and $m = 16$ is divisible by 4 and yet n is not divisibly by m.

3 **D**

- Item A is false. If $m = 3$ and $n = 2$ then $mn = 6$ is even, though m is not even.
- Item B is false. If $m = 1$ and $n = 3$ then $m + n = 4$ is even, even though neither m nor n is even.
- Item C is false. If $m = 1$ and $n = 2$ then $m + n = 3$ is odd, while $mn = 2$ is even.
- Item D is true. If mn is odd then both m and n are odd. Therefore $m + n$ is even.
- Item E is false. Note that $m + n$ and $m - n$ will both be odd, or both be even.

4 **C** To form the converse, we switch the hypothesis (If n is even) and the conclusion (then $n + 3$ is odd). This gives "If $n + 3$ is odd, then n is even".

5 **E**

- Item A is true. If $a > b$, then $a - b > 0$. Therefore $\dfrac{1}{a-b} > 0$.
- Item B is true. If $a > b$ then $\dfrac{a}{b} - \dfrac{b}{a} = \dfrac{a^2 - b^2}{ab} > 0$.
- Item C is true. If $a > b$ then $a + b > b + b = 2b$.
- Item D is true. If $a > b$ then $a + 3 > b + 2$.
- Item E may be false. For example, if $a = 3$ and $b = 2$ then $a > b$ while $2a = 6 = 2b$.

6 **E** Since,

$$mn - n = 12$$

$$n(m - 1) = 12$$

Clearly $n = \pm1, \pm2, \pm3, \pm4, \pm6, \pm12$. And for each of these twelve values of n we can easily find a corresponding value of m.

7 B Since $9n^2 - 4 = (3n - 2)(3n + 2)$, the number will always be composite, unless $3n - 2 = 1$. This implies that $n = 1$, in which case $9n^2 - 4 = 5$. So there is only one such value of n.

8 B

- Item A is true. For any four consecutive numbers, two will be even, two will be odd. So the sum will be even.
- Item B may be false. For example, $1 + 2 + 3 + 4 = 10$ is not divisible by 4.
- Item C is true. For any four consecutive numbers, one will be divisible by 3.
- Item D is true. For any four consecutive numbers, one will be divisible by 4, and another will be divisible by 2. Therefore the product will be divisible by 8.
- Item E is true. For any four consecutive numbers, one will be divisible by 4, and another will be divisible by 2. Furthermore, one number will be divisible by 3. Therefore the product will be divisible by 24.

9 A To form the contrapositive, we negate the assumption and the conclusion and then interchange them. Therefore, the contrapositive is: If you lose the game, then you don't know the rules or you are overconfident. Note that the negation of "you know the rules **and** you are not overconfident" is "you do not know the rules **or** you are overconfident".

10 B By distributing, we find that

$$\begin{aligned} A \cap (A' \cup B) &= (A \cap A') \cup (A \cap B) \\ &= \varnothing \cup (A \cap B) \\ &= A \cap B. \end{aligned}$$

11 E We can expand using De Morgan's Laws to give

$$\begin{aligned} ((x \wedge y)' \vee z)' &= (x \wedge y)'' \wedge z' \\ &= (x \wedge y) \wedge z' \\ &= x \wedge y \wedge z'. \end{aligned}$$

Note that the brackets in the final line can be omitted because of associativity.

12 C There is only one row with a 1 in the rightmost column. This corresponds to $x \wedge y' = 1$. Therefore $f(x, y) = x \wedge y'$.

13 D The contrapositive will be equivalent. To form the contrapositive, we negate the assumption and the conclusion and then interchange them. Therefore, the contrapositive is: "If it is an elephant, then it does not fly".

14 A For such questions, it is easiest to work from right to left to build the expression. The top path corresponds to $x \wedge (x' \vee y)$. The bottom path is just z. These are combined as $(x \wedge (x' \vee y)) \vee z$.

15 B We first use De Morgan's Laws and then distribute to give:

$$\begin{aligned} x \vee (x \vee y)' &= x \vee (x' \wedge y') \\ &= (x \vee x') \wedge (x \vee y') \\ &= 1 \wedge (x \vee y') \\ &= x \vee y'. \end{aligned}$$

16 B

- $P \Rightarrow Q$ is not a contradiction is it is not false for all truth values of P and Q.
- $\neg P \wedge P$ is a contradiction as it is false for all truth values of P.
- $\neg P \vee P$ is not contradiction as it is true for all truth values of P. It is actually a tautology.
- $P \vee Q$ is not a contradiction is it is not false for all truth values of P and Q.
- $P \wedge Q$ is not a contradiction is it is not false for all truth values of P and Q.

17 A Translated into works, we could say A or B, and not A, implies B. Only the first sentence has this form.

18 E We let A be the statement "I am 18". Let B be the statement *I am eligible to vote*. Therefore the proposition can be written in symbolic form as

$$[(A \Rightarrow B) \wedge A] \Rightarrow B.$$

19 E

- $A \vee \neg A$ is a tautology as it is true for all truth values of A.
- $A \vee \neg A$ is a tautology as it is true for all truth values of A.
- $A \vee B \Leftrightarrow B \vee A$ is a tautology as the statement $A \vee B$ is equivalent to $B \vee A$.
- $A \wedge B \Rightarrow B$ is a tautology as it is true for all truth values of A and B.
- $A \vee B \Rightarrow A \wedge B$ is not a tautology. This statement is not true when A is true and B is false.

20 B The input for the OR gate is X and Y then there is a not gate. This gives $\neg(X \vee Y)$. This along with Z are inputs for the AND gate. This gives $\neg(X \vee Y) \wedge Z$.

21 A

22 E This algorithm will print the product of all of the number of the form $2n - 1$ where n ranges from 1 to 6. These are the odd numbers: $1, 3, 5, 7, 9$ and 11.

23 E The function added 1 to the sum for every time 12 is divisible by i, where i varies from 1 to 12. Therefore this function counts the number of divisors of 12. Since the divisors of 12 are $1, 2, 3, 4, 6$ and 12, there are exactly 6 such divisors.

24 D As x ranges from 1 to 2 and y ranges from 1 to 3, this algorithm will print the sum of all of the values given in this table.

+	1	2
1	2	3
2	3	4
3	4	5

This is 2 + 3 + 3 + 4 + 4 + 5 = 21.

25 A Five people can be arranged in a line in 5! ways.

26 C There are two vowels $\{O, A\}$ and four consonants $\{H, B, R, T\}$. If the arrangement begins with a vowel then there are two choices for the first letter. The remaining five letters can be arranged in 5! ways. Using the multiplication principle, there are $2 \times 5! = 240$ arrangements in total.

27 C There are five choices for the first digit, four the second, three for the third and two for the fourth. This gives a total of $5 \times 4 \times 3 \times 2$ different numbers.

28 A There are six digits in total, of which a group of 3 are alike and a further group of 3 are alike. Therefore, they can be arranged in

$$\frac{6!}{3! \times 3!}$$

different ways.

29 B Sam has $2n$ coins in total, of which a group of n are alike and a further group of n are alike. Therefore, they can be arranged in

$$\frac{(2n)!}{n! \times n!} = \frac{(2n)!}{(n!)^2}$$

different ways.

30 D Mark is still to select two more flavours out of the nine remaining options. This can be done in ${}^9C_2$ different ways.

31 D One must choose two out of four Labour members and two out of five Liberal members. Using the multiplication principle, this can be done in

$${}^4C_2 \times^5 C_2$$

different ways

32 D A set with ten elements (friends!) has 2^10 subsets (of friends). This includes the empty set. However, because we are inviting at least one friend, the empty set must be excluded. This leaves $2^10 - 1$ subsets.

33 A Create 3 holes for each of the different utensils, (K,F,S). Clearly selecting 9 items and placing them in their corresponding hole may not be sufficient, as you could get 3 of each type. However, if 10 are selected then, since $10 = 3 \times 3 + 1$, by the generalised pigeonhole principle there must be some hole with at least 4 utensils. Therefore the smallest number of items is 10.

34 E There are three possible remainders when a number is divided by 3. Label three holes with each of these remainders:

$$\boxed{0}\,\boxed{1}\,\boxed{2}$$

If 15 integers are written on the board, then placed in their corresponding box, then this may not be sufficient - you could get 5 of each remainder. However, if 16 integers are written on the board then, since $16 = 5 \times 3 + 1$, by the generalised pigeonhole principle there must be some hole with at least 6 integers. Therefore the smallest number of integers is 16.

35 B Let A and B be the sets comprising of multiples of 2 and 5 respectively. Clearly $A \cap B$ consists of the multiples of 2 and 5, that is, multiples of 10. Therefore, $|A| = 30$, $|B| = 12$ and $|A \cap B| = 6$. We then use the Inclusion Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 30 + 12 - 6 \\ &= 36. \end{aligned}$$

Solutions to extended-response questions

1 a If $a + b$ is even then either a and b are both odd or a and b are both even. If $b + c$ is odd then either b is odd and c is even or b is even and c is odd. Therefore, one of these two statement must be true:
Statement 1: a is odd and b is odd and c is even.
Statement 2: a is even and b is even and c is odd.
Therefore, we can't determine whether a, b or c are even or odd. For instance, the numbers $a = b = 1$ and $c = 2$ satisfy the given conditions, as do the numbers $a = b = 2$ and $c = 1$.

b If we additionally know that $a + b + c$ is even then the second statement above cannot be true, as $a + b + c$ would be odd. Therefore, the first statement must be true. Therefore, a is odd and b is odd and c is even.

2 a We first show that $a = 2$. If $a = 1$ then the left hand side is too large. If $a \geq 2$ then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Therefore, $c = 2$ and

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{2}.$$

We now show that $b = 3$. If $b \geq 4$ then

$$\frac{1}{b} + \frac{1}{c} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore, $b = 3$ and

$$\frac{1}{c} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Therefore, $c = 6$. We have obtained just one set of values:

$$(a, b, c) = (2, 3, 6).$$

b We first show that $a = 1$. If $a \geq 2$ then the left hand side is too small since,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} < \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$$

Therefore, $a = 1$ so that we now require that

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 1.$$

We now show that $b = 2$. If $b \geq 3$ then

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Therefore, $b = 2$ so that we now require that

$$\frac{1}{c} + \frac{1}{d} > \frac{1}{2}.$$

We now show that $c = 3$. If $c \geq 4$ then

$$\frac{1}{c} + \frac{1}{d} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore, $c = 3$ so that we now require that

$$\frac{1}{d} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Therefore, either $d = 4$ or $d = 5$. We have obtained just two sets of values:

$$(a, b, c, d) = (1, 2, 3, 4) \text{ or } (a, b, c, d) = (1, 2, 3, 5).$$

3 Suppose that $b > a$. Then

$$\begin{aligned}
\frac{a+c}{b+c} - \frac{a}{b} &= \frac{b(a+c)}{b(b+c} - \frac{a(b+c)}{b(b+c)} \\
&= \frac{b(a+c) - a(b+c)}{b(b+c)} \\
&= \frac{ab + bc - ab - ac}{b(b+c)} \\
&= \frac{bc - ac}{b(b+c)} \\
&= \frac{c(b-a)}{b(b+c)} \\
&> 0
\end{aligned}$$

Note that the last line follows from the fact that each term in the fraction is positive. Therefore,

$$\frac{a+c}{b+c} > \frac{a}{b},$$

as required.

4 **a** Since $2^9 = 512 < 10^3$ and $2^{10} = 1024 > 10^3$, the smallest such n will be 10.

b Since,

$$\begin{aligned}
2^{100} &= (2^{10})^{10} \\
&> (10^3)^{10} \\
&= 10^{30},
\end{aligned}$$

we know that 2^{100} must have at least 31 digits.

c As there are at least 31 digits, and 10 different digits, there must be some digit that occurs at least 4 times.

5 **a** Since the newspaper has 100 pages and each sheet includes 4 pages, the stack must contain $100 \div 4 = 25$ sheets. The 25th sheet includes pages $49, 50, 51$ and 52.

b The least two numbers have increased by 6 from 1 and 2 to 7 and 8. The last two pages will decrease by 6 from 99 and 100 to 93 and 94.

c Suppose the newspaper is make up of n sheets of paper. Then the kth sheet of paper will include pages $2k-1, 2k, 4n-2k+1, 4n-2k+2$. The sum of these numbers is

$$2k-1+2k+4n-2k+1+4n-2k+2 = 8n+2.$$

Therefore, the sum of page numbers on each sheet depends only on the total number of sheets.

d From the previous question, we see that

$$\begin{aligned} 8n+2 &= 11+12+33+34 \\ 8n+2 &= 90 \\ 8n &= 88 \\ n &= 11. \end{aligned}$$

There are 11 sheets of paper. Therefore, there are $11 \times 4 = 44$ pages.

6 a The smallest number of coins that Sam would need to do this is

$$0+1+2+3+4+5+6 = 21.$$

Sam has only 20 coins, so this is impossible.

b The calculation above shows that Sam would need 21 coins.

c Since

$$0+1+2+3+4+5+6+7+8+9 = 45,$$

Sam could fill 10 pockets with $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9 coins. His last five coins could go in the pocket containing 9 coins. Each pocket would then have a different number of coins. We now show that it is impossible for him to fill more than 10 pockets with a different number of coins in each. Arrange these numbers from smallest to largest. The smallest eleven numbers are no less than $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ respectively, and the sum of these numbers is

$$0+1+2+3+4+5+6+7+8+9+10 = 55 > 50.$$

7 a If the first digit is n and the second digit is 5 then the last two digits of its square will be 25 and the first two digits will be $n \times (n+1)$.

b Since $7 \times 8 = 56$, from the observed pattern we expect that $75^2 = 5625$. You can easily check that this is true.

c Each number is of the form $10n + 5$. We square this number to obtain

$$\begin{aligned}(10n+5)^2 &= 100n^2 + 100n + 25\\ &= 100n(n+1) + 25\end{aligned}$$

This shows that the first two digits will be $n(n + 1)$ and the last two digits will be 25.

8 **a** Note that

$$1 + 2 + \cdots + 10 = \frac{10 \times 11}{2} = 55.$$

If the blocks could somehow be used to build two towers of the same height, then each would be $55 \div 2 = 27.5$ cm tall. This is impossible, as each block has an integer side length.

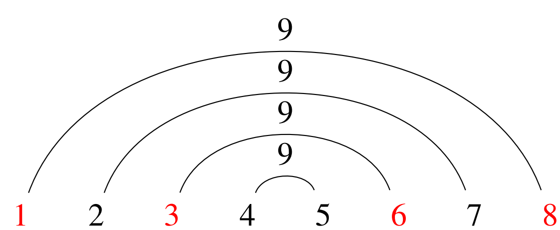
b If $n = 4k + 1$ or $n = 4k + 2$ then you cannot build two towers of the same height. First suppose $n = 4k + 1$. Then note that

$$\begin{aligned}1 + 2 + \cdots + (4k+1) &= \frac{(4k+1)(4k+2)}{2}\\ &= (4k+1)(2k+1).\end{aligned}$$

Since this is odd, it is not divisible by 2. Similarly, if $n = 4k + 2$ then,

$$\begin{aligned}1 + 2 + \cdots + (4k+2) &= \frac{(4k+2)(4k+3)}{2}\\ &= (2k+1)(4k+3).\end{aligned}$$

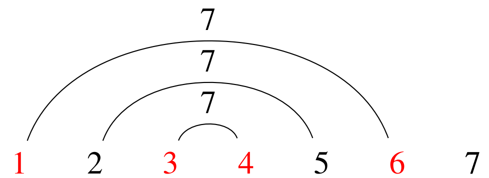
Since this is odd, it is not divisible by 2.
We will prove $n = 4k$ or $n = 4k - 1$ then we can build two towers of the same height.
When $n = 4k$, this is easy. We indicate how this can be done with an example that is easily generalised. Let $n = 8$. By pairing 1 with 8, then 2 with 7, etc., we see that each pair has the same sum.



If follows that we can make two towers whose heights are the same. For example,

$$1 + 3 + 6 + 8 = 2 + 4 + 5 + 7.$$

When $n = 4k - 1$, we do something similar. We indicate how this can be done with a an example that is easily generalised. Let $n = 7$. By pairing 1 with 6, 2 with 5 etc., we see that each pair has the same sum, 7. Notice that 7 does not belong to a pair.



Once again, using this diagram we can make two towers whose heights are the same. For example,

$$1 + 3 + 4 + 6 = 2 + 5 + 7.$$

9 **a** Suppose that a is odd and b is odd. Then $a = 2k + 1$ and $b = 2m + 1$ where $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4k^2 + 2k + 2m + 1 \\ &= 2(2k^2 + k + m) + 1 \\ &= 2n + 1, \text{ where } n = 2k^2 + k + m \in \mathbb{Z}. \end{aligned}$$

We see that ab is odd.

b $\boxed{P(n)}$

If $n \in \mathbb{N}$ and a is odd then a^n is odd

$\boxed{P(1)}$

If $n = 1$ then $a^1 = a$ is odd, by assumption. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that a^k is odd.

$\boxed{P(k+1)}$

Since

$$a^{k+1} = a^k \times a$$

is the product of two odd numbers, a^{k+1} will be odd. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c Assume, on the contrary, that $3^{\frac{n}{m}} = 2$ where $n, m \in \mathbb{N}$. Then raising both sides to the power of m gives,

$$\begin{aligned} \left(3^{\frac{n}{m}}\right)^m &= 2^m \\ 3^n &= 2^m \end{aligned}$$

We have proved that the left hand side is odd. However, the right hand side is even. This is a contradiction.

10 **a** Expanding the left hand side gives

$$n^4 + 6n^3 + 11n^2 + 6n + 1 = a^2n^4 + 2abn^3 + (2ac + b^2)n^2 + 2bcn + c^2.$$

We then equate coefficients. Since a is positive and $a^2 = 1$, clearly $a = 1$. Likewise $c = 1$. Finally, as $2bc = 6$, we know that $b = 3$.

b Let the four consecutive numbers be $n, n + 1, n + 2$ and $n + 4$. Then when 1 is added to their product, we obtain

$$n(n + 1)(n + 2)(n + 3) + 1.$$

If we expand this expression, we obtain

$$n^4 + 6n^3 + 11n^2 + 6n + 1.$$

From the previous question, we know that this is equal to

$$n^4 + 6n^3 + 11n^2 + 6n + 1 = (n^2 + 3n + 1)^2.$$

c Let $n = 5$ in the previous question, so that

$$\begin{aligned} 5 \times 6 \times 7 \times 8 + 1 &= (5^2 + 3 \times 5 + 1)^2 \\ &= 41^2 \end{aligned}$$

11 **a** At each step we subtract if *tally* remains non-negative, otherwise we add the term. This gives:

n		1	2	3	4	5	6	7	8	9	10
tally	0	1	3	0	4	9	3	10	2	11	1

b The required pseudocode is given as:

```
tally ← 0
for i from 1 to 10
    if tally − i > 0
        tally ← tally − i
    else
        tally ← tally + i
    end if
end for
print tally
```

c One possibility is to consider the reordered list:

$$1, 4, 2, 3, 5, 8, 6, 7, 8, 9, 10$$

This works, because after every subgroup of four numbers, the tally will equal to zero. That is,

$$(1 + 4 - 2 - 3) + (5 + 8 - 6 - 7) + 9 + 10 = 10$$

d The previous question suggestions how we can achieve a net tally of 0. Note that there are n subgroups of 4 numbers:

$$(1,2,3,4),(5,6,7,8),\ldots,(4n-3,4n-2,4n-1,4n)$$

Each subgroup of four numbers $k, k+1, k+2, k+3$ is rearranged as $k, k+3, k+1, k+2$. Each subgroup then adds to zero since

$$n+(n+3)-(n+1)-(n+2)=0.$$

12 **a** The truth table can be found as follows.

A	B	$A \vee B$	$\neg(A \vee B)$
1	1	1	0
1	0	1	0
0	1	1	0
0	0	0	1

b The truth table can be found as follows.

A	B	$A \wedge B$	$\neg(A \wedge B)$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	1

c Show an implementation of each of the following using only nand gates.

i We input A twice into the same nand gate as shown below. The truth table table is shown below. Note that $\neg(A \wedge A)$ is equivalent to $\neg A$, as can be seen in the highlighted columns.

A	$A \wedge A$	$\neg(A \wedge A)$	$\neg A$
1	1	0	0
0	0	1	1

ii We input A and B into a nand gate to produce $\neg(A \wedge B)$. From the previous question, we know how to construct a not gate. Therefore we can input $\neg(A \wedge B)$ twice into a second nand gate to produce

$$\neg(\neg(A \wedge B)) = A \wedge B.$$

iii We first input A twice into a nand gate to produce $\neg A$. We do the same for B, to yield $\neg B$. We then input these into a nand gate. We can show that this gives $A \vee B$ by using De Morgan's Laws:

$$\begin{aligned}\neg((\neg A) \wedge (\neg B)) &= (\neg(\neg A)) \vee (\neg(\neg B))\\ &= A \vee B\end{aligned}$$

d Show an implementation of each of the following using only nor gates.

i Similarly to the previous question, we input A twice into the same nor gate as shown below. The truth table table is shown below. Note that $\neg(A \vee A)$ is equivalent to $\neg A$, as can be seen in the highlighted columns.

A	$A \vee A$	$\neg(A \vee A)$	$\neg A$
T	T	F	F
F	F	T	T

ii We first input A and B into a nor gate to produce $\neg(A \vee B)$. From the previous question, we know how to construct a not gate. Therefore we can input $\neg(A \vee B)$ twice into a second nnor gate to produce

$$\neg(\neg(A \vee B)) = A \vee B.$$

iii We first input A twice into a nor gate to produce $\neg A$. We do the same for B, to yield $\neg B$. We then input these into a nor gate. We can show that this gives $A \wedge B$ by using De Morgan's Laws:

$$\begin{aligned} \neg((\neg A) \vee (\neg B)) &= (\neg(\neg A)) \wedge (\neg(\neg B)) \\ &= A \wedge B. \end{aligned}$$

13 **a** Even though there are no restrictions, the first digit cannot be 0. Therefore, there are six choices for the first digit. There are then six choices for the second digit (including 0), five for the third and so on. Using the multiplication principle, there are

$$6 \times 6 \times 5 \times 4 \times 3 = 2160$$

different numbers.

b If the number is divisible by 10 then the last digit must be 0. Therefore, there is only one choice for this digit. There are then six choices for the first digit, five for the second and so on. Using the multiplication principle, there are

$$6 \times 5 \times 4 \times 3 \times 1 = 360$$

different numbers.

c If the number is odd, then the last digit must be one of three options: $1, 3$ or 5. The first digit cannot be 0, and obviously can't be equal to the last digit. Therefore, there are five choices for the first digit. There are then five choices for the second digit (including 0), four for the third and so on. Using the multiplication principle, there are

$$5 \times 5 \times 4 \times 3 \times 3 = 900$$

different numbers.

d There are a total 2160 numbers, of which 900 are odd. The remaining $2160 - 900 = 1260$ will be even.

14 a There are eight workers in total, from which four are to be selected. This can be done in

$$^{8}C_4 = 70$$

different ways.

b We must select two of three men and two of five women. Using the multiplication principle this can be done in

$$^{3}C_2 \times ^{5}C_2 = 30$$

different ways.

c If the group must contain Mike and Sonia then we need only select two more workers from the six that remain. This can be done in

$$^{6}C_2 = 15$$

different ways.

d If the group cannot contain both Mike and Sonia, then we need only evaluate the total number of selections, then subtract those selections that contain both Mike and Sonia. This gives

$$70 - 15 = 55$$

different selections.

15 a There are six items in total, of which a group of three are alike and another group of three are alike. These can be arranged in

$$\frac{6!}{3! \times 3!} = 20$$

different ways.

b There must be at least one red flag between each black flag. Denote black and red flags by the letters B and R respectively. Then consider the sequence BRBRB. This arrangement isolates the black flags using two red flags. The third red flag can be inserted anywhere, giving four different arrangements:

RBRBRB,BRRBRB,BRBRRB,BRBRBR.

c We list all of the possibilities in the table below. In the first two columns we write down the numbers of red and black flags respectively.

R	B	arrangements
1	0	1
0	1	1
0	2	1
2	0	1
1	1	2
3	0	1
0	3	1
1	2	3
2	1	3
1	3	4
3	1	4
2	2	6
2	3	10
3	2	10
3	3	20

This gives a total of 68 different arrangements.

16 **a** There are seven letters in total, of which a group of three Gs are alike and another group of two As are alike. These can be arranged in

$$\frac{7!}{3! \times 2!} = 420$$

different ways.

b There are three cases to consider, each of which gives the same number of arrangements.

Case 1: If the arrangement begins and ends with A then there are now just five letters to arrange, of which a group of three Gs are alike. These can be arranged in

$$\frac{5!}{3!} = 20$$

different ways.

Case 2: If the arrangement begins with A and ends with E then there are now just five letters to arrange, of which a group of three Gs are alike. These can be arranged in

$$\frac{5!}{3!} = 20$$

different ways.

t **Case 3:** If the arrangement begins with E and ends with A then there are now just five letters to arrange, of which a group of three Gs are alike. These can be arranged in

$$\frac{5!}{3!} = 20$$

different ways

Therefore the total number of arrangements will be 20 + 20 + 20 = 60.

c There are three cases to consider.
Case 1: If the arrangement begins and ends with a G then there are now just five letters to arrange, of which a group of two As are alike. These can be arranged in

$$\frac{5!}{2!} = 60$$

different ways.
Case 2: If the arrangement begins with B and ends with a G then there are now just five letters to arrange, of which a group of two Gs are alike and a group of two As are alike. These can be arranged in

$$\frac{5!}{2!2!} = 30$$

different ways.
Case 3: If the arrangement begins with a G and ends with B then there are now just five letters to arrange, of which a group of two Gs are alike and a group of two As are alike. These can be arranged in

$$\frac{5!}{2!2!} = 30$$

different ways.
Therefore the total number of arrangements will be 60 + 30 + 30 = 120.

d We group together all of the vowels {A,A,E} and all of the consonants {B,G,G,G}. There are now two groups to arrange .This can be done in 2 ways. We then arrange within each group. The first group can be arranged in $\frac{3!}{2!} = 3$ different ways, and the second group can be arranged in $\frac{4!}{3!} = 4$ different ways. Using the multiplication principle, the total number of different arrangements will be,

$$2 \times 3 \times 4 = 24.$$

17 a There are many ways to answer this question, each giving the same answer.
Method 1: There are

$$^{25}C_2 = 300$$

ways of selecting two of twenty-five people to shake hands.
Method 2: The first person shakes hands with 24 others, the second with 23 and so on. This gives the total number of handshakes as

$$24 + 23 + \cdots + 1 = 300.$$

Method 3: Each of the 25 people shakes hands with 24 others, but this double counts each handshake. Therefore the total number of handshakes is

$$\frac{25 \times 24}{2} = 300.$$

b This question can be done by trial and error. Here's an algebraic solution. Suppose that there are n people in the first group and $25 - n$ people in the second group. Then,

$$\begin{aligned}
{}^{n}C_2 + {}^{25-n}C_2 &= 150 \\
\frac{n!}{2!(n-2)!} + \frac{(25-n)!}{2!(23-n!} &= 150 \\
\frac{n(n-1)(n-2)!}{2!(n-2)!} + \frac{(25-n)(24-n)(23-n)!}{2!(23-n!} &= 150 \\
\frac{n(n-1)}{2} + \frac{(25-n)(24-n)}{2} &= 150 \\
n(n-1) + (25-n)(24-n) &= 300 \\
n^2 - 25n + 150 &= 0 \\
(n-10)(n-15) &= 0 \\
n &= 10, 15.
\end{aligned}$$

Therefore, the number of people in each group is 10 and 15.

c If we tried to count the total number of handshakes then each of the 25 people shakes hands with exactly 3 others. This double counts each handshake, so the total number of handshakes is

$$\frac{25 \times 3}{2} = \frac{75}{2},$$

which not a whole number.

18 a The required table is shown below.

n	0	1	2	3	4	5
x_n	0	0.111	0.051	0.061	0.055	0.056
y_n	0	0.182	0.151	0.168	0.165	0.167

b Solving these equations simultaneously gives $x = \frac{1}{18}$ and $y = \frac{1}{6}$. The difference between teh exact values and the approximate values is quite small. Accurate to three decimal places, we find that

$$\left|\frac{1}{18} - x_5\right| \approx 0.0005$$

$$\left|\frac{1}{6} - y_5\right| \approx 0.0001$$

That is, the approximate answers are already quite accurate.

c We rearrange each of the equations as follows

$$8x + y = 5 \Rightarrow x = \frac{1}{8}(5 - y)$$

$$2x + 13y = 4 \Rightarrow y = \frac{1}{13}(4 - 2x)$$

This leads to the numerical system,

$$x_{n+1} = \frac{1}{8}(5 - y_n)$$

$$y_{n+1} = \frac{1}{13}(4 - 2x_n).$$

We then use this to complete the table shown below.

n	0	1	2	3	4	5
x_n	0	0.625	0.587	0.599	0.598	0.598
y_n	0	0.308	0.212	0.217	0.216	0.216

d By solving these simultaneous equations (using any method) we find that $x = -1$ and $y = -1$.

e We rearrange each of the equations as follows

$$-2x + 3y = -1 \Rightarrow x = \frac{1}{2}(1 + 3y)$$

$$3x - 2y = -1 \Rightarrow y = \frac{1}{2}(1 + 3x)$$

This leads to the numerical system,

$$x_{n+1} = \frac{1}{2}(1 + 3y_n)$$

$$y_{n+1} = \frac{1}{2}(1 + 3x_n).$$

We then use this to complete the table shown below.

n	0	1	2	3	4	5
x_n	0	0.500	1.250	2.375	4.063	6.594
x_y	0	0.500	1.250	2.375	4.063	6.594

f For each n, we first note that $x_n = y_n$. Moreover, each value appears to be growing without bound.

g You can give an informal proof of this, but it's better to give a formal proof using mathematical induction. Since $x_n = y_n$ for all n, we first note that

$$x_{n+1} = \frac{1}{2}(1 + 3x_n).$$

For the base case, we let $n = 1$. We see that

$$\begin{aligned} x_1 &= \tfrac{1}{2}(1 + 3x_0) \\ &= \tfrac{1}{2} \\ &= \tfrac{1}{2}(\tfrac{3}{2})^0. \end{aligned}$$

Therefore the base case $n = 1$ is true. When $n = k$, we assume that $x_k \geq \frac{1}{2}(\frac{3}{2})^{k-1}$. Finally, when $n = k + 1$, we have

$$\begin{aligned} x_{k+1} &= \frac{1}{2}(1 + 3x_k) \\ &\geq \frac{1}{2}(1 + 3 \cdot \tfrac{1}{2}(\tfrac{3}{2})^{k-1}) \\ &= \frac{1}{2} + \tfrac{1}{2}(\tfrac{3}{2})^{k}) \\ &\geq \tfrac{1}{2}(\tfrac{3}{2})^{k} \end{aligned}$$

as required. Therefore, by the principle of mathematical induction, $x_n \geq \frac{1}{2}(\frac{3}{2})^{n-1}$ for all $n \in \mathbb{N}$.

19 **a** Four points can be selected from twelve in ${}^{12}C_4 = 495$ ways.

b Two points can be selected from twelve in ${}^{12}C_2 = 66$ ways. From this, subtract the 6 pairs that are diametrically opposite. This gives a total of $66 - 6 = 60$.

c Pick any two vertices that are not diametrically opposite. These two points, and the two points that are diametrically opposite, will lie on a rectangle.

d Selecting any two points that are not diametrically opposite will define the edge of a rectangle as described in the previous question. This can be done in 60 ways. However, there are four edges that give the same rectangle. Therefore, the total number of rectangles will be $60 \div 4 = 15$.

e There are a total of 495 choices of 4 points, and of these 15 are rectangles. Therefore, the probability of selecting a rectangle is

$$\frac{15}{495} = \frac{1}{33}$$

.

20 **a** **i** $(1, 0) \wedge (0, 1) = (1 \wedge 0, 0 \wedge 1) = (0, 0)$

ii $(1, 0) \vee (0, 1) = (1 \vee 0, 0 \vee 1) = (1, 1)$

iii $(1, 1) \wedge (0, 1) = (1 \wedge 0, 1 \wedge 1) = (0, 1)$

iv $(0, 0) \vee (0, 1) = (0 \vee 0, 0 \vee 1) = (0, 1)$

v $(1,0)' = (1',0') = (0,1)$

vi $(1,1)' = (1',1') = (0,0)$

b We define $\mathbb{B}^n = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n \in \mathbb{B}\}$. Each a_i is either 0 or 1, there are 2 choices for each a_i. Therefore there are 2^n elements in this set.

c **i** The eight elements in $\mathbb{B}^3$ are:

$$(0,0,0), (0,0,1), (0,1,0), (0,1,1)$$
$$(1,0,0), (1,0,1), (1,1,0), (1,1,1).$$

ii We find that

$$(1,0,0) \wedge (0,1,0) = (1 \wedge 0, 0 \wedge 1, 0 \wedge 0) = (0,0,0)$$

and

$$(1,0,0) \vee (0,0,1) = (1 \vee 0, 0 \vee 0, 0 \vee 1) = (1,0,1)$$

iii There is more than one answer available for this question. We give one example for each element in $\mathbb{B}^3$.

$$\begin{aligned}
(0,0,0) &= (1,0,0) \wedge (0,1,0) \\
(0,0,1) &= (0,0,1) \wedge (0,0,1) \\
(0,1,0) &= (0,1,0) \wedge (0,1,0) \\
(0,1,1) &= (0,1,0) \vee (0,0,1) \\
(1,0,0) &= (1,0,0) \wedge (1,0,0) \\
(1,0,1) &= (1,0,0) \vee (0,0,1) \\
(1,1,0) &= (1,0,0) \vee (0,1,1) \\
(1,1,1) &= (0,0,1) \vee (0,1,0) \vee (1,0,0)
\end{aligned}$$

d In Extended response question 2 in Chapter7, we considered the Boolean algebra B of all factors of 30. Find a suitable correspondence between the elements of B and the elements of $\mathbb{B}^3$. You want to show the structures are similar. For example, we can make these correspondences:

$$2 \Leftrightarrow (1,0,0) \qquad 3 \Leftrightarrow (0,1,0) \qquad 5 \Leftrightarrow (0,0,1)$$

From this, we see that $6 \Leftrightarrow (1,1,0)$. Consider the operations on both Boolean algebras.

e The number 6 has 4 divisors. Discuss the correspondence between $\mathbb{B}^2$ and the Boolean algebra of divisors of 6.

f The number 210 has 16 factors. Discuss the correspondence between $\mathbb{B}^4$ and the Boolean algebra of divisors of 210.

Solutions to Investigations

1
- $3 = 1 + 2$
- 4 - not possible
- $5 = 2 + 3$
- $6 = 1 + 2 + 3$
- $7 = 3 + 4$
- 8 - not possible
- $9 = 4 + 5 = 2 + 3 + 4$
- $10 = 3 + 7 = 1 + 2 + 3 + 4$
- $11 = 5 + 6$
- $15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5$

Conjecture 1 All natural numbers ≥ 3 except powers of 2 can be expressed as a sum of consecutive integers.
Some partial results:
If n is odd then $n = 2k + 1$ for some $k \in \mathbb{N}$
Observe $2k + 1 = k + k + 1$
For example, $27 = 2 \times 13 + 1 = 13 + 14$
If n is divisible by 3, then $n = 3k$,for some $k \in \mathbb{N}$. In this case take $k > 1$
$n = k - 1 + k + k + 1$
If n is divisible by 5, then $n = 5k$,for some $k \in \mathbb{N}$. In this case take $k > 2$
$n = k - 2 + k - 1 + k + k + 1 + k + 2$
For example, $15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5$ Obviously, you can extend this argument for divisible by any odd number.
Conjecture 2 Any natural number which is not a power of 2 can be expressed as a sum of consecutive integers and the number of ways this can be done is the number of odd divisors greater than 1.

b Let N be a sum of consecutive natural numbers.
We use the fact that a sum of consecutive natural numbers can be expressed as a difference of arithmetic sequences each with first term 1.

$$N = \frac{1}{2}n(n+1) - \frac{1}{2}m(m+1)$$
$$8N = (2n+1)^2 - (2m+1)^2$$
$$= (2n+2m+2)(2n-2m)$$
$$2N = (m+n+1)(n-m)$$
Assume $n - m$ is even. There exists a $k \in \mathbb{N}$ such that $n = m + 2k$.
Therefore $m + n + 1 = m + m + 2k + 1 = 2(m+n) + 1$. That is it is odd. The sum of consecutive natural numbers can't be a power of 2. Try and tighten the whole argument.

2 a Let a, b be the two numbers. Then if $a + b + ab = 71, ab + a + b + 1 = 72$
Hence $(a+1)(b+1) = 72$
$(a+1)(b+1) = 2^3 \times 3^2$
$(a+1)(b+1) = 2 \times 36$
$(a+1)(b+1) = 4 \times 18$
$(a+1)(b+1) = 8 \times 9$
$(a+1)(b+1) = 6 \times 12$
$(a+1)(b+1) = 3 \times 24$
For (a, b) we have the pairs $(1, 35), (3, 17), (7, 8), (5, 11), (2, 23)$. Of course the reversed ordered pairs satisfy this.

b Let a, b, c be the three numbers. Proceed as before so that we take the two numbers to be $a + b + ab$ and c. For convenciene take $x = a + b + ab$
Then, $x + c + cx = 71$
$x + c + cx + 1 = 72$
$(x+1)(c+1) = 72$ $(a+1)(b+1)(c+1) = 72$
We again use the observation that $(a+1)(b+1)(c+1) = 2^3 \times 3^2$
$2 \times 2 \times 18 = 72$
$4 \times 2 \times 9 = 72$
$6 \times 3 \times 4 = 72$
$8 \times 3 \times 3 = 72$ In this way we can find the possible values a, b and c

c You can separate the prime factors to obtain

$$2 \times 2 \times 2 \times 3 \times 3 = 72$$

. So take $a = b = c = 1$ and $d = e = 2$
The sequence goes
$$3, 1, 2, 2 \to 7, 2, 2 \to 23, 2 \to 71$$
The order of performing the operation doesn't matter. This should be shown.

3 ■ Let the first two be a and b. The next ones will then be

$$a + b, a + 2b, (a + b) + (a + 2b) = 2a + 3b$$

and

$$(a + 2b) + (2a + 3b) = 3a + 5b.$$

We now add all of them, to get:

$$a + b + (a + b) + (a + 2b) + (2a + 3b) + (3a + 5b) = 8a + 12b.$$

If we divide the result by 4 we get

$$\frac{8a + 12b}{4} = 2a + 3b.$$

■ The terms $F_3, F_6, F_9, \ldots$ are even. You can prove this by induction.
F_3 is even.
Assume F_K is even.

$$\begin{aligned} F_{K+3} &= F_{K+2} + F_{K+1} \\ &= F_{K+1} + F_K + F_{K+1} \\ &= 2F_{K+1} + F_K \end{aligned}$$

Therefore F_{K+3} is even.

■ Terms $F_4, F_8, F_{12}, \ldots$

■ This approaches the golden ratio

■ $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ (Prove by induction)

4 a i $\star\star\star\star|\star\star\star\star\star\star|\star$

$|\star\star\star\star\star\star\star|\star\star\star\star$

$||\star\star\star\star\star\star\star\star\star\star$

Note that the bars divide the stars into 3 cells. In the third example there are two empty cells to the left of the symbol. We are looking at the way of dividing 10 stars into 3 cells. The stars are indistinguishable as are the bars. Hence there are

$$\frac{12!}{2!10!}$$

ways of organising them. In general this can be thought of at

$$^{n+k-1}C_n =^{n+k1} C_{k-1}$$

ways of organising n stars into k cells.
For 2 stars and 3 cells.

$||\star\star$ $(0, 0, 2)$

$|\star|\star \quad (0,1,1)$

$\star||\star \quad (1,0,1)$

$|\star\star| \quad (0,2,0)$

$\star|\star| \quad (1,1,0)$

$\star\star|| \quad (2,0,0)$

ii It is the number of ways that 10 stars and 2 bars can be arranged. There are:

$${}^{12}C_2 = 66 \text{ ways of distributing 10 chocolates among 3 children}$$

iii

$${}^{11}C_3 = 165 \text{ ways of distributing 8 chocolates among 4 children}$$

iv

$${}^{n+k-1}C_{k-1} \text{ ways of distributing } n \text{ chocolates among } k \text{ children}$$

v The number of places is reduced

$${}^{n-1}C_{k-1} \text{ ways of distributing } n \text{ chocolates among } k \text{ children in this way}$$

b There are 3 stars and 3 cells. Therefore there are

$${}^{3+3-1}C_{3-1} = {}^5C_2 = 10 \text{ ways}$$

We can illustrate these as only ten:

$\circ\circ\circ|\ \ | \quad (3,0,0)$

$|\circ\circ\circ| \quad (0,3,0)$

$|\ \ |\circ\circ\circ \quad (0,0,3)$

$\circ|\circ|\circ \quad (1,1,1)$

$\circ\circ|\circ| \quad (2,1,0)$

$\circ\circ||\circ \quad (2,0,1)$

$|\circ\circ|\circ \quad (0,2,1)$

$|\circ|\circ\circ \quad (0,1,2)$

$\circ|\circ\circ| \quad (1,2,0)$

$\circ | | \circ \circ \quad (1, 0, 2)$

c This includes 0. We can apply the stars and bars mode. Here there are 10 stars and 4 cells (3 bars). Therefore

$$^{10}C_3 = 120 \text{ ways}$$

For example:
$\circ \circ | \circ \circ \circ \circ | \circ \circ \circ | \circ$
corresponds to the sum $2 + 4 + 3 + 1 = 10$
$| \circ \circ | \circ \circ \circ \circ \circ \circ \circ | \circ$
corresponds to the sum $0 + 2 + 7 + 1 = 10$

d 36 ways . The numbers to work with are 1,3,5,7,9,11,13,15. We cannot use 0. Systematic listing succeeds here, together with noticing that sums such as $1 + 3 + 13$ can be arranged in 6 ways and sums such as $3 + 3 + 11$ can be arranged in 6 ways

e You can consider sequences of the form $RRDDRD \dots DR$ with m R's and n D's. The total number will be $^{m+n-1}C_{n-1}$. Explore this further.

f These are left to the reader.

Chapter 11 – Matrices

Solutions to Exercise 11A

1 **a** Number of rows × number of columns = 2×2

b Number of rows × number of columns = 2×3

c Number of rows × number of columns = 1×4

d Number of rows × number of columns = 4×1

2 **a** There will be 5 rows and 5 columns to match the seating. Every seat of both diagonals is occupied, and so the diagonals will all be ones, and the rest of the numbers, representing unoccupied seats, will all be 0.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b If all seats are occupied, then every number in the matrix will be 1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

3 **a** $i = j$ for the leading diagonal only, so the leading diagonal will be all ones, and the rest of the numbers 0.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

4 We can present this as a table with the girls on the top row, and the boys on the bottom row, in order of year level, i.e. years 7, 8, 9, 10, 11 and 12 going from left to right.

$$\begin{bmatrix} 200 & 180 & 135 & 110 & 56 & 28 \\ 110 & 117 & 98 & 89 & 53 & 33 \end{bmatrix}$$

Alternatively, girls and boys could be the two columns, and year levels could run down from year 7 to 12, in order. This would give:

$$\begin{bmatrix} 200 & 110 \\ 180 & 117 \\ 135 & 98 \\ 110 & 89 \\ 56 & 53 \\ 28 & 33 \end{bmatrix}$$

5 **a** Matrices are equal only if they

have the same number of rows and columns, and all pairs of corresponding entries are equal. The first two matrices have the same dimensions, but the top entries are not equal, so the matrices cannot be equal.
The last two matrices have the same dimensions and equal first (left) entries, so they will be equal if $x = 4$.
Thus, $\begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ if $x = 4$.

b The first two matrices cannot be equal because corresponding entries are not equal, nor can the second and third for the same reason.
The last matrix cannot equal any of the others because it has different dimensions. The only two that can be equal are the first and third.
$\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$ if $x = 4$

c All three matrices have the same dimensions and all corresponding numerical entries are equal. They could all be equal.

$$\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$$
$$\text{if } x = 0, y = 2$$

6 a The entry corresponding to x is 2, and the entry corresponding to y is 3, so $x = 2$ and $y = 3$.

b The entry corresponding to x is 3, and the entry corresponding to y is 2, so $x = 3$ and $y = 2$.

c The entry corresponding to x is 4, and the entry corresponding to y is -3, so $x = 4$ and $y = -3$.

d The entry corresponding to x is 3, and the entry corresponding to y is -2, so $x = 3$ and $y = -2$.

7 Write it as set out, with each row representing players A, B, C, D and E respectively, and columns showing points, rebounds and assists respectively.

$$\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$$

Solutions to Exercise 11B

1 Add the corresponding entries.

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 1+3 \\ -2+0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Double each entry.

$$2\mathbf{X} = \begin{bmatrix} 2 \times 1 \\ 2 \times -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Multiply each entry in **Y** by 4 and add the corresponding entry for **X**.

$$4\mathbf{Y} + \mathbf{X} = \begin{bmatrix} 4 \times 3 + 1 \\ 4 \times 0 + -2 \end{bmatrix} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$$

Subtract corresponding entries.

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 1-3 \\ -2-0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Multiply each entry by −3.

$$-3\mathbf{A} = \begin{bmatrix} -3 \times 1 & -3 \times -1 \\ -3 \times 2 & -3 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix}$$

Add **B** to the previous answer.

$$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -7 & -7 \end{bmatrix}$$

2

$$2\mathbf{A} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$$

$$-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ 0 & -6 \end{bmatrix}$$

$$-6\mathbf{A} = \begin{bmatrix} -6 & 6 \\ 0 & -12 \end{bmatrix}$$

3 a As the matrices have the same dimensions, corresponding terms can be added. They will simply be added in the opposite order.
Since the commutative law holds true for numbers, all corresponding entries in the added matrices terms will be equal, so the matrices will be equal.

b As the matrices have the same dimensions, corresponding terms can be added. The first matrix will add the first two numbers, then the third, and the second matrix will add the second and third numbers frst, then add the result to the first number.
Since the associative law holds true for numbers, all corresponding entries in the added matrices terms will be equal, so the matrices will be equal.

4 a Multiply each entry by 2.

$$2\mathbf{A} = \begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix}$$

b Multiply each entry by 3.

$$3\mathbf{B} = \begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix}$$

c Add answers to **a** and **b**.

$$2\mathbf{A} + 3\mathbf{B} = \begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix} + \begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 \\ 8 & -1 \end{bmatrix}$$

d Subtract **a** from **b**.

$$3\mathbf{B} - 2\mathbf{A} = \begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -13 \\ 16 & 7 \end{bmatrix}$$

5 a Add corresponding entries.

$$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}+\begin{bmatrix}-1 & 1\\2 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$$

b Triple entries in **Q**, then add to corresponding entries in **P**.

$$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}+\begin{bmatrix}-3 & 3\\6 & 0\end{bmatrix}=\begin{bmatrix}-2 & 3\\6 & 3\end{bmatrix}$$

c Double entries in **P**, then subtract **Q** and add **R**.

$$\begin{bmatrix}2 & 0\\0 & 6\end{bmatrix}-\begin{bmatrix}-1 & 1\\2 & 0\end{bmatrix}+\begin{bmatrix}0 & 4\\1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3 & 3\\-1 & 7\end{bmatrix}$$

6 a If $2\mathbf{A}-3\mathbf{X}=\mathbf{B}$, then $2\mathbf{A}-\mathbf{B}=3\mathbf{X}$

$3\mathbf{X}=2\mathbf{A}-\mathbf{B}$

$$\mathbf{X}=\frac{2}{3}\mathbf{A}-\frac{1}{3}\mathbf{B}$$

$$=\frac{2}{3}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}-\frac{1}{3}\begin{bmatrix}0 & -10\\-2 & 17\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{3}\times 3-\frac{1}{3}\times 0 & \frac{2}{3}\times 1-\frac{1}{3}\times -10\\ \frac{2}{3}\times -1-\frac{1}{3}\times 2 & \frac{2}{3}\times 4-\frac{1}{3}\times -17\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4\\0 & -3\end{bmatrix}$$

b If $3\mathbf{A}+2\mathbf{Y}=2\mathbf{B}$ then $2\mathbf{Y}=2\mathbf{B}-3\mathbf{A}$

$\mathbf{Y}=\mathbf{B}-1\frac{1}{2}\mathbf{A}$

$$=\begin{bmatrix}0 & -10\\-2 & 17\end{bmatrix}-1\frac{1}{2}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}0-\frac{3}{2}\times 3 & -10-\frac{3}{2}\times 1\\ -2-\frac{3}{2}\times -1 & 17-\frac{3}{2}\times 4\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{9}{2} & -\frac{23}{2}\\ -\frac{1}{2} & 11\end{bmatrix}$$

7

X + **Y**

$$=\begin{bmatrix}150+160 & 90+90 & 100+120 & 50+40\\100+100 & 0+0 & 75+50 & 0+0\end{bmatrix}$$

$$=\begin{bmatrix}310 & 180 & 220 & 90\\200 & 0 & 125 & 0\end{bmatrix}$$

The matrix represents the total production at two factories in two successive weeks.

Solutions to Exercise 11C

1 $\mathbf{AX} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1\times 2 + -2\times -1 \\ -1\times 2 + 3\times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$\mathbf{BX} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3\times 2 + 2\times -1 \\ 1\times 2 + 1\times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\mathbf{AY} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1\times 1 + -2\times 3 \\ -1\times 1 + 3\times 3 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1\times 2 + 0\times -1 \\ 0\times 2 + 1\times -1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\mathbf{AC} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1\times 2 + -2\times 1 & 1\times 1 + -2\times 1 \\ -1\times 2 + 3\times 1 & -1\times 1 + 3\times 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2\times 1 + 1\times -1 & 2\times -2 + 1\times 3 \\ 1\times 1 + 1\times -1 & 1\times -2 + 1\times 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Use $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 0\times 2 + -1\times -1 \\ 1\times 2 + 2\times -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Use $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2\times 4 + 1\times 1 \\ 1\times 4 + 1\times 1 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

$\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1\times 1 + -2\times 0 & 1\times 0 + -2\times 1 \\ -1\times 1 + 3\times 0 & -1\times 0 + 3\times 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1\times 3 + 0\times 1 & 1\times 2 + 0\times 1 \\ 0\times 3 + 1\times 1 & 0\times 2 + 1\times 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 3 + -2\times 1 & 1\times 2 + -2\times 1 \\ -1\times 3 + 3\times 1 & -1\times 2 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 1 + 2\times -1 & 3\times -2 + 2\times 3 \\ 1\times 1 + 1\times -1 & 1\times -2 + 1\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1 + -2\times -1 & 1\times -2 + -2\times 3 \\ -1\times 1 + 3\times -1 & -1\times -2 + 3\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$$

$$\mathbf{B}^2 = \mathbf{BB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 3 + 2\times 1 & 3\times 2 + 2\times 1 \\ 1\times 3 + 1\times 1 & 1\times 2 + 1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Use $\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\mathbf{A}(\mathbf{CA})$

$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1 + -2\times 0 & 1\times -1 + -2\times 1 \\ -1\times 1 + 3\times 0 & -1\times -1 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

Use $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$$\mathbf{A}^2\mathbf{C} = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 2 + -8\times 1 & 3\times 1 + -8\times 1 \\ -4\times 2 + 11\times 1 & -4\times 1 + 11\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$$

2 a A product is defined only if the number of columns in the first matrix equals the number of rows of the second.

A has 2 columns and **Y** has 2 rows, so **AY** is defined.

Y has 1 column and **A** has 2 rows, so **YA** is not defined.

X has 1 column and **Y** has 2 rows, so **XY** is not defined.

X has 1 column and 2 rows, so $\mathbf{X}^2$ is not defined.

C has 2 columns and **I** has 2 rows, so **CI** is defined.

X has 1 column and **I** has 2 rows, so **XI** is not defined.

3 $\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\times 0 + 0\times -3 & 2\times 0 + 0\times 2 \\ 0\times 0 + 0\times -3 & 0\times 0 + 0\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4 $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

No, because **Q.2** part **b** shows that **AB** can equal **O**, and $\mathbf{A} \neq \mathbf{O}, \mathbf{B} \neq \mathbf{O}$.

5 One possible answer is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

6 $\mathbf{LX} = [2 \ \ -1]\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$= [2 \times 2 + -1 \times -3] = [7]$

$\mathbf{XL} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}[2 \ \ -1]$

$= \begin{bmatrix} 2 \times 2 & 2 \times -1 \\ -3 \times 2 & -3 \times -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

7 A product is defined only if the number of columns in the first matrix equals the number of rows of the second.
This can only happen if $m = n$, in which case both products will be defined.

8

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \begin{bmatrix} a \times d + b \times -c & a \times -b + b \times a \\ c \times d + d \times -c & c \times -b + d \times a \end{bmatrix}$

$= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For the equations to be equal, all corresponding entries must be equal, therefore $ad - bc = 1$.
When written in reverse order, we get

$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} d \times a + -b \times c & d \times b + -b \times d \\ -c \times a + a \times c & -c \times b + a \times d \end{bmatrix}$

$= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

since $ad - bc = 1$.

9 We can use any values of a, b, c and d as long as $ad - bc = 1$.
For example, $a = 5, d = 2, b = 3, c = 3$ satisfy $ad - bc = 1$ and give

$\mathbf{AB} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Other values could be chosen.

10 One possible answer.

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$

$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+0 & 2+1 \\ 4+2 & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 6 & 6 \end{bmatrix}$

$\mathbf{B} + \mathbf{C} = \begin{bmatrix} 0+-1 & 1+2 \\ 2+-2 & 3+1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 \times -1 + 2 \times 0 & 1 \times 3 + 2 \times 4 \\ 4 \times -1 + 3 \times 0 & 4 \times 3 + 3 \times 4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 0 + 2 \times 2 & 1 \times 1 + 2 \times 3 \\ 4 \times 0 + 3 \times 2 & 4 \times 1 + 3 \times 4 \end{bmatrix}$

$= \begin{bmatrix} 4 & 7 \\ 6 & 13 \end{bmatrix}$

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times -1 + 2\times -2 & 1\times 2 + 2\times 1 \\ 4\times -1 + 3\times -2 & 4\times 2 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 \\ -10 & 11 \end{bmatrix}$$

$$\mathbf{AB} + \mathbf{AC} = \begin{bmatrix} 4 & 7 \\ 6 & 13 \end{bmatrix} + \begin{bmatrix} -5 & 4 \\ -10 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + -5 & 7 + 4 \\ 6 + -10 & 13 + 11 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$$

$(\mathbf{B} + \mathbf{C})\mathbf{A}$

$$= \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1\times 1 + 3\times 4 & -1\times 2 + 3\times 3 \\ 0\times 1 + 4\times 4 & 0\times 2 + 4\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

11 For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

12 a $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 5\times 1 + 12\times 2 \\ 2.50\times 1 + 3.00\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$$

1×5 min plus 2×12 min means 29 min for one milkshake and two banana splits.
The total cost is \$8.50.

b

$$\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 1 + 12\times 2 & 5\times 2 + 12\times 1 & 5\times 0 + 12\times 1 \\ 2.5\times 1 + 3\times 2 & 2.5\times 2 + 3\times 1 & 2.5\times 0 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$$

The matrix shows that John spent 29 min and \$8.50, one friend spent 22 min and \$8.00 (2 milkshakes and 1 banana split) while the other friend spent 12 min and \$3.00 (no milkshakes and 1 banana split).

13 $\mathbf{A}^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}$,
$\mathbf{A}^8 = \begin{bmatrix} -527 & 336 \\ -336 & -527 \end{bmatrix}$

14 $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$,
$\mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Solutions to Exercise 11D

1 a $\det(\mathbf{A}) = 2 \times 2 - 1 \times 3$

$= 1$

b $\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

c $\det(\mathbf{B}) = -2 \times 2 - -2 \times 3$

$= 2$

d $\mathbf{B}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$

2 a Determinant $= 3 \times -1 - -1 \times 4 = 1$

$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$

b Determinant $= 3 \times 4 - 1 \times -2 = 14$

$\mathbf{A}^{-1} = \frac{1}{14}\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{7} & -\frac{1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$

c Determinant $= 1 \times k - 0 \times 0 = k$

$\mathbf{A}^{-1} = \frac{1}{k}\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$

d Determinant $= \cos\theta \times \cos\theta$

$- -\sin\theta \times \sin\theta$

$= 1$

since $\cos^2\theta + \sin^2\theta = 1$

$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

3 a $\det(\mathbf{A}) = 2 \times -1 - 1 \times 0 = -2$

$\mathbf{A}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$

$\det(\mathbf{B}) = 1 \times 1 - 0 \times 3 = 1$

$\mathbf{B}^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b

$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 1 + 1 \times 3 & 2 \times 0 + 1 \times 1 \\ 0 \times 1 + -1 \times 3 & 0 \times 0 + -1 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$

$\det(\mathbf{AB}) = 5 \times -1 - 1 \times -3 = -2$

$(\mathbf{AB})^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 3 & 5 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

c $\mathbf{A}^{-1}\mathbf{B}^{-1}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \times 1 + \frac{1}{2} \times -3 & \frac{1}{2} \times 0 + \frac{1}{2} \times 1 \\ 0 \times 1 + -1 \times -3 & 0 \times 0 + -1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times \frac{1}{2} + 0 \times 0 & 1 \times \frac{1}{2} + 0 \times -1 \\ -3 \times \frac{1}{2} + 1 \times 0 & -3 \times \frac{1}{2} + 1 \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

4 **a** $\det(\mathbf{A}) = 4 \times 1 - 3 \times 2 = -2$

$$\mathbf{A}^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$$

b If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1} \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$\therefore \mathbf{IX} = \mathbf{X}$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \times 3 + \frac{3}{2} \times 1 & -\frac{1}{2} \times 4 + \frac{3}{2} \times 6 \\ 1 \times 3 + -2 \times 1 & 1 \times 4 + -2 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$$

c If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, multiply both sides from the right by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}\mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y}$$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times -\frac{1}{2} + 4 \times 1 & 3 \times \frac{3}{2} + 4 \times -2 \\ 1 \times -\frac{1}{2} + 6 \times 1 & 1 \times \frac{3}{2} + 6 \times -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$$

5 **a** If $\mathbf{AX} + \mathbf{B} = \mathbf{C}$ then $\mathbf{AX} = \mathbf{C} - \mathbf{B}$

$$\therefore \quad \mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$\det(\mathbf{A}) = 3 \times 6 - 2 \times 1 = 16$

$$\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$$

If $\mathbf{AX} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$, multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$\therefore \mathbf{IX} = \mathbf{X}$$

$$= \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} \times -1 + -\frac{1}{8} \times 0 & \frac{3}{8} \times 5 + -\frac{1}{8} \times 4 \\ -\frac{1}{16} \times -1 + \frac{3}{16} \times 0 & -\frac{1}{16} \times 5 + \frac{3}{16} \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$$

b IF $\mathbf{YA} + \mathbf{B} = \mathbf{C}$ then $\mathbf{YA} = \mathbf{C} - \mathbf{B}$

$$\therefore \mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

From part **a**, $\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$

If $\mathbf{YA} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$, multiply both sides from the right by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}\mathbf{A}^{-1}$$

$$\mathbf{YI} = \mathbf{Y}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times \frac{3}{8} + 5 \times -\frac{1}{16} & -1 \times -\frac{1}{8} + 5 \times \frac{3}{16} \\ 0 \times \frac{3}{8} + 4 \times -\frac{1}{16} & 0 \times -\frac{1}{8} + 4 \times \frac{3}{16} \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} -\frac{11}{16} & \frac{17}{16} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

6 $\mathbf{A}^{-1} = \dfrac{1}{a^2 + b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. which always exists

7 Suppose $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$

and $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$

Then

$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}$

8 $\mathbf{A}$ must be $\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$.

$\det(\mathbf{A}) = a_{11} \times a_{22} - 0 \times 0 = a_{11}a_{22}$

$\det(\mathbf{A}) \neq 0$ since $a_{11} \neq 0$ and $a_{22} \neq 0$ and the product of two non-zero numbers cannot be zero.

$\therefore$ $\mathbf{A}$ is regular.

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22}}\begin{bmatrix} a_{22} & 0 \\ 0 & a_{11} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$$

9 If $\mathbf{A}$ is invertible, it will have an inverse, $\mathbf{A}^{-1}$. Multiply both sides of the equation $\mathbf{AB} = 0$ from the left by $\mathbf{A}^{-1}$.

$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}0$

$\therefore \quad \mathbf{IB} = 0$

$\mathbf{B} = 0$

10

$$\begin{aligned} \det(\mathbf{A}) &= x^2 - (2x+1)^2 \\ &= x^2 - 4x^2 - 4x - 1 \\ &= -3x^2 - 4x - 1 \end{aligned}$$

$\det(\mathbf{A}) = 0$

$\Rightarrow 3x^2 + 4x + 1 = 0$

$(3x+1)(x+1) = 0$

$x = -\frac{1}{3}$ or $x = -1$

Matrix $\mathbf{A}$ will have an inverse for $x \in \mathbb{R}\backslash\{-\frac{1}{3}, -1\}$

11 a $\mathbf{A}^{-1} = \frac{1}{-3a+8}\begin{bmatrix} -3 & -4 \\ 2 & a \end{bmatrix}$

$\mathbf{A}^{-1} = \mathbf{A}$

$$\frac{1}{-3a+8}\begin{bmatrix} -3 & -4 \\ 2 & a \end{bmatrix} = \begin{bmatrix} a & 4 \\ -2 & -3 \end{bmatrix}$$

We have four equations from equal entries:

$\frac{-3}{-3a+8} = a \ldots (1)$

$\frac{-4}{-3a+8} = 4 \ldots (2)$

$\frac{2}{-3a+8} = -2 \ldots (3)$

$\frac{a}{-3a+8} = -3 \ldots (4)$

Start with equation (2)

$-4 = -12a + 32 \Leftrightarrow a = 3$

Check in the other equations: In (1) when $a = 3$:

LHS = 3 and RHS= 3

In (3) when $a = 3$:

LHS = −2 and RHS = −2

In (4) when $a = 3$:

LHS = −3 and RHS = −3

b Let $\mathbf{A}$ be any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If the determinant is n, then the inverse of $\mathbf{A}$ is given by $\frac{1}{n}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{n}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$a = \frac{d}{n} \text{ and } d = \frac{a}{n}$$

Substituting for d, $a = \frac{a \div n}{n} = \frac{a}{n^2}$

This gives $n^2 = 1$, or $n = \pm 1$.

If $n = 1$, $a = d$ and $-b = b$, which gives $b = 0$ and similarly $c = 0$.

$\det(\mathbf{A}) = ad = a^2 = 1$

This leads to two matrices, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

If $n = -1$, $a = -d$; there are no restrictions on b and c but the determinant $= ad - bc = -1$.

$\therefore \quad a^2 + bc = 1$ (since $a = -d$)

If $b = 0$, $a = \pm 1$, giving $\begin{bmatrix} \pm 1 & 0 \\ c & \mp 1 \end{bmatrix}$, which can be written $\begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}$.

If $b \neq 0$, $a^2 + bc = 1$ gives $c = \dfrac{1 - a^2}{b}$, giving $\begin{bmatrix} a & b \\ \dfrac{1 - a^2}{b} & -a \end{bmatrix}$, which includes the cases $\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}$ when $a = \pm 1$.

12 $a = \pm\sqrt{2}$

13 $\det(\mathbf{A}) = n^2 + 2n - (n^2 + 2n + 1)$

$= -1$

Therefore

$$\mathbf{A}^{-1} = -1\begin{bmatrix} n+2 & -n-1 \\ -n-1 & n \end{bmatrix}$$

$$= \begin{bmatrix} -n-2 & n+1 \\ n+1 & -n \end{bmatrix}$$

All entries are integers.

14 $\det(\mathbf{A}) = n^2 + 3n - (n^2 + 3n + 2)$

$= -2$

Therefore

$$\mathbf{A}^{-1} = -\frac{1}{2}\begin{bmatrix} n+3 & -n-1 \\ -n-2 & n \end{bmatrix}$$

If n is even then $n + 3$ is odd.

If n is odd we are finished

15

$$\det(\mathbf{A}) = \frac{1}{n^2 + 2n} - \frac{1}{n^2 + 2n + 1}$$

$$= \frac{1}{n(n+1)^2(n+2)}$$

Therefore

$$\mathbf{A}^{-1} = n(n+1)^2(n+2)\begin{bmatrix} \frac{1}{n+2} & -\frac{1}{n} \\ -\frac{1}{n+1} & \frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} n(n+1)^2 & -(n+1)^2(n+2) \\ -n(n+1)(n+2) & (n+1)^2(n+2 \end{bmatrix}$$

All the entries are integers

Solutions to Exercise 11E

1 First find the inverse of **A**.

$\det(\mathbf{A}) = 3 \times -1 - -1 \times 4 = 1$

$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$

a If $\mathbf{AX} = \mathbf{K}$ then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$

$\therefore \mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \times -1 + 1 \times 2 \\ -4 \times -1 + 3 \times 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

b If $\mathbf{AX} = \mathbf{K}$ then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$

$\therefore \mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \times -2 + 1 \times 3 \\ -4 \times -2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 17 \end{bmatrix}$$

2 a $\begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

Determinant $= -2 \times 1 - 4 \times 3 = -14$

$$\text{Inverse} = \frac{1}{-14}\begin{bmatrix} 1 & -4 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{14} \times 6 + \frac{2}{7} \times 1 \\ \frac{3}{14} \times 6 + \frac{1}{7} \times 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ \frac{10}{7} \end{bmatrix}$$

$x = -\frac{1}{7},\ y = \frac{10}{7}$

b $\begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Determinant $= -1 \times 4 - 2 \times -1 = -2$

$$\text{Inverse} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \times -1 + 1 \times 2 \\ -\frac{1}{2} \times -1 + \frac{1}{2} \times 2 \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{3}{2} \end{bmatrix}$$

$x = 4,\ y = \frac{3}{2}$ or 1.5

c $\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Determinant $= \frac{1}{2} \times \dfrac{1}{4} - \dfrac{1}{3} \times \dfrac{1}{3} = \dfrac{1}{72}$

Inverse $= 72\begin{bmatrix} \frac{1}{4} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 18 & -24 \\ -24 & 36 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 & -24 \\ -24 & 36 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -6 \\ 12 \end{bmatrix}$

$x = -6,\ y = 12$

d $\begin{bmatrix} \frac{1}{20} & \frac{1}{21} \\ \frac{1}{21} & \frac{1}{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Determinant $= \frac{1}{20} \times \dfrac{1}{22} - \dfrac{1}{21} \times \dfrac{1}{21} = \dfrac{1}{194040}$

Inverse $= 194040\begin{bmatrix} \frac{1}{22} & -\frac{1}{21} \\ -\frac{1}{21} & \frac{1}{20} \end{bmatrix}$

$= \begin{bmatrix} 8820 & -9240 \\ -9240 & 9702 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8820 & -9240 \\ -9240 & 9702 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -420 \\ 462 \end{bmatrix}$

$x = -420,\ y = 462$

3 Solve the simultaneous equations

$2x - 3y = 7$

$3x + y = 5$

$\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

Determinant $= 2 \times 1 - -3 \times 3 = 11$

Inverse $= \dfrac{1}{11}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{11}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix}$

$= \dfrac{1}{11}\begin{bmatrix} 1 \times 7 + 3 \times 5 \\ -3 \times 7 + 2 \times 5 \end{bmatrix}$

$= \dfrac{1}{11}\begin{bmatrix} 22 \\ -11 \end{bmatrix}$

$x = 2,\ y = -1$

The point of intersection is $(2,\ -1)$.

4 If x is the number of books they are buying and y is the number of CDs they are buying, then the following equations apply.

$4x + 4y = 120$

$5x + 3y = 114$

$\begin{bmatrix} 4 & 4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 120 \\ 114 \end{bmatrix}$

Determinant $= 4 \times 3 - 4 \times 5 = -8$

Inverse $= \dfrac{1}{-8}\begin{bmatrix} 3 & -4 \\ -5 & 4 \end{bmatrix} = \dfrac{1}{8}\begin{bmatrix} -3 & 4 \\ 5 & -4 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{8}\begin{bmatrix} -3 & 4 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 120 \\ 114 \end{bmatrix}$

$= \dfrac{1}{8}\begin{bmatrix} -3 \times 120 + 4 \times 114 \\ 5 \times 120 + -4 \times 114 \end{bmatrix}$

$= \dfrac{1}{8}\begin{bmatrix} 96 \\ 144 \end{bmatrix}$

$x = 12,\ y = 18$

One book costs \$12, a CD costs \$18.

5 **a** $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

b $\det(\mathbf{A}) = 2 \times -6 - -3 \times 4 = 0$, so the matrix is non-invertible.

c Yes. For example $x = 0,\ y = -1$ is an obvious solution.

d You should notice that the second equation is simply the first with both sides multiplied by 2.

There is an infinite number of solutions to these equations, just as there is an infinite number of ordered pairs that make $2x - 3y = 3$ a true equation.

6 **a** $\mathbf{A}^{-1}\mathbf{C}$

b $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C}$

c $\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$

d $\mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$

e $\mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

f $(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1}$

Solutions to Exercise 11F

1 a Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$\mathbf{AA}^{-1} = \mathbf{I}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ 5g & 5h & 5i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore a = 1, b = 0, c = 0, d = 0$

$e = \dfrac{1}{2}, f = 0, g = 0, h = 0, i = \dfrac{1}{5}$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

b Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

$$\mathbf{AA}^{-1} = \mathbf{I}$$

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a + 4d + 5g & b + 4e + 5h & c + 4f + 5i \\ 2d + 3g & 2e + 3h & 2f + 3i \\ 5g & 5h & 5i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore i = \dfrac{1}{5}, h = 0, g = 0, f = -\dfrac{3}{10}$

$e = \dfrac{1}{2}, d = 0, c = \dfrac{1}{5}, b = -2, a = 1$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -2 & \frac{1}{5}, \\ 0 & \frac{1}{2} & -\frac{3}{10} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

2 $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 5 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 5 & -2 & 3 \\ -7 & 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3 $\mathbf{AB} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $\quad \mathbf{A}^{-1} = \dfrac{1}{7}\mathbf{B}$

4 $\mathbf{A}^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$; $\quad \mathbf{A}^{-1} = \dfrac{1}{9}\mathbf{A}$

5 $\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $\quad \mathbf{A}^{-1} = \dfrac{1}{4}\mathbf{A}$

6 a $\begin{bmatrix} 2 & 1 & -10 \\ 3 & 2 & -17 \\ -5 & -3 & 28 \end{bmatrix}$

b $\frac{1}{29}\begin{bmatrix} 8 & -13 & 14 \\ 2 & 4 & -11 \\ -9 & 11 & 6 \end{bmatrix}$

c $\frac{1}{37}\begin{bmatrix} 6 & 4 & -7 & -17 \\ -13 & -21 & 46 & 43 \\ 8 & 30 & -34 & -35 \\ -4 & -15 & 17 & 36 \end{bmatrix}$

d $\frac{1}{37}\begin{bmatrix} 6 & -13 & 8 & -4 \\ 4 & -21 & 30 & -15 \\ -7 & 46 & -34 & 17 \\ -17 & 43 & -35 & 36 \end{bmatrix}$

7 a Determinant= $9(2-6) - 1(1-4) + 3(3-4)$
$= -36 + 3 - 3 = -36$

b Determinant= $1(0-0) - 3(0-5) + 2(0-7)$
$= 15 - 14 = 1$

8 a i -2

ii -2

b i -4

ii -16

9 a $\det(\mathbf{A}) = 1(2-4) - 2(2-6) + p(4-6) = 6 - 2p$

b Does not have an inverse when $p = 3$

10 a $\det(\mathbf{A}) = 1(2p-4) - 2(2p-2p) + p(4-2p)$
$= 2p - 4 + 4p - 2p^2$
$= -2p^2 + 6p - 4$
$= -2(p^2 - 3p + 2)$

b $\det(\mathbf{A}) = 0 \Rightarrow p^2 - 3p + 2 = 0$
$\Rightarrow (p-1)(p-2) = 0$
$\Rightarrow p = 1$ or $p = 2$

Solutions to Exercise 11G

1 a i $2x + 3y - z = 12$

$2y + z = 7$

$2y - z = 5$

Write the equation in the form

$\mathbf{AX} = \mathbf{B}$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \\ 5 \end{bmatrix}$$

$\mathbf{A}^{-1} =$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} & -\frac{5}{8} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 4 & -1 & -5 \\ 0 & 2 & 2 \\ 0 & 4 & -4 \end{bmatrix}$$

$\mathbf{AX} = \mathbf{B}$

$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

$$X = \frac{1}{8}\begin{bmatrix} 4 & -1 & -5 \\ 0 & 2 & 2 \\ 0 & 4 & -4 \end{bmatrix}\begin{bmatrix} 12 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$x = 2, y = 3, z = 1$

b $x + 2y + 3z = 13$

$-x - y + 2z = 2$

$-x + 3y + 4z = 26$

Write the equation in the form

$\mathbf{AX} = \mathbf{B}$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \\ -1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 26 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \\ -1 & 3 & 4 \end{bmatrix}^{-1} =$$

$$\frac{1}{26}\begin{bmatrix} 2 & -17 & -7 \\ 6 & 1 & 5 \\ 4 & 5 & -1 \end{bmatrix}$$

$\mathbf{AX} = \mathbf{B}$

$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

$$X = \frac{1}{18}\begin{bmatrix} 10 & -1 & -7 \\ -2 & -7 & 5 \\ 4 & 5 & -1 \end{bmatrix}\begin{bmatrix} 13 \\ 2 \\ 26 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

$x = -3, y = 5, z = 2$

c $x + y = 5$

$y + z = 7$

$z + x = 12$

Write the equation in the form

$\mathbf{AX} = \mathbf{B}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} =$$

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$\mathbf{AX} = \mathbf{B}$

$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$$

$x = 5, y = 0, z = 7$

d $x - y - z = 0$

$5x + 20z = 50$

$10y - 20z = 30$

Write the equation in the form

$\mathbf{AX} = \mathbf{B}$

$$\begin{bmatrix} 1 & -1 & -1 \\ 5 & 0 & 20 \\ 0 & 10 & -20 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \\ 30 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 5 & 0 & 20 \\ 0 & 10 & -20 \end{bmatrix}^{-1} =$$

$$\frac{1}{70}\begin{bmatrix} 40 & 6 & 4 \\ -20 & 4 & 5 \\ -10 & 2 & -1 \end{bmatrix}$$

$\mathbf{AX} = \mathbf{B}$

$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

$$= \frac{1}{70}\begin{bmatrix} 40 & 6 & 4 \\ -20 & 4 & 5 \\ -10 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 50 \\ 30 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

$x = 6, y = 5, z = 1$

e $x = 5, y = 2, z = 4, w = -1$

2 a $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \\ -1 & -4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 17 \end{bmatrix}$

b det(**A**) = 0, so **A** is non-invertible

c **i** $-y + 5z = 15,\ -y + 5z = 15$

ii The two equations are the same

iii $y = 5\lambda - 15$

iv $x = 43 - 13\lambda$

Solutions to technology-free questions

1 a

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix}1 & 0\\2 & 3\end{bmatrix} + \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\2 & 4\end{bmatrix}$$

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix}1 & 0\\2 & 3\end{bmatrix} - \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 0\\2 & 2\end{bmatrix}$$

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \begin{bmatrix}0 & 0\\2 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\2 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\12 & 8\end{bmatrix}$$

b

$$\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0\\8 & 9\end{bmatrix}$$

$$\mathbf{B}^2 = \mathbf{BB} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\mathbf{A}^2 - \mathbf{B}^2 = \begin{bmatrix}1 & 0\\8 & 9\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\8 & 8\end{bmatrix}$$

2 Find the inverse of $\begin{bmatrix}3 & 4\\6 & 8\end{bmatrix}$.

Determinant $= 3 \times 8 - 4 \times 6 = 0$

This is a non-invertible matrix.

If $\mathbf{A} = \begin{bmatrix}x\\y\end{bmatrix}$, then this corresponds to the simultaneous equations:

$3x + 4y = 8$

$6x + 8y = 16$

The second equation is equivalent to the first, as it is obtained by multiplying both sides of the first by 2.

Thus if $x = a$,

$3a + 4y = 8$

$$4y = 8 - 3a$$

$$y = 2 - \frac{3a}{4}$$

The matrices may be expressed as $\begin{bmatrix}a\\2 - \frac{3a}{4}\end{bmatrix}$.

3 a For a product to exist, the number of columns of the first matrix must equal the number of rows of the second. This is true only for **AC**, **CD** and **BE**, so these products exist.

b

$$\mathbf{DA} = \begin{bmatrix}2 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix} = \begin{bmatrix}2\times1+4\times3\\2\times2+4\times-1\end{bmatrix} = \begin{bmatrix}14 & 0\end{bmatrix}$$

$$\det(\mathbf{A}) = 1\times-1-2\times3 = -7$$

$$\mathbf{A}^{-1} = \frac{1}{-7}\begin{bmatrix}-1 & -2\\-3 & 1\end{bmatrix} = \frac{1}{7}\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix} \text{ or } \begin{bmatrix}\frac{1}{7} & \frac{2}{7}\\\frac{3}{7} & -\frac{1}{7}\end{bmatrix}$$

4 $\mathbf{AB} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$

$= \begin{bmatrix} 1\times 1 + -2\times 1 + 1\times 3 & 1\times -4 + -2\times -6 + 1\times -8 \\ -5\times 1 + 1\times 1 + 2\times 3 & -5\times -4 + 1\times -6 + 2\times -8 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$

$\det(\mathbf{C}) = 1\times 4 - 2\times 3 = -2$

$\mathbf{C}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5 Find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Determinant $= 1\times 4 - 2\times 3 = -2$

$$\text{Inverse} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

Multiply by the inverse on the right:

$$\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$$

6 $\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

7 The determinant must be zero.

$$1\times x - 2\times 4 = 0$$

$$x - 8 = 0$$

$$x = 8$$

8 a i $\mathbf{MM} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$$

ii $\mathbf{MMM} = \mathbf{MM}(\mathbf{M})$

$$= \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$$

iii Determinant $= 2\times 3 - -1\times 1 = 7$

$$\mathbf{M}^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

b $\mathbf{M}^{-1}\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{M}^{-1}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 14 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x = 2,\ y = 1$

Solutions to multiple-choice questions

1 B The dimension is number of rows by number of columns, i.e. 4×2.

2 E The matrices cannot be added as they have different dimensions.

3 C

$$\mathbf{D}-\mathbf{C} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -3--3 & 1-1 \\ 2-1 & 3-0 & -1--2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

4 E Multiply every entry by -1.

$$-\mathbf{M} = -\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$$

5 C

$$2\mathbf{M} - 2\mathbf{N} = 2 \times \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 \\ -6 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 8 \\ 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$$

6 A $\mathbf{A} + \mathbf{B}$ will have the same dimension as $\mathbf{A}$ and $\mathbf{B}$, i.e. $m \times n$.

7 E The number of columns of **Q** is not the same as the number of rows of **P**, so they cannot be multiplied.

8 A Determinant $= 2 \times 1 - 2 \times -1$

$$= 4$$

9 E Determinant $= 1 \times -2 - -1 \times 1$

$$= -1$$

$$\text{Inverse} = \frac{1}{-1}\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

10 D

$$\mathbf{NM} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 0 + 2 \times -3 & 0 \times -2 + 2 \times 1 \\ 3 \times 0 + 1 \times -3 & 3 \times -2 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}$$

Solutions to extended-response questions

1 a i The equations $2x - 3y = 3$ and $4x + y = 5$ can be written as

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

ii Determinant of $\mathbf{A} = 2 \times 1 - 4 \times (-3)$

$$= 2 + 12$$

$$= 14$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$$

iii $$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 18 \\ -2 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 9 \\ -1 \end{bmatrix}$$

Therefore $x = \frac{9}{7}$ and $y = -\frac{1}{7}$.

iv The two lines corresponding to the equations intersect at $\left(\frac{9}{7}, -\frac{1}{7}\right)$.

b i The equations $2x + y = 3$ and $4x + 2y = 8$ can be written as

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

ii Determinant of $\mathbf{A} = 2 \times 2 - 4 \times 1$

$$= 4 - 4$$

$$= 0$$

Since the determinant of $\mathbf{A}$ equals zero, $\mathbf{A}$ is a non-invertible matrix and the inverse $\mathbf{A}^{-1}$ does not exist.

c The two lines corresponding to the equations are parallel.

2 a The 2×3 matrix is: $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$

The rows correspond to the semesters and the columns to the forms of assessment.

b The percentages of the three components can be represented in the 3×1 matrix:
$$\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$$

c Multiplying the two matrices gives the semester scores.
$$\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 79.2 \\ 80.4 \end{bmatrix}$$
Notice that multiplication of a 2×3 matrix by a 3×1 matrix results in a 2×1 matrix.

d For Chemistry the result is given by the following multiplication.
$$\begin{bmatrix} 86 & 82 & 84 \\ 81 & 80 & 70 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 83.8 \\ 75.2 \end{bmatrix}$$

e The aggregate of the four marks is 318.6. This is below 320.

f Three marks will be required to obtain an aggregate of marks above 320.

3 **a** The part-time and full-time teachers required for the 4 terms can be shown in a 4×2 matrix. The columns are for the two types of teachers and the rows for the different terms. Hence the matrix is: $\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix}$

b The full-time teachers are paid \$70 an hour and the part-time teachers \$60. This can be represented in the 2×1 matrix: $\begin{bmatrix} 70 \\ 60 \end{bmatrix}$

c The product these two matrices gives the cost per hour for each term.
$$\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} 70 \\ 60 \end{bmatrix} = \begin{bmatrix} 820 \\ 800 \\ 1040 \\ 1020 \end{bmatrix}$$
The cost per hour for term 1 is \$820.
The cost per hour for term 2 is \$800.
The cost per hour for term 3 is \$1040.
The cost per hour for term 4 is \$1020.

d For the technical, catering and cleaning staff, the matrix for the 4 terms is the 4×3 matrix: $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$

e The rate per hour can be represented in the 3×1 matrix: $\begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix}$

f The cost per hour is given by the product.

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix} = \begin{bmatrix} 270 \\ 270 \\ 480 \\ 480 \end{bmatrix}$$

The cost per hour for term 1 is \$270.
The cost per hour for term 2 is \$270.
The cost per hour for term 3 is \$480.
The cost per hour for term 4 is \$480.

g The total cost per hour is given by the sum of the matrices.

$$\begin{bmatrix} 820 \\ 800 \\ 1040 \\ 1020 \end{bmatrix} + \begin{bmatrix} 270 \\ 270 \\ 480 \\ 480 \end{bmatrix} = \begin{bmatrix} 1090 \\ 1070 \\ 1520 \\ 1500 \end{bmatrix}$$

The cost per hour for term 1 is \$1090.
The cost per hour for term 2 is \$1070.
The cost per hour for term 3 is \$1520.
The cost per hour for term 4 is \$1500.

4 Suppose Brad, Flynn and Lina employ x, y and z workers respectively. The there contractors need to supply the warehouse with 310 dresses, 175 slacks and 175 shirts, so x, y and z must satisfy the matrix equation

$$\begin{bmatrix} 3 & 6 & 10 \\ 3 & 4 & 5 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 310 \\ 175 \\ 175 \end{bmatrix}$$

which is in the for $\mathbf{AX} = \mathbf{B}$, there A is the 3×3 matrix, $\mathbf{X}$ is the column matrix of the variables and $\mathbf{B}$ is the column matrix of the numbers required.
The solution is given by: $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$
Use a calculator to find A^{-1}, then multiply by $\mathbf{B}$ to find $\mathbf{X}$.

$$A^{-1} = \frac{1}{20}\begin{bmatrix} -10 & 30 & -10 \\ -5 & 5 & 5 \\ 10 & -18 & 2 \end{bmatrix}$$

$$\mathbf{X} = \frac{1}{20}\begin{bmatrix} -10 & 30 & -10 \\ -5 & 5 & 5 \\ 10 & -18 & 2 \end{bmatrix}\begin{bmatrix} 310 \\ 175 \\ 175 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 10 \\ 15 \end{bmatrix}$$

So Brad need 20 workers, Flynn need 10 workers and Lina need 15 workers.

5 **a** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

$\det(\mathbf{A}) = ad - bc$ and $\det(\mathbf{B}) = eh - fg$.

Then $\det(\mathbf{A})\det(\mathbf{B}) = (ad - bc)(eh - fg)$

$$= adeh + bcfg - adfg - bceh$$

Furthermore $\mathbf{AB} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$.

and $\det(\mathbf{AB}) = adeh + bcfg - adfg - bceh$

$\therefore \det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

b A 2×2 matrix is invertible if and only if its determinant is non-zero. Hence if **A** and **B** are invertible then so is **AB**

6 True for $n = 1$

Assume true for k

That is, $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}$

To prove true for $k + 1$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^k \times \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & (3^k - 2^k) \\ 0 & 2^{k+1} \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

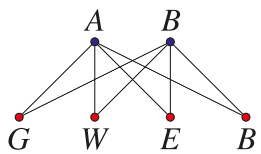
$$= \begin{bmatrix} 3^{k+1} & 3^k \times 1 + 2(3^k - 2^k \\ 0 & 2^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 2^{k+1} \\ 0 & 2^{k+1} \end{bmatrix}$$

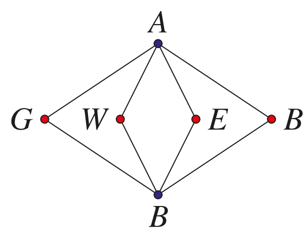
Chapter 12 – Graph Theory

Solutions to Exercise 12A

1 a



b We can move vertex B beneath the utilities to avoid the intersecting edges.



c Each of the two houses has an edges connecting each of the four utilities. This gives the adjacency matrix shown below.

$$\begin{array}{c} \\ A \\ B \\ G \\ W \\ E \\ B \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & G & W & E & B \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2 a i There are three edges connected to Town A. Therefore $\deg(A) = 3$.

ii There are two edges connected to Town A. Therefore $\deg(B) = 2$.

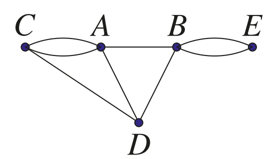
iii There is one edge connected to Town H. Therefore $\deg(H) = 1$.

b

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ H \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & H \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

c No, the graph is not simple as there are two edges joining C and D

3 a

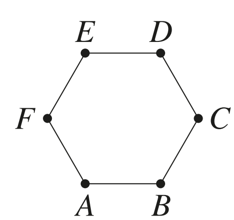


b

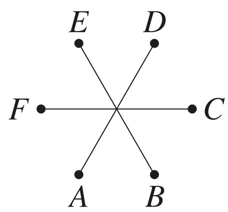
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

c The graph is not sample as there are two edges joining A and C, and also B and E.

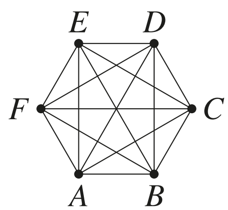
4 a



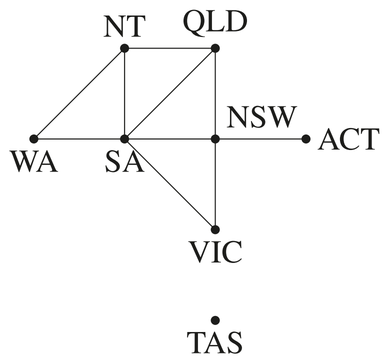
b



c



5



6 **a**

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

b

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

c

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

d

$$\begin{array}{c} \\ A \end{array}\begin{array}{c} A \\ \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

e

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

f

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

g

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

h

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

7 a, b, c, e, f, h

8 **a** There is a loop at vertex v_1.

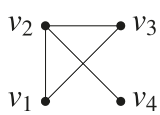
b There are two edges connecting vertices v_1 and v_2.

9 **a** The degree of vertex v_i in a simple graph can be found by adding the entries in row i (or column i).

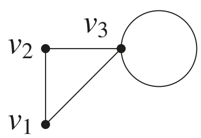
b The total degree of a simple graph can be found by adding all of the entries in the adjacency matrix.

c The number of edges will be half of the sum of the all of the entries in the adjacency matrix.

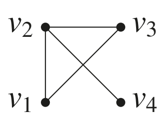
10 **a**



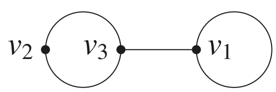
b



c



d



11 **a** One possible isomorphism is

$A \leftrightarrow Z$

$B \leftrightarrow W$

$C \leftrightarrow X$

$D \leftrightarrow Y$

b There is one vertex of degree three in each graph so we must have $A \leftrightarrow Z$. Each of the other vertices simply connects to the vertex of degree three. Therefore B can be identified with any three of the remaining nodes W, X or Y. There will then be two choices for C and then one for D, giving a total of $3 \times 2 \times 1$ choices in all.

c One possible isomorphism is

$A \leftrightarrow X$

$C \leftrightarrow Y$

$D \leftrightarrow W$

$B \leftrightarrow Z$

There are many more possibilities.

d Three possible reasons are:

- Graph G has a vertex of degree 3 and Graph I does not.
- Graph G has a node of degree 1 and Graph I does not.
- Graph G has vertex connected to every other vertex and Graph I does not.

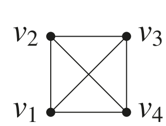
There are many more possibilities.

e Three possible reasons are:

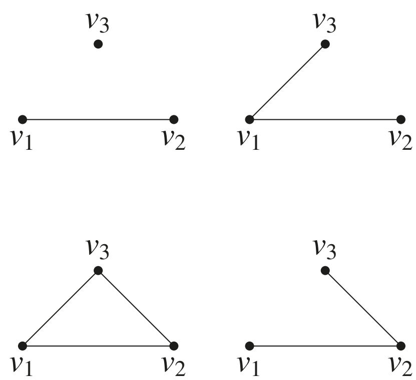
- Graph K has an isolated vertex and Graph I does not.
- Graph K has three vertices of degree 2 and Graph I has four.
- Graph K has three vertices of degree 2 and Graph I has four.

There are many more possibilities.

12



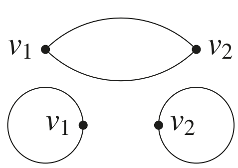
13



14 **a** The total degree of any graph is equal to twice the number of edges. Therefore the total degree must be even.

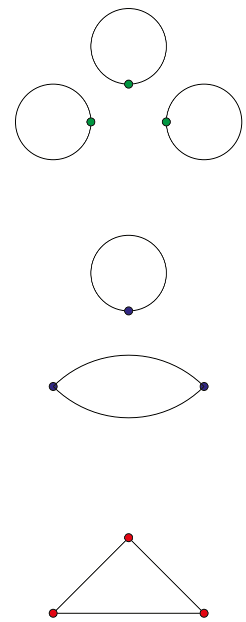
b If a graph had an odd number of vertices of odd degree, then the total degree of the graph would equal to the sum of an odd number of odd numbers. This is will always be an odd number, which contradicts the fact that the total degree of a graph will be even.

15 **a**

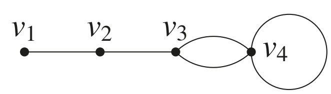


b The total degree of the graph would be 9, which contradicts the fact that the total degree must be even.

c

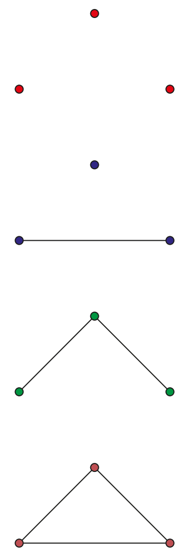


16 **a**

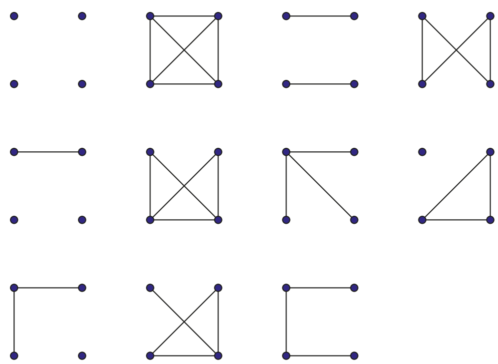


b The sum of the degrees is $2 + 1 + 2 + 2 = 7$, which contradicts the fact that the sum of the degrees must be even. Therefore no such graph exists.

17 Recall that a simple graph has no loops or multiple edges. Therefore a simple graph with three vertices can have at most three edges, or else there will be a multiple edge. Therefore it may have either 0, 1, 2 or 3 edges. There are four non-isomorphic possibilities, and these are shown below. Your solutions may look different.



18

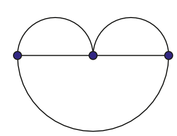


19 If G has an isolated vertex then there is nothing to prove. So we can suppose that G has no isolated vertex. Suppose, by way of contradiction, that G has no vertex of degree 1. Then the degree of each of the n vertices must be at least 2. Therefore, using the handshaking lemma,

$$\begin{aligned} & 2(n-1) \\ &= \text{twice the number of edges} \\ &= \text{sum of the degrees of each vertex} \\ &\geq 2n \end{aligned}$$

This is impossible, since $2n < 2(n-1)$. We conclude that G has some vertex of degree 1.

20 **a**



b If this weren't true, then each of the three vertices would have degree less than or equal to 3. Therefore, the total degree would be less than or equal to $3 \times 3 = 9$, which is a contradiction.

Solutions to Exercise 12B

1 a Vertices v_1 and v_2 have odd degree, so Euler trails must begin and end at either of these vertices. We have:

$$(v_1, v_2, v_3, v_4, v_2)$$
$$(v_1, v_2, v_4, v_3, v_2)$$
$$(v_2, v_3, v_4, v_2, v_1)$$
$$(v_2, v_4, v_3, v_2, v_1).$$

b The graph has no Euler circuit since it has vertices with odd degree (v_1 and v_2).

2 a Vertices v_1 and v_3 have odd degree. Therefore it has no Euler circuit.

b The graph has exactly two vertices of odd degree. Therefore it has an Euler trail beginning and ending at either of these two vertices.

3 a This graph has no Euler circuit as it has vertices with odd degree. It has an Euler trail. One example is

$$v_3, v_4, v_1, v_3, v_2, v_1.$$

b This graph has no Euler circuit as it has vertices with odd degree. It has an Euler trail. One example is

$$v_1, v_2, v_3, v_4, v_1, v_5, v_4, v_2, v_5.$$

c This graph has an Euler trail that begins and ends at the vertices v_2 and v_4 with odd degree. One example is

$$v_2, v_3, v_4, v_1, v_2, v_6, v_7, v_8, v_5, v_6, v_8, v_4$$

d This graph has no Euler circuit has it has vertices with odd degree. This graph has an Euler circuit. One example is

$$v_1, v_3, v_5, v_4, v_2, v_3, v_5, v_6.$$

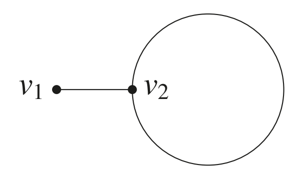
e This graph has an Euler circuit. One example is

$$v_1, v_2, v_3, v_4, v_5, v_3, v_1.$$

f This graph has an Euler circuit. One example is

$$v_5, v_1, v_2, v_3, v_4, v_6, v_5, v_2, v_4, v_5.$$

4 A graph with a vertex with degree 1 cannot have an Euler circuit. Each vertex must have even degree. A graph with a degree 1 vertex can have an Euler trail. The graph below has one vertex of degree 1 and another of degree 3. An Euler trail is v_1, v_2, v_2.



5 a Yes. This polyhedron has exactly two vertices of degree 3. There is a Euler trail starting at one of these and finishing at the other.

b No. This polyhedron has six vertices of degree 3. It cannot be traced without lifting your pencil nor tracing the same edge twice.

c Yes. Every vertex of this polyhedron has degree four. Therefore it has an Euler circuit.

6 We represent each room with a vertex. Two vertices are joined if there is a door connecting the two room that these vertices represent. This gives the graph shown below.

Every vertex has even degree. Therefore this graph has an Euler circuit. One such circuit is $r_1, r_2, r_3, r_5, r_2, r_4, r_5, r_6, r_3, r_1$.

7 **a** Each of the four vertices have odd degree. We require two vertices of even degree to have an Euler trail. We can connect any two of these vertices with **one** edge to achieve this.

b Each of the four vertices have odd degree. We require each of the vertices to have even degree. We can connect any two two pairs of different vertices with **two** edges to achieve this.

8 **a** The triangular grid graph T_4 is shown below.

b When $n = 1$ we have just one vertex of degree 0. This graph has an Euler circuit of length 0. When $n \geq 2$, each vertex of the triangular grid graph has degree $2, 4$ or 6. As each of these numbers is even, there is an Euler circuit.

9 **a** The only grid graphs that have an Euler trial are of size:

$1 \times m$ where $m \geq 1$

$m \times 1$ where $m \geq 1$

$2 \times 2, 2 \times 3, 3 \times 2.$

b To have an Euler circuit our graph must have no vertices of odd. The only grid graphs that avoid having a vertex of odd degree are of size 1×1 (the trail begins and ends at the single vertex) and 2×2.

10 Suppose there is some Euler trail that does not begin and end at the two vertices v_1 and v_2 of odd degree. Then this trail must pass through each of these vertices. Any Euler trail leading into either of these vertices must also exit the vertex. Moreover, the Euler trail must include **every** edge leading into and out of either of these vertices. Therefore the edges leading into and out of these vertices can be paired, which means they must have even degree.

11 Suppose there is some Euler trail that is not a circuit. Then this trail starts and ends at two different vertices v_1 and v_2. If we connect these two vertices by adding an edge, then the graph has an Euler circuit. Therefore each vertex has even degree. If we then delete the added edge, we see that vertices v_1 and v_2 have odd degree. This contradicts the fact that every vertex of the graph was assumed to be even.

Solutions to Exercise 12C

1 A Hamiltonian path must visit every vertex exactly once. There are many answers these two questions. We list just one example.

a $v_1, v_2, v_3, v_8, v_7, v_6, v_5, v_4$

b $v_6, v_1, v_4, v_5, v_8, v_3, v_2, v_7$

2 A Hamiltonian path must visit every vertex exactly once. There are many answers for each of these questions we list just one example.

a v_1, v_2, v_3, v_4

b $v_3, v_5, v_6, v_4, v_1, v_2$

c v_1, v_3, v_2, v_4

d $v_1, v_2, v_3, v_4, v_5, v_6, v_7$

3 A Hamiltonian cycle must start and end at the same vertex and visit every other vertex exactly once. There are many answers for each of these questions we list just one example.

a v_1, v_4, v_3, v_2, v_1

b $v_1, v_2, v_3, v_5, v_6, v_4, v_1$

c $v_1, v_5, v_2, v_3, v_4, v_1$

d $v_1, v_2, v_5, v_7, v_6, v_4, v_3, v_1$

4 **a** The graph cannot have a Hamiltonian cycle as it has two vertices of degree 1, namely v_1 and v_3. For a Hamiltonian cycle to exist, every vertex needs to be of degree 2 or more, as every path leading into a vertex must lead out of the vertex.

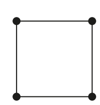
b There are just two Hamiltonian paths: v_1, v_4, v_2, v_5, v_3 and v_3, v_5, v_2, v_4, v_1

c If we add an edge joining v_1 and v_3 we can find a Hamiltonian path. For example, $v_1, v_4, v_2, v_5, v_3, v_1$

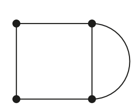
5 **a** Any Hamiltonian cycle contains at most two edges meeting at v_3: the edge going to v_3 and the next edge which goes away from v_3. The other two edges cannot be used. Deleting any pair of edges at v_3 either leaves a disconnected graph or a linear graph, and neither of these has a Hamiltonian cycle.

b There are 8 Hamiltonian paths. Each path can start at any vertex excluding v_3. There are 4 choice. The path must finish at either of 2 vertices on the other side of the of the graph. Therefore, there are 4×2 Hamiltonian paths.

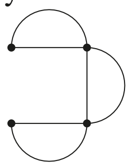
6 **a** This has a Hamiltonian cycle and Euler circuit:



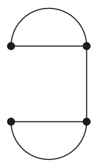
b This has a Hamiltonian cycle but no Euler circuit:



c This has an Euler circuit but no Hamiltonian cycle:

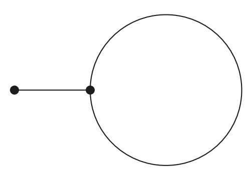


d This has neither Euler circuit nor Hamiltonian cycle:



7 a **False.** Every vertex must have degree at least 2. This is because for any edge leading into the vertex, there must be any edge leading out of the vertex.

b **True.** Exactly one vertex in the graph below has degree 1. This graph clearly has a Hamiltonian path.



c **True.** Exactly two vertices in the graph below have degree 1. This graph clearly has a Hamiltonian path.

d **False.** Suppose there are three vertices of degree 1. Then one of these three vertices is not at the start or end of any given Hamiltonian path. But then the degree of this vertex must be at least 2, as for any edge leading into the vertex, there must be any edge leading out of the vertex.

e **True**. As the vertex of degree 2 has two edges, any path passing into and out of this vertex must use these two edges.

8 a We must find a Hamiltonian cycle. Vertices A, B, G and H all have degree 2. Therefore the two edges that connect to any of these vertices must be part of any Hamiltonian cycle. Therefore, the Hamiltonian cycle must include paths $(F, A, C), (G, B, E), (D, G, B)$ and (D, H, I), or their reversals. Piecing these together, the cycle must include the paths (I, H, D, G, B, E) and (F, A, C). Finally, as F is adjacent to E and C is adjacent to I, we piece these together to obtain the Hamiltonian cycle $I, H, D, G, G, E, F, A, C, I$.

b We must find a Hamiltonian path. There are two vertices of degree one. Any such path must begin and end with these. Therefore the path must include (A, G) and (H, C). Vertices B, E and I all have degree 2. Therefore the two edges that connect to any of these vertices must be part of any Hamiltonian path. Therefore, the Hamiltonian path must include paths $(F, B, D), (D, E, G)$ and (F, I, C), or their reversals. Piecing these together, we obtain the Hamiltonian path $A, G, E, D, B, F, I, C, H$.

9 **a** There are many way of doing this. One solution is given below.

b Any cycle contains edges with only two alternating colours. The edge with the third colour are not included in the cycle.

c Let v be the total number of vertices and let e be the total number of edges. Each of these has degree three. Therefore, the total degree of the graph will be $3v$. By the handshaking lemma, the total degree of the graph is equal to twice the number of edges. Therefore $3v = 2e$. As the right-hand side is even, the left-hand side must also be even. However, as 3 is odd, it must be true that v is even.

d By the previous result, the 3-regular graph must contain an even number of vertices, so the Hamiltonian cycle must be of even length; colour the edges of this cycle by alternating between just two of the three colours. At each vertex, there must be one edge at each vertex that is not included in the Hamiltonian cycle. We colour this edge with the third unused colour.

Solutions to Exercise 12D

1 **a** To find the number of walks of length 2 we must refer to the matrix

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \mathbf{A}^2 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 5 & 4 & 2 & 2 \\ 4 & 5 & 2 & 2 \\ 2 & 2 & 9 & 0 \\ 2 & 2 & 0 & 9 \end{pmatrix} \end{array}$$

i We refer to the entry in row 1 and column 2. This is 4.

ii We refer to the entry in row 3 and column 4. This is 9.

iii We refer to the entry in row 1 and column 1. This is 5.

iv We refer to the entry in row 4 and column 2. This is 2.

b To find the number of walks of length 3 we must refer to the matrix

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \mathbf{A}^2 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 8 & 9 & 20 & 2 \\ 9 & 8 & 20 & 2 \\ 20 & 20 & 8 & 9 \\ 2 & 2 & 9 & 0 \end{pmatrix} \end{array}$$

i We refer to the entry in row 1 and column 2. This is 9.

ii We refer to the entry in row 3 and column 4. This is 9.

iii We refer to the entry in row 1 and column 1. This is 8.

iv We refer to the entry in row 4 and column 2. This is 2.

2 **a** The adjacency matrix is

$$\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \mathbf{A} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \end{array}$$

b To find the number of walks of length 2 we must refer to the matrix

$$\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \mathbf{A}^2 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{pmatrix} 10 & 2 & 6 \\ 2 & 13 & 3 \\ 6 & 3 & 5 \end{pmatrix} \end{array}$$

i We refer to the entry in row 1 and column 2. This is 2.

ii We refer to the entry in row 2 and column 2. This is 13.

iii We refer to the entry in row 3 and column 1. This is 6.

c We must sum the entries in row 1. This gives $10 + 2 + 6 = 18$ paths that start at vertex v_1.

d We must sum the entries in column 3. This gives $6 + 3 + 5 = 14$ walks that end at vertex v_3.

e To find the number of walks of length 3 we must refer to the matrix

$$\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \mathbf{A}^3 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{pmatrix} 12 & 42 & 14 \\ 42 & 12 & 28 \\ 14 & 28 & 12 \end{pmatrix} \end{array}$$

i We refer to the entry in row 1 and column 2. This is 42.

ii We refer to the entry in row 2 and column 2. This is 12.

iii We refer to the entry in row 3 and column 1. This is 14.

f We must sum the diagonal entries. This gives $12 + 12 + 12 = 36$ walks of length 3 that begin and end at the same vertex.

3 a The adjacency matrix of the graph is below. Remember that loops contribute just 1 to the adjacency matrix.

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

b To find the number of walks of length 3 we must refer to the matrix

$$\mathbf{A^3} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 1 & 4 & 1 & 5 \\ 4 & 0 & 4 & 2 \\ 1 & 4 & 1 & 5 \\ 5 & 2 & 5 & 5 \end{pmatrix} \end{array}$$

There is no walk of length 3 from vertex v_2 to v_2.

c To find the number of walks of length 3 we must refer to the matrix

$$\mathbf{A^4} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 9 & 2 & 9 & 7 \\ 2 & 8 & 2 & 10 \\ 9 & 2 & 9 & 7 \\ 7 & 10 & 7 & 15 \end{pmatrix} \end{array}$$

Since there are no zero entries, there is a walk of length 4 between any two pairs of vertices.

4 a The adjacency matrix of this graph is

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

b The are 4 walks of length 2 from any vertex to itself. There are 3 walks of length 2 from any vertex to a different vertex. Therefore,

$$\mathbf{A^2} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 4 & 3 & 4 & 4 & 4 \\ 4 & 4 & 3 & 4 & 4 \\ 4 & 4 & 4 & 3 & 4 \\ 4 & 4 & 4 & 4 & 3 \end{pmatrix} \end{array}$$

c There are $4 \times 3 = 12$ walks of length 3 from any vertex to itself. There are $4 + 3 \times 3 = 13$ walks of length 3 from any vertex to a different vertex. Therefore

$$\mathbf{A^3} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 12 & 13 & 13 & 13 & 13 \\ 12 & 13 & 12 & 12 & 12 \\ 12 & 12 & 13 & 12 & 12 \\ 12 & 12 & 12 & 13 & 13 \\ 12 & 12 & 12 & 12 & 13 \end{pmatrix} \end{array}$$

5 a The adjacency matrix of this graph is

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

b The entry in row 1 and column 1 of

$\mathbf{A}^3$ is the number of walks of length 3 from vertex 1 to vertex 1. There is no such walk, so this entry is 0.

c The entry in row 1 and column 2 of $\mathbf{A}^4$ is the number of walks of length 4 from vertex 1 to vertex 2. There is no such walk, so this entry is 0.

d The entry in row i and column i of $\mathbf{A}^n$ is the number of walks of length n from vertex i to itself. If n is odd, then there is no such walk, so this entry is 0.

6 **a** The adjacency matrix is

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \end{array}$$

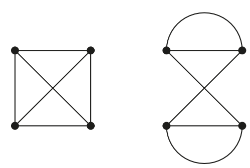
b The resulting matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 104 & 105 & 76 \\ 104 & 105 & 76 \\ 76 & 76 & 57 \end{pmatrix} \end{array}$$

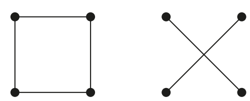
c As the entry in row 1 and column 2 is 104, this is the number of walks of length less than or equal to 4 from vertex v_1 to v_2.

Solutions to Exercise 12E

1 Two possibilities are shown below.



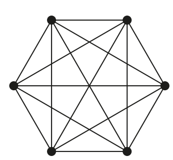
2 The cycle graph C_4 is shown, alongside its complement. Note that vertices are joined in the complement if and only if they are not joined in the original graph.



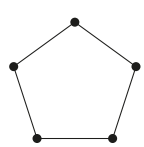
3 **a** Each of the 6 teams plays 5 others. Each match is between two teams, so the total number of matches will be

$$\frac{6 \times 5}{2} = 15.$$

b The complete graph K_6 is shown below.



4 **a** Each of the five teams can play two others as shown below:



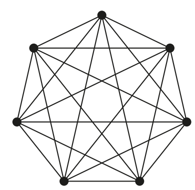
b If each of the five teams could play three others, then the total degree of the corresponding graph would be 5×3. The number of edges would then half of this number, which is not possible.

5 **a** The complete graph K_7 has

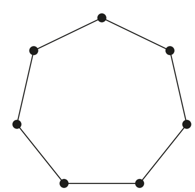
$$\frac{7 \times 6}{2} = 21$$

edges.

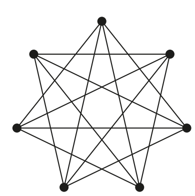
b The complete graph K_7 is shown below:



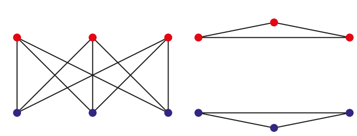
c The cycle graph C_7 is shown below:



The complement of the cycle graph C_7 is shown below:



6 **a** The complete bipartite graph $K_{3,3}$ is shown below, next to its complement



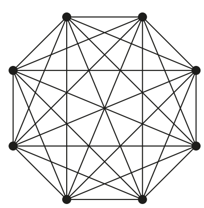
b This statement is not true. The

complement of $K_{3,3}$ consists of two disjoint cycle graphs C_3, and is not bipartite.

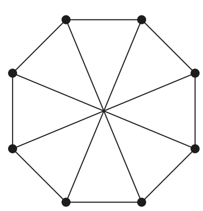
7 The maximum number of handshakes that can take place between 8 people is

$$\frac{8 \times 7}{2} = 28.$$

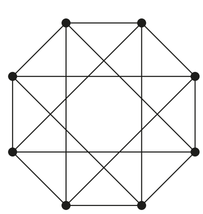
This corresponds to the number of edges in the complete graph K_8 shown below:



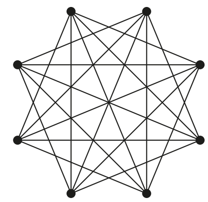
8 **a** A regular graph with eight vertices and degree 3 is shown below.



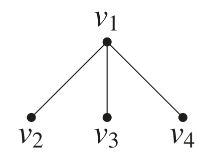
b A regular graph with eight vertices and degree 4 is shown below.



c A regular graph with eight vertices and degree 5 is shown below.



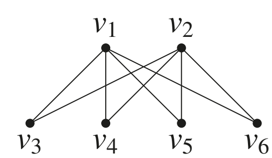
9 **a** The graph of $K_{1,3}$ is shown below.



The adjacency matrix of $K_{1,3}$ is shown below.

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

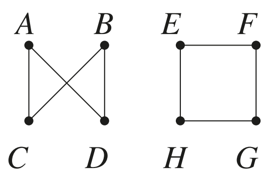
b The graph of $K_{2,4}$ is shown below.



The adjacency matrix of $K_{2,4}$ is shown below.

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

10 The graphs $K_{2,2}$ and C_4 are shown below.

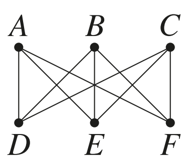


One suitable isomorphism is

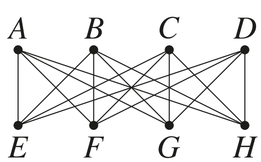
$$A \leftrightarrow E, D \leftrightarrow F, B \leftrightarrow G, C \leftrightarrow H.$$

There are other possibilities.

11 **a** The graph of $K_{3,3}$ is shown below.

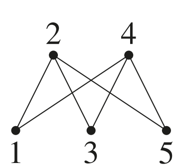


An example of a Hamiltonian cycle in $K_{3,3}$ is A, D, B, E, C, F, A. The graph of $K_{4,4}$ is shown below.



An example of a Hamiltonian cycle in $K_{4,4}$ is $A, E, B, F, C, G, D, H, A$

b The graph of $K_{2,3}$ is shown below. The top row of vertices has been labelled with even numbers and the bottom row with odd numbers. Note that every even number is only connected to odd numbers, and vice versa.



This graph cannot have a Hamiltonian cycle. To see this we note that any such cycle would have to alternate between even and odd numbers. However, this is not possible, as there is unequal number of even and odd numbers.

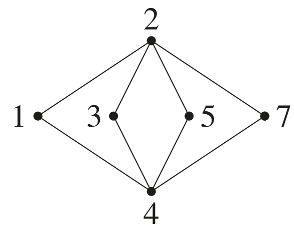
12 As in the previous question, we label the bottom row of vertices with odd numbers and the top row with even numbers. If $m = n$ then there are m odd numbers in the bottom row $\{1, 3, \cdots, 2m - 1\}$ and m even numbers is the both row $\{2, 4, \cdots, 2m\}$. A Hamiltonian cycle can then be found by simply listing the vertices in numerical order before looping back to the first vertex:

$$(1, 2, 3, \ldots, 2n - 1, 2n, 1).$$

If $m \neq n$ then there are m odd numbers in the bottom row and n even numbers is the top row. Any Hamiltonian cycle would then have to consist of an alternating list of even and odd numbers. However, this is not possible, as there is an unequal number of odd and even numbers.

13 **a** Recall that a graph has an Euler circuit if, and only if, the degree of every vertex is even. The graph $K_{m,n}$ has m vertices of degree n and n vertices of degree m. Therefore, the degree of every vertex in $K_{m,n}$ is even if, and only if, m and n are both even.

b It is easiest to find an Euler circuit in $K_{2,4}$ when it is drawn as shown below.



An Euler circuit is

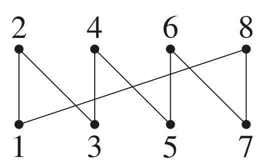
$$(1, 2, 3, 4, 5, 2, 7, 4, 1).$$

There are many others.

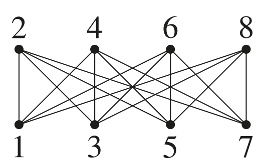
14 a We can label the vertices of the cycle graph C_8 with the numbers

$$\{1, 2, 3, 4, 5, 6, 7, 8\}$$

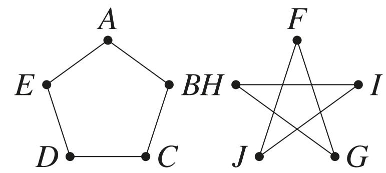
Note that every odd vertex is joined to two even vertices and every even vertex is joined to two even vertices. Therefore we have to disjoint subsets $\{1, 3, 5, 7\}$ and $\{2, 4, 6, 8\}$. The graph is sketch as shown below.



b Every vertex and edge in C_8 shown above appears in the graph of $K_{4,4}$, which is shown below.



15 a The cycle graph C_5 and its complement are shown below. It is clear that the complement is also a cycle graph five five vertices.

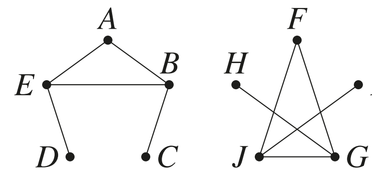


One suitable isomorphism is given by

$$A \leftrightarrow F, B \leftrightarrow G, C \leftrightarrow H, D \leftrightarrow I, E \leftrightarrow J.$$

There are many other re-labelings.

b The simple graph with five vertices show below is also isomorphic to its complement.

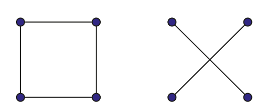


Note that each graph consists of a triangle with two adjoining edges. One suitable relabelling is

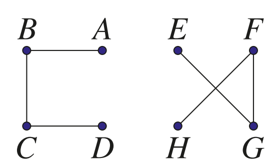
$$A \leftrightarrow F, B \leftrightarrow G, C \leftrightarrow H, D \leftrightarrow I, E \leftrightarrow J.$$

There is actually just one other possible relabelling. Can you find it?

c The cycle graph C_4 cannot be self-complemented as C_4 has 4 edges while its complement has just 2. This is shown below.



d The graph with four vertices shown below is isomorphic to its complement.



Both graph are simple linear graphs.

One suitable relabelling is

$$A \leftrightarrow E, B \leftrightarrow G, C \leftrightarrow F, D \leftrightarrow H,$$

There is just one other possible relabelling. Can you find it?

e Suppose a self-complemented graph has n vertices. If the graph is isomorphic to its complement then the graph and its complement have the same number of edges. The graph and its complement have no edges in common, and their union is equal to the complete graph K_n. The complete graph K_n has $\frac{n(n-1)}{2}$ edges. Therefore the graph has half this number:

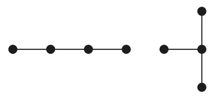
$$\frac{1}{2} \times \frac{n(n-1)}{2} = \frac{n(n-1)}{4}.$$

f A graph with $n = 1$ vertex will be self-complementary. Now consider a self-complementary graph with $n > 1$ vertices. By the previous question, this will have $\frac{n(n-1)}{4}$ edges. The number of edges must a whole number. As the highest common factor of n and $n - 1$ is 1, either n is either divisible by 4 or $n - 1$ is divisible by 4. Therefore, either $n = 4k$ or $n = 4k + 1$, where k is a positive integer, or zero.

16 Suppose G is disconnected. We need to show that $\overline{G}$ is connected. Suppose that a and b are any two vertices in $\overline{G}$. We need to show that there is a path in $\overline{G}$ from a to b. If (a, b) is not an edge in G, then it is an edge in $\overline{G}$. This gives a path in $\overline{G}$ from a to b. On the other hand, suppose (a, b) is an edge in G. Since G is disconnected, we can find a vertex c that is not connected to either a or b. Therefore the edges (a, c) and (c, b) are not in G. However, this means that they are edges in $\overline{G}$. Therefore, a, c, b is a path from a to b in $\overline{G}$.

Solutions to Exercise 12F

1 There are two trees with four vertices:

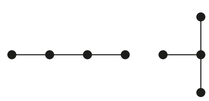


2 **a** There is one tree with one vertex:

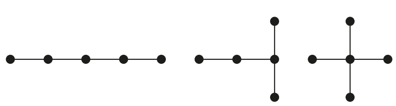
There is one tree with two vertices:

There is one tree with three vertices:

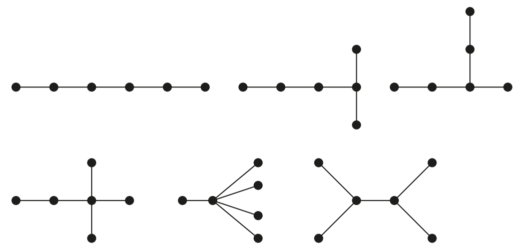
There are two trees with four vertices:



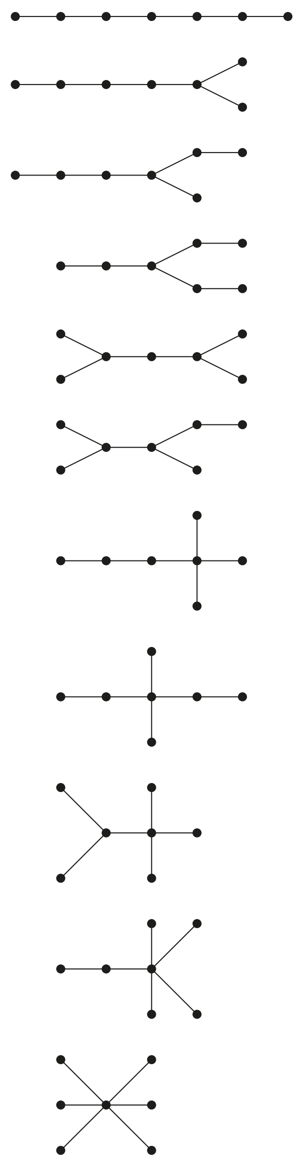
There are three trees with five vertices. These can be found by systematically adding edges the previous two trees and identifying which of these are different.



b There are six trees with six vertices. These can be found by systematically adding edges to the trees with five vertices and identifying which of these are different. Your may look different, but they should be isomorphic to these given below.

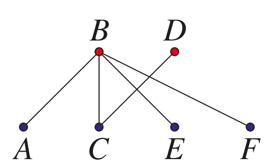


c There are eleven trees with seven vertices. These can be found by systematically adding edges to the trees with five vertices and identifying which of these are different. Your may look different, but they should be isomorphic to these given below.

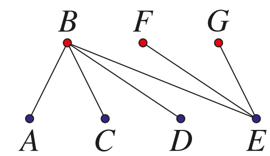


3 We can find the disjoint subsets by colouring the vertices with colours that alternate between red and blue.

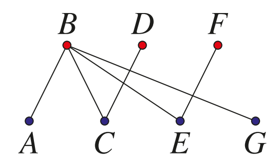
a The disjoint sets are $\{A, C, E, F\}$ and $\{B, D\}$. The graph is shown below.



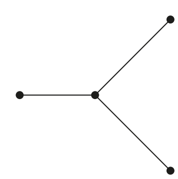
b The disjoint sets are $\{A, C, D, E\}$ and $\{B, F, G\}$. The graph is shown below.



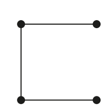
c The disjoint sets are $\{A, C, E, G\}$ and $\{B, D, F\}$. The graph is shown below.



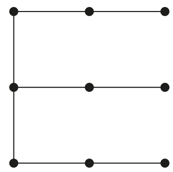
4 **a** By deleting the right most edge belonging to the cycle gives this spanning tree. There are other possibilities.



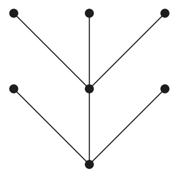
b By deleting the right most edge and the diagonal we obtain a spanning tree. There are other possibilities.



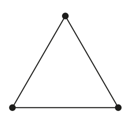
c By sequentially deleting edges belonging to cycles, we obtain the following spanning tree. There are many other possibilities.

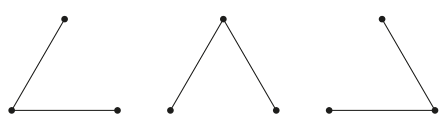


d By sequentially deleting edges belonging to cycles, we obtain the following spanning tree. There are many other possibilities.

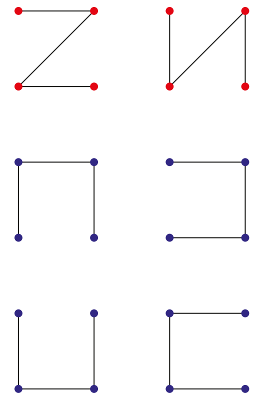


5 **a** The graph of C_3 is shown below. We can delete any of the three edges to give a spanning tree. Each of these are isomorphic, as each is just a linear graph with three vertices.

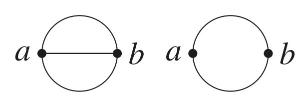




b We obtain six spanning trees. One group of isomorphic trees are shown in red and the other other group is shown in blue.



6 Consider any tree. Suppose that the addition of the edge (a, b) from vertex a to vertex b forms two different cycles. These two cycles are shown below on the left. Note that there may be other vertices in these cycles. These have not been included. If we delete this edge, then this would restore the original tree. Note that this leaves a cycle intact. However this means at the original tree has a cycle, which is a contradiction.



7 **a** Consider any path of maximal length in this tree. This path is not a cycle as a tree has no cycles. The two endpoints of this path must have degree 1, for if they did not have degree 1 then we can create a longer path by including the additional vertices to which the endpoints are connected. This would contradict the fact that the path is of maximal length

b $\boxed{P(n)}$

Let $P(n)$ be the proposition that a tree with n vertices has $n - 1$ edges.

$\boxed{P(1)}$

For the base case, we consider a tree with $n = 1$ vertices. This tree has $1 - 1 = 0$ edges. Therefore the base case is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true, so that every tree with k vertices has $k - 1$ edges.

$\boxed{P(k+1)}$

We now prove that $P(k+1)$ is true. Consider any tree with $k + 1$ vertices. Delete one vertex of degree 1 along with the edges that meets this vertex. This new tree has k vertices. By assumption, it has $k - 1$ edges. Restore the graph by adding the deleted vertex and edge. There are now $(k - 1) + 1 = (k + 1) - 1$ edges. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

8 Take any two vertices u and v in the same connected graph. Since the graph is connected, there is a walk between vertices u and v. Consider the walk of minimal length. Suppose this walk visited some vertex w twice. Then this walk is of the form

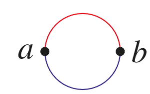
$$u, \ldots, w, \ldots, w, \ldots, v.$$

We can delete the part of the walk from w to itself. This gives the path

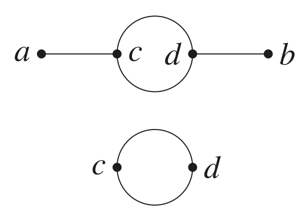
$$u, \ldots, w, \ldots, v.$$

The length of the new walk is less than the length the original walk. This contradicts the fact that the walk had minimal length.

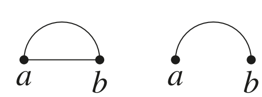
9 **a** Suppose the connected graph is not a tree. Then it must have a cycle. Suppose the cycle contains vertices a and b. Therefore there are two path from a to b: one going way way around the cycle and another going the other way around the cycle.



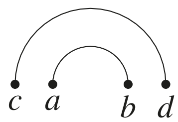
b Suppose there are two different paths from vertex a to vertex b. Then there are two vertices c and d at which these paths first diverge and then first converge. We can then construct a cycle from c by going along the first path to d and then returning to b along the second path. This contradicts the fact that the tree has no cycles.



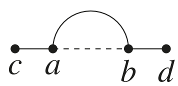
c Suppose the connected graph is not a tree. Then the tree has a cycle. Delete any edge from this cycle. Suppose this connects vertices a and b. We will show that the resulting graph is still connected.



Take any two vertices in the graph c and d in the graph. As the original graph is connected, there is some path from c to d. If the path does not contain the deleted edge, then there is still a path from c to d. This is shown below.

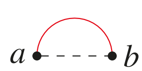


If the path does contain the deleted edge, then we can travel around the *other* part of the circuit containing vertices a and b. This is shown below.

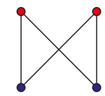


Therefore, when an edge belonging to a cycle is deleted, the graph is still connected.

d Suppose that deleting some edge does not disconnect the graph. Suppose the deleted edge joins points a and b. The deleted edge is shown as a dashed line below. As the graph is not disconnected, there is some other path from a to b. This is shown below in red. However, this means that the graph has a cycle consisting of the red path and the deleted edge. This contradicts the fact that the graph is a tree.



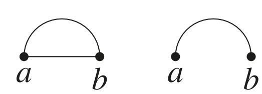
10 a There are many examples. The bipartite graph $K_{2,2}$ is not a tree as it has a cycle. Indeed, it is actually isomorphic to the cycle graph C_4.



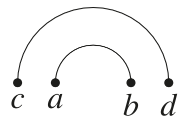
b We show that each vertex in the tree can be placed in one of two sets E or O and that every edge in the tree joins a vertex in E to a vertex in O. Start with any vertex v in the tree. Put vertex v in set E. Take any other vertex u in the tree. There is a unique path from v to u. If the length of this path is even, place the vertex in E. If the length of the path is odd, place it in O. Note that any vertex that joins u will be placed in the alternate set. Therefore, every edge in the three connects a vertex in O to a vertex in E.

11 Recall that there is a path joining any two vertices in a connected graph.

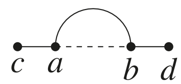
a The argument here is the same as in **8 c**. We repeat the argument here. Delete any edge from a cycle. Suppose this edge connects vertices a and b. We will show that the resulting graph is still connected.



Take any two vertices in the graph c and d in the graph. As the original graph is connected, there is some path from c to d. If the path does not contain the deleted edge, then there is still a path from c to d. This is shown below.



If the path does contain the deleted edge, then we can travel around the *other* part of the circuit containing vertices a and b. This is shown below.



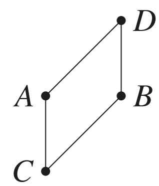
Therefore, when an edge belonging to a cycle is deleted, the graph is still connected.

b If the graph is already a tree then the graph itself is a spanning tree. If the graph is not a tree, then it has cycles. Delete one edges from each cycle so that there are no more cycles. The resulting cycle-free graph still contains all of the vertices from G. Moreover, by part **a**, the graph is still connected. Therefore it is a spanning tree for G.

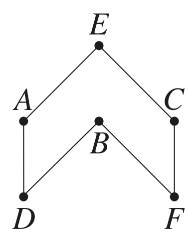
Solutions to Exercise 12G

1 There are many ways to answer this question. We have only drawn one solution and yours may look different.

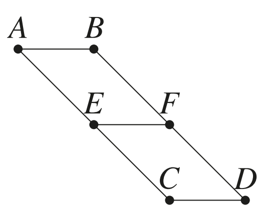
a We have moved vertex D upwards to eliminate the crossing.



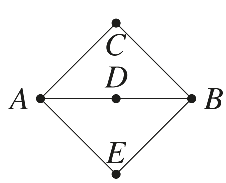
b We have moved vertex D upwards to eliminate the crossing.



c We have moved vertices C and D downwards to eliminate the crossing.



d We have rotated C and E clockwise to eliminate the crossing. We also moved D upwards so that the graph was more symmetrically drawn.



2 For each of these questions remember to count the outside unbounded face.

a As $v = 8, e = 12$ and $f = 6$,

$$v - e + f = 8 - 12 + 6 = 2,$$

as required.

b As $v = 6, e = 12$ and $f = 8$,

$$v - e + f = 6 - 12 + 8 = 2,$$

as required.

c As $v = 7, e = 12$ and $f = 7$,

$$v - e + f = 7 - 12 + 7 = 2,$$

as required.

d As $v = 7, e = 9$ and $f = 4$,

$$v - e + f = 7 - 9 + 4 = 2,$$

as required.

3 **a** If v, e and f are all odd, then $v - e + f$ will also be odd. This contradicts the fact that 2 is even.

b Suppose that $v = 4a, e = 4b$ and $f = 4c$ for integers a, b, c. Then

$$\begin{aligned} v - e + f &= 2 \\ 4a - 4b + 4c &= 2 \\ 4(a - b + c) &= 2 \\ 2(a - b + c) &= 1. \end{aligned}$$

The left-hand side is even, while the right-hand side is odd. This is a contradiction.

4 **a** For this grid graph, $v = 12, e = 17$ and $f = 7$ so that

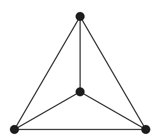
$$v - e + f = 12 - 17 + 7 = 2,$$

as required.

b For the $m \times n$ grid graph, we find that $v = mn$, $e = 2mn - m - n - 1$ and $f = (m-1)(n-1)$. Therefore,

$$\begin{aligned} v - e + f \\ &= mn \\ &- (2mn - m - n - 1) \\ &+ (m-1)(n-1) \\ &= mn \\ &- 2mn + m + n + 1 \\ &+ (mn - m - n + 1) \\ &= 2. \end{aligned}$$

5 a Shown below is a simple, connected and planar graph with 4 vertices and 6 edges. Your graph may look different.

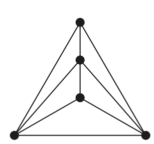


b If the graph were planar, then $e \leq 3v - 6$. However, for this graph

$$3v - 6 = 3(4) - 6 = 6 < e.$$

Therefore, no such graph exists.

6 a Show below is a simple, connected and planar graph with 5 vertices and 9 edges. Your graph may look different.



b If the graph were planar, then $e \leq 3v - 6$. However, for this graph

$$3v - 6 = 3(5) - 6 = 9 < e.$$

Therefore, no such graph exists.

7 a For a cube, $v = 8$, $e = 12$ and $f = 6$ so that

$$v - e + f = 8 - 12 + 6 = 2,$$

as required. For a tetrahedron, $v = 4$, $e = 6$ and $f = 4$ so that

$$v - e + f = 4 - 6 + 4 = 2,$$

as required.

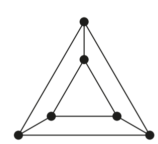
b We find that

$$\begin{aligned} v - e + f &= 2 \\ v - 30 + 12 &= 2 \\ v - 18 &= 2 \\ v &= 20 \end{aligned}$$

c We find that

$$\begin{aligned} v - e + f &= 2 \\ 12 - e + 20 &= 2 \\ 32 - e &= 2 \\ e &= 30 \end{aligned}$$

8 a The polyhedron graph of the triangular prism is shown below. Your graph may look different.

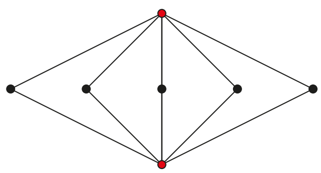


b For this graph, we have $v = 6$, $e = 9$ and $f = 5$. Remember to count the unbounded outside face! Therefore,

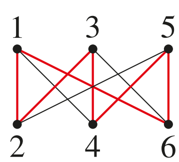
$$v - e + f = 6 - 9 + 5 = 2,$$

as expected.

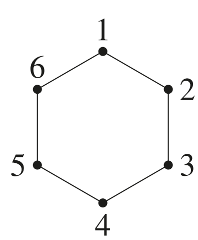
9 We draw the complete bipartite graph $K_{2,m}$ so that the two red vertices in the first set are drawn on either side of the set of m vertices, which we have drawn in black.



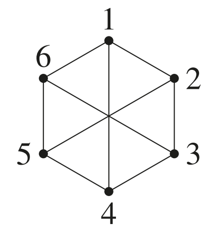
10 **a** The graph $K_{3,3}$ is shown below. The red edges show that C_6 is a subgraph. The graph of $K_{3,3}$ is shown below.



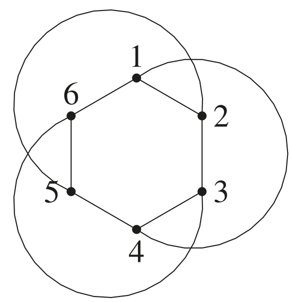
Shown below is the subgraph C_6 drawn as a planar graph.



To form $K_{3,3}$, we still need to add three edges: $(1, 4), (3, 6), (5, 2)$. At least two of the edges must be either inside the loop or outside of the loop. Any pair of these edges inside of the loop cross, as shown below.



Any pair of these edges outside of the loop also cross, as shown below.



This shows that $K_{3,3}$ is now planar.

b In the house utility problem, we need to connect each of three houses which we label as $\{1, 3, 5\}$ to three utilities, which we label as $\{2, 4, 6\}$. The previous question show that we cannot do this without the connecting pipes crossing.

c If $m, n \geq 3$, then $K_{3,3}$ is a subgraph of $K_{m,n}$. If $K_{m,n}$ is planar, then every subgraph would also be planar. However, the subgraph $K_{3,3}$ is not planar, which is a contradiction.

11 **a** If K_5 is a planar graph, then it would be a simple connected and planar graph. Therefore we would have $e \leq 3v - 6$. However, the number of edges in K_5

$$e = \frac{5 \times 4}{2} = 10$$

and the othernumber of vertices is

$v = 5$. Therefore

$$3v - 6 = 3(5) - 6 = 9,$$

which is less than the number of edges. Therefore, K_5 is not a planar graph.

b If $n \geq 5$, then K_5 is a subgraph of K_n. If K_n is planar, then every subgraph would also be planar. However, the subgraph K_5 is not planar, as we showed in the previous example, which is a contradiction.

12 **a** Here is one way of thinking about this question. Suppose there are x squares. Let's count the number of right angles belonging to the x squares. Each square has 4 right angles, therefore there are $4x$ right angles. At each of the v vertices, there are 3 right angles. Therefore there are $3v$ right angles. Since

$$4x = 3v \Leftrightarrow x = \frac{3v}{4}.$$

b Let's count the number of sixty degree angles belonging to the y triangles. Each triangle has 3 angles, therefore there are $3y$ sixty degree angles. At each of the v vertices, there is 1 sixty degree angle. Therefore there are v sixty degree angles. Since

$$3y = v \Leftrightarrow y = \frac{v}{3}.$$

c The total number of edges belonging to squares is

$$4 \times \frac{3v}{4} = 3v$$

The total number of edges belonging to trianlges is

$$3 \times \frac{v}{3} = v.$$

However, each edge belongs to two faces, therefore the total number of edges must be

$$e = \frac{3v + v}{2} = 2v.$$

d Using Euler's formula,

$$\begin{aligned} v - e + f &= 2 \\ v - (2v) + (\tfrac{3v}{4} + \tfrac{v}{3}) &= 2 \\ \tfrac{v}{12} &= 2 \\ v &= 24 \end{aligned}$$

Therefore, there are 24 vertices, 48 edges and 26 faces.

e We do this question in the same way as the previous three parts. Let v be the total number of vertices.
Step 1. Count hexagons. Suppose there are x squares. Let's count the number of angles belonging to the x hexagons. Each hexagon has 6 angles, therefore there are $6x$ angles belonging to hexagons. At each of the v vertices, there are 2 angle belonging to hexagons. Therefore, there are $2v$ angles belonging to hexagons. Since

$$2v = 6x \Leftrightarrow x = \frac{v}{3}.$$

Step 2. Count pentagons. Let's count the number of angles belonging to the y pentagons. Each pentagon has 5 angles, therefore there are $5y$ angles belonging to pentagons. At each of the v vertices, there is 1 angle belonging to a pentagon. Therefore, there are v angles belonging to pentagons. Since

$$5y = v \Leftrightarrow y = \frac{v}{5}.$$

Step 3. Count edges. The total number of edges belonging to hexagons

$$6 \times \frac{v}{6} = v$$

The total number of edges belonging to pentagons

$$5 \times \frac{2v}{5} = 2v.$$

However, each edges belongs to two faces, therefore the total number of edges must be

$$e = \frac{2v + v}{2} = \frac{3v}{2}.$$

Step 4. Use Euler's formula. We find that

$$\begin{aligned} v - e + f &= 2 \\ v - \tfrac{3v}{2} + (\tfrac{v}{3} + \tfrac{v}{5}) &= 2 \\ \tfrac{v}{30} &= 2 \\ v &= 60 \end{aligned}$$

Therefore, there are 60 vertices, 90 edges and 32 faces.

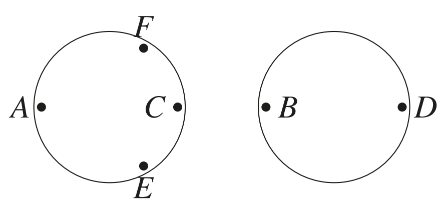
Solutions to Exercise 12H

1 a We will start at vertex A and proceed clockwise around the first cycle until we reach a vertex belonging to the second cycle. We then proceed clockwise all of the way around the second cycle before continuing on the first cycle. This gives the Euler circuit

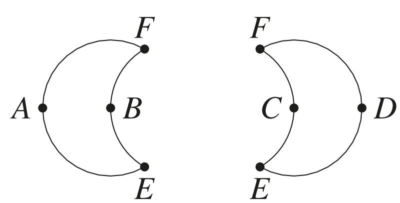
$$(A, F, C, E, B, F, D, E, A).$$

There are many other possibilities.

b There are two different cycle splittings. The first of these is shown below.



The second of these is shown below.

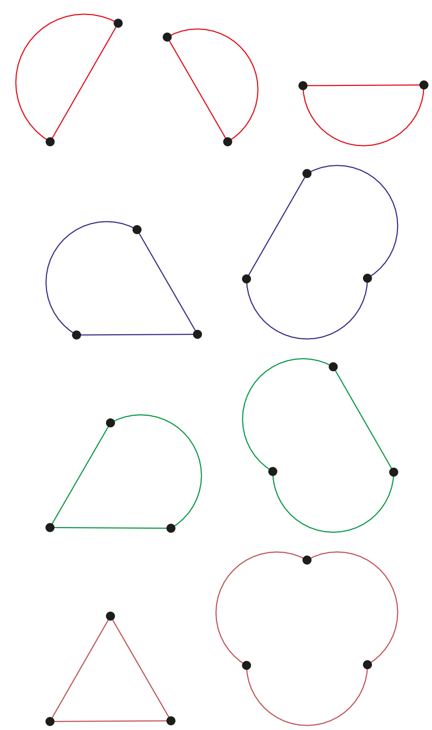


.

2 a We will start at vertex A and proceed clockwise around the first cycle until we reach a vertex belonging to the second cycle. We then proceed clockwise all of the way around the second cycle before continuing on the first cycle. This gives the Euler circuit

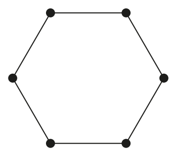
$$(A, B, C, A, B, C, A).$$

b We have drawn the four different cycle splittings in a different colour below. To save effort we have not labelled all of the vertices.

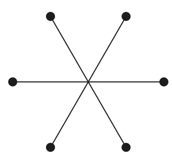


Solutions to Technology-free questions

1 **a** When six people seated at round table shake hands with those sitting on either side we can represent this using the graph below.

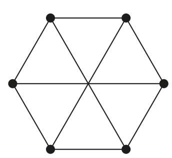


When six people seated at round table shake hands with those sitting opposite can represent this using the graph below.



i

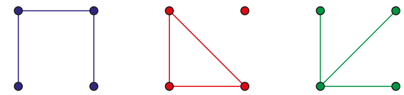
ii When six people seated at round table shake hands with three others we can represent this using the graph below.



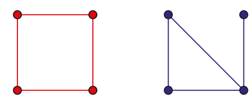
b If seven people were to shake hand with three other people, then by the handshaking lemma, the corresponding graph would have $\frac{7\times 3}{2} = \frac{21}{2}$ edges, which is not possible.

2 **a** A **simple graph** is a graph with no loops or multiple edges.

b There are only three non-isomorphic graphs with four vertices and three edges. Even though your graphs may look different to these, make sure that they are isomorphic to those shown here.

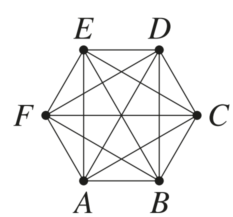


c There are only two non-isomorphic graphs with four vertices and four edges. Even though your graphs may look different to these, make sure that they are isomorphic to those shown here.



d Any simple graph with four vertices must be a subgraph of the complete graph K_4, which has 6 edges. So for any graph with 7 edges, at least two of these edges must connect the same two vertices. Therefore the graph wouldn't be simple.

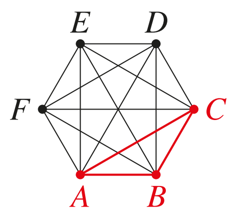
3 **a** The complete graph K_6 is shown below.



b The Hamiltonian path must begin at A, end at B and visit every other vertex exactly once. Therefore, the path must be of the form A, W, X, Y, Z, B

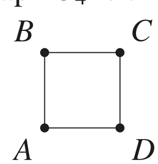
where the middle four letters is some arrangement of the letters C, D, E, F. There are $4 \times 3 \times 2 \times 1 = 24$ ways of arranging these letters.

c The triangle subgraph graph is shown in red below:



d There is a triangle subgraph for every choice of 3 vertices from the 6 available. There are ${}^6C_3 = 20$ of these.

4 **a** The cycle graph C_4 is shown below.



There are 2 paths of length 2 are there from vertex A to vertex C (one going clockwise, the other anti-clockwise).

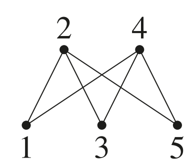
b There are no paths of length 3 are there from vertex A to vertex C. Any path starting at A with odd length will end at either B or D.

c The adjacency matrix of C_4 is shown below:

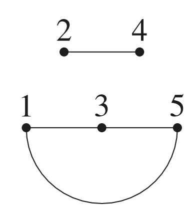
$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

d Consider paths of length 99 from vertex A to vertex A. There are no such paths, as any path starting at A with odd length will end at either B or D. The same argument is true of all vertices in C_4. Therefore the diagonal entries of $\mathbf{A}^{99}$ are all zero.

5 **a** The graph of $K_{2,3}$ is shown below. The top row of vertices has been labelled with even numbers and the bottom row with odd numbers. Note that every even number is only connected to odd numbers, and vice versa.



b Below we have sketch the complement of $K_{2,3}$. Now every odd number is only connected to every odd number and every even number is only connected to every even number.



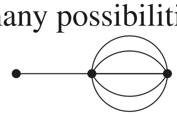
c The quickest way to see this is to note that the complement of $K_{m,n}$ is the union of the complete graph K_m and the complete graph K_n. As K_m has $\frac{m(m-1)}{2}$ edges and K_n has $\frac{n(n-1)}{2}$. Adding these together gives the result.

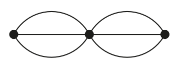
6 a Let v be the number of vertices so that $2v$ is the number of edges and $2v - 1$ is the number of faces. By Euler's formula,

$$\begin{aligned} v - e + f &= 2 \\ v - (2v) + (2v - 1) &= 2 \\ v - 1 &= 2 \\ v &= 3. \end{aligned}$$

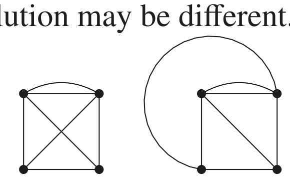
Therefore the number of vertices is 3, the number of edges is 6 and the number of faces is 5.

b Two examples are shown below. There are many possibilities.





7 a To show that this is planar we can elongate one of the diagonals so that it wraps around the graph instead. You solution may be different.

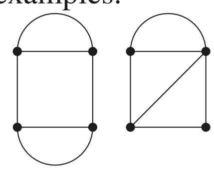


b For this graph $v = 4, e = 7$ and $f = 5$ so that

$$v - e + f = 4 - 7 + 5 = 2,$$

as expected.

8 a The two graphs below have 4 vertices, 6 edges and 4 faces. There are many examples.



b There are many ways to show that these are not isomorphic. We give only three.

- Second graph has a vertex of degree 2. The first graph has no vertex of degree 2.
- Every vertex on the first graph has degree 3 (i.e. it is regular). The second graph of vertices with different degrees.
- The first graph has no Euler trail, as it has two vertices with odd degree. The second graph has an Euler trail, as it has exactly two vertices of odd degree.

Solutions to multiple-choice questions

1 C This is **false** since the total degree of a graph is an even number. If there was an odd number of vertices with odd degree then the total degree would be odd.

2 D The graph will have an Euler circuit if (and only if) each of the vertices has an odd degree. We could determine the degree of each vertex by adding the entries in its corresponding row. We find that $\deg(v_1) = 2, \deg(v_2) = 3, \deg(v_3) = 3$ and $\deg(v_4) = 2$. Therefore vertices v_2 and v_3 have odd degree. Therefore we add edge (v_2, v_3).

3 D The Euler train **must** begin and end at the vertices with odd degree. Only vertices v_2 and v_3 have odd degree.

4 E For Graphs A, B, C and D, every vertex has even degree. Graph E has two vertices of odd degree, so it does not have an Euler circuit.

5 E We can find the original graph by finding the complement of the graph drawn. To find the complement, we join any two vertices that are not connected. After doing this, we obtain $K_{3,3}$ as the original graph.

6 B Each of the vertices has even degree degree except vertices B and E. Therefore we must add another edge connecting these vertices.

7 B Each vertex of the complete graph K_4 has odd degree, and likewise for K_6. Each vertex of $K_{1,3}$ has odd degree, and likewise for $K_{3,3}$. Therefore these graph do not have an Euler circuit. On the other hand, each of the 5 vertices in the complete graph K_5 is connected to the 4 other vertices. Therefore each vertex has even degree so that this graph has an Euler circuit.

8 A Any Hamiltonian cycle must alternate between vertices in one disjoint set and the other. Therefore the two disjoint sets must have the same same. Therefore, we require that $m = n$.

9 B We first have to form the adjacency matrix. This gives:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

We then calculate

$$\mathbf{A}^6 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 91 & 65 & 90 & 65 \\ 65 & 58 & 65 & 58 \\ 90 & 65 & 91 & 65 \\ 65 & 58 & 65 & 58 \end{pmatrix} \end{array}$$

Therefore, the number of paths of length 6 from vertex A to vertex A is 91.

10 A The union of a simple graph and its complement will be the complete graph K_7. The complete graph has $\frac{7\times 6}{2} = 21$ edges. Therefore the complement must have $21 - 11 = 10$

edges.

11 D The complete bipartite graph $K_{m,n}$ has mn vertices and $m + n$ vertices. Therefore $mn = 24$ and $m + n = 10$. That is, we want to whole numbers whose sum is 10 and whose product is 24. The two numbers are 4 and 6 so that the complete graph is $K_{4,6}$.

12 D A tree is a cycle-free connected graph. Graphs A, B and C each have a cycle. Graph E is disconnected. Graph D is cycle-free and connected. Therefore it is a tree.

13 E Item **A** is true as every tree is a connected graph by definition.
Item **B** is true as no tree has a cycle by definition.
Item **C** is true as every tree is a bipartite graph. This is true because we can colour the vertices of a tree using two alternating colours.
Item **D** is true as adding an edge to a tree will create a cycle. This is because there is a path between any two vertices in a tree. Adding an edge will then complete a cycle.
Item **E** is false as a tree does *not* have more edges than vertices. A tree always has more more vertex than it has edges.
f

14 D Every connected graph with 6 vertices has a spanning tree with $6 - 1 = 5$ edges. The graph has 13 edges. Therefore $13 - 5 = 8$ edges must be deleted.

Solutions to extended-response questions

1 a The adjacency matrix $\mathbf{A}$ of this graph is

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

b Every walk of length 3 from vertex A to itself will be of the form A, X, Y, A. The are four choices for X (namely B, C, D, E) leaving only 3 choices for Y. This gives a total of $4 \times 3 = 12$ choices.

c **i** The number of walks of length 1 from any vertex to itself is 0. Therefore each of the diagonal entries of A is 0. Therefore, $\text{tr}\,(\mathbf{A}) = 0$

ii The number of walks of length 2 from one vertex to itself is given by the corresponding diagonal entry of $\mathbf{A}^2$. If these are all added, then we get the total number of walks of length 2 from a vertex to itself.

Every walk of length 2 is of the form X, Y, X, and this corresponds to the edge (X, Y). Now consider any edge (X, Y). This corresponds to exactly two walks of length 2, namely X, Y, X and Y, X, Y. Therefore the total number of walks of length 2 is twice the number of edges.

iii The number of walks of length 3 from one vertex to itself is given by the corresponding diagonal entry of $\mathbf{A}^3$. If these are all added, then we get the total number of walks of length three from a vertex to itself.

Every walk of length 3 is of the form X, Y, Z, X and this corresponds to the triangle XYZ in the graph. However, there are 6 walks of length 3 around this triangle. That is,

$(X, Y, Z, X), (Y, Z, X, Y), (Z, X, Y, Z)$

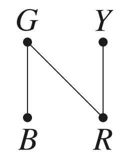
and the walks going in the opposite direction:

$(X, Z, Y, X), (Y, X, Z, Y), (Z, Y, Z, Z)$

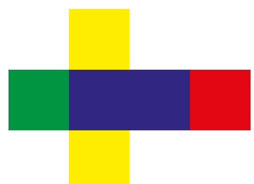
f Therefore the total number of walks of length 3 is six times the number of triangles in the graph.

d Suppose by way of contradiction that this graph can be two coloured. Suppose that the trace of $\mathbf{A}^n \neq 0$ where n is odd. Then (at least) one of the entries along the diagonal of $\mathbf{A}^n$ is not zero. These means that there is an odd length path from some vertex to itself. Along this path the colours would alternate between two colours. But this is not possible, as the path has an odd length (i.e. the vertex would have two different colours).

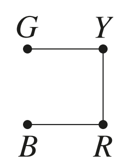
2 a As a green face is opposite a blue face we draw an edge from G to B. As a green face is opposite a red face, an edge is drawn from G to R. As a red face is opposite a yellow face, we draw an edge from R to Y. This gives the graph shown below.



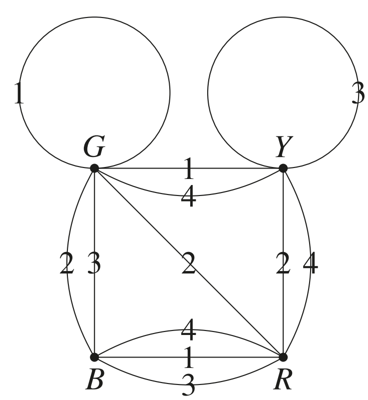
b A green face is opposite a blue face. Another blue face is opposite a red face. A yellow face is opposite a second yellow face. An example of a coloured net for Cube 3 is show below.



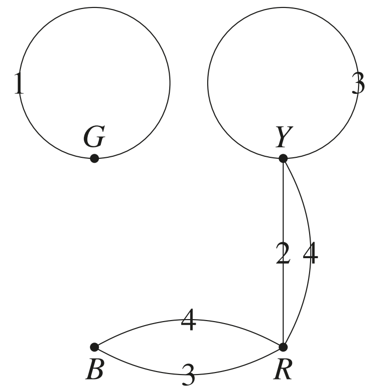
c From the graph we know that a green face is opposite one of the yellow faces and the blue face is opposite one of the red faces. This leave one of the yellow faces and one of the red faces, and these must be paired. Therefore we add edge (Y, R) to complete the graph.



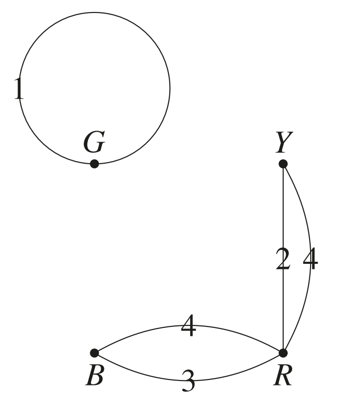
d The union of the four graphs is shown below.



e If the loop labelled 1 that joins G to G is used, then no other edge joining G can be used, and no other edge labelled 1 can be used. This leaves the graph shown below.

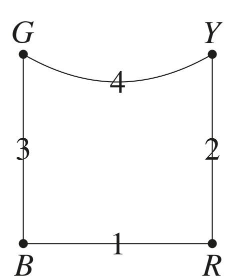


Only one edge is labelled 2, so this must be used. Therefore we can't use the loop labelled 3, since the degree of vertex Y must be be 2. Therefore we delete this loop to give the graph below.

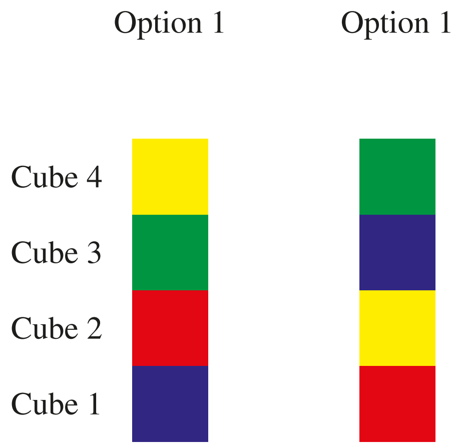


There is only one edge labelled 3, so this must be used too. Now me must select which of the edges labelled 4 to use. However, no matter which of the two edges labelled 4 that we choose, the degree of vertex R will be 3.

f We need to identify a subgraph with four edges labelled 1, 2, 3 and 4 and four vertices of degree two. One such subgraph is shown below. One can also show that there are no others.



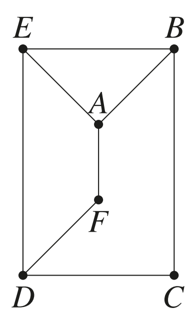
g We consider the back of the tower. We see that Cube 1 must be red or blue, Cube 2 must be red or yellow, Cube 3 must be green or blue and Cube 4 must be yellow or green. The two possible colourings of the back of the tower are shown below. Note that the front of the tower must be coloured with the alternate choice.



3 A planar graph G is described by the adjacency matrix below.

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

a There are many mays to draw this graph. All that matters is that the vertices are connected in the correct way. Moreover, as the graph is planar, its edges should not cross.

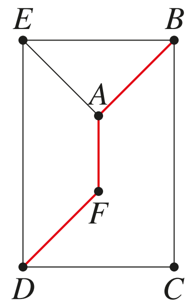


b We see that $v = 6$, $e = 8$ and $f = 4$ (remember to include the unbounded face). Therefore,

$$v - e + f = 6 - 8 + 4 = 2,$$

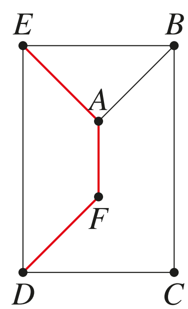
as required.

c Suppose G had a Hamiltonian cycle that included edge (A, B). Since the degree of vertex F is 2, the cycle must also include D, F, A (or the reversal of this). Piecing these together, the cycle must include D, F, A, B (or the reversal of this).

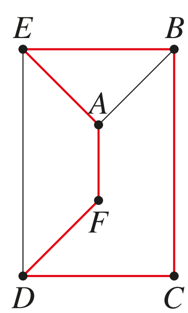


However, vertices E and C are on either side of this path, so it is impossible for the cycle to include both of these points.

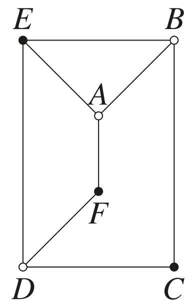
d We see from the previous question that the cycle cannot contain edge (A, B). Therefore the cycle must contain edge (A, E). Since the degree of vertex F is 2, the cycle must also include D, F, A (or the reversal of this). Piecing these, together, the cycle must include D, F, A, E (or the reversal of this).



The cycle must include vertices B and C and so the cycle must also include edges (E, B) and (B, C). Piecing these together gives the Hamiltonian cycle D, F, A, E, B, C, D. Note that the cycle can start at any one or the vertices, and can also be written in reverse. This gives a total of $6 \times 2 = 12$ Hamiltonian cycles.

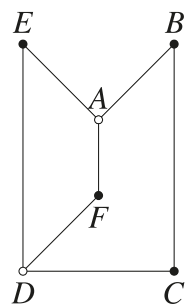


e We first colour E black. Vertices A, B and D are connected to E. We colour these white. Vertices F and C are both connected to white vertices. We colour these black.

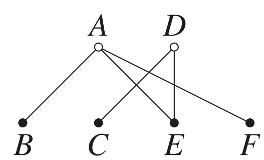


We note that the adjacent vertices A and B both have the same colour. This shows that the graph is not bipartite. This is because, if the graph were bipartite, then two vertices would be connected if (and only if) they were differently coloured.

f If we remove edge (E, B) then vertex B would not be coloured white in the second round of colourings. We would instead obtain the coloured graph shown below.



g The two sets of vertices that make this a bipartite graph are then the white vertices $\{A, D\}$ and the black vertices $\{B, E, C, F\}$. The graph can also be drawn to make this fact more obvious. This is shown below.



Chapter 13 – Revision of Chapters 11-12

Solutions to technology-free questions

1 **a** **AB** is not defined since the product of a 2×2 and a 3×1 matrix is not defined.
AC is defined since the product of a 2×2 and a 2×1 matrix is a 2×1 matrix
CD is defined since the product of a 2×1 and a 1×2 matrix is a 2×2 matrix
BE is defined since the product of a 1×3 and a 3×1 matrix is a 1×1 matrix

b $\mathbf{DA} = \begin{bmatrix} -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -12 \end{bmatrix}$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{1(-1) - (4)(2)} \begin{bmatrix} -1 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{-9} \begin{bmatrix} -1 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

2 **a**

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 18 & -24 \end{bmatrix}.$$

b Firstly,

$$\mathbf{A}^2 = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -10 \\ 10 & 12 \end{bmatrix},$$

$$\mathbf{B}^2 = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 3 & 28 \end{bmatrix},$$

Therefore,

$$\mathbf{A}^2 - \mathbf{B}^2 = \begin{bmatrix} -3 & -10 \\ 10 & 12 \end{bmatrix} - \begin{bmatrix} 7 & 9 \\ 3 & 28 \end{bmatrix} = \begin{bmatrix} -10 & -19 \\ 7 & -16 \end{bmatrix}.$$

3 The matrix is not invertible if and only if its determinant is zero.

$$\det \begin{bmatrix} 2 & 4 \\ 8 & x \end{bmatrix} = 0$$

$$2 \times x - 4 \times 8 = 0$$

$$2x - 32 = 0$$

$$x = 16$$

4 Suppose that $\mathbf{A} = \begin{bmatrix} x \\ y \end{bmatrix}$. This matrix equation is equivalent to the pair of equations,

$$3x - y = 5, \quad (1)$$

$$-6x + 2y = 10. \quad (2)$$

Notice that equation (1) is equivalent to equation (2). Therefore, we really have one equation,

$$3x - y = 5.$$

There are infinitely many solutions to this equation. Let $x = t \in \mathbb{R}$. Then

$$y = 3x - 5 = 3t - 5.$$

Therefore

$$\mathbf{A} = \begin{bmatrix} t \\ 3t - 5 \end{bmatrix}.$$

5 $\mathbf{AB} = \begin{bmatrix} -1 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & -6 \\ -3 & -8 \end{bmatrix} =$

$\begin{bmatrix} -9 & -8 \\ -15 & 10 \end{bmatrix}$

$$\mathbf{C}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{(-1)(-4) - (2)(3)} \begin{bmatrix} -4 & -2 \\ -3 & -1 \end{bmatrix}$$
$$= \frac{1}{-2} \begin{bmatrix} -4 & -2 \\ -3 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

6 **a** $\mathbf{B} + \mathbf{XA} = \mathbf{C}$

$\mathbf{XA} = \mathbf{C} - \mathbf{B}$

$\mathbf{X} = (\mathbf{C} - \mathbf{B})\mathbf{A}^{-1}$

b $\mathbf{B}(\mathbf{X} + \mathbf{A}) = \mathbf{C}$

$\mathbf{X} + \mathbf{A} = \mathbf{B}^{-1}\mathbf{C}$

$\mathbf{X} = \mathbf{B}^{-1}\mathbf{C} - \mathbf{A}$

c $\mathbf{AX} + \mathbf{BA} = \mathbf{A}$

$\mathbf{AX} = \mathbf{A} - \mathbf{BA}$

$\mathbf{X} = \mathbf{A}^{-1}(\mathbf{A} - \mathbf{BA})$

$= \mathbf{I} - \mathbf{A}^{-1}\mathbf{BA}$

d $\mathbf{X} = -\mathbf{A}$

e $\mathbf{X} = \frac{1}{2}\mathbf{B}$

f $\mathbf{X} = \mathbf{A}^{-1}(\mathbf{A} - \mathbf{I}) = \mathbf{I} - \mathbf{A}^{-1}$

7 **a** $\begin{bmatrix} 3 \\ -2 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ -10 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

c

$\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 20 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4 & 24 \\ 7 & 4 \end{bmatrix}$

8 $\det(\mathbf{A}) = w(w + 1) + 2w + 5$

$= w^2 + w + 2w + 5$

$= w^2 + 3w + 5$

It is given that:

$\det(\mathbf{A}) = 15$

$\Rightarrow w^2 + 3w + 5 = 15$

$\Rightarrow w^2 + 3w - 10 = 0$

$\Rightarrow (w + 5)(w - 2) = 0$

$\Rightarrow w = -5$ or $w = 2$

9 $\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ 5 \end{bmatrix} = \begin{bmatrix} x & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$2x + 5 = x + 8$

$x = 3$

10 $a = 0, b = -1$

11 $(\mathbf{AB})\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1}$

$= \mathbf{AIA}^{-1}$

$= \mathbf{AA}^{-1}$

$= \mathbf{I}$

Hence $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is an inverse of $\mathbf{AB}$. The inverse is unique.

12 a $\mathbf{AB} = \begin{bmatrix} 1 & -2 \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & 4 \\ b & 2 \end{bmatrix}$

$= \begin{bmatrix} 1-2b & 0 \\ ab & 2a \end{bmatrix}$

b If $\mathbf{B} = \mathbf{A}^{-1}$

$1 - 2b = 1 \Rightarrow b = 0$

$2a = 1 \Rightarrow a = \dfrac{1}{2}$

13 If $\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$

$= \begin{bmatrix} a+c & b \\ a^2+c & ab \end{bmatrix}$

If

$\mathbf{AB} = \mathbf{O}$

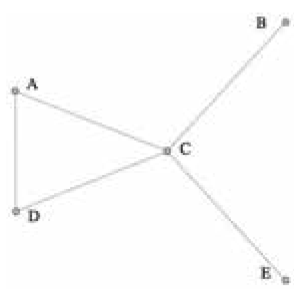
$b = 0, a + c = 0, a^2 + c = 0$

Hence, $c^2 + c = 0 \Rightarrow c(c+1) = 0 \Rightarrow c = 0$ or $c = -1$

Possible triples are:

$(0, 0, 0), (1, 0, -1)$

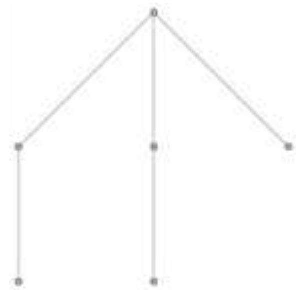
14 a



b

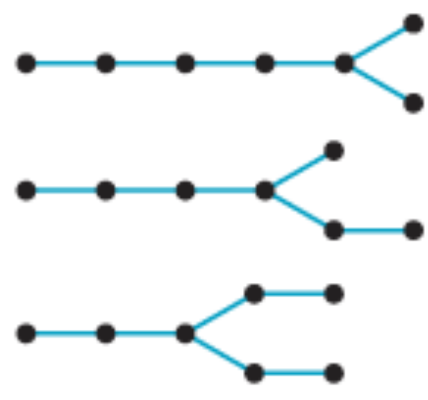
$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 0 & 1 & 1 & 0 \\ B & 0 & 0 & 1 & 0 & 0 \\ C & 1 & 1 & 0 & 1 & 1 \\ D & 1 & 0 & 1 & 0 & 0 \\ E & 0 & 0 & 1 & 0 & 0 \end{array}$$

15



- 3 vertices of degree 1
- 2 vertices of degree 2
- 1 vertex of degree 3

16



17 Let n be the number of vertices.
Therefore total degree $= 4n$.
Number of edges $= 12$.
Total degree $= 2\times$ number of edges.
Therefore $4n = 24 \Rightarrow n = 6$

18 a Let $\mathbf{K}_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

b $\mathbf{K}_4^3 = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$

There are 7 paths of length 3 between

diffent vertices.

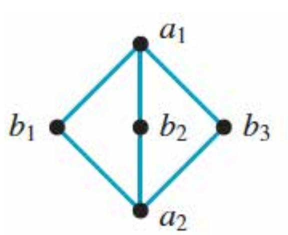
19 **a** Note that the cycle graph C_n is a subgraph of the complete graph K_n. As C_n has a Hamiltonian cycle (just travel once around the graph!), it follows that K_n has one too.

b There are $n!$ ways of picking n vertices. Order does matter.

20 Pick two non-isomorphic trees with 7 vertices. They have 7 vertices, 6 edges and 1 face.

21 Consider each line segment as a vertex and say that they are adjacent if they intersect. If each line segment intersects with 3 others the total degree of the graph is $9 \times 3 = 27$. But Total degree $= 2\times$ number of edges so that the number of edges in the graph is $\frac{27}{2}$ which is impossible.

22



23 Total degree $= 6 \times 4 = 24$. Therefore number of edges $= 12$. Use Euler's formula:

$$\begin{aligned} v - e + f &= 2 \\ 6 - 12 + f &= 2 \\ -6 + f &= 2 \\ f &= 8 \end{aligned}$$

It has 8 faces.

24 Let v be the number of vertices. Total degree $\geq 5v$
Therefore $2 \times e \geq 5v$.
Therefore, $e \geq \frac{5v}{2}$.
But for a simple connected planar graph $e \leq 3v - 6$
Therefore $\frac{5v}{2} \leq 3v - 6 \Rightarrow v \geq 12$

25 f

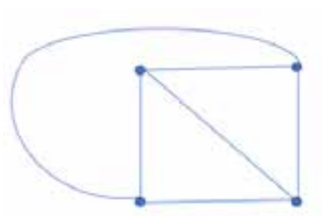
A Hamilton circuit is $1, 3, 5, 2, 4, 1$ (or its reversal).

b

A Hamilton circuit is 1, 3, 5, 2, 6, 4, 1. There are many others.

c An algorithm for constructing the Hamilton cycle is: Go through the odd numbers in increasing order, go to vertex 2 and then the largest even integer, then proceed through the even vertices in decreasing order to 4, and then return to 1.

26 a



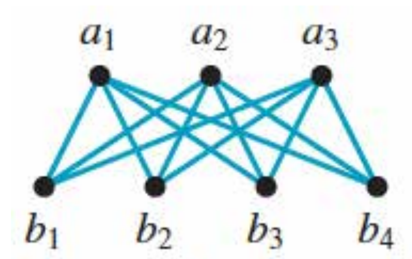
b

c If a planar graph has n vertices and a face which is not triangular then each of those face can be divided into triangles. Therefore the planar graph with most faces with n vertices has triangular faces. Every triangle has 3 edges and each edge belongs to two triangles. Therefore $e = \frac{3f}{2}$.
Hence $n - \frac{3f}{2} + f = 2 \Rightarrow f = 2n - 4$.
Therefore, the graph G has at most $2n - 4$ faces.

d Each face can be divided into triangles.

27 The dodecahedron graph has 20 vertices, 30 edges and 12 faces.

28 a



b Four vertices have odd degree (b_1, b_2, b_3, b_4)

c One edge (for example $\{b_1, b_2\}$

d Two edges (for example $\{b_1, b_2\}, \{b_3, b_4\}$

Solutions to multiple-choice questions

1 A

$$\mathbf{P}^2 = 4I$$
$$\frac{1}{4}\mathbf{PP} = I$$
$$\left(\frac{1}{4}\mathbf{P}\right)\mathbf{P} = I.$$

Therefore,

$$\mathbf{P}^{-1} = \frac{1}{4}\mathbf{P}.$$

2 B $\mathbf{RS} = \begin{bmatrix}5(0) + (3)(-1) + (1)(2)\end{bmatrix} = \begin{bmatrix}-1\end{bmatrix}$

3 E $\det \mathbf{A} = (9)(5) - (8)(-11) = 133$

4 A The product of an 1×3 matrix by a 3×1 matrix will be a 1×1 matrix.

5 B

$$\mathbf{AX} + \mathbf{B} = \mathbf{C}$$
$$\mathbf{AX} = \mathbf{C} - \mathbf{B}$$
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$$
$$= \begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}\begin{bmatrix}3 & 5\\ 2 & 2\end{bmatrix}$$
$$= \begin{bmatrix}-1 & 1\\ 4 & 0\end{bmatrix}$$

6 C Since $\mathbf{PQR} = \begin{bmatrix}7 & 0\\ 0 & 56\end{bmatrix}$, there are 2 zero entries.

7 A

$$\mathbf{X}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$
$$= \frac{1}{3(-2) - (5)(-1)}\begin{bmatrix}-2 & -5\\ 1 & 3\end{bmatrix}$$
$$= \frac{1}{-1}\begin{bmatrix}-2 & -5\\ 1 & 3\end{bmatrix}$$
$$= \begin{bmatrix}2 & 5\\ -1 & -3\end{bmatrix}$$

8 B $\det \mathbf{A} = ad - bc = (4)(4) - (6)(2) = 4$

9 D

$$\mathbf{S}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$
$$= \frac{1}{5(2) - (7)(2)}\begin{bmatrix}2 & -7\\ -2 & 5\end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix}-2 & 7\\ 2 & -5\end{bmatrix}$$

10 C $v = 8, e = 13.$
Euler's formula is $v - e + f = 2$
Therefore, $8 - 13 + f = 2$
$\Rightarrow f = 7$

11 D A tree with n vertices has $n - 1$ edges. Therefore 11 edges.

12 B All vertices must have even degree. Add edges between v_1 and v_2 and between v_3 and v_4

13 B If there are v vertices and e edges then Euler's formula gives $v - e = -15$.
Number of edges $= \frac{1}{2}$ total degree sum.
Total degree sum $= \frac{4v}{2} + \frac{5v}{2} = \frac{9v}{2}$

Therefore $\frac{9v}{4}$ edges.

$$v - \frac{9v}{4} = -15$$
$$\frac{5v}{4} = 15$$
$$v = 12$$

14 D The adjacency matrix

$$\begin{array}{c} \\ \mathbf{M} = \begin{array}{c} A \\ B \\ C \end{array} \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ \mathbf{M}^3 = \begin{array}{c} A \\ B \\ C \end{array} \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 4 & 6 & 6 \\ 6 & 4 & 12 \\ 6 & 12 & 4 \end{pmatrix} \end{array}$$

15 D The complete graph K_{10} has 45 edges. Therefore $\overline{G}$ will have $45 - 24 = 21$ edges

16 E *UVSPTQRU*

17 A The complete graph K_n has $\frac{n^2 - n}{2}$ edges. C_n has n edges. Therefore $\overline{C_n}$ has $\frac{n^2 - n}{2} - n = \frac{n^2 - 3n}{2}$ edges.

18 B The graph C_6 has vertex 1 connected to vertex 2 and vertex 6, vertex 2 connected to vertex 1 and vertex 3, and so on.

19 E $\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

$$\mathbf{M}^4 = \begin{bmatrix} 6 & 4 & 1 & 1 & 4 \\ 4 & 6 & 4 & 1 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 1 & 4 & 6 & 4 \\ 4 & 1 & 1 & 4 & 6 \end{bmatrix}$$

The number of paths of length 4 from a vertex back to itself is 6

Solutions to extended-response questions

1 a i $\mathbf{A}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & d^2 + bc \end{bmatrix}$

ii $3\mathbf{A} = 3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$

b i

$$\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$$

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & d^2 + bc \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & d^2 + bc \end{bmatrix} = \begin{bmatrix} 3a - 1 & 3b \\ 3c & 3d - 1 \end{bmatrix}$$

Equating the top right entries gives $ab + bd = 3b$. Since $b \neq 0$, this implies that

$$a + d = 1.$$

ii If $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$ then

$$3\mathbf{A} - \mathbf{A}^2 = \mathbf{I}$$

$$\mathbf{A}(3\mathbf{I} - \mathbf{A}) = \mathbf{I}$$

so this implies that $\mathbf{A}$ has an inverse and that $\mathbf{A}^{-1} = 3\mathbf{I} - \mathbf{A}$. Therefore,

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 3 - a & -b \\ -c & 3 - d \end{bmatrix}$$

Equating top right entries gives

$$\frac{-b}{ad - bc} = -b.$$

Since $b \neq 0$, this implies that

$$\det(\mathbf{A}) = ad - bc = 1.$$

c Since $ad - bc = 1$, we know that $\mathbf{A}^{-1}$ exists and that

$$\mathbf{A}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Now since $a + d = 3$, we know that

$$\begin{aligned}3\mathbf{I} - \mathbf{A} &= \begin{bmatrix} 3-a & -b \\ -c & 3-d \end{bmatrix} \\ &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \mathbf{A}^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{A}(3\mathbf{I} - \mathbf{A}) &= \mathbf{I} \\ 3\mathbf{A} - \mathbf{A}^2 &= \mathbf{I} \\ \mathbf{A}^2 &= 3\mathbf{A} - \mathbf{I},\end{aligned}$$

as required.

2 a Let $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

i
$$\begin{aligned}\mathrm{Tr}(\mathbf{X} + \mathbf{Y}) &= \mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \\ &= \mathrm{Tr}\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \\ &= a + e + d + h \\ \mathrm{Tr}(\mathbf{X}) + \mathrm{Tr}(\mathbf{Y}) &= \mathrm{Tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \mathrm{Tr}\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= a + d + e + h\end{aligned}$$

Hence $\mathrm{Tr}(\mathbf{X} + \mathbf{Y}) = \mathrm{Tr}(\mathbf{X}) + \mathrm{Tr}(\mathbf{Y})$

ii
$$\begin{aligned}\mathrm{Tr}(-\mathbf{X}) &= \mathrm{Tr}\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \\ &= -a - d \\ -\mathrm{Tr}(\mathbf{X}) &= -\mathrm{Tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= -(a + d) \\ &= -a - d\end{aligned}$$

Hence $\mathrm{Tr}(-\mathbf{X}) = -\mathrm{Tr}(\mathbf{X})$

iii $\mathrm{Tr}(\mathbf{XY}) = \mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$

$= \mathrm{Tr}\begin{bmatrix} ae + bg & af + & bh \\ ce + dg & cf + & dh \end{bmatrix}$

$= ae + bg + cf + dh$

$\mathrm{Tr}(\mathbf{YX}) = \mathrm{Tr}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$= \mathrm{Tr}\begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix}$

$= ae + cf + bg + dh$

Hence $\mathrm{Tr}(\mathbf{XY}) = \mathrm{Tr}(\mathbf{YX})$

b $\mathrm{Tr}(\mathbf{XY} - \mathbf{YX}) = \mathrm{Tr}(\mathbf{XY}) - \mathrm{Tr}(\mathbf{YX})$

$= 0$ as $\mathrm{Tr}(\mathbf{XY}) = \mathrm{Tr}(\mathbf{YX})$

$\mathrm{Tr}(\mathbf{I}) = \mathrm{Tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= 2$

As $\mathrm{Tr}(\mathbf{XY} - \mathbf{YX}) \neq \mathrm{Tr}(\mathbf{I})$

$\mathbf{XY} - \mathbf{YX} \neq \mathbf{I}$ for any $\mathbf{X}, \mathbf{Y}$

3 a i $\begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$

ii $\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$

iii $\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$

iv The one takes a little more work. We find that

$$\frac{1}{4}\begin{bmatrix} 1-\cos\theta & \sin\theta \\ \sin\theta & 1+\cos\theta \end{bmatrix}^2$$

$$= \frac{1}{4}\begin{bmatrix} (1-\cos\theta)^2+\sin^2\theta & \sin\theta(1-\cos\theta)+\sin\theta(1+\cos\theta) \\ \sin\theta(1-\cos\theta)+\sin\theta(1+\cos\theta) & \sin^2\theta+(1+\cos\theta)^2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1-2\cos\theta+\cos^2\theta+\sin^2\theta & 2\sin\theta \\ 2\sin\theta & \sin^2\theta+1+2\cos\theta+\cos^2\theta \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2-2\cos\theta & 2\sin\theta \\ 2\sin\theta & 2+2\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-\cos\theta & \sin\theta \\ \sin\theta & 1+\cos\theta \end{bmatrix}$$

b **i** $\frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}^2 = \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$

ii $\frac{1}{36}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & -2 \end{bmatrix}^2 = \frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & -2 \end{bmatrix}$

c From above, $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$ are idempotent.. However,

$\mathbf{AB} = \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix}$ but $(\mathbf{AB})^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \mathbf{AB}$

d
$$\mathbf{A}^2 = \mathbf{A}$$
$$\Rightarrow \det(\mathbf{A}^2) = \det(\mathbf{A})$$
$$\Rightarrow \det(\mathbf{A})(\det(\mathbf{A}) - 1) = 0$$
$$\Rightarrow \det(\mathbf{A}) = 0 \text{ or } \det(\mathbf{A}) = 1$$

e Let $\mathbf{A}^{-1}$ be the inverse of $\mathbf{A}$

Then $\mathbf{A}^2 = \mathbf{A}$
$$\Rightarrow \mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}\mathbf{A}$$
$$\Rightarrow \mathbf{A} = \mathbf{I}$$

f $(\mathbf{I}-\mathbf{A})^2 = \mathbf{I} - 2\mathbf{A} + \mathbf{A}^2$

$= \mathbf{I} - 2\mathbf{A} + \mathbf{A}$

$= \mathbf{I} - \mathbf{A}$

g Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\mathbf{A}^2 = \mathbf{A}$ implies

$a^2 + bc = a \ldots (1)$

$ab + bd = b \ldots (2)$

$ac + dc = c \ldots (3)$

$cb + d^2 = d \ldots (4)$

Consider (2)

$b(a+d) - b = 0$

$b(a+d-1) = 0$

$b = 0$ or $a + d = 1$

Consider (3)

$c(a+d) - c = 0$

$c = 0$ or $a + d = 1$

Subtract (4) from (1) to obtain

$a^2 - d^2 = a - d \Rightarrow (a-d)(a+d) = a - d \Rightarrow a = d$ or $a + d = 1$ The equations (1) to (4) define all 2×2 matrices which are idempotent. We consider 3 cases to generate more examples besides the obvious

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Cases

Case 1: $b = 0$

From (1) $a^2 - a = 0 \Rightarrow a = 0$ or $a = 1$

From (4) $d = 0$ or $d = 1$

This gives examples such as:

$\begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$

Case 2: $c = 0$

From (1) $a^2 - a = 0 \Rightarrow a = 0$ or $a = 1$

From (4) $d = 0$ or $d = 1$

This gives examples such as:

$\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$

Case 3: $a + d = 1$ This gives examples such as: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Choose a and d such that $a + d = 1$ and choose $c = a$ and $d = b$ then

$\begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix}$ is an idempotent matrix

4 a i $\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ii $\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

iii $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b $\mathbf{A}^2 = \mathbf{I}$

$\Rightarrow \det(\mathbf{A}^2) = \det(\mathbf{I})$

$\Rightarrow (\det(\mathbf{A}))^2 = 1$

$\Rightarrow \det(\mathbf{A}) = \pm 1$

c Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $\mathbf{A}^2 = \mathbf{I}$

$a^2 + bc = 1 \ldots (1)$

$b(a + d) = 0 \ldots (2)$

$c(a + d) = 0 \ldots (3)$

$cb + d^2 = 1 \ldots (4)$

From (2) $b = 0$ or $a + d = 0$

If $b = 0$ then from (1) and (4) $a = \pm 1$ and $d = \pm 1$ If $a = 1$ and $d = -1$, c can be any value. $\begin{bmatrix} 1 & 0 \\ 5 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Similarly if $a = -1$ and $d = 1$

$\begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

From (3) $c = 0$ or $a + d = 0$

If $c = 0$ then from (1) and (4) $a = \pm 1$ and $d = \pm 1$ If $a = 1$ and $d = -1$, b can be any value. $\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Similarly if $a = -1$ and $d = 1$

$$\begin{bmatrix}-1 & 5\\ 0 & 1\end{bmatrix}^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

If $a = 1$ and $d = 1$ then b and c are 0.

If $a = -1$ and $d = -1$ then b and c are 0.

If $a = -d$ then $bc = 1 - a^2$. For example,

$$\begin{bmatrix}-3 & -2\\ 4 & 3\end{bmatrix}^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

For example if $\mathbf{A} = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$ then $\mathbf{A}^2 = \mathbf{I}$

d First assume $\mathbf{A}$ is involutory. That is $\mathbf{A}^2 = \mathbf{I}$

$$\begin{aligned}\left[\frac{1}{2}(\mathbf{A}+\mathbf{I})\right]^2 &= \frac{1}{4}(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I})\\ &= \frac{1}{4}(2\mathbf{A} + 2\mathbf{I})\\ &= \frac{1}{2}(\mathbf{A}+\mathbf{I})\end{aligned}$$

Conversely assume $\left[\frac{1}{2}(\mathbf{A}+\mathbf{I})\right]^2 = \left[\frac{1}{2}(\mathbf{A}+\mathbf{I})\right]$

$$\left[\frac{1}{2}(\mathbf{A}+\mathbf{I})\right]^2 = \frac{1}{2}(\mathbf{A}+\mathbf{I})$$

$$\frac{1}{2}(\mathbf{A}+\mathbf{I})^2 = \mathbf{A}+\mathbf{I}$$

$$\frac{1}{2}(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}) = \mathbf{A}+\mathbf{I}$$

$$\frac{1}{2}(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}) = \mathbf{A}+\mathbf{I}$$

$$\mathbf{A}^2 = \mathbf{I}^2$$

5 **a** $\mathbf{A} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$

$$\mathbf{A} - m\mathbf{I} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} - \begin{bmatrix}m & 0\\ 0 & m\end{bmatrix} = \begin{bmatrix}a-m & b\\ c & d-m\end{bmatrix}$$

$$\begin{aligned}\det\mathbf{A} &= (a-m)(d-m) - bc\\ &= ad - am - md + m^2 - bc\\ &= m^2 - (a+d)m + ad - bc\end{aligned}$$

b $m^2 - (a+d)m + ad - bc = (m-\lambda_1)(m-\lambda_2)$

$m^2 - (a+d)m + ad - bc = m^2 - (\lambda_1+\lambda_2)m + \lambda_1\lambda_2$

$\Rightarrow (a+d) = \lambda_1 + \lambda_2$ and $\lambda_1\lambda_2 = ad - bc = \det\mathbf{A}$

c Suppose $\mathbf{A} + b = c + d = 1$

$ad - bc = a(1 - c) - (1 - a)c = a - ac - c + ac = a - c$

$a + d = a + 1 - c$

Therefore $\lambda_1 + \lambda_2 = a - c + 1 \ldots (1)$ and $\lambda_1\lambda_2 = a - c \ldots (2)$

$\lambda_1 + \lambda_2 = \lambda_1\lambda_2 + 1$

$(\lambda_1 - 1)(\lambda_2 - 1) = 0$

Therefore $\lambda_1 = 1$ or $\lambda_2 = 1$

Hence from (1), $1 + \lambda_2 = a - c + 1 \Rightarrow \lambda_2 = a - c$

d **i** $\mathbf{A} - m\mathbf{I} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} - \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} -3 - m & 4 \\ 6 & -5 - m \end{bmatrix}$

$$\begin{aligned} \det \mathbf{A} &= (-3 - m)(-5 - m) - 24 \\ &= 15 + 8m + m^2 - 24 \\ &= m^2 + 8m - 9 \\ &= (m + 9)(m - 1) \end{aligned}$$

$\det \mathbf{A} = 0 \Rightarrow m = -9$ or $m = 1$

ii $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

implies

$-3x + 4y = x$ and $6x - 5y = y$

That is $y = x$

Solutions are of the form (x, x)

iii $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -9\begin{bmatrix} x \\ y \end{bmatrix}$

implies

$-3x + 4y = -9x$ and $6x - 5y = -9y$

That is, $6x + 4y = 0 \Rightarrow y = -\dfrac{3x}{2}$

Solutions are of the form $\left(x, -\dfrac{3x}{2}\right)$

e First return to **c**

Suppose $a + b = c + d = 6$

$ad - bc = a(6 - c) - (6 - a)c = 6a - ac - 6c + ac = 6(a - c)$

$a + d = a + 6 - c$

Therefore $\lambda_1 + \lambda_2 = a - c + 6 \ldots (1)$ and $\lambda_1\lambda_2 = 6a - 6c \ldots (2)$

$\lambda_1 + \lambda_2 = \dfrac{1}{6}\lambda_1\lambda_2 + 6$

$(\lambda_1 - 6)(\lambda_2 - 6) = 0$

Therefore $\lambda_1 = 6$ or $\lambda_2 = 6$

Hence from (1), $6 + \lambda_2 = a - c + 6 \Rightarrow \lambda_2 = a - c$

6 a You cannot draw such a graph because total degree is $1 + 2 + 3 + 4 + 5 = 15$, which is odd.

b i The total degree sum is $2 + 2 + 2 + 2 + 4 + 6 = 18$. There are 9 edges. One vertex has degree 6 and there are 5 other vertices, There must a loop or multiple edges. The graph is not simple.

ii It has Euler circuit since all vertices are even.

c i The minimum possible degree sum will be for a tree. With 6 vertices there are 5 edges and therefore total dgree sum = 10.

ii All the vertices must be of even degree. A vertex could have degree 2. For example, every vertex in the cycle graph C_6 has degree 2. The degree could also be 4. In fact, the graph shown below in **d ii** has vertices of degree 4. No vertex can have degree 6 or more, since the graph has no loops or multiple edeges.

d i All vertices are even. The sum of the degrees is twice the number of edges. Therefore the degree sum must be $2 \times 10 = 20$. The graph has vertices with degrees, $4, 4, 4, 4, 2, 2$. Illustrated below is the only graph with these properties.

ii

7 a **Method 1. Count faces**. Each of the e edges contributes 2 faces (the face on either side of the edge). However, this triple counts the faces, as every triangular face has three edges. Therefore $f = \frac{2e}{3}$, in which case $3f = 2e$.
Method 2. Count edges. Each of the f triangular faces contributes 3 edges. However, this double counts edges, since every edge belongs to 2 of the triangular faces. Therefore the total number of edges will be $e = \frac{3f}{2}$. That is, $2e = 3f$.

b Using Euler's formula we have

$$\begin{aligned} v - e + f &= 2 \\ 3v - 3e + 3f &= 6 \\ 3v - 3e + 2e &= 6 \qquad \text{(since } 2e = 3f\text{)} \\ e &= 3v - 6. \end{aligned}$$

c Now suppose that $e = 3v - 6$. Then using Euler's formula again, we obtain

$$\begin{aligned} v - e + f &= 6 \\ 3v - 3e + 3f &= 6 \\ 3v - 6 &= 3e - 3f \\ e &= 3e - 3f \qquad \text{(since } e = 3v - 6\text{)} \\ 2e &= 3f. \end{aligned}$$

d Suppose the graph has at least one face bounded by more than three edges. Then $e > \frac{3f}{2}$, in which case $2e > 3f$, which is a contradiction.

e Recall that any convex polyhedron can be considered as a simple planar graph. As $v = 12$ and $f = 20$, by Euler's formula we find that $e = 2 - v - f = 2 - 12 - 20 = 30$. Therefore

$$3v - 6 = 3(12) - 6 = 30 = e.$$

By the previous question, we can conclude that the convex polyhedron has only triangular faces.

Solutions to Problem-solving and modelling investigations

1 a Let $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$\mathbf{Q}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} f_3 & f_2 \\ f_2 & f_1 \end{bmatrix}$

$\mathbf{Q}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} f_4 & f_3 \\ f_3 & f_2 \end{bmatrix}$

$\mathbf{Q}^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} f_5 & f_4 \\ f_4 & f_3 \end{bmatrix}$

Conjecture: $\mathbf{Q}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$

We prove it by using mathematical induction:

Proof It is true for $n = 2$: $\mathbf{Q}^2 = \begin{bmatrix} f_3 & f_2 \\ f_2 & f_1 \end{bmatrix}$

Assume true for $n = k$. That is $\mathbf{Q}^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}$

$$\begin{aligned} \mathbf{Q}^{k+1} &= \mathbf{Q}^k\mathbf{Q} \\ &= \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{bmatrix} \\ &= \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix} \end{aligned}$$

The result is proved by mathematical induction.

b $\det(\mathbf{Q}^2) = 2 - 1 = 1$

$\det(\mathbf{Q}^3) = 3 - 4 = -1$

$\det(\mathbf{Q}^4) = 10 - 9 = 1$

Conjecture: $\mathbf{Q}^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

$\det(\mathbf{Q}^n) = f_{n+1}f_{n-1} - f_n^2$

Using induction we can move through the even powers starting at $\mathbf{Q}^2$ by multiplying by $\mathbf{Q}^2$:

Assume k even and that $\det(\mathbf{Q}^k) = 1$:

$\mathbf{Q}^{k+2} = \mathbf{Q}^k\mathbf{Q}^2$ and thus $\det(\mathbf{Q}^{k+2}) = \det(\mathbf{Q}^k)\det(\mathbf{Q}^2) = 1 \times 1 = 1$

Similarly we can move through the odd powers starting at $\mathbf{Q}^3$ by multiplying by $\mathbf{Q}^2$:

Assume k odd and that $\det(\mathbf{Q}^k) = -1$:

$\mathbf{Q}^{k+2} = \mathbf{Q}^k\mathbf{Q}^2$ and thus $\det(\mathbf{Q}^{k+2}) = \det(\mathbf{Q}^k)\det(\mathbf{Q}^2) = -1 \times 1 = -1$

c $\mathbf{Q}^{n+1}\mathbf{Q}^{n} = \mathbf{Q}^{2n+1}$

Hence, we have

$$\begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}\begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{2n+2} & f_{2n+1} \\ f_{2n+1} & f_{2n} \end{bmatrix}$$

That is

$$\begin{bmatrix} f_{n+2}f_{n+1} + f_{n+1}f_n & f_{n+2}f_n + f_{n+1}f_{n-1} \\ (f_{n+1})^2 + (f_n)^2 & f_{n+1}f_n + f_nf_{n-1} \end{bmatrix} = \begin{bmatrix} f_{2n+2} & f_{2n+1} \\ f_{2n+1} & f_{2n} \end{bmatrix}$$

Hence

$$(f_{n+1})^2 + (f_n)^2 = f_{2n+1}$$

d $\mathbf{Q}^{m}\mathbf{Q}^{n-1} = \mathbf{Q}^{m+n-1}$

Hence we have:

$$\begin{bmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{bmatrix}\begin{bmatrix} f_n & f_{n-1} \\ f_{n-1} & f_{n-2} \end{bmatrix} = \begin{bmatrix} f_{m+n} & f_{m+n-1} \\ f_{m+n-1} & f_{m+n-2} \end{bmatrix}$$

That is

$$\begin{bmatrix} f_{m+1}f_n + f_mf_{n-1} & f_{m+1}f_{n-1} + f_mf_{n-2} \\ (f_mf_n + f_{m-1}f_{n-1} & f_mf_{n-1} + f_{m-1}f_{n-2} \end{bmatrix} = \begin{bmatrix} f_{m+n} & f_{m+n-1} \\ f_{m+n-1} & f_{m+n-2} \end{bmatrix}$$

Therefore

$$f_{m+n} = f_{m+1}f_n + f_mf_{n-1}$$

e
$$\begin{aligned} \det(\mathbf{Q} - x\mathbf{I}) &= \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1-x & 1 \\ 1 & -x \end{bmatrix}\right) \\ &= x^2 - x - 1 \end{aligned}$$

So if $x^2 - x - 1 = 0$, then $x = \dfrac{1 \pm \sqrt{5}}{2}$.

f Open ended investigation

g Open ended investigation

2 **a** Pr(Green tea on a Wednesday $= \dfrac{3}{5} \times \dfrac{3}{5} + \dfrac{2}{5} \times \dfrac{3}{4} = \dfrac{33}{50}$

b **i** The probability of green tea on Monday is 1.

ii $\begin{bmatrix} \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{4} \end{bmatrix} \times \mathbf{S_n} = \begin{bmatrix} \frac{3}{5} \times \Pr(G_n) + \frac{3}{4} \times \Pr(J_n) \\ \frac{2}{5} \times \Pr(G_n) + \frac{2}{4} \times \Pr(J_n) \end{bmatrix}$

$= \mathbf{S_{n+1}}$

$= \begin{bmatrix} \Pr(G_{n+1}) \\ \Pr(J_{n+1}) \end{bmatrix}$

c $\mathbf{S_2} = \begin{bmatrix} \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$

$\mathbf{S_3} = \begin{bmatrix} \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{33}{50} \\ \frac{17}{50} \end{bmatrix}$

d $\mathbf{S_2} = \mathbf{TS_1}$

$\mathbf{S_3} = \mathbf{TS_2}$

$= \mathbf{TTS_1}$

$= \mathbf{T}^2\mathbf{S_1}$

Similarly

$\mathbf{S_4} = \mathbf{TS_3}$

$= \mathbf{T}^3\mathbf{S_1}$

An induction argument can be used to show $\mathbf{S_n} = \mathbf{T}^{n-1}\mathbf{S_1}$

e $\mathbf{S_{20}} = \mathbf{T}^{19}\mathbf{S_1} \approx \begin{bmatrix} 0.6522 \\ 0.3478 \end{bmatrix}$

f $\mathbf{S_{200}} = \mathbf{T}^{199}\mathbf{S_1} \approx \begin{bmatrix} 0.6522 \\ 0.3478 \end{bmatrix}$

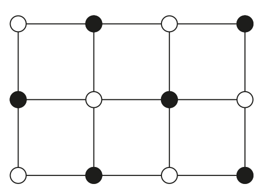
g
$$\begin{bmatrix} \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \frac{3}{5}a + \frac{3}{4}b \\ \frac{2}{5}a + \frac{1}{4}b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$
$$\Rightarrow 12a + 15b = 20a \ldots (1)$$
$$\text{and } 8a + 5b = 20b \ldots (2)$$
Also
$$a + b = 1$$
$$\Rightarrow -8a + 15(1 - a) = 0$$
$$\Rightarrow -23a = -15$$
$$\Rightarrow a = \frac{15}{23} \approx 0.6522$$
$$\Rightarrow b = \frac{8}{23} \approx 0.3478$$

h $\begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{62}{125} \\ \frac{63}{125} \end{bmatrix}$

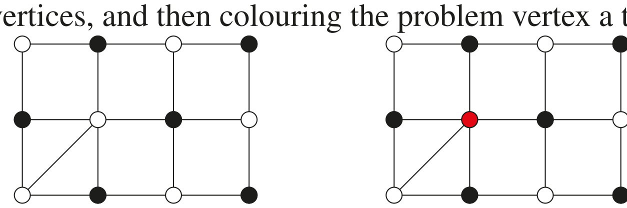
i Open ended investigation

3 **a** The outer three vertices require three colours, since each is connected to two others. Therefore the central vertex requires a fourth colour, as is it connected to three differently coloured vertices.

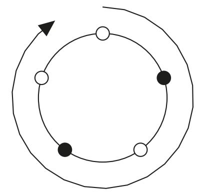
b Every $m \times n$ grid graph can be 2-coloured in a checkered fashion of alternating black and white vertices.



c The graph shown has a triangular subgraph that requires 3 colours. This graph can be 3 coloured by first colouring $m \times n$ subgraph in a checkered fashion of alternating black and white vertices, and then colouring the problem vertex a third colour.



d Suppose the graph is 2 coloured. Suppose, by way of contradiction, that the graph has a cycle of odd length. The cycle is a list of vertices whose colours must alternate. However, as the cycle has odd length, the first and last vertex in the list must be the same colour, which is a contradiction.

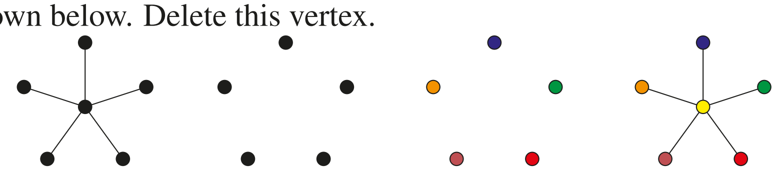


e If the degree of every vertex is greater than or equal to 6 then, by the Handshaking Lemma,

$$\begin{aligned} 2e &= \text{sum of degrees of all vertices} \\ &\geq 6v \\ \implies e &\geq 3v \\ \implies e &> 3v - 6 \end{aligned}$$

This contradicts the result that $e \leq 3v - 6$. Therefore each of the vertex degrees cannot exceed 5. Hence the graph must have at least one vertex with degree less than or equal to 5.

f Let $P(n)$ be the statement that a graph with n vertices is 6-colourable. The base case $P(1)$ is obviously true. If the graph has 1 vertex, then it is 6-colourable (in fact, it is 1-colourable!). Now assume $P(k)$ is is true from some particular value of k. We need to show that $P(k+1)$ is also true. Consider any graph with $n = k + 1$ vertices. By part **e**, this graph has some vertex with degree less than or equal to 5. This part of the graph is shown below. Delete this vertex.



The graph that remains can be 6-coloured, as we assumed that $P(k)$ is true. Restore the deleted vertex. The vertex is connected to only 5 other vertices. Each of these is coloured with at most 5 colours. Therefore at least one of the 6 colours remains to colour the restored vertex. Therefore $P(k+1)$ is also true. We conclude that $P(n)$ is true for all $n \in \mathbb{N}$ by mathematical induction.

Chapter 14 – Simulation, sampling and sampling distributions

Solutions to Exercise 14A

1 a $k + 2k + 3k + 4k + 5k = 1$

$15k = 1$

$k = \frac{1}{15}$

b $\Pr(X \geq 3) = 3k + 4k + 5k$

$= 12k$

$= \frac{12}{15}$

$= \frac{4}{5}$

2 a $E(X) = 1 \times 0.1 + 3 \times 0.3 + 5 \times 0.3 + 7 \times 0.3 = 4.6$

b $E(X) = -1 \times 0.25 + 0 \times 0.25 + 1 \times 0.25 + 2 \times 0.25 = 0.5$

c $E(X) = 0 \times 0.18 + 1 \times 0.22 + 2 \times 0.26 + 3 \times 0.21 + 4 \times 0.13 = 1.89$

d $E(X) = -3 \times 0.1 - 2 \times 0.1 - 1 \times 0.2 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.1 + +3 \times 0.1$
$= 0$

3

Profit	−\$10 000	\$0	\$10 000	\$20 000
probability	0.1	0.3	0.5	0.1

E(Profit)$= 10\,000 \times 0.1 + 0 \times 0.3 + 10\,000 \times 0.5 + 20\,000 \times 0.1$
$= 6000$
Expected profit is \$6000.

4

Win amount	\$8	−\$2
probability	$\frac{1}{6}$	$\frac{5}{6}$

Expected profit $= \frac{1}{6} \times 8 - 2 \times \frac{5}{6}$
$= -\frac{1}{3}$
A loss of 33 cents

5 a $E(X) = 1 \times 0.1 + 3 \times 0.3 + 5 \times 0.3 + 7 \times 0.3$
$= 4.6$
$E(X^2) = 1 \times 0.1 + 9 \times 0.3 + 25 \times 0.3 + 49 \times 0.3$
$= 25$
$\text{Var}(X) = E(X^2) - (E(X))^2$
$= 25 - 4.6^2$
$= 3.84$

b $E(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4}$
$= \frac{6}{4}$
$= \frac{3}{2}$
$E(X^2) = 0 \times \frac{1}{4} + 1 \times \frac{1}{4} + 4 \times \frac{1}{4} + 9 \times \frac{1}{4}$
$= \frac{14}{4}$
$= \frac{7}{2}$
$\text{Var}(X) = E(X^2) - (E(X))^2$
$= \frac{7}{2} - \frac{9}{4}$
$= \frac{5}{4}$

6 a $p+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=1$

$p+\frac{15}{16}=1$

$p=\frac{1}{16}$

b $\mathrm{E}(X)=1\times\frac{1}{2}+2\times\frac{1}{4}+3\times\frac{1}{8}+4\times\frac{1}{16}$

$=\frac{13}{8}=1.625$

c $\mathrm{E}(X^2)=1\times\frac{1}{2}+4\times\frac{1}{4}+9\times\frac{1}{8}+16\times\frac{1}{16}$

$=\frac{29}{8}$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$

$=\frac{29}{8}-\left(\frac{9}{4}\right)^2$

$=\frac{63}{64}=0.98435$

d $\mathrm{sd}(X)=\sqrt{\frac{63}{64}}=\frac{\sqrt{63}}{8}\approx 0.9922$

7 a $4k+3k+2k+k=1$

$10k=1$

$k=\frac{1}{10}$

b $\mathrm{E}(X)=1\times\frac{3}{10}+2\times\frac{2}{10}+3\times\frac{1}{10}$

$=1$

c $\mathrm{E}(X^2)=1\times\frac{3}{10}+4\times\frac{2}{10}+9\times\frac{1}{10}$

$=\frac{20}{10}=2$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$

$=2-1$

$=1$

d $\mathrm{sd}(X)=1$

Solutions to Exercise 14B

1 **a** Consider $Z = X_1 + X_2$. It can take values $-4, -2, 0, 48, 50, 100$. The values are obtained by looking at all possible sums.

- $\Pr(Z = -4) = \Pr(X_1 = -2 \text{ and } X_2 = -2)$
  $= \Pr(X_1 = -2) \times \Pr(X_2 = -2)$
  $= 0.8 \times 0.8$
  $= 0.64$

- $\Pr(Z = -2) = \Pr(X_1 = -2 \text{ and } X_2 = 0 \text{ or } X_1 = 0 \text{ and } X_2 = -2)$
  $= 2 \times \Pr(X_1 = -2) \times \Pr(X_2 = 0)$
  $= 2 \times 0.8 \times 0.15$
  $= 0.24$

- $\Pr(Z = 0) = \Pr(X_1 = 0 \text{ and } X_2 = 0$
  $= \Pr(X_1 = 0) \times \Pr(X_2 = 0)$
  $= 0.15 \times 0.15$
  $= 0.0225$

- $\Pr(Z = 48) = \Pr(X_1 = 50 \text{ and } X_2 = -2 \text{ or } X_1 = -2 \text{ and } X_2 = 50$
  $= 2 \times \Pr(X_1 = 50) \times \Pr(X_2 = -2)$
  $= 2 \times 0.05 \times 0.8$
  $= 0.08$

- $\Pr(Z = 50) = \Pr(X_1 = 50 \text{ and } X_2 = 0 \text{ or } X_1 = 0 \text{ and } X_2 = 50$
  $= 2 \times \Pr(X_1 = 50) \times \Pr(X_2 = 0)$
  $= 2 \times 0.05 \times 0.15$
  $= 0.015$

- $\Pr(Z = 100) = \Pr(X_1 = 50 \text{ and } X_2 = 50)$
  $= \Pr(X_1 = -2) \times \Pr(X_2 = -2)$
  $= 0.05 \times 0.05$
  $= 0.0025$

z	-4	-2	0	48	50	100
$\Pr(Z = z)$	0.64	0.24	0.0225	0.08	0.015	0.0025

b $\Pr(Z \geq 50) = 0.015 + 0.0025 = 0.0175$

2 **a** We can enter the probabilities in a grid. For example the pronability of a 1 on the first and 5 on the second or vice versa are in the enries (1,5) and (5,1) on the grid.

The possible sums are 2, 3, . . . , 10, 11, 12. They are shown in brackets in the table. The associated probability is also given.

.	**1**	**2**	**3**	**4**	**5**	**6**
1	0.04 (2)	0.04(3)	0.04(4)	0.04(5)	0.02(6)	0.02(7)
2	0.04(3)	0.04(4)	0.04(5)	0.04(6)	0.02(7)	0.02(8)
3	0.04(4)	0.04(5)	0.04(6)	0.04(7)	0.02(8)	0.02(9)
4	0.04(5)	0.04(6)	0.04(7)	0.04(8)	0.02(9)	0.02(10)
5	0.02(6)	0.02(7)	0.02(8)	0.02(9)	0.01(10)	0.01(11)
6	0.02(7)	0.02(8)	0.02(9)	0.02(10)	0.01(11)	0.01(12)

From the table we can constuct the distribution table.

$\Pr(Z = 2) = 0.04$,

$\Pr(Z = 3) = 0.04 + 0.04 = 0.08$,

$\Pr(Z = 4) = 0.04 + 0.04 + 0.04 = 0.12$

$\Pr(Z = 5) = 4 \times 0.04 = 0.16$

$\vdots \qquad \vdots$

$\vdots \qquad \vdots$

$\Pr(Z = 11) = 0.01 + 0.01 = 0.02$

$\Pr(Z = 12) = 0.01$

These can be btained by adding along the evident diagonal.

z	2	3	4	5	6	7
$\Pr(Z = z)$	0.04	0.08	0.12	0.16	0.16	0.16
z	8	9	10	11	12	
$\Pr(Z = z)$	0.12	0.08	0.05	0.02	0.01	

b $\Pr(Z > 10) = 0.02 + 0.01 = 0.03$

3 a $\Pr(X = 1) = \Pr(X = 2) = \Pr(X = 3)$

$= \Pr(X = 4) = \Pr(X = 5) = 0.2$

$E(X) = (1 + 2 + 3 + 4 + 5) \times 0.2 = 3$

$E(X^2) = (1 + 4 + 9 + 16 + 25) \times 0.2 = 11$

$\mathrm{Var}(X) = E(X^2) - E(X)^2$

$= 2$

$\mathrm{sd}(X) = \sqrt{2}$

b

.	**1**	**2**	**3**	**4**	**5**
1	0.04 (2)	0.04(3)	0.04(4)	0.04(5)	0.04(6)
2	0.04(3)	0.04(4)	0.04(5)	0.04(6)	0.04(7)
3	0.04(4)	0.04(5)	0.04(6)	0.04(7)	0.04(8)
4	0.04(5)	0.04(6)	0.04(7)	0.04(8)	0.04(9)
5	0.04(6)	0.04(7)	0.04(8)	0.04(9)	0.04(10))

Let $Z = X_1 + X_2$

z	2	3	4	5	6
$\Pr(Z = z)$	0.04	0.08	0.12	0.16	0.2
z	7	8	9	10	
$\Pr(Z = z)$	0.16	0.12	0.08	0.04	

c The probability that the sum is even $= \Pr(Z = 2) + \Pr(Z = 4) + \Pr(Z = 6) + \Pr(Z = 8) + \Pr(Z = 10)$
$= 0.52$

d $E(Z) = 2 \times 0.04 + 3 \times 0.08 + 4 \times 0.12 + 5 \times 0.16 + 6 \times 0.2 + 7 \times 0.16 + 8 \times 0.12 + 9 \times 0.08 + 10 \times 0.04$
$= 6$
$E(Z^2) = 4 \times 0.04 + 9 \times 0.08 + 16 \times 0.12 + 25 \times 0.16 + 36 \times 0.2 + 49 \times 0.16 + 64 \times 0.12 + 81 \times 0.08 + 100 \times 0.04$
$= 40$
$\mathrm{Var}(Z) = E(Z^2) - E(Z)^2$
$= 4$

4 a $E(X) = -5 \times 0.9 + 5 \times 0.03 + 50 \times 0.01$
$= -3.85$
$E(X^2) = 25 \times 0.9 + 25 \times 0.03 + 2500 \times 0.01$
$= 48.25$
$\mathrm{Var}(X) = E(X^2) - E(X)^2$
$= 33.4275$
$\mathrm{sd}(X) = 5.7816\ldots$

b

	−5	**0**	**5**	**50**
−5	0.81(−10)	0.054(−5)	0.027 (0)	0.009(45)
0	0.054(-5)	0.036(0)	0.0018(5)	0.0006(50)
5	0.027(0)	0.0018(5)	0.0009(10)	0.0003(55)
50	0.009(45)	0.0006(50)	0.0003(55)	0.0001(100)

z	-10	-5	0	5	10	45	50	55	100
$\Pr(Z = z)$	0.81	0.108	0.0576	0.0036	0.0009	0.018	0.0012	0.0006	0.0001

c $E(Z) = -10 \times 0.81 - 5 \times 0.108 + 5 \times 0.0036 + 10 \times 0.0009 + 45 \times 0.018 + 50 \times 0.0012 + 55 \times 0.006 + 100 \times 0.0001$
$= -7.70$
$E(Z^2) = 100 \times 0.81 + 25 \times 0.108 + 25 \times 0.0036 + 100 \times 0.0009 + 45^2 \times 0.018 + 50^2 \times 0.0012 + 55^2 \times 0.006 + 100^2 \times 0.0001$

$= 126.145$
$\text{Var}(Z) = \text{E}(Z^2) - \text{E}(Z)^2$
$= 66.855$
$\text{sd}(Z) = 8.176\ldots$

5 **a** $\text{E}(X_1 + X_2 + X_3 + X_4) = 4 \times 100 = 400$

b $\text{Var}(X_1 + X_2 + X_3 + X_4) = 4 \times 16 = 64$

c $\text{sd}(X_1 + X_2 + X_3 + X_4) = \sqrt{64} = 8$

6 **a** $\text{E}(X_1 + X_2 + X_3) = 3 \times 30 = 90$

b $\text{Var}(X_1 + X_2 + X_3) = 3 \times 7 = 21$

c $\text{sd}(X_1 + X_2 + X_3 + X_4) = \sqrt{21} = 4.583$

7 $\text{E}(X_1 + X_2 + X_3) = 3 \times (-3.85) = -11.55$
$\text{Var}(X_1 + X_2 + X_3) = 3 \times 33.4275 = 100.283$
$\text{sd}(X_1 + X_2 + X_3 + X_4) = 10.014$

8 **a** **i** $\text{E}(4X) = 4 \times 100 = 400$

ii $\text{Var}(4X) = 16 \times 16 = 256$

iii $\text{sd}(4X) = 16$

b The means are the same but variance is 4 times greater.

9 **a** $\text{E}(10X) = 10 \times 3.4 = 34$

b $\text{Var}(10X) = 100 \times 1.2 = 120$

c $\text{sd}(10X) = 10.954$

10 **a** $\text{E}(X) = 1 \times 0.38 + 2 \times 0.11 + 3 \times 0.01$
$= 0.63$
$\text{E}(X^2) = 1 \times 0.38 + 4 \times 0.11 + 9 \times 0.01$
$= 0.91$
$\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2$

$= 0.5131$

$\mathrm{sd}(X) = 0.7163\ldots$

b E(number of dogs for 10 households)$= 10 \times 0.63 = 6.3$

Var(number of dogs for 10 households)$= 10 \times 0.513 = 5.13$

sd(number of dogs for 10 households)$= \sqrt{5.13} \approx 2.265$

c $\mathrm{E}(40X) = 40 \times 0.63 = 25.2$

$\mathrm{Var}(40X) = 1600 \times 0.0513 = 820.8$

$\mathrm{sd}(40X) = \sqrt{820.8} \approx 28.6496$

Solutions to Exercise 14C

1 No, people who do not use email will not be included in the sample.

2 No, people who use the restaurant at different times of the day, or during the week, will not be included in the sample.

3 a Yes, because every student in the school had the sample probability of being included in the sample.

b 2.7

4 a One sample chosen was 102, 133, 87, 107, 75.

b $\bar{x} = 101.8$

5 a The population of Australia.

b $\mu = 4$.

c $\bar{x} = 3.5$

Solutions to Exercise 14D

1 45.6

2 Total wages for the 150 cybersecurity engineers = $50 \times 3250 + 100 \times 3070 = 469500$
An estimate of $\mu = 469500 \div 150 = \$3130$

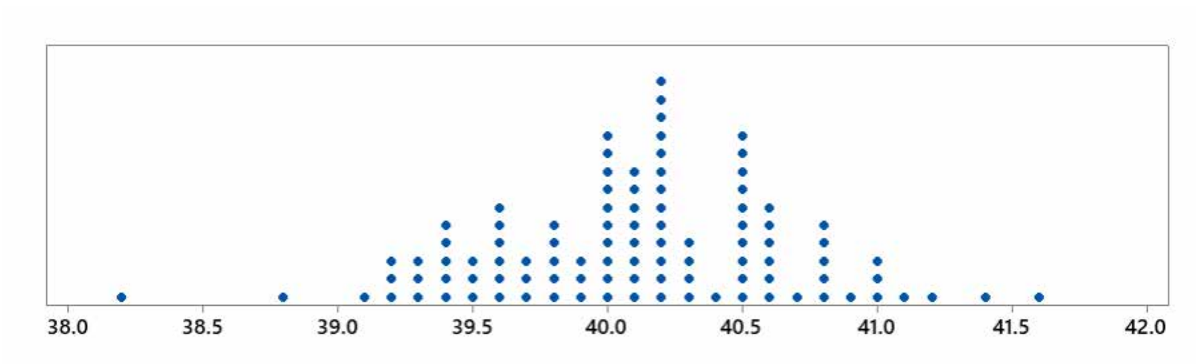
3 **a** $\Pr(\bar{X} \geq 25) = 2 \div 100 = 0.02$

b $\Pr(\bar{X} \leq 23) = 1 \div 100 = 0.01$

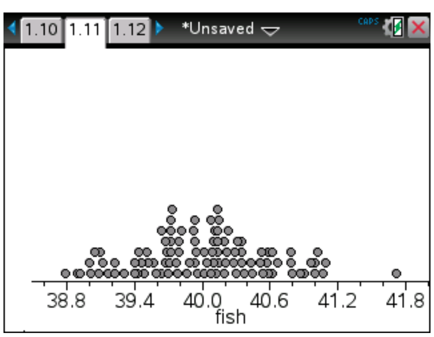
4 **a** $\Pr(\bar{X} \geq 163 = 4 \div 100 = 0.04$

b $\Pr(\bar{X} \leq 158 = 5 \div 100 = 0.05$

5 **a**



Using a calculator

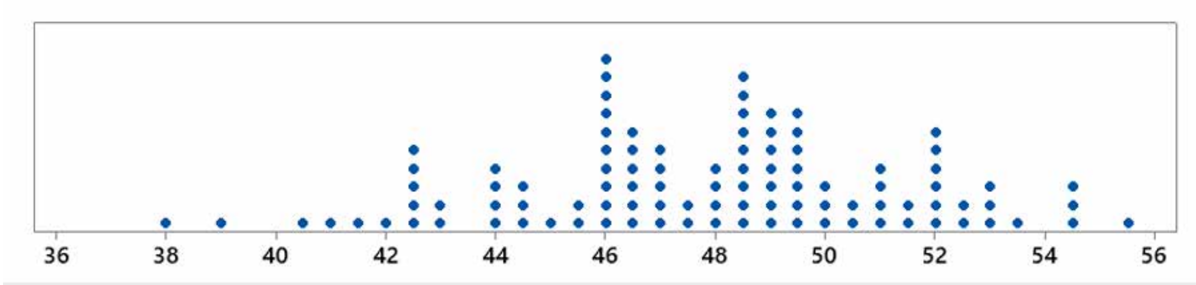


b Answers will vary

i $\Pr(\bar{X} \geq 41) = 0.04$

ii $\Pr(\bar{X} \leq 39) = 0.02$

6a,b One simulation gave the following dotplot (answers will differ).



c From the dotplot given (answers will differ):

i $\Pr(\bar{X} \geq 55) = 0.01$

ii $\Pr(\bar{X} \leq 40) = 0.02$

7 a $E(X_1 + \ldots + X_{25}) = 25 \times -1.10 = -27.5$,
$\text{Var}(X_1 + \ldots + X_{25}) = 25 \times 57.09 = 1427.25$

b $E(\bar{X}) = \mu = -1.10$,
$\text{Var}(\bar{X}) = 57.09 \div 25 = 2.2836$

8 a $E(X_1 + \ldots + X_{10}) = 10 \times 0.63 = 6.3$,
$\text{Var}(X_1 + \ldots + X_{10}) = 10 \times 0.5131 = 5.131$

b $E(\bar{X}) = \mu = 0.63$,
$\text{Var}(\bar{X}) = 0.5131 \div 10 = 0.0513$

9 a $E(\bar{X}) = 30$, $\text{sd}(\bar{X}) = 1.4$

b $E(\bar{X}) = 30$, $\text{sd}(\bar{X}) = 0.14$

c $E(\bar{X}) = 30$, $\text{sd}(\bar{X}) = 0.014$

10 a $E(\bar{X}) = 16.77$, $\text{sd}(\bar{X}) = 0.7748$

b $E(\bar{X}) = 16.77$, $\text{sd}(\bar{X}) = 0.245$

c $E(\bar{X}) = 16.77$, $\text{sd}(\bar{X}) = 0.0775$

11

$p.$	-5	0	35
$\Pr(P = p)$	0.75	0.2	0.05

a $E(P) = -5 \times 0.75 + 35 \times 0.05 = -2$

$E(P^2) = 25 \times 0.75 + 35^2 \times 0.05 = 80$

$\text{Var}(X) = E(X^2) - E(X)^2$

$= 80 - 4 = 76$

Therefore $\text{sd}(P) = \sqrt{76} \approx 8.718$

b **i** $E(\bar{P}) = -2$, $\text{sd}(\bar{P}) = 2.757$

ii $E(\bar{P}) = -2$, $\text{sd}(\bar{P}) = 0.872$

iii $E(\bar{P}) = -2$, $\text{sd}(\bar{P}) = 0.276$

Solutions to technology-free questions

1 a $p + \frac{3}{4} = 1 \Rightarrow p = \frac{1}{4}$

b $E(X) = (1 + 2 + 3 + 4) \times \frac{1}{4} = \frac{5}{2}$

c $E(X)^2 = (1 + 4 + 9 + 16) \times \frac{1}{4} = \frac{15}{2}$

$\text{Var}(X) = E(X^2) - E(X)^2$

$= \frac{15}{2} - \frac{25}{4}$

$= \frac{5}{4}$

2 a $k + 2k + 3k + 2k + 2k = 1$

$10k = 1$

$k = \frac{1}{10}$

b

$E(X) = -1 \times 0.1 + 1 \times 0.3 + 2 \times 0.2 + 3 \times 0.2$

$= 1.2$

c

$E(X)^2 = 1 \times 0.1 + 1 \times 0.3 + 4 \times 0.2 + 9 \times 0.2$

$= 3$

$\text{Var}(X) = E(X^2) - E(X)^2$

$= 3 - 1.44$

$= 1.56$

3 a $E(X_1 + X_2 + X_3 + X_4) = 4 \times 50 = 200$

b $\text{Var}(X_1 + X_2 + X_3 + X_4) = 4 \times 25 = 100$

c $\text{sd}(X_1 + X_2 + X_3 + X_4) = 10$

4 a $E(10X) = 10 \times 30 = 300$

b $\text{Var}(10X) = 100 \times 16 = 1600$

c $\text{sd}(10X) = 40$

5 No, this sample (people already interested in yoga) is not representative of the population

6 a People with Type II diabetes

b Population is too large and dispersed to use for such an experiment.

c Unknown

d $\bar{x} = 1.5$

5 Estimate $= \frac{1.58 + 1.62}{2} = 1.6$

6 a $E(\bar{X}) = 10, \ \text{sd}(\bar{X}) = \frac{2}{3}$

b $E(\bar{X}) = 10, \ \text{sd}(\bar{X}) = \frac{2}{5}$

c $E(\bar{X}) = 10, \ \text{sd}(\bar{X}) = \frac{1}{5}$

Solutions to multiple-choice questions

1 C

$$\begin{aligned}E(X) &= -1 \times p + 0 \times p + 1 \times (1 - 2p)\\ &= -p + 1 - 2p\\ &= 1 - 3p\end{aligned}$$

2 E $$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - [E(X)]^2\\ &= 2.89 - 2.25\\ &= 0.64\end{aligned}$$

$\therefore \mathrm{sd}(X) = 0.8$

3 D The posible values are:
$-1 + 0 = -1$,
$-1 + 25 = 24$,
$(-1) + (-1) = -2$,
$0 + 25 = 25$,
$0 + 0 = 0$,
$25 + 25 = 50$

4 E There are two ways of getting \$25, 25 and 0 or 0 and 25.
$\Pr(\text{Sum} = 25) = 2 \times 0.2 \times 0.1 = 0.04$

5 D

$$\begin{aligned}E(X) &= -1 \times 0.2 + 0 \times 0.3 + 1 \times 0.3 + 2 \times 0.2\\ &= 0.5\end{aligned}$$

$E(X_1 + X_2 + X_3 + X_4) = 4 \times 0.5 = 2$

6 C $E(5X) = 5 \times 0.5 = 2.5$

7 E

$$\begin{aligned}E(X^2) &= 1 \times 0.2 + 0 \times 0.3 + 1 \times 0.3 + 4 \times 0.2\\ &= 1.3\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - [E(X)]^2\\ &= 1.3 - 0.25\\ &= 1.05\end{aligned}$$

$\therefore \mathrm{Var}(5X) = 25 \times 1.05 = 26.25$

8 B A sample statistic

9 C A population parameter

10 A We use sample statistics to estimate parameters

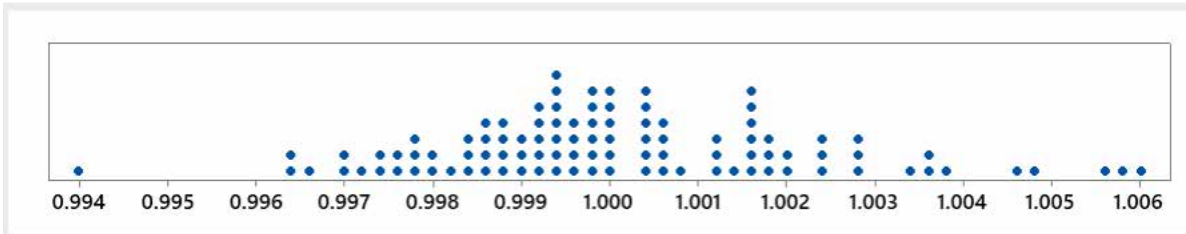
11 B Describes how a statistic's value changes from sample to sample.

12 E Thus increasing the sample size will result in a decrease in the variability of the sample estimates, as we have seen from the sampling dsitributions.

13 C $E(\bar{X}) = 8$; $\mathrm{sd}(\bar{X}) = 2.5 \div 10 = 0.25$

Solutions to extended-response questions

1 **a** One simulation gave the following dotplot (answers will differ).

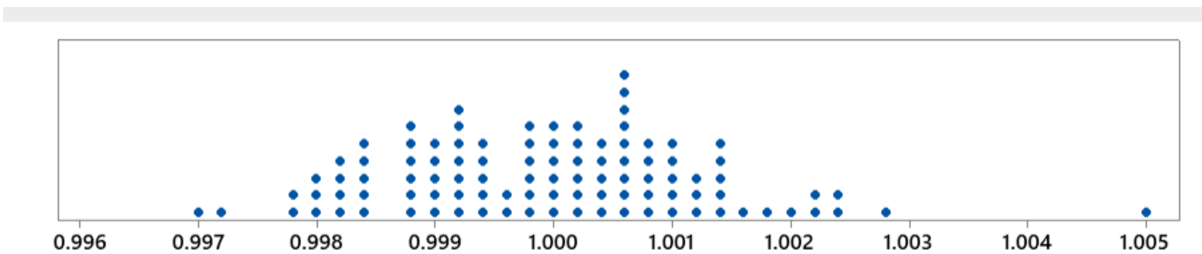


b From the dotplot given (answers will differ):

i 0.09

ii 0.01

c One simulation gave the following dotplot (answers will differ).

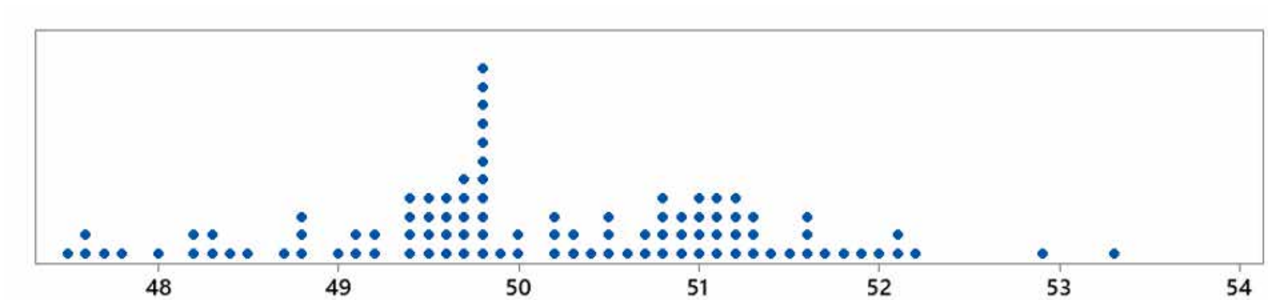


d From the dotplot given (answers will differ):

i 0

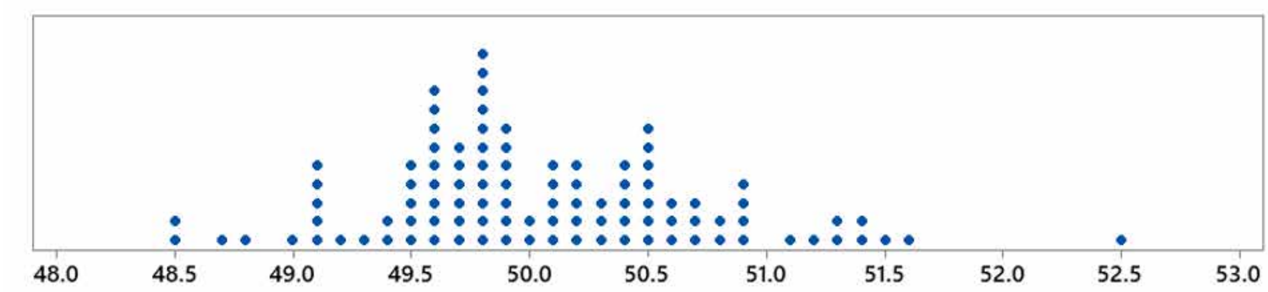
ii 0

2 **a** **ii** One simulation gave the following dotplot (answers will differ).



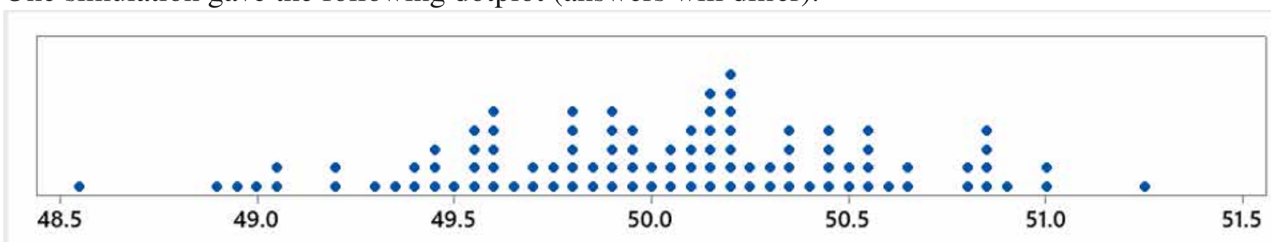
iii For this sampling distribution mean = 50.125, sd = 1.220 (answers will differ).

b **ii** One simulation gave the following dotplot (answers will differ).



iii For this sampling distribution mean = 50.055, sd = 0.708 (answers will differ).

c **ii** One simulation gave the following dotplot (answers will differ).



iii For this sampling distribution mean = 50.0165, sd = 0.526 (answers will differ).

3 **a** $E(X) = 0 \times 0.1 + 1 \times 0.6 + 2 \times 0.3$

$= 1.2$

$E(X^2) = 0 \times 0.1 + 1 \times 0.6 + 4 \times 0.3$

$= 1.8$

$\text{Var}(X) = E(X^2) - E(X)^2$

$= 0.36$

$\text{sd}(X) = 0.6$

$E(X) = 1.2$, $\text{Var}(X) = 0.36$, $\text{sd}(X) = 0.6$

b **i**

z	0	1	2	3	4
$p(z)$	0.01	0.12	0.42	0.36	0.09

ii $\Pr(X_1 + X_2 > 3) = \Pr(X_1 + X_2 = 4) = 0.09$

iii $E(X_1 + X_2) = 2.4$, $\text{Var}(X_1 + X_2) = 0.72$

iv $E(\bar{X}) = 1.2$, $\text{Var}(\bar{X}) = 0.18$

c **i** $E(X_1 + \ldots + X_7) = 8.4$, $\text{Var}(X_1 + \ldots + X_7) = 2.52$

ii $E(\bar{X}) = 1.2$, $\text{Var}(\bar{X}) = 0.051$

Chapter 15 – Trigonometric ratios and applications

Solutions to Exercise 15A

1 a $\frac{x}{5} = \cos 35^\circ$

$x = 5 \times 0.8191$

$= 4.10$ cm

b $\frac{x}{10} = \sin 45^\circ$

$x = 10 \times 0.0871$

$= 0.87$ cm

c $\frac{x}{8} = \tan 20.16^\circ$

$x = 8 \times 0.3671$

$= 2.94$ cm

d $\frac{x}{7} = \tan 30^\circ 15'$

$x = 7 \times 0.9661$

$= 4.08$ cm

e $\tan x^\circ = \frac{10}{15}$

$= 0.666$

$x = 33.69^\circ$

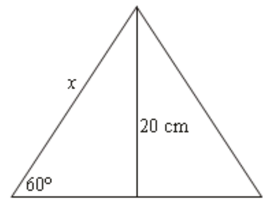
f $\frac{10}{x} = \tan 40^\circ$

$10 = x \times 0.8390$

$x = \frac{10}{0.8390}$

$= 11.92$ cm

2

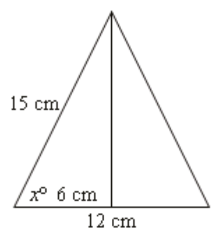


$\frac{20}{x} = \sin 60^\circ$

$20 = x \times \frac{\sqrt{3}}{2}$

$x = \frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3}$ cm

3



$\cos x^\circ = \frac{6}{15} = 0.4$

$x^\circ = 66.42^\circ$

The third angle $= 180^\circ - 2 \times 66.42^\circ$

$= 47.16^\circ$

4 $\frac{h}{20} = \tan 49^\circ$

$x = 20 \times 1.1503$

≈ 23 m

5 a $\sin \angle ACB = \frac{1}{6}$

$\angle ACB = 9.59^\circ$

b $BC^2 = 6^2 - 1^2 = 35$

$BC = \sqrt{35}$ m

$= 5.92$ m

6 a $\cos\theta = \dfrac{10}{20} = 0.5$

$\theta = 60°$

b $\dfrac{PQ}{20} = \sin 60°$

$PQ = 20 \times 0.866$

$= 17.32$ m

7 a $\dfrac{3}{L} = \sin 26°$

where L m is the length if the ladder

$3 = L \times 0.4383$

$L = \dfrac{3}{0.4383}$

$= 6.84$ m

b $\dfrac{3}{h} = \tan 26°$

where h m is the height above the ground.

$3 = h \times 0.4877$

$h = \dfrac{3}{0.4877}$

$= 6.15$ m

8 $\sin\theta = \dfrac{13}{60} = 0.21666\ldots$

$\theta = 12.51°$

9 $\dfrac{h}{200} = \sin 66°$

$x = 200 \times 0.9135$

$= 182.7$ m

10 $\dfrac{400}{d} = \sin 16°$

$400 = d \times 0.2756$

$d = \dfrac{400}{0.2756}$

$= 1451$ m

11 Since the diagonals are equal in length, the rhombus must be a square.

a $AC^2 = BC^2 + BA^2 = 2BC^2$

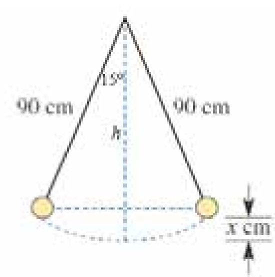
$100 = 2BC^2$

$BC^2 = 50$

$BC = \sqrt{50} = 5\sqrt{2}$ cm

b As the rhombus is a square, $\angle ABC = 90°$.

12 Find the vertical height, h cm.



$\dfrac{h}{90} = \cos 15°$

$h = 90 \times 0.9659$

$h = 86.93$ cm

$x = 90 - 86.93 = 3.07$ cm

13 $\dfrac{15}{\left(\dfrac{L}{2}\right)} = \sin 52.5°$

$$15 = \frac{L}{2} \times 0.7933$$

$$L = \frac{30°}{0.7933}$$

$$= 37.8 \text{ cm}$$

14 $\dfrac{w}{50} = \tan 32°$

$$w = 50 \times 0.6248$$

$$= 31.24 \text{ cm}$$

15 $h^2 + 1.7^2 = 4.7^2$

$$h^2 = 4.7^2 - 1.7^2$$

$$= 19.2$$

$$h = 4.38 \text{ m}$$

16 $\dfrac{50}{d} = \sin 60°$

$$50 = d \times 0.866$$

$$d = \frac{50}{0.866}$$

$$= 57.74 \text{ m}$$

17 Let length of the flagpole be l

$$\sin 60° = \frac{l}{l+2}$$

$$\frac{\sqrt{3}}{2} = \frac{l}{l+2}$$

$$(l+2)\frac{\sqrt{3}}{2} = l$$

$$(\tfrac{\sqrt{3}}{2} - 1)l = -\sqrt{3}$$

$$l = \frac{\sqrt{3}}{\dfrac{-\sqrt{3}}{2} - 1}$$

$$l = \frac{2\sqrt{3}}{2 - \sqrt{3}}$$

18 Let h be the length of the hypotenuse and y be the length of the opposite. Then Perimeter = 10 $\Rightarrow x + h + y = 10$

$$\cos 30° = \frac{x}{h}$$

$$h = \frac{x}{\cos 30°} = \frac{x}{\dfrac{\sqrt{3}}{2}} = \frac{2x}{\sqrt{3}}$$

$$\tan 30° = \frac{y}{x}$$

$$y = x \tan 30° = \frac{1}{\sqrt{3}}x$$

$$x + \frac{x}{\cos 30°} + x \tan 30° = 10$$

$$x + \frac{2x}{\sqrt{3}} + \frac{1}{\sqrt{3}}x = 10$$

$$(\sqrt{3} + 1)x = 10$$

$$x = \frac{10}{\sqrt{3}+1} = 5(\sqrt{3} - 1)$$

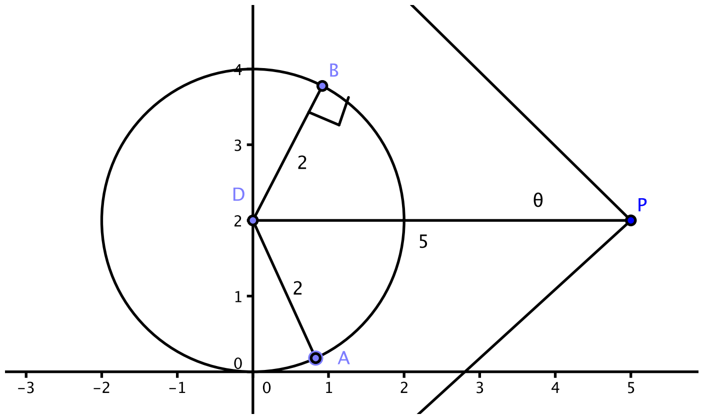
$$x \approx 3.66$$

19 In $\triangle PDB$, Let $\angle PBD = \theta$

Then $\sin\theta = \dfrac{2}{5}$

Hence $\theta = 23.578...^\circ$

$\angle APB = 2\theta \approx 47.16^\circ$



Solutions to Exercise 15B

1 **a** $\dfrac{x}{\sin 50°} = \dfrac{10}{\sin 70°}$

$$x = \frac{10 \times \sin 50°}{\sin 70°}$$

$$= 8.15 \text{ cm}$$

b $\dfrac{y}{\sin 37°} = \dfrac{6}{\sin 65°}$

$$x = \frac{6 \times \sin 37°}{\sin 65°}$$

$$= 3.98 \text{ cm}$$

c $\dfrac{x}{\sin 100°} = \dfrac{5.6}{\sin 28°}$

$$x = \frac{5.6 \times \sin 100°}{\sin 28°}$$

$$= 11.75 \text{ cm}$$

d $x = 180° - 38° - 90°$

$$= 52°$$

$$\frac{x}{\sin 52°} = \frac{12}{\sin 90°}$$

$$x = \frac{12 \times \sin 52°}{\sin 90°}$$

$$= 9.46 \text{ cm}$$

2 **a** $\dfrac{\sin\theta}{7} = \dfrac{\sin 72°}{8}$

$$\sin\theta = \frac{7 \times \sin 72°}{8}$$

$$= 0.8321$$

$$\theta = 56.32°$$

In this case θ cannot be obtuse. Since it is opposite a smaller side.

b $\dfrac{\sin\theta}{8.3} = \dfrac{\sin 42°}{9.4}$

$$\sin\theta = \frac{8.3 \times \sin 42°}{9.4}$$

$$= 0.5908$$

$$\theta = 36.22°$$

In this case θ cannot be obtuse. Since it is opposite a smaller side.

c $\dfrac{\sin\theta}{8} = \dfrac{\sin 108°}{10}$

$$\sin\theta = \frac{8 \times \sin 108°}{10}$$

$$= 0.7608$$

$$\theta = 49.54°$$

In this case θ cannot be obtuse. Since the given angle is obtuse.

d $\dfrac{\sin\theta}{9} = \dfrac{\sin 38°}{8}$

$$\sin\theta = \frac{9 \times \sin 38°}{8}$$

$$= 0.6929$$

$$\theta = 43.84° \text{ or } 180 - 43.84$$

$$= 131.16°$$

$$\theta = 180 - 43.84 - 38 = 98.16°$$

$$\text{or } 180 - 136.16 - 38 = 5.84°$$

3 a
$$\begin{aligned} A &= 180^\circ - 59^\circ - 73^\circ \\ &= 48^\circ \end{aligned}$$
$$\frac{b}{\sin 59^\circ} = \frac{12}{\sin 48^\circ}$$
$$\begin{aligned} b &= \frac{12 \times \sin 59^\circ}{\sin 48^\circ} \\ &= 13.84 \text{ cm} \end{aligned}$$
$$\frac{c}{\sin 73^\circ} = \frac{12}{\sin 48^\circ}$$
$$\begin{aligned} c &= \frac{12 \times \sin 73^\circ}{\sin 48^\circ} \\ &= 15.44 \text{ cm} \end{aligned}$$

b
$$\begin{aligned} C &= 180^\circ - 75.3^\circ - 48.25^\circ \\ &= 56.45^\circ \end{aligned}$$
$$\frac{a}{\sin 75.3^\circ} = \frac{5.6}{\sin 48.25^\circ}$$
$$\begin{aligned} a &= \frac{5.6 \times \sin 75.3^\circ}{\sin 48.25^\circ} \\ &= 7.26 \text{ cm} \end{aligned}$$
$$\frac{c}{\sin 56.45^\circ} = \frac{5.6}{\sin 48.25^\circ}$$
$$\begin{aligned} c &= \frac{5.6 \times \sin 56.45^\circ}{\sin 48.25^\circ} \\ &= 6.26 \text{ cm} \end{aligned}$$

c
$$\begin{aligned} B &= 180^\circ - 123.2^\circ - 37^\circ \\ &= 19.8^\circ \end{aligned}$$
$$\frac{b}{\sin 19.8^\circ} = \frac{11.5}{\sin 123.2^\circ}$$
$$\begin{aligned} b &= \frac{11.5 \times \sin 19.8^\circ}{\sin 123.2^\circ} \\ &= 4.66 \text{ cm} \end{aligned}$$
$$\frac{c}{\sin 37^\circ} = \frac{11.5}{\sin 123.2^\circ}$$
$$\begin{aligned} c &= \frac{11.5 \times \sin 37^\circ}{\sin 123.2^\circ} \\ &= 8.27 \text{ cm} \end{aligned}$$

d
$$\begin{aligned} C &= 180^\circ - 23^\circ - 40^\circ \\ &= 117^\circ \end{aligned}$$
$$\frac{b}{\sin 40^\circ} = \frac{15}{\sin 23^\circ}$$
$$\begin{aligned} b &= \frac{15 \times \sin 40^\circ}{\sin 23^\circ} \\ &= 24.68 \text{ cm} \end{aligned}$$
$$\frac{c}{\sin 117^\circ} = \frac{15}{\sin 23^\circ}$$
$$\begin{aligned} c &= \frac{15 \times \sin 117^\circ}{\sin 23^\circ} \\ &= 34.21 \text{ cm} \end{aligned}$$

e
$$\begin{aligned} C &= 180^\circ - 10^\circ - 140^\circ \\ &= 30^\circ \end{aligned}$$
$$\frac{a}{\sin 10^\circ} = \frac{20}{\sin 140^\circ}$$
$$\begin{aligned} a &= \frac{20 \times \sin 10^\circ}{\sin 140^\circ} \\ &= 5.40 \text{ cm} \end{aligned}$$
$$\frac{c}{\sin 30^\circ} = \frac{20}{\sin 140^\circ}$$
$$\begin{aligned} c &= \frac{20 \times \sin 30^\circ}{\sin 140^\circ} \\ &= 15.56 \text{ cm} \end{aligned}$$

4 a
$$\frac{\sin B}{17.6} = \frac{\sin 48.25^\circ}{15.3}$$
$$\begin{aligned} \sin B &= \frac{17.6 \times \sin 48.25^\circ}{15.3} \\ &= 0.8582 \end{aligned}$$
$$\begin{aligned} B &= 59.12^\circ \text{ or } 180^\circ - 59.12^\circ \\ &= 120.88^\circ \end{aligned}$$

$$\begin{aligned} A &= 180° - 48.25° - 59.12° \\ &= 72.63° \\ &\quad \text{or } 180 - 48.25° - 120.88° \\ &= 10.87° \end{aligned}$$

$$\frac{15.3}{\sin 48.25°} = \frac{a}{\sin 72.63°} \text{ or } \frac{a}{\sin 10.87°}$$

$$\begin{aligned} a &= \frac{15.3 \times \sin 72.63°}{\sin 48.25°} \\ &\quad \text{or } \frac{15.3 \times \sin 10.87°}{\sin 48.25°} \\ &= 19.57 \text{ cm or } 3.87 \text{ cm} \end{aligned}$$

b

$$\frac{\sin C}{4.56} = \frac{\sin 129°}{7.89}$$

$$\begin{aligned} \sin C &= \frac{4.56 \times \sin 129°}{7.89} \\ &= 0.4991 \\ C &= 26.69° \\ A &= 180° - 129° - 26.69° \\ &= 24.31° \end{aligned}$$

$$\frac{a}{\sin 24.31°} = \frac{7.89}{\sin 129°}$$

$$\begin{aligned} a &= \frac{7.89 \times \sin 24.31°}{\sin 129°} \\ &= 4.18 \text{ cm} \end{aligned}$$

c

$$\frac{\sin B}{14.8} = \frac{\sin 28.35°}{8.5}$$

$$\begin{aligned} \sin B &= \frac{14.8 \times \sin 28.35°}{85} \\ &= 0.8268 \end{aligned}$$

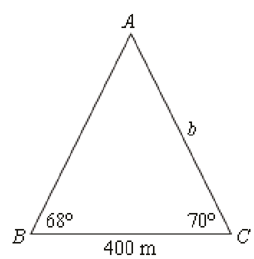
$$B = 55.77° \text{ or } 180 - 55.77 = 124.23°$$

$$\begin{aligned} C &= 180° - 55.77° - 28.35° = 95.88° \\ &\quad \text{or } 180° - 124.23° - 28.35° \\ &= 27.42° \end{aligned}$$

$$\frac{8.5}{\sin 28.35°} = \frac{c}{\sin 95.88°} \text{ or } \frac{c}{\sin 27.42°}$$

$$\begin{aligned} c &= \frac{8.5 \times \sin 95.88°}{\sin 28.35°} \\ &\quad \text{or } \frac{8.5 \times \sin 27.42°}{\sin 28.35°} \\ &= 17.81 \text{ cm or } 8.24 \text{ cm} \end{aligned}$$

5



$$\begin{aligned} A &= 180° - 68° - 70° \\ &= 42° \end{aligned}$$

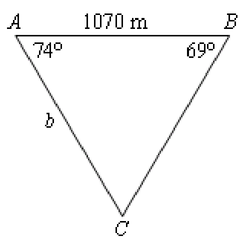
$$\frac{b}{\sin 68°} = \frac{400}{\sin 42°}$$

$$\begin{aligned} b &= \frac{400 \times \sin 68°}{\sin 42°} \\ &= 554.26 \text{ m} \end{aligned}$$

6

$$\angle APB = 46.2^\circ - 27.6^\circ$$
$$= 18.6^\circ \text{ (exterior angle property)}$$
$$\frac{a}{\sin 27.6^\circ} = \frac{34}{\sin 18.6^\circ}$$
$$PB = a = \frac{34 \times \sin 27.6^\circ}{\sin 18.6^\circ}$$
$$= 49.385 \text{ m}$$
$$\frac{h}{PB} = \sin 46.2^\circ$$
$$h = 49.385 \times 0.7217$$
$$= 35.64 \text{ m}$$

7



$$C = 180^\circ - 69^\circ - 74^\circ$$
$$= 37^\circ$$
$$\frac{b}{\sin 69^\circ} = \frac{1070}{\sin 37^\circ}$$
$$b = \frac{1070 \times \sin 69^\circ}{\sin 37^\circ}$$
$$= 1659.86 \text{ m}$$

8 **a**

$$X = 180^\circ - 120^\circ - 20^\circ$$
$$= 40^\circ$$
$$\frac{AX}{\sin 20^\circ} = \frac{50}{\sin 40^\circ}$$
$$= \frac{50 \times \sin 20^\circ}{\sin 40^\circ}$$
$$= 26.60 \text{ m}$$

b

$$Y = 180^\circ - 109^\circ - 32^\circ$$
$$= 39^\circ$$
$$\frac{AY}{\sin 109^\circ} = \frac{50}{\sin 39^\circ}$$
$$AY = \frac{50 \times \sin 109^\circ}{\sin 39^\circ}$$
$$= 75.12 \text{ m}$$

9 **a** By the sine rule we find that

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$
$$= \frac{\sin A}{\sin C} + \frac{\sin B}{\sin C}$$
$$= \frac{\sin A + \sin B}{\sin C}.$$

b This one is pretty much the same:

$$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$
$$= \frac{\sin A}{\sin C} - \frac{\sin B}{\sin C}$$
$$= \frac{\sin A - \sin B}{\sin C}.$$

Solutions to Exercise 15C

1 $$\begin{aligned} BC^2 &= a^2 \\ &= b^2 + c^2 - 2bc\cos A \\ &= 15^2 + 10^2 - 2 \times 15 \times 10 \\ &\quad \times \cos 15^\circ \\ &= 325 - 300 \times \cos 15^\circ \\ &= 35.222 \\ BC &= 5.93 \text{ cm} \end{aligned}$$

2 $$\begin{aligned} \angle ABC &= \angle B \\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{5^2 + 8^2 - 10^2}{2 \times 5 \times 8} \\ &= -0.1375 \\ \therefore\ \angle ABC &\approx 97.90^\circ \\ \angle ACB &= \angle C \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ac} \\ &= \frac{5^2 + 10^2 - 8^2}{2 \times 5 \times 10} \\ &= 0.61 \\ \therefore\ \angle ACB &\approx 52.41^\circ \end{aligned}$$

3 **a** $$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos a \\ &= 16^2 + 30^2 - 2 \times 16 \times 30 \\ &\quad \times \cos 60^\circ \\ &= 1156 - 960 \times \cos 60^\circ \\ &= 676 \\ a &= 26 \end{aligned}$$

b $$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 14^2 + 12^2 - 2 \times 14 \times 12 \\ &\quad \times \cos 53^\circ \\ &= 340 - 336 \times \cos 53^\circ \\ &= 137.7901 \\ a &\approx 11.74 \end{aligned}$$

c $$\begin{aligned} \angle ABC &= \angle B \\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{27^2 + 46^2 - 35^2}{2 \times 27 \times 46} \\ &= 0.6521 \\ \therefore\ \angle ABC &\approx 49.29^\circ \end{aligned}$$

d $$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 17^2 + 63^2 - 2 \times 17 \\ &\quad \times 63 \times \cos 120^\circ \\ &= 4258 - 2142 \times \cos 120^\circ \\ &= 5329 \\ b &= 73 \end{aligned}$$

e $$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 31^2 + 42^2 - 2 \times 31 \\ &\quad \times 42 \times \cos 140^\circ \\ &= 2642 - 2604 \times \cos 140^\circ \\ &= 4719.77 \\ c &\approx 68.70 \end{aligned}$$

f $\angle BCA = \angle C$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{10^2 + 12^2 - 9^2}{2 \times 10 \times 12}$$

$$= 0.6791$$

$$\therefore \ \angle BCA \approx 47.22^\circ$$

g $c^2 = a^2 + b^2 - 2ab\cos C$

$$= 11^2 + 9^2 - 2 \times 11 \times 9 \times \cos 43.2^\circ$$

$$= 202 - 198 \times \cos 43.2^\circ$$

$$= 57.6642$$

$$c \approx 7.59$$

h $\angle CBA = \angle B$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{8^2 + 15^2 - 10^2}{2 \times 8 \times 15}$$

$$= 0.7875$$

$$\therefore \ \angle ABC \approx 38.05^\circ$$

4 $c^2 = a^2 + b^2 - 2ab\cos C$

$$= 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 20^\circ$$

$$= 52 - 48 \times \cos 20^\circ$$

$$= 6.8947$$

$$c \approx 2.626 \text{ km}$$

5 $AB^2 = a^2 + b^2 - 2ab\cos O$

$$= 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 30^\circ$$

$$= 52 - 48 \times \cos 30^\circ$$

$$= 10.4307$$

$$AB \approx 3.23 \text{ km}$$

6 Label the points suitably: A and B are the hooks, and C is the 70° angle.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$BD^2 = 42^2 + 54^2 - 2 \times 42 \times 54 \times \cos 70^\circ$$

$$= 4680 - 4536 \times \cos 70^\circ$$

$$= 3128.5966$$

$$BD \approx 55.93 \text{ cm}$$

7 **a** $\angle B = 180^\circ - 48^\circ = 132^\circ$

$$AC^2 = a^2 + c^2 - 2ac\cos B$$

$$= 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos 132^\circ$$

$$= 41 - 40 \times \cos 132^\circ$$

$$= 67.7652$$

$$AC \approx 8.23 \text{ cm}$$

b $BD^2 = b^2 + d^2 - 2bd\cos A$

$$= 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos 48^\circ$$

$$= 41 - 40 \times \cos 48^\circ$$

$$= 14.2347$$

$$BD \approx 3.77 \text{ cm}$$

8 **a** Use $\triangle ABD$.

$$BD^2 = b^2 + d^2 - 2bd\cos A$$

$$= 6^2 + 4^2 - 2 \times 6 \times 4 \times \cos 92^\circ$$

$$= 52 - 48 \times \cos 92^\circ$$

$$= 53.6751$$

$$BD \approx 7.326 \text{ cm}$$

b $\angle D = \angle BDC$

$$\frac{\sin D}{5} = \frac{\sin 88^\circ}{7.3263}$$

$$\sin D = \frac{5 \times \sin 88^\circ}{7.3263}$$

$$= 0.6820$$

$$D = 43.0045^\circ$$

$$B = 180^\circ - 88^\circ - 43.0045^\circ$$

$$= 48.9954^\circ$$

$$\frac{b}{\sin 48.9954^\circ} = \frac{7.3263}{\sin 88^\circ}$$

$$b = \frac{7.3263 \times \sin 48.9956^\circ}{\sin 88^\circ}$$

$$\approx 5.53 \text{ cm}$$

9 a $\cos \angle AO'B = \dfrac{6^2 + 6^2 - 8^2}{2 \times 6 \times 6}$

$$= 0.111$$

$$\angle AO'B \approx 83.62^\circ$$

b $\cos \angle AOB = \dfrac{7.5^2 + 7.5^2 - 8^2}{2 \times 7.5 \times 7.5}$

$$= 0.43111$$

$$\angle AOB \approx 64.46^\circ$$

10 a Treat AB as c.

$$c^2 = a^2 + b^2 - 2ab \cos O$$

$$AB^2 = 70^2 + 90^2 - 2 \times 70 \times 90 \times \cos 65^\circ$$

$$= 13\,000 - 12\,600 \times \cos 65^\circ$$

$$= 7675.0099$$

$$AB \approx 87.61 \text{ m}$$

b $\cos \angle B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$$= \frac{70^2 + 87.6071^2 - 90^2}{2 \times 70 \times 87.6071}$$

$$= 0.3648$$

$$\angle AOB \approx 68.6010^\circ$$

Now use $\triangle OCB$.

Let $CB = a,\ OB = b,\ OC = c$.

$$CB = \frac{AB}{2} = 43.80$$

$$c^2 = a^2 + b^2 - 2ab \cos O$$

$$OC^2 = 43.8035^2 + 70^2 - 2 \times 43.8035 \times 70 \times 0.3648$$

$$= 4581.24$$

$$OC \approx 67.7 \text{ m}$$

Solutions to Exercise 15D

1 **a**
$$\begin{aligned}\text{Area} &= \frac{1}{2}ab\sin C\\ &= \frac{1}{2}\times 6\times 4\times \sin 70^\circ\\ &= 11.28\text{ cm}^2\end{aligned}$$

b
$$\begin{aligned}\text{Area} &= \frac{1}{2}yz\sin X\\ &= \frac{1}{2}\times 5.1\times 6.2\times \sin 72.8^\circ\\ &= 15.10\text{ cm}^2\end{aligned}$$

c
$$\begin{aligned}\text{Area} &= \frac{1}{2}nl\sin M\\ &= \frac{1}{2}\times 3.5\times 8.2\times \sin 130^\circ\\ &= 10.99\text{ cm}^2\end{aligned}$$

d
$$\angle C = 180 - 25 - 25 = 130^\circ$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}ab\sin C\\ &= \frac{1}{2}\times 5\times 5\times \sin 130^\circ\\ &= 9.58\text{ cm}^2\end{aligned}$$

2 **a** Use the cosine rule to find $\angle B$. (Any angle will do.)
$$\begin{aligned}\cos\angle B &= \frac{a^2 + c^2 - b^2}{2ac}\\ &= \frac{3.2^2 + 4.1^2 - 5.9^2}{2\times 3.1\times 4.1}\\ &= -0.2957\\ \angle B &= 107.201^\circ\end{aligned}$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}ac\sin B\\ &= \frac{1}{2}\times 3.2\times 4.1\\ &\quad\times \sin 107.201^\circ\\ &\approx 6.267\text{ cm}^2\end{aligned}$$

b Use the sine rule to fmd $\angle C$.
$$\begin{aligned}\frac{\sin C}{7} &= \frac{\sin 100^\circ}{9}\\ \sin C &= \frac{7\times\sin 100^\circ}{9}\\ &= 0.7659\\ C &= 49.992^\circ\\ \angle A &= 180^\circ - 100^\circ - 49.992^\circ\\ &= 30.007^\circ\end{aligned}$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}bc\sin A\\ &= \frac{1}{2}\times 9\times 7\times \sin 30.007^\circ\\ &\approx 15.754\text{ cm}^2\end{aligned}$$

c
$$\begin{aligned}E &= 180^\circ - 65^\circ - 66^\circ\\ &= 60^\circ\end{aligned}$$
$$\begin{aligned}\frac{e}{\sin 60^\circ} &= \frac{6.3}{\sin 55^\circ}\\ e &= \frac{6.3\times\sin 60^\circ}{\sin 55^\circ}\\ &= 6.6604\text{ cm}\end{aligned}$$
$$\begin{aligned}\text{Area} &= \frac{1}{2}ef\sin D\\ &= \frac{1}{2}\times 6.6604\times 6.3\times \sin 65^\circ\\ &\approx 19.015\text{ cm}^2\end{aligned}$$

d Use the cosine rule to find $\angle D$.

$$\begin{aligned}\cos\angle D &= \frac{e^2 + f^2 - d^2}{2ef}\\ &= \frac{5.1^2 + 5.7^2 - 5.9^2}{2 \times 5.1 \times 5.7}\\ &= -0.4074\\ \angle D &= 65.95^\circ\\ \text{Area} &= \frac{1}{2}ef\sin D\\ &= \frac{1}{2} \times 5.1 \times 5.7 \times \sin 65.95^\circ\\ &\approx 13.274\text{ cm}^2\end{aligned}$$

e

$$\begin{aligned}\frac{\sin I}{12} &= \frac{\sin 24^\circ}{5}\\ \sin I &= \frac{12 \times \sin 24^\circ}{5}\\ &= 0.9671\\ I &= 77.466^\circ \text{ or } 180^\circ - 74.466^\circ\\ &= 102.533^\circ\\ G &= 180^\circ - 24^\circ - 108.533^\circ\\ &\quad\text{or } 180^\circ - 24^\circ - 77.466^\circ\\ &= 53.466^\circ \text{ or } 78.534^\circ\\ \text{Area} &= \frac{1}{2}hi\sin G\\ &= \frac{1}{2} \times 5 \times 12 \times \sin 53.466^\circ\\ &\quad\text{or } \frac{1}{2} \times 5 \times 12 \times \sin 78.534^\circ\\ &\approx 24.105\text{ cm}^2 \text{ or } 29.401\text{ cm}^2\end{aligned}$$

Note that although the diagram is drawn as if I is obtuse, you should not make this assumption. Diagrams are not neccessarily drawn to scale.

f

$$\begin{aligned}I &= 180 - 10 - 19\\ &= 151^\circ\\ \frac{i}{\sin 151^\circ} &= \frac{4}{\sin 19^\circ}\\ i &= \frac{4 \times \sin 151^\circ}{\sin 19^\circ}\\ &= 5.9564\\ \text{Area} &= \frac{1}{2}ih\sin G\\ &= \frac{1}{2} \times 5.9564 \times 4 \times \sin 10^\circ\\ &\approx 2.069\text{ cm}^2\end{aligned}$$

Solutions to Exercise 15E

1 $l = \frac{105}{360} \times 2\pi r$

$= \frac{105}{360} \times 2 \times \pi \times 25$

≈ 45.81 cm

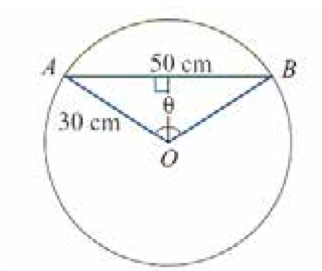
2 **a** $\theta = \frac{50}{30°} = \frac{5}{3}$ radians

$= \frac{5}{3} \times \frac{180}{\pi}$ degrees

$= 95.4929°$

$= 95°30'$

b



$\sin\frac{\theta}{2} = \frac{25}{30°} = 0.8333$

$\frac{\theta}{2} = 56.4426°$

$\theta = 112.885°$

$= 112°53'$

3 **a** Set your calculator to radian mode.

$\sin\frac{\theta}{2} = \frac{3}{7} = 0.4285$

$\frac{\theta}{2} = 0.4429$

$\theta = 0.8858$

$l = r\theta$

$= 7 \times 0.8858$

$= 6.20$ cm

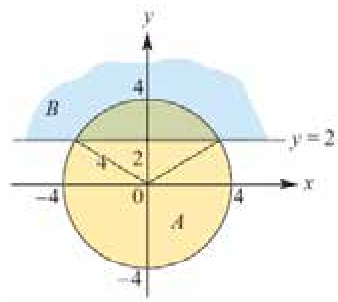
b This represents the minor segment area.

$A = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \times 7^2 \times (0.8858 - \sin\ 0.8858)$

$= 2.73\ \text{cm}^2$

4 A represents the interior of a circle of radius 4, centre the origin.



$\cos\frac{\theta}{2} = \frac{2}{4} = \frac{1}{2}$

$\frac{\theta}{2} = \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$

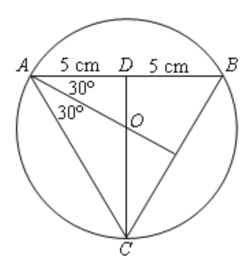
$A \cap B$ is a segment where $r = 4,\ \theta = \frac{2\pi}{3}$

$A = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \times 4^2 \times \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= 9.83\ \text{cm}^2$

5



Altitude $CD = 5\tan 60°$

$= 5\sqrt{3}$ cm

$OD = 5\tan 30°$

$= \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$ cm

Radius $= 5\sqrt{3} - \frac{5\sqrt{3}}{3}$

$= \frac{15\sqrt{3} - 5\sqrt{3}}{3}$

$= \frac{10\sqrt{3}}{3}$ cm

$\angle AOD = 60°$

$\therefore\ \angle AOB = 120° = \frac{2\pi}{3}$ radians

Area = 3 × segment area

$= \frac{3}{2} \times r^2 \times (\theta - \sin\theta)$

$= \frac{3}{2} \times \frac{300}{9} \times \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= 50\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

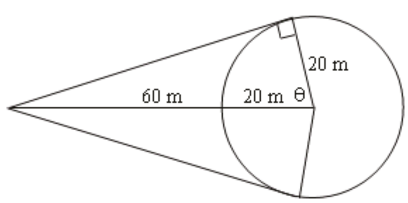
$= 61.42$ cm^2

6 a $C = 2\pi r$

$= 2 \times \pi \times 20$

$= 40\pi \approx 125.66$ m

b



$\cos\theta = \frac{20}{20 + 60} = 0.25$

$\theta = 1.3181$ radians

$2\theta = 2.6362$

Proportion visible $= \frac{2.6362}{2\pi}$

$= 0.41956$

$\approx 41.96\%$

7 a Use fractions of an hour (minutes).

$l = \frac{25}{60} \times 2\pi r$

$= \frac{25}{60} \times 2 \times \pi \times 4$

$= \frac{10\pi}{3} \approx 10.47$ m

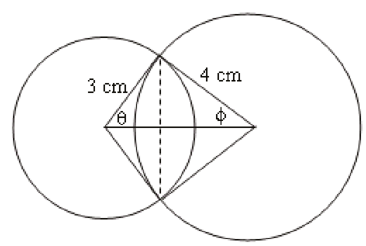
b Angle $= \frac{25}{60} \times 2\pi = \frac{5\pi}{6}$

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 4^2 \times \frac{5\pi}{6}$

$= \frac{20\pi}{3} \approx 20.94$ m^2

8



The required area is the sum of two segments.

Let the left area be A_1 and the right area A_2.

$\tan\theta = \frac{4}{3}$

$\theta = 0.9272$

$2\theta = 1.8545$

$A_1 = \frac{1}{2} \times 3^2 \times (1.8545 - \sin\ 1.8545)$

$= 4.0256$

$$\tan\phi = \frac{3}{4}$$

$$\phi = 0.6435$$

$$2\phi = 1.2870$$

$$A_2 = \frac{1}{2} \times 4^2 \times (1.2870 - \sin\ 1.2870)$$

$$= 2.6160$$

Total area = 4.0256 + 2.6160

$= 6.64\ \text{cm}^2$

9

$$A = \frac{1}{2}r^2\theta = 63$$

$$r^2\theta = 126$$

$$\theta = \frac{126}{r^2}$$

$$P = r + r + r\theta = 32$$

$$2r + r \times \frac{126}{r^2} = 32$$

$$2r + \frac{126}{r} = 32$$

$$2r^2 + 126 = 32r$$

$$2r^2 - 32r + 126 = 0$$

$$r^2 - 16r + 63 = 0$$

$$(r - 7)(r - 9) = 0$$

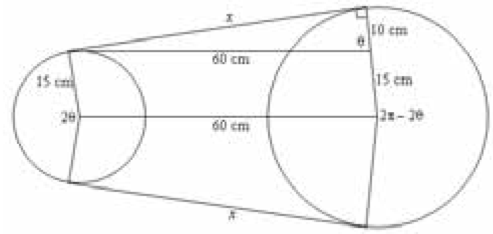
$$r = 7 \text{ or } 9 \text{ cm}$$

$$\theta = \frac{126}{r^2}$$

$$\text{When} r = 7,\ \theta = \frac{126}{7^2} = \left(\frac{18}{7}\right)^c$$

$$\text{When} r = 9,\ \theta = \frac{126}{9^2} = \left(\frac{14}{9}\right)^c$$

10 The following diagram can be deduced from the data:



$$x^2 = 60^2 - 10^2 = 3500$$

$$x = 10\sqrt{35}$$

$$\cos\theta = \frac{10}{60} = \frac{1}{6}$$

$$\theta = 1.4033$$

$$2\theta = 2.8066$$

$$2\pi - 2\theta = 3.4764$$

Length of belt on left wheel:

$l = r\theta$

$= 15 \times 2.8066 = 42.1004$

Length of belt on right wheel:

$l = r\theta$

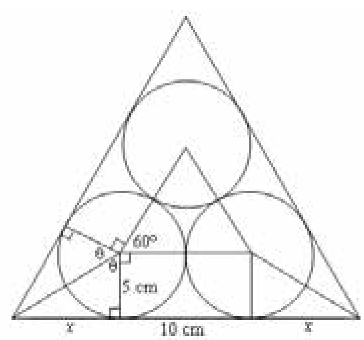
$= 25 \times 3.4764 = 86.9122$

$$\text{Total} = 12 \times 10\sqrt{25} + 42.1004$$

$$+ 86.9112$$

$$\approx 247.33 \text{ cm}$$

11 a



The balls can be enclosed as in the diagram above.

$$2\theta = 360 - 90 - 60 - 90$$
$$= 120°$$
$$\theta = 60°$$
$$\frac{x}{5} = \tan 60° = \sqrt{3}$$
$$x = 5\sqrt{3}$$
$$\text{Perimeter} = 6 \times 5\sqrt{3} + 3 \times 10$$
$$\approx 81.96 \text{ cm}$$

b Height of large triangle
$$= (2x + 10) \times \sin 60°$$
$$= (10\sqrt{3} + 10) \times \frac{\sqrt{3}}{2}$$
$$= 15 + 5\sqrt{3} \text{ cm}$$

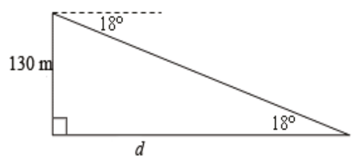
Area of large triangle
$$= \frac{1}{2}(10\sqrt{3} + 10)(15 + 5\sqrt{3})$$
$$\approx 173.2050 \text{ cm}^2$$

Area of three discs = 10 cm triangle − half a circle

Height of 10 cm triangle
$$= 10 \times \sin 60°$$
$$= 5\sqrt{3} \text{ cm}$$
$$\text{Area} = \frac{1}{2} \times 10 \times 5\sqrt{3} - \frac{1}{2} \times \pi \times 5^2$$
$$= 50\sqrt{3} - 12.5\pi$$
$$\approx 4.03 \text{ cm}^2$$

Solutions to Exercise 15F

1

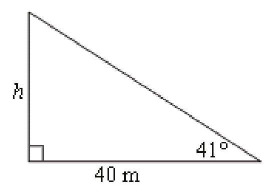


$$\frac{130°}{d} = \tan 18°$$

$$d = \frac{130°}{\tan 18°}$$

$$= 400.10 \text{ m}$$

2

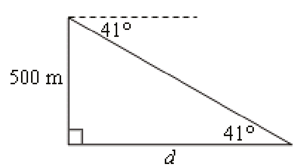


$$\frac{h}{40} = \tan 41°$$

$$h = 40 \times 0.869$$

$$= 34.77 \text{ m}$$

3

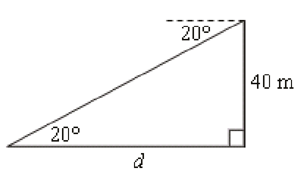


$$\frac{500}{d} = \tan 41°$$

$$d = \frac{500}{\tan 41°}$$

$$= 575.18 \text{ m}$$

4

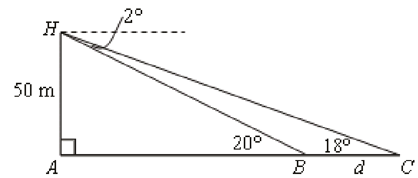


$$\frac{40}{d} = \tan 20°$$

$$d = \frac{40}{\tan 20°}$$

$$= 109.90 \text{ m}$$

5



$$\frac{50}{AB} = \tan 20°$$

$$AB = \frac{50}{\tan 20°}$$

$$= 137.373 \text{ m}$$

$$\frac{50}{AC} = \tan 18°$$

$$AC = \frac{50}{\tan 18°}$$

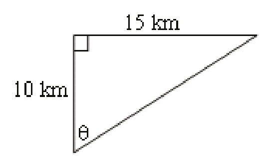
$$= 153.884 \text{ m}$$

$$d = AC - AB$$

$$= 153.884 - 137.373$$

$$\approx 16.51 \text{ m}$$

6

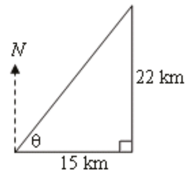


$$\tan\theta = \frac{15}{10} = 1.5$$

$$\theta \approx 56°$$

The bearing is 056°.

7 **a**



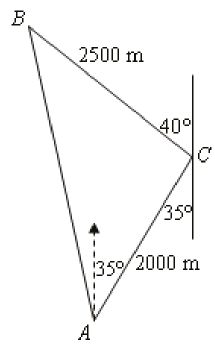
$$\tan\theta = \frac{22}{15} = 1.466$$

$$\theta = 55.713°$$

The bearing is $90° - \theta \approx 034°$.

b $180° + 34° = 214°$

8



a Use the cosine rule, where

$\angle C = 180 - 40 - 35 = 105°$

$$\begin{aligned} AB^2 &= c^2 \\ &= a^2 + b^2 - 2ab\cos C \\ &= 2500^2 + 2000^2 \\ &\quad - 2 \times 2500 \times 2000 \times \cos 105° \\ &= 12\,838\,190.4510 \end{aligned}$$

$AB = 3583.04$ m

b $\dfrac{2500}{\sin A} = \dfrac{3583.04}{\sin 105°}$

$A = 42.38°$

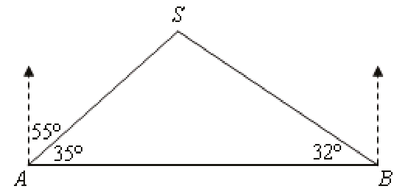
$\therefore$ bearing of B from A

$= (360 - 42.38 + 35)°$

$\approx 353°$

9 $207° - 180° = 027°$

10

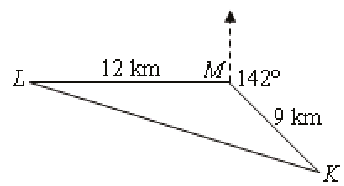


$\angle SAB = 90° - 55° = 35°$

$\angle SBA = 302° - 270° = 32°$

$\angle ASB = 180° - 35° - 32° = 113°$

11



$$\begin{aligned} \angle LMK &= 360° - 90° - 142° \\ &= 128° \end{aligned}$$

First, use the cosine rule to find LK.

$$\begin{aligned} LK^2 &= m^2 \\ &= k^2 + l^2 - 2kl\cos M \\ &= 12^2 + 9^2 - 2 \times 12 \times 9 \times \cos 128° \\ &= 357.9829 \end{aligned}$$

$LK = 18.920$

It is easier to use the sine rule to find $\angle MLK$.

$$\frac{\sin L}{9} = \frac{\sin 128°}{18.920}$$

$$\sin L = \frac{\sin 128° \times 9}{18.920}$$

$$= 0.3748$$

$$\begin{aligned} \angle MLK &= \angle L \\ &\approx 22.01° \end{aligned}$$

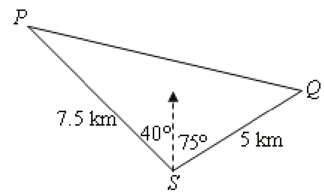
12 **a** $\angle BAN = 360° - 346° = 14°$

$\angle BAC = 14° + 35° = 49°$

b Use the cosine rule:

$$\begin{aligned}BC^2 &= a^2\\ &= b^2 + c^2 = 2bc\cos A\\ &= 340^2 + 160^2 - 2\times 340\\ &\quad \times 160\times\cos 49^\circ\\ &= 69\,820.7776\\ BC &= 264.24\text{ km}\end{aligned}$$

13



Use the cosine rule:

$$\begin{aligned}\angle PSQ &= 115^\circ\\ PQ^2 &= s^2\\ &= p^2 + q^2 - 2pq\cos A\\ &= 5^2 + 7.5^2 - 2\times 5\\ &\quad \times 7.5\times\cos 115^\circ\\ &= 112.9464\\ PQ &= 10.63\text{ km}\end{aligned}$$

Solutions to Exercise 15G

1 a $FH^2 = 12^2 + 5^2$
$= 169$
$FH = 13$ cm

b $BH^2 = 13^2 + 8^2$
$= 233$
$BH = \sqrt{233} \approx 15.26$ cm

c $\tan \angle BHF = \dfrac{8}{13}$
$= 0.615$
$\angle BHF = 31.61°$

d $\angle BGH = 90°$ and $BH = \sqrt{233}$
$\cos \angle BGH = \dfrac{12}{\sqrt{233}}$
$= 0.786$
$\angle BGH = 38.17°$

2 a $AB = 2EF$
$EF = 4$ cm

b $\tan \angle VEF = \dfrac{VE}{EF}$
$= \dfrac{12}{4} = 3$
$\angle VEF = 71.57°$

c $VE^2 = 4^2 + 12^2$
$= 160$
$VE = \sqrt{160}$
$= 4\sqrt{10} \approx 12.65$ cm

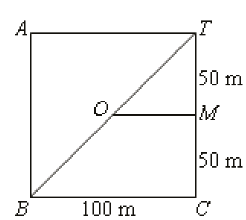
d All sloping sides are equal in length. Choose VA.

$VA^2 = VE^2 + EA^2$
$= 160 + 4^2 = 176$
$VA = \sqrt{176}$
$= 4\sqrt{11} \approx 13.27$ cm

e $\angle VAD = \angle VAE$
$\tan \angle VAE = \dfrac{VE}{EA}$
$= \dfrac{4\sqrt{10}}{4}$
$= \sqrt{10} \approx 3.162$
$\angle VAE = 72.45°$

f Area of a triangular face
$= \dfrac{1}{2} \times AD \times VE$
$= \dfrac{1}{2} \times 8 \times 4\sqrt{10}$
$= 16\sqrt{10}$ cm^2
Area of base $= 8 \times 8 = 64$ cm^2
Surface area $= 4 \times 16\sqrt{10} + 64$
≈ 266.39 cm^2

3 First, sketch the square base, and find the height h of the tree. Mark M as the mid-point of TC and O as the centre of the square.



$OM = TM = 50$ m

$$OT^2 = 50^2 + 50^2 = 5000$$

$$OT = \sqrt{5000} \text{ m}$$

$$\frac{h}{\sqrt{5000}} = \tan 20°$$

$$h = \sqrt{5000} \times \tan 20°$$

$$= 25.7365$$

At A and C,

$$\tan\theta = \frac{25.7365}{100} = 0.2573$$

$$\theta = 14.43°$$

At B, $TB = 2 \times OT = 2\sqrt{5000}$ m

$$\tan\theta = \frac{25.7365}{\sqrt{5000}} = 0.1819$$

$$\theta = 10.31°$$

4 **a** $\angle ABC = 180° - 90° - 45°$

$$= 45°$$

ABC is isosceles, and

$CB = AC = 85$ m.

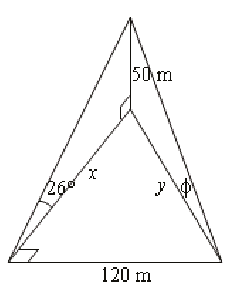
b $$\frac{XB}{BC} = \sin 32°$$

$$\frac{XB}{85} = \sin 32°$$

$$XB = 85 \times \sin 32°$$

$$= 45.04 \text{ m}$$

5



$$\frac{50}{x} = \tan 26°$$

$$x = \frac{50}{\tan 26°}$$

$$= 102.515 \text{ m}$$

$$y^2 = x^2 + 120^2$$

$$= 24\,909.364$$

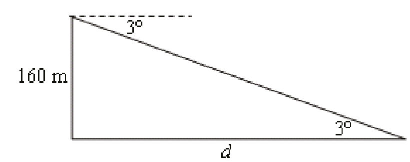
$$y = \sqrt{24\,909.364}$$

$$= 157.827 \text{ m}$$

$$\tan\phi = \frac{50}{y} = 0.316$$

$$\phi = 17.58°$$

6 From the top of the cliff:



For the first buoy:

$$\frac{160}{d} = \tan 3°$$

$$d = \frac{160}{\tan 3°}$$

$$= 3052.981 \text{ m}$$

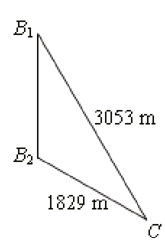
For the second buoy

$$\frac{160}{d} = \tan 5°$$

$$d = \frac{160}{\tan 5°}$$

$$= 1828.808 \text{ m}$$

From the cliff:



$\angle C = 337 - 308 = 29°$

Use the cosine rule.

$$c^2 = 3052.981^2 + 1828.808^2 - 2 \times 3052.981 \times 1828.808 \times \cos 29^\circ$$
$$= 2\,898\,675.1436$$
$$c = 1702.55 \text{ m}$$

7 a $AC^2 = 12^2 + 5^2 = 169$

$AC = 13$ cm

$\tan \angle ACE = \dfrac{6}{13}$

$= 0.4615$

$\angle ACE = 24.78^\circ$

b Triangle HDF is identical (congruent) to triangle AEC.

$\therefore \angle HFD = \angle ACE$

$\angle HDF = 90^\circ - 24.28^\circ$

$= 65.22^\circ$

c $CH^2 = 12^2 + 6^2 = 180$

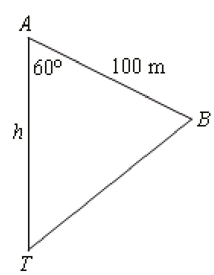
$CH = \sqrt{180}$

$\tan \angle ECH = \dfrac{EH}{CH}$

$= \dfrac{5}{\sqrt{180}} = 0.3726$

$\angle ECH = 20.44^\circ$

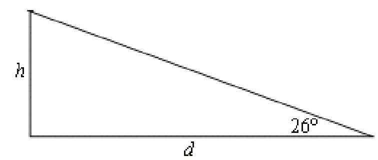
8 Looking from above, the following diagram applies.



Because the angle of elevation is 45°, AT will equal the height of the tower, h m. Use the cosine rule.

$$BT^2 = h^2 + 100^2 - 2 \times h \times 100 \times \cos 60^\circ$$
$$= h^2 + 100^2 - 200h \times \frac{1}{2}$$
$$= h^2 - 100h + 100^2$$

From point B:

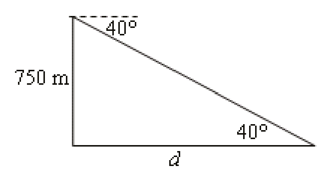


$$\frac{h}{d} = \tan 26^\circ$$
$$d = \frac{h}{\tan 26^\circ}$$
$$= 2.050h$$
$$\therefore\quad 2.050^2h^2 = h^2 - 100h + 100^2$$
$$4.2037h^2 = h^2 - 100h + 10\,000$$
$$3.2037h^2 + 100h = 10\,000$$

Using the quadratic formula:

$h \approx 42.40$ m

9 Find the horizontal distance of A from the balloon.



$$\frac{750}{d} = \tan 40^\circ$$
$$d = \frac{750}{\tan 40^\circ}$$
$$= 893.815 \text{ m}$$

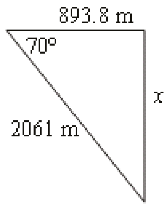
The distance of B from the balloon may be calculated in the same way:

$$\frac{750}{d} = \tan 20^\circ$$

$$d = \frac{750}{\tan 20^\circ}$$

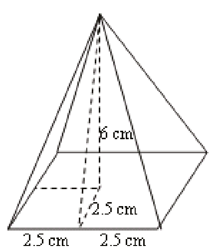
$$= 2060.608 \text{ m}$$

Draw the view from above and use the cosine rule.



$$x^2 = 893.8152 + 2060.6082$$
$$- 2 \times 893.815 \times 2060.608$$
$$\times \cos 70^\circ$$
$$= 3\,785\,143.5836$$
$$x = 1945.54 \text{ m}$$

10 a Find the length of an altitude:



$$a^2 = 2.5^2 + 6^2 = 42.45$$
$$a \approx 6.5 \text{ cm}$$

The sloping edges are also the hypotenuse of a right-angled triangle.

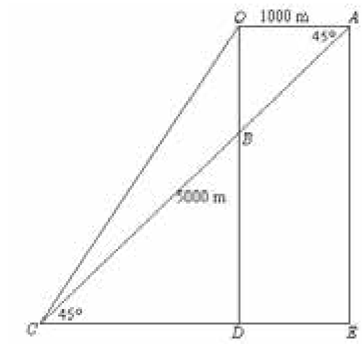
$$s^2 = 2.5^2 + 6.5^2 = 48.5$$
$$s \approx 6.96 \text{ cm}$$

b Area $= \frac{1}{2} \times 5 \times 6.5$

$$= 16.25 \text{ cm}^2$$

11 a Distance $= 300 \times \frac{1}{60} = 5$ km

b Looking from above:



$$AE = 5000 \times \sin 45^\circ$$
$$= \frac{5000}{\sqrt{2}} \approx 3535.433$$
$$CE = 5000 \times \sin 45^\circ$$
$$= \frac{5000}{\sqrt{2}} \approx 3535.433$$
$$CD = CE - DE$$
$$= 3535.533 - 1000$$
$$= 2535.533$$
$$\tan \angle COD = \frac{2535.533}{3535.533}$$
$$= 0.7171$$
$$\angle COD = 35.65^\circ$$
$$\text{Bearing} = 180^\circ + 35.65^\circ$$
$$= 215.65^\circ$$

c Let the angle of elevation be θ.

$$OC^2 = 3535.533^2 + 2535.533^2$$
$$= 18\,928\,932$$
$$OC = 4350.739$$
$$\tan \theta = \frac{500}{4350.739}$$
$$= 0.1149$$
$$\theta = 6.56^\circ = 6^\circ 33'$$

Solutions to Exercise 15H

1 **a** Area $ABFE = AB \times GC$

$= 4a \times a = 4a^2$ units

Area $BCGF = BC \times GC$

$= 3a \times a = 3a^2$ units

Area $ABCD = AB \times BC$

$= 4a \times 3a = 12a^2$ units

b This is equivalent to $\angle FAB$.

$$\tan \angle FAB = \frac{FB}{AB}$$
$$= \frac{a}{4a} = 0.25$$
$$\angle FAB = 14.04°$$

c $$\tan \angle GBC = \frac{GC}{BC}$$
$$= \frac{a}{3a} = 0.333$$
$$\angle GBC = 18.43°$$

d $$AC = \sqrt{(4a)^2 + (3a)^2}$$
$$= \sqrt{25a^2} = 5a$$
$$\tan \angle GAC = \frac{GC}{AC}$$
$$= \frac{a}{5a} = 0.2$$
$$\angle GAC = 11.31°$$

2 **a** Let the altitude of triangle FAB be a.

$$s = \sqrt{a^2 + a^2}$$
$$= \sqrt{2a^2} = a\sqrt{2}$$
$$\sin \angle VA0 = \frac{OV}{VA}$$
$$= \frac{a}{a\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}} = 0.577$$
$$\angle VA0 = 35.26°$$

b This will be the slope of the altitude.

$$\sin \theta = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\theta = 45°$$

3 **a** $$BE = \sqrt{5^2 + 12^2}$$
$$= \sqrt{169} = 13$$

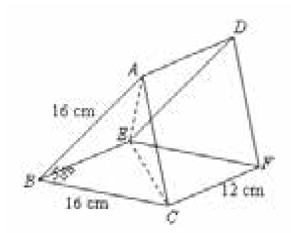
Triangle BEF is isosceles, so

$BE = EF$

$$BF = \sqrt{13^2 + 13^2}$$
$$= \sqrt{338}$$
$$BD = \sqrt{338 - 5^2}$$
$$= \sqrt{313}$$
$$\text{Gradient of} BF = \frac{DF}{DB}$$
$$= \frac{5}{\sqrt{313}} \approx 0.28$$

b $$\tan \angle FBD = \frac{5}{\sqrt{313}}$$
$$= 0.2826$$
$$\angle FBD = 15.78°$$

4



a Use the cosine rule.

$$AC^2 = b^2$$
$$= a^2 + c^2 - 2ac\cos B$$
$$= 16^2 + 16^2 - 2 \times 16 \times 16 \times \cos 58°$$
$$= 240.681$$
$$AC \approx 15.51 \text{ cm}$$

Hint: Keep the exact value in your calculator for part **c**.

b $AE = \sqrt{16^2 + 12^2}$

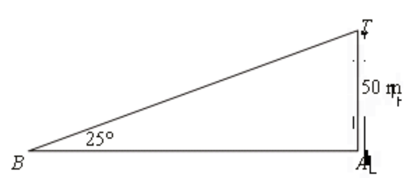
$= \sqrt{400} = 20$ cm

c $AE = CE = 20$ cm

Use the cosine rule in triangle AEC.

$$\cos E = \frac{a^2 + c^2 - e^2}{2ac}$$
$$= \frac{20^2 + 20^2 - 240.681}{2 \times 20 \times 20}$$
$$= 0.699$$
$$\angle AEC = 45.64°$$

5 **a** First calculate the distances of B and C from the tower.



i $\dfrac{50}{AB} = \tan 25°$

$$AB = \frac{50}{\tan 25°}$$
$$= 107.225 \approx 107 \text{ m}$$

ii Likewise,

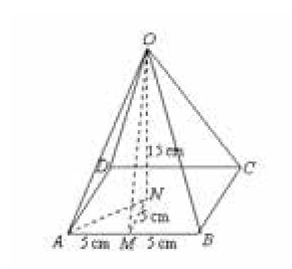
$$AC = \frac{50}{\tan 30°}$$
$$= 86.602 \approx 87 \text{ m}$$

iii Use Pythagoras' theorem:

$$BC = \sqrt{107.225^2 + 86.602^2}$$
$$\approx 138 \text{ m}$$

b
$$MA = \frac{1}{2}AB$$
$$= \frac{25}{\tan 25°}$$
$$= 53.612$$
$$\tan \angle TMA = \frac{50}{53.612}$$
$$= 0.9326$$
$$\angle TMA = 43.00°$$

6 Let M be the midpoint of AB.



a $OM = \sqrt{5^2 + 15^2}$

$= \sqrt{250}$

$OA = \sqrt{250 + 5^2}$

$= \sqrt{275} = 5\sqrt{11}$ cm

b $\sin \angle OAM = \dfrac{15}{5\sqrt{11}}$

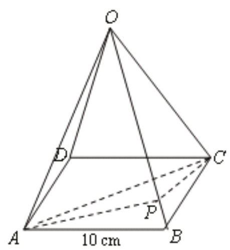
$= \dfrac{3}{\sqrt{11}} = 0.9045$

$\angle OAM = 64.76°$

c $\tan \angle OMN = \dfrac{15}{5} = 3$

$\angle OMN = 71.57°$

d Draw the perpendiculars from C and A to meet OB at the common point P.



Find $\angle AOB$ using the cosine rule in triangle AOB.

$\cos \angle AOB = \dfrac{(5\sqrt{11})^2 + (5\sqrt{11})^2 - 10^2}{2 \times 5\sqrt{11} \times 5\sqrt{11}}$

$= \dfrac{275 + 275 - 100}{550}$

$= \dfrac{450}{550}$

$= 0.8181$

$\angle AOB = 35.096°$

$\sin \angle AOP = \dfrac{AP}{OA}$

$0.5749 = \dfrac{AP}{5\sqrt{11}}$

$AP = 5\sqrt{11} \times 0.5749$

$= 9.534$

$AC = \sqrt{10^2 + 10^2}$

$= \sqrt{200} = 10\sqrt{2}$

Use the cosine rule in triangle APC to find the required angle, $\angle APC$.

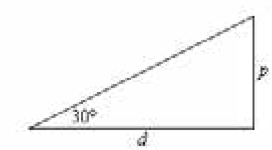
$\cos \angle APC = \cos P$

$= \dfrac{9.534^2 + 9.534^2 - 200}{2 \times 9.534 \times 9.634}$

$= -0.1$

$\angle APC = 95.74°$

7 Let the height of the post be p.



The distance away of the first corner is given by:

$\dfrac{p}{d} = \tan 30°$

$d = \dfrac{p}{\tan 30°}$

$= p\sqrt{3}$

Likewise, the distance away of the second corner is given by

$d = \dfrac{p}{\tan 45°}$

$= p$

The distance of the diagonal from the post is

$\sqrt{(p\sqrt{3})^2 + p^2} = \sqrt{3p^2 + p^2}$

$= \sqrt{4p^2}$

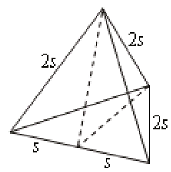
$= 2p$

The elevation from the diagonally opposite corner is

$\tan \theta = \dfrac{p}{2p} = \dfrac{1}{2}$

$\theta = 26.57°$

8 a Let each side of the tetrahedron be $2s$.



Height of altitudes $= \sqrt{(2s)^2 - s^2}$

$= \sqrt{3s^2}$

$= \sqrt{3}s$

Use the cosine rule:

$$\cos\theta = \frac{(2s)^2 + (\sqrt{3}s)^2 - (\sqrt{3}s)^2}{2 \times 2s \times \sqrt{3}s}$$

$$= \frac{4s^2}{4\sqrt{3}s^2}$$

$$= \frac{1}{\sqrt{3}}$$

$\theta = 54.74°$

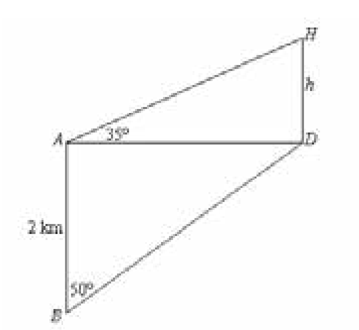
b Use the cosine rule:

$$\cos\theta = \frac{(\sqrt{3}s)^2 + (\sqrt{3}s)^2 - (2s)^2}{2 \times \sqrt{3}s \times \sqrt{3}s}$$

$$= \frac{2s^2}{6s^2} = \frac{1}{3}$$

$\theta = 70.53°$

9



$$\frac{AD}{2} = \tan 50°$$

$$AD = 2\tan 50°$$

$$\frac{h}{AD} = \tan 35°$$

$$h = AD\tan 35°$$

$$= 2\tan 50° \tan 35°$$

$$= 1.6689 \approx 1.67 \text{ km}$$

10 a This is the hypotenuse of right-angled triangle ABF.

$AF = \sqrt{100^2 + 100^2}$

≈ 141.42 m

b $\sin\theta = \dfrac{AD}{AF}$

$$\frac{AD}{AE} = \sin 30°$$

$$= \frac{1}{2}$$

$$AD = \frac{1}{2}AE$$

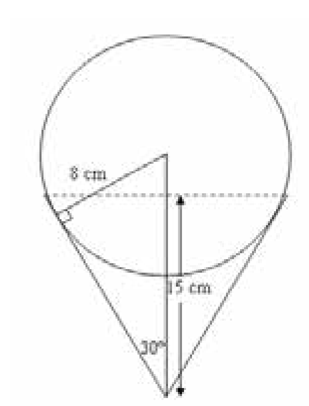
$= 50$ m

$$\therefore \quad \sin\theta = \frac{50}{141.42}$$

$= 0.3535$

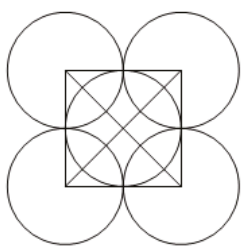
$\theta = 20.70°$

11

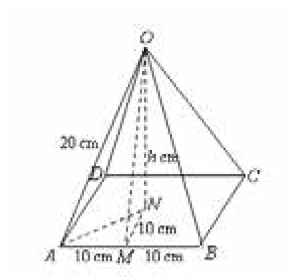


$$\frac{8}{\text{Height}} = \sin 30°$$
$$= \frac{1}{2}$$
$$\text{Height} = \frac{8}{0.5}$$
$$= 16 \text{ cm}$$

12 Joining the centres of the four balls to each other (excluding diagonal balls), and to the top ball, will form a square pyramid with each side 20 cm. Each line will go through the point where the spheres just touch each other.
The diagram shows the view from above:



Find the height of this square pyramid.



$$OM = \sqrt{20^2 - 10^2}$$
$$= \sqrt{300} \text{ cm}$$
$$ON = \sqrt{OM^2 - 10^2}$$
$$= \sqrt{300 - 100}$$
$$= \sqrt{200} \approx 14.14 \text{ cm}$$
There will be the radius of the top sphere above this, and of the bottom spheres below.

The height of the top will be
14.14 + 10 + 10 = 34.14 cm.

13 a The diagonal of the cube will be the diameter of the sphere.
Applying Pythagoras' rule twice gives
$$d = \sqrt{3a^2}$$
$$= a\sqrt{3} \text{ cm}$$
$$r = \frac{a\sqrt{3}}{2} \text{ cm}$$

b The diameter will be the length of one side of the cube.
$$r = \frac{a}{2} \text{ cm}$$

14 a Let the required angle be θ.
$$\tan\theta = \frac{AB}{BD}$$
$$= \frac{20}{40} = 0.5$$
$$\theta = 26.57°$$

b Let the required angle be ϕ.

$$\angle BED = 90°$$

$$\tan \angle BCD = \frac{BD}{BC}$$
$$= \frac{40}{30°} = \frac{4}{3}$$
$$\angle BCD = 53.130°$$
$$\frac{BE}{BC} = \sin \angle BCE$$
$$= \sin 53.130° = 0.8$$
$$BE = 30 \times 0.8$$
$$= 24 \text{ m}$$
$$\tan \phi = \frac{AB}{BE}$$
$$= \frac{20}{24} = 0.8333$$
$$\phi = 39.81°$$

Note: BE may also be found using similar triangles, and/or noticing that triangles CBD and CBE are 3–4–5 triangles.

c Let the required angle be α.

$$CD = \sqrt{30^2 + 40^2}$$
$$= 50 \text{ m}$$
$$CE = 25 \text{ m}$$

Use the cosine rule to find BE.

$$\cos c = \frac{CB}{CD}$$
$$= \frac{30°}{50} = 0.6$$
$$BE^2 = CB^2 + CE^2 - 2$$
$$\times CB \times CE \times \cos C$$
$$= 30^2 + 25^2 - 2$$
$$\times 30 \times 25 \times 0.6$$
$$= 625$$
$$BE = 25 \text{ m}$$
$$\tan \alpha = \frac{AB}{BE}$$
$$= \frac{20}{25} = 0.8$$
$$\alpha = 38.66°$$

Solutions to technology-free questions

1 **a** $a^2 = b^2 + c^2 - 2bc\cos A$

$$6^2 = x^2 + 10^2 - 2x \times 10 \times \frac{\sqrt{3}}{2}$$

$$x^2 - 10\sqrt{3}x + 64 = 0$$

$$x = \frac{10\sqrt{3} \pm \sqrt{300 - 4 \times 1 \times 64}}{2}$$

$$= \frac{10\sqrt{3} \pm \sqrt{44}}{2}$$

$$= \frac{10\sqrt{3} \pm 2\sqrt{11}}{2}$$

$$= 5\sqrt{3} \pm \sqrt{11}$$

b $\dfrac{\sin y}{10} = \dfrac{\sin 30^\circ}{6}$

$$\sin y = \frac{10 \times \sin 30^\circ}{6}$$

$$= \frac{10}{12} = \frac{5}{6}$$

$$y = \sin^{-1}\left(\frac{5}{6}\right)$$

$$\text{or } 180^\circ - \sin^{-1}\left(\frac{5}{6}\right)$$

Since both answers to **a** are positive, this must be an ambiguous case.

2 **a** Triangle is isosceles, so $\angle B = 30^\circ$ and $\angle C = 120^\circ$

b $\dfrac{AB}{2} = 40\cos 30^\circ$

$$AB = 80\cos 30^\circ$$

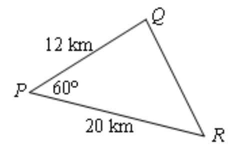
$$= 40\sqrt{3}\text{ cm}$$

c $\dfrac{CM}{40} = \sin 30^\circ$

$$CM = 40 \times \sin 30^\circ$$

$$= 20\text{ cm}$$

3



$$QR^2 = 12^2 + 20^2 - 2 \times 12 \times 20 \times \cos 60^\circ$$

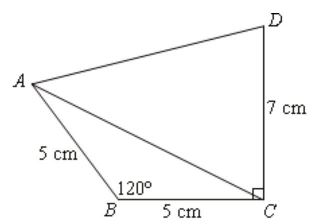
$$= 144 + 400 - 240$$

$$= 304$$

$$QR = \sqrt{304}$$

$$= \sqrt{16 \times 19} = 4\sqrt{19}\text{ km}$$

4



a Use the cosine rule.

$$AC^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \times \cos 120^\circ$$

$$= 25 + 25 + 25$$

$$= 75$$

$$AC = \sqrt{75} = 5\sqrt{3}\text{ cm}$$

b Area $= \dfrac{1}{2} \times 5 \times 5 \times \sin 120^\circ$

$$= \frac{25\sqrt{3}}{4}\text{ cm}^2$$

c In isosceles triangle ABC,

$\angle ACB = \angle BAC$

$$= \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$$

$\angle ACD = 90^\circ - 30^\circ = 60^\circ$

$$\text{Area of } ADC = \frac{1}{2} \times 7 \times AC \times \sin 60^\circ$$

$$= \frac{1}{2} \times 7 \times 5\sqrt{3} \times \frac{\sqrt{3}}{2}$$

$$= \frac{105}{4}\ \text{cm}^2$$

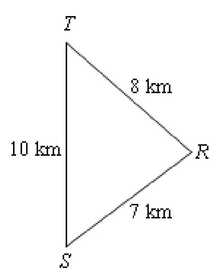
d $$\text{Total area} = \frac{25\sqrt{3}}{4} + \frac{105}{4}$$

$$= \frac{25\sqrt{3} + 105}{4}$$

$$= \frac{5(5\sqrt{3} + 21)}{4}\ \text{cm}^2$$

5 $x = 180^\circ - 37^\circ = 143^\circ$

6



$$\cos S = \frac{10^2 + 7^2 - 8^2}{2 \times 10 \times 7}$$

$$= \frac{85}{140} = \frac{17}{28}$$

7 First note that $AB = c = 5$ cm $\angle BAC = A = 60^\circ$ and $AC = b = 6$ cm, so the angle is included. So start by finding $a = BC$ by the cosine rule.

$$a^2 = b^2 + c^2 2bc\cos A$$

$$= 36 + 25 - 60\cos 60^\circ$$

$$= 36 + 25 - 30$$

$$= 31$$

$$a = \sqrt{31}$$

Now use the sine rule.

$$\frac{\sin B}{6} = \frac{\sin 60^\circ}{\sqrt{31}}$$

$$\sin \angle ABC = \frac{6 \sin 60^\circ}{\sqrt{31}}$$

$$= \frac{3\sqrt{3}}{\sqrt{31}} = \frac{3\sqrt{93}}{31}$$

8 $$A = \frac{1}{2}r^2\theta$$

$$33 = \frac{1}{2} \times 6^2 \times \theta$$

$$= 18\theta$$

$$\theta = \frac{33}{18} = \frac{11}{6}\ \text{(radians)}$$

9 **a** **i** $\angle TAB = 90^\circ - 60^\circ = 30^\circ$

ii $\angle ATB = 180^\circ - 30^\circ - (90^\circ + 45^\circ)$

$$= 15^\circ$$

b $$\frac{AT}{\sin 135^\circ} = \frac{300}{\sin 15^\circ}$$

$$AT = \sin 135^\circ \times 300 \times \frac{4}{\sqrt{6} - \sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \times \frac{1200}{\sqrt{6} - \sqrt{2}}$$

$$= \frac{1200}{\sqrt{12} - 2}$$

$$= \frac{1200}{2\sqrt{3} - 2} = \frac{600}{\sqrt{3} - 1}$$

$= \dfrac{600}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \dfrac{600(\sqrt{3}+1)}{3-1}$

$= 300(\sqrt{3}+1)$ m

For BT

$AT \sin 30° = 150(\sqrt{3}+1)$

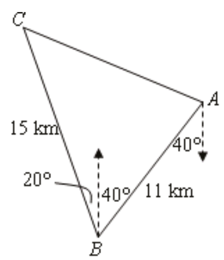
Let X be the point due east of A and south of T.

Then $TX = BX = 150(\sqrt{3}+1)$

Use Pythagoras's theorem.

$\therefore BT = 150(\sqrt{6}+\sqrt{2})$

10



Use the cosine rule.

$AC^2 = b^2$

$= a^2 + c^2 - 2ac \cos B$

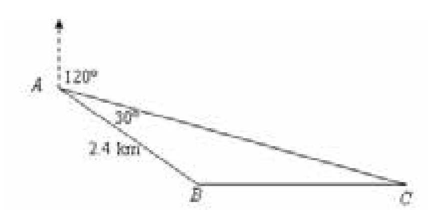
$= 11^2 + 15^2 - 2 \times 11 \times 15 \cos 60°$

$= 121 + 225 - 165$

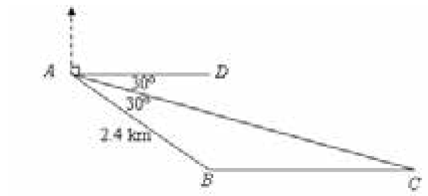
$= 181$

$AC = \sqrt{181}$ km

11 a



Draw a line AD in an easterly direction from A (parallel to BC).



$\angle DAC = 30°$

$\angle ACB = \angle DAC = 30°$

$\angle ABC = 180° - 30 - 30$

$= 120°$

$\therefore \quad BC = 2.4$ km

Use the cosine rule to find AC.

$AC^2 = b^2$

$= a^2 + c^2 - 2ac \cos B$

$= 2.4^2 + 2.4^2 - 2 \times 2.4$

$\times 2.4 \times \cos 120°$

$= 5.76 + 5.76 + 5.76 = 17.28$

$AC = \sqrt{1728}$

$= \sqrt{5.76 \times 3}$

$= 2.4\sqrt{3}$ or $\dfrac{12\sqrt{3}}{5}$ km

b Speed$= (AB + BC) \div \dfrac{1}{12} = 57.6$ km/h

12 $l = r\theta$

$30 = 12\theta$

$\theta = \dfrac{30°}{12} = \left(\dfrac{5}{2}\right)^c$

$A = \dfrac{1}{2} \times 12^2 \times \dfrac{5}{2}$

$= 180$ cm^2

13 The reflex angle $= 2\pi - 2$

$\approx 2 \times 3.14 - 2$

≈ 4.28 radians

Arc length $\approx 5 \times 4.28$

$= 21.4$ cm

14 **a** Draw a perpendicular from O to bisect AB at D

$\sin \angle AOD = \dfrac{12}{13}$

$\angle AOD = \sin^{-1} \dfrac{12}{13}$

$\angle AOB = 2\sin^{-1} \dfrac{12}{13}$

arc $AB = r\theta$

$= 13 \times 2\sin^{-1} \dfrac{12}{13}$

$= 26\sin^{-1} \dfrac{12}{13}$

b Reflex $\angle AOB = 2\pi - 2\sin^{-1} \dfrac{12}{13}$

area $= \dfrac{1}{2} \times 13^2 \times \left(2\pi - 2\sin^{-1} \dfrac{12}{13}\right)$

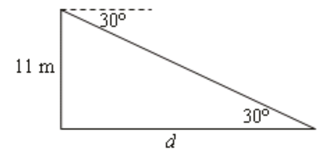
$= 169\left(\pi - \sin^{-1} \dfrac{12}{13}\right)$ cm^2

Note: the perpendicular distance from O to AB can be calculated to be 5 cm using Pythagoras' theorem, and so

$\sin^{-1} \dfrac{12}{13} = \cos^{-1} \dfrac{5}{13} = \tan^{-1} \dfrac{12}{5}$.

Either these three angles may be used interchangeably.

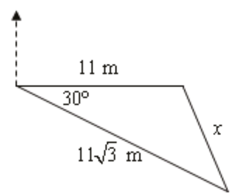
15 First calculate the distance of each boat from the cliff.
The first boat will form a right-angled isosceles triangle and is 11 m from the cliff.



For the second boat,

$\dfrac{11}{d} = \tan 30° = \dfrac{1}{\sqrt{3}}$

$d = 11\sqrt{3}$ m



Use the cosine rule.

$x^2 = 11^2 + (11\sqrt{3})^2 - 2 \times 11$

$\times 11\sqrt{3} \times \cos 30°$

$= 121 + 363 - 363$

$= 121$

$x = 11$ m

Solutions to multiple-choice questions

1 D Use the sine rule.

$$\frac{\sin Y}{y} = \frac{\sin X}{x}$$

$$\frac{\sin Y}{18} = \frac{\sin 62^\circ}{21}$$

$$\sin Y = 18 \times \frac{\sin 62^\circ}{21}$$

$$= 0.7568$$

$$Y = 49.2^\circ$$

2 C Use the cosine rule.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 30^2 + 21^2 - 2 \times 30 \times 21 \times \frac{51}{53}$$

$$= 128.547$$

$$c \approx 11$$

3 C Use the cosine rule.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{5.2^2 + 6.8^2 - 7.3^2}{2 \times 5.2 \times 6.8}$$

$$= 0.2826$$

$$C \approx 74^\circ$$

4 B Area $= \frac{1}{2}bc\sin A$

$$= \frac{1}{2} \times 5 \times 3 \times \sin 30^\circ$$

$$= 3.75 \text{ cm}^2$$

5 A The other angles in the (isosceles) triangle are both

$$\frac{180^\circ - 130^\circ}{2} = 25^\circ.$$

Use the sine rule.

$$\frac{10}{\sin 130^\circ} = \frac{r}{\sin 25^\circ}$$

$$r = \frac{10 \times \sin 25^\circ}{\sin 130^\circ}$$

$$\approx 5.52 \text{ cm}$$

6 A First find the angle at the centre using the cosine rule.

$$\cos C = \frac{6^2 + 6^2 - 5^2}{2 \times 6 \times 6}$$

$$= 0.6527$$

$$C = 49.248^\circ = 0.8595^c$$

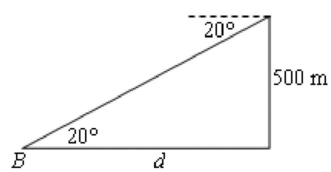
Segment area

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

$$= \frac{1}{2} \times 6^2 \times (0.8595 - \sin 0.8595)$$

$$\approx 1.8 \text{ cm}^2$$

7 D



$$\frac{500}{d} = \tan 20^\circ$$

$$d = \frac{500}{\tan 20^\circ}$$

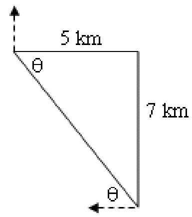
$$\approx 1374 \text{ m}$$

8 B $\tan\theta = \frac{80}{1300}$

$$= 0.0615$$

$$\theta = 3.521^\circ \approx 4^\circ$$

9 C



$$\tan\theta = \frac{7}{5} = 1.4$$

$$\theta = 54°$$

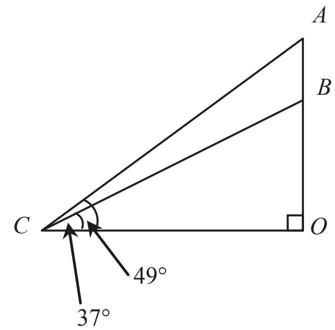
Bearing = 270° + 54° = 324°

10 A 215° − 180° = 035°

Solutions to extended-response questions

1 **a** $\angle ACB = 12°$, $\angle CBO = 53°$, $\angle CBA = 127°$

b $\angle CAB = 41°$



The sine rule applied to triangle ABC gives

$$\frac{CB}{\sin 41°} = \frac{60}{\sin 12°}$$

$$\therefore \quad CB = \frac{60 \sin 41°}{\sin 12°}$$

$$= 189.33, \text{ correct to two decimal places}$$

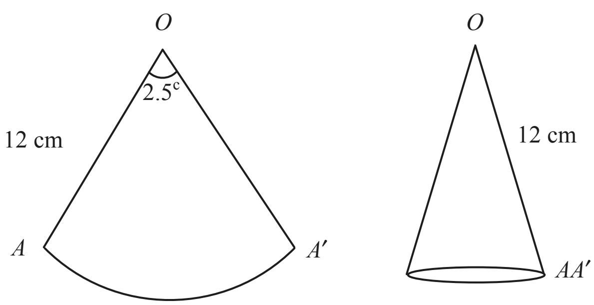
c

$$\frac{OB}{CB} = \sin 37°$$

$$\therefore \quad OB = CB \sin 37°$$

$$= 113.94 \text{ m}$$

2 **a**



The circumference of the circular base $= 2.5 \times 12$

$= 30$ cm

Therefore $2\pi r = 30$

Solve for r, the radius of the base.

$$r = \frac{30°}{2\pi}$$

$$= 4.77 \text{ cm, correct to two decimal places}$$

b Curved surface area of the cone = area of the sector

$$= \frac{1}{2} \times 144 \times 2.5$$

$$= 180 \text{ cm}^2$$

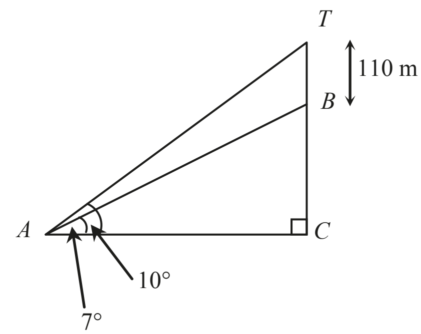
c The diameter length is required.

$$\text{Diameter} = 2r = \frac{30^\circ}{\pi} = 9.55 \text{ cm}$$

3 **a** $\angle TAB = 3^\circ,\ \angle ABT = 97^\circ$

$$\angle ATB = (83 - 3)^\circ = 80^\circ$$

b The sine rule applied to triangle ATB gives



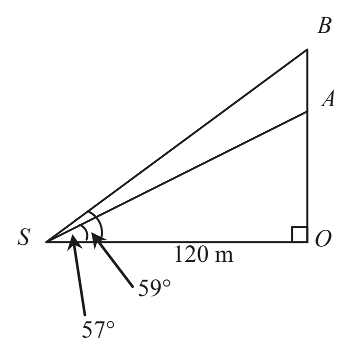
$$\frac{AB}{\sin 80^\circ} = \frac{110}{\sin 3^\circ}$$

$$\therefore \quad CB = \frac{110 \sin 80^\circ}{\sin 3^\circ} = 2069.87$$

c $CB = AB \sin 7^\circ$

$$= 252.25 \text{ m}$$

4 **a** In right-angled triangle AOS



$$\frac{OA}{120} = \tan 57^\circ$$

$$\therefore \quad OA = 120 \tan 57^\circ = 184.78 \text{ m, correct to two decimal places}$$

b In right-angled triangle SOB

$$\frac{OB}{120} = \tan 59^\circ$$

$$\therefore \quad OB = 120 \tan 59^\circ = 199.71 \text{ m, correct to two decimal places}$$

c The distance $AB = OB - OA = 14.93$ m, correct to two decimal places.

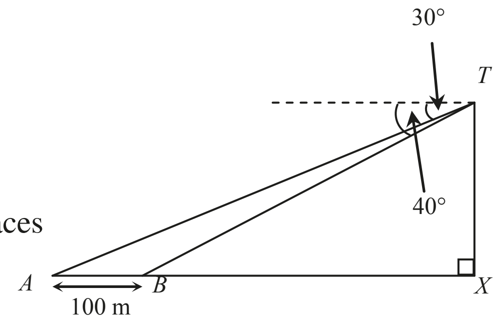
5 a $\angle ATB = 10°$

In triangle ATB the sine rule gives

$$\frac{100}{\sin 10°} = \frac{AT}{\sin 140°}$$

$$\therefore \quad AT = \frac{100 \sin 140°}{\sin 10°}$$

$$= 370.17 \text{ m, correct to two decimal places}$$



b Applying the sine rule again gives

$$\frac{BT}{\sin 30°} = \frac{100}{\sin 10°}$$

$$\therefore \quad BT = 287.94 \text{ m, correct to two decimal places}$$

c In right-angled-triangle TBX

$$\frac{XT}{BT} = \sin 40°$$

$$\therefore \quad XT = BT \sin 40°$$

$$= 185.08 \text{ m, correct to two decimal places}$$

6 a Applying Pythagoras' theorem in triangle VBA

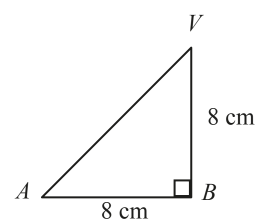
$$VA^2 = 8^2 + 8^2$$

$$= 64 + 64$$

$$= 128$$

$$\therefore \quad VA = 8\sqrt{2}$$

The distance VA is $8\sqrt{2}$ cm.



b Applying Pythagoras' theorem in triangle VBC

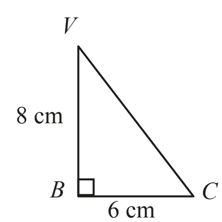
$$VC^2 = 8^2 + 6^2$$

$$= 64 + 36$$

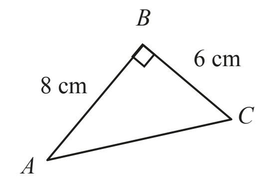
$$= 100$$

$$\therefore \quad VC = 10$$

The distance VC is 10 cm.



c Applying Pythagoras' theorem in triangle ABC



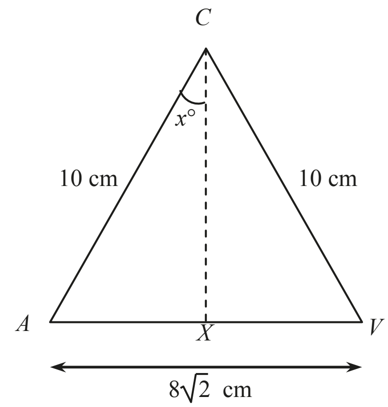
$$\begin{aligned} AC^2 &= 8^2 + 6^2 \\ &= 64 + 36 \\ &= 100 \end{aligned}$$

$\therefore \quad AC = 10$

The distance AC is 10 cm.

d Triangle VCA is isosceles with $VC = AC$

In right-angled triangle CXA



$$\begin{aligned} \sin x^\circ &= \frac{4\sqrt{2}}{10} \\ &= \frac{2\sqrt{2}}{5} \end{aligned}$$

Therefore $x^\circ = 34.4490\ldots^\circ$

and $\angle ACV = 68.899\ldots^\circ$

$= 68.9^\circ$, correct to one decimal place

7 Let L be the perimeter of triangle ABC and α, β and γ the angles at A, B and C respectively.

The sine rule gives: $\dfrac{AB}{\sin\gamma} = \dfrac{AC}{\sin\beta} = \dfrac{BC}{\sin\alpha}$

Let $AB = x$

$$\frac{x}{\sin\gamma} = \frac{AC}{\sin\beta} = \frac{BC}{\sin\alpha}$$

Therefore $AC = \dfrac{x\sin\beta}{\sin\gamma}$ and $BC = \dfrac{x\sin\alpha}{\sin\gamma}$

Next

$$\begin{aligned} L &= AB + AC + BC \\ &= x + \frac{x\sin\beta}{\sin\gamma} + \frac{x\sin\alpha}{\sin\gamma} \\ &= x\left(1 + \frac{\sin\beta}{\sin\gamma} + \frac{\sin\alpha}{\sin\gamma}\right) \\ &= x\left(\frac{\sin\gamma + \sin\beta + \sin\alpha}{\sin\gamma}\right) \end{aligned}$$

$$\therefore x = \frac{L\sin\gamma}{\sin\gamma + \sin\beta + \sin\alpha}$$

Area $= \dfrac{1}{2}AC \times AB \times \sin\alpha$

$$= \frac{1}{2}\frac{L\sin\gamma}{\sin\gamma + \sin\beta + \sin\alpha} \times \frac{L\sin\gamma}{\sin\gamma + \sin\beta + \sin\alpha} \times \frac{\sin\beta}{\sin\gamma} \times \sin\alpha$$

$$= \frac{L^2 \sin\alpha \sin\beta \sin \sin\gamma}{2(\sin\gamma + \sin\beta + \sin\alpha)^2}$$

Chapter 16 – Trigonometric identities

Solutions to Exercise 16A

1 a $\cot\frac{3\pi}{4} = \dfrac{\cos\frac{3\pi}{4}}{\sin\frac{3\pi}{4}}$

$= -\frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}}$

$= -1$

b $\operatorname{cosec}\frac{5\pi}{4} = \dfrac{1}{\sin\frac{5\pi}{4}}$

$= -\dfrac{1}{\frac{1}{\sqrt{2}}}$

$= -\sqrt{2}$

c $\sec\frac{5\pi}{6} = \dfrac{1}{\cos\frac{5\pi}{6}}$

$= \dfrac{1}{\frac{-\sqrt{3}}{2}}$

$= -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

d $\operatorname{cosec}\frac{\pi}{2} = \dfrac{1}{\sin\frac{\pi}{2}}$

$= \frac{1}{1} = 1$

e $\sec\frac{4\pi}{3} = \dfrac{1}{\cos\frac{4\pi}{3}}$

$= \dfrac{1}{-\frac{1}{2}} = -2$

f $\operatorname{cosec}\frac{13\pi}{6} = \dfrac{1}{\sin\frac{13\pi}{6}}$

$= \dfrac{1}{\sin\frac{\pi}{6}}$

$= \dfrac{1}{\frac{1}{2}} = 2$

g $\cot\frac{7\pi}{3} = \dfrac{\cos\frac{7\pi}{3}}{\sin\frac{7\pi}{3}}$

$= \frac{1}{2} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

h $\sec\frac{5\pi}{3} = \dfrac{1}{\cos\frac{5\pi}{3}}$

$= \dfrac{1}{\frac{1}{2}} = 2$

2 a $\cot 135^\circ = \dfrac{\cos 135^\circ}{\sin 135^\circ}$

$= -\frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}}$

$= -1$

b $\sec 150^\circ = \dfrac{1}{\cos 150^\circ}$

$= \dfrac{1}{-\frac{\sqrt{3}}{2}}$

$= -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

c $\operatorname{cosec} 90^\circ = \dfrac{1}{\sin 90^\circ}$

$= \dfrac{1}{1} = 1$

d $\cot 240^\circ = \dfrac{\cos 240^\circ}{\sin 240^\circ}$

$= -\dfrac{1}{2} \div -\dfrac{\sqrt{3}}{2}$

$= \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

e $\sec 225^\circ = \dfrac{1}{\cos 225^\circ}$

$= \dfrac{1}{-\dfrac{1}{\sqrt{2}}}$

$= -\sqrt{2}$

f $\sec 330^\circ = \dfrac{1}{\cos 330^\circ}$

$= \dfrac{1}{\dfrac{\sqrt{3}}{2}}$

$= \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

g $\cot 315^\circ = \dfrac{\cos 315^\circ}{\sin 315^\circ}$

$= \dfrac{1}{\sqrt{2}} \div -\dfrac{1}{\sqrt{2}}$

$= -1$

h $\operatorname{cosec} 300^\circ = \dfrac{1}{\sin 300^\circ}$

$= \dfrac{1}{\dfrac{-\sqrt{3}}{2}}$

$= -\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$

i $\cot 420^\circ = \dfrac{\cos 420^\circ}{\sin 420^\circ}$

$= \dfrac{\cos 60^\circ}{\sin 60^\circ}$

$= \dfrac{1}{2} \div \dfrac{\sqrt{3}}{2}$

$= \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

3 **a** $\operatorname{cosec} x = 2$

$\sin x = \dfrac{1}{2}$

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

b $\cot x = \sqrt{3}$

$\tan x = \dfrac{1}{\sqrt{3}}$

$x = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$

c $\sec x = -\sqrt{2}$

$\cos x = -\dfrac{1}{\sqrt{2}}$

$x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

d $\operatorname{cosec} x = \sec x$

$\sin x = \cos x$

$\tan x = 1$

$x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

4 **a** $\cos\theta = \dfrac{1}{\sec\theta}$

$= -\dfrac{8}{17}$

b $\cos^2\theta + \sin^2\theta = 1$

$$\frac{64}{289} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{225}{289}$$

$$\sin\theta = \frac{15}{17} \text{ (Since } \sin\theta > 0)$$

c $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{15}{17} \div -\frac{8}{17}$$

$$= -\frac{15}{8}$$

5 $1 + \tan^2\theta = \sec^2\theta$

$$\sec^2\theta = 1 + \frac{49}{576} = \frac{625}{576}$$

$$\sec\theta = \frac{25}{24} \text{ (since } \cos\theta > 0)$$

$$\cos\theta = \frac{24}{25}$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = -\frac{7}{24}$$

$$\sin\theta = -\frac{7}{24} \times \frac{24}{25}$$

$$= -\frac{7}{25}$$

6 $1 + \tan^2\theta = \sec^2\theta$

$\sec^2\theta = 1 + 0.16 = 1.16$

$$\sec\theta = -\sqrt{\frac{116}{100}}$$

(Since θ is in the 3rd quadrant)

$$= -\sqrt{\frac{29}{25}}$$

$$= -\frac{\sqrt{29}}{5}$$

7 $\cot\theta = \dfrac{3}{4}$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\sec^2\theta = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\sec\theta = -\frac{5}{3} (\cos\theta < 0)$$

$$\cos\theta = -\frac{3}{5}$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{4}{3}$$

$$\sin\theta = \frac{4}{3} \times -\frac{3}{5}$$

$$= -\frac{4}{5}$$

$$\frac{\sin\theta - 2\cos\theta}{\cot\theta - \sin\theta} = \frac{-\frac{4}{5} - -\frac{6}{5}}{\frac{3}{4} - -\frac{4}{5}}$$

$$= \frac{2}{5} \div \frac{31}{20}$$

$$= \frac{2}{5} \times \frac{20}{31} = \frac{8}{31}$$

8 $\cos^2\theta + \sin^2\theta = 1$

$$\frac{4}{9} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{5}{9}$$

$$\sin\theta = -\frac{\sqrt{5}}{3}\left(\frac{3\pi}{2} < \theta < 2\pi\right)$$

$$\tan\theta = -\frac{\sqrt{5}}{3} \div \frac{2}{3} = -\frac{\sqrt{5}}{2}$$

$$\cot\theta = -\frac{2}{\sqrt{5}}$$

$$\frac{\tan\theta - 3\sin\theta}{\cos\theta - 2\cot\theta} = \frac{-\frac{\sqrt{5}}{2} - \left(-\sqrt{5}\right)}{\frac{2}{3} - \left(-\frac{4}{\sqrt{5}}\right)}$$

$$= \frac{\sqrt{5}}{2} \div \frac{2\sqrt{5} + 12}{3\sqrt{5}}$$

$$= \frac{\sqrt{5}}{2} \times \frac{3\sqrt{5}}{2\sqrt{5} + 12}$$

$$= \frac{15}{4(\sqrt{5} + 6)} \times \frac{6 - \sqrt{5}}{6 - \sqrt{5}}$$

$$= \frac{15(6 - \sqrt{5})}{4 \times (36 - 5)}$$

$$= \frac{15(6 - \sqrt{5})}{124}$$

9 **a** $(1 - \cos^2\theta)(1 + \cot^2\theta)$

$= \sin^2\theta \times (1 + \cot^2\theta)$

$= \sin^2\theta \times \operatorname{cosec}^2\theta$

$= 1$

b $\cos^2\theta \tan^2\theta + \sin^2\theta \cot 2\theta$

$= \cos^2\theta \times \dfrac{\sin^2\theta}{\cos^2\theta} + \sin^2\theta \times \dfrac{\cos^2\theta}{\sin^2\theta}$

$= \sin^2\theta + \cos^2\theta$

$= 1$, provided $\sin\theta \neq 0$ and $\cos\theta \neq 0$

c In cases like this, it is a good strategy to start with the more complicated expression.

$$\frac{\tan\theta + \cot\phi}{\cot\theta + \tan\phi}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\phi}{\sin\phi}}{\frac{\cos\theta}{\sin\theta} + \frac{\sin\phi}{\cos\phi}}$$

$$= \frac{\frac{\sin\theta\sin\phi + \cos\phi\cos\theta}{\cos\theta\sin\phi}}{\frac{\cos\phi\cos\theta + \sin\theta\cos\phi}{\cos\phi\sin\theta}}$$

$$= \frac{\sin\theta\sin\phi + \cos\phi\cos\theta}{\cos\theta\sin\phi}$$

$$\times \frac{\cos\phi\sin\theta}{\cos\phi\cos\theta + \sin\theta\cos\phi}$$

$$\frac{\tan\theta + \cot\phi}{\cot\theta + \tan\phi} = \frac{\cos\phi\sin\theta}{\cos\theta\sin\phi}$$

$$= \frac{\sin\theta}{\cos\theta} \times \frac{\cos\phi}{\sin\phi}$$

$$= \frac{\sin\theta}{\cos\theta} \div \frac{\sin\phi}{\cos\phi}$$

$$= \frac{\tan\theta}{\tan\phi}$$

This is provided $\cot\theta + \tan\phi \neq 0$ and the tangent and cotangent are defined.

d $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$

$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$

$\quad + \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$

$= 2\sin^2\theta + 2\cos^2\theta$

$= 2$

There are no restrictions on θ.

e $\dfrac{1+\cot^2\theta}{\cot\theta\operatorname{cosec}\theta} = \dfrac{\operatorname{cosec}^2\theta}{\cot\theta\operatorname{cosec}\theta}$

$$= \frac{\operatorname{cosec}\theta}{\cot\theta}$$

$$= \frac{1}{\sin\theta} \times \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

Conditions: $\sin\theta \neq 0, \cos\theta \neq 0$

f $\sec\theta + \tan\theta = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{1+\sin\theta}{\cos\theta} \times \frac{1-\sin\theta}{1-\sin\theta}$$

$$= \frac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta(1-\sin\theta)}$$

$$= \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}$$

$$= \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)}$$

$$= \frac{\cos\theta}{1-\sin\theta}$$

Conditions: $\cos\theta \neq 0$ (includes $\sin\theta \neq 1$)

Solutions to Exercise 16B

1 Different angles may be used.

a $\cos 15^\circ = \cos(60^\circ - 45^\circ)$

$= \cos 60^\circ \cos 45^\circ$

$+ \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{2} + \sqrt{6}}{4}$

b $\cos 105^\circ = \cos(45 + 60)^\circ$

$= \cos 45^\circ \cos 60^\circ$

$- \sin 45^\circ \sin 60^\circ$

$= \frac{\sqrt{2}}{2} \times \frac{1}{2} - \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{2} - \sqrt{6}}{4}$

2 a

$\sin 165^\circ = \sin(120 + 45)^\circ$

$= \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

b $\tan 75^\circ = \tan(45 + 30)^\circ$

$= \frac{\tan(45)^\circ + \tan(30)^\circ}{1 - \tan(45)^\circ \tan(30)^\circ}$

$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}$

$= \frac{3 + \sqrt{3}}{3 - \sqrt{3}}$

$= 2 + \sqrt{3}$

3 Different angles may be used.

a $\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

$= \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6}$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

b $\sin \frac{11\pi}{12} = \sin\left(\pi - \frac{\pi}{12}\right)$

$= \sin \frac{\pi}{12}$

$\sin \frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

c $\tan\left(-\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$

$= \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(-\frac{\pi}{3}\right)}$

$= \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$

$= \sqrt{3} - 2$

4 $\cos^2 u = 1 - \sin^2 u$

$= 1 - \frac{144}{169} = \frac{25}{169}$

$\cos u = \pm\frac{5}{13}$

$\cos^2 v = 1 - \sin^2 v$

$= 1 - \frac{9}{25} = \frac{16}{25}$

$\cos v = \pm\frac{4}{5}$

$\sin(u + v) = \sin u \cos v + \cos u \sin v$

$= \pm\frac{3}{5} \times \frac{5}{13} \pm \frac{4}{5} \times \frac{12}{13}$

$= \frac{\pm 15 \pm 48}{65}$

There are four possible answers:

$\frac{63}{65}, \frac{33}{65}, -\frac{33}{65}, -\frac{63}{65}$

5 **a** $\sin\left(\theta + \frac{\pi}{6}\right) = \sin\theta \cos \frac{\pi}{6} + \cos\theta \sin \frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} \sin\theta + \frac{1}{2} \cos\theta$

b $\cos\left(\pi - \frac{\pi}{4}\right)$

$= \cos\phi \cos \frac{\pi}{4} + \sin\phi \sin \frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} \cos\phi + \frac{1}{\sqrt{2}} \sin\phi$

$= \frac{1}{\sqrt{2}}(\cos\phi + \sin\phi)$

c $\cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}(\cos\theta - \sqrt{3}\sin\theta)$

d $\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta \cos \frac{\pi}{4} - \cos\theta \sin \frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} \sin\theta - \frac{1}{\sqrt{2}} \cos\theta$

$= \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

6 **a** $\sin(v + (u - v)) = \sin u$

b $\cos((u + v) - v) = \cos u$

7 $\cos^2\theta = 1 - \sin^2\theta$

$= 1 - \frac{9}{25} = \frac{16}{25}$

$\cos\theta = -\frac{4}{5}$

(Since $\cos\theta < 0$)

$$\sin^2\phi = 1 - \cos^2\phi$$

$$= 1 - \frac{25}{169} = \frac{144}{169}$$

$$\sin\phi = \frac{12}{13}$$

(Since $\sin\theta > 0$)

a $\cos 2\phi = \cos^2\phi - \sin^2\phi$

$$= \frac{25}{169} - \frac{144}{169}$$

$$= -\frac{119}{169}$$

b $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right)$$

$$= \frac{24}{25}.$$

c $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\dfrac{\sin\theta}{\cos\theta}}{1-\left(\dfrac{\sin\theta}{\cos\theta}\right)^2}$$

$$= \frac{2\times\dfrac{-3/5}{-4/5}}{1-\left(\dfrac{-3/5}{-4/5}\right)^2}$$

$$= \frac{2\times\dfrac{3}{4}}{1-\dfrac{9}{16}}$$

$$= \frac{24}{7}$$

d $\sec 2\phi = \dfrac{1}{\cos 2\phi}$

$$= -\frac{169}{119}$$

e $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= -\frac{3}{5}\times -\frac{5}{13} + -\frac{4}{5}\times\frac{12}{13}$$

$$= \frac{14-48}{65}$$

$$= -\frac{33}{65}$$

f $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= -\frac{4}{5}\times -\frac{5}{13} + -\frac{3}{5}\times\frac{12}{13}$$

$$= \frac{20-36}{65}$$

$$= -\frac{16}{65}$$

g $\operatorname{cosec}(\theta+\phi) = \dfrac{1}{\sin(\theta+\phi)}$

$$= -\frac{65}{33}$$

h $\cot^2 2\theta = \operatorname{cosec}^2 2\theta - 1$

$$= \frac{25^2}{24^2} - 1$$

$$= \frac{49}{576}$$

$$\therefore \cot 2\theta = \frac{7}{24}$$

8 Question 8

9 $\cos\alpha = -\frac{4}{5}$

$$\cos^2\beta = 1 - \sin^2\beta$$
$$= 1 - \frac{576}{625} = \frac{29}{625}$$
$$\cos\beta = -\frac{7}{25}$$
$$\cos^2\alpha = 1 - \sin^2\alpha$$
$$= 1 - \frac{9}{25} = \frac{16}{25}$$

a $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$$= \frac{16}{25} - \frac{9}{25}$$
$$= \frac{7}{25}$$

b $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$$= \frac{3}{5} \times -\frac{7}{25} - -\frac{4}{5} \times \frac{24}{25}$$
$$= \frac{75}{125} = \frac{3}{5}$$

c We have that $\tan\alpha = -\frac{3}{4}$ and $\tan\beta = -\frac{24}{7}$. Therefore,
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$= \frac{-\frac{3}{4} - \frac{24}{7}}{1 - \frac{3}{4} \times \frac{24}{7}}$$
$$= \frac{117}{44}$$

d $\sin 2\beta = 2\sin\beta\cos\beta$
$$= 2 \times \frac{7}{25} \times -\frac{24}{25}$$
$$= -\frac{336}{625}$$

10 a $\sin 2\theta = 2\sin\theta\cos\theta$
$$= 2 \times -\frac{\sqrt{3}}{2} \times \frac{1}{2}$$
$$= -\frac{\sqrt{3}}{2}$$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$$= \frac{1}{4} - \frac{3}{4}$$
$$= -\frac{1}{2}$$

11 a $(\sin\theta - \cos\theta)^2$
$$= \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$$
$$= 1 - \sin 2\theta$$

b $\sin^4\theta - \cos^4\theta$
$$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$$
$$= \cos 2\theta \times 1$$
$$= \cos 2\theta$$

12 a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$
$$= \sqrt{2}\left(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)$$
$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right)$$
$$= \sin\theta - \cos\theta$$

b $\cos\left(\theta - \frac{\pi}{3}\right) = \cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}$
$$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$
$$\cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$
Add the last two equations:

$$\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$$

c $\tan\left(\theta + \frac{\pi}{4}\right)\tan\left(\theta - \frac{\pi}{4}\right)$

$$= \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}} \cdot \frac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta\tan\frac{\pi}{4}}$$

$$= \frac{\tan\theta + 1}{1 - \tan\theta}\frac{\tan\theta - 1}{1 + \tan\theta}$$

$$= \frac{\tan^2\theta - 1}{1 - \tan^2\theta}$$

$$= -1$$

d $\cos\left(\theta + \frac{\pi}{6}\right) = \cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6}$

$$= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta$$

$$\sin\left(\theta + \frac{\pi}{3}\right) = \sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$$

Add the two equations:

$$\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$$

e $\tan\left(\theta + \frac{\pi}{4}\right) = \dfrac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}}$

$$= \frac{\tan\theta + 1}{1 - \tan\theta}$$

f $\dfrac{\sin(u+v)}{\cos u\cos v}$

$$= \frac{\sin u\cos v + \cos u\sin v}{\cos u\cos v}$$

$$= \frac{\sin u\cos v}{\cos u\cos v} + \frac{\cos u\sin v}{\cos u\cos v}$$

$$= \tan u + \tan v$$

g

$$\frac{\tan u + \tan v}{\tan u - \tan v} = \frac{\dfrac{\sin u}{\cos u} + \dfrac{\sin v}{\cos v}}{\dfrac{\sin u}{\cos u} - \dfrac{\sin v}{\cos v}}$$

$$= \frac{\dfrac{\sin u\cos v + \cos u\sin v}{\cos u\cos v}}{\dfrac{\sin u\cos v - \cos u\sin v}{\cos u\cos v}}$$

$$= \frac{\sin(u+v)}{\sin(u-v)}$$

h $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= (1 - \sin^2\theta) - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

i $\sin 4\theta = \sin 3\theta\cos\theta + \sin\theta\cos 3\theta$

$$= (3\sin\theta - 4\sin^3\theta)\cos\theta$$
$$+ \sin\theta(4\cos^3\theta - 3\cos\theta)$$
$$= 3\sin\theta\cos\theta - 4\sin^3\theta\cos\theta$$
$$+ 4\cos^3\theta\sin\theta - 3\cos\theta\sin\theta$$
$$= 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$$

j $\dfrac{1 - \sin 2\theta}{\sin\theta - \cos\theta}$

$$= \frac{1 - \sin 2\theta}{\sin\theta - \cos\theta} \times \frac{\sin\theta - \cos\theta}{\sin\theta - \cos\theta}$$

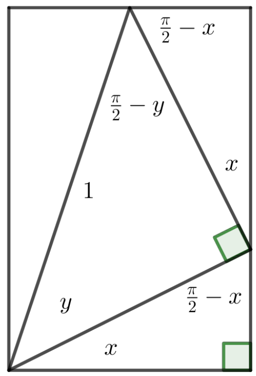
$$= \frac{(1 - \sin 2\theta)(\sin\theta - \cos\theta)}{\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta}$$

$$= \frac{(1 - \sin 2\theta)(\sin\theta - \cos\theta)}{1 - 2\sin\theta\cos\theta}$$

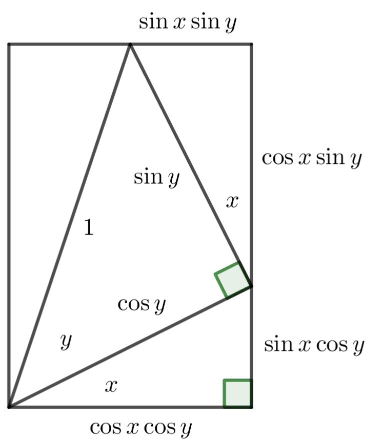
$$= \frac{(1 - \sin 2\theta)(\sin\theta - \cos\theta)}{1 - \sin 2\theta}$$

$$= \sin\theta - \cos\theta$$

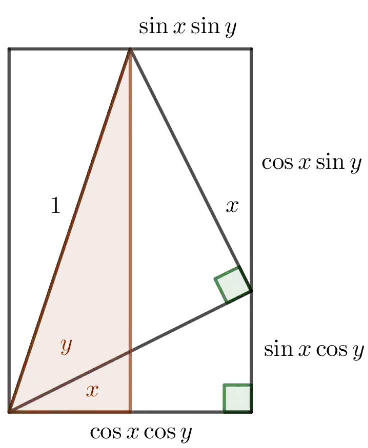
13 a i We first find each of the angles in the diagram.



Using these angles, we find each of the side lengths in the diagram. Thes are shown below.



Now consider the shaded right-angled triangle shown below.



The height of the shaded triangle can then be found two different ways. First, the height of the triangle is $1 \times \sin(x+y) = \sin(x+y)$. We can also find this height as the sum of two side lengths: $\sin x \cos y + \cos x \sin y$. Equating the two results gives

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

ii Using the same diagram. The base lengh of the shaded triangle is $1 \times \cos(x+y) = \cos(x+y)$. We can also find this length as the difference of two side lengths: $\cos x \cos y - \sin x \sin y$ Equating the two results gives

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

b **i** Many adaptations of the diagram are possible. However, it is perhaps smarter to simply note that

$$\begin{aligned}\sin(x-y) &= \sin(x+(-y)) \\ &= \sin(x)\cos(-y) + \cos(x)\sin(-y) \\ &= \sin(x)\cos(y) - \cos(x)\sin(y)\end{aligned}$$

ii Likewise,

$$\begin{aligned}\cos(x-y) &= \cos(x+(-y)) \\ &= \cos x \cos(-y) - \sin x \sin(-y) \\ &= \sin(x)\cos(y) + \cos(x)\sin(y)\end{aligned}$$

Solutions to Exercise 16C

1 **a** Maximum $= \sqrt{4^2 + 3^2} = 5$

Minimum $= -5$

b Maximum $= \sqrt{3+1} = 2$

Minimum $= -2$

c Maximum $= \sqrt{1+1} = \sqrt{2}$

Minimum $= -\sqrt{2}$

d Maximum $= \sqrt{1+1} = \sqrt{2}$

Minimum $= -\sqrt{2}$

e Maximum $= \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$

Minimum $= -2\sqrt{3}$

f Maximum $= \sqrt{1+3} = 2$

Minimum $= -2$

g Maximum $= \sqrt{1+3} + 2 = 4$

Minimum $= -\sqrt{1+3} + 2 = 0$

h Maximum $= 5 + \sqrt{3^2 + 2^2}$

$= 5 + \sqrt{13}$

Minimum $= 5 - \sqrt{3^2 + 2^2}$

$= 5 - \sqrt{13}$

2 **a**

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\cos\alpha = \frac{1}{\sqrt{2}};\ \sin\alpha = -\frac{1}{\sqrt{2}}$$

$$\alpha = -\frac{\pi}{4}$$

$$\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 1$$

$$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x - \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$x = \frac{\pi}{2}, \pi$$

b

$$r = \sqrt{3+1} = 2$$

$$\cos\alpha = \frac{\sqrt{3}}{2};\ \sin\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$2\sin\left(x + \frac{\pi}{6}\right) = 1$$

$$\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{5}, \frac{7\pi}{6}$$

$$x = 0, \frac{2\pi}{3}, 2\pi$$

c
$$r = \sqrt{3+1} = 2$$
$$\cos\alpha = \frac{1}{2};\ \sin\alpha = -\frac{\sqrt{3}}{2}$$
$$\alpha = -\frac{\pi}{3}$$
$$2\sin\left(x - \frac{\pi}{3}\right) = -1$$
$$\sin\left(c - \frac{\pi}{3}\right) = -\frac{1}{2}$$
$$x - \frac{\pi}{3} = -\frac{\pi}{6},\ \frac{7\pi}{6}$$
$$x = \frac{\pi}{6},\ \frac{3\pi}{2}$$

d
$$r = \sqrt{9+3} = \sqrt{12}$$
$$= 2\sqrt{3}$$
$$\cos\alpha = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$
$$\sin\alpha = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$$
$$\alpha = \frac{\pi}{6}$$
$$2\sqrt{3}\cos\left(x + \frac{\pi}{6}\right) = 3$$
$$\cos\left(x + \frac{\pi}{6}\right) = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$
$$x + \frac{\pi}{6} = \frac{\pi}{6},\ \frac{11\pi}{6},\ \frac{13\pi}{6}$$
$$x = 0,\ \frac{5\pi}{3},\ 2\pi$$

e
$$r = \sqrt{4^2 + 3^2}$$
$$= \sqrt{25} = 5$$
$$\cos\alpha = \frac{4}{5};\ \sin\alpha = \frac{3}{5}$$
$$\alpha \approx 36.87^\circ$$
$$5\sin(\theta + 36.87) \approx 5$$
$$\sin(\theta + 36.87) \approx 1$$
$$\theta + 36.87 \approx 90^\circ$$
$$\theta \approx 53.13^\circ$$

f
$$r = \sqrt{8+4} = \sqrt{12} = 2\sqrt{3}$$
$$\cos\alpha = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$
$$\sin\alpha = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$
$$\alpha \approx -35.26^\circ$$
$$2\sqrt{3}\sin(\theta - 35.26) \approx 3$$
$$\sin(\theta - 35.26) \approx \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$
$$\theta - 35.26 \approx 60^\circ, 120^\circ$$
$$\theta \approx 95.26^\circ, 155.26^\circ$$

3
$$r = \sqrt{3+1} = 2$$
$$\cos\alpha = \frac{\sqrt{3}}{2};\ \sin\alpha = \frac{1}{2}$$
$$\alpha = \frac{\pi}{6}$$
$$2\cos\left(2x + \frac{\pi}{6}\right)$$

4 $r = \sqrt{1+1} = \sqrt{2}$

$$\cos\alpha = -\frac{1}{\sqrt{2}};\ \sin\alpha = -\frac{1}{\sqrt{2}}$$

$$\alpha = \frac{5\pi}{4}$$

$$\sqrt{2}\sin\left(3x - \frac{5\pi}{4}\right)$$

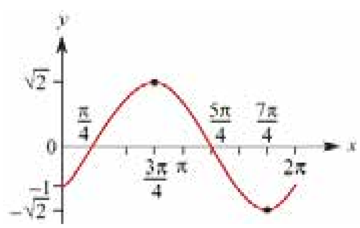
5 a $r = \sqrt{1+1} = \sqrt{2}$

$$\cos\alpha = \frac{1}{\sqrt{2}};\ \sin\alpha = -\frac{1}{\sqrt{2}}$$

$$\alpha = -\frac{\pi}{4}$$

$$f(x) = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

The graph will have amplitude $\sqrt{2}$, period 2π, and be translated $\frac{\pi}{4}$ units right.



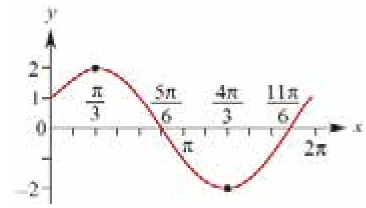
b $r = \sqrt{3+1} = 2$

$$\cos\alpha = \frac{\sqrt{3}}{2},\ \sin\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$f(x) = 2\sin\left(x + \frac{\pi}{6}\right)$$

The graph will have amplitude 2, period 2π, and be translated $\frac{\pi}{6}$ units left.



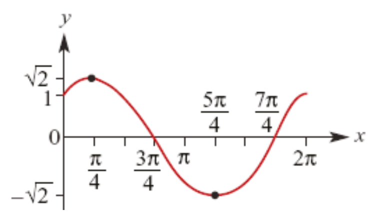
c $r = \sqrt{1+1} = \sqrt{2}$

$$\cos\alpha = \frac{1}{\sqrt{2}};\ \sin\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = \frac{\pi}{4}$$

$$f(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

The graph will have amplitude $\sqrt{2}$, period 2π, and be translated $\frac{\pi}{4}$ units left.



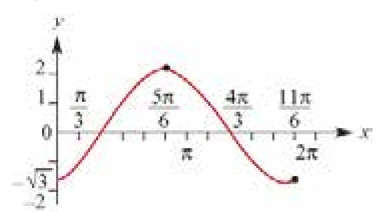
d $r = \sqrt{1+3} = 2$

$$\cos\alpha = \frac{1}{2};\ \sin\alpha = -\frac{\sqrt{3}}{2}$$

$$\alpha = -\frac{\pi}{3}$$

$$f(x) = 2\sin\left(x - \frac{\pi}{3}\right)$$

The graph will have amplitude 2, period 2π, and be translated $\frac{\pi}{3}$ units right.



Solutions to Exercise 16D

1 **a** $\sin(5\pi t) + \sin(\pi t)$

b $\frac{1}{2}(\sin 50° - \sin 10°)$

c $\sin(\pi x) + \sin\left(\frac{\pi x}{2}\right)$

d $\sin(A) + \sin(B + C)$

2 $\cos(\theta) - \cos(5\theta)$

3 $2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

$= \sin\left(\frac{A-B+A+B}{2}\right) + \sin\left(\frac{A-B-A-B}{2}\right)$

$= \sin A + \sin(-B)$

$= \sin A - \sin B$

4 $2\sin(75°)\sin(15°)$

$= \cos(75-15)° - \cos(75+15)°$

$= \cos 60° - \cos(90)°$

$= \frac{1}{2} \therefore \sin(75°)\sin(15°) = \frac{1}{4}$

5 **a** $2\sin 39° \cos 17°$

b $2\cos 39° \cos 17°$

c $2\cos 39° \sin 17°$

d $-2\sin 39° \sin 17°$

6 **a** $2\sin(4A)\cos(2A)$

b $2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$

c $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{7x}{2}\right)$

d $-2\sin(2A)\sin(A)$

7 $$\begin{aligned}\text{LHS} &= \sin A + 2\sin 3A + \sin 5A\\ &= \sin A + \sin 3A + \sin 3A + \sin 5A\\ &= 2\sin 2A\cos(-A) + 2\sin 4A\cos(-A)\\ &= 2\cos A(\sin 2A + \sin 4A)\\ &= 2\cos A(2\sin 3A\cos A)\\ &= 4\cos^2 A\sin 3A\\ &= \text{RHS}\end{aligned}$$

8 $$\begin{aligned}\text{LHS} &= \sin(\alpha+\beta)\sin(\alpha-\beta) + \sin(\beta+\gamma)\sin(\beta-\gamma) + \sin(\gamma+\alpha)\sin(\gamma-\alpha)\\ &= \frac{1}{2}(\cos(2\beta) - \cos(2\alpha) + \cos(2\gamma) - \cos(2\beta) + \cos(2\alpha) - \cos(2\gamma))\\ &= 0\\ &= \text{RHS}\end{aligned}$$

9 $$\begin{aligned}\text{LHS} &= \cos 70^\circ + \sin 40^\circ\\ &= \cos 70^\circ + \cos 50^\circ\\ &= 2\cos 60^\circ\cos 10^\circ\\ &= \cos 10^\circ\\ &= \text{RHS}\end{aligned}$$

10 $$\begin{aligned}\text{LHS} &= \cos 20^\circ + \cos 100^\circ + \cos 140^\circ\\ &= \cos 20^\circ + 2\cos 120^\circ\cos(-20)^\circ\\ &= \cos 20^\circ - \cos(-20)^\circ\\ &= 0\\ &= \text{RHS}\end{aligned}$$

11 **a** First,

$$\cos 5x + \cos x = 0$$
$$\Rightarrow 2\cos(3x)\cos(x) = 0$$
$$\Rightarrow \cos(3x) = 0 \text{ or } \cos(x) = 0.$$

Solving for $x \in [-\pi, \pi]$ gives $x = -\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{6}\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

b First,

$$\begin{aligned} \cos 5x - \cos x &= 0 \\ \Rightarrow -2\sin(3x)\sin(x) &= 0 \\ \Rightarrow \sin(3x) = 0 \text{ or } \sin(x) &= 0 \end{aligned}$$

Solving for $x \in [-\pi, \pi]$ gives $x = -\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

c First,

$$\begin{aligned} \sin 5x + \sin x &= 0 \\ \Rightarrow 2\sin(3x)\cos(x) &= 0 \\ \Rightarrow \sin(3x) = 0 \text{ or } \cos(x) &= 0 \end{aligned}$$

Solving for $x \in [-\pi, \pi]$ gives $x = -\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$

d First,

$$\begin{aligned} \sin 5x - \sin x &= 0 \\ \Rightarrow 2\sin(x)\cos(3x) &= 0 \\ \Rightarrow \sin(x) = 0 \text{ or } \cos(3x) &= 0 \end{aligned}$$

Solving for $x \in [-\pi, \pi]$ gives $x = -\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

12 a First,

$$\begin{aligned} \cos 2\theta - \sin\theta &= 0 \\ \Rightarrow 1 - 2\sin^2\theta - \sin\theta &= 0 \\ \Rightarrow 2\sin^2\theta + \sin\theta - 1 &= 0 \\ \Rightarrow (\sin\theta + 1)(2\sin\theta - 1) &= 0 \\ \Rightarrow \sin\theta = -1 \text{ or } \sin\theta &= \frac{1}{2} \end{aligned}$$

Solving for $x \in [0, \pi]$ gives $x = \frac{\pi}{6}, \frac{5\pi}{6}$

b First,

$$\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$$
$$\Rightarrow 2\sin(\theta)\cos(4\theta) + \sin(\theta) = 0$$
$$\Rightarrow \sin(\theta)(2\cos(4\theta) + 1) = 0$$
$$\Rightarrow \sin(\theta) = 0 \text{ or } \cos(4\theta) = -\frac{1}{2}$$

Solving for $x \in [0, \pi]$ gives $x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

c First,

$$\sin(7\theta) - \sin(\theta) = \sin(3\theta)$$
$$\Rightarrow 2\sin(3\theta)\cos(4\theta) = \sin(3\theta)$$
$$\Rightarrow 2\sin(3\theta)\cos(4\theta) - \sin(3\theta) = 0$$
$$\Rightarrow \sin(3\theta)(2\cos(4\theta) - 1) = 0$$
$$\Rightarrow \sin(3\theta) = 0 \text{ or } \cos(4\theta) = \frac{1}{2}$$

Solving for $x \in [0, \pi]$ gives $x = 0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$

d First,

$$\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$$
$$\Rightarrow \cos(7\theta) + \cos(3\theta) - \cos(5\theta) = 0$$
$$\Rightarrow 2\cos(5\theta)\cos(2\theta) - \cos(5\theta) = 0$$
$$\Rightarrow \cos(5\theta)(2\cos(2\theta) - 1) = 0$$
$$\Rightarrow \cos(5\theta) = 0 \text{ or } \cos(2\theta) = \frac{1}{2}$$

Solving for $x \in [0, \pi]$ gives $x = \frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

13 LHS $= \dfrac{\sin A + \sin B}{\cos A + \cos B}$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)}$$

$$= \tan\left(\frac{A+B}{2}\right)$$

$= \text{RHS}$

14 LHS $= 4\sin(A+B)\sin(B+C)\sin(C+A)$

$= 2(\cos(A-C) - \cos(A+2B+C)\sin(C+A)$

$= 2\cos(A-C)\sin(C+A) - 2\cos(A+2B+C)\sin(C+A)$

$= \sin 2A + \sin 2C - (\sin(2A+2B+2C) + \sin(-2B))$

$= \sin 2A + \sin 2C + \sin 2B - \sin(2A+2B+2C)$

$= \text{RHS}$

15 LHS $= \dfrac{\cos 2A - \cos 2B}{\sin(2A-2B)}$

$$= \frac{2\sin\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2B-2A}{2}\right)}{\sin(2A-2B)}$$

$$= \frac{2\sin(A+B)\sin(B-A)}{2\sin(A-B)\cos(A-B)}$$

$$= -\frac{\sin(A+B)}{\cos(A-B)}$$

$= \text{RHS}$

16 **a** LHS $= \dfrac{\sin(A) + \sin(3A) + \sin(5A)}{\cos(A) + \cos(3A) + \cos(5A)}$

$= \dfrac{2\sin 3A \cos 2A + \sin 3A}{2\cos 3A \cos 2A + \cos 3A}$

$= \dfrac{\sin 3A(2\cos 2A + 1)}{\cos 3A(2\cos 2A + 1)}$

$= \tan 3A$

= RHS

b RHS $= \cos(A + B)\cos(A - B)$

$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$= \cos^2 A \cos^2 B - (1 - \cos^2 A)(1 - \cos^2 B)$

$= \cos^2 A \cos^2 B - (1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B)$

$= \cos^2 A + \cos^2 B - 1$

= LHS

c LHS $= \cos^2(A - B) - \cos^2(A + B)$

$= (\cos(A - B) - \cos(A + B))(\cos(A - B) + \cos(A + B))$

$= 2\cos A \cos B \times 2\sin A \sin B$

$= \sin 2A \sin 2B$

= RHS

d LHS $= \cos^2(A - B) - \sin^2(A + B)$

$= \cos^2(A - B) - (1 - \cos^2(A + B))$

$= \cos 2A \cos 2B$ by **16b**

= RHS

17

$$
\begin{aligned}
\text{Let } S &= \sin x + \sin 3x + \sin 5x + \cdots + \sin 99x \\
\text{Then } 2\sin xS &= 2\sin^2 x + 2\sin x \sin 3x + 2\sin x \sin 5x + 2\sin x \sin 7x + \cdots + 2\sin x \sin 99x \\
&= 2\sin^2 x + \cos 2x - \cos 4x + \cos 4x - \cos 6x + \cos 6x - \cos 8x + \cdots + \cos 98x - \cos 100x \\
&= 2\sin^2 x + \cos 2x - \cos 100x \\
&= 2\sin^2 x + 1 - 2\sin^2 x - \cos 100x \\
&= 1 - \cos 100x \\
\therefore S &= \frac{1 - \cos 100x}{2\sin x}
\end{aligned}
$$

Solutions to technology-free questions

1 a $\sec\theta + \operatorname{cosec}\theta\cot\theta$

$$= \frac{1}{\cos\theta} + \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$
$$= \frac{1}{\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1}{\cos\theta} + \frac{\cos^2\theta}{\sin^2\theta}$$
$$= \frac{1}{\cos\theta} + \frac{\cos^2\theta}{\sin^2\theta}$$
$$= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta\sin^2\theta}$$
$$= \frac{1}{\cos\theta\sin^2\theta}$$
$$= \sec\theta\operatorname{cosec}^2\theta$$

b $(\sec\theta - \sin\theta)(\sec\theta + \sin\theta)$

$= \sec^2\theta - \sin^2\theta$

$= (1 + \tan^2\theta) - (1 - \cos^2\theta)$

$= \tan^2\theta + \cos^2\theta.$

Now divide both sides by $(\sec\theta + \sin\theta)$ to give the result.

2 a $\operatorname{cosec}^2\theta = 4$

$$\frac{1}{\sin^2\theta} = 4$$
$$\sin^2\theta = \frac{1}{4}$$
$$\sin\theta = \pm\frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

b $\operatorname{cosec}(2\theta) = 2$

$$\frac{1}{\sin(2\theta)} = 2$$
$$\sin(2\theta) = \frac{1}{2}$$
$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

c

$$\sec(3\theta) = \frac{2\sqrt{3}}{3}$$
$$\frac{1}{\cos(3\theta)} = \frac{2\sqrt{3}}{3}$$
$$\cos(3\theta) = \frac{3}{2\sqrt{3}}$$
$$\cos(3\theta) = \frac{\sqrt{3}}{2}$$
$$3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{35\pi}{6}$$
$$\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$$

d $\operatorname{cosec}^2(2\theta) = 1$

$$\frac{1}{\sin^2(2\theta)} = 1$$
$$\sin^2(2\theta) = 1$$
$$\sin(2\theta) = \pm 1$$
$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

e $\cot^2(\theta) = 3$

$$\frac{1}{\tan^2(\theta)} = 3$$

$$\tan^2(\theta) = \frac{1}{3}$$

$$\tan(\theta) = \pm\frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

f $\cot(2\theta) = -1$

$$\frac{1}{\tan(2\theta)} = -1$$

$$\tan(2\theta) = -1$$

$$2\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

$$\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$$

g $\operatorname{cosec}(3\theta) = -1$

$$\frac{1}{\sin(3\theta)} = -1$$

$$\sin(3\theta) = -1$$

$$3\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

h $\operatorname{cosec}(3\theta) = -1$

$$\frac{1}{\sin(3\theta)} = -1$$

$$\sin(3\theta) = -1$$

$$3\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

i $\sec(2\theta) = \sqrt{2}$

$$\frac{1}{\cos(2\theta)} = \sqrt{2}$$

$$\cos(2\theta) = \frac{1}{\sqrt{2}}$$

$$2\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$$

$$\theta = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$$

3

$$\tan\theta = 2\sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$

$$\sin\theta = 2\sin\theta\cos\theta$$

$$\sin\theta - 2\sin\theta\cos\theta = 0$$

$$\sin\theta(1 - 2\cos\theta) = 0$$

$$\sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$

$$\theta = 0°, 60°, 180°, 300°, 360°$$

4 $\cos^2 A = 1 - \sin^2 A$

$$= 1 - \frac{25}{169} = \frac{144}{169}$$

$$\cos A = \frac{12}{13} \text{ (Since A is acute)}$$

$$\cos^2 B = 1 - \sin^2 B$$

$$= 1 - \frac{64}{289} = \frac{225}{289}$$

$$\cos B = \frac{15}{17} \text{ (Since B is acute)}$$

a $\cos(A + B)$

$$= \cos A \cos B - \sin A \sin B$$

$$= \frac{12}{13} \times \frac{15}{17} - \frac{5}{13} \times \frac{8}{17}$$

$$= \frac{140}{221}$$

b $\sin(A + B)$

$= \sin A \cos B + \cos A \sin B$

$= \dfrac{5}{13} \times \dfrac{15}{17} + \dfrac{12}{13} \times \dfrac{8}{17}$

$= \dfrac{171}{221}$

c
$$\tan A = \frac{\sin A}{\cos A} = \frac{5}{12}$$
$$\tan B = \frac{\sin B}{\cos B} = \frac{8}{15}$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$= \left(\frac{5}{12} + \frac{8}{15}\right)$$
$$\div \left(1 - \frac{5}{12} \times \frac{8}{15}\right)$$
$$= \frac{57}{60} \div \frac{7}{9}$$
$$= \frac{19}{20} \times \frac{9}{7}$$
$$= \frac{171}{140}$$

5 **a** $\cos 80° \cos 20° + \sin 80° \sin 20°$

$= \cos(80° - 20°)$

$= \cos 60° = \dfrac{1}{2}$

b $\dfrac{\tan 15° + \tan 30°}{1 - \tan 15° \tan 30°} = \tan(15° + 30°)$

$= \tan(45°)$

$= 1$

6 **a** $\sin A \cos B + \cos A \sin B = \sin(A + B)$

$= \sin \dfrac{\pi}{2} = 1$

b $\cos A \cos B - \sin A \sin B = \cos(A + B)$

$= \cos \dfrac{\pi}{2} = 0$

7 **a** $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A(1 - \sin^2 B)$

$\quad - (1 - \sin^2 A) \sin^2 B$

$= \sin^2 A - \sin^2 A \sin^2 B$

$\quad - \sin^2 B + \sin^2 A \sin^2 B$

$= \sin^2 A - \sin^2 B$

b Left side

$$= \frac{\sin^2 \theta + (1 + \cos \theta)^2}{\sin \theta(1 + \cos \theta)}$$
$$= \frac{\sin^2 \theta + 1 + 2\cos \theta + \cos^2 \theta}{\sin \theta(1 + \cos \theta)}$$
$$= \frac{2 + 2\cos \theta}{\sin \theta(1 + \cos \theta)}$$
$$= \frac{2(1 + \cos \theta)}{\sin \theta(1 + \cos \theta)}$$
$$= \frac{2}{\sin \theta}$$

c Left side

$$= \frac{\sin \theta(1 - 2\sin^2 \theta)}{\cos \theta(2\cos^2 \theta - 1)}$$
$$= \frac{\sin \theta(1 - \sin^2 \theta - \sin^2 \theta)}{\cos \theta(\cos^2 \theta + \cos^2 \theta - 1)}$$
$$= \frac{\sin \theta(\cos^2 \theta - \sin^2 \theta)}{\cos \theta(\cos^2 \theta - (1 - \cos^2 \theta))}$$
$$= \frac{\sin \theta(\cos^2 \theta - \sin^2 \theta)}{\cos \theta(\cos^2 \theta - \sin^2 \theta)}$$
$$= \frac{\sin \theta}{\cos \theta}$$
$$= \tan \theta$$

8 $\cos^2 A = 1 - \sin^2 A$

$$= 1 - \frac{5}{9} = \frac{4}{9}$$

$$\cos A = -\frac{2}{3} \text{ (Since } A \text{ is obtuse)}$$

a $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{4}{9} - \frac{5}{9}$$

$$= -\frac{1}{9}$$

b $\sin 2A = 2 \sin A \cos A$

$$= 2 \times \frac{\sqrt{5}}{3} \times -\frac{2}{3}$$

$$= -\frac{4\sqrt{5}}{9}$$

c $\sin 4A = 2 \sin 2A \cos 2A$

$$= 2 \times -\frac{4\sqrt{5}}{9} \times -\frac{1}{9}$$

$$= \frac{8\sqrt{5}}{81}$$

9 **a** Left side $= \dfrac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}}$

$$= \frac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}} \times \frac{\cos^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{\cos 2\theta}{1} = \cos 2\theta$$

b Left side

$$= \sqrt{2r^2(1 - \cos\theta)}$$

$$= \sqrt{2r^2(1 - (1 - \sin^2\frac{\theta}{2})}$$

$$= 2r \sin\left(\frac{\theta}{2}\right)$$

10 Using the appropriate compound angle formula gives
$\tan 15° = \tan (60 - 45)°$

$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$$

$$= 2 - \sqrt{3}$$

11 **a** Express in the form $r \sin(x + \alpha) = 1$.

$$r = \sqrt{1 + 1} = \sqrt{2}$$

$$\cos\alpha = \frac{1}{\sqrt{2}}; \ \sin\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = \frac{\pi}{4}$$

$$\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) = 1$$

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$$

$$x = 0, \ \frac{\pi}{2}, \ 2\pi$$

b $2\sin\frac{x}{2}\cos\frac{x}{2} = -\frac{1}{2}$

$$\sin\left(2\times\frac{x}{2}\right) = -\frac{1}{2}$$

$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

c $3\times\dfrac{2\tan x}{1-\tan^2 x} = 2\tan x$

$$2\tan x\left(\frac{3}{1-\tan^2 x} - 1\right) = 0$$

$$2\tan x\left(\frac{3-(1-\tan^2 x)}{1-\tan^2 x}\right) = 0$$

$\tan x = 0$ (since$2 + \tan^2 x \neq 0$)

$$x = 0, \pi, 2\pi$$

d $\sin^2 x - \cos^2 x = 1$

$$\cos 2x = -1$$

$$2x = \pi, 3\pi$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

e $\sin(3x - x) = \dfrac{\sqrt{3}}{2}$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

f $\cos\left(2x - \frac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

$$2x - \frac{\pi}{3} = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$$

$$2x = \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{19\pi}{6}, \frac{21\pi}{6}$$

$$x = \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$$

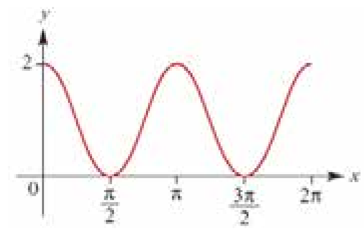
12 **a** $y = 2\cos^2 x$ The

$= \cos^2 x + (1 - \sin^2 x)$

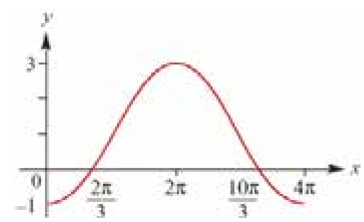
$= \cos^2 x - \sin^2 x + 1$

$= \cos 2x + 1$

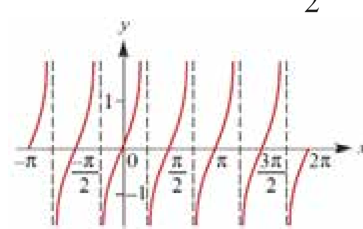
graph of $y = \cos 2x$ (amplitude 1, period π) raised 1 unit.



b The graph is $y = 1 - 2\sin\left(\frac{\pi}{2} - \frac{x}{2}\right) = 1 - 2\cos\frac{\pi}{2}$. It is $y = 2\cos\frac{x}{2}$ (period 4π) reflected in the x-axis and raised 1 unit.



c The normal tangent graph, but with period $\frac{\pi}{2}$.



13 Given $\tan A = 2$ and $\tan(\theta + A) = 4$ we find that

$$\tan(\theta) = \tan((\theta + A) - A)$$
$$= \frac{\tan(\theta + A) - \tan A}{1 + \tan(\theta + A)\tan A}$$
$$= \frac{4 - 2}{1 + 4 \times 2}$$
$$= \frac{2}{9}$$

14 a $r = \sqrt{4 + 81} = \sqrt{85}$

$\cos\alpha = \frac{2}{\sqrt{85}}$; $\sin\alpha = \frac{9}{\sqrt{85}}$

$\sqrt{85}\cos(\theta - \alpha)$, where

$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$

b i $\sqrt{85}$

ii $\cos(\theta - \alpha) = 1$

$$\theta - \alpha = 0$$
$$\theta = -\alpha$$
$$\cos\theta = \cos\alpha$$
$$= \frac{2}{\sqrt{85}}$$

iii Solve $\sqrt{85}\cos(\theta + \alpha) = 1$.

$$\cos(\theta - \alpha) = \frac{1}{\sqrt{85}}$$
$$\theta - \alpha = \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$$
$$\theta = \alpha + \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$$
$$= \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$$
$$+ \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$$

15 a $\sin 4\theta + \sin 2\theta = 0$

$$2\sin 3\theta \sin\theta = 0$$
$$\therefore \sin 3\theta = 0 \text{ or } \sin\theta = 0$$
$$\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$$

b $\sin 2\theta - \sin\theta = 0$

$$2\sin\frac{\theta}{2}\cos\frac{3\theta}{2} = 0$$
$$\therefore \sin\frac{\theta}{2} = 0 \text{ or } \cos\frac{3\theta}{2} = 0$$
$$\theta = 0, \frac{\pi}{3}, \pi$$

16 $\text{LHS} = \frac{\cos A - \cos B)}{\sin A + \sin B}$

$$= \frac{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$
$$= \frac{-\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}$$
$$= \tan\left(\frac{B-A}{2}\right)$$
$$= \text{RHS}$$

Solutions to multiple-choice questions

1 A
$$\begin{aligned}\text{cosec}x - \sin x &= \frac{1}{\sin x} - \sin x\\ &= \frac{1-\sin^2 x}{\sin x}\\ &= \frac{\cos^2 x}{\sin x}\\ &= \cos x \times \frac{\cos x}{\sin x}\\ &= \cos x \cot x\end{aligned}$$

2 A
$$\cos x = -\frac{1}{3}$$
$$\cos^2 x + \sin^2 x = 1$$
$$\left(-\frac{1}{3}\right)^2 + \sin^2 x = 1$$
$$\begin{aligned}\sin^2 x &= 1 - \frac{1}{9} = \frac{8}{9}\\ \sin x &= \pm\sqrt{\frac{8}{9}}\\ &= -\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\end{aligned}$$

3 B
$$\sec\theta = \frac{b}{a}$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\begin{aligned}\tan^2\theta &= \frac{b^2}{a^2} = 1\\ &= \frac{b^2-a^2}{a^2}\\ \tan\theta &= \frac{\sqrt{b^2-a^2}}{a}\end{aligned}$$
(Since $\tan\theta > 0$)

4 A First we have that

$$\tan\alpha = \frac{x}{2}$$
$$\tan(\theta+\alpha) = \frac{x+4}{2}$$

Therefore

$$\begin{aligned}\tan(\theta) &= \tan((\theta+\alpha)-\alpha)\\ &= \frac{\tan(\theta+\alpha) - \tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha}\\ &= \frac{\frac{x+4}{2} - \frac{x}{2}}{1+\frac{x+4}{2}\frac{x}{2}}\\ &= \frac{2}{1+\frac{x(x+4)}{4}}\\ &= \frac{8}{4+4x+x^2}\\ &= \frac{8}{(x+2)^2}\end{aligned}$$

5 C $\sin A = \sqrt{1-t^2}$
(Since $\sin A > 0$)
$$\begin{aligned}\cos^2 B &= 1-\sin^2 B\\ &= 1-t^2\\ \cos B &= -\sqrt{1-t^2}\end{aligned}$$
Since $\cos B < 0$)
$$\begin{aligned}\sin(B+A) &= \sin B\cos A + \cos B \sin A\\ &= t\times t + \left(-\sqrt{1-t^2}\right)\times\sqrt{1-t^2}\\ &= t^2 - (1-t^2)\\ &= 2t^2 - 1\end{aligned}$$

6 E $\dfrac{\sin 2A}{\cos 2A - 1}$

$$= \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A - 1}$$

$$= \frac{2\sin A\cos A}{-\sin^2 A - (1 - \cos^2 A)}$$

$$= \frac{2\sin A\cos A}{-\sin^2 A - \sin^2 A}$$

$$= \frac{2\sin A\cos A}{-2\sin^2 A}$$

$$= \frac{\cos A}{\sin A}$$

$$= -\cot A$$

7 E $(1 + \cot x)^2 + (1 - \cot x)^2$

$$= 1 + 2\cot x + \cot^2 x + 1$$
$$- 2\cot x + \cot^2 x$$
$$= 2 + 2\cot^2 x$$
$$= 2(1 + \cot^2 x)$$
$$= 2\text{cosec}^2 x$$

8 A $\sin 2A = 2\sin A\cos A$

$$m = 2\sin A \times n$$
$$\sin A = \frac{m}{2n}$$
$$\tan A = \frac{\sin A}{\cos A}$$
$$= \frac{m}{2n} \times \frac{1}{n}$$
$$= \frac{m}{2n^2}$$

9 D $r = \sqrt{1+1} = \sqrt{2}$

$$\cos\alpha = \frac{1}{\sqrt{2}};\ \sin\alpha = -\frac{1}{\sqrt{2}}$$

A positive angle must be chosen,

$$\therefore \quad \alpha = \frac{7\pi}{4}$$

$$\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$$

10 E Using a product-to-sum identity gives

$$\sin 25^\circ \cos 75^\circ = \frac{1}{2}(\sin(100^\circ) + \sin(-50^\circ))$$
$$= \frac{1}{2}(\sin(100^\circ) - \sin(50^\circ)).$$

Solutions to extended-response questions

1 a $P = AD + DC + CB + BA$

$= 2AO + BA + 2AO + BA$

$= 4AO + 2BA$

$= 4 \times 5\cos\theta + 2 \times 5\sin\theta$

$= 20\cos\theta + 10\sin\theta$, as required.

b
$$\begin{aligned} a = 20,\ b = 10 \text{ and } R &= \sqrt{a^2 + b^2} \\ &= \sqrt{20^2 + 10^2} \\ &= \sqrt{500} \\ &= 10\sqrt{5} \\ \text{Now } \cos\alpha &= \frac{a}{R} \\ &= \frac{20}{10\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} \\ &= \frac{2\sqrt{5}}{5} \\ \text{Also } \sin\alpha &= \frac{b}{R} \\ &= \frac{10}{10\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \\ &= \frac{\sqrt{5}}{5} \end{aligned}$$

Hence, $0 < \alpha < 90$ and $\alpha^\circ = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)^\circ = (26.565\,05\ldots)^\circ$

$$\begin{aligned} \text{Hence } P &= R\cos(\theta - \alpha) \\ &= 10\sqrt{5}\cos(\theta - \alpha) \text{ where } \alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) \end{aligned}$$

When $P = 16$,

$10\sqrt{5}\cos(\theta - \alpha) = 16$

$\therefore \quad \cos(\theta - \alpha) = \dfrac{16}{10\sqrt{5}}$

$\therefore \quad (\theta - \alpha)^\circ = \cos^{-1}\left(\dfrac{8}{5\sqrt{5}}\right)^\circ$

$\therefore \quad \theta^\circ = \cos^{-1}\left(\dfrac{8}{5\sqrt{5}}\right)^\circ + \cos^{-1}\left(\dfrac{2}{\sqrt{5}}\right)^\circ$

When $P = 16, \theta = 70.88^\circ$

c Area of rectangle $= AB \times AD$

$= 5\sin\theta \times 2AO$

$= 5\sin\theta \times 2 \times 5\cos\theta$

$= 50\sin\theta\cos\theta$

$= 25 \times 2\sin\theta\cos\theta$

$= 25\sin 2\theta$

$\therefore \quad k\sin 2\theta = 25\sin 2\theta$

$\therefore \quad k = 25$

d Area is a maximum when $\sin 2\theta = 1$

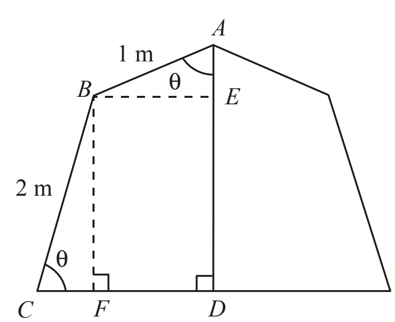
$\therefore \quad 2\theta = 90^\circ$

$\therefore \quad \theta = 45^\circ$

2 a $AD = AE + ED$

$= \cos\theta + BF$

$= \cos\theta + 2\sin\theta$



b $a = 1,\ b = 2$ and $R = \sqrt{a^2 + b^2}$

$$= \sqrt{1^2 + 2^2}$$

$$= \sqrt{5}$$

$$\text{Now } \cos\alpha = \frac{a}{R}$$

$$= \frac{1}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{5}$$

$$\text{Also } \sin\alpha = \frac{b}{R}$$

$$= \frac{2}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5}$$

Hence, $0 < \alpha < 90$ and $\alpha^\circ = \cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right)^\circ = (63.434\,94\ldots)^\circ$

Hence $AD = \sqrt{5}\cos(\theta - 63)^\circ$

c The maximum length of AD is $\sqrt{5}$ metres.
When $AD = \sqrt{5}$,

$$\sqrt{5}\cos(\theta - 63)^\circ = \sqrt{5}$$

$$\therefore \quad \cos(\theta - 63)^\circ = 1$$

$$\therefore \quad \theta - 63 = 0$$

$$\therefore \quad \theta = 63$$

d When $AD = 2.15$,

$$\sqrt{5}\cos(\theta - \alpha)^\circ = 2.15$$

$$\therefore \quad \cos(\theta - \alpha)^\circ = \frac{2.15}{\sqrt{5}}$$

$$\therefore \quad (\theta - \alpha)^\circ = \cos^{-1}\left(\frac{2.15}{\sqrt{5}}\right)^\circ$$

$$= (15.948\,46\ldots)^\circ$$

$$\therefore \quad \theta = (15.948 + 63.435)^\circ$$

The value of θ, for which $\theta > \alpha$, is 79.38°.

3 a $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\theta}{\sec^2\theta}$$
$$= \cos^2\theta\,(1-\tan^2\theta)$$
$$= \cos^2\theta - \frac{\cos^2\theta\sin^2\theta}{\cos^2\theta}$$
$$= \cos^2\theta - \sin^2\theta$$

Hence, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$, as required.

b i From **a**, $\cos\left(2\times 67\frac{1}{2}^\circ\right) = \dfrac{1-\tan^2\left(67\frac{1}{2}^\circ\right)}{1+\tan^2\left(67\frac{1}{2}^\circ\right)}$

$$\therefore \quad \cos 135^\circ = \frac{1-x^2}{1+x^2} \text{ where } x = \tan\left(67\frac{1}{2}^\circ\right)$$
$$\therefore \quad -\cos 45^\circ = \frac{1-x^2}{1+x^2}$$
$$\therefore \quad -\frac{1}{\sqrt{2}} = \frac{1-x^2}{1+x^2}$$
$$\therefore \quad -\sqrt{2} = \frac{1+x^2}{1-x^2}$$
$$\therefore \quad 1+x^2 = -\sqrt{2}(1-x^2)$$
$$\therefore \quad 1+x^2 = \sqrt{2}x^2 - \sqrt{2}, \text{ as required.}$$

ii $1+x^2 = \sqrt{2}x^2 - \sqrt{2}$

$$\therefore 1+\sqrt{2} = \sqrt{2}x^2 - x^2$$
$$= x^2(\sqrt{2}-1)$$
$$\therefore \quad x^2 = \frac{1+\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$
$$= \frac{\sqrt{2}+1+2+\sqrt{2}}{2-1}$$
$$= 3+2\sqrt{2} \qquad \ldots \boxed{1}$$

Given $\tan\left(67\frac{1}{2}^\circ\right) = a + b\sqrt{2}$

$$\therefore \quad x = a + b\sqrt{2} \text{ where } x = \tan\left(67\frac{1}{2}^\circ\right)$$

$$\therefore \quad x^2 = (a + b\sqrt{2})^2$$

$$= a^2 + 2\sqrt{2}ab + 2b^2$$

$$= (a^2 + 2b^2) + (2ab)\sqrt{2} \qquad \ldots \boxed{2}$$

Equating $\boxed{1}$ and $\boxed{2}$

$$a^2 + 2b^2 = 3 \qquad \ldots \boxed{3}$$

$$2ab = 2$$

$$ab = 1$$

As a and b are integers, $a = 1, b = 1$ or $a = -1, b = -1$ and $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$

Note: An alternative method is to note

$$x^2 = \frac{\sqrt{2} + 1}{\sqrt{2} - 1}$$

$$= \frac{(\sqrt{2} + 1)^2}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$$

$$= (\sqrt{2} + 1)^2$$

$\therefore x = \pm(\sqrt{2} + 1)$

When $b = -1, a = -1$,

$a + b\sqrt{2} = -1 - \sqrt{2}$

When $b = 1, a = 1$,

$a + b\sqrt{2} = 1 + \sqrt{2}$

But $\tan\left(67\frac{1}{2}^\circ\right) > 0$,

$$\therefore a + b\sqrt{2} = \sqrt{2} + 1$$

$$= 1 + \sqrt{2}$$

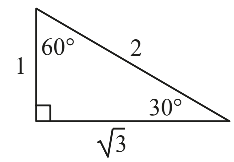
$$\therefore \quad a = 1,\ b = 1$$

c $\tan\left(7\frac{1}{2}^\circ\right) = \tan\left(67\frac{1}{2}^\circ - 60^\circ\right)$

$$= \frac{\tan\left(67\frac{1}{2}^\circ\right) - \tan(60^\circ)}{1 + \tan\left(67\frac{1}{2}^\circ\right)\tan(60^\circ)}$$

$$= \frac{1 + \sqrt{2} - \sqrt{3}}{1 + (1 + \sqrt{2})\sqrt{3}}$$

$$= \frac{1 + \sqrt{2} - \sqrt{3}}{1 + \sqrt{3} + \sqrt{6}}$$



$\tan 60^\circ = \sqrt{3}$

4

5 **a** **i** $\angle CBA = \pi - \frac{2\pi}{5} = \frac{3\pi}{5}$

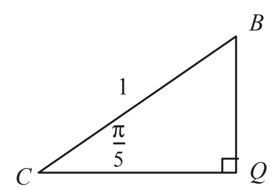
$\angle BCA = \frac{1}{2}\left(\pi - \frac{3\pi}{5}\right)$ as $\angle BCA = \angle BAC$ ($\triangle ABC$ is isosceles)

$= \frac{1}{2} \times \frac{2\pi}{5} = \frac{\pi}{5}$, as required.

ii $CA = 2CQ$

$= 2\cos\frac{\pi}{5}$

The length of CA is $2\cos\frac{\pi}{5}$ units.



b **i** $\angle DCP = \angle BCD - \angle BCA$

$= \angle CBA - \angle BCA$

$= \frac{3\pi}{5} - \frac{\pi}{5} = \frac{2\pi}{5}$, as required.

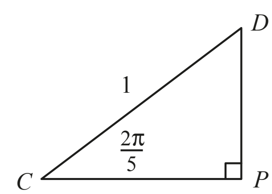
ii $AC = 2CP + PR$

$= 2\cos\frac{2\pi}{5} + DE$

$= 2\cos\frac{2\pi}{5} + 1$

But $AC = 2\cos\frac{\pi}{5}$ (from **a ii**)

$\therefore \quad 2\cos\frac{\pi}{5} = 2\cos\frac{2\pi}{5} + 1$, as required.



iii

$$2\cos\frac{\pi}{5} = 2\cos\frac{2\pi}{5} + 1$$

$$\therefore \quad 2\cos\frac{2\pi}{5} = 2\cos\frac{\pi}{5} - 1$$

$$\therefore \quad \cos\frac{2\pi}{5} = \cos\frac{\pi}{5} - \frac{1}{2}$$

$$\therefore \quad 2\cos^2\frac{\pi}{5} - 1 = \cos\frac{\pi}{5} - \frac{1}{2}$$

$$\therefore \quad 2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0 \text{ or equivalently } 4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0$$

iv

$$2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0$$

$$\therefore \quad 2\left(\cos^2\frac{\pi}{5} - \frac{1}{2}\cos\frac{\pi}{5} - \frac{1}{4}\right) = 0$$

$$\therefore \quad 2\left(\cos^2\frac{\pi}{5} - \frac{1}{2}\cos\frac{\pi}{5} + \frac{1}{16} - \frac{5}{16}\right) = 0$$

$$\therefore \quad 2\left(\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 - \frac{5}{16}\right) = 0$$

$$\therefore \quad 2\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 - \frac{5}{8} = 0$$

$$\therefore \quad 2\left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 = \frac{5}{8}$$

$$\therefore \quad \left(\cos\frac{\pi}{5} - \frac{1}{4}\right)^2 = \frac{5}{16}$$

$$\therefore \quad \cos\frac{\pi}{5} - \frac{1}{4} = \pm\frac{\sqrt{5}}{4}$$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1}{4} \pm \frac{\sqrt{5}}{4}$$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1-\sqrt{5}}{4}, \ \frac{1+\sqrt{5}}{4}$$

but $\cos\frac{\pi}{5} > 0$, as $0 < \frac{\pi}{5} < \frac{\pi}{2}$

$$\therefore \quad \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$$

6 a i LHS $= \cos\theta$

$$= \cos\left(2 \times \frac{\theta}{2}\right)$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$\text{RHS} = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

$$= \frac{1 - \tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$= \cos^2\frac{\theta}{2}\left(1 - \tan^2\frac{\theta}{2}\right)$$

$$= \cos^2\frac{\theta}{2} - \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

Therefore LHS = RHS.

Hence $\cos\theta = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$, as required.

ii $\text{RHS} = \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

$$= \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$= \cos^2\frac{\theta}{2} \times 2\tan\frac{\theta}{2}$$

$$= \frac{2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \sin\left(2\times\frac{\theta}{2}\right)$$

$$= \sin\theta$$

$$= \text{LHS}$$

Hence $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$, as required.

b $$8\cos\theta - \sin\theta = 4$$

$$\therefore \quad 8\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right) - \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = 4$$

$$\therefore \quad 8\left(1-\tan^2\frac{\theta}{2}\right) - 2\tan\frac{\theta}{2} = 4\left(1+\tan 2\frac{\theta}{2}\right)$$

$$\therefore \quad 8 - 8\tan^2\frac{\theta}{2} - 2\tan\frac{\theta}{2} = 4 + 4\tan^2\frac{\theta}{2}$$

$$\therefore \quad 12\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - 4 = 0$$

$$\therefore \quad 6\tan^2\frac{\theta}{2} + \tan\frac{\theta}{2} - 2 = 0$$

$$\therefore \quad \left(3\tan\frac{\theta}{2} + 2\right)\left(2\tan\frac{\theta}{2} - 1\right) = 0$$

$$\therefore \quad 3\tan\frac{\theta}{2}+2 = 0 \text{ or } 2\tan\frac{\theta}{2} - 1 = 0$$

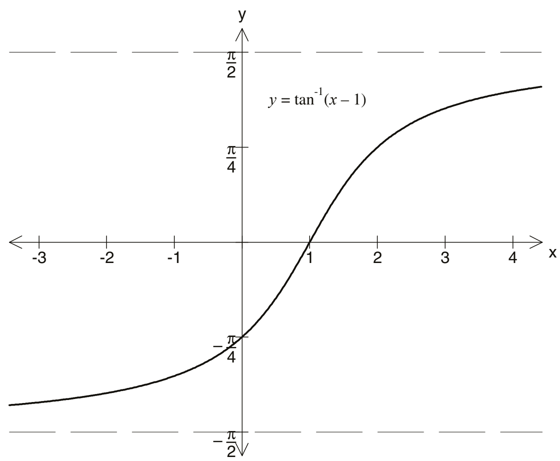
$$\therefore \quad \tan\frac{\theta}{2} = \frac{-2}{3} \quad \tan\frac{\theta}{2} = \frac{1}{2}$$

Chapter 17 – Graphing functions and relations

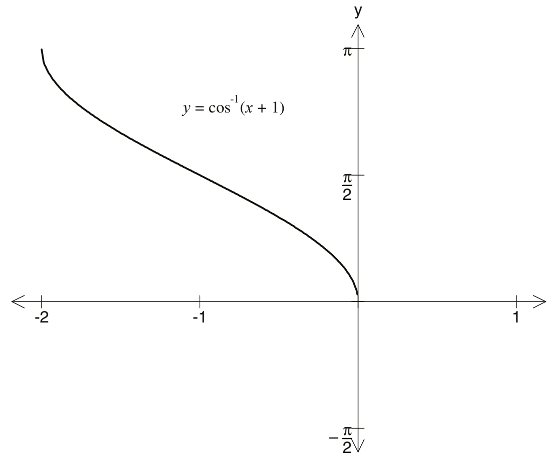
Solutions to Exercise 17A

1 a The graph of $y = \tan^{-1}(x - 1)$ is a translation of the graph of $y = \tan^{-1}(x)$, one unit in the positive direction of the x axis. The x axis intercept is at 1, the y axis intercept is at $\tan^{-1}(-1) = -\dfrac{\pi}{4}$, the asymptotes remain the same: $y = \dfrac{\pi}{2}$ and $y = -\dfrac{\pi}{2}$.

The range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and the domain is R.



b The graph of $y = \cos^{-1}(x + 1)$ is a translation of the graph of $y = \cos^{-1}(x)$ one unit in the negative direction of the x axis. The domain is $[-2, 0]$, the range is $[0, \pi]$



c The graph of $y = 2\sin^{-1}\left(x + \dfrac{1}{2}\right)$ is a dilation of factor 2 from the x axis of the graph of $y = \sin^{-1}\left(x + \dfrac{1}{2}\right)$.

That is why the range of the function $y = 2\sin^{-1}\left(x + \dfrac{1}{2}\right)$ is

$\left[2 \times \left(-\dfrac{\pi}{2}\right), 2 \times \dfrac{\pi}{2}\right] = [-\pi, \pi]$.

The graph of $y = \sin^{-1}\left(x + \dfrac{1}{2}\right)$ is a translation of the graph of $y = \sin^{-1}(x)$, $\dfrac{1}{2}$ unit in the negative direction of the x axis. Therefore the domain of the function $y = 2\sin^{-1}\left(x + \dfrac{1}{2}\right)$ is

$\left[-1 - \dfrac{1}{2}, 1 - \dfrac{1}{2}\right] = \left[-\dfrac{3}{2}, \dfrac{1}{2}\right]$.

When $x = -\frac{3}{2}, y = 2\sin^{-1}\left(-\frac{3}{2} + \frac{1}{2}\right)$

$= 2\sin^{-1}(-1)$

$= 2 \times -\frac{\pi}{2}$

$= -\pi$

When $x = \frac{1}{2}, y = 2\sin^{-1}\left(\frac{1}{2} + \frac{1}{2}\right)$

$= 2\sin^{-1}(1)$

$= 2 \times \frac{\pi}{2}$

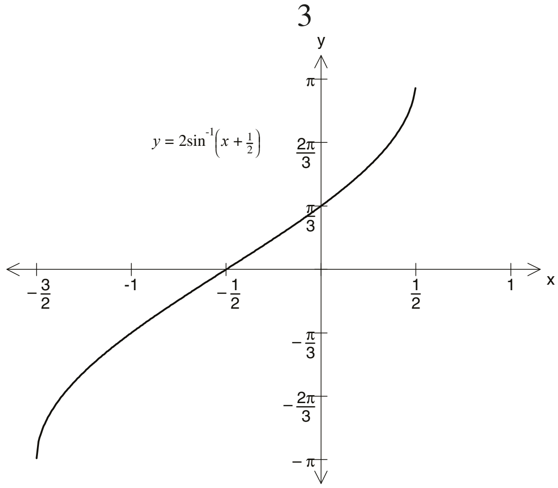
$= \pi$

x axis intercept is $x = -\frac{1}{2}$

y axis intercept is $y = 2\sin^{-1}\left(\frac{1}{2}\right)$

$= 2 \times \frac{\pi}{6}$

$= \frac{\pi}{3}$



d The graph of $y = 2\tan^{-1}(x) + \frac{\pi}{2}$ is obtained from the graph of $y = \tan^{-1}(x)$, by a dilation of factor 2 from the x axis followed by a translation of $\frac{\pi}{2}$ units in the positive direction of the y axis. Therefore the domain of the function $y = 2\tan^{-1}(x) + \frac{\pi}{2}$ is R, and the range is $\left(2 \times -\frac{\pi}{2} + \frac{\pi}{2}, 2 \times \frac{\pi}{2} + \frac{\pi}{2}\right) = \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

The asymptotes are at $y = -\frac{\pi}{2}$ and $y = \frac{3\pi}{2}$

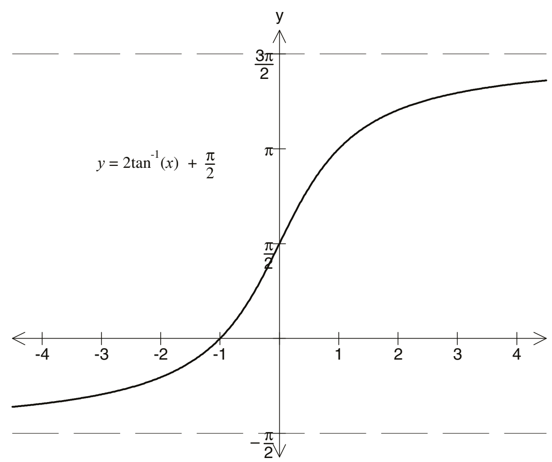
y axis intercept is $2\tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$

x axis intercept can be found from the equation

$2\tan^{-1}(x) = -\frac{\pi}{2}$

$\Rightarrow \tan^{-1}(x) = -\frac{\pi}{4}$

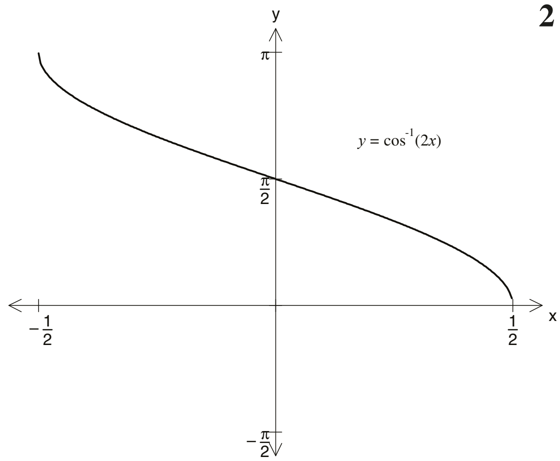
$\Rightarrow x = \tan\left(-\frac{\pi}{4}\right) = -1$



e The graph of $y = \cos^{-1}(2x)$ is obtained from the graph of $y = \cos^{-1}(x)$ by a dilation of factor $\frac{1}{2}$ from the y axis.

The domain of the function $y = \cos^{-1}(2x)$ is $\left[-1 \times \frac{1}{2}, 1 \times \frac{1}{2}\right] = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

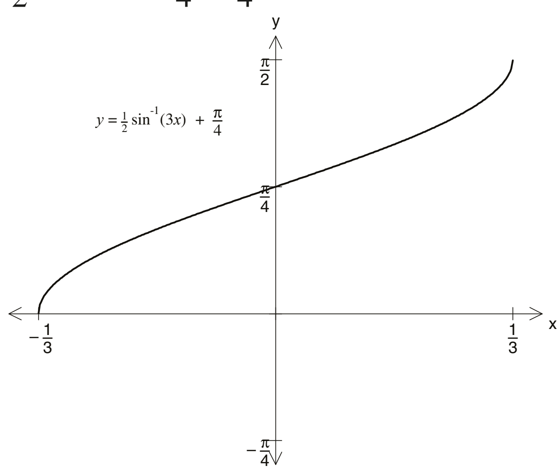
The range is $[0, \pi]$.



f The graph of $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$ is a consequence of a dilation of the graph of $y = \sin^{-1}(x)$ of factor $\frac{1}{3}$ from the y axis, then a dilation of $y = \sin^{-1}(3x)$ of factor $\frac{1}{2}$ from the x axis and then a translation $\frac{\pi}{4}$ units in the positive direction of the y axis. Therefore the domain of the function $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$ is $\left[-\frac{1}{3}, \frac{1}{3}\right]$ and the range is

$$\left[-\frac{\pi}{2} \times \frac{1}{2} + \frac{\pi}{4}, \frac{\pi}{2} \times \frac{1}{2} + \frac{\pi}{4}\right] = \left[0, \frac{\pi}{2}\right].$$

The y axis intercept is at $\frac{1}{2}\sin^{-1}(0) + \frac{\pi}{4} = \frac{\pi}{4}$.



2 a Evaluating $\sin^{-1} 1$ is equivalent to solving the equation $\sin x = 1$.

$$\sin\frac{\pi}{2} = 1$$

$$\Rightarrow \sin^{-1} 1 = \frac{\pi}{2}$$

b $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ because $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$\Rightarrow \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

c $\sin^{-1} 0.5 = \frac{\pi}{6}$ because $\sin\frac{\pi}{6} = 0.5$

d Evaluating $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is equivalent to solving the equation $\cos x = -\frac{\sqrt{3}}{2}$.

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos\left(\pi - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

e $\cos^{-1} 0.5 = \frac{\pi}{3}$ because $\cos\frac{\pi}{3} = 0.5$

f $\tan^{-1} 1 = \frac{\pi}{4}$ because $\tan\frac{\pi}{4} = 1$

g $\tan^{-1}(-\sqrt{3}) = -\tan\sqrt{3} = -\frac{\pi}{3}$ because $\tan\frac{\pi}{3} = \sqrt{3}$

h $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$ because $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

i $\cos^{-1}(-1) = \pi - \cos^{-1} 1 = \pi - 0 = \pi$

3 **a** $\sin(\cos^{-1} 0.5) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b $\sin^{-1}\left(\cos \frac{5\pi}{6}\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{5\pi}{6}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$

$= -\frac{\pi}{3}$

c $\tan\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right) = \tan\left(-\frac{\pi}{4}\right) = -1$

d $\cos(\tan^{-1} 1) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

e $\tan^{-1}\left(\sin \frac{5\pi}{2}\right) = \tan^{-1}\left(\sin\left(2\pi + \frac{\pi}{2}\right)\right)$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

f $\tan(\cos^{-1} 0.5) = \tan \frac{\pi}{3} = \sqrt{3}$

g $\cos^{-1}\left(\cos \frac{7\pi}{3}\right) = \cos^{-1}\left(\cos\left(2\pi + \frac{\pi}{3}\right)\right)$

$= \cos^{-1}\left(\cos \frac{\pi}{3}\right)$

$= \frac{\pi}{3}$

h $\sin^{-1}\left(\sin \frac{-2\pi}{3}\right) = \sin^{-1}\left(\sin\left(-\pi + \frac{\pi}{3}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$

$= -\frac{\pi}{3}$

i $\tan^{-1}\left(\tan \frac{11\pi}{4}\right) = \tan^{-1}\left(\tan\left(3\pi - \frac{\pi}{4}\right)\right)$

$= \tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

$= -\frac{\pi}{4}$

j $\cos^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

k $\cos^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right) = \cos^{-1}(-1) = \pi$

l $\sin^{-1}\left(\cos \frac{-3\pi}{4}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

4 $f: \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to R, f(x) = \sin x$

a The range of $f(x) = \sin x$ is $[-1, 1]$
Therefore the domain of f^{-1} is $[-1, 1]$
The range of f^{-1} is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ as it is a given domain of $f(x)$.
Therefore $f^{-1}(x) = \pi - \sin^{-1}(x)$

b **i** $f\left(\frac{\pi}{2}\right) = 1$

ii $f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

iii $f\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$

iv $f^{-1}(-1) = \pi - \sin^{-1}(-1)$

$$= \pi - \left(-\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{2}$$

v $f^{-1}(0) = \pi - \sin^{-1}(0) = \pi$

vi $f^{-1}(0.5) = \pi - \sin^{-1}(0.5)$

$$= \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

5 a The domain of $\sin^{-1}(x)$ is $[-1, 1]$

$\Rightarrow \quad -1 \le 2 - x \le 1$

$-3 \le -x \le -1$

$1 \le x \le 3$

Therefore the domain of $\sin^{-1}(2 - x)$ is $[1, 3]$ The range is unchanged at $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b The domain of $\sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad -\frac{\pi}{2} \le x + \frac{\pi}{4} \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{3\pi}{4} \le x \le \frac{\pi}{4}$

Therefore the domain of $\sin\left(x + \frac{\pi}{4}\right)$ is $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

The range is unchanged at $[-1, 1]$.

c As in **a**, the domain of $\sin^{-1}(2x + 4)$ can be defined from the inequality

$-1 \le 2x + 4 \le 1$

$-5 \le 2x \le -3$

$-\frac{5}{2} \le x \le -\frac{3}{2}$

The domain of $\sin^{-1}(2x + 4)$ is $\left[-\frac{5}{2}, -\frac{3}{2}\right]$, the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

d As in **b**, the domain of $\sin\left(3x - \frac{\pi}{3}\right)$ can be defined from the inequality

$-\frac{\pi}{2} \le 3x - \frac{\pi}{3} \le \frac{\pi}{2}$

$-\frac{\pi}{2} + \frac{\pi}{3} \le 3x \le \frac{\pi}{2} + \frac{\pi}{3}$

$-\frac{\pi}{6} \le 3x \le \frac{5\pi}{6}$

$-\frac{\pi}{18} \le x \le \frac{5\pi}{18}$

So the domain of $\sin\left(3x - \frac{\pi}{3}\right)$ is $\left[-\frac{\pi}{18}, \frac{5\pi}{18}\right]$, the range is $[-1, 1]$.

e The domain of $\cos x$ is $[0, \pi]$

$\Rightarrow 0 \le x - \frac{\pi}{6} \le \pi$

$\frac{\pi}{6} \le x \le \frac{7\pi}{6}$

Therefore the domain of $\cos\left(x - \frac{\pi}{6}\right)$ is $\left[\frac{\pi}{6}, \frac{7\pi}{6}\right]$, the range is $[-1, 1]$.

f The domain of $\cos^{-1}(x)$ is $[-1, 1]$

$\Rightarrow -1 \le x + 1 \le 1$

$-2 \le x \le 0$

Therefore the domain of $\cos^{-1}(x + 1)$ is $[-2, 0]$ The range is unchanged at $[0, \pi]$.

g As in **f**, $-1 \le x^2 \le 1$

$\Rightarrow -1 \le x \le 1$

$\Rightarrow$ the domain of $\cos^{-1}(x^2)$ is $[-1, 1]$

However, when $x \in [-1, 1]$, $x^2 \in [0, 1]$, so the range of $\cos^{-1}(x^2)$ is $\left[0, \frac{\pi}{2}\right]$.

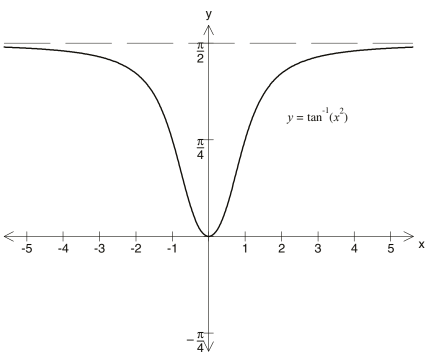
h As in **e**, $0 \le 2x + \dfrac{2\pi}{3} \le \pi$

$$-\frac{2\pi}{3} \le 2x \le \frac{\pi}{3}$$

$$-\frac{\pi}{3} \le x \le \frac{\pi}{6}$$

Therefore the domain of $\cos\left(2x + \dfrac{2\pi}{3}\right)$ is $\left[-\dfrac{\pi}{3}, \dfrac{\pi}{6}\right]$, the range is $[-1, 1]$.

i The domain of $\tan^{-1}(x)$ is R, so the domain of $\tan^{-1}(x^2)$ is also R. However when $x \in R$, $x^2 \in R^+ \cup \{0\}$, therefore the range of $\tan^{-1}(x^2)$ is $\left[0, \dfrac{\pi}{2}\right)$



j The domain of $\tan(x)$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

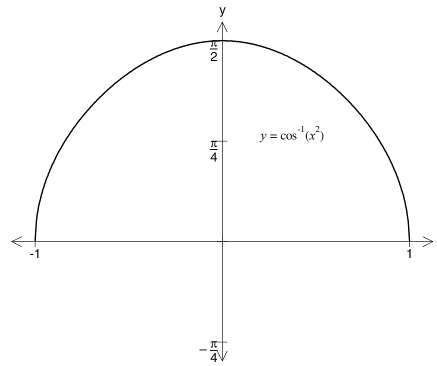
$$\therefore \quad -\frac{\pi}{2} < 2x - \frac{\pi}{2} < \frac{\pi}{2}$$

$$0 < 2x < \pi$$

$$0 < x < \frac{\pi}{2}$$

Therefore the domain of $\tan\left(2x - \dfrac{\pi}{2}\right)$ is $\left(0, \dfrac{\pi}{2}\right)$, the range is R.

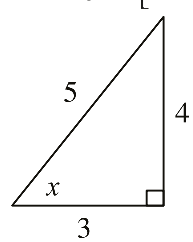
k Both the domain and the range of $\tan^{-1}(2x + 1)$ are the same as those of $\tan^{-1}(x)$: the domain is R, the range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$



l The domain of $\tan x$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, so the domain of $\tan x^2$ is $\left(-\sqrt{\dfrac{\pi}{2}}, \sqrt{\dfrac{\pi}{2}}\right)$.

At the same time $x^2 \in \left[0, \dfrac{\pi}{2}\right)$, therefore the range of $\tan x^2$ is $R^+ \cup \{0\}$.

6 a $\sin^{-1}\dfrac{4}{5} \in \left[0, \dfrac{\pi}{2}\right]$

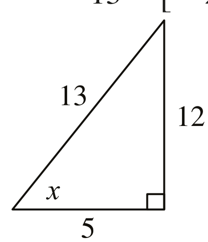


Using a trigonometirc ratio, $\sin x = \dfrac{4}{5}$

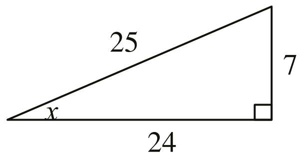
$$\Rightarrow \quad x = \sin^{-1}\frac{4}{5}$$

$$\Rightarrow \quad \cos\left(\sin^{-1}\frac{4}{5}\right) = \cos(x) = \frac{3}{5}$$

b $\cos^{-1}\dfrac{5}{13} \in \left[0, \dfrac{\pi}{2}\right]$



c $\tan^{-1}\frac{7}{24} \in \left[0, \frac{\pi}{2}\right]$



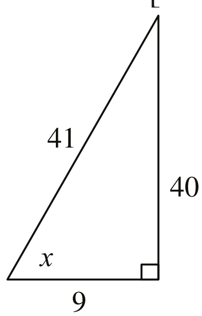
Using a trigonometirc ratio,

$\tan x = \frac{7}{24}$

$$\Rightarrow \quad x = \tan^{-1}\frac{7}{24}$$

$$\Rightarrow \quad \cos\left(\tan^{-1}\frac{7}{24}\right) = \cos(x) = \frac{24}{25}$$

d $\sin^{-1}\frac{40}{41} \in \left[0, \frac{\pi}{2}\right]$



Using a trigonometirc ratio,

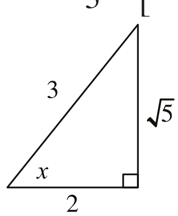
$\sin x = \frac{40}{41}$

$$\Rightarrow \quad x = \sin^{-1}\frac{40}{41}$$

$$\Rightarrow \quad \tan\left(\sin^{-1}\frac{40}{41}\right) = \tan(x) = \frac{40}{9}$$

e $\tan\left(\cos^{-1}\frac{1}{2}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

f $\cos^{-1}\frac{2}{3} \in \left[0, \frac{\pi}{2}\right]$



Using a trigonometirc ratio,

$\cos x = \frac{5}{13}$

$$\Rightarrow \quad x = \cos^{-1}\frac{5}{13}$$

$$\Rightarrow \quad \tan\left(\cos^{-1}\frac{5}{13}\right) = \tan(x) = \frac{12}{5}$$

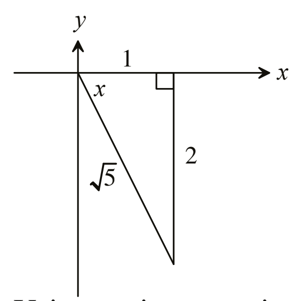
Using a trigonometirc ratio,

$\cos x = \frac{2}{3}$

$$\Rightarrow \quad x = \cos^{-1}\frac{2}{3}$$

$$\Rightarrow \quad \sin\left(\cos^{-1}\frac{2}{3}\right) = \sin(x) = \frac{\sqrt{5}}{3}$$

g $\tan^{-1}(-2) \in \left[-\frac{\pi}{2}, 0\right]$



Using a trigonometirc ratio,

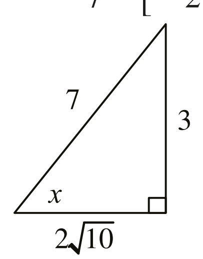
$\tan x = \frac{-2}{1}$

$$\Rightarrow \quad x = \tan^{-1}\frac{-2}{1}$$

$$\Rightarrow \quad \sin(\tan^{-1}(-2)) = \sin(x) = \frac{-2}{\sqrt{5}}$$

$$= \frac{-2\sqrt{5}}{5}$$

h $\sin^{-1}\frac{3}{7} \in \left[0, \frac{\pi}{2}\right]$



Using a trigonometirc ratio, $\sin x = \frac{3}{7}$

$$\Rightarrow \qquad x = \sin^{-1}\frac{3}{7}$$

$$\Rightarrow \quad \cos\left(\sin^{-1}\frac{3}{7}\right) = \cos(x) = \frac{2\sqrt{10}}{7}$$

7 $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{5}{13}, \alpha \in \left[0, \frac{\pi}{2}\right]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$

a **i** $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{9}{25}}$

$$= \frac{4}{5}$$

ii $\cos\beta = \sqrt{1-\frac{25}{169}} = \frac{12}{13}$

b **i** To prove the equality we have to prove that $\sin(\alpha-\beta) = \frac{16}{65}$

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta \\ &\quad - \cos\alpha\sin\beta \\ &= \frac{3}{5}\times\frac{12}{13} - \frac{4}{5}\times\frac{5}{13} \\ &= \frac{36-20}{65} \\ &= \frac{16}{65}\end{aligned}$$

ii As in **i**,

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta \\ &\quad - \sin\alpha\sin\beta \\ &= \frac{4}{5}\times\frac{12}{13} - \frac{3}{5}\times\frac{5}{13} \\ &= \frac{48-15}{65} \\ &= \frac{33}{65}\end{aligned}$$

8 **a** $\sin^{-1}(-0.5) = -\frac{\pi}{6}$

However, the domain of $\cos x$ is $[0, \pi]$, so $\cos\left(-\frac{\pi}{6}\right)$ does not exist.

b $\cos^{-1}(-0.2) \in \left(\frac{\pi}{2}, \pi\right) \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

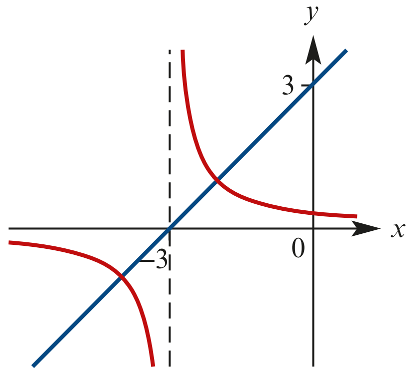
So $\sin(\cos^{-1}(-0.2))$ does not exist.

c $\tan^{-1}(-1) = -\frac{\pi}{4} \notin [0, \pi]$.

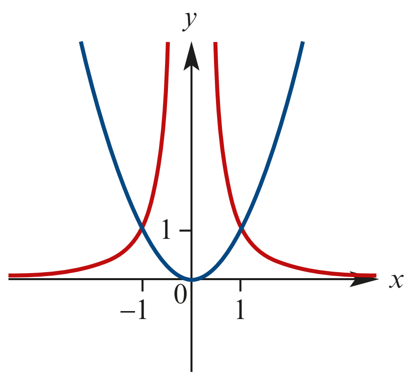
So $\cos(\tan^{-1}(-1))$ does not exist.

Solutions to Exercise 17B

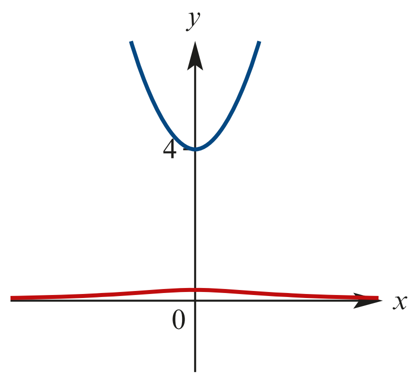
1 a



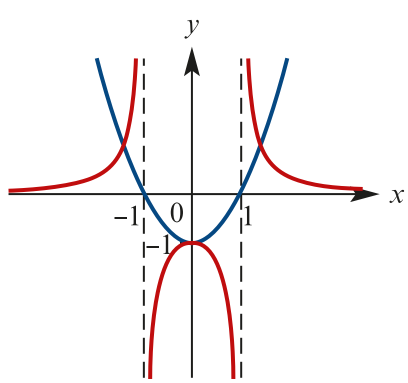
b



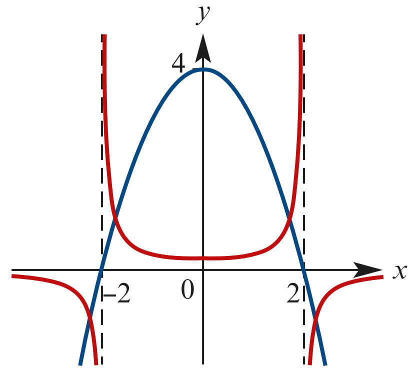
c



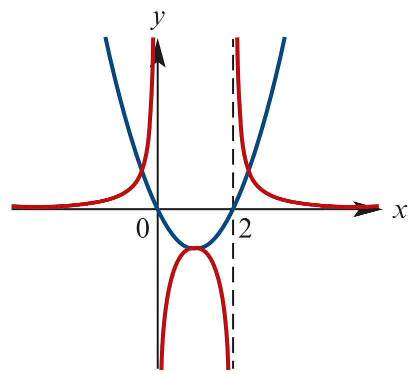
d



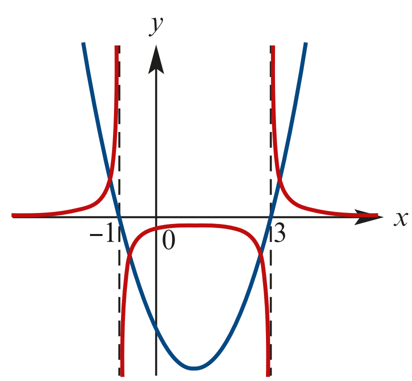
e



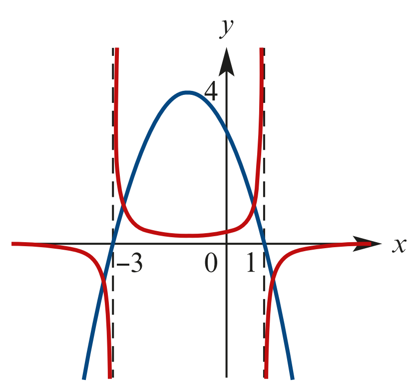
f



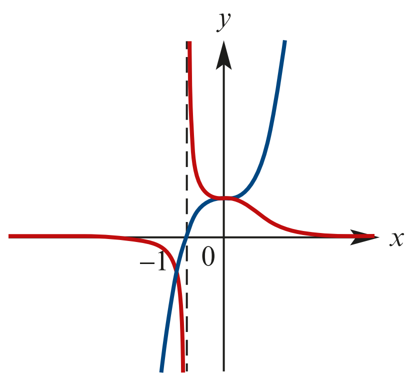
g



h

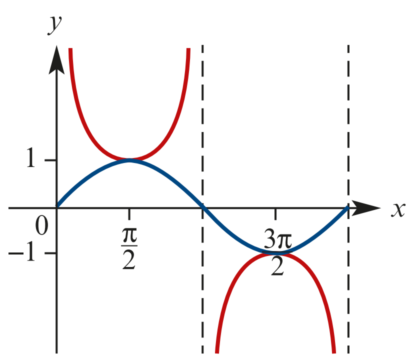


i



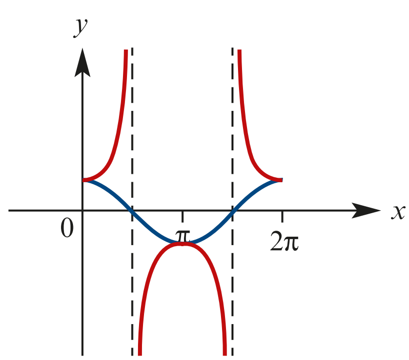
y
x
−1
0

2 a



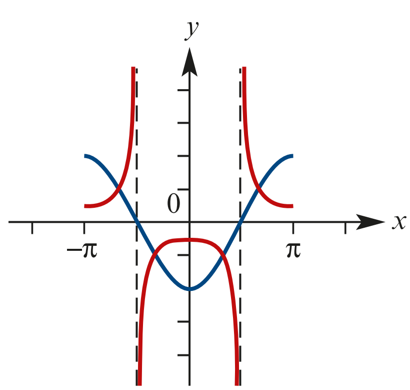
y
x
1
0
−1
π/2
3π/2

b



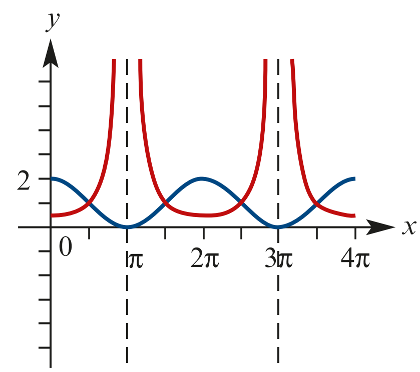
y
x
0
π
2π

c



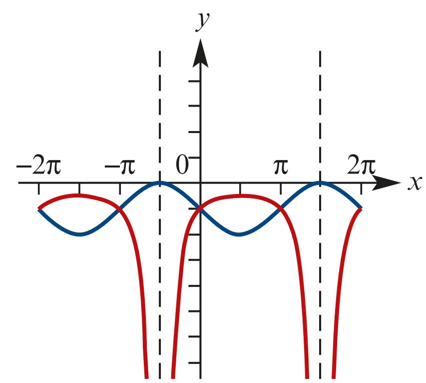
y
x
0
−π
π

d



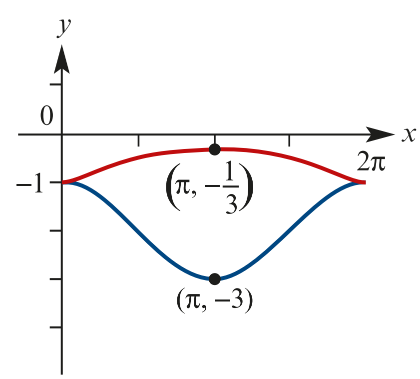
y
x
2
0
π
2π
3π
4π

e



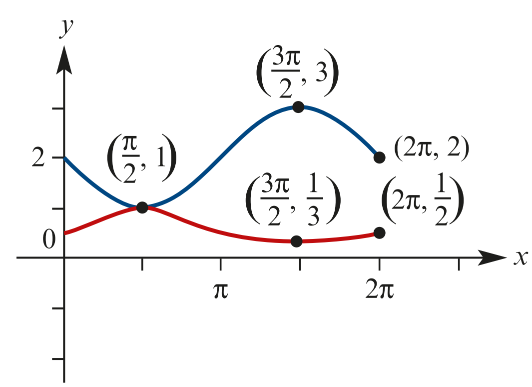
y
x
−2π
−π
0
π
2π

f



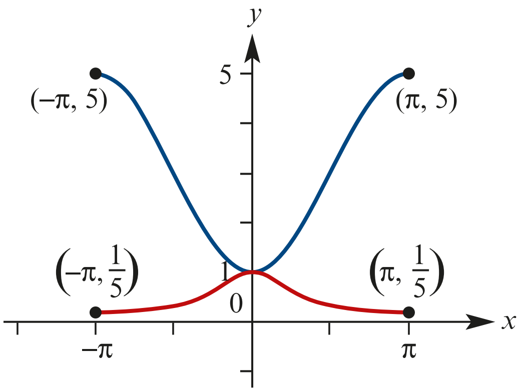
y
x
0
−1
2π
(π, −1/3)
(π, −3)

g



y
x
(3π/2, 3)
(π/2, 1)
(2π, 2)
2
(3π/2, 1/3)
(2π, 1/2)
0
π
2π

h

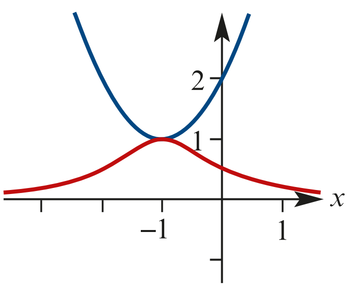


3 **a** We complete the square so that

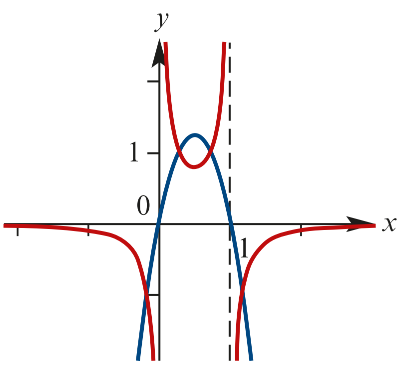
$$f(x) = x^2 + 2x + 2$$
$$= (x^2 + 2x + 1) - 1 + 2$$
$$= (x + 1)^2 + 1.$$

Therefore, a minimum turning point is located at point $(-1, 1)$.

b



4 **a**



b To find points of intersection we solve two equations: $f(x) = 1$ and $f(x) = -1$. If $f(x) = 1$ then

$$5x(1 - x) = 1.$$

Solving this quadratic equation (using the quadratic equation or your calculator) gives

$$x = \frac{5 \pm \sqrt{5}}{10}.$$

Since $f(x) = 1$, the coordinates are

$$\left(\frac{5 \pm \sqrt{5}}{10}, 1\right).$$

If $f(x) = 1$ then

$$5x(1 - x) = -1.$$

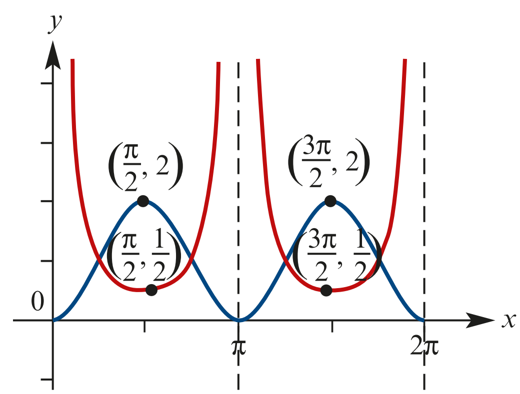
Solving this quadratic equation gives

$$x = \frac{5 \pm 3\sqrt{5}}{10}.$$

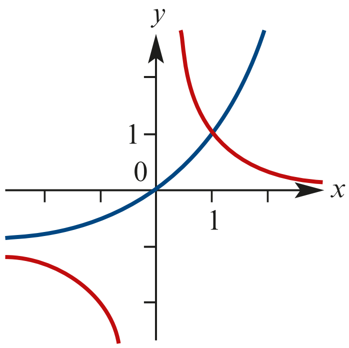
Since $f(x) = -1$, the coordinates are

$$\left(\frac{5 \pm 3\sqrt{5}}{10}, -1\right).$$

5 Notice that $y = 2\sin^2 x$ will have the same x-intercepts as $y = 2\sin x$ but will be non-negative for all values of x.



6



7 **a** We complete the square so that

$$f(x) = x^2 + 2kx + 1$$
$$= (x^2 + 2x + k^2) - k^2 + 1$$
$$= (x + k)^2 + 1 - k^2.$$

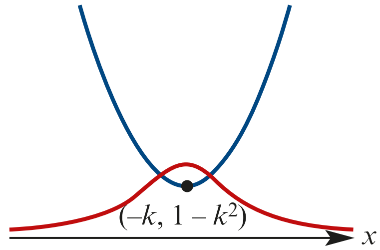
Therefore, a minimum turning point is located at point $(-k, 1 - k^2)$.

b **i** The graph of $y = f(x)$ will have no x-intercept provided $1 - k^2 > 0$. This means that $-1 < k < 1$.

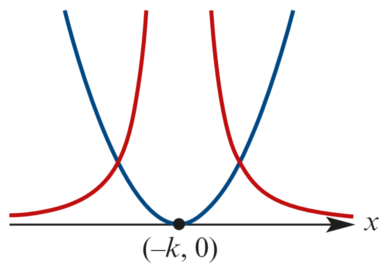
ii The graph of $y = f(x)$ will have one x-intercept provided $1 - k^2 - 0$. This means that $k = \pm 1$.

iii The graph of $y = f(x)$ will have two x-intercepts provided $1 - k^2 < 0$. This means that $k > 1$ or $k < -1$.

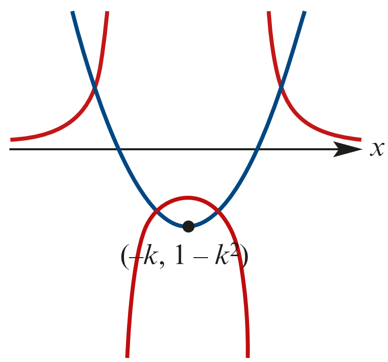
c **i**



ii

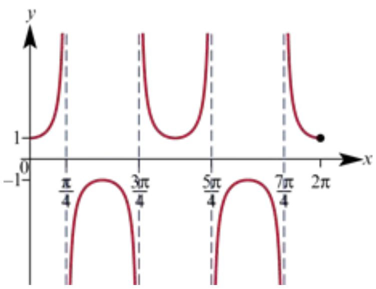


iii

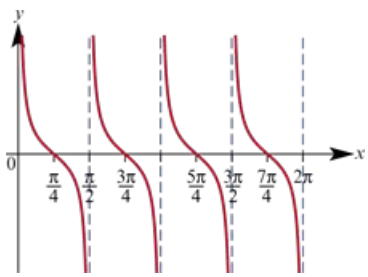


Solutions to Exercise 17C

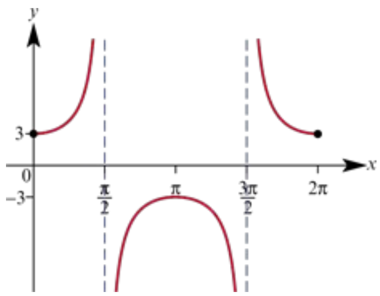
1 a



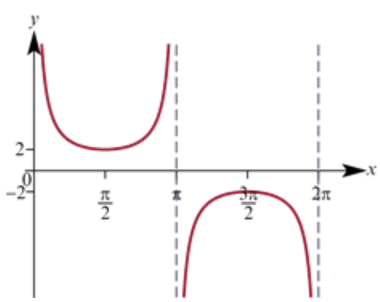
b



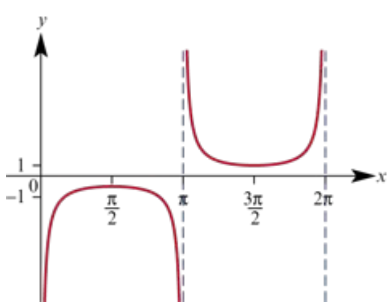
c



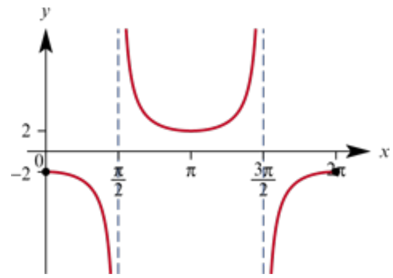
d



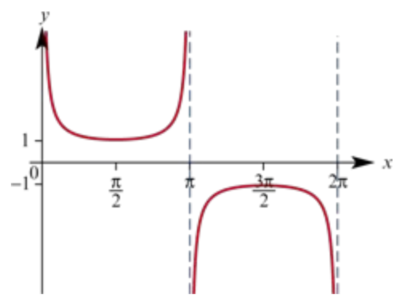
e



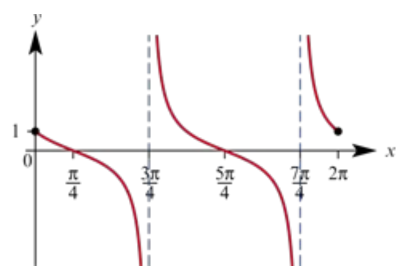
f



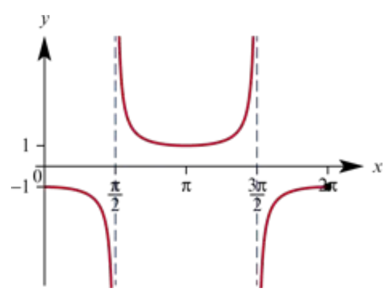
2 a



b



c



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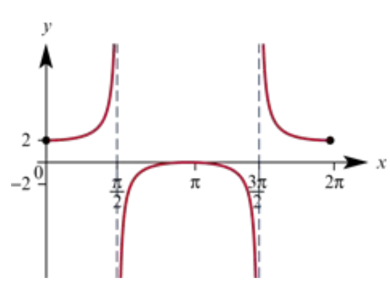
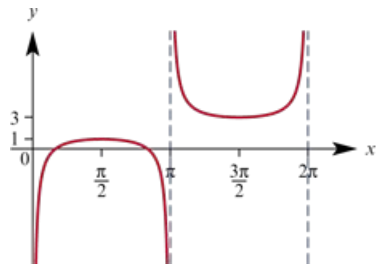
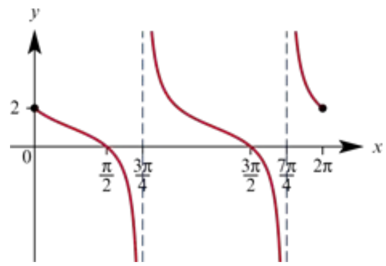


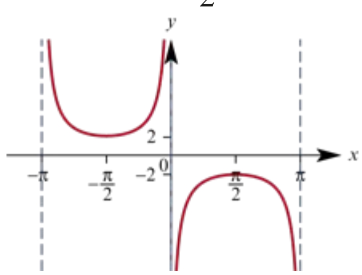
e



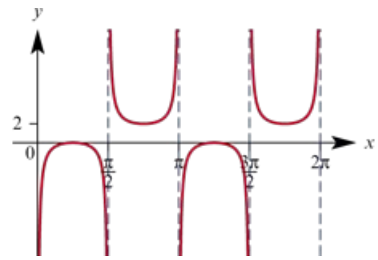
f



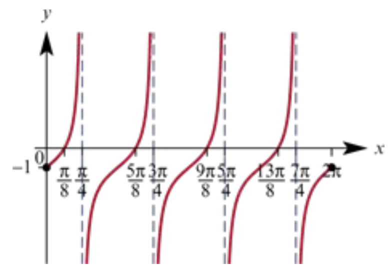
3
- Reflection in the x-axis
- Dilation of factor 2 from the x-axis
- Translation $\dfrac{\pi}{2}$ units to the right



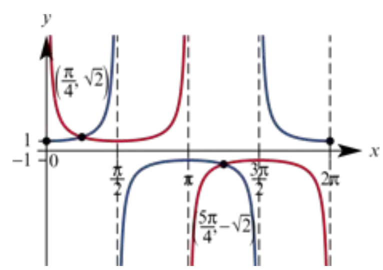
4
- Reflection in the y-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation 1 unit up



5
- Reflection in the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\dfrac{\pi}{4}$ units to the right and 1 unit down



6



Solutions to Exercise 17D

1 **a** 8

b 8

c 2

d −2

e −2

f 4

2 **a**
$$\begin{aligned}|x-1| &= 2\\ x-1 &= \pm 2\\ x = 3 &\text{ or } x=-1\end{aligned}$$

b
$$\begin{aligned}|2x-3| &= 4\\ 2x-3 &= \pm 4\\ 2x = 7 &\text{ or } 2x=-1\\ x = \frac{7}{2} &\text{ or } x=-\frac{1}{2}\end{aligned}$$

c
$$\begin{aligned}|5x-3| &= 9\\ 5x-3 &= \pm 9\\ 5x = 12 &\text{ or } 5x=-6\\ x = \frac{12}{5} &\text{ or } x=-\frac{6}{5}\end{aligned}$$

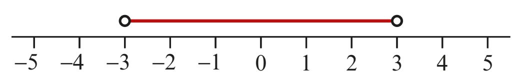
d
$$\begin{aligned}|x-3| &= 9\\ x-3 &= \pm 9\\ x = 12 &\text{ or } x=-6\end{aligned}$$

e
$$\begin{aligned}|x-3| &= 4\\ x-3 &= \pm 4\\ x = 7 &\text{ or } x=-1\end{aligned}$$

f
$$\begin{aligned}|3x+4| &= 8\\ 3x+4 &= \pm 8\\ 3x = 4 &\text{ or } 3x=-12\\ x = \frac{4}{3} &\text{ or } x=-4\end{aligned}$$

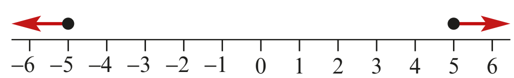
g
$$\begin{aligned}|5x+11| &= 9\\ 5x+11 &= \pm 9\\ 5x = -2 &\text{ or } 5x=-20\\ x = -\frac{2}{5} &\text{ or } x=-4\end{aligned}$$

3 **a**



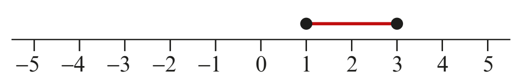
$(-3, 3)$

b



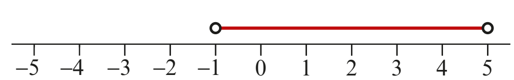
Answer:$(-\infty, -5] \cup [5, \infty)$

c



$$\begin{aligned}|x-2| \le 1 &\Leftrightarrow -1 \le x-2 \le 1\\ &\Leftrightarrow 1 \le x \le 3\end{aligned}$$
Answer:$[1, 3]$

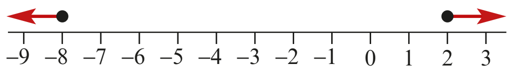
d



$$\begin{aligned}|x-2| < 3 &\Leftrightarrow -3 < x-2 < 3\\ &\Leftrightarrow -1 < x < 5\end{aligned}$$

Answer:$(-1, 5)$

e

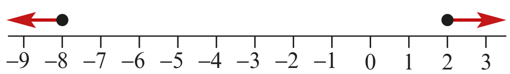


$$|x + 3| \geq 5 \Leftrightarrow x + 3 \geq 5 \text{ or } x + 3 \leq -5$$

$$\Leftrightarrow x \geq 2 \text{ or } x \leq -8$$

Answer:$(-\infty, -8] \cup [2, \infty)$

f

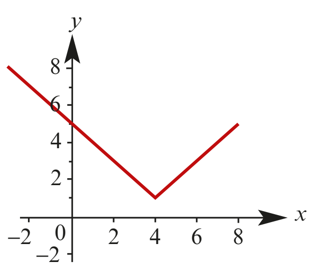


$$|x + 2| \leq 1 \Leftrightarrow -1 < x + 2 < 1$$

$$\Leftrightarrow -3 < x < -1$$

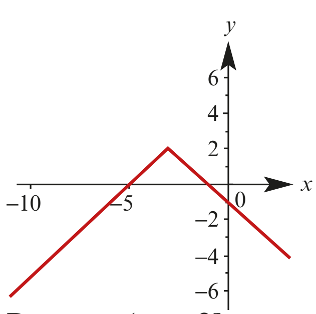
Answer: $[-3, -1]$

4 **a**



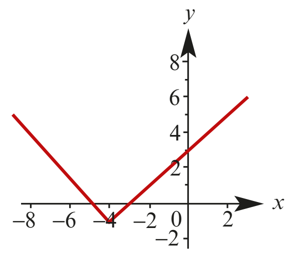
Range = $[1, \infty)$

b



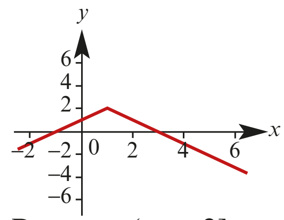
Range = $(-\infty, 2]$

c



Range = $[-1, \infty)$

d



Range = $(-\infty, 2]$

5 **a** $|x| \leq 5 \Leftrightarrow -5 \leq x \leq 5$

Answer: $\{x : -5 \leq x \leq 5\}$

b $|x| \geq 2 \Leftrightarrow x \geq 2 \text{ or } x \leq -2$

Answer: $\{x : x \leq -2\} \cup \{x : x \geq 2\}$

c $|2x - 3| \leq 1 \Leftrightarrow -1 < 2x - 3 < 1$

$$\Leftrightarrow 2 \leq 2x \leq 4$$

$$\Leftrightarrow 1 \leq x \leq 2$$

Answer: $\{x : 1 \leq x \leq 2\}$

d $|5x - 2| < 31 \Leftrightarrow -3 < 5x - 2 < 3$

$$\Leftrightarrow -1 \leq 5x \leq 5$$

$$\Leftrightarrow -\frac{1}{5} < x < 1$$

Answer: $\{x : -\frac{1}{5} < x < 1\}$

e

$$|-x + 3| \geq 7 \Leftrightarrow -x + 3 \geq 7 \text{ or } -x + 3 \leq -7$$

$$\Leftrightarrow -x \geq 4 \text{ or } -x \leq -10$$

$$\Leftrightarrow x \leq -4 \text{ or } x \geq 10$$

Answer: $\{x : x \leq -4\} \cup \{x : x \geq 10\}$

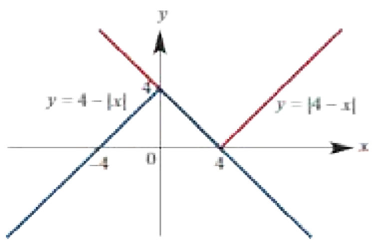
f $|-x+2| \le 1 \Leftrightarrow -1 < -x+2 < 1$

$\Leftrightarrow -3 \le -x \le -1$

$\Leftrightarrow 1 \le x \le 3$

Answer: $\{x : 1 \le x \le 3\}$

6 a

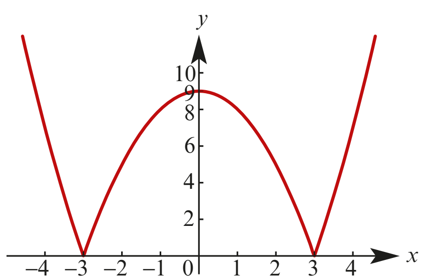


b $|f(x)| = f(|x|)$

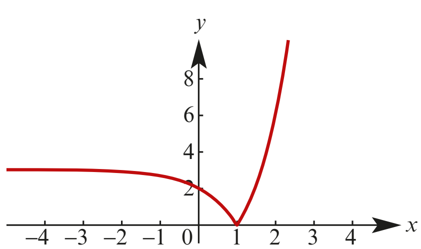
$|4-x| = 4-|x|$

$x \in [0, 4]$

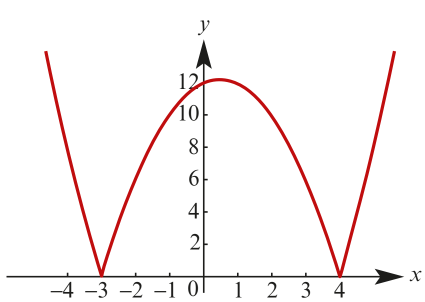
7 a



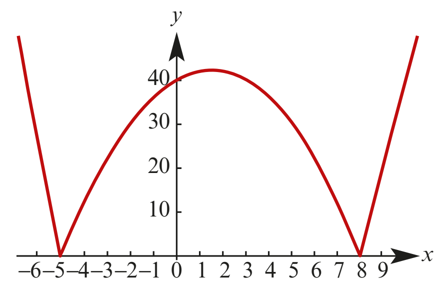
b



c



d



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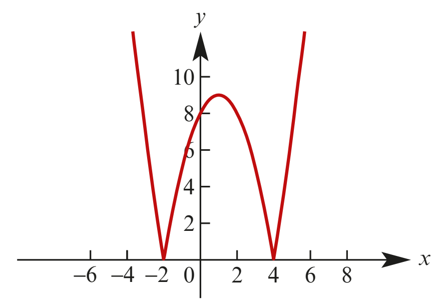

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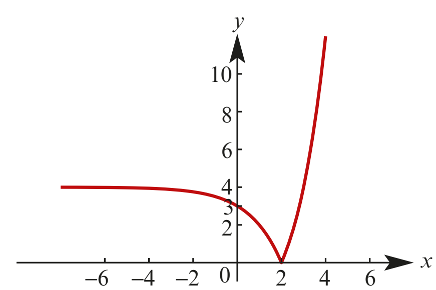
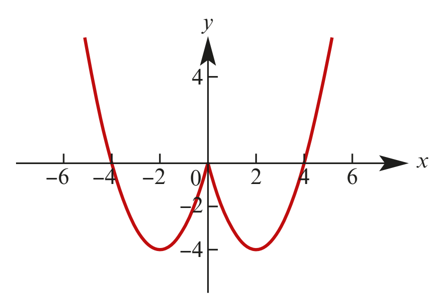
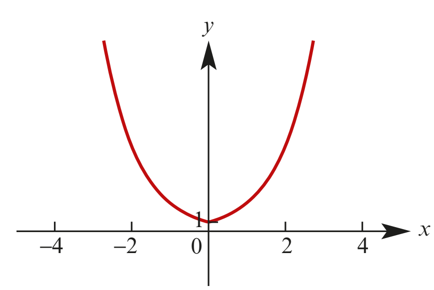


8 a



$$y = \begin{cases} x^2 - 4x & x \ge 0 \\ x^2 + 4x & x < 0 \end{cases}$$

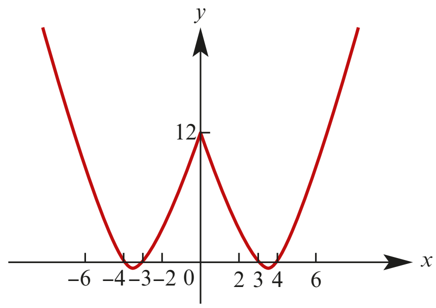
b



$$y = \begin{cases} 3^{x} & x \geq 0 \\ 3^{-x} & x < 0 \end{cases}$$

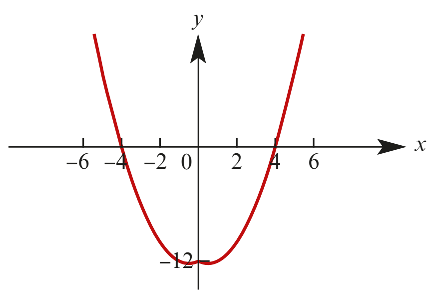
$$y = \begin{cases} -3^{x} & x \geq 0 \\ -3^{-x} & x < 0 \end{cases}$$

c



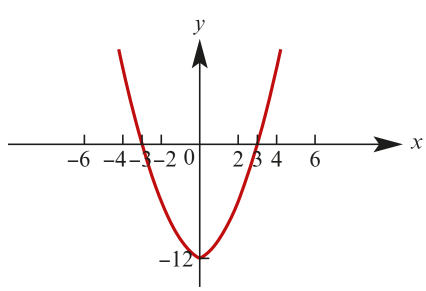
$$y = \begin{cases} x^2 - 7x + 12 & x \geq 0 \\ x^2 + 7x + 12 & x < 0 \end{cases}$$

d



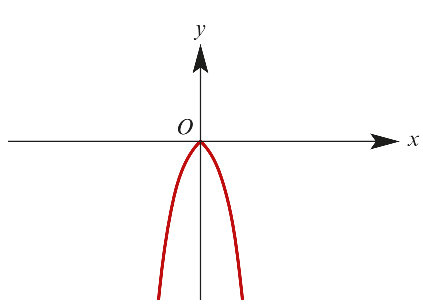
$$y = \begin{cases} x^2 - x - 12 & x \geq 0 \\ x^2 + x - 12 & x < 0 \end{cases}$$

e



$$y = \begin{cases} x^2 + x - 12 & x \geq 0 \\ x^2 - x - 12 & x < 0 \end{cases}$$

f



9 a Case 1 $x < 0$ or $x > 2$

$$x^2 - 2x = \frac{1}{2}$$

$$2x^2 - 4x - 1 = 0$$

$$x = \frac{4 \pm \sqrt{24}}{4}$$

$$= \frac{2 \pm \sqrt{6}}{2}$$

Case 2 $0 \leq x \leq 2$

$$x^2 - 2x = -\frac{1}{2}$$

$$2x^2 - 4x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{8}}{4}$$

$$= \frac{2 \pm \sqrt{2}}{2}$$

b Case 1 $x < 0$ or $x > 2$

$$x^2 - 2x = 1$$

$$x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

$$= 1 \pm \sqrt{2}$$

Case 2 $0 \leq x \leq 2$

$$x^2 - 2x = -1$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0x \quad = 1$$

c Case 1 $x < 0$ or $x > 2$

$$x^2 - 2x = 8$$

$$x^2 - 2x - 8 = 0$$

$$(x - 4)(x + 2) = 0x \quad = 4 \text{ or } x = -2$$

Case 2 $0 \leq x \leq 2$

$$x^2 - 2x = -8$$
$$x^2 - 2x + 8 = 0$$
no solution

d $3 - \sqrt{17}, 3 - \sqrt{17}, 4$

e $-2, 8$

f $3 - \sqrt{2}, 3 - \sqrt{2}, 3$

10 We use an algebraic approach but using graphs to help simplifies it somewhat.

a Consider Cases:
Crucial points are -2 and 4
Case 1 :$x \geq 4$
$x - 4 - (x + 2) = 6$
No soln
Case 2: $-2 \leq x \leq 4$
$4 - x - (x + 2) = 6$
$2 - 2x = 6$
$-2x = 4$
$x = -2$
Case 3: $x \leq -2$
$4 - x - (-x - 2) = 6$
$6 = 6$
Always true
Solution: $(-\infty, -2]$

b Consider Cases:
Crucial points are $\frac{5}{2}$ and 4
Case 1 :$x \geq 4$
$2x - 5 - (x - 4) = 10$
$x - 1 = 10$
$x = 11$ (Solution)
Case 2: $\frac{5}{2} \leq x \leq 4$
$2x - 5 - (4 - x) = 10$
$3x - 9 = 6$
$x = 5$ (No solution)
Case 3: $x \leq \frac{5}{2}$
$5 - 2x - (4 - x) = 10$
$1 - x = 10$
$x = -9$ (Solution)
Therefore $x = 11$ or $x = -9$

c Use a calculator
$x = \frac{5}{4}$ or $x = \frac{15}{4}$

11 $f(x) = |x - a| + b$
Given, $f(3) = 3$ and $f(-1) = 3$
The symmetry of f gives us that $a = 1$
Hence $b = 1$

12 **a** $|ab|^2 = (ab)^2 = a^2b^2 = |a|^2|b|^2$.

b We find that

$$\begin{aligned}|a + b|^2 &= (a + b)^2 \\ &= a^2 + 2(ab) + b^2 \\ &= |a|^2 + 2(ab) + |b|^2 \\ &\leq |a|^2 + 2|ab| + |b|^2 \\ &= |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2\end{aligned}$$

This result is called the *triangle inequality*. We will use this result for the following question.

13 **a** We use **b** from the previous question:

$$|x-y| = |x+(-y)| \leq |x|+|-y| = |x|+|y|.$$

b We use **b** from the previous question:

$$|x| = |(x - y) + y| \leq |x - y| + |y|.$$

Therefore, $|x| - |y| \leq |x - y|$.

c We use **b** from the previous question twice:

$$|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|.$$

Solutions to Exercise 17E

1 We know that the point $P(x, y)$ satisfies,

$$QP = 4$$

$$\sqrt{(x-1)^2 + (y-(-2))^2} = 4$$

$$(x-1)^2 + (y+2)^2 = 4^2.$$

This is a circle with centre $(1, -2)$ and radius 4.

2 We know that the point $P(x, y)$ satisfies,

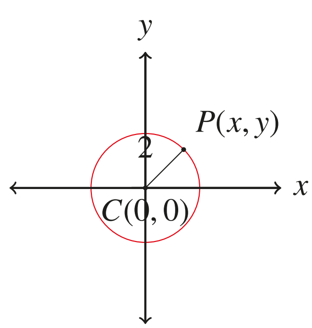
$$QP = 5$$

$$\sqrt{(x-(-4))^2 + (y-3)^2} = 5$$

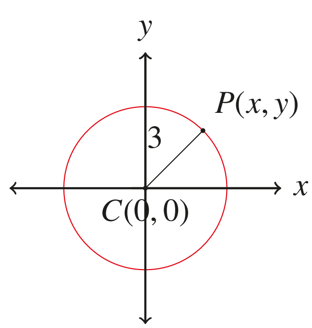
$$(x+4)^2 + (y-3)^2 = 5^2.$$

This is a circle with centre $(-4, 3)$ and radius 5.

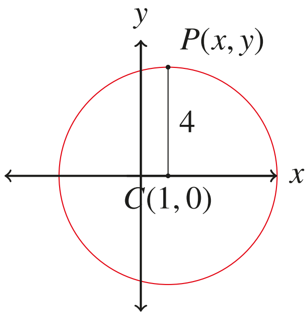
3 a



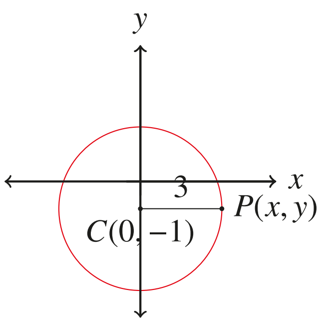
b



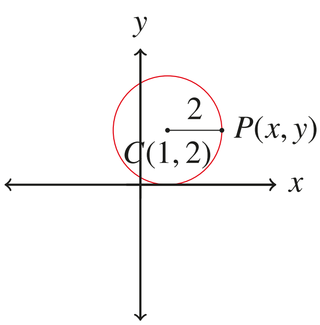
c



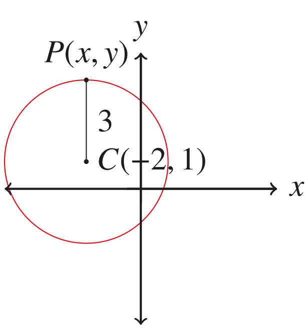
d



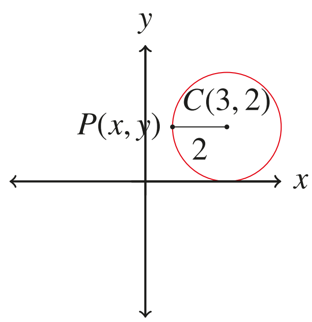
e



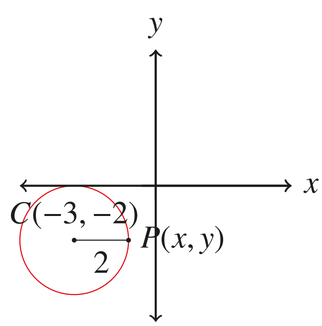
f



g



h



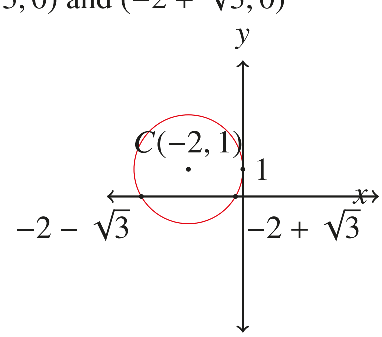
4 We first complete the square in both variables. This gives

$$\begin{aligned} x^2 + 4x + y^2 - 2y &= -1 \\ (x^2 + 4x + 4) - 4 + (y^2 - 2y + 1) - 1 &= -1 \\ (x + 2)^2 + (y - 1)^2 &= 4 \end{aligned}$$

Therefore the centre of the circle is $(-2, 1)$ and its radius is 2. The y-intercept is $(0, 1)$. To find the x-intercepts we let $y = 0$ so that

$$\begin{aligned} (y + 2)^2 + 1 &= 4 \\ (y + 2)^2 &= 3 \\ y &= -2 \pm \sqrt{3} \end{aligned}$$

The x-intercepts are $(-2 - \sqrt{3}, 0)$ and $(-2 + \sqrt{3}, 0)$

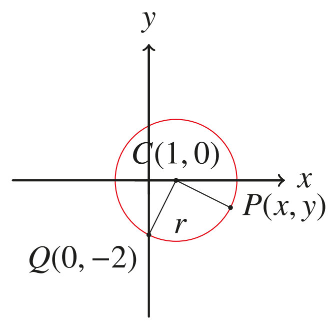


5 **a** To find the radius we note that

$$\begin{aligned} r &= CQ \\ &= \sqrt{(0-1)^2 + (-2-0)^2} \\ &= \sqrt{5}. \end{aligned}$$

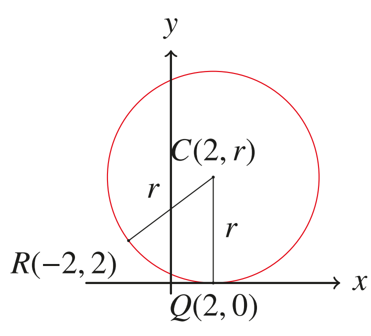
We let $P(x, y)$ be a point on the circle. Then

$$\begin{aligned} CP &= r \\ \sqrt{(x-1)^2 + (y-0)^2} &= \sqrt{5} \\ (x-1)^2 + y^2 &= 5. \end{aligned}$$



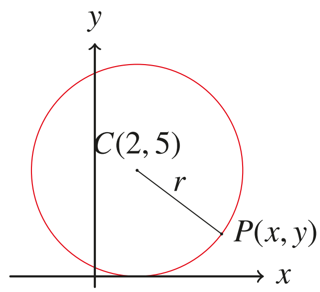
b Let the centre of the circle be $C(2, r)$. Then

$$\begin{aligned} RC &= QC \\ \sqrt{(2-(-2))^2 + (r-2)^2} &= r \\ 16 + (r-2)^2 &= r^2 \\ 16 + r^2 - 4r + 4 &= r^2 \\ 4r &= 20 \\ r &= 5. \end{aligned}$$



The centre of the circle is $C(2, 5)$ and its radius is 5. Let $P(x, y)$ be a point on the circle. Then

$$
\begin{aligned}
CP &= r \\
\sqrt{(x-2)^2 + (y-5)^2} &= 5 \\
(x-2)^2 + (y-5)^2 &= 5^2.
\end{aligned}
$$



6 **a** We know that the point $P(x, y)$ satisfies,

$$
\begin{aligned}
QP &= RP \\
\sqrt{(x-(-1))^2 + (y-(-1))^2} &= \sqrt{(x-1)^2 + (y-1)^2} \\
(x+1)^2 + (y+1)^2 &= (x-1)^2 + (y-1)^2 \\
x^2 + 2x + 1 + y^2 + 2y + 1 &= x^2 - 2x + 1 + y^2 - 2y + 1 \\
2x + 2y + &= -2x - 2y \\
y &= -x
\end{aligned}
$$

b The above line has gradient -1. The straight line through $Q(-1, -1)$ and $R(1, 1)$ has equation $y = x$ and thus has gradient 1. Because the product of the two gradients is -1, the two lines are perpendicular. Lastly, the midpoint of points Q and R is $(0, 0)$. This point is on the line $y = -x$ since if $x = 0$ then $y = 0$.

7 **a** We know that the point $P(x, y)$ satisfies,

$$
\begin{aligned}
QP &= RP \\
\sqrt{(x-0)^2 + (y-2)^2} &= \sqrt{(x-1)^2 + y^2} \\
x^2 + (y-2)^2 &= (x-1)^2 + y^2 \\
x^2 + y^2 - 4y + 4 &= x^2 - 2x + 1 + y^2 \\
-4y + 4 &= -2x + 1 \\
y &= \frac{x}{2} + \frac{3}{4}
\end{aligned}
$$

b The above line has gradient $\frac{1}{2}$. The straight line through $Q(0,2)$ and $R(1,0)$ has gradient

$$m = \frac{0-2}{1-0} = -2$$

and equation

$$y = -2x + 2.$$

Because the product of the two gradients is -1, the two lines are perpendicular. Lastly, the midpoint of points Q and R is $(\frac{1}{2}, 1)$. This point is on the line

$$y = \frac{x}{2} + \frac{3}{4}$$

since if $x = \frac{1}{2}$ then

$$y = \frac{1}{4} + \frac{3}{4} = 1.$$

8 Since $P(x,y)$ is equidistant from points $Q(0,1)$ and $R(2,3)$ we know that

$$\begin{aligned}
QP &= RP \\
\sqrt{x^2 + (y-1)^2} &= \sqrt{(x-2)^2 + (y-3)^2} \\
x^2 + (y-1)^2 &= (x-2)^2 + (y-3)^2 \\
x^2 + y^2 - 2y + 1 &= x^2 - 4x + 4 + y^2 - 6y + 9 \\
-2y + 1 &= -4x - 6y + 13 \\
4y + 4x &= 12 \\
y &= -x + 3 \qquad (1)
\end{aligned}$$

We also know that $P(x,y)$ is 3 units away from $S(3,3)$. Therefore $P(x,y)$ must lie on the circle whose equation is

$$(x-3)^2 + (y-3)^2 = 3^2. \qquad (2)$$

Substituting equation (1) into equation (2) gives

$$\begin{aligned}
(x-3)^2 + (-x+3-3)^2 &= 9 \\
(x-3)^2 + x^2 &= 9 \\
x^2 - 6x + 9 + x^2 &= 9 \\
2x^2 - 6x &= 0 \\
2x(x-3) &= 0
\end{aligned}$$

Therefore, either $x = 0$ or $x = 3$. Substituting $x = 0$ into (1) gives $y = 3$. Substituting $x = 3$ into (1) gives $y = 0$. Therefore, there are two answers: coordinates $(0,3)$ and $(3,0)$.

9 Since $P(x, y)$ is equidistant from points $Q(0, 1)$ and $R(2, 0)$ we know that

$$\begin{aligned}
QP &= RP \\
\sqrt{x^2 + (y-1)^2} &= \sqrt{(x-2)^2 + y^2} \\
x^2 + y^2 - 2y + 1 &= x^2 - 4x + 4 + y^2 \\
-2y + 1 &= -4x + 4 \\
-2y &= -4x + 3 \\
4x - 2y &= 3. \qquad (1)
\end{aligned}$$

Since $P(x, y)$ is equidistant from points $S(-1, 0)$ and $T(0, 2)$ we know that

$$\begin{aligned}
SP &= TP \\
\sqrt{(x+1)^2 + y^2} &= \sqrt{x^2 + (y-2)^2} \\
x^2 + 2x + 1 + y^2 &= x^2 + y^2 - 4y + 4 \\
2x + 1 &= -4y + 4 \\
-4y &= 2x - 3 \\
2x + 4y &= 3. \qquad (2)
\end{aligned}$$

Solving equations (1) and (2) simultaneously gives $x = \dfrac{9}{10}$ and $y = \dfrac{3}{10}$.

10 Since the treasure is 10 metres from a tree stump located at coordinates $T(0, 0)$, it lies on the circle whose equation is

$$x^2 + y^2 = 10^2. \qquad (1)$$

Since the treasure is 2 metres from a rock at coordinates $R(6, 10)$, it lies on the circle whose equation is

$$(x-6)^2 + (y-10)^2 = 2^2 \qquad (2)$$

Solving equations (1) and (2) simultaneously or by using your calculator gives two possible coordinates: either $(6, 8)$ or $\left(\dfrac{72}{17}, \dfrac{154}{17}\right)$.

11 **a** Since $P(x, y)$ is equidistant from points $R(4, 5)$ and $S(6, 1)$ we know that

$$\begin{aligned}
RP &= SP \\
\sqrt{(x-4)^2 + (y-5)^2} &= \sqrt{(x-6)^2 + (y-1)^2} \\
x^2 - 8x + 16 + y^2 - 10y + 25 &= x^2 - 12x + 36 + y^2 - 2y + 1 \\
-8x + 16 - 10y + 25 &= -12x + 36 - 2y + 1 \\
8y - 4x &= 4 \\
2y - x &= 1 \qquad (1)
\end{aligned}$$

b Since $P(x, y)$ is equidistant from points $S(6, 1)$ and $T(1, -4)$ we know that

$$\begin{aligned} SP &= TP \\ \sqrt{(x-6)^2 + (y-1)^2} &= \sqrt{(x-1)^2 + (y+4)^2} \\ x^2 - 12x + 36 + y^2 - 2y + 1 &= x^2 - 2x + 1 + y^2 + 8y + 16 \\ -12x + 36 - 2y + 1 &= -2x + 1 + 8y + 16 \\ x + y &= 2 \qquad (1) \end{aligned}$$

c Solving equations (1) and (2) simultaneously gives $x = 1$ and $y = 1$ so that the point required is $P(1, 1)$.

d The centre of the circle is $P(1, 1)$ and its radius will be the distance from $P(1, 1)$ to $R(4, 5)$. This is

$$r = PR = \sqrt{(4-1)^2 + (5-1)^2} = 5.$$

Therefore the equation of the circle must be

$$(x-1)^2 + (y-1)^2 = 5^2.$$

12 Let the point be $P(x, y)$. The gradient of AB is

$$\frac{5-1}{2-0} = 2.$$

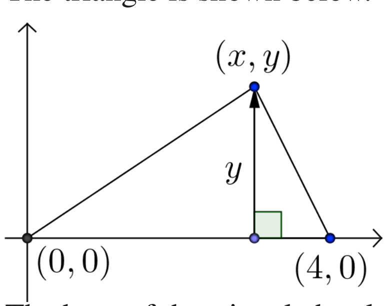
The gradient of BP is

$$\frac{y-5}{x-2}.$$

Equating the two gradients gives,

$$\begin{aligned} \frac{y-5}{x-2} &= 2 \\ y - 5 &= 2(x-2) \\ y &= 2x + 1. \end{aligned}$$

13 The triangle is shown below.



The base of the triangle has length 4 and its height is y. Therefore,

$$A = \frac{bh}{2}$$
$$12 = \frac{4y}{2}$$
$$y = 6.$$

14 **a** Let the point be $P(x, y)$. Then as the distance from P to the origin is equal to the sum of its x and y coordinates,

$$\sqrt{x^2 + y^2} = x + y$$
$$x^2 + y^2 = (x + y)^2$$
$$x^2 + y^2 = x^2 + 2xy + y^2$$
$$2xy = 0$$

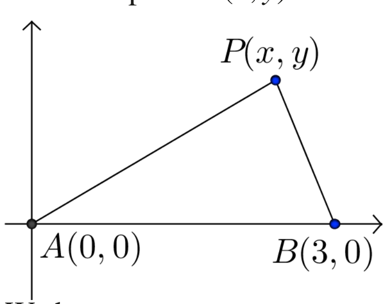
Therefore either $x = 0$ or $y = 0$. This is just both coordinate axes.

b Let the point be $P(x, y)$. Then as the distance from P to the origin is equal to the square of the sum of its x and y coordinates,

$$x^2 + y^2 = x + y$$
$$x^2 - x + y^2 - y = 0$$
$$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4} = 0$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

This is a circle with centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ with radius $\frac{1}{\sqrt{2}}$.

15 Consider point $P(x, y)$. The triangle is shown below.



We have

$$AP = \sqrt{x^2 + y^2},$$

and

$$BP = \sqrt{(x - 3)^2 + y^2}.$$

Since $AP : BP = 2$, we have

$$\frac{AP}{BP} = 2$$

$$\frac{\sqrt{x^2 + y^2}}{\sqrt{(x-3)^2 + y^2}} = 2$$

$$\frac{x^2 + y^2}{x^2 - 6x + 9 + y^2} = 4$$

$$x^2 + y^2 = 4(x^2 - 6x + 9 + y^2)$$

$$x^2 + y^2 = 4x^2 - 24x + 36 + 4y^2$$

$$3x^2 - 24x + 36 + 3y^2 = 0$$

$$x^2 - 8x + y^2 = -12$$

$$(x^2 - 8x + 16) - 16 + y^2 = -12$$

$$(x-4)^2 + y^2 = 4$$

This is a circle of radius 2 and centre $(4, 0)$.

16 The distance from the point $P(x, y)$ to the line $y = 3$ is 2. Therefore,

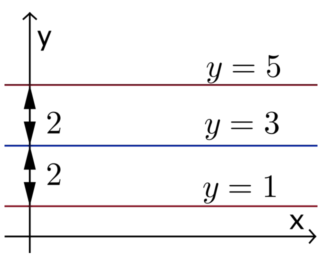
$$|y - 3| = 2$$

$$y - 3 = \pm 2$$

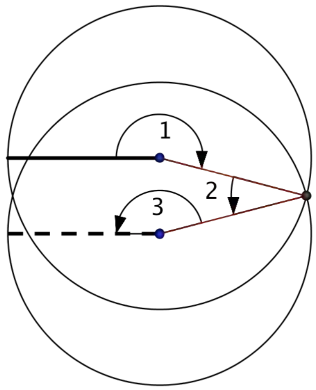
$$y = 3 \pm 2$$

$$y = 1 \text{ or } y = 5$$

This pair of lines are shown in red on the diagram below.



17 To solve this problem, draw two circles whose radii are equal to the length of the pipe, and whose centres are the endpoints of the pipe. The pipe can then be moved in a minimum of 3 moves. These are indicated on the diagram below.


1
2
3

Solutions to Exercise 17F

1 We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{x^2 + (y-3))^2} = \sqrt{(y-(-3))^2}$$

$$x^2 + (y-3)^2 = (y+3)^2$$

$$x^2 + y^2 - 6y + 9 = y^2 + 6y + 9$$

$$x^2 - 12y = 0$$

$$y = \frac{x^2}{12}.$$

Therefore, the set of points is a parabola whose equation is $y = \frac{x^2}{12}$.

2 We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{x^2 + (y-(-4))^2} = \sqrt{(y-2)^2}$$

$$x^2 + (y+4)^2 = (y-2)^2$$

$$x^2 + y^2 + 8y + 16 = y^2 - 4y + 4$$

$$x^2 + 12y = -12$$

$$y = -\frac{x^2}{12} - 1.$$

Therefore, the set of points is a parabola whose equation is

$$y = -\frac{x^2}{12} - 1.$$

3 We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{(x-2)^2 + y^2} = \sqrt{(x-(-4))^2}$$

$$(x-2)^2 + y^2 = (x+4)^2$$

$$x^2 - 4x + 4 + y^2 = x^2 + 8x + 16$$

$$-12x + y^2 = 12$$

$$x = \frac{y^2}{12} - 1.$$

Therefore, the set of points is a (sideways) parabola whose equation is $x = \frac{y^2}{12} - 1$.

4 **a** We know that the point $P(x, y)$ satisfies,

$$FP = RP$$

$$\sqrt{(x-c)^2 + y^2} = \sqrt{(x-(-c))^2}$$

$$(x-c))^2 + y^2 = (x+c)^2$$

$$x^2 - 2cx + c^2 + y^2 = x^2 + 2cx + c^2$$

$$y^2 - 2cx = +2cx$$

$$y^2 = 4cx$$

$$x = \frac{y^2}{4c}.$$

Therefore, the set of points is a (sideways) parabola whose equation is $x = \frac{y^2}{4c}$.

b The parabola with equation $x = -\frac{y^2}{4c}$ has focus $F(0, c)$ and directrix $x = -c$. For the parabola $x = 3y^2$, we have $\frac{1}{4c} = 3$ so that $c = \frac{1}{12}$. Therefore, its focus is $(1/12, 0)$ and its directrix is at $x = -1/12$.

5 **a** We know that the point $P(x, y)$ satisfies,

$$FP = RP$$
$$\sqrt{(x-a)^2+(y-b)^2} = \sqrt{(y-c)^2}$$
$$(x-a)^2+(y-b)^2 = (y-c)^2$$
$$x^2-2ax+a^2-2by+b^2 = -2cy+c^2$$
$$x^2-2ax+a^2+b^2-c^2 = 2by-2cy$$
$$x^2-2ax+a^2+b^2-c^2 = (2b-2c)y$$

Solving for y gives,

$$y = \frac{1}{2b-2c}(x^2-2ax+a^2+b^2-c^2).$$

b Let $a = 1, b = 2$ and $c = 3$ in the above equation. This gives,

$$y = \frac{1}{2b-2c}(x^2-2ax+a^2+b^2-c^2)$$
$$= -\frac{1}{2}(x^2-2x-4).$$

6 Since the parabola has a vertical line of symmetry, its directrix will be a horizontal line, $y = c$. The point $P(7, 9)$ is on the parabola. Therefore, the distance from $P(7, 9)$ to the focus $F(1, 1)$ is the same as the distance from $P(x, y)$ to the line $y = c$. Therefore,

$$FP = RP$$
$$\sqrt{(7-1)^2+(9-1)^2} = \sqrt{(9-c)^2}$$
$$6^2+8^2 = (9-c)^2$$
$$(9-c)^2 = 100$$
$$9-c = \pm 10$$
$$c = 9 \pm 10$$
$$= -1, 19$$

Therefore, there are two possibilities for the equation of the directrix: $y = -1$ and $y = 19$.

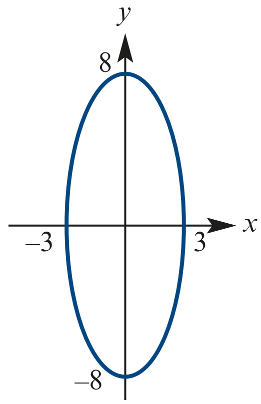
7 As the focus lies on the line of symmetry, we can suppose that the coordinates of the focus are $(2, a)$. The distance from the focus $(2, a)$ to $P(1, 1)$ is the same as the distance from the line $y = 3$ to the point $P(1, 1)$. Therefore,

$$FP = RP$$
$$\sqrt{(1-2)^2+(1-a)^2} = 2$$
$$1+(1-a)^2 = 4$$
$$(1-a)^2 = 3$$
$$1-a = \pm\sqrt{3}$$
$$a = 1 \pm \sqrt{3}$$

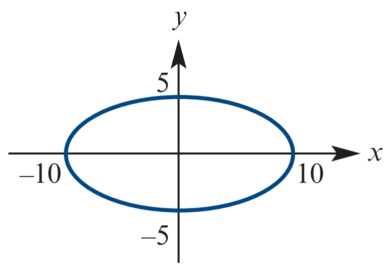
Therefore. the coordinates of the focus are either $(2, 1+\sqrt{3})$ or $(2, 1-\sqrt{3})$

Solutions to Exercise 17G

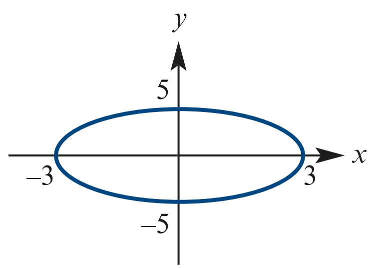
1 a



b

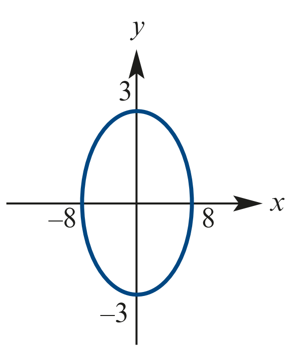


c

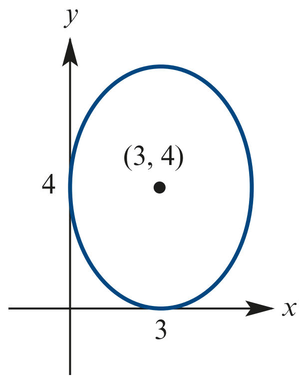


d Dividing both sides of the expression by 225 gives

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

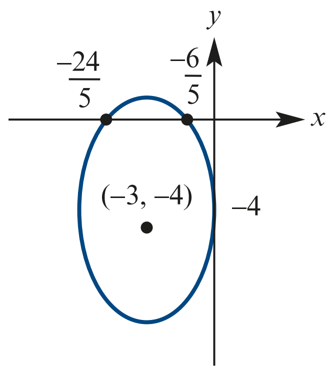


2 **a**



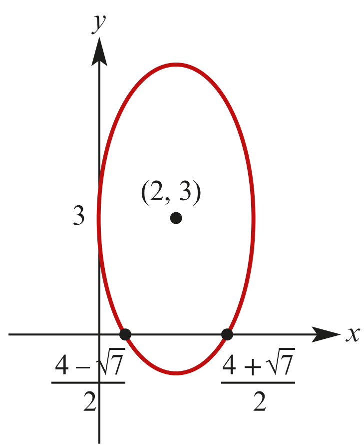
y
(3, 4)
4
x
3

b



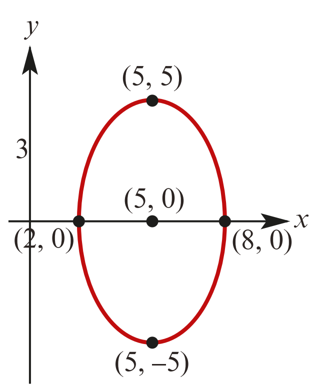
y
$\frac{-24}{5}$
$\frac{-6}{5}$
x
(−3, −4)
−4

c



y
(2, 3)
3
x
$\frac{4-\sqrt{7}}{2}$
$\frac{4+\sqrt{7}}{2}$

d



y
(5, 5)
3
(5, 0)
x
(2, 0)
(8, 0)
(5, −5)

3 **a** The equation can be found by noting that the x-intercepts are $x = \pm 5$ and the y-intercepts are $y = \pm 4$. Therefore, the equation must be

$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$$

or $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1.$

b The centre of the ellipse is $(2, 0)$. Therefore, the equation of the ellipse must be

$$\frac{(x-2)^2}{3^2} + \frac{y^2}{2^2} = 1$$

or $\dfrac{(x-2)^2}{9} + \dfrac{y^2}{4} = 1.$

c The centre of the ellipse is $(-1, 1)$. Therefore, the equation of the ellipse must be

$$\frac{(x+1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$$

or $\dfrac{(x+1)^2}{4} + (y-1)^2 = 1.$

4 Let (x, y) be the coordinates of point P. If $AP + BP = 4$ then,

$$\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4,$$

$$\sqrt{(x-1)^2 + y^2} = 4 - \sqrt{(x+1)^2 + y^2}.$$

Squaring both sides gives,

$$(x-1)^2 + y^2 = 16 - 8\sqrt{(x+1)^2 + y^2} + (x+1)^2 + y^2.$$

Now expand and simplify to obtain

$$x^2 - 2x + 1 + y^2 = 16 - 8\sqrt{(x+1)^2 + y^2} + x^2 + 2x + 1 + y^2$$

$$-2x = 16 - 8\sqrt{(x+1)^2 + y^2} + 2x,$$

$$4x + 16 = 8\sqrt{(x+1)^2 + y^2}$$

$$x + 4 = 2\sqrt{(x+1)^2 + y^2}$$

Squaring both sides again gives

$$x^2 + 8x + 16 = 4(x^2 + 2x + 1 + y^2).$$

Simplifying yields

$$12 = 3x^2 + 4y^2 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{3} = 1.$$

This is an ellipse with centre the origin, and intercepts at $x = \pm 2$ and $y = \pm\sqrt{3}$.

5 Let (x, y) be the coordinates of point P. If $AP + BP = 6$ then,

$$\sqrt{x^2 + (y-2)^2} + \sqrt{x^2 + (y+2)^2} = 6,$$

$$\sqrt{x^2 + (y-2)^2} = 6 - \sqrt{x^2 + (y+2)^2}$$

Squaring both sides gives,

$$x^2 + (y-2)^2 = 36 - 12\sqrt{x^2 + (y+2)^2} + x^2 + (y+2)^2.$$

Now expand and simplify to obtain

$$\begin{aligned} x^2 + y^2 - 4y + 4 & \\ &= 36 - 12\sqrt{x^2 + (y+2)^2} + x^2 + y^2 + 4y + 4 \\ -4y &= 36 - 12\sqrt{x^2 + (y+2)^2} + 4y \\ 8y + 36 &= 12\sqrt{x^2 + (y+2)^2} \\ 2y + 9 &= 3\sqrt{x^2 + (y+2)^2} \end{aligned}$$

Squaring both sides again gives

$$4y^2 + 36y + 81 = 9(x^2 + y^2 + 4y + 4).$$

Simplifying yields

$$9x^2 + 5y^2 = 45 \quad \text{or} \quad \frac{x^2}{5} + \frac{y^2}{9} = 1.$$

This is an ellipse with centre the origin, and intercepts at $x = \pm\sqrt{5}$ and y-intercepts $y = \pm 3$.

6 Let (x, y) be the coordinates of point P. If $FP = \frac{1}{2}MP$ then

$$\sqrt{(x-2)^2 + y^2} = \frac{1}{2}\sqrt{(x+4)^2}.$$

Squaring both sides gives

$$\begin{aligned} (x-2)^2 + y^2 &= \frac{1}{4}(x+4)^2 \\ 4(x^2 - 4x + 4) + 4y^2 &= x^2 + 8x + 16 \\ 4x^2 - 16x + 16 + 4y^2 &= x^2 + 8x + 16 \\ 3x^2 - 24x + 4y^2 &= 0. \end{aligned}$$

Completing the square gives,

$$\begin{aligned} 3(x^2 - 8x) + 4y^2 &= 0 \\ 3((x^2 - 8x + 16) - 16) + 4y^2 &= 0 \\ 3((x-4)^2 - 16) + 4y^2 &= 0 \\ 3(x-4)^2 + 4y^2 &= 48 \end{aligned}$$

Or equivalently,

$$\frac{(x-4)^2}{16} + \frac{y^2}{12} = 1.$$

7 The transformation is defined by the rule $(x, y) \to (5x, 3y)$. Therefore let $x' = 5x$ and $y' = 3y$ where (x', y') is the image of (x, y) under the transformation. Hence $x = \frac{x'}{5}$ and $x = \frac{y'}{3}$. The equation

$$x^2 + y^2 = 1$$

becomes,

$$\frac{(x')^2}{25} + \frac{(y')^2}{9} = 1$$

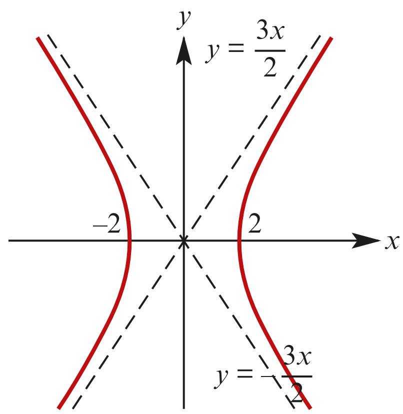
Ignoring the apostrophes gives,

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

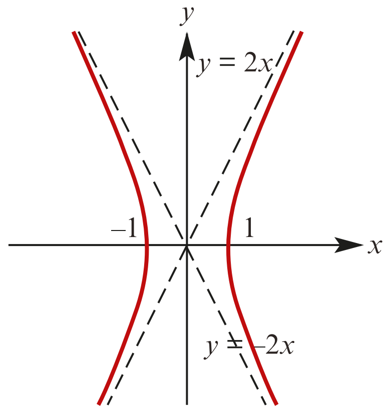
This is an ellipse with centre the origin, with intercepts at $(\pm 5, 0)$ and $(0, \pm 3)$.

Solutions to Exercise 17H

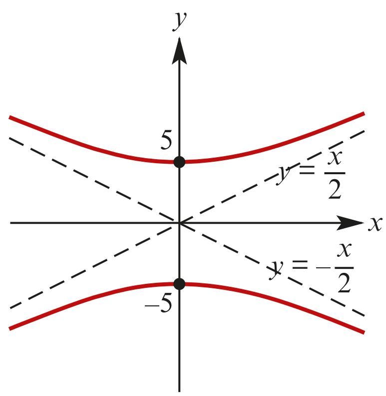
1 a



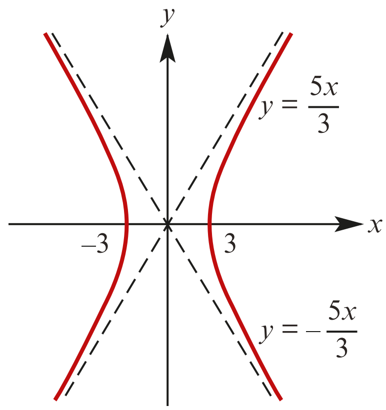
b



c

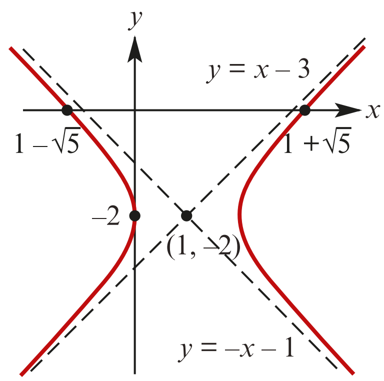


d



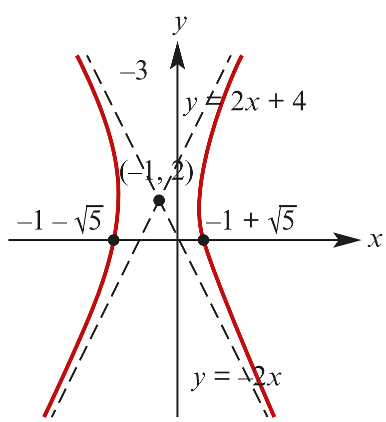
y
x
$y = \frac{5x}{3}$
–3
3
$y = -\frac{5x}{3}$

2 a



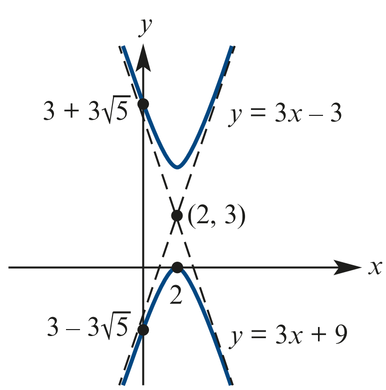
y
x
$y = x - 3$
$1 - \sqrt{5}$
$1 + \sqrt{5}$
–2
(1, –2)
$y = -x - 1$

b



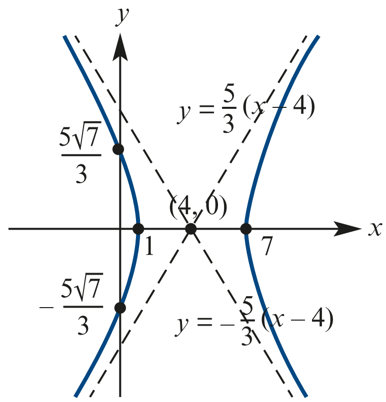
y
x
–3
$y = 2x + 4$
(–1, 2)
$-1 - \sqrt{5}$
$-1 + \sqrt{5}$
$y = -2x$

c



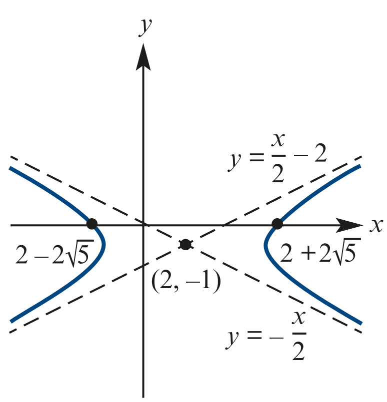
y
x
$3 + 3\sqrt{5}$
$y = 3x - 3$
(2, 3)
2
$3 - 3\sqrt{5}$
$y = 3x + 9$

d



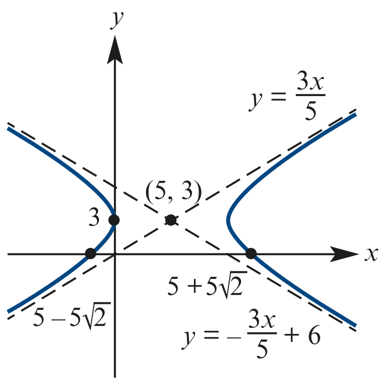
e For this question, we must first complete the square in both x and y variables. This gives,

then
$$x^2 - 4y^2 - 4x - 8y - 16 = 0$$
$$(x^2 - 4x) - 4(y^2 + 2y) - 16 = 0$$
$$(x^2 - 4x + 4 - 4) - 4(y^2 + 2y + 1 - 1) - 16 = 0$$
$$((x - 2)^2 - 4) - 4((y + 1)^2 - 1) - 16 = 0$$
$$(x - 2)^2 - 4 - 4(y + 1)^2 + 4 - 16 = 0$$
$$(x - 2)^2 - 4(y + 1)^2 = 16$$
$$\frac{(x - 2)^2}{16} - \frac{(y + 1)^2}{4} = 1$$



f For this question, we must first complete the square in both x and y variables. This gives,

$$9x^2 - 25y^2 - 90x + 150y = 225$$
$$9(x^2 - 10x) - 25(y^2 - 6y) = 225$$
$$9(x^2 - 10x + 25 - 25) - 25(y^2 - 6y + 9 - 9) = 225$$
$$9((x - 5)^2 - 25) - 25((y - 3)^2 - 9) = 225$$
$$9(x - 5)^2 - 225 - 25(y - 3)^2 + 225 = 225$$
$$9(x - 5)^2 - 25(y - 3)^2 = 225$$
$$\frac{(x - 5)^2}{25} - \frac{(y - 3)^2}{9} = 1$$



3 Let (x, y) be the coordinates of point P. If $AP - BP = 6$, then

$$\sqrt{(x-4)^2 + y^2} - \sqrt{(x+4)^2 + y^2} = 3$$
$$\sqrt{(x-4)^2 + y^2} = 6 + \sqrt{(x+4)^2 + y^2}.$$

Squaring both sides gives

$$(x-4)^2 + y^2 = 36 + 12\sqrt{(x+4)^2 + y^2} + (x+4)^2 + y^2$$

Expanding and simplifying

$$x^2 - 8x + 16 + y^2 = 36 + 12\sqrt{(x+4)^2 + y^2} + x^2 + 8x + 16 + y^2$$
$$-16x - 36 = 12\sqrt{(x+2)^2 + y^2}$$
$$-4x - 9 = 3\sqrt{(x+4)^2 + y^2}$$

Note that this only holds if $x \le -\frac{9}{4}$. Squaring both sides agains gives,

$$16x^2 + 72x + 81 = 9(x^2 + 8x + 16 + y^2)$$

Expanding and simplifying yields

$$16x^2 + 72x + 81 = 9x^2 + 72x + 144 + 9y^2$$

$$7x^2 - 9y^2 = 63$$

$$\frac{x^2}{9} - \frac{y^2}{7} = 1, \quad x \le -\frac{9}{4}.$$

4 Let (x, y) be the coordinates of point P. If $AP - BP = 4$, then

$$\sqrt{(x+3)^2 + y^2} - \sqrt{(x-3)^2 + y^2} = 4$$

$$\sqrt{(x+3)^2 + y^2} = 4 + \sqrt{(x-3)^2 + y^2}$$

Squaring both sides gives

$$(x+3)^2 + y^2 = 16 + 8\sqrt{(x-3)^2 + y^2} + (x-3)^2 + y^2.$$

Expanding and simplifying

$$x^2 + 6x + 9 + y^2$$

$$= 16 + 8\sqrt{(x-3)^2 + y^2} + x^2 - 6x + 9 + y^2$$

$$12x - 16 = 8\sqrt{(x-3)^2 + y^2}$$

$$3x - 4 = 2\sqrt{(x-3)^2 + y^2}$$

Note that this only holds if $x \ge \frac{4}{3}$. Squaring both sides again gives,

$$9x^2 - 24x + 16 = 4(x^2 - 6x + 9 + y^2)$$

Expanding and simplifying yields

$$9x^2 - 24x + 16 = 4x^2 - 24x + 36 + 4y^2$$

$$5x^2 - 4y^2 = 20$$

5 Let (x, y) be the coordinates of point P. If $FP = 2MP$

$$\sqrt{(x-5)^2 + y^2} = 2\sqrt{(x+1)^2}$$

Squaring both sides

$$
\begin{aligned}
(x-5)^2 + y^2 &= 4(x+1)^2 \\
x^2 - 10x + 25 + y^2 &= 4(x^2 + 2x + 1) \\
x^2 - 10x + 25 + y^2 &= 4x^2 + 8x + 4 \\
0 &= 3x^2 + 18x - y^2 - 21
\end{aligned}
$$

Completing the square gives,

$$0 = 3(x^2 + 6x) - y^2 - 21$$

$$0 = 3(x^2 + 6x + 9 - 9) - y^2 - 21$$

$$0 = 3((x+3)^2 - 9) - y^2 - 21$$

$$0 = 3(x+3)^2 - y^2 - 48$$

$$\frac{(x+3)^2}{16} - \frac{y^2}{48} = 1$$

This is a hyperbola with centre $(-3, 0)$

6 Let (x, y) be the coordinates of point P. If $FP = 2MP$

$$\sqrt{x^2 + (y+1)^2} = 2\sqrt{(y+4)^2}$$

Squaring both sides

$$
\begin{aligned}
x^2 + (y+1)^2 &= 4(y+4)^2 \\
x^2 + y^2 + 2y + 1 &= 4(y^2 + 8y + 16) \\
x^2 + y^2 + 2y + 1 &= 4y^2 + 32y + 64 \\
0 &= 3y^2 + 30y - x^2 + 63
\end{aligned}
$$

Completing the square gives,

$$0 = 3(y^2 + 10y) - x^2 + 63$$

$$0 = 3(y^2 + 10y + 25 - 25) - x^2 + 63$$

$$0 = 3((y+5)^2 - 25) - x^2 + 63$$

$$0 = 3(y+5)^2 - 75 - x^2 + 63$$

$$0 = 3(y+5)^2 - x^2 - 12$$

$$\frac{(y+5)^2}{4} - \frac{x^2}{12} = 1.$$

This is a hyperbola with centre $(0, -5)$

Solutions to Exercise 17I

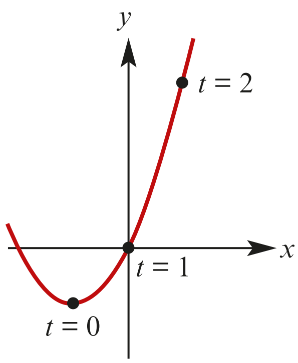
1 **a** From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$y = (x+1)^2 - 1$$
$$= x^2 + 2x.$$

b To sketch the curve it helps to write $y = x(x+2)$. This is a parabola with intercepts at $x = 0$ and $x = -2$. To label the points corresponding to $t = 0, 1, 2, 3$, we first complete the table shown below.

t	0	1	2
$x = t - 1$	−1	0	1
$y = t^2 - 1$	−1	0	3

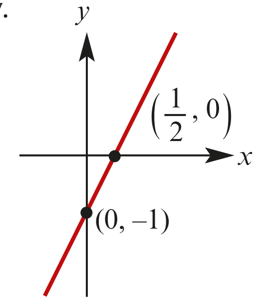
The curve and the required points are shown below.



2 **a** From the first equation we know that $t = x - 1$. Substitute this into the second equation to get

$$y = 2t + 1$$
$$= 2(x-1) + 1$$
$$= 2x - 2 + 1$$
$$= 2x - 1.$$

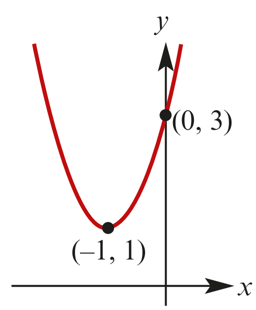
We obtain straight line whose equation is $y = 2x - 1$, and whose graph is shown below.



b From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$y = 2t^2 + 1$$
$$= 2(x+1)^2 + 1.$$

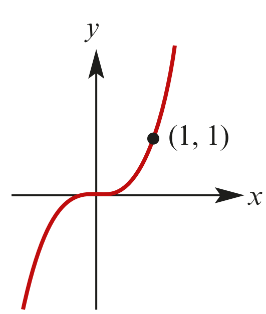
We obtain a parabola whose equation is $y = 2(x+1)^2 + 1$, and whose graph is shown below.



c For this question, we note that $y = (t^2)^3$. Therefore,

$$y = (t^2)^3 = x^3.$$

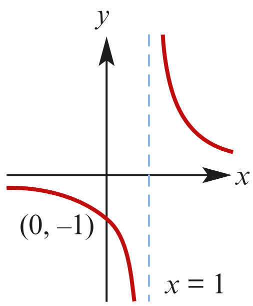
This is clearly a cubic equation whose graph is shown below.



d From the first equation we know that $t = x - 2$. Substitute this into the second equation to get

$$y = \frac{1}{t+1} = \frac{1}{x-2+1} = \frac{1}{x-1}$$

We obtain a hyperbola whose equation is $y = \dfrac{1}{x-1}$, and whose graph is shown below.



3 **a** We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x}{2} = \cos t \text{ and } \frac{y}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

Multiplying both sides by 2^2 gives the cartesian equation as

$$x^2 + y^2 = 2^2,$$

which is a circle centred at the origin on radius 2.

b We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x+1}{3} = \cos t \text{ and } \frac{y-2}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-2}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1,$$

which is an ellipse centred at the point $(-1, 2)$.

c We divide both sides of the equation by 9 so that the equation becomes,

$$\left(\frac{x+3}{3}\right)^2 + \left(\frac{y-2}{2}\right)^2.$$

We then let

$$\cos t = \frac{x+3}{3} \text{ and } \sin t = \frac{y-2}{3}.$$

Therefore, the required equations are

$$x = 3\cos t - 3 \text{ and } y = 3\sin t + 2.$$

d We write this equation as

$$\left(\frac{x+2}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2.$$

We then let

$$\cos t = \frac{x+2}{3} \text{ and } \sin t = \frac{y-1}{2}.$$

so that

$$x = 3\cos t - 2 \text{ and } y = 2\sin t + 1.$$

4 The gradient of the line through points A and B is

$$m = \frac{4-(-2)}{1-(-1)} = \frac{6}{2} = 3.$$

Therefore, the line has equation

$y - 4 = 3(x - 1)$

$y = 3x + 1.$

We can simply let $x = t$ so that $y = 3t + 1$. Note that this is not the only possible answer.

5 **a** We rearrange each equation to isolate $\sec t$ and $\tan t$ respectively. This gives

$$\frac{x-1}{2} = \sec t \text{ and } \frac{y+2}{3} = \tan t.$$

Therefore,

$$\left(\frac{x-1}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = \sec^2 t - \tan^2 t$$
$$= 1.$$

b We can let

$$\sec t = x - 2 \text{ and } \tan t = \frac{y+1}{2}$$

giving

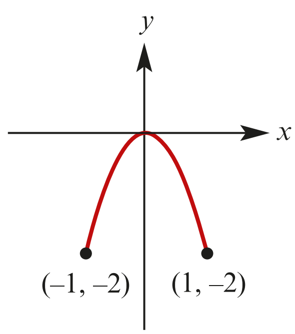
$$x = \sec t + 2 \text{ and } y = 2\tan t - 1.$$

6 a From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= -2t^2 + 4t - 2 \\ &= -2(x+1)^2 + 4(x+1) - 2 \\ &= -2(x^2 + 2x + 1) + 4x + 4 - 2 \\ &= -2x^2 - 4x - 2 + 4x + 4 - 2 \\ &= -2x^2 \end{aligned}$$

Moreover, since $0 \le t \le 2$, we know that $-1 \le x \le 1$.

b We sketch the curve over the domain $-1 \le x \le 1$.



7 The cartesian equation of the circle is

$$x^2 + y^2 = 1. \quad (1)$$

It is a little harder to find the cartesian equation of the straight line. Solving both equations for t gives,

$$t = \frac{x-6}{3} \text{ and } t = \frac{y-8}{4}.$$

Therefore,

$$\frac{x-6}{4} = \frac{y-8}{4}$$

$$4(x-6) = 3(y-8)$$

$$4x - 24 = 3y - 24$$

$$y = \frac{4x}{3} \quad (2)$$

Solving equations (1) and (2) simultaneously gives $x = -\frac{3}{5}$ and $x = \frac{3}{5}$. Substituting these two values into the equation $y = \frac{4x}{3}$ gives $y = -\frac{4}{5}$ and $x = \frac{4}{5}$ respectively. Therefore, the required coordinates are $(-3/5, -4/5)$ and $(3/5, 4/5)$.

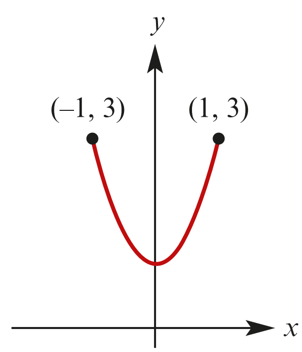
8 a We substitute $x = \sin t$ into the second equation to give,

$$\begin{aligned} y &= 2\sin^2 t + 1 \\ &= 2x^2 + 1. \end{aligned}$$

b Since the domain is the set of possible x-values and $x = \sin t$ where $0 \le t \le 2\pi$, the domain will be $-1 \le x \le 1$.

c Since the domain is the set of x such that $-1 \le x \le -1$, the range must be the set of y such that $1 \le y \le 3$.

d The curve is sketched below over the interval $-1 \le x \le 1$.



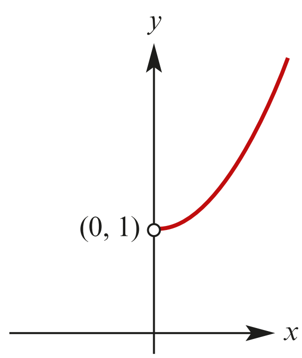
9 **a** We substitute $x = 2^t$ into the second equation to give,

$$\begin{aligned} y &= 2^{2t} + 1 \\ &= (2^t)^2 + 1 \\ &= x^2 + 1. \end{aligned}$$

b The domain is the set of possible x-values. Since $x = 2^t$ and $t \in \mathbb{R}$, we know that the domain will be $x > 0$.

c Since the domain is the set of all x such that $x > 0$, the range must be the set of y such that $y > 1$.

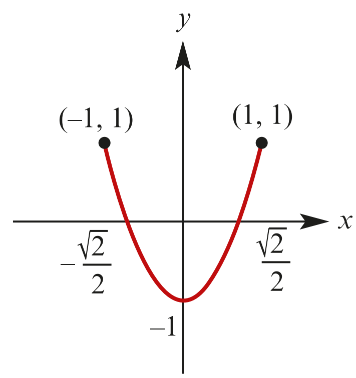
d The curve is sketched below over the interval $x > 0$.



10 Here, we must make use of the identity $\cos^2 t + \sin^2 t = 1$. Since $x = \cos t$ we have,

$$\begin{aligned} y &= 1 - 2\sin^2 t \\ &= 1 - 2(1 - \cos^2 t) \\ &= 1 - 2 + 2\cos^2 t \\ &= -1 + 2x^2 \end{aligned}$$

The domain is the set of possible x-values. Since $x = \cos t$ and $0 \le t \le 2\pi$, we know that the domain will be $-1 \le x \le 1$. We sketch the curve over this interval.

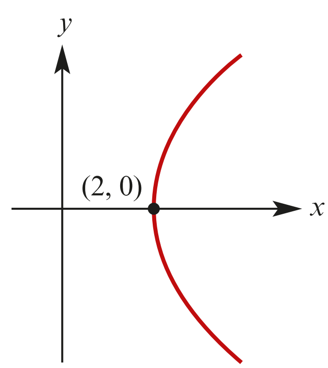


11 **a** We substitute $x = 2^t + 2^{-t}$ and $y = 2^t - 2^{-t}$ into the left hand side of the cartesian equation. This gives,

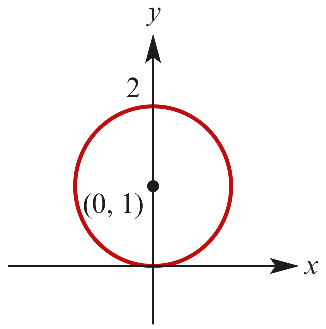
$$\begin{aligned} \text{LHS} &= \frac{x^2}{4} - \frac{y^2}{4} \\ &= \frac{(2^t + 2^{-t})^2}{4} - \frac{(2^t - 2^{-t})^2}{4} \\ &= \frac{2^{2t} + 2 + 2^{-2t}}{4} - \frac{(2^{2t} - 2 + 2^{-2t})}{4} \\ &= \frac{2^{2t} + 2 + 2^{-2t} - 2^{2t} + 2 - 2^{-2t}}{4} \\ &= \frac{4}{4} \\ &= 1 \\ &= \text{RHS}, \end{aligned}$$

as required.

b The curves is one side of a hyperbola centred at the origin.



12 **a** This is the equation of a circle of radius 1 centred at $(0, 1)$. Its graph is shown below.



b Since $x = \cos t$ and $y - 1 = \sin t$, we have

$$x^2 + (y-1)^2 = \cos^2 t + \sin^2 t = 1.$$

c We will find the points of intersection of the line,

$$y = 2 - tx \quad (1)$$

and the circle,

$$x^2 + (y-1)^2 = 1. \quad (2)$$

Substituting equation (1) into equation (2), we find that,

$$\begin{aligned} x^2 + (2 - tx - 1)^2 &= 1 \\ x^2 + (1 - tx)^2 &= 1 \\ x^2 + 1 - 2tx + t^2x^2 &= 1 \\ (1 + t^2)x^2 - 2tx &= 0 \\ x((1 + t^2)x - 2t) &= 0 \end{aligned}$$

Since $x \neq 0$, we see that

$$x = \frac{2t}{1 + t^2}.$$

We can find y by substituting this into equation (1). This gives,

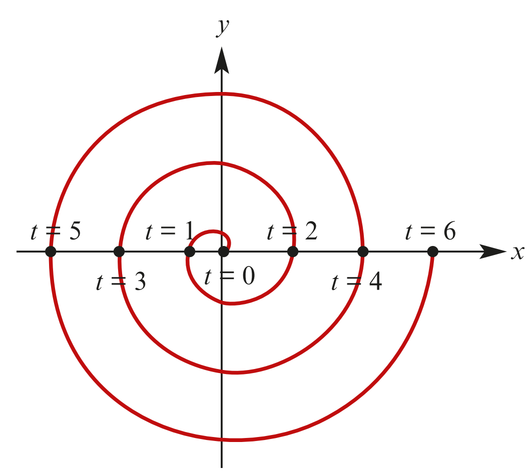
$$\begin{aligned} y &= 2 - tx \\ &= 2 - \frac{2t^2}{1 + t^2} \\ &= \frac{2(1 + t^2)}{1 + t^2} - \frac{2t^2}{1 + t^2} \\ &= \frac{2}{1 + t^2}. \end{aligned}$$

d To verify that these equations parameterise the same circle we note that

$$\begin{aligned} & x^2 + (y-1)^2 \\ &= \left(\frac{2t}{1 + t^2}\right)^2 + \left(\frac{2}{1 + t^2} - 1\right)^2 \\ &= \left(\frac{2t}{1 + t^2}\right)^2 + \left(\frac{2}{1 + t^2} - \frac{1 + t^2}{1 + t^2}\right)^2 \\ &= \left(\frac{2t}{1 + t^2}\right)^2 + \left(\frac{1 - t^2}{1 + t^2}\right)^2 \\ &= \frac{4t^2}{(1 + t^2)^2} + \frac{(1 - t^2)^2}{(1 + t^2)^2} \\ &= \frac{4t^2}{(1 + t^2)^2} + \frac{(1 - 2t^2 + t^4)}{(1 + t^2)^2} \\ &= \frac{t^4 + 2t^2 + 1}{(1 + t^2)^2} \\ &= \frac{(1 + t^2)^2}{(1 + t^2)^2} \\ &= 1, \end{aligned}$$

as required.

13 a



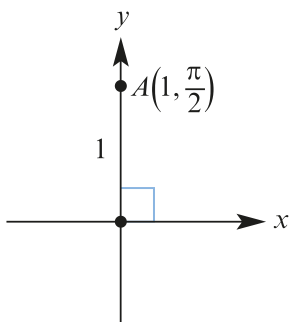
b The points corresponding to $t = 0, 1, 2, 3, 4, 5, 6$ are all on the x-axis. The values of t correspond to the number of half turns through which the spiral has turned.

Solutions to Exercise 17J

1 a We have

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 1\cos\pi/2 & &= 1\sin\pi/2 \\ &= 0 & &= 1 \end{aligned}$$

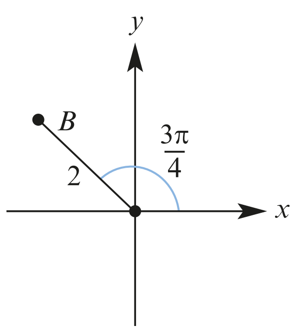
so that the cartesian coordinates are (0, 1).



b We have

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 2\cos 3\pi/4 & &= 2\sin 3\pi/4 \\ &= -\sqrt{2} & &= \sqrt{2} \end{aligned}$$

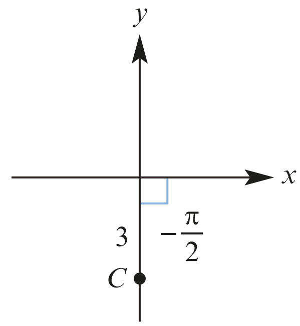
so that the cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$.



c We have

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 3\cos(-\pi/2) & &= 3\sin(-\pi/2) \\ &= 0 & &= -3 \end{aligned}$$

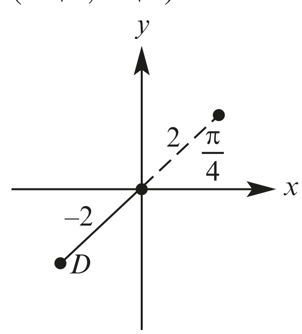
Te cartesian coordinates are (0, −3).



d We have

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= -2\cos\pi/4 & &= -2\sin\pi/4 \\ &= -\sqrt{2} & &= -\sqrt{2} \end{aligned}$$

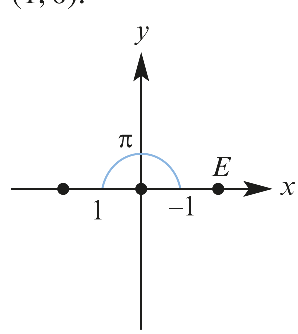
so that the cartesian coordinates are $(-\sqrt{2}, -\sqrt{2})$.



e We have

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= -1\cos\pi & &= -1\sin\pi \\ &= 1 & &= 0 \end{aligned}$$

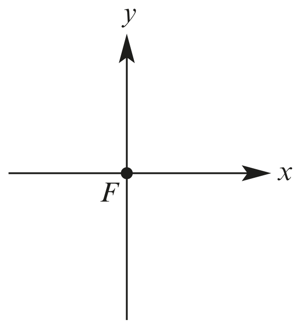
so that the cartesian coordinates are (1, 0).



f We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 0\cos\pi/4 \qquad = 0\sin\pi/4$$
$$= 0 \qquad = 0$$

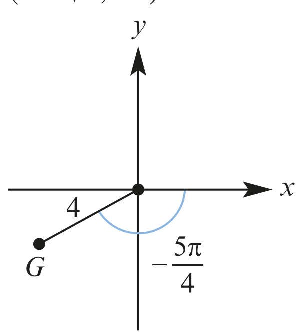
so that the cartesian coordinates are $(0, 0)$.



g We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 4\cos -5\pi/6 \qquad = 4\sin -5\pi/6$$
$$= -2\sqrt{3} \qquad = -2$$

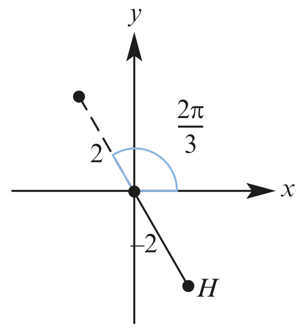
so that the cartesian coordinates are $(-2\sqrt{3}, -2)$.



h We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= -2\cos 2\pi/3 \qquad = -2\sin 2\pi/3$$
$$= 1 \qquad = -\sqrt{3}$$

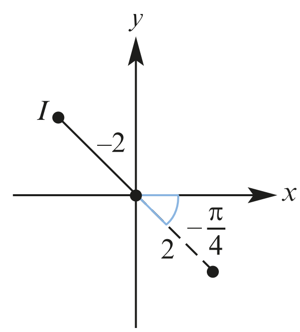
so that the cartesian coordinates are $(1, -\sqrt{3})$.



i We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= -2\cos(-\pi/4) \qquad = -2\sin(-\pi/4)$$
$$= -\sqrt{2} \qquad = \sqrt{2}$$

so that the cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$.



2 **a** $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
$\theta = \tan^{-1} -1 = -\dfrac{\pi}{4}$
The point has polar coordinates $[\sqrt{2}, -\pi/4]$. We could also let$r = -\sqrt{2}$ and add π to the found angle, giving coordinate $[-\sqrt{2}, 3\pi/4]$.

b $r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$
$\theta = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$
The point has polar coordinates $[2, \pi/3]$. We could also let $r = -2$ and add π to the found angle, giving coordinate $[-2, 4\pi/3]$.

c $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$
$\theta = \tan^{-1} -1 = -\dfrac{\pi}{4}$

The point has polar coordinates $[2\sqrt{2}, -\pi/4]$. We could also let $r = -2\sqrt{2}$ and add π to the found angle, giving $[-2\sqrt{2}, 3\pi/4]$.

d $r = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2} = 2$

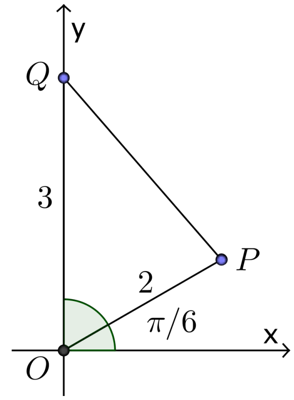
$\theta = -\dfrac{3\pi}{4}$

The point has polar coordinates $[2, -3\pi/4]$. We could also let $r = -2$ and add π to the found angle, giving coordinate$[-2, \pi/4]$.

e Clearly, $r = 3$ and $\theta = 0$ so that the point has polar coordinates $[3, 0]$. We could also let$r = -3$ and add π to the found angle, giving coordinate $[-3, \pi]$.

f Clearly, $r = 2$ and $\theta = -\dfrac{\pi}{2}$ so that the point has polar coordinates $[2, -\dfrac{\pi}{2}]$. We could also let$r = -2$ and add π to the found angle, giving coordinate $[-2, \pi/2]$.

3 Points P and Q are shown on the diagram below.

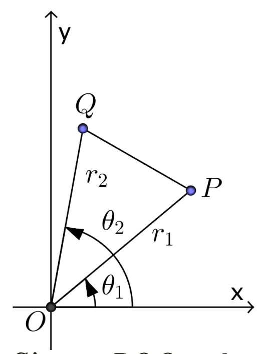


Since $\angle POQ = \dfrac{\pi}{3}$, we can use the cosine rule to find that

$$PQ^2 = OP^2 + OQ^2 - 2 \cdot OP \cdot OQ \cos(\pi/3)$$
$$= 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \frac{1}{2}$$
$$= 4 + 9 - 6$$
$$= 7.$$

Therefore, $PQ = \sqrt{7}$.

4 Points P and Q are shown on the diagram below.

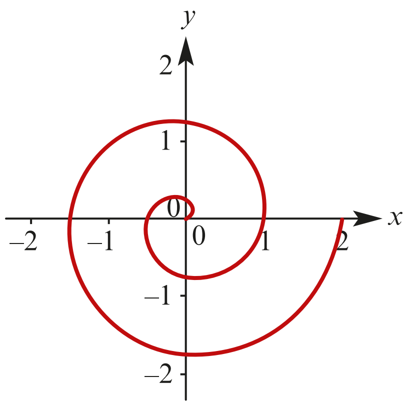


Since $\angle POQ = \theta_2 - \theta_1$, we can use the cosine rule to find that $PQ^2 = {r_1}^2 + {r_2}^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$. Therefore,

$$PQ = \sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$

Solutions to Exercise 17K

1



2 **a** Since $x = r\cos\theta$, this equation becomes

$$x = 4$$
$$r\cos\theta = 4$$
$$r = \frac{4}{\cos\theta}$$

b Since $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$y = x^2$$
$$r\sin\theta = r^2\cos^2\theta$$
$$r\cos^2\theta = \sin\theta$$
$$r = \frac{\sin\theta}{\cos^2\theta}$$
$$= \tan\theta\sec\theta$$

c This is just a circle of radius 3 centred at the origin and so has equation $r = 3$. We can check this by letting $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$x^2 + y^2 = 9$$
$$r^2\cos^2\theta + r^2\sin^2\theta = 9$$
$$r^2(\cos^2\theta + \sin^2\theta) = 9$$
$$r^2 = 9$$

In fact, since we are allowing negative r values we could take either $r = 3$ or $r = -3$ as the equation of this circle.

d Since $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$x^2 - y^2 = 9$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2(\cos^2\theta - \sin^2\theta) = 1$$
$$r^2\cos 2\theta = 1$$
$$r^2 = \frac{1}{\cos 2\theta}$$

e Since $x = r\cos\theta$ and $y = r\sin\theta$ this equation becomes

$$2x - 3y = 5$$
$$2r\cos\theta - 3r\sin\theta = 5$$
$$r(2\cos\theta - 3\sin\theta) = 5$$
$$r = \frac{5}{2\cos\theta - 3\sin\theta}$$

3 **a** The trick here is to first multiply both sides of the expression through by $\cos\theta$ to get

$$r\cos\theta = 2$$

Since $r\cos\theta = x$, this equation simply becomes,

$$x = 2.$$

b Since $r = 2$, this is just a circle of radius 2 centred at the origin. Its cartesian equation will then be simply

$$x^2 + y^2 = 2^2.$$

c Here, for all values of r the angle is constant and equal to $\pi/4$. This corresponds to the straight line through the origin, $y = x$. To see this algebraically, note that

$$\frac{y}{x} = \tan(\pi/4) = 1.$$

Therefore, $y = x$.

d Rearranging the equation we find that

$$\frac{4}{3\cos\theta - 2\sin\theta} = r$$

$$r(3\cos\theta - 2\sin\theta) = 4$$

$$3r\cos\theta - 2r\sin\theta = 4 \quad (1)$$

Then since $x = r\cos\theta$ and $y = r\sin\theta$, equation (1) becomes

$$3x - 2y = 4.$$

4 a The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 6r\cos\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, equation (1) becomes,

$$x^2 + y^2 = 6x$$

$$x^2 - 6x + y^2 = 0$$

(completing the square)

$$(x^2 - 6x + 9) - 9 + y^2 = 0$$

$$(x - 3)^2 + y^2 = 9.$$

This is a circles whose centre is $(3, 0)$ and whose radius is 3.

b The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 4r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, equation (1) becomes,

$$x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y = 0$$

(completing the square)

$$x^2 + (y^2 - 4y + 4) - 4 = 0$$

$$x^2 + (y - 2)^2 = 4.$$

This is a circle whose centre is $(0, 2)$ and whose radius is 2.

c The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 2r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, equation (1) becomes,

$$x^2 + y^2 = -6x$$

$$x^2 + 6x + y^2 = 0$$

(completing the square)

$$(x^2 + 6x + 9) - 9 + y^2 = 0$$

$$(x + 3)^2 + y^2 = 9$$

This is a circle whose centre is $(-3, 0)$ and whose radius is 3.

d The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 2r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, equation (1) becomes,

$$x^2 + y^2 = -8y$$

$$x^2 + y^2 + 8y = 0$$

$$x^2 + (y^2 + 8y + 16) - 16 = 0$$

(completing the square)

$$x^2 + (y + 4)^2 = 16.$$

This is a circle whose centre is $(0, -4)$ and whose radius is 4.

5 The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 2ar\cos\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\cos\theta = x$, equation (1) becomes,

$$x^2 + y^2 = 2ax$$

$$x^2 - 2ax + y^2 = 0$$

$$(x^2 - 2ax + a^2) - a^2 + y^2 = 0$$

(completing the square)

$$(x-a)^2 + y^2 = a^2.$$

This is a circle whose centre is $(a, 0)$ and whose radius is a.

6 a The trick here is to first multiply both sides of the expression through by $\cos\theta$ to obtain,

$r\cos\theta = a$

$$x = a,$$

which is the equation of a vertical line.

b Let $y = r\sin\theta$ so that

$r\sin\theta = a$

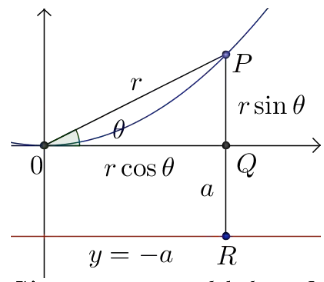
$$r = \frac{a}{\sin\theta}.$$

7 a The distance from P to the line is

$RP = RQ + QP$

$= a + r\sin\theta.$

b Consider the complete diagram shown below.



Since we are told that $OP = RP$, this implies that

$OP = RP$ as required.

$$r = a + r\sin\theta$$

$$r - r\sin\theta = a$$

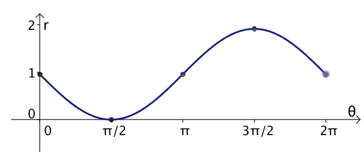
$$r(1 - \sin\theta) = a$$

$$r = \frac{a}{1-\sin\theta},$$

8 a To help sketch this curve we first graph the function

$$r = 1 - \sin\theta$$

as shown below. This allows us to see how r changes as θ increases.

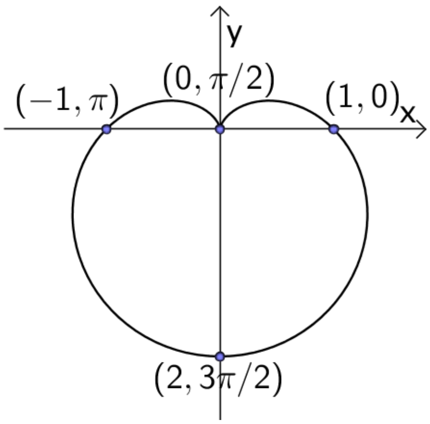


Note that:

- As angle θ increases from 0 to $\pi/2$, the radius r decreases from 1 to 0.
- As angle θ increases from $\pi/2$ to π, the radius r increases from 0 to 1.
- As angle θ increases from π to $3\pi/2$, the radius r increases from 1 to 2.
- As angle θ increases from $3\pi/2$ to 2π, the radius r decreases from 2 to 1.

This gives the graph shown

below. The points are labelled using polar coordinates.



b The trick, once again, is to multiply both sides of the equation through by r. This gives,

$$r^2 = r - r\sin\theta \text{ as}$$

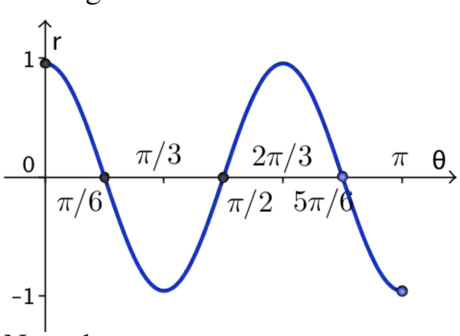
$$x^2 + y^2 = r - y$$

$$x^2 + y^2 + y = r$$

$$x^2 + y^2 + y = \sqrt{x^2 + y^2}$$

$(x^2 + y^2 + y)^2 = x^2 + y^2$, required.

9 a To help sketch this curve we first graph the function $r = \cos 3\theta$ as shown below. This allows us to see how r changes as θ increases.



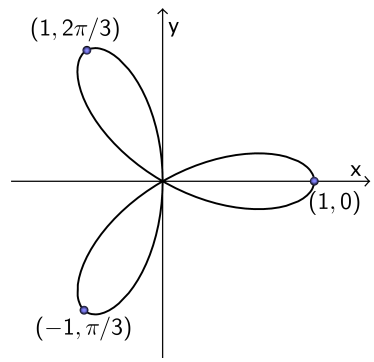
Note that:

- As angle θ increases from 0 to $\pi/6$, the radius r varies from 1 to 0.

- As angle θ increases from $\pi/6$ to $\pi/3$, the radius r varies from 0 to -1.

- As angle θ increases from $\pi/3$ to $\pi/2$, the radius r varies from -1 to 0.

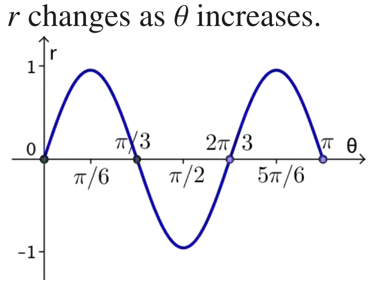
Continuing in this manner, we obtain the following graph shown below. Note that the labelled points are polar coordinates.



b To help sketch this curve we first graph the function

$$r = \cos 3\theta$$

as shown below. This allows us to see how r changes as θ increases.

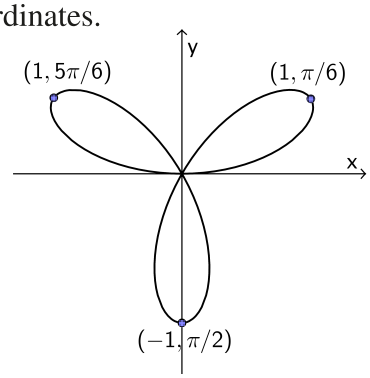


Note that:

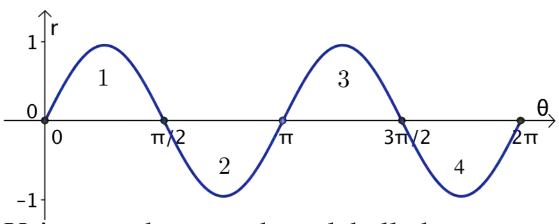
- As angle θ increases from 0 to $\pi/6$, the radius r varies from 0 to 1.

- As angle θ increases from $\pi/6$ to $\pi/3$, the radius r varies from 1 to 0.

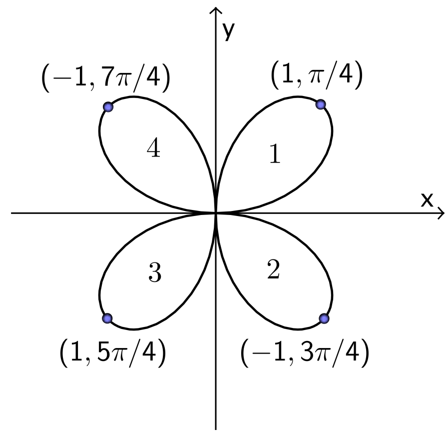
Continuing in this manner, we obtain the following graph shown below. Note that the labelled points are polar coordinates.



10 **a** To help sketch this curve we first graph the function $r = \sin 2\theta$ as shown below. This allows us to see how r changes as θ increases.



Using numbers, we have labelled how each section of this graph corresponds to a each section in the rose below. Note that the labelled points are polar coordinates.



b Since $\sin 2\theta = 2\sin\theta\cos\theta$, we have

$$\begin{aligned} r &= \sin 2\theta \\ r &= 2\sin\theta\cos\theta \\ r^3 &= 2 \cdot r\sin\theta \cdot r\cos\theta \\ r^3 &= 2xy \\ \left(x^2+y^2\right)^{\frac{3}{2}} &= 2xy \\ (x^2+y^2)^3 &= 4x^2y^2, \end{aligned}$$

as required.

Solutions to technology-free questions

1 **a** $\sin^{-1}(1) = \frac{\pi}{2}$ since $\sin\frac{\pi}{2} = 1$

b $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$

c $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$ since $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

d $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

e In two steps we find that

$$\cos\left(\sin^{-1}\left(-\tfrac{1}{2}\right)\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

f In two steps we find that

$$\tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right) = \tan^{-1}(1)) = \frac{\pi}{4}.$$

2 **a** To find the implied domain note that $\sin^{-1}(x+1)$ is defined

$$\Leftrightarrow \quad -1 \le x+1 \le 1$$

$$\Leftrightarrow \quad -2 \le x \le 0$$

Therefore the domain is $[-2, 0]$. The range will be the same as the range of $y = \sin^{-1}(x)$. That is, $[-\frac{\pi}{2}, \frac{\pi}{2}]$

b To find the implied domain note that $\cos^{-1}\left(x+\dfrac{1}{2}\right)$ is defined

$\Leftrightarrow \quad -1 \leq x + \frac{1}{2} \leq 1$

$\Leftrightarrow \quad -\frac{3}{2} \leq x \leq \frac{1}{2}$

Therefore the domain is $[-\frac{3}{2}, \frac{1}{2}]$. To find the range of the transformed function we dilate the original domain $[0, \pi]$ by factor of 2 than translate the result by $-\pi$. Therefore, the range is $[-\pi, \pi]$.

c Note that the domain of $\tan^{-1}(x)$ is $\mathbb{R}$. Therefore implied domain of $y = -2\tan^{-1}(x) + \dfrac{\pi}{4}$ is also $\mathbb{R}$. To find the range of the transformed function we dilate the original domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ by factor of 2 than translate the result by $\frac{\pi}{4}$. Therefore, the range is $[-\frac{3\pi}{4}, \frac{5\pi}{4}]$.

3 a $|-9| = 9$

b $\left|-\dfrac{1}{400}\right| = \dfrac{1}{400}$

c $|9-5| = |4| = 4$

d $|5-9 = |-4| = 4|$

e $|\pi - 3| = \pi - 3$ (since $\pi > 3$)

f $|\pi - 4| = 4 - \pi$ (since $\pi < 4$)

4 There are two cases to consider.
Case 1. If $f(x) \geq 0$ then $f(x) = x^2 - 3x$. Therefore

$$\begin{aligned} f(x) &= x \\ x^2 - 3x &= x \\ x^2 - 4x &= 0 \\ x(x-4) &= 0 \end{aligned}$$

$\Rightarrow x = 0, 4.$

Case 1. If $f(x) < 0$ then $f(x) = -(x^2 - 3x)$. Therefore

$$\begin{aligned} f(x) &= x \\ -(x^2 - 3x) &= x \\ -x^2 + 3x &= x \\ x^2 - 2x &= 0 \\ x(x-4) &= 0 \end{aligned}$$

$\Rightarrow x = 0, 2.$

Combining the two cases, we find that there are three solutions: $x = 0, 2, 4$.

5 We first sketch the graph of $y = x^2 - 4x$ and then use this to sketch the three required graphs. This is shown below.

a

Range $(f) = [0, \infty)$

b

Range $(f) = [-3, \infty)$

c

Range $(f) = (-\infty, 3]$

6 **a** Note that

$$\begin{aligned}|n^2 - 9| &= |(n-3)(n+3)| \\ &= |n-3||n+3|\end{aligned}$$

If this this prime then

$$|n-3| = 1 \text{ or } |n+3| = 1$$
$$\Rightarrow n-3 = \pm 1 \text{ or } n+3 = \pm 1$$
$$\Rightarrow n = 2, 4 \text{ or } n = -2, -4$$

We need to see that we obtain a prime for each of these values. We obtain:

$$n = \pm 2 \Rightarrow |n^2-9| = |-5| = 5$$
$$n = \pm 4 \Rightarrow |n^2-9| = |7| = 7$$

b **i** If $x > 0$ then $|x| = x$. Therefore

$$x^2 + 5|x| - 6 = 0$$
$$x^2 + 5x - 6 = 0$$
$$(x+6)(x-1) = 0$$
$$x = -6, 1$$

If $x > 0$ then $|x| = -x$. Therefore

$$x^2 + 5|x| - 6 = 0$$
$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$
$$x = -1, 6$$

There are four solutions is total: $x = \pm 1, \pm 6$.

ii If $x \geq 0$ then $|x| = x$. Therefore

$$x + |x| = 0$$
$$x + x = 0$$
$$2x = 0$$
$$\Rightarrow x = 0$$

If $x < 0$ then $|x| = -x$. Therefore

$$x - x = 0$$
$$\Rightarrow 0 = 0$$

This equation is satisfied by **all** negative numbers. Overall, we find that $x \in (-\infty, 0]$.

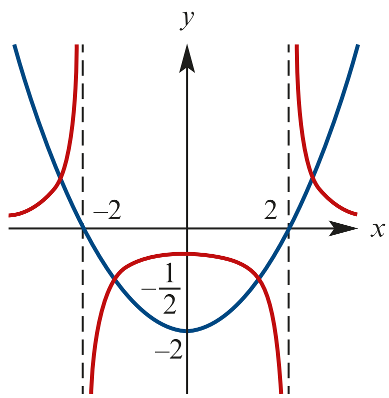
c We find that

$$5 - |x| < 4$$
$$-|x| < -1$$
$$|x| > 1$$
$$x < -1 \text{ or } x > 1.$$

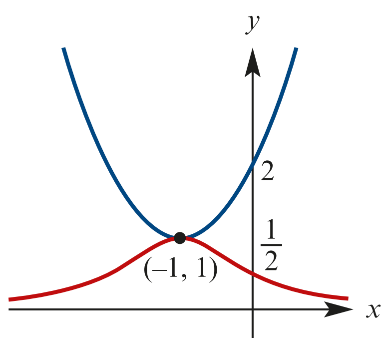
We can also describe these numbers using interval notation:

$x \in (-\infty, -1] \cup (1, \infty]$

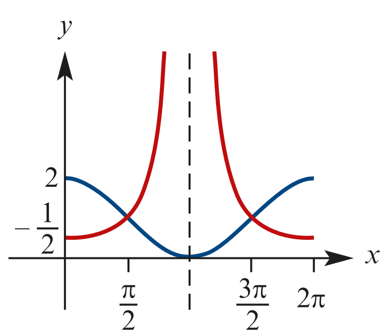
7 **a**



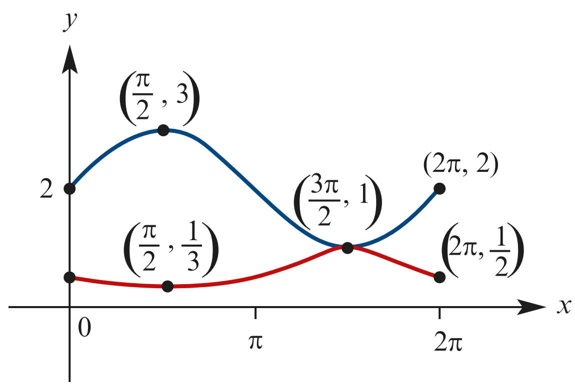
b



c

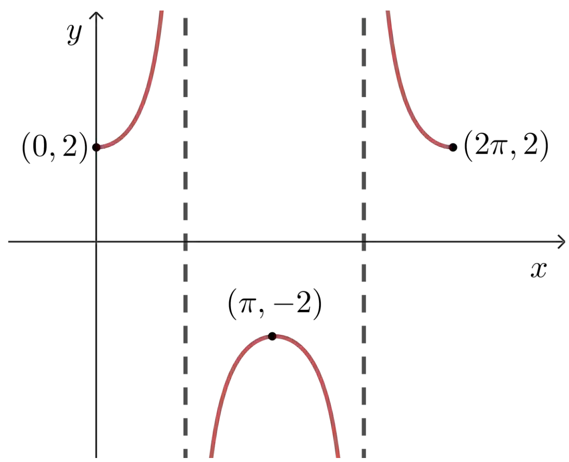


d



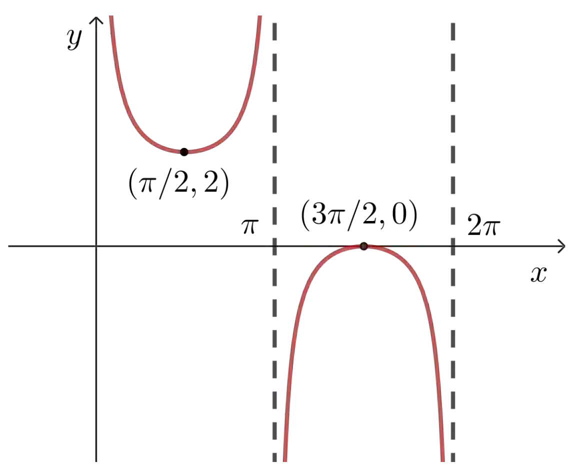
y
$\left(\frac{\pi}{2}, 3\right)$
2
$\left(\frac{3\pi}{2}, 1\right)$
$(2\pi, 2)$
$\left(\frac{\pi}{2}, \frac{1}{3}\right)$
$\left(2\pi, \frac{1}{2}\right)$
x
0
π
2π

8 a



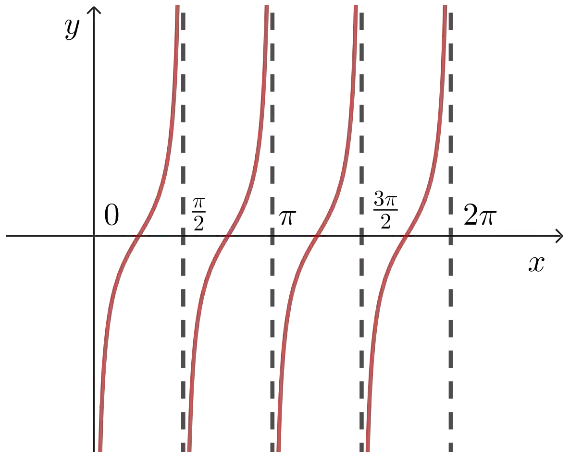
y
$(0, 2)$
$(2\pi, 2)$
x
$(\pi, -2)$

b



y
$(\pi/2, 2)$
π
$(3\pi/2, 0)$
2π
x

c



9 We know that the point $P(x, y)$ satisfies,

$$AP = 6$$

$$\sqrt{(x-3)^2 + (y-2)^2} = 6$$

$$(x-3)^2 + (y-2)^2 = 6^2.$$

This is a circle with centre $(3, 2)$ and radius 6 units.

10 **a**

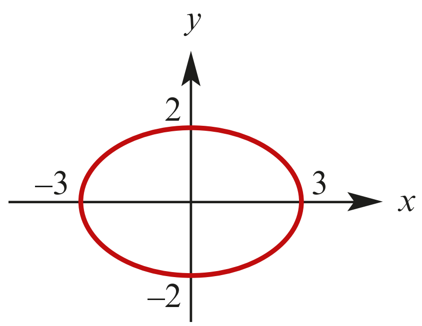
b

11 We complete the square to find that

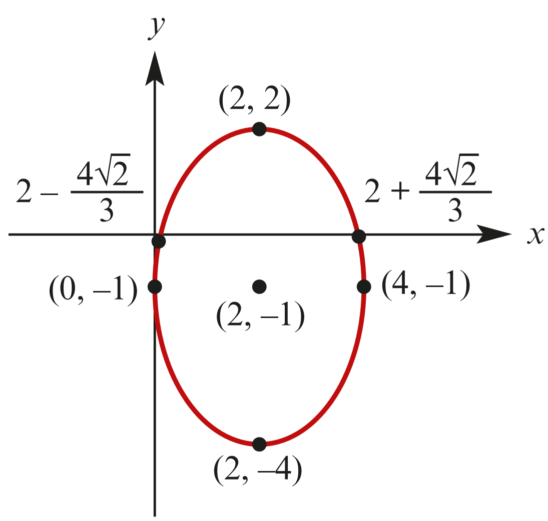
$$
\begin{aligned}
x^2 + 4x + y^2 - 8y &= 0\\
[(x^2 + 4x + 4) - 4] + [(y^2 - 8y + 16) - 16] &= 0\\
(x + 2)^2 - 4 + (y - 4)^2 - 16 &= 0\\
(x + 2)^2 + (y - 4)^2 &= 20.
\end{aligned}
$$

This is the equation of a circle with centre $(-2, 4)$ and radius $\sqrt{20}$ units.

12 **a**



b



13 We complete the square to find that,

$$
\begin{aligned}
x^2 + 4x + 2y^2 &= 0\\
(x^2 + 4x + 4) - 4 + 2y^2 &= 0\\
(x + 2)^2 + 2y^2 &= 4\\
\frac{(x + 2)^2}{4} + \frac{y^2}{2} &= 1
\end{aligned}
$$

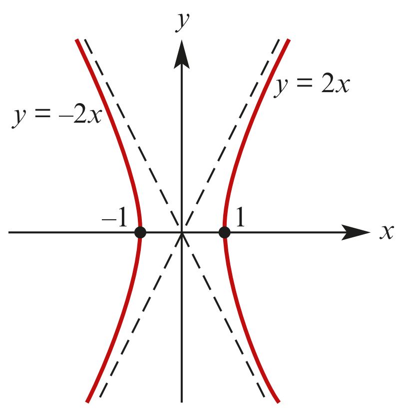
The centre is then $(-2, 0)$. To find the x-intercepts we let $y = 0$. Therefore,

$$\begin{aligned}\frac{(x+2)^2}{4} &= 1\\ (x+2)^2 &= 4\\ x+2 &= \pm 2\\ x &= -4, 0\end{aligned}$$

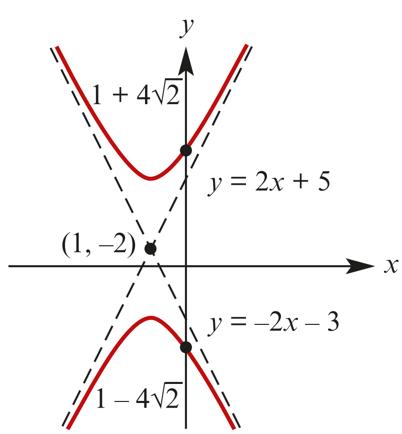
To find the y-intercepts we let $x = 0$ (in the original equation). Therefore,

$$\begin{aligned}2y^2 &= 0\\ y &= 0.\end{aligned}$$

14 **a**



b



15 We know that the point $P(x, y)$ satisfies,

$$KP = 2MP$$

$$\sqrt{(x-(-2))^2 + (y-5)^2} = 2\sqrt{(x-1)^2}$$

$$(x+2)^2 + (y-5)^2 = 4(x-1)^2$$

$$x^2 + 4x + 4 + y^2 - 10y + 25 = 4(x^2 - 2x + 1)$$

$$x^2 + 4x + 4 + y^2 - 10y + 25 = 4x^2 - 8x + 4$$

$$3x^2 - 12x - y^2 + 10y - 25 = 0$$

Completing the square then gives,

$$3(x^2 - 4x) - (y^2 - 10y) - 25 = 0$$

$$3(x^2 - 4x + 4 - 4) - (y^2 - 10y + 25 - 25) - 25 = 0$$

$$3((x-2)^2 - 4) - ((y-5)^2 + 25) - 25 = 0$$

$$3(x-2)^2 - 12 - (y-5)^2 - 25 - 25 = 0$$

$$3(x-2)^2 - (y-5)^2 = 12$$

$$\frac{(x-2)^2}{4} - \frac{(y-5)^2}{12} = 1$$

Therefore, the set of points is a hyperbola with centre $(2, 5)$.

16 **a** From the first equation we know that $t = \dfrac{x+1}{2}$. Substitute this into the second equation to get

$$\begin{aligned} y &= 6 - 4t \\ &= 6 - 4\frac{x+1}{2} \\ &= 6 - 2(x+1) \\ &= 6 - 2x - 2 \\ &= 4 - 2x \end{aligned}$$

We obtain straight line whose equation is $y = 4 - 2x$.

b We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x}{2} = \cos t \text{ and } \frac{y}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{x^2}{2^2} + \frac{y^2}{2^2} = 1$$

$$x^2 + y^2 = 2^2$$

which is a circle of radius 2 centred at the origin.

c We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x-1}{3} = \cos t \text{ and } \frac{y+1}{5} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{5}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x-1)^2}{3^2} + \frac{(y+1)^2}{5^2} = 1,$$

which is an ellipse centred at the point $(1, -1)$.

d Since $x = \cos t$, we have,

$$\begin{aligned} y &= 3\sin^2 t - 2 \\ &= 3(1 - \cos^2 t) - 2 \\ &= 3 - 3\cos^2 t - 2 \\ &= 1 - 3\cos^2 t \\ &= 1 - 3x^2 \end{aligned}$$

Note that this does not give the entire parabola. Since $x = \cos t$, the domain will be $-1 \le x \le 1$. Therefore, the cartesian equation of the curve is

$$y = 1 - 3x^2, \text{ where } -1 \le x \le 1.$$

17 a From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= 2t^2 - 1 \\ &= 2(x+1)^2 - 1. \end{aligned}$$

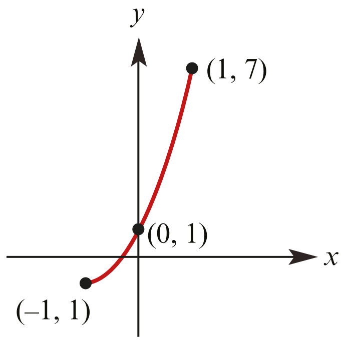
b Since $0 \le t \le 2$ and $x = t - 1$, we know that $-1 \le x \le 1$.

c The parabola has a minimum at $(-1, -1)$. It increases after this point. The maximum value of y is obtained when $x = 1$. Therefore,

$$y = 2(1+1)^2 - 1 = 7.$$

The range is the interval $-1 \le y \le 7$.

d We sketch the curve over the domain $-1 \le x \le 1$.



18 We have

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 2\cos 3\pi/4 & &= 2\sin 3\pi/4 \\ &= -\sqrt{2} & &= \sqrt{2} \end{aligned}$$

so that the cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$.

19 $r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$

$\theta = \tan^1 \dfrac{-2\sqrt{3}}{2} = \tan^{-1} - \sqrt{3} = -\dfrac{\pi}{3}$

The point has polar coordinates $[4, -\pi/3]$. We could also let $r = -4$ and add π to the found angle, giving coordinate $[-4, 2\pi/3]$.

20 Since $x = r\cos\theta$ and $y = r\sin\theta$ the equation becomes,

$$\begin{aligned} 2x + 3y &= 5 \\ 2r\cos\theta + 3r\sin\theta &= 5 \\ r(2\cos\theta + 3\sin\theta) &= 5 \end{aligned}$$

Therefore the polar equation is,

$$r = \frac{5}{2\cos\theta + 3\sin\theta}.$$

21 The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 6r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r \sin\theta = y$, equation (1) becomes,

$$x^2 + y^2 = 6y$$

$$x^2 + y^2 - 6y = 0$$

$$x^2 + (y^2 - 6y + 9) - 9 = 0 \qquad \text{(completing the square)}$$

$$x^2 + (y - 3)^2 = 9.$$

This is a circle whose centre is $(0, 3)$ and whose radius is 3, as required.

Solutions to multiple-choice questions

1 A The graph shown can be obtained by translating the graph of $y = \cos^{-1} x$ by 1 unit to the left and $\frac{\pi}{2}$ units down. This corresponds to the equation $y = \cos^{-1}(x+1) - \frac{\pi}{2}$.

2 D The graph shown can be obtained by first reflecting the graph of $y = |x|$ in the x-axis, and then translating the result by 2 unit to the right and 3 units up. This corresponds to the equation $y = -|x-2| + 3$.

3 B The graph will have two vertical asymptotes provided that the denominator has two x-intercepts. Therefore the discriminant of the quadratic must satisfy,

$$\begin{aligned} \Delta &> 0 \\ b^2 - 4ac &> 0 \\ 64 - 4(1)k &> 0 \\ 64 - 4k &> 0 \\ k &< 16. \end{aligned}$$

4 B

5 A We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} AP &= BP \\ \sqrt{(x-2)^2 + (y+5)^2} &= \sqrt{(x+4)^2 + (y-1)^2} \\ (x-2)^2 + (y+5)^2 &= (x+4)^2 + (y-1)^2 \\ x^2 - 4x + 4 + y^2 + 10y + 25 & \\ &= x^2 + 8x + 16 + y^2 - 2y + 1 \\ y &= x - 1 \end{aligned}$$

Therefore, the set of points is a straight line with equation $y = x - 1$. Alternatively, one could also just find the perpendicular bisector of line AB. This will give the same equation for about the same effort.

6 D One can answer this question either by reasoning geometrically, or by finding the equation of the parabola. Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= MP \\ \sqrt{x^2 + (y-2)^2} &= \sqrt{(y-(-2))^2} \\ x^2 + (y-2)^2 &= (y+2)^2 \\ x^2 + y^2 - 4y + 4 &= y^2 + 4y + 4 \\ x^2 &= 8y \\ y &= \frac{x^2}{8}. \end{aligned}$$

Clearly **A**,**B** and **C** are true. The point $(2, 1)$ does not lie on the parabola since when $x = 2$,

$$y = \frac{x^2}{8} = \frac{2^2}{8} \neq 1.$$

The point $(4, 2)$ does lie on the parabola since when $x = 4$,

$$y = \frac{x^2}{8} = \frac{4^2}{8} = 2.$$

7 D To find the x-intercepts we let $y = 0$ to find that

$$\begin{aligned} \frac{x^2}{25} &= 1 \\ x^2 &= 25 \\ x &= \pm 5. \end{aligned}$$

This gives coordinates $(\pm 5, 0)$.

8 D The hyperbola is centred at the point $(2, 0)$. This means that we can exclude options **A**,**C** and **E**, each of which are centred at the point $(-2, 0)$. The x-intercepts of the hyperbola occur when $x = -7$ and $x = 11$. We let $y = 0$ in option **B** and **D**, and see that only option **D** has the correct intercepts.

9 C The graph of

$$\frac{y^2}{9} - \frac{x^2}{4} = 1$$

is centred at the point $(0, 0)$. If we translate this by 3 units to the left and 2 units up we obtain the given equation. It will now be centred at the point $(-3, 2)$.

10 C We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x-1}{4} = \cos t \text{ and } \frac{y+1}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x-1}{4}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x-1)^2}{4^2} + \frac{(y+1)^2}{2^2} = 1.$$

To find the x-intercepts, we let $y = 0$. Solving for x gives,

$$\begin{aligned}
\frac{(x-1)^2}{4^2} + \frac{(0+1)^2}{2^2} &= 1 \\
\frac{(x-1)^2}{4^2} + \frac{1}{4} &= 1 \\
\frac{(x-1)^2}{4^2} &= \frac{3}{4} \\
(x-1)^2 &= 12 \\
x - 1 &= \pm\sqrt{12} \\
x &= 1 \pm 2\sqrt{3}
\end{aligned}$$

11 E **Option A:** These points are in quadrants 1 and 2 respectively and so cannot represent the same point.
Option B: These are located on the y-axis, but on opposite sides.
Option C: These points are in quadrants 1 and 4 respectively so cannot represent the same point.
Option D: These points are in quadrants 1 and 3 respectively so cannot represent the same point.
Option E: These coordinates do represent the same point. Recall that the coordinate $(-1, 7\pi/6)$ means that we locate direction $7\pi/6$, then move 1 unit in the opposite direction. This is the same as moving 1 unit in the direction $\pi/6$.

12 B The trick is to multiply both sides of the equation through by r. This gives,

$$\begin{aligned}
r^2 &= r + r\cos\theta \\
x^2 + y^2 &= r + x \\
x^2 + y^2 - x &= r \\
x^2 + y^2 - x &= \sqrt{x^2 + y^2} \\
(x^2 + y^2 - x)^2 &= x^2 + y^2,
\end{aligned}$$

as required.

Solutions to extended-response questions

1 **a** We know that

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

Therefore we let $\alpha = \tan^{-1}(x)$ and $\beta = \tan^{-1}(y)$ so that

$$\begin{aligned}\tan(\tan^{-1}(x)+\tan^{-1}(y)) &= \frac{\tan(\tan^{-1}(x))+\tan(\tan^{-1}(y))}{1-\tan(\tan^{-1}(x))\tan(\tan^{-1}(y))}\\ &= \frac{x+y}{1-xy}.\end{aligned}$$

Therefore

$$\tan^{-1}(x)+\tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

b Letting $x = \dfrac{1}{2}$ and $y = \dfrac{1}{3}$ gives

$$\begin{aligned}\tan^{-1}(\tfrac{1}{2})+\tan^{-1}(\tfrac{1}{3}) &= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)\\ &= \tan^{-1}\left(\frac{\frac{5}{6}}{1-\frac{1}{6}}\right)\\ &= \tan^{-1}(1)\\ &= \frac{\pi}{4}.\end{aligned}$$

2 **a** $mx+2=0$

$$\Rightarrow x = -\frac{2}{m}.$$

b $-\dfrac{2}{m} < -2 \Rightarrow -2 < -2m \Rightarrow m < 1$

c **i** The perpendicular line has gradient $-\dfrac{1}{m}$. Therefore equation:

$$y-2 = -\frac{1}{m}(x)$$

$$y = -\frac{x}{m}+2$$

ii If $m > 1, -\dfrac{x}{m}+2 = |mx+2|$

Assume $x < -\dfrac{2}{m}$ for the oher arm of the graph.

$$-\frac{x}{m} + 2 = -mx - 2$$

$$-x + 2m = -m^2x - 2m$$

$$x(1 - m^2) = 4m$$

$$x = \frac{4m}{1 - m^2}$$

Hence

$$y = -m \times \frac{4m}{1 - m^2} - 2$$

That is $y = \dfrac{-4m^2}{1 - m^2} - 2$

$$= \frac{-4m^2 - 2(1 - m^2)}{1 - m^2}$$

$$= \frac{-2m^2 - 2}{1 - m^2}$$

$$= \frac{2(1 + m^2)}{m^2 - 1}$$

Coordinates are:

$$\left(\frac{4m}{1 - m^2}, \frac{2(1 + m^2)}{m^2 - 1}\right)$$

iii If $m = 1$, ℓ is parallel to $y = -x + 2$. There is no second point of intersection.

iv if they meet when $x = -\dfrac{3}{2}$

$$\left|m \times \left(-\frac{3}{2}\right) + 2\right| = \frac{3}{2m} + 2$$

First consider:

$$\left(-\frac{3m}{2}\right) + 2 = \frac{3}{2m} + 2$$

$$-\frac{3m}{2} = \frac{3}{2m}$$

$$-6m^2 = 6$$

$$m^2 = -1$$

Hence no solution.

Now:

$$\left(\frac{3m}{2}\right) - 2 = \frac{3}{2m} + 2$$

$$\frac{3m}{2} = \frac{3}{2m} + 4$$

$$3m^2 = 3 + 8m$$

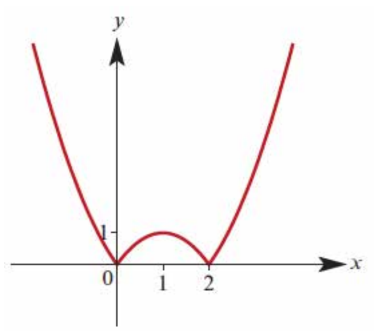
$$3m^2 - 8m - 3 = 0$$

$$(3m + 1)(m - 3) = 0$$

$$m = 3$$

since $m > 0$

3 a i



ii $|x^2 - ax| = 0$

$\Rightarrow x^2 - ax = 0$

$x(x - a) = 0$

$x = 0$ or $x = a$

iii The graph of $y = x^2 - ax$ is a parabola with local minimum at $\left(\frac{a}{2}, -\frac{a^2}{4}\right)$.

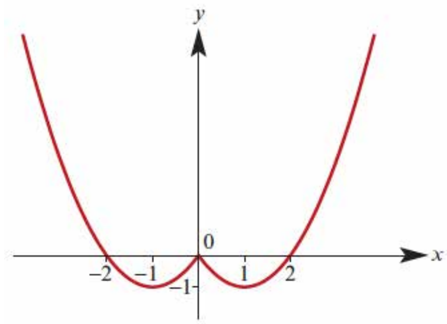
Therefore $y = f(x)$ has a local maximum at $\left(\frac{a}{2}, \frac{a^2}{4}\right)$

iv If $f(-1) = 4$ $\quad |1 + a| = 4$

$\therefore 1 + a = 4$ if $a > -1 \Rightarrow a = 3$

and $-1 - a = 4$ if $a < -1 \Rightarrow a = -5$

b **i**



ii $|x|^2 - a|x| = 0$

$\Rightarrow |x|(|x| - a) = 0$

$x = 0$ or $x = a$ or $x = -a$

iii If $x > 0, g(x) = x^2 - ax$ and local minimum is at $\left(\frac{a}{2}, -\frac{a^2}{4}\right)$

If $x < 0, g(x) = ax - x^2$ and local minimum is at $\left(-\frac{a}{2}, -\frac{a^2}{4}\right)$

iv If $g(-1) = 4$

$1 - a = 4$

$a = -3$

c Assume $a > 0$

$$f(x) = g(x)$$

$$|x^2 - ax| = x^2 - a|x|$$

Case 1 : $x > a$

$x^2 - ax = x^2 - ax$

Hence the graphs coincide for (a, ∞)

Case 2 : $0 \le x \le a$

$ax - x^2 = x^2 - ax$

$$2x^2 - 2ax = 0$$

$$2x(x - a) = 0$$

$$x = 0 \text{ or } x = a$$

Case 3 : $x < 0$

$$\begin{aligned} x^2 - ax &= x^2 + ax \\ 2ax &= 0 \\ x &= 0 \\ x &= 0 \end{aligned}$$

Therefore $f(x) = g(x)$ for $x = 0$ and $x \geq a$.

d Assume $a < 0$

$$\begin{aligned} f(x) &= g(x) \\ |x^2 - ax| &= x^2 - a|x| \end{aligned}$$

Case 1 : $x > 0$

$$x^2 - ax = x^2 - ax$$

Hence the graphs coincide for $(0, \infty)$

Case 2 : $a \leq x \leq 0$

$$\begin{aligned} ax - x^2 &= x^2 + ax \\ 2x^2 &= 0 \\ x &= 0 \end{aligned}$$

Case 3 : $x < a$

$$\begin{aligned} x^2 - ax &= x^2 + ax \\ 2ax &= 0 \\ x &= 0 \\ x &= 0 \end{aligned}$$

Therefore $f(x) = g(x)$ for $x \geq 0$.

4 a We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} AP &= BP \\ \sqrt{x^2 + (y-3)^2} &= \sqrt{(x-6)^2 + y^2} \\ x^2 + (y-3)^2 &= (x-6)^2 + y^2 \\ x^2 + y^2 - 6y + 9 &= x^2 - 12x + 36 + y^2 \\ -6y + 9 &= -12x + 36 \\ y &= 2x - \frac{9}{2} \end{aligned}$$

Therefore, the set of points is a straight line with equation $y = 2x - \frac{9}{2}$.

b We know that the point $P(x, y)$ satisfies,

$$AP = 2BP$$

$$\sqrt{x^2 + (y-3)^2} = 2\sqrt{(x-6)^2 + y^2}$$

$$x^2 + (y-3)^2 = 4[(x-6)^2 + y^2]$$

$$x^2 + y^2 - 6y + 9 = 4[x^2 - 12x + 36 + y^2]$$

$3x^2 - 48x + 3y^2 + 6y + 135 = 0$

Completing the square then gives,

$$\begin{aligned}
3x^2 - 48x + 3y^2 + 6y + 135 &= 0\\
3(x^2 - 16x) + 3(y^2 + 2y) + 135 &= 0\\
3[(x^2 - 16x + 64) - 64] + 3[(y^2 + 2y + 1) - 1] + 135 &= 0\\
3[(x-8)^2 - 64] + 3[(y+1)^2 - 1] + 135 &= 0\\
3(x-8)^2 + 3(y+1)^2 &= 60\\
(x-8)^2 + (y+1)^2 &= 20
\end{aligned}$$

This defines a circle with centre $(8, -1)$ and radius $\sqrt{20}$.

5 **a** Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$\begin{aligned}
FP &= MP\\
\sqrt{x^2 + (y-4)^2} &= \sqrt{(y-(-2))^2}\\
x^2 + (y-4)^2 &= (y+2)^2\\
x^2 + y^2 - 8y + 16 &= y^2 + 4y + 4\\
12y &= x^2 + 12\\
y &= \frac{x^2}{12} + 1.
\end{aligned}$$

Therefore, the set of points is a parabola.

b Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$
\begin{aligned}
FP &= \frac{1}{2}MP \\
\sqrt{x^2 + (y-4)^2} &= \frac{1}{2}\sqrt{(y-(-2))^2} \\
x^2 + (y-4)^2 &= \frac{1}{4}(y+2)^2 \\
4[x^2 + (y-4)^2] &= (y+2)^2 \\
4(x^2 + y^2 - 8y + 16) &= y^2 + 4y + 4 \\
4x^2 + 4y^2 - 32y + 64 &= y^2 + 4y + 4 \\
4x^2 + 3y^2 - 36y + 60 &= 0
\end{aligned}
$$

Completing the square then gives,

$$
\begin{aligned}
4x^2 + 3y^2 - 36y + 60 &= 0 \\
4x^2 + 3[y^2 - 12y + 20] &= 0 \\
4x^2 + 3[(y^2 - 12y + 36) - 36 + 20] &= 0 \\
4x^2 + 3[(y-6)^2 - 16] &= 0 \\
4x^2 + 3(y-6)^2 &= 48 \\
\frac{x^2}{12} + \frac{(y-6)^2}{16} &= 1
\end{aligned}
$$

Therefore, the set of points is an ellipse.

c Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$
\begin{aligned}
FP &= 2MP \\
\sqrt{x^2 + (y-4)^2} &= 2\sqrt{(y-(-2))^2} \\
x^2 + (y-4)^2 &= 4(y+2)^2 \\
x^2 + (y-4)^2 &= 4(y+2)^2 \\
x^2 + y^2 - 8y + 16 &= 4(y^2 + 4y + 4) \\
x^2 + y^2 - 8y + 16 &= 4y^2 + 16y + 16 \\
x^2 - 3y^2 - 24y &= 0
\end{aligned}
$$

Completing the square then gives,

$$x^2 - 3y^2 - 24y = 0$$
$$x^2 - 3[y^2 + 8y] = 0$$
$$x^2 - 3[y^2 + 8y + 16 - 16] = 0$$
$$x^2 - 3[(y+4)^2 - 16] = 0$$
$$3(y+4)^2 - x^2 = 48$$
$$\frac{(y+4)^2}{16} - \frac{x^2}{48} = 1$$

Therefore, the set of points is a hyperbola.

6 **a** Since $x = 10t$, we know that $t = \dfrac{x}{10}$. We substitute this into the second equation to give

$$\begin{aligned} y &= 20t - 5t^2 \\ &= 20\left(\frac{x}{10}\right) - 5\left(\frac{x}{10}\right)^2 \\ &= 2x - 5\frac{x^2}{100} \\ &= 2x - \frac{x^2}{20} \end{aligned}$$

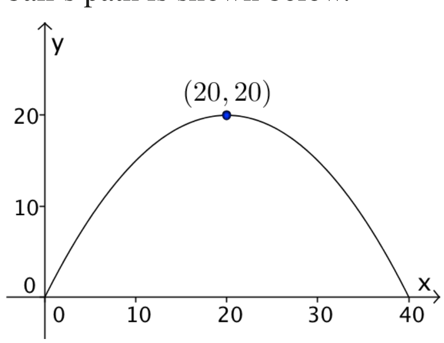
It will also help later if we consider the factorised expression. That is,

$$y = \frac{1}{20}x(40 - x).$$

b The equation of the ball's path is

$$y = \frac{1}{20}x(40 - x).$$

We note that the x-intercepts are $x = 0, 40$. The turning point will be located half-way between at $x = 20$. When $x = 20$, we find that $y = 20$. The graph of the ball's path is shown below.

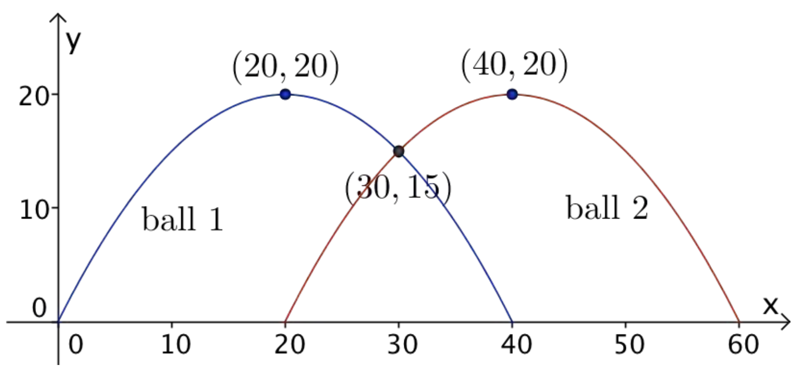


c The maximum height reached by the balls is 20 metres, and occurs when $x = 20$.

d Since $x = 10t$, we know that $t = \dfrac{60 - x}{10}$. We substitute this into the second equation to give

$$
\begin{aligned}
y &= 20t - 5t^2 \\
&= 20\left(\frac{60 - x}{10}\right) - 5\left(\frac{60 - x}{10}\right)^2 \\
&= 120 - 2x - 5\frac{(60 - x)^2}{100} \\
&= 120 - 2x - \frac{(60 - x)^2}{20} \\
&= 120 - 2x - \frac{(3600 - 120x + x^2)^2}{20} \\
&= 120 - 2x - 180 + 6x - \frac{x^2}{20} \\
&= 4x - 60 - \frac{x^2}{20} \\
&= -\frac{1}{20}(x^2 - 80x + 1200) \\
&= -\frac{1}{20}(x - 20)(x - 60)
\end{aligned}
$$

e The second ball's path has been included on the diagram below. The point of intersection has been identified in the following question.



f To find where the paths meet, we solve the following pair of equations simultaneously (or using your calculator),

$$y = -\frac{1}{20}(x - 20)(x - 60) \quad (1)$$

$$y = \frac{1}{20}x(40 - x) \quad (2)$$

This gives a solution of $x = 30$ and $y = 15$.

g Note: just because the paths cross does *not* automatically mean that the balls collide. For this to happen, they must be at the same point at the same *time*. For the first ball, when $x = 30$, we find that

$$t = \frac{x}{10} = \frac{30}{10} = 3.$$

For the second ball, when $x = 30$, we find that

$$t = \frac{60 - x}{10} = \frac{60 - 30}{10} = 3.$$

So the balls are at the same position at the same time. Therefore, they collide.

7 a First $OA = \sqrt{1 + m^2}$, $OB = \sqrt{1 + n^2}$, $AB = |m - n|$. If $\angle AOB = 90°$, then by Pythagoras' theorem we find that

$$\begin{aligned}
OA^2 + OB^2 &= AB^2 \\
(1 + m^2) + (1 + n^2) &= (m - n)^2 \\
2 + m^2 + n^2 &= m^2 - 2mn + n^2 \\
2 &= -2mn \\
mn &= -1
\end{aligned}$$

as required.

b Let $\angle AOB = \theta$. Suppose $mn = -1$. Using the cosine rule, we find that

$$\begin{aligned}
AB^2 &= OA^2 + OB^2 - 2 \cdot OA \cdot OB \cdot \cos\theta \\
(m - n)^2 &= (1 + m^2) + (1 + n^2) - 2\sqrt{1 + n^2}\sqrt{1 + m^2} \cdot \cos\theta \\
m^2 - 2mn + n^2 &= 2 + m^2 + n^2 - 2\sqrt{1 + n^2}\sqrt{1 + m^2} \cdot \cos\theta \\
-2mn &= 2 + -2\sqrt{1 + n^2}\sqrt{1 + m^2} \cdot \cos\theta \\
2 &= 2 + -2\sqrt{1 + n^2}\sqrt{1 + m^2} \cdot \cos\theta \\
0 &= 2\sqrt{1 + n^2}\sqrt{1 + m^2} \cdot \cos\theta
\end{aligned}$$

Since $1 + m^2 > 0$ and $1 + n^2 > 0$ we must have $\cos\theta = 0$. Therefore $\theta = 90°$, as required.

c Consider point $P(x, y)$. If $AP \perp BP$, then $m_{AP}m_{BP} = -1$. Therefore

$$\begin{aligned}
\frac{y-4}{x-0} \cdot \frac{y-10}{x-8} &= -1 \\
\frac{(y-4)(y-10)}{x(x-8)} &= -1 \\
y^2 - 14y + 40 &= -x(x-8) \\
y^2 - 14y + x^2 - 8x &= -40 \\
(y^2 - 14y + 49) - 49 + (x^2 - 8x + 16) - 16 &= -40 \\
(y^2 - 7)^2 + (x-4)^2 - 16 &= 25
\end{aligned}$$

8 The ladder is initially vertical with its midpoint located 3 metres up the wall at coordinate $(0, 3)$. The ladder comes to a rest lying horizontally. Its midpoint is located 3 metres to the right of the wall at coordinate $(3, 0)$. So if the midpoint is to move along a circular path then it must be along the circle

$$x^2 + y^2 = 3^2 \quad (1)$$

To check that this is indeed true, we suppose that the ladder is t units from the base of the wall. Then by Pythagoras' theorem, the ladder reaches

$$s = \sqrt{6^2 - t^2} = \sqrt{36 - t^2}$$

units up the wall. The midpoint of the ladder will then be

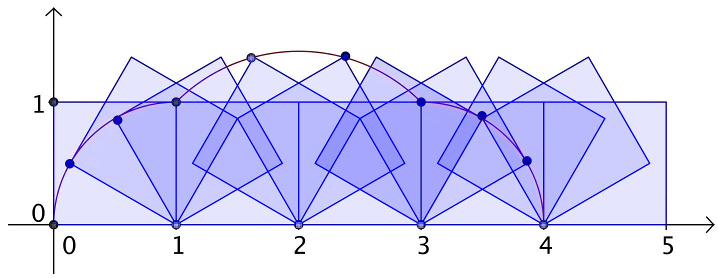
$$P\left(\frac{t}{2}, \frac{\sqrt{36 - t^2}}{2}\right).$$

We just need to check that this point lies on the circle whose equation is (1). Indeed,

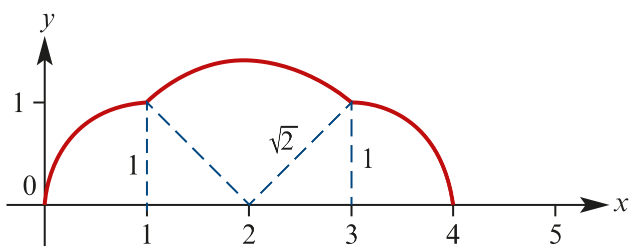
$$\begin{aligned}
x^2 + y^2 &= \left(\frac{t}{2}\right)^2 + \left(\frac{\sqrt{36 - t^2}}{2}\right)^2 \\
&= \frac{t^2}{4} + \frac{36 - t^2}{4} \\
&= \frac{36}{4} \\
&= 9 \\
&= 3^2
\end{aligned}$$

Therefore point P lies on the circle whose equation is $x^2 + y^2 = 3^2$.

9 **a** The (rather complicated) path of the point is shown in red below.



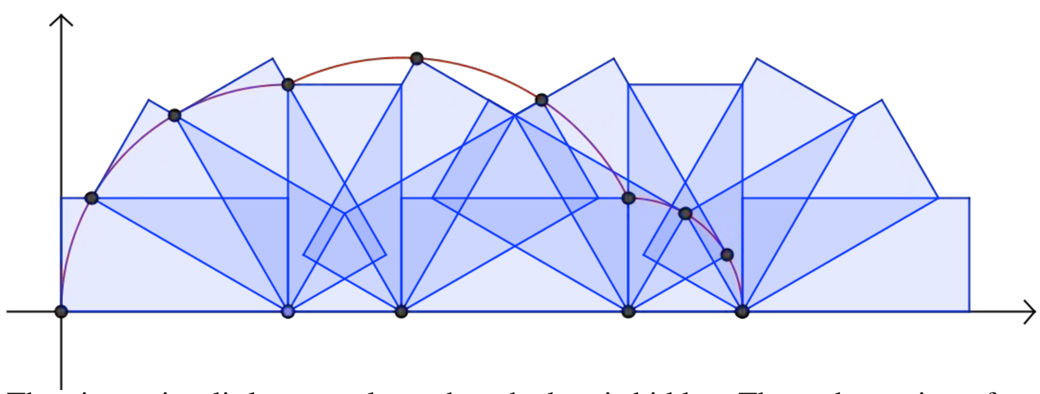
The picture is a little more clear when the box is hidden. The path consists of a quarter circle of of radius 1, another quarter circle of radius $\sqrt{2}$ (the diagonal length) and then another quarter circle of length 1.



b The total distance covered by point P will be

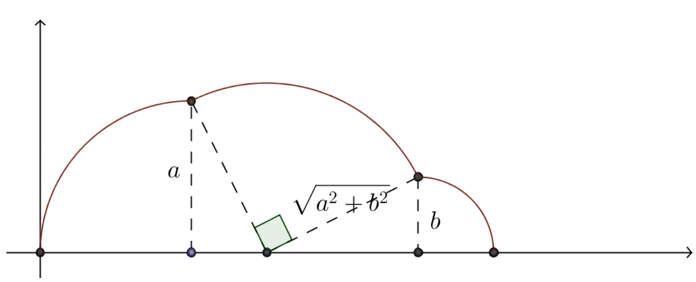
$$D = \frac{1}{4}(2\pi \cdot 1 + 2\pi \cdot \sqrt{2} + 2\pi \cdot 1) = \frac{1}{2}(2\pi + \pi\sqrt{2}).$$

c The (rather complicated) path of the point is shown in red below.



The picture is a little more clear when the box is hidden. The path consists of a quarter circle of of radius a, another quarter circle of radius $\sqrt{a^2 + b^2}$ (the diagonal length) and then another quarter circle of length b.

d



The total distance covered by point P will then be

$$D = \frac{1}{4}(2\pi a + 2\pi \cdot \sqrt{a^2 + b^2} + 2\pi b) = \frac{\pi}{2}(a + \sqrt{a^2 + b^2} + b).$$

The area consists of three quarter circles and two triangles. The total area will then be.

$$\begin{aligned} A &= \frac{1}{4}(\pi a^2 + \pi(\sqrt{a^2 + b^2})^2 + \pi b^2) + 2 \times \frac{1}{2}ab \\ &= \frac{1}{4}(2\pi a^2 + 2\pi b^2) + ab \\ &= \frac{\pi}{2}(a^2 + b^2) + ab. \end{aligned}$$

Chapter 18 – Complex numbers

Solutions to Exercise 18A

1 **a** $\text{Re}(z) = a = 2$

$\text{Im}(z) = b = 3$

b $\text{Re}(z) = a = 4$

$\text{Im}(z) = b = 5$

c $\text{Re}(z) = a = \frac{1}{2}$

$\text{Im}(z) = b = -\frac{3}{2}$

d $\text{Re}(z) = a = -4$

$\text{Im}(z) = b = 0$

e $\text{Re}(z) = a = 0$

$\text{Im}(z) = b = 3$

f $\text{Re}(z) = a = \sqrt{2}$

$\text{Im}(z) = b = -2\sqrt{2}$

2 **a** $2a - 3bi = 4 + 6i$

$2a = 4$

$a = 2$

$-3bi = 6i$

$b = -2$

b $a + b = 5$

$b = 5 - a$

$-2ab = -12$

$ab = 6$

$a(5 - a) = 6$

$5a - a^2 = 6$

$a^2 - 5a + 6 = 0$

$(a - 2)(a - 3) = 0$

When $a = 2$

$b = 5 - 2 = 3$

When $a = 3$

$b = 5 - 3 = 2$

c $2a + bi = 10$

$= 10 + 0i$

$2a = 10$

$a = 5$

$b = 0$

d $3a = 2$

$a = \frac{2}{3}$

$a - b = 1$

$\frac{2}{3} - b = 1$

$b = \frac{2}{3} - 1 = -\frac{1}{3}$

3 **a** $(2 - 3i) + (4 - 5i) = 2 + 4 - 3i - 5i$

$= 6 - 8i$

b $(4 + i) + (2 - 2i) = 4 + 2 + i - 2i$

$= 6 - i$

c $(-3 - i) - (3 + i) = -3 - 3 - i - i$

$= -6 - 2i$

d $(2-\sqrt{2}i)+(5-\sqrt{8}i)$

$= 2+5-\sqrt{2}i-\sqrt{8}i$

$= 7-\sqrt{2}i-2\sqrt{2}i$

$= 7-3\sqrt{2}i$

e $(1-i)-(2i+3) = 1-3-i-2i$

$= -2-3i$

f $(2+i)-(-2-i) = 2+2+i+i$

$= 4+2i$

g $4(2-3i)-(2-8i)$

$= 8-2-12i+8i$

$= 6-4i$

h $-(5-4i)+(1+2i)$

$= -5+1+4i+2i$

$= -4+6i$

i $5(i+4)+3(2i-7)$

$= 20-21+5i+6i$

$= -1+11i$

j $\frac{1}{2}(4-3i)-\frac{3}{2}(2-i)$

$= 2-3-\frac{3}{2}i+\frac{3}{2}i$

$= -1$

4 a $\sqrt{-16} = \sqrt{16\times -1}$

$= 4i$

b $2\sqrt{-9} = 2\sqrt{9\times -1}$

$= 6i$

c $\sqrt{-2} = \sqrt{2\times -1}$

$= \sqrt{2}i$

d $i^3 = i^2 \times i$

$= -i$

e $i^{14} = i^{4\times 3+2}$

$= -1$

f $i^{20} = i^{4\times 5}$

$= 1$

g $-2i\times i^3 = -2i^4$

$= -2$

h $4i^4\times 3i^2 = 4\times 3\times i^4\times i^2$

$= 12i^6$

$= -12$

i $\sqrt{8}i^5\times\sqrt{-2} = \sqrt{8}i^4\times i\times\sqrt{2}i$

$= \sqrt{16}\times 1\times -1$

$= -4$

5 a $i(2-i) = 2i-i^2$

$= 2i-(-1)$

$= 1+2i$

b $i^2(3-4i) = -1(3-4i)$

$= -3+4i$

c $\sqrt{2}i(i-\sqrt{2}) = \sqrt{2}i^2-2i$

$= -\sqrt{2}-2i$

d $-\sqrt{3}(\sqrt{-3}+\sqrt{2}) = -\sqrt{3}(\sqrt{3}i+\sqrt{2})$

$= -3i-\sqrt{6}$

$= -\sqrt{6}-3i$

Solutions to Exercise 18B

1 a $(4+i)^2 = 16 + 8i + i^2$
$= 15 + 8i$

b $(2-2i)^2 = 4 - 8i + 4i^2$
$= -8i$

c $(3+2i)(2+4i) = 6 + 12i + 4i + 8i^2$
$= -2 + 16i$

d $(-1-i)^2 = 1 + 2i + i^2$
$= 2i$

e $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i) = 2 - 3i^2$
$= 2 + 3$
$= 5$

f $(5-2i)(-2+3i)$
$= -10 + 15i + 4i - 6i^2$
$= -4 + 19i$

2 a We first expand to find that

$wz = (3+2i)(2+4i)$
$= 6 + 12i + 4i + 8i^2$
$= 6 + 16i - 8$
$= -2 + 16i$

Therefore, $\text{Re}(wz) = -2$

b We first expand to find that

$wz = (4+5i)(3-2i)$
$= 12 - 8i + 15i - 10i^2$
$= 12 - 8i + 15i + 10$
$= 22 + 7i$

Therefore, $\text{Im}(wz) = 7$

3 a $z = 2 - 5i$
$\bar{z} = 2 + 5i$

b $z = -1 + 3i$
$\bar{z} = -1 - 3i$

c $z = \sqrt{5} - 2i$
$\bar{z} = \sqrt{5} + 2i$

d $z = 0 - 5i$
$\bar{z} = 0 + 5i = 5i$

4 a $z \cdot \bar{z} = (3+4i)(3-4i)$
$= 3^2 - (4i)^2$
$= 9 + 16$
$= 26$

b $z \cdot \bar{z} = (1+i)(1-i)$
$= 1^2 - i^2$
$= 1 + 1$
$= 2$

c $z \cdot \bar{z} = (2-3i)(2+3i)$
$= 2^2 - (3i)^2$
$= 4 + 9$
$= 13$

d $z \cdot \bar{z} = (\sqrt{2} + \sqrt{3}i)(\sqrt{2} - \sqrt{3}i)$
$= (\sqrt{2})^2 - (\sqrt{3}i)^2$
$= 2 + 3$
$= 5$

5 a $\bar{z}_1 = 2 + i$

b $\bar{z}_2 = -3 - 2i$

c $z_1 z_2 = (2 - i)(-3 + 2i)$

$= -6 + 4i + 3i - 2i^2$

$= -4 + 7i$

d $\overline{z_1 z_2} = -4 - 7i$

e $\bar{z}_1 \bar{z}_2 = (2 + i)(-3 - 2i)$

$= -6 - 4i - 3i - 2i^2$

$= -4 - 7i$

f $z_1 + z_2 = (2 - i) + (-3 + 2i)$

$= -1 + i$

g $\overline{z_1 + z_2} = -1 - i$

h $\bar{z}_1 + \bar{z}_2 = (2 + i) + (-3 - 2i)$

$= -1 - i$

6 **a** $|wz| = |(1 + i)(3 - 4i)|$

$= |7 - i|$

$= \sqrt{7^2 + (-1)^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$

b $|w||z| = |1 + i||3 - 4i|$

$= \sqrt{1^2 + 1^2}\sqrt{3^2 + (-4)^2}$

$= \sqrt{2}\sqrt{25}$

$= 5\sqrt{2}$

c $|w + z| = |(1 + i) + (3 - 4i)|$

$= |4 - 3i|$

$= \sqrt{4^2 + (-3)^2}$

$= \sqrt{25}$

$= 5$

d $|3w - 2z| = |3(1 + i) - 2(3 - 4i)|$

$= |-3 + 11i|$

$= \sqrt{(-3)^2 + 11^2}$

$= \sqrt{130}$

7 **a** $\bar{z} = 2 + 4i$

b $z\bar{z} = (2 - 4i)(2 + 4i)$

$= 4 - 16i^2$

$= 20$

c $z + \bar{z} = (2 - 4i) + (2 + 4i)$

$= 4$

d $z(z + \bar{z}) = 4z$

$= 8 - 16i$

e $z - \bar{z} = (2 - 4i) - (2 + 4i)$

$= -8i$

f $i(z - \bar{z}) = i \times -8i$

$= -8i^2 = 8$

g $z^{-1} = \dfrac{1}{2-4i}$

$= \dfrac{1}{2-4i} \times \dfrac{2+4i}{2+4i}$

$= \dfrac{2+4i}{4-16i^2}$

$= \dfrac{2+4i}{20}$

$= \dfrac{1}{10}(1+2i)$

h $\dfrac{z}{i} = \dfrac{z}{i} \times \dfrac{i}{i}$

$= \dfrac{i(2-4i)}{-1}$

$= -1 \times (2i - 4i^2)$

$= -4 - 2i$

8 $(a+bi)(2+5i) = 2a + 5ai + 2bi - 5b$

$= 3 - i$

$2a - 5b = 3$

$5a + 2b = -1$

Multiply the first equation by 2 and the second equation by 5.

$4a - 10b = 6$ ①

$25a + 10b = -5$ ②

① + ②:

$29a = 1$

$a = \dfrac{1}{29}$

$\dfrac{2}{29} - 5b = 3$

$5b = \dfrac{2}{29} - 3$

$= -\dfrac{85}{29}$

$b = -\dfrac{17}{29}$

9 **a** $\dfrac{2-i}{4+1} = \dfrac{2-i}{4+1} \times \dfrac{4-i}{4-i}$

$= \dfrac{8 - 2i - 4i + i^2}{16 - i^2}$

$= \dfrac{7-6i}{17}$

$= \dfrac{7}{17} - \dfrac{6}{17}i$

b $\dfrac{3+2i}{2-3i} = \dfrac{3+2i}{2-3i} \times \dfrac{2+3i}{2+3i}$

$= \dfrac{6 + 9i + 4i + 6i^2}{4 - 9i^2}$

$= \dfrac{13i}{13} = i$

c $\dfrac{4+3i}{1+i} = \dfrac{4+3i}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{4 - 4i + 3i - 3i^2}{1 - i^2}$

$= \dfrac{7-i}{2}$

$= \dfrac{7}{2} - \dfrac{1}{2}i$

d $\dfrac{2-2i}{4i} = \dfrac{2-2i}{4i} \times \dfrac{i}{i}$

$= \dfrac{2i-2i^2}{-4}$

$= \dfrac{2+2i}{-4}$

$= \dfrac{-1-i}{2}$

$= -\dfrac{1}{2} - \dfrac{1}{2}i$

e $\dfrac{1}{2-3i} = \dfrac{1}{2-3i} \times \dfrac{2+3i}{2+3i}$

$= \dfrac{2+3i}{4-9i^2}$

$= \dfrac{2+3i}{13}$

$= \dfrac{2}{13} + \dfrac{3}{13}i$

f $\dfrac{i}{2+6i} = \dfrac{i}{2+6i} \times \dfrac{2-6i}{2-6i}$

$= \dfrac{2i+6}{4-36i^2}$

$= \dfrac{2i+6}{40}$

$= \dfrac{3}{20} + \dfrac{1}{20}i$

10 $(3-i)(a+bi) = 3a + 3bi - ai + b$

$= 6 - 7i$

$3a + b = 6$ ①

$-a + 3b = -7$

$-3a + 9b = -21$ ②

① + ②:

$10b = -15$

$b = -\dfrac{3}{2}$

$3a - \dfrac{3}{2} = 6$

$3a = 6 + \dfrac{3}{2} = \dfrac{15}{2}$

$a = \dfrac{5}{2}$

11 a $z = \dfrac{42i}{2-i}$

$= \dfrac{42i}{2+i} \times \dfrac{2+i}{2+i}$

$= \dfrac{84i + 42i^2}{4-i^2}$

$= \dfrac{-42+84i}{5}$

$= -\dfrac{42}{5} + \dfrac{84i}{5}$

b $z = \dfrac{-2-i}{1+3i}$

$= \dfrac{-2-i}{1+3i} \times \dfrac{1-3i}{1-3i}$

$= \dfrac{-2+6i-i+3i^2}{1-9i^2}$

$= \dfrac{-5+5i}{10}$

$= -\dfrac{1}{2}(1-i)$

c $z = \dfrac{1+i}{5+3i}$

$= \dfrac{1+i}{5+3i} \times \dfrac{5-3i}{5-3i}$

$= \dfrac{5-3i+5i-3i^2}{25-9i^2}$

$= \dfrac{8+2i}{34}$

$= \dfrac{1}{17}(4+i)$

d $z = \dfrac{5+2i}{2(4-7i)}$

$= \dfrac{5+2i}{2(4-7i)} \times \dfrac{4+7i}{4+7i}$

$= \dfrac{20+35i+8i+14i^2}{2(16-49i^2)}$

$= \dfrac{6+43i}{130}$

$= \dfrac{1}{130}(6+43i)$

e $z = \dfrac{4}{1+i}$

$= \dfrac{4}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{4-4i}{1-i^2}$

$= \dfrac{4-4i}{2}$

$= 2-2i$

12 Expanding the left-hand side gives

$$(a+bi)^2 = -5+12i$$
$$a^2+2abi+(bi)^2 = -5+12i$$
$$a^2+2abi-b^2 = -5+12i$$
$$(a^2-b^2)+2abi = -5+12i$$

Equating the real and imaginary parts on both sides gives

$a^2-b^2 = -5$ and $2ab = 12$.

We see that $b = \dfrac{6}{a}$. Substituting this into the first equation and solving gives

$$a^2-b^2 = -7$$
$$a^2 - \frac{36}{a^2} = -5$$
$$a^4 - 36 = -5a^2$$
$$a^4+5a^2-36 = 0$$
$$(a^2+9)(a^2-4) = 0$$
$$(a^2+9)(a-2)(a+2) = 0$$

If $a = 2$ then $b = -3$. If $a = -2$, then $b = 3$.

13 Simplifying the left-hand side gives

$\dfrac{1}{a+3i} + \dfrac{1}{a-3i}$

$= \dfrac{a-3i}{(a+3i)(a-3i)} + \dfrac{a+3i}{(a+3i)(a-3i)}$

$= \dfrac{a-3i}{a^3+9} + \dfrac{a+3i}{a^2+9}$

$= \dfrac{2a}{a^2+9}$

Therefore,

$$\frac{2a}{a^2+9} = \frac{4}{13}$$
$$\frac{a}{a^2+9} = \frac{2}{13}$$
$$13a = 2a^2+18$$
$$2a^2-13a+18 = 0$$
$$(a-2)(2a-9) = 0$$
$$a = 2, \frac{9}{2}$$

14 **a** If $z = \bar{z}$ then

$a+bi = a-bi$

$\Rightarrow 2bi = 0$

$\Rightarrow b = 0.$

Therefore, z is a real number.

b We find that

$$\begin{aligned} z+\overline{z} &= (a+bi)+(a-bi) \\ &= a+bi+a-bi \\ &= 2a \in \mathbb{R}. \end{aligned}$$

c We find that

$$\begin{aligned} \frac{1}{z}+\frac{1}{\overline{z}} &= \frac{1}{a+bi}+\frac{1}{a-bi} \\ &= \frac{(a-bi)+(a+bi)}{(a+bi)(a-ib)} \\ &= \frac{2a}{a^2+b^2} \in \mathbb{R}. \end{aligned}$$

15 We note that the denominator of z is the conjugate of the numerator of z. Therefore, it will help to let $w = a + bi$ so that $\overline{w} = a - ib$ and $z = \dfrac{w}{\overline{w}}$. Therefore

$$\begin{aligned} \frac{z^2+1}{2z} &= \frac{\left(\dfrac{w}{\overline{w}}\right)^2+1}{2\dfrac{w}{\overline{w}}} \\ &= \frac{\dfrac{w^2}{\overline{w}^2}+1}{2\dfrac{w}{\overline{w}}} \\ &= \frac{w^2+\overline{w}^2}{2w\overline{w}} \\ &= \frac{(a+ib)^2+(a-ib)^2}{2(a^2+b^2)} \\ &= \frac{(a^2+2abi-b^2)+(a^2-2abi-b^2)}{2(a^2+b^2)} \\ &= \frac{2(a^2-b^2)}{2(a^2+b^2)} \\ &= \frac{a^2-b^2}{a^2+b^2}, \end{aligned}$$

which is a real number.

Solutions to Exercise 18C

1 $A = 3 + i$

$B = 2i$

$C = -3 - 4i$

$D = 2 - 2i$

$E = -3$

$F = -1 - i$

2 $A = 3 - 4i$

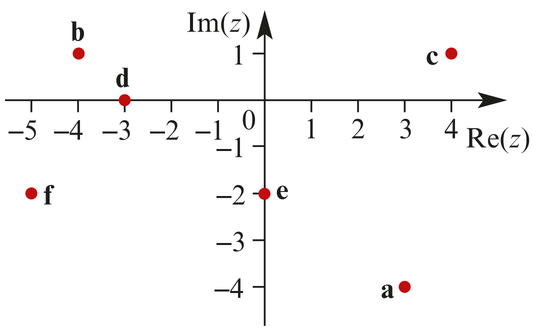
$B = -4 + i$

$C = 4 + i$

$D = -3$

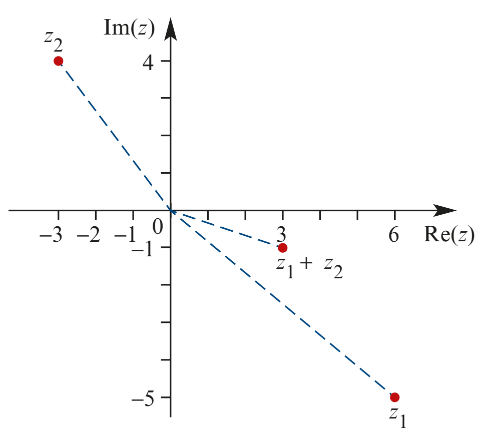
$E = -2i$

$F = -5 - 2i$



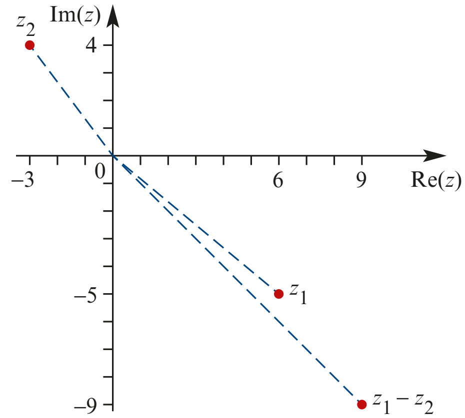
3 **a** $z_1 + z_2 = (6 - 5i) + (-3 + 4i)$

$= 3 - i$



b $z_1 - z_2 = (6 - 5i) - (-3 + 4i)$

$= 9 - 9i$



4 **a** $A : z = 1 + 3i$

b $B : \bar{z} = 1 - 3i$

c $C : z^2 = 1 + 6i + 9i^2$

$= -8 + 6i$

d $D : -z = -(1 + 3i)$

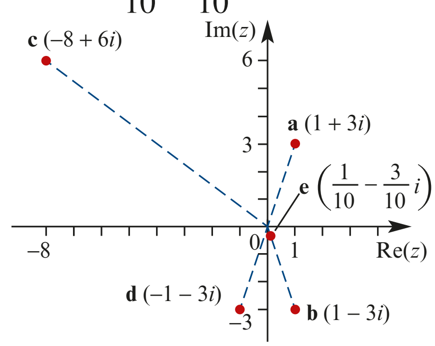
$= -1 - 3i$

e $E : \dfrac{1}{z} = \dfrac{1}{1 + 3i}$

$= \dfrac{1}{1 + 3i} \times \dfrac{1 - 3i}{1 - 3i}$

$= \dfrac{1 - 3i}{1 + 9i^2}$

$= \dfrac{1}{10} - \dfrac{3}{10}i$



5 **a** $A : z = 2 - 5i$

b $B : zi = i(2 - 5i)$

$$= 2i - 5i^2$$

$$= 5 + 2i$$

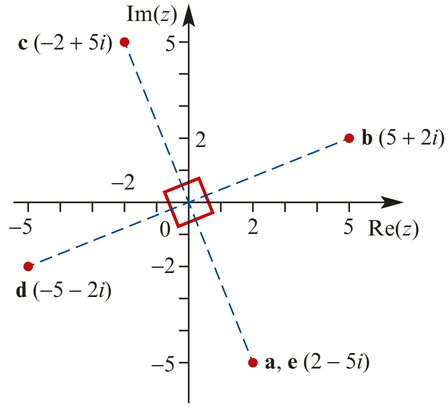
c $C : zi^2 = -z = -2 + 5i$

d $D : zi^3 = -iz$

$$= -i(2 - 5i)$$

$$= -5 - 2i$$

e $E : zi^4 = z = 2 - 5i$



Solutions to Exercise 18D

1 **a**
$$\begin{aligned} z^2 + 1 &= 0 \\ z^2 - i^2 &= 0 \\ (z - i)(z + i) &= 0 \\ \Rightarrow z &= \pm i \end{aligned}$$

b
$$\begin{aligned} z^2 + 9 &= 0 \\ z^2 - (3i)^2 &= 0 \\ (z - 3i)(z + 3i) &= 0 \\ \Rightarrow z &= \pm 3i \end{aligned}$$

c
$$\begin{aligned} z^2 + 16 &= 0 \\ z^2 - (4i)^2 &= 0 \\ (z - 4i)(z + 4i) &= 0 \\ \Rightarrow z &= \pm 4i \end{aligned}$$

d
$$\begin{aligned} 4z^2 + 25 &= 0 \\ (2z)^2 - (5i)^2 &= 0 \\ (2z - 5i)(2z + 5i) &= 0 \\ \Rightarrow z &= \pm\frac{5i}{2} \end{aligned}$$

e
$$\begin{aligned} z^2 + 2 &= 0 \\ z^2 - (\sqrt{2}i)^2 &= 0 \\ (z - \sqrt{2}i)(z + \sqrt{2}i) &= 0 \\ \Rightarrow z &= \pm\sqrt{2}i \end{aligned}$$

f
$$\begin{aligned} 2(z^2 + 4) &= 0 \\ 2(z^2 - (2i)^2) &= 0 \\ 2(z - 2i)(z + 2i) &= 0 \\ \Rightarrow z &= \pm 2i \end{aligned}$$

g
$$\begin{aligned} 3(z^2 + 25) &= 0 \\ 2(z^2 - (5i)^2) &= 0 \\ 2(z - 5i)(z + 5i) &= 0 \\ \Rightarrow z &= \pm 5i \end{aligned}$$

h
$$\begin{aligned} 4z^2 + 1 &= 0 \\ (2z)^2 - i^2 &= 0 \\ (2z - i)(2z + i) &= 0 \\ \Rightarrow z &= \pm\frac{i}{2} \end{aligned}$$

i
$$\begin{aligned} 16z^2 + 9 &= 0 \\ (4z)^2 - (3i)^2 &= 0 \\ (4z - 3i)(4z + 3i) &= 0 \\ \Rightarrow z &= \pm\frac{3i}{4} \end{aligned}$$

j
$$\begin{aligned} z^2 + 3 &= 0 \\ z^2 - (\sqrt{3}i)^2 &= 0 \\ (z - \sqrt{3}i)(z + \sqrt{3}i) &= 0 \\ \Rightarrow z &= \pm\sqrt{3}i \end{aligned}$$

k
$$\begin{aligned} 2z^2 + 10 &= 0 \\ 2(z^2 + 5) &= 0 \\ 2(z^2 - (\sqrt{5}i)^2) &= 0 \\ 2(z - \sqrt{5}i)(z + \sqrt{5}i) &= 0 \\ \Rightarrow z &= \pm\sqrt{5}i \end{aligned}$$

l
$$\begin{aligned} (z + 1)^2 + 1 &= 0 \\ (z + 1)^2 - i^2 &= 0 \\ (z + 1 - i)(z + 1 + i) &= 0 \\ \Rightarrow z &= -1 \pm i \end{aligned}$$

m
$$\begin{aligned}(z-2)^2+5&=0\\(z-2)^2-(\sqrt{5}i)^2&=0\\(z-2-\sqrt{5}i)(z-2+\sqrt{5}i)&=0\\\Rightarrow z&=2\pm\sqrt{5}i\end{aligned}$$

n
$$\begin{aligned}(z+3)^2+3&=0\\(z+3)^2-(\sqrt{3}i)^2&=0\\(z+3-\sqrt{3}i)(z+3+\sqrt{3}i)&=0\\\Rightarrow z&=-3\pm\sqrt{3}i\end{aligned}$$

o
$$\begin{aligned}(z-2)^2+4&=0\\(z-2)^2-(2i)^2&=0\\(z-2-2i)(z-2+2i)&=0\\\Rightarrow z&=2\pm 2i\end{aligned}$$

2 a
$$\begin{aligned}z^2+2z+3&=0\\(z^2+2z+1)-1+3&=0\\(z+1)^2+2&=0\\(z+1)^2-(\sqrt{2}i)^2&=0\\(z+1-\sqrt{2})(z+1+\sqrt{2}i)&=0\end{aligned}$$
The solutions are $z=-1\pm\sqrt{2}i$

b
$$\begin{aligned}z^2-4z+5&=0\\(z^2-4z+4)-4+5&=0\\(z-2)^2+1&=0\\(z-2)^2-i^2&=0\\(z-2-i)(z-2+i)&=0\end{aligned}$$
The solutions are $z=2\pm i$

c
$$\begin{aligned}z^2+6z+12&=0\\(z^2+6z+9)-9+12&=0\\(z+3)^2+3&=0\\(z+3)^2-(\sqrt{3}i)^2&=0\\(z+3-\sqrt{3}i)(z+3+\sqrt{3}i)&=0\end{aligned}$$
The solutions are $z=-3\pm\sqrt{3}i$

d
$$\begin{aligned}2z^2-8z+10&=0\\z^2-4z+5&=0\\(z^2-4z+4)-4+5&=0\\(z-2)^2+1&=0\\(z-2)^2-i^2&=0\\(z-2-i)(z-2+i)&=0\end{aligned}$$
The solutions are $z=2\pm i$

e
$$\begin{aligned}3z^2+2z+1&=0\\z^2+z+\tfrac{2}{3}z+\tfrac{1}{3}&=0\\z^2+z+\tfrac{2}{3}z+\tfrac{1}{9}-\tfrac{1}{9}+\tfrac{1}{3}&=0\\(z+\tfrac{1}{3})^2+\tfrac{2}{9}&=0\\(z+\tfrac{1}{3})^2-(\tfrac{\sqrt{2}}{3}i)^2&=0\\(z+\tfrac{1}{3}-\tfrac{\sqrt{2}}{3}i)(z+\tfrac{1}{3}+\tfrac{\sqrt{2}}{3}i)&=0\end{aligned}$$
The solutions are $z=-\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$

f
$$\begin{aligned}2z^2+2z+1&=0\\z^2+z+\tfrac{1}{2}&=0\\(z^2+z+\tfrac{1}{4})-\tfrac{1}{4}+\tfrac{4}{4}&=0\\(z+\tfrac{1}{2})^2+\tfrac{3}{4}&=0\\(z+\tfrac{1}{2})^2-(\tfrac{\sqrt{3}i}{2})^2&=0\\(z+\tfrac{1}{2}-\tfrac{\sqrt{3}i}{2})(z+\tfrac{1}{2}+\tfrac{\sqrt{3}i}{2})&=0\end{aligned}$$
The solutions are $z=-\frac{1}{2}\pm\frac{\sqrt{3}i}{2}$

3 a $z = \dfrac{-3 \pm \sqrt{9-12}}{2}$

$= \dfrac{-3 \pm \sqrt{-3}}{2}$

$= \dfrac{-3 \pm \sqrt{3}i}{2}$

b $z = \dfrac{4 \pm \sqrt{16-20}}{2}$

$= \dfrac{4 \pm \sqrt{-4}}{2}$

$= \dfrac{4 \pm 2i}{2}$

$= 2 \pm i$

c $z = \dfrac{-6 \pm \sqrt{36-48}}{2}$

$= \dfrac{-6 \pm \sqrt{-12}}{2}$

$= \dfrac{-6 \pm 2\sqrt{3}i}{2}$

$= -3 \pm \sqrt{3}i$

d $z = \dfrac{4 \pm \sqrt{16-32}}{2}$

$= \dfrac{4 \pm \sqrt{-16}}{2}$

$= \dfrac{4 \pm 4i}{2}$

$= 2 \pm 2i$

e $z = \dfrac{-2 \pm \sqrt{4-12}}{6}$

$= \dfrac{-2 \pm \sqrt{-8}}{6}$

$= \dfrac{-2 \pm 2\sqrt{2}i}{6}$

$= \dfrac{-1 \pm \sqrt{2}i}{3}$

f $z = \dfrac{\sqrt{2} \pm \sqrt{2-8}}{4}$

$= \dfrac{\sqrt{2} \pm \sqrt{-6}}{4}$

$= \dfrac{\sqrt{2} \pm \sqrt{6}i}{4}$

4 a
$$\begin{aligned} z^2 + 4 &= 0 \\ z^2 - 4i^2 &= 0 \\ (z-2i)(z+2i) &= 0 \\ z &= \pm 2i \end{aligned}$$

b
$$\begin{aligned} 2x^2 + 18 &= 0 \\ z^2 + 9 &= 0 \\ z^2 - 9i^2 &= 0 \\ (z-3i)(z+3i) &= 0 \\ z &= \pm 3i \end{aligned}$$

c
$$\begin{aligned} 3z^2 + 15 &= 0 \\ z^2 + 5 &= 0 \\ z^2 - 5i^2 &= 0 \\ (z-\sqrt{5}i)(z+\sqrt{5}i) &= 0 \\ z &= \pm\sqrt{5}i \end{aligned}$$

d $(z-2)^2 = -16$

$z - 2 = \pm 4i$

$z = 2 \pm 4i$

e $(z+1)^2 = -49$

$z + 1 = \pm 7i$

$z = -1 \pm 7i$

f Complete the square.

$z^2 - 2z + 1 + 2 = 0$

$(z-1)^2 - 2i^2 = 0$

$(z - 1 - \sqrt{2}i)(z - 1 + \sqrt{2}i) = 0$

$z = 1 \pm \sqrt{2}i$

g Use the quadratic formula.

$$\begin{aligned} z &= \frac{-3 \pm \sqrt{9-12}}{2} \\ &= \frac{-3 \pm \sqrt{-3}}{2} \\ &= \frac{1}{2}(-3 \pm \sqrt{3}i) \end{aligned}$$

h Use the quadratic formula.

$$\begin{aligned} z &= \frac{-5 \pm \sqrt{25-32}}{4} \\ &= \frac{-5 \pm \sqrt{-7}}{4} \\ &= \frac{1}{4}(-5 \pm \sqrt{7}i) \end{aligned}$$

i Use the quadratic formula.

$3z^2 - z + 2 = 0$

$$\begin{aligned} z &= \frac{1 \pm \sqrt{1-24}}{6} \\ &= \frac{1 \pm \sqrt{-23}}{6} \\ &= \frac{1}{6}(1 \pm \sqrt{23}i) \end{aligned}$$

j Complete the square.

$$\begin{aligned} z^2 - 2z + 5 &= 0 \\ (z - 1 - 2i)(z - 1 + 2i) &= 0 \\ z &= 1 \pm 2i \end{aligned}$$

k Use the quadratic formula.

$2z^2 - 6z + 10 = 0$

$z^2 - 3z + 5 = 0$

$$\begin{aligned} z &= \frac{3 \pm \sqrt{9-20}}{2} \\ &= \frac{3 \pm \sqrt{-11}}{2} \\ &= \frac{1}{2}(3 \pm \sqrt{11}i) \end{aligned}$$

l Complete the square.

$$\begin{aligned} z^2 - 6z + 14 &= 0 \\ z^2 - 6z + 9 + 5 &= 0 \\ (z-3)^2 - 5i^2 &= 0 \\ (z - 3 - \sqrt{5}i)(z - 3 + \sqrt{5}i) &= 0 \\ z &= 3 \pm \sqrt{5}i \end{aligned}$$

5 **a** $(z - (1+i))(z - (1-i))$

$= (z - 1 - i))(z - 1 + i)$

$= (z-1)^2 - i^2$

$= z^2 - 2z + 1 + 1$

$= z^2 - 2z + 2$

Therefore $a = 1, b = -2, c = 2$.

b $(z - (-2 - 5i))(z - (-2 + 5i))$

$= (z + 2 + 5i)(z + 2 - 5i)$

$= (z+2)^2 - (5i)^2$

$= z^2 + 4z + 4 + 25$

$= z^2 + 4z + 29$

Therefore $a = 1, b = 4, c = 29$.

6 If a, b, c are consecutive positive integers then we can write these as $n-1, n, n+1$ where $n \geq 2$. Looking to the discriminant of the equation

$(n-1)z^2 + nz + (n+1) = 0$ gives

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= n^2 - 4(n-1)(n+1) \\
&= n^2 - 4(n-1)(n+1) \\
&= n^2 - 4(n^2 - 1) \\
&= 4 - 3n^2 \\
&< 0
\end{aligned}$$

since $n \geq 2$. Therefore this quadratic equation does not have real solutions.

Solutions to Exercise 18E

1 Let $P(z) = z^3 + 2z^2 - 3z - 10$ so that

$$\begin{aligned} P(2) &= 2^3 + 2(2)^2 - 3(2) - 10 \\ &= 8 + 8 - 6 - 10 \\ &= 0. \end{aligned}$$

Therefore $(z - 2)$ is a factor. Either by inspection, or polynomial division, we find that

$$\begin{aligned} &z^3 + 2z^2 - 3z - 10 \\ &= (z-2)(z^2 + 4z + 5) \\ &= (z-2)((z^2 + 4z + 4) - 4 + 5) \\ &= (z-2)((z+2)^2 + 1) \\ &= (z-2)((z+2)^2 - i^2) \\ &= (z-2)(z+2-i)(z+2+i). \end{aligned}$$

Therefore, if $P(z) = 0$, then $z = 2, -2 \pm i$.

2 Let $P(z) = z^3 + 3z^2 + 4z + 2$ so that

$$\begin{aligned} P(-1) &= (-1)^3 + 3(-1)^2 + 4(-1) + 2 \\ &= -1 + 3 - 4 + 2 \\ &= 0. \end{aligned}$$

Therefore $(z + 1)$ is a factor. Either by inspection, or polynomial division, we find that

$$\begin{aligned} &z^3 + 3z^2 + 4z + 2 \\ &= (z+1)(z^2 + 2z + 2) \\ &= (z+1)((z^2 + 2z + 1) - 1 + 2) \\ &= (z+1)((z+1)^2 + 1) \\ &= (z+1)((z+1)^2 - i^2) \\ &= (z+1)(z+1-i)(z+1+i). \end{aligned}$$

Therefore, if $P(z) = 0$, then $z = 1, 1 \pm i$.

3 We are given that $z = 3 - 2i$ is a solution. As the polynomial has real coefficients, the complex conjugate $\bar{z} = 3 + 2i$ will also be a solution. Therefore the polynomial has monic factors $(z - 3 + 2i)$ and $(z - 3 - 2i)$. The product of these will also be a factor:

$$\begin{aligned} &(z - 3 + 2i)(z - 3 - 2i) \\ &= ((z-3) + 2i)((z-3) - 2i) \\ &= (z-3)^2 - (2i)^2 \\ &= z^2 - 6z + 9 + 4 \\ &= z^2 - 6z + 13 \end{aligned}$$

The remaining factor can be found by inspection, or by polynomial division. This gives,

$$z^3 - 9z^2 + 31z - 39 = (z-3)(z^2 - 6z + 13).$$

All three solutions are then $z = 3, 3 \pm 2i$.

4 We are given that $z = 1 - \sqrt{2}i$ is a solution. As the polynomial has real coefficients, the complex conjugate $\bar{z} = 1 + \sqrt{2}i$ will also be a solution. Therefore the polynomial has monic factors $(z - 1 + \sqrt{2}i)$ and $(z - 1 - \sqrt{2}i)$. The product of these will also be a factor:

$$\begin{aligned} &(z - 1 + \sqrt{2}i)(z - 1 - \sqrt{2}i) \\ &= ((z-1) + \sqrt{2}i)((z-1) - \sqrt{2}i) \\ &= (z-1)^2 - (\sqrt{2}i)^2 \\ &= z^2 - 2z + 1 + 2 \\ &= z^2 - 2z + 3 \end{aligned}$$

The remaining factor can be found by inspection, or by polynomial division. This gives,

$$z^3 - 4z^2 + 7z - 6 = (z-2)(z^2 - 2z + 3).$$

All three solutions are then
$z = 2, 1 \pm \sqrt{2}i$.

5 Let $P(z) = z^3 - 3z^2 + 4z - 12$ so that

$$\begin{aligned} P(2i) &= (2i)^3 - 3(2i)^2 + 4(2i) - 12 \\ &= -8i + 12 + 8i - 12 \\ &= 0. \end{aligned}$$

Therefore $z = 2i$ is a solution, and $(z - 2i)$ is a factor of the cubic. As the polynomial has real coefficients, $(z + 2i)$ will is also factor. By multiplying these, we find a quadratic factor,

$$\begin{aligned} (z - 2i)(z + 2i) &= z^2 - (\sqrt{2}i)^2 \\ &= z^2 + 4 \end{aligned}$$

The remaining linear factor can be found by inspection, or by polynomial division. This gives,

$$z^3 - 3z^2 + 4z - 12 = (z - 3)(z^2 + 4).$$

All three solutions are then $z = 3, \pm 2i$.

6 Let $P(z) = z^4 + z^3 + 7z^2 + 9z - 18$ so that

$$\begin{aligned} P(3i) &= (3i)^4 + (3i)^3 + 7(3i)^2 + 9(3i) - 18 \\ &= 81 - 27i - 63 + 27i - 18 \\ &= 0. \end{aligned}$$

Therefore $z = 3i$ is a solution and $(z - 3i)$ is a factor of the quartic. As the polynomial has real coefficients, $(z + 3i)$ will is also factor. By multiplying these, we find a quadratic factor,

$$\begin{aligned} (z - 3i)(z + 3i) &= z^2 - (3i)^2 \\ &= z^2 + 9 \end{aligned}$$

The remaining quadratic factor can be found by inspection, or by polynomial division. This gives,

$$\begin{aligned} &z^4 + z^3 + 7z^2 + 9z - 18 \\ &= (z^2 + z - 2)(z^2 + 9) \\ &= (z + 2)(z - 1)(z^2 + 9) \end{aligned}$$

All three solutions are then
$z = -2, 1, \pm 3i$.

7 **a** Let $P(z) = z^3 - z^2 + z - 1$ so that

$$\begin{aligned} P(1) &= 1^3 - 1^2 + 1 - 1 \\ &= 1 - 1 + 1 - 1 \\ &= 0. \end{aligned}$$

Therefore $(z - 1)$ is a factor. Either by inspection, or polynomial division, we find that

$$\begin{aligned} z^3 - z^2 + z - 1 &= (z - 1)(z^2 + 1) \\ &= (z - 1)(z^2 - i^2) \\ &= (z - 1)(z - i)(z + i) \end{aligned}$$

Therefore, if $P(z) = 0$, then $z = 1, \pm i$.

b Let $P(z) = z^3 - z^2 + 3z + 5$ so that

$$\begin{aligned} P(-1) &= (-1)^3 - (-1)^2 + 3(-1) + 5 \\ &= -1 - 1 - 3 + 5 \\ &= 0. \end{aligned}$$

Therefore $(z + 1)$ is a factor. Either by inspection, or polynomial division, we find that

$$\begin{aligned} &z^3 - z^2 + 3z + 5 \\ &= (z + 1)(z^2 - 2z + 5) \\ &= (z + 1)((z^2 - 2z + 1) - 1 + 5) \\ &= (z + 1)((z - 1)^2 + 4) \\ &= (z + 1)((z - 1)^2 - (2i)^2) \\ &= (z + 1)(z - 1 - 2i)(z - 1 + 2i). \end{aligned}$$

Therefore, if $P(z) = 0$, then
$z = 1, 1 \pm 2i$.

c Let $P(z) = z^3 - 2z + 4$ so that

$$\begin{aligned} P(-2) &= (-2)^3 - 2(-2) + 4 \\ &= -8 + 4 + 4 \\ &= 0. \end{aligned}$$

Therefore $(z + 2)$ is a factor. Either by inspection, or polynomial division, we find that

$$\begin{aligned} &z^3 - 2z + 4 \\ &= (z + 2)(z^2 - 2z + 2) \\ &= (z + 2)((z^2 - 2z + 1) - 1 + 2) \\ &= (z + 2)((z - 1)^2 + 1) \\ &= (z + 2)((z - 1)^2 - i^2) \\ &= (z + 2)(z - 1 - i)(z - 1 + i). \end{aligned}$$

Therefore, if $P(z) = 0$, then $z = 2, 1 \pm i$.

d Let $P(z) = z^3 + 3z^2 - 6z - 36$ so that

$$\begin{aligned} P(3) &= 3^3 + 3(3^2) - 6(3) - 36 \\ &= 27 + 27 - 18 - 36 \\ &= 0. \end{aligned}$$

Therefore $(z - 3)$ is a factor. Either by inspection, or polynomial division, we find that

$$\begin{aligned} &z^3 + 3z^2 - 6z - 36 \\ &= (z - 3)(z^2 + 6z + 12) \\ &= (z - 3)((z^2 + 6z + 9) - 9 + 12) \\ &= (z - 3)((z + 3)^2 + 3) \\ &= (z - 3)((z + 3)^2 - (\sqrt{3}i)^2) \\ &= (z - 3)(z + 3 - \sqrt{3}i)(z + 3 + \sqrt{3}i) \end{aligned}$$

Therefore, if $P(z) = 0$, then $z = 3, -3 \pm \sqrt{3}i$.

8 As each of the coefficients are real, since $z_1 = 1 + i$ is a solution so too is $z_3 = 1 - i$. Therefore $(z - 1 - i), (z - 1 + i)$ and $(z - 3)$ are factors of the polynomial. Therefore

$$\begin{aligned} &(z - 1 - i)(z - 1 + i)(z - 3) \\ &= (z^2 - 2z + 1)(z - 3) \\ &= z^3 - 5z^2 + 8z - 6 \end{aligned}$$

in which case $a = -5, b = 8$ and $c = -6$.

9 As each of coefficients are real, $z_1 = 1 - 2i$ and $z_2 = 1 + 2i$ are both roots of this polynomial and $(z - 1 + 2i)$ and $(z - 1 - 2i)$ are both factors. Their product will be a quadratic factor:

$$\begin{aligned} (z - 1 + 2i)(z - 1 - 2i) &= (z - 1)^2 - (2i)^2 \\ &= z^2 - 2z + 1 + 4 \\ &= z^2 - 2z + 5 \end{aligned}$$

All the remains is the final linear factor. By inspection, we see that

$$(2z - 1)(z^2 - 2z + 5) = 2z^3 - 5z^2 + cz - 5.$$

By expanding the left-hand side we find that $c = 12$.

10 **a** The imaginary factors must occur in complex conjugate pairs. So one example will be

$$\begin{aligned} P(z) &= (z - i)(z + i)(z - 2i)(z + 2i) \\ &= (z^2 + 1)(z^2 + 4) \\ &= z^4 + 5z^2 + 4. \end{aligned}$$

b One pair of factors must be complex conjugate pair. So one example will be

$$\begin{aligned} P(z) &= z(z - 1)(z - i)(z + i) \\ &= (z^2 - z)(z^2 + 1) \\ &= z^4 - z^3 + z^2 - z. \end{aligned}$$

11 If z is a solution then $\overline{z}$ is a solution. If z is not a real number, then these two solutions are distinct. This is because if $\overline{z} = z$, then z is a real number. Therefore every cubic has either:

- two non-real solutions of multiplicity 1 and one real solution of multiplicity 1. For example,

$$P(z) = z(z - i)(z + i) = z^3 + z$$

- one real solution of multiplicity 1 and one real solution of multiplicity 2. For example,

$$P(z) = z^2(z - 1) = z^3 - z^2.$$

- one real solution of multiplicity 3. For example,

$$P(z) = z^3$$

- three real solutions of multiplicity 1. For example,

$$P(z) = z(z - 1)(z + 1) = z^3 - z.$$

12 **a** Every equation of degree 4 has 4 solutions, counting multiplicity. Therefore, if $P(z) = 0$ has exactly one real solution z_1, then it has three distinct complex solutions, z_2, z_3 and z_4. As 3 is odd, at least one of these does not belong to a conjugate pair. This contradicts the conjugate root theorem.

b Every equation of degree 4 has 4 solutions, counting multiplicity. Therefore, if $P(z) = 0$ has exactly three real solutions z_1, z_2, z_3, then it has one distinct complex solution, z_4. However, as the coefficients are real, the conjugate of z_4 must also be a solution. The conjugate is not equal to any of z_1, z_2 or z_3, for otherwise z_4 is a real number. Therefore there are more than 4 solutions, which is a contradiction.

13 **a** We will prove that LHS = RHS. We find that

$$\begin{aligned}
\text{LHS} &= \overline{zw}\\
&= \overline{(a + bi)(c + di)}\\
&= \overline{ac + adi + bci + bdi^2}\\
&= \overline{(ac - bd) + (ad + bc)i}\\
&= (ac - bd) - (ad + bc)i
\end{aligned}$$

We also find that

$$\begin{aligned}
\text{RHS} &= \overline{z}\,\overline{w}\\
&= \overline{(a + bi)}\,\overline{(c + di)}\\
&= (a - bi)(c - di)\\
&= ac - adi - bci + bdi^2\\
&= (ac - bd) - (ad + bc)i
\end{aligned}$$

Since LHS = RHS, the proof is complete.

b We will prove that LHS = RHS. We find that

$$\begin{aligned}
\text{LHS} &= \overline{z + w}\\
&= \overline{(a + bi) + (c + di)}\\
&= \overline{(a + c) + (b + d)i}\\
&= (a + c) - (b + d)i
\end{aligned}$$

We also find that

$$\begin{aligned}\text{RHS} &= \overline{w} + \overline{z} \\ &= \overline{(a+bi)} + \overline{(c+di)} \\ &= (a-bi) + (c-di) \\ &= (a+c) - (b+d)i\end{aligned}$$

Since LHS = RHS, the proof is complete.

c We will prove that LHS = RHS. We find that

$$\begin{aligned}\text{LHS} &= \overline{cz} \\ &= \overline{c(a+bi)} \\ &= \overline{(ca) + (cb)i} \\ &= (ca) - (cb)i \\ &= c(a-bi) \\ &= c\,\overline{z} \\ &= \text{RHS}\end{aligned}$$

d $\boxed{P_n}$ Let P_n be the statement that

$$\overline{z^n} = \overline{z}^{\,n}.$$

We need to show that P_n is true for all $n \in \mathbb{N}$.

$\boxed{P_1}$ For the base case, we let $n = 1$. Then $\overline{z^1} = \overline{z} = \overline{z}^{\,1}$. Therefore, P_1 is true.

$\boxed{P_k}$ We now assume that P_k is true. Therefore

$$\overline{z^k} = \overline{z}^{\,k}.$$

$\boxed{P_{k+1}}$ We find that

$$\begin{aligned}\overline{z^{k+1}} &= \overline{z^k\,z} \\ &= \overline{z^k}\,\overline{z} \quad \text{(by part } \mathbf{a}\ P_k) \\ &= \overline{z}^{\,k}.\overline{z} \quad \text{(by } P_k) \\ &= \overline{z}^{\,k+1}.\end{aligned}$$

Therefore P_{k+1} is true whenever P_k is true. Moreover P_1 is true. Therefore P_n is true for all $n \in \mathbb{N}$, by mathematical induction.

e We find that

$$\begin{aligned}a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0 &= 0 \\ \overline{a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0} &= \overline{0} \\ \overline{a_nz^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_0} &= \overline{0} \quad \text{(by part } \mathbf{b}) \\ \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_0} &= \overline{0} \quad \text{(by part } \mathbf{a}) \\ \overline{a_n}\,\overline{z}^{\,n} + \overline{a_{n-1}}\,\overline{z}^{\,n-1} + \cdots + \overline{a_0} &= \overline{0} \quad \text{(by part } \mathbf{d}) \\ a_n\overline{z}^{\,n} + a_{n-1}\overline{z}^{\,n-1} + \cdots + a_0 &= 0 \quad \text{(by part } \mathbf{c})\end{aligned}$$

This means that $\overline{z}$ is a solution of the same equation.

Solutions to Exercise 18F

1 a The point is in the first quadrant.

$$r = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= \sqrt{1+3} = 2$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$
$$\Rightarrow 1 + \sqrt{3}i = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$$

b The point is in the fourth quadrant.

$$r = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = -\frac{\pi}{4}$$
$$\Rightarrow 1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$$

c The point is in the second quadrant.

$$r = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2}$$
$$= \sqrt{16} = 4$$
$$\cos\theta = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$
$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$
$$\Rightarrow -2\sqrt{3} + 2i = 4\operatorname{cis}\left(\frac{5\pi}{6}\right)$$

d The point is in the third quadrant.

$$r = \sqrt{4^2 + 4^2}$$
$$= \sqrt{32} = 4\sqrt{2}$$
$$\cos\theta = -\frac{4}{4\sqrt{2}} = -\frac{1}{2}$$
$$\theta = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$
$$\Rightarrow -4 - 4i = 4\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

e The point is in the fourth quadrant.

$$r = \sqrt{12^2 + 12^2 \times 3}$$
$$= \sqrt{4 \times 144} = 24$$
$$\cos\theta = -\frac{12}{24}$$
$$= -\frac{1}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\Rightarrow 12 - 12\sqrt{3}\,i = 24\operatorname{cis}\left(-\frac{\pi}{3}\right)$$

f The point is in the second quadrant.

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$
$$= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$
$$\cos\theta = -\frac{1}{2} \div \frac{1}{\sqrt{2}}$$
$$= -\frac{1}{2} \times \sqrt{2} = -\frac{1}{\sqrt{2}}$$
$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\Rightarrow -\frac{1}{2} + \frac{1}{2}i = \frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{3\pi}{4}\right)$$

2 a
$$3\operatorname{cis}\frac{\pi}{2} = 3\cos\frac{\pi}{2} + 3i\sin\frac{\pi}{2}$$
$$= 3i$$

b $\sqrt{2} \text{ cis } \frac{\pi}{3} = \sqrt{2} \cos \frac{\pi}{3} + \sqrt{2} i \sin \frac{\pi}{3}$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} i$$

$$= \frac{\sqrt{2}}{2}(1 + \sqrt{3}\, i)$$

c $2 \text{ cis } \frac{\pi}{6} = 2 \cos \frac{\pi}{6} + 2i \sin \frac{\pi}{6}$

$$= \sqrt{3} + i$$

d $5 \text{ cis } \frac{3\pi}{4} = 5 \cos \frac{3\pi}{4} + 5i \sin \frac{3\pi}{4}$

$$= -\frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}} i$$

$$= -\frac{5\sqrt{2}}{2}(1 - i)$$

e $12 \text{ cis } \frac{5\pi}{6} = 12 \cos \frac{5\pi}{6} + 12i \sin \frac{5\pi}{6}$

$$= -6\sqrt{3} + 6i$$

$$= -6(\sqrt{3} - i)$$

f $3\sqrt{2} \text{ cis } \left(-\frac{\pi}{4}\right) = 3\sqrt{2} \cos \left(-\frac{\pi}{4}\right)$

$$+ 3\sqrt{2} i \sin\left(-\frac{\pi}{4}\right)$$

$$= 3 - 3i$$

$$= 3(1 - i)$$

g $5 \text{ cis } \frac{4\pi}{3} = 5 \cos \frac{4\pi}{3} + 5i \sin \frac{4\pi}{3}$

$$= -\frac{5}{2} - \frac{5\sqrt{3}}{2} i$$

$$= -\frac{5}{2}(1 + \sqrt{3} i)$$

h $5 \text{ cis } \left(-\frac{2\pi}{3}\right) = 5 \cos\left(-\frac{2\pi}{3}\right)$

$$+ 5i \sin\left(-\frac{2\pi}{3}\right)$$

$$= -\frac{5}{2} - \frac{5\sqrt{3}}{2} i$$

$$= -\frac{5}{2}(1 + \sqrt{3} i)$$

3 $z_1 z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$

a $\left(2 \text{ cis } \frac{\pi}{6}\right) \cdot \left(3 \text{ cis } \frac{\pi}{12}\right)$

$$= 6 \text{ cis } \left(\frac{\pi}{6} + \frac{\pi}{12}\right)$$

$$= 6 \text{ cis } \frac{\pi}{4}$$

$$= 6 \cos \frac{\pi}{4} + 6i \sin \frac{\pi}{4}$$

$$= \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}} i$$

$$= 3\sqrt{2}(1 + i)$$

b $\left(4 \text{ cis } \frac{\pi}{12}\right) \cdot \left(3 \text{ cis } \frac{\pi}{4}\right)$

$$= 12 \text{ cis } \left(\frac{\pi}{12} + \frac{\pi}{4}\right)$$

$$= 12 \text{ cis } \frac{\pi}{3}$$

$$= 12 \cos \frac{\pi}{3} + 12i \sin \frac{\pi}{3}$$

$$= 6 + 6\sqrt{3} i$$

$$= 6(1 + \sqrt{3} i)$$

c $\left(\operatorname{cis}\frac{\pi}{4}\right)\cdot\left(5\operatorname{cis}\frac{5\pi}{12}\right)$

$= 5\operatorname{cis}\left(\frac{\pi}{4}+\frac{5\pi}{12}\right)$

$= 5\operatorname{cis}\frac{2\pi}{3}$

$= 5\cos\frac{2\pi}{3}+5i\sin\frac{2\pi}{3}$

$= -\frac{5}{2}+\frac{5\sqrt{3}}{2}i$

$= -\frac{5}{2}(1-\sqrt{3}i)$

d $\left(12\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)\cdot\left(3\operatorname{cis}\frac{2\pi}{3}\right)$

$= 36\operatorname{cis}\left(-\frac{\pi}{3}+\frac{2\pi}{3}\right)$

$= 36\operatorname{cis}\frac{\pi}{3}$

$= 36\cos\frac{\pi}{3}+36i\sin\frac{\pi}{3}$

$= 18+18\sqrt{3}i$

$= 18(1+\sqrt{3}i)$

e $\left(12\operatorname{cis}\frac{5\pi}{6}\right)\cdot\left(3\operatorname{cis}\frac{\pi}{2}\right)$

$= 36\operatorname{cis}\left(\frac{5\pi}{6}+\frac{\pi}{2}\right)$

$= 36\operatorname{cis}\frac{4\pi}{3}$

$= 36\cos\frac{4\pi}{3}+36i\sin\frac{4\pi}{3}$

$= -18-18\sqrt{3}i$

$= -18(1+\sqrt{3}i)$

f $\left(\sqrt{2}\operatorname{cis}\pi\right)\cdot\left(\sqrt{3}\operatorname{cis}\left(-\frac{3\pi}{4}\right)\right)$

$= \sqrt{6}\operatorname{cis}\left(\pi-\frac{3\pi}{4}\right)$

$= \sqrt{6}\operatorname{cis}\frac{\pi}{4}$

$= \sqrt{6}\cos\frac{\pi}{4}+\sqrt{6}i\sin\frac{\pi}{4}$

$= \sqrt{3}+\sqrt{3}i$

$= \sqrt{3}(1+i)$

g $\dfrac{10\operatorname{cis}\frac{\pi}{4}}{5\operatorname{cis}\frac{\pi}{12}} = \frac{10}{5}\operatorname{cis}\left(\frac{\pi}{4}-\frac{\pi}{12}\right)$

$= 2\operatorname{cis}\frac{\pi}{6}$

$= 2\cos\frac{\pi}{6}+2i\sin\frac{\pi}{6}$

$= \sqrt{3}+i$

h $\dfrac{12\operatorname{cis}\left(-\frac{\pi}{3}\right)}{3\operatorname{cis}\frac{2\pi}{3}} = \frac{12}{3}\operatorname{cis}\left(-\frac{\pi}{3}-\frac{2\pi}{3}\right)$

$= 4\operatorname{cis}(-\pi)$

$= 4\cos(-\pi)+4i\sin(-\pi)$

$= -4+0=-4$

i $\dfrac{12\sqrt{8}\operatorname{cis}\frac{3\pi}{4}}{3\sqrt{2}\operatorname{cis}\frac{\pi}{12}} = \frac{12\sqrt{8}}{3\sqrt{2}}\operatorname{cis}\left(\frac{3\pi}{4}-\frac{\pi}{12}\right)$

$= 8\operatorname{cis}\frac{2\pi}{3}$

$= 8\cos\frac{2\pi}{3}+8i\sin\frac{2\pi}{3}$

$= -4+4\sqrt{3}i$

$= -4(1-\sqrt{3}i)$

j $\dfrac{20 \text{ cis}\left(-\frac{\pi}{6}\right)}{8 \text{ cis } \frac{5\pi}{6}}$

$= \dfrac{20}{8} \text{ cis}\left(-\dfrac{\pi}{6} - \dfrac{5\pi}{6}\right)$

$= \dfrac{5}{2} \text{ cis }(-\pi)$

$= \dfrac{5}{2}\cos(-\pi) + \dfrac{5}{2} i \sin(-\pi)$

$= -\dfrac{5}{2} + 0$

$= -\dfrac{5}{2}$

4 a The point $(5, 2)$ corresponds to the complex number $z = 5 + 2i$. We need to rotate z by $\frac{\pi}{3}$ anticlockwise. Therefore we multiply z by $1\text{cis}\,(\frac{\pi}{3})$. We find that

$(5 + 2i)\text{cis}\,(\frac{\pi}{3})$

$= (5 + 2i)(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}))$

$= (5 + 2i)(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$

$= \frac{5-2\sqrt{3}}{2} + i\frac{5\sqrt{3}+2}{2}$

Therefore, the point $(2, 3)$ is rotated to $\left(\frac{5-2\sqrt{3}}{2}, \frac{5\sqrt{3}+2}{2}\right)$.

b The point $(3, 2)$ corresponds to the complex number $z = 3 + 2i$. We need to rotate z by $\frac{\pi}{4}$ clockwise. Therefore we multiply z by$1\text{cis}\,(-\frac{\pi}{4})$. We find that

$(3 + 2i)\text{cis}(-\frac{\pi}{4})$

$= (3 + 2i)(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))$

$= (2 + 3i)(\frac{\sqrt{3}}{2} - \frac{i}{2})$

$= \frac{2\sqrt{3}+3}{2} + i\frac{3\sqrt{3}-2}{2}$

Therefore, the point $(3, 2)$ is rotated to $\left(\frac{2\sqrt{3}+3}{2}, \frac{3\sqrt{3}-2}{2}\right)$.

c The point (x, y) corresponds to the complex number $z = x + yi$. We need to rotate z by θ anticlockwise. Therefore we multiply $z = x + yi$ by $1\text{cis}\,\theta$. We find that

$(x + yi)\text{cis}\theta$

$= (x + yi)(\cos\theta + i\sin\theta)$

$= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$

Therefore, the point (x, y) is rotated to $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

5 We find that

$$\begin{aligned}
z_1 z_2 &= (r_1\text{cis}\,\theta_1)(r_2\text{cis}\,\theta_2)\\
&= (r_1 r_2)(\text{cis}\,\theta_1)(\text{cis}\,\theta_2)\\
&= (r_1 r_2)(\cos\theta_1 + i\sin(\theta_1))(\cos\theta_2 + i\sin(\theta_2)\\
&= r_1 r_2[\cos(\theta_1)\cos(\theta_2) + i\sin(\theta_1)\cos(\theta_2)\\
&\quad + i\cos(\theta_1)\sin(\theta_2) + i^2\sin(\theta_1)\sin(\theta_2)]\\
&= r_1 r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2))\\
&\quad + i(\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\sin(\theta_2))]\\
&= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]\\
&= r_1 r_2\text{cis}\,(\theta_1 + \theta_2)
\end{aligned}$$

Solutions to Exercise 18G

1 Recall that the distance from z to w is $|z - w|$.

a $|z - w| = |(1 + i) - (4 + 5i)|$
$= |-3 - 4i|$
$= \sqrt{(-3)^2 + (-4)^2}$
$= \sqrt{25}$
$= 5$

b $|z - w| = |(3 - 4i) - (2 - 3i)|$
$= |1 - i|$
$= \sqrt{1^2 + (-1)^2}$
$= \sqrt{2}$

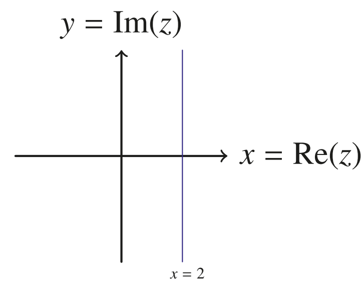
c $|z - w| = |(4 - 6i) - (-1 + 6i)|$
$= |5 - 12i|$
$= \sqrt{5^2 + (-12)^2}$
$= \sqrt{169}$
$= 13$

d $|z - w| = |2 - (-2i)|$
$= |2 + 2i|$
$= \sqrt{2^2 + 2^2}$
$= \sqrt{8}$
$= 2\sqrt{2}$

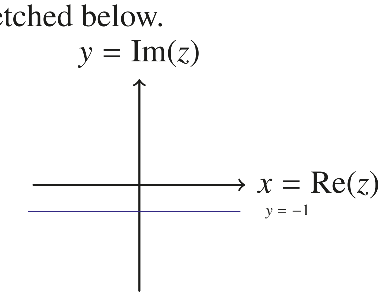
e $|z - w| = |10i - (-3i)|$
$= |13i|$
$= 13$

f $|z - w| = |\sqrt{2} - 2i|$
$= \sqrt{(\sqrt{2})^2 + (-2)^2}$
$= \sqrt{2 + 4}$
$= \sqrt{6}$

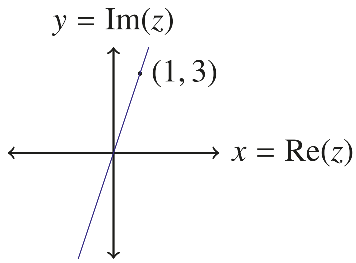
2 **a** If $z = x + yi$ then
$\text{Re}(z) = 2$
$x = 2.$
This vertical line has been sketched below.

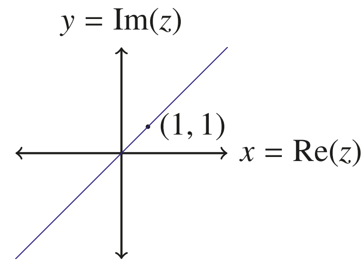


b If $z = x + yi$ then
$\text{Im}(z) = -1$
$y = -1.$
This horizontal line has been sketched below.



c If $z = x + yi$ then
$\text{Im}(z) = 3\,\text{Re}(z)$
$y = 3x.$
This line has been sketched below.

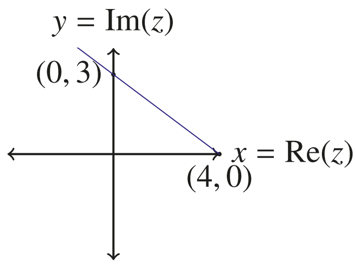




d If $z = x + yi$ then

$$\begin{aligned} 3\,\mathrm{Re}(z) + 4\,\mathrm{Im}(z) &= 12 \\ 3x + 4y &= 12. \end{aligned}$$

This line has been sketched below.



e This relation describes the set of complex numbers z that are equidistant to the complex numbers 1 and i. This will clearly be the line $y = x$. We can also see this algebraically. Let $z = x + yi$ so that

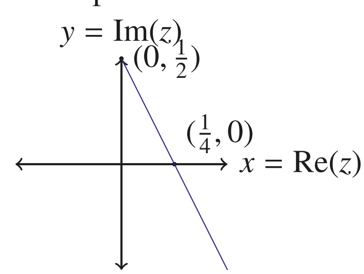
$$\begin{aligned} |z - 1| &= |z - i| \\ |x + yi - 1| &= |x + yi - i| \\ |(x - 1) + iy| &= |x + (y - 1)i| \\ \sqrt{(x-1)^2 + y^2} &= \sqrt{x^2 + (y-1)^2} \\ x^2 - 2x + 1 + y^2 &= x^2 + y^2 - 2y + 1 \\ -2x &= -2y \\ y &= x. \end{aligned}$$

This set of points is shown below.

f We can rewrite this as $|z - (1 + i)| = |z - (-1)|$. Therefore this relation describes the set of complex numbers z that are equidistant to the complex numbers $1 + i$ and -1. The cartesian equation must therefore be $y = -2x + \frac{1}{2}$. We can also show this algebraically. Let $z = x + yi$ so that

$$\begin{aligned} |z - (1 + i)| &= |z + 1| \\ |x + yi - 1 - i| &= |x + yi + 1| \\ |(x - 1) + (y - 1)i| &= |(x + 1) + yi| \\ \sqrt{(x-1)^2 + (y-1)^2} &= \sqrt{(x+1)^2 + y^2} \\ -2x - 2y + 1 &= 2x \\ y &= -2x + \tfrac{1}{2} \end{aligned}$$

This set of points is shown below.

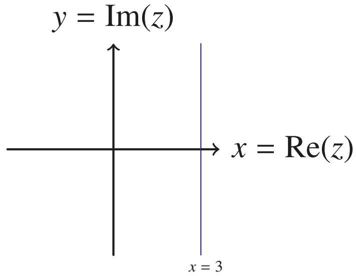


g If $z = x + yi$ then

$$\begin{aligned} z + \bar{z} &= 6 \\ (x + yi) + (x - iy) &= 6 \\ 2x &= 6 \\ x &= 3. \end{aligned}$$

This vertical line has been sketched

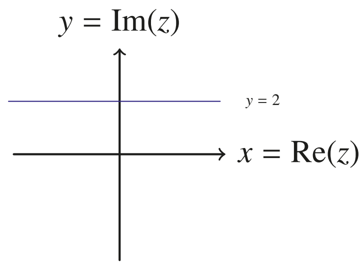
below.



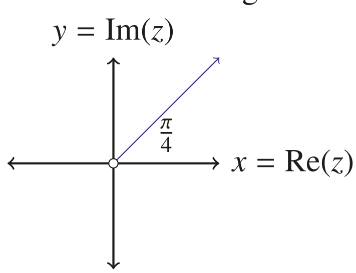
h If $z = x + yi$ then

$$
\begin{aligned}
z - \bar{z} &= 4i \\
(x + yi) - (x - iy) &= 4i \\
2iy &= 4i \\
y &= 2.
\end{aligned}
$$

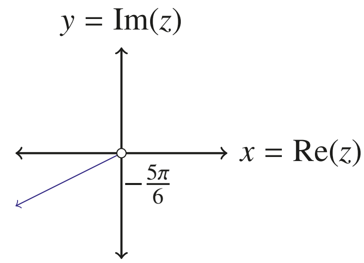
This horizontal line has been sketched below.



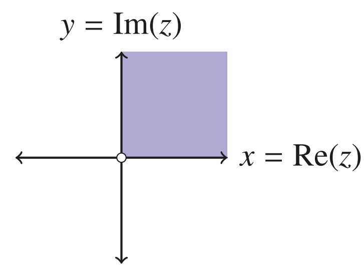
3 a This is a ray emerging from the origin at an angle of $\frac{\pi}{4}$ to the x-axis. Note that Arg(0) is not defined so the ray does not include the origin.



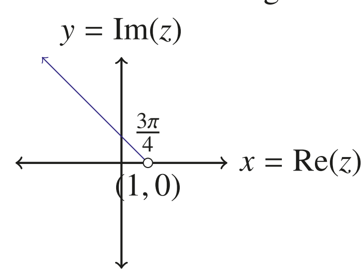
b This is a ray emerging from the origin at an angle of $-\frac{5\pi}{6}$ to the x-axis. Note that Arg(0) is not defined so the ray does not include the origin.



c This is the set of points whose principal argument is between 0 and $\frac{\pi}{2}$, inclusive. Loosely speaking, this is the first quadrant, with the origin removed.



d The ray extending from the origin is translated 1 unit to the right.

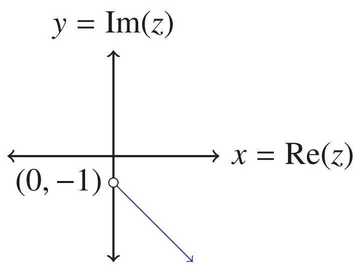


To show this algebraically, we can let

$z = x + yi$ so that

$$\begin{aligned}
\operatorname{Arg}(z - 1) &= \frac{3\pi}{4} \\
\operatorname{Arg}(x + yi - 1) &= \frac{3\pi}{4} \\
\operatorname{Arg}((x - 1) + yi) &= \frac{3\pi}{4} \\
\tan^{-1}\left(\frac{y}{x-1}\right) &= \frac{3\pi}{4} \\
\frac{y}{x-1} &= \tan\left(\frac{3\pi}{4}\right) \\
\frac{y}{x-1} &= -1 \\
y &= 1 - x \text{ (given } x < 1)
\end{aligned}$$

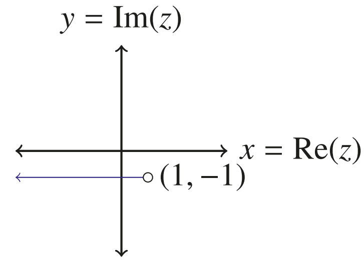
e A ray extending from the origin is translated 1 unit down.



To show this algebraically, we can let $z = x + yi$ so that

$$\begin{aligned}
\operatorname{Arg}(z + i) &= -\frac{\pi}{4} \\
\operatorname{Arg}(x + yi + i) &= -\frac{\pi}{4} \\
\operatorname{Arg}(x + (y + 1)i) &= -\frac{\pi}{4} \\
\tan^{-1}\left(\frac{y+1}{x}\right) &= -\frac{\pi}{4} \\
\frac{y+1}{x} &= \tan\left(-\frac{\pi}{4}\right) \\
\frac{y+1}{x} &= -1 \\
y &= -x - 1 \text{ (given } x > 0)
\end{aligned}$$

f A ray extending from the origin is translated 1 unit right and 1 unit down.



To show this algebraically, we can let $z = x + yi$ so that

$$\begin{aligned}
\operatorname{Arg}(z - 1 + i) &= \pi \\
\operatorname{Arg}(x + yi - 1 + i) &= \pi \\
\operatorname{Arg}((x - 1) + (y + 1)i) &= \pi \\
\tan^{-1}\left(\frac{y+1}{x-1}\right) &= \pi \\
\frac{y+1}{x-1} &= \tan(\pi) \\
\frac{y+1}{x-1} &= 0 \\
y &= -1 \text{ (given } x < 1)
\end{aligned}$$

4 By letting $z = x + yi$ we find that

$$\begin{aligned}
|z - 2| &= 1 \\
|x + yi - 2| &= 1 \\
|(x - 2) + yi| &= 1 \\
\sqrt{(x-2)^2 + y^2} &= 1 \\
(x - 2)^2 + y^2 &= 1,
\end{aligned}$$

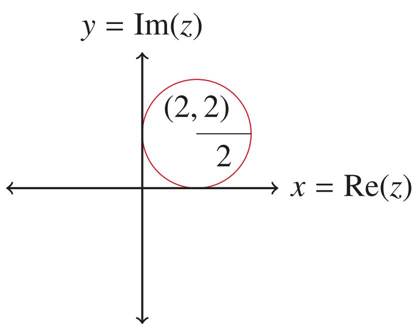
as required.

5 By letting $z = x + yi$ we find that

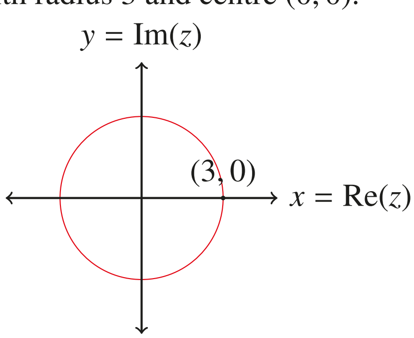
$$\begin{aligned} |z| &= |z - 2 - 2i| \\ |x + yi| &= |x + yi - 2 - 2i| \\ |x + yi| &= |(x - 2) + (y - 2)i| \\ \sqrt{x^2 + y^2} &= \sqrt{(x - 2)^2 + (y - 2)^2} \\ x^2 + y^2 &= x^2 - 4x + 4 + y^2 - 4y + 4 \\ 0 &= -4x - 4y + 8 \\ y &= 2 - x, \end{aligned}$$

as required.

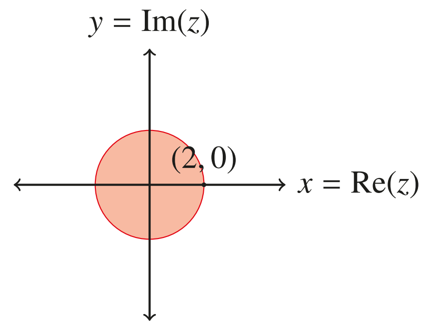
6 The required set of points is a circle of radius 2 centred at the point $(2, 2)$. This is shown below.



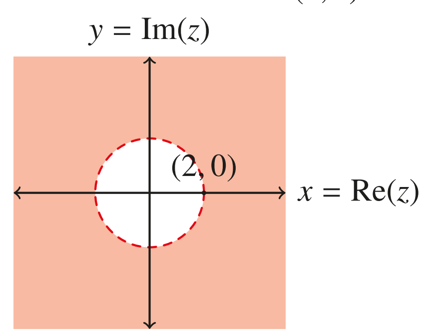
7 **a** This is the set of points z that are 3 units from the origin. That is, a circle with radius 3 and centre $(0, 0)$.



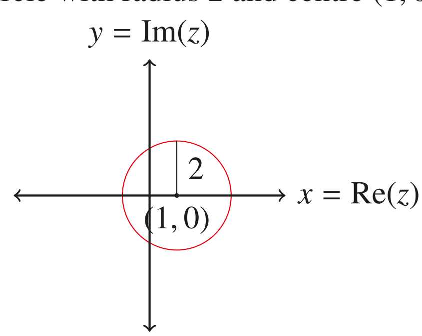
b This is the set of points z that are less than or equal to 2 units from the origin. That is, the points on or inside the circle with radius 2 and centre $(0, 0)$.



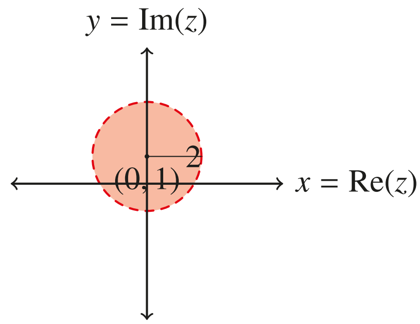
c This is the set of points z that are greater than 2 units from the origin. That is, the points outside of a circle with radius 2 and centre $(0, 0)$.



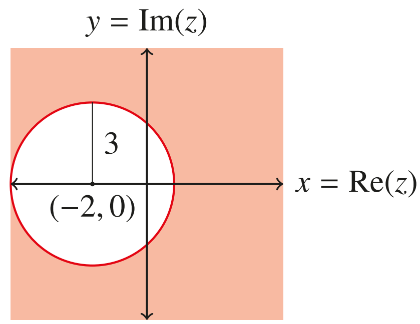
d This is the set of points z that are 2 units from the point $w = 1$. That is, a circle with radius 2 and centre $(1, 0)$.



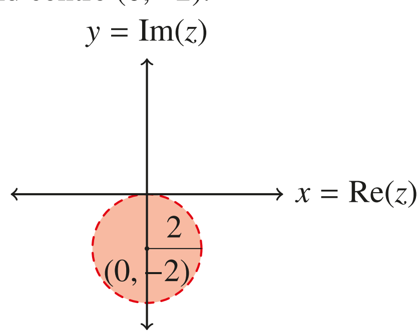
e This is the set of points z that are less than 2 units from the point i. That is, points inside of a circle with radius 2 and centre $(0, 1)$.



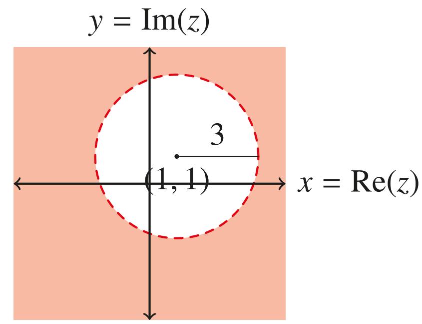
f We rewrite this as $|z - (-2)| \geq 3$. This is the set of points z that are greater than or equal to 3 units from $w = -2$. That is, points on or outside of a circle with radius 3 and centre $(-2, 0)$.



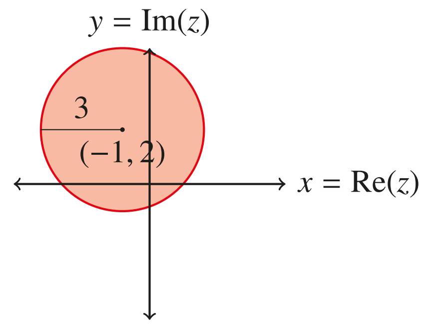
g We rewrite this as $|z - (-2i)| < 2$. This is the set of points z that are less than 2 units from $w = -2i$. That is, points inside of a circle with radius 2 and centre $(0, -2)$.



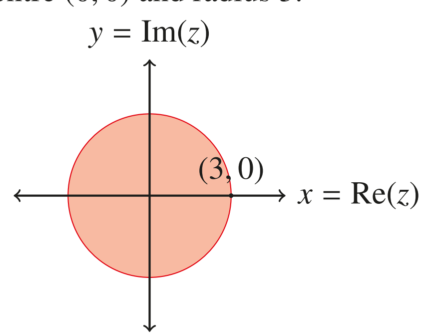
h This is the set of points z that are greater than 3 units from $w = 1 + i$. That is, points outside of a circle with radius 3 and centre $(1, 1)$.



i We rewrite this as $|z - (-1 + 2i)| \leq 3$. This is the set of points z that are less than or equal to 3 units from $w = -1 + 2i$. That is, points on or inside of a circle with radius 3 and centre $(-1, 2)$.



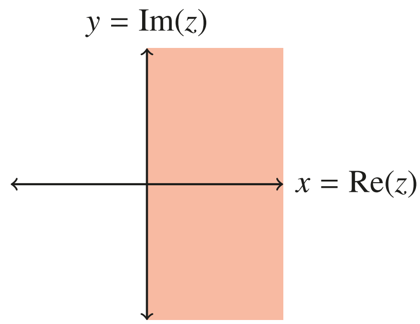
8 **a** Set R consists of the set of points less than or equal to 3 units from the origin. That is, a closed disc with centre $(0, 0)$ and radius 3.



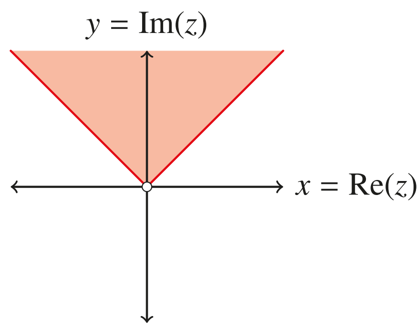
b Let $z = x + yi$ so that

$$\mathrm{Re}(z) \geq 0$$
$$x \geq 0$$

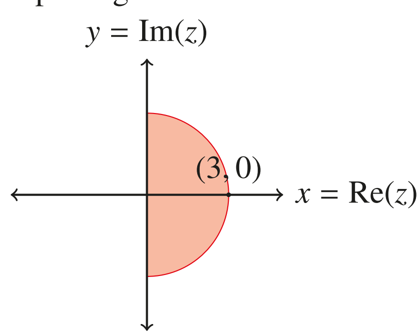
This gives the half plane shown below.



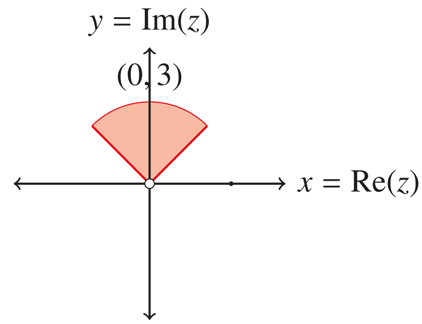
c Set T consists of all complex numbers whose principle argument ranges from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$. This gives the wedge-shaped region shown below.



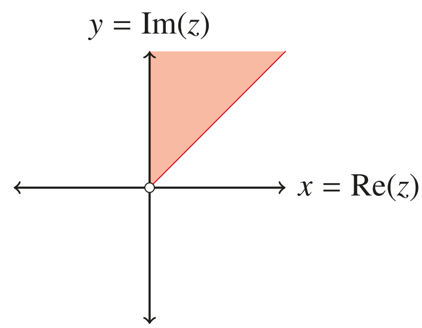
d The intersection of the circle and the half-plane gives a semi-circle



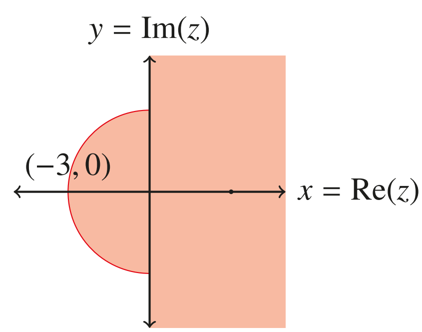
e The intersection of the circle and the wedge-shaped region is the quarter-circle shown below.



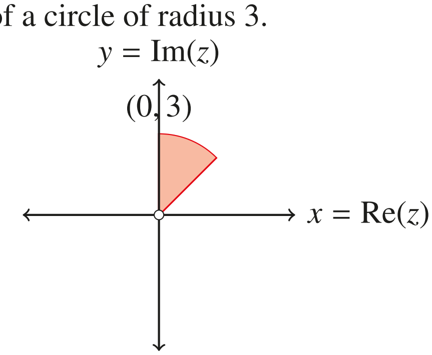
f The intersection of the wedge and the half-plane is shown below.



g The union $R \cup S$ is shown below.



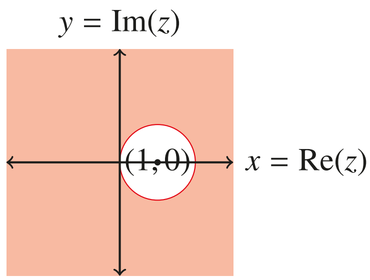
h The intersection of the three regions $R \cap S \cap T$ is shown. It is one eighth of a circle of radius 3.



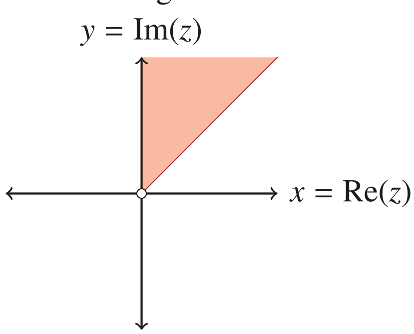
9 **a** We let $z = x + yi$ so that

$$\begin{aligned} z + \overline{z} &\leq |z|^2 \\ (x+yi) + \overline{x+yi} &\leq |x+yi|^2 \\ (x+yi) + (x-yi) &\leq x^2 + y^2 \\ 2x &\leq x^2 + y^2 \\ x^2 - 2x + y^2 &\geq 0 \\ (x^2 - 2x + 1) - 1 + y^2 &\geq 0 \\ (x-1)^2 + y^2 &\geq 1 \end{aligned}$$

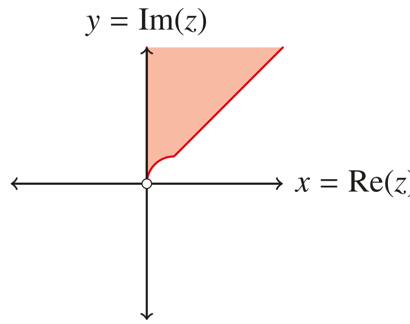
b Set S is the set of points on or outside of a circle with centre $(1, 0)$ and radius 1.



c The set T consists of the set of points whose principal argument lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. This is sketched below. The origin must be omitted.



d The intersection $S \cap T$ is shown below.



10 Let $z = x + yi$ so that

$$\begin{aligned} |z + 2i| &= |2iz - 1| \\ |(x+yi) + 2i| &= |2i(x+yi) - 1| \\ |x + (y+2)i| &= |(-2y-1) + 2xi| \\ \sqrt{x^2 + (y+2)^2} &= \sqrt{(-2y-1)^2 + (2x)^2} \\ x^2 + y^2 + 4y + 4 &= 4y^2 + 4y + 1 + y^2 + 4x^2 \\ x^2 + y^2 &= 1 \end{aligned}$$

Therefore, this is a circle of radius 1 centred at the origin $(0, 0)$.

11 Let $z = x + yi$ so that

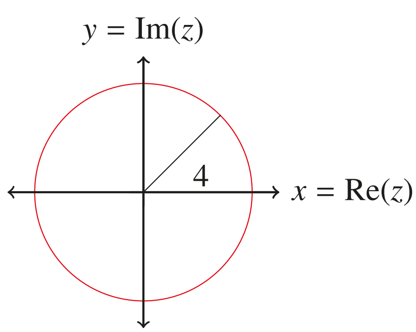
$$\begin{aligned} 2|z - i| &= |z + \overline{z} + 2| \\ 2|(x+yi) - i| &= |(x+yi) + (x-yi) + 2| \\ 2|x + (y-1)i| &= |2x + 2| \\ |x + (y-1)i| &= |x + 1| \\ \sqrt{x^2 + (y-1)^2} &= \sqrt{(x+1)^2} \\ x^2 + (y-1)^2 &= (x+1)^2 \\ x^2 + (y-1)^2 &= x^2 + 2x + 1 \\ y^2 - 2y &= 2x \\ x &= \frac{1}{2}(y^2 - 2y) \end{aligned}$$

Note that this is just a parabola turned 90° to the more familiar orientation.

12 **a** There is more than one solution to this problem. We will use the fact that $z\bar{z} = |z|^2$. Squaring both sides of the relation gives

$$\begin{aligned}
|z+16| &= 4|z+1| \\
|z+16|^2 &= 16|z+1|^2 \\
(z+16)\overline{(z+16)} &= 16(z+1)\overline{(z+1)} \\
(z+16)(\bar{z}+16) &= 16(z+1)(\bar{z}+1) \\
z\bar{z} + 16(z+\bar{z}) + 16^2 &= 16z\bar{z} + 16(z+\bar{z}) + 16 \\
|z|^2 + 16^2 &= 16|z|^2 + 16 \\
15|z|^2 &= 16^2 - 16 \\
15|z|^2 &= 16(16-1) \\
15|z|^2 &= (16)(15) \\
|z|^2 &= 16 \\
|z| &= 4
\end{aligned}$$

b



13 Let $z = x + yi$ so that

$$\begin{aligned}
|z|^2 &= 9|z+8|^2 \\
|x+yi|^2 &= 9|x+yi+8|^2 \\
|x+yi|^2 &= 9|(x+8)+yi|^2 \\
x^2+y^2 &= 9((x+8)^2+y^2)
\end{aligned}$$

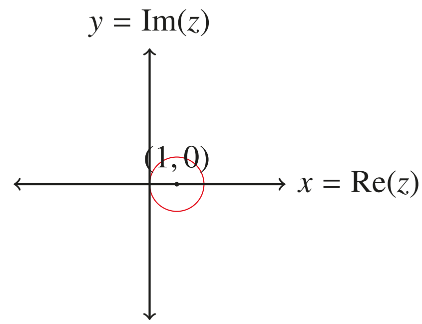
$$x^2 + 18x + y^2 + 72 = 0$$

By completing the square this gives

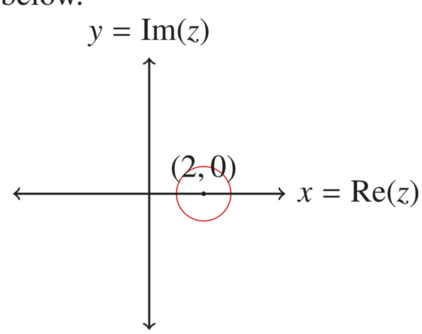
$$(x+9)^2 + y^2 = 3^2.$$

The centre of the circle is $(-9, 0)$ and its radius is 3.

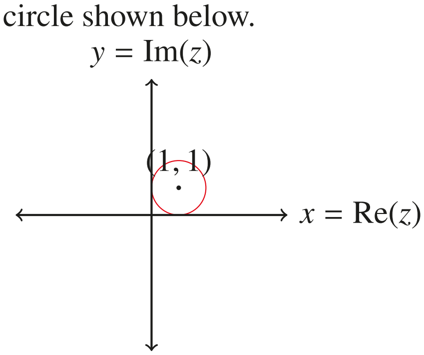
14 a S is a circle with centre $(1, 0)$ and radius 1 this is shown below.



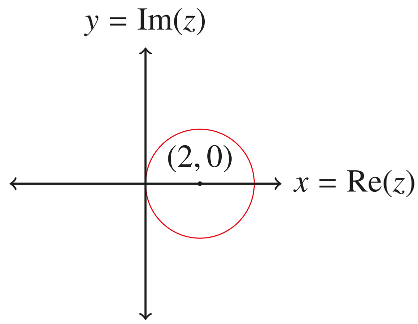
b i To each point in S we add 1. This will translate circle S by 1 unit to the right giving the circle shown below.



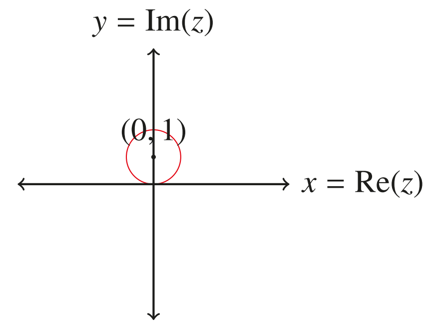
ii To each point in S we add i. This will translate circle S by 1 unit in the vertical direction giving the circle shown below.



iii Each point in S is scaled by a factor of 2. This will give a circle whose centre is $(2, 0)$ and radius 2.



iv Multiplication by i will rotate each point in S by $\frac{\pi}{2}$, i.e. a quarter turn in the anticlockwise direction. This will give a circle with centre $(0, 1)$ and radius 1.



Solutions to short-answer questions

1 a $2z_1 + 3z_2 = 2m + 2ni + 3p + 3qi$
$= (2m + 3p) + (2n + 3q)i$

b $\bar{z}_2 = p - qi$

c $z_1\,\bar{z}_2 = (m + ni)(p - qi)$
$= mp + npi - mqi - nqi^2$
$= (mp + nq) + (np - mq)i$

d $\dfrac{z_1}{z_2} = \dfrac{m + ni}{p + qi}$
$= \dfrac{m + ni}{p + qi} \times \dfrac{p - qi}{p - qi}$
$= \dfrac{mp + npi - mqi - nqi^2}{p^2 + q^2}$
$= \dfrac{(mp + nq) + (np - mq)i}{p^2 + q^2}$

e $z_1 + \bar{z}_1 = (m + ni) + (m - ni)$
$= 2m$

f
$(z_1 + z_2)(z_1 - z_2)$
$= z_1^2 - z_2^2$
$= m^2 + 2mni + n^2 i^2 - (p^2 + 2pqi + q^2 i^2)$
$= m^2 + 2mni - n^2 - (p^2 + 2pqi - q^2)$
$= (m^2 - n^2 - p^2 + q^2) + (2mn - 2pq)i$

g $\dfrac{1}{z_1} = \dfrac{1}{m + ni}$
$= \dfrac{1}{m + ni} \times \dfrac{m - ni}{m - ni}$
$= \dfrac{m - ni}{m^2 + n^2}$

h $\dfrac{z_2}{z_1} = \dfrac{p + qi}{m + ni}$
$= \dfrac{p + qi}{m + ni} \times \dfrac{m - ni}{m - ni}$
$= \dfrac{mp + nq + (mq - np)i}{m^2 + n^2}$

i $\dfrac{3z_1}{z_2} = \dfrac{3(m + ni)}{p + qi}$
$= \dfrac{3(m + ni)}{p + qi} \times \dfrac{p - qi}{p - qi}$
$= \dfrac{3(mp + npi - mqi - nqi^2)}{p^2 + q^2}$
$= \dfrac{3[(mp + nq) + (np - mq)i]}{p^2 + q^2}$

2 a $A : z = 1 - \sqrt{3}i$

b $B : z^2 = (1 - \sqrt{3}i)^2$
$= 1 - 2\sqrt{3}i + 3i^2$
$= -2 - 2\sqrt{3}i$

c $C : z^3 = z^2 \times z$
$= (-2 - 2\sqrt{3}i)(1 - \sqrt{3}i)$
$= -2 + 2\sqrt{3}i - 2\sqrt{3}i + 6i^2$
$= -8$

d $D : \dfrac{1}{z} = \dfrac{1}{1 - \sqrt{3}i}$
$= \dfrac{1}{1 - \sqrt{3}i} \times \dfrac{1 + \sqrt{3}i}{1 + \sqrt{3}i}$
$= \dfrac{1 + \sqrt{3}i}{4}$

e $E : \bar{z} = 1 + \sqrt{3}i$

f $F: \frac{1}{\bar{z}} = \frac{1}{1+\sqrt{3}i}$

$$= \frac{1}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$$

$$= \frac{1-\sqrt{3}i}{4}$$

Note: use existing diagram from answers

3 a The point is in the first quadrant.

$$r = \sqrt{1^2+1^2}$$
$$= \sqrt{2}$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4}$$
$$\Rightarrow 1+i = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$

b The point is in the fourth quadrant.

$$r = \sqrt{1+3}$$
$$= 2$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\Rightarrow 1-\sqrt{3}i = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$$

c The point is in the first quadrant.

$$r = \sqrt{12+1}$$
$$= \sqrt{13}$$
$$\tan\theta = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}$$
$$\Rightarrow 2\sqrt{3}+i = \sqrt{13}\operatorname{cis}\left(\tan^{-1}\frac{\sqrt{3}}{6}\right)$$

d The point is in the first quadrant.

$$r = \sqrt{18+18}$$
$$= \sqrt{36} = 6$$
$$\cos\theta = \frac{3\sqrt{2}}{6} = \frac{1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4}$$
$$\Rightarrow 3\sqrt{2}+3\sqrt{2}i = 6\operatorname{cis}\left(\frac{\pi}{4}\right)$$

e The point is in the third quadrant.

$$r = \sqrt{18+18}$$
$$= \sqrt{36} = 6$$
$$\cos\theta = -\frac{3\sqrt{2}}{6} = -\frac{1}{\sqrt{2}}$$
$$\theta = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$
$$\Rightarrow -3\sqrt{2}-3\sqrt{2}i = 6\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

f The point is in the fourth quadrant.

$$r = \sqrt{3+1}$$
$$= 2$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$
$$\theta = -\frac{\pi}{6}$$
$$\Rightarrow \sqrt{3}-i = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$$

4 a

$$x = -2\cos\left(\frac{\pi}{3}\right)$$
$$= -1$$
$$y = -2\sin\left(\frac{\pi}{3}\right)$$
$$= -\sqrt{3}$$
$$\Rightarrow z = -1-\sqrt{3}i$$

b $$x = 3\cos\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$
$$y = 3\sin\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$
$$\Rightarrow z = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

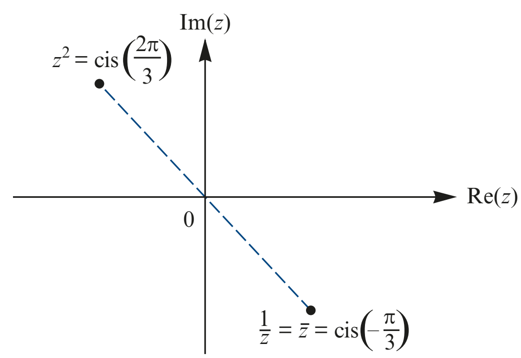
c $$x = 3\cos\left(\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$$
$$y = 3\sin\left(\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$
$$\Rightarrow z = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

d $$x = -3\cos\left(-\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$
$$y = -3\sin\left(-\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}$$
$$\Rightarrow z = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

e $$x = 3\cos\left(-\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{2}$$
$$y = 3\sin\left(-\frac{5\pi}{6}\right) = -\frac{3}{2}$$
$$\Rightarrow z = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

f $$x = \sqrt{2}\cos\left(-\frac{\pi}{4}\right) = 1$$
$$y = \sqrt{2}\,\sin\left(-\frac{\pi}{4}\right) = -1$$
$$\Rightarrow z = 1 - i$$

5



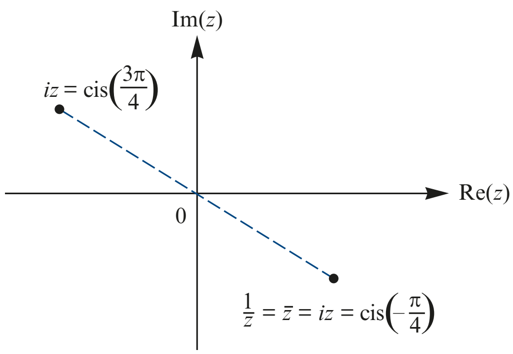
a $z^2 = \text{cis}\left(\frac{2\pi}{3}\right)$

b $\bar{z} = \text{cis}\left(-\frac{\pi}{3}\right)$

c $\frac{1}{z} = \text{cis}\left(-\frac{\pi}{3}\right)$

d $\text{cis}\left(\frac{2\pi}{3}\right)$

6



a $iz = \text{cis}\left(\dfrac{3\pi}{4}\right)$

b $\bar{z} = \text{cis}\left(-\dfrac{\pi}{4}\right)$

c $\dfrac{1}{z} = \text{cis}\left(-\dfrac{\pi}{4}\right)$

d $-iz = \text{cis}\left(-\dfrac{\pi}{4}\right)$

7 **a** By rewriting this as the difference of two squares we find that

$$\begin{aligned} z^2 + 4 &= 0 \\ z^2 - (2i)^2 &= 0 \\ (z - 2i)(z + 2i) &= 0 \\ z &= \pm 2i. \end{aligned}$$

b By taking out the HCF and then rewriting this as the difference of two squares we find that

$$\begin{aligned} 3z^2 + 9 &= 0 \\ 3(z^2 + 3) &= 0 \\ 3(z^2 - (\sqrt{3}i)^2) &= 0 \\ 3(z - \sqrt{3}i)(z + \sqrt{3}i) &= 0z \quad = \pm\sqrt{3}i \end{aligned}$$

c It is most efficient to complete the square here. This gives

$$\begin{aligned} z^2 + 4z + 5 &= 0 \\ (z^2 + 4z + 4) - 4 + 5 &= 0 \\ (z + 2)^2 + 1 &= 0 \\ (z + 2)^2 - i^2 &= 0 \\ (z + 2 - i)(z + 2 + i)) &= 0 \\ z &= -2 \pm i \end{aligned}$$

d We use the quadratic formula, giving

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(4)}}{4} \\ &= \frac{3 \pm \sqrt{-23)}}{4} \\ &= \frac{3 \pm \sqrt{23}i}{4}. \end{aligned}$$

8 If $z = 2$ then

$$2^3 - 2(2^2) + 4(2) - 8 = 8 - 8 + 8 - 8 = 0.$$

Therefore $z = 2$ is a solution and $(z - 2)$ is a factor. By either polynomial division or by inspection we find that

$$\begin{aligned} &z^3 - 2z^2 + 4z - 8 \\ &= (z - 2)(z^2 + 4) \\ &= (z - 2)(z^4 - (2i)^2) \\ &= (z - 2)(z - 2i)(z + 2i). \end{aligned}$$

From this we find that the remaining two solutions are $z = \pm 2i$.

9 **a** If $z = i$ then

$$\begin{aligned} &12i^3 - 11i^2 + 12i - 11 \\ &= -12i + 11 + 12i - 11 \\ &= 0. \end{aligned}$$

Therefore $z = i$ is a solution

b The cubic has real coefficients. Therefore, if $z = i$ is solution, then so too is the conjugate $z = \bar{i}$. Therefore $z - i$ and $z + i$ are factors. Multiplying these gives,

$$(z - i)(z + i) = z^2 + 1.$$

We can find the remaining linear factor by polynomial division or by inspection, giving

$$\begin{aligned} 12z^3 - 11z^2 + 12z - 11 &= 0 \\ (z^2 + 1)(12z - 11) &= 0 \\ z &= \pm i, \frac{11}{12}. \end{aligned}$$

c This polynomial will also have a solution $z = i$ since

$$\begin{aligned} &ni^3 - (n-1)i^2 + ni - (n-1) \\ &= -ni + (n-1) + ni - (n-1) \\ &= 0. \end{aligned}$$

The cubic has real coefficients. Therefore, if $z = i$ is solution, then so too is the conjugate $z = \bar{i}$. Therefore $z - i$ and $z + i$ are factors. Multiplying these gives,

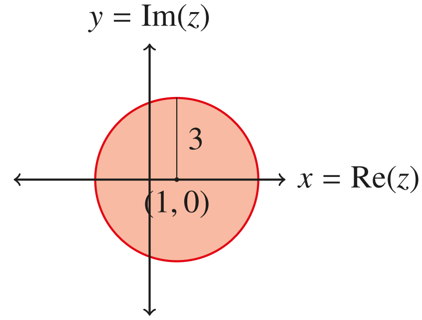
$$(z - i)(z + i) = z^2 + 1.$$

We can find the remaining linear factor by polynomial division or by inspection, giving

$$\begin{aligned} nz^3 - (n-1)z^2 + nz - (n-1) &= 0 \\ (z^2 + 1)(nz - (n-1)) &= 0 \\ z &= \pm i, \frac{n-1}{n}. \end{aligned}$$

The first two solutions listed above are complex, and so cannot be integers. The final solution listed above is an integer if and only if $n - 1 = 0$. That is, if and only if $n = 1$.

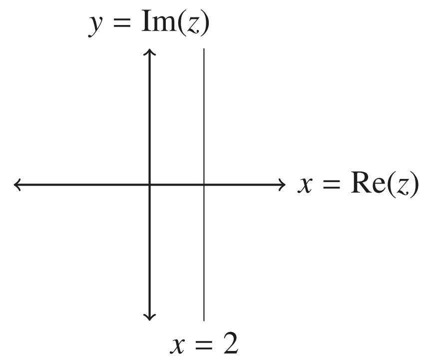
10 **a** If $|z - 1| \le 3$, then z is less than or equal to 3 units from 1. Therefore every point lies on or inside of a circle of radius 3 centred at $(1, 0)$.



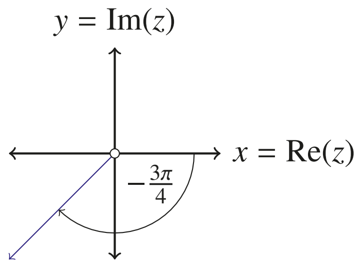
b Let $z = x + yi$ so that

$$\begin{aligned} z + \bar{z} &= 4 \\ (x + yi) + (x - yi) &= 4 \\ 2x &= 4 \\ x &= 2. \end{aligned}$$

This straight vertical line is graphed below.



c This describes the set of complex numbers whose principal argument is $-\frac{3\pi}{4}$. Therefore we draw a ray at an angle of $-\frac{3\pi}{4}$ to the positive direction of the x-axis. The ray does not include the origin as we cannot assign this point an angle.



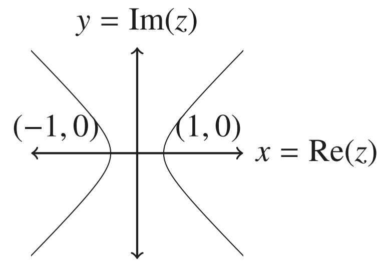
11 **a** We find that

$$\begin{aligned} z^2 &= (x + yi)^2 \\ &= x^2 + 2xyi + y^2 i^2 \\ &= (x^2 - y^2) + (2xy)i \end{aligned}$$

b We find that

$$\begin{aligned} \mathrm{Re}(z^2) &= 1 \\ x^2 - y^2 &= 1 \end{aligned}$$

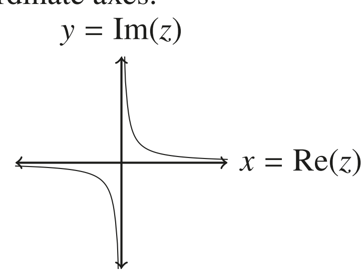
This is a hyperbola with axial intercepts at $(1, 0)$ and $(-1, 0)$ and asymptotes $y = \pm x$.



c We find that

$$\begin{aligned} \mathrm{Im}(z^2) &= 4 \\ 2xy &= 1 \\ y &= \frac{1}{2x} \end{aligned}$$

This is a hyperbola without axial intercepts. The asymptotes are the coordinate axes.



Solutions to multiple-choice questions

1 C $\dfrac{1}{2-u} = \dfrac{1}{1-i}$

$= \dfrac{1}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{1+i}{2}$

$= \dfrac{1}{2} + \dfrac{1}{2}i$

2 D $i = \operatorname{cis}\dfrac{\pi}{2}$, so the point will be rotated by $\dfrac{\pi}{2}$.

3 C $|z| = 5$

$\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$

$= \dfrac{1}{5}$

4 D $(x + yi)^2 = x^2 + 2xyi + y^2i^2$

$= (x^2 - y^2) + 2xyi$

Therefore

$x^2 - y^2 = 0$ and $2xy = -32$.

Therefore

$x^2 - y^2 = 0 \Rightarrow y = \pm x$

If $y = x$ then

$2xy = -32$

has no solution. If $y = -x$, then

$2xy = -32$

$-2x^2 = -32$

$x^2 = 16$

$x = \pm 4$

Therefore, $x = 4, y = -4$ or $x = 4, y = -4$.

5 D Completing the square gives,

$z^2 + 6z + 10 = z^2 + 6z + 9 + 1$

$= (z + 3)^2 + 1$

$= (z + 3)^2 - i^2$

$= (z + 3 - i)(z + 3 + i)$.

6 E $\dfrac{1}{1-i} = \dfrac{1}{1-i}\dfrac{1+i}{1+i}$

$= \dfrac{1+i}{2}$

$= \dfrac{1}{2} + \dfrac{1}{2}i$

Therefore,

$|z| = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2} = \dfrac{1}{\sqrt{2}}$

and

$\theta = \dfrac{\pi}{4}$.

7 D $\dfrac{z - 2i}{z - (3 - 2i)} = 2$

$z - 2i = 2(z - (3 - 2i))$

$z - 2i = 2z - 2(3 - 2i)$

$z = 2(3 - 2i) - 2i$

$= 6 - 6i$

8 D $z^2(1 + i) = 2 - 2i$

$z^2 = \dfrac{2 - 2i}{1 + i}$

$= \dfrac{(2 - 2i)(1 - i)}{2}$

$= (1 - i)^2$

$= (-1 + i)^2$

9 B $\Delta = b^2 - 4ac$

$$\begin{aligned} &= (8i)^2 - 4(2+2i)(-4(1-i)) \\ &= 64i^2 + 16(2+2i)(1-i) \\ &= -64 + 32(1+i)(1-i) \\ &= -64 + 32(1-i^2) \\ &= -64 + 32 \times 2 \\ &= 0 \end{aligned}$$

10 D $\operatorname{Arg}(1+ai) = \dfrac{\pi}{6}$

$$\begin{aligned} \tan^{-1} a &= \frac{\pi}{6} \\ a &= \tan\left(\frac{\pi}{6}\right) \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

11 A If $z = 3 + 4i$ is a solution of the equation $z^2 + bz + c = 0$, then the two solutions are $z = 3 \pm 4i$. Therefore the quadratic has factors $(z - 3 - 4i)$ and $(z - 3 + 4i)$. We multiply these together to give

$$\begin{aligned} &(z-3-4i)(z-3+4i) \\ &= ((z-3)-4i)((z-3)+4i) \\ &= (z-3)^2 - (4i)^2 \\ &= z^2 - 6z + 9 + 16 \\ &= z^2 - 6z + 25 \end{aligned}$$

Therefore $b = -6$ and $c = 25$.

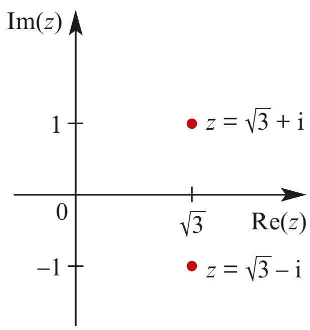
Solutions to extended-response questions

1 **a** $z^2 - 2\sqrt{3}z + 4 = 0$

Completing the square gives

$$z^2 - 2\sqrt{3}z + 3 + 1 = 0$$
$$\Rightarrow (z - \sqrt{3})^2 + 1 = 0$$
$$\Rightarrow (z - \sqrt{3})^2 - i^2 = 0$$
$$\Rightarrow (z - \sqrt{3} + i)(z - \sqrt{3} - i) = 0$$
$$\Rightarrow z = \sqrt{3} \pm i$$

b **i**



ii $|\sqrt{3} + i| = |\sqrt{3} - i| = 2$

The circle has centre the origin and radius 2.

The cartesian equation is $x^2 + y^2 = 4$.

iii The circle passes through $(0, 2)$ and $(0, -2)$. The corresponding complex numbers are $2i$ and $-2i$. So $a = 2$

2 $|z| = 6$

a **i**
$$|(1 + i)z| = |1 + i||z| = \sqrt{2} \times 6 = 6\sqrt{2}$$

ii
$$|(1 + i)z - z| = |z + iz - z| = |iz| = |i|\,|z| = 6$$

b A is the point corresponding to z, and $|OA| = 6$.

B is the point corresponding to $(1 + i)z$, and $|OB| = 6\sqrt{2}$.

From part **b**, $|AB| = |(1+i)z - z|$

$$= 6$$

Therefore ΔOAB is isosceles.

Note also that

$|OA|^2 + |AB|^2 = 6^2 + 6^2 = 72$

and $\quad |OB|^2 = (6\sqrt{2})^2$

$$= 72$$

The converse of Pythagoras' theorem gives the triangle is right-angled at A.

3 $z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

a $1 + z = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

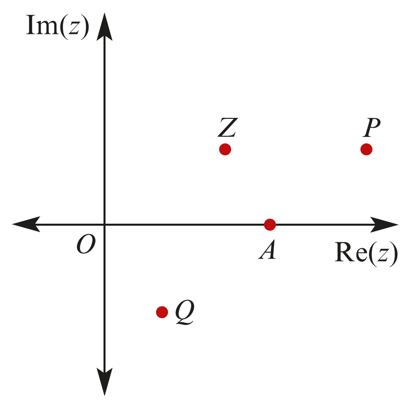
$$= \left(1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}i$$

$$= \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

and $1 - z = 1 - \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}i$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}}\frac{1}{\sqrt{2}}i$$



b $|OP|^2 = \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2}(2 + 2\sqrt{2} + 1 + 1)$

$= 2 + \sqrt{2}$

$|OQ|^2 = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2}(2 - 2\sqrt{2} + 1 + 1)$

$= 2 - \sqrt{2}$

$|QP| = |-1 + z + 1 + z|$

$= |2z|$

$= 2|z|$

$= 2$

and $|QP|^2 = 4$

Therefore $|QP|^2 = |OP|^2 + |OQ|^2$

By the converse of Pythagoras' theorem $\angle POQ$ is a right angle, i.e. $\angle POQ = \frac{\pi}{2}$

Now $\frac{|OP|}{|OQ|} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$

$= \sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}} \times \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$

$= \frac{2+\sqrt{2}}{\sqrt{2}}$

$= \sqrt{2} + 1$

4 For this question we will use the fact that $|z|^2 = z\bar{z}$. This is easy to prove.

a $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2$

$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$

b $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2})$

$$= (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$

$$= z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - \overline{z_1}z_2$$

$$= |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$$

c Since

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$$

and

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$$

we can add these two
equations to give,

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$$

This result has a geometric interpretation. By interpreting complex numbers z_1 and z_2 as vectors, we obtain a parallelogram with diagonals whose vectors are $z_1 + z_2$ and $z_1 - z_2$. This result then shows that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals

5 **a** For this question we will use the fact that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$. This is easy to prove if you haven't already seen it done.

i $\overline{\overline{z_1} z_2} = \overline{\overline{z_1}}\ \overline{z_2}$

$$= z_1\overline{z_2}$$

ii First note that $z + \overline{z} = 2\mathrm{Re}\,(z)$. Now using part (i) we have

$$z_1\overline{z_2} + \overline{z_1}z_2 = \overline{\overline{z_1}z_2} + \overline{z_1}z_2$$

$$= 2\mathrm{Re}\,(\overline{z_1}z_2),$$

which is a real number.

iii First note that $z - \overline{z} = 2i\ \mathrm{Im}\,(z)$. Now using part (i) we have

$$z_1\overline{z_2} - \overline{z_1}z_2 = \overline{\overline{z_1}z_2} - \overline{z_1}z_2$$
$$= 2i\ \mathrm{Im}\ (\overline{z_1}z_2),$$
which is an imaginary number.

iv Adding the results of the two previous questions gives

$$\begin{aligned}(z_1\overline{z_2} + \overline{z_1}z_2)^2 + (z_1\overline{z_2} - \overline{z_1}z_2)^2 &= (2\mathrm{Re}\ (\overline{z_1}z_2))^2 - (2i\ \mathrm{Im}\ (\overline{z_1}z_2))^2\\ &= 4(\mathrm{Re}\ (\overline{z_1}z_2))^2 + 4(\mathrm{Im}\ (\overline{z_1}z_2))^2\\ &= 4((\mathrm{Re}\ (\overline{z_1}z_2))^2 + (\mathrm{Im}\ (\overline{z_1}z_2))^2)\\ &= 4|\overline{z_1}z_2|^2\\ &= 4|\overline{z_1}||z_2|^2\\ &= 4|z_1||z_2|^2\\ &= 4|z_1z_2|^2.\end{aligned}$$

b
$$\begin{aligned}(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (z_1 + z_2)\overline{(z_1 + z_2)}\\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (z_1 + z_2)(\overline{z_1} + \overline{z_2})\\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2)\\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - (|z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2)\\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 - |z_1|^2 - |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)\\ &= 2|z_1||z_2| - (z_1\overline{z_2} + \overline{z_1}z_2)\\ &= 2|z_1||z_2| - 2\mathrm{Re}\ (\overline{z_1}z_2)\\ &= 2|\overline{z_1}||z_2| - 2\mathrm{Re}\ (\overline{z_1}z_2)\\ &= 2|\overline{z_1}z_2| - 2\mathrm{Re}\ (\overline{z_1}z_2)\\ &\geq 0\end{aligned}$$

c This question simply requires a trick:

$$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|.$$

Therefore,

$$|z_1 - z_2| \geq |z_1| - |z_2|.$$

6 $z = \text{cis}\theta$

a $z + 1 = \text{cis}\theta + 1$

$= \cos\theta + i\sin\theta + 1$

$= (1 + \cos\theta) + i\sin\theta$

$$|z + 1| = \sqrt{(1 + \cos\theta)^2 + \sin^2\theta}$$

$$= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \sqrt{1 + 2\cos\theta + 1}$$

$$= \sqrt{2 + 2\cos\theta}$$

$$= \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}$$

$$= 2\cos\left(\frac{\theta}{2}\right) \text{ since } 0 \le \frac{\theta}{2} \le \frac{\pi}{2}.$$

To find the argument, we find that

$$\frac{\sin\theta}{1 + \cos\theta} = \frac{\sin\theta}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}$$

so that $\text{Arg}(z + 1) = \dfrac{\theta}{2}$.

b $z - 1 = \text{cis}\,\theta - 1$

$$= \cos\theta + i\sin\theta - 1$$

$$= (\cos\theta - 1) + i\sin\theta$$

$$|z - 1| = \sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$$

$$= \sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$= \sqrt{2 - 2\cos\theta}$$

$$= \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}$$

$$= 2\sin\left(\frac{\theta}{2}\right) \text{ since } 0 \le \frac{\theta}{2} \le \frac{\pi}{2}.$$

To find the argument, we evaluate

$$\frac{\sin\theta}{\cos\theta - 1} = -\frac{\sin\theta}{1 - \cos\theta}$$

$$= -\frac{\sin\theta}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$= -\cot\left(\frac{\theta}{2}\right)$$

$$= \tan\left(\frac{\theta}{2} + \frac{\pi}{2}\right)$$

so that $\text{Arg}(z - 1) = \frac{\pi}{2} + \frac{\theta}{2}$.

c $$\left|\frac{z-1}{z+1}\right| = \frac{|z-1|}{|z+1|}$$
$$= \frac{2\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}$$
$$= \tan\left(\frac{\theta}{2}\right)$$
$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \operatorname{Arg}(z-1) - \operatorname{Arg}(z+1)$$
$$= \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}$$
$$= \frac{\pi}{2}$$

7 **a** $\Delta = b^2 - 4ac$

b The equation has no real solutions if and only if

$b^2 - 4ac < 0$.

c If $b^2 - 4ac$ then we can assume that
$$z_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } z_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a}.$$
It follows that P_1 has coordinates
$$\left(\frac{-b}{2a}, \frac{\sqrt{4ac - b^2}}{2a}\right)$$
and P_2 has coordinates
$$\left(\frac{-b}{2a}, -\frac{\sqrt{4ac - b^2}}{2a}\right).$$

i $$z_1 + z_2 = -\frac{b}{a}$$
$$|z_1| = |z_2| = \sqrt{\left(\frac{-b}{2a}\right)^2 + \left(\frac{\sqrt{4ac - b^2}}{2a}\right)^2}$$
$$= \sqrt{\frac{b^2}{4a^2} + \frac{4ac - b^2}{4a^2}}$$
$$= \sqrt{\frac{c}{a}}$$

ii To find $\angle P_1OP_2$ it will also help to find

$$z_1 - z_2 = \frac{i\sqrt{4ac - b^2}}{a}$$

$$|z_1 - z_2| = \frac{\sqrt{4ac - b^2}}{|a|}$$

Therefore, with reference to the diagram below, we use the cosine law to show that

$$P_1P_2 = OP_1^2 + OP_2^2 - 2 \cdot OP_1 \cdot OP_2 \cdot \cos\theta$$

$$\frac{4ac - b^2}{a^2} = \frac{c}{a} + \frac{c}{a} - 2\frac{c}{a}\cos\theta$$

$$\frac{4ac - b^2}{a^2} = \frac{2c}{a} - 2\frac{c}{a}\cos\theta$$

$$\frac{4ac - b^2}{a^2} = \frac{2c}{a}(1 - \cos\theta)$$

$$\frac{4ac - b^2}{a} = 2c(1 - \cos\theta)$$

$$1 - \cos\theta = \frac{4ac - b^2}{2ac}$$

$$\cos\theta = 1 - \frac{4ac - b^2}{2ac}$$

$$\cos\theta = \frac{b^2 - 2ac}{2ac.}$$

Therefore

$$\cos(\angle P_1OP_2) = \frac{b^2 - 2ac}{2ac}.$$

8 **a** It's perhaps fastest to simply use the quadratic formula here:

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= \frac{-1 \pm i\sqrt{3}}{2} \end{aligned}$$

so that

$$z_1 = \frac{-1 + i\sqrt{3}}{2} \text{ and } z_2 = \frac{-1 - i\sqrt{3}}{2}.$$

b We prove the first equality. The proof for the second is similar. We have

$$\begin{aligned} z_2^2 &= \left(\frac{-1 - i\sqrt{3}}{2}\right)^2 \\ &= \frac{1}{4}(1 + i\sqrt{3})^2 \\ &= \frac{1}{4}(1 + 2i\sqrt{3} + 3i^2) \\ &= \frac{1}{4}(-2 + 2i\sqrt{3}) \\ &= \frac{-1 + i\sqrt{3}}{2} \\ &= z_1, \end{aligned}$$

as required.

c First consider $z_1 = \dfrac{-1 + i\sqrt{3}}{2}$. The point is in the second quadrant.

$$\begin{aligned} r &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)} \\ &= \sqrt{\frac{1}{4} + \frac{3}{4}} \\ &= 1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3} \end{aligned}$$

$$\Rightarrow \frac{-1 + i\sqrt{3}}{2} = 1 \text{ cis}\left(\frac{2\pi}{3}\right).$$

Now consider $z_2 = \dfrac{-1 - i\sqrt{3}}{2}$. The point is in the third quadrant.

$$\begin{aligned} r &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)} \\ &= \sqrt{\frac{1}{4} + \frac{3}{4}} \\ &= 1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= -\frac{2\pi}{3} \end{aligned}$$

$$\Rightarrow \frac{-1 - i\sqrt{3}}{2} = 1 \operatorname{cis}\left(-\frac{2\pi}{3}\right).$$

d Plot points O, P_1 and P_2. From this, you will see that

$$\begin{aligned} A &= \frac{bh}{2} \\ &= \frac{\sqrt{3} \times \frac{1}{2}}{2} \\ &= \frac{\sqrt{3}}{4}. \end{aligned}$$

Chapter 19 – Revision of chapters 15-18

Solutions to Technology-free questions

1 **a** Since
$$\cos^2 A + \sin^2 A = 1,$$
we see that
$$\begin{aligned}\cos^2 A &= 1 - \sin^2 A\\ &= 1 - \left(\frac{3}{5}\right)^2\\ &= 1 - \frac{9}{25}\\ &= \frac{16}{25}.\end{aligned}$$
Therefore, $\cos A = \pm\frac{4}{5}$. However, as A is acute, we can reject the negative solution, giving $\cos A = \frac{4}{5}$. Therefore,
$$\sec A = \frac{1}{\cos A} = \frac{5}{4}.$$

b Using the result from the previous question we have,
$$\begin{aligned}\cot A &= \frac{\cos A}{\sin A}\\ &= \frac{\frac{4}{5}}{\frac{3}{5}}\\ &= \frac{4}{3}.\end{aligned}$$

c Since
$$\cos^2 B + \sin^2 B = 1,$$
we see that $\sin^2 B = 1 - \cos^2 A$
$$\begin{aligned}&= 1 - \left(-\frac{1}{2}\right)^2\\ &= 1 - \frac{1}{4}\\ &= \frac{3}{4}.\end{aligned}$$

Therefore, $\sin B = \pm\frac{\sqrt{3}}{2}$. However, as B is obtuse, we can reject the negative, giving $\sin B = \frac{\sqrt{3}}{2}$. It follows that,
$$\begin{aligned}\cot B &= \frac{\cos A}{\sin A}\\ &= \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\\ &= -\frac{\sqrt{3}}{3}.\end{aligned}$$

d Using work from the previous question, we have,
$$\begin{aligned}\operatorname{cosec} B &= \frac{1}{\sin B}\\ &= \frac{1}{\frac{\sqrt{3}}{2}}\\ &= \frac{2\sqrt{3}}{3}.\end{aligned}$$

2 Since $\cos A = 2\cos^2\frac{A}{2} - 1$, we know that
$$\begin{aligned}2\cos^2\frac{A}{2} - 1 &= \cos A\\ 2\cos^2\frac{A}{2} - 1 &= \frac{1}{3}\\ 2\cos^2\frac{A}{2} &= \frac{4}{3}\\ \cos^2\frac{A}{2} &= \frac{2}{3}\\ \cos\frac{A}{2} &= \pm\sqrt{\frac{2}{3}}\end{aligned}$$

Or equivalently,

$$\cos\frac{A}{2} = \pm\frac{\sqrt{6}}{3}.$$

3 We have,

$$\begin{aligned}
\text{LHS} &= \frac{1}{1+\sin A} + \frac{1}{1-\sin A} \\
&= \frac{1-\sin A}{(1+\sin A)(1-\sin A)} \\
&\quad + \frac{1+\sin A}{(1+\sin A)(1-\sin A)} \\
&= \frac{2}{1-\sin^2 A} \\
&= \frac{2}{\cos^2 A} \\
&= 2\sec^2 A \\
&= \text{RHS}.
\end{aligned}$$

4 **a** $w + z = (3+2i) + (3-2i)$
$= 6$

b $w - z = (3+2i) - (3-2i)$
$= 3 + 2i - 3 + 2i$
$= 4i$

c $wz = (3+2i)(3-2i)$
$= 3^2 - (2i)^2$
$= 9 + 4$
$= 13$

d
$$\begin{aligned}
w^2 + z^2 &= (3+2i)^2 + (3-2i)^2 \\
&= 9 + 12i + (2i)^2 + 9 - 12i + (2i)^2 \\
&= 18 + 4i^2 + 4i^2 \\
&= 10
\end{aligned}$$

e Using a previous result, we see that

$$\begin{aligned}
(w+z)^2 &= 6^2 \\
&= 36.
\end{aligned}$$

f Using a previous result, we see that

$$\begin{aligned}
(w-z)^2 &= (4i)^2 \\
&= -16.
\end{aligned}$$

g $w^2 - z^2 = (w-z)(w+z)$
$= 4i \times 6$
$= 24i$

h Using the previous question,

$$\begin{aligned}
(w-z)(w+z) &= w^2 - z^2 \\
&= 24i
\end{aligned}$$

5 **a** $w + z = (1-2i) + (2-3i)$
$= 3 - 5i$

b $w - z = (1-2i) - (2-3i)$
$= 1 - 2i - 2 + 3i$
$= -1 + i$

c $wz = (1-2i)(2-3i)$
$= 2 - 3i - 4i + 6i^2$
$= 2 - 7i - 6$
$= -4 - 7i$

d
$$\begin{aligned}
\frac{w}{z} &= \frac{1-2i}{2-3i} \\
&= \frac{(1-2i)}{(2-3i)}\frac{(2+3i)}{(2+3i)} \\
&= \frac{2+3i-4i-6i^2}{2^2-(3i)^2} \\
&= \frac{2-i-6}{4+9} \\
&= \frac{8-i}{13}
\end{aligned}$$

e $iw = i(1-2i)$
$= i - 2i^2$
$= 2 + i$

f $\frac{i}{w} = \frac{i}{1-2i}$
$= \frac{i}{(1-2i)}\frac{(1+2i)}{(1+2i)}$
$= \frac{i+2i^2}{1^2-(2i)^2}$
$= \frac{-2+i}{1+4}$
$= \frac{-2+i}{5}$

g $\frac{w}{i} = \frac{1-2i}{i}$
$= \frac{1-2i}{i}\frac{i}{i}$
$= \frac{i-2i^2}{i^2}$
$= \frac{2+i}{-1}$
$= -2-i$

h $\frac{z}{w} = \frac{2-3i}{1-2i}$
$= \frac{(2-3i)}{(1-2i)}\frac{(1+2i)}{(1+2i)}$
$= \frac{2+4i-3i-6i^2}{1^2-(2i)^2}$
$= \frac{2+i+6}{1+4}$
$= \frac{8+i}{5}$

i $\frac{w}{w+z} = \frac{1-2i}{3-5i}$
$= \frac{(1-2i)}{(3-5i)}\frac{(3+5i)}{(3+5i)}$
$= \frac{3+5i-6i-10i^2}{3^2-(5i)^2}$
$= \frac{3-i+10}{9+25}$
$= \frac{13-i}{34}$

j $(1+i)w = (1+i)(1-2i)$
$= 1-2i+i-2i^2$
$= 1-i+2$
$= 3-i$

k $\frac{w}{1+i} = \frac{1-2i}{1+i}$
$= \frac{(1-2i)}{(1+i)}\frac{(1-i)}{(1-i)}$
$= \frac{1-i-2i+2i^2}{1^2-i^2}$
$= \frac{1-3i-2}{1+1}$
$= \frac{-1-3i}{2}$

l $w^2 = (1-2i)^2$
$= 1-4i+(2i)^2$
$= 1-4i-4$
$= -3-4i$

6 **a** $z^2 + 49 = z^2 - (7i)^2$
$= (z-7i)(z+7i)$

b Here, we must complete this square, giving,

$$
\begin{aligned}
z^2 - 2z + 10 &= (z^2 - 2z + 1) - 1 + 10\\
&= (z-1)^2 + 9\\
&= (z-1)^2 - (3i)^2\\
&= (z - 1 - 3i)(z - 1 + 3i)
\end{aligned}
$$

c Here, we must complete this square. Factor out the 9 first, so that

$$
\begin{aligned}
9z^2 - 6z + 5 &= 9(z^2 - \frac{2}{3}z + \frac{5}{9})\\
&= 9\left(\left(z^2 - \frac{2}{3}z + \frac{1}{9}\right) - \frac{1}{9} + \frac{5}{9}\right)\\
&= 9\left(\left(z - \frac{1}{3}\right)^2 + \frac{4}{9}\right)\\
&= 9\left(\left(z - \frac{1}{3}\right)^2 - \left(\frac{2}{3}i\right)^2\right)\\
&= 9\left(z - \frac{1}{3} - \frac{2}{3}i\right)\left(z - \frac{1}{3} + \frac{2}{3}i\right)
\end{aligned}
$$

d Here, we must complete this square. Factor out the 4 first, so that

$$
\begin{aligned}
4z^2 + 12z + 13 &= 4(z^2 + 3z + \frac{13}{4})\\
&= 4\left(\left(z^2 + 3z + \frac{9}{4}\right) - \frac{9}{4} + \frac{13}{4}\right)\\
&= 4\left(\left(z + \frac{3}{2}\right)^2 + 1\right)\\
&= 4\left(\left(z + \frac{3}{2}\right)^2 - i^2\right)\\
&= 4\left(z + \frac{3}{2} - i\right)\left(z + \frac{3}{2} + i\right)
\end{aligned}
$$

7 a We need to find real numbers x and y such that

$$
\begin{aligned}
(x + iy)^2 &= 3 - 4i\\
x^2 + 2xyi + (iy)^2 &= 3 - 4i\\
(x^2 - y^2) + 2xyi &= 3 - 4i
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x^2 - y^2 &= 3 \quad (1)\\
2xy &= -4 \quad (2).
\end{aligned}
$$

Solving equation (2) for y gives $y = -\frac{2}{x}$, and substituting this into equation (1) gives

$$
\begin{aligned}
x^2 - \left(-\frac{2}{x}\right)^2 &= 3\\
x^2 - \frac{4}{x^2} &= 3\\
x^4 - 4 &= 3x^2\\
x^4 - 3x^2 - 4 &= 0\\
(x^2 - 4)(x^2 + 1) &= 0\\
x &= \pm 2.
\end{aligned}
$$

Moreover, if $x = 2$, then $y = -1$ while if $x = -2$ then $y = 1$. Therefore the two square roots are $2 - i$ and $-2 + i$.

b We have

$$
\begin{aligned}
z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
&= \frac{-(4+3i) \pm \sqrt{(4+3i)^2 - 4(2-i)(-1+3i)}}{2(2-i)}\\
&= \frac{-4 - 3i \pm \sqrt{16 + 24i + (3i)^2 - 4(-2 + 6i + i - 3i^2)}}{4 - 2i}\\
&= \frac{-4 - 3i \pm \sqrt{16 + 24i - 9 - 4(-2 + 7i + 3)}}{4 - 2i}\\
&= \frac{-4 - 3i \pm \sqrt{7 + 24i - 4(1 + 7i)}}{4 - 2i}\\
&= \frac{-4 - 3i \pm \sqrt{7 + 24i - 4 - 28i}}{4 - 2i}\\
&= \frac{-4 - 3i \pm \sqrt{3 - 4i}}{4 - 2i} \qquad (1)
\end{aligned}
$$

From the previous question, we know that either $\sqrt{3 - 4i} = 2 - i$ or $\sqrt{3 - 4i} = -2 + i$. We can subgraph-stitute either of these into (1) to find that $z = -i$ or $z = 1 - i$.

8 If $z = -1 + i$ is a solution then so is the conjugate $z = -1 - i$. Let w be the third solution. As the sum of the solutions is 4 we know that

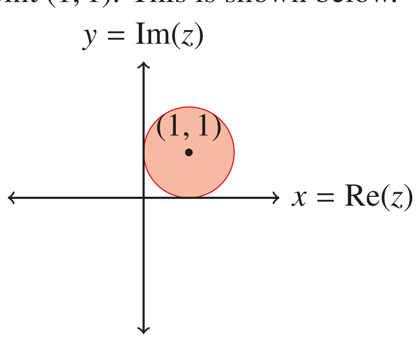
$$\begin{aligned}(-1+i)+(-1-i)+w &= 4\\ -2+w &= 4\\ w &= 6\end{aligned}$$

Therefore we now know the three linear factors of the cubic, which we multiply to give

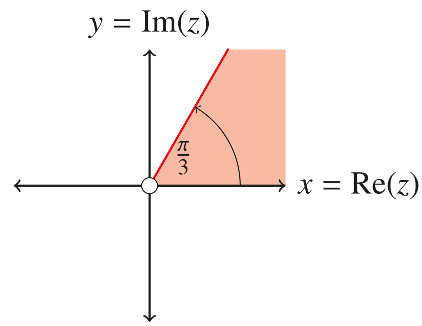
$$\begin{aligned}&(z+1-i)(z+1+i)(z-6)\\ &= ((z+1)-i)((z+1)+i)(z-6)\\ &= ((z+1)^2-i^2)(z-6)\\ &= ((z^2+2z+1)+1)(z-6)\\ &= (z^2+2z+2)(z-6)\\ &= z^3-4z-10z-12.\end{aligned}$$

Therefore $a = -4, b = -10$ and $c = -12$.

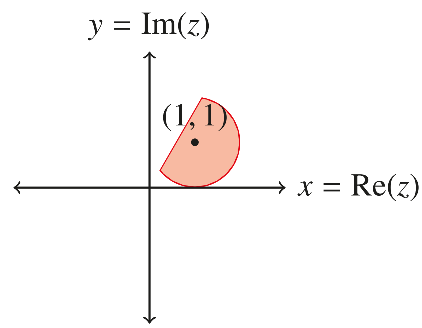
9 **a** Region S is the set of of points on or inside a circle or radius 1 centred at point $(1, 1)$. This is shown below.



b The set T consists of the set of points whose principal argument lies between 0 and $\frac{\pi}{3}$. This is sketched below. The origin must be omitted.



c The region $S \cap T$ is the circle segment shown.



10 **a**

b

c

11 **a** The sketch of these graphs is shown below.

b To find the point of intersection we have to solve the equation

$$x^2 - 4 = |x| + 2$$
$$x^2 - 6 = |x|$$

If $x \geq 0$, then $|x| = x$ so that

$$\begin{aligned} x^2 - 6 &= x \\ x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0 \\ x &= 3. \end{aligned}$$

Note: we take only the positive solution. This gives the point $(3, 5)$.

If $x < 0$, then $|x| = -x$ so that

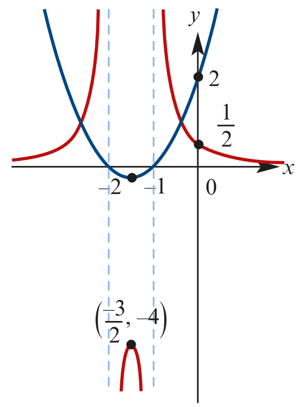
$$\begin{aligned} x^2 - 6 &= -x \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ x &= -3. \end{aligned}$$

Note: we take only the negative solution. This gives the point $(-3, 5)$

12 **a**

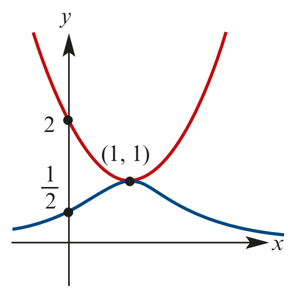
b

13 a



y
x
2
$\frac{1}{2}$
−2
−1
0
$\left(\frac{-3}{2}, -4\right)$

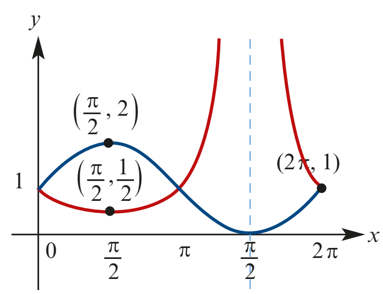
b



y
x
2
$\frac{1}{2}$
(1, 1)

c

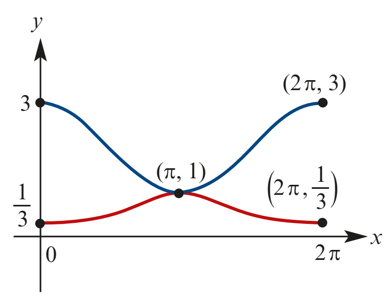
c



y
x
1
$\left(\frac{\pi}{2}, 2\right)$
$\left(\frac{\pi}{2}, \frac{1}{2}\right)$
$(2\pi, 1)$
0
$\frac{\pi}{2}$
π
$\frac{\pi}{2}$
2π

d

d



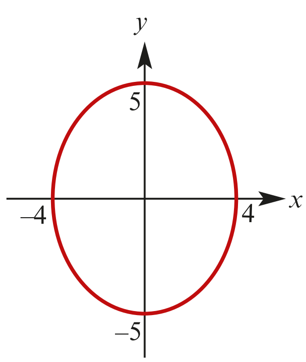
y
x
3
$\frac{1}{3}$
$(2\pi, 3)$
$(\pi, 1)$
$\left(2\pi, \frac{1}{3}\right)$
0
2π

14 a

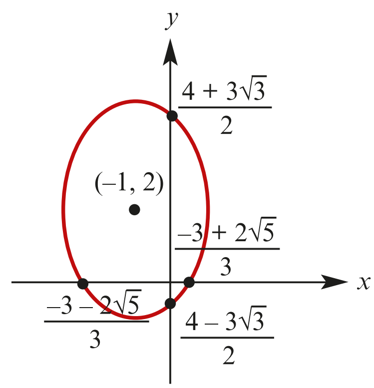
b

c

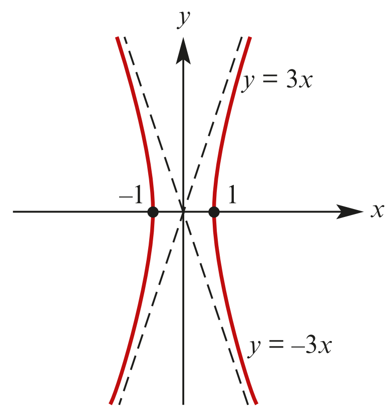
15 a



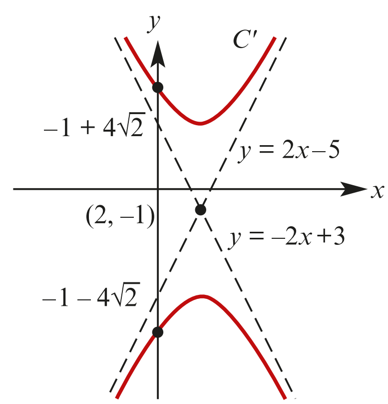
b



16 a



b



17 We know that the point $P(x, y)$ satisfies,

$$AP = BP$$

$$\sqrt{(x-2)^2+(y-2)^2} = \sqrt{(x-3)^2+(y-4)^2}$$

$$(x-2)^2+(y-2)^2 = (x-3)^2+(y-4)^2$$

$$-4x+4-4y+4 = -6x+9-8y+16$$

$$2x+4y = 17$$

18 Let (x, y) be the coordinates of point P.

If $FP = \frac{1}{2}MP$ then

$$\sqrt{x^2 + (y-1)^2} = \frac{1}{2}\sqrt{(x-(-3))^2}.$$

Squaring both sides gives

$$x^2 + (y-1)^2 = \frac{1}{4}(x+3)^2$$
$$4x^2 + 4(y-1)^2 = x^2 + 6x + 9$$
$$3x^2 - 6x + 4(y-1)^2 = 9$$
$$3x^2 - 6x + 4(y-1)^2 = 9$$

Completing the square

$$3x^2 - 6x + 4(y-1)^2 = 9$$
$$3(x^2 - 2x) + 4(y-1)^2 = 9$$
$$3((x^2 - 2x + 1) - 1) + 4(y-1)^2 = 9$$
$$3((x-1)^2 - 1) + 4(y-1)^2 = 9$$
$$3(x-1)^2 + 4(y-1)^2 = 12$$

or equivalently $\dfrac{(x-1)^2}{4} + \dfrac{(y-1)^2}{3} = 1.$

This is an ellipse with centre $(1, 1)$.

19 We know that the point $P(x, y)$ satisfies,

$$FP = RP$$
$$\sqrt{x^2 + (y-1)^2} = \sqrt{(y-(-3))^2}$$
$$x^2 + (y-1)^2 = (y+3)^2$$
$$x^2 + y^2 - 2y + 1 = y^2 + 6y + 9$$
$$x^2 - 2y + 1 = 6y + 9$$
$$8y = x^2 - 8$$
$$y = \frac{x^2}{8} - 1$$

Therefore, the set of points is a parabola whose equation is $y = \dfrac{x^2}{8} - 1$

20 **a** Since $x = 2t + 1$ and $y = 2 - 3t$ we solve both equations for t to find that

$$t = \frac{x-1}{2} \text{ and } t = \frac{2-y}{3}.$$

Eliminating t then gives

$$\frac{x-1}{2} = \frac{2-y}{3}$$
$$3(x-1) = 2(2-y)$$
$$3x - 3 = 4 - 2y$$
$$3x + 2y = 7.$$

b Since

$$x^2 + y^2 = \cos^2 2t + \sin^2 2t$$
$$= 1,$$

these equations parameterise a circle with centre $(0, 0)$ and radius 1.

c Solving each equation for the $\cos t$ and $\sin t$ respectively gives,

$$\cos t = \frac{x-2}{2} \text{ and } \sin t = \frac{y-3}{3}.$$

Therefore,

$$\left(\frac{x-2}{2}\right)^2 + \left(\frac{y-3}{3}\right)^2 = \cos^2 t + \sin^2 t$$
$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1.$$

d Solving each equation for the $\tan t$ and $\sec t$ respectively gives,

$$\tan t = \frac{x}{2} \text{ and } \sec t = \frac{y}{3}.$$

Therefore,

$$\left(\frac{y}{3}\right)^2 - \left(\frac{x}{2}\right)^2 = \sec^2 t - \tan^2 t$$
$$\frac{y^2}{9} - \frac{x^2}{4} = 1.$$

21 **a** Since $x = t - 1$ we know that $t = x + 1$. Substitute this into the second parametric equation to give,

$$y = 1 - 2t^2$$
$$= 1 - 2(x+1)^2.$$

b Since $0 \le t \le 2$ we have

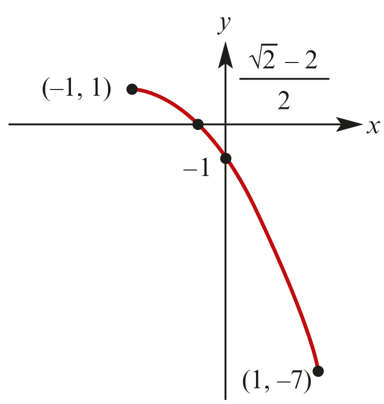
$$0 \le x + 1 \le 2$$
$$-1 \le x \le 1$$

c Since $0 \le t \le 2$, we have that

$$-7 \le 1 - 2t^2 \le 1.$$

Therefore the range is $-7 \le y \le 1$.

d



22 We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 2\cos 7\pi/6 \qquad = 2\sin 7\pi/6$$
$$= -\sqrt{3} \qquad = -1$$

so that the cartesian coordinates are $(-\sqrt{3}, -1)$.

23 Finding r first gives,

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}.$$

Since

$$\tan\theta = \frac{-2}{2} = -1,$$

we can assume that $\theta = -\frac{\pi}{4}$ so that the point has polar coordinates

$$\left[2\sqrt{2}, -\frac{\pi}{4}\right].$$

We could also let $r = -2\sqrt{2}$ and add π to the previously found angle, giving

$$\left[2\sqrt{2}, \frac{3\pi}{4}\right].$$

24 **a** Since $r = 5$ and $r^2 = x^2 + y^2$ we know that

$$x^2 + y^2 = 5^2.$$

This is a circle of radius 5 centred at the origin.

b Since $\tan\theta = \frac{y}{x}$ we know that

$$\frac{y}{x} = \tan\left(\frac{\pi}{3}\right)$$
$$\frac{y}{x} = \sqrt{3}$$
$$y = \sqrt{3}x.$$

c Since $y = r\sin\theta$, we know that

$$r = \frac{3}{\sin\theta}$$
$$r\sin\theta = 3$$
$$y = 3.$$

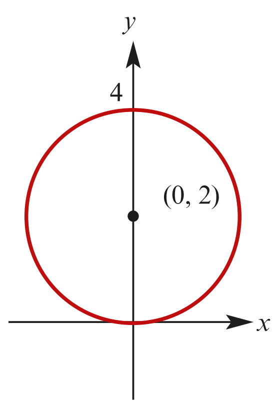
d Since $x = r\cos\theta$ and $y = r\sin\theta$, we know that

$$\frac{2}{3\sin\theta + 4\cos\theta} = r$$
$$3r\sin\theta + 4r\cos\theta = 2$$
$$3y + 4x = 2.$$

e Since $\sin(2\theta) = 2\sin\theta\cos\theta$, we have

$$\begin{aligned} r^2 &= \frac{1}{\sin(2\theta)} \\ r^2\sin(2\theta) &= 1 \\ 2r^2\sin\theta\cos\theta &= 1 \\ 2(r\sin\theta)(r\cos\theta) &= 1 \\ 2yx &= 1 \\ y &= \frac{1}{2x}. \end{aligned}$$

25 a



b You can start with the polar equation and show that it has the given cartesian equation or visa versa. We start with $r = 4\sin\theta$. Multiplying both sides by r gives,

$$\begin{aligned} r^2 &= 4r\sin\theta \text{ as} \\ x^2 + y^2 &= 4x \\ x^2 - 4x + y^2 &= 0 \\ (x^2 - 4x + 4) - 4 + y^2 &= 0 \\ (x-2)^2 + y^2 &= 2^2, \end{aligned}$$

required.

Solutions to multiple-choice questions

1 B Since

$$\frac{c}{\sin 38°} = \frac{58}{\sin 130°}$$

it follow that

$$c = \frac{58 \sin 38°}{\sin 130°}.$$

2 B We first must find $\cos A$ and $\cos B$. Since both angle are acute, we know that

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{12}{13},$$

$$\cos B = \sqrt{1 - \sin^2 B} = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \frac{15}{17}.$$

Therefore,

$$\begin{aligned}\sin(A - B) &= \sin A \cos B - \cos A \sin B \\ &= \frac{5}{13}\frac{15}{17} - \frac{12}{13}\frac{8}{17} \\ &= -\frac{21}{221}.\end{aligned}$$

3 D We can find the area of the triangle using the formula

$$A = \frac{1}{2} bc \sin A.$$

You can find side a using the cosine rule,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Once you've found a you can fine angle B using the sine rule,

$$\frac{\sin B}{b} = \frac{\sin A}{a}.$$

Therefore, you can find all three options.

4 E We first must find $\cos A$ and $\cos B$. Since both angle are acute, we know that

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\tfrac{5}{13}\right)^2} = \frac{12}{13},$$

$$\cos B = \sqrt{1 - \sin^2 B} = \sqrt{1 - \left(\tfrac{8}{17}\right)^2} = \frac{15}{17}.$$

Therefore,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12},$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}.$$

Therefore,

$$\begin{aligned}\tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{\frac{5}{12} + \frac{8}{15}}{1 - \frac{5}{12}\frac{8}{15}} \\ &= \frac{171}{140}.\end{aligned}$$

5 A Since angle A is the angle between the given sides, the area will be given by

$$A = \frac{1}{2} \times 6 \times 7 \sin 48°.$$

6 D If $\cos\theta = c$ and θ is acute then

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - c^2}.$$

Therefore,

$$\begin{aligned}\cot\theta &= \frac{\cos\theta}{\sin\theta} \\ &= \frac{c}{\sqrt{1 - c^2}}.\end{aligned}$$

7 E As the arc length L is given by the formula

$$L = \frac{\pi r \theta°}{180°},$$

we can rearrange this for $\theta°$, giving,

$$\begin{aligned}\theta° &= \frac{180°L}{\pi r}\\ &= \frac{180° \times 3}{\pi \times 4}\\ &\approx 43°.\end{aligned}$$

Item B gives the closest answer.

8 D We have,

$$\begin{aligned}\cos A \cos B - \sin A \sin B &= \cos(A + B)\\ &= \cos\frac{\pi}{2}\\ &= 0.\end{aligned}$$

9 E We first must find $\cos A$. Since A is obtuse, we know that

$$\begin{aligned}\cos A &= -\sqrt{1 - \sin^2 A}\\ &= -\sqrt{1 - \left(\tfrac{\sqrt{5}}{3}\right)^2}\\ &= -\frac{2}{3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sin(2A) &= 2\sin A\cos A\\ &= -2 \times \frac{\sqrt{5}}{3} \times \frac{2}{3}\\ &= -\frac{4\sqrt{5}}{9}\end{aligned}$$

10 C The area of the sector will be

$$\begin{aligned}A &= \frac{\theta}{360°}\pi r^2\\ &= \frac{60°}{360°} \times \pi \times 5^2\\ &\approx 13.09 \text{ cm}^2\end{aligned}$$

11 E Considering right ΔVOE, we have

$$\begin{aligned}\tan\theta &= \frac{VO}{OE}\\ &= \frac{100}{40}\\ &\frac{5}{2}.\end{aligned}$$

Therefore,

$$\theta = \tan^{-1}\tfrac{5}{2} \approx 68°.$$

12 E If $\cos\theta = c$ and θ is acute then

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - c^2}.$$

Therefore,

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta\\ &= 2c\sqrt{1 - c^2}.\end{aligned}$$

13 D We can use the sum to product identities to give

$$\begin{aligned}\cos(3x) + \cos(x) &= 2cos(\tfrac{3x+x}{2})\cos(\tfrac{3x-x}{2})\\ &= 2\cos(\tfrac{4x}{2})\cos(\tfrac{2x}{2})\\ &= 2\cos(2x)\cos(x)\end{aligned}$$

14 C This is most efficiently solved using your calculator, giving, 41.50° and 244.67°.

15 B We simply find the area of the circle segment,

$$\begin{aligned}A &= \frac{\pi r^2\theta}{360} - \frac{1}{2}r^2\sin\theta\\ &= \frac{\pi \times 45^2 \times 110}{360} - \frac{1}{2} \times 45^2 \sin 110°\\ &\approx 992 \text{ cm}^2\end{aligned}$$

16 C $8\sin\theta\cos^3\theta - 8\sin^3\theta\cos\theta$

$$\begin{aligned}&= 8\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)\\ &= 4\sin(2\theta)\cos(2\theta)\\ &= 2\sin(4\theta).\end{aligned}$$

17 **B** We have,

$$z = vw$$
$$= 4\,\text{cis}(-0.3\pi) \times 5\text{cis}(-0.6\pi)$$
$$= 20\,\text{cis}(-0.3\pi + (-0.6\pi))$$
$$= 20\,\text{cis}(-0.9\pi)$$

so that $\text{Arg}z = -0.9\pi$.

18 **D** $2\,\text{cis}\left(\frac{2\pi}{3}\right) = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$

$$= 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2})$$
$$= -1 + \sqrt{3}i$$

19 **E** This complex number is in the third quadrant. Moreover, since

$$\tan\theta = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}},$$

Therefore, $\theta = -\frac{5\pi}{6}$.

20 **C** $uv = 3\,\text{cis}\left(\frac{\pi}{2}\right) \cdot 5\,\text{cis}\left(\frac{2\pi}{3}\right)$

$$= 15\,\text{cis}\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)$$
$$= 15\,\text{cis}\left(\frac{3\pi}{6} + \frac{4\pi}{6}\right)$$
$$= 15\,\text{cis}\left(\frac{7\pi}{6}\right)$$
$$= 15\,\text{cis}\left(\frac{7\pi}{6} - 2\pi\right)$$
$$= 15\,\text{cis}\left(-\frac{5\pi}{6}\right)$$

21 **C** The modulus is given by

$$|12 - 5i| = \sqrt{12^2 + (-5)^2} = 13.$$

22 **A**
- z^2 need not be real. For example, $(1 + i)^2 = 2i$ is not a real number.
- Since $z\bar{z} = |z|^2$, this will always be a real number.
- Since $z^{-1}z = 1$, this will always be a real number.
- $\text{Im}z$ is the coefficient of i, and so will be real.
- Since $z + \bar{z} = 2\,\text{Re}(z)$, this will be real.

23 **C** $\bar{z} = -14 + 7i$.

24 **E** Factorising the expression gives,

$$3z^2 + 9 = 3(z^2 + 3)$$
$$= 3(z^2 - (\sqrt{3}i)^2)$$
$$= 3(z - \sqrt{3}i)(z + \sqrt{3}i).$$

25 **C** Expanding the brackets gives,

$$(1 + 2i)^2 = 1 + 4i + (2i)^2$$
$$= 1 + 4i - 4$$
$$= -3 + 4i.$$

26 **D** Set S is a circle of radius r centred at the point $C(1, -2)$. The point $z = 4 + 2i$ corresponds to the point $Z(4, 2)$. As Z is on the circle, to find the value of r we find distance CZ. This gives

$$r = CZ$$
$$= \sqrt{(4 - 1)^2 + (2 - (-2))^2}$$
$$= 5.$$

27 **E**
- The answer is not A as this equation has solutions $z = \pm\sqrt{2}$.
- The answer is not B as this equation has solutions $z = \pm\sqrt{2}i$.

- The answer is not C as this equation has solutions $z = \pm 2$.

- The answer is not D as when $z = 2i$ we find that

$$z^3 - 3z^2 + 4z - 11$$
$$= (2i)^3 - 3(2i)^2 + 4(2i) - 11$$
$$= 1.$$

- The answer is E as when $z = 2i$ we find that

$$z^3 - 3z^2 + 4z - 12$$
$$= (2i)^3 - 3(2i)^2 + 4(2i) - 11$$
$$= 0.$$

28 C The graph shown can be obtained from the graph of $y = |x|$ by a reflection in the x-axis followed by a translation 2 units to the right and 2 units up. Therefore the required equation is $y = -|x - 2| + 2$.

29 B The range of $f(x) = \sin^{-1}(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By translating this graph up by $\frac{\pi}{2}$ we will obtain the correct range. Therefore $b = \frac{\pi}{2}$. Therefore the answer is either B, C or D. If $a = -\frac{1}{2}$ then the range can be found by solving

$$-1 \le -\frac{1}{2}x + 2 \le 1$$
$$-3 \le -\frac{1}{2}x \le -1$$
$$2 \le x \le 6$$

This is the only value of a that gives the correct range. Therefore $a = -\frac{1}{2}$ and $b = \frac{\pi}{2}$.

30 A As the graph has asymptote at $x = \pm 2$, the denominator of the function must be equal to zero when $x = \pm 2$. This leaves only items A and B. For item B, if $x = 0$ then $y = -\dfrac{1}{4}$. This does not agree with the given graph, leaving only item A.

31 B The graph of $y = a\sin(x) + b$ needs to have two x-intercepts. This will happen provided that the amplitude a exceeds the vertical translation term b. That is, $a > b$.

32 C Since $f(x) = \sec(2x) = \dfrac{1}{\cos(2x)}$, the graph of f will have local minimum turning points precisely where $y = \cos(2x)$ has local maximum turning points. These occur where $x = -\pi, 0, \pi$.

33 B Since the distance from fixed point A to point $P(x, y)$ is a constant, the set of points must be a circle.

34 B Since $AP = BP$ for each point $P(x, y)$, the line $y = x + 1$ is the perpendicular bisector of line AB. The line the perpendicular bisector of points $A(0, 0)$ and $B(-1, 1)$, but none of the other pairs.

35 C

- Item A is false as the axis of symmetry will be $x = 0$.

- Item B is false. The parabola will not go through the origin as the distance from $(0, 0)$ to $F(0, 2)$ is 2 while the distance from $(0, 0)$ to $y = -4$ is 4.

- Item C is true. The distance from $F(0, 2)$ to $(0, -1)$ is 3 is equal to the distance from $y = -4$ to $(0, -1)$.

- Item D false. The distance from $F(0, 2)$ to $(1, 2)$ is not equal to the distance from $y = -4$ to $(1, 2)$.

- Item E is false. This cannot be the equation of the parabola, as the parabola must go through the point $(0, -1)$ and so has a y-intercept of -1.

36 B The hyperbola has x-intercpets at $x = \pm 1$. The ellipse will have x-intercepts at $x = \pm a$. Therefore, to have four points of intersection we require that $a > 1$.

37 D To find the centre of the hyperbola, we can find the point of intersection of the asymptotes. To find this, we solve,

$$\begin{aligned} 2x + 1 &= -2x + 1 \\ 4x &= 0 \\ x &= 0. \end{aligned}$$

Therefore, $y = 1$ and the centre is $(0, 1)$. This leaves items A,B and D. The graph has no x-axis intercept. Therefore we can exclude items A and B. This leaves item D.

38 A Since $x = 1 + t$, we know that $t = x - 1$. Substituting $t = x - 1$ into the second equation gives,

$$\begin{aligned} y &= \frac{1 - t}{1 + t} \\ &= \frac{1 - (x - 1)}{1 + (x - 1)} \\ &= \frac{2 - x}{x} \\ &= \frac{2}{x} - 1. \end{aligned}$$

39 E We can write this equation as

$$\left(\frac{x - 1}{2}\right)^2 + \left(\frac{y + 1}{3}\right)^2 = 1.$$

So we can set

$$\cos t = \frac{x - 1}{2},$$

$$\sin t = \frac{y + 1}{3},$$

so that

$$x = 2\cos(t) + 1,$$

$$y = 3\sin(t) - 1.$$

40 D Notice that in each of the items the x-coordinate is 5. So solving $5 = 2t - 3$ for t gives $t = 4$. Now let $t = 4$ in the second equation to give

$$y = 4^2 - 3 \times 4 = 16 - 12 = 4.$$

Therefore the coordinates are $(5, 4)$.

41 A Since $y = r\sin\theta$ and $x = r\cos\theta$, we obtain, $r^2\cos^2\theta = r\sin\theta$

$$\begin{aligned} r &= \frac{\sin\theta}{\cos^2\theta} \\ &= \sec\theta\tan\theta. \end{aligned}$$

Solutions to extended-response questions

1 **a** **i** We denote $\angle BCA$ by C then using the sine rule, we obtain,

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$
$$\frac{\sin C}{80} = \frac{\sin 30^\circ}{60}$$
$$\sin C = \frac{80 \sin 30^\circ}{60}$$
$$= \frac{2}{3}$$

Note that $\sin^{-1}\left(\frac{2}{3}\right) \approx 41.81^\circ$, which is acute. We require an obtuse angle, so that

$$\angle BCA = 180^\circ - 41.81^\circ = 138.19^\circ.$$

Therefore,

$$\angle ABC = 180 - 30 - 138.19 = 11.81^\circ.$$

ii Using the answers to the previous question to obtain

$$\angle BC'A = 180 - 138.98 = 41.81^\circ,$$
$$\angle ABC' = 180 - 30 - 41.81 = 108.19^\circ.$$

b **i** We denote AC by b. Therefore,

$$\frac{b}{\sin B} = \frac{60}{\sin 30^\circ}$$
$$\frac{b}{\sin 11.81^\circ} = \frac{60}{\sin 30^\circ}$$
$$b = \frac{60 \sin 11.81^\circ)}{\sin 30^\circ}$$
$$\approx 24.56.$$

ii Since $\angle ABC' \approx 108.19^\circ$., we again use the sine rule to find that

$$\frac{AC}{\sin 108.19^\circ} = \frac{60}{\sin 30^\circ}$$
$$AC = \frac{60 \sin 108.19^\circ}{\sin 30^\circ}$$
$$\approx 114.00.$$

iii Subtracting the two previous answers gives,

$$CC' = AC' - AC = 114.00 - 24.56 = 89.44.$$

c **i** The area of the triangle will be

$$A = \frac{1}{2} \times 60 \times 60 \times \sin 96.38$$
$$\approx 1788.85.$$

ii The area of the sector will be

$$A = \frac{\theta}{360^\circ}\pi r^2$$
$$= \frac{96.38^\circ}{360^\circ} \times \pi \times 60^2$$
$$\approx 3027.87.$$

iii The area of the shaded segment will be,

$$A = \frac{\pi r^2 \theta}{360} - \frac{1}{2} r^2 \sin\theta$$
$$= \frac{\pi \times 60^2 \times 96.38}{360} - \frac{1^{\,2}}{2} \sin 96.38$$
$$\approx 1239.01$$

2 **a** Since triangle AEB is isosceles, $\angle EBA = \angle EAB = \theta$. By the exterior angle theorem,

$$\angle BED = \angle EBA + \angle EAB$$
$$= \theta + \theta$$
$$= 2\theta.$$

b Using the cosine rule, we have

$$1 = 1^2 + DE^2 - 2 \times 1 \times DE \cos(2\theta)$$
$$0 = DE^2 - 2DE\cos(2\theta)$$
$$0 = DE(DE - 2\cos(2\theta))$$

Since $DE \neq 0$, this implies that $DE - 2\cos(2\theta) = 0$. Therefore, $DE = 2\cos(2\theta)$, as required.

c **i** Firstly, we show that $\angle DBC = 3\theta$. By the exterior angle theorem, applied to triangle ADB, we know that

$$\angle DBC = \angle BAD + \angle BDA$$
$$= \theta + 2\theta$$
$$= 3\theta.$$

Therefore, considering right angled triangle BCD, we obtain

$$\sin(3\theta) = \frac{DC}{1}$$
$$DC = \sin(3\theta),$$

as required.

ii Applying the sine rule to triangle ADB gives

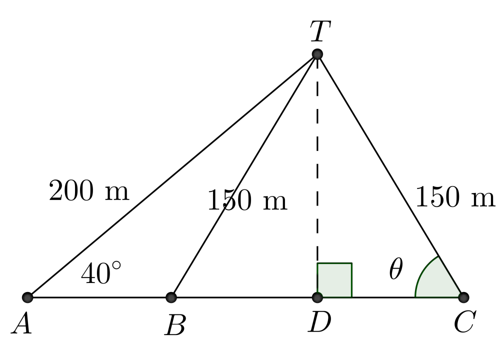
$$\frac{AD}{\sin(180-3\theta)} = \frac{BD}{\sin\theta}$$
$$\frac{AD}{\sin(3\theta)} = \frac{1}{\sin\theta}$$
$$AD = \frac{\sin(3\theta)}{\sin\theta},$$

as required.

d Since $AD = AE + DE$ we know that

$$\frac{\sin(3\theta)}{\sin\theta} = 1 + 2\cos(2\theta)$$
$$\begin{aligned}\sin(3\theta) &= \sin\theta + 2\cos(2\theta)\sin\theta\\ &= \sin\theta + 2(1-2\sin^2\theta)\sin\theta\\ &= \sin\theta + 2\sin\theta - 4\sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

3 a Consider the diagram below.



We let $\theta = \angle ACT$. We can find this angle using the sine rule applied to triangle ACT. This gives,

$$\frac{\sin\theta}{200} = \frac{\sin 40°}{150}$$
$$\sin\theta = \frac{200\sin 40°}{150}$$
$$\approx 58.99°.$$

Now draw a line through T perpendicular to BC. By considering right-angled triangle DCT we can find length DC as

$$\begin{aligned}\cos\theta &= \frac{DC}{150}\\ DC &= 150\cos\theta\\ &\approx 150\cos 58.99^\circ\\ &\approx 77.28 \text{ m.}\end{aligned}$$

Therefore the distance between B and C is,

$$BC = 2DC \approx 154.57 \text{ m.}$$

b **i** By considering right angled triangle BAT, we know that

$$\begin{aligned}\tan 32^\circ &= \frac{10}{AB}\\ AB &= \frac{10}{\tan 32}\\ &\approx 16.00 \text{ m.}\end{aligned}$$

ii By considering right angled triangle DAT, we know that

$$\begin{aligned}\tan 19^\circ &= \frac{10}{AD}\\ AD &= \frac{10}{\tan 19}\\ &\approx 29.04 \text{ m.}\end{aligned}$$

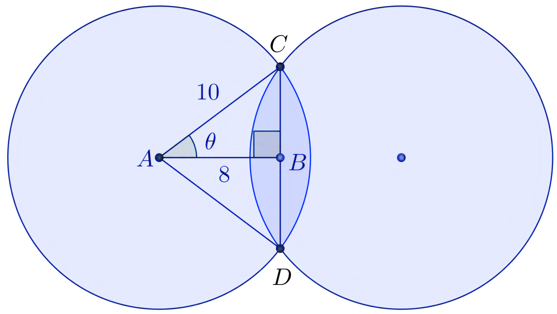
iii We first need to establish the length of AC. Using Pythagoras' theorem, we obtain,

$$AC = \sqrt{BA^2 + DA^2} \approx 33.16 \text{ m.}$$

Therefore, the angle of depression can be found as,

$$\begin{aligned}\tan\theta &= \frac{10}{AC}\\ &\approx 0.30157\\ \theta &\approx \tan^{-1}(0.30157)\\ &\approx 16.78^\circ\end{aligned}$$

c Consider right-angled triangle ABC on the diagram below.



We first find angle θ in radians,

$$\cos\theta = \frac{8}{10}$$

$$\theta = \cos^{-1}\frac{4}{5}$$

$$\approx 0.6435.$$

Therefore,

$$\angle DAC \approx 2 \times 0.6435 = 1.2870.$$

To find the area common to both circles, we simply have to calculate twice the area of minor segment CD. This is given by,

$$A = 2 \times \frac{1}{2} \times 10^2 \times (1.2870 - \sin 1.2870)$$

$$\approx 32.7 \text{ cm}^2.$$

4 **a** We first calculate $\theta = \angle TSO$. Using the sine rule, we obtain

$$\frac{\sin\theta}{6400} = \frac{\sin 120^\circ}{8000}$$

$$\sin\theta = \frac{6400 \sin 120^\circ}{8000}$$

$$\approx 0.6928$$

$$\theta \approx 43.8538^\circ$$

Therefore

$$\angle TOS \approx 180 - 120 - 43.8538 = 16.1462^\circ.$$

The satellite completes one orbit every two hours. Therefore, the time in minutes after 12 p.m. will be

$$\frac{16.1462}{360} \times 2 \times 60 = 5.38 \text{ min}.$$

Therefore the time will be approximately 12.05.

b As the satellite rotates, $\angle TOS$ increases. After 6 minutes, the satellite will have rotated by

$$\angle TOS = \frac{6}{120} \times 360^\circ = 18^\circ.$$

We apply the cosine law to find that

$$\begin{aligned} TS &= \sqrt{6400^2 + 8000^2 - 2 \times 6400 \times 8000 \times \cos 18^\circ} \\ &\approx 2752 \text{ km.} \end{aligned}$$

c Let $\angle STO = \theta$. Then using the sine rule, we obtain,

$$\begin{aligned} \frac{\sin\theta}{8000} &= \frac{\sin 18^\circ}{2572} \\ \sin\theta &= \frac{8000 \sin 18^\circ}{2572} \\ &\approx 0.8984 \end{aligned}$$

As θ is obtuse, we obtain $\theta \approx 116.0507^\circ$. Therefore, the angle above the horizon will be approximately,

$$116.0507^\circ - 90^\circ \approx 26.1^\circ.$$

5 a Since the diagonals of a parallelogram bisect one another, applying the cosine law to triangle DEC gives

$$\begin{aligned} x^2 &= \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 - 2\left(\frac{p}{2}\right)\left(\frac{q}{2}\right)\cos\theta \\ &= \frac{p^2}{4} + \frac{q^2}{4} - \frac{pq}{2}\cos\theta \\ x &= \sqrt{\frac{p^2}{4} + \frac{q^2}{4} - \frac{pq}{2}\cos\theta}. \end{aligned}$$

b Since the diagonals of a parallelogram bisect one another, applying the cosine law to triangle DEA gives

$$\begin{aligned} y^2 &= \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 - 2\left(\frac{p}{2}\right)\left(\frac{q}{2}\right)\cos(180 - \theta) \\ &= \frac{p^2}{4} + \frac{q^2}{4} + \frac{pq}{2}\cos\theta \\ y &= \sqrt{\frac{p^2}{4} + \frac{p^2}{4} + \frac{pq}{2}\cos\theta}. \end{aligned}$$

Note that we used the fact that $\cos(180 - \theta) = -\cos(\theta)$.

c We have

$$x^2 + y^2 = \frac{p^2}{4} + \frac{q^2}{4} - \frac{pq}{2}\cos\theta + \frac{p^2}{4} + \frac{q^2}{4} + \frac{pq}{2}\cos\theta$$
$$= \frac{p^2}{4} + \frac{q^2}{4} + \frac{p^2}{4} + \frac{q^2}{4}$$
$$= \frac{p^2}{2} + \frac{q^2}{2}$$

Therefore

$$2(x^2 + y^2) = p^2 + q^2.$$

d Using the result from the previous question, we have,

$$q^2 + 13^2 = 2(8^2 + 6^2)$$
$$q^2 + 169 = 2(64 + 36)$$
$$q^2 + 169 = 200$$
$$q^2 = 31$$
$$q = \sqrt{31} \text{ cm}.$$

6 a Let $\beta = \angle XOB$. Then,

$$\cos\beta = \frac{32}{40}$$
$$\beta = \cos^{-1}\left(\frac{4}{5}\right)$$
$$\approx 0.6435.$$

Therefore, $\angle AOB = 2\beta \approx 1.29$.

b **i** We use the formula

$$L = r\theta$$
$$\approx 40 \times 1.29$$
$$= 51.48 \text{ cm}.$$

ii We first find the segment area above the surface of the water. This is given by,

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$
$$\approx \frac{1}{2} \times 40^2(1.29 - sin(1.29))$$
$$\approx 261.60 \text{ cm}^2.$$

We subtract this from the area of the full circle to give

$$A \approx \pi \times 40^2 - 261.60 \approx 4764.95 \text{ cm}^2.$$

iii The percentage of the log beneath the surface will be given by

$$\frac{4764.95}{\pi \times 40^2} \times 100\% \approx 94.80\%.$$

7 a We need to show that exactly one number is less than or equal to zero or two numbers are less than or equal to zero. If all three numbers are less than or equal to zero then

$|a| = -a$ and $|b| = -b$ and $|c| = -c$.

Therefore,

$$\begin{aligned} |a| + |b| + |c| &= 14 \\ \Rightarrow -a - b - c &= 14 \\ \Rightarrow a + b + c &= -14 \\ \Rightarrow |a + b + c| &= 14. \end{aligned}$$

This contradicts the fact that $|a+b+c| = 2$. Likewise, if all three numbers are greater than or equal to zero then

$|a| = a$ and $|b| = b$ and $|c| = c$.

Therefore,

$$\begin{aligned} |a| + |b| + |c| &= 14 \\ \Rightarrow a + b + c &= 14 \\ \Rightarrow |a + b + c| &= 14. \end{aligned}$$

This also contradicts the fact that $|a + b + c| = 2$.

b There are two cases to consider.

Case 1. If $a \le 0 \le b \le c$, then the three equations become

$$\begin{aligned} |a| + |b| + |c| = 14 &\Rightarrow -a + b + c = 14 \\ |a + b + c| = 2 &\Rightarrow a + b + c = \pm 2 \\ |abc| = 72 &\Rightarrow abc = 72 \end{aligned}$$

We can solve these using technology (or by hand) to give two solutions:

$$a = -6, b = 4 - 2\sqrt{7}, c = 4 + 2\sqrt{7}$$

$$a = -8, b = 3 - 3\sqrt{2}, c = 3 + 3\sqrt{2}$$

Case 2. If $a \le b \le 0 \le c$, then the three equations become

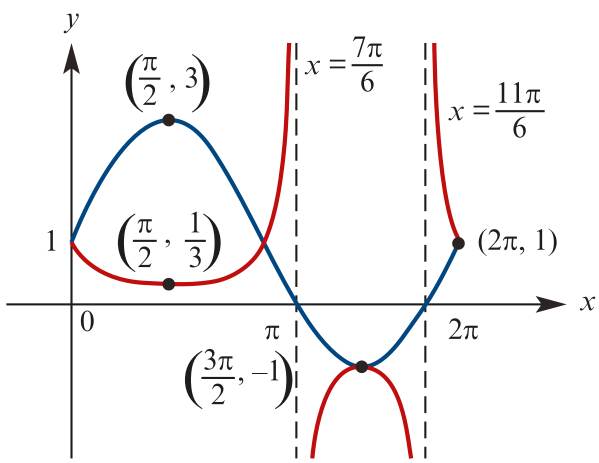
$$\begin{aligned} |a| + |b| + |c| = 14 &\Rightarrow -a - b + c = 14 \\ |a + b + c| = 2 &\Rightarrow a + b + c = \pm 2 \\ |abc| = 72 &\Rightarrow abc = 72 \end{aligned}$$

We can also solve these using technology (or by hand) to give two solutions:

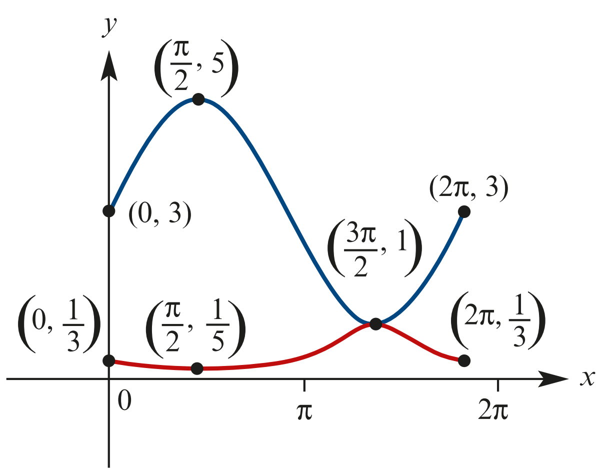
$a = -6, b = -2, c = 6,$

$a = -3, b = -3, c = 8.$

8 a i

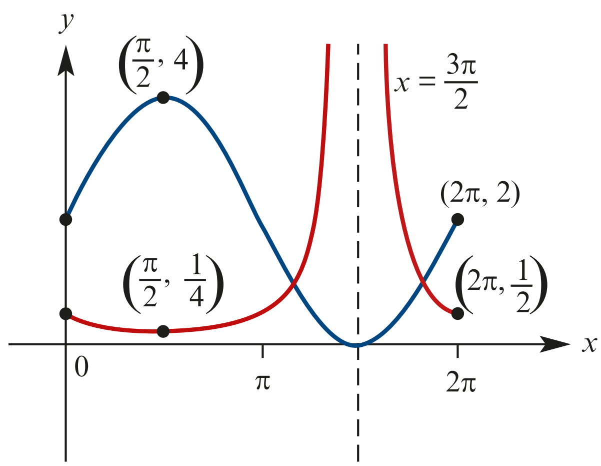


ii



b The graph will have just one vertical asymptote provided the graph of $f(x) = 2\sin x + k$ intersects the x-axis once only. This will only happen if $k = 2$.

c



9 **a** We know that the point $P(x, y)$ satisfies,

$$\begin{aligned}
AP &= BP \\
\sqrt{(x-1)^2 + (y-2)^2} &= \sqrt{(x-2)^2 + (y+2)^2} \\
(x-1)^2 + (y-2)^2 &= (x-2)^2 + (y+2)^2 \\
x^2 - 2x + 1 + y^2 - 4y + 4 &= x^2 - 4x + 4 + y^2 + 4y + 4 \\
y &= \frac{x}{4} - \frac{3}{8}.
\end{aligned}$$

b The gradient of the line AB is

$$m = \frac{-2-2}{2-1} = -4.$$

Since $-4 \times \frac{1}{4} = -1$, the two lines are perpendicular. The midpoint of segment AB is $M(3/2, 0)$, and this is on the line $y = \frac{x}{4} - \frac{3}{8}$ since when $x = \frac{3}{2}$,

$$y = \frac{1}{4} \cdot \frac{3}{2} - \frac{3}{8} = 0.$$

c The shortest distance from the town to the road will be AM where $M(3/2, 0)$ is the midpoint of AB. This distance is

$$AM = \sqrt{\left(\frac{3}{2} - 1\right)^2 + (0-2)^2} = \frac{\sqrt{17}}{2} \text{ km}.$$

10 **a** There are various ways to do this question. We will find the cartesian equation corresponding to this pair of parametric equations. Solving each equation for t gives,

$$t = \frac{x+1}{4} \text{ and } t = \frac{3-y}{3}.$$

Eliminating t then gives,

$$\frac{3-y}{3} = \frac{x+1}{4}$$

$$3x + 4y = 9.$$

Now simply note that each of the points $(3, 0)$ and $(-1, 3)$ lie on the line since

$$3 \times 3 + 4 \times 0 = 9, \text{ and}$$

$$3 \times -1 + 4 \times 3 = 9.$$

b If we substitute $x = 4t - 1$ and $y = 3 - 3t$ into the equation for the circle we obtain,

$$\begin{aligned}(4t-1)^2 + (3-3t)^2 &= 4\\ 16t^2 - 8t + 1 + 9 - 18t + 9t^2 &= 4\\ 16t^2 - 8t + 1 + 9 - 18t + 9t^2 &= 4\\ 25t^2 - 26t + 6 &= 0.\end{aligned}$$

We simply need to show that this equation has a solution. You can find the solutions, but it's easier to show show that the discriminant is positive. We have,

$$\Delta = b^2 - 4ac = (-26)^2 - 4 \times 25 \times 6 > 0.$$

c We first find the cartesian equation of the line. Its gradient is

$$m = \frac{0-4}{3-(-1)} = \frac{-4}{4} = -1.$$

The equation of the line will then be

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - 0 &= -1(x-3)\\ y &= -x + 3.\end{aligned}$$

We can then let $x = t$ so that $y = -t + 3$.

d Substitute $x = t$ so that $y = -t + 3$ into the equation for the circle, giving,

$$\begin{aligned}t^2 + (-t+3)^2 &= 4\\ t^2 + t^2 - 6t + 9 &= 4\\ 2t^2 - 6t + 5 &= 0\end{aligned}$$

We simply need to show that this equation has no solution. We look at the discriminant,

$$\Delta = b^2 - 4ac = (-6)^2 - 4 \times 2 \times 5 = -4 < 0.$$

e This can be done without using parametric equations. Simply find the equation of the line through D and B. Its gradient will be

$$m = \frac{0-k}{3-(-1)} = -\frac{k}{4}.$$

The equation of the line will then be

$$y - y_1 = -\frac{k}{4}(x - x_1)$$
$$y - 0 = -\frac{k}{4}(x - 3)$$
$$y = -\frac{k}{4}x + \frac{3k}{4}.$$

Substituting this into the equation for the circle gives,

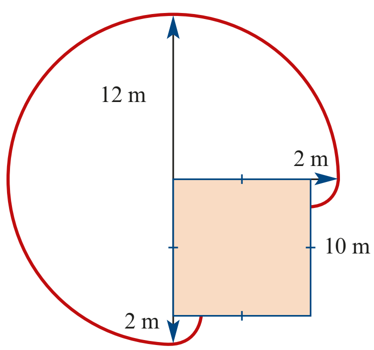
$$x^2 + \left(\frac{k}{4}x + \frac{3k}{4}\right)^2 = 4.$$

Using technology or by expanding and solving by hand, this equation has solutions

$$x = \frac{-3k^2 - 4\sqrt{64 - 5k^2}}{k^2 + 16}.$$

These solutions will not exist if $64 - 5k^2 < 0$. That is, if $k > \dfrac{8}{\sqrt{5}}$ or $k < -\dfrac{8}{\sqrt{5}}$

11 a



b The area comprises a three-quarter circle of radius 12 m, and two quarter circles of radius 2 m. The total area will then be

$$A = \frac{3}{4} \times \pi \times 12^2 + 2 \times \frac{1}{4} \times \pi \times 2^2$$
$$= 110\pi \text{ m}^2.$$

c **Case 1.** If $x \leq 2$ then the area comprises

- A half circle of radius 12 ,
- a quarter circle of radius $12 - x$,
- a quarter circle of radius $12 - 10 - x = 2 - x$ and,
- a quarter circle of radius $12 - (10 - x) = 2 + x$.

The total area will then be given by the expression

$$A = \frac{1}{2} \times \pi \times 12^2 + \frac{1}{4} \times \pi \times (12 - x)^2 + \frac{1}{4}\pi \times (2 - x)^2 + \frac{1}{4}\pi \times (2 + x)^2$$
$$= \frac{3\pi x^2}{4} - 6\pi x + 110\pi.$$

Case 2. If $2 < x \leq 5$ then the area comprises

- a half circle of radius 12 ,
- a quarter circle of radius $12 - x$, and
- a quarter circle of radius $12 - (10 - x) = 2 + x$.

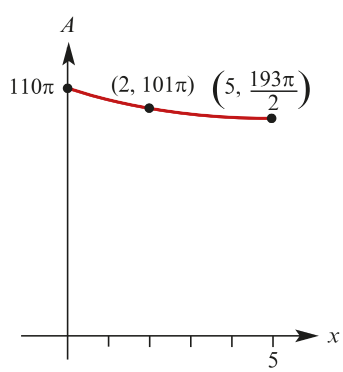
The total area will then be given by the expression

$$A = \frac{1}{2} \times \pi \times 12^2 + \frac{1}{4} \times \pi \times (12 - x)^2 + \frac{1}{4}\pi \times (2 + x)^2$$
$$= \frac{\pi x^2}{2} - 5\pi x + 109\pi.$$

Therefore the area is given by the hybrid function,

$$A(x) = \begin{cases} \dfrac{3\pi x^2}{4} - 6\pi x + 110\pi, & 0 \leq x \leq 2 \\ \dfrac{\pi x^2}{2} - 5\pi x + 109\pi, & 2 < x \leq 5. \end{cases}$$

d

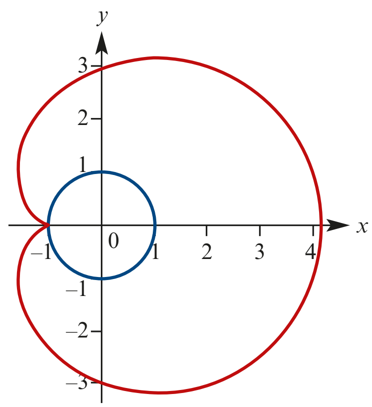


e **i** The function has a global maximum at $x = 0$, which corresponds to the corner of the shed.

ii The function has a global minimum at $x = 5$, which corresponds to the middle of one side.

12 **a** The length of the rope is π, is exactly the same as the arc length from S to the opposite side of the circle.

b The curve is shown below. The right hand side is a semicircle.



c **i** The arc length SQ will simply be

$$L = r\theta$$
$$= 1 \times \theta$$
$$= \theta.$$

ii $PQ = \pi - \text{Arc}(SQ) = \pi - \theta$

iii Since $\angle RQO = \angle SOQ = \theta$, we know that $\angle PQR = 90^\circ - \theta$. Therefore,

$$\angle RPQ = 180^\circ - (90^\circ - \theta) = \theta.$$

iv Considering right-angled triangle RPQ we have

$$\sin\theta = \frac{RQ}{PQ}$$
$$RQ = PQ\sin\theta$$
$$= (\pi - \theta)\sin\theta.$$

v Considering right-angled triangle RPQ we have

$$\cos\theta = \frac{RP}{PQ}$$
$$RP = PQ\cos\theta$$
$$= (\pi - \theta)\cos\theta.$$

d First note the coordinates of Q are $Q(\cos\theta, \sin\theta)$. Therefore the x-coordinate of point P will be given by the expression

$$x = \cos\theta - RQ$$
$$= \cos\theta - (\pi - \theta)\sin\theta.$$

The y-coordinate of point P will be given by the expression,

$$\begin{aligned} y &= \sin\theta + RP \\ &= \sin\theta + (\pi - \theta)\cos\theta. \end{aligned}$$

13 a Since Arg z and Arg w are acute , $0 \le \text{Arg}\, z + \text{Arg}\, w \le \pi$ Hence in this case we can write Arg (wz) = Arg (z) + Arg (w)

b i $\text{Arg}\,(2+i) = \tan^{-1}\left(\frac{1}{2}\right)$

ii $\text{Arg}\,(3+i) = \tan^{-1}\left(\frac{1}{3}\right)$

iii $\text{Arg}\,(5+5i) = \tan^{-1}\left(\frac{5}{5}\right) = \frac{\pi}{4}$

c $(2+i)(3+i) = 6 + 5i - 1 = 5 + 5i$

d We therefore find that

$$\begin{aligned} \text{Arg}\,((2+i)(3+i)) &= \text{Arg}\,(5+5i) \\ \text{Arg}\,(2+i) + \text{Arg}\,(3+i) &= \text{Arg}\,(5+5i) \\ \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) &= \frac{\pi}{4}, \end{aligned}$$

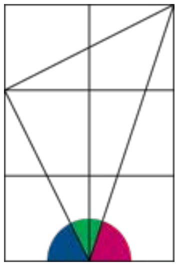
as required.

e

$$\begin{aligned} (3+i)^2(7+i) &= (8+6i)(7+i) \\ &= 56 + 50i - 6 \\ &= 50 - 50i \\ \Rightarrow \text{Arg}\,((3+i)^2(7+i)) &= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) \\ &= \frac{\pi}{4} \end{aligned}$$

f

$$\begin{aligned} (1+i)(1+2i)(1+3i) &= (-1+3i(1+3i) \\ &= -1 + 9i^2 \\ &= -10 \\ \Rightarrow \text{Arg}\,((1+i)(1+2i)(1+3i)) &= \tan^{-1}\left(1\right) + \tan^{-1}\left(2\right) + \tan^{-1}\left(3\right) \\ &= \pi \end{aligned}$$

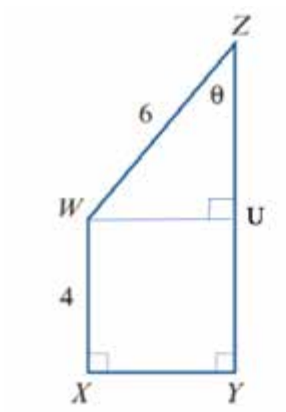
g 

The blue angle = $\tan^{-1}\left(2\right)$

The red angle = $\tan^{-1}\left(3\right)$

The lengths of the sides of the triangle with the green angle are $\sqrt{5}$, $\sqrt{5}$ and $\sqrt{10}$. Therefore, this triangle is a right-angled isosceles triangle so that the green angle is $\frac{\pi}{4}$. Therefore, we can see that

$$\tan^{-1}\left(1\right) + \tan^{-1}\left(2\right) + \tan^{-1}\left(3\right) = \pi$$

14 a 

Area of triangle $WUZ = \frac{1}{2} \times 6\cos\theta \times 6\sin\theta = 18\sin\theta\cos\theta = 9\sin 2\theta$
Area of rectangle $WUYX = 4 \times 6\sin\theta$
Area of $XYZW = 9\sin 2\theta + 24\sin\theta$

b Perimeter of $XYZW = 8 + 6 + 6\cos\theta + 6\sin\theta = 14 + 6(\cos\theta + \sin\theta)$

c Perimeter$= 14 + 6\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$.
Maximum is 20 and occurs when $\theta = \frac{\pi}{4}$

d $14 + 6\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = 21$

$$6\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = 7$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \tfrac{7}{6\sqrt{2}}$$

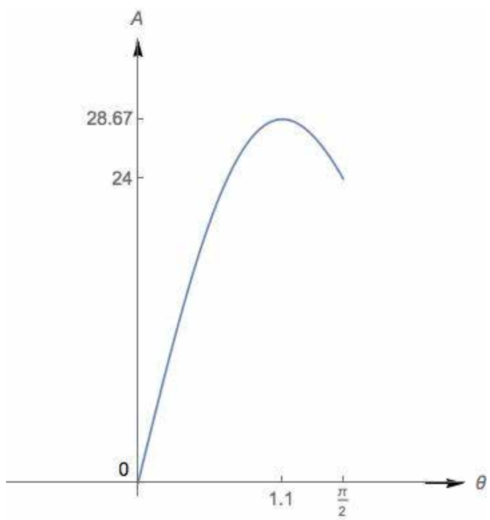
$$\theta + \frac{\pi}{4} = \sin^{-1}(\tfrac{7}{6\sqrt{2}}) \text{ or } \pi - \sin^{-1}(\tfrac{7}{6\sqrt{2}})$$

$$\theta \approx 0.1845 \text{ or } 1.3861$$

e When $\theta = \frac{\pi}{4}, A = \frac{24}{\sqrt{2}} + 9 = 12\sqrt{2} + 9$

f Maximum value of 28.67 when $\theta = 1.1$

g

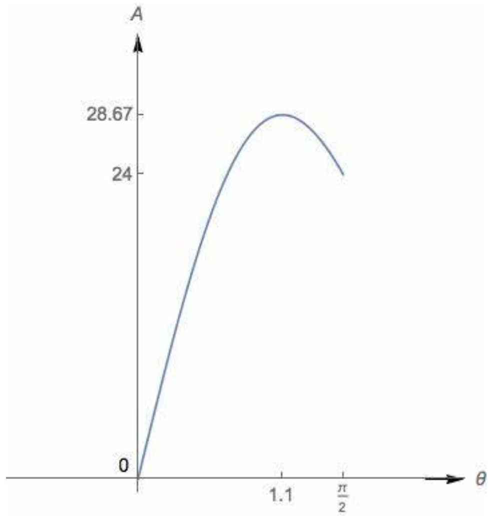


h Area of rectangle $WUYX = 4x$

$ZU = \sqrt{36 - x^2}$

Area of triangle $WZU = \frac{1}{2} \times x \times \sqrt{36 - x^2}$

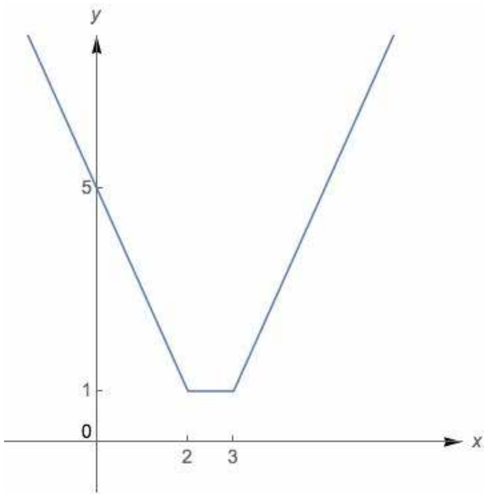
$$\Rightarrow A = 4x + \frac{1}{2}x\sqrt{36 - x^2}$$

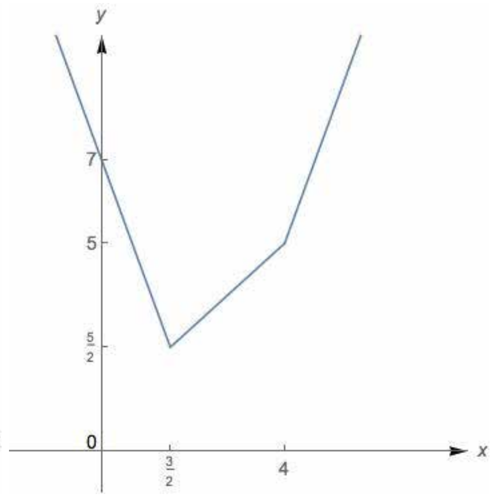


Maximum is ≈ 28.67 when $x \approx 5.36$

Investigations

1 a i





ii $y = |ax + b| + |cx + d|$

Assume $-\dfrac{d}{c} > -\dfrac{b}{a}$

Case 1: $x > -\dfrac{d}{c}$

$y = (a + c)x + (b + d)$

Case 2: $-\dfrac{b}{a} < x < -\dfrac{d}{c}$

$y = (a - c)x + (b - d)$

Case 3: $x < -\dfrac{b}{a}$

$y = -(a + c)x - (b + d)$

$y = |ax + b| - |cx + d|$

Assume $-\dfrac{d}{c} > -\dfrac{b}{a}$

Case 1: $x > -\dfrac{d}{c}$

$y = (a - c)x + (b - d)$

Case 2: $-\dfrac{b}{a} < x < -\dfrac{d}{c}$

$y = (a + c)x + (b + d)$

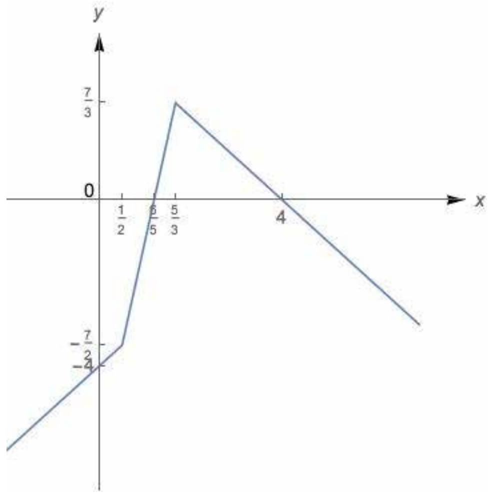
Case 3: $x < -\dfrac{b}{a}$

$y = (c - a)x + d - b$

We illustrate the second family

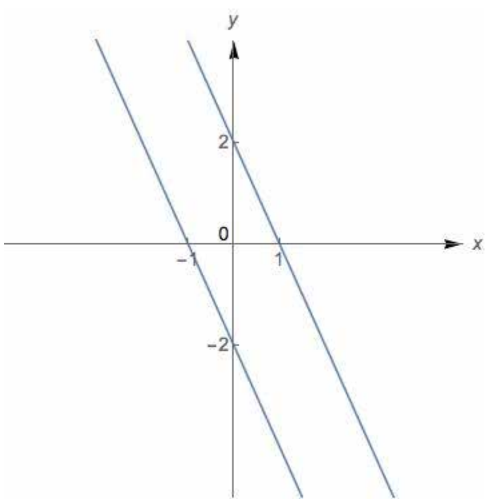
$f(x) = |2x - 1| - |3x - 5|$

$$f(x) = \begin{cases} -x+4 & x > \dfrac{5}{3} \\ 5x-6 & \dfrac{1}{2} \le x \le \dfrac{5}{3} \\ x-4 & x < \dfrac{1}{2} \end{cases}$$



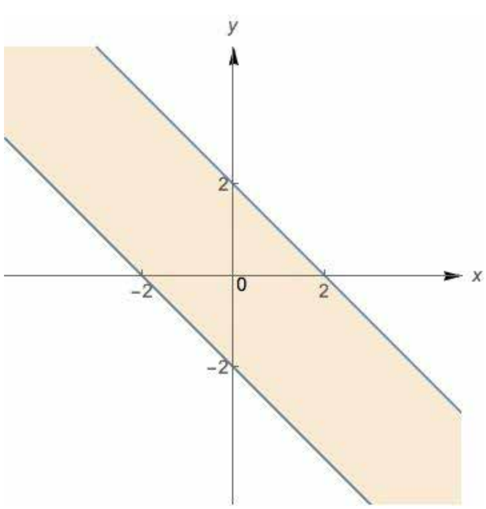
There is more to investigate here.

b **i**



ii Parallel lines are formed

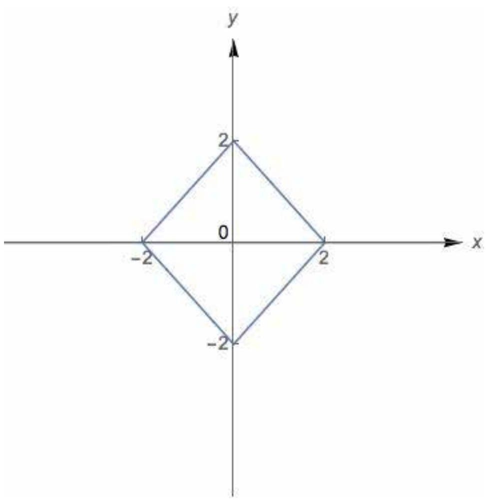
c **i**



ii Similar results are found.

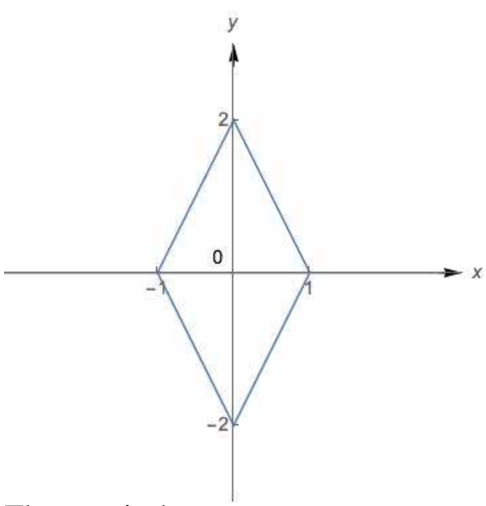
d **i** $|x| + |y| = 2$

$$\Leftrightarrow \begin{cases} x + y = 2 & x \geq 0, y \geq 0 \\ x - y = 2 & x \geq 0, y \leq 0 \\ -x + y = 2 & x \leq 0, y \geq 0 \\ -x - y = 2 & x \leq 0, y \leq 0 \end{cases}$$



Area $= (2\sqrt{2})^2 = 8$

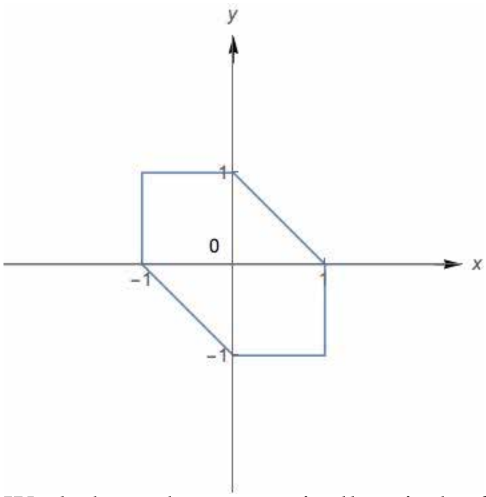
ii



This is the graph of $|2x| + |y| = 2$.

The area is 4.

e Area = 3



Work through systematically wit the four quadrants.

For example:

If $x > 0$ and $y < 0$ we have $x - y + |x + y| \leq 2$

If $y > -x$ this becomes:

$x - y + x + y \leq 2$

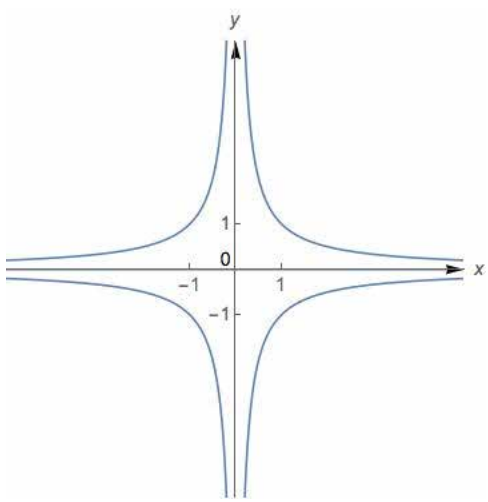
That is $x \leq 1$

If $y \leq -x$ it becomes $x - y - x - y \leq 2$

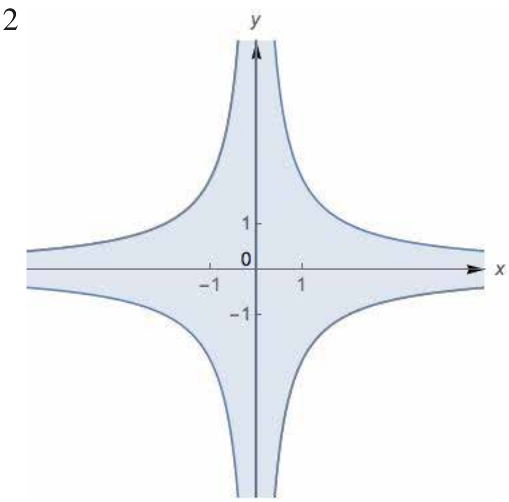
That is $y \geq 1$

Continue in this way through the 4 quadrants.

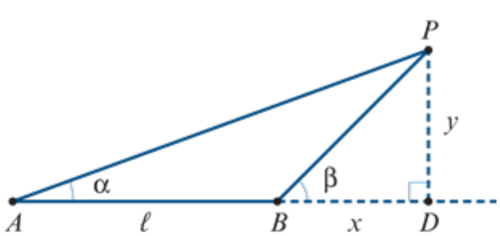
f $|xy| = 1$



$|xy| \leq 2$



2



a $\tan\beta = \dfrac{y}{x}$ and $\tan\alpha = \dfrac{y}{x+\ell}$

$$(x+\ell)\tan\alpha = x\tan\beta$$

$$x\tan\alpha + \ell\tan\alpha = x\tan\beta$$

$$\ell\tan\alpha = x(\tan\beta - \tan\alpha)$$

$$\therefore x = \frac{\ell\tan\alpha}{\tan\beta - \tan\alpha}$$

b $x = \ell \times \dfrac{\sin\alpha}{\cos\alpha} \div \left(\dfrac{\sin\beta}{\cos\beta} - \dfrac{\sin\alpha}{\cos\alpha}\right)$

$$= \ell \times \frac{\sin\alpha}{\cos\alpha} \div \frac{\sin\beta - \sin\alpha\cos\beta}{\cos\beta\cos\alpha}$$

$$= \ell \times \frac{\sin\alpha}{\cos\alpha} \times \frac{\cos\beta\cos\alpha}{\sin(\beta-\alpha)}$$

$$= \frac{\ell\sin\alpha\cos\beta}{\sin(\beta-\alpha)}$$

$$y = x\tan\beta = \frac{\ell\sin\alpha\cos\beta}{\sin(\beta-\alpha)} \times \frac{\sin\beta}{\cos\beta}$$

$$\therefore y = \frac{\ell\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$$

c d

Investigate what errors in measurement of angles cause in the calculation of x and y. The form tells us to also consider the difference in sizes of α and β. Consider percentage errors. For example a 2.5% error in the measurement of α can cause a 27% error in the value of x for $\alpha = 40°$ and $\beta = 45°$

3 a i $z^2 + az + a^2 = 0$

$$z^2 + az + \frac{a^2}{4} + a - \frac{a^2}{4} = 0$$

$$\left(z + \frac{a}{2}\right)^2 + \frac{3a^2}{4} = 0$$

$$\left(z + \frac{a}{2}\right)^2 - \left(\frac{\sqrt{3}a}{2}i\right)^2 = 0$$

$$\therefore z = -\frac{a}{2} \pm \frac{\sqrt{3}a}{2}i$$

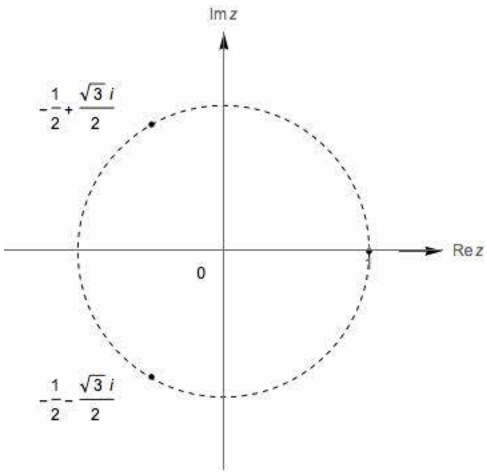
ii $z^3 - a^3 = (z-a)(z^2 + az + a^2)$

$\therefore z^3 = a^3$ has solutions

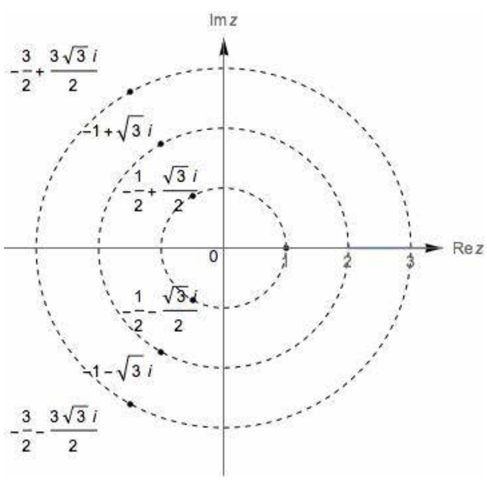
$z = a$ or $z = -\dfrac{a}{2} \pm \dfrac{\sqrt{3}a}{2}i$

iii If $a = 1$, the solutions are $z = 1, z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

In polar form $z = \text{cis } 0, \text{cis } \frac{2\pi}{3}, \text{cis}\left(-\frac{2\pi}{3}\right)$



iv



v The cube roots of 1,2 and 3 are found in $\mathbb{C}$. On the circles they are separated by an 'angular distance' of $\frac{2\pi}{3}$

b

$$z^2 + az + a^2 = 0$$

$$z^2 + az - \frac{a^2}{4} + a - \frac{a^2}{4} = 0$$

$$\left(z - \frac{a}{2}\right)^2 + \frac{3a^2}{4} = 0$$

$$\left(z - \frac{a}{2}\right)^2 - \left(\frac{\sqrt{3}a}{2}i\right)^2 = 0$$

$$\therefore z = \frac{a}{2} \pm \frac{\sqrt{3}a}{2}i$$

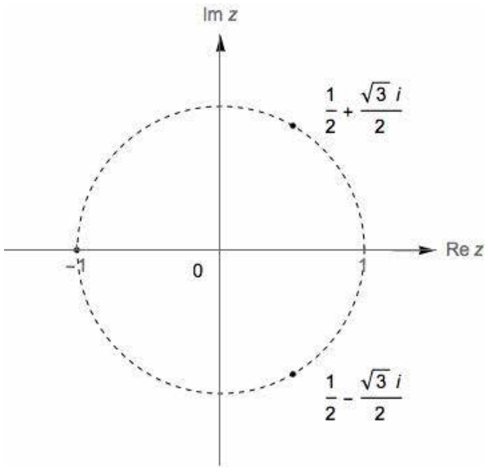
$$z^3 + a^3 = (z + a)(z^2 - az + a^2)$$

$\therefore z^3 = -a^3$ has solutions

$z = -a$ or $z = \dfrac{a}{2} \pm \dfrac{\sqrt{3}a}{2}i$

If $a = 1$, the solutions are $z = 1, z = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

In polar form $z = \text{cis}\,\pi, \text{cis}\,\dfrac{\pi}{3}, \text{cis}\left(-\dfrac{\pi}{3}\right)$



c $$z^2 + aiz + a^2 = 0$$

$$z^2 + aiz - \frac{a^2}{4} + a - \frac{a^2}{4} = 0$$

$$\left(z - \frac{ai}{2}\right)^2 - \frac{3a^2}{4} = 0$$

$$\left(z - \frac{ai}{2}\right)^2 - \left(\frac{\sqrt{3}a}{2}\right)^2 = 0$$

$$\therefore z = \pm\frac{\sqrt{3}a}{2} + \frac{ai}{2}$$

$z^3 + a^3 i = z^3 - (ai)^3 = (z - ai)(z^2 + aiz + a^2)$

$\therefore z^3 = -a^3$ has solutions

$z = ai$ or $z = \pm\dfrac{\sqrt{3}a}{2} + \dfrac{ai}{2}$

If $a = 1$, the solutions are $z = i, z = \pm\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

In polar form $z = \text{cis}\,\dfrac{\pi}{2}, \text{cis}\,\dfrac{\pi}{6}, \text{cis}\left(\dfrac{5\pi}{6}\right)$

d Solutions are $z = -ai$ or $z = \pm\dfrac{\sqrt{3}a}{2} - \dfrac{ai}{2}$

e

$$z^4 + 1 = 0$$

$$z^4 + 2z^2 + 1 - 2z^2 = 0$$

$$(z^2 + 1)^2 - 2z^2 = 0$$

$$(z^2 + \sqrt{2}z + 1)(z^2 - \sqrt{2}z - 1) = 0$$

Using the quadratic formula for both factors.

$$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}i$$

Chapter 20 – Transformations of the plane

Solutions to Exercise 20A

1 **a** $(2, -4) \to (2 + (-4), 2 - (-4)) = (-2, 6)$

b $(2, -4) \to (2(2) + 3(-4), 3(2) - 4(-4)) = (-8, 22)$

c $(2, -4) \to (3(2) - 5(-4), 2) = (26, 2)$

d $(2, -4) \to (-4, -2)$

2 **a** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \times 2 + 1 \times 3 \\ 1 \times 2 + 0 \times 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Therefore $(2, 3) \to (3, 2)$.

b $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \times 2 + 0 \times 3 \\ 0 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$

Therefore $(2, 3) \to (-4, 9)$.

c $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 2 \times 3 \\ 0 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$

Therefore $(2, 3) \to (8, 3)$.

d $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 1 \times 3 \\ 1 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

Therefore $(2, 3) \to (7, 11)$.

3 **a** The linear transformation can be written as

$$x' = 2x + 3y$$
$$y' = 4x + 5y$$

so the transformation matrix is

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

b The linear transformation can be written as

$$x' = 11x - 3y$$
$$y' = 3x - 8y$$

so the transformation matrix is

$$\begin{bmatrix} 11 & -3 \\ 3 & -8 \end{bmatrix}.$$

c The linear transformation can be written as

$$x' = 2x + 0y$$
$$y' = x - 3y$$

so the transformation matrix is

$$\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}.$$

d The linear transformation can be written as

$$x' = 0x + 1y$$
$$y' = -1x + 0y$$

so the transformation matrix is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

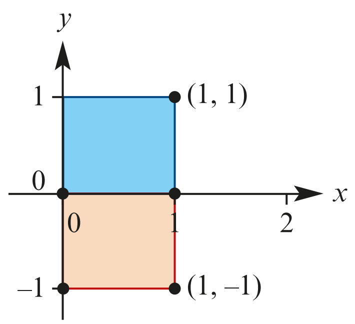
4 For each of these questions we multiply the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex.

a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$

The columns then give the required points:

$$(0, 0), (0, -1), (1, 0), (1, -1).$$

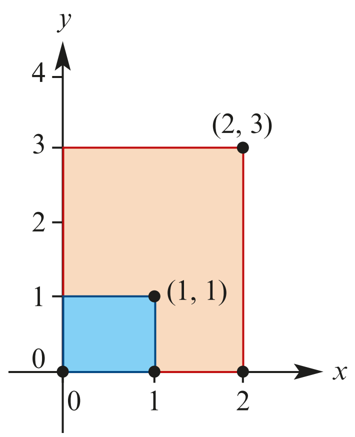
The square is shown in blue, and its image in red.



b $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The columns then give the required points:

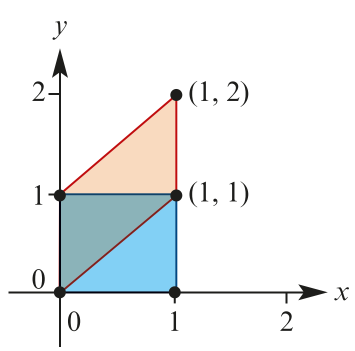
$(0, 0), (2, 0), (0, 3), (2, 3)$. The square is shown in blue, and its image in red.



c $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The columns then give the required points:

$(0, 0), (1, 1), (0, 1), (1, 2)$. The square is shown in blue, and its image in red.

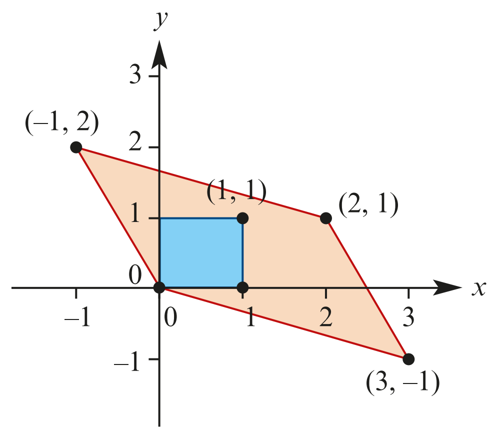


d $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =$

$\begin{bmatrix} 0 & -1 & 3 & 2 \\ 0 & 2 & -1 & 1 \end{bmatrix}$

The columns then give the required points:

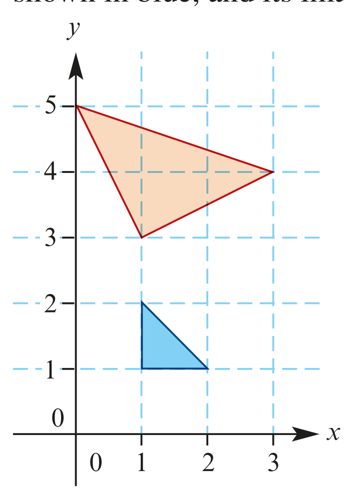
$(0, 0), (-1, 2), (3, -1), (2, 1)$. The original triangle is shown in blue, and its image in red.



5 We multiply the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex. This gives, $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 5 & 4 \end{bmatrix}$

The columns then give the required points:

$(1, 3), (0, 5), (3, 4)$. The square is shown in blue, and its image in red.



6 The image of $(1, 0)$ is $(3, 4)$. The image of $(0, 1)$ is $(5, 6)$. Write these images as the column of a matrix, $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

Therefore $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$
so that $(-2, 4) \to (14, 16)$.

7 The image of $(1, 0)$ is $(-3, 2)$. The image of $(0, 1)$ is $(1, -1)$. Write these images as the column of a matrix, $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$.

Therefore $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$
so that $(2, 3) \to (-3, 1)$.

8 **a** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$.

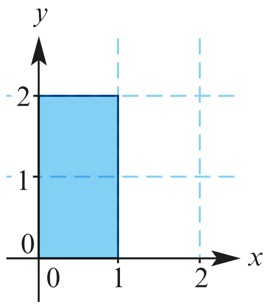
b $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$.

c $\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}$.

Solutions to Exercise 20B

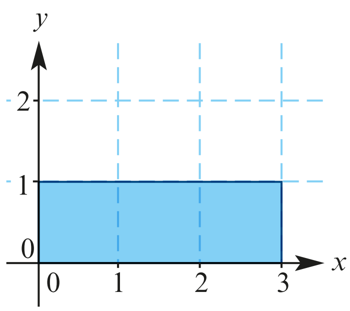
1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

ii



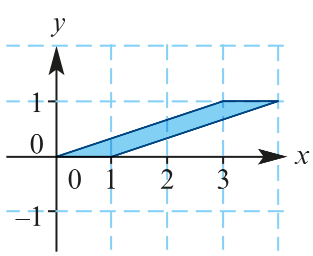
b i $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

ii



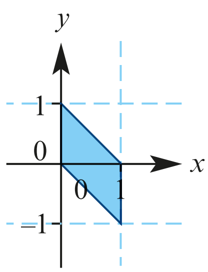
c i $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

ii



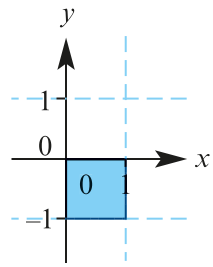
d i $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

ii



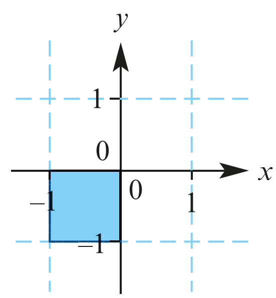
e i $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

ii



f i $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

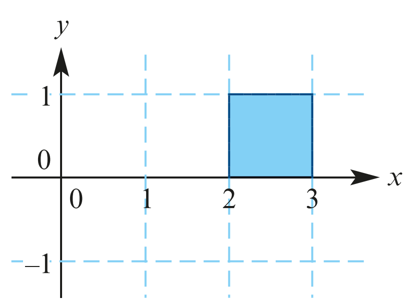
ii



2 a i

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix},$$

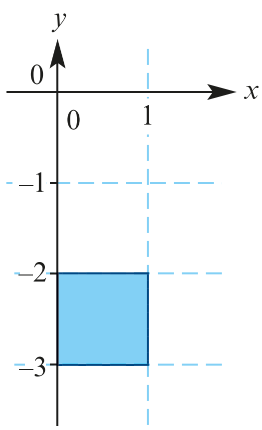
ii



b i

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y-3 \end{bmatrix},$$

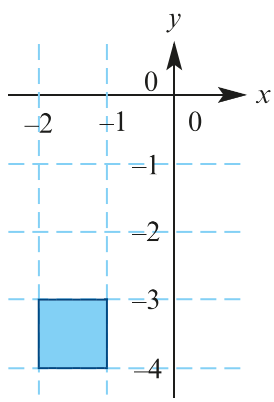
ii



c **i**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x-2 \\ y-4 \end{bmatrix},$$

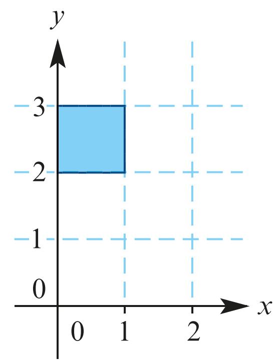
ii



d **i**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y+2 \end{bmatrix},$$

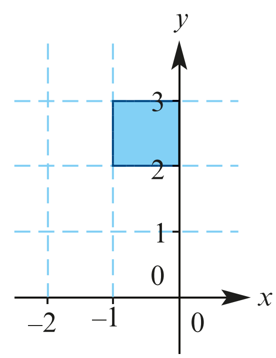
ii



e **i**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+2 \end{bmatrix},$$

ii



Solutions to Exercise 20C

1 a $\begin{bmatrix} \cos 270 & -\sin 270 \\ \sin 270 & \cos 270 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$

$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

c $\begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

d $\begin{bmatrix} \cos(-135°) & -\sin(-135°) \\ \sin(-135°) & \cos(-135°) \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

2 a The rotation matrix is
$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Therefore

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

so that $(2, 3) \to (-3, 2)$.

b The rotation matrix is

$\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Therefore

$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

so that $(2, 3) \to \left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

3 a $\begin{bmatrix} \cos 90° & \sin 90° \\ \sin 90° & -\cos 90° \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \cos 120° & \sin 120° \\ \sin 120° & -\cos 120° \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c $\begin{bmatrix} \cos(-60°) & \sin(-60°) \\ \sin(-60°) & -\cos(-60°) \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

d $\begin{bmatrix} \cos 30° & \sin 30° \\ \sin 30° & -\cos 30° \end{bmatrix}$

$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

4 a Since

$$\tan\theta = 3 = \frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 3 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{10}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = \frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1\\ &= \frac{2}{10} - 1\\ &= -\frac{4}{5},\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}}\\ &= \frac{3}{5}.\end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}-\frac{4}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{bmatrix}.$$

b Since

$$\tan\theta = 5 = \frac{5}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 5 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{26}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{26}} \text{ and } \sin\theta = \frac{5}{\sqrt{26}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{\sqrt{26}}\right)^2 - 1\\ &= \frac{2}{26} - 1\\ &= -\frac{12}{13},\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{1}{\sqrt{26}}\frac{5}{\sqrt{26}}\\ &= \frac{5}{13}.\end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}-\frac{12}{13} & \frac{5}{13}\\ \frac{5}{13} & \frac{12}{13}\end{bmatrix}.$$

c Since $\tan\theta = \frac{2}{3}$,

we draw a right angled triangle with opposite and adjacent lengths 2 and 3 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{13}$. Therefore,

$$\cos\theta = \frac{3}{\sqrt{13}} \text{ and } \sin\theta = \frac{2}{\sqrt{13}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{3}{\sqrt{13}}\right)^2 - 1\\ &= \frac{18}{13} - 1\\ &= \frac{5}{13},\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{2}{\sqrt{13}}\frac{3}{\sqrt{13}}\\ &= \frac{12}{13}.\end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}.$$

d Since

$$\tan\theta = -3 = -\frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 3 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{10}$. Therefore, since $-90° < \theta < 0°$,

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = -\frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1 \\ &= \frac{2}{10} - 1 \\ &= -\frac{4}{5}, \end{aligned}$$

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= -2\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}} \\ &= -\frac{3}{5}. \end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

5 a Since

$$\tan\theta = m = \frac{m}{1},$$

we draw a right angled triangle with opposite and adjacent lengths m and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{m^2+1}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{m^2+1}} \quad \text{We then use the}$$

$$\sin\theta = \frac{m}{\sqrt{m^2+1}}.$$

double angle formulas to show that

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{m^2+1}}\right)^2 - 1 \\ &= \frac{2}{m^2+1} - 1 \\ &= \frac{1-m^2}{m^2+1}, \end{aligned}$$

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\frac{m}{\sqrt{m^2+1}}\frac{1}{\sqrt{m^2+1}} \\ &= \frac{2m}{m^2+1}. \end{aligned}$$

Therefore the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & -\frac{m^2-1}{m^2+1} \end{bmatrix}.$$

b The gradient of the line is $m = 6$. Substituting this into the matrix found above, the reflection matrix is

$$\begin{aligned} \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & -\frac{m^2-1}{m^2+1} \end{bmatrix} &= \begin{bmatrix} \frac{1-6^2}{6^2+1} & \frac{2\times 6}{6^2+1} \\ \frac{2\times 6}{6^2+1} & -\frac{6^2-1}{6^2+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-35}{37} & \frac{12}{37} \\ \frac{12}{37} & -\frac{35}{37} \end{bmatrix} \\ &= \frac{1}{37}\begin{bmatrix} -35 & 12 \\ 12 & 35 \end{bmatrix} \end{aligned}$$

Therefore the image of (1, 1) can be found by evaluating,

$$\frac{1}{37}\begin{bmatrix} -35 & 12 \\ 12 & 35 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} -23 \\ 47 \end{bmatrix}$$

so that

$$(1,1) \to \left(\frac{-23}{37}, \frac{47}{37}\right).$$

6 a $\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

b To find the image of the unit square we evaluate

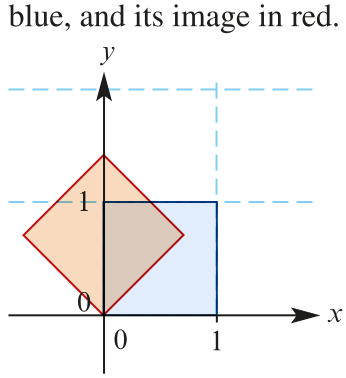
$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \end{bmatrix}.$$

The columns then give the required points:

$$(0,0), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(0, \sqrt{2}\right).$$

The square is shown in blue, and its image in red.



c To find the overlapping region, we subtract the area of the small upper isosceles triangle from the right half of the red square. The base and height of the small isosceles triangle is $\sqrt{2}-1$ so that the overlapping area is $A = \frac{1}{2} - \frac{1}{2}(\sqrt{2}-1)^2$

$$= \frac{1}{2} - \frac{1}{2}(2 - 2\sqrt{2} + 1)$$

$$= \frac{1}{2} - \frac{1}{2}(3 - 2\sqrt{2})$$

$$= \frac{1}{2} - \frac{3}{2} + \sqrt{2}$$

$$= \sqrt{2} - 1.$$

7 a There is no real need to use the rotation matrix for this question. Let O be the origin. We know that length $OA = 1$. Therefore, lengths $OB = 1$ and $OC = 1$. Therefore,

$$B = (\cos 120°, \sin 120°) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$C = (\cos 240°, \sin 240°) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

b Triangle ABC is clearly equilateral.

c Its lines of symmetry will be

$y = x\tan 60° = \sqrt{3}x$

$y = 0$

$y = x\tan 300° = -\sqrt{3}x$

Solutions to Exercise 20D

1 The matrix that will reflect the plane in the y-axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix that will dilate the result by a factor of 3 from the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

2 The matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The matrix that will reflect the result in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

3 **a** The matrix that will reflect the plane in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The matrix that will reflect the plane in the y-axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The matrix that will rotate the plane by 180° clockwise is given by

$$\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the same as the matrix found above.

4 **a** T_1 : The matrix that will reflect the plane in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

T_2 : The matrix that will dilate the result by a factor of 2 from the y-axis is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the matrix of T_1 followed by T_2 will be

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The matrix of T_2 followed by T_1 will be

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

c No. The order of transformation does not matter in this instance, since the two matrices are the same.

5 **a** T_1 : The matrix that will rotate the plane by 90° clockwise is given by

$$\begin{bmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

T_2 : The matrix that will reflect the plane in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the matrix of T_1 followed by T_2 will be

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b The matrix of T_2 followed by T_1 will be

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

c Yes. The order of transformation does matter in this instance, as the two matrices are different (the first gives the reflection in the y-axis, the second a reflection in the x-axis).

6 a
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -x - 3 \\ y + 5 \end{bmatrix}$$

Therefore, the transformation is $(x, y) \to (-x - 3, y + 5)$.

b
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix}\right)$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x - 3 \\ y + 5 \end{bmatrix}$$
$$= \begin{bmatrix} -x + 3 \\ y + 5 \end{bmatrix}$$

Therefore, the transformation is $(x, y) \to (-x + 3, y + 5)$.

c Yes. The order of transformation does matter in this instance, as the rule for each composition is different.

7 a This is a reflection in the x-axis followed by a dilation from the y-axis by a factor of 2 (or visa versa):

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

b This is a reflection in the x-axis followed by a dilation from the x-axis by a factor of 3 (or visa versa):

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}.$$

c This is a reflection in the line $y = x$ followed by a dilation from the x-axis by a factor of 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Alternatively, it is a dilation from the y-axis by a factor of 2 followed by a reflection in the line $y = x$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

d This is a reflection in the line $y = -x$ followed by a dilation from the y-axis by a factor of 2:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}.$$

Alternatively, it is a dilation from the x-axis by a factor of 2 followed by a reflection in the line $y = -x$:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}.$$

8 a The required matrix is

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}.$$

b The above matrix corresponds to a rotation by angle $\theta = 90°$ since

$$\begin{bmatrix}\cos 90° & -\sin 90°\\ \sin 90° & \cos 90°\end{bmatrix}=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}.$$

9 We require that:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}=\begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & sin\theta\end{bmatrix}$$

For these two matrices to be equal, we required that

$$-\sin\theta = \sin\theta$$
$$2\sin\theta = 0$$
$$\sin\theta = 0$$

$\theta = 180°k,\ \text{where } k \in \mathbb{Z}.$

10 a Matrix A^2 will rotate the plane by angle 2θ.

b $$A^2=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$
$$=\begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$$

c Since A^2 will rotate the plane by angle 2θ, another expression for A^2 is

$$A^2=\begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}.$$

Equating the two expressions for A^2 gives

$$\begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}=\begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}.$$

Therefore,

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\sin 2\theta = 2\sin\theta\cos\theta.$

11 a $$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}$$
$$=\begin{bmatrix}y\\x\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}$$
$$=\begin{bmatrix}y+1\\x+2\end{bmatrix}$$

Therefore, the rule can be written in the form $(x, y) \to (y + 1, x + 2)$ or in the form $\begin{matrix}x' = y+1\\ y' = x+2.\end{matrix}$

b We have,

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}y+1\\x+2\end{bmatrix}$$
$$=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x+2\\y+1\end{bmatrix}$$
$$=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\left(\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}\right)$$

This shows the the transformation can be expressed as a reflection in the line $y = x$ followed by a translation in the direction of vector $\begin{bmatrix}2\\1\end{bmatrix}$.

12 a $$\begin{bmatrix}\cos(60) & -\sin(60)\\ \sin(60) & \cos(60)\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

b $\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

c A 60° rotation followed by a −45° rotation will give a 15° rotation. Therefore, the required matrix is.

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{2} \end{bmatrix}$$

The rotation matrix of 15° is also given by the expression

$$\begin{bmatrix} \cos(15) & -\sin(15) \\ \sin(15) & \cos(15) \end{bmatrix}.$$

d Comparing the entries of these two matrices gives

$$\cos 15° = \frac{\sqrt{2}+\sqrt{6}}{4},$$

$$\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4},$$

13 The matrix that will reflect the plane in the line $y = x \tan\phi$ is

$$\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}.$$

The matrix that will reflect the plane in the line $y = x \tan\theta$ is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi & \cos 2\theta \sin 2\phi - \sin 2\theta \cos 2\phi \\ \sin 2\theta \cos 2\phi - \cos 2\theta \sin 2\phi & \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi & -(\cos 2\theta \sin 2\phi - \cos 2\theta \sin 2\phi) \\ \cos 2\theta \sin 2\phi - \cos 2\theta \sin 2\phi & \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta - 2\phi) & -\sin(2\theta - 2\phi) \\ \sin(2\theta - 2\phi) & \cos(2\theta - 2\phi). \end{bmatrix}$$

This is a rotation matrix corresponding to angle $2\theta - 2\phi$.

Solutions to Exercise 20E

1 **a** $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{4\times 1 - 1\times 3}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$
$$= \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

b $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{3\times(-4) - 2\times 1}\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix}$$
$$= -\frac{1}{14}\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{14} \end{bmatrix}.$$

c $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{0\times 4 - 3\times(-2)}\begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$$

d $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{(-1)\times 5 - 3\times(-4)}\begin{bmatrix} 5 & -3 \\ 4 & -1 \end{bmatrix}$$
$$= \frac{1}{7}\begin{bmatrix} 5 & -3 \\ 4 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$$

2 **a** Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix},$$

the inverse transformation will have matrix
$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{5\times(-1) - (-2)\times 2}\begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$
$$= \frac{1}{-1}\begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ 2 & -5 \end{bmatrix}$$
Therefore the rule of the inverse transformation is
$(x, y) \to (x - 2y, 2x - 5y)$

b Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},$$

the inverse transformation will have matrix

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1 \times 0 - (-1) \times 1}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Therefore, the rule of the inverse transformation is $(x, y) \to (y, -x + y)$.

3 a We need to solve the following equation for X. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore, $(-1, 1) \to (1, 1)$.

b We need to solve the following equation for X. $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Therefore $(-\frac{1}{2}, 1) \to (1, 1)$.

4 We need to find a matrix A such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This can be written as a single equation, which we then solve to give
$$A\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}.$$

5 This can be solved in one step by solving the following equation for X.
$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} X = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$X = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & -1 \end{bmatrix}$$

The vertices are then given by the columns of matrix X. These are $(0, 0), (-1, -2), (1, 1)$ and $(0, -1)$.

6 a The dilation matrix is
$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
$$= \frac{1}{k \times 1 - 0 \times 0}\begin{bmatrix} 1 & -0 \\ -0 & k \end{bmatrix}$$
$$= \frac{1}{k}\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$
$$= \begin{bmatrix} 1/k & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix will dilate each point

from the y-axis by a factor of $1/k$.

7 **a** The shear matrix is
$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix
$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1 \times 1 - k \times 0}\begin{bmatrix} 1 & -k \\ -0 & 1 \end{bmatrix}$$
$$= \frac{1}{1}\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}.$$
This matrix will shear each point from the x-direction by a factor of $-k$.

8 **a** The reflection matrix is
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The inverse transformation will have matrix
$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1 \times (-1) - 0 \times 0}\begin{bmatrix} -1 & -0 \\ -0 & 1 \end{bmatrix}$$
$$= -1\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= A.$$
This is expected, since two reflections in the same axis will return the point (x, y) to its original position.

9 **a** The reflection matrix is
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

b The inverse transformation will have matrix
$$A^{-1}$$
$$= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{-\cos^2\theta - \sin^2\theta}\begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$
$$= \frac{1}{-1}\begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$
$$= A.$$
This is expected, since two reflections in the same axis will return the point (x, y) to its original position.

Solutions to Exercise 20F

1 a The matrix of the transformation is
$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.$$
Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,
$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-y\end{bmatrix}.$$
Therefore, $x' = x$ and $y' = -y$.
Rearranging gives $x = x'$ and $y = -y''$. Therefore, $y = 3x + 1$ becomes, $-y'' = 3x' + 1$ We now
$$y' = -3x' - 1.$$
ignore the apostrophes, so that the transformed equation is
$$y = -3x - 1.$$

b The matrix of the transformation is
$$\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}.$$
Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,
$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\y\end{bmatrix}.$$
Therefore, $x' = 2x$ and $y' = y$.
Rearranging gives $x = \frac{x'}{2}$ and $y = y''$.
Therefore $y = 3x + 1$ becomes,
$$y' = \frac{3x'}{2} + 1.$$
We now ignore the apostrophes, so that the transformed equation is
$$y = \frac{3x}{2} + 1.$$

c The matrix of the transformation is
$$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}.$$
Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,
$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\3y\end{bmatrix}.$$
Therefore, $x' = 2x$ and $y' = 3y$.
Rearranging gives $x = \frac{x'}{2}$ and $y = \frac{y'}{3}$.
Therefore $y = 3x + 1$ becomes,
$\frac{y'}{3} = 3\left(\frac{x'}{2}\right) + 1$ We now ignore the
$$y' = \frac{9x'}{2} + 3.$$
apostrophes, so that the transformed equation is
$$y = \frac{9x}{2} + 3.$$

d The matrix of the transformation is
$$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}.$$
Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,
$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\-y\end{bmatrix}.$$
Therefore,$x' = -x$ and $y' = -y$.
Rearranging gives $x = -x'$ and $y = -y'$. Therefore $y = 3x + 1$ becomes, $-y' = 3(-x') + 1$
$$-y' = -3x' + 1$$
$$y' = 3x' - 1.$$
We now ignore the apostrophes, so that the transformed equation is
$$y = 3x - 1.$$

e The matrix of the transformation is
$$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ 3y \end{bmatrix}.$$

Therefore, $x' = -x$ and $y' = 3y$. Rearranging gives $x = -x'$ and $y = \frac{y'}{3}$. Therefore $y = 3x + 1$ becomes,

$$\frac{y'}{3} = 3(-x') + 1$$
$$\frac{y'}{3} = -3x' + 1$$
$$y' = -9x' + 3.$$

We now ignore the apostrophes, so that the transformed equation is

$$y = -9x + 3.$$

f The matrix of the transformation is

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Therefore, $x' = -y$ and $y' = x$. Rearranging gives $x = y'$ and $y = -x'$. Therefore $y = 3x + 1$ becomes,

$$-x' = 3y' + 1$$
$$3y' = -x' - 1$$
$$y' = \frac{-x' - 1}{3}.$$

We now ignore the apostrophes, so that the transformed equation is

$$y = \frac{-x - 1}{3}.$$

g Firstly, the rotation matrix is

$$\begin{bmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The reflection matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, if (x', y') be the co-ordinates of the image of (x, y), then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

Therefore, $x' = y$ and $y' = x$. Therefore, $y = 3x + 1$ becomes,

$$x' = 3y' + 1$$
$$3y' = x' - 1$$
$$y' = \frac{x' - 1}{3}.$$

We now ignore the apostrophes, so that the transformed equation is

$$y = \frac{x - 1}{3}.$$

2 a If (x', y') be the coordinates of the image of (x, y), then $x' = 2x$ and $y' = 3y$.

Rearranging gives $x = \frac{x'}{2}$ and $y = \frac{y'}{3}$. Therefore $y = 2 - 3x$ becomes,

$$\frac{y'}{3} = 2 - 3\left(\frac{x'}{2}\right)$$
$$y' = 6 - \frac{9x'}{2}.$$

We now ignore the apostrophes, so that the transformed equation is

$$y = 6 - \frac{9x}{2}.$$

b If (x', y') be the coordinates of the image of (x, y), then $x' = -y$ and $y' = x$.

Rearranging gives $x = y'$ and $y = -x'$. Therefore, $y = 2 - 3x$ becomes,

$$\begin{aligned}-x' &= 2 - 3y'\\ 3y' &= x' + 2\\ y' &= \frac{x'+2}{3}.\end{aligned}$$

We now ignore the apostrophes, so that the transformed equation is

$$y = \frac{x+2}{3}.$$

c Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} x' + 2y' \\ y' \end{bmatrix}$$

so that $x = x' + 2y'$ and $y = y'$. Therefore, $y = 2 - 3x$ becomes $y' = 2 - 3(x' + 2y')$. We solve the equation for y' in terms of x',

$$\begin{aligned}y' &= 2 - 3(x' + 2y')\\ y' &= 2 - 3x' - 6y'\\ 7y' &= 2 - 3x'\\ y' &= \frac{2-3x'}{7}.\end{aligned}$$

The transformed equation is

$$y = \frac{2-3x}{7}.$$

d Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 2x' - 5y' \\ -x' + 3y' \end{bmatrix}$$

so that $x = 2x' - 5y'$ and $y = -x' + 3y'$. Therefore, $y = 2 - 3x$ becomes $x' + 3y' = 2 - 3(2x' - 5y')$. We solve the equation for for y' in terms of x',

$$\begin{aligned}x' + 3y' &= 2 - 3(2x' - 5y')\\ x' + 3y' &= 2 - 6x' + 15y'\\ 12y' &= 7x' - 2\\ y' &= \frac{7x'-2}{12}.\end{aligned}$$

The transformed equation is

$$y = \frac{7x-2}{12}.$$

3 There are many answers. We find a matrix that maps the x-intercept of the first line to the x-intercept of the second line, and likewise for the y-intercepts. Then

$$(1, 0) \to (2, 0) \text{ and } (0, 1) \to (0, 2).$$

Since we have found the images of the standard unit vectors, the matrix that will achieve this result is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

4 There are many answers. Let's find the matrix that maps the x-intercept of the first line to the x-intercept of the second line, and likewise for the y-intercepts. Then

$$(-1, 0) \rightarrow (3, 0) \text{ and } (0, 1) \rightarrow (0, 6).$$

The matrix that will achieve this results is

$$\begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}.$$

5 Let (x', y') be the coordinates of the image of (x, y). Then the rule for the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$$

$$= \begin{bmatrix} x-1 \\ -y-2 \end{bmatrix}$$

Therefore,
$x' = x - 1$ and $y' = -y - 2$ so that $x = x' + 1$ and $y = -y' - 2$.
Therefore, the equation $y = x^2 - 1$ becomes $-y' - 2 = (x' + 1)^2 - 1$.
Therefore, t $-y' - 2 = (x' + 1)^2 - 1$

$$-y' = (x' + 1)^2 + 1$$

$$y' = -(x' + 1)^2 - 1$$

The transformed equation is

$$y = -(x + 1)^2 - 1.$$

6 Let (x', y') be the coordinates of the image of (x, y). Then the rule for the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} \text{ Therefore,}$$

$$= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -x+2 \\ y-3 \end{bmatrix}$$

$$x' = -x + 2 \text{ and} y' = y - 3$$

so that

$$x = -x' + 2 \text{ and } y = y' + 3.$$

Therefore, the equation $y = (x - 1)^2$ becomes $y' + 3 = (-x' + 2 - 1)^2$.
Therefore,

$$y' + 3 = (-x' + 2 - 1)^2$$

$$y' = (-x' + 1)^2 - 3$$

$$= (-(x' - 1))^2 - 3$$

$$= (x' - 1)^2 - 3$$

The transformed equation is

$$y = (x - 1)^2 - 3.$$

7 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3y \\ x \end{bmatrix}$$

The transformation is defined by the rule $(x, y) \rightarrow (-3y, x)$. Therefore let $x' = -3y$ and $y' = x$ where (x', y') is the image of (x, y) under the transformation. Hence $x = y'$ and $y = -\frac{x'}{3}$. The equation

$$x^2 + y^2 = 1$$

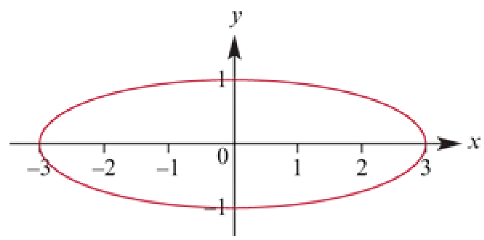
becomes,

$$(y')^2 + \frac{(-x')^2}{9} = 1$$

Ignoring the apostrophes gives,

$$y^2 + \frac{x^2}{9} = 1$$

This is an ellipse with centre the origin, with intercepts at $(\pm 3, 0)$ and $(0, \pm 1)$.



8 Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$

$$= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{dx' - by'}{ad - bc} \\ \dfrac{-cx' + ay'}{ad - bc} \end{bmatrix}$$

so that

$$x = \frac{dx' - by'}{ad - bc} \text{ and } y = \frac{-cx' + ay'}{ad - bc}.$$

Therefore $px + qy = r$ becomes,

$$p\frac{dx' - by'}{ad - bc} + q\frac{-cx' + ay'}{ad - bc} = r.$$

which, although horribly ugly, is most definitely the equation of a line.

9 The matrix of the transformation is

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}.$$

Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} x' + y' \\ -x' + y' \end{bmatrix}$$

so that

$$x = \frac{1}{\sqrt{2}}(x' + y'),$$

$$y = \frac{1}{\sqrt{2}}(-x' + y').$$

Therefore, $y = \dfrac{1}{x}$ becomes,

$$\frac{1}{\sqrt{2}}(-x' + y') = \frac{1}{\dfrac{1}{\sqrt{2}}(x' + y')}.$$

Ignoring the apostrophes, and simplifying this expression gives,

$$\frac{1}{\sqrt{2}}(x + y)\frac{1}{\sqrt{2}}(-x + y) = 1$$

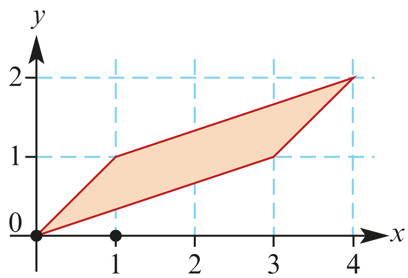
$$\frac{1}{2}(x + y)(y - x) = 1$$

$$y^2 - x^2 = 2$$

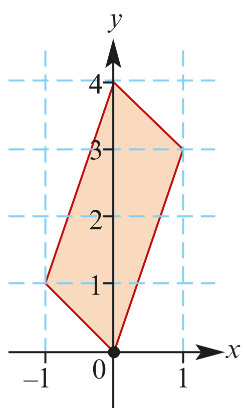
This is the required equation.

Solutions to Exercise 20G

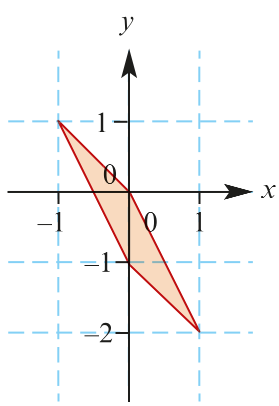
1 **a** The area will be given by $|\det B| = |3 \times 1 - 1 \times 1| = |2| = 2.$



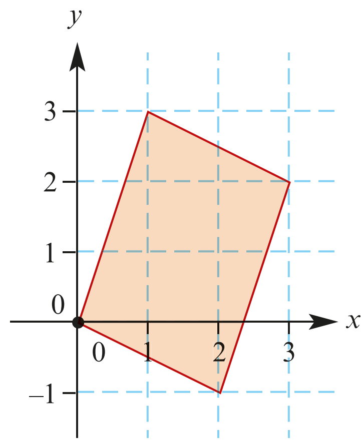
b The area will be given by $|\det B| = |(-1) \times 3 - 1 \times 1| = |-4| = 4.$



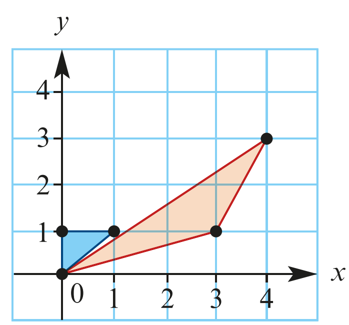
c The area will be given by $|\det B| = |1 \times 1 - (-1) \times (-2)| = |-1| = 1.$



d The area will be given by $|\det B| = |2 \times 3 - 1 \times (-1)| = |6 + 1| = 7.$



2 **a** The original triangle is shown in blue, and its image is in red.



b The area of the original triangle is $\frac{1}{2}$. Therefore the area of the image will be given by,

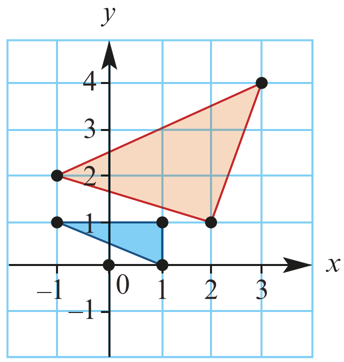
Area of Image = $|\det B| \times$ Area of Region

$$= |1 \times 1 - 3 \times 2| \times \frac{1}{2}$$

$$= |-5| \times \frac{1}{2}$$

$$= 2.5.$$

3 **a** The original triangle is shown in blue, and its image is in red.



b The area of the original triangle is 1. Therefore the area of the image will be given by,

$$\text{Area of Image} = |\det B| \times \text{Area of Region}$$
$$= |2 \times 3 - 1 \times 1| \times 1$$
$$= 5.$$

4 Since the original area is 1 and the area of the image is 6, we have,

$$|\det B| \times 1 = 6$$
$$|m \times m - 2 \times (-1)| = 6$$
$$|m^2 + 2| = 6$$
$$m^2 + 2 = 6 \text{ (since } m^2 + 2 > 0)$$
$$m^2 = 4$$
$$m = \pm 2.$$

5 The original area is 1 and the area of the image is 2. Therefore,

$$\text{Area of Image} = |\det B| \times \text{Area of region}$$
$$2 = |m \times m - m \times 1| \times 1$$
$$2 = |m^2 - m|$$

Therefore, either

$$m^2 - m = 2 \text{ or } m^2 - m = -2.$$

In the first case, we have

$$m^2 - m - 2 = 0$$
$$(m - 2)(m + 1) = 0$$
$$m = -1, 2.$$

In the second case, we have $m^2 - m + 2 = 0$. This has no solutions since the discriminant of the quadratic equation is

$$\Delta = b^2 - 4ac$$
$$= 1^2 - 4(1)(2)$$
$$= 1 - 8 < 0.$$

6 **a** **i** If

$$B = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

then

$$|\det B| = |1 \times 1 - k \times 0| = 1.$$

ii If

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

then

$$|\det B| = |\cos\theta\cos\theta - (-\sin\theta)\sin\theta|$$
$$= |\cos^2\theta + \sin^2\theta|$$
$$= 1.$$

iii If

$$B = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

then

$$|\det B| = |\cos 2\theta(-\cos 2\theta) - \sin 2\theta \sin 2\theta|$$
$$= |-(\cos^2 2\theta + \sin^2 2\theta)|$$
$$= |-1|$$
$$= 1$$

b **i** This transformation is a dilation by a factor k away from the y-axis and a factor of $1/k$ away from the x-axis.

ii We have,

$$|\det B| = |k \times 1/k - 0 \times 0|$$
$$= 1$$

7 a We have,

$$|\det B| = |x \times (x+2) - 1 \times (-2)|$$
$$= |x^2 + 2x + 2|$$
$$= |(x^2 + 2x + 1) + 1|$$

(completing the square)

$$= |(x+1)^1 + 1|$$
$$= (x+1)^2 + 1.$$

b The area will be a minimum at the turning point of the parabola whose equation is $y = (x+1)^2 + 1$. This occurs when $x = -1$.

8 We require that $|\det B| > 2$

$$|4m - 6| > 2.$$

Therefore, either $4m - 6 > 2$ or $4m - 6 < -2$. In the first case, $m > 2$. In the second case, $m < 1$. Therefore $m > 2$ or $m < 1$.

9 Since $(1,0) \to (1,0)$ we can assume that the matrix is of the form $\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$. Since the area is $\frac{1}{2}$, we know that

$$|1 \times c - b \times 0| = \frac{1}{2}$$
$$|c| = \frac{1}{2}$$
$$c = \pm\frac{1}{2}$$

Since $(0,1) \to (b,c)$, one corner of the rhombus will be given by the second column (written as a coordinate). Moreover, since it is a rhombus, the distance from $(0,0)$ to (b,c) is 1. Therefore,

$$b^2 + c^2 = 1^2$$
$$b^2 + \left(\frac{1}{2}\right)^2 = 1$$
$$b^2 = \frac{3}{4}$$
$$b = \pm\frac{\sqrt{3}}{2}$$

so that the required matrix is

$$\begin{bmatrix} 1 & \pm\frac{\sqrt{3}}{2} \\ 0 & \pm\frac{1}{2} \end{bmatrix}.$$

10 a We can assume that $(1,0) \to (a,c)$ and $(0,1) \to (b,d)$. Therefore, the required matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

b The area of the original triangle is $\frac{1}{2}$. Therefore, the area of the image will be given by,

Area of Image = $|\det B| \times$ Area of Region

$$= |a \times d - b \times c| \times \frac{1}{2}$$
$$= \frac{1}{2}|ad - bc|$$

c If a, b, c, d are all rational numbers then so too is $\frac{1}{2}|ad - bc|$.

d We will assume that the triangle has vertices $O(0,0), A(a,c)$ and $B(b,d)$.Then the area of the triangle is

$$\frac{1}{2}|ad - bc|. \quad (1)$$

We will find another expression for the area. Since the triangle is equilateral,

$$OB = OA = \sqrt{a^2 + c^2}$$

Using Pythagoras' Theorem, we can show that

$$MB^2 + OM^2 = OB^2$$

$$MB^2 + \left(\frac{1}{2}OA\right)^2 = OA^2$$

$$MB^2 + \frac{1}{4}OA^2 = OA^2$$

$$MB^2 = \frac{3}{4}OA^2$$

$$MB = \frac{\sqrt{3}}{2}OA$$

$$= \frac{\sqrt{3}\sqrt{a^2 + c^2}}{2}.$$

Therefore, another expression for the area is

$$A = \frac{1}{2} \times OA \times MB$$

$$= \frac{1}{2} \times \sqrt{a^2 + c^2} \times \frac{\sqrt{3}\sqrt{a^2 + c^2}}{2}$$

$$= \frac{\sqrt{3}(a^2 + b^2)}{4} \qquad (2)$$

Equating equations (1) and (2) gives,

$$\frac{\sqrt{3}(a^2 + b^2)}{4} = \frac{1}{2}|ad - bc|$$

$$\sqrt{3} = \frac{2|ad - bc|}{a^2 + b^2}$$

Since a, b, c and d are all rational numbers, the right hand side of the above expression is rational. This contradicts the fact that $\sqrt{3}$ is irrational.

Solutions to Exercise 20H

1 Firstly, the matrix that will rotate the plane by 90° clockwise is given by

$$\begin{bmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

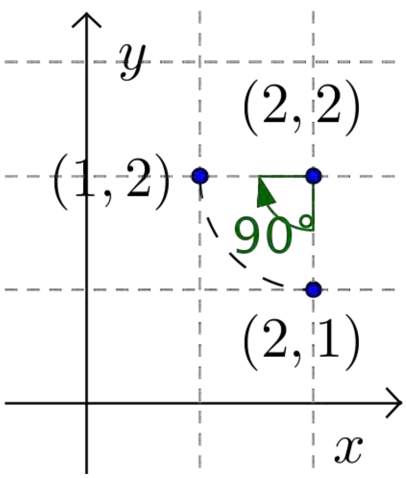
Therefore, the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x-2 \\ y-2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} y-2 \\ -x+2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} y \\ -x+4 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(2, 1)$. Let $x = 2$ and $y = 1$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ -2+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore, $(2, 1) \to (1, 2)$, as expected from the diagram shown below.



2 Firstly, the matrix that rotates the plane by 180° about the origin is

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

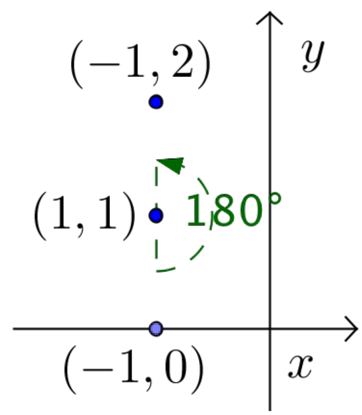
Therefore the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x+1 \\ y-1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -x-1 \\ -y+1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -x-2 \\ -y+2 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(-1, 0)$. Let $x = -1$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -(-1)-2 \\ -0+2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Therefore, $(-1, 0) \to (-1, 2)$, as expected from the diagram shown below.



3 **a** Firstly, the matrix that reflects the plane in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

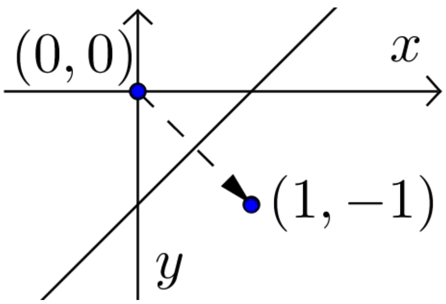
Therefore, the required transformation is

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} y+1 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}. \end{aligned}$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore, $(0, 0) \to (1, -1)$, as expected from the diagram shown below.



b Firstly, the matrix that reflects the plane in the line $y = -x$ is given by

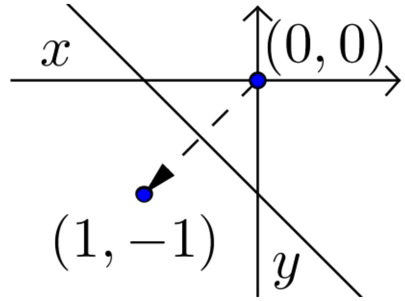
$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, the required transformation is
$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y-1 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y-1 \\ -x-1 \end{bmatrix}.\end{aligned}$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0-1 \\ -0-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Therefore, $(0, 0) \to (-1, -1)$, as expected from the diagram shown below.

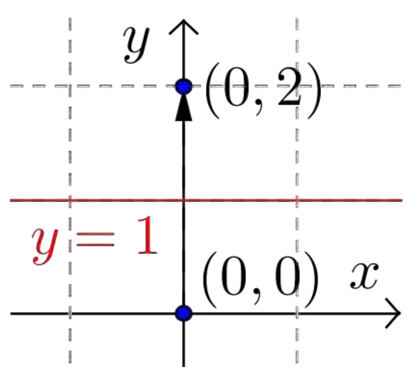


c We will translate the plane 1 unit down so that we can then reflect the plane in the line $y = 0$, that is, the x-axis. We then return the plane to its original position by translating the plane 1 unit up.
Therefore, the required transformation is
$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -y+2 \end{bmatrix}.\end{aligned}$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ -0+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Therefore, $(0, 0) \to (0, 2)$, as expected from the diagram shown below.



d We will translate the plane 2 units right so that we can then reflect the plane in the line $x = 0$, that is, the y-axis. We then return the plane to its original position by translating the plane 2 units left.

Therefore, the required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x+2 \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

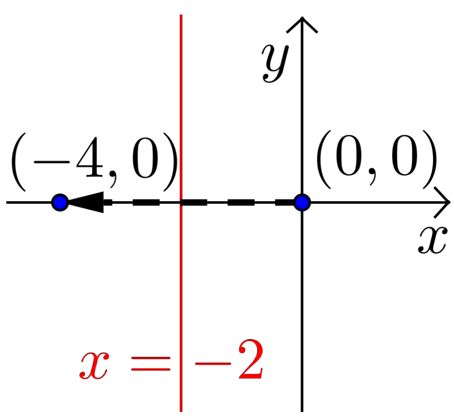
$$= \begin{bmatrix} -x-2 \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -x-4 \\ y \end{bmatrix}.$$

We check our answer by finding the image of the point $(0, 0)$. Let $x = 0$ and $y = 0$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0-4 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

Therefore, $(0, 0) \to (-4, 0)$, as expected from the diagram shown below.



4 We will rotate the plane clockwise by angle θ, dilate the point (x, y) by a factor of k from the y-axis, then return the plane to its original position by rotating by angle θ anticlockwise.

Therefore, the required matrix will be

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + k\sin^2\theta & \cos\theta\sin\theta - k\cos\theta\sin\theta \\ \cos\theta\sin\theta - k\cos\theta\sin\theta & \sin^2\theta + k\cos^2\theta \end{bmatrix}.$$

5 We will rotate the plane clockwise by angle θ, project the point (x, y) onto the x-axis, then return the plane to its original position by rotating by angle θ anticlockwise.

Therefore, the required matrix will be

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

6 The transformation that reflects the plane in the line $y = x + 1$ is given by,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} y-1 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} y-1 \\ x+1 \end{bmatrix}.$$

If we then want to reflect the result in the the line $y = x$ we would multiply by the reflection matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This gives a transformation whose rule is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y-1 \\ x+1 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}.$$

This corresponds to a translation defined by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solutions to technology-free questions

1 a We let $x = 2$ and $y = 3$ so that

$(2, 3) \to (2\times2+3, -2+2\times3) = (7, 4)$.

b The matrix of the transformation is given by the coefficients in the rule, that is,

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

c The fastest way to find the image of the unit square is to evaluate

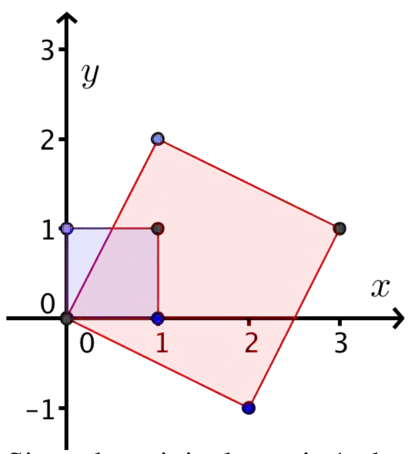
$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$$

The columns then give the required points:

$$(0, 0), (2, -1), (1, 2), (3, 1)$$

The square is shown in blue, and its image in red.



Since the original area is 1, the area of the image will be Area $= |ad - bc| = |2 \times 2 - 1 \times (-1)| = 5$

d Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

the inverse transformation will have matrix

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{2\times2-1\times(-1)}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Therefore, the rule of the inverse transformation is

$$(x, y) \to \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

c $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

e $\begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

f $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

3 a Since

$$\tan\theta = 3 = \frac{3}{1},$$

we draw a right angled triangle with opposite and adjacent lengths 5 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{3^2+1^2} = \sqrt{10}$. Therefore,

$$\cos\theta = \frac{1}{\sqrt{10}} \text{ and } \sin\theta = \frac{3}{\sqrt{10}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{\sqrt{10}}\right)^2 - 1\\ &= \frac{2}{10} - 1\\ &= -\frac{4}{5},\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\frac{1}{\sqrt{10}}\frac{3}{\sqrt{10}}\\ &= \frac{3}{5}.\end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}-\frac{4}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{bmatrix}.$$

b The image of the point $(2,4)$ can be found by evaluating,

$$\frac{1}{5}\begin{bmatrix}-4 & 3\\ 3 & 4\end{bmatrix}\begin{bmatrix}2\\ 4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4\\ 22\end{bmatrix}.$$

Therefore, $(2,4) \to \left(\frac{4}{5}, \frac{22}{5}\right)$.

4 a The matrix that will reflect the plane in the x-axis is given by

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The matrix that will reflect the plane in the line $y = -x$ is given by

$$\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

b The matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

The dilation matrix by a factor of 2 from the x-axis is given by

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}.$$

c The matrix that will reflect the plane in the line $y = x$ is given by

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

The matrix that will skew the result by a factor of 2 from the x-axis is given by

$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}.$$

5 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ There-

$$= \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} x-3 \\ -y+4 \end{bmatrix}$$

fore, the transformation is $(x, y) \to (x-3, -y+4)$.

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x-3 \\ y+4 \end{bmatrix}$$

$$= \begin{bmatrix} x-3 \\ -y-4 \end{bmatrix}$$

Therefore, the transformation is $(x, y) \to (x-3, -y-4)$.

6 a The required matrix is

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

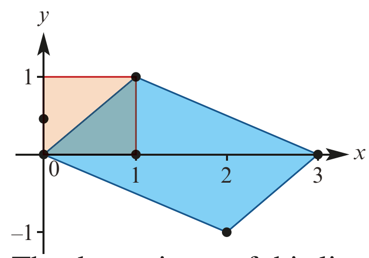
$$= \frac{1}{1\times 1 - 0\times k}\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}.$$

This matrix will shear each point in the y-direction by a factor of $-k$.

7 a The unit square is shown in red, and its image in blue.



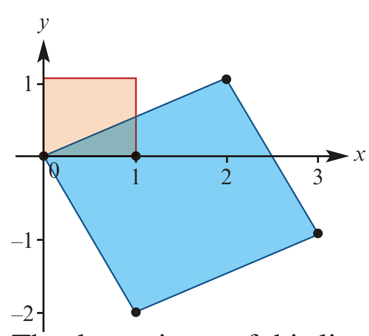
The determinant of this linear transformation is

$$\det B = 2\times 1 - 1\times(-1) = 2+1 = 3.$$

The unit square has area 1 square unit, so to find the area of its image we evaluate:

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= 3\times 1 \\ &= 3 \text{ square units.} \end{aligned}$$

b The unit square is shown in red, and its image in blue.



The determinant of this linear transformation is

$$\det B = 2\times(-2) - 1\times 1 = -4-1 = -5.$$

The unit square has area 1 square unit, so to find the area of its image we evaluate:

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= |-5|\times 1 \\ &= 5 \text{ square units.} \end{aligned}$$

8 a We do this as a sequence of three steps:

- translate the plane so that the origin is the centre of rotation.
- rotate the plane about the origin by 90° anticlockwise.
- translate the plane back to its original position.

Firstly the rotation matrix is

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the overall transformation of

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y-1 \\ x-1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -y \\ x-2 \end{bmatrix} \end{aligned}$$

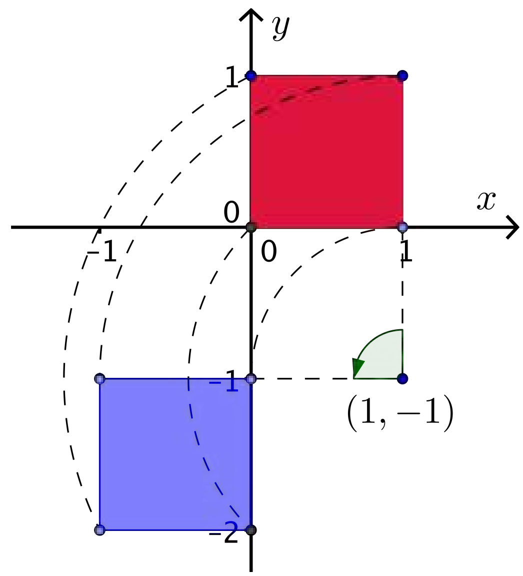
b To find the image of the point $(2, -1)$.

Let $x = 2$ and $y = -1$ so that

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -y \\ x-2 \end{bmatrix} \\ &= \begin{bmatrix} -(-1) \\ 2-2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore $(2, -1) \to (1, 0)$.

c The unit square is shown in red, and its image after the rotation is in blue.



Solutions to multiple-choice questions

1 B The point $(2, -1)$ maps to the point $(2\times 2-3\times(-1), -2+4\times(-1)) = (7, -6)$.

2 D The required transformation is $(x, y) \to (-y, -x)$, which corresponds to matrix

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

3 A The matrix that will dilate the plane by a factor of 2 from the y-axis is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix that will reflect the result in the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

4 D The location of the negative entry suggests that this should be a reflection matrix. Indeed, if $\theta = 30°$ then,

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

This corresponds to a reflection in the line $y = x\tan 30°$.

5 C Firstly, matrix that will rotate the plane by $90°$ anticlockwise is given by

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the required transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x+2 \\ y-3 \end{bmatrix}$$

$$= \begin{bmatrix} -y+3 \\ x+2 \end{bmatrix}$$

Therefore, the transformation is $(x, y) \to (-y+3, x+2)$.

6 A Note that this matrix is equal to the product:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This corresponds to a rotation by $180°$ (or, equivalently, a reflection through the origin) followed by a dilation by a factor of 2 from the x-axis.

7 D Note that this matrix corresponds to a reflection in both the x and y axes. So we draw the graph of $y = (x-1)^2$, then reflect this in each axis. Alternatively, you can show that the transformed graph has equation $y = -(x+1)^2$.

8 E We simply need to find the matrix that has a determinant of 2. Only the last matrix has this property.

9 D Matrix R is a rotation matrix of $40°$. Therefore, matrix R^n is a rotation matrix of $40m°$. Since a rotation by any multiple of 360° corresponds to the identity matrix, we need to find the smallest value of m such that $40m$ is a multiple of 360°. Therefore, $m = 9$.

Solutions to extended-response questions

1 **a** The required rotation matrix is

$$\begin{bmatrix}\cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

b The required rotation matrix is

$$\begin{bmatrix}\cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}.$$

c A 45° rotation followed by a 30° rotation will give a 75° rotation. Therefore, the required matrix is

$$\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}}\end{bmatrix}.$$

d The rotation matrix of 75° is also given by the expression

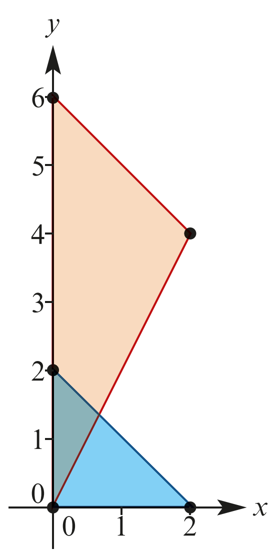
$$\begin{bmatrix}\cos 75^\circ & -\sin 75^\circ \\ \sin 75^\circ & \cos 75^\circ\end{bmatrix}.$$

Comparing the entries of these two matrices gives

$$\cos 75^\circ = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4},$$

$$\sin 75^\circ = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$

2 **a** The triangle is shown in blue and its image in red.



b The area of the original triangle is

$$\frac{bh}{2} = \frac{2 \times 2}{2} = 2.$$

Therefore the area of the image will be given by,

$$\begin{aligned}\text{Area of Image} &= |\det B| \times \text{Area of Region}\\ &= |1 \times 3 - 0 \times 2| \times 2\\ &= 3 \times \frac{1}{2}\\ &= 6 \text{ square units.}\end{aligned}$$

c When the red figure is revolved around the y-axis, we obtain a figure that is the compound of two cones. The upper cone has base radius $r_1 = 2$ and height $h_1 = 2$. The lower cone has base radius $r = 2$ and height $h = 4$. Therefore, the total volume will be

$$\begin{aligned}V &= \frac{1}{3}\pi r_1^2 h_1 + \frac{1}{3}\pi r_2^2 h_2\\ &= \frac{1}{3} \times \pi 2^2 \times 2 + \frac{1}{3} \times \pi 2^2 \times 4\\ &= 8\pi \text{ cubic units.}\end{aligned}$$

3 **a** The matrix of the transformation is obtained by reading off the coefficients in the rule for the linear transformation. That is,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

b This transformation is a shear by a factor of 1 in the x direction.

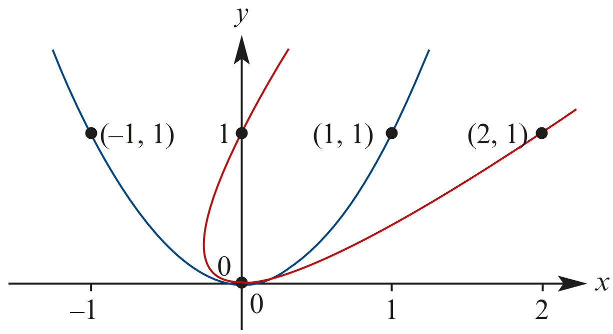
c The image of the points can be found in one step by evaluating,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The columns then give the required points:

$$(0, 0), (2, 1), (0, 1).$$

d The image will be a sheared parabola, shown in red. The original parabola is shown in blue.



4 a The matrix of the transformation is

$$\begin{bmatrix}\cos 45° & -\sin 45° \\ \sin 45° & \cos 45°\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

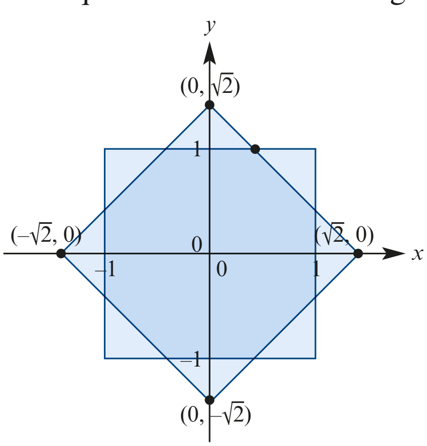
To find the image of the point (1, 1) we multiply,

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 \\ 1\end{bmatrix} = \begin{bmatrix}0 \\ \sqrt{2}\end{bmatrix}.$$

Therefore $(1, 1) \to (0, \sqrt{2})$. Since this matrix will rotate the square by 45° anticlockwise, the four points must be:

$$(0, \sqrt{2}), (\sqrt{2}, 0), (0, -\sqrt{2}), (-\sqrt{2}, 0).$$

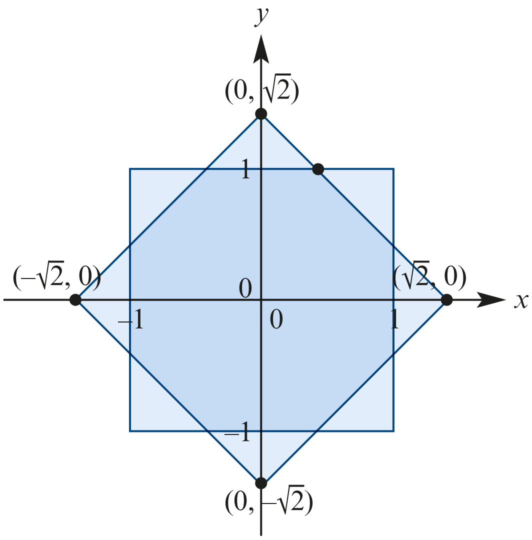
b The square and its rotated image are shown below.



c The area of the shape can be found in many ways. We will find the coordinates of point *A* shown in the above diagram. Point *A* is the intersection of the lines

$$y = 1 \text{ and } x + y = \sqrt{2}.$$

Solving this pair of equations gives $x = \sqrt{2} - 1$ and $y = 1$ so that the required point is $A(\sqrt{2}-1, 1)$. The area of the figure is the sum of one square and four triangles, one of which is indicated in red below.



Since point A has coordinates $(\sqrt{2}-1, 1)$, the area of each triangle is

$$\begin{aligned} A &= \frac{bh}{2} \\ &= \frac{(2\sqrt{2}-2)(\sqrt{2}-1)}{2} \\ &= 3-2\sqrt{2}. \end{aligned}$$

Therefore, the total area will be $A = 1 + 4 \times (3-2\sqrt{2})$

$$= 13 - 8\sqrt{2} \text{ square units.}$$

5 a i $\text{Rot}(\theta)\text{Rot}(\phi)$

$$\begin{aligned} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\theta) \\ \sin\theta\cos\phi + \cos\theta\sin\theta & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \\ &= \text{Rot}(\theta+\phi) \end{aligned}$$

ii Ref(θ)Ref(ϕ)

$$=\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta\cos 2\phi+\sin 2\theta\sin 2\phi & -(\sin 2\theta\cos 2\phi-\cos 2\theta\sin 2\theta) \\ \sin 2\theta\cos 2\phi-\cos 2\theta\sin 2\theta & \cos 2\theta\cos 2\phi+\sin 2\theta\sin 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta-2\phi) & -\sin(2\theta-2\phi) \\ \sin(2\theta-2\phi) & \cos(2\theta-2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi))\end{bmatrix}$$

$$=\text{Rot}(2(\theta-\phi))$$

iii Rot(θ)Ref(ϕ)

$$=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta\cos 2\phi-\sin\theta\sin 2\phi & \sin\theta\cos 2\phi+\cos\theta\sin 2\theta \\ \sin\theta\cos 2\phi+\cos\theta\sin 2\theta & -(\cos\theta\cos 2\phi-\sin\theta\sin 2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\theta+2\phi) & \sin(\theta+2\phi) \\ \sin(\theta+2\phi) & -\cos(\theta+2\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\phi+\theta/2)) & \sin(2(\phi+\theta/2)) \\ \sin(2(\phi+\theta/2)) & -\cos(2(\phi+\theta/2))\end{bmatrix}$$

$$=\text{Ref}(\phi+\theta/2)$$

iv Ref(θ)Rot(ϕ)

$$=\begin{bmatrix}\cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta\cos\phi+\sin 2\theta\sin\phi & \sin 2\theta\cos\phi-\cos 2\theta\sin\theta \\ \sin 2\theta\cos\phi-\cos 2\theta\sin\theta & -(\cos 2\theta\cos\phi+\sin 2\theta\sin\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta-\phi) & \sin(2\theta-\phi) \\ \sin(2\theta-\phi) & -\cos(2\theta-\phi)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\theta-\phi/2)) & \sin(2(\theta-\phi/2)) \\ \sin(2(\theta-\phi/2)) & -\cos(2(\theta-\phi/2))\end{bmatrix}$$

$$=\text{Ref}(\theta-\phi/2)$$

b **i** The composition of two rotations is a rotation.

ii The composition of two reflections is a rotation.

iii The composition of a reflection followed by a rotation is a reflection.

iv The composition of a rotation followed by a reflection is a reflection.

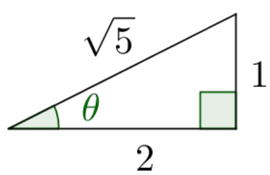
c Evaluating from left to right we have,

$$\begin{aligned}
&\mathrm{Rot}(60^\circ)\mathrm{Ref}(60^\circ)\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
&= (\mathrm{Rot}(60^\circ)\mathrm{Ref}(60^\circ))\,\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
&= \mathrm{Ref}(60^\circ + 30^\circ)\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
&= \mathrm{Ref}(60^\circ)\mathrm{Ref}(60^\circ)\mathrm{Rot}(60^\circ)\\
&= (\mathrm{Ref}(60^\circ)\mathrm{Ref}(60^\circ))\,\mathrm{Rot}(60^\circ)\\
&= \mathrm{Rot}(2(90^\circ - 60^\circ))\mathrm{Rot}(60^\circ)\\
&= \mathrm{Rot}(60^\circ)\mathrm{Rot}(60^\circ)\\
&= \mathrm{Rot}(120^\circ)\\
&= \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{bmatrix}\\
&= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

6 **a** Since

$$\tan\theta = \frac{1}{2},$$

we draw a right angled triangle with opposite and adjacent lengths 1 and 2 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{5}$.



Therefore,

$$\cos\theta = \frac{2}{\sqrt{5}} \text{ and } \sin\theta = \frac{1}{\sqrt{5}}.$$

We then use the double angle formulas to show that

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{2}{\sqrt{5}}\right)^2 - 1 = \frac{8}{5} - 1 = \frac{3}{5},$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{1}{\sqrt{5}}\frac{2}{\sqrt{5}} = \frac{4}{5}.$$

Therefore the required matrix is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

b The image of the point $A(-3, 1)$ is found by evaluating the matrix product,

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.$$

Therefore, the required point is $A'(-1, -3)$.

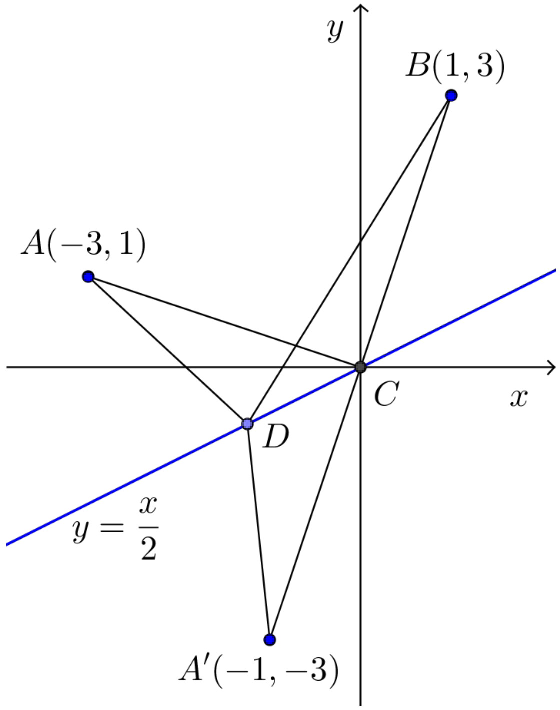
c Using the distance formula we find that

$$\begin{aligned} A'B &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - (-1))^2 + (3 - (-3))^2} \\ &= \sqrt{2^2 + 6^2} \\ &= \sqrt{40} \\ &= 2\sqrt{10}. \end{aligned}$$

d The line $y = \frac{x}{2}$ is the perpendicular bisects of line AA'. Therefore, $CA = CA'$, so that triangle ACA' is isosceles.

e Referring to the diagram below we have:

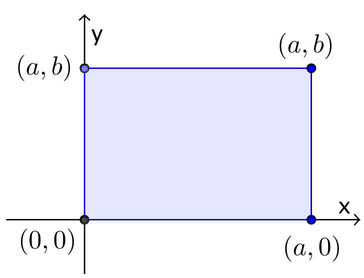
$$\begin{aligned} AD + DB &= A'D + DB \quad \text{(triangle } ADA' \text{ is isosceles)} \\ &> A'B \quad \text{(the side length of a triangle is always less than the sum of the other two)} \\ &= A'C + CB \\ &= AC + CB \quad \text{(triangle } ACA' \text{ is isosceles)} \end{aligned}$$



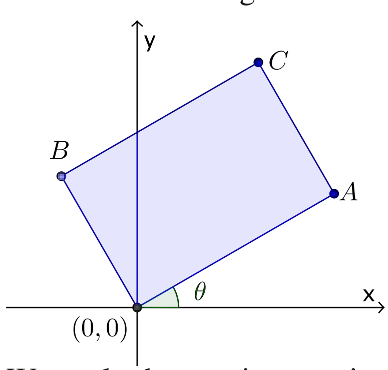
f The above calculation shows that $AC + CB$ is the shortest distance from A to B via the line. Therefore the shortest distance is

$$AC + CB = A'C + CB = A'B = 2\sqrt{10}.$$

7 **a** The rectangle is shown below.



b The rotated rectangle is shown below.



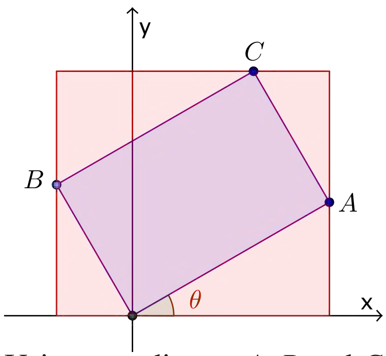
We apply the rotation matrix to the coordinate of the original rectangle to find the following co-ordinates:

$$A(a\cos\theta, a\sin\theta),$$

$$B(-b\sin\theta, b\cos\theta),$$

$$C(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta).$$

c The rectangle described is shown in red in the diagram below.



Using coordinates A, B and C found in the previous question, we can find the area of the triangle. Its width is equal to

$$a\cos\theta + b\sin\theta,$$

and its height is equal to

$$a\sin\theta + b\cos\theta.$$

Therefore, its area is

$$\begin{aligned}
A &= (a\cos\theta + b\sin\theta)(a\sin\theta + b\cos\theta)\\
&= a^2\cos\theta\sin\theta + ab\cos^2\theta + ab\sin^2\theta + b^2\cos\theta\sin\theta\\
&= (a^2+b^2)\cos\theta\sin\theta + ab(\cos^2\theta + \sin^2\theta)\\
&= (a^2+b^2)\cos\theta\sin\theta + ab(\cos^2\theta + \sin^2\theta)\\
&= (a^2+b^2)\cos\theta\sin\theta + ab\\
&= \frac{(a^2+b^2)}{2}\sin 2\theta + ab
\end{aligned}$$

d For θ between 0 and 90°, the maximum value of $\sin 2\theta$ occurs when $\theta = \frac{\pi}{4}$.
Therefore, the maximum area will be

$$\begin{aligned}
A &= \frac{(a^2+b^2)}{2} + ab \quad \text{as required.}\\
&= \frac{(a^2+2ab+b^2)}{2}\\
&= \frac{(a+b)^2}{2},
\end{aligned}$$

8 **a** Line L_1 is perpendicular to the line $y = mx$ and so has gradient $-\frac{1}{m}$. Moreover, it goes through the point $(1, 0)$. Therefore, its equation can be easily found:

$$\begin{aligned} y - 0 &= -\frac{1}{m}(x - 1) \\ y &= -\frac{x}{m} + \frac{1}{m} \\ &= \frac{1}{m} - \frac{x}{m}. \end{aligned}$$

To find where the line intersects the unit circle, we substitute $y = \dfrac{1}{m} - \dfrac{x}{m}$ into the equation for the circle, $x^2 + y^2 = 1$ and solve. This gives,

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(\frac{1}{m} - \frac{x}{m}\right)^2 &= 1 \\ x^2 + \frac{1}{m^2} - \frac{2x}{m^2} + \frac{x^2}{m^2} &= 1 \\ m^2x^2 + 1 - 2x + x^2 &= m^2 \\ (m^2 + 1)x^2 - 2x + (1 - m^2) &= 0. \end{aligned}$$

Since we already know that $(x - 1)$ is a factor of this polynomial, we can find the other factor by inspection. This gives,

$$(x - 1)\left((m^2 + 1)x - (1 - m^2)\right) = 0$$

so that

$$x = 1 \text{ or } x = \frac{1 - m^2}{1 + m^2}.$$

Substituting $x = \dfrac{1 - m^2}{1 + m^2}$ into the equation of the line gives

$$\begin{aligned} y &= \frac{1}{m} - \frac{x}{m} \\ &= \frac{1}{m} - \frac{1 - m^2}{m(1 + m^2)} \\ &= \frac{1 + m^2}{m(1 + m^2)} - \frac{1 - m^2}{m(1 + m^2)} \\ &= \frac{2m^2}{m(1 + m^2)} \\ &= \frac{2m}{1 + m^2} \end{aligned}$$

Therefore the other point of intersection is

$$\left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right).$$

b Line L_2 is perpendicular to the line $y = mx$ and so has gradient $-\dfrac{1}{m}$. Moreover, it goes through the point $(0, 1)$. Therefore, its equation can be easily found:

$$y - 1 = -\frac{1}{m}(x - 0)$$
$$y = 1 - \frac{x}{m}$$
To find where the line intersects the unit circle, we substitute $y = 1 - \dfrac{x}{m}$ into the equation for the circle, $x^2 + y^2 = 1$ and solve. This gives,
$$x^2 + y^2 = 1$$
$$x^2 + \left(1 - \frac{x}{m}\right)^2 = 1$$
$$x^2 + 1 - \frac{2x}{m} + \frac{x^2}{m^2} = 1$$
$$m^2x^2 + m^2 - 2mx + x^2 = m^2$$
$$(1 + m^2)x^2 - 2mx = 0.$$
We factorise this expression to give
$$x\big((1 + m^2)x - 2m\big) = 0$$
so that
$$x = 0 \text{ or } x = \frac{2m}{1 + m^2}.$$
Substituting $x = \dfrac{2m}{1 + m^2}$ into the equation of the line gives
$$\begin{aligned} y &= 1 - \frac{x}{m} \\ &= 1 - \frac{2m}{m(1 + m^2)} \\ &= 1 - \frac{2}{(1 + m^2)} \\ &= \frac{1 + m^2}{1 + m^2} - \frac{2}{(1 + m^2)} \\ &= \frac{m^2 - 1}{1 + m^2} \end{aligned}$$
Therefore, the other point of intersection is
$$\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right).$$

c When reflected in the line $y = mx$, the point $(1, 0)$ maps to
$$\left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$$
while the point $(0, 1)$ maps to
$$\left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right).$$

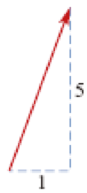
We write these points as the columns of a matrix to give,

$$\begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} \\ \dfrac{2m}{1+m^2} & \dfrac{m^2-1}{1+m^2} \end{bmatrix}.$$

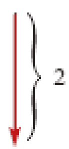
Chapter 21 – Vectors

Solutions to Exercise 21A

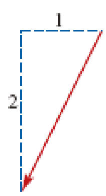
1 a $\begin{bmatrix}1\\5\end{bmatrix}$ is the vector "1 across to the right and 5 up."



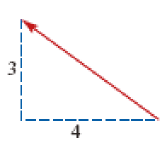
b $\begin{bmatrix}0\\-2\end{bmatrix}$ is the vector "2 down."



c $\begin{bmatrix}-1\\-2\end{bmatrix}$ is the vector "1 across to the left and 2 down."



d $\begin{bmatrix}-4\\3\end{bmatrix}$ is the vector "4 across to the left and 3 up."



2 $\boldsymbol{u} = \begin{bmatrix}6-1\\6-5\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}$

$a = 5, b = 1$

3 $\boldsymbol{v} = \begin{bmatrix}2--1\\-10-5\end{bmatrix} = \begin{bmatrix}-3\\15\end{bmatrix}$

$a = 3, b = -15$

4 a $\overrightarrow{OA} = \begin{bmatrix}1-0\\-2-0\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$

b $\overrightarrow{AB} = \begin{bmatrix}3-1\\0--2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

c $\overrightarrow{BC} = \begin{bmatrix}2-3\\-3-0\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix}$

d $\overrightarrow{CO} = -\overrightarrow{OC} = \begin{bmatrix}-2\\3\end{bmatrix}$

e $\overrightarrow{CB} = -\overrightarrow{BC} = \begin{bmatrix}1\\3\end{bmatrix}$

5 a i $\boldsymbol{a}+\boldsymbol{b} = \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\-3\end{bmatrix}$

$= \begin{bmatrix}1+1\\2+-3\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$

ii $2\boldsymbol{c}-\boldsymbol{a} = 2\times\begin{bmatrix}-2\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}$

$= \begin{bmatrix}-4-1\\2-2\end{bmatrix} = \begin{bmatrix}-5\\0\end{bmatrix}$

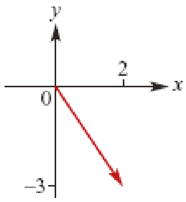
iii $\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c} = \begin{bmatrix}2\\-1\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix}$

$= \begin{bmatrix}2--2\\-1-1\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$

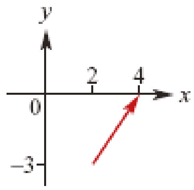
b $\boldsymbol{a}+\boldsymbol{b} = \begin{bmatrix}2\\-1\end{bmatrix} = -\boldsymbol{c}$

$\therefore$ $\boldsymbol{a} + \boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

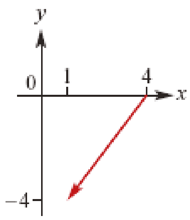
6 a



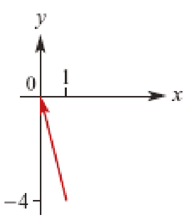
b



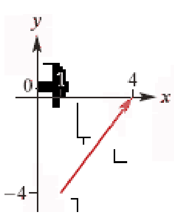
c



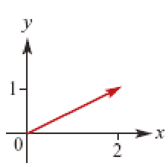
d



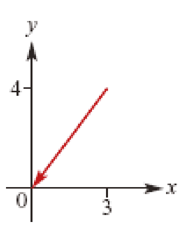
e



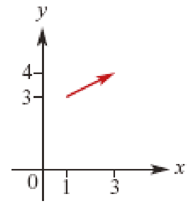
7 a



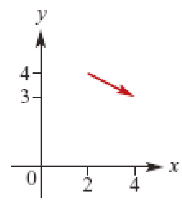
b



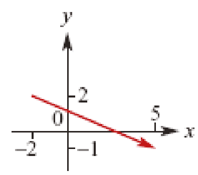
c



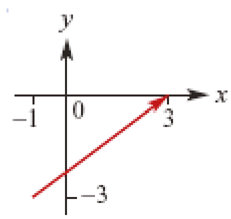
d



e



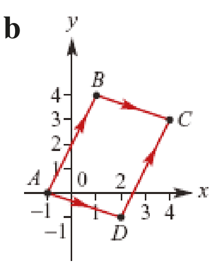
f



8 From the graphs above it can be seen that **a** and **c** are parallel.

9

a & b



c **i** $\overrightarrow{AB} = \begin{bmatrix} 1 - -1 \\ 4 - 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} 4 - 2 \\ 3 - -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\therefore\ \overrightarrow{AB} = \overrightarrow{DC}$

ii $\overrightarrow{BC} = \begin{bmatrix} 4 - -1 \\ 3 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\overrightarrow{AD} = \begin{bmatrix} 2 - -1 \\ -1 - 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\therefore\ \overrightarrow{BC} = \overrightarrow{AD}$

d $ABCD$ is a parallelogram.

10 $m\begin{bmatrix} 3 \\ -3 \end{bmatrix} + n\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3m \\ -3m \end{bmatrix} + \begin{bmatrix} 2n \\ 4n \end{bmatrix}$

$= \begin{bmatrix} 3m & +2n \\ -3m & +4n \end{bmatrix} = \begin{bmatrix} -19 \\ 61 \end{bmatrix}$

$$\begin{aligned} 3m + 2n &= -19 \\ 6m + 4n &= -38 \quad ① \\ -3m + 4n &= 61 \quad ② \end{aligned}$$

① − ②:

$$\begin{aligned} 9m &= -99 \\ m &= -11 \\ -33 + 2n &= -19 \\ 2n &= -19 + 33 \\ &= 14 \\ n &= 7 \end{aligned}$$

11 a i $\overrightarrow{MD} = \overrightarrow{MA} + \overrightarrow{AD}$

$= \frac{1}{2}\overrightarrow{BA} + \boldsymbol{b}$

$= -\frac{1}{2}\overrightarrow{AB} + \boldsymbol{b}$

$= \boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

ii $\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DN}$

$= \frac{1}{2}\overrightarrow{BA} + \boldsymbol{b} + \frac{1}{2}\overrightarrow{DN}$

$= -\frac{1}{2}\overrightarrow{AB} + \boldsymbol{b} + \frac{1}{2}\overrightarrow{DC}$

$= -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

$= \boldsymbol{b}$

b $\overrightarrow{MN} = \overrightarrow{AD}$
(both are equal to $\boldsymbol{b}$)

12 a $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

$= -\boldsymbol{b} + \boldsymbol{a} = \boldsymbol{a} - \boldsymbol{b}$

$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN}$

$= -\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

b $\overrightarrow{MN}$ is half the length of $\overrightarrow{CB}$, is parallel to $\overrightarrow{CB}$ and in the opposite direction to $\overrightarrow{CB}$.

13 a $\overrightarrow{CD} = \overrightarrow{AF} = \boldsymbol{a}$

b $\overrightarrow{ED} = \overrightarrow{AB} = \boldsymbol{b}$

c The regular hexagon can be divided into equilateral triangles, showing that
$\overrightarrow{BE} = 2\overrightarrow{AF} = 2\boldsymbol{a}$.

d Likewise, $\overrightarrow{FC} = 2\overrightarrow{AB} = 2\boldsymbol{b}$

e $\overrightarrow{FA} = -\overrightarrow{AF} = -\boldsymbol{a}$

f $\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB}$

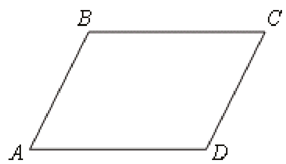
$= -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$

g $\overrightarrow{FE} = \overrightarrow{FA} + \overrightarrow{AB} + \overrightarrow{BE}$

$= -\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{a}$

$= \boldsymbol{a} + \boldsymbol{b}$

14



a $\overrightarrow{DC} = \overrightarrow{AB} = \boldsymbol{a}$

b $\overrightarrow{DA} = -\overrightarrow{BC} = -\boldsymbol{b}$

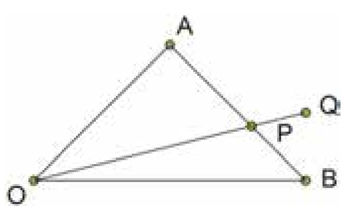
c $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$

d $\overrightarrow{CA} = -\overrightarrow{AC} = -\boldsymbol{a} - \boldsymbol{b}$

e $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

$= -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$

15



a $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \boldsymbol{a} - \boldsymbol{b}$

b $\overrightarrow{AB} = -\overrightarrow{BA} = \boldsymbol{b} - \boldsymbol{a}$

$\overrightarrow{PB} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})$

c $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$

$= \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

d $\overrightarrow{PQ} = \frac{1}{3}\overrightarrow{OP}$

$= \frac{1}{3} \times \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

$= \frac{1}{9}(\boldsymbol{a} + 2\boldsymbol{b})$

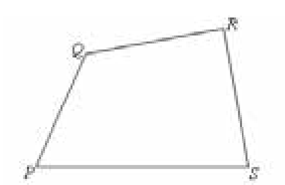
e $\overrightarrow{BP} = -\overrightarrow{PB} = \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$

$\overrightarrow{BQ} = \overrightarrow{BP} + \overrightarrow{PQ}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{9}(\boldsymbol{a} + 2\boldsymbol{b})$

$= \frac{1}{9}(4\boldsymbol{a} - \boldsymbol{b})$

16



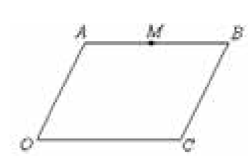
a $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \boldsymbol{u} + \boldsymbol{v}$

b $\overrightarrow{QS} = \overrightarrow{QR} + \overrightarrow{RS} = \boldsymbol{v} + \boldsymbol{w}$

c $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS}$

$= \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$

17



a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \boldsymbol{u} + \boldsymbol{v}$

$$\overrightarrow{AM} = \overrightarrow{MB}$$

$$= \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\boldsymbol{v}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

$$= \boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$$

b $\overrightarrow{CM} = \overrightarrow{CB} + \overrightarrow{BM}$

$$= \boldsymbol{u} + \frac{1}{2}\overrightarrow{BA}$$

$$= \boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$$

c $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$

$$= \frac{2}{3}\left(\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}\right)$$

$$= \frac{2}{3}\boldsymbol{u} - \frac{1}{3}\boldsymbol{v}$$

d $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$

$$= \boldsymbol{v} + \left(\frac{2}{3}\boldsymbol{u} - \frac{1}{3}\boldsymbol{v}\right)$$

$$= \frac{2}{3}\boldsymbol{u} + \frac{2}{3}\boldsymbol{v}$$

$$= \frac{2}{3}(\boldsymbol{u} + \boldsymbol{v}) = \frac{2}{3}\overrightarrow{OB}$$

Since OP is parallel to OB and they share a common point O, OP must be on the line OB. Hence P is on $\overrightarrow{OB}$

e Using the result from part **d**,
$OP : PB = 2 : 1$.

Solutions to Exercise 21B

1 $\overrightarrow{AB} = (3\boldsymbol{i} - 5\boldsymbol{j}) - (\boldsymbol{i} + 2\boldsymbol{j})$

$= 3\boldsymbol{i} - 5\boldsymbol{i} - \boldsymbol{i} - 2\boldsymbol{j}$

$= 2\boldsymbol{i} - 7\boldsymbol{j}$

2

a $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= 5\boldsymbol{i} + 6\boldsymbol{j}$

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -5\boldsymbol{i} + 6\boldsymbol{j}$

c $\overrightarrow{BA} = -\overrightarrow{AB}$

$= 5\boldsymbol{i} - 6\boldsymbol{j}$

3 a $|5\boldsymbol{i}| = \sqrt{5^2} = 5$

b $|-2\boldsymbol{j}| = \sqrt{(-2)^2} = 2$

c $|3\boldsymbol{i} + 4\boldsymbol{j}| = \sqrt{3^2 + 4^2}$

$= \sqrt{9 + 16} = 5$

d $|-5\boldsymbol{i} + 12\boldsymbol{j}| = \sqrt{(-5)^2 + 12^2}$

$= \sqrt{25 + 144} = 13$

4 a $\boldsymbol{u} - \boldsymbol{v} = (7\boldsymbol{i} + 8\boldsymbol{j}) - (2\boldsymbol{i} - 4\boldsymbol{j})$

$= 7\boldsymbol{i} + 8\boldsymbol{j} - 2\boldsymbol{i} + 4\boldsymbol{j}$

$= 5\boldsymbol{i} + 12\boldsymbol{j}$

$|\boldsymbol{u} - \boldsymbol{v}| = |5\boldsymbol{i} + 12\boldsymbol{j}|$

$= \sqrt{25 + 144}$

$= 13$

b $x\boldsymbol{u} + y\boldsymbol{v} = x(7\boldsymbol{i} + 8\boldsymbol{j}) + y(2\boldsymbol{i} - 4\boldsymbol{j})$

$= 7x\boldsymbol{i} + 8x\boldsymbol{j} + 2y\boldsymbol{i} - 4y\boldsymbol{j}$

$= 44\boldsymbol{j}$

$7x + 2y = 0$

$14x + 4y = 0$ ①

$8x - 4y = 44$ ②

① + ② :

$22x = 44$

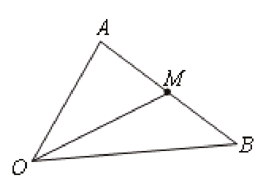
$x = 2$

$7 \times 2 + 2y = 0$

$2y = -14$

$y = -7$

5



$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -10\boldsymbol{i} + (4\boldsymbol{i} + 5\boldsymbol{j})$

$= -6\boldsymbol{i} + 6\boldsymbol{j}$

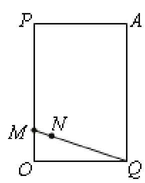
$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$

$= -3\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= 10\boldsymbol{i} + \left(-3\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}\right)$

$= 7\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

6 

a **i** $\overrightarrow{OM} = \frac{1}{5}\overrightarrow{OP}$

$= \frac{2}{5}\boldsymbol{i}$

ii $\overrightarrow{MQ} = \overrightarrow{MO} + \overrightarrow{OQ}$

$= -\overrightarrow{OM} + \overrightarrow{OQ}$

$= -\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}$

iii $\overrightarrow{MN} = \frac{1}{6}\overrightarrow{MQ}$

$= \frac{1}{6}\left(-\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}\right)$

$= -\frac{1}{15}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$

iv $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$= \frac{2}{5}\boldsymbol{i} + \left(-\frac{1}{15\boldsymbol{i}} + \frac{1}{6}\boldsymbol{j}\right)$

$= \frac{1}{3}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$

v $\overrightarrow{OA} = \overrightarrow{OP} + \overrightarrow{PA}$

$= 2\boldsymbol{i} + \boldsymbol{j}$

b **i** $\overrightarrow{ON} = \frac{1}{3}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$

$= \frac{1}{6}(2\boldsymbol{i} + \boldsymbol{j})$

$= \frac{1}{6}\overrightarrow{OA}$

Since *ON* is parallel to *OA* and they share a common point *O*, *ON* must be on the line *OA*. Hence *N* is on *OA*.

ii 1:5

7 $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \boldsymbol{i} + 3\boldsymbol{j}$

$\overrightarrow{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} = 5\boldsymbol{i} - \boldsymbol{j}$

$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{i} - \boldsymbol{j}$

$= 4\boldsymbol{i} - 4\boldsymbol{j}$

$|\overrightarrow{AB}| = \sqrt{4^2 + (-4)^2}$

$= \sqrt{16 + 16}$

$= \sqrt{32} = 4\sqrt{2}$ units

8 a $2\boldsymbol{i} + 3\boldsymbol{j} = 2l\boldsymbol{i} + 2k\boldsymbol{j}$

$$2\boldsymbol{j} = 2$$
$$l = 1$$
$$2k = 3$$
$$k = \frac{3}{2}$$

b $x - 1 = 5$

$$x = 6$$
$$y = x - 4$$
$$= 2$$

c $x + y = 6 \quad ①$

$x - y = 0 \quad ②$

① + ② :

$$2x = 6$$
$$x = 3$$
$$3 + y = 6$$
$$y = 3$$

d $k = 3 + 2l$

$$k = -2 - l$$
$$3 + 2l = -2 - l$$
$$3l = -5$$
$$l = -\frac{5}{3}$$
$$k = -2 - -\frac{5}{3}$$
$$= -2 + \frac{5}{3}$$
$$= -\frac{1}{3}$$

9 a $\overrightarrow{AB} = \begin{bmatrix} 5-2 \\ 1-3 \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$
$$= 3\boldsymbol{i} - 2\boldsymbol{j}$$

b $|\overrightarrow{AB}| = \sqrt{3^2 + (-2)^2}$

$$= \sqrt{9 + 4}$$
$$= \sqrt{13}$$

10 a $\overrightarrow{AB} = \boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{i}$

$$= -2\boldsymbol{i} + 4\boldsymbol{j}$$

b $\overrightarrow{AC} = -3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{i}$

$$= -6\boldsymbol{i} + \boldsymbol{j}$$

c $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$$= -6\boldsymbol{i} + \boldsymbol{j} - (-2\boldsymbol{i} + 4\boldsymbol{j})$$
$$= -4\boldsymbol{i} - 3\boldsymbol{j}$$
$$|\overrightarrow{BC}| = \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{16 + 9}$$
$$= 5$$

11 a Let $D = (a, b)$.

$$\overrightarrow{AB} = -5\boldsymbol{i} + 3\boldsymbol{j}$$
$$\overrightarrow{CD} = (a + 1)\boldsymbol{i} + b\boldsymbol{j}$$
$$a + 1 = -5$$
$$a = -6$$
$$b = 3$$

D is $(-6, 3)$.

b Let $F = (c, d)$.

$$\overrightarrow{BC} = -\boldsymbol{i} - 4\boldsymbol{j}$$

$$\overrightarrow{AF} = (c-5)\boldsymbol{i} + (d-1)\boldsymbol{j}$$

$$c - 5 = -1$$

$$c = 4$$

$$d - 1 = -4$$

$$d = -3$$

F is $(4, -3)$.

c Let $G = (e, f)$.

$$\overrightarrow{AB} = -5\boldsymbol{i} + 3\boldsymbol{j}$$

$$2\overrightarrow{GC} = 2(-1-e)\boldsymbol{i} + 2(-f)\boldsymbol{j}$$

$$2(-1-e) = -5$$

$$e = \frac{3}{2}$$

$$-2f = 3$$

$$f = -\frac{3}{2}$$

G is $\left(\frac{3}{2}, -\frac{3}{2}\right)$.

12 $\overrightarrow{OA} = -\overrightarrow{AO}$

$$= -\boldsymbol{i} - 4\boldsymbol{j}$$

A is $(-1, -4)$.

B is $(-2, 2)$.

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

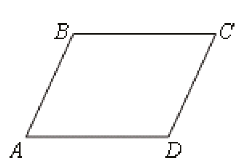
$$\overrightarrow{OC} = \overrightarrow{BC} + \overrightarrow{OB}$$

$$= 2\boldsymbol{i} + 8\boldsymbol{j} + (-2\boldsymbol{i} + 2\boldsymbol{j})$$

$$= 10\boldsymbol{j}$$

C is $(0, 10)$

13



a **i** $2\boldsymbol{i} - \boldsymbol{j}$

ii $-5\boldsymbol{i} + 4\boldsymbol{j}$

iii $\boldsymbol{i} + 7\boldsymbol{j}$

iv $6\boldsymbol{i} + 3\boldsymbol{j}$

v $\overrightarrow{AD} = \overrightarrow{BC}$

$$= 6\boldsymbol{i} + 3\boldsymbol{j}$$

b $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$$\overrightarrow{OD} = \overrightarrow{AD} + \overrightarrow{OA}$$

$$= 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{i} - \boldsymbol{j}$$

$$= 8\boldsymbol{i} + 2\boldsymbol{j}$$

D is $(8, 2)$.

14 **a** $\overrightarrow{OP} = 12\boldsymbol{i} + 5\boldsymbol{j}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= 18\boldsymbol{i} + 13\boldsymbol{j} - 12\boldsymbol{i} - 5\boldsymbol{j}$$

$$= 6\boldsymbol{i} + 8\boldsymbol{j}$$

b $|\overrightarrow{RQ}| = |\overrightarrow{OP}|$

$$= \sqrt{12^2 + 5^2}$$

$$= 13$$

$$|\overrightarrow{OR}| = |\overrightarrow{PQ}|$$

$$= \sqrt{6^2 + 8^2}$$

$$= 10$$

15 **a** **i** $|\overrightarrow{AB}| = |2\boldsymbol{i} - 5\boldsymbol{j}|$

$$= \sqrt{2^2 + 5^2} = \sqrt{29}$$

ii $|\overrightarrow{BC}| = |10\boldsymbol{i} + 4\boldsymbol{j}|$

$= \sqrt{10^2 + 4^2}$

$= \sqrt{116} = 2\sqrt{29}$

iii $|\overrightarrow{CA}| = |12\boldsymbol{i} - \boldsymbol{j}|$

$= \sqrt{12^2 + 1^2} = \sqrt{145}$

b $AB^2 + BC^2 = 29 + 116$

$= 145 = AC^2$

$\therefore$ ABC is a right-angled triangle.

16 a **i** $\overrightarrow{AB} = -\boldsymbol{i} - 3\boldsymbol{j}$

ii $\overrightarrow{BC} = 4\boldsymbol{i} + 2\boldsymbol{j}$

iii $\overrightarrow{CA} = -3\boldsymbol{i} + \boldsymbol{j}$

b **i** $|\overrightarrow{AB}| = \sqrt{1^2 + 3^2}$

$= \sqrt{10}$

ii $|\overrightarrow{BC}| = \sqrt{4^2 + 2^2}$

$= \sqrt{20} = 2\sqrt{5}$

iii $|\overrightarrow{CA}| = \sqrt{3^2 + 1^2}$

$= \sqrt{10}$

c $AB = CA$

$= \sqrt{10}$

$AB^2 + CA^2 = 10 + 10$

$= 20 = BC^2$

$\therefore$ ABC is an isosceles right-angled triangle.

17 a **i** $\overrightarrow{OA} = -3\boldsymbol{i} + 2\boldsymbol{j}$

ii $\overrightarrow{OB} = 7\boldsymbol{j}$

iii $\overrightarrow{BA} = -3\boldsymbol{i} - 5\boldsymbol{j}$

iv $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BA}$

$= \frac{1}{2}(-3\boldsymbol{i} - 5\boldsymbol{j})$

b $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$

$\overrightarrow{OD} = 7\boldsymbol{j} + -\frac{3}{2}\boldsymbol{i} - \frac{5}{2}\boldsymbol{j}$

$= -\frac{3}{2}\boldsymbol{i} + \frac{9}{2}\boldsymbol{j}$

$M = \left(-\frac{3}{2}, \frac{9}{2}\right)$

18 a $a = 3\boldsymbol{i} + 4\boldsymbol{j}$

$|a| = \sqrt{3^2 + 4^2}$

$= 5$

$\hat{a} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

b $b = 3\boldsymbol{i} - \boldsymbol{j}$

$|b| = \sqrt{3^2 + (-1)^2}$

$= \sqrt{10}$

$\hat{b} = \frac{1}{\sqrt{10}}(3\boldsymbol{i} - \boldsymbol{j})$

c $c = -\boldsymbol{i} + \boldsymbol{j}$

$|c| = \sqrt{(-1)^2 + 1^2}$

$= \sqrt{2}$

$\hat{c} = \frac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$

d $d = \boldsymbol{i} - \boldsymbol{j}$

$\hat{d} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

e $e = \frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}$

$$|e| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{9}}$$

$$= \sqrt{\frac{13}{36}}$$

$$= \frac{\sqrt{13}}{6}$$

$$\hat{e} = \frac{6}{\sqrt{13}}\left(\frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}\right)$$

$$= \frac{1}{\sqrt{13}}(3\boldsymbol{i} + 2\boldsymbol{j})$$

f $f = 6\boldsymbol{i} - 4\boldsymbol{j}$

$$|f| = \sqrt{6^2 + (-4)^2}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

$$\hat{f} = \frac{1}{2\sqrt{13}}(6\boldsymbol{i} - 4\boldsymbol{j})$$

$$= \frac{1}{\sqrt{13}}(3\boldsymbol{i} - 2\boldsymbol{j})$$

Solutions to Exercise 21C

1 Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j}, \boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$

a $\boldsymbol{a} \cdot \boldsymbol{a} = 17$

b $\boldsymbol{b} \cdot \boldsymbol{b} = 13$

c $\boldsymbol{c} \cdot \boldsymbol{c} = 8$

d $\boldsymbol{a} \cdot \boldsymbol{b} = -10$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{i} - 4\boldsymbol{j}) \cdot (\boldsymbol{j}) = -4$

f $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

$$\begin{aligned} &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c} \\ &= 17 + 6 - 10 - 10 \\ &= 3 \end{aligned}$$

g $\boldsymbol{a} + 2\boldsymbol{b} = 5\boldsymbol{i} + 2\boldsymbol{j}$

$3\boldsymbol{c} - \boldsymbol{b} = -8\boldsymbol{i} - 9\boldsymbol{j}$

$\therefore (\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b}) = -58$

2 Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}, \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j}$

a $\boldsymbol{a} \cdot \boldsymbol{a} = 5$

b $\boldsymbol{b} \cdot \boldsymbol{b} = 13$

c $\boldsymbol{a} \cdot \boldsymbol{b} = 8$

d $\boldsymbol{a} \cdot \boldsymbol{c} = -5$

e $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) = 13$

3 $|\boldsymbol{a}| = 5$ and $|\boldsymbol{b}| = 6$

a $\boldsymbol{a} \cdot \boldsymbol{b} = 5 \times 6 \cos 45°$

$$\begin{aligned} &= 30 \times \frac{1}{\sqrt{2}} \\ &= 15\sqrt{2} \end{aligned}$$

b $\boldsymbol{a} \cdot \boldsymbol{b} = 5 \times 6 \cos 135°$

$$\begin{aligned} &= 30 \times -\frac{1}{\sqrt{2}} \\ &= -15\sqrt{2} \end{aligned}$$

4 **a**

$$\begin{aligned} (\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 3\boldsymbol{b}) &= \boldsymbol{a} \cdot \boldsymbol{a} + 4\boldsymbol{a} \cdot \boldsymbol{b} + 4(\boldsymbol{b} \cdot \boldsymbol{b}) \\ &= |\boldsymbol{a}|^2 + 4\boldsymbol{a} \cdot \boldsymbol{b} + 4|\boldsymbol{b}|^2 \end{aligned}$$

b

$$\begin{aligned} &|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2 \\ &= (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) + (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) \\ &= (\boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{a} - 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}) \\ &= 4\boldsymbol{a} \cdot \boldsymbol{b} \end{aligned}$$

c

$$\begin{aligned} \boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{a} + \boldsymbol{b} &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a} \\ &= |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 \end{aligned}$$

d

$$\begin{aligned} \frac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} &= \frac{|\boldsymbol{a}|^2 + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} \\ &= |\boldsymbol{a}| \end{aligned}$$

5 **a** $\overrightarrow{AB} = -2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{i} + 3\boldsymbol{j}$

$= -3\boldsymbol{i} + \boldsymbol{j}$

b $|\overrightarrow{AB}| = \sqrt{9 + 1} = \sqrt{10}$

c $\boldsymbol{a} \cdot \overrightarrow{AB} = |\boldsymbol{a}||\overrightarrow{AB}| \cos\theta$

$$\begin{aligned} \therefore -4 &= \sqrt{10} \times 2\sqrt{2} \cos\theta \\ \therefore \cos\theta &= -\frac{4}{2\sqrt{20}} \\ \therefore \theta &= 116.57° \end{aligned}$$

6 $\overrightarrow{CD} = -\boldsymbol{c} + \boldsymbol{d}$

Let θ be the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$

$\boldsymbol{c} \cdot \boldsymbol{d} = |\boldsymbol{c}||\boldsymbol{d}| \cos\theta$

$\therefore \cos\theta = \dfrac{4}{5 \times 7}$

Using the cosine rule.

$|\overrightarrow{CD}|^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \cos\theta$

$= 25 + 49 - 2 \times 5 \times 7 \times \dfrac{4}{35}$

$= 66$

$\therefore |\overrightarrow{CD}| = \sqrt{66}$

7 a $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (5\boldsymbol{i} + x\boldsymbol{j}) = -6$

$5 + 2x = -6$

$2x = -11$

$x = -\dfrac{11}{2}$

b $(x\boldsymbol{i} + 7\boldsymbol{j}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j}) = 10$

$-4x + 7x = 10$

$3x = 10$

$x = \dfrac{10}{3}$

c $(x\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j}) = x$

$-2x - 3 = x$

$-3 = 3x$

$-1 = x$

d $x(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} + x\boldsymbol{j}) = 6$

$x(2 + 3x) = 6$

$2x + 3x^2 = 6$

$3x^2 + 2x - 6 = 0$

$x = \dfrac{-2 \pm \sqrt{76}}{6}$

8 a $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$= -4\boldsymbol{i} - 4\boldsymbol{j} + q(2\boldsymbol{i} + 5\boldsymbol{j})$

$= (2q - 1)\boldsymbol{i} + (5q - 4)\boldsymbol{j}$

b

$\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$

$\Rightarrow ((2q - 1)\boldsymbol{i} + (5q - 4)\boldsymbol{j}) \cdot (2\boldsymbol{i} + 5\boldsymbol{j}) = 0$

$\Rightarrow 4q - 2 + 25q - 20 = 0$

$\Rightarrow 29q - 22 = 0$

$\Rightarrow q = \dfrac{22}{29}$

c $\overrightarrow{OP} = q\boldsymbol{b} = \dfrac{22}{9}(2\boldsymbol{i} + 5\boldsymbol{j})$

Cooordinates of P are $\left(\dfrac{44}{29}, \dfrac{110}{29}\right)$

9 a $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (\boldsymbol{i} - 4\boldsymbol{j}) = \sqrt{5} \times \sqrt{17} \cos\theta$

$-7 = \sqrt{85} \cos\theta$

$\cos\theta = -\dfrac{7}{\sqrt{85}}$

$\theta = 139.40°$

b $-2\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 2\boldsymbol{j}) = \sqrt{5} \times \sqrt{8} \cos\theta$

$2 = \sqrt{40} \cos\theta$

$\cos\theta = \dfrac{2}{\sqrt{40}}$

$\theta = 71.57°$

c $2\boldsymbol{i} - \boldsymbol{j}) \cdot (4\boldsymbol{i} = \sqrt{5} \times 4 \cos\theta$

$8 = 4\sqrt{5} \cos\theta$

$\cos\theta = \dfrac{2}{\sqrt{5}}$

$\theta = 26.57°$

d $7\boldsymbol{i}+\boldsymbol{j})\cdot(-\boldsymbol{i}+\boldsymbol{i})+ = \sqrt{50}\times\sqrt{2}\cos\theta$

$$-6 = 10\cos\theta$$

$$\cos\theta = -\frac{3}{5}$$

$$\theta = 126.87°$$

10 $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

If $\boldsymbol{a}$ and $\boldsymbol{a}$ are non-zero vectors, then

$\boldsymbol{a}\cdot\boldsymbol{b} = 0 \Leftrightarrow \cos\theta = 0$

11 a $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$$

$$= \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$$

$$= \frac{3}{2}\boldsymbol{i}$$

b $\boldsymbol{a}\cdot\overrightarrow{OM} = |\boldsymbol{a}|\overrightarrow{OM}|\cos(\angle AOM)$

$$\cos(\angle AOM) = \frac{\frac{3}{2}}{\sqrt{2}\times\frac{3}{2}}$$

$$\therefore \angle AOM = 45°$$

c $\overrightarrow{MB}\cdot\overrightarrow{MO} = |\overrightarrow{MB}||\overrightarrow{MO}|\cos(\angle BMO)$

$$\cos(\angle BMO) = \frac{-\frac{3}{4}}{\frac{\sqrt{5}}{2}\times\frac{3}{2}}$$

$$\cos(\angle BMO) = -\frac{1}{\sqrt{5}}$$

$$\angle BMO = 116.57°$$

12 a i $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$$

$$= \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$$

$$= \frac{1}{2}(3\boldsymbol{i}+4\boldsymbol{j})$$

ii $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{c}-\boldsymbol{a})$$

$$= \frac{1}{2}(\boldsymbol{c}+\boldsymbol{b})$$

$$= \frac{1}{2}(\boldsymbol{i}+6\boldsymbol{j})$$

b $\overrightarrow{OM}\cdot\overrightarrow{ON} = |\overrightarrow{ON}||\overrightarrow{OM}|\cos(\angle MON)$

$$\cos(\angle MON) = \frac{\frac{27}{4}}{\frac{5}{2}\times\frac{\sqrt{37}}{2}}$$

$$\cos(\angle MON) = \frac{1}{\sqrt{5}}$$

$$\angle BMO = 27.41°$$

c $\overrightarrow{OM}\cdot\overrightarrow{OC} = |\overrightarrow{OM}||\overrightarrow{OC}|\cos(\angle MOC)$

$$\cos(\angle MOC) = \frac{9}{\frac{5}{2}\times\sqrt{40}}$$

$$\cos(\angle MOC) = \frac{9}{5\sqrt{10}}$$

$$\angle BMO = 55.30°$$

Solutions to Exercise 21D

1 a $|\boldsymbol{a}| = \sqrt{1+9} = \sqrt{10}$

$\therefore \hat{\boldsymbol{a}} = \dfrac{1}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$

b $|\boldsymbol{b}| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\therefore \hat{\boldsymbol{a}} = \dfrac{1}{2\sqrt{2}}(2\boldsymbol{i} + 2\boldsymbol{j}) = \dfrac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j})$

c $\boldsymbol{c} = \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \boldsymbol{i} - \boldsymbol{j}$

$\therefore \hat{\boldsymbol{c}} = \dfrac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

2 a i $\hat{\boldsymbol{a}} = \dfrac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

ii $|\boldsymbol{b}| = \sqrt{2}$

b $\dfrac{\sqrt{2}}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

3 a i $\hat{\boldsymbol{a}} = \dfrac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

ii $\hat{\boldsymbol{b}} = \dfrac{1}{13}(5\boldsymbol{i} + 12\boldsymbol{j})$

b Let $\overrightarrow{OA'} = \hat{\boldsymbol{a}}$ and $\overrightarrow{OB'} = \hat{\boldsymbol{b}}$

Then $\triangle A'OB'$ is isosceles. Therefore the angle bisector of $\angle AOB$ passes through the midpoint of $A'B'$.

Let M be the midpoint of $A'B'$

Then

$$\begin{aligned}\overrightarrow{OM} &= \frac{1}{2}(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}})\\ &= \frac{1}{2}\left(\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j}) + \frac{1}{13}(5\boldsymbol{i} + 12\boldsymbol{j})\right)\\ &= \frac{8}{65}(4\boldsymbol{i} + 7\boldsymbol{j})\end{aligned}$$

$\therefore$the unit vector in the direction of

$\overrightarrow{OM}$ is: $= \dfrac{1}{\sqrt{65}}(4\boldsymbol{i} + 7\boldsymbol{j})$

4 a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

$$\begin{aligned}\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} &= \frac{1-12}{17}(\boldsymbol{i} - 4\boldsymbol{j})\\ &= -\frac{11}{17}(\boldsymbol{i} - 4\boldsymbol{j})\end{aligned}$$

b $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

$$\begin{aligned}\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} &= \frac{1+12}{17}(\boldsymbol{i} - 4\boldsymbol{j})\\ &= \frac{13}{17}(\boldsymbol{i} - 4\boldsymbol{j})\end{aligned}$$

c The vector resolute is $\boldsymbol{b}$

5 a $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|} = \dfrac{2}{1} = 2$

b $\dfrac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{c}|} = \dfrac{3-2}{\sqrt{5}} = \dfrac{1}{\sqrt{5}}$

c $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|} = \dfrac{2\sqrt{3}}{\sqrt{7}}$

d $\dfrac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{c}|} = \dfrac{-1-4\sqrt{5}}{\sqrt{17}}$

6 a $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i}$ and $\boldsymbol{w} = \boldsymbol{j}$

b $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{w} = \boldsymbol{i} - \boldsymbol{j}$

c $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{w} = -\boldsymbol{i} + \boldsymbol{j}$

7 a $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b} = 2(\boldsymbol{i} + \boldsymbol{j})$

b Let $\overrightarrow{OC} = 2(\boldsymbol{i} + \boldsymbol{j})$

$\overrightarrow{OC}$ is the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

$\therefore \overrightarrow{CA}$ is a vector perpendiculr to $\overrightarrow{OB}$

$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$ Therefore

$= -2(\boldsymbol{i} + \boldsymbol{j}) + (\boldsymbol{i} + 3\boldsymbol{j})$

$= -\boldsymbol{i} + \boldsymbol{j}$

the unit vector is $\dfrac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$

8 a $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \dfrac{3}{2}(\boldsymbol{i} - \boldsymbol{j})$

b $\boldsymbol{a} - \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = 4\boldsymbol{i} + \boldsymbol{j} - \dfrac{3}{2}(\boldsymbol{i} - \boldsymbol{j})$

$= \dfrac{1}{2}(8\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{i} + 3\boldsymbol{j})$

$= \dfrac{1}{2}(5\boldsymbol{i} + 5\boldsymbol{j})$

c Distance $= |\dfrac{1}{2}(5\boldsymbol{i} + 5\boldsymbol{j})| = \dfrac{5\sqrt{2}}{2}$

9 $\overrightarrow{OA} = \boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OB} = \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$

$\overrightarrow{OC} = \boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$

a i $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{i} + \boldsymbol{j}$

$= \boldsymbol{i} - \boldsymbol{j}$

ii $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{i} - 3\boldsymbol{j}$

$= \boldsymbol{i} - 5\boldsymbol{j}$

b The vector resolute $= \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AC} \cdot \overrightarrow{AC}}\overrightarrow{AC}$

$= \dfrac{1+5}{26}(\boldsymbol{i} - 5\boldsymbol{j})$

$= \dfrac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$

c

The shortest distance $= \overrightarrow{AB} - \dfrac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$

$= \dfrac{3}{13}(10\boldsymbol{i} + 2\boldsymbol{j})$

The shortest distance is the height of triangle ABC wherethe base is taken as AC

Therefore height= $|\dfrac{3}{13}(10\boldsymbol{i} + 2\boldsymbol{j})| = \dfrac{1}{13}\sqrt{104}$

The area of the triangle

$= \dfrac{1}{2} \times \dfrac{1}{13}\sqrt{104} \times \sqrt{26}$

$=2$

Solutions to Exercise 21E

1

2 a **i** $\overrightarrow{OR} = \frac{4}{5}\overrightarrow{OP}$

$= \frac{4}{5}\boldsymbol{p}$

ii $\overrightarrow{RP} = \frac{1}{5}\overrightarrow{OP}$

$= \frac{1}{5}\boldsymbol{p}$

iii $\overrightarrow{PO} = -\boldsymbol{p}$

iv $\overrightarrow{PS} = \frac{1}{5}\overrightarrow{PQ}$

$= \frac{1}{5}(\boldsymbol{q} - \boldsymbol{p})$

v $\overrightarrow{RS} = \overrightarrow{RP} + \overrightarrow{PS}$

$= \frac{1}{5}\boldsymbol{p} + \frac{1}{5}(\boldsymbol{q} - \boldsymbol{p})$

$= \frac{1}{5}\boldsymbol{q}$

b They are parallel (and $OQ = 5RS$).

c A trapezium (one pair of parallel lines).

d The area of triangle POQ is 25 times the area of $PRS = 125 \text{ cm}^2$.

$\therefore$ area of $ORSQ = 125 - 5$

$= 120 \text{ cm}^2$

3 a $AP = \frac{2}{3}AB$ and $CQ = \frac{6}{7}CB$.

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$

$\overrightarrow{OQ} = \overrightarrow{OC} + \overrightarrow{CQ}$

$= \overrightarrow{OC} + \frac{6}{7}\overrightarrow{CB}$

$= k\boldsymbol{a} + \frac{6}{7}(\boldsymbol{b} - k\boldsymbol{a})$

$= \frac{k}{7}\boldsymbol{a} + \frac{6}{7}\boldsymbol{b}$

b **i** OPQ is a straight line if $OP = nOQ$.

$\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b} = n\left(\frac{k}{7}\boldsymbol{a} + \frac{6}{7}\boldsymbol{b}\right)$

$= \frac{nk}{7}\boldsymbol{a} + \frac{6n}{7}\boldsymbol{b}$

$\frac{2}{3} = \frac{6n}{7}$

$n = \frac{14}{18} = \frac{7}{9}$

$\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b} = \frac{7}{9}\left(\frac{k}{7}a + \frac{6}{7}b\right)$

$= \frac{k}{9}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$

$\frac{k}{9} = \frac{1}{3}$

$k = 3$

ii From part i

$$\overrightarrow{OP} = \frac{7}{9}\overrightarrow{OQ}$$
$$= \frac{7}{9}(OP + PQ)$$
$$= \frac{7}{9}OP + \frac{7}{9}PQ$$
$$\frac{2}{9}OP = \frac{7}{9}PQ$$
$$2OP = 7PQ$$
$$\frac{OP}{PQ} = \frac{7}{2}$$

c $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$$= -\boldsymbol{b} + k\boldsymbol{a}$$
$$= 3\boldsymbol{a} - \boldsymbol{b}, \text{ since } k = 3$$
$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$$
$$= -\frac{1}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} + \frac{7}{3}\boldsymbol{a}$$
$$= 2\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}$$
$$= \frac{2}{3}(3\boldsymbol{a} - \boldsymbol{b})$$
$$= \frac{2}{3}\overrightarrow{BC}$$

Hence PR is parallel to BC

4 a i $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$

$$= \frac{1}{3}(6\boldsymbol{i} - 1.5\boldsymbol{j})$$
$$= 2\boldsymbol{i} - 0.5\boldsymbol{j}$$
$$\overrightarrow{AB} = 3\boldsymbol{i} - 6\boldsymbol{j}$$
$$\overrightarrow{AE} = \frac{1}{4}(3\boldsymbol{i} - 5\boldsymbol{j})$$
$$= -0.75\boldsymbol{i} - 1.25\boldsymbol{j}$$
$$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$$
$$= 3\boldsymbol{i} + 3.5\boldsymbol{j} + 0.75\boldsymbol{i} - 1.25\boldsymbol{j}$$
$$= 3.75\boldsymbol{i} + 2.25\boldsymbol{j}$$
$$= \frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}$$

ii $\overrightarrow{ED} = 2\boldsymbol{i} - 0.5\boldsymbol{j} - \left(\frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}\right)$

$$= -\frac{6}{4}\boldsymbol{i} - \frac{11}{4}\boldsymbol{j}$$
$$|\overrightarrow{ED}| = \sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{11}{4}\right)^2}$$
$$= \sqrt{\frac{49 + 121}{16}}$$
$$= \sqrt{\frac{170}{16}}$$
$$= \frac{\sqrt{170}}{4}$$

b i $\overrightarrow{OX} = \frac{15p}{4}\boldsymbol{i} + \frac{9p}{4}\boldsymbol{j}$

ii $\overrightarrow{AD} = 2\boldsymbol{i} - 0.5\boldsymbol{j} - (3\boldsymbol{i} + 3.5\boldsymbol{j})$

$$= -\boldsymbol{i} - 4\boldsymbol{j}$$
$$\overrightarrow{XD} = -q\boldsymbol{i} - 4q\boldsymbol{j}$$
$$\overrightarrow{OD} = \overrightarrow{OX} + \overrightarrow{OD}$$
$$\overrightarrow{OX} = \overrightarrow{OD} - \overrightarrow{XD}$$
$$= 2\boldsymbol{i} = 0.5\boldsymbol{j} - (-q\boldsymbol{i} - 4q\boldsymbol{j})$$
$$= (q + 2)\boldsymbol{i} + (4q - 0.5)\boldsymbol{j}$$

c $(q+2)\boldsymbol{i} + (4q-0.5)\boldsymbol{j} = \frac{15p}{4}\boldsymbol{i} + \frac{9p}{4}\boldsymbol{j}$

$$q + 2 = \frac{15p}{4}$$

$$4q + 8 = 15p \qquad ①$$

$$4q - 0.5 = \frac{9p}{4} \qquad ②$$

$$① - ②:\ 8.5 = \frac{51p}{4}$$

$$p = \frac{8.5 \times 4}{51}$$

$$= \frac{2}{3}$$

$$q + 2 = \frac{15p}{4}$$

$$= \frac{10}{4} = \frac{5}{2}$$

$$q = \frac{1}{2}$$

5 a $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p}$

$$= \overrightarrow{PM} + \overrightarrow{MQ}$$

$$\overrightarrow{MQ} = \frac{\beta}{\alpha}\overrightarrow{PM}$$

$$\therefore\ \overrightarrow{PQ} = \overrightarrow{PM} + \frac{\beta}{\alpha}\overrightarrow{PM}$$

$$= \frac{\alpha+\beta}{\alpha}\overrightarrow{PM}$$

$$\overrightarrow{PM} = \frac{\alpha}{\alpha+\beta}\overrightarrow{PQ}$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$$

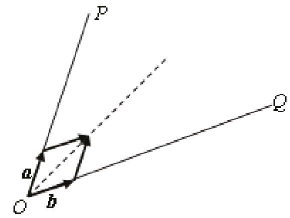
$$= \boldsymbol{p} + \frac{\alpha}{\alpha + \boldsymbol{ba}}(\boldsymbol{q} - \boldsymbol{p})$$

$$= \frac{\alpha+\beta}{\alpha+\beta}\boldsymbol{p} + \frac{\alpha}{\alpha + ba}(\boldsymbol{q} - \boldsymbol{p})$$

$$= \frac{\alpha+\beta-\alpha}{\alpha+\beta}\boldsymbol{p} + \frac{\alpha}{\alpha+\beta}\boldsymbol{q}$$

$$= \frac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha+\beta}$$

b i



It can be seen from the parallelogram formed by adding $\boldsymbol{a}$ and $\boldsymbol{b}$ that $\boldsymbol{a} + \boldsymbol{b}$ will lie on the bisector of angle POQ.
Hence any multiple, $\lambda(\boldsymbol{a} + \boldsymbol{b})$, will also lie on this bisector.

ii If $\boldsymbol{p} = k\boldsymbol{a}$ and $\boldsymbol{q} = l\boldsymbol{b}$, then

$$\overrightarrow{OM} = \frac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha+\beta}$$

$$= \frac{\beta k\boldsymbol{a} + \alpha l\boldsymbol{b}}{\boldsymbol{a}n + \beta}$$

If M is the bisector of $\angle POQ$,

$OM = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$

$\therefore\ \alpha l = \beta k$

Divide both sides by βl:

$$\frac{\alpha}{\beta} = \frac{k}{l}$$

6 Let $OABC$ be a rhombus.
Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$
We note that $|\boldsymbol{a}| = |\boldsymbol{c}|$

a i $\overrightarrow{AB} = \boldsymbol{c}$

ii $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \boldsymbol{a} + \boldsymbol{c}$

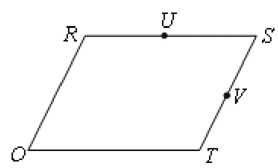
iii $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\boldsymbol{a} + \boldsymbol{c}$

b $\overrightarrow{OB} \cdot \overrightarrow{AC} = (\boldsymbol{a} + \boldsymbol{c}) \cdot (-\boldsymbol{a} + \boldsymbol{c})$

$$= -\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c}$$

$$= -|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2$$

$$= 0$$

7 

a $s = \overrightarrow{OS}$

$= \overrightarrow{OR} + \overrightarrow{RS}$

$= \overrightarrow{OR} + \overrightarrow{OT}$

$= \boldsymbol{r} + \boldsymbol{t}$

b $\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS}$

$= \boldsymbol{t} - \boldsymbol{s}$

$\boldsymbol{v} = \overrightarrow{OV}$

$= \overrightarrow{OS} + \overrightarrow{SV}$

$= \overrightarrow{OS} + \frac{1}{2}\overrightarrow{ST}$

$= \boldsymbol{s} - \frac{1}{2}(\boldsymbol{t} - \boldsymbol{s})$

$= \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

c Similarly:

$\boldsymbol{u} = \overrightarrow{OU}$

$= \overrightarrow{OS} + \overrightarrow{SU}$

$= \overrightarrow{OS} + \frac{1}{2}\overrightarrow{SR}$

$= \boldsymbol{s} - \frac{1}{2}(\boldsymbol{r} - \boldsymbol{s})$

$= \frac{1}{2}(\boldsymbol{s} + \boldsymbol{r})$

$\therefore\ \boldsymbol{u} + \boldsymbol{v} = \frac{1}{2}(\boldsymbol{s} + \boldsymbol{r}) + \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

$= \frac{1}{2}(2\boldsymbol{s} + \boldsymbol{r} + \boldsymbol{t})$

$2\boldsymbol{u} + 2\boldsymbol{v} = 2\boldsymbol{s} + \boldsymbol{r} + \boldsymbol{t}$

We may also express $\boldsymbol{u}$ as

$\boldsymbol{u} = \overrightarrow{OR} + \overrightarrow{RU}$

$= \overrightarrow{OR} + \frac{1}{2}\overrightarrow{RS}$

$= \overrightarrow{OR} + \frac{1}{2}\overrightarrow{OT}$

$= \boldsymbol{r} + \frac{1}{2}\boldsymbol{t}$

$\therefore\ \boldsymbol{u} + \boldsymbol{v} = 1 + \frac{1}{2}\boldsymbol{t} + \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

$= \frac{1}{2}(\boldsymbol{s} + 2\boldsymbol{r} + 2\boldsymbol{t})$

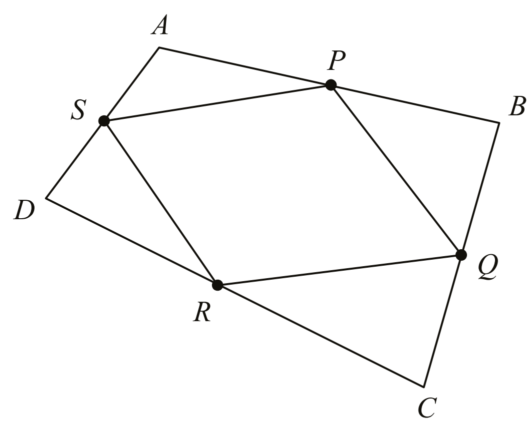
$2\boldsymbol{u} + 2\boldsymbol{v} = \boldsymbol{s} + 2\boldsymbol{r} + 2\boldsymbol{t}$

Add the two expressions for $2\boldsymbol{u} + 2\boldsymbol{v}$:

$4\boldsymbol{u} + 4\boldsymbol{v} = 3\boldsymbol{s} + 3\boldsymbol{r} + 3\boldsymbol{t}$

$= 3(\boldsymbol{s} + \boldsymbol{r} + \boldsymbol{t})$

8 Required to prove that if the midpoints of the sides of a quadrilateral are joined then a parallelogram if formed.



ABCD is a quadrilateral. *P*, *Q*, *R* and *S* are the midpoints of the sides *AB*, *BC*, *CD* and *DA* respectively.

$\overrightarrow{AS} = \frac{1}{2}\overrightarrow{AD}$

$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{SP} = \overrightarrow{AP} - \overrightarrow{AS}$

$= \frac{1}{2}\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AD}$

$= \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{AD})$

$= \frac{1}{2}\overrightarrow{DB}$

$\therefore \overrightarrow{SP} = \frac{1}{2}\overrightarrow{DB}$

Similarly,

$\overrightarrow{CR} = \frac{1}{2}\overrightarrow{CD}$

$\overrightarrow{CQ} = \frac{1}{2}\overrightarrow{CB}$

$\overrightarrow{RQ} = \overrightarrow{RC} + \overrightarrow{CQ}$

$= \frac{1}{2}\overrightarrow{CB} - \frac{1}{2}\overrightarrow{CD}$

$= \frac{1}{2}(\overrightarrow{CB} - \overrightarrow{CD})2$

$= \frac{1}{2}\overrightarrow{DB}$

$\therefore \overrightarrow{RQ} = \frac{1}{2}\overrightarrow{DB}$

Thus $\overrightarrow{SP} = \overrightarrow{RQ}$ meaning $SP \parallel RQ$ and $SP = RQ$

Hence $PQRS$ is a parallelogram.

9 Consider the square $OACB$.

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

They are of equal magnitude. That is, $|\boldsymbol{a}| = |\boldsymbol{b}|$.

The diagonals are $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$

$|\boldsymbol{a} + \boldsymbol{b}|^2 = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$

$= \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$

$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$

$|\boldsymbol{a} - \boldsymbol{b}|^2 = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$ The

$= \boldsymbol{a} \cdot \boldsymbol{a} - 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$

$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$

diagonals are of equal length

Let M be the midpoint of diagonal $\overrightarrow{OC}$.

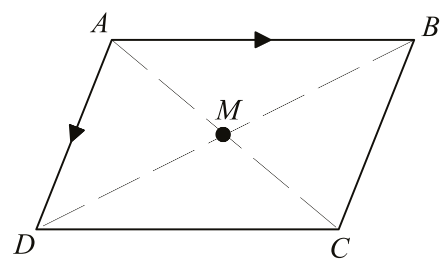
Then $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.

Let N be the midpoint of diagonal $\overrightarrow{BA}$.

Then $\overrightarrow{ON} = \overrightarrow{OB} + \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.

Therefore $M = N$. The diagonals bisect each other

10 Required to prove that the diagonals of a parallelogram bisect each other.



$ABCD$ is a parallelogram.

Let $\overrightarrow{AD} = \boldsymbol{a}$

Let $\overrightarrow{AB} = \boldsymbol{b}$

Let M be the midpoint of AC.

$\overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{a}$

$\Rightarrow \overrightarrow{AM} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

$\overrightarrow{BM} = -\overrightarrow{AB} + \overrightarrow{AM}$

$= -\boldsymbol{b} + \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$

$\overrightarrow{MD} = -\overrightarrow{AM} + \overrightarrow{AD}$

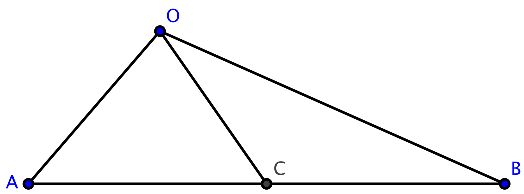
$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{a}$

$= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$

$= \overrightarrow{BM}$

Thus M is the midpoint BD.
Therefore the diagonals of a parallelogram bisect each other.

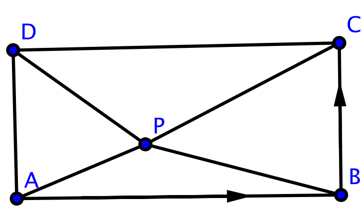
11



$$\overrightarrow{OC} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OB})$$
$$= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b})$$
$$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$
$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$
$$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$
$$4\overrightarrow{OC} \cdot \overrightarrow{OC} = \boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{b} \cdot \boldsymbol{a}$$
$$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{b}.\boldsymbol{a}$$
$$4\overrightarrow{AC} \cdot \overrightarrow{AC} = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{b}.\boldsymbol{a}$$

Therefore
$$4|\overrightarrow{OC}|^2 + 4|\overrightarrow{AC}|^2 = 2|\boldsymbol{a}|^2 + 2|\boldsymbol{b}|^2$$
$$\therefore 2|\overrightarrow{OC}|^2 + 2|\overrightarrow{AC}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$$

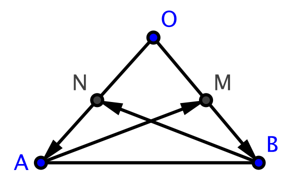
12



For rectangle $ABCD$
Let $\overrightarrow{AB} = \boldsymbol{x}$ and $\overrightarrow{BC} = \boldsymbol{y}$
Then there exist real numbers $0 < \lambda < 1$ and $0 < \mu < 1$ such that:

$$\overrightarrow{PB} = \lambda\boldsymbol{x} + \mu\boldsymbol{y}$$
$$\overrightarrow{PC} = \lambda\boldsymbol{x} + (1-\mu)\boldsymbol{y}$$
$$\overrightarrow{PD} = -(1-\lambda)\boldsymbol{x} + (1-\mu)\boldsymbol{y}$$
$$\overrightarrow{PA} = -(1-\lambda)\boldsymbol{x} - \mu\boldsymbol{y}$$
$$|\overrightarrow{PB}|^2 + |\overrightarrow{PD}|^2$$
$$= \lambda^2|\boldsymbol{x}|^2 + \mu^2|\boldsymbol{y}|^2 + (1-\lambda)^2|\boldsymbol{x}|^2 + (1-\mu)^2|\boldsymbol{y}|^2$$
$$|\overrightarrow{PA}|^2 + |\overrightarrow{PC}|^2$$
$$= (1-\lambda)^2|\boldsymbol{x}|^2 + \mu^2|\boldsymbol{y}|^2 + \lambda^2|\boldsymbol{x}|^2 + (1-\mu)^2|\boldsymbol{y}|^2$$
$$\therefore |\overrightarrow{PB}|^2 + |\overrightarrow{PD}^2| = |\overrightarrow{PA}|^2 + |\overrightarrow{PC}|^2$$

13



Let $OA = OB$
Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$
Let M be the midpoint of OB and N be the midpoint of OA.

$$\overrightarrow{AM} = \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB}$$
$$= -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$
$$\overrightarrow{BN} = \overrightarrow{BO} + \frac{1}{2}\overrightarrow{OA}$$
$$= -\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$$
$$|\overrightarrow{AM}|^2 = (-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}) \cdot (-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b})$$
$$= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} + \frac{1}{4}\boldsymbol{b} \cdot \boldsymbol{b}$$
$$= |\boldsymbol{a}|^2 + \frac{1}{4}|\boldsymbol{b}|^2$$
$$|\overrightarrow{BN}|^2 = (\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}) \cdot (\frac{1}{2}\boldsymbol{a} - \boldsymbol{b})$$
$$= \frac{1}{4}\boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$$
$$= \frac{1}{4}|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$$

But $|\boldsymbol{a}|| = |\boldsymbol{b}|$.
Hence $|\overrightarrow{BN}| = |\overrightarrow{AM}|$

14 Consider $\triangle ABC$. Let the altitudes from A to BC and B to AC meet at O.
Let$\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.
Then
$(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0 \ldots (1)$.
$(\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} = 0 \ldots (2)$.
Subtract (1) from (2)

$$\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} - (\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$$

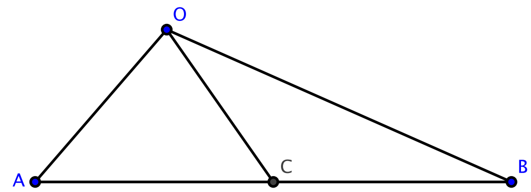
$$\therefore \boldsymbol{c} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{a} = 0$$

$$\therefore \boldsymbol{c} \cdot \boldsymbol{b} - \boldsymbol{c} \cdot \boldsymbol{a} = 0$$

$$\therefore \boldsymbol{c} \cdot (\boldsymbol{b} - \boldsymbol{a}) = 0$$

Therefore OC is the altitude from C to AB

15



$$\overrightarrow{OC} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OB})$$

$$= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b})$$

$$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$

$$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$

$$4\overrightarrow{OC} \cdot \overrightarrow{OC} = \boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{b} \cdot \boldsymbol{a}$$

$$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{b}.\boldsymbol{a}$$

$$4\overrightarrow{AC} \cdot \overrightarrow{AC} = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{b}.\boldsymbol{a}$$

Therefore
$4|\overrightarrow{OC}|^2 + 4|\overrightarrow{AC}|^2 = 2|\boldsymbol{a}|^2 + 2|\boldsymbol{b}|^2$
$\therefore 2|\overrightarrow{OC}|^2 + 2|\overrightarrow{AC}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$

Solutions to Exercise 21F

1 **a** $\overrightarrow{AB} = (2\boldsymbol{i} - 4\boldsymbol{j}) - (3\boldsymbol{i} + 7\boldsymbol{j})$
$= -\boldsymbol{i} - 11\boldsymbol{j}$

b $\overrightarrow{AB} = (3\boldsymbol{i} - 2\boldsymbol{j}) - (-2\boldsymbol{i} + 4\boldsymbol{j})$
$= 5\boldsymbol{i} - 6\boldsymbol{j}$

c $\overrightarrow{AB} = (4\boldsymbol{i} + 6\boldsymbol{j}) - (3\boldsymbol{i} + \boldsymbol{j})$
$= \boldsymbol{i} + 5\boldsymbol{j}$

d $\overrightarrow{AB} = (3\boldsymbol{i} - 4\boldsymbol{j}) - (3\boldsymbol{i} + 7\boldsymbol{j})$
$= -11\boldsymbol{j}$

e $\overrightarrow{AB} = (2\boldsymbol{i} - 7\boldsymbol{j}) - (2\boldsymbol{i} - 7\boldsymbol{j})$
$= 4\boldsymbol{i}$

f $\overrightarrow{AB} = (11\boldsymbol{i} + 5\boldsymbol{j}) - (5\boldsymbol{i} - 6\boldsymbol{j})$
$= 6\boldsymbol{i} + 11\boldsymbol{j}$

2 12.58 km on a bearing of 341.46°

3 7.74 km on a bearing of 071.17°

4 **a** $\sqrt{25 + 16} = \sqrt{41}$ m/s

b $\sqrt{9 + 16} = 5$ m/s

c $\sqrt{1 + 16} = \sqrt{17}$ m/s

d $\sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}$ m/s

e $\sqrt{25 + 144} = 13$ m/s

f $\sqrt{49 + 121} = \sqrt{170}$ m/s

5 **a** $\overrightarrow{OA'} = \overrightarrow{OA} + 5(5\boldsymbol{i} + 12\boldsymbol{j})$
$= (-\boldsymbol{i} + 2\boldsymbol{j}) + (25\boldsymbol{i} + 60\boldsymbol{j})$
$= 24\boldsymbol{i} + 62\boldsymbol{j}$

b $\overrightarrow{OA'} = \overrightarrow{OA} + t(5\boldsymbol{i} + 12\boldsymbol{j})$
$= (-\boldsymbol{i} + 2\boldsymbol{j}) + (5t\boldsymbol{i} + 12t\boldsymbol{j})$
$= (5t - 1)\boldsymbol{i} + (12t + 2)\boldsymbol{j}$

6
Displacement $= \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= (-15\boldsymbol{i} + 24\boldsymbol{j}) + (-5\boldsymbol{i} - 4\boldsymbol{j})$
$= (-20)\boldsymbol{i} + 20\boldsymbol{j})$ m

Velocity $= \dfrac{1}{5}((-20)\boldsymbol{i} + 20\boldsymbol{j})$
$= -4\boldsymbol{i} + 4\boldsymbol{j}$ m/s

7 **a** **i** $\overrightarrow{OB'} = \overrightarrow{OB} + 4(7\boldsymbol{i} + 24\boldsymbol{j})$
$= (-2\boldsymbol{i} + 3\boldsymbol{j}) + (28\boldsymbol{i} + 96\boldsymbol{j})$
$= 26\boldsymbol{i} + 99\boldsymbol{j}$

ii $\overrightarrow{OB'} = \overrightarrow{OB} + t(7\boldsymbol{i} + 24\boldsymbol{j})$
$= (-2\boldsymbol{i} + 3\boldsymbol{j}) + (7t\boldsymbol{i} + 24t\boldsymbol{j})$
$= (7t - 2)\boldsymbol{i} + (24t + 3)\boldsymbol{j}$

b **i** Distance from origin
$= \sqrt{26^2 + 99^2}$
≈ 102.36 m

ii Distance from origin
$= \sqrt{(7t - 2)^2 + (24t + 3)^2}$ m

8 a Displacement

$$= \overrightarrow{AB}$$

$$= \overrightarrow{AO} + \overrightarrow{OB}$$

$$= (-5\boldsymbol{i} - 2\boldsymbol{j}) + (-5\boldsymbol{i} - 3\boldsymbol{j})$$

$$= (-10)\boldsymbol{i} - 5\boldsymbol{j}) \text{ m}$$

$$\text{Velocity} = \frac{1}{10}((-10)\boldsymbol{i} - 5\boldsymbol{j})$$

$$= -\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} \text{ m/s}$$

b Speed $= |-\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}|$

$$= \frac{\sqrt{5}}{2} \text{ m/s}$$

9 Position of A after t seconds:
$\overrightarrow{OA} = t(\boldsymbol{i} + 2\boldsymbol{j})$
Position of B:
$\overrightarrow{OB} = (t - 2)(\frac{6}{\sqrt{5}}(\boldsymbol{i} + 2\boldsymbol{j}))$
They meet when
$\overrightarrow{OA} = \overrightarrow{OB}$
That is when:
$t(\boldsymbol{i} + 2\boldsymbol{j}) = (t - 2)(\frac{6}{\sqrt{5}}(\boldsymbol{i} + 2\boldsymbol{j}))$

$$\Rightarrow t = \frac{6}{\sqrt{5}}(t - 2)$$

$$\Rightarrow t = \frac{6}{\sqrt{5}}(t - 2)$$

$$\sqrt{5}t = 6t - 12$$

$$12 = (6 - \sqrt{5})t$$

$$t = \frac{12}{6 - \sqrt{5}}$$

$$= \frac{12(6 + \sqrt{5})}{31} \text{ seconds}$$

Therefore position vector is:

$$\overrightarrow{OA} = \overrightarrow{OB} = \frac{12(6 + \sqrt{5})}{31}(\boldsymbol{i} + 2\boldsymbol{j})$$

10 a Position of first particle at time t seconds:
$t(\boldsymbol{i} + 2\boldsymbol{j}$
Position of second particle at time t seconds:
$20\boldsymbol{j} + \boldsymbol{v}t$
where $\boldsymbol{v}$ m/s is the constant velocity.
Let $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$
The particles meet at right angles when
$t(2\boldsymbol{i} + \boldsymbol{j}) = 20\boldsymbol{j} + (a\boldsymbol{i} + b\boldsymbol{j})t \ldots (1)$
and $(2\boldsymbol{i} + \boldsymbol{j}) \cdot (a\boldsymbol{i} + b\boldsymbol{j}) = 0 \ldots (2)$
From (2), $-2a = b$
From (1)
$2t = ta$ and $t = 20 + bt$
Therefore $a = 2$ and $b = -4$
They meet when $t = 20 - 4t \Rightarrow t = 4$
Position vector is:
$4(2\boldsymbol{i} + \boldsymbol{j}) = 8\boldsymbol{i} + 4\boldsymbol{j}$

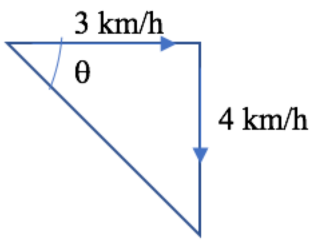
b From above, $\boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j}$ m/s

11 a Position of first particle at time t seconds:
$10\boldsymbol{j} + 2t\boldsymbol{i}$
Position of second particle at time t seconds:
$20\boldsymbol{i} + \boldsymbol{v}t$
where $\boldsymbol{v}$ m/s is the constant velocity.
Let $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$
The particles meet at right angles when
$10\boldsymbol{j} + 2t\boldsymbol{i} = 20\boldsymbol{i} + (a\boldsymbol{i} + b\boldsymbol{j})t \ldots (1)$
and $2\boldsymbol{i} \cdot (a\boldsymbol{i} + b\boldsymbol{j}) = 0 \ldots (2)$
From (2), $a = 0$
Substituting in (1)
$10\boldsymbol{j} + 2t\boldsymbol{i} = 20\boldsymbol{i} + bt\boldsymbol{j} \ldots (1')$
Therefore $2t = 20 \Rightarrow t = 10$
and
therefore they meet at the point with position vector $20\boldsymbol{i} + 10\boldsymbol{j}$

b The velocity vector is $0\boldsymbol{i} + \boldsymbol{j} = \boldsymbol{j}$ m/s

Solutions to Exercise 21G

1

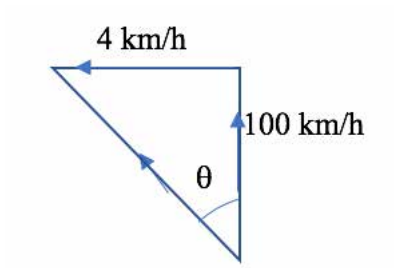


$\theta = \tan^{-1}\left(\dfrac{4}{3}\right) \approx 53.13°$

Bearing $(90 + 53.13)° = 143.13°$

Speed = $\sqrt{3^2 + 4^2} = 5$ km/h

2



Speed= $\sqrt{4^2 + 100^2} \approx 100.08$ km/h

$\theta = \tan^{-1}\left(\dfrac{1}{25}\right) \approx 2.29°$

Bearing $(360 - 2.29)° = 357.71°$

3 **a** Velocity of A relative to B

$100 - 80 = 20$ km/h west

b Velocity of A relative to B

$100 + 80 = 180$ km/h west

4 Velocity of the ball = $45 + 2 = 47$ m/s North

5 Velocity of the bird relative to the sea= $15 - 5 = 10$ m/s

6 **a** Velocity of A relative to

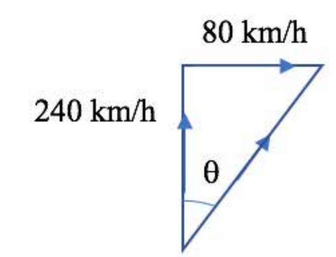
$B = 60 - 40 = 20$ km/h North.

b Velocity of B relative to

$A = 40 - 60 = -20$ km/h North

$= 20$ km/h South.

7

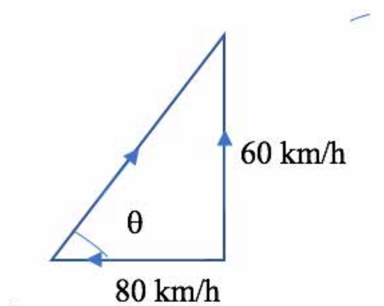


Speed = $\sqrt{240^2 + 80^2} \approx 252.98$ km/h

$\theta = \tan^{-1}\left(\dfrac{1}{3}\right) \approx 18.43°$

Bearing: 18.43°

8



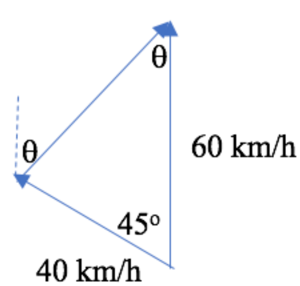
Speed of A relative to

$B = \sqrt{60^2 + 80^2} = 100$

$\theta = \tan^{-1}\left(\dfrac{3}{4}\right) \approx 36.87°$

Bearing: $(90 - 36.87)° = 53.13°$

9



Speed of A relative to B:
Using the cosine rule.
$|\boldsymbol{v}|^2 = 60^2 + 40^2 - 2 \times 40 \times 60 \cos 45°$

$= 3600 + 1600 - 4800 \times \dfrac{1}{\sqrt{2}}$

$|\boldsymbol{v}| \approx 42.5$ km/h

$\dfrac{40}{\sin\theta} = \dfrac{|\boldsymbol{v}|}{\sin 45°}$

$\sin\theta = \dfrac{40 \sin 45°}{|\boldsymbol{v}|}$

$= 0.6655\ldots$

$\theta = 41.73°$

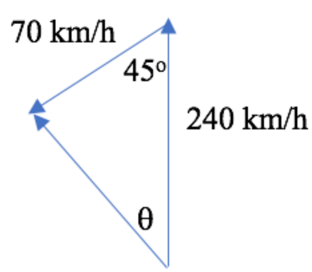
Velocity of P relative to Q is 42.5 km/h with bearing 41.73°

10 **a** $\boldsymbol{v} = \boldsymbol{v_B} - \boldsymbol{v_A} = (5\boldsymbol{i} - 7\boldsymbol{j}) - ((4\boldsymbol{i} - 3\boldsymbol{j})$
$= \boldsymbol{i} - 4\boldsymbol{j}$ m/s

b $|\boldsymbol{v}| = \sqrt{17} \approx 4.12$ m/s

11 Speed of bird relative to sea
$= \sqrt{15^2 + 5^2 - 2 \times 5 \times 15 \cos 18°}$
≈ 10.36 km/h

12



Using the cosine rule:
$|\boldsymbol{v_T}|^2 = 70^2 + 240^2 - 2 \times 70 \times 240 \cos 45°$
$\therefore |\boldsymbol{v_T}| \approx 196.83$ km/h

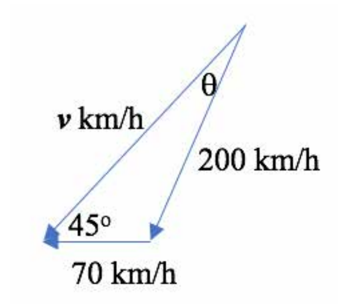
$\dfrac{|\boldsymbol{v_T}|}{\sin 45} = \dfrac{70}{\sin\theta}$

$\therefore \sin\theta = \dfrac{70 \sin 45°}{|\boldsymbol{v_T}|}$

$\theta \approx 14.56°$

Bearing is $(360 - 14.56)° = 345.44°$

13 We want to ensure that the plane's' true velocity $\boldsymbol{v}$ is south-west.



Using the sine rule.

$\dfrac{70}{\sin\theta} = \dfrac{200}{\sin 45°}$

$\sin\theta = \dfrac{70 \sin 45°}{200}$

$\theta \approx 14.33°$

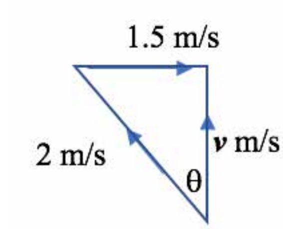
Bearing is:
$(180 + 90 - (45 + 14.63))° = 210°$.
Using the sine rule to find the magnitude of the true velocity:
The third angle of the triangle
$= (180 - 45 - 14.63)° = 120.67°$

$\dfrac{|\boldsymbol{v}|}{\sin 120.67°} = \dfrac{200}{\sin 45°}$

$|\boldsymbol{v}| = \dfrac{200 \sin 120.67°}{\sin 45°}$

≈ 243.28 km/h

14 **a**



$|\boldsymbol{v}| = \sqrt{4 - 1.5^2} \approx 1.32$m/s

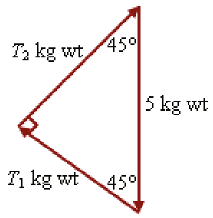
$\sin\theta = \dfrac{1.5}{2}$

$\theta \approx 48.59°$

b Speed $= \dfrac{60}{\sqrt{1.75}} \approx 45.36$ seconds

Solutions to Exercise 21H

1 Rearrange into a triangle of forces.



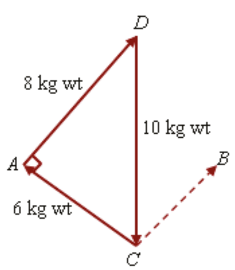
Using trigonometry,

$T_1 = T_2$

$= 5\sin 45°$

$= \dfrac{5\sqrt{2}}{2}$ kg wt

2 Rearrange into a triangle of forces.



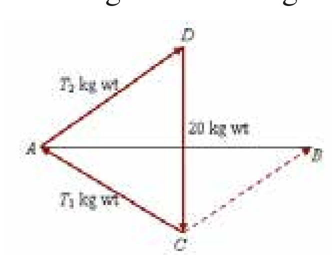
$\angle ACB = \angle ACD + \angle ADC$

These angles can be calculated using the cosine rule, but the student should notice that ΔACD is a 'doubled' 3-4-5 triangle with $\angle CAD = 90°$.

$\therefore ACB = \angle ACD + \angle ADC$

$= 180 - 90 = 90°$

3 Rearrange into a triangle of forces.



Using the cosine rule in the triangle in the original diagram, it is clear that:

$$\cos\angle CAB = \frac{15^2 + 10^2 - 12^2}{2 \times 15 \times 10}$$

$$= 0.6033$$

$$\angle CAB = 52.89°$$

$$\angle ADC = 90 - \angle CAB$$

$$= 37.11°$$

$$\cos\angle CBA = \frac{15^2 + 12^2 - 10^2}{2 \times 15 \times 12}$$

$$= 0.7472$$

$$\angle CBA = 41.65°$$

$$\angle ACD = 90 - \angle CBA$$

$$= 48.35°$$

$$\angle CAD = 180 - 37.11 - 48.35$$

$$= 94.54°$$

Use the sine rule to find T_1 and T_2.

$$\frac{T_1}{\sin\angle ACD} = \frac{20}{\sin\angle CAD}$$

$$T_1 = \frac{20 \times \sin 48.35°}{\sin 94.54°}$$

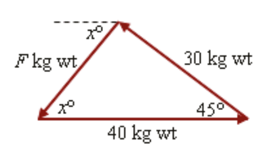
$$\approx 14.99 \text{ kg wt}$$

$$\frac{T_2}{\sin\angle ADC} = \frac{20}{\sin\angle CAD}$$

$$T_2 = \frac{20 \times \sin 37.11°}{\sin 94.54°}$$

$$\approx 12.10 \text{ kg wt}$$

4 Rearrange into a triangle of forces.



Using the cosine rule,

$F^2 = 40^2 + 30^2 - 2 \times 30 \times 40 \times \cos 45°$

$= 802.94$

$F \approx 28.34$ kg wt

Using the cosine rule,

$$\cos x = \frac{F^2 + 40^2 - 30^2}{2 \times F \times 40}$$

$$= 0.663$$

$$x \approx 48.5°$$

W 48.5° S or S 41.5° W

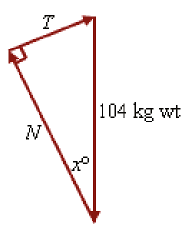
5 The angle between the plane and the horizontal is given by

$$\tan x = \frac{5}{12}$$

$$= 0.4167$$

$$x \approx 22.619°$$

Rearrange into a triangle of forces.



$$T = 104 \sin x$$

$$= 40 \text{ kg wt}$$

Note: The hypotenuse is 13, so

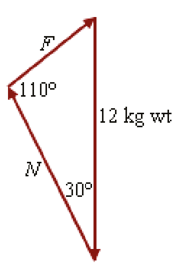
$\sin x = \dfrac{5}{13}$ and $\cos x = \dfrac{12}{13}$.

$$N = 104 \cos x$$

$$= 96 \text{ kg wt}$$

6 Note that F will be acting at 50° to the horizontal and 70° to N, which becomes 110° when the force vectors joined head to tail.

Rearrange into a triangle of forces.



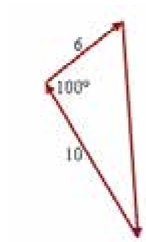
Use the sine rule.

$$\frac{F}{\sin 30°} = \frac{12}{\sin 110°}$$

$$F = \frac{12 \times \sin 30°}{\sin 110°} \approx 6.39 \text{ kg wt}$$

7 In each case, the particle will be in equilibrium if the forces add to zero. Draw the first two forces, and calculate the third force required for equilibrium.

a



Use the cosine rule to calculate the magnitude of the third force.

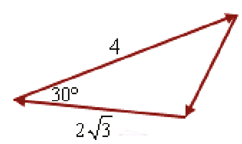
$$F^2 = 10^2 + 6^2 - 2 \times 10 \times 6 \cos 100°$$

$$= 156.837$$

$$F \approx 12.52 \text{ kg wt}$$

This is not the force in the diagram, so these forces will not be in equilibrium.

b



Use the cosine rule to calculate the magnitude of the third force.

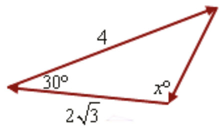
$$F^2 = 4^2 + (2\sqrt{3})^2 - 2 \times 4$$

$$\times 2\sqrt{3} \times \cos 30°$$

$$= 4$$

$$F = 2 \text{ kg wt}$$

It has the same magnitude as the third force in the diagram.



Use the sine rule to find x.

$$\frac{\sin x}{4} = \frac{\sin 30°}{2}$$

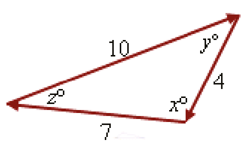
$$\sin x = \frac{0.5 \times 4}{2} = 1$$

$$x = 90°$$

This vector is at the same angle with the $2\sqrt{3}$ vector as in the original diagram.

∴ the vectors will be in equilibrium.

8 Draw the triangle of forces and use the cosine rule to find the three angles. When the vectors are placed tail to tail, the angles between them will be the supplements of the angles in the triangle.



$$\cos x = \frac{7^2 + 4^2 - 10^2}{2 \times 7 \times 4}$$

$$= -0.625$$

$$x \approx 128.68°$$

Angle between vectors is

$180° - 128.68° = 51.32°$

$$\cos y = \frac{10^2 + 4^2 - 7^2}{2 \times 10 \times 4}$$

$$= 0.8375$$

$$y \approx 33.12°$$

Angle between vectors is

$180° - 33.12° = 146.88°$

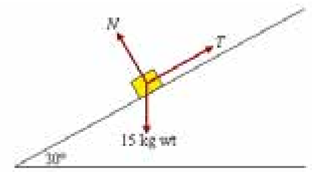
$$z \approx 180° - 128.68° - 33.12°$$

$$= 18.2°$$

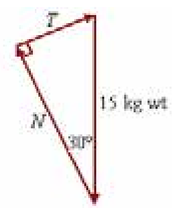
Angle between vectors is

$180° - 18.2° = 161.8°$

9 a



Draw the triangle of forces.



$$T = 15 \sin 30°$$

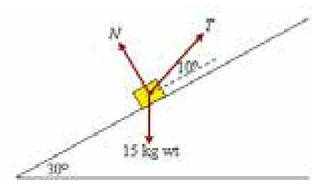
$$= 7.5 \text{ kg wt}$$

b The situation will be the same, except that the 30° angle will now be 40°.

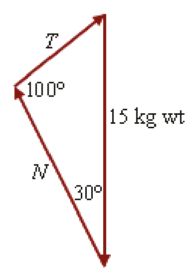
$$T = 15 \sin 40°$$

$$\approx 9.64 \text{ kg wt}$$

c The angle between T and N is now 80°.



Draw the triangle of forces.

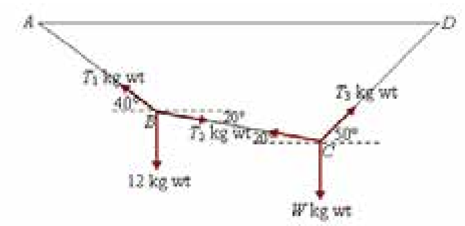


Use the sine rule.

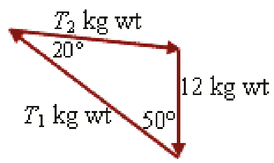
$$\frac{T}{\sin 30°} = \frac{15}{\sin 100°}$$

$$T = \frac{15 \times 0.5}{\sin 100°}$$

$$\approx 7.62 \text{ kg wt}$$

10 

Draw the triangle of forces for point B.



Use the sine rule to find T_1 and T_2.

$$\frac{T_1}{\sin 110^\circ} = \frac{12}{\sin 20^\circ}$$

$$T_1 = \frac{12 \times \sin 110^\circ}{\sin 20^\circ}$$

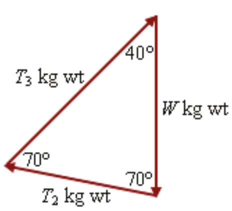
$$\approx 32.97 \text{ kg wt}$$

$$\frac{T_2}{\sin 50^\circ} = \frac{12}{\sin 20^\circ}$$

$$T_2 = \frac{12 \times \sin 50^\circ}{\sin 20^\circ}$$

$$\approx 26.88 \text{ kg wt}$$

Now draw the triangle of forces for point C.



Use the sine rule to find T_3.

$$\frac{T_3}{\sin 70^\circ} = \frac{T_2}{\sin 40^\circ}$$

$$T_3 = \frac{26.88 \times \sin 70^\circ}{\sin 40^\circ}$$

$$\approx 39.29 \text{ kg wt}$$

Since the triangle is isosceles,

$W = T_3 \approx 39.29$ kg wt

The mass of W is 39.29 kg.

11 $F \cos 40^\circ = 10$ kg wt

$$F = \frac{10}{\cos 40^\circ}$$

$$\approx 13.05 \text{ kg wt}$$

12 Resolve in the direction of F.

$F - 10 \cos 55^\circ = 0$

$$F = 5.74 \text{ kg wt}$$

13 First resolve vertically to find N.

$N \cos 25^\circ - 8 = 0$

$$N = \frac{8}{\cos 25^\circ}$$

$$\approx 8.83 \text{ kg wt}$$

Keep the exact value of N in your calculator.

Resolve horizontally.

$F - N \sin 25^\circ = 0$

$$F = N \sin 25^\circ$$

$$\approx 3.73 \text{ kg wt}$$

$F - N \sin 25^\circ = 0$

$F = N \sin 25^\circ \approx 3.73$ kg wt

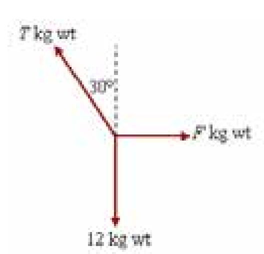
14 Resolve parallel to the plane, i.e. perpendicular to N.

F is at an angle of 34° to the plane.

$F \cos 34^\circ - 10 \sin 20^\circ = 0$

$$F = \frac{10 \sin 20^\circ}{\cos 34^\circ}$$

$$\approx 4.13 \text{ kg wt}$$

15 

Resolve vertically:

$T\cos 30° - 12 = 0$

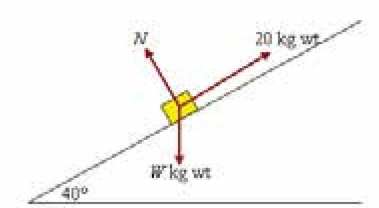
$$T = \frac{12}{\cos 30°}$$

Resolve horizontally:

$F - T\sin 30° = 0$

$$F = T\sin 30°$$
$$= \frac{12\sin 30°}{\cos 30°}$$
$$\approx 6.93 \text{ kg wt}$$

16



Resolve parallel to the plane.

$20 - W\sin 40° = 0$

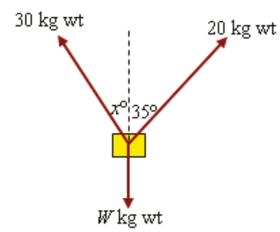
$$W = \frac{20}{\sin 40°}$$
$$\approx 31.11 \text{ kg wt}$$

The force W exerts on the plane is the part of its weight resolved perpendicular to the plane.

$F = W\cos 40°$

$$= \frac{20\cos 40°}{\sin 40°}$$
$$= 23.84 \text{ kg wt}$$

17



First resolve horizontally so only one unknown is involved.

$30\sin x - 20\sin 35° = 0$

$$\sin x = \frac{20\sin 35°}{30}$$
$$= 0.382$$
$$x \approx 22°29'$$

Keep the exact value in your calculator and resolve vertically.

$0 = W - 20\cos 35° - 30\cos 22.481°$

$W = 20\cos 35° + 30\cos 22.481°$

≈ 44.10 kg wt

Solutions to Exercise 21I

1 a $\boldsymbol{a} - \boldsymbol{b} = (\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$
$= -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

b $3\boldsymbol{b} - 2\boldsymbol{a} + \boldsymbol{c} = 3(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) - 2(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + (-\boldsymbol{i} + \boldsymbol{k})$
$= 6\boldsymbol{i} - 3\boldsymbol{j} + 9\boldsymbol{k} - 2\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k} - \boldsymbol{i} + \boldsymbol{k}$
$= 3\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$

c $|\boldsymbol{b}| = \sqrt{2^2 + (-1)^2 + 3^2}$
$= \sqrt{4 + 1 + 9}$
$= \sqrt{14}$

d $|\boldsymbol{b} + \boldsymbol{c}| = |(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) + (-\boldsymbol{i} + \boldsymbol{k})|$
$= |\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}|$
$= \sqrt{1^2 + (-1)^2 + 4^2}$
$= \sqrt{18} = 3\sqrt{2}$

e $3(\boldsymbol{a} - \boldsymbol{b}) + 2\boldsymbol{c} = 3((\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})) + 2(-\boldsymbol{i} + \boldsymbol{k})$
$= 3(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) - 2\boldsymbol{i} + 2\boldsymbol{k}$
$= -3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k} - 2\boldsymbol{i} + 2\boldsymbol{k}$
$= -5\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$

2 a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$
$= 2\boldsymbol{j} + 2\boldsymbol{k}$

b $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{OD}$
$= \boldsymbol{i} + 2\boldsymbol{j}$

c $\overrightarrow{OG} = \overrightarrow{OC} + \overrightarrow{OD}$
$= \boldsymbol{i} + 2\boldsymbol{k}$

d $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OD}$
$= \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

e $\overrightarrow{ED} = -\overrightarrow{OA}$
$= -2\boldsymbol{j}$

f $\overrightarrow{EG} = -\overrightarrow{OA} + \overrightarrow{OC}$
$= -2\boldsymbol{j} + 2\boldsymbol{k}$

g $\overrightarrow{CE} = -\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{OD}$
$= \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

h $\overrightarrow{BD} = -\overrightarrow{OC} - \overrightarrow{OA} + \overrightarrow{OD}$
$= \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$

3 a i $|\boldsymbol{a}| = \sqrt{3^2 + 1^2 + 1^2}$
$= \sqrt{11}$

$\hat{\boldsymbol{a}} = \dfrac{1}{\sqrt{11}}(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$
$= \dfrac{3}{\sqrt{11}}\boldsymbol{i} + \dfrac{1}{\sqrt{11}}\boldsymbol{j} - \dfrac{1}{\sqrt{11}}\boldsymbol{k}$

ii $-2\hat{\boldsymbol{a}} = -\dfrac{6}{\sqrt{11}}\boldsymbol{i} - \dfrac{2}{\sqrt{11}}\boldsymbol{j} + \dfrac{2}{\sqrt{11}}\boldsymbol{k}$

b $5\hat{\boldsymbol{a}} = \dfrac{15}{\sqrt{11}}\boldsymbol{i} + \dfrac{5}{\sqrt{11}}\boldsymbol{j} - \dfrac{5}{\sqrt{11}}\boldsymbol{k}$

4 $|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + 5^2}$

$= \sqrt{27} = 3\sqrt{3}$

$|\boldsymbol{b}| = \sqrt{2^2 + 1^2 + 3^2}$

$= \sqrt{14}$

$\boldsymbol{c} = \dfrac{|\boldsymbol{a}|}{|\boldsymbol{b}|}\boldsymbol{a}$

$= \dfrac{\sqrt{14}}{3\sqrt{3}}(\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$

$= \dfrac{\sqrt{42}}{9}(\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$

5 a $\overrightarrow{PQ} = \boldsymbol{i} - 3\boldsymbol{j}$

b $|\overrightarrow{PQ}| = \sqrt{1^2 + 3^2 + 0^2}$

$= \sqrt{10}$

c $\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$

$= \overrightarrow{OP} + \dfrac{1}{2}\overrightarrow{PQ}$

$= \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \dfrac{1}{2}\boldsymbol{i} - \dfrac{3}{2}\boldsymbol{j}$

$= \dfrac{3}{2}\boldsymbol{i} + \dfrac{1}{2}\boldsymbol{j} - \boldsymbol{k}$

6 a $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$

$= \boldsymbol{i} + 3\boldsymbol{j}$

$\overrightarrow{OM} = \dfrac{1}{3}\overrightarrow{OE}$

$= \dfrac{1}{3}\boldsymbol{i} + \boldsymbol{j}$

$\overrightarrow{BF} = \overrightarrow{OD}$

$= \boldsymbol{i}$

$\overrightarrow{BN} = \dfrac{1}{2}\overrightarrow{BF}$

$= \dfrac{1}{2}\boldsymbol{i}$

$\overrightarrow{ON} = \overrightarrow{OC} + \overrightarrow{CB} + \overrightarrow{BN}$

$= \dfrac{1}{2}\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$

$= \dfrac{1}{2}\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k} - \left(\dfrac{1}{3}\boldsymbol{i} + \boldsymbol{j}\right)$

$= \dfrac{1}{6}\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

b $|\overrightarrow{MN}| = \sqrt{\left(\dfrac{1}{6}\right)^2 + 2^2 + 2^2}$

$= 4\sqrt{\dfrac{1 + 144 + 144}{36}}$

$= \sqrt{\dfrac{289}{36}}$

$= \dfrac{17}{6}$

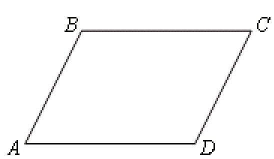
Solutions to technology-free questions

1 a $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ if $\boldsymbol{a} = k\boldsymbol{b}$, where k is a constant. $7\boldsymbol{i} + 6\boldsymbol{j} = k(2\boldsymbol{i} + x\boldsymbol{j})$

$$2k = 7$$
$$k = \frac{7}{2}$$
$$kx = 6$$
$$\frac{7x}{2} = 6$$
$$x = \frac{12}{7}$$

b
$$|\boldsymbol{a}| = \sqrt{7^2 + 6^2}$$
$$= \sqrt{85}$$
$$|\boldsymbol{b}| = \sqrt{2^2 + x^2}$$
$$= |\boldsymbol{a}| = \sqrt{85}$$
$$\therefore\ x^2 + 4 = 85$$
$$x^2 = 81$$
$$x = \pm 9$$

2



$$A = (2, -1)$$
$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$
$$= 5\boldsymbol{i} + 3\boldsymbol{j}$$
$$B = (5, 3)$$
$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$= \overrightarrow{AB} + \overrightarrow{AD}$$
$$= \boldsymbol{i} + 9\boldsymbol{j}$$
$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{i} + 9\boldsymbol{j}$$
$$= 3\boldsymbol{i} + 8\boldsymbol{j}$$
$$C = (3, 8)$$
$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$
$$= 4\boldsymbol{j}$$
$$D = (0, 4)$$

3
$$\boldsymbol{a} + p\boldsymbol{b} + q\boldsymbol{c} = (2 + 2p - q)\boldsymbol{i} + (-3 - 4p - 4q)\boldsymbol{j} + (1 + 5p + 2q)\boldsymbol{k}$$

To be parallel to the x-axis,
$$\boldsymbol{a} + p\boldsymbol{b} + q\boldsymbol{c} = k\boldsymbol{i}$$
$$1 + 5p + 2q = 0$$
$$2 + 10p + 4q = 0 \qquad (1)$$
$$-3 - 4p - 4q = 0 \qquad (2)$$
$(1) + (2)$:
$$-1 + 6p = 0$$
$$p = \frac{1}{6}$$
$$1 + \frac{5}{6} + 2q = 0$$
$$2q = -\frac{11}{6}$$
$$q = -\frac{11}{12}$$

4 a
$$\overrightarrow{PQ} = (3\boldsymbol{i} - 7\boldsymbol{j} + 12\boldsymbol{k}) - (2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k})$$
$$= \boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k}$$
$$|\overrightarrow{PQ}| = \sqrt{1^2 + 5^2 + 8^2}$$
$$= \sqrt{90} = 3\sqrt{10}$$

b $\dfrac{1}{3\sqrt{10}}(\boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k})$

5 $\overrightarrow{AB} = 4\boldsymbol{i} + 8\boldsymbol{j} + 16\boldsymbol{k}$

$\overrightarrow{AC} = x\boldsymbol{i} + 12\boldsymbol{j} + 24\boldsymbol{k}$

For A, B and C to be collinear, we need $\overrightarrow{AC} = k\overrightarrow{AB}$.

$$x\boldsymbol{i} + 12\boldsymbol{j} + 24\boldsymbol{k} = k(4\boldsymbol{i} + 8\boldsymbol{j} + 16\boldsymbol{k})$$
$$8k = 12$$
$$k = 1.5$$
$$x = 4k$$
$$= 6$$

6 **a** $\overrightarrow{OA} = \sqrt{4^2 + 3^2}$

$= 5$

Unit vector $= \dfrac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$

b $\overrightarrow{OC} = \dfrac{16}{5}\overrightarrow{OA}$

$= \dfrac{16}{5} \times \dfrac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$

$= \dfrac{16}{25}(4\boldsymbol{i} + 3\boldsymbol{j})$

7 **a** **i** $\overrightarrow{SQ} = \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{a} + \boldsymbol{b}$

ii $\overrightarrow{TQ} = \dfrac{1}{3}\overrightarrow{SQ}$

$= \dfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

iii $\overrightarrow{RQ} = -2\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{b} - \boldsymbol{a}$

iv $\overrightarrow{PT} = \overrightarrow{PQ} + \overrightarrow{QT}$

$= \overrightarrow{PQ} - \overrightarrow{TQ}$

$= \boldsymbol{a} - \dfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

$= \dfrac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$

v $\overrightarrow{TR} = \overrightarrow{TQ} + \overrightarrow{QR}$

$= \overrightarrow{TQ} - \overrightarrow{RQ}$

$= \dfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b}) - (\boldsymbol{b} - \boldsymbol{a})$

$= \dfrac{1}{3}(4\boldsymbol{a} - 2\boldsymbol{b})$

$= \dfrac{2}{3}(2\boldsymbol{a} - \boldsymbol{b})$

b $2\overrightarrow{PT} = \overrightarrow{TR}$

P, T and R are collinear.

8 $\boldsymbol{a} = \boldsymbol{b}$

a **i** $-s\boldsymbol{j} = 2\boldsymbol{j}$

$s = -2$

ii $5\boldsymbol{i} = t\boldsymbol{i}$

$t = 5$

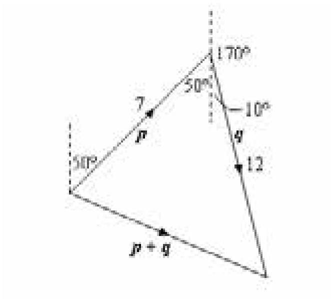
iii $2\boldsymbol{k} = u\boldsymbol{k}$

$u = 2$

b $\hat{\boldsymbol{a}} = \sqrt{5^2 + 2^2 + 2^2}$

$= \sqrt{25 + 4 + 4}$

$= \sqrt{33}$

9 

Use the cosine rule

$|\boldsymbol{p} + \boldsymbol{q}|^2 = 7^2 + 12^2$

$\quad - 2 \times 7 \times 12 \times \cos 60°$

$= 109$

$|\boldsymbol{p} + \boldsymbol{q}| = \sqrt{109}$

10 a $\boldsymbol{a} + 2\boldsymbol{b} = (5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

$\quad + 2 \times (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= 11\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

b $|\boldsymbol{a}| = \sqrt{5^2 + 2^2 + 1^2}$

$= \sqrt{30}$

c $\hat{\boldsymbol{a}} = \dfrac{1}{\sqrt{30}}(5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

d $\boldsymbol{a} - \boldsymbol{b} = (5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) - (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= 2\boldsymbol{i} + 4\boldsymbol{j}$

11 a $\overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB}$

$= (3\boldsymbol{i} + 4\boldsymbol{j}) - (4\boldsymbol{i} - 6\boldsymbol{j})$

$= -\boldsymbol{i} + 10\boldsymbol{j}$

$C = (-1, 10)$

b $\boldsymbol{i} + 24\boldsymbol{j} = h(3\boldsymbol{i} + 4\boldsymbol{j}) + k(4\boldsymbol{i} - 6\boldsymbol{j})$

$3h + 4k = 1$

$4h - 6k = 24$

Multiply the first equation by 3 and the second equation by 2.

$9h + 12k = 3$ ①

$8h - 12k = 48$ ②

① + ②:

$17h = 51$

$h = 3$

$9 + 4k = 1$

$k = -2$

12 $m\boldsymbol{p} + n\boldsymbol{q} = 3m\boldsymbol{i} + 7m\boldsymbol{j} + 2n\boldsymbol{i} - 5n\boldsymbol{j}$

$= 8\boldsymbol{i} + 9\boldsymbol{j}$

$3m + 2n = 8$

$7m - 5n = 9$

Multiply the first equation by 5 and the second equation by 2.

$15m + 10n = 40$ ①

$14m - 10n = 18$ ②

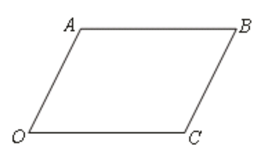
① + ② :

$29m = 58$

$m = 2$

$6 + 2n = 8$

$n = 1$

13 a 

$\boldsymbol{b} = \overrightarrow{OB}$

$= \overrightarrow{OA} + \overrightarrow{AB}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= \boldsymbol{a} + c$

b $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$BC = \boldsymbol{c} - \boldsymbol{b}$

$$AB : BC = 3 : 2$$

$$\frac{AB}{BC} = \frac{3}{2}$$

$$2AB = 3BC$$

$$2(\boldsymbol{b} - \boldsymbol{a}) = 3(\boldsymbol{c} - \boldsymbol{b})$$

$$2\boldsymbol{b} - 2\boldsymbol{a} = 3\boldsymbol{c} - 3\boldsymbol{b}$$

$$5\boldsymbol{b} = 2\boldsymbol{a} + 3\boldsymbol{c}$$

$$\boldsymbol{b} = \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$$

14 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}, \boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$

a $\boldsymbol{a} \cdot \boldsymbol{a} = 13$

b $\boldsymbol{b} \cdot \boldsymbol{b} = 10$

c $\boldsymbol{c} \cdot \boldsymbol{c} = 8$

d $\boldsymbol{a} \cdot \boldsymbol{b} = -11$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = (2\boldsymbol{i} - 3\boldsymbol{j}) \cdot (-3\boldsymbol{i} + \boldsymbol{j}) = -9$

f

$$\begin{aligned}(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c}) &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c} \\ &= 13 + 2 - 11 - 4 \\ &= 0\end{aligned}$$

g $\boldsymbol{a} + 2\boldsymbol{b} = 3\boldsymbol{j}$

$3\boldsymbol{c} - \boldsymbol{b} = -5\boldsymbol{i} - 9\boldsymbol{j}$

$\therefore (\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b}) = -27$

15 $\overrightarrow{OA} = \boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$

$\overrightarrow{OB} = \boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j}$

$\overrightarrow{OC} = \boldsymbol{c} = -5\boldsymbol{i} + 3\boldsymbol{j}$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{i} + 5\boldsymbol{j} \\ &= -\boldsymbol{i} + 4\boldsymbol{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} \\ &= -3\boldsymbol{i} - 5\boldsymbol{j} - 5\boldsymbol{i} + 3\boldsymbol{j} \\ &= -8\boldsymbol{i} - 2\boldsymbol{j}\end{aligned}$$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 8 - 8 = 0.$

Hence there is a right angle at B.

16 $\boldsymbol{p} = 5\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} + t\boldsymbol{j}$

a If $\boldsymbol{p} + \boldsymbol{q}$ is parallel to $\boldsymbol{p} + \boldsymbol{q}$ there exists a non-zeo real number k such that.

$k(\boldsymbol{p} + \boldsymbol{q}) = \boldsymbol{p} - \boldsymbol{q}$.

That is,

$k(7\boldsymbol{i} + (3 + t)\boldsymbol{j} = 3\boldsymbol{i} + (3 - t)\boldsymbol{j}$.

Hence $7k = 3$

$$k = \frac{3}{7}$$

$$k(3 + t) = (3 - t)$$

$$\therefore 3(3 + t) = 7(3 - t)$$

$$\therefore 9 + 3t = 21 - 7t$$

$$10t = 12$$

$$t = \frac{6}{5}$$

b

$$\begin{aligned}\boldsymbol{p} - 2\boldsymbol{q} &= 5\boldsymbol{i} + 3\boldsymbol{j} - 2(2\boldsymbol{i} + t\boldsymbol{j}) \\ &= \boldsymbol{i} + (3 - 2t)\boldsymbol{j} \\ \boldsymbol{p} + 2\boldsymbol{q} &= 5\boldsymbol{i} + 3\boldsymbol{j} + 2(2\boldsymbol{i} + t\boldsymbol{j}) \\ &= 9\boldsymbol{i} + (3 + 2t)\boldsymbol{j}\end{aligned}$$

Since the vectors are perpendicular

$$(\boldsymbol{i} + (3 - 2t)\boldsymbol{j}) \cdot (9\boldsymbol{i} + (3 + 2t)\boldsymbol{j}) = 0$$

$$9 + (3 - 2t)(3 + 2t) = 0$$

$$9 + (9 - 4t^2) = 0$$

$$4t^2 = 18$$

$$t^2 = \frac{9}{2}$$

$$t = \pm\frac{3}{\sqrt{2}}$$

c $$|\boldsymbol{p} - \boldsymbol{q}| = |3\boldsymbol{i} + (3 - t)\boldsymbol{j}|$$

$$= \sqrt{9 + (3 - t)^2}$$

$$|\boldsymbol{q}| = |2\boldsymbol{i} + t\boldsymbol{j}|$$

$$= \sqrt{4 + t^2}$$

If $|\boldsymbol{p} - \boldsymbol{q}| = |\boldsymbol{q}|$

then $9 + (3 - t)^2 = 4 + t^2$

$\therefore 9 + 9 - 6t + t^2 = 4 + t^2$

$$14 - 6t = 0$$

$$t = \frac{7}{3}$$

17 $\overrightarrow{OA} = \boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OB} = \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OC} = \boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$

a **i** $\overrightarrow{AB} = -\boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{i}$

ii $\overrightarrow{AC} = -\boldsymbol{a} + \boldsymbol{c} = -5\boldsymbol{j}$

b The vector resolute $= \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AC} \cdot \overrightarrow{AC}}\overrightarrow{AC}$

$= 0$

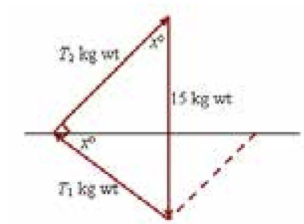
c 1

18 **a** Speed = $\sqrt{1.6^2 + 1.2^2} = 2$ m/s

b She swims 60 m in at 2m/s. It takes her 30 seconds

c She arrives at the opposite bank 36 m downstream

19 Note that the two strings form a 3-4-5 triangle. Draw the triangle of forces.



Note:

$\sin x = \dfrac{6}{10} = \dfrac{3}{5}$; $\cos x = \dfrac{8}{10} = \dfrac{4}{5}$

$T_1 = 15 \sin x$

$= 15 \times \dfrac{3}{5} = 9$ kg wt

$T_2 = 15 \cos x$

$= 15 \times \dfrac{4}{5} = 12$ kg wt

20 The force exerted on the body by the plane will be perpendicular to the plane. Resolve parallel to the plane, so the component this force will be zero.
The hypotenuse of the marked triangle is

$h = \sqrt{12^2 + 6^2}$

$= \sqrt{180} = 6\sqrt{5}$ cm

If x is the angle of the plane to the horizontal,

$\sin x = \dfrac{6}{6\sqrt{5}} = \dfrac{1}{\sqrt{5}}$

$\cos x = \dfrac{12}{6\sqrt{5}} = \dfrac{2}{\sqrt{5}}$

Resolving,

$T - 70\sin x = 0$

$$T = 70\sin x$$
$$= 70 \times \frac{1}{\sqrt{5}}$$
$$= \frac{70\sqrt{5}}{5} = 14\sqrt{5}\text{ kg wt}$$

Resolving perpendicular to the plane,

$N - 70\cos x = 0$

$$N = 70\cos x$$
$$= 70 \times \frac{2}{\sqrt{5}}$$
$$= \frac{140\sqrt{5}}{5} = 28\sqrt{5}\text{ kg wt}$$

21 The force exerted on the body by the plane will be perpendicular to the plane. Resolve parallel to the plane, so the component of this force will be zero.

$F\cos 30° - 15\sin 30° = 0$

$$\frac{F\sqrt{3}}{2} = 15 \times \frac{1}{2}$$
$$F = \frac{15}{\sqrt{3}}$$
$$= \frac{15\sqrt{3}}{3}$$
$$= 5\sqrt{3}\text{ kg wt}$$

Solutions to multiple-choice questions

1 C $v = \begin{bmatrix} 3-1 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$a = 2, b = 4$

2 C $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

$= -\overrightarrow{AC} + \overrightarrow{AB}$

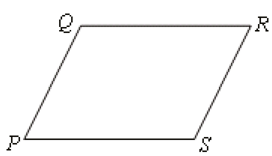
$= \boldsymbol{u} - \boldsymbol{v}$

3 A $2\boldsymbol{a} - 3\boldsymbol{b} = 2\begin{bmatrix} 3 \\ -2 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 6 - -3 \\ -4 - 9 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ -13 \end{bmatrix}$

4 B



$\overrightarrow{SQ} = \overrightarrow{SR} + \overrightarrow{RQ}$

$= \overrightarrow{PQ} + -\overrightarrow{QR}$

$= \boldsymbol{p} - \boldsymbol{q}$

5 B $|3\boldsymbol{i} - 5\boldsymbol{j}| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34}$

6 A $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= (\boldsymbol{i} - 2\boldsymbol{j}) - (2\boldsymbol{i} + 3\boldsymbol{j})$

$= -\boldsymbol{i} - 5\boldsymbol{j}$

7 C $|\overrightarrow{AB}| = |-\boldsymbol{i} - 5\boldsymbol{j}|$

$= \sqrt{(-1)^2 + (-5)^2}$

$= \sqrt{1 + 25}$

$= \sqrt{26}$

8 D $|\boldsymbol{a}| = \sqrt{2^2 + 3^2}$

$= \sqrt{13}$

$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{13}}(2\boldsymbol{i} + 3\boldsymbol{j})$

9 A $\boldsymbol{v} = \frac{100}{5}(3\boldsymbol{i} - 4\boldsymbol{j}) + (-5\boldsymbol{i} + 20\boldsymbol{j})$

$= 60\boldsymbol{i} - 80\boldsymbol{j}) - 5\boldsymbol{i} + 20\boldsymbol{j}$

$= 55\boldsymbol{i} - 60\boldsymbol{j})$

10 A

$-20\boldsymbol{i} + -4\boldsymbol{i} + 3\boldsymbol{j} = -24\boldsymbol{i} + 3\boldsymbol{j}$

11 E $50 \cos 60° = 50 \times \frac{1}{2}$

$= 25$ N

12 C Use Pythagoras' theorem.

Resultant $= \sqrt{5^2 + 4^2}$

$= \sqrt{41}$ kg wt

13 B The forces act at right angles.

Complete a triangle of forces.

$7^2 + a^2 = 9^2$

$a^2 = 32a = 4\sqrt{2}$ kg wt

14 B The angle between the forces when they are head to tail will be 120°.

Use the cosine rule.

$F^2 = 20^2 + 20^2 - 2 \times 20$

$\times 20 \times \cos 120°$

$= 400 + 400 - 800 \times -\frac{1}{2}$

$= 1200$

$F = \sqrt{1200}$

$= 20\sqrt{3}$ kg wt

Solutions to extended-response questions

1 $\begin{bmatrix}1\\0\end{bmatrix}$ is in the east direction and $\begin{bmatrix}0\\1\end{bmatrix}$ in the north direction.

a $\overrightarrow{OP} = -32\begin{bmatrix}0\\1\end{bmatrix} - 31\begin{bmatrix}1\\0\end{bmatrix}$

$= \begin{bmatrix}-31\\-32\end{bmatrix}$

b The ship is travelling parallel to the vector $\boldsymbol{u} = \begin{bmatrix}4\\3\end{bmatrix}$ with speed 20 km/h.

The unit vector in the direction of $\boldsymbol{u}$ is $\frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}$.

The vector $\overrightarrow{PR} = \frac{20}{5}\begin{bmatrix}4\\3\end{bmatrix}$

$= \begin{bmatrix}16\\12\end{bmatrix}$

The position vector ofthe ship is

$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= \begin{bmatrix}-31\\-32\end{bmatrix} + \begin{bmatrix}16\\12\end{bmatrix}$

$= \begin{bmatrix}-15\\-20\end{bmatrix}$

$= -5\begin{bmatrix}3\\4\end{bmatrix}$

c $|\overrightarrow{OR}| = 5\sqrt{3^2 + 4^2}$

$= 25$

When the ship reaches R, it is 25 km from the lighthouse, and therefore the lighthouse is visible from the ship.

2 $\boldsymbol{p} = 3\boldsymbol{i} + \boldsymbol{j}$ and $q = -2\boldsymbol{i} + 4\boldsymbol{j}$

a $\therefore\ |\boldsymbol{p} - \boldsymbol{q}| = |3\boldsymbol{i} + \boldsymbol{j} - (-2\boldsymbol{i} + 4\boldsymbol{j})|$

$= |5\boldsymbol{i} - 3\boldsymbol{j}|$

$= \sqrt{25 + 9}$

$= \sqrt{34}$

b $|\boldsymbol{p}| = \sqrt{9+1}$

$= \sqrt{10}$

and $|\boldsymbol{q}| = \sqrt{4+16}$

$= 2\sqrt{5}$

$\therefore\ |\boldsymbol{p}| - |\boldsymbol{q}| = \sqrt{10} - 2\sqrt{5}$

c $3\boldsymbol{i} + \boldsymbol{j} + 2(-2\boldsymbol{i} + 4\boldsymbol{j}) + \boldsymbol{r} = \boldsymbol{0}$

$3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{i} + 8\boldsymbol{j} + \boldsymbol{r} = \boldsymbol{0}$

$-\boldsymbol{i} + 9\boldsymbol{j} + \boldsymbol{r} = \boldsymbol{0}$

Hence $\boldsymbol{r} = \boldsymbol{i} - 9\boldsymbol{j}$

3 $\boldsymbol{a} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 7 \\ 9 \\ 7 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$

a $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = k\boldsymbol{d}$

$$\therefore \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \\ 7 \end{bmatrix} = k\begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 13 \\ 6 \\ 1 \end{bmatrix} = k\begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$$

Therefore $k = \frac{1}{2}$ and $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = \frac{1}{2}\boldsymbol{d}$

b $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{d}$

$$\therefore\ x\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$$

The following equations are formed:

$-2x + 11y = 26$ . . . ①

$x + 7y = 12$ . . . ②

$2x + 3y = 2$ . . . ③

Add ① and ③

$14y = 28$

$\therefore\ y = 2$

Substitute in ③

$2x + 6 = 2$

$\therefore\ x = -2$

Equation ② must be checked

$-2 + 14 = 12$

Therefore $-2\boldsymbol{a} + 2\boldsymbol{b} = \boldsymbol{d}$.

c $p\boldsymbol{a} + q\boldsymbol{b} - r\boldsymbol{c} = \mathbf{0}$

From parts **a** and **b**

$$\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = \frac{1}{2}\boldsymbol{d} \qquad \ldots ①$$

$$-2\boldsymbol{a} + 2\boldsymbol{b} = \boldsymbol{d} \qquad \ldots ②$$

From ① $\quad 2\boldsymbol{a} + 4\boldsymbol{b} - 2\boldsymbol{c} = \boldsymbol{d}$

Therefore from ②

$$-2\boldsymbol{a} + 2\boldsymbol{b} = 2\boldsymbol{a} + 4\boldsymbol{b} - 2\boldsymbol{c}$$

$\therefore\ 4\boldsymbol{a} + 2\boldsymbol{b} - 2\boldsymbol{c} = \mathbf{0}$

Hence $p = 4, q = 2$ and $r = 2$. (Other answers are possible e.g. $p = 2, q = 1,$ $r = -1$)

4 a $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \begin{bmatrix} 5 \\ 8 \end{bmatrix} + \begin{bmatrix} 20 \\ -15 \end{bmatrix}$$

$$= \begin{bmatrix} 25 \\ -7 \end{bmatrix}$$

The coordinates of Q are $(25, -7)$.

$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$$= \begin{bmatrix} -25 \\ 7 \end{bmatrix} + \begin{bmatrix} 32 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 24 \end{bmatrix}$$

b $\overrightarrow{RS} = \overrightarrow{QP}$

$$= \begin{bmatrix} -20 \\ 15 \end{bmatrix}$$

$\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{RS}$

$$= \begin{bmatrix} 32 \\ 17 \end{bmatrix} + \begin{bmatrix} -20 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 32 \end{bmatrix}$$

Hence the coordinates of S are $(12, 32)$.

5 **a** $\overrightarrow{OP} = 4\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$

The coordinates of P are $(12, 4)$.

b $\overrightarrow{PM} = \overrightarrow{PO} + \overrightarrow{OM}$

$$= \begin{bmatrix} -12 \\ -4 \end{bmatrix} + \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} k - 12 \\ -4 \end{bmatrix}$$

c

$$|\overrightarrow{OP}| = \sqrt{12^2 + 4^2}$$

$$= \sqrt{160}$$

$$= 4\sqrt{10}$$

Now $|\overrightarrow{OM}| = k$

and, from part **b**, $\overrightarrow{PM} = \begin{bmatrix} k - 12 \\ -4 \end{bmatrix}$

$$\therefore\ |\overrightarrow{PM}| = \sqrt{(k - 12)^2 + 16}$$

For triangle OPM to be right-angled at P, Pythagoras' theorem has to be satisfied.

$$\text{i.e. } |\overrightarrow{OP}|^2 + |\overrightarrow{PM}|^2 = |\overrightarrow{OM}|^2$$

$$\therefore\ 160 + (k - 12)^2 + 16 = k^2$$

$$\therefore\ 160 + k^2 - 24k + 160 = k^2$$

$$\therefore\ 24k = 320$$

$$\therefore\ 3k = 40$$

$$\therefore\ k = \frac{40}{3}$$

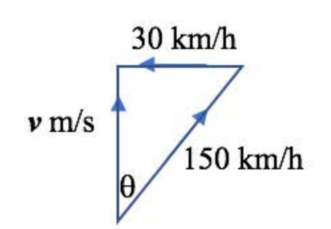
d If M has coordinates $(9, 0)$ then,
if $\angle OPX = \alpha°, \tan\alpha° = 3$
and if $\angle MPX = \beta°, \tan\beta° = \dfrac{3}{4}$
$\therefore$ Angle $\theta = \alpha - \beta$

$$= \tan^{-1}(3) - \tan^{-1}\left(\frac{3}{4}\right)$$

$$= 34.7°,\ \text{correct to one decimal place}$$

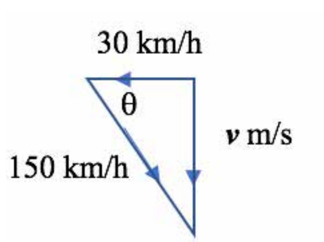
6 **a** Going out the true speed = 150 − 30 = 120 km/h
Returning true speed = 150 + 30 = 180 km/h
Total Time taken = $\dfrac{180}{180} + \dfrac{180}{120} = \dfrac{5}{2}$ hours

b



$\sin\theta = \dfrac{1}{5} \Rightarrow \theta \approx 11.54°$
Using Pythagoras' theorem:
True velocity = $\sqrt{150^2 - 30^2} \approx 146.97$ km/h
Bearing is 11.54°

c



$\cos\theta = \dfrac{1}{5} \Rightarrow \theta \approx 78.40°$

Therefore Bearing = $(78.46 + 90)° = 168.46°$

7 **a** $\boldsymbol{v} = \boldsymbol{v_A} - \boldsymbol{v_B}$

$= 12\boldsymbol{i} + 16\boldsymbol{j} - (8\boldsymbol{i} + \alpha\boldsymbol{j}$

$= 4\boldsymbol{i} + (16 - \alpha)\boldsymbol{j}$

b Consider boat B to be at the origin.
Position of boat A is $-10\boldsymbol{i}$
For collision:
$-10\boldsymbol{i} + t(12\boldsymbol{i} + 16\boldsymbol{j}) = t(8\boldsymbol{i} + \alpha\boldsymbol{j})$
Therfore:
$-10 + 12t = 8t$
$16t = \alpha t$
$\Rightarrow$ Collision when $t = 2.5$
Hence $\alpha = 16$

c Time between sighting and collision is 16 hours.

Chapter 22 – Revision of Chapters 20-21

Solutions to Technology-free questions

1 a $(2, 3) \to (2 \times 2 + 3, -2 - 2 \times 3) = (7, -8)$

b The entries of the matrix are the coefficients of x and y in the transformation, $B = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$.

c The area of the original square is 1. Therefore, the image will have area,

$$\begin{aligned} \text{area of image} &= |\det B| \times \text{original area} \\ &= |2(-2) - (1)(-1)| \times 1 \\ &= 3. \end{aligned}$$

d The inverse transformation will have matrix

$$\begin{aligned} B^{-1} &= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{2(-2) - (1)(-1)}\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{-3}\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix} \end{aligned}$$

Therefore, the rule for the inverse transformation is $(x, y) \to (\frac{2}{3}x + \frac{1}{3}y, -\frac{1}{3}x - \frac{2}{3}y)$

2 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

e

$$\begin{bmatrix}\cos 45° & -\sin 45° \\ \sin 45° & \cos 45°\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

f $$\begin{bmatrix}\cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

g $$\begin{bmatrix}-1 & 0 \\ 0 & -1\end{bmatrix}$$

h $$\begin{bmatrix}\cos(60°) & \sin(60°) \\ \sin(60°) & -\cos(60°)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$$

3 a Since $\tan\theta = 4 = \frac{4}{1}$,
we draw a right angled triangle with opposite and adjacent lengths 4 and 1 respectively. Pythagoras' Theorem gives the hypotenuse as $\sqrt{17}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{17}} \text{ and } \sin\theta = \frac{4}{\sqrt{17}}.$$

We then use the double angle formulas to show that

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{\sqrt{17}}\right)^2 - 1 \\ &= \frac{2}{17} - 1 \\ &= -\frac{15}{17}, \\ \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\frac{4}{\sqrt{17}}\frac{1}{\sqrt{17}} \\ &= \frac{8}{17}.\end{aligned}$$

Therefore, the required matrix is

$$\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}-\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17}\end{bmatrix}.$$

b

$$\begin{bmatrix}-\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17}\end{bmatrix}\begin{bmatrix}2 \\ 4\end{bmatrix} = \begin{bmatrix}\frac{2}{17} \\ \frac{76}{17}\end{bmatrix}$$

Therefore, the image is the point $\left(\frac{2}{17}, \frac{76}{17}\right)$.

4 a The matrix that will reflect the plane in the y-axis is given by

$$\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}.$$

The matrix that will dilate the result by a factor of 2 from the x-axis is given by

$$\begin{bmatrix}1 & 0 \\ 0 & 2\end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix}1 & 0 \\ 0 & 2\end{bmatrix}\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & 2\end{bmatrix}.$$

b The matrix that will rotate the plane by 90° anticlockwise is given by

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The matrix that will reflect the result in the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

c The matrix that will reflect the plane in the line $y = -x$ is given by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The matrix that will shear the result by a factor of 2 in the x-direction is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the matrix of the composition transformation is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}.$$

5 **a**

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -x+2 \\ y-1 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \to (-x + 2, y - 1)$.

b

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x+2 \\ y-1 \end{bmatrix} \\ &= \begin{bmatrix} -x-2 \\ y-1 \end{bmatrix} \end{aligned}$$

Therefore, the transformation is $(x, y) \to (-x - 2, y - 1)$.

6 **a**

The area of the original square is 1. Therefore, the image will have area,

$$\begin{aligned} \text{area of image} &= |\det B| \times \text{original area} \\ &= |1(1) - (2)(-1)| \times 1 \\ &= 3. \end{aligned}$$

b

The area of the original square is 1. Therefore, the image will have area,

$$\begin{aligned}\text{area of image} &= |\det B| \times \text{original area}\\ &= |2(-2) - (1)(1)| \times 1\\ &= 5.\end{aligned}$$

7 **a** Firstly, the matrix that will reflect the plane in the line $y = x$ is given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Therefore the required transformation is

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y+1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\\ &= \begin{bmatrix} y+1 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\\ &= \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}.\end{aligned}$$

b To find the image of $(0, 0)$ we let $x = 0$ and $y = 0$ so that $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore, $(0, 0) \to (1, 1)$, as expected.

c

8 $\boldsymbol{a} = 2\boldsymbol{i} + 6\boldsymbol{j}$

a $|2\boldsymbol{i} + 6\boldsymbol{j}| = \sqrt{4+36} = 2\sqrt{10}$

b $\hat{\boldsymbol{a}} = \dfrac{1}{2\sqrt{10}}(2\boldsymbol{i} + 6\boldsymbol{j})$

c $8\hat{\boldsymbol{a}} = \dfrac{4}{\sqrt{10}}(2\boldsymbol{i} + 6\boldsymbol{j})$

d $-2\hat{\boldsymbol{a}} = -\dfrac{1}{\sqrt{10}}(2\boldsymbol{i} + 6\boldsymbol{j})$

9 **a** $\boldsymbol{a} \cdot \boldsymbol{a} = (2)(2) + (-3)(-3) = 4 + 9 = 13$

b $\boldsymbol{b} \cdot \boldsymbol{b} = (-2)(-2) + (3)(3) = 4 + 9 = 13$

c $\boldsymbol{a} \cdot \boldsymbol{a} = (-3)(-3) + (-2)(-2) = 4 + 9 = 13.$

d $\boldsymbol{a} \cdot \boldsymbol{b} = (2)(-2) + (-3)(3) = -4 - 9 = -13.$

e
$$\begin{aligned}\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) &= (2\boldsymbol{i} - 3\boldsymbol{j}) \cdot (-5\boldsymbol{i} + \boldsymbol{j})\\ &= (2)(-5) + (-3)(1)\\ &= -13\end{aligned}$$

f
$$\begin{aligned}(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c}) &= \boldsymbol{0} \cdot (-\boldsymbol{i} - 5\boldsymbol{j})\\ &= 0\end{aligned}$$

g
$$\begin{aligned}(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b}) &= (-2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (-7\boldsymbol{i} - 9\boldsymbol{j})\\ &= (-2)(-7) + (3)(-9)\\ &= -13\end{aligned}$$

10 a

$$m\overrightarrow{OA} + n\overrightarrow{BC} = 2\boldsymbol{i} + 10\boldsymbol{j}$$
$$m(4\boldsymbol{i} + 2\boldsymbol{j}) + n(9\boldsymbol{i} - \boldsymbol{j}) = 2\boldsymbol{i} + 10\boldsymbol{j}$$
$$(4m\boldsymbol{i} + 2m\boldsymbol{j}) + (9n\boldsymbol{i} - n\boldsymbol{j}) = 2\boldsymbol{i} + 10\boldsymbol{j}$$
$$(4m + 9n)\boldsymbol{i} + (2m - n)\boldsymbol{j} = 2\boldsymbol{i} + 10\boldsymbol{j}$$

Therefore

$4m + 9n = 2$ and $2m - n = 10.$

These simultaneous equations have solution

$$m = \frac{46}{11} \text{ and } n = -\frac{18}{11}.$$

b Since

$$\overrightarrow{OB} = -\boldsymbol{i} + 7\boldsymbol{j},$$
$$\overrightarrow{CD} = (p - 8)\boldsymbol{i} - 8\boldsymbol{j},$$

We have,

$$\overrightarrow{OB} \cdot \overrightarrow{CD} = 0$$
$$(-\boldsymbol{i} + 7\boldsymbol{j}) \cdot ((p - 8)\boldsymbol{i} - 8\boldsymbol{j}) = 0$$
$$(-1)(p - 8) + (7)(-8) = 0$$
$$-p + 8 - 56 = 0$$
$$p = -48.$$

c Since

$$\overrightarrow{AD} = (p - 4)\boldsymbol{i} - 4\boldsymbol{j},$$

we have,

$$|\overrightarrow{AD}| = \sqrt{17}$$
$$\sqrt{(p - 4)^2 + (-4)^2} = \sqrt{17}$$
$$(p - 4)^2 + 16 = 17$$
$$(p - 4)^2 = 1$$
$$p - 4 = \pm 1$$
$$p = 3, 5.$$

11 a Speed$= \sqrt{16^2 + 9^2} \approx 18.36$ m/s

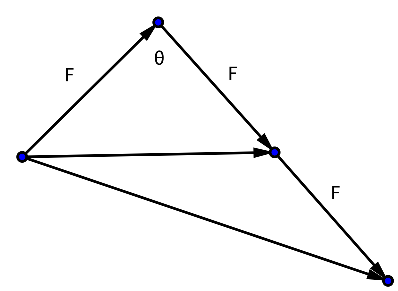
Angle to the bank Bearing

$= \tan^{-1}\left(\frac{16}{9}\right) \approx 60.646°$

b $\frac{136}{16} = 8.5$ seconds

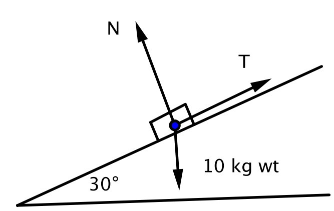
c Distance down the river$= 9 \times 8.5 = 765$ m

12



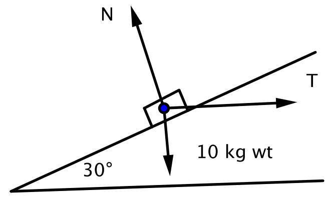
In the 'top triangle'
$36 = 2F^2 - 2F^2 \cos\theta \ldots (1)$
In the 'large triangle'
$121 = 5F^2 - 4F^2 \cos\theta \ldots (2)$
Multiply (1) by 2 and subtract from (2)
$49 = F^2$
$F = 7$
Substitute in (1)
$36 = 98 - 98 \cos\theta$
$\cos\theta = -\frac{31}{49}$

13 a



Resolve parallel to the plane.
$T = 10 \cos 60° = 5$
The tension in the string is 5 kg wt
$N = 10 \sin 60° = 5\sqrt{3}$ kg wt

b

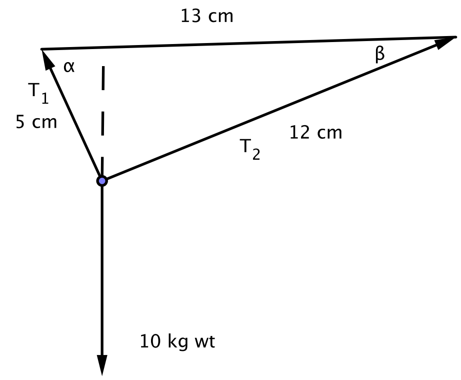


Resolve parallel to the plane.

$T\cos 30^\circ = 10\cos 60^\circ$

$T = \dfrac{10}{\sqrt{3}} = \dfrac{10\sqrt{3}}{3}$ $N = \dfrac{20\sqrt{3}}{3}$

14



The triangle has sides 5 cm, 12 cm and 13cm and is therefore a right-angled triangle.

$\cos\alpha = \dfrac{5}{13}$ and $\cos\beta = \dfrac{12}{13}$

Resolving vertically

$T_2\cos\alpha + T_1\cos\beta = 10\ldots(1)$

Resolving horizontally

$T_2\cos\beta = T_1\cos\alpha\ldots(2)$

$\therefore 12T_1 + 5T_2 = 130$ and $5T_1 - 12T_2 = 0$

$\therefore T_1 = \dfrac{120}{13}$ kg wt and $T_2 = \dfrac{50}{13}$ kg wt

Solutions to multiple-choice questions

1 A We can think of this as a translation of (a, b) to the line $x = m$ by translating the point by $m - a$ units in the x-direction, then a further $m - a$ units in the x-direction. The x-coordinate will then be

$$a + (m - a) + (m - a) = 2m - a.$$

the y-coordinate is unchanged.

2 E If the line $x + y = 4$ is dilated from the y-axis by a factor of $\frac{1}{2}$ its new equation will be $y + 2x = 4$. We reflect its intercepts $(0, 4)$ and $(2, 0)$ in the line $x = 4$ to the points $(8, 4)$ and $(6, 0)$ respectively. The straight line through these two points has equation $y = 2x - 12$.

3 B The required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 3 \\ -(y + 2) \end{bmatrix}.$$

Therefore, $x' = x + 3$ and $y' = -y - 2$. Solving for x and y gives,

$$x = x' - 3 \text{ and } y = -y' - 2$$

so that $y = x^2$ becomes $-y' - 2 = (x' - 3)^2$. Solving for y' gives,

$$y' = -(x' - 3)^2 - 2.$$

Deleting the dash symbols leaves $y = -(x - 3)^2 - 2$, which corresponds to item B.

4 C The required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{x}{3} \\ 2y \end{bmatrix}.$$

Therefore, $x' = \frac{x}{3}$ and $y' = 2y$. Solving for x and y gives,

$$x = 3x' \text{ and } y = \frac{y'}{2}$$

so that $y = 2^x$ becomes $\frac{y'}{2} = 2^{3x'}$. Solving for y' gives,

$$y' = 2 \times 2^{3x'}.$$

Deleting the dash symbols leaves $y = 2 \times 2^{3x}$.

5 D A reflection in the line $x = 2$ is given by the rule

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 - x \\ y \end{bmatrix}.$$

If we then perform the translation we obtain the transformation,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 - x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 - x \\ y + 3 \end{bmatrix}$$

6 D The matrix of the transformation is

$$B = \begin{bmatrix} 4 & 3 \\ 4 & 5 \end{bmatrix}.$$

The inverse transformation will have matrix

$$B^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{4(4) - (5)(3)} \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix},$$

which corresponds to item D

7 D A rotation by 35° clockwise then 15° anticlockwise is a rotation by 20° clockwise. The has transformation matrix,

$$\begin{bmatrix} \cos(-20)^\circ & -\sin(-20)^\circ \\ \sin(-20)^\circ & \cos(-20)^\circ \end{bmatrix} = \begin{bmatrix} \cos 20^\circ & \sin 20^\circ \\ -\sin 20^\circ & \cos 20^\circ \end{bmatrix}$$

8 B An anticlockwise rotation by angle θ is given by the matrix,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This cannot be a reflection because of the location of the negative entry.

9 B Since

$$|\boldsymbol{a}| = \sqrt{3^2 + 4^2} = 5,$$

the unit vector will be

$$\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j}).$$

10 D

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) - (2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}) \\ &= \boldsymbol{i} + 8\boldsymbol{j} \end{aligned}$$

11 B

$$\begin{aligned} \boldsymbol{a} - \boldsymbol{b} &= (2\boldsymbol{i} + 4\boldsymbol{j}) - (3\boldsymbol{i} - 2\boldsymbol{j}) \\ &= -\boldsymbol{i} + 6\boldsymbol{j} \end{aligned}$$

12 A $|\boldsymbol{a}| = \sqrt{2^2 + (-1)^2 + (4)^2} = \sqrt{21}$

13 B

$$\begin{aligned} \overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BO} + \overrightarrow{OC} + \overrightarrow{CD} \\ &= \boldsymbol{c} + -\boldsymbol{b} + \boldsymbol{c} + -\boldsymbol{b} \\ &= 2\boldsymbol{c} - 2\boldsymbol{b} \\ &= 2(\boldsymbol{c} - \boldsymbol{b}) \end{aligned}$$

14 D

$$\begin{aligned} 2\boldsymbol{r} - \boldsymbol{s} &= 2(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) - (-\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) \\ &= 5\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k} \end{aligned}$$

15 A

16 B Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel if

$$\begin{aligned} \boldsymbol{u} &= c\boldsymbol{v} \\ \boldsymbol{i} + a\boldsymbol{j} - 5\boldsymbol{k} &= cb\boldsymbol{i} - 3c\boldsymbol{j} + 6c\boldsymbol{k} \end{aligned}$$

Equating coefficients gives $c = -\frac{5}{6}$ and

$$a = -3c = -3 \times -\frac{5}{6} = \frac{5}{2},$$

$$b = 1 \div c = -\frac{6}{5}.$$

17 C

$$\begin{aligned} \boldsymbol{x} &= s\boldsymbol{a} + t\boldsymbol{b} \\ \boldsymbol{i} + 5\boldsymbol{j} &= 3s\boldsymbol{i} + 4s\boldsymbol{j} + 2t\boldsymbol{i} - t\boldsymbol{j} \\ \boldsymbol{i} + 5\boldsymbol{j} &= (3s + 2t)\boldsymbol{i} + (4s - t)\boldsymbol{j} \end{aligned}$$

Therefore, $3s + 2t = 1$ and $4s - t = 5$. Solving these simultaneous equations gives,

$$s = 1, t = -1.$$

18 B

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OC} + \overrightarrow{CB} \\ &= -\boldsymbol{a} + \boldsymbol{c} + \frac{1}{3}\boldsymbol{a} \\ &= \boldsymbol{c} - \frac{2}{3}\boldsymbol{a} \end{aligned}$$

19 B

$$\begin{aligned} \boldsymbol{c} &= \overrightarrow{OC} \\ &= \overrightarrow{OB} + \overrightarrow{BA} + \overrightarrow{AC} \\ &= \boldsymbol{b} + (\boldsymbol{a} - \boldsymbol{b}) + 2(\boldsymbol{a} - \boldsymbol{b}) \\ &= \boldsymbol{b} + 3(\boldsymbol{a} - \boldsymbol{b}) \\ &= 3\boldsymbol{a} - 2\boldsymbol{b} \end{aligned}$$

20 C $20 - (-20) + 2$

21 A Resolve perpendicular to F_2. The angle between F_1 and F_2 extended back is $100 + 120 - 180 = 40^\circ$.

$$F_1 \sin 40^\circ - 8 \sin 60^\circ = 0$$

$$F_1 = \frac{8 \sin 60^\circ}{\sin 40^\circ}$$

$$\approx 10.78 \text{ kg wt}$$

$$F_2 \sin 40^\circ - 8 \sin 80^\circ = 0$$

$$F_2 = \frac{8 \sin 80^\circ}{\sin 40^\circ}$$

$$\approx 12.26 \text{ kg wt}$$

22 **D** Resolve perpendicular to F_1.
The angle between F_2 and F_1 extended back is $100 + 120 - 180 = 40^\circ$.
The angle between the 8 kg wt force and F_1 extended back is $120 - 40 = 80^\circ$.

23 **B** Resolve perpendicular to the plane.

$$N - 10 \cos 25^\circ = 0$$

$$N = 10 \cos 25^\circ$$

$$\approx 9.06 \text{ kg wt}$$

24 **A** Resolve parallel to the plane.

$$F - 10 \sin 25^\circ = 0$$

$$F = 10 \sin 25^\circ$$

$$\approx 4.23 \text{ kg wt}$$

Solutions to extended-response questions

1 a This is a translation 6 units to the right and 3 units up, so the transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} x + 6 \\ y + 3 \end{bmatrix}$$

b

c

d The required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} x - 3 \\ 2y + 2 \end{bmatrix}$$

Therefore, $x' = x - 3$ and $y' = 2y + 2$. Solving for x and y gives,

$$x = x' + 3 \text{ and } y = \frac{y' - 2}{2}.$$

Substituting these into the equation $y = x^2$ gives,

$$\frac{y' - 2}{2} = (x' + 3)^2$$
$$y' - 2 = 2(x' + 3)^2$$
$$y' = 2(x' + 3)^2 + 2$$

so that the image has equation $y = 2(x + 3)^2 + 2$.

e This function can be obtained by a sequence of 3 transformations:

- a dilation by a factor of 2 from the x-axis then,
- a reflection in the x-axis then,
- a translation by the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

The required transformation has rule,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} x + 3 \\ -2y + 4 \end{bmatrix}.$$

2 a The matrix corresponds to a rotation by angle,

$$\theta = \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{4}{5}\right).$$

b i The circle has centre (0, 1) and radius 1. Therefore its equation is

$$x^2 + (y - 1)^2 = 1 \qquad (1)$$

ii The rotation will change the centre of the circle, but not its radius. To find the image of the centre, we evaluate,

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}.$$

Therefore, the circle has centre $\left(-\frac{4}{5}, \frac{3}{5}\right)$ and equation,

$$\left(x + \frac{4}{5}\right)^2 + \left(y - \frac{3}{5}\right)^2 = 1 \qquad (2)$$

c Expanding and simplifying equations (1) and (2) gives,

$$x^2 + y^2 - 2y = 0 \qquad (3)$$

$$x^2 + \frac{8x}{5} + y^2 - \frac{6y}{5} = 0 \qquad (4)$$

Subtract (4) from (3) to give $y = -2x$. Substitute $y = -2x$ equation (3) to obtain

$$\begin{aligned} x^2 + (4x^2) + 4x &= 0 \\ 5x^2 + 4x &= 0 \\ x(5x + 4) &= 0 \\ x &= 0, -\frac{4}{5} \\ y &= 0, \frac{8}{5}. \end{aligned}$$

so that $(0, 0)$ and $\left(-\frac{4}{5}, \frac{8}{5}\right)$ are the required points.

3 a

$$R = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

b The inverse matrix will simply be a rotation matrix by $\frac{\pi}{4}$ in the clockwise direction,

$$R^{-1} = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

c The point $A(a, b)$ is rotated by $\frac{\pi}{4}$ anticlockwise to the point $A'(1, 1)$. Since OA' is at an angle $\frac{\pi}{4}$ to the x-axis, the original point A must be on the x-axis. Moreover, since

$$OA' = \sqrt{1^2 + 1^2} = \sqrt{2}$$

, we know that $OA = \sqrt{2}$. Therefore the required coordinates are $A(\sqrt{2}, 0)$.

d

$$R\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = R^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

e i

$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = R^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \\ -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{bmatrix}$$

ii Since

$$x = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

$$y = -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

The image of $y = x^2$ is

$$-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' = (\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y')^2$$

$$\sqrt{2}(y' - x') = (x' + y')^2.$$

Therefore the image has equation $\sqrt{2}(y - x) = (x + y)^2$.

4 a

b The acute angle between the x-axis and $y = x$ is $\theta_1 = \frac{\pi}{4}$. The acute angle between the x-axis and the line $y = 2x$ is $\theta_2 = \tan^{-1} 2$. The acute angle between the two lines will then be

$$\theta = \tan^{-1} 2 - \frac{\pi}{4}.$$

That is $a = 2$ and $b = \frac{\pi}{4}$.

c We need to evaluate $\cos\theta$ and $\sin\theta$. We have

$$\begin{aligned}\cos\theta &= \cos(\tan^{-1} 2 - \frac{\pi}{4})\\ &= \cos(\tan^{-1} 2)\cos\frac{\pi}{4} + \sin(\tan^{-1} 2)\sin\frac{\pi}{4}\\ &= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}}\frac{1}{\sqrt{2}}\\ &= \frac{3}{\sqrt{10}}\end{aligned}$$

Likewise,

$$\sin\theta = \frac{1}{\sqrt{10}}.$$

Therefore, the required matrix is

$$\begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$

5 **a** **i** $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Therefore, $(1, 3) \to (3, 1)$.

ii Reflecting point (a, b) in the line $y = x$ simply switches the $x-$ and y-coordinates. Therefore the images has coordinates $A'(3, 1), B'(5, 1), C'(3, 3)$.

iii

b **i** Since $x' = y$ and $y' = x$, the equation $y = x^2 - 2$ simply becomes $x' = (y')^2 - 2$. Ignoring the dashes, gives the equation $x = y^2 - 2$.

ii Substitute $y = x$ into $y = x^2 - 2$ to give

$$\begin{aligned} x^2 - 2 &= x \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x &= -1, 2. \end{aligned}$$

As $y = x$, the coordinates are $(-1, -1)$ and $(2, 2)$

iii Substituting $y = x^2 - 2$ into $x = y^2 - 2$ gives,

$$\begin{aligned} (x^2 - 2)^2 - 2 &= x \\ x^4 - 4x^2 + 4 - 2 &= x \\ x^4 - 4x^2 - x + 2 &= 0 \end{aligned}$$

iv When $x = \frac{1}{2}(-1 + \sqrt{5})$,

$$y = x^2 - 2 = \frac{1}{2}(-1 - \sqrt{5})$$

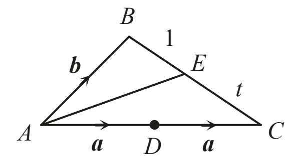
, and when $x = \frac{1}{2}(-1 - \sqrt{5})$,

$$y = x^2 - 2 = \frac{1}{2}(-1 + \sqrt{5})$$

Therefore, the points of intersection are:

$$(-1, -1), (2, 2), (\tfrac{1}{2}(-1 + \sqrt{5}), \tfrac{1}{2}(-1 - \sqrt{5})), (\tfrac{1}{2}(-1 - \sqrt{5}), \tfrac{1}{2}(-1 + \sqrt{5})).$$

6 a $\overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE}$



$$= 2\overrightarrow{AD} + \frac{t}{t+1}$$

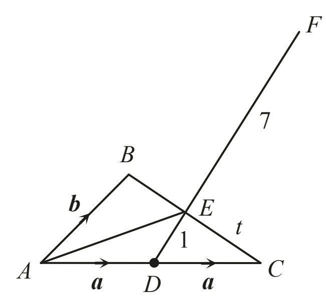
$$= 2\boldsymbol{a} + \frac{t}{t+1}(\boldsymbol{b} - 2\boldsymbol{a})$$

$$= \frac{2(t+1)}{t+1}\boldsymbol{a} + \frac{t}{t+1}\boldsymbol{b} - \frac{2t}{t+1}\boldsymbol{a}$$

$$= \frac{1}{t+1}((2t + 2 - 2t)\boldsymbol{a} + t\boldsymbol{b})$$

$$= \frac{1}{t+1}(2\boldsymbol{a} + t\boldsymbol{b})$$

b $\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$



$$= \boldsymbol{a} + \frac{1}{8}\overrightarrow{DF}$$

$$= \boldsymbol{a} + \frac{1}{8}(\overrightarrow{AF} - \overrightarrow{AD})$$

$$= \boldsymbol{a} + \frac{1}{8}\overrightarrow{AF} - \frac{1}{8}\boldsymbol{a}$$

$$= \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$$

c $\overrightarrow{AE} = \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$

$\therefore\ 8\overrightarrow{AE} = 7\boldsymbol{a} + \overrightarrow{AF}$

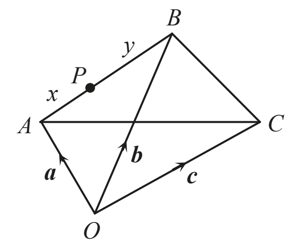
$$\begin{aligned}
\therefore\ \overrightarrow{AF} &= 8\overrightarrow{AE} - 7\boldsymbol{a} \\
&= \frac{8}{t+1}(2\boldsymbol{a} + t\boldsymbol{b}) - 7\boldsymbol{a} \\
&= \frac{1}{t+1}(16\boldsymbol{a} + 8t\boldsymbol{b} - 7(t+1)\boldsymbol{a}) \\
&= \frac{1}{t+1}(16\boldsymbol{a} + 8t\boldsymbol{b} - (7t+7)\boldsymbol{a}) \\
&= \frac{1}{t+1}((9-7t)\boldsymbol{a} + 8t\boldsymbol{b}) \\
&= \frac{9-7t}{1+t}\boldsymbol{a} + \frac{8t}{1+t}\boldsymbol{b}, \text{ as required.}
\end{aligned}$$

d If A, B and F are collinear, then $\overrightarrow{AF} = k\,\overrightarrow{AB}, k > 0$

$$\begin{aligned}
&= k\boldsymbol{b} \\
&= 0\boldsymbol{a} + k\boldsymbol{b}
\end{aligned}$$

$$\begin{aligned}
\therefore\ \frac{9-7t}{1+t} &= 0 \\
\therefore\ 9 - 7t &= 0 \\
\therefore\ t &= \frac{9}{7}
\end{aligned}$$

7 **a** Assume P divides AB in the ratio $x : y$.

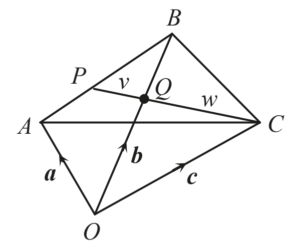


$$\begin{aligned}
\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\
&= \boldsymbol{a} + \frac{x}{x+y}\overrightarrow{AB}\\
&= \boldsymbol{a} + \frac{x}{x+y}(\overrightarrow{OB} - \overrightarrow{OA})\\
&= \frac{x+y}{x+y}\boldsymbol{a} + \frac{x}{x+y}(\boldsymbol{b} - \boldsymbol{a})\\
&= \frac{1}{x+y}((x+y-x)\boldsymbol{a} + x\boldsymbol{b})\\
&= \frac{y}{x+y}\boldsymbol{a} + \frac{x}{x+y}\boldsymbol{b}\\
&= m\boldsymbol{a} + n\boldsymbol{b} \text{ where } m = \frac{y}{x+y},\ n = \frac{x}{x+y}, m, n \geq 0
\end{aligned}$$

$$\begin{aligned}
\text{and } m + n &= \frac{y}{x+y} + \frac{x}{x+y}\\
&= 1, \text{as required.}
\end{aligned}$$

b
$$\begin{aligned}
\overrightarrow{PC} &= -\overrightarrow{AP} - \overrightarrow{OA} + \overrightarrow{OC}\\
&= -n(\boldsymbol{b} - \boldsymbol{a}) - \boldsymbol{a} + \boldsymbol{c}\\
&= -n\boldsymbol{b} + n\boldsymbol{a} - \boldsymbol{a} + \boldsymbol{c}\\
&= (n-1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c}
\end{aligned}$$

c Assume Q divides PC in the ratio $v : w$.

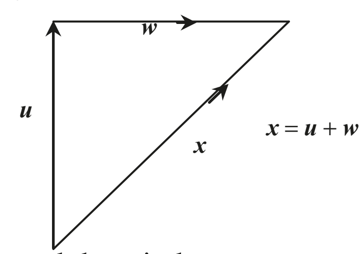


$$\begin{aligned}
\overrightarrow{OQ} &= \overrightarrow{OA} + \overrightarrow{AP} + \overrightarrow{PQ} \\
&= \boldsymbol{a} + n(\boldsymbol{b} - \boldsymbol{a}) + \frac{v}{v+w}\overrightarrow{PC} \\
&= \boldsymbol{a} + n\boldsymbol{b} - n\boldsymbol{a} + \frac{v}{v+w}((n-1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c}) \\
&= \frac{1}{v+w}((v+w)\boldsymbol{a} + n(v+w)\boldsymbol{b} - n(v+w)\boldsymbol{a} + v(n-1)\boldsymbol{a} - nv\boldsymbol{b} + v\boldsymbol{c}) \\
&= \frac{1}{v+w}((v+w-nv-nw+vn-v)\boldsymbol{a} + (nv+nw-nv)\boldsymbol{b} + v\boldsymbol{c}) \\
&= \frac{1}{v+w}((w-nw)\boldsymbol{a} + nw\boldsymbol{b} + v\boldsymbol{c}) \\
&= \frac{w(1-n)}{v+w}\boldsymbol{a} + \frac{nw}{v+w}\boldsymbol{b} + \frac{v}{v+w}\boldsymbol{c} \\
&= \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \gamma\boldsymbol{c}
\end{aligned}$$

where $\lambda = \dfrac{w(1-n)}{v+w}, \mu = \dfrac{nw}{v+w}, \gamma = \dfrac{v}{v+w}, \lambda, \mu, \gamma \geq 0$

$$\begin{aligned}
\text{and } \lambda + \mu + \gamma &= \frac{w(1-n) + nw + v}{v+w} \\
&= \frac{w - nw + nw + v}{v+w} \\
&= \frac{v+w}{v+w} = 1, \text{ as required.}
\end{aligned}$$

8 **a** Let $\boldsymbol{x}$ be the (proper) velocity of the wind relative to a stationary object.
Let $\boldsymbol{u}$ be the man's velocity, 4 km in a northerly direction.
Let $\boldsymbol{w}$ be the apparent velocity of the wind.

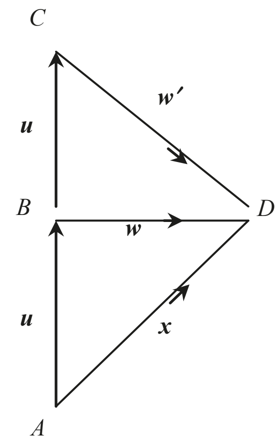


When the man doubles his speed the wind appears to come from the north west.

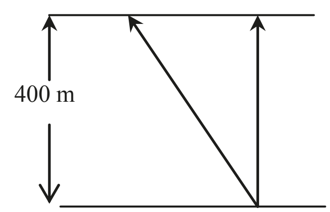
Let $\boldsymbol{w}'$ be the new apparent velocity of the wind.
The new velocity is $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$.
The second vector diagram is superimposed on the first.
The vertices are labelled to describe the triangles.
The triangle BCD is isosceles as $\angle CBD$ is a right angle and $\angle BCD = 45°$.
$|\boldsymbol{u}| = 4$ and therefore $|\boldsymbol{w}| = 4$.
By Pythagoras' theorem,

$$|\boldsymbol{x}|^2 = 4^2 + 4^2$$

and so, $|\boldsymbol{x}| = 4\sqrt{2}$
and the direction that it blows from is south west.



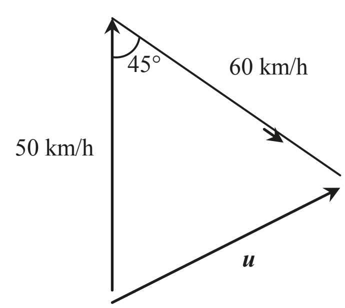
b The northerly component of the swimmer's velocity is 2000 m/h.
The river is 400 m wide.
It takes $\frac{400}{2000} = \frac{1}{5}$ hour to reach the north bank.
The river is flowing from east to west at 1 km/h = 1000 m/h.
Hence in 1hour the swimmer has gone $\frac{1}{5} \times 1000 =$ 200 m downstream.



c Let $\boldsymbol{u}$ be the true velocity of the wind.
The cosine rule can be used to determine the magnitude of the velocity.

$$|\boldsymbol{u}|^2 = 50^2 + 60^2 - 2 \times 50 \times 60 \cos 45°$$
$$= 2500 + 3600 - 6000 \cos 45°$$
$$= 6100 - 3000\sqrt{2}$$

So $|\boldsymbol{u}| = 43.1$ km/h (correct to one decimal place).



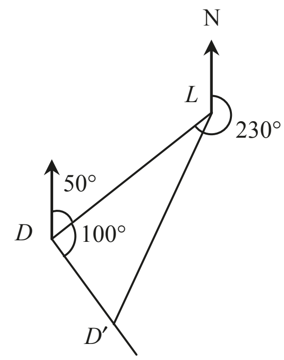
For the direction to be determined the sine rule is used.

$$\frac{60}{\sin\alpha} = \frac{|\boldsymbol{u}|}{\sin 45°}$$
$$\therefore \sin\alpha = \frac{60 \sin 45°}{|\boldsymbol{u}|}$$

Therefore, $\alpha = 79.88°$
The true velocity of the wind is 43.1 km/h blowing at a bearing of 080° (correct to the nearest degree).

d Let $\angle DLD' = \alpha°$.

From the diagram and using the sine rule



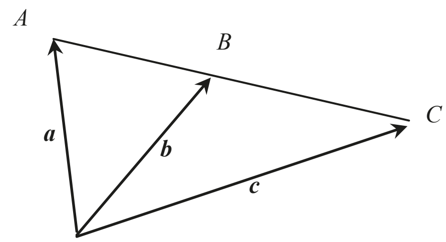
$$\frac{5}{\sin\alpha} = \frac{35}{\sin 100°}$$

$$\therefore\ \sin\alpha = \frac{5\sin 100°}{35}$$

$$\therefore\ \alpha \approx 8.1°$$

This represents a bearing of $230° - 8.1° = 221.9°$, or $222°$ correct to the nearest degree.

9 **a** $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}, \overrightarrow{OC} = \boldsymbol{c}$



Since A, B and C are collinear,

$\overrightarrow{AC} = k\overrightarrow{AB}$.

$$\begin{aligned}\boldsymbol{c} &= \overrightarrow{OC}\\ &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \overrightarrow{OA} + k\overrightarrow{AB}\\ &= \overrightarrow{OA} + k(\overrightarrow{AO} + \overrightarrow{OB})\\ &= (1-k)\boldsymbol{a} + k\boldsymbol{b}\end{aligned}$$

So if $\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b}$, and O does not lie on the line ABC, then $\alpha = 1 - k$ and $\beta = k$ and $\alpha + \beta = 1$.

b **i** Let N be the midpoint of YZ.

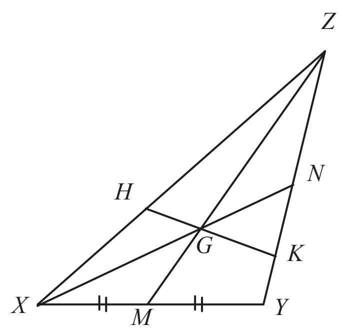
As G lies on ZM,

$\overrightarrow{ZG} = k\overrightarrow{ZM}$ for some non-zero real number k.

Similarly,

$\overrightarrow{XG} = l\overrightarrow{XN}$ for some non-zero real number l.

$\overrightarrow{ZG}$ will be found in two different ways to obtain simultaneous equations in k and l.



$$\begin{aligned}
\overrightarrow{ZG} &= k\overrightarrow{ZM} \\
&= k(\overrightarrow{ZX} + \overrightarrow{XM}) \\
&= k\overrightarrow{ZX} + \frac{1}{2}k\overrightarrow{XY} \\
&= k\overrightarrow{ZX} + \frac{1}{2}k(\overrightarrow{XZ} + \overrightarrow{ZY}) \\
&= \frac{1}{2}k\overrightarrow{ZX} + \frac{1}{2}k\overrightarrow{ZY}
\end{aligned}$$

$$\begin{aligned}
\text{Also } \overrightarrow{ZG} &= \overrightarrow{ZX} + \overrightarrow{XG} \\
&= \overrightarrow{ZX} + l\overrightarrow{XN} \\
&= \overrightarrow{ZX} + l(\overrightarrow{XZ} + \overrightarrow{ZN}) \\
&= \overrightarrow{ZX} + l\left(-\overrightarrow{ZX} + \frac{1}{2}\overrightarrow{ZY}\right) \\
&= (1 - l)\overrightarrow{ZX} + \frac{1}{2}l\overrightarrow{ZY}
\end{aligned}$$

Thus $\overrightarrow{ZG} = \frac{1}{2}k\overrightarrow{ZX} + \frac{1}{2}k\overrightarrow{ZY} = (1 - l)\overrightarrow{ZX} + \frac{1}{2}l\overrightarrow{ZY}$

is not parallel, to $\overrightarrow{ZY}$

Hence equating coefficients,

$\frac{1}{2}k = 1 - l$ and $\frac{1}{2}k = \frac{1}{2}l$

$\therefore\ l = k$ and $\frac{1}{2}k = 1 - k$

$$\therefore\ k = l = \frac{2}{3}$$

Thus $\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM}$.

ii
$$\begin{aligned}
\overrightarrow{ZG} &= \frac{2}{3}\overrightarrow{ZM} \\
&= \frac{2}{3}(\overrightarrow{ZX} + \overrightarrow{XM}) \\
&= \frac{2}{3}\overrightarrow{ZX} + \frac{1}{3}\overrightarrow{XY} \\
&= \frac{2}{3}\overrightarrow{ZX} + \frac{1}{3}(\overrightarrow{XZ} + \overrightarrow{ZY}) \\
\therefore\ \overrightarrow{ZG} &= \frac{2}{3}\overrightarrow{ZX} - \frac{1}{3}\overrightarrow{ZX} + \frac{1}{3}\overrightarrow{ZY} \\
&= \frac{1}{3}\overrightarrow{ZX} + \frac{1}{3}\overrightarrow{ZY}
\end{aligned}$$

But $\overrightarrow{ZH} = h\overrightarrow{ZX}, \overrightarrow{ZK} = k\overrightarrow{ZY}$

So $\overrightarrow{ZX} = \frac{1}{h}\overrightarrow{ZH}$ and $\overrightarrow{ZY} = \frac{1}{k}\overrightarrow{ZK}$

$$\text{So } \overrightarrow{ZG} = \frac{1}{3h}\overrightarrow{ZH} + \frac{1}{3k}\overrightarrow{ZK}$$

iii Since H, G and K are collinear and $\overrightarrow{ZG} = \frac{1}{3h}\overrightarrow{ZH} + \frac{1}{3k}\overrightarrow{ZK}$, from part **a**

$\frac{1}{3h} + \frac{1}{3k} = 1$

and $\frac{1}{h} + \frac{1}{k} = 3$

iv If $h = k$ then $\frac{2}{h} = 3$

Hence $h = k = \frac{2}{3}$

This means $\frac{ZH}{ZX} = \frac{ZK}{ZY}$ and so triangles ZHK and ZXY are similar triangles and HK is parallel to XY.

(Also HK is parallel to XY implies $h = k = \frac{2}{3}$.)

v If $h = k$ then $h = k = \frac{2}{3}$. Triangles ZHK and ZXY are similar.

$\therefore$ Area of$\triangle ZHK = \frac{4}{9}$(Area of$\triangle ZXY$)

$= \frac{4}{9}$ cm ^{2}

vi If $k = 2h$,

then $\frac{1}{h} + \frac{1}{2h} = 3$

$\therefore \frac{3}{2h} = 3$

and $h = \frac{1}{2}$

Thus $ZH = \frac{1}{2}ZX$ and H is the midpoint of ZX. This means that HG is the median and in this case K coincides Y.

vii If H lies on the line segment ZX and K lies on the line segment ZY, then $0 \le h \le 1$ and $0 \le k \le 1$.

Now $\quad \dfrac{1}{h} + \dfrac{1}{k} = 3 \qquad \ldots ①$

so $\quad \dfrac{1}{h} = 3 - \dfrac{1}{k}$

$\therefore \quad \dfrac{1}{h} = \dfrac{3k - 1}{k}$

$\therefore \quad h = \dfrac{k}{3k - 1} \qquad \ldots ②$

From ①, $\quad \dfrac{1}{k} = 3 - \dfrac{1}{h}$

$\quad = \dfrac{3h - 1}{h}$

$\therefore \quad k = \dfrac{h}{3h - 1} \qquad \ldots ③$

Now $\quad 0 < h \le 1$

$\therefore$ from ②, $\quad 0 < \dfrac{k}{3k - 1} \le 1$

$\therefore \quad 0 < k \le 3k - 1$

Consider $\quad 3k - 1 \ge k$

$\therefore \quad 2k \ge 1$

$\therefore \quad k \ge \dfrac{1}{2}$

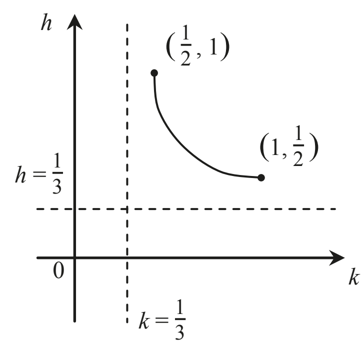
Hence $\quad \dfrac{1}{2} \le k \le 1$

Similarly $\quad 0 < k \le 1$

$\therefore$ from ③, $\quad 0 < \dfrac{h}{3h - 1} \le 1$

$\therefore \quad h \ge \dfrac{1}{2}$

Hence $\quad \dfrac{1}{2} \le h \le 1$



The graph of h against k is part of a hyperbola as shown.

viii Let the area of $\triangle XYZ$ be 1 cm ^{2}.

Then, as $\triangle ZKX$ and $\triangle XYZ$ have bases along ZY and have the same height,

$$\frac{\text{area of } \triangle ZKX}{\text{area of } \triangle ZYX} = \frac{ZK}{ZY}$$

$$\begin{aligned}\therefore \text{ area of } \triangle ZKX &= \frac{ZK}{ZY} \times \text{ area of } \triangle XYZ \\ &= \frac{kZY}{ZY} \times 1, \text{ since } ZK = kZY \\ &= k\end{aligned}$$

Also, as $\triangle ZHK$ and $\triangle ZKX$ have bases along ZX and have the same height,

$$\begin{aligned}\frac{\text{area of } \triangle ZHK}{\text{area of } \triangle ZKX} &= \frac{ZH}{ZX} \\ &= \frac{hZX}{ZX}, \text{ since } ZH = hZX \\ &= h\end{aligned}$$

$$\therefore \text{ area of } \triangle ZHK = h \times \text{ area of } \triangle ZKX$$

$$\therefore A = h\,k$$

Using equation ② in part **vii**,

$$\begin{aligned}A &= \frac{k}{3k-1} \times k \\ &= \frac{k^2}{3k-1}\end{aligned}$$

Now, using long division, or the propFrac command of a CAS calculator, A can be expressed as

$$A = \frac{1}{3}k + \frac{1}{9} + \frac{1}{9(3k-1)}$$

Thus the graph of A against k has an asymptote with equation $A = \frac{1}{3}k + \frac{1}{9}$.

In part **vii** it was established that $k \in \left[\frac{1}{2}, 1\right]$.

Using a CAS calculator, the minimum is at $k = \frac{2}{3}$ and then $A = \frac{4}{9}$ which appears to be $\left(\frac{2}{3}, \frac{4}{9}\right)$

To check this algebraically, first note that for $k > \frac{1}{3}$, $3k - 1 > 0$, so $A = \frac{k^2}{3k-1}$ is always positive.

Also $\left(k - \frac{2}{3}\right)^2 \geq 0$

$\therefore \quad k^2 - \frac{4}{3}k + \frac{4}{9} \geq 0$

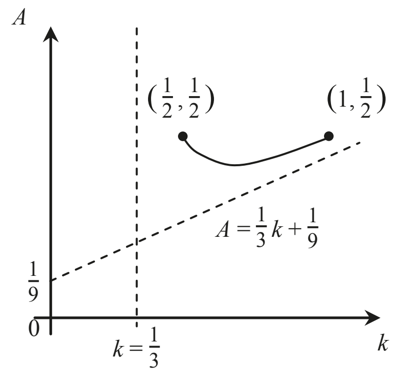
$\therefore \quad k^2 \geq \frac{4}{3}k - \frac{4}{9}$

$\therefore \quad k^2 \geq \frac{4}{9}(3k - 1)$

Now $A = \dfrac{k^2}{3k - 1}$ and so

$$A \geq \frac{\frac{4}{9}(3k - 1)}{3k - 1}$$

$\therefore \quad A \geq \frac{4}{9}$



Chapter 23 – Kinematics

Solutions to Exercise 23A

1 a When $t = 0, x = 12$.
12 cm to the right of O

b When $t = 5, x = 5^2 - 7 \times 5 + 12$
$= 2$
2 cm to the right of O

c $v = \dfrac{dx}{dt}$
$= 2t - 7$
When $t = 0, v = -7$.
7cm/s to the left

d $v = 0$ when $2t - 7 = 0$
$t = 3.5$
When $t = 3.5$,
$x = 3.5^2 - 7 \times 3.5 + 12$
$= -0.25$
$t = 3.5$; the particle is 0.25 cm to the left of O.

e Average velocity
$= \dfrac{\text{change in position}}{\text{change in time}}$
$= \dfrac{2 - 12}{5}$
$= -2$ cm/s

f Average speed $= \dfrac{\text{distance travelled}}{\text{change in time}}$
For the first 3.5 s, the particle has travelled 12.25 cm.
From 3.5 s to 5 s, the particle has travelled $2 - (-0.25) = 2.25$ cm.

Average speed $= \dfrac{12.25 + 2.25}{5}$
$= \dfrac{14.5}{5}$
$= 2.9$ cm/s

2 a $v = \dfrac{dx}{dt}$
$= 2t - 7$
$v = 0$ when $2t - 7 = 0$
$t = 3.5$ s

b $a = \dfrac{dv}{dt}$
$= 2$ m/s^2

c When $t = 0, x = 10$.
When $t = 3.5, x = 3.5^2 - 7 \times 3.5 + 10$
$= -2.25$
For the first 3.5 s, the particle has travelled 12.25 m.
When $t = 5, x = 5^2 - 7 \times 5 + 10$
$= 0$
From 3.5 s to 5 s, the particle has travelled 2.25 m.
Distance travelled $= 12.25 + 2.25$
$= 14.5$ m

d $v = 2t - 7 = -2$
$2t = 5$
$t = 2.5$
$x = 2.5^2 - 7 \times 2.5 + 19$
$= -1.25$
After 2.5 s, when the particle is 1.25 m left of O.

3 **a** When $t = 0, x = -3$.

$v = \frac{dx}{dt}$

$= 3t^2 - 22t + 24$

When $t = 0, v = 24$.

3 cm to the left of O and moving at 24 cm/s to the right.

b $v = \frac{dx}{dt}$

$= 3t^2 - 22t + 24$

c $v = 0$ when

$3t^2 - 22t + 24 = 0$

$(3t - 4)(t - 6) = 0$

$t = \frac{4}{3}$ or 6

After $\frac{4}{3}$ s and after 6 s

d When $t = \frac{4}{3}$,

$x = \left(\frac{4}{3}\right)^3 - 11 \times \left(\frac{4}{3}\right)^2 + 24 \times \left(\frac{4}{3}\right) - 3$

$= \frac{64}{27} - \frac{176}{9} \times \frac{3}{3} + 32 - 3$

$= -\frac{464}{27} + 29$

$= 11\,\frac{22}{27}$

When $t = 6$,

$x = 6^3 - 11 \times 6^2 \times 6 - 3$

$= -39$

39 cm to the left of O and $11\,\frac{22}{27}$ cm to the right of O

e $v < 0$ when $(3t - 4)(t - 6) = 0$

This is a parabola with a minimum value.

$\therefore \quad v < 0$ when $\frac{4}{3} < t < 6$

Length of time $= 6 - \frac{4}{3}$

$= \frac{14}{3}$

$= 4\frac{2}{3}$ s

f $a = \frac{dv}{dt}$

$= 6t - 22$ m/s^2

g $6t - 22 = 0$

$t = \frac{22}{6} = \frac{11}{3}$

$v = 3t^2 - 22t + 24$

$= 3 \times \left(\frac{11}{3}\right)^2 - 22 \times \frac{11}{3} + 24$

$= \frac{121}{3} - \frac{242}{3} + 24$

$= 16\,\frac{2}{3}$

$x = \left(\frac{11}{3}\right)^3$

$= 11 \times \left(\frac{11}{3}\right)^2 + 24 \times \frac{11}{3} - 3$

$= \frac{1331}{27} - \frac{1331}{9} \times \frac{3}{3} + 88 - 3$

$= -13\,\frac{16}{27}$

The acceleration is zero after $\frac{11}{3}$ s, when the velocity is $16\,\frac{1}{3}$ cm/s to the left and its position is $13\,\frac{16}{27}$ cm left of O.

4 **a** $v = 6t^2 - 10t + 4$

When $v = 0$:

$$6t^2 - 10t + 4 = 0$$
$$3t^2 - 5t + 2 = 0$$
$$(3t - 2)(t - 1) = 0$$
$$t = \frac{2}{3} \text{ or } 1$$

$a = 12t - 10$

$t = \frac{2}{3}$:

$$a = 12 \times \frac{2}{3} - 10$$
$$= -2$$

$t = 1$:

$$a = 12 \times 1 - 10$$
$$= 2$$

Velocity is zero after $\frac{2}{3}$ s when the acceleration is 2 cm/s^2 to the left, and after 1 s when the acceleration is 2 cm/s^2 to the right.

b $a = 12t - 10$

$$= 0$$
$$t = \frac{10}{12} = \frac{5}{6}$$

Find v when $a = \frac{5}{6}$:

$v = 6t^2 - 10t + 4$

$$= 6 \times \left(\frac{5}{6}\right)^2 - 10 \times \frac{5}{5} + 4$$
$$= \frac{25}{6} - \frac{50}{6} + 4 = -\frac{1}{6}$$

Acceleration is zero after $\frac{5}{6}$ s, at which time the velocity is $\frac{1}{6}$ cm/s to the left.

5 The particle passes through O when $x = 0$.

$t^3 - 13t^2 + 46t - 48 = 0$

Trial and error will give $x = 0$ when $t = 2$.

This means $(t - 2)$ is a factor of $t^3 - 13t^2 + 46t - 48$.

$t^3 - 13t^2 + 46t - 48$

$$= (t - 2)(t^2 - 11t + 24)$$
$$= 0$$

Factorising the quadratic gives

$(t - 2)(t - 3)(t - 8) = 0$

$$t = 2, 3 \text{ or } 8$$
$$v = \frac{dx}{dt}$$
$$= 3t^2 - 26t + 46$$
$$a = \frac{dv}{dt}$$
$$= 6t - 26$$

$t = 2$:

$$v = 3 \times 4 - 26 \times 2 + 46$$
$$= 6 \text{ cm/s}$$
$$a = 6 \times 2 - 26$$
$$= -14 \text{ cm}^2\text{/s}$$

$t = 3$:

$$v = 3 \times 9 - 26 \times 3 + 46$$
$$= -5 \text{ cm/s}$$
$$a = 6 \times 3 - 26$$
$$= -8 \text{ cm}^2\text{/s}$$

$t = 8$:

$$v = 3 \times 64 - 26 \times 8 + 46$$
$$= 30 \text{ cm/s}$$
$$a = 6 \times 8 - 26$$
$$= -22 \text{ cm}^2\text{/s}$$

6 **a** They will be at the same position when

$$t^2 - 2t - 2 = t + 2$$

$$t^2 - 3t - 4 = 0$$

$$(t - 4)(t + 1) = 0$$

$$t = 4 \text{ or } -1$$

After 4 s, or 1 s before the start.

(Note: In some cases, motion is not considered before $t = 0$, and negative values of t may be discarded.)

b The velocities are 1 cm/s and $2t - 2$ cm/s.

$$2t - 2 = 1$$

$$2t = 3$$

$$t = \frac{3}{2}$$

After $\frac{3}{2}$ s.

Solutions to Exercise 23B

1 a $x = 2t^2 - 6t + c$

When $t = 0, x = 0$.

$\therefore\ 0 = 0 - 0 + c$

$c = 0$

$x = 2t^2 - 6t$

b $t = 3$

$x = 2 \times 3^2 - 6 \times 3$

$= 0$

It will be at the origin, O.

c Consider when $v = 0$:

$4t - 6 = 0$

$t = \dfrac{3}{2}$

$x = 2 \times \left(\dfrac{3}{2}\right)^2 - 6 \times \dfrac{3}{2}$

$= -4\dfrac{1}{2}$

The particle will travel $4\dfrac{1}{2}$ cm to the left of the origin and back, for a total of 9 cm.

d Average velocity

$= \dfrac{\text{change in position}}{\text{change in time}}$

$= \dfrac{0}{3} = 0 \text{ cm/s}$

e Average speed $= \dfrac{\text{distance travelled}}{\text{change in time}}$

$= \dfrac{9}{3} = 3 \text{ cm/s}$

2 a $x = t^3 - 4t^2 + 5t + c$

When $t = 0, x = 4$.

$\therefore\ 4 = 0 - 0 + 0 + c$

$c = 4$

$x = t^3 - 4t^2 + 5t + 4$

$a = \dfrac{dv}{dt}$

$= 6t - 8$

b $3t^2 - 8t + 5 = 0$

$(3t - 5)(t - 1) = 0$

$t = \dfrac{5}{3}$ or 1

When $t = \dfrac{5}{3}$,

$x = \left(\dfrac{5}{3}\right)^3 - 4 \times \left(\dfrac{5}{3}\right)^2 + 5 \times \dfrac{5}{3} + 4$

$= \dfrac{125}{27} - \dfrac{100}{9} + \dfrac{25}{3} + 4$

$= 5\dfrac{23}{27}$

When $t = 1$,

$x = 1^3 - 4 \times 1^2 + 5 \times 1 + 4$

$= 6$

c When $t = \dfrac{5}{3}$,

$a = 6 \times \dfrac{5}{3} - 8$

$= 2 \text{ m/s}^2$

When $t = 1$,

$a = 6 \times 1 - 8$

$= -2 \text{ m/s}^2$

3 $v = 10t + c$

$x = 5t^2 + ct + d$

When $t = 2$:

$x = 5 \times 2^2 + 3c + d = 0$

$2c + d = -20$ ①

When $t = 3$:

$x = 5 \times 3^2 + 3c + 2 = 25$

$3c + d = -20$ ②

② − ① : $c = 0$

$d = -20$

$x = 5t^2 - 20$

When $t = 0$, $x = -20$

20 m to the left of O

4 $a = 2t - 3$

$v = t^2 - 3t + c$

When $t = 0$, $v = 3$.

$3 = 0 - 0 + c$

$c = 3$

$v = t^2 - 3t + 3$

$$x = \frac{t^3}{3} - \frac{3t^2}{2} + 3t + d$$

When $t = 0$, $x = 2$.

$2 = 0 - 0 + 0 + d$

$d = 2$

$$x = \frac{t^3}{3} - \frac{3t^2}{2} + 3t + 2$$

When $t = 10$,

$$x = \frac{10^3}{3} - \frac{3 \times 10^2}{2} + 3 \times 10 + 2$$

$$= \frac{2000 - 900}{6} + 32$$

$$= 215\frac{1}{3}\text{ m}$$

$v = t^2 - 3t - 3$

$= 10^2 - 3 \times 10 + 3$

$= 73$ m/s

5 **a** $a = -10$

$v = -10t + c$

When $t = 0$, $v = 25$.

$25 = 0 + c$

$c = 25$

$v = -10t + 25$

b $v = -10t + 25$

$x = -5t^2 + 25t + d$

When $t = 0$, $x = 0$.

(Define the point of projection as $x = 0$, the origin.)

$0 = 0 + 0 + d$

$d = 0$

$x = -5t^2 + 25t$

c Maximum height occurs when $v = 0$.

$v = -10t + 25 = 0$

$$t = \frac{25}{10} = \frac{5}{2}$$

2.5 s after projection

d When $t = 2.5$,

$x - 5t^2 + 25t$

$= -5 \times 2.5^2 + 25 \times 2.5$

$= 31.25$ m

e $x = -5t^2 + 25t = 0$

$-5t(t-5) = 0$

$t = 5$ ($t = 0$ is the start)

6 Define $t = 0$ as the moment the lift passes the 50th floor.

$a = \frac{1}{9}t - \frac{5}{9}$

$v = \frac{1}{18}t^2 - \frac{5}{9}t + c$

$-8 = 0 - 0 + c$

$c = -8$

$v = \frac{1}{18}t^2 - \frac{5}{9} - 8$

$x = \frac{1}{54}t^3 - \frac{5}{18}t^2 - 8t + d$

$50 \times 6 = 0 - 0 - 0 + d$

$d = 300$

$v = 0$ when

$\frac{1}{18}t^2 - \frac{5}{9}t - 8 = 0$

$t^2 - 10t - 8 \times 18 = 0$

$(t-18)(t+8) = 0$

$t = 18$

$x = \frac{1}{54}t^3 - \frac{5}{18}t^2$

$- 8t + 300$

$= \frac{1}{54} \times 18^3 - \frac{5}{18}18^2$

$- 8 \times 18 + 300$

$= 174$

$\frac{174}{6} = 29$

It will stop on the 29th floor.

Solutions to Exercise 23C

1 $s = 30, u = 0, a = 1.5$

$$s = ut + \frac{1}{2}at^2$$

$$30 = \frac{1}{2} \times 1.5 \times t^2$$

$$t^2 = 40$$

$$t = \sqrt{40}$$

$$= 2\sqrt{10} \text{ s}$$

2 $u = 25, v = 0, t = 3$

$$s = \frac{1}{2}(u + v)t$$

$$= \frac{1}{2}(25 + 0) \times 3$$

$$= 37.5 \text{ m}$$

3 a For constant acceleration,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

$$= \frac{27}{9} = 3 \text{ m/s}^2$$

b $u = 30, v = 50, a = 3$

$$v = u + at$$

$$50 = 30 + 3t$$

$$3t = 20$$

$$t = \frac{20}{3} = 6\frac{2}{3} \text{ s}$$

c $s = ut + \frac{1}{2}at^2$

$$= \frac{1}{2} \times 3 \times 15^2$$

$$= 337.5 \text{ m}$$

d $200 \text{ km/h} = 200 \div 3.6$

$$= \frac{500}{9} \text{ m/s}$$

$$u = 0, v = \frac{500}{9}, a = 3$$

$$v = u + at$$

$$\frac{500}{9} = 0 + 3t$$

$$3t = \frac{500}{9}$$

$$t = \frac{500}{27}$$

$$= 18\frac{14}{27} \text{ s}$$

4 a $45 \text{ km/h} = 45 \div 3.6$

$$= 12.5 \text{ m/s}$$

For constant acceleration,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

$$= \frac{12.5}{5} = 2.5 \text{ m/s}^2$$

b $s = ut + \frac{1}{2}at^2$

$$= \frac{1}{2} \times 2.5 \times 5^2$$

$$= 31.25 \text{ m}$$

5 a $90 \text{ km/h} = 90 \div 3.6$

$$= 25 \text{ m/s}$$

$$u = 0, v = 25, a = 0.5$$

$$v = u + at$$

$$25 = 0 + 0.5t$$

$$0.5t = 25$$

$$t = \frac{2.5}{05} = 50 \text{ s}$$

b $s = ut + \frac{1}{2}at^2$

$= \frac{1}{2} \times 0.5 \times 50^2$

$= 625$ m

6 **a** 54 km/h $= 54 \div 3.6$

$= 15$ m/s

$u = 15, a = -0.25, s = 250$

$s = ut + \frac{1}{2}at^2$

$250 = 15t + \frac{1}{2} \times -0.25t^2$

Multiply both sides by 8:

$2000 = 120t - t^2$

$t^2 - 120t + 2000 = 0$

$(t - 20)(t - 100) = 0$

$t = 100$ represents the train changing velocity and retuming to this point.

$\therefore$ $t = 20$ s

b $v = u + at$

$= 15 + -0.25 \times 20$

$= 10$ m/s

$= 10 \times 3.6 = 36$ km/h

7 **a** $v = u + at$

$= 20 + -9.8 \times 4$

$= -19.2$ m/s

b $s = ut + \frac{1}{2}at^2$

$= 20 \times 4 + \frac{1}{2} \times -9.8 \times 4^2$

$= 1.6$ m

8 **a** $v = u + at$

$= -20 + -9.8 \times 4$

$= -59.2$ m/s

b $s = ut + \frac{1}{2}at^2$

$= -20 \times 4 + \frac{1}{2} \times -9.8 \times 4^2$

$= -158.4$ m

9 **a** $u = 49, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 49t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 49t - 4.9t^2$

$0 = 4.9t(10 - t)$

$t = 10$ s

b $u = 49, s = 102.9, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$102.9 = 49t + \frac{1}{2} \times -9.8 \times t^2$

$102.9 = 49t - 4.9t^2$

$0 = 4.9t^2 - 49t + 102.9$

Divide by 4.9:

$t^2 - 10t + 21 = 0$

$(t - 3)(t - 7) = 0$

At both 3 s (going up) and 7 s (going down).

10 **a** $v = u + at$

$= 4.9 - 9.8t$

$= 4.9(1 - 2t)$

b $s = ut + \frac{1}{2}at^2$

$= 4.9t + \frac{1}{2} \times -9.8 \times t^2$

$= 4.9t - 4.9t^2$

$= 4.9t(1 - t)$ m/s

This is his displacement from the initial 3 m height.

$\therefore h = 4.9t(1 - t) + 3$ m

c From part a, the diver's velocity is zero when

$4.9(1 - 2t) = 0$

$t = \frac{1}{2} = 0.5$

The maximum height reached is

$h = 4.9(0.5)(1 - 0.5) + 3$

$= 4.9 \times 0.25 + 3$

$= 4.225$

d The diver reaches the water when $h = 0$, so:

$4.9t(1 - t) + 3 = 0$

$49t - 49t^2 + 30 = 0$

$49t^2 - 49t - 30 = 0$

$(7t + 3)(7t - 10) = 0$

$t = \frac{10}{7}$ s

Since $t > 0$

11 a Maximum height occurs when $v = 0$.

$u = 19.6, a = -9.8, v = 0$

$v = u + at$

$0 = 19.6 - 9.8t$

$t = \frac{19.6}{98} = 2$ s

b $s = ut + \frac{1}{2}at^2$

$= 19.6 \times 2 + \frac{1}{2} \times -9.8 \times 2^2$

$= 19.6$ m

So the maximum height from the foot of the cliff is $19.6 + 24.5 = 44.1$ m.

c $u = 19.6, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 19.6t - 4.9t^2$

$0 = 4.9t(4 - t)$

$t = 4$ s

d $u = 19.6, s = -24.5, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$-24.5 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$-24.5 = 19.6t - 4.9t^2$

$0 = 4.9t^2 - 19.6t - 24.5$

Divide by 4.9:

$t^2 - 4t - 5 = 0$

$(t - 5)(t + 1) = 0$

$t = 5$ s

12 Let the distance between P and Q be x m.

$$u = 20, v = 40, s = x$$

$$v^2 = u^2 + 2as$$

$$1600 = 400 + 2ax$$

$$2ax = 1200$$

$$a = \frac{1200}{2x} = \frac{600}{x}$$

At the halfway mark,

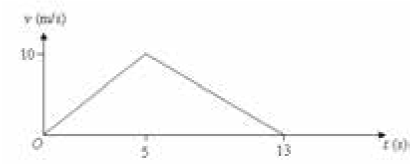
$$u = 20, a = \frac{600}{x}, s = \frac{x}{2}$$

$$v^2 = u^2 + 2as = 400 + 2 \times \frac{600}{x} \times \frac{x}{2} = 1000$$

$$v = \sqrt{1000} = 10\sqrt{10} \text{ m/s}$$

Solutions to Exercise 23D

1 Draw the velocity–time graph.



Distance travelled = area under graph

$$= \frac{1}{2} \times 10 \times 13$$

$$= 65 \text{ m}$$

2 Draw the velocity–time graph.

a The area can be calculated using the trapezium formula, or as the sum of two triangles and a rectangle.

$$A = \frac{1}{2}(a + b)h$$

$$= \frac{1}{2} \times (25 + 50) \times 15$$

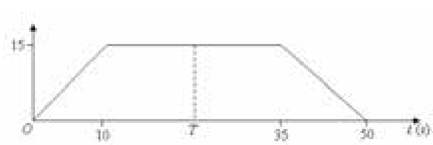
$$= 562.5 \text{ m}$$

b $A = \frac{1}{2}(a + b)h$

$$= \frac{1}{2} \times (25 + 35) \times 15$$

$$= 450 \text{ m}$$

c Let the halfway point be at time T as below.



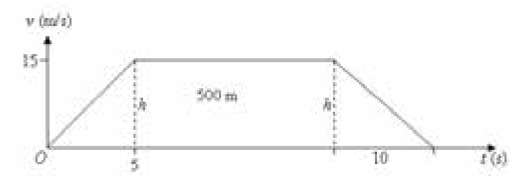
$$\frac{1}{2} \times 10 \times 15 + 15(T - 10) = \frac{562.5}{2}$$

$$75 + 15T - 150 = 281.25$$

$$15T = 356.25$$

$$T = 23.75 \text{ s}$$

3



Since the total distance travelled is 1 km or 1000 m, the combined areas of the two triangles will equal a distance of 500 m.

$$\frac{1}{2} \times 5 \times h + \frac{1}{2} \times 10 \times h = 500$$

$$5h + 10h = 1000$$

$$15h = 1000$$

$$h = \frac{1000}{15}$$

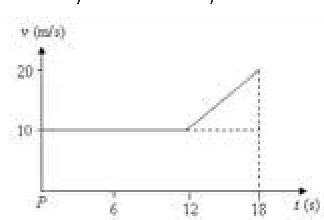
$$= 66\frac{2}{3}$$

Maximum speed $= 66\frac{2}{3}$ m/s

4 36 km/h = 36 ÷ 3.6

= 10 m/s.

72 km/h = 20 m/s.

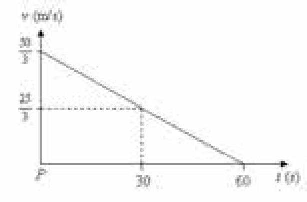


$$\text{Distance} = A = 18 \times 10 + \frac{1}{2} \times 6 \times 10$$

$$= 210 \text{ m}$$

5 60 km/h = 60 ÷ 3.6

$$= \frac{50}{3} \text{ m/s}$$



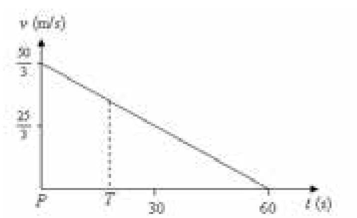
a Distance $= A = \frac{1}{2} \times 60 \times \frac{50}{3}$

$= 500$ m

b Distance $= A = \frac{1}{2} \times \left(\frac{50}{3} + \frac{25}{3}\right) \times 30$

$= 375$ m

c Let the required time be T s.



It is easier to work with the triangle on the right.

This triangle will have area

$= 500 \div 2$

$= 250$

Its base $= (60 - T)$

The sloping line has gradient

$= -\frac{50}{3} \div 60$

$= -\frac{50}{180} = \frac{5}{18}$

$\therefore$ the triangle's height $= \frac{5}{18}(60 - T)$

$$\frac{1}{2} \times (60 - T) \times \frac{5}{18}(60 - T) = 250$$

$$\frac{5}{36}(60 - T)^2 = 250$$

$$(60 - T)^2 = 250 \times \frac{36}{5}$$

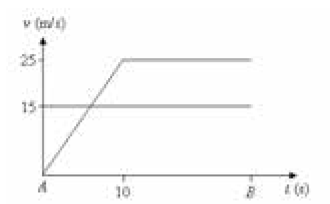
$$= 1800$$

$$60 - T = \sqrt{1800}$$

$$\approx 42.43$$

$$T \approx 17.57 \text{ s}$$

6 Let the common time be T s and the distance x m.



a For the first car, $x = 15t$

For the second car,

$x = \frac{1}{2} \times 10 \times 25 + 25(t - 10)$

$= 125 + 25t - 250$

$= 25t - 125$

$= 15t$

$10t = 125$

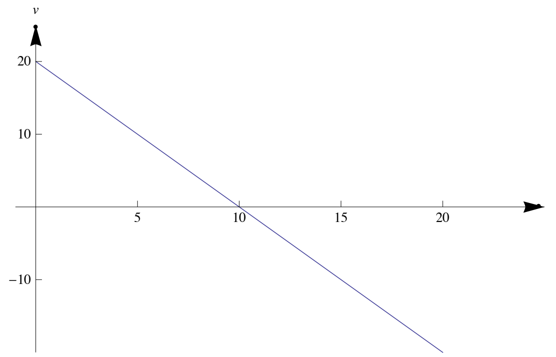
$t = 12.5$ s

b $x = 15t$

$= 15 \times 12.5$

$= 187.5$ m

7 a



b The particle moves to the right for the first 10 seconds. Its position at time t is given by
$s = 20t - t^2$
It slows for the first ten seconds. At time $t = 10$, it is 100 m to the right of its starting point. It then heads to the right for 4 seconds. When $t = 14$ it is 84 m from its starting point.

c Total distance travelled = 100 + 16 = 116m.

d It is 84m to the right of its starting point.

8 a For the first ten seconds of motion

$$\text{acceleration} = \frac{10-0}{10-0} = 1 \text{ m/s}^2$$

b From $t = 20$ to $t = 30$ the

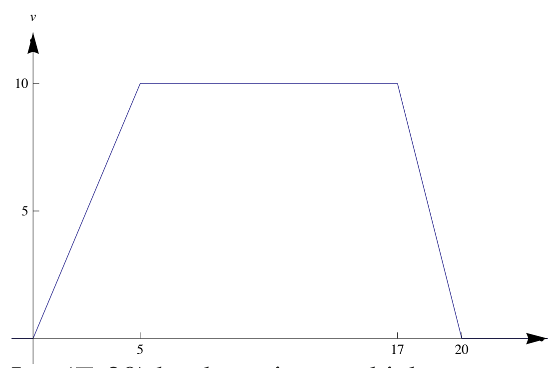
$$\text{acceleration} = \frac{-15-10}{30-20} = -\frac{5}{2} \text{ m/s}^2$$

c The equation of the line through (20, 10) and (30, −15) is
$v - 10 = -\frac{5}{2}(t - 20)$ which can be written as $v = -\frac{5}{2}t + 60$.
When $v = 0, t = 24$

$$\begin{aligned}\text{Distance travelled in the first 24 s} &= 5(10 + 24)\\ &= 170\text{m}\\ \text{Distance travelled in next 6 s} &= 3 \times 15\\ &= 45\text{m}\\ \therefore \text{total distance} &= 45 + 170\\ &= 215\text{m}\end{aligned}$$

d Displacement= 170 − 45 = 125 m to the right of its starting point.

9 a

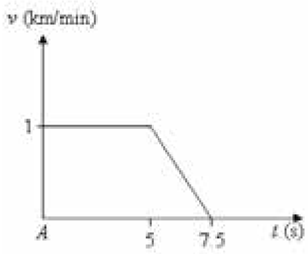


Let $(T, 20)$ be the point at which the constant acceleration ends. The motion ends at $(20, 0)$.
Considering the area of the trapezium:
$5(20 + (T - 5)) = 160$
$\therefore T - 15 = 32$
$\therefore T = 17$

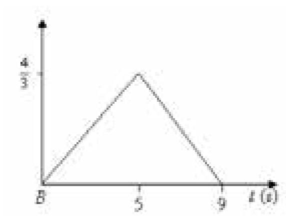
b $\text{acceleration} = \dfrac{10}{17-20} = -\dfrac{10}{3} \text{ m/s}^2$

10 Convert the speeds to km/min.
60 km/h = 1 km/min

80 km/h = $\frac{4}{3}$ km/min
Treat each train separately.
The first train:



The second train:



First train distance

$= 5 \times 1 + \frac{1}{2} \times 2.5 \times 1$

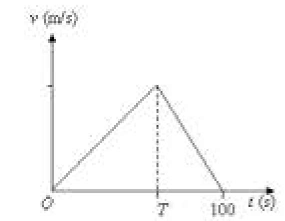
$= 6.25$ km

Second train distance $= \frac{1}{2} \times 9 \times \frac{4}{3}$

$= 6$ km

Since the trains have together travelled less than 14 km, they will not crash.

11 **a** The maximum speed will be the height of the triangle.



$$\frac{1}{2} \times 100 \times h = 800$$

$$50h = 800$$

$$h = 16$$

Maximum speed $= 16$ m/s

$= 16 \times 3.6$

$= 57.6$ km/h

b The slope of the deceleration is twice as steep as the slope of the acceleration.

Since the heights are equal, the acceleration run will be twice as long as the deceleration run.

$$T = \frac{2}{3} \times 100$$

$$= 66\frac{2}{3} \text{ s}$$

$$= 1 \text{ min } 6\frac{2}{3} \text{ seconds}$$

c Taking the acceleration section,

the gradient $= a = 16 \div 66\frac{2}{3}$

$= \frac{48}{200}$

$= 0.24 \text{ m/s}^2$

Solutions to short-answer questions

1 a When $t = 0, x = -5$.
5 cm to the left of O

b When $t = 3, x = 3^2 - 4 \times 3 - 5$
$= -8$
8 cm to the left of O

c $v = \dfrac{dx}{dt}$
$= 2t - 4$
When $t = 0, -4$ cm/s

d $v = 0$ when $2t - 4 = 0$
$t = 2$
When $t = 2, x = 2^2 - 4 \times 2 - 5$
$= -9$
At 2 s, 9 cm to the left of O

e Average velocity
$= \dfrac{\text{change in position}}{\text{change in time}}$
$= \dfrac{-8 - (-5)}{3} = -1$ cm/s
1 cm/s to the left

f Distance travelled = distance from $t = 0$ to $t = 2$ (when $v = 0$), plus distance from $t = 2$ to $t = 3$
So distance travelled $= 4 + 1$
$= 5$ cm
Average speed $= \dfrac{\text{distance travelled}}{\text{change in time}}$
$= \dfrac{5}{3} = 1\dfrac{2}{3}$ cm/s
(Note: Average velocity has a direction and hence a sign, but average speed does not.)

2 a $v = \dfrac{dx}{dt}$
$= 3t^2 - 4t$
$a = \dfrac{dv}{dt} = 6t - 4$
When $t = 0, x = 8, v = 0$ and $a = -4$.
8 cm to the right of O, stationary and accelerating at 4 cm/s^2 to the left.

b $v = 0$ when
$3t^2 - 4t = 0t(3t - 4) \quad = 0$
$t = 0$ or $\dfrac{4}{3}$
$t = 0 : x = 8$ and $a = -4$
So 8 cm to the right, -4 cm/s^2
$t = \dfrac{4}{3} : x = \dfrac{64}{27} - \dfrac{32}{9} + 8 = 6\dfrac{22}{27}$
$a = 8 - 4 = 4$
So $6\dfrac{22}{27}$ cm to the right, 4 cm/s^2

3 a Solve $-2t^3 + 3t^2 + 12t + 7 = 0$
Using factors of 7, $t = -1$ gives
$-2 \times (-1)^3 + 3 \times (-1)^2 + 12 \times -1$
$+7 = 0$
Dividing by $(t + 1)$,
$-2t^3 + 3tg^2 + 12t + 7$
$= -(t + 1)(2t^2 - 5t - 7)$
$= -(t + 1)(t + 1)(2t - 7)$
$= 0$
$t = 3.5$, as $t = -1$ is usually discarded.
$v = \dfrac{dx}{dt}$
$= -6t^2 + 6t + 12$
$a = \dfrac{dv}{dt} = -12t + 6$
When $t = 3.5$

$$v = -6 \times 3.5^2 + 6 \times 3.5 + 12$$
$$= -40.5 \text{ cm/s}$$
$$a = \frac{dv}{dt}$$
$$= -12 \times 3.5 + 6$$
$$= -36 \text{ cm/s}^2$$

b $v = 0$

$$-6t^2 + 6t + 12 = 0$$
$$t^2 - t - 2 = 0$$
$$(t + 1)(t - 2) = 0$$
$$t = 2$$

After 2s (discarding $t = -1$)

c Distance travelled in first 2 seconds

$$= (-2 \times 2^3 + 3 \times 2^2 + 12 \times 2 + 7)$$
$$- (-0 + 0 + 0 + 7)$$
$$= 20 \text{ cm}$$

Distance travelled from $t = 2$ to $t = 3$ is

$$|(-2 \times 3^2 + 3 \times 32 + 12 \times 3 + 7)$$
$$- (-2 \times 2^3 + 3 \times 2^2 + 12 \times 2 + 7)|$$
$$= |16 - 27|$$
$$= 11 \text{ cm}$$

Distance travelled in first 3 s

$$= 20 + 11$$
$$= 31 \text{ cm}$$

4 a i $x_1\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2$

$$= \frac{1}{8} - \frac{1}{4}$$
$$= -\frac{1}{8}$$

$\frac{1}{8}$ cm to the left

ii $a_1(t) = \frac{d^2x}{dt^2}$

$$= 6t - 2$$
$$a_1\left(\frac{1}{2}\right) = 6 \times \frac{1}{2} - 2$$
$$= 1 \text{ cm/s}^2$$

iii $v_2(t) = \frac{dx}{dt} = 2t$

$$v_2 = 2 \times \frac{1}{2}$$
$$= 1 \text{ cm/s}$$

b i $x_1(t) = x_2(t)$

$$t^3 - t^2 = t^2$$
$$t^3 - 2t^2 = 0$$
$$t^2(t - 2) = 0$$
$$t = 0 \text{ and } 2$$

The particles will have the same position at the start and after 2 s.

ii Let the distance between the particles be $y = |t^3 - 2t^2|$.
Define $y = t^3 - 2t^2$:

$$\frac{dy}{dt} = 3t^2 - 4t$$
$$= t(3t - 4)$$
$$= 0 \text{ when } t = 0 \text{ and } \frac{4}{3}$$

When $t = 0, y = 0$.
When $t = \frac{4}{3}, y = \frac{64}{27} - \frac{32}{9}$

$$= -1\frac{5}{27}$$

When $t = 2, y = 8 - 2 \times 4$

$$= 0$$

The maximum distance the particles are apart in the first 2 s is $\frac{32}{27} = 1\frac{5}{27}$ cm

5 a $a = 6t$

$v = 3t^2 + c$
When $t = 0, v = 0$.
$0 = 0 + c$

$c = 0$

$\therefore \quad v = 3t^2$

When $t = 2, v = 3 \times 4$

$= 12$ m/s

b $v = 3t^2$

$x = t^3 + d$
When $t = 0, x = 0$.
$0 = 0 + d$

$d = 0$

$x = t^3$
Since the particle starts at the origin, its displacement is $s = x = t^3$.

6 a $a = 3 - 2t$

$v = 3t^2 - t^2 + c$
When $t = 0, v = 4$.
$4 = 0 - 0 + c$

$c = 4$

$v = 3t - t^2 + 4 = 0$

$-(t^2 - 3t - 4) = 0$

$-(t - 4)(t + 1) = 0$

$t = 4$

After 4 s

b $v = 3t - t^2 + 4$

$x = \frac{3t^2}{2} - \frac{t^3}{3} + 4t + d$
When $t = 0, x = 0$.
$0 = 0 - 0 + 0 + d$

$d = 0$

$x = \frac{3t^2}{2} - \frac{t^3}{3} + 4t$

When $t = 4, x = \frac{3 \times 4^2}{2} - \frac{4^3}{3}$

$+ 4 \times 4$

$= 18\frac{2}{3}$

$18\frac{2}{3}$ m to the right

c When $t = 4, a = 3 - 2 \times 4$

$= -5$ m/s^2

d $a = 3 - 2t = 0$

$t = 1.5$ s

e When $t = 1.5$,
$v = 3t - t^2 + 4$

$= 3 \times 1.5 - 1.5^2 + 4$

$= 6.25$ m/s

7 a $s = \frac{2t^3}{3} - \frac{3t^4}{4} + c$
When $t = 0, s = 0$.
$0 = 0 - 0 + c$

$c = 0$

$s = \frac{2t^3}{3} - \frac{3t^4}{4}$

When $t = 1, x = \frac{2 \times 1^3}{3} - \frac{3 \times 1^4}{4}$

$= \frac{2}{3} - \frac{3}{4} = \frac{1}{12}$

$\frac{1}{12}$ m to the left.

b When $t = 1, v = 2 - 3$

$= -1$ m/s

c $a = \frac{dv}{dt}$

$= 4t - 9t^2$
When $t = 1, a = 4 \times 1 - 9 \times 1^2$

$= -5$ m/s^2

8 a $v = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$

$a = \frac{dv}{dt}$

$= \frac{1}{2} \times (-2t^{-3}) = -\frac{1}{t^3}$

b $v = \frac{1}{2}t^{-2}$

$s = -\frac{1}{2}t^{-1} + c$

When $t = 1, s = 0$.

$0 = -\frac{1}{2} \times 1^{-1} + c$

$0 = -\frac{1}{2} + c$

$c = \frac{1}{2}$

$s = \frac{1}{2} - \frac{1}{2t}$

9 a $a = \frac{dv}{dt}$

$= 3t^2 - 22t + 24$

b Solve for $v = 0$.

$t^3 - 11t + 24t = t(t - 3)(t - 8)$

Since motion is only defined for $t \geq 0$, it cannot be said to change direction at $t = 0$.

$\therefore \quad t = 3$

$a = 3 \times 3^2 - 22 \times 3 + 24$

$= -15 \text{ m/s}^2$

c $v = t^3 - 11t^2 + 24t$

$x = \frac{t^4}{4} - \frac{11t^3}{3} + 12t^2 + c$

When $t = 0, x = 0$

$0 = 0 - 0 + 0 + c$

$c = 0$

$x = \frac{t^4}{4} - \frac{11t^3}{3} + 12t^2$

When $t = 5$, $x = \frac{5^4}{4} - \frac{11 \times 5^3}{3} + 12 \times 5^2$

$= -2\frac{1}{2}$

When $t = 3$, $x = \frac{3^4}{4} - \frac{11 \times 3^3}{3} + 12 \times 3^2$

$= 29\frac{1}{4}$

When $t = 0, x = 0$.

Total distance

$= 29\frac{1}{4} + \left(29\frac{1}{4} + 2\frac{1}{12}\right)$

$= 60\frac{7}{12}$ m

$2\frac{1}{12}$ m left of O, $60\frac{7}{12}$ m

10 $u = 20, v = 0, t = 4$

$s = \frac{1}{2}(u + v)t$

$= \frac{1}{2} \times 20 \times 4$

$= 40$ m

11 a $u = 0, v = 30, t = 12$

$v = u + at$

$30 = 12a$

$a = \frac{30}{12}$

$= 2.5 \text{ m/s}^2$

b $u = 30, v = 50, a = 2.5$

$v = u + at$

$50 = 30 + 2.5t$

$2.5t = 20$

$t = 8$ s

c $s = ut + \frac{1}{2}at^2$

$= 0 + \frac{1}{2} \times 2.5 \times 20^2$

$= 500$ m

d 100 km/h $= 100 \div 3.6$

$= \frac{250}{9}$ m/s

$u = 0, v = \frac{250}{9}, a = 2.5$

$v = u + at$

$\frac{250}{9} = 2.5t$

$t = \frac{250}{9 \times 2.5}$

$= 11\frac{1}{9}$ s

12 a 100 km/h $= 100 \div 3.6$

$= \frac{50}{3}$ m/s

$u = 0, v = \frac{50}{3}, a = 0.4$

$v = u + at$

$\frac{50}{3} = 0.4t$

$t = \frac{50}{3 \times 0.4}$

$= 41\frac{2}{3}$ s

b $s = \frac{1}{2}(u + v)t$

$= \frac{1}{2} \times \frac{50}{3} \times \frac{125}{3}$

$= 347\frac{2}{9}$ m

13 a $u = 35, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 3.5t - 4.9t^2$

$0.7t(50 - 7t) = 0$

$t = \frac{50}{7} = 7\frac{1}{7}$ s

≈ 7.143 s

b $u = 35, s = 60, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$60 = 35t - 49t^2$

$4.9t^2 - 35t + 60 = 0$

$49t - 250t + 600 = 0$

$(7t - 20)(7t - 30) = 0$

$t = 2\frac{6}{7}$ or $4\frac{2}{7}$

After $2\frac{6}{7}$ s (going up) and $4\frac{2}{7}$ s (going down)

14 a Maximum height occurs when $v = 0$.
$u = 19.6, a = -9.8, v = 0$

$v = u + at$

$0 = 19.6 - 9.8t$

$t = \frac{19.6}{9.8} = 2$ s

b $s = ut + \frac{1}{2}at^2$

$= 19.6 \times 2 + \frac{1}{2} \times -9.8 \times 2^2$

$= 19.6$ m

With respect to ground level,
height = 19.6 + 20 = 39.6 m

c $u = 19.6, s = 0, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$0 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 19.6t - 4.9t^2$

$0 = 4.9t(4 - t)$

$t = 4$ s

d $u = 19.6, s = -20, a = -9.8$

$s = ut + \frac{1}{2}at^2$

$-20 = 19.6t + \frac{1}{2} \times -9.8 \times t^2$

$-20 = 19.6t - 4.9t^2$

$4.9t^2 - 19.t - 20 = 0$

$49t^2 - 196t - 200 = 0$

$\triangle = b^2 - 4ac$

$= 196^2 - 4 \times 49 \times -200$

$= 77\,616$

$\sqrt{\triangle} \approx 278.596$

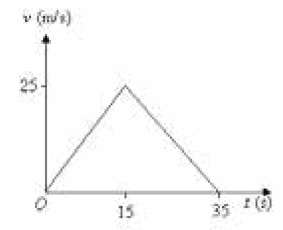
Since the discriminant is irrational, solve using the quadratic formula:

$t = \dfrac{196 \pm 278.596}{98}$

≈ 4.84 or -0.84

≈ 4.84 s (since $t > 0$)

15

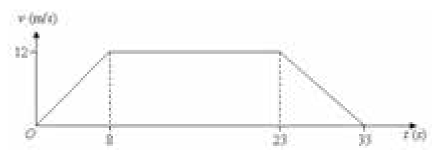


Distance = area

$= \frac{1}{2} \times 35 \times 25$

$= 437.5$ m

16



a Distance = trapezium area

$= \frac{1}{2} \times (33 + 15) \times 12$

$= 288$ m

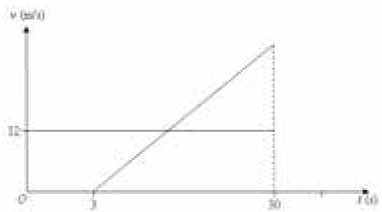
b Halfway point is 144 m.
The car has travelled
$\frac{1}{2} \times 8 \times 12 = 48$ m in the first 8 s.
It must travel 144 − 48 = 96 m at 12 m/s.
This will take 96 ÷ 12 = 8 s.
Total of 16 s.

17

Since the vehicle travels 1 km = 1000 m, adding the two triangles together should give an area equal to a distance of 200 m. The triangles have a combined base of 25.

$$A = \frac{1}{2} \times 25 \times V$$
$$= 200$$
$$V = \frac{200 \times 2}{25}$$
$$= 16 \text{ m/s}$$

18 After 3 s, the first car has travelled $12 \times 3 = 36$ m.



Let the second car's final velocity be V m/s. The two areas will be equal.

$$\frac{1}{2} \times 27 \times V = 12 \times 30$$
$$= 360$$
$$V = \frac{2 \times 360}{27}$$
$$= \frac{80}{3}$$

For constant acceleration,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \frac{80}{3 \times 27} = \frac{80}{81} \text{ m/s}^2$$

19 **a** $v = \frac{10^2}{4} - 3 \times 10 + 5$

$$= 0 \text{ m/s}$$

b $a = \frac{dv}{dt}$

$$= \frac{2t}{4} - 3$$
$$= \frac{t}{2} - 3$$

When $t = 0$, $a = -3$ m/s^2.

c Minimum velocity occurs when $a = 0$.

$$\frac{t}{2} - 3 = 0$$
$$t = 6$$

When $t = 6$,

$$v = \frac{6^2}{4} - 3 \times 6 + 5$$
$$= -4 \text{ m/s}$$

d $v = \frac{t^2}{4} - 3t + 5$

$$x = \frac{t^3}{12} - \frac{3t^2}{2} + 5t + c$$

When $t = 0$, $x = 0$.

$$0 = 0 - 0 + 0 + c$$
$$c = 0$$
$$x = \frac{t^3}{12} - \frac{3t^2}{2} + 5t$$

Check for change of direction of velocity.

$$v = 0 \text{ if } \frac{t^2}{4} - 3t + 5 = 0$$
$$t^2 - 12t + 20 = 0$$
$$(t - 2)(t - 10) = 0$$
$$t = 2 \text{ or } 10$$

There will be no change of direction of velocity in the first 2 s.

When $t = 2$,

$$x = \frac{2^3}{12} - \frac{3 \times 2^2}{2} + 5 \times 2$$
$$= \frac{2}{3} - 6 + 10$$
$$= 4\frac{2}{3} \text{ m}$$

e When $t = 3$,

$$x = \frac{3^3}{12} - \frac{3 \times 3^3}{2} + 5 \times 3$$
$$= \frac{9}{4} - \frac{27}{2} + 15$$
$$= 3\frac{3}{4} \text{ m}$$

Distance travelled in the third second

$= 4\frac{2}{3} - 3\frac{3}{4}$

$= \frac{11}{12}$ m (to the left)

20 a $a = 2 - 2t$

$v = 2t = t^2 + c$

When $t = 3, v = 5$.

$5 = 2 \times 3 - 3^2 + c$

$5 = -3 + c$

$c = 8$

$v = 2t - t^2 + 8$

b $v = 2t - t^2 + 8$

$x = t^2 - \frac{t^3}{3} + 8t + d$

When $t = 0, x = 0$.

$0 = 0 - 0 + 0 + d$

$d = 0$

$x = t^2 - \frac{t^3}{3} + 8t$

21 a $a = 4 - 4t$

$v = 4t - 2t^2 + c$

When $t = 0, v = 6$.

$6 = 0 - 0 + c$

$c = 6$

$v = 4t - 2t^2 + 6$

$= 6 + 4t - 2t^2$

b Minimum velocity occurs when $a = 0$.

i $4 - 4t = 0$

$t = 1$

$v = 6 + 4t - 2t^2$

$= 6 + 4 \times 1 - 2 \times 1^2$

$= 8$ m/s

ii $6 + 4t - 2t^2 = 6$

$4t - 2t^2 = 0$

$2t(2 - t) = 0$

So the velocity of P is again 6 m/s after 2 s.

iii $6 + 4t - 2t^2 = 0$

$-2t^2 + 4t + 6 = 0$

$t^2 - 2t - 3 = 0$

$(t - 3)(t + 1) = 0$

$t = 3$

$x = -\frac{2t^3}{3} + 2t^2 + 6t + d$

$x = 0$ when $t = 0$

$\therefore d = 0$

$x = -\frac{2t^3}{3} + 2t^2 + 6t$

When $t = 3$,

$x = -\frac{2 \times 3^3}{3} + 2 \times 3^2 + 6 \times 3$

$= 18$ m

22 a When $t = 0, a = 27$ m/s^2.

b $a = 27 - 4t^2$

$v = 27t - \frac{4t^3}{3} + c$

When $t = 0, v = 5$.

$5 = 0 - 0 + c$

$c = 5$

$v = 27t - \frac{4t^3}{3} + 5$

When $t = 3$, $v = 27 \times 3 - \frac{4 \times 3^3}{3} + 5$

$= 50$ m/s

c $v = 27t - \frac{4t^3}{3} + 5 = 5$

$27t - \frac{4t^3}{3} = 0$

$81t - 4t^3 = 0$

$t(81 - 4t^2) = 0$

$t(9 - 2t)(9 + 2t) = 0$

$t = 4.5$ s

23 a $a = 3 - 3t$

$v = 3t = \frac{3t^2}{2} + c$

When $t = 0$, $v = 2$.

$2 = 0 - 0 + c$

$c = 2$

$v = 3t - \frac{3t^2}{2} + 2$

When $t = 4$, $v = 3 \times 4 - \frac{3 \times 4^2}{2} + 2$

$= -10$ m/s

b $v = 3t - \frac{3t^2}{2} + 2$

$x = \frac{3t^2}{2} - \frac{t^3}{2} + 2t + d$

When $t = 0$, $x = 0$.

$0 = 0 - 0 + 0 + d$

$d = 0$

$x = \frac{3t^2}{2} - \frac{t^3}{2} + 2t$

When $t = 4$, $x = \frac{3 \times 4^2}{2} - \frac{4^3}{2} + 24$

$= 24 - 32 + 8$

$= 0$

24 a $t^2 - 10t + 24 = 0$

$(t - 4)(t - 6) = 0$

$t = 4$ and 6

b $v = t^2 - 10t + 24$

$x = \frac{t^3}{5} - 5t^2 + 24t + c$

When $t = 0$, $x = 0$.

$0 = 0 - 0 + 0 + c$

$c = 0$

$x = \frac{t^3}{3} - 5t^2 + 24t$

When $t = 3$, $x = \frac{3^3}{3} - 5 \times 3^2$

$+ 24 \times 3$

$= 36$ m

c $a = 2t - 10 < 0$

$2t < 10$

$t < 5$

Since $t \geq 0, 0 \leq t < 5$

Solutions to multiple-choice questions

1 A When $t = 0, x = 0$

2 E When $t = 0, x = 0$.

When $t = 2, x = -2^3 + 7 \times 2^2 - 12 \times 2 = -4$

Average velocity

$$= \frac{\text{change in position}}{\text{change in time}}$$

$$= -\frac{4}{2}$$

$$= -2 \text{ cm/s}$$

3 C $v = 4t - 3t^2 + c$

When $t = 0, v = -1$

$-1 = 0 - 0 + c$

$c = -1$

$v = 4t - 3t^2 - 1$

When $t = 1, v = 4 \times 1 - 3 \times 1^2 - 1 = 0 \text{ m/s}$

4 C $u = 0, s = 90, a = 1.8$

$$s = ut + \frac{1}{2}at^2$$

$$90 = \frac{1}{2} \times 1.8 \times t^2$$

$$90 = 0.9t^2$$

$$t^2 = 100$$

$$t = 10 \text{ s}$$

5 E $60 \text{ km/h} = 60 \div 3.6 = \frac{50}{3} \text{ m/s}$

$u = 0, v = \frac{50}{3}, t = 4$

$v = u + at$

$$\frac{50}{3} = 4a$$

$$a = \frac{50}{12} = \frac{25}{6} \text{ m/s}^2$$

6 C $60 \text{ km/h} = 60 \div 3.6 = \frac{50}{3} \text{ m/s}$

$u = 0, v = \frac{50}{3}, t = 4$

$$s = \frac{1}{2}(u + v)t$$

$$= \frac{1}{2} \times \frac{50}{3} \times 4$$

$$= \frac{100}{3} \text{ m}$$

7 D Distance

= area under graph

= triangle + trapezium + triangle

$$= \frac{1}{2} \times 4 \times 10 + \frac{1}{2} \times (10 + 25) \times 2 + \frac{1}{2} \times 9 \times 25$$

$$= 20 + 25 + 112.5$$

$$= 167.5 \text{ m}$$

8 E $u = 0, a = 9.8, s = 40$

$$v^2 = u^2 + 2as$$

$$= 0 + 2 \times 9.8 \times 40$$

$$= 784$$

$$v = \sqrt{784} = 28 \text{ m/s}$$

9 A $u = 20, v = 0, a = -4$

$$v = u + at$$
$$0 = 20 - 4t$$
$$t = 5$$
$$s = \frac{1}{2}(u + v)t$$
$$= \frac{1}{2} \times 20 \times 5$$
$$= 50 \text{ m}$$

10 D $v = 6t^2 - 5t + c$

When $t = 0, v = 1$.

$$1 = 0 - 0 + c$$
$$c = 1$$
$$v = 6t^2 - 5t + 1$$

When $t = 1, v = 6 \times 1^2 - 5 \times 1 + 1$

$$= 2 \text{ m/s}$$

Solutions to extended-response questions

1 a When $t = 0, x = -\frac{7}{3}$

Initial displacement is $\frac{7}{3}$ cm to the left of O.

b $v = t^2 - 4t + 4$

When $t = 0, v = 4$

Initial velocity is 4 cm/s.

c $a = 2t - 4$

When $t = 3, a = 2(3) - 4 = 2$

Acceleration after three seconds is 2 cm/s^2.

d When $v = 0$,

$t^2 - 4t + 4 = 0$

$\therefore (t-2)^2 = 0$

$\therefore t = 2$

Velocity is zero after two seconds.

e When $v = 0, t = 2$

$$\begin{aligned}\therefore x &= \frac{1}{3}(2)^3 - 2(2)^2 + 4(2) - \frac{7}{3}\\ &= \frac{8}{3} - 8 + 8 - \frac{7}{3}\\ &= \frac{1}{3}\end{aligned}$$

When the velocity is zero, the particle is $\frac{1}{3}$ cm to the right of O.

f When $x = 0, \frac{1}{3}t^3 - 2t^2 + 4t - \frac{7}{3} = 0$

Try $t = 1$

$$\begin{aligned}\text{LHS} &= \frac{1}{3}(1)^3 - 2(1)^2 + 4(1) - \frac{7}{3}\\ &= \frac{1}{3} - 2 + 4 - \frac{7}{3}\\ &= 0\end{aligned}$$

$\therefore$ LHS = RHS and $t = 1$

The displacement is zero after one second.

Also $3P(t) = t^3 - 6t^2 + 12t - 7 = (t-1)(t^2 - 5t + 7)$

and $t^2 - 5t + 7$ is irreducible since $\Delta = 25 - 4 \times 7 < 0$

2 **a**
$$x = t^4 + 2t^2 - 8t$$
$$v = \frac{dx}{dt}$$
$$= 4t^3 + 4t - 8$$

When $t = 0, v = -8$
Since the initial velocity is negative, the particle moves first to the left.

b When $v = 0, 4t^3 + 4t - 8 = 0$ After one second, the particle is instantaneously at rest.
$$4(t^3 + t - 2) = 0$$
$$\therefore 4(t - 1)(t^2 + t + 2) = 0$$
$$\therefore t = 1$$
For $t > 1, t - 1 > 0$ and $t^2 + t + 2 > 0$
$$\therefore 4(t - 1)(t^2 + t + 2) > 0$$
$$\therefore v > 0$$
Hence at one second the particle has travelled the greatest distance to the left.

c As $v > 0$ when $t > 1$, the particle always moves to the right for $t > 1$.

3 **a** The rocket crashes when $h = 0$
i.e.$6t^2 - t^3 = 0$
$$t^2(6 - t) = 0$$
$$t = 0 \text{ or } 6$$
$t = 6$ since $t = 0$ represents take-off.
$$v = \frac{dv}{dh}$$
$$= 12t - 3t^2$$
When $t = 6, v = 12(6) - 3(6)^2$
$$= 72 - 108$$
$$= -36$$
The rocket crashes after six seconds with a velocity of -36 m/s.

b When $v = 0, 12t - 3t^2 = 0$

$$\therefore 3t(4 - t) = 0$$

$$\therefore t = 0 \text{ or } 4$$

When $t = 4$, $h = 6(4)^2 - (4)^3$

$$= 96 - 64$$

$$= 32$$

The speed of the rocket is zero at take-off and after four seconds. The maximum height of the rocket is 32 metres after four seconds.

c $$a = \frac{dv}{dt}$$

$$= 12 - 6t$$

When $a < 0, 12 - 6t < 0$

$$\therefore 12 < 6t$$

$$\therefore 2 < t$$

The acceleration becomes negative after two seconds.

4
- $x(1) - x(0) = 15, 1$
- $x(2) - x(1) = 5.3$ difference -9.8
- $x(3) - x(2) = -4.5$ difference -9.8
- $x(4) - x(3) = -14.3$ difference -9.8
- $x(5) - x(4) = -24.1$ difference -9.8
- $x(6) - x(5) = -33.9$ difference -9.8
- $x(7) - x(6) = -43.7$ difference -9.8
- $x(8) - x(7) = -53.5$ difference -9.8
- $x(9) - x(8) = -63.3$ difference -9.8
- $x(10) - x(9) = -73.1$ difference -9.8

The body has a constant acceleration of 9.8 m/s^2 which is the acceleration due to gravity.

5 a Let $a = -g$ (m/s^2), $v = 0$ (m/s)

Using $v = u + at$,

$$t = \frac{v - u}{a}$$
$$= \frac{0 - u}{-g}$$
$$= \frac{u}{g},$$

as required.

b When $t = \frac{u}{g}, v = 0$

$$s = \frac{1}{2}(u + v)t$$
$$= \frac{1}{2}(u + 0)\frac{u}{g}$$
$$= \frac{u^2}{2g}$$

The particle will have travelled $\frac{2u^2}{2g} = \frac{u^2}{g}$ metres to return to its point of projection.

Consider the path of the particle from its highest point when its velocity is zero, until it returns to the point of projection $\frac{u^2}{2g}$ downwards.

Then $u = 0, s = \frac{u^2}{2g}, a = g$

and $s = ut + \frac{1}{2}at^2$

$$\therefore = 0 \times t + gt^2$$
$$\therefore \frac{u^2}{2g} = gt^2$$
$$\therefore t^2 = \frac{u^2}{g^2}$$
$$\therefore t = \frac{u}{g} (t = -\frac{u}{g} \quad \text{is discounted as } t > 0)$$

Hence the total time taken is $\frac{u}{g} + \frac{u}{g} = \frac{2u}{g}$ seconds, as required.

c For the return downwards, $u = 0, t = \frac{u}{g}, a = g$

$$v = u + at$$
$$= 0 + g \times \frac{u}{g}$$
$$= u$$

Hence the speed of returning to the point of projection is u m/s.

6 Consider the throw of the stone to its maximum height. $u = 14, a = 9.8, v = 0$

$$t = \frac{v - u}{a}$$
$$= \frac{0 - 14}{-9.8}$$
$$= \frac{10}{7}$$

It therefore takes $2 \times \frac{10}{7} = \frac{20}{7}$ seconds for the stone to reach the top of the mine shaft on its descent.

From this point,

$u = -14, a = -9.8, s = ut + \frac{1}{2}at^2$

$\therefore s = 14t - 4.9t^2 \ldots (1)$

When the stone reaches the top of the mine shaft, the lift has been descending for $\frac{20}{7} + 5 = \frac{55}{7}$ seconds and has travelled $\frac{55}{7} \times 3.5 = 27.5$ metres.

From this point,

$s = -27.5 - 3.5t \qquad (\text{for the lift}) \ldots (2)$

Equating (1) and (2) to find the point of impact.

$$-14t - 4.9t^2 = 27.5 - 3.5t$$
$$\therefore 4.9t^2 + 10.5t - 27.5 = 0$$
$$\therefore t = \frac{-10.5 \pm \sqrt{10.5^2 - 4 \times 4.9 \times (-27.5)}}{2 \times 4.9}$$
$$= 1.42857\ldots$$

(the negative solution is not practical)

$$\text{When } t = 1.42857\ldots,$$
$$s = -27.5 - 3.5 \times 1.42857$$
$$= -32.85013$$

Hence the depth of the lift when the stone hits it is 33 metres, to the nearest metre.

7 **a** $90 \text{ km/h} = 90 \times \frac{5}{18} \text{ m/s}$

$$= 25 \text{ m/s}$$
$$v = -\frac{25}{5}t + 25$$
$$\therefore v = -5t + 25, \qquad 0 \le t \le 5$$

b Distance travelled $= \text{area under the graph}$

$$= \frac{1}{2} \times 25 \times 5$$

$$= 62.5$$

The distance travelled in five seconds is 62.5 metres.

8 $$x = 3t^4 - 4t^3 + 24t^2 - 48t$$

$$v = \frac{dx}{dt}$$

$$= 12t^3 - 12t^2 + 48t - 48$$

When $t = 0, v = -48$

Since $v < 0$, the particle moves at first to the left.

When $v = 0, 12t^3 - 12t^2 + 48t - 48 = 0$

$\therefore 12(t^3 - t^2 + 4t - 4) = 0$

$\therefore 12(t - 1)(t^2 + 4) = 0$

$\therefore t = 1$

When $t = 1, x = 3(1)^4 - 4(1)^3 + 24(1)^2 - 48(1)$

$$= 3 - 4 + 24 - 48$$

$$= -25$$

The particle comes to rest at $(1, -25)$

When $t > 1, t - 1 > 0$ and $t^2 + 4 > 0$

$\therefore 12(t - 1)(t^2 + 4) > 0$

$\therefore v > 0$

Since $v > 0$, the particle always moves to the right for $t > 1$.

9 For the first particle, $s = ut - \frac{1}{2}gt^2$ where $a = -g$

For the second particle, $s = u(t - T) - \frac{1}{2}g(t - T)^2$

The particles collide when

a **i**

$$ut - gt^2 = u(t - T) - \frac{1}{2}g(t - T)^2$$
$$= ut - uT - \frac{1}{2}gt^2 + gtT - \frac{1}{2}gT^2$$
$$\therefore 0 = -uT + gtT - \frac{1}{2}gT^2$$
$$= T(-u + gt - \frac{1}{2}gT)$$
$$\therefore -u + gt - \frac{1}{2}gT = 0 \qquad (T \neq 0)$$
$$\therefore gt = u + \frac{1}{2}gT$$
$$\therefore t = \frac{u}{g} + \frac{T}{2} \text{as required.}$$

ii When $t = \dfrac{u}{g} + \dfrac{T}{2}$

$$s = u(\frac{u}{g} + \frac{T}{2}) - g\left(\frac{u}{g} + \frac{T}{2}\right)^2$$
$$= \frac{u^2}{g} + \frac{uT}{2} - \frac{1}{2}g\left(\frac{u^2}{g^2} + \frac{uT}{g} + \frac{T^2}{4}\right)$$
$$= \frac{u^2}{g} + \frac{uT}{2} - \frac{u^2}{2g} - \frac{uT}{2} - \frac{gT^2}{8}$$
$$= \frac{u^2}{2g} - \frac{gT^2}{8}$$
$$= \frac{4u^2 - g^2T^2}{8g}, \text{ as required.}$$

When $T = \dfrac{2u}{g}$, $s = \dfrac{4u^2 - g^2\left(\frac{2u}{g}\right)^2}{8g}$

$$= \frac{4u^2 - 4u^2}{8g}$$
$$= 0$$

b This is the case when the second particle is projected upward at the instant the first particle lands. Hence there is no collision.

c If $T > \dfrac{2u}{g}$, the second particle is projected upward after the first particle has landed, hence no collision.