CAMBRIDGE

SPECIALIST MATHEMATICS

VCE UNITS 1 & 2

CAMBRIDGE SENIOR MATHEMATICS **VCE**

SECOND EDITION

DAVID TREEBY | MICHAEL EVANS | DOUGLAS WALLACE
GARETH AINSWORTH | KAY LIPSON

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Contents

Introduction and overview

Cambridge Specialist Mathematics VCE Units 1&2 Second Edition provides a complete teaching and learning resource for the VCE Study Design **to be first implemented in 2023**. It has been written with understanding as its chief aim, and with ample practice offered through the worked examples and exercises. The work has been trialled in the classroom, and the approaches offered are based on classroom experience and the responses of teachers to earlier editions of this book and the requirements of the new Study Design.

The course is designed as preparation for Specialist Mathematics Units 3 and 4.

Specialist Mathematics Units 1 and 2 provide an introductory study of topics in proof, logic, sequences, **algorithms and pseudocode**, graph theory, algebra, functions, statistics, complex numbers, and vectors and their applications in a variety of practical and theoretical contexts. Techniques of proof are discussed in Chapter 6 and the concepts discussed there are employed in the following chapters and in Specialist Mathematics Units 3 and 4.

Chapter 1 provides an opportunity for students to revise and strengthen their algebra; and this is revisited in Chapter 3, where polynomial identities and partial fractions are introduced.

We have also written online appendices to support teachers and students to better develop their programming capabilities using **both the programming language Python and the inbuilt capabilities of students' CAS calculators**. Additional material on kinematics has also been placed in an online appendix.

Five extensive revision chapters are placed at key stages throughout the book. These provide technology-free multiple-choice and extended-response questions.

The first four revision chapters contain material suitable for **student investigations**, a feature of the new course. The Study Design suggests that '[a]n Investigation comprises one to two weeks of investigation into one or two practical or theoretical contexts or scenarios based on content from areas of study and application of key knowledge and key skills for the outcomes'. We have aimed to provide strong support for teachers in the development of these investigations.

The TI-Nspire calculator examples and instructions have been completed by Peter Flynn, and those for the Casio ClassPad by Mark Jelinek, and we thank them for their helpful contributions.

Overview of the print book

1 Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercise questions.

2 Section summaries provide important concepts in boxes for easy reference.

3 Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.

4 Questions that suit the use of a CAS calculator to solve them are identified within exercises.

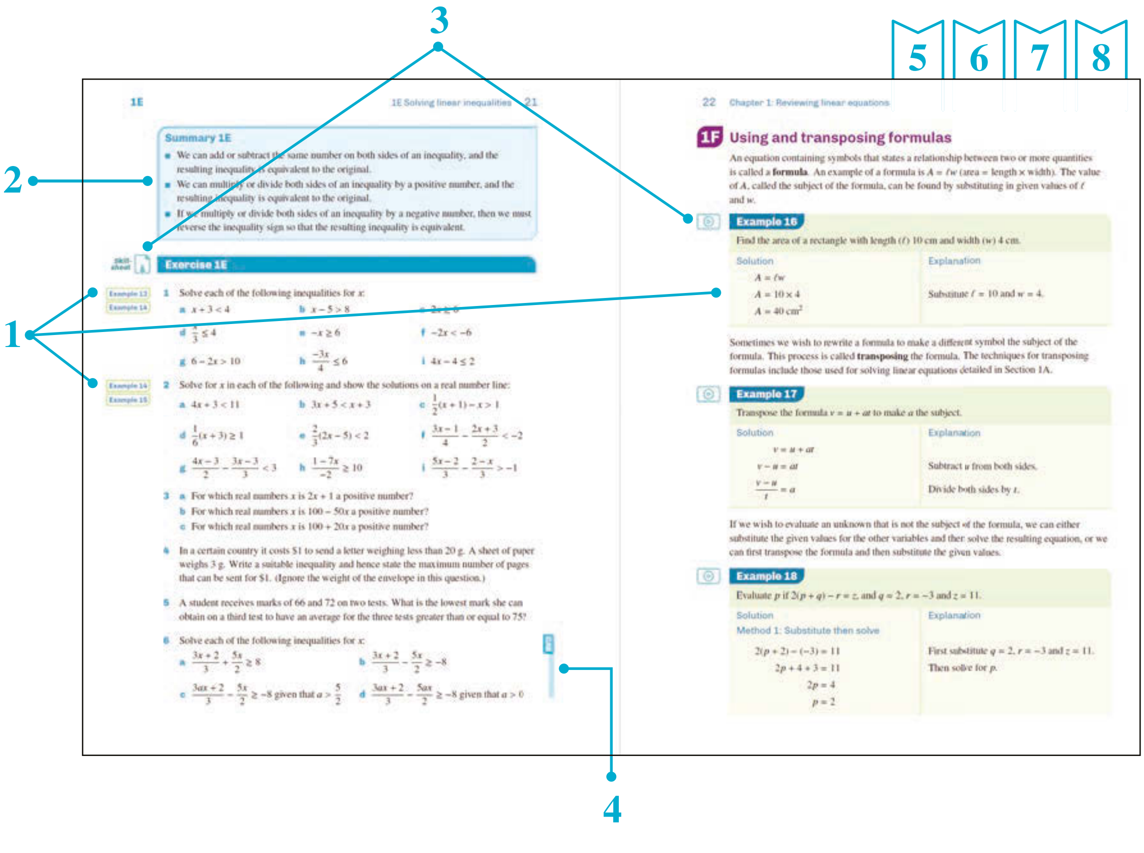
5 Chapter reviews contain a chapter summary and technology-free, multiple-choice, and extended-response questions.

6 Revision chapters provide comprehensive revision and preparation for assessment, including new practice Investigations.

7 The glossary includes page numbers of the main explanation of each term.

8 In addition to coverage within chapters, print and online appendices provide additional support for learning and applying algorithms and pseudocode, including the use of Python and TI-Nspire and Casio ClassPad for coding.

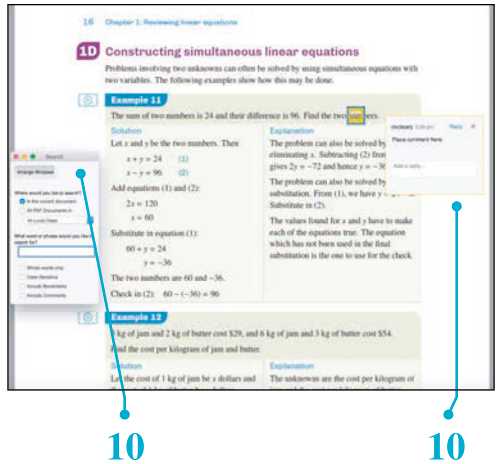
Numbers refer to descriptions above.



Overview of the downloadable PDF textbook

9 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

10 PDF annotation and search features are enabled.



Overview of the Interactive Textbook

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

11 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

12 **Workspaces** for all questions, which can be enabled or disabled by the teacher, allow students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing done on paper.

13 **Self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.

14 All worked examples have **video versions** to encourage independent learning.

15 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

16 An expanded and revised set of **Desmos interactives** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

17 The **Desmos graphics calculator, scientific calculator**, and **geometry tool** are also embedded for students to use for their own calculations and exploration.

18 **Revision of prior knowledge** is provided with links to diagnostic tests and Year 10 HOTmaths lessons.

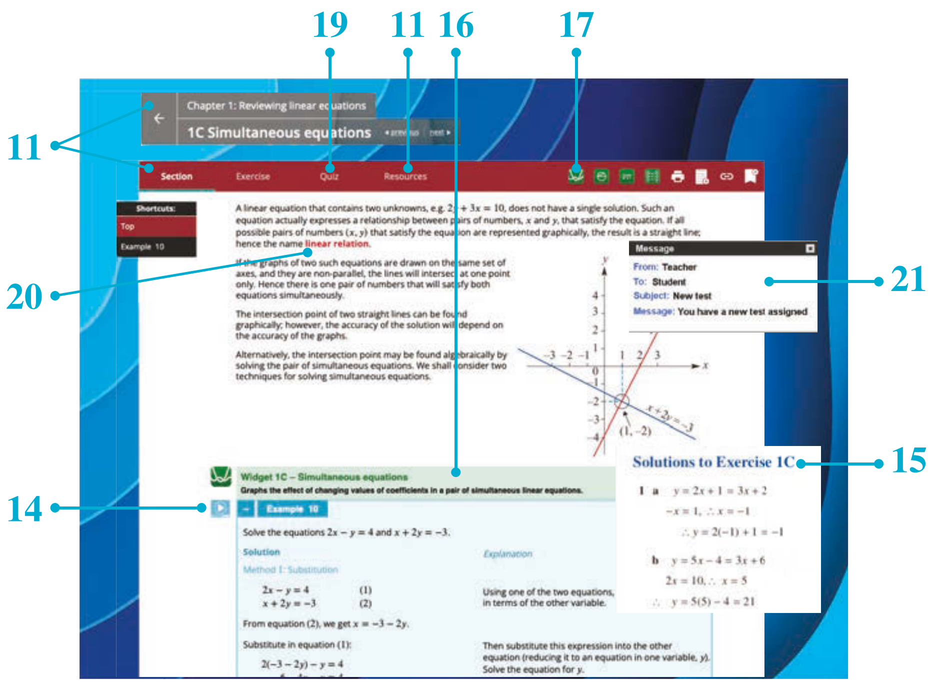
19 **Quick quizzes** containing automarked multiple-choice questions have been thoroughly expanded and revised, enabling students to check their understanding.

20 **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

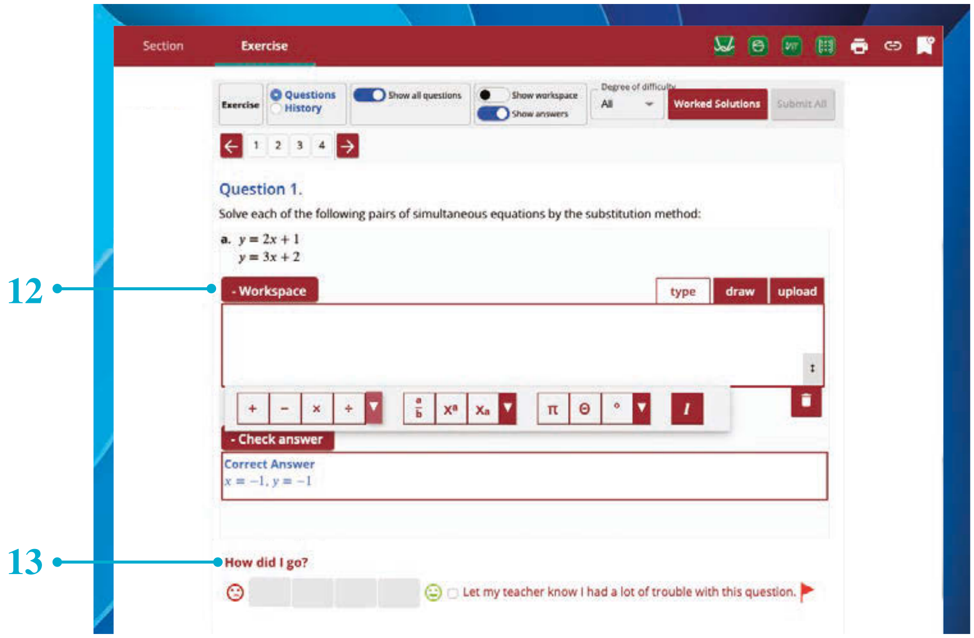
21 Messages from the teacher assign tasks and tests.

INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

A selection of features is shown. Numbers refer to the descriptions on pages xi–xii. HOTmaths platform features are updated regularly



WORKSPACES AND SELF-ASSESSMENT



Overview of the Online Teaching Suite powered by the HOTmaths platform

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

22 The HOTmaths learning management system with class and student analytics and reports, and communication tools.

23 Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had diffculty.

24 A HOTmaths-style test generator.

25 An expanded and revised suite of chapter tests, assignments and sample investigations.

26 Editable curriculum grids and teaching programs.

27 A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite includes a comprehensive bank of VCAA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- VCAA marking scheme
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

Acknowledgements

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1

Reviewing algebra

Objectives

- To revise the **index laws**.
- To express numbers using **scientific notation**.
- To solve problems with **linear equations** and **simultaneous linear equations**.
- To use **substitution** and **transposition** with formulas.
- To add and multiply algebraic fractions.
- To solve **literal equations**.
- To solve **simultaneous literal equations**.

Algebra is the language of mathematics. Algebra helps us to state ideas more simply. It also enables us to make general statements about mathematics, and to solve problems that would be difficult to solve otherwise.

We know by basic arithmetic that $9 \times 7 + 2 \times 7 = 11 \times 7$. We could replace the number 7 in this statement by any other number we like, and so we could write down infinitely many such statements. These can all be captured by the algebraic statement $9x + 2x = 11x$, for any number x. Thus algebra enables us to write down general statements.

Formulas enable mathematical ideas to be stated clearly and concisely. An example is the well-known formula for compound interest. Suppose that an initial amount P is invested at an interest rate R, with interest compounded annually. Then the amount, A_n, that the investment is worth after n years is given by $A_n = P(1 + R)^n$.

In this chapter we review some of the techniques which you have met in previous years. Algebra plays a central role in Specialist Mathematics at Years 11 and 12. It is important that you become fluent with the techniques introduced in this chapter and in Chapter 4.

1A Indices

This section revises algebra involving indices.

Review of index laws

For all non-zero real numbers a and b and all integers m and n:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $\frac{1}{a^{-n}} = a^n$
- $a^0 = 1$

Rational indices

If a is a positive real number and n is a natural number, then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. For example: $9^{\frac{1}{2}} = \sqrt{9} = 3$.

If n is odd, then we can define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$.

In both cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In general, the expression a^x can be defined for rational indices, i.e. when $x = \frac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

To employ this definition, we will always first write the fractional power in simplest form.

Note: The index laws hold for rational indices m and n whenever both sides of the equation are defined (for example, if a and b are positive real numbers).

Example 1

Simplify each of the following:

a $x^2 \times x^3$ **b** $\frac{x^4}{x^2}$ **c** $x^{\frac{1}{2}} \div x^{\frac{4}{5}}$ **d** $(x^3)^{\frac{1}{2}}$

Solution	Explanation
a $x^2 \times x^3 = x^{2+3} = x^5$	$a^m \times a^n = a^{m+n}$
b $\frac{x^4}{x^2} = x^{4-2} = x^2$	$\frac{a^m}{a^n} = a^{m-n}$
c $x^{\frac{1}{2}} \div x^{\frac{4}{5}} = x^{\frac{1}{2}-\frac{4}{5}} = x^{-\frac{3}{10}}$	$\frac{a^m}{a^n} = a^{m-n}$
d $(x^3)^{\frac{1}{2}} = x^{\frac{3}{2}}$	$(a^m)^n = a^{mn}$

Example 2

Evaluate:

a $125^{\frac{2}{3}}$ **b** $\left(\frac{1000}{27}\right)^{\frac{2}{3}}$

Solution	Explanation
a $125^{\frac{2}{3}} = \left(125^{\frac{1}{3}}\right)^2 = 5^2 = 25$	$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$
b $\left(\frac{1000}{27}\right)^{\frac{2}{3}} = \left(\left(\frac{1000}{27}\right)^{\frac{1}{3}}\right)^2 = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$	$\left(\frac{1000}{27}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1000}{27}} = \frac{10}{3}$

Example 3

Simplify $\dfrac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$.

Solution	Explanation
$\dfrac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} = \dfrac{(x^2y^3)^{\frac{1}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} = \dfrac{x^{\frac{2}{4}}y^{\frac{3}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$	$(ab)^n = a^nb^n$
$= x^{\frac{2}{4}-\frac{1}{2}}y^{\frac{3}{4}-\frac{2}{3}}$	$\dfrac{a^m}{a^n} = a^{m-n}$
$= x^0y^{\frac{1}{12}}$	
$= y^{\frac{1}{12}}$	$a^0 = 1$

Summary 1A

- **Index laws**
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^nb^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $a^{-n} = \frac{1}{a^n}$
  - $\frac{1}{a^{-n}} = a^n$
  - $a^0 = 1$
- **Rational indices**
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Exercise 1A

Example 1

1 Simplify each of the following using the appropriate index laws:

a $x^3 \times x^4$ **b** $a^5 \times a^{-3}$ **c** $x^2 \times x^{-1} \times x^2$ **d** $\dfrac{y^3}{y^7}$

e $\dfrac{x^8}{x^{-4}}$ **f** $\dfrac{p^{-5}}{p^2}$ **g** $a^{\frac{1}{2}} \div a^{\frac{2}{3}}$ **h** $(a^{-2})^4$

i $(y^{-2})^{-7}$ **j** $(x^5)^3$ **k** $(a^{-20})^{\frac{3}{5}}$ **l** $\left(x^{-\frac{1}{2}}\right)^{-4}$

m $(n^{10})^{\frac{1}{5}}$ **n** $2x^{\frac{1}{2}} \times 4x^3$ **o** $(a^2)^{\frac{5}{2}} \times a^{-4}$ **p** $\dfrac{1}{x^{-4}}$

q $\left(2n^{-\frac{2}{5}}\right)^5 \div (4^3 n^4)$ **r** $x^3 \times 2x^{\frac{1}{2}} \times -4x^{-\frac{3}{2}}$

s $(ab^3)^2 \times a^{-2}b^{-4} \times \dfrac{1}{a^2b^{-3}}$ **t** $(2^2p^{-3} \times 4^3p^5 \div (6p^{-3}))^0$

Example 2

2 Evaluate each of the following:

a $25^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $\left(\dfrac{16}{9}\right)^{\frac{1}{2}}$ **d** $16^{-\frac{1}{2}}$

e $\left(\dfrac{49}{36}\right)^{-\frac{1}{2}}$ **f** $27^{\frac{1}{3}}$ **g** $144^{\frac{1}{2}}$ **h** $64^{\frac{2}{3}}$

i $9^{\frac{3}{2}}$ **j** $\left(\dfrac{81}{16}\right)^{\frac{1}{4}}$ **k** $\left(\dfrac{23}{5}\right)^0$ **l** $128^{\frac{3}{7}}$

3 Use your calculator to evaluate each of the following, correct to two decimal places:

a 4.35^2 **b** 2.4^5 **c** $\sqrt{34.6921}$ **d** $(0.02)^{-3}$ **e** $\sqrt[3]{0.729}$

f $\sqrt[4]{2.3045}$ **g** $(345.64)^{-\frac{1}{3}}$ **h** $(4.568)^{\frac{2}{5}}$ **i** $\dfrac{1}{(0.064)^{-\frac{1}{3}}}$

4 Simplify each of the following, giving your answer with positive index:

a $\dfrac{a^2b^3}{a^{-2}b^{-4}}$ **b** $\dfrac{2a^2(2b)^3}{(2a)^{-2}b^{-4}}$ **c** $\dfrac{a^{-2}b^{-3}}{a^{-2}b^{-4}}$

d $\dfrac{a^2b^3}{a^{-2}b^{-4}} \times \dfrac{ab}{a^{-1}b^{-1}}$ **e** $\dfrac{(2a)^2 \times 8b^3}{16a^{-2}b^{-4}}$ **f** $\dfrac{2a^2b^3}{8a^{-2}b^{-4}} \div \dfrac{16ab}{(2a)^{-1}b^{-1}}$

5 Write $\dfrac{2^n \times 8^n}{2^{2n} \times 16}$ in the form 2^{an+b}.

6 Write $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$ as a power of 6.

7 Simplify each of the following:

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}}$ **b** $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}}$ **c** $2^{\frac{2}{3}} \times 2^{\frac{5}{6}} \times 2^{-\frac{2}{3}}$

d $\left(2^{\frac{1}{3}}\right)^2 \times \left(2^{\frac{1}{2}}\right)^5$ **e** $\left(2^{\frac{1}{3}}\right)^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$

Example 3

8 Simplify each of the following:

a $\sqrt[3]{a^3b^2} \div \sqrt[3]{a^2b^{-1}}$ **b** $\sqrt{a^3b^2} \times \sqrt{a^2b^{-1}}$ **c** $\sqrt[5]{a^3b^2} \times \sqrt[5]{a^2b^{-1}}$

d $\sqrt{a^{-4}b^2} \times \sqrt{a^3b^{-1}}$ **e** $\sqrt{a^3b^2c^{-3}} \times \sqrt{a^2b^{-1}c^{-5}}$ **f** $\sqrt[5]{a^3b^2} \div \sqrt[5]{a^2b^{-1}}$

g $\dfrac{\sqrt{a^3b^2}}{a^2b^{-1}c^{-5}} \times \dfrac{\sqrt{a^{-4}b^2}}{a^3b^{-1}} \times \sqrt{a^3b^{-1}}$

1B Standard form

Often when dealing with real-world problems, the numbers involved may be very small or very large. For example:

- The distance from Earth to the Sun is approximately 150 000 000 kilometres.
- The mass of an oxygen atom is approximately 0.000 000 000 000 000 000 000 026 grams.

To help deal with such numbers, we can use a more convenient way to express them. This involves expressing the number as a product of a number between 1 and 10 and a power of 10 and is called **standard form** or **scientific notation**.

These examples written in standard form are:

- 1.5×10^8 kilometres
- 2.6×10^{-23} grams

Multiplication and division with very small or very large numbers can often be simplified by first converting the numbers into standard form. When simplifying algebraic expressions or manipulating numbers in standard form, a sound knowledge of the index laws is essential.

Example 4

Write each of the following in standard form:

a 3 453 000 **b** 0.00675

Solution

a $3\,453\,000 = 3.453 \times 10^6$ **b** $0.00675 = 6.75 \times 10^{-3}$

Example 5

Find the value of $\dfrac{32\,000\,000 \times 0.000\,004}{16\,000}$.

Solution

$$\begin{aligned}\frac{32\,000\,000 \times 0.000\,004}{16\,000} &= \frac{3.2 \times 10^7 \times 4 \times 10^{-6}}{1.6 \times 10^4} \\ &= \frac{12.8 \times 10^1}{1.6 \times 10^4} \\ &= 8 \times 10^{-3} \\ &= 0.008\end{aligned}$$

Significant figures

When measurements are made, the result is recorded to a certain number of significant figures. For example, if we say that the length of a piece of ribbon is 156 cm to the nearest centimetre, this means that the length is between 155.5 cm and 156.5 cm. The number 156 is said to be correct to three significant figures. Similarly, we may record π as being 3.1416, correct to five significant figures.

When rounding off to a given number of significant figures, first identify the last significant digit and then:

- if the next digit is 0, 1, 2, 3 or 4, round down
- if the next digit is 5, 6, 7, 8 or 9, round up.

It can help with rounding off if the original number is first written in scientific notation.

So $\pi = 3.141\,592\,653\ldots$ is rounded off to 3, 3.1, 3.14, 3.142, 3.1416, 3.14159, etc. depending on the number of significant figures required.

Writing a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 600 is recorded to one, two or three significant figures. However, when written in scientific notation as 6.00×10^2, 6.0×10^2 or 6×10^2, it is clear how many significant figures are recorded.

Example 6

Evaluate $\dfrac{\sqrt[5]{a}}{b^2}$ if $a = 1.34 \times 10^{-10}$ and $b = 2.7 \times 10^{-8}$.

Solution

$$\begin{aligned}\frac{\sqrt[5]{a}}{b^2} &= \frac{\sqrt[5]{1.34 \times 10^{-10}}}{(2.7 \times 10^{-8})^2}\\ &= \frac{(1.34 \times 10^{-10})^{\frac{1}{5}}}{2.7^2 \times (10^{-8})^2}\\ &= 1.45443\ldots \times 10^{13}\\ &= 1.45 \times 10^{13} \quad \text{to three significant figures}\end{aligned}$$

Many calculators can display numbers in scientific notation. The format will vary from calculator to calculator. For example, the number $3\,245\,000 = 3.245 \times 10^6$ may appear as 3.245E6 or 3.245^{06}.

Using the TI-Nspire

Insert a **Calculator** page, then use [on] > **Settings** > **Document Settings** and change the **Exponential Format** field to **Scientific**. If you want this change to apply only to the current page, select OK to accept the change. Select **Current** to return to the current page.

Using the Casio ClassPad

The ClassPad calculator can be set to express decimal answers in various forms, including scientific notation with a fixed number of significant figures. Go to **Settings** and select **Basic Format**. You can then choose from the various number formats available.

Note: The number format **Normal 2** only defaults to scientific notation for numbers greater than or equal to 10^{10}, or less than or equal to 10^{-10}.

Summary 1B

- A number is said to be in **scientific notation** (or **standard form**) when it is written as a product of a number between 1 and 10 and an integer power of 10.
  For example: $6547 = 6.547 \times 10^3$ and $0.789 = 7.89 \times 10^{-1}$
- Writing a number in scientific notation makes it clear how many **significant figures** have been recorded.
- When rounding off to a given number of significant figures, first identify the last significant digit and then:
  - if the next digit is 0, 1, 2, 3 or 4, round down
  - if the next digit is 5, 6, 7, 8 or 9, round up.

Exercise 1B

Example 4

1 Express each of the following numbers in standard form:

a 47.8 **b** 6728 **c** 79.23 **d** 43 580
e 0.0023 **f** 0.000 000 56 **g** 12.000 34 **h** 50 million
i 23 000 000 000 **j** 0.000 000 0013 **k** 165 thousand **l** 0.000 014 567

2 Express each of the following in scientific notation:

a X-rays have a wavelength of 0.000 000 01 cm.
b The mass of a hydrogen atom is 0.000 000 000 000 000 000 000 001 67 g.
c Visible light has wavelength 0.000 05 cm.
d One nautical mile is 1853.18 m.
e A light year is 9 461 000 000 000 km.
f The speed of light is 29 980 000 000 cm/s.

3 Express each of the following as an ordinary number:

a The star Sirius is approximately 8.128×10^{13} km from Earth.
b A single red blood cell contains 2.7×10^8 molecules of haemoglobin.
c The radius of an electron is 2.8×10^{-13} cm.

4 Write each of the following in scientific notation, correct to the number of significant figures indicated in the brackets:

a 456.89 (4) **b** 34567.23 (2) **c** 5679.087 (5)
d 0.04536 (2) **e** 0.09045 (2) **f** 4568.234 (5)

Example 5

5 Find the value of:

a $\dfrac{324\,000 \times 0.000\,0007}{4000}$ **b** $\dfrac{5\,240\,000 \times 0.8}{42\,000\,000}$

Example 6

6 Evaluate the following correct to three significant figures:

a $\dfrac{\sqrt[3]{a}}{b^4}$ if $a = 2 \times 10^9$ and $b = 3.215$ **b** $\dfrac{\sqrt[4]{a}}{4b^4}$ if $a = 2 \times 10^{12}$ and $b = 0.05$

1C Solving linear equations and simultaneous linear equations

Many problems may be solved by first translating them into mathematical equations and then solving the equations using algebraic techniques. An equation is solved by finding the value or values of the variables that would make the statement true.

Linear equations are simple equations that can be written in the form $ax + b = 0$. There are a number of standard techniques that can be used for solving linear equations.

Example 7

a Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

b Solve $\frac{x-3}{2} - \frac{2x-4}{3} = 5$.

Solution

a Multiply both sides of the equation by the lowest common multiple of 3 and 5:

$$\frac{x}{5} - 2 = \frac{x}{3}$$

$$\frac{x}{5} \times 15 - 2 \times 15 = \frac{x}{3} \times 15$$

$$3x - 30 = 5x$$

$$3x - 5x = 30$$

$$-2x = 30$$

$$x = \frac{30}{-2}$$

$$\therefore \quad x = -15$$

b Multiply both sides of the equation by the lowest common multiple of 2 and 3:

$$\frac{x-3}{2} - \frac{2x-4}{3} = 5$$

$$\frac{x-3}{2} \times 6 - \frac{2x-4}{3} \times 6 = 5 \times 6$$

$$3(x-3) - 2(2x-4) = 30$$

$$3x - 9 - 4x + 8 = 30$$

$$3x - 4x = 30 + 9 - 8$$

$$-x = 31$$

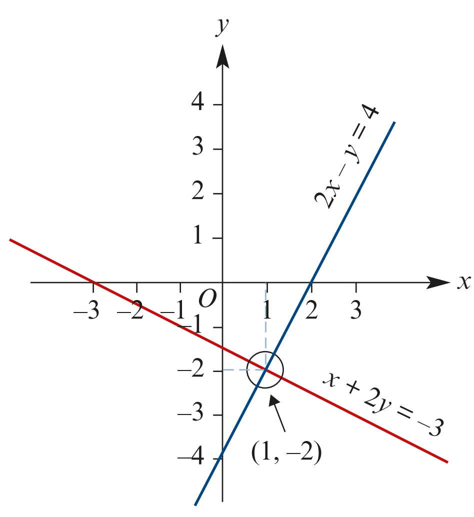
$$x = \frac{31}{-1}$$

$$\therefore \quad x = -31$$

Simultaneous linear equations

The intersection point of two straight lines can be found graphically; however, the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively, the intersection point may be found algebraically by solving the pair of simultaneous equations. We shall consider two techniques for solving simultaneous equations.



Example 8

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

Method 1: Substitution

$$2x - y = 4 \quad (1)$$
$$x + 2y = -3 \quad (2)$$

From equation (2), we get $x = -3 - 2y$.

Substitute in equation (1):

$$2(-3 - 2y) - y = 4$$
$$-6 - 4y - y = 4$$
$$-5y = 10$$
$$y = -2$$

Substitute the value of y into (2):

$$x + 2(-2) = -3$$
$$x = 1$$

Check in (1): LHS $= 2(1) - (-2) = 4$

RHS $= 4$

Method 2: Elimination

$$2x - y = 4 \quad (1)$$
$$x + 2y = -3 \quad (2)$$

To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

When we multiply equation (2) by 2, the pair of equations becomes:

$$2x - y = 4 \quad (1)$$
$$2x + 4y = -6 \quad (2')$$

Subtract (2′) from (1):

$$-5y = 10$$
$$y = -2$$

Now substitute for y in equation (2) to find x, and check as in the substitution method.

Explanation

Using one of the two equations, express one variable in terms of the other variable.

Then substitute this expression into the other equation (reducing it to an equation in one variable, y). Solve the equation for y.

Substitute this value for y in one of the equations to find the other variable, x.

A check can be carried out with the other equation.

If one of the variables has the same coefficient in the two equations, we can eliminate that variable by subtracting one equation from the other.

It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same in the two equations.

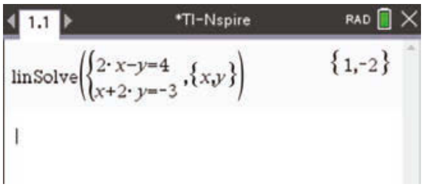
Note: This example shows that the point $(1, -2)$ is the point of intersection of the graphs of the two linear relations.

Using the TI-Nspire

Method 1: Using a Calculator application

Simultaneous linear equations can be solved in a **Calculator** application.

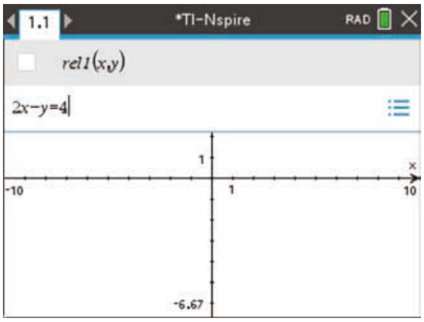
- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- Complete the pop-up screen.
- Enter the equations as shown to give the solution to the simultaneous equations $2x - y = 4$ and $x + 2y = -3$.
- Hence the solution is $x = 1$ and $y = -2$.

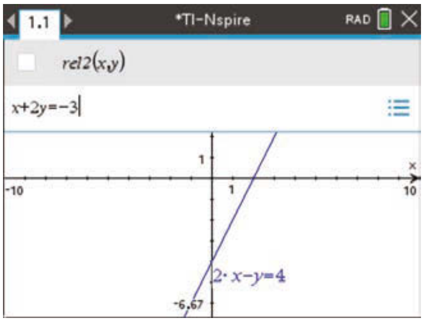


Method 2: Using a Graphs application

Simultaneous linear equations can also be solved graphically in a **Graphs** application.

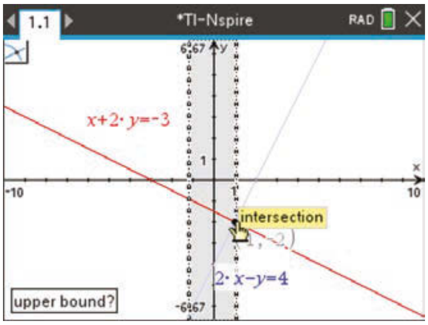
- Equations of the form $a \cdot x + b \cdot y = c$ can be entered directly using (menu) > **Graph Entry/Edit** > **Relation**. (Alternatively, you can use (menu) > **Graph Entry/Edit** > **Equation Templates** > **Line** > **Line Standard** $\boldsymbol{a \cdot x + b \cdot y = c}$.)

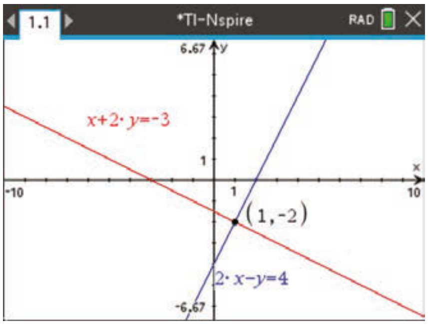




Note: If the entry line is not visible, press (tab). Pressing (enter) will hide the entry line. If you want to add more equations, use ▼ to add the next equation.

- The intersection point can be found using (menu) > **Analyze Graph** > **Intersection**.
- Move the cursor to the left of the intersection point (lower bound) and click. Then move the cursor to the right of the intersection point (upper bound). Click to paste the coordinates to the screen.



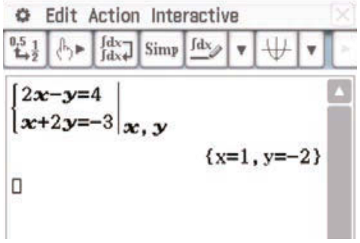


Note: Alternatively, use (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)** and click on both graphs.

Using the Casio ClassPad

To solve the simultaneous equations algebraically:

- Open the Main $\sqrt{\alpha}$ application and turn on the keyboard.
- In Math1, tap the simultaneous equations icon {.
- Enter the two equations as shown.
- Type x, y in the bottom-right square to indicate the variables.
- Tap EXE.

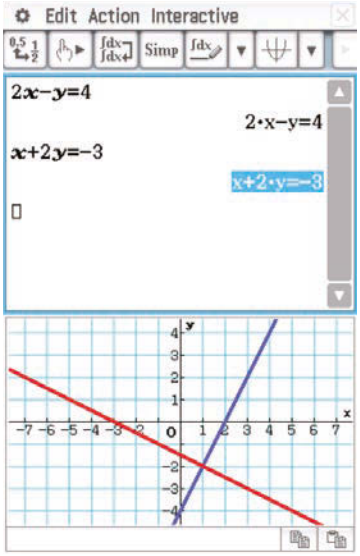


There are two methods for solving simultaneous equations graphically.

Method 1

In the Main $\sqrt{\alpha}$ application:

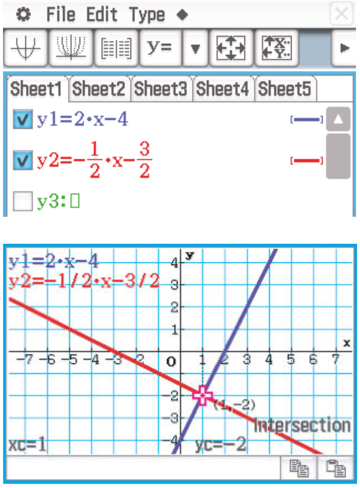
- Enter the equation $2x - y = 4$ and tap EXE.
- Enter the equation $x + 2y = -3$ and tap EXE.
- Select the graph icon from the toolbar to insert a graph window. An appropriate window can be set by selecting **Zoom** > **Quick** > **Quick Standard**.
- Highlight each equation and drag it into the graph window.
- To find the point of intersection, go to **Analysis** > **G-Solve** > **Intersection**.



Method 2

For this method, the equations need to be rearranged to make y the subject. In this form, the equations are $y = 2x - 4$ and $y = -\frac{1}{2}x - \frac{3}{2}$.

- Open the menu Menu; select **Graph & Table**.
- Tap in the working line of $y1$ and enter $2x - 4$.
- Tap in the working line of $y2$ and enter $-\frac{1}{2}x - \frac{3}{2}$.
- Tick the boxes for $y1$ and $y2$.
- Select the graph icon from the toolbar.
- Go to **Analysis** > **G-Solve** > **Intersection**.
- If necessary, the view window settings can be adjusted by tapping the view window icon or by selecting **Zoom** > **Quick** > **Quick Standard**.



Summary 1C

- An equation is solved by finding the value or values of the variables that would make the statement true.
- A linear equation is one in which the variable is to the first power.
- There are often several different ways to solve a linear equation. The following steps provide some suggestions:
  1. Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
  2. Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.
- Methods for solving simultaneous linear equations in two variables by hand:

  Substitution

  - Make one of the variables the subject in one of the equations.
  - Substitute for that variable in the other equation.

  Elimination

  - Choose one of the two variables to eliminate.
  - Obtain the same or opposite coefficients for this variable in the two equations. To do this, multiply both sides of one or both equations by a number.
  - Add or subtract the two equations to eliminate the chosen variable.

Exercise 1C

Example 7a

1 Solve the following linear equations:

a $3x + 7 = 15$ **b** $8 - \frac{x}{2} = -16$ **c** $42 + 3x = 22$

d $\frac{2x}{3} - 15 = 27$ **e** $5(2x + 4) = 13$ **f** $-3(4 - 5x) = 24$

g $3x + 5 = 8 - 7x$ **h** $2 + 3(x - 4) = 4(2x + 5)$ **i** $\frac{2x}{5} - \frac{3}{4} = 5x$

j $6x + 4 = \frac{x}{3} - 3$

Example 7b

2 Solve the following linear equations:

a $\frac{x}{2} + \frac{2x}{5} = 16$ **b** $\frac{3x}{4} - \frac{x}{3} = 8$

c $\frac{3x - 2}{2} + \frac{x}{4} = -8$ **d** $\frac{5x}{4} - \frac{4}{3} = \frac{2x}{5}$

e $\frac{x - 4}{2} + \frac{2x + 5}{4} = 6$ **f** $\frac{3 - 3x}{10} - \frac{2(x + 5)}{6} = \frac{1}{20}$

g $\frac{3 - x}{4} - \frac{2(x + 1)}{5} = -24$ **h** $\frac{-2(5 - x)}{8} + \frac{6}{7} = \frac{4(x - 2)}{3}$

Example 8

3 Solve each of the following pairs of simultaneous equations:

a $3x + 2y = 2$
$2x - 3y = 6$

b $5x + 2y = 4$
$3x - y = 6$

c $2x - y = 7$
$3x - 2y = 2$

d $x + 2y = 12$
$x - 3y = 2$

e $7x - 3y = -6$
$x + 5y = 10$

f $15x + 2y = 27$
$3x + 7y = 45$

1D Solving problems with linear equations

Many problems can be solved by translating them into mathematical language and using an appropriate mathematical technique to find the solution. By representing the unknown quantity in a problem with a symbol (called a pronumeral or a variable) and constructing an equation from the information, the value of the unknown can be found by solving the equation.

Before constructing the equation, each variable and what it stands for (including the units) should be stated. All the elements of the equation must be in units of the same system.

Example 9

For each of the following, form the relevant linear equation and solve it for x:

a The length of the side of a square is $(x - 6)$ cm. Its perimeter is 52 cm.

b The perimeter of a square is $(2x + 8)$ cm. Its area is 100 cm^2.

Solution

a Perimeter = 4 × side length

Therefore

$$4(x - 6) = 52$$
$$x - 6 = 13$$

and so $x = 19$

b The perimeter of the square is $2x + 8$.

The length of one side is $\dfrac{2x + 8}{4} = \dfrac{x + 4}{2}$.

Thus the area is

$$\left(\frac{x + 4}{2}\right)^2 = 100$$

As the side length must be positive, this gives the linear equation

$$\frac{x + 4}{2} = 10$$

Therefore $x = 16$.

Example 10

An athlete trains for an event by gradually increasing the distance she runs each week over a five-week period. If she runs an extra 5 km each successive week and over the five weeks runs a total of 175 km, how far did she run in the first week?

Solution

Let the distance run in the first week be x km.

Then the distance run in the second week is $x + 5$ km, and the distance run in the third week is $x + 10$ km, and so on.

The total distance run is $x + (x + 5) + (x + 10) + (x + 15) + (x + 20)$ km.

$$\begin{aligned} \therefore \quad 5x + 50 &= 175 \\ 5x &= 125 \\ x &= 25 \end{aligned}$$

The distance she ran in the first week was 25 km.

Example 11

A man bought 14 books at a sale. Some cost \$15 each and the remainder cost \$12.50 each. In total he spent \$190. How many \$15 books and how many \$12.50 books did he buy?

Solution

Let n be the number of books costing \$15.
Then $14 - n$ is the number of books costing \$12.50.

$$\begin{aligned} \therefore \quad 15n + 12.5(14 - n) &= 190 \\ 15n + 175 - 12.5n &= 190 \\ 2.5n + 175 &= 190 \\ 2.5n &= 15 \\ n &= 6 \end{aligned}$$

He bought 6 books costing \$15 and 8 books costing \$12.50.

Summary 1D

Steps for solving a word problem with a linear equation

- Read the question carefully and write down the known information clearly.
- Identify the unknown quantity that is to be found.
- Assign a variable to this quantity.
- Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
- Solve the equation.
- Write a sentence answering the initial question.

Skill-sheet

Exercise 1D

Example 9

1 For each of the cases below, write down a relevant equation involving the variables defined, and solve the equation for parts **a**, **b** and **c**.

- **a** The length of the side of a square is $(x - 2)$ cm. Its perimeter is 60 cm.
- **b** The perimeter of a square is $(2x + 7)$ cm. Its area is 49 cm^2.
- **c** The length of a rectangle is $(x - 5)$ cm. Its width is $(12 - x)$ cm. The rectangle is twice as long as it is wide.
- **d** The length of a rectangle is $(2x + 1)$ cm. Its width is $(x - 3)$ cm. The perimeter of the rectangle is y cm.
- **e** n people each have a meal costing $\$p$. The total cost of the meal is $\$Q$.
- **f** S people each have a meal costing $\$p$. A 10% service charge is added to the cost. The total cost of the meal is $\$R$.
- **g** A machine working at a constant rate produces n bolts in 5 minutes. It produces 2400 bolts in 1 hour.
- **h** The radius of a circle is $(x + 3)$ cm. A sector subtending an angle of 60° at the centre is cut off. The arc length of the minor sector is a cm.

Example 10

2 Bronwyn and Noel have a women's clothing shop in Summerland. Bronwyn manages the shop and her sales are going up steadily over a particular period of time. They are going up by \$500 per week. If over a five-week period her sales total \$17 500, how much did she earn in the first week?

Example 11

3 Bronwyn and Noel have a women's clothing shop in Summerland. Sally, Adam and baby Lana came into the shop and Sally bought dresses and handbags. The dresses cost \$65 each and the handbags cost \$26 each. Sally bought 11 items and in total she spent \$598. How many dresses and how many handbags did she buy?

4 A rectangular courtyard is three times as long as it is wide. If the perimeter of the courtyard is 67 m, find the dimensions of the courtyard.

5 A wine merchant buys 50 cases of wine. He pays full price for half of them, but gets a 40% discount on the remainder. If he paid a total of \$2260, how much was the full price of a single case?

6 A real-estate agent sells 22 houses in six months. He makes a commission of \$11 500 per house on some and \$13 000 per house on the remainder. If his total commission over the six months was \$272 500, on how many houses did he make a commission of \$11 500?

7 Three boys compare their marble collections. The first boy has 14 fewer than the second boy, who has twice as many as the third. If between them they have 71 marbles, how many does each boy have?

8 Three girls are playing Scrabble. At the end of the game, their three scores add up to 504. Annie scored 10% more than Belinda, while Cassie scored 60% of the combined scores of the other two. What did each player score?

9 A biathlon event involves running and cycling, where the cycling component takes up the most time. Kim can cycle 30 km/h faster than she can run. If Kim spends 48 minutes running and a third as much time again cycling in an event that covers a total distance of 60 km, how fast can she run?

10 The mass of a molecule of a certain chemical compound is 2.45×10^{-22} g. If each molecule is made up of two carbon atoms and six oxygen atoms and the mass of an oxygen atom is one-third that of a carbon atom, find the mass of an oxygen atom.

11 Mother's pearl necklace fell to the floor. One-sixth of the pearls rolled under the fridge, one-third rolled under the couch, one-fifth of them behind the bookcase, and nine were found at her feet. How many pearls are there?

12 Parents say they don't have favourites, but everyone knows that's a lie. A father distributes $96 to his three children according to the following instructions: The middle child receives $12 less than the oldest, and the youngest receives one-third as much as the middle child. How much does each receive?

13 Kavindi has achieved an average mark of 88% on her first four maths tests. What mark would she need on her fifth test to increase her average to 90%?

14 In a particular class, 72% of the students have black hair. Five black-haired students leave the class, so that now 65% of the students have black hair. How many students were originally in the class?

15 Two tanks are being emptied. Tank A contains 100 litres of water and tank B contains 120 litres of water. Water runs from Tank A at 2 litres per minute, and water runs from tank B at 3 litres per minute. After how many minutes will the amount of water in the two tanks be the same?

16 Suppose that candle A is initially 10 cm tall and burns out after 2 hours. Candle B is initially 8 cm tall and burns out after 4 hours. Both candles are lit at the same time. Assuming 'constant burning rates':

a When is the height of candle A the same as the height of candle B?

b When is the height of candle A half the height of candle B?

c When is candle A 1 cm taller than candle B?

17 A motorist drove 320 km in $\frac{10}{3}$ hours. He drove part of the way at an average speed of 100 km/h and the rest of the way at an average speed of 90 km/h. What is the distance he travelled at 100 km/h?

18 Jarmila travels regularly between two cities. She takes $\frac{14}{3}$ hours if she travels at her usual speed. If she increases her speed by 3 km/h, she can reduce her time taken by 20 minutes. What is her usual speed?

1E Solving problems with simultaneous linear equations

When the relationship between two quantities is linear, we can find the constants which determine this linear relationship if we are given two sets of information satisfying the relationship. Simultaneous linear equations enable this to be done.

Another situation in which simultaneous linear equations may be used is where it is required to find the point of the Cartesian plane which satisfies two linear relations.

Example 12

There are two possible methods for paying gas bills:

Method A A fixed charge of \$25 per quarter + 50c per unit of gas used

Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

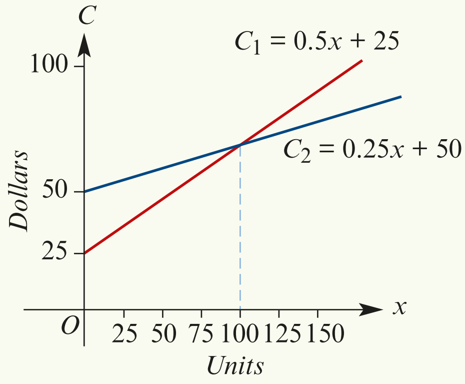
Let C_1 = charge (\$) using method A

C_2 = charge (\$) using method B

x = number of units of gas used

Then $C_1 = 25 + 0.5x$

$C_2 = 50 + 0.25x$



From the graph we see that method B is cheaper if the number of units exceeds 100.

The solution can be obtained by solving simultaneous linear equations:

$$C_1 = C_2$$

$$25 + 0.5x = 50 + 0.25x$$

$$0.25x = 25$$

$$x = 100$$

Example 13

If 3 kg of jam and 2 kg of butter cost \$29, and 6 kg of jam and 3 kg of butter cost \$54, find the cost per kilogram of jam and butter.

Solution

Let the cost of 1 kg of jam be x dollars and the cost of 1 kg of butter be y dollars.

Then $3x + 2y = 29$ (1)

and $6x + 3y = 54$ (2)

Multiply (1) by 2: $6x + 4y = 58$ (1′)

Subtract (1′) from (2): $-y = -4$

$y = 4$

Substitute in (2): $6x + 3(4) = 54$

$6x = 42$

$x = 7$

Jam costs \$7 per kilogram and butter costs \$4 per kilogram.

Summary 1E

Steps for solving a word problem with simultaneous linear equations

- Read the question carefully and write down the known information clearly.
- Identify the two unknown quantities that are to be found.
- Assign variables to these two quantities.
- Form expressions in terms of x and y (or other suitable variables) and use the other relevant information to form the two equations.
- Solve the system of equations.
- Write a sentence answering the initial question.

Exercise 1E

Example 12

1 A car hire firm offers the option of paying \$108 per day with unlimited kilometres, or \$63 per day plus 32 cents per kilometre travelled. How many kilometres would you have to travel in a given day to make the unlimited-kilometres option more attractive?

2 Company A will cater for your party at a cost of \$450 plus \$40 per guest. Company B offers the same service for \$300 plus \$43 per guest. How many guests are needed before Company A's charge is less than Company B's?

Example 13

3 A basketball final is held in a stadium which can seat 15 000 people. All the tickets have been sold, some to adults at \$45 and the rest for children at \$15. If the revenue from the tickets was \$525 000, find the number of adults who bought tickets.

4 A contractor employed eight men and three boys for one day and paid them a total of \$2240. Another day he employed six men and eighteen boys for \$4200. What was the daily rate he paid each man and each boy?

5 The sum of two numbers is 212 and their difference is 42. Find the two numbers.

6 A chemical manufacturer wishes to obtain 700 litres of a 24% acid solution by mixing a 40% solution with a 15% solution. How many litres of each solution should be used?

7 Two children had 220 marbles between them. After one child had lost half her marbles and the other had lost 40 marbles, they had an equal number of marbles. How many did each child start with and how many did each child finish with?

8 An investor received \$31 000 interest per annum from a sum of money, with part of it invested at 10% and the remainder at 7% simple interest. She found that if she interchanged the amounts she had invested she could increase her return by \$1000 per annum. Calculate the total amount she had invested.

9 Each adult paid \$30 to attend a concert and each student paid \$20. A total of 1600 people attended. The total paid was \$37 000. How many adults and how many students attended the concert?

10 Twelve teams play in a soccer league. Every team plays against each of the other teams once. In each game, the winner gets 3 points and the loser gets 0 points; in the case of a draw, the two teams get 1 point each. At the end of the season, the total number of points given to all teams was 180. How many games finished in a draw?

1F Substitution and transposition of formulas

An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A, the subject of the formula, may be found by substituting a given value of r and the value of π.

Example 14

Using the formula $A = \pi r^2$, find the value of A correct to two decimal places if $r = 2.3$ and $\pi = 3.142$ (correct to three decimal places).

Solution

$$\begin{aligned} A &= \pi r^2 \\ &= 3.142(2.3)^2 \\ &= 16.62118 \end{aligned}$$

$\therefore \quad A = 16.62$ correct to two decimal places

The formula $A = \pi r^2$ can also be transposed to make r the subject.

When transposing a formula, follow a similar procedure to solving a linear equation. Whatever has been done to the variable required is ‘undone’.

Example 15

a Transpose the formula $A = \pi r^2$ to make r the subject.

b Hence find the value of r correct to three decimal places if $A = 24.58$ and $\pi = 3.142$ (correct to three decimal places).

Solution

a $A = \pi r^2$

$\frac{A}{\pi} = r^2$

$r = \sqrt{\frac{A}{\pi}}$

b $r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{24.58}{3.142}}$

$= 2.79697\ldots$

$r = 2.797$ correct to three decimal places

Summary 1F

- A formula relates different quantities: for example, the formula $A = \pi r^2$ relates the radius r with the area A of the circle.
- The variable on the left is called the subject of the formula: for example, in the formula $A = \pi r^2$, the subject is A.
- To calculate the value of a variable which is not the subject of a formula:

  Method 1 Substitute the values for the known variables, then solve the resulting equation for the unknown variable.

  Method 2 Rearrange to make the required variable the subject, then substitute values.

Exercise 1F

Example 14

1 Substitute the specified values to evaluate each of the following, giving the answers correct to two decimal places:

a v if $v = u + at$ and $u = 15, a = 2, t = 5$

b I if $I = \frac{PrT}{100}$ and $P = 600, r = 5.5, T = 10$

c V if $V = \pi r^2 h$ and $r = 4.25, h = 6$

d S if $S = 2\pi r(r + h)$ and $r = 10.2, h = 15.6$

e V if $V = \frac{4}{3}\pi r^2 h$ and $r = 3.58, h = 11.4$

f s if $s = ut + \frac{1}{2}at^2$ and $u = 25.6, t = 3.3, a = -1.2$

g T if $T = 2\pi\sqrt{\frac{\ell}{g}}$ and $\ell = 1.45, g = 9.8$

h f if $\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$ and $v = 3, u = 7$

i c if $c^2 = a^2 + b^2$ and $a = 8.8, b = 3.4$

j v if $v^2 = u^2 + 2as$ and $u = 4.8, a = 2.5, s = 13.6$

Example 15

2 Transpose each of the following to make the symbol in brackets the subject:

a $v = u + at$ (a)

b $S = \frac{n}{2}(a + \ell)$ (ℓ)

c $A = \frac{1}{2}bh$ (b)

d $P = I^2R$ (I)

e $s = ut + \frac{1}{2}at^2$ (a)

f $E = \frac{1}{2}mv^2$ (v)

g $Q = \sqrt{2gh}$ (h)

h $-xy - z = xy + z$ (x)

i $\frac{ax + by}{c} = x - b$ (x)

j $\frac{mx + b}{x - b} = c$ (x)

3 The formula $F = \frac{9C}{5} + 32$ is used to convert temperatures given in degrees Celsius (C) to degrees Fahrenheit (F).

a Convert 28 degrees Celsius to degrees Fahrenheit.

b Transpose the formula to make C the subject and find C if $F = 135°$.

4 The sum, S, of the interior angles of a polygon with n sides is given by the formula $S = 180(n - 2)$.

a Find the sum of the interior angles of an octagon.

b Transpose the formula to make n the subject and hence determine the number of sides of a polygon whose interior angles add up to 1260°.

5 The volume, V, of a right cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.

a Find the volume of a cone with radius 3.5 cm and height 9 cm.

b Transpose the formula to make h the subject and hence find the height of a cone with base radius 4 cm and volume 210 cm^3.

c Transpose the formula to make r the subject and hence find the radius of a cone with height 10 cm and volume 262 cm^3.

6 For a particular type of sequence of numbers, the sum (S) of the terms in the sequence is given by the formula

$$S = \frac{n}{2}(a + \ell)$$

where n is the number of terms in the sequence, a is the first term and ℓ is the last term.

a Find the sum of such a sequence of seven numbers whose first term is −3 and whose last term is 22.

b What is the first term of such a sequence of 13 numbers whose last term is 156 and whose sum is 1040?

c How many terms are there in the sequence $25 + 22 + 19 + \cdots + (-5) = 110$?

1G Algebraic fractions

The principles involved in addition, subtraction, multiplication and division of algebraic fractions are the same as for simple numerical fractions.

Addition and subtraction

To add or subtract, all fractions must be written with a common denominator.

Example 16

Simplify:

a $\dfrac{x}{3} + \dfrac{x}{4}$

b $\dfrac{2}{x} + \dfrac{3a}{4}$

c $\dfrac{5}{x+2} - \dfrac{4}{x-1}$

d $\dfrac{4}{x+2} - \dfrac{7}{(x+2)^2}$

Solution

a $$\frac{x}{3} + \frac{x}{4} = \frac{4x+3x}{12} = \frac{7x}{12}$$

b $$\frac{2}{x} + \frac{3a}{4} = \frac{8+3ax}{4x}$$

c $$\frac{5}{x+2} - \frac{4}{x-1} = \frac{5(x-1)-4(x+2)}{(x+2)(x-1)} = \frac{5x-5-4x-8}{(x+2)(x-1)} = \frac{x-13}{(x+2)(x-1)}$$

d $$\frac{4}{x+2} - \frac{7}{(x+2)^2} = \frac{4(x+2)-7}{(x+2)^2} = \frac{4x+1}{(x+2)^2}$$

Multiplication and division

Before multiplying and dividing algebraic fractions, it is best to factorise numerators and denominators where possible so that common factors can be readily identified.

Example 17

Simplify:

a $\dfrac{3x^2}{10y^2} \times \dfrac{5y}{12x}$

b $\dfrac{2x-4}{x-1} \times \dfrac{x^2-1}{x-2}$

c $\dfrac{x^2-1}{2x-2} \times \dfrac{4x}{x^2+4x+3}$

d $\dfrac{x^2+3x-10}{x^2-x-2} \div \dfrac{x^2+6x+5}{3x+3}$

Solution

a $$\frac{3x^2}{10y^2} \times \frac{5y}{12x} = \frac{x}{8y}$$

b $\dfrac{2x-4}{x-1} \times \dfrac{x^2-1}{x-2} = \dfrac{2(x-2)}{x-1} \times \dfrac{(x-1)(x+1)}{x-2}$

$= 2(x+1)$

c $\dfrac{x^2-1}{2x-2} \times \dfrac{4x}{x^2+4x+3} = \dfrac{(x-1)(x+1)}{2(x-1)} \times \dfrac{4x}{(x+1)(x+3)}$

$= \dfrac{2x}{x+3}$

d $\dfrac{x^2+3x-10}{x^2-x-2} \div \dfrac{x^2+6x+5}{3x+3} = \dfrac{(x+5)(x-2)}{(x-2)(x+1)} \times \dfrac{3(x+1)}{(x+1)(x+5)}$

$= \dfrac{3}{x+1}$

More examples

The following two examples involve algebraic fractions and rational indices.

Example 18

Express $\dfrac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x}$ as a single fraction.

Solution

$$\frac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x} = \frac{3x^3 + 3x^2\sqrt{4-x}\,\sqrt{4-x}}{\sqrt{4-x}}$$

$$= \frac{3x^3 + 3x^2(4-x)}{\sqrt{4-x}}$$

$$= \frac{12x^2}{\sqrt{4-x}}$$

Example 19

Express $(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}}$ as a single fraction.

Solution

$$(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}} = (x-4)^{\frac{1}{5}} - \frac{1}{(x-4)^{\frac{4}{5}}}$$

$$= \frac{(x-4)^{\frac{1}{5}}(x-4)^{\frac{4}{5}} - 1}{(x-4)^{\frac{4}{5}}}$$

$$= \frac{x-5}{(x-4)^{\frac{4}{5}}}$$

Summary 1G

- **Simplifying algebraic fractions**
  - First factorise the numerator and denominator.
  - Then cancel any factors common to the numerator and denominator.
- **Adding and subtracting algebraic fractions**
  - First obtain a common denominator and then add or subtract.
- **Multiplying and dividing algebraic fractions**
  - First factorise each numerator and denominator completely.
  - Then complete the calculation by cancelling common factors.

Skill-sheet

Exercise 1G

Example 16

1 Simplify each of the following:

a $\frac{2x}{3} + \frac{3x}{2}$ **b** $\frac{3a}{2} - \frac{a}{4}$ **c** $\frac{3h}{4} + \frac{5h}{8} - \frac{3h}{2}$

d $\frac{3x}{4} - \frac{y}{6} - \frac{x}{3}$ **e** $\frac{3}{x} + \frac{2}{y}$ **f** $\frac{5}{x-1} + \frac{2}{x}$

g $\frac{3}{x-2} + \frac{2}{x+1}$ **h** $\frac{2x}{x+3} - \frac{4x}{x-3} - \frac{3}{2}$ **i** $\frac{4}{x+1} + \frac{3}{(x+1)^2}$

j $\frac{a-2}{a} + \frac{a}{4} + \frac{3a}{8}$ **k** $2x - \frac{6x^2-4}{5x}$ **l** $\frac{2}{x+4} - \frac{3}{x^2+8x+16}$

m $\frac{3}{x-1} + \frac{2}{(x-1)(x+4)}$ **n** $\frac{3}{x-2} - \frac{2}{x+2} + \frac{4}{x^2-4}$

o $\frac{5}{x-2} + \frac{3}{x^2+5x+6} + \frac{2}{x+3}$ **p** $x - y - \frac{1}{x-y}$

q $\frac{3}{x-1} - \frac{4x}{1-x}$ **r** $\frac{3}{x-2} + \frac{2x}{2-x}$

Example 17

2 Simplify each of the following:

a $\frac{x^2}{2y} \times \frac{4y^3}{x}$ **b** $\frac{3x^2}{4y} \times \frac{y^2}{6x}$ **c** $\frac{4x^3}{3} \times \frac{12}{8x^4}$

d $\frac{x^2}{2y} \div \frac{3xy}{6}$ **e** $\frac{4-x}{3a} \times \frac{a^2}{4-x}$ **f** $\frac{2x+5}{4x^2+10x}$

g $\frac{(x-1)^2}{x^2+3x-4}$ **h** $\frac{x^2-x-6}{x-3}$ **i** $\frac{x^2-5x+4}{x^2-4x}$

j $\frac{5a^2}{12b^2} \div \frac{10a}{6b}$ **k** $\frac{x-2}{x} \div \frac{x^2-4}{2x^2}$ **l** $\frac{x+2}{x(x-3)} \div \frac{4x+8}{x^2-4x+3}$

m $\frac{2x}{x-1} \div \frac{4x^2}{x^2-1}$ **n** $\frac{x^2-9}{x+2} \times \frac{3x+6}{x-3} \div \frac{9}{x}$ **o** $\frac{3x}{9x-6} \div \frac{6x^2}{x-2} \times \frac{2}{x+5}$

3 Express each of the following as a single fraction:

a $\dfrac{1}{x-3}+\dfrac{2}{x-3}$ **b** $\dfrac{2}{x-4}+\dfrac{2}{x-3}$ **c** $\dfrac{3}{x+4}+\dfrac{2}{x-3}$

d $\dfrac{2x}{x-3}+\dfrac{2}{x+4}$ **e** $\dfrac{1}{(x-5)^2}+\dfrac{2}{x-5}$ **f** $\dfrac{3x}{(x-4)^2}+\dfrac{2}{x-4}$

g $\dfrac{1}{x-3}-\dfrac{2}{x-3}$ **h** $\dfrac{2}{x-3}-\dfrac{5}{x+4}$ **i** $\dfrac{2x}{x-3}+\dfrac{3x}{x+3}$

j $\dfrac{1}{(x-5)^2}-\dfrac{2}{x-5}$ **k** $\dfrac{2x}{(x-6)^3}-\dfrac{2}{(x-6)^2}$ **l** $\dfrac{2x+3}{x-4}-\dfrac{2x-4}{x-3}$

Example 18

4 Express each of the following as a single fraction:

a $\sqrt{1-x}+\dfrac{2}{\sqrt{1-x}}$ **b** $\dfrac{2}{\sqrt{x-4}}+\dfrac{2}{3}$ **c** $\dfrac{3}{\sqrt{x+4}}+\dfrac{2}{\sqrt{x+4}}$

d $\dfrac{3}{\sqrt{x+4}}+\sqrt{x+4}$ **e** $\dfrac{3x^3}{\sqrt{x+4}}-3x^2\sqrt{x+4}$ **f** $\dfrac{3x^3}{2\sqrt{x+3}}+3x^2\sqrt{x+3}$

Example 19

5 Simplify each of the following:

a $(6x-3)^{\frac{1}{3}}-(6x-3)^{-\frac{2}{3}}$ **b** $(2x+3)^{\frac{1}{3}}-2x(2x+3)^{-\frac{2}{3}}$ **c** $(3-x)^{\frac{1}{3}}-2x(3-x)^{-\frac{2}{3}}$

1H Literal equations

A literal equation in x is an equation whose solution will be expressed in terms of pronumerals rather than numbers.

For the equation $2x+5=7$, the solution is the number 1.

For the literal equation $ax+b=c$, the solution is $x=\dfrac{c-b}{a}$.

Literal equations are solved in the same way as numerical equations. Essentially, the literal equation is transposed to make x the subject.

Example 20

Solve the following for x:

a $px-q=r$ **b** $ax+b=cx+d$ **c** $\dfrac{a}{x}=\dfrac{b}{2x}+c$

Solution

a
$$\begin{aligned} px-q&=r\\ px&=r+q\\ \therefore\quad x&=\frac{r+q}{p}\end{aligned}$$

b
$$\begin{aligned} ax+b&=cx+d\\ ax-cx&=d-b\\ x(a-c)&=d-b\\ \therefore\quad x&=\frac{d-b}{a-c}\end{aligned}$$

c Multiply both sides by $2x$:
$$\begin{aligned} \frac{a}{x}&=\frac{b}{2x}+c\\ 2a&=b+2xc\\ 2a-b&=2xc\\ \therefore\quad x&=\frac{2a-b}{2c}\end{aligned}$$

Simultaneous literal equations

Simultaneous literal equations are solved by the same methods that are used for solving simultaneous equations, i.e. substitution and elimination.

Example 21

Solve each of the following pairs of simultaneous equations for x and y:

a $y = ax + c$

$y = bx + d$

b $ax - y = c$

$x + by = d$

Solution

a Equate the two expressions for y:

$$ax + c = bx + d$$
$$ax - bx = d - c$$
$$x(a - b) = d - c$$

Thus $$x = \frac{d - c}{a - b}$$

and $$y = a\left(\frac{d - c}{a - b}\right) + c = \frac{ad - ac + ac - bc}{a - b} = \frac{ad - bc}{a - b}$$

b We will use the method of elimination, and eliminate y.

First number the two equations:

$$ax - y = c \quad (1)$$
$$x + by = d \quad (2)$$

Multiply (1) by b:

$$abx - by = bc \quad (1')$$

Add (1′) and (2):

$$abx + x = bc + d$$
$$x(ab + 1) = bc + d$$
$$\therefore \quad x = \frac{bc + d}{ab + 1}$$

Substitute in (1):

$$y = ax - c = a\left(\frac{bc + d}{ab + 1}\right) - c = \frac{ad - c}{ab + 1}$$

Summary 1H

- An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.
- The methods for solving linear literal equations or simultaneous linear literal equations are exactly the same as when the coefficients are given numbers.

Exercise 1H

Example 20

1 Solve each of the following for x:

a $ax + n = m$

b $ax + b = bx$

c $\dfrac{ax}{b} + c = 0$

d $px = qx + 5$

e $mx + n = nx - m$

f $\dfrac{1}{x + a} = \dfrac{b}{x}$

g $\dfrac{b}{x - a} = \dfrac{2b}{x + a}$

h $\dfrac{x}{m} + n = \dfrac{x}{n} + m$

i $-b(ax + b) = a(bx - a)$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

k $\dfrac{x}{a} - 1 = \dfrac{x}{b} + 2$

l $\dfrac{x}{a - b} + \dfrac{2x}{a + b} = \dfrac{1}{a^2 - b^2}$

m $\dfrac{p - qx}{t} + p = \dfrac{qx - t}{p}$

n $\dfrac{1}{x + a} + \dfrac{1}{x + 2a} = \dfrac{2}{x + 3a}$

2 For the simultaneous equations $ax + by = p$ and $bx - ay = q$, show that $x = \dfrac{ap + bq}{a^2 + b^2}$ and $y = \dfrac{bp - aq}{a^2 + b^2}$.

3 For the simultaneous equations $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$, show that $x = y = \dfrac{ab}{a + b}$.

Example 21

4 Solve each of the following pairs of simultaneous equations for x and y:

a $ax + y = c$
$x + by = d$

b $ax - by = a^2$
$bx - ay = b^2$

c $ax + by = t$
$ax - by = s$

d $ax + by = a^2 + 2ab - b^2$
$bx + ay = a^2 + b^2$

e $(a + b)x + cy = bc$
$(b + c)y + ax = -ab$

f $3(x - a) - 2(y + a) = 5 - 4a$
$2(x + a) + 3(y - a) = 4a - 1$

5 Write s in terms of a only in the following pairs of equations:

a $s = ah$
$h = 2a + 1$

b $s = ah$
$h = a(2 + h)$

c $as = a + h$
$h + ah = 1$

d $as = s + h$
$ah = a + h$

e $s = h^2 + ah$
$h = 3a^2$

f $as = a + 2h$
$h = a - s$

g $s = 2 + ah + h^2$
$h = a - \dfrac{1}{a}$

h $3s - ah = a^2$
$as + 2h = 3a$

1I Using a CAS calculator for algebra

Using the TI-Nspire

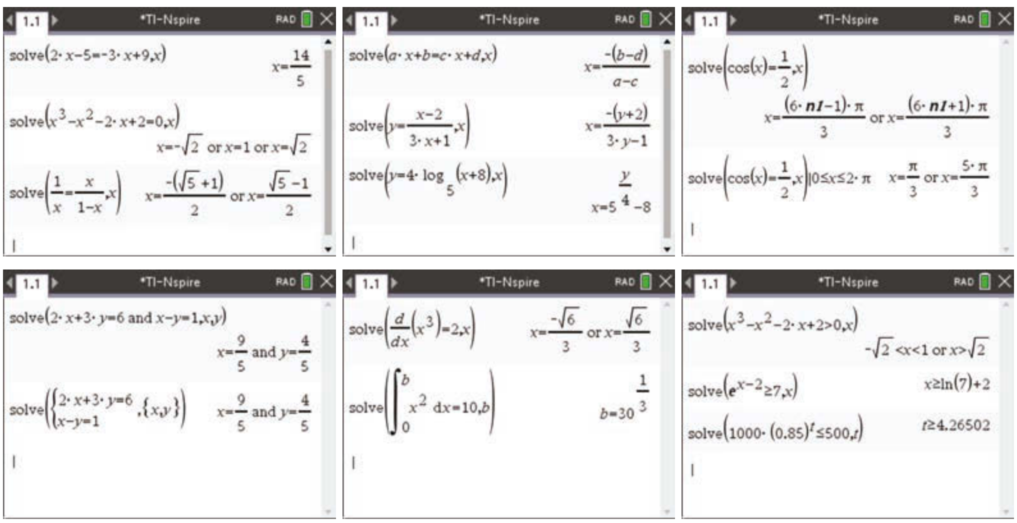
This section demonstrates the basic algebra commands of the TI-Nspire. To access these commands, open a **Calculator** application (home on > **New** > **Add Calculator**) and select menu > **Algebra**. The three main commands are solve, factor and expand.

1: Solve

This command is used to solve equations, simultaneous equations and some inequalities.

An approximate (decimal) answer can be obtained by pressing ctrl enter or by including a decimal number in the expression.

The following screens illustrate its use.

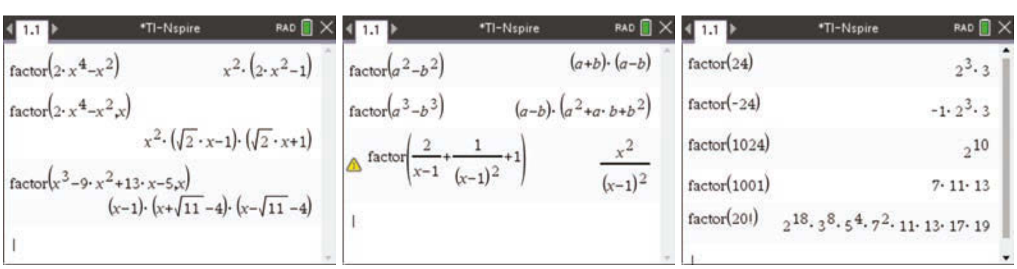


2: Factor

This command is used for factorisation.

Factorisation over the rational numbers is obtained by not specifying the variable, whereas factorisation over the real numbers is obtained by specifying the variable.

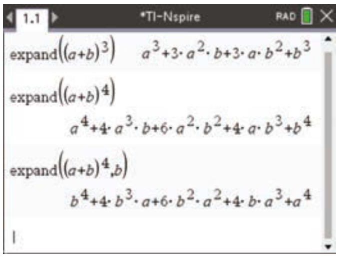
The following screens illustrate its use.

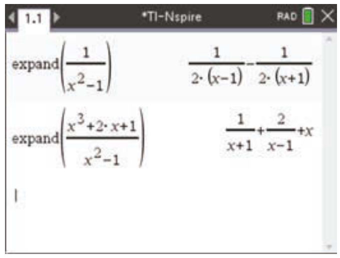


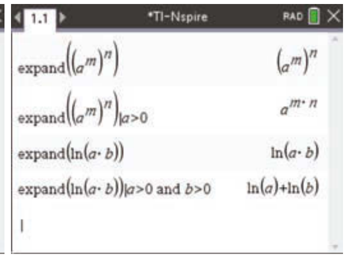
3: Expand

This command is used for expanding out expressions.

By specifying the variable, the expanded expression will be ordered in decreasing powers of that variable. Symbolic expressions can only be expanded for an appropriate domain.







Using the Casio ClassPad

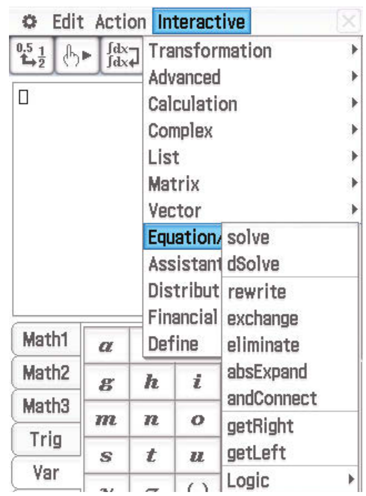
This section explores the Main $\sqrt{\alpha}$ application.

The **Interactive** menu is easiest to use with the stylus and the soft keyboards Math1, Math2 and Math3.

Solve

The **solve** command can be used to solve equations and inequalities. It can be accessed from the menu **Interactive** > **Equation/Inequality** or by tapping the icon solve(from the Math1 or Math3 keyboard.

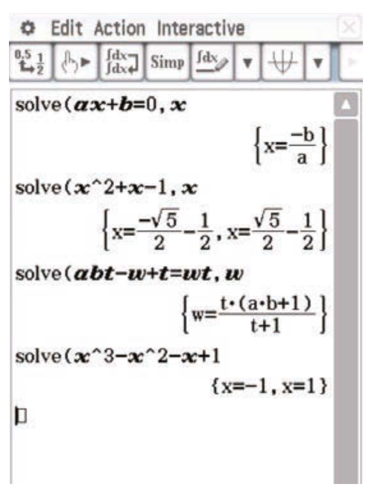
The variables x, y and z are found on the hard keyboard. Other variables should be entered using the Var keyboard. Variables are shown in bold italics.



Note: The abc keyboard is used for typing text. If you use the abc keyboard for variables, then you must type $a \times x$, for example, because ax will be treated as a single variable.

Examples:

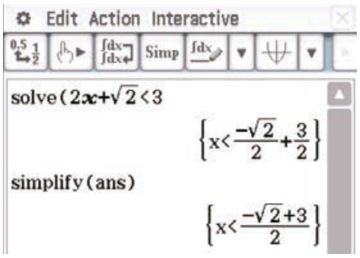
- To solve $ax + b = 0$ for x, first tap solve(in the Math1 keyboard. Enter $ax + b = 0, x$ as shown. (The variables and the comma are found in the Var keyboard.) Then tap EXE.
- Solve $x^2 + x - 1 = 0$ for x. Note that '= 0' has been omitted in this example. It is not necessary to enter the right-hand side of an equation if it is zero.
- Solve $abt - w + t = wt$ for w.
- Solve $x^3 - x^2 - x + 1 = 0$ for x. Note that ', x' has been omitted in this example. This is because the default setting is to solve for the variable x.



Note: Alternatively, the **solve** command can be accessed from the **Interactive** menu. To solve $ax + b = 0$ for x, enter the equation $ax + b = 0$ and highlight it with the stylus. Go to **Interactive** > **Equation/Inequality** > **solve**. The default setting is to solve for the variable x. Tap OK.

More examples:

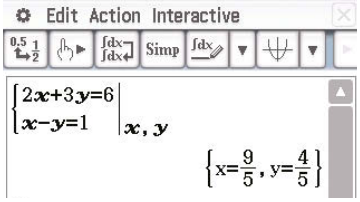
- Solve $2x + \sqrt{2} < 3$ for x.



Note: For the square root, use √■ from Math1. The inequality signs (<, >, ≤, ≥) are in Math3.

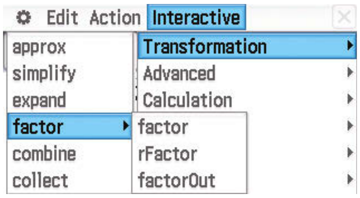
- To simplify the answer, tap Simp in the toolbar.

- To solve a pair of simultaneous equations, tap {■ from the Math1 keyboard and enter the equations and variables as shown.
- For more than two equations, tap {■ until the required number of equations is displayed.



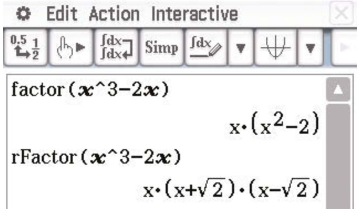
Factor

To factorise is to write an expression as a product of simpler expressions. The **factor** and **rFactor** commands are accessed from the menu **Interactive** > **Transformation** > **factor**.



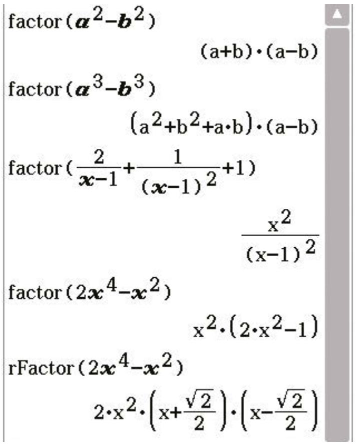
Examples:

- To factorise $x^3 - 2x$ over the rational numbers, use **factor**.
- To factorise over the real numbers, use **rFactor**.



More examples:

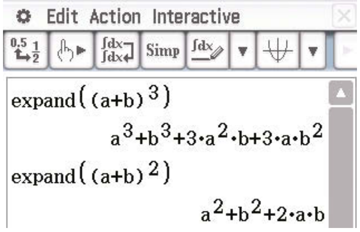
- Factorise $a^2 - b^2$.
- Factorise $a^3 - b^3$.
- Factorise $\dfrac{2}{x-1} + \dfrac{1}{(x-1)^2} + 1$.
- Factorise $2x^4 - x^2$ over the rationals.
- Factorise $2x^4 - x^2$ over the reals.



The **factor** command can also be used to give the prime decomposition (factors) of integers.

Expand

An expression can be expanded out by using **Interactive** > **Transformation** > **expand**.



Examples:

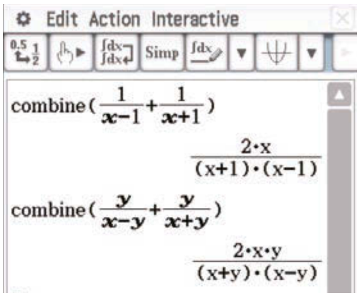
- Expand $(a+b)^3$.
- Expand $(a+b)^2$.

Approximate

Switch mode in the status bar to Decimal. If an answer is given in Standard (exact) mode, it can be converted by highlighting the answer and tapping in the toolbar.

Combining fractions

The **combine** command returns the answer as a single fraction with the denominator in factorised form.



Examples:

- Enter and highlight $\dfrac{1}{x-1}+\dfrac{1}{x+1}$. Then select **Interactive** > **Transformation** > **combine**.
- Enter and highlight $\dfrac{y}{x-y}+\dfrac{y}{x+y}$. Then select **Interactive** > **Transformation** > **combine**.

Exercise 1I

This exercise provides practice in some of the skills associated with a CAS calculator. Other exercises in this chapter can be attempted with CAS, but it is recommended that you also use this chapter to develop your 'by hand' skills.

1 Solve each of the following equations for x:

a $\dfrac{a(a-x)}{b}-\dfrac{b(b+x)}{a}=x$

b $2(x-3)+(x-2)(x-4)=x(x+1)-33$

c $\dfrac{x+a}{x+b}=1-\dfrac{x}{x-b}$

d $\dfrac{x+a}{x-c}+\dfrac{x+c}{x-a}=2$

2 Factorise each of the following:

a $x^2y^2-x^2-y^2+1$

b $x^3-2-x+2x^2$

c $a^4-8a^2b-48b^2$

d $a^2+2bc-(c^2+2ab)$

3 Solve each of the following pairs of simultaneous equations for x and y:

a $axy+b=(a+c)y$

$bxy+a=(b+c)y$

b $x(b-c)+by-c=0$

$y(c-a)-ax+c=0$

Review

Chapter summary

Assign-ment

Nrich

- **Indices**
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^n b^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $a^{-n} = \frac{1}{a^n}$
  - $a^0 = 1$
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
- A number is expressed in **standard form** or **scientific notation** when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 1.5×10^8
- **Linear equations**

  First identify the steps done to construct an equation; the equation is then solved by 'undoing' these steps. This is achieved by doing 'the opposite' in 'reverse order'.

  e.g.: Solve $3x + 4 = 16$ for x.

  Note that x has been multiplied by 3 and then 4 has been added.

  Subtract 4 from both sides: $3x = 12$

  Divide both sides by 3: $x = 4$
- An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A, the subject of the formula, may be found by substituting a given value of r and the value of π.

  A formula can be transposed to make a different variable the subject by using a similar procedure to solving linear equations, i.e. whatever has been done to the variable required is 'undone'.
- A **literal equation** is solved using the same techniques as for a numerical equation: transpose the literal equation to make the required variable the subject.

Technology-free questions

1 Simplify the following:

a $(x^3)^4$ **b** $(y^{-12})^{\frac{3}{4}}$ **c** $3x^{\frac{3}{2}} \times -5x^4$ **d** $(x^3)^{\frac{4}{3}} \times x^{-5}$

2 Express the product $32 \times 10^{11} \times 12 \times 10^{-5}$ in standard form.

3 Simplify the following:

a $\frac{3x}{5} + \frac{y}{10} - \frac{2x}{5}$ **b** $\frac{4}{x} - \frac{7}{y}$ **c** $\frac{5}{x+2} + \frac{2}{x-1}$

d $\frac{3}{x+2} + \frac{4}{x+4}$ **e** $\frac{5x}{x+4} + \frac{4x}{x-2} - \frac{5}{2}$ **f** $\frac{3}{x-2} - \frac{6}{(x-2)^2}$

4 Simplify the following:

a $\frac{x+5}{2x-6} \div \frac{x^2+5x}{4x-12}$ **b** $\frac{3x}{x+4} \div \frac{12x^2}{x^2-16}$

c $\frac{x^2-4}{x-3} \times \frac{3x-9}{x+2} \div \frac{9}{x+2}$ **d** $\frac{4x+20}{9x-6} \times \frac{6x^2}{x+5} \div \frac{2}{3x-2}$

5 The human body can produce 2.5 million red blood cells per second. If a person donates 500 mL of blood, how long will it take to replace the red blood cells if a litre of blood contains 5×10^{12} red blood cells?

6 The Sun is approximately 1.5×10^8 km from Earth and a comet is approximately 3×10^6 km from Earth. How many times further from Earth than the comet is the Sun?

7 Swifts Creek Soccer Team has played 54 matches over the past three seasons. They have drawn one-third of their games and won twice as many games as they have lost. How many games have they lost?

8 An online bookshop sells three types of books: crime, science fiction and romance. In one week they sold a total of 420 books. They sold 10% more crime than science fiction, while sales of romance constituted 50% more than the combined sales of crime and science fiction. How many of each type of book did they sell?

9 The volume, V, of a cylinder is given by the formula $V = \pi r^2 h$, where r is the radius of the base and h is the height of the cylinder.

a Find the volume of a cylinder with base radius 5 cm and height 12 cm.

b Transpose the formula to make h the subject and hence find the height of a cylinder with a base radius of 5 cm and a volume of 585 cm^3.

c Transpose the formula to make r the subject and hence find the radius of a cylinder with a height of 6 cm and a volume of 768 cm^3.

10 Solve for x:

a $xy + ax = b$

b $\dfrac{a}{x} + \dfrac{b}{x} = c$

c $\dfrac{x}{a} = \dfrac{x}{b} + 2$

d $\dfrac{a - dx}{d} + b = \dfrac{ax + d}{b}$

11 Simplify:

a $\dfrac{p}{p+q} + \dfrac{q}{p-q}$

b $\dfrac{1}{x} - \dfrac{2y}{xy - y^2}$

c $\dfrac{x^2 + x - 6}{x+1} \times \dfrac{2x^2 + x - 1}{x+3}$

d $\dfrac{2a}{2a+b} \times \dfrac{2ab + b^2}{ba^2}$

12 A is three times as old as B. In three years' time, B will be three times as old as C. In fifteen years' time, A will be three times as old as C. What are their present ages?

13 **a** Solve the following simultaneous equations for a and b:

$$a - 5 = \frac{1}{7}(b + 3) \qquad b - 12 = \frac{1}{5}(4a - 2)$$

b Solve the following simultaneous equations for x and y:

$$(p - q)x + (p + q)y = (p + q)^2$$
$$qx - py = q^2 - pq$$

14 A man has to travel 50 km in 4 hours. He does it by walking the first 7 km at x km/h, cycling the next 7 km at $4x$ km/h and motoring the remainder at $(6x + 3)$ km/h. Find x.

Review

15 Simplify each of the following:

a $2n^2 \times 6nk^2 \div (3n)$

b $\dfrac{8c^2x^3y}{6a^2b^3c^3} \div \dfrac{\frac{1}{2}xy}{15abc^2}$

16 Solve the equation $\dfrac{x+5}{15} - \dfrac{x-5}{10} = 1 + \dfrac{2x}{15}$.

Multiple-choice questions

1 For non-zero values of x and y, if $5x + 2y = 0$, then the ratio $\dfrac{y}{x}$ is equal to

A $-\dfrac{5}{2}$ **B** $-\dfrac{2}{5}$ **C** $\dfrac{2}{5}$ **D** 1 **E** $\dfrac{5}{4}$

2 The solution of the simultaneous equations $3x + 2y = 36$ and $3x - y = 12$ is

A $x = \dfrac{20}{3},\ y = 8$ **B** $x = 2,\ y = 0$ **C** $x = 1,\ y = -3$

D $x = \dfrac{20}{3},\ y = 6$ **E** $x = \dfrac{3}{2},\ y = -\dfrac{3}{2}$

3 The solution of the equation $t - 9 = 3t - 17$ is

A $t = -4$ **B** $t = \dfrac{11}{2}$ **C** $t = 4$ **D** $t = 2$ **E** $t = -2$

4 If $m = \dfrac{n-p}{n+p}$, then $p =$

A $\dfrac{n(1-m)}{1+m}$ **B** $\dfrac{n(m-1)}{1+m}$ **C** $\dfrac{n(1+m)}{1-m}$ **D** $\dfrac{n(1+m)}{m-1}$ **E** $\dfrac{m(n-1)}{m+1}$

5 $\dfrac{3}{x-3} - \dfrac{2}{x+3} =$

A 1 **B** $\dfrac{x+15}{x^2-9}$ **C** $\dfrac{15}{x-9}$ **D** $\dfrac{x+3}{x^2-9}$ **E** $-\dfrac{1}{9}$

6 $9x^2y^3 \div (15(xy)^3)$ is equal to

A $\dfrac{9x}{15}$ **B** $\dfrac{18xy}{5}$ **C** $\dfrac{3y}{5x}$ **D** $\dfrac{3x}{5}$ **E** $\dfrac{3}{5x}$

7 Transposing the formula $V = \dfrac{1}{3}h(\ell + w)$ gives $\ell =$

A $\dfrac{hw}{3V}$ **B** $\dfrac{3V}{h} - w$ **C** $\dfrac{3V - 2w}{h}$ **D** $\dfrac{3Vh}{2} - w$ **E** $\dfrac{1}{3}h(V + w)$

8 $\dfrac{(3x^2y^3)^2}{2x^2y} =$

A $\dfrac{9}{2}x^2y^7$ **B** $\dfrac{9}{2}x^2y^5$ **C** $\dfrac{9}{2}x^6y^7$ **D** $\dfrac{9}{2}x^6y^6$ **E** $\dfrac{9}{2}x^2y^4$

9 If X is 50% greater than Y and Y is 20% less than Z, then

A X is 30% greater than Z **B** X is 20% greater than Z **C** X is 20% less than Z

D X is 10% less than Z **E** X is 10% greater than Z

10 The average of two numbers is $5x + 4$. One of the numbers is x. The other number is

A $4x + 4$ **B** $9x + 8$ **C** $9x + 4$ **D** $10x + 8$ **E** $3x + 1$

11 $\dfrac{4}{(x+3)^2} + \dfrac{2x}{x+1}$ is equal to

A $\dfrac{8x}{(x+3)^2(x+1)}$ **B** $\dfrac{2(3x^2 + x + 18)}{(x+3)^2(x+1)}$ **C** $\dfrac{3x^2 + 13x + 18}{(x+3)^2(x+1)}$

D $\dfrac{2(3x^2 + 13x + 18)}{(x+3)^2(x+1)}$ **E** $\dfrac{2(x^3 + 6x^2 + 11x + 2)}{(x+3)^2(x+1)}$

Extended-response questions

1 Jack cycles home from work, a distance of $10x$ km. Benny leaves at the same time and drives the $40x$ km to his home.

a Write an expression in terms of x for the time taken for Jack to reach home if he cycles at an average speed of 8 km/h.

b Write an expression in terms of x for the time taken for Benny to reach home if he drives at an average speed of 70 km/h.

c In terms of x, find the difference in times of the two journeys.

d If Jack and Benny arrive at their homes 30 minutes apart:

i find x, correct to three decimal places

ii find the distance from work of each home, correct to the nearest kilometre.

2 Sam's plastic dinghy has sprung a leak and water is pouring in the hole at a rate of 27 000 cm^3 per minute. He grabs a cup and frantically starts bailing the water out at a rate of 9000 cm^3 per minute. The dinghy is shaped like a circular prism (cylinder) with a base radius of 40 cm and a height of 30 cm.

a How fast is the dinghy filling with water?

b Write an equation showing the volume of water, V cm^3, in the dinghy after t minutes.

c Find an expression for the depth of water, h cm, in the dinghy after t minutes.

d If Sam is rescued after 9 minutes, is this before or after the dinghy has completely filled with water?

3 Henry and Thomas Wong collect basketball cards. Henry has five-sixths the number of cards that Thomas has. The Wright family also collect cards. George Wright has half as many cards again as Thomas, Sally Wright has 18 fewer than Thomas, and Zeb Wright has one-third the number Thomas has.

a Write an expression for each child's number of cards in terms of the number Thomas has.

b The Wright family owns six more cards than the Wong family. Write an equation representing this information.

c Solve the equation from part **b** and use the result to find the number of cards each child has collected.

4 The gravitational force between two objects, F N, is given by the formula

$$F = \frac{6.67 \times 10^{-11} m_1 m_2}{r^2}$$

where m_1 and m_2 are the masses (in kilograms) of the two objects and r is the distance (in metres) between them.

a What is the gravitational force between two objects each weighing 200 kg if they are 12 m apart? Express the answer in standard form (to two significant figures).

b Transpose the above formula to make m_1 the subject.

c The gravitational force between a planet and an object 6.4×10^6 m away from the centre of the planet is found to be 2.4×10^4 N. If the object has a mass of 1500 kg, calculate the approximate mass of the planet, giving the answer in standard form (to two significant figures).

5 A water storage reservoir is 3 km wide, 6 km long and 30 m deep. (The water storage reservoir is assumed to be a cuboid.)

a Write an equation to show the volume of water, V m^3, in the reservoir when it is d metres full.

b Calculate the volume of water, V_F m^3, in the reservoir when it is completely filled.

The water flows from the reservoir down a long pipe to a hydro-electric power station in a valley below. The amount of energy, E J, that can be obtained from a full reservoir is given by the formula

$$E = kV_F h$$

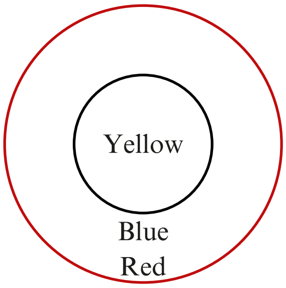
where k is a constant and h m is the length of the pipe.

c Find k, given that $E = 1.06 \times 10^{15}$ when $h = 200$, expressing the answer in standard form correct to three significant figures.

d How much energy could be obtained from a full reservoir if the pipe was 250 m long?

e If the rate of water falling through the pipe is 5.2 m^3/s, how many days without rain could the station operate before emptying an initially full reservoir?

6 A new advertising symbol is to consist of three concentric circles as shown, with the outer circle having a radius of 10 cm. It is desired that the three coloured regions cover the same area. Find the radius of the innermost circle in the figure shown.



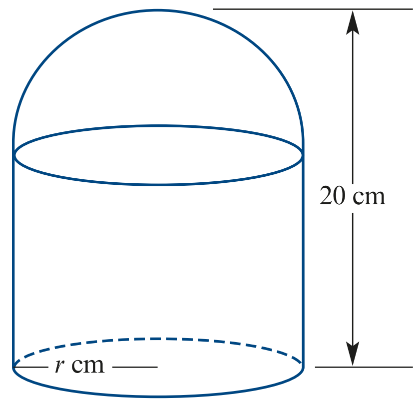
7 Temperatures in Fahrenheit (F) can be converted to Celsius (C) by the formula

$$C = \frac{5}{9}(F - 32)$$

Find the temperature which has the same numerical value in both scales.

8 A cyclist goes up a long slope at a constant speed of 15 km/h. He turns around and comes down the slope at a constant speed of 40 km/h. Find his average speed over a full circuit.

9 A container has a cylindrical base and a hemispherical top, as shown in the figure. The height of the container is 20 cm and its capacity is to be exactly 2 litres. Let r cm be the radius of the base.



a Express the height of the cylinder, h cm, in terms of r.

b **i** Express the volume of the container in terms of r.

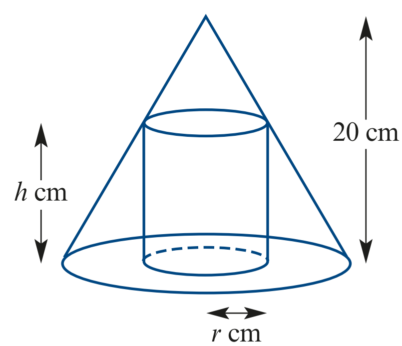
ii Find r and h if the volume is 2 litres.

10 **a** Two bottles contain mixtures of wine and water. In bottle A there is two times as much wine as water. In bottle B there is five times as much water as wine. Bottle A and bottle B are used to fill a third bottle, which has a capacity of 1 litre. How much liquid must be taken from each of bottle A and bottle B if the third bottle is to contain equal amounts of wine and water?

b Repeat for the situation where the ratio of wine to water in bottle A is $1 : 2$ and the ratio of wine to water in bottle B is $3 : 1$.

c Generalise the result for the ratio $m : n$ in bottle A and $p : q$ in bottle B.

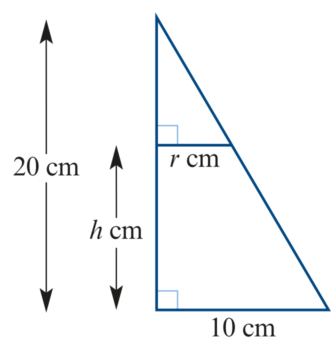
11 A cylinder is placed so as to fit into a cone as shown in the diagram.
The cone has a height of 20 cm and a base radius of 10 cm. The cylinder has a height of h cm and a base radius of r cm.



a Use similar triangles to find h in terms of r.

b The volume of the cylinder is given by the formula $V = \pi r^2 h$. Find the volume of the cylinder in terms of r.

c Use a CAS calculator to find the values of r and h for which the volume of the cylinder is 500 cm^3.



2

Number systems and sets

Objectives

- To understand and use **set notation**, including the symbols $\in$, $\subseteq$, $\cup$, $\cap$, $\emptyset$ and ξ.
- To be able to identify sets of numbers, including the natural numbers, integers, rational numbers, irrational numbers and real numbers.
- To understand and use **interval notation**.
- To know and apply the rules for working with **surds**, including:
  - simplification of surds
  - rationalisation of surds.
- To know and apply the definitions of **factor**, **prime number**, **highest common factor** and **lowest common multiple**.
- To be able to solve problems with sets.

This chapter introduces set notation and discusses numbers and their properties. Set notation is used widely in mathematics and in this book it is employed where appropriate. In fact, students who study mathematics at university will come to learn that all of mathematics is built upon the theory of sets.

In this chapter we discuss natural numbers, integers and rational numbers, and then continue on to consider the algebra of surds and the real numbers in general. We will use numbers and their properties to illustrate proof techniques in Chapter 6.

In the final section of this chapter, we solve various problems using sets. Here, the words ‘and’ and ‘or’ play a special role in the process of combining sets. The ideas introduced in this section provide some background for our study of logic in Chapter 7.

2A Set notation

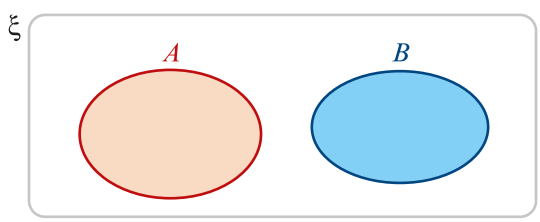
A **set** is a name given to any collection of things or numbers. There must be a way of deciding whether any particular object is a member of the set or not. This may be done by referring to a list of the members of the set or a statement describing them.

For example: $A = \{-3, 3\} = \{ x : x^2 = 9 \}$

Note: $\{ x : \ldots \}$ is read as 'the set of all x such that . . . '.

- The symbol $\in$ means 'is a member of' or 'is an element of'.
  For example: $3 \in$ {prime numbers} is read '3 is a member of the set of prime numbers'.
- The symbol $\notin$ means 'is not a member of' or 'is not an element of'.
  For example: $4 \notin$ {prime numbers} is read '4 is not a member of the set of prime numbers'.
- Two sets are **equal** if they contain exactly the same elements, not necessarily in the same order. For example: if A = {prime numbers less than 10} and $B = \{2, 3, 5, 7\}$, then $A = B$.
- The set with no elements is called the **empty set** and is denoted by $\varnothing$.
- The **universal set** will be denoted by ξ. The universal set is the set of all elements which are being considered.
- If all the elements of a set B are also elements of a set A, then the set B is called a **subset** of A. This is written $B \subseteq A$. For example: $\{a, b, c\} \subseteq \{a, b, c, d, e, f, g\}$ and $\{3, 9, 27\} \subseteq$ {multiples of 3}. We note also that $A \subseteq A$ and $\varnothing \subseteq A$.

Venn diagrams are used to illustrate sets. For example, the diagram on the right shows two subsets A and B of a universal set ξ such that A and B have no elements in common. Two such sets are said to be **disjoint**.



The union of two sets

The set of all the elements that are members of set A or set B (or both) is called the **union** of A and B. The union of A and B is written $A \cup B$.

Example 1

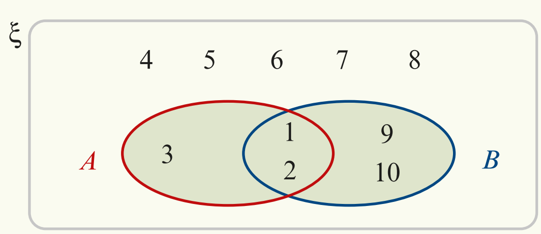
Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$ and $B = \{1, 2, 9, 10\}$.

Find $A \cup B$ and illustrate on a Venn diagram.

Solution

$A \cup B = \{1, 2, 3, 9, 10\}$

The shaded area illustrates $A \cup B$.



The intersection of two sets

The set of all the elements that are members of both set A and set B is called the **intersection** of A and B. The intersection of A and B is written $A \cap B$.

Example 2

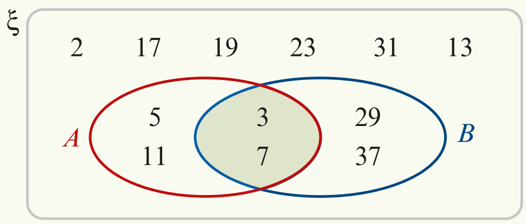
Let ξ = {prime numbers less than 40}, $A = \{3, 5, 7, 11\}$ and $B = \{3, 7, 29, 37\}$.

Find $A \cap B$ and illustrate on a Venn diagram.

Solution

$A \cap B = \{3, 7\}$

The shaded area illustrates $A \cap B$.



The complement of a set

The **complement** of a set A is the set of all elements of ξ that are not members of A. The complement of A is denoted by A'.

If ξ = {students at Highland Secondary College} and A = {students with blue eyes}, then A' is the set of all students at Highland Secondary College who do not have blue eyes.

Similarly, if $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8, 10\}$.

Example 3

Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

A = {odd numbers} = $\{1, 3, 5, 7, 9\}$

B = {multiples of 3} = $\{3, 6, 9\}$

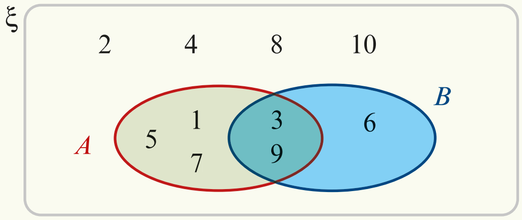
a Show these sets on a Venn diagram.

b Use the diagram to list the following sets:

i A' **ii** B' **iii** $A \cup B$ **iv** the complement of $A \cup B$, i.e. $(A \cup B)'$ **v** $A' \cap B'$

Solution

a



b From the diagram:

i $A' = \{2, 4, 6, 8, 10\}$

ii $B' = \{1, 2, 4, 5, 7, 8, 10\}$

iii $A \cup B = \{1, 3, 5, 6, 7, 9\}$

iv $(A \cup B)' = \{2, 4, 8, 10\}$

v $A' \cap B' = \{2, 4, 8, 10\}$

Finite and infinite sets

When all the elements of a set may be counted, the set is called a **finite** set. For example, the set A = {months of the year} is finite. The number of elements of a set A will be denoted $|A|$. In this example, $|A| = 12$. If C = {letters of the alphabet}, then $|C| = 26$.

Sets which are not finite are called **infinite** sets. For example, the set of real numbers, $\mathbb{R}$, and the set of integers, $\mathbb{Z}$, are infinite sets.

Summary 2A

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- The **empty set** is denoted by $\varnothing$ and the **universal set** by ξ.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The **complement** of A, denoted by A', is the set of all elements of ξ that are not in A.
- If two sets A and B have no elements in common, we say that they are **disjoint** and write $A \cap B = \varnothing$.

Exercise 2A

Example 1
Example 2
Example 3

1 Let $\xi = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 5\}$ and $B = \{2, 4\}$.
Show these sets on a Venn diagram and use the diagram to find:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

2 Let ξ = {natural numbers less than 17}, P = {multiples of 3} and Q = {even numbers}.
Show these sets on a Venn diagram and use it to find:

a P' **b** Q' **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

3 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, A = {multiples of 4} and B = {even numbers}.
Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

4 Let $\xi = \{p, q, r, s, t, u, v, w\}$, $X = \{r, s, t, w\}$ and $Y = \{q, s, t, u, v\}$.
Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cap Y'$ **d** $X' \cup Y'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

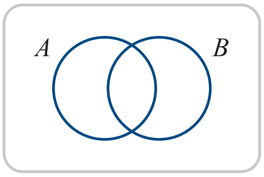
Which two sets are equal?

5 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, X = {factors of 12} and Y = {even numbers}.
Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cup Y'$ **d** $(X \cap Y)'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

6 Draw this diagram six times.



Use shading to illustrate each of the following sets:

a A' **b** B' **c** $A' \cap B'$

d $A' \cup B'$ **e** $A \cup B$ **f** $(A \cup B)'$

7 Let ξ = {different letters in the word *GENERAL*},

A = {different letters in the word *ANGEL*},

B = {different letters in the word *LEAN*}

Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cap B$ **d** $A \cup B$ **e** $(A \cup B)'$ **f** $A' \cup B'$

8 Let ξ = {different letters in the word *MATHEMATICS*}

A = {different letters in the word *ATTIC*}

B = {different letters in the word *TASTE*}

Show ξ, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cap B$ **d** $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

9 Let $C = \{1, 2, 3, 4, 5\}$.

a How many subsets of C have exactly two elements?

b How many subsets of C have exactly three elements?

c Your answers to parts **a** and **b** should be the same. Can you give a reason why this is to be expected?

2B Sets of numbers

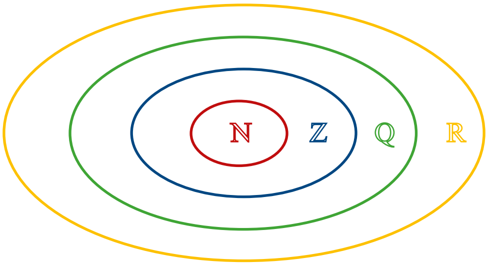
The elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational**. Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, π, $\pi + 2$ and $\sqrt{6} + \sqrt{7}$. These numbers cannot be written in the form $\frac{p}{q}$, for integers p, q; the decimal representations of these numbers do not terminate or repeat.

- The set of natural numbers is denoted by $\mathbb{N}$.
- The set of integers is denoted by $\mathbb{Z}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of real numbers is denoted by $\mathbb{R}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



We can use set notation to describe subsets of the real numbers.

For example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x \geq 3 \}$ is the set of all real numbers greater than or equal to 3
- $\{ 2n : n \in \mathbb{Z} \}$ is the set of all even integers.

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. That is,

$$\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$$

This set is known as the **Cartesian product** of $\mathbb{R}$ with itself.

Rational numbers

Every rational number can be expressed as a terminating or recurring decimal.

To find the decimal representation of a rational number $\frac{m}{n}$, perform the division $m \div n$.

For example, to find the decimal representation of $\frac{3}{7}$, divide $3.0000000\ldots$ by 7.

$$\begin{array}{r|l} & 0.\ \ 4\ \ 2\ \ 8\ \ 5\ \ 7\ \ 1\ \ 4\ \ldots \\ \hline 7 & 3.\ {}^{3}0\ {}^{2}0\ {}^{6}0\ {}^{4}0\ {}^{5}0\ {}^{1}0\ {}^{3}0\ \ldots \end{array}$$

Therefore $\frac{3}{7} = 0.\dot{4}2857\dot{1}$.

Theorem

Every rational number can be written as a terminating or recurring decimal.

Proof Consider any two natural numbers m and n. At each step in the division of m by n, there is a remainder. If the remainder is 0, then the division algorithm stops and the decimal is a terminating decimal.

If the remainder is never 0, then it must be one of the numbers $1, 2, 3, \ldots, n - 1$. (In the above example, $n = 7$ and the remainders can only be 1, 2, 3, 4, 5 and 6.) Hence the remainder must repeat after at most $n - 1$ steps.

Further examples:

$$\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\dot{3}, \quad \frac{1}{7} = 0.\dot{1}4285\dot{7}$$

Theorem

A real number has a terminating decimal representation if and only if it can be written as

$$\frac{m}{2^{\alpha} \times 5^{\beta}}$$

for some $m \in \mathbb{Z}$ and some $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

Proof Assume that $x = \dfrac{m}{2^\alpha \times 5^\beta}$ with $\alpha \geq \beta$. Multiply the numerator and denominator by $5^{\alpha-\beta}$. Then

$$x = \frac{m \times 5^{\alpha-\beta}}{2^\alpha \times 5^\alpha} = \frac{m \times 5^{\alpha-\beta}}{10^\alpha}$$

and so x can be written as a terminating decimal. The case $\alpha < \beta$ is similar.

Conversely, if x can be written as a terminating decimal, then there is $m \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{0\}$ such that $x = \dfrac{m}{10^\alpha} = \dfrac{m}{2^\alpha \times 5^\alpha}$.

The method for finding a rational number $\dfrac{m}{n}$ from its decimal representation is demonstrated in the following example.

Example 4

Write each of the following in the form $\dfrac{m}{n}$, where m and n are integers:

a 0.05 **b** $0.\dot{4}2857\dot{1}$

Solution

a $0.05 = \dfrac{5}{100} = \dfrac{1}{20}$

b We can write

$$0.\dot{4}2857\dot{1} = 0.428571428571\ldots \qquad (1)$$

Multiply both sides by 10^6:

$$0.\dot{4}2857\dot{1} \times 10^6 = 428571.428571428571\ldots \qquad (2)$$

Subtract (1) from (2):

$$0.\dot{4}2857\dot{1} \times (10^6 - 1) = 428571$$

$$\therefore \quad 0.\dot{4}2857\dot{1} = \frac{428571}{10^6 - 1} = \frac{3}{7}$$

Real numbers

The set of real numbers is made up of two important subsets: the **algebraic numbers** and the **transcendental numbers**.

An algebraic number is a solution to a polynomial equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad \text{where } a_0, a_1, \ldots, a_n \text{ are integers}$$

Every rational number is algebraic. This is because a rational number $\dfrac{p}{q}$, where p and q are integers, is the solution of the equation $qx - p = 0$.

The irrational number $\sqrt{2}$ is algebraic, as it is a solution of the equation $x^2 - 2 = 0$.

It can be shown that π is not an algebraic number; it is a transcendental number. The proof is too difficult to be given here.

A proof that $\sqrt{2}$ is irrational is presented in Chapter 6.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$

$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$

$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$

$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 5.

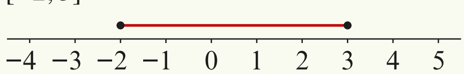
Example 5

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-2, 4)$ **d** $(-3, \infty)$ **e** $(-\infty, 2]$

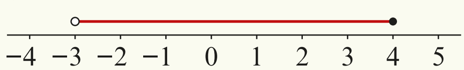
Solution

a $[-2, 3]$



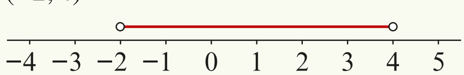
Explanation: The square brackets indicate that the endpoints are included; this is shown with closed circles.

b $(-3, 4]$



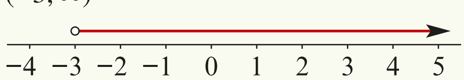
Explanation: The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

c $(-2, 4)$



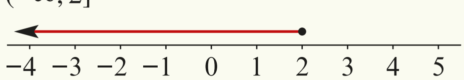
Explanation: Both brackets are round; the endpoints are not included.

d $(-3, \infty)$



Explanation: The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as ‘infinity’. The left endpoint is not included.

e $(-\infty, 2]$



Explanation: The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as ‘negative infinity’. The right endpoint is included.

Note: The ‘closed’ circle (●) indicates that the number is included.
The ‘open’ circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{ x : x > 0 \}$
- Negative real numbers: $\mathbb{R}^- = \{ x : x < 0 \}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Summary 2B

- Sets of numbers
  - Natural numbers: $\mathbb{N}$
  - Integers: $\mathbb{Z}$
  - Rational numbers: $\mathbb{Q}$
  - Real numbers: $\mathbb{R}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$

$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$

$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$

$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Exercise 2B

1 **a** Is the sum of two rational numbers always rational?
b Is the product of two rational numbers always rational?
c Is the quotient of two rational numbers always rational (if defined)?

2 **a** Is the sum of two irrational numbers always irrational?
b Is the product of two irrational numbers always irrational?
c Is the quotient of two irrational numbers always irrational?

Example 4

3 Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

a 0.45 **b** $0.\dot{2}$ **c** $0.\dot{2}\dot{7}$
d 0.12 **e** $0.\dot{3}\dot{6}$ **f** $0.\dot{2}8571\dot{4}$

4 Give the decimal representation of each of the following rational numbers:

a $\frac{2}{7}$ **b** $\frac{5}{11}$ **c** $\frac{7}{20}$ **d** $\frac{4}{13}$ **e** $\frac{1}{17}$

Example 5

5 Illustrate each of the following intervals of real numbers:

a $[-1, 4]$ **b** $(-2, 2]$ **c** $(-\infty, 3]$ **d** $(-1, 5)$ **e** $(-2, \infty)$

6 Write each of the following sets using interval notation:

a $\{ x : x < 3 \}$ **b** $\{ x : x \ge -3 \}$ **c** $\{ x : x \le -3 \}$
d $\{ x : x > 5 \}$ **e** $\{ x : -2 \le x < 3 \}$ **f** $\{ x : -2 \le x \le 3 \}$
g $\{ x : -2 < x \le 3 \}$ **h** $\{ x : -5 < x < 3 \}$

2C Surds

A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.

Note: $\sqrt{a}$ is taken to mean the positive square root.

In general, a **surd of order *n*** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.

For example:

- $\sqrt{7}$, $\sqrt{24}$, $\sqrt{\frac{9}{7}}$, $\sqrt{\frac{1}{2}}$ are quadratic surds
- $\sqrt{9}$, $\sqrt{16}$, $\sqrt{\frac{9}{4}}$ are *not* surds
- $\sqrt[3]{7}$, $\sqrt[3]{15}$ are surds of order 3
- $\sqrt[4]{100}$, $\sqrt[4]{26}$ are surds of order 4

Quadratic surds hold a prominent position in school mathematics. For example, the solutions of quadratic equations often involve surds:

$$x = \frac{1+\sqrt{5}}{2} \text{ is a solution of the quadratic equation } x^2 - x - 1 = 0.$$

Some well-known values of trigonometric functions involve surds. For example:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Exact solutions are often required in Mathematical Methods Units 3 & 4 and Specialist Mathematics Units 3 & 4.

Properties of square roots

The following properties of square roots are often used.

For positive numbers a and b:

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ e.g. $\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ e.g. $\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{\sqrt{9}} = \frac{\sqrt{7}}{3}$

Properties of surds

Simplest form

If possible, a factor which is the square of a rational number is ‘taken out’ of a square root. When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.

Example 6

Write each of the following in simplest form:

a $\sqrt{72}$ **b** $\sqrt{28}$ **c** $\sqrt{\dfrac{700}{117}}$ **d** $\sqrt{\dfrac{99}{64}}$

Solution

a $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$

b $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

c $\sqrt{\dfrac{700}{117}} = \dfrac{\sqrt{700}}{\sqrt{117}} = \dfrac{\sqrt{7 \times 100}}{\sqrt{9 \times 13}}$

$= \dfrac{10}{3}\sqrt{\dfrac{7}{13}}$

d $\sqrt{\dfrac{99}{64}} = \dfrac{\sqrt{99}}{\sqrt{64}} = \dfrac{\sqrt{9 \times 11}}{8}$

$= \dfrac{3\sqrt{11}}{8}$

Like surds

Surds which have the same 'irrational factor' are called **like surds**.

For example: $3\sqrt{7}$, $2\sqrt{7}$ and $\sqrt{7}$ are like surds.

The sum or difference of two like surds can be simplified:

- $m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p}$
- $m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$

Example 7

Express each of the following as a single surd in simplest form:

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

c $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

$= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3} - \sqrt{11^2 \times 3}$

$= 7\sqrt{3} + 6\sqrt{3} - 11\sqrt{3}$

$= 2\sqrt{3}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

$= \sqrt{3} + \sqrt{5} + 2\sqrt{5} + 3\sqrt{3} - 3\sqrt{5} - 4\sqrt{3}$

$= 0\sqrt{3} + 0\sqrt{5}$

$= 0$

c $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

$= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2}$

$= 8\sqrt{2} - 6\sqrt{2}$

$= 2\sqrt{2}$

Rationalising the denominator

In the past, a labour-saving procedure with surds was to **rationalise** any surds in the denominator of an expression. This is still considered to be a neat way of expressing final answers.

For $\sqrt{5}$, a rationalising factor is $\sqrt{5}$, as $\sqrt{5} \times \sqrt{5} = 5$.

For $1 + \sqrt{2}$, a rationalising factor is $1 - \sqrt{2}$, as $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1$.

For $\sqrt{3} + \sqrt{6}$, a rationalising factor is $\sqrt{3} - \sqrt{6}$, as $(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6}) = 3 - 6 = -3$.

Example 8

Rationalise the denominator of each of the following:

a $\dfrac{1}{2\sqrt{7}}$ **b** $\dfrac{1}{2 - \sqrt{3}}$ **c** $\dfrac{1}{\sqrt{3} - \sqrt{6}}$ **d** $\dfrac{3 + \sqrt{8}}{3 - \sqrt{8}}$

Solution

a $\dfrac{1}{2\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{\sqrt{7}}{14}$

b $\dfrac{1}{2 - \sqrt{3}} \times \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}} = \dfrac{2 + \sqrt{3}}{4 - 3}$

$= 2 + \sqrt{3}$

c $\dfrac{1}{\sqrt{3} - \sqrt{6}} \times \dfrac{\sqrt{3} + \sqrt{6}}{\sqrt{3} + \sqrt{6}} = \dfrac{\sqrt{3} + \sqrt{6}}{3 - 6}$

$= -\dfrac{1}{3}(\sqrt{3} + \sqrt{6})$

d $\dfrac{3 + \sqrt{8}}{3 - \sqrt{8}} = \dfrac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \times \dfrac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}}$

$= \dfrac{9 + 12\sqrt{2} + 8}{9 - 8}$

$= 17 + 12\sqrt{2}$

Example 9

Expand the brackets in each of the following and collect like terms, expressing surds in simplest form:

a $(3 - \sqrt{2})^2$ **b** $(3 - \sqrt{2})(1 + \sqrt{2})$

Solution

a $(3 - \sqrt{2})^2$

$= (3 - \sqrt{2})(3 - \sqrt{2})$

$= 3(3 - \sqrt{2}) - \sqrt{2}(3 - \sqrt{2})$

$= 9 - 3\sqrt{2} - 3\sqrt{2} + 2$

$= 11 - 6\sqrt{2}$

b $(3 - \sqrt{2})(1 + \sqrt{2})$

$= 3(1 + \sqrt{2}) - \sqrt{2}(1 + \sqrt{2})$

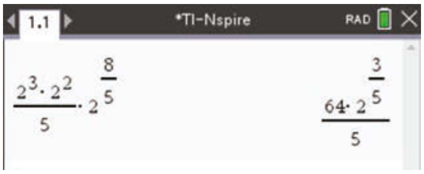
$= 3 + 3\sqrt{2} - \sqrt{2} - 2$

$= 1 + 2\sqrt{2}$

Using the TI-Nspire

Expressions on the screen can be selected using the up arrow ▲. This returns the expression to the entry line and modifications can be made.

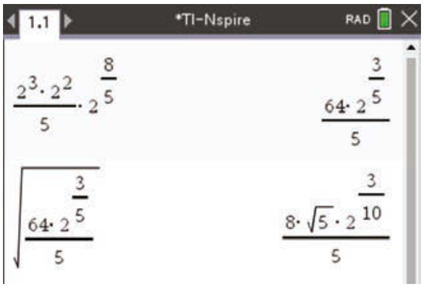
For example:



- Evaluate $\frac{2^3 \cdot 2^2}{5} \cdot 2^{\frac{8}{5}}$ as shown.

Note: The fraction template can be accessed using ctrl ÷.

- To find the square root of this expression, first type ctrl x^2. Then move upwards by pressing the up arrow ▲, so that the expression is highlighted.

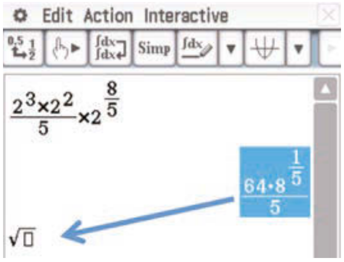


- Press enter to paste this expression into the square root sign.
- Press enter once more to evaluate the square root of this expression.

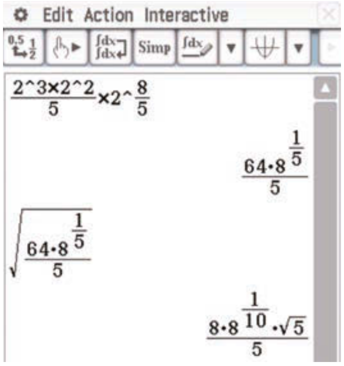
Using the Casio ClassPad

Expressions on the screen can be selected using the stylus. Highlight and drag the expression to the next entry line, where modifications can be made.

For example:



- Using templates from the Math1 keyboard, enter the expression $\frac{2^3 \times 2^2}{5} \times 2^{\frac{8}{5}}$.
- Tap EXE to evaluate.
- In the next entry line, tap $\sqrt{\blacksquare}$ from the Math1 keyboard.
- Highlight the expression and drag to the square root sign.
- Tap EXE to evaluate.



Note: Alternatively, highlight the expression and select **Edit** > **Copy**. Then tap the cursor in the desired position and select **Edit** > **Paste**.

Skill-sheet

Exercise 2C

Example 6

1 Express each of the following in terms of the simplest possible surds:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{27}$ **d** $\sqrt{50}$
e $\sqrt{45}$ **f** $\sqrt{1210}$ **g** $\sqrt{98}$ **h** $\sqrt{108}$
i $\sqrt{25}$ **j** $\sqrt{75}$ **k** $\sqrt{512}$

Example 7

2 Simplify each of the following:

a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$ **b** $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
c $\sqrt{28} + \sqrt{175} - \sqrt{63}$ **d** $\sqrt{1000} - \sqrt{40} - \sqrt{90}$
e $\sqrt{512} + \sqrt{128} + \sqrt{32}$ **f** $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

3 Simplify each of the following:

a $\sqrt{75} + \sqrt{108} + \sqrt{14}$ **b** $\sqrt{847} - \sqrt{567} + \sqrt{63}$
c $\sqrt{720} - \sqrt{245} - \sqrt{125}$ **d** $\sqrt{338} - \sqrt{288} + \sqrt{363} - \sqrt{300}$
e $\sqrt{12} + \sqrt{8} + \sqrt{18} + \sqrt{27} + \sqrt{300}$ **f** $2\sqrt{18} + 3\sqrt{5} - \sqrt{50} + \sqrt{20} - \sqrt{80}$

Example 8

4 Express each of the following with rational denominators:

a $\dfrac{1}{\sqrt{5}}$ **b** $\dfrac{1}{\sqrt{7}}$ **c** $-\dfrac{1}{\sqrt{2}}$ **d** $\dfrac{2}{\sqrt{3}}$ **e** $\dfrac{3}{\sqrt{6}}$

f $\dfrac{1}{2\sqrt{2}}$ **g** $\dfrac{1}{\sqrt{2}+1}$ **h** $\dfrac{1}{2-\sqrt{3}}$ **i** $\dfrac{1}{4-\sqrt{10}}$ **j** $\dfrac{2}{\sqrt{6}+2}$

k $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ **l** $\dfrac{3}{\sqrt{6}-\sqrt{5}}$ **m** $\dfrac{1}{3-2\sqrt{2}}$

Example 9

5 Express each of the following in the form $a + b\sqrt{c}$:

a $\dfrac{2}{3-2\sqrt{2}}$ **b** $(\sqrt{5}+2)^2$ **c** $(1+\sqrt{2})(3-2\sqrt{2})$ **d** $(\sqrt{3}-1)^2$

e $\sqrt{\dfrac{1}{3}} - \dfrac{1}{\sqrt{27}}$ **f** $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}$ **g** $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ **h** $\dfrac{\sqrt{8}+3}{\sqrt{18}+2}$

6 Expand and simplify each of the following:

a $(2\sqrt{a}-1)^2$ **b** $(\sqrt{x+1}+\sqrt{x+2})^2$

7 Since $8 = \sqrt{64}$ and $3\sqrt{7} = \sqrt{63}$, it is easy to see that $8 > 3\sqrt{7}$. Using the same idea, order these numbers from smallest to largest: $7,\ 3\sqrt{5},\ 5\sqrt{2},\ 4\sqrt{3}$.

8 For real numbers a and b, we have $a > b$ if and only if $a - b > 0$. Use this to state the larger of:

a $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$ **b** $2\sqrt{6} - 3$ and $7 - 2\sqrt{6}$

9 For positive real numbers a and b, we have $a > b$ if and only if $a^2 - b^2 > 0$. Use this to state the larger of:

a $\frac{2}{\sqrt{3}}$ and $\frac{3}{\sqrt{2}}$ **b** $\frac{\sqrt{7}}{3}$ and $\frac{\sqrt{5}}{2}$ **c** $\frac{\sqrt{3}}{7}$ and $\frac{\sqrt{5}}{5}$ **d** $\frac{\sqrt{10}}{2}$ and $\frac{8}{\sqrt{3}}$

10 Find the values of b and c for a quadratic function $f(x) = x^2 + bx + c$ such that the solutions of the equation $f(x) = 0$ are:

a $\sqrt{3},\ -\sqrt{3}$ **b** $2\sqrt{3},\ -2\sqrt{3}$ **c** $1 - \sqrt{2},\ 1 + \sqrt{2}$

d $2 - \sqrt{3},\ 2 + \sqrt{3}$ **e** $3 - 2\sqrt{2},\ 3 + 2\sqrt{2}$ **f** $4 - 7\sqrt{5},\ 3 + 2\sqrt{5}$

11 Express $\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

12 **a** Show that $a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$.

b Express $\frac{1}{1 - 2^{\frac{1}{3}}}$ with a rational denominator.

13 Evaluate $\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \cdots + \frac{1}{\sqrt{24} + \sqrt{25}}$.

2D Natural numbers

Factors and composites

The factors of 8 are 1, 2, 4 and 8.

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

The factors of 5 are 1 and 5.

A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.

A natural number greater than 1 is a **prime number** if its only factors are itself and 1.

The prime numbers less than 100 are:

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
53	59	61	67	71	73	79	83	89	97					

A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

Prime decomposition

Expressing a composite number as a product of powers of prime numbers is called **prime decomposition**. For example:

$$3000 = 3 \times 2^3 \times 5^3$$

$$2294 = 2 \times 31 \times 37$$

This is useful for finding the factors of a number. For example, the prime decomposition of 12 is given by $12 = 2^2 \times 3$. The factors of 12 are

$$1, \quad 2, \quad 2^2 = 4, \quad 3, \quad 2 \times 3 = 6 \quad \text{and} \quad 2^2 \times 3 = 12$$

This property of natural numbers is described formally by the following crucial theorem.

Fundamental theorem of arithmetic

Every natural number greater than 1 either is a prime number or can be represented as a product of prime numbers. Furthermore, this representation is unique apart from rearrangement of the order of the prime factors.

Example 10

Give the prime decomposition of 17 248 and hence list the factors of this number.

Solution

The prime decomposition can be found by repeated division, as shown on the right.

2	17 248
2	8624
2	4312
2	2156
2	1078
7	539
7	77
11	11
	1

The prime decomposition of 17 248 is

$$17\,248 = 2^5 \times 7^2 \times 11$$

Therefore each factor must be of the form

$$2^\alpha \times 7^\beta \times 11^\gamma$$

where $\alpha = 0, 1, 2, 3, 4, 5$, $\beta = 0, 1, 2$ and $\gamma = 0, 1$.

The factors of 17 248 can be systematically listed as follows:

1	2	2^2	2^3	2^4	2^5
7	2×7	$2^2 \times 7$	$2^3 \times 7$	$2^4 \times 7$	$2^5 \times 7$
7^2	2×7^2	$2^2 \times 7^2$	$2^3 \times 7^2$	$2^4 \times 7^2$	$2^5 \times 7^2$
11	2×11	$2^2 \times 11$	$2^3 \times 11$	$2^4 \times 11$	$2^5 \times 11$
7×11	$2 \times 7 \times 11$	$2^2 \times 7 \times 11$	$2^3 \times 7 \times 11$	$2^4 \times 7 \times 11$	$2^5 \times 7 \times 11$
$7^2 \times 11$	$2 \times 7^2 \times 11$	$2^2 \times 7^2 \times 11$	$2^3 \times 7^2 \times 11$	$2^4 \times 7^2 \times 11$	$2^5 \times 7^2 \times 11$

Highest common factor

The **highest common factor** of two natural numbers a and b is the largest natural number that is a factor of both a and b. It is denoted by HCF(a, b).

For example, the highest common factor of 15 and 24 is 3. We write HCF$(15, 24) = 3$.

Note: The highest common factor is also called the **greatest common divisor**.

Using prime decomposition to find HCF

Prime decomposition can be used to find the highest common factor of two numbers.

For example, consider the numbers 140 and 110. Their prime factorisations are

$$140 = 2^2 \times 5 \times 7 \quad \text{and} \quad 110 = 2 \times 5 \times 11$$

A number which is a factor of both 140 and 110 must have prime factors which occur in both these factorisations. The highest common factor of 140 and 110 is $2 \times 5 = 10$.

Next consider the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the highest common factor, we take the *lower power* of each prime factor:

$$\text{HCF}(396\,000, 1\,960\,200) = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11$$

Example 11

a Find the highest common factor of 528 and 3168.

b Find the highest common factor of 3696 and 3744.

Solution

a $528 = 2^4 \times 3 \times 11$

$3168 = 2^5 \times 3^2 \times 11$

$\therefore \text{HCF}(528, 3168) = 2^4 \times 3 \times 11$

$= 528$

b $3696 = 2^4 \times 3 \times 7 \times 11$

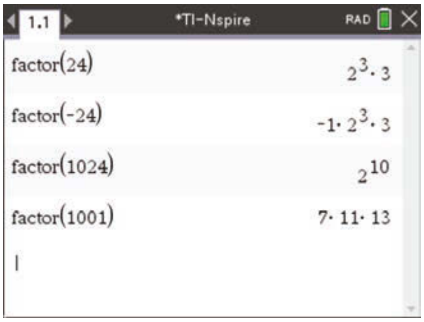
$3744 = 2^5 \times 3^2 \times 13$

$\therefore \text{HCF}(3696, 3744) = 2^4 \times 3$

$= 48$

Using the TI-Nspire

- The prime decomposition of a natural number can be obtained using menu > **Algebra** > **Factor** as shown.

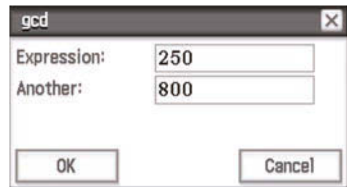


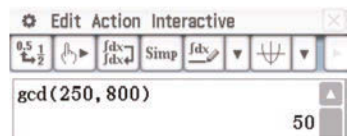
- The highest common factor of two numbers (also called their *greatest common divisor*) can be found by using the command **gcd( )** from (menu) > **Number** > **Greatest Common Divisor**, or by just typing it in, as shown.

Note: Nested **gcd( )** commands may be used to find the greatest common divisor of several numbers.

Using the Casio ClassPad

- To find the highest common factor of two numbers, go to **Interactive** > **Calculation** > **gcd/lcm** > **gcd**.
- Enter the required numbers in the two lines provided, and tap OK.





Lowest common multiple

- A natural number a is a **multiple** of a natural number b if there exists a natural number k such that $a = kb$.
- The **lowest common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b. It is denoted by $\text{LCM}(a, b)$.

For example: $\text{LCM}(24, 36) = 72$ and $\text{LCM}(256, 100) = 6400$.

Using prime decomposition to find LCM

Consider again the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the lowest common multiple, we take the *higher power* of each prime factor:

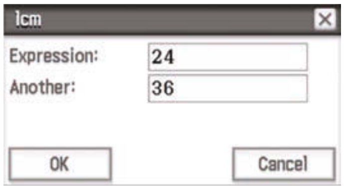
$$\text{LCM}(396\,000, 1\,960\,200) = 2^5 \cdot 3^4 \cdot 5^3 \cdot 11^2$$

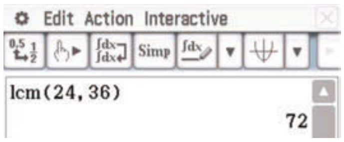
Using the TI-Nspire

The lowest common multiple of two numbers (also called their *least common multiple*) can be found by using the command **lcm( )** from (menu) > **Number** > **Least Common Multiple**, or by just typing it in, as shown.

Using the Casio ClassPad

- To find the lowest common multiple of two numbers, go to **Interactive** > **Calculation** > **gcd/lcm** > **lcm**.
- Enter the required numbers in the two lines provided, and tap OK.





Summary 2D

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b is the largest natural number that is a factor of both a and b. It is denoted by $\text{HCF}(a, b)$.
- The **lowest common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b. It is denoted by $\text{LCM}(a, b)$.

Exercise 2D

Example 10

1 Give the prime decomposition of each of the following numbers:

a 60 **b** 676 **c** 228 **d** 900 **e** 252
f 6300 **g** 68 640 **h** 96 096 **i** 32 032 **j** 544 544

Example 11

2 Find the highest common factor of each of the following pairs of numbers:

a 4361, 9281 **b** 999, 2160 **c** 5255, 716 845
d 1271, 3875 **e** 804, 2358

3 **a** List all the factors of 18 and all the factors of 36.
b Why does 18 have an even number of factors and 36 an odd number of factors?
c Find the smallest number greater than 100 with exactly three factors.

4 A woman has three children and two of them are teenagers, aged between 13 and 19. The product of their three ages is 1050. How old is each child?

5 By using prime decomposition, find a natural number n such that $22^2 \times 55^2 = 10^2 \times n^2$.

6 Find the smallest natural number n such that $60n$ is a square number.
Hint: First find the prime decomposition of 60.

7 The natural number n has exactly eight different factors. Two of these factors are 15 and 21. What is the value of n?

8 Let n be the smallest of three natural numbers whose product is 720. What is the largest possible value of n?

9 When all eight factors of 30 are multiplied together, the product is 30^k. What is the value of k?

10 A bell rings every 36 minutes and a buzzer rings every 42 minutes. If they sound together at 9 a.m., when will they next sound together?

11 The LCM of two numbers is $2^5 \times 3^3 \times 5^3$ and the HCF is $2^3 \times 3 \times 5^2$. Find all the possible numbers.

2E Problems involving sets

Sets can be used to help sort information, as each of the following examples demonstrates. Recall that, if A is a finite set, then the number of elements in A is denoted by $|A|$.

Example 12

Two hundred and eighty students each travel to school by either train or tram or both. Of these students, 150 travel by train, and 60 travel by both train and tram.

a Show this information on a Venn diagram.

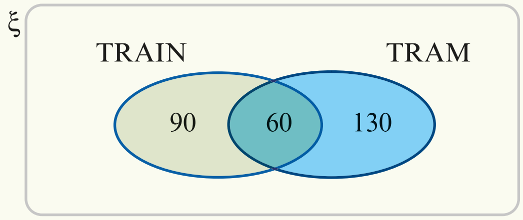
b Hence find the number of students who travel by:

i tram

ii train but not tram

iii just one of these modes of transport.

Solution

a 

$|\xi| = 280$

b **i** $|\text{TRAM}| = 130 + 60 = 190$

ii $|\text{TRAIN} \cap (\text{TRAM})'| = 90$

iii $|\text{TRAIN} \cap (\text{TRAM})'| + |(\text{TRAIN})' \cap \text{TRAM}| = 90 + 130 = 220$

Example 13

An athletics team has 18 members. Each member competes in at least one of three events: sprints (S), jumps (J) or hurdles (H). Every hurdler also jumps or sprints. The following additional information is available:

$$|S| = 11, \quad |J| = 10, \quad |J \cap H' \cap S'| = 5, \quad |J' \cap H' \cap S| = 5 \quad \text{and} \quad |J \cap H'| = 7$$

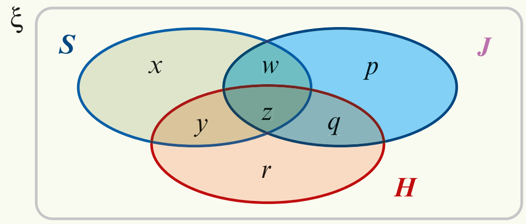
a Draw a Venn diagram.

b Find:

i $|H|$ **ii** $|S \cap H \cap J|$ **iii** $|S \cup J|$ **iv** $|S \cap J \cap H'|$

Solution

a Assign a variable to the number of members in each region of the Venn diagram.



The information in the question can be summarised in terms of these variables:

$x + y + z + w = 11$	as $\vert S\vert = 11$	(1)
$p + q + z + w = 10$	as $\vert J\vert = 10$	(2)
$x + y + z + w + p + q + r = 18$	as all members compete	(3)
$p = 5$	as $\vert J \cap H' \cap S'\vert = 5$	(4)
$x = 5$	as $\vert J' \cap H' \cap S\vert = 5$	(5)
$r = 0$	as every hurdler also jumps or sprints	(6)
$w + p = 7$	as $\vert J \cap H'\vert = 7$	(7)

From (4) and (7): $w = 2$.
Equation (3) now becomes

$$5 + y + z + 2 + 5 + q = 18$$

$$\therefore \quad y + z + q = 6 \qquad (8)$$

Equation (1) becomes

$$y + z = 4$$

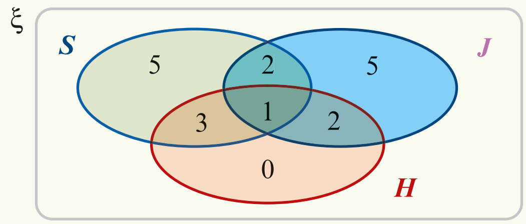
Therefore from (8): $q = 2$.
Equation (2) becomes

$$5 + 2 + z + 2 = 10$$

$$\therefore \quad z = 1$$

$$\therefore \quad y = 3$$

The Venn diagram can now be completed as shown.



b **i** $|H| = 6$ **ii** $|S \cap H \cap J| = 1$ **iii** $|S \cup J| = 18$ **iv** $|S \cap J \cap H'| = 2$

Skill-sheet

Exercise 2E

Example 12

1 There are 28 students in a class, all of whom take either History or Economics or both. Of the 14 students who take History, five also take Economics.

a Show this information on a Venn diagram.

b Hence find the number of students who take:

i Economics **ii** History but not Economics **iii** just one of these subjects.

2 a Draw a Venn diagram to show three sets A, B and C in a universal set ξ. Enter numbers in the correct parts of the diagram using the following information:

$$|A \cap B \cap C| = 2, \quad |A \cap B| = 7, \quad |B \cap C| = 6,$$

$$|A \cap C| = 8, \quad |A| = 16, \quad |B| = 20, \quad |C| = 19, \quad |\xi| = 50$$

b Use the diagram to find:

i $|A' \cap C'|$ **ii** $|A \cup B'|$ **iii** $|A' \cap B \cap C'|$

3 In a border town in the Balkans, 60% of people speak Bulgarian, 40% speak Greek and 20% speak neither. What percentage of the town speak both Bulgarian and Greek?

4 At an international conference there were 105 delegates. Seventy spoke English, 50 spoke French and 50 spoke Japanese. Twenty-five spoke English and French, 15 spoke French and Japanese and 30 spoke Japanese and English.

a How many delegates spoke all three languages?

b How many spoke Japanese only?

5 A restaurant serves lunch to 350 people. It offers three desserts: profiteroles, gelati and fruit. Forty people have all three desserts, 70 have gelati only, 50 have profiteroles only and 60 have fruit only. Forty-five people have fruit and gelati only, 30 people have gelati and profiteroles only and 10 people have fruit and profiteroles only. How many people do not have a dessert?

Example 13

6 Forty travellers were questioned about the various methods of transport they had used the previous day. Every traveller used at least one of the following methods: car (C), bus (B), train (T). Of these travellers:

- eight had used all three methods of transport
- four had travelled by bus and car only
- two had travelled by car and train only
- the number (x) who had travelled by train only was equal to the number who had travelled by bus and train only.

If 20 travellers had used a train and 33 had used a bus, find:

a the value of x

b the number who travelled by bus only

c the number who travelled by car only.

7 Let ξ be the set of all integers and let

$$X = \{ x : 21 < x < 37 \}, \quad Y = \{ 3y : 0 < y \leq 13 \}, \quad Z = \{ z^2 : 0 < z < 8 \}$$

a Draw a Venn diagram representing these sets.

b **i** Find $X \cap Y \cap Z$. **ii** Find $|X \cap Y|$.

8 A number of students bought red, green and black pens. Three bought one of each colour. Of the students who bought two colours, three did not buy red, five did not buy green and two did not buy black. The same number of students bought red only as bought red with other colours. The same number bought black only as bought green only. More students bought red and black but not green than bought black only. More bought only green than bought green and black but not red. How many students were there and how many pens of each colour were sold?

9 For three subsets B, M and F of a universal set ξ,

$$|B \cap M| = 12, \quad |M \cap F \cap B| = |F'|, \quad |F \cap B| > |M \cap F|,$$

$$|B \cap F' \cap M'| = 5, \quad |M \cap B' \cap F'| = 5, \quad |F \cap M' \cap B'| = 5, \quad |\xi| = 28$$

Find $|M \cap F|$.

10 A group of 80 students were interviewed about which sports they play. It was found that 23 do athletics, 22 swim and 18 play football. If 10 students do athletics and swim only, 11 students do athletics and play football only, six students swim and play football only and 46 students do none of these activities on a regular basis, how many students do all three?

11 At a certain secondary college, students have to be proficient in at least one of the languages Italian, French and German. In a particular group of 33 students, two are proficient in all three languages, three in Italian and French only, four in French and German only and five in German and Italian only. The number of students proficient in Italian only is x, in French only is x and in German only is $x + 1$. Find x and then find the total number of students proficient in Italian.

12 At a certain school, 201 students study one or more of Mathematics, Physics and Chemistry. Of these students: 35 take Chemistry only, 50% more students study Mathematics only than study Physics only, four study all three subjects, 25 study both Mathematics and Physics but not Chemistry, seven study both Mathematics and Chemistry but not Physics, and 20 study both Physics and Chemistry but not Mathematics. Find the number of students studying Mathematics.

Chapter summary

Assign-ment

Nrich

Sets

- Set notation

$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
ξ	the universal set
$\varnothing$	the empty set
$A \subseteq B$	A is a subset of B
$A \cup B$	the union of A and B consists of all elements that are in either A or B or both
$A \cap B$	the intersection of A and B consists of all elements that are in both A and B
A'	the complement of A consists of all elements of ξ that are not in A
$\lvert A \rvert$	the number of elements in a finite set A

- Sets of numbers

$\mathbb{N}$	Natural numbers	$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers	$\mathbb{R}$	Real numbers

Surds

- A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.
- A **surd of order *n*** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.
- When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.
- Surds which have the same 'irrational factor' are called **like surds**. The sum or difference of two like surds can be simplified:

$$m\sqrt{p} + n\sqrt{p} = (m+n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m-n)\sqrt{p}$$

Natural numbers

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b is the largest natural number that is a factor of both a and b. It is denoted by $\text{HCF}(a, b)$.
- The **lowest common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b. It is denoted by $\text{LCM}(a, b)$.

Technology-free questions

1 Express the following as fractions in their simplest form:

a $0.0\dot{7}$ **b** $0.\dot{4}\dot{5}$ **c** 0.005 **d** 0.405 **e** $0.2\dot{6}$ **f** $0.\dot{1}71428\dot{5}$

2 Express 504 as a product of powers of prime numbers.

3 Express each of the following with a rational denominator:

a $\dfrac{2\sqrt{3}-1}{\sqrt{2}}$ **b** $\dfrac{\sqrt{5}+2}{\sqrt{5}-2}$ **c** $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

4 Express $\dfrac{3+2\sqrt{75}}{3-\sqrt{12}}$ in the form $a+b\sqrt{3}$, where $a, b \in \mathbb{Q} \setminus \{0\}$.

5 Express each of the following with a rational denominator:

a $\dfrac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$ **b** $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$

6 In a class of 100 students, 55 are girls, 45 have blue eyes, 40 are blond, 25 are blond girls, 15 are blue-eyed blonds, 20 are blue-eyed girls, and five are blue-eyed blond girls. Find:

a the number of blond boys

b the number of boys who are neither blond nor blue-eyed.

7 A group of 30 students received prizes in at least one of the subjects of English, Mathematics and French. Two students received prizes in all three subjects. Fourteen received prizes in English and Mathematics but not French. Two received prizes in English alone, two in French alone and five in Mathematics alone. Four received prizes in English and French but not Mathematics.

a How many received prizes in Mathematics and French but not English?

b How many received prizes in Mathematics?

c How many received prizes in English?

8 Fifty people are interviewed. Twenty-three people like Brand X, 25 like Brand Y and 19 like Brand Z. Eleven like X and Z. Eight like Y and Z. Five like X and Y. Two like all three. How many like none of them?

9 Three rectangles A, B and C overlap (intersect). Their areas are 20 cm^2, 10 cm^2 and 16 cm^2 respectively. The area common to A and B is 3 cm^2, that common to A and C is 6 cm^2 and that common to B and C is 4 cm^2. How much of the area is common to all three if the total area covered is 35 cm^2?

10 Express $\sqrt{112}-\sqrt{63}-\dfrac{224}{\sqrt{28}}$ in simplest form.

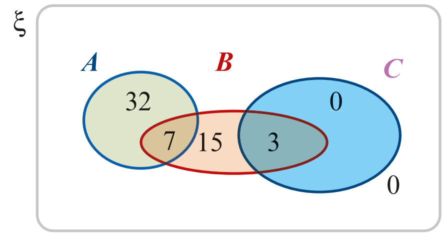
11 If $\dfrac{\sqrt{7}-\sqrt{3}}{x} = \dfrac{x}{\sqrt{7}+\sqrt{3}}$, find the values of x.

12 Express $\dfrac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}} + \dfrac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ in the form $a\sqrt{5} + b\sqrt{6}$.

13 Simplify $\sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\dfrac{48}{25}}$.

14 Using the result that $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$, determine the square root of $17 + 6\sqrt{8}$.

15 A, B and C are three sets and $\xi = A \cup B \cup C$. The number of elements in the regions of the Venn diagram are as shown. Find:



a the number of elements in $A \cup B$

b the number of elements in C

c the number of elements in $B' \cap A$.

Multiple-choice questions

1 $\dfrac{4}{3 + 2\sqrt{2}}$ expressed in the form $a + b\sqrt{2}$ is

A $12 - 8\sqrt{2}$ **B** $3 + 2\sqrt{2}$ **C** $\dfrac{3}{17} - \dfrac{8}{17}\sqrt{2}$

D $\dfrac{3}{17} + \dfrac{8}{17}\sqrt{2}$ **E** $12 + 8\sqrt{2}$

2 The prime decomposition of 86 400 is

A $2^5 \times 3^2 \times 5$ **B** $2^6 \times 3^3 \times 5^2$ **C** $2^7 \times 3^3 \times 5$ **D** $2^7 \times 3^3 \times 5^2$ **E** $2^6 \times 3^3 \times 5^3$

3 $(\sqrt{6} + 3)(\sqrt{6} - 3)$ is equal to

A $3 - 12\sqrt{6}$ **B** $-3 - 6\sqrt{6}$ **C** $-3 + 6\sqrt{6}$ **D** -3 **E** 3

4 For the Venn diagram shown, ξ is the set of natural numbers less than 20, A is the set of natural numbers less than 10, and B is the set of natural numbers less than 20 that are divisible by 3. The set $B' \cap A$ is

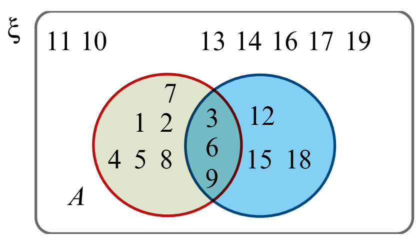
A $\{3, 6, 9\}$

B $\{12, 15, 18\}$

C $\{10, 11, 13, 14, 16, 17, 19\}$

D $\{1, 2, 4, 5, 7, 8\}$

E $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18\}$



5 $(3, \infty) \cap (-\infty, 5] =$

A $(-\infty, 3)$ **B** $(-\infty, 5]$ **C** $(3, 5]$ **D** $\mathbb{R}$ **E** $[3, 5]$

6 A bell is rung every 6 minutes and a gong is sounded every 14 minutes. If these occur together at a particular time, then the smallest number of minutes until the bell and the gong are again heard simultaneously is

A 10 **B** 20 **C** 72 **D** 42 **E** 84

7 If X is the set of multiples of 2, Y the set of multiples of 7, and Z the set of multiples of 5, then $X \cap Y \cap Z$ can be described as

A the set of multiples of 2 **B** the set of multiples of 70 **C** the set of multiples of 35
D the set of multiples of 14 **E** the set of multiples of 10

8 In a class of students, 50% play football, 40% play tennis and 30% play neither. The percentage that plays both is

A 10 **B** 20 **C** 30 **D** 50 **E** 40

9 $\dfrac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} =$

A $5+2\sqrt{7}$ **B** $13+2\sqrt{6}$ **C** $13-2\sqrt{42}$ **D** $1+2\sqrt{42}$ **E** $13-2\sqrt{13}$

10 There are 40 students in a class, all of whom take either Literature or Economics or both. Twenty take Literature and five of these also take Economics. The number of students who take only Economics is

A 20 **B** 5 **C** 10 **D** 15 **E** 25

11 The number of factors that the integer $2^p 3^q 5^r$ has is

A $\dfrac{(p+q+r)!}{p!\,q!\,r!}$ **B** pqr **C** $p+q+r$
D $(p+1)(q+1)(r+1)$ **E** $p+q+r+1$

12 The number of pairs of integers (m, n) which satisfy the equation $m+n = mn$ is

A 1 **B** 2 **C** 3 **D** 4 **E** more than 4

Extended-response questions

1 **a** Show that $(\sqrt{x}+\sqrt{y})^2 = x+y+2\sqrt{xy}$.

b Substitute $x = 3$ and $y = 5$ in the identity from part **a** to show that

$$\sqrt{3}+\sqrt{5} = \sqrt{8+2\sqrt{15}}$$

c Use this technique to find the square root of:

i $14+2\sqrt{33}$ (Hint: Use $x = 11$ and $y = 3$.) **ii** $15-2\sqrt{56}$ **iii** $51-36\sqrt{2}$

2 In this question, we consider the set $\{a+b\sqrt{3} : a, b \in \mathbb{Q}\}$. In Chapter 18, the set $\mathbb{C}$ of complex numbers is introduced, where $\mathbb{C} = \{a+b\sqrt{-1} : a, b \in \mathbb{R}\}$.

a If $(2+3\sqrt{3})+(4+2\sqrt{3}) = a+b\sqrt{3}$, find a and b.

b If $(2+3\sqrt{3})(4+2\sqrt{3}) = p+q\sqrt{3}$, find p and q.

c If $\dfrac{1}{3+2\sqrt{3}} = a+b\sqrt{3}$, find a and b.

d Solve each of the following equations for x:

i $(2+5\sqrt{3})x = 2-\sqrt{3}$ **ii** $(x-3)^2-3 = 0$ **iii** $(2x-1)^2-3 = 0$

e Explain why every rational number is a member of $\{a+b\sqrt{3} : a, b \in \mathbb{Q}\}$.

3 A **Pythagorean triple** (x, y, z) consists of three natural numbers x, y, z such that $x^2 + y^2 = z^2$. For example: $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples.
A Pythagorean triple is in simplest form if x, y, z have no common factor. Up to swapping x and y, all Pythagorean triples in simplest form may be generated by:

$$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2 \qquad \text{where } m, n \in \mathbb{N}$$

For example, if $m = 2$ and $n = 1$, then $x = 4$, $y = 3$ and $z = 5$.

a Find the Pythagorean triple for $m = 5$ and $n = 2$.

b Verify that, if $x = 2mn$, $y = m^2 - n^2$ and $z = m^2 + n^2$, where $m, n \in \mathbb{N}$, then $x^2 + y^2 = z^2$.

4 The factors of 12 are 1, 2, 3, 4, 6, 12.

a How many factors does each of the following numbers have?

i 2^3 **ii** 3^7

b How many factors does 2^n have?

c How many factors does each of the following numbers have?

i $2^3 \cdot 3^7$ **ii** $2^n \cdot 3^m$

d Every natural number greater than 1 may be expressed as a product of powers of primes; this is called prime decomposition. For example: $1080 = 2^3 \times 3^3 \times 5$.
Let x be a natural number greater than 1 and let

$$x = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$$

be its prime decomposition, where each $\alpha_i \in \mathbb{N}$ and each p_i is a prime number.
How many factors does x have? (Answer to be given in terms of α_i.)

e Find the smallest number which has eight factors.

5 **a** Give the prime decompositions of 1080 and 25 200.

b Use your answer to part **a** to find the lowest common multiple of 1080 and 25 200.

c Carefully explain why, if m and n are integers, then $mn = \text{LCM}(m, n) \times \text{HCF}(m, n)$.

d **i** Find four consecutive even numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

ii Find four consecutive natural numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

6 Consider the universal set ξ as the set of all students enrolled at Sounion Secondary College. Let B denote the set of students taller than 180 cm and let A denote the set of female students.

a Give a brief description of each of the following sets:

i B' **ii** $A \cup B$ **iii** $A' \cap B'$

b Use a Venn diagram to show $(A \cup B)' = A' \cap B'$.

c Hence show that $A \cup B \cup C = (A' \cap B' \cap C')'$, where C is the set of students who play sport.

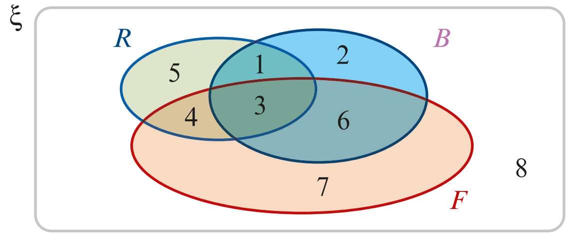
Review

7 Use Venn diagrams to illustrate:

a $|A \cup B| = |A| + |B| - |A \cap B|$

b $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

8 **a** The Venn diagram shows the set ξ of all students enrolled at Argos Secondary College. Set R is the set of all students with red hair. Set B is the set of all students with blue eyes. Set F is the set of all female students.



The numbers on the diagram are to label the eight different regions.

i Identify the region in the Venn diagram which represents male students who have neither red hair nor blue eyes.

ii Describe the gender, hair colour and eye colour of students represented in region 1 of the diagram.

iii Describe the gender, hair colour and eye colour of students represented in region 2 of the diagram.

b It is known that, at Argos Secondary College, 250 students study French (F), Greek (G) or Japanese (J). Forty-one students do not study French. Twelve students study French and Japanese but not Greek. Thirteen students study Japanese and Greek but not French. Thirteen students study only Greek. Twice as many students study French and Greek but not Japanese as study all three. The number studying only Japanese is the same as the number studying both French and Greek.

i How many students study all three languages?

ii How many students study only French?

9 There are three online news services (A, B and C) based in a certain city. In a sample of 500 people from this city, it was found that:

- nobody subscribes to both A and C
- a total of 100 people subscribe to A
- 205 people subscribe only to B
- of those who subscribe to C, exactly half of them also subscribe to B
- 35 people subscribe to A and B but not C
- 35 people don't subscribe to any of the news services at all.

a Draw a Venn diagram showing the number of subscribers for each possible combination of A, B and C.

b How many people in the sample were subscribers of C?

c How many people in the sample subscribe to A only?

d How many people are subscribers of A, B and C?

3

Sequences and series

Objectives

- To explore **sequences** of numbers and their **recurrence relations**.
- To recognise **arithmetic sequences**, and to find their terms, recurrence relations and numbers of terms.
- To calculate the sum of the terms in an **arithmetic series**.
- To recognise **geometric sequences**, and to find their terms, recurrence relations and numbers of terms.
- To calculate the sum of the terms in a **geometric series**.
- To work with sequences defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where r and d are constants.
- To calculate the sum of the terms in an **infinite geometric series**.
- To apply sequences and series to solving problems.

The following are examples of sequences of numbers:

a 1, 3, 5, 7, 9, . . . **b** 10, 7, 4, 1, −2, . . . **c** 0.6, 1.7, 2.8, . . . , 9.4

d $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ **e** 0.1, 0.11, 0.111, 0.1111, 0.11111, . . .

Each sequence is a list of numbers, with order being important. Sequence **c** is an example of a finite sequence, and the others are infinite sequences.

For some sequences of numbers, we can give a rule for getting from one number to the next:

- a rule for sequence **a** is: add 2
- a rule for sequence **b** is: subtract 3
- a rule for sequence **c** is: add 1.1
- a rule for sequence **d** is: multiply by $\frac{1}{3}$

In this chapter, we will develop algebraic techniques for studying sequences like these. We will also look at various applications of sequences.

3A Introduction to sequences

The numbers of a sequence are called its **terms**. The nth term of a sequence is denoted by the symbol t_n. So the first term is t_1, the 12th term is t_{12}, and so on.

Recurrence relations

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation**, a **recursive formula** or an **iterative rule**. For example:

- The sequence $1, 3, 5, 7, 9, \dots$ may be defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ may be defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

Example 1

Use the recurrence relation to find the first four terms of the sequence

$$t_1 = 3, \quad t_n = t_{n-1} + 5$$

Solution

$$t_1 = 3$$
$$t_2 = t_1 + 5 = 8$$
$$t_3 = t_2 + 5 = 13$$
$$t_4 = t_3 + 5 = 18$$

The first four terms are 3, 8, 13, 18.

Example 2

Find the recurrence relation for the following sequence:

$$9, -3, 1, -\frac{1}{3}, \dots$$

Solution

$$-3 = -\frac{1}{3} \times 9 \qquad \text{i.e. } t_2 = -\frac{1}{3}t_1$$

$$1 = -\frac{1}{3} \times (-3) \qquad \text{i.e. } t_3 = -\frac{1}{3}t_2$$

The sequence is defined by $t_1 = 9$ and $t_n = -\frac{1}{3}t_{n-1}$.

A sequence may also be defined explicitly by a rule that is stated in terms of n. For example:

- The rule $t_n = 2n$ defines the sequence of even numbers: $t_1 = 2, \ t_2 = 4, \ t_3 = 6, \dots$
- The rule $t_n = 2n - 1$ defines the sequence of odd numbers: $t_1 = 1, \ t_2 = 3, \ t_3 = 5, \dots$
- The rule $t_n = 2^{n-1}$ defines the sequence of powers of 2: $t_1 = 1, \ t_2 = 2, \ t_3 = 4, \dots$

For an infinite sequence, there is a term t_n of the sequence for each natural number n. Therefore we can consider an infinite sequence to be a function whose domain is the natural numbers. For example, we can write $t\colon \mathbb{N} \to \mathbb{R}$, $t_n = 2n + 3$.

Example 3

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

Solution

$$t_1 = 2(1) + 3 = 5$$
$$t_2 = 2(2) + 3 = 7$$
$$t_3 = 2(3) + 3 = 9$$
$$t_4 = 2(4) + 3 = 11$$

The first four terms are 5, 7, 9, 11.

Example 4

Find a rule for the nth term of the sequence 1, 4, 9, 16 in terms of n.

Solution

$$t_1 = 1 = 1^2$$
$$t_2 = 4 = 2^2$$
$$t_3 = 9 = 3^2$$
$$t_4 = 16 = 4^2$$
$$\therefore \quad t_n = n^2$$

Example 5

At a particular school, the number of students studying Specialist Mathematics increases each year. There are presently 40 students studying Specialist Mathematics.

a Set up the recurrence relation if the number is increasing by five students each year.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the number of students expected to be studying Specialist Mathematics at the school in five years' time.

Solution

a $t_n = t_{n-1} + 5$

b $t_1 = 40$

$$t_2 = t_1 + 5 = 45 = 40 + 1 \times 5$$
$$t_3 = t_2 + 5 = 50 = 40 + 2 \times 5$$

Therefore $t_n = 40 + (n-1) \times 5$

$= 35 + 5n$

c Five years from now implies $n = 6$:

$$t_6 = 40 + 5 \times 5 = 65$$

Sixty-five students will be studying Specialist Mathematics in five years.

Example 6

The height of a sand dune is increasing by 10% each year. It is currently 4 m high.

a Set up the recurrence relation that describes the height of the sand dune.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the height of the sand dune seven years from now.

Solution

a $t_n = t_{n-1} \times 1.1$

b $t_1 = 4$

$t_2 = 4 \times 1.1 = 4.4$

$t_3 = 4 \times (1.1)^2 = 4.84$

Therefore $t_n = 4 \times (1.1)^{n-1}$

c Seven years from now implies $n = 8$:

$$t_8 = 4 \times (1.1)^7 \approx 7.795$$

The sand dune will be 7.795 m high in seven years.

Using a calculator with explicitly defined sequences

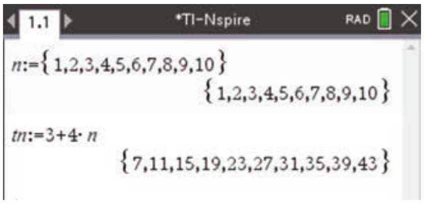
Example 7

Use a calculator to generate the first 10 terms of the sequence of numbers defined by the rule $t_n = 3 + 4n$.

Using the TI-Nspire

Sequences defined in terms of n can be investigated in a **Calculator** application.

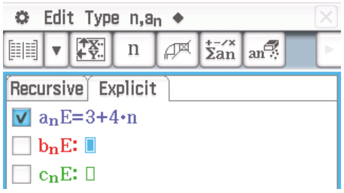
- To generate the first 10 terms of the sequence defined by the rule $t_n = 3 + 4n$, complete as shown. The assignment symbol := is accessed using ctrl (templates key).



Note: Assigning (storing) the resulting list as *tn* enables the sequence to be graphed. If preferred, the variable *tn* can be entered as t_n using the subscript template $\square_{\square}$, which is accessed via (templates key).

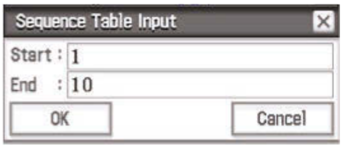
Using the Casio ClassPad

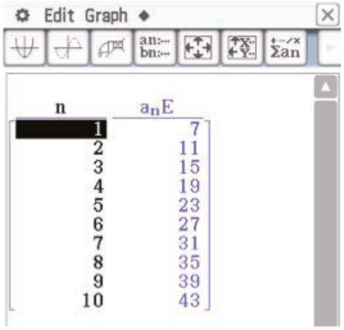
- Open the menu Menu; select **Sequence** Sequence.
- Ensure that the **Explicit** window is activated.
- Tap the cursor next to a_nE and enter $3 + 4n$.



Note: The variable n can be entered by tapping on n in the toolbar. Alternatively, it can be obtained from the Var or abc keyboard.

- Tap to view the sequence values.
- Tap to open the Sequence Table Input window and complete as shown below; tap OK.





- Tap Resize to see the sequence of numbers.

Using a calculator with recursively defined sequences

Example 8

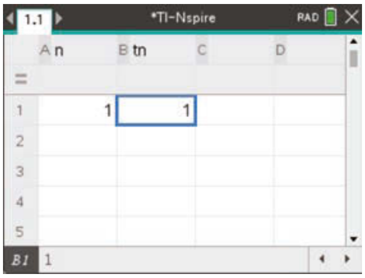
Use a calculator to generate the sequence defined by the recurrence relation

$$t_n = t_{n-1} + 3, \quad t_1 = 1$$

and plot the graph of the sequence against n.

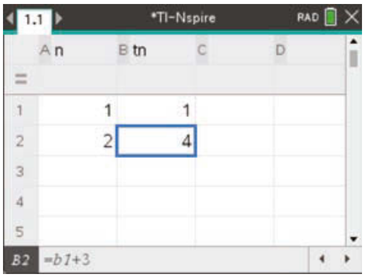
Using the TI-Nspire

- In a **Lists & Spreadsheet** page, name the first two lists n and tn respectively.
- Enter 1 in cell A1 and enter 1 in cell B1.

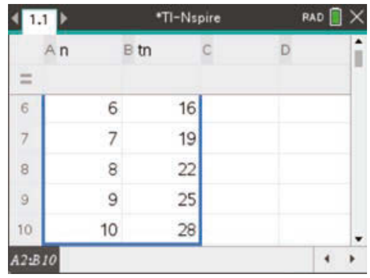


Note: If preferred, the variable tn can be entered as t_n using the subscript template, which is accessed via.

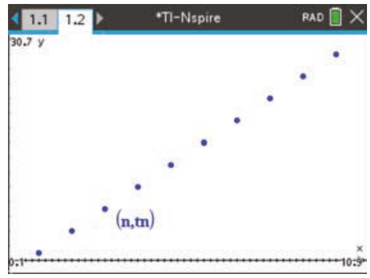
- Enter $= a1 + 1$ in cell A2 and enter $= b1 + 3$ in cell B2.



- Highlight the cells A2 and B2 using shift and the arrows.
- Use menu > **Data** > **Fill** and arrow down to row 10.
- Press enter to populate the lists.

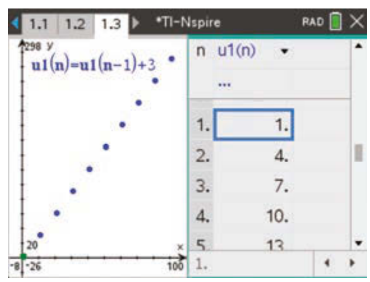


- To graph the sequence, open a **Graphs** application (ctrl I > **Add Graphs**).
- Graph the sequence as a scatter plot using menu > **Graph Entry/Edit** > **Scatter Plot**. Enter the list variables as n and tn in their respective fields.
- Set an appropriate window using menu > **Window/Zoom** > **Zoom – Data**.



Note: It is possible to see the coordinates of the points: menu > **Trace** > **Graph Trace**. The scatter plot can also be graphed in a **Data & Statistics** page.

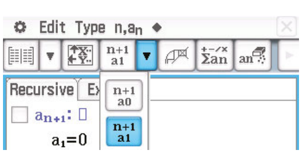
- Alternatively, the sequence can be graphed directly in the sequence plotter (menu > **Graph Entry/Edit** > **Sequence** > **Sequence**).
- Enter the rule $u1(n) = u1(n - 1) + 3$ and the initial value 1. Change **nStep** to 10.
- Set an appropriate window using menu > **Window/Zoom** > **Zoom – Fit**.
- Use ctrl T to show a table of values.

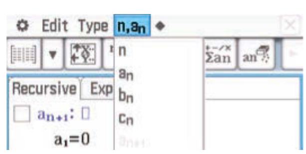


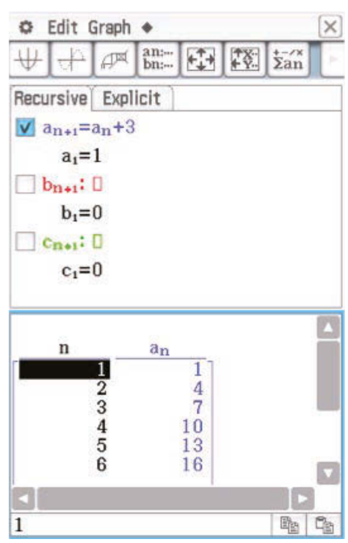
Using the Casio ClassPad

- Open the menu Menu; select **Sequence**.
- Ensure that the **Recursive** window is activated.
- Select the setting $\frac{n+1}{a1}$ as shown below.
- Tap the cursor next to a_{n+1} and enter $a_n + 3$.

Note: The symbol a_n can be found in the dropdown menu in the toolbar as shown below.

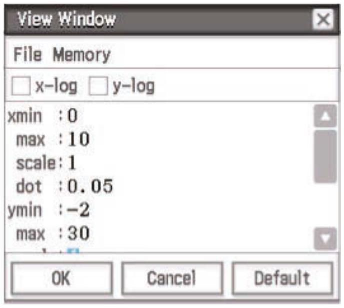


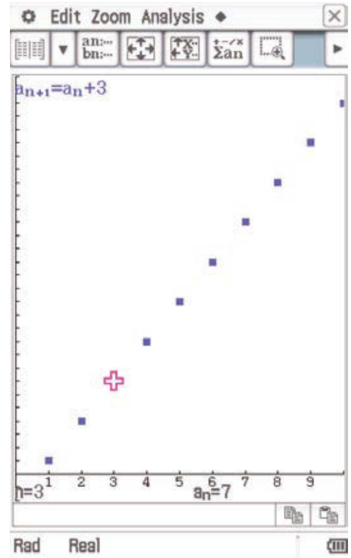




- Enter 1 for the value of the first term, a_1.
- Tick the box. Tap the table icon to view the sequence values.

- Tap [graph icon] to view the graph.
- Tap Resize [icon] and then [icon]. Set the View Window as shown below.





- Select **Analysis** > **Trace** and use the cursor ► to view each value in the sequence.

Note: Selecting [icon] instead of [icon] will produce a line graph, where successive sequence points are connected by line segments.

Summary 3A

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation** and we say that the sequence has been defined **recursively**. For example:

- The sequence $1, 3, 5, 7, 9, \ldots$ is defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ is defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

Exercise 3A

Example 1

1 In each of the following, a recursive definition for a sequence is given. List the first five terms.

a $t_1 = 3,\ t_n = t_{n-1} + 4$ **b** $t_1 = 5,\ t_n = 3t_{n-1} + 4$ **c** $t_1 = 1,\ t_n = 5t_{n-1}$

d $t_1 = -1,\ t_n = t_{n-1} + 2$ **e** $t_{n+1} = 2t_n + t_{n-1},\ t_1 = 1,\ t_2 = 3$

Example 2

2 For each of the following sequences, find a recurrence relation:

a $3, 6, 9, 12, \ldots$ **b** $1, 2, 4, 8, \ldots$ **c** $3, -6, 12, -24, \ldots$

d $4, 7, 10, 13, \ldots$ **e** $4, 9, 14, 19, \ldots$

Example 3

3 Each of the following is a rule for a sequence. In each case, find t_1, t_2, t_3, t_4.

a $t_n = \frac{1}{n}$ **b** $t_n = n^2 + 1$ **c** $t_n = 2n$ **d** $t_n = 2^n$

e $t_n = 3n + 2$ **f** $t_n = (-1)^n n^3$ **g** $t_n = 2n + 1$ **h** $t_n = 2 \times 3^{n-1}$

Example 4

4 For each of the following sequences, find a possible rule for t_n in terms of n:

a $3, 6, 9, 12, \ldots$ **b** $1, 2, 4, 8, \ldots$ **c** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

d $3, -6, 12, -24, \ldots$ **e** $4, 7, 10, 13, \ldots$ **f** $4, 9, 14, 19, \ldots$

5 Consider the sequence for which $t_n = 3n + 1$. Find t_{n+1} and t_{2n}.

Example 5

6 Hamish collects football cards. He currently has 15 and he adds three to his collection every week.

a Set up the recurrence relation that will generate the number of cards Hamish has in any given week.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the number of cards Hamish should have after another 12 weeks.

Example 6

7 Isobel can swim 100 m in 94.3 s. She aims to reduce her time by 4% each week.

a Set up the recurrence relation that generates Isobel's time for the 100 m in any given week.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the time in which Isobel expects to be able to complete the 100 m after another 8 weeks.

8 Stephen is a sheep farmer with a flock of 100 sheep. He wishes to increase the size of his flock by both breeding and buying new stock. He estimates that 80% of his sheep will produce one lamb each year and he intends to buy 20 sheep to add to the flock each year. Assuming that no sheep die:

a Write a recurrence relation for the expected number of sheep at the end of each year. (For this question, it will help to take the initial term $t_0 = 100$.)

b Calculate the number of sheep at the end of each of the first five years.

9 Alison invests \$2000 at the beginning of the year. At the beginning of each of the following years, she puts a further \$400 into the account. Compound interest of 6% p.a. is paid on the investment at the end of each year.

a Write down the amount of money in the account at the end of each of the first three years.

b Set up a recurrence relation to generate the sequence for the investment. (Let t_1 be the amount in the account at the end of the first year.)

c With a calculator or spreadsheet, use the recurrence relation to find the amount in the account after 10 years.

Example 7

10 For each of the following, use a CAS calculator to find the first six terms of the sequence and plot the graph of these terms against n:

a $t_n = 3n - 2$ **b** $t_n = 5 - 2n$ **c** $t_n = 2^{n-2}$ **d** $t_n = 2^{6-n}$

Example 8 **11** For each of the following, use a CAS calculator to find the first six terms of the sequence and plot the graph of these terms against n:

a $t_n = (t_{n-1})^2, \ t_1 = 1.1$

b $t_n = \frac{2}{3}t_{n-1}, \ t_1 = 27$

c $t_n = 2t_{n-1} + 5, \ t_1 = -1$

d $t_n = 4 - t_{n-1}, \ t_1 = -3$

12 **a** For a sequence for which $t_n = 2^{n-1}$, find t_1, t_2 and t_3.

b For a sequence for which $u_n = \frac{1}{2}(n^2 - n) + 1$, find u_1, u_2 and u_3.

c What do you notice?

d Find t_4 and u_4.

13 Assume that $S_n = an^2 + bn$, for constants $a, b \in \mathbb{R}$. Find S_1, S_2, S_3 and $S_{n+1} - S_n$.

14 For the sequence defined by $t_1 = 1$ and $t_{n+1} = \frac{1}{2}\left(t_n + \frac{2}{t_n}\right)$, find t_2, t_3 and t_4.

The terms of this sequence are successive rational approximations of a real number. Can you recognise the number?

15 The Fibonacci sequence is defined by $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N}$. Use the rule to find F_3, F_4 and F_5. Show that $F_{n+2} = 2F_n + F_{n-1}$ for all $n \in \mathbb{N} \setminus \{1\}$.

3B Arithmetic sequences

A sequence in which each successive term is found by adding a fixed amount to the previous term is called an **arithmetic sequence**. That is, an arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is a constant.

For example: $2, 5, 8, 11, 14, 17, \dots$ is an arithmetic sequence.

The nth term of an arithmetic sequence is given by

$$t_n = a + (n-1)d$$

where a is the first term and d is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

Example 9

Find the 10th term of the arithmetic sequence $-4, -1, 2, 5, \dots$.

Solution

$a = -4$, $d = 3$

$$t_n = a + (n-1)d$$

$$\therefore \quad t_{10} = -4 + (10-1) \times 3 = 23$$

Example 10

If 41 is the nth term in the arithmetic sequence $-4, -1, 2, 5, \dots$, find the value of n.

Solution

$a = -4$, $d = 3$

$$t_n = a + (n-1)d = 41$$
$$\therefore \quad -4 + (n-1) \times 3 = 41$$
$$3(n-1) = 45$$
$$n - 1 = 15$$
$$n = 16$$

Hence 41 is the 16th term of the sequence.

Example 11

The 12th term of an arithmetic sequence is 9 and the 25th term is 100. Find a and d, and hence find the 8th term.

Solution

An arithmetic sequence has rule

$$t_n = a + (n-1)d$$

Since the 12th term is 9, we have

$$9 = a + 11d \qquad (1)$$

Since the 25th term is 100, we have

$$100 = a + 24d \qquad (2)$$

To find a and d, we solve the two equations simultaneously.

Subtract (1) from (2):

$$91 = 13d$$
$$\therefore \quad d = 7$$

From (1), we have

$$9 = a + 11(7)$$
$$\therefore \quad a = -68$$

Therefore

$$t_8 = a + 7d$$
$$= -68 + 7 \times 7$$
$$= -19$$

The 8th term of the sequence is -19.

Alternative

Alternatively, you can find the common difference d using

$$d = \frac{t_m - t_n}{m - n}$$

and then find the first term a using

$$a = t_n - (n-1)d$$

Example 12

A national park has a series of huts located at regular intervals along one of its mountain trails. The first hut is 5 km from the start of the trail, the second is 8 km from the start, the third 11 km and so on.

a How far from the start of the trail is the sixth hut?

b How far is it from the sixth hut to the twelfth hut?

Solution

The distances of the huts from the start of the trail form an arithmetic sequence with $a = 5$ and $d = 3$.

a For the sixth hut:

$$t_6 = a + 5d$$
$$= 5 + 5 \times 3 = 20$$

The sixth hut is 20 km from the start of the trail.

b For the twelfth hut:

$$t_{12} = a + 11d$$
$$= 5 + 11 \times 3 = 38$$

The distance from the sixth hut to the twelfth hut is $t_{12} - t_6 = 38 - 20 = 18$ km.

Arithmetic mean

The **arithmetic mean** of two numbers a and b is defined as $\frac{a+b}{2}$.

If the numbers a, c, b are consecutive terms of an arithmetic sequence, then

$$c - a = b - c$$
$$2c = a + b$$
$$\therefore \quad c = \frac{a+b}{2}$$

That is, the middle term c is the arithmetic mean of a and b.

Summary 3B

- An **arithmetic sequence** has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is a constant. Each successive term is found by adding a fixed amount to the previous term. For example: $2, 5, 8, 11, \ldots$
- The nth term of an arithmetic sequence is given by

  $$t_n = a + (n-1)d$$

  where a is the first term and d is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

Exercise 3B

1 For the arithmetic sequence where $t_n = a + (n-1)d$, find the first four terms given that:

a $a = 0,\ d = 2$ **b** $a = -3,\ d = 5$

c $a = d = -\sqrt{5}$ **d** $a = 11,\ d = -2$

Example 9

2 **a** If an arithmetic sequence has a first term of 5 and a common difference of −3, find the 13th term.

b If an arithmetic sequence has a first term of −12 and a common difference of 4, find the 10th term.

c For the arithmetic sequence with $a = 25$ and $d = -2.5$, find the ninth term.

d For the arithmetic sequence with $a = 2\sqrt{3}$ and $d = \sqrt{3}$, find the fifth term.

3 Find the rule of the arithmetic sequence whose first few terms are:

a $3, 7, 11$ **b** $3, -1, -5$ **c** $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{11}{2}$ **d** $5 - \sqrt{5},\ 5,\ 5 + \sqrt{5}$

Example 10

4 In each of the following, t_n is the nth term of an arithmetic sequence:

a If 54 is the nth term in the sequence $6, 10, 14, 18, \dots$, find the value of n.

b If −16 is the nth term in the sequence $5, 2, -1, -4, \dots$, find the value of n.

c Find n if $t_1 = 16$, $t_2 = 13$ and $t_n = -41$.

d Find n if $t_1 = 7$, $t_2 = 11$ and $t_n = 227$.

Example 11

5 For an arithmetic sequence with fourth term 7 and thirtieth term 85, find the values of a and d, and hence find the seventh term.

6 If an arithmetic sequence has $t_3 = 18$ and $t_6 = 486$, find the rule for the sequence, i.e. find t_n.

7 For the arithmetic sequence with $t_7 = 0.6$ and $t_{12} = -0.4$, find t_{20}.

8 The number of laps that a swimmer swims each week follows an arithmetic sequence. In the 5th week she swims 24 laps and in the 10th week she swims 39 laps. How many laps does she swim in the 15th week?

9 For an arithmetic sequence, find t_6 if $t_{15} = 3 + 9\sqrt{3}$ and $t_{20} = 38 - \sqrt{3}$.

Example 12

10 A small company producing wallets plans an increase in output. In the first week it produces 280 wallets. The number of wallets produced each week is to be increased by 8 per week until the weekly number produced reaches 1000.

a How many wallets are produced in the 50th week?

b In which week does the production reach 1000?

11 An amphitheatre has 25 seats in row A, 28 seats in row B, 31 seats in row C, and so on.

a How many seats are there in row P?

b How many seats are there in row X?

c Which row has 40 seats?

12 The number of people who go to see a movie over a period of a week follows an arithmetic sequence. On the first day only three people go to the movie, but on the sixth day 98 people go. Find the rule for the sequence and hence determine how many attend on the seventh day.

13 An arithmetic sequence contains 10 terms. If the first is 4 and the tenth is 30, what is the eighth term?

14 The number of goals kicked by a team in the first six games of a season follows an arithmetic sequence. If the team kicked 5 goals in the first game and 15 in the sixth, how many did they kick in each of the other four games?

15 The first term of an arithmetic sequence is a and the mth term is 0. Find the rule for t_n for this sequence.

16 Find the arithmetic mean of:

a 8 and 15 **b** $\frac{1}{2\sqrt{2}-1}$ and $\frac{1}{2\sqrt{2}+1}$

17 Find x if $3x - 2$ is the arithmetic mean of $5x + 1$ and 11.

18 If a, $4a - 4$ and $8a - 13$ are successive terms of an arithmetic sequence, find a.

19 If $t_m = n$ and $t_n = m$, prove that $t_{m+n} = 0$. (Here t_m and t_n are the mth and nth terms of an arithmetic sequence.)

20 If a, $2a$ and a^2 are consecutive terms of an arithmetic sequence, find a ($a \neq 0$).

21 Show that there is no infinite arithmetic sequence whose terms are all prime numbers.

3C Arithmetic series

The sum of the terms in a sequence is called a **series**. If the sequence is arithmetic, then the series is called an **arithmetic series**.

The symbol S_n is used to denote the sum of the first n terms of a sequence. That is,

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

Writing this sum in reverse order, we have

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + d) + a$$

Adding these two expressions together gives

$$2S_n = n(2a + (n - 1)d)$$

Therefore

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big)$$

Since the last term $\ell = t_n = a + (n - 1)d$, we can also write

$$S_n = \frac{n}{2}(a + \ell)$$

That is, the sum of the first n terms of an arithmetic sequence is equal to n times the mean of the first and last terms.

Example 13

For the arithmetic sequence 2, 5, 8, 11, . . . , calculate the sum of the first 14 terms.

Solution

$a = 2$, $d = 3$, $n = 14$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\therefore \quad S_{14} = \frac{14}{2}(2 \times 2 + 13 \times 3) = 301$$

Example 14

For the arithmetic sequence 27, 23, 19, 15, . . . , −33, find:

a the number of terms

b the sum of the terms.

Solution

a $a = 27$, $d = -4$, $\ell = t_n = -33$

$$\begin{aligned} t_n &= a + (n-1)d \\ -33 &= 27 + (n-1)(-4) \\ -60 &= (n-1)(-4) \\ 15 &= n - 1 \\ n &= 16 \end{aligned}$$

There are 16 terms in the sequence.

b $a = 27$, $\ell = t_n = -33$, $n = 16$

$$S_n = \frac{n}{2}(a + \ell)$$

$$\begin{aligned} \therefore \quad S_{16} &= \frac{16}{2}(27 - 33) \\ &= -48 \end{aligned}$$

The sum of the terms is −48.

Example 15

For the arithmetic sequence 3, 6, 9, 12, . . . , calculate:

a the sum of the first 25 terms

b the number of terms in the series if $S_n = 1395$.

Solution

a $a = 3$, $d = 3$, $n = 25$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\begin{aligned} \therefore \quad S_{25} &= \frac{25}{2}\big(2(3) + (24)(3)\big) \\ &= 975 \end{aligned}$$

b $a = 3$, $d = 3$, $S_n = 1395$

$$\begin{aligned} S_n = \frac{n}{2}\big(2a + (n-1)d\big) &= 1395 \\ \frac{n}{2}\big(2(3) + (n-1)(3)\big) &= 1395 \\ n(6 + 3n - 3) &= 2790 \\ 3n + 3n^2 &= 2790 \\ 3n^2 + 3n - 2790 &= 0 \\ n^2 + n - 930 &= 0 \\ (n-30)(n+31) &= 0 \end{aligned}$$

Therefore $n = 30$, since $n > 0$.

Hence there are 30 terms in the series.

Example 16

A hardware store sells nails in a range of packet sizes. Packet A contains 50 nails, packet B contains 75 nails, packet C contains 100 nails, and so on.

a Find the number of nails in packet J.

b Lachlan buys one each of packets A to J. How many nails in total does Lachlan have?

c Assuming he buys one of each packet starting at A, how many packets does he need to buy to have a total of 1100 nails?

Solution

a $a = 50$, $d = 25$

$$t_n = a + (n-1)d$$

For packet J, we take $n = 10$:

$$\begin{aligned} t_{10} &= 50 + 9 \times 25 \\ &= 275 \end{aligned}$$

Packet J contains 275 nails.

b $a = 50$, $d = 25$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\begin{aligned} \therefore \quad S_{10} &= \frac{10}{2}\big(2 \times 50 + 9 \times 25\big) \\ &= 1625 \end{aligned}$$

Packets A to J contain 1625 nails.

c $a = 50$, $d = 25$, $S_n = 1100$

$$\begin{aligned} S_n = \frac{n}{2}\big(2a + (n-1)d\big) &= 1100 \\ \frac{n}{2}\big(2(50) + (n-1)(25)\big) &= 1100 \\ n(100 + 25n - 25) &= 2200 \\ 25n^2 + 75n - 2200 &= 0 \\ n^2 + 3n - 88 &= 0 \\ (n+11)(n-8) &= 0 \end{aligned}$$

Thus $n = 8$, since $n > 0$. If Lachlan buys one each of the first eight packets (A to H), he will have exactly 1100 nails.

Example 17

The sum of the first 10 terms of an arithmetic sequence is $48\frac{3}{4}$. If the fourth term is $3\frac{3}{4}$, find the first term and the common difference.

Solution

$$t_4 = a + 3d = 3\tfrac{3}{4}$$

$$\therefore \quad a + 3d = \frac{15}{4} \qquad (1)$$

$$S_{10} = \frac{10}{2}(2a + 9d) = 48\tfrac{3}{4}$$

$$\therefore \quad 10a + 45d = \frac{195}{4} \qquad (2)$$

Solve equations (1) and (2) simultaneously:

$(1) \times 40$: $40a + 120d = 150$

$(2) \times 4$: $40a + 180d = 195$

$$60d = 45$$

$$\therefore \quad d = \frac{3}{4}$$

Substitute in (1) to obtain $a + 3\left(\frac{3}{4}\right) = \frac{15}{4}$ and therefore $a = \frac{3}{2}$.

The first term is $1\frac{1}{2}$ and the common difference is $\frac{3}{4}$.

Summary 3C

The sum of the first n terms of an arithmetic sequence

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

is given by

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

Skill-sheet

Exercise 3C

Example 13

1 For each arithmetic sequence, find the specified sum:

a $8, 13, 18, \dots$ find S_{12} **b** $-3.5, -1.5, 0.5, \dots$ find S_{10}

c $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}}, \dots$ find S_{15} **d** $-4, 1, 6, \dots$ find S_8

2 Greg goes fishing every day for a week. On the first day he catches seven fish and each day he catches three more than the previous day. How many fish did he catch in total?

3 Find the sum of the first 16 multiples of 5.

4 Find the sum of all the even numbers between 1 and 99.

Example 14

5 For the arithmetic sequence $-3, 1, 5, 9, \dots, 49$, find:

a the number of terms **b** the sum of the terms.

6 For the arithmetic sequence $24, 20, 16, 12, \dots, -52$, find:

a the number of terms **b** the sum of the terms.

7 For the arithmetic sequence $\frac{1}{2}, 2, \frac{7}{2}, 5, \dots, 17$, find:

a the number of terms **b** the sum of the terms.

Example 15

8 For the sequence 4, 8, 12, . . . , find:

a the sum of the first 9 terms

b the value of n such that $S_n = 180$.

9 There are 110 logs to be put in a pile, with 15 logs in the bottom layer, 14 in the next, 13 in the next, and so on. How many layers will there be?

10 The sum of the first m terms of an arithmetic sequence with first term −5 and common difference 4 is 660. Find m.

11 Evaluate $54 + 48 + 42 + \cdots + (-54)$.

Example 16

12 Dora's walking club plans 15 walks for the summer. The first walk is a distance of 6 km, the last walk is a distance of 27 km, and the distances of the walks form an arithmetic sequence.

a How far is the 8th walk?

b How far does the club plan to walk in the first five walks?

c Dora's husband, Alan, can only complete the first n walks. If he walks a total of 73.5 km, how many walks does he complete?

d Dora goes away on holiday and misses the 9th, 10th and 11th walks, but completes all other walks. How far does Dora walk in total?

13 Liz has to proofread 500 pages of a new novel. She plans to read 30 pages on the first day and to increase the number of pages she reads by five each day.

a How many days will it take her to complete the proofreading?

She has only five days to complete the task. She therefore decides to read 50 pages on the first day and to increase the number she reads by a constant amount each day.

b By how many should she increase the number of pages she reads each day if she is to meet her deadline?

14 An assembly hall has 50 seats in row A, 54 seats in row B, 58 seats in row C, and so on. That is, there are four more seats in each row.

a How many seats are there in row J?

b How many seats are there altogether if the back row is row Z?

On a particular day, the front four rows are reserved for parents (and there is no other seating for parents).

c How many parents can be seated?

d How many students can be seated?

The hall is extended by adding more rows following the same pattern.

e If the final capacity of the hall is 3410, how many rows were added?

15 A new golf club is formed with 40 members in its first year. Each following year, the number of new members exceeds the number of retirements by 15. Each member pays \$120 p.a. in membership fees. Calculate the amount received from fees in the first 12 years of the club's existence.

Example 17 **16** For the arithmetic sequence with $t_2 = -12$ and $S_{12} = 18$, find a, d, t_6 and S_6.

17 The sum of the first 10 terms of an arithmetic sequence is 120, and the sum of the first 20 terms is 840. Find the sum of the first 30 terms.

18 If $t_6 = 16$ and $t_{12} = 28$, find S_{14}.

19 For an arithmetic sequence, find t_n if:

a $t_3 = 6.5$ and $S_8 = 67$

b $t_4 = \dfrac{6}{\sqrt{5}}$ and $S_5 = 16\sqrt{5}$

20 For the sequence with $t_n = bn$, where $b \in \mathbb{R}$, find:

a $t_{n+1} - t_n$

b $t_1 + t_2 + \cdots + t_n$

21 For a sequence where $t_n = 15 - 5n$, find t_5 and find the sum of the first 25 terms.

22 An arithmetic sequence has a common difference of d and the sum of the first 20 terms is 25 times the first term. Find the sum of the first 30 terms in terms of d.

23 The sum of the first n terms of a particular sequence is given by $S_n = 17n - 3n^2$.

a Find an expression for the sum of the first $(n - 1)$ terms.

b Find an expression for the nth term of the sequence.

c Show that the sequence is arithmetic and find a and d.

24 Three consecutive terms of an arithmetic sequence have a sum of 36 and a product of 1428. Find the three terms.

25 **a** Prove that the sum of the first n odd numbers $1, 3, 5, \dots, 2n - 1$ is equal to n^2.

b In 1615, Galileo proved that

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots$$

Let S_n be the sum of the first n odd numbers.

i Prove that $S_{2n} - S_n = 3S_n$.

ii Hence prove Galileo's result.

26 Show that the sum of the first $2n$ terms of an arithmetic sequence is n times the sum of the two middle terms.

27 Find the sum of the numbers between 1 and 120 inclusive that are multiples of 2 or 3.

28 Find all arithmetic sequences consisting of four positive integers whose sum is 100.

29 How many triangles have three angles that are positive integers in an arithmetic sequence?

3D Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed amount is called a **geometric sequence**. That is, a geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is a constant.

For example: $2, 6, 18, 54, \dots$ is a geometric sequence.

The nth term of a geometric sequence is given by

$$t_n = ar^{n-1}$$

where a is the first term and r is the **common ratio** of successive terms, that is, $r = \frac{t_k}{t_{k-1}}$, for all $k > 1$.

Example 18

Find the 10th term of the geometric sequence $2, 6, 18, \dots$.

Solution

$a = 2,\ r = 3$

$$t_n = ar^{n-1}$$
$$\therefore \quad t_{10} = 2 \times 3^{10-1}$$
$$= 39\,366$$

Example 19

For a geometric sequence, the first term is 18 and the fourth term is 144. Find the common ratio.

Solution

$a = 18,\ t_4 = 144$

$$t_4 = 18 \times r^{4-1} = 144$$
$$18r^3 = 144$$
$$r^3 = 8$$
$$\therefore \quad r = 2$$

The common ratio is 2.

Example 20

For a geometric sequence 36, 18, 9, . . . , the nth term is $\frac{9}{16}$. Find the value of n.

Solution

$a = 36$, $r = \frac{1}{2}$

$$t_n = 36 \times \left(\frac{1}{2}\right)^{n-1} = \frac{9}{16}$$

$$\left(\frac{1}{2}\right)^{n-1} = \frac{9}{576}$$

$$\left(\frac{1}{2}\right)^{n-1} = \frac{1}{64}$$

$$\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{6}$$

$$n - 1 = 6$$

$$\therefore \quad n = 7$$

Example 21

The third term of a geometric sequence is 10 and the sixth term is 80. Find the common ratio and the first term.

Solution

$$t_3 = ar^2 = 10 \qquad (1)$$

$$t_6 = ar^5 = 80 \qquad (2)$$

Divide (2) by (1):

$$\frac{ar^5}{ar^2} = \frac{80}{10}$$

$$r^3 = 8$$

$$\therefore \quad r = 2$$

Substitute in (1):

$$a \times 4 = 10$$

$$\therefore \quad a = \frac{5}{2}$$

The common ratio is 2 and the first term is $\frac{5}{2}$.

Example 22

Georgina draws a pattern consisting of a number of equilateral triangles. The first triangle has sides of length 4 cm and the side length of each successive triangle is one and a half times the side length of the previous one.

a What is the side length of the fifth triangle?

b Which triangle has a side length of $45\frac{9}{16}$ cm?

Solution

a $a = 4,\ r = \dfrac{3}{2}$

$$t_n = ar^{n-1}$$

$$\therefore\quad t_5 = ar^4 = 4 \times \left(\frac{3}{2}\right)^4 = 20\tfrac{1}{4}$$

The fifth triangle has a side length of $20\frac{1}{4}$ cm.

b $a = 4,\ r = \dfrac{3}{2},\ t_n = 45\frac{9}{16}$

$$t_n = ar^{n-1} = 45\tfrac{9}{16}$$

$$4 \times \left(\frac{3}{2}\right)^{n-1} = \frac{729}{16}$$

$$\left(\frac{3}{2}\right)^{n-1} = \frac{729}{64} = \left(\frac{3}{2}\right)^6$$

Therefore $n - 1 = 6$ and so $n = 7$.

The seventh triangle has a side length of $45\frac{9}{16}$ cm.

Geometric mean

The **geometric mean** of two positive numbers a and b is defined as $\sqrt{ab}$.

If positive numbers a, c, b are consecutive terms of a geometric sequence, then

$$\frac{c}{a} = \frac{b}{c}$$

$$\therefore\quad c = \sqrt{ab}$$

Summary 3D

- A **geometric sequence** has a recurrence relation of the form $t_n = rt_{n-1}$, where r is a constant. Each successive term is found by multiplying the previous term by a fixed amount. For example: 2, 6, 18, 54, . . .
- The nth term of a geometric sequence is given by

  $$t_n = ar^{n-1}$$

  where a is the first term and r is the **common ratio** of successive terms, that is, $r = \dfrac{t_k}{t_{k-1}}$, for all $k > 1$.

Exercise 3D

1 For a geometric sequence $t_n = ar^{n-1}$, find the first four terms given that:

a $a = 3,\ r = 2$ **b** $a = 3,\ r = -2$

c $a = 10\,000,\ r = 0.1$ **d** $a = r = 3$

Example 18

2 Find the specified term in each of the following geometric sequences:

a $\frac{15}{7}, \frac{5}{7}, \frac{5}{21}, \dots$ find t_6 **b** $1, -\frac{1}{4}, \frac{1}{16}, \dots$ find t_5

c $\sqrt{2}, 2, 2\sqrt{2}, \dots$ find t_{10} **d** $a^x, a^{x+1}, a^{x+2}, \dots$ find t_6

3 Find the rule for the geometric sequence whose first few terms are:

a $3, 2, \frac{4}{3}$ **b** $2, -4, 8, -16$ **c** $2, 2\sqrt{5}, 10$

Example 19

4 Find the common ratio for the following geometric sequences:

a the first term is 2 and the sixth term is 486

b the first term is 25 and the fifth term is $\frac{16}{25}$

Example 20

5 A geometric sequence has first term $\frac{1}{4}$ and common ratio 2. Which term of the sequence is 64?

6 If t_n is the nth term of the following geometric sequences, find n in each case:

a $2, 6, 18, \dots$ $t_n = 486$ **b** $5, 10, 20, \dots$ $t_n = 1280$

c $768, 384, 192, \dots$ $t_n = 3$ **d** $\frac{8}{9}, \frac{4}{3}, 2, \dots$ $t_n = \frac{27}{4}$

e $-\frac{4}{3}, \frac{2}{3}, -\frac{1}{3}, \dots$ $t_n = \frac{1}{96}$

Example 21

7 The 12th term of a geometric sequence is 2 and the 15th term is 54. Find the 7th term.

8 A geometric sequence has $t_2 = \frac{1}{2\sqrt{2}}$ and $t_4 = \sqrt{2}$. Find t_8.

9 The number of fish in the breeding tanks of a fish farm follow a geometric sequence. The third tank contains 96 fish and the sixth tank contains 768 fish.

a How many fish are in the first tank? **b** How many fish are in the 10th tank?

Example 22

10 An algal bloom is growing in a lake. The area it covers triples each day.

a If it initially covers an area of 10 m^2, what is the area it will cover after one week?

b If the lake has a total area of 200 000 m^2, how long before the entire lake is covered?

11 A ball is dropped from a height of 2 m and continues to bounce so that it rebounds to three-quarters of the height from which it previously falls. Find the height it rises to on the fifth bounce.

12 An art collector has a painting that is increasing in value by 8% each year. If the painting is currently valued at \$2500:

a How much will it be worth in 10 years?

b How many years before its value exceeds \$100 000?

13 The Tour de Moravia is a cycling event which lasts for 15 days. On the first day the cyclists must ride 120 km, and each successive day they ride 90% of the distance of the previous day.

a How far do they ride on the 8th day?

b On which day do they ride 30.5 km?

14 A child negotiates a new pocket-money deal with her unsuspecting father in which she receives 1 cent on the first day of the month, 2 cents on the second, 4 cents on the third, 8 cents on the fourth, and so on, until the end of the month. How much would the child receive on the 30th day of the month? (Give your answer to the nearest thousand dollars.)

15 The first three terms of a geometric sequence are $4, 8, 16$. Find the first term in the sequence which exceeds 2000.

16 The first three terms of a geometric sequence are $3, 9, 27$. Find the first term in the sequence which exceeds 500.

17 The number of 'type A' apple bugs present in an orchard is estimated to be 40 960, and the number is reducing by 50% each week. At the same time it is estimated that there are 40 'type B' apple bugs, whose number is doubling each week. After how many weeks will there be the same number of each type of bug?

18 Find the geometric mean of:

a 5 and 720 **b** 1 and 6.25 **c** $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$ **d** x^2y^3 and x^6y^{11}

19 The fourth, seventh and sixteenth terms of an arithmetic sequence also form consecutive terms of a geometric sequence. Find the common ratio of the geometric sequence.

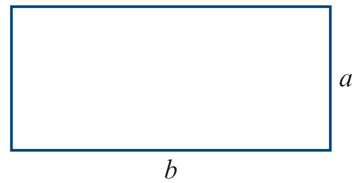
20 Consider the geometric sequence $1, a, a^2, a^3, \dots$. Suppose that the sum of two consecutive terms in the sequence gives the next term in the sequence. Find a.

21 A bottle contains 1000 mL of pure ethanol. Then 300 mL is removed and the bottle is topped up with 300 mL of pure water. The mixture is stirred.

a What is the volume of ethanol in the bottle if this process is repeated five times in total?

b How many times should the process be repeated for there to be less than 1 mL of ethanol in the bottle?

22 The rectangle shown has side lengths a and b.



a Find the side length of a square with the same perimeter. Comment.

b Find the side length of a square with the same area. Comment.

3E Geometric series

The sum of the terms in a geometric sequence is called a **geometric series**. An expression for S_n, the sum of the first n terms of a geometric sequence, can be found using a similar method to that used for arithmetic series.

Let $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad (1)$

Then $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad (2)$

Subtract (1) from (2):

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n - 1)$$

Therefore

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

For values of r such that $-1 < r < 1$, it is often more convenient to use the equivalent formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

which is obtained by multiplying both the numerator and the denominator by -1.

Example 23

Find the sum of the first nine terms of the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$.

Solution

$a = \frac{1}{3}$, $r = \frac{1}{3}$, $n = 9$

$$\therefore \quad S_9 = \frac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^9\right)}{1 - \frac{1}{3}}$$

$$= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^9\right)$$

$$\approx 0.499975$$

Example 24

For the geometric sequence $1, 3, 9, \ldots$, find how many terms must be added together to obtain a sum of 1093.

Solution

$a = 1$, $r = 3$, $S_n = 1093$

$$S_n = \frac{a(r^n - 1)}{r - 1} = 1093$$

$$\frac{1(3^n - 1)}{3 - 1} = 1093$$

$$3^n - 1 = 1093 \times 2$$

$$\therefore \quad 3^n = 2187$$

A CAS calculator can be used to find $n = 7$.

Seven terms are required to give a sum of 1093.

Example 25

In the 15-day Tour de Moravia, the cyclists must ride 120 km on the first day, and each successive day they ride 90% of the distance of the previous day.

a How far do they ride in total to the nearest kilometre?

b After how many days will they have ridden half that distance?

Solution

a $a = 120$, $r = 0.9$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\therefore \quad S_{15} = \frac{120(1 - (0.9)^{15})}{1 - 0.9}$$

$$= 952.93$$

They ride 953 km.

b $a = 120$, $r = 0.9$, $S_n = 476.5$

$$S_n = \frac{a(1 - r^n)}{1 - r} = 476.5$$

$$\frac{120(1 - (0.9)^n)}{1 - 0.9} = 476.5$$

$$1 - 0.9^n = \frac{476.5 \times 0.1}{120}$$

$$1 - 0.9^n = 0.3971$$

$$0.9^n = 1 - 0.3971$$

$$\therefore \quad 0.9^n = 0.6029$$

A CAS calculator can be used to find $n \approx 4.8$. Thus they pass the halfway mark on the fifth day.

Summary 3E

The sum of the first n terms of a geometric sequence

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Skill-sheet

Exercise 3E

Example 23

1 Find the specified sum for each of the following geometric series:

a $5 + 10 + 20 + \cdots$ find S_{10}

b $1 - 3 + 9 - \cdots$ find S_6

c $-\frac{4}{3} + \frac{2}{3} - \frac{1}{3} + \cdots$ find S_9

2 Find:

a $2 - 6 + 18 - \cdots + 1458$

b $-4 + 8 - 16 + \cdots - 1024$

c $6250 + 1250 + 250 + \cdots + 2$

Example 24

3 For the geometric sequence $3, 6, 12, \ldots$, find how many terms must be added together to obtain a sum of 3069.

4 For the geometric sequence $24, -12, 6, \ldots$, find how many terms must be added together to obtain a sum of $16\frac{1}{8}$.

Example 25

5 Gerry owns a milking cow. On the first day he milks the cow, it produces 600 mL of milk. On each successive day, the amount of milk increases by 10%.

a How much milk does the cow produce on the seventh day?

b How much milk does it produce in the first week?

c After how many days will it have produced a total in excess of 10 000 mL?

6 On Monday, William spends 20 minutes playing the piano. On Tuesday, he spends 25 minutes playing, and on each successive day he increases the time he spends playing in the same ratio.

a For how many minutes does he play on Friday?

b How many minutes in total does he play from Monday to Friday?

c On which day of the following week will his total time playing pass 15 hours?

7 A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound, it rises to two-thirds of the height of the previous rebound. What total distance will it have travelled when it strikes the ground for the 10th time?

8 An insurance broker makes \$15 000 commission on sales in her first year. Each year, she increases her sales by 5%.

a How much commission would she make in her fifth year?

b How much commission would she make in total over 5 years?

9 For a geometric sequence with nth term t_n:

a if $t_3 = 20$ and $t_6 = 160$, find S_5 **b** if $t_3 = \sqrt{2}$ and $t_8 = 8$, find S_8.

10 **a** How many terms of the geometric sequence where $t_1 = 1, t_2 = 2, t_3 = 4, \dots$ must be taken for $S_n = 255$?

b Let $S_n = 1 + 2 + 4 + \cdots + 2^{n-1}$. Find $\{ n : S_n > 1\,000\,000 \}$.

11 Find $1 - x^2 + x^4 - x^6 + x^8 - \cdots + x^{2m}$, where m is even.

12 A sheet of A4 paper is about 0.05 mm thick. The paper is torn in half, and each half is again torn in half, and this process is repeated for a total of 40 times.

a How high will the stack of paper be if the pieces are placed one on top of the other?

b How many times would the process have to be repeated for the stack to first reach the moon, 384 400 km away?

13 Which would you prefer: \$1 million given to you every week for a year, or 1c in the first week, 2c in the second, 4c in the third, and so on, until the end of the year?

14 For a particular geometric sequence, the sum of the first 50 terms is 60 and the sum of the first 100 terms is 80. Find the common ratio for this geometric sequence.

3F Applications of geometric sequences

Compound interest

One application of geometric sequences is **compound interest**. Compound interest is interest calculated at regular intervals on the total of the amount originally invested and the amount accumulated in the previous years.

For example, assume that \$1000 is invested at 10% per annum, compounded annually. Then the value of the investment increases by 10% each year.

1 After 1 year, the investment will have grown to $\$1000 \times 1.1 = \$1100$.

2 After 2 years, the investment will have grown to $\$1100 \times 1.1 = \$1000 \times 1.1^2 = \$1210$.

3 After 3 years, the investment will have grown to $\$1210 \times 1.1 = \$1000 \times 1.1^3 = \$1331$.

The value of the investment after n years will be $\$1000 \times 1.1^n$.

Compound interest

Suppose that $\$P$ is invested at an interest rate of $R\%$ per annum, compounded annually. Then the value of the investment after n years, $\$A_n$, is given by

$$A_n = Pr^n, \quad \text{where } r = 1 + \frac{R}{100}$$

Example 26

Marta invests \$2500 at 7% p.a. compounded annually.

a Find the value of her investment after 5 years.

b Find how long it takes until her investment is worth \$10 000.

Solution

The value after n years is $A_n = Pr^n$, where $P = 2500$ and $r = 1.07$.

a
$$A_5 = Pr^5 = 2500(1.07)^5 = 3506.38$$

The value of the investment after 5 years is \$3506.38.

b
$$A_n = Pr^n = 10\,000$$
$$2500(1.07)^n = 10\,000$$
$$1.07^n = 4$$
$$\log_{10}(1.07^n) = \log_{10} 4$$
$$n\log_{10}(1.07) = \log_{10} 4$$
$$n = \frac{\log_{10} 4}{\log_{10}(1.07)}$$
$$\therefore \quad n \approx 20.49$$

By the end of the 21st year, the investment will be worth over \$10 000.

Note: For part **b**, the number of years can also be found by trial and error or by using a CAS calculator.

Compound interest using a recurrence relation

Example 26 can also be solved using a spreadsheet. Let $\$A_n$ be the value of the investment at the end of the nth year. These values can be found recursively by

$$A_0 = 2500 \quad \text{and} \quad A_n = 1.07 \times A_{n-1}$$

To find the values using a spreadsheet:

- In cell A1, enter the value 0.
- In cell A2, enter the formula = A1 + 1.
- In cell B1, enter the initial value 2500.
- In cell B2, enter the formula = 1.07 ∗ B1.
- Fill down in columns A and B.

	Year	Amount
	A	B
1	0	2500.00
2	1	2675.00
3	2	2862.25
4	3	3062.61
5	4	3276.99
6	5	3506.38

At the end of year 5, the amount is \$3506.38. By filling down further in columns A and B, you can find the year that the amount reaches \$10 000.

Depreciation

Depreciation occurs when the value of an asset reduces as time passes. For example, suppose that a new car is bought for \$20 000, and that its value depreciates by 10% each year.

1. After 1 year, the car's value will have fallen to $\$20\,000 \times 0.9 = \$18\,000$.
2. After 2 years, the car's value will have fallen to $\$20\,000 \times 0.9^2 = \$16\,200$.
3. After 3 years, the car's value will have fallen to $\$20\,000 \times 0.9^3 = \$14\,580$.

The value of the car after n years will be $\$20\,000 \times 0.9^n$.

Depreciation

Suppose that an asset has initial value $\$P$ and that its value depreciates at a rate of $R\%$ per annum. Then the value of the asset after n years, $\$D_n$, is given by

$$D_n = Pr^n, \quad \text{where } r = 1 - \frac{R}{100}$$

Example 27

A machine bought for \$15 000 depreciates at the rate of $12\frac{1}{2}\%$ per annum.

a What will be the value of the machine after 9 years?

b After how many years will its value drop below 10% of its original cost?

Solution

The value after n years is $D_n = Pr^n$, where $P = 15\,000$ and $r = 1 - 0.125 = 0.875$.

a
$$\begin{aligned} D_9 &= Pr^9 \\ &= 15\,000 \times (0.875)^9 \\ &= 4509.87 \end{aligned}$$

The value of the machine after 9 years is \$4509.87.

b We want to find the smallest value of n for which $D_n < 0.1P$.

$$\begin{aligned} D_n = Pr^n &< 0.1P \\ 15\,000 \times (0.875)^n &< 0.1 \times 15\,000 \\ 0.875^n &< 0.1 \end{aligned}$$

A calculator gives the solution $n > 17.24$. The value will drop below 10% of the original cost during the 18th year.

Regular payments

We now look at situations where compound interest is combined with equal payments at regular intervals of time. Examples include superannuation contributions, loan repayments and annuities. We will focus on yearly payments, but regular payments are also often made weekly, monthly or quarterly.

In the following two examples, we find the solutions using geometric series. Alternatively, the solutions can be found using a spreadsheet. In the next section, we will see another more general approach that can also be used in these situations.

Regular deposits into a savings account

Example 28

Sophie plans to retire in 15 years. She decides to deposit \$6000 into a bank account at the start of each year until her retirement. The interest rate is 6% p.a. compounded annually. What will be the account balance when Sophie retires at the end of the 15th year?

Solution

- The first \$6000 is in the bank for 15 years and so contributes $6000(1.06)^{15}$.
- The second \$6000 is in the bank for 14 years and so contributes $6000(1.06)^{14}$.
- The third \$6000 is in the bank for 13 years and so contributes $6000(1.06)^{13}$.

⋮

- The final \$6000 is in the bank for one year and so contributes $6000(1.06)^{1}$.

The final amount in the account is

$$6000(1.06)^{15} + 6000(1.06)^{14} + 6000(1.06)^{13} + \cdots + 6000(1.06)^{1}$$

This is a geometric series with $a = 6000(1.06)$, $r = 1.06$ and $n = 15$.

$$S_{15} = \frac{a(r^{15} - 1)}{r - 1} = \frac{6000(1.06)(1.06^{15} - 1)}{1.06 - 1} = 148\,035.17$$

Sophie will have \$148 035.17 in her bank account. She has contributed \$90 000.

In general, if $\$P$ is deposited at the start of each year into an account earning compound interest of $R\%$ p.a., then the account balance after n years, $\$A_n$, is given by

$$A_n = \frac{Pr(r^n - 1)}{r - 1}, \quad \text{where } r = 1 + \frac{R}{100}$$

Loan repayments

Example 29

Luke borrows \$50 000 and undertakes to repay \$6000 at the end of each year. Interest of 10% p.a. is charged on the unpaid debt.

a How much does he owe after the 8th repayment?

b How long does it take to pay off the loan?

Solution

Let $\$D_n$ be the amount still owing after the nth repayment.

a
- Amount owing after the 1st repayment:

$$D_1 = 50\,000 \times 1.1 - 6000$$

- Amount owing after the 2nd repayment:

$$\begin{aligned} D_2 &= D_1 \times 1.1 - 6000 \\ &= (50\,000 \times 1.1 - 6000) \times 1.1 - 6000 \\ &= 50\,000 \times 1.1^2 - 6000(1 + 1.1) \end{aligned}$$

- Amount owing after the 3rd repayment:

$$\begin{aligned} D_3 &= D_2 \times 1.1 - 6000 \\ &= (50\,000 \times 1.1^2 - 6000(1 + 1.1)) \times 1.1 - 6000 \\ &= 50\,000 \times 1.1^3 - 6000(1 + 1.1 + 1.1^2) \end{aligned}$$

Following this pattern, the amount owing after the 8th repayment is

$$\begin{aligned} D_8 &= 50\,000 \times 1.1^8 - 6000(1 + 1.1 + 1.1^2 + \cdots + 1.1^7) \\ &= 50\,000 \times 1.1^8 - 6000\left(\frac{1.1^8 - 1}{0.1}\right) \\ &= 50\,000 \times 1.1^8 - 60\,000 \times (1.1^8 - 1) \\ &= 38\,564.11 \end{aligned}$$

After the 8th repayment, he owes \$38 564.11.

b After the nth repayment, the amount owing is

$$\begin{aligned} D_n &= 50\,000 \times 1.1^n - 60\,000 \times (1.1^n - 1) \\ &= 60\,000 - 10\,000 \times 1.1^n \end{aligned}$$

We want to find when the debt is zero:

$$\begin{aligned} 60\,000 - 10\,000 \times 1.1^n &= 0 \\ 1.1^n &= 6 \end{aligned}$$

Using a calculator gives $n \approx 18.8$. It takes 19 years to pay off the loan.

In general, if \$$P$ is borrowed at an interest rate of R% p.a. and a repayment of \$$Q$ is made at the end of each year, then the amount owing after n years, \$$D_n$, is given by

$$D_n = Pr^n - \frac{Q(r^n - 1)}{r - 1}, \quad \text{where } r = 1 + \frac{R}{100}$$

Loan repayments using a recurrence relation

Example 29 can be solved using a spreadsheet. Let \$$D_n$ be the amount owing at the end of the nth year. Then

$$D_0 = 50\,000 \quad \text{and} \quad D_n = 1.1 \times D_{n-1} - 6000$$

To find the values using a spreadsheet:

- In cell A1, enter the value 0.
- In cell A2, enter the formula = A1 + 1.
- In cell B1, enter the initial value 50 000.
- In cell B2, enter the formula = 1.1 ∗ B1 − 6000.
- Fill down in columns A and B.

After 8 years, the debt is \$38 564.11. By filling down further, you can find the year that the debt reaches zero.

	Year	Owing
	A	B
1	0	50 000.00
2	1	49 000.00
3	2	47 900.00
4	3	46 690.00
5	4	45 359.00
6	5	43 894.90
7	6	42 284.39
8	7	40 512.83
9	8	38 564.11

Summary 3F

- **Compound interest**
  Suppose that $\$P$ is invested at an interest rate of $R\%$ per annum, compounded annually. Then the value of the investment after n years, $\$A_n$, is given by

$$A_n = Pr^n, \quad \text{where } r = 1 + \frac{R}{100}$$

- **Depreciation**
  Suppose that an asset has initial value $\$P$ and that its value depreciates at a rate of $R\%$ per annum. Then the value of the asset after n years, $\$D_n$, is given by

$$D_n = Pr^n, \quad \text{where } r = 1 - \frac{R}{100}$$

- Situations involving regular payments can be investigated by:
  - finding a pattern in the calculation of the first few values and then using a geometric series
  - finding the recurrence relation for the values and using a spreadsheet.

Exercise 3F

Example 26

1 \$5000 is invested at 6% p.a. compounded annually.

a Find the value of the investment after 6 years.

b Find how long it will take for the original investment to double in value.

2 How much would need to be invested at 8.5% p.a., compounded annually, to yield a return of \$8000 after 12 years?

3 The profits of a cosmetics company have been increasing by 15% per annum since its formation. The profit in the first year was \$60 000.

a Find a formula for the profit in the nth year.

b In which year did the annual profit first exceed \$1 200 000?

c Find a formula for the total profit over the first n years.

Example 27

4 A car bought for \$65 000 depreciates at the rate of 15% per annum.

a What will be the value of the car after 3 years?

b After how many years will its value drop below 50% of its original cost?

5 What annual compound interest rate would be required to triple the value of an investment of \$200 in 10 years?

6 The value of a car is \$40 000 when new. If its value depreciates by 15% each year, after how many years will its value be less than \$10 000?

Example 28 **7** At the beginning of each year, an investor deposits \$25 000 into a fund that pays 5% p.a. compounded annually. How much is the investment worth after 10 years?

8 I wish to accumulate \$100 000 over 20 years at 10% p.a. compounded annually. What should be the amount of my annual payments?

9 Chen pays \$20 000 into an investment fund at the start of each year, and the fund earns compound interest at a rate of 6% p.a.

a How much is the investment worth at the end of 10 years?

b After how many years will the value of the investment be over \$200 000?

Example 29 **10** Daniel borrows \$100 000 and undertakes to repay \$10 000 at the end of each year. Interest of 5% p.a. is charged on the unpaid debt.

a How much does he owe after the 10th repayment?

b How long does it take to pay off the loan?

11 Grace lends \$50 000 on the condition that she is repaid the money in 15 equal yearly installments. If she receives interest at the rate of 4% p.a., what is the amount of each installment?

12 Andrew invests \$1000 at 20% simple interest for 10 years. Bianca invests her \$1000 at 12.5% compound interest for 10 years. At the end of 10 years, whose investment is worth more?

13 By sampling, it is estimated that there are 20 000 trout in a lake. Assume that the trout population, left untouched, would increase by 15% per annum. It is known that 2000 trout per year are removed by fishing.

a How many trout are there in the lake after:

i 1 year **ii** 2 years **iii** 3 years?

b Write a recurrence relation that gives the number of trout in the lake after n years in terms of the number of trout in the lake after $n - 1$ years.

c Write a formula for the number of trout in the lake after n years in terms of n.

d Find the number of trout in the lake after 15 years.

14 When Emma retired from work at the start of January, she invested a lump sum of \$300 000 at an interest rate of 10% p.a. compounded annually. She now uses this account to pay herself an annuity of \$40 000 at the end of December every year.

a What is the amount left in the account at the end of:

i the first year **ii** the second year **iii** the third year?

b Write a recurrence relation that gives the account balance after n years in terms of the account balance after $n - 1$ years.

c Write a formula for the account balance after n years in terms of n.

d For how many years will Emma be able to pay herself an annuity of \$40 000 before the account balance becomes too low?

3G Recurrence relations of the form $t_n = rt_{n-1} + d$

Throughout this chapter, we have studied two useful types of sequences:

- Arithmetic sequences are defined by recurrence relations of the form $t_n = t_{n-1} + d$.
- Geometric sequences are defined by recurrence relations of the form $t_n = rt_{n-1}$.

In the previous section, we studied sequences based on regular payments, where each step involves a percentage increase or decrease together with a fixed-amount increase or decrease.

In this section, we consider a generalisation of all these types of sequences. We shall study sequences defined by a recurrence relation of the form

$$t_n = rt_{n-1} + d$$

where r and d are constants.

Note: The case where $r = 1$ corresponds to an arithmetic sequence.
The case where $d = 0$ corresponds to a geometric sequence.

Example 30

Find the first four terms of the sequence defined by the recurrence relation

$$t_n = 2t_{n-1} + 1, \quad t_1 = 10$$

Solution

$$t_1 = 10$$
$$t_2 = 2 \times 10 + 1 = 21$$
$$t_3 = 2 \times 21 + 1 = 43$$
$$t_4 = 2 \times 43 + 1 = 87$$

Example 31

The sequence $5, 13, 37, \ldots$ is generated by a recurrence relation of the form

$$s_n = rs_{n-1} + d, \quad s_1 = 5$$

Find the values of r and d.

Solution

From the first three terms of this sequence, we have a system of linear equations:

$$13 = 5r + d \qquad (1)$$
$$37 = 13r + d \qquad (2)$$

Subtract (1) from (2):

$$24 = 8r$$
$$r = 3$$

Substitute in equation (1):

$$d = -2$$

Finding the *n*th term of the sequence

We can derive a general formula for the *n*th term of a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$.

For a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where $r \neq 1$, the *n*th term is given by

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

where t_1 is the first term.

Proof Use the recurrence relation to obtain an equation for each term from t_n down to t_2:

$$\begin{aligned} t_n &= rt_{n-1} + d \\ t_{n-1} &= rt_{n-2} + d \\ t_{n-2} &= rt_{n-3} + d \\ &\vdots \\ t_3 &= rt_2 + d \\ t_2 &= rt_1 + d \end{aligned}$$

Now multiply the equations by increasing powers of r:

$$\begin{aligned} t_n &= rt_{n-1} + d && \text{(multiply by } r^0\text{)} \\ rt_{n-1} &= r^2t_{n-2} + rd && \text{(multiply by } r^1\text{)} \\ r^2t_{n-2} &= r^3t_{n-3} + r^2d && \text{(multiply by } r^2\text{)} \\ &\vdots && \vdots \\ r^{n-3}t_3 &= r^{n-2}t_2 + r^{n-3}d && \text{(multiply by } r^{n-3}\text{)} \\ r^{n-2}t_2 &= r^{n-1}t_1 + r^{n-2}d && \text{(multiply by } r^{n-2}\text{)} \end{aligned}$$

Add these equations:

$$\begin{aligned} &t_n + rt_{n-1} + r^2t_{n-2} + \cdots + r^{n-3}t_3 + r^{n-2}t_2 \\ &= rt_{n-1} + r^2t_{n-2} + r^3t_{n-3} + \cdots + r^{n-2}t_2 + r^{n-1}t_1 + d + rd + r^2d + \cdots + r^{n-3}d + r^{n-2}d \end{aligned}$$

Cancelling gives

$$t_n = r^{n-1}t_1 + d + rd + r^2d + \cdots + r^{n-3}d + r^{n-2}d$$

Hence

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

Note: You should check that this formula for t_n satisfies the recurrence relation $t_n = rt_{n-1} + d$.

Example 32

Find a formula for the nth term of the sequence defined by the recurrence relation

$$t_n = 2t_{n-1} + 1, \quad t_1 = 10$$

Solution

We will use the general formula

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

Here $r = 2$ and $d = 1$. Hence

$$\begin{aligned} t_n &= 2^{n-1} \times 10 + \frac{1 \times (2^{n-1} - 1)}{2 - 1} \\ &= 10 \times 2^{n-1} + 2^{n-1} - 1 \\ &= 11 \times 2^{n-1} - 1 \end{aligned}$$

Note: You can check this formula for the first four terms of the sequence, which were found in Example 30.

Example 33

Consider the sequence defined by the recurrence relation

$$t_n = \frac{1}{2}t_{n-1} + 8, \quad t_1 = 5$$

a Find a formula for the nth term of the sequence.

b Determine the first four terms of the sequence.

c Describe what happens to t_n for large values of n.

Solution

a We will use the general formula

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

Here $r = \frac{1}{2}$ and $d = 8$. Hence

$$\begin{aligned} t_n &= \left(\frac{1}{2}\right)^{n-1} \times 5 + \frac{8\left(\left(\frac{1}{2}\right)^{n-1} - 1\right)}{\frac{1}{2} - 1} \\ &= 5\left(\frac{1}{2}\right)^{n-1} - 16\left(\left(\frac{1}{2}\right)^{n-1} - 1\right) \\ &= -11\left(\frac{1}{2}\right)^{n-1} + 16 \\ &= 16 - \frac{11}{2^{n-1}} \end{aligned}$$

b $t_1 = 16 - 11 = 5$

$$t_2 = 16 - \frac{11}{2} = \frac{21}{2} = 10\tfrac{1}{2}$$

$$t_3 = 16 - \frac{11}{4} = \frac{53}{4} = 13\tfrac{1}{4}$$

$$t_4 = 16 - \frac{11}{8} = \frac{117}{8} = 14\tfrac{5}{8}$$

c We use the formula for t_n from part **a**:

$$t_n = 16 - \frac{11}{2^{n-1}}$$

As n becomes very large, the number $\frac{11}{2^{n-1}}$ becomes very small and so $t_n \to 16$.

Consider a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where $r \neq 1$. Then the general formula that we obtained for t_n can be rewritten into a rule of the form

$$t_n = Ar^{n-1} + B$$

for constants A and B. The following example shows how this observation can be used.

Example 34

The sequence $5, 16, 38, \ldots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the nth term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants A and B.

Solution

From the first three terms, we have

$$t_1 = A + B = 5 \qquad (1)$$

$$t_2 = Ar + B = 16 \qquad (2)$$

$$t_3 = Ar^2 + B = 38 \qquad (3)$$

Subtract equation (1) from both (2) and (3):

$$A(r-1) = 11 \qquad (4)$$

$$A(r^2-1) = 33 \qquad (5)$$

Divide (5) by (4):

$$\frac{r^2-1}{r-1} = 3$$

$$\frac{(r+1)(r-1)}{r-1} = 3$$

$$r + 1 = 3$$

$$r = 2$$

From (4): $A = 11$

From (1): $B = -6$

The formula for the nth term is

$$t_n = 11 \times 2^{n-1} - 6$$

Example 35

At the start of the year, a lake in a national park is estimated to contain 10 000 trout. Experience with this lake shows that, if left to natural factors, the trout numbers in the lake will increase on average by 20% per year. On this basis, the park authorities give permission for 1800 trout to be taken from the lake each year by anglers.

a Write down a recurrence relation that can be used to model the number of trout in the lake at the start of each year. (Apply the percentage increase before the fixed decrease.)

b Using this model, how many trout will the lake contain after 10 years?

c When will the population of trout in the lake exceed 20 000?

d Suppose instead that the park authorities allow 2200 trout to be fished from the lake each year.

  i Write down a recurrence relation that can be used to model the number of trout in the lake at the start of each year.

  ii Give an expression for the number of trout in the lake at the start of the nth year.

  iii When will trout disappear from the lake?

e Suppose instead that the park authorities allow 2000 trout to be fished from the lake each year. Describe what will happen.

Solution

a Let P_n be the number of trout in the lake at the start of the nth year. Then

$$P_{n+1} = 1.2P_n - 1800, \quad P_1 = 10\,000$$

b We can use the general formula for the nth term of the sequence:

$$\begin{aligned} P_n &= 1.2^{n-1} \times 10\,000 + \frac{(-1800)(1.2^{n-1} - 1)}{1.2 - 1} \\ &= 10\,000 \times 1.2^{n-1} - 9000(1.2^{n-1} - 1) \\ &= 1000 \times 1.2^{n-1} + 9000 \end{aligned}$$

After 10 years:

$$\begin{aligned} P_{11} &= 1000 \times 1.2^{10} + 9000 \\ &\approx 15\,192 \end{aligned}$$

c We want to find the smallest value of n for which $P_n > 20\,000$:

$$\begin{aligned} P_n &> 20\,000 \\ 1000 \times 1.2^{n-1} + 9000 &> 20\,000 \\ 1000 \times 1.2^{n-1} &> 11\,000 \\ 1.2^{n-1} &> 11 \end{aligned}$$

Using your calculator gives $n > 14.152\ldots$.
By the start of the 15th year, the population of trout will be over 20 000.

d **i** Let Q_n be the number of trout in the lake at the start of the nth year. Then

$$Q_{n+1} = 1.2Q_n - 2200, \quad Q_1 = 10\,000$$

ii The general formula gives

$$\begin{aligned} Q_n &= 1.2^{n-1} \times 10\,000 + \frac{(-2200)(1.2^{n-1} - 1)}{1.2 - 1} \\ &= 10\,000 \times 1.2^{n-1} - 11\,000(1.2^{n-1} - 1) \\ &= 11\,000 - 1000 \times 1.2^{n-1} \end{aligned}$$

iii We want to find the smallest value of n for which $Q_n \leq 0$:

$$\begin{aligned} Q_n &\leq 0 \\ 11\,000 - 1000 \times 1.2^{n-1} &\leq 0 \\ 1.2^{n-1} &\geq 11 \\ n &\geq 15 \end{aligned}$$

Trout will disappear from the lake during the 14th year.

e Let R_n be the number of trout in the lake at the start of the nth year. Then

$$R_{n+1} = 1.2R_n - 2000, \quad R_1 = 10\,000$$

This gives

$$R_2 = 1.2 \times 10\,000 - 2000 = 10\,000$$

$$R_3 = 1.2 \times 10\,000 - 2000 = 10\,000$$

We can see that $R_n = 10\,000$ for all n. The model predicts that the trout population will stay at 10 000.

First-order linear recurrence relations

In general, a **first-order linear recurrence relation** has the form $t_n = f(n)\,t_{n-1} + g(n)$, where f and g are functions. In this section, we consider the special case that f and g are constant functions. However, we look at two examples where the function g is not constant in Exercise 3G (Questions 11 and 12).

Summary 3G

- Let $t_1, t_2, t_3, \ldots$ be a sequence defined by a recurrence relation of the form

  $$t_n = rt_{n-1} + d$$

  where r and d are constants. Then the nth term of the sequence is given by

  $$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1} \qquad \text{(provided } r \neq 1\text{)}$$

Skill-sheet

Exercise 3G

Example 30

1 Calculate the first four terms of the sequence defined by each of the following recurrence relations:

a $t_n = 3t_{n-1} + 4, \quad t_1 = 6$

b $s_n = 6s_{n-1} + 2, \quad s_1 = 1$

c $t_{n+1} = 3t_n - 4, \quad t_1 = 6$

d $u_{n+1} = 4u_n + 1, \quad u_1 = 2$

2 **a** The sequence $2, 6, 26, \ldots$ is defined by the recurrence relation $t_n = 5t_{n-1} + d$, with $t_1 = 2$. Find the value of d.

b The sequence $500, 650, 875, \ldots$ is defined by the recurrence relation $t_n = rt_{n-1} - 100$, with $t_1 = 500$. Find the value of r.

c The sequence $1000, 100, -80, \ldots$ is defined by the recurrence relation $T_n = 0.2T_{n-1} + d$, with $T_1 = 1000$. Find the value of d.

d The sequence $a, 22, 90, \ldots$ is defined by the recurrence relation $s_n = 4s_{n-1} + 2$, with $s_1 = a$. Find the value of a.

Example 31

3 **a** The sequence $2, 5, 11, \ldots$ is defined by the recurrence relation $t_n = rt_{n-1} + d$, with $t_1 = 2$. Find the values of r and d.

b The sequence $512, 192, 32, \ldots$ is defined by the recurrence relation $v_n = rv_{n-1} + d$, with $v_1 = 512$. Find the values of r and d.

c The sequence $a, 10, 55, \ldots$ is defined by the recurrence relation $t_{n+1} = 5t_n + d$, with $t_1 = a$. Find the values of a and d.

d The sequence $200, 500, 1400, \ldots$ is defined by the recurrence relation $t_{n+1} = rt_n + d$, with $t_1 = 200$. Find the values of r and d.

4 A sequence is defined recursively by $a_1 = k$ and $a_n = 5a_{n-1} + 3$.

a Find a_2 and a_3 in terms of k.

b Find the sum of the first four terms of the sequence in terms of k.

Example 32

5 For each of the following recurrence relations, determine an expression for the nth term of the sequence in terms of n:

a $t_n = 2t_{n-1} - 6, \quad t_1 = 7$

b $t_n = 2t_{n-1} - 2, \quad t_1 = 1$

c $t_{n+1} = \frac{1}{2}t_n + 10, \quad t_1 = 20$

d $t_{n+1} = \frac{1}{2}t_n + 14, \quad t_1 = 20$

e $t_{n+1} = \frac{1}{2}t_n - 10, \quad t_1 = 20$

f $t_{n+1} = \frac{1}{2}t_n + \frac{1}{2}, \quad t_1 = 1$

Example 33

6 Consider the sequence defined by the recurrence relation

$$t_n = \frac{1}{2}t_{n-1} + 5, \quad t_1 = 6$$

a Find a formula for the nth term of the sequence.

b Determine the first four terms of the sequence.

c Describe what happens to t_n for large values of n.

7 Consider the sequence defined by the recurrence relation

$$t_n = -\frac{1}{2}t_{n-1} + 5, \quad t_1 = 6$$

a Find a formula for the nth term of the sequence.

b Determine the first four terms of the sequence.

c Describe what happens to t_n for large values of n.

Example 34

8 The sequence $7, 31, 103, \ldots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the nth term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants A and B.

9 The sequence $16, 5, -\frac{1}{2}, \ldots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the nth term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants A and B.

10 The sequence $t_1, t_2, t_3, \ldots$ is defined by a recurrence relation of the form $t_n = rt_{n-1} + d$. The nth term of this sequence is given by the formula $t_n = 3 \times 2^n - 4$.

a Find the values of r and d.

b Determine the smallest value of n for which $t_n > 1000$.

c Determine an expression for $t_{n+1} - t_n$. Hence show that $t_{n+1} > t_n$ for all $n \in \mathbb{N}$.

11 Using first principles, find a formula for the nth term of the sequence defined by the recurrence relation $t_n = t_{n-1} + 2n$, with $t_1 = 5$.

12 Using first principles, find a formula for the nth term of the sequence defined by the recurrence relation $t_n = t_{n-1} + 2n + 1$, with $t_1 = 5$.

13 A sequence satisfies the recurrence relation $a_n = 4a_{n-1} - 1$.

a Given that $a_2 = 43$, find a_1 and a_3.

b Find a formula for a_n in terms of n.

14 Consider the sequence defined by $s_1 = 1$ and $2s_n + s_{n-1} = 6$.

a Determine a formula for the nth term of the sequence.

b Determine the first five terms of the sequence.

c Plot the graph of this sequence (s_n against n) for the first five terms.

d Describe the behaviour of the sequence as n becomes large.

15 Wild deer are causing a problem in a nature reserve. When counted at the start of the year, there were 1356 deer in the reserve. Under normal conditions, the deer population grows at a rate of 22% per year. To reduce deer numbers in the reserve, the rangers recommend that hunters be allowed to take 250 deer from the reserve at the end of each year.

a Write down a recurrence relation to describe this situation. Let N_n represent the number of deer in the nature reserve at the start of the nth year.

b What impact does allowing hunters to take 250 deer per year have on the growth of the deer population?

Example 35

16 The change in population of a country town is dependent on births, deaths and the arrival of new residents. The birth rate is 9% per annum and the death rate is 0.5% per annum. A constant number of 250 new residents move into the town each year. At the start of 2010, the number of residents of this town was 3000.

a Set up a recurrence relation to describe the population of the town at the start of each year. (Apply the percentage change before the fixed increase.)

b Find a formula for the population of the town at the start of the nth year.

c Determine the population of the town at the start of 2020.

d In which year does the population of the town pass 5050?

17 A farmer decides to keep a herd of goats. The herd is started with 200 goats. Each year, 3% of the goats die through old age, 20% produce one offspring and 5% produce two offspring. The farmer sells 20 goats at the end of each year.

a Set up a recurrence relation to describe the herd size at the start of each year.

b Find a formula for the herd size at the start of the nth year.

c Determine the herd size after 10 years.

18 You borrow \$15 000 from the bank. You plan to pay off the loan at \$400 per month. Interest at a rate of 8.4% per annum is charged monthly on the amount still owing.

a Write down a recurrence relation of the form $A_{n+1} = rA_n - d$, $A_1 = a$, that can be used to describe the amount, $\$A_n$, that you owe at the start of the nth month.

b Write down an expression for A_n in terms of n.

c How long will it take you to pay off the loan?

19 Consider the sequence $t_1, t_2, t_3, \ldots$ defined by the recurrence relation

$$t_n = 0.6t_{n-1} + 60, \qquad t_1 = 32$$

a Find a rule for t_n in terms of n.

b Find $t_{n+1} - t_n$ in terms of n. Hence prove that the sequence is increasing. (That is, show that $t_{n+1} \geq t_n$ for all $n \in \mathbb{N}$.)

c Determine the smallest value of n for which $t_{n+1} - t_n \leq 0.001$.

d Prove that $t_n \leq 150$ for all $n \in \mathbb{N}$.

e Prove that $t_n \to 150$ as $n \to \infty$.

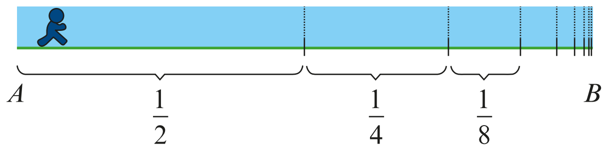
f Now consider the sequence $s_1, s_2, s_3, \ldots$ defined by the recurrence relation

$$s_n = 0.6s_{n-1} + d, \qquad s_1 = 32$$

What is the value of d if $s_n \to 200$ as $n \to \infty$?

3H Zeno's paradox and infinite geometric series

A runner wants to go from point A to point B. To do this, he would first have to run half the distance, then half the remaining distance, then half the remaining distance, and so on.



The Greek philosopher Zeno of Elea, who lived about 450 BC, argued that since the runner has to complete an infinite number of stages to get from A to B, he cannot do this in a finite amount of time, and so he cannot reach B. In this section we see how to resolve this paradox.

Infinite geometric series

If a geometric sequence has a common ratio with magnitude less than 1, that is, if $-1 < r < 1$, then each successive term is closer to zero. For example, consider the sequence

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$$

In Example 23 we found that the sum of the first 9 terms is $S_9 \approx 0.499975$. The sum of the first 20 terms is $S_{20} \approx 0.49999999986$. We might conjecture that, as we add more and more terms of the sequence, the sum will get closer and closer to 0.5, that is, $S_n \to 0.5$ as $n \to \infty$.

An infinite series $t_1 + t_2 + t_3 + \cdots$ is said to be **convergent** if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. This limit is called the **sum to infinity** of the series.

If $-1 < r < 1$, then the infinite geometric series $a + ar + ar^2 + \cdots$ is convergent and the sum to infinity is given by

$$S_\infty = \frac{a}{1-r}$$

Proof We know that

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

As $n \to \infty$, we have $r^n \to 0$ and so $\frac{ar^n}{1-r} \to 0$. Hence $S_n \to \frac{a}{1-r}$ as $n \to \infty$.

Resolution of Zeno's paradox Assume that the runner is travelling at a constant speed and that he takes 1 minute to run half the distance from A to B. Then he takes $\frac{1}{2}$ minute to run half the remaining distance, and so on. The total time taken is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This is an infinite geometric series, and the formula gives $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

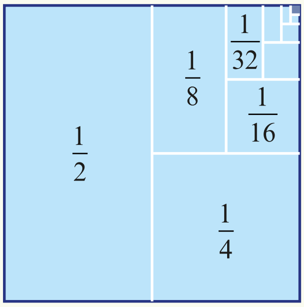
This fits with our common sense: If the runner takes 1 minute to cover half the distance, then he will take 2 minutes to cover the whole distance.

Example 36

Find the sum to infinity of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution

$a = \frac{1}{2}$, $r = \frac{1}{2}$ and so $S_\infty = \dfrac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$



Note: This result is illustrated by the unit square shown. Divide the square in two, then divide one of the resulting rectangles in two, and so on. The sum of the areas of the rectangles equals the area of the square.

Example 37

A square has a side length of 40 cm. A copy of the square is made so that the area of the copy is 80% of the original. The process is repeated so that each time the area of the new square is 80% of the previous one. If this process is repeated indefinitely, find the total area of all the squares.

Solution

The area of the first square is $40^2 = 1600$ cm^2.

We have $a = 1600$ and $r = 0.8$, giving

$$S_\infty = \frac{1600}{1 - 0.8} = 8000 \text{ cm}^2$$

Example 38

Express the recurring decimal $0.\dot{3}\dot{2}$ as the ratio of two integers.

Solution

$$0.\dot{3}\dot{2} = 0.32 + 0.0032 + 0.000032 + \cdots$$

We have $a = 0.32$ and $r = 0.01$, giving

$$S_\infty = \frac{0.32}{0.99} = \frac{32}{99}$$

i.e. $0.\dot{3}\dot{2} = \dfrac{32}{99}$

Skill-sheet

Exercise 3H

Example 36

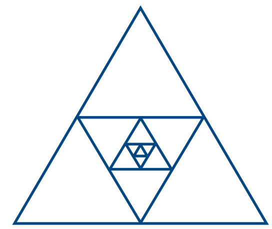
1 Find:

a $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

b $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

Example 37

2 An equilateral triangle has perimeter p cm. The midpoints of the sides are joined to form another triangle, and this process is repeated. Find the perimeter and area of the nth triangle, and find the limits as $n \to \infty$ of the sum of the perimeters and the sum of the areas of the first n triangles.



3 A rocket is launched into the air so that it reaches a height of 200 m in the first second. Each subsequent second it gains 6% less height. Find how high the rocket will climb.

4 A man can walk 3 km in the first hour of a journey, but in each succeeding hour walks half the distance covered in the preceding hour. Can he complete a journey of 6 km? Where does this problem cease to be realistic?

5 A frog standing 10 m from the edge of a pond sets out to jump towards it. Its first jump is 2 m, its second jump is $1\frac{1}{2}$ m, its third jump is $1\frac{1}{8}$ m, and so on. Show that the frog will never reach the edge of the pond.

6 A stone is thrown so that it skips across the surface of a lake. If each skip is 30% less than the previous skip, how long should the first skip be so that the total distance travelled by the stone is 40 m?

7 A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound it rises two-thirds of the height of the previous rebound. If it continues to bounce indefinitely, what is the total distance it will travel?

Example 38

8 Express each of the following periodic decimals as the ratio of a pair of integers:

a $0.\dot{4}$ **b** $0.0\dot{3}$ **c** $10.\dot{3}$ **d** $0.0\dot{3}\dot{5}$ **e** $0.\dot{9}$ **f** $4.\dot{1}$

9 A geometric series has first term a and common ratio $\dfrac{a-b}{a}$, where $a > 0$.

a Show that the geometric series is convergent if $0 < b < 2a$.

b Find the sum to infinity of this series.

10 For each of the following, state the condition under which the geometric series is convergent and find the sum to infinity in this case:

a $1 + x + x^2 + x^3 + \cdots$

b $a + a(2a-1) + a(2a-1)^2 + \cdots$

c $\dfrac{3x-1}{x} + 1 + \dfrac{x}{3x-1} + \cdots$

d $\dfrac{3x^2-1}{x^2} + 1 + \dfrac{x^2}{3x^2-1} + \cdots$

11 The sum of the first four terms of a geometric series is 30 and the sum to infinity is 32. Find the first two terms.

12 Find the third term of a geometric sequence that has a common ratio of $-\dfrac{1}{4}$ and a sum to infinity of 8.

13 Find the common ratio of a geometric sequence with first term 5 and sum to infinity 15.

14 For any number $x > 2$, show that there is an infinite geometric series such that $a = 2$ and the sum to infinity is x.

Review

Chapter summary

Assign-ment

Nrich

- The nth term of a sequence is denoted by t_n.
- A **recurrence relation** enables each subsequent term to be found from previous terms. A sequence specified in this way is said to be defined **recursively**.

  e.g. $t_1 = 1, \quad t_n = t_{n-1} + 2$
- A sequence may also be defined by a rule that is stated in terms of n.

  e.g. $t_n = 2n - 1$

Arithmetic sequences and series

- An **arithmetic sequence** has a rule of the form

  $$t_n = a + (n-1)d$$

  where a is the first term and d is the **common difference** (i.e. $d = t_k - t_{k-1}$ for all $k > 1$).
- The sum of the terms in an arithmetic sequence is called an **arithmetic series**.
- The sum of the first n terms of an arithmetic sequence is given by

  $$S_n = \frac{n}{2}\big(2a + (n-1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

Geometric sequences and series

- A **geometric sequence** has a rule of the form

  $$t_n = ar^{n-1}$$

  where a is the first term and r is the **common ratio** (i.e. $r = \frac{t_k}{t_{k-1}}$ for all $k > 1$).
- The sum of the terms in a geometric sequence is called a **geometric series**.
- For $r \neq 1$, the sum of the first n terms of a geometric sequence is given by

  $$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$
- For $-1 < r < 1$, the sum S_n approaches a limiting value as $n \to \infty$, and the series is said to be **convergent**. This limit is called the **sum to infinity** and is given by

  $$S_\infty = \frac{a}{1 - r}$$

Applications of geometric sequences

- Compound interest Suppose that $\$P$ is invested at an interest rate of $R\%$ per annum, compounded annually. Then the value of the investment after n years, $\$A_n$, is given by

  $$A_n = Pr^n, \quad \text{where } r = 1 + \frac{R}{100}$$
- Depreciation Suppose that an asset has initial value $\$P$ and that its value depreciates at a rate of $R\%$ per annum. Then the value of the asset after n years, $\$D_n$, is given by

  $$D_n = Pr^n, \quad \text{where } r = 1 - \frac{R}{100}$$

Recurrence relations of the form $t_n = rt_{n-1} + d$

- Let $t_1, t_2, t_3, \dots$ be a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where r and d are constants. Then the nth term of the sequence is given by

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1} \qquad \text{(provided } r \neq 1\text{)}$$

Note that this formula for t_n can be written in the form $t_n = Ar^{n-1} + B$, where A and B are constants.

Technology-free questions

1 Find the first six terms of the following sequences:

a $t_1 = 3,\ t_n = t_{n-1} - 4$

b $t_1 = 5,\ t_n = 2t_{n-1} + 2$

2 Find the first six terms of the following sequences:

a $t_n = 2n$

b $t_n = -3n + 2$

3 Nick invests \$5000 at 5% p.a. compound interest at the beginning of the year. At the beginning of each of the following years, he puts a further \$500 into the account.

a Write down the amount of money in the account at the end of each of the first two years.

b Set up a recurrence relation to generate the sequence for the investment.

4 The 4th term of an arithmetic sequence is 19 and the 7th term is 43. Find the 20th term.

5 For the arithmetic sequence with $t_5 = 0.35$ and $t_9 = 0.15$, find t_{14}.

6 For the arithmetic sequence with $t_6 = -24$ and $t_{14} = 6$, find S_{10}.

7 For the arithmetic sequence $-5, 2, 9, \dots$, find $\{ n : S_n = 402 \}$.

8 The 6th term of a geometric sequence is 9 and the 10th term is 729. Find the 4th term.

9 One thousand dollars is invested at 3.5% p.a. compounded annually. Find the value of the investment after n years.

10 A sequence is defined by the recurrence relation $t_n = 2t_{n-1} - 3$ and $t_1 = 4$. Find a formula for the nth term of the sequence in terms of n.

11 The first term of a geometric sequence is 9 and the third term is 4. Find the possible values for the second and fourth terms.

12 The sum of three consecutive terms of a geometric sequence is 24 and the sum of the next three terms is also 24. Find the sum of the first 12 terms.

13 Find the sum of the first eight terms of the geometric sequence with first term 6 and common ratio −3.

14 Find the sum to infinity of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

15 The numbers x, $x + 4$, $2x + 2$ are three successive terms of a geometric sequence. Find the value of x.

Multiple-choice questions

1 The first three terms of the sequence defined by the rule $t_n = 3n + 2$ are

A 1, 2, 3 **B** 2, 4, 6 **C** 5, 7, 9 **D** 5, 8, 11 **E** 5, 8, 10

2 If $t_1 = 3$ and $t_{n+1} = t_n + 3$, then t_4 is

A 4 **B** 12 **C** 9 **D** 15 **E** 14

3 For the arithmetic sequence $10, 8, 6, \ldots$, we have $t_{10} =$

A −8 **B** −10 **C** −12 **D** 10 **E** 8

4 For the arithmetic sequence $10, 8, 6, \ldots$, we have $S_{10} =$

A 10 **B** 0 **C** −10 **D** 20 **E** −20

5 If 58 is the nth term of the arithmetic sequence $8, 13, 18, \ldots$, then $n =$

A 12 **B** 11 **C** 10 **D** 5 **E** 3

6 The sixth term of the geometric sequence $12, 8, \frac{16}{3}, \ldots$ is

A $\frac{16}{3}$ **B** $\frac{128}{27}$ **C** $\frac{64}{81}$ **D** $\frac{128}{81}$ **E** $\frac{256}{81}$

7 For the sequence $8, 4, 2, \ldots$, we have $S_6 =$

A $\frac{1}{4}$ **B** $15\frac{1}{2}$ **C** $15\frac{7}{8}$ **D** 15 **E** $15\frac{3}{4}$

8 For the sequence $8, 4, 2, \ldots$, we have $S_\infty =$

A $\frac{1}{2}$ **B** 0 **C** 16 **D** 4 **E** ∞

9 If \$2000 is invested at 5.5% p.a. compounded annually, the value of the investment after 6 years is

A \$13 766.10 **B** \$11 162.18 **C** \$2550.00 **D** \$2613.92 **E** \$2757.69

10 If $S_\infty = 37.5$ and $r = \frac{1}{3}$, then a equals

A $\frac{2}{3}$ **B** 12.5 **C** $16\frac{2}{3}$ **D** 25 **E** 56.25

11 A sequence satisfies the recurrence relation $t_n = 4t_{n-1} - 5$. If $t_2 = 19$, then $t_3 =$

A 17 **B** 71 **C** 84 **D** 214 **E** 279

12 A sequence is generated by the rule $t_{n+1} = \frac{1}{2}t_n + 2$, for $n \in \mathbb{N}$. If the third term of the sequence is 12, then the first term is

A 6 **B** 10 **C** 20 **D** 24 **E** 36

13 The first five terms of the sequence defined by the recurrence relation $t_n = 2t_{n-1} - 6$, with $t_0 = 6$, are

A 6, 0, 0, 0, 0 **B** 6, 6, 6, 6, 6 **C** 6, 12, 6, 12, 6
D 6, 0, −6, −12, −18 **E** 6, 12, 18, 30, 54

Extended-response questions

1 A do-it-yourself picture-framing kit is available in various sizes. Size 1 contains 0.8 m of moulding, size 2 contains 1.5 m, size 3 contains 2.2 m, and so on.

a Form the sequence of lengths of moulding.

b Is the sequence of lengths of moulding an arithmetic sequence?

c Find the length of moulding contained in the largest kit, size 12.

2 A firm proposes to sell coated seeds in packs containing the following number of seeds: 50, 75, 100, 125, ...

a Is this an arithmetic sequence?

b Find a formula for the nth term.

c Find the number of seeds in the 25th size packet.

3 A number of power poles are to be placed in a straight line between two towns, A and B, which are 32 km apart. The first is placed 5 km from town A, and the last is placed 3 km from town B. The poles are placed so that the intervals starting from town A and finishing at town B are

$$5,\ 5-d,\ 5-2d,\ 5-3d,\ \dots,\ 5-6d,\ 3$$

There are seven poles. How far is the fifth pole from town A, and how far is it from town B?

4 A firm makes nylon thread in the following deniers (thicknesses):

$$2, 9, 16, 23, 30, \dots$$

a Find the denier number, D_n, of the firm's nth thread in order of increasing thickness.

A request came in for some very heavy 191 denier thread, but this turned out to be one stage beyond the thickest thread made by the firm.

b How many different thicknesses does the firm make?

5 A new house appears to be slipping down a hillside. The first year it slipped 4 mm, the second year 16 mm, and the third year 28 mm. If it goes on like this, how far will it slip during the 40th year?

6 Anna sends 16 Christmas cards the first year, 24 the second year, 32 the next year, and so on. How many Christmas cards will she have sent altogether after 10 years if she keeps increasing the number sent each year in the same way?

7 Each time Lee rinses her hair after washing it, the result is to remove a quantity of shampoo from her hair. With each rinse, the quantity of shampoo removed is one-tenth of that removed by the previous rinse.

a If Lee washes out 90 mg of shampoo with the first rinse, how much will she have washed out altogether after six rinses?

b How much shampoo do you think was present in her hair at the beginning?

8 A prisoner is trapped in an underground cell, which is inundated by a sudden rush of water. The water comes up to a height of 1 m, which is one-third of the height of the ceiling (3 m). After an hour another inundation occurs, and the water level in the cell rises by $\frac{1}{3}$ m. After a second hour another inundation raises the water level by $\frac{1}{9}$ m. If this process continues for 6 hours, write down:

a the amount the water level will rise at the end of the sixth hour

b the total height of the water level then.

If this process continues, do you think the prisoner, who cannot swim, will drown? Why?

9 After an undetected leak in a storage tank, the staff at an experimental station were subjected to 500 curie hours of radiation the first day, 400 the second day, 320 the third day, and so on. Find the number of curie hours they were subjected to:

a on the 14th day

b during the first five days of the leak.

10 A rubber ball is dropped from a height of 81 m. Each time it strikes the ground, it rebounds two-thirds of the distance through which it has fallen.

a Find the height that the ball reaches after the sixth bounce.

b Assuming that the ball continues to bounce indefinitely, find the total distance travelled by the ball.

11 In payment for loyal service to the king, a wise peasant asked to be given one grain of rice for the first square of a chessboard, two grains for the second square, four for the third square, and so on for all 64 squares of the board. The king thought that this seemed fair and readily agreed, but was horrified when the court mathematician informed him how many grains of rice he would have to pay the peasant. How many grains of rice did the king have to pay? (Leave your answer in index form.)

12 **a** In its first month of operation, a cement factory, A, produces 4000 tonnes of cement. In each successive month, production rises by 250 tonnes per month. This growth in production is illustrated for the first five months in the table shown.

Month number (n)	1	2	3	4	5
Cement produced (tonnes)	4000	4250	4500	4750	5000

i Find an expression, in terms of n, for the amount of cement produced in the nth month.

ii Find an expression, in terms of n, for the total amount of cement produced in the first n months.

iii In which month is the amount of cement produced 9250 tonnes?

iv In month m, the amount of cement produced is T tonnes. Find m in terms of T.

v The total amount of cement produced in the first p months is 522 750 tonnes. Find the value of p.

b A second factory, B, commences production at exactly the same time as the first. In its first month it produces 3000 tonnes of cement. In each successive month, production increases by 8%.

i Find an expression for the total amount of cement produced by this factory after n months.

ii Let Q_A be the total amount of cement produced by factory A in the first n months and Q_B the total amount of cement produced by factory B in the first n months. Find an expression in terms of n for $Q_B - Q_A$ and find the smallest value of n for which $Q_B - Q_A \geq 0$.

13 An endangered species of bird has a total population of 350 and lives only in a certain area known as the Old Swamp. Scientists who are monitoring the situation decide to relocate 30 birds per year from the Old Swamp to the New Swamp. The initial bird populations of the Old Swamp and the New Swamp are thus 320 and 30 respectively, and it is assumed that each population increases by 15% per year.

a Find the bird population of each area after one year, just after the second set of 30 birds has been relocated.

b The size, P_n, of the bird population at the Old Swamp after n years (just after 30 birds have been relocated for that year) is given by the recurrence relation

$$P_0 = 320 \quad \text{and} \quad P_n = 1.15P_{n-1} - 30, \quad \text{for } n \in \mathbb{N}$$

Write down a similar recurrence relation for the size, Q_n, of the bird population at the New Swamp after n years.

c Find expressions for P_n and Q_n in terms of n.

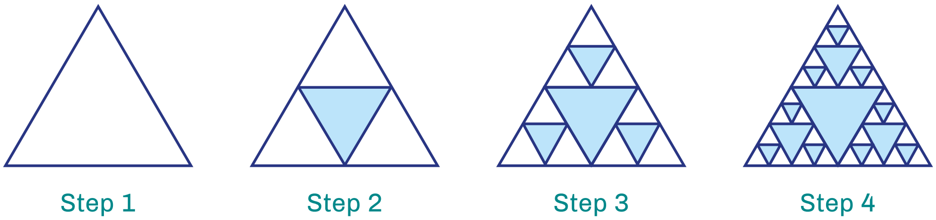
d Hence, or otherwise, predict:

i the bird populations of the Old Swamp and the New Swamp after five years

ii the number of years (to the nearest whole number) that pass before the bird populations of the two areas are most nearly equal.

Review

14 The following diagrams show the first four steps in forming the Sierpiński triangle.



The diagrams are produced in the following way:

Step 1 Start with an equilateral triangle of side length 1 unit.

Step 2 Subdivide it into four smaller congruent equilateral triangles and colour the central one blue.

Step 3 Repeat Step 2 with each of the smaller white triangles.

Step 4 Repeat again.

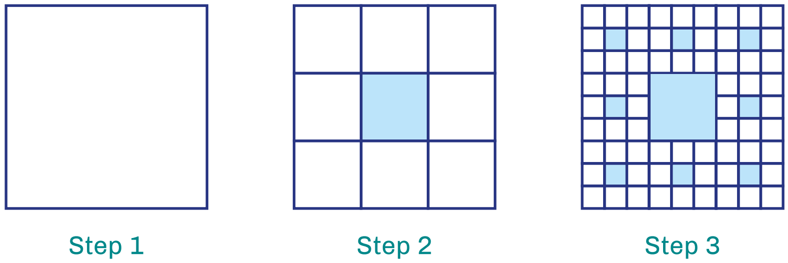
a How many white triangles are there in the nth diagram (that is, after Step n)?

b What is the side length of a white triangle in the nth diagram?

c What fraction of the area of the original triangle is still white in the nth diagram?

d Consider what happens as n approaches infinity.

15 The Sierpiński carpet is formed from a unit square in a way similar to the Sierpiński triangle. The following diagrams show the first three steps.



a How many white squares are there in the nth diagram (that is, after Step n)?

b What is the side length of a white square in the nth diagram?

c What fraction of the area of the original square is still white in the nth diagram?

d Consider what happens as n approaches infinity.

4

Additional algebra

Objectives

- ► To understand equality of **polynomials**.
- ► To use **equating coefficients** to solve problems.
- ► To solve **quadratic equations** by various methods.
- ► To use quadratic equations to solve problems involving **rates**.
- ► To resolve rational algebraic expressions into **partial fractions**.
- ► To find the coordinates of the points of intersection of straight lines with parabolas, circles and rectangular hyperbolas.

In this chapter we first consider equating coefficients of polynomial functions, and then apply this technique to establish partial fractions.

In Chapter 1 we added and subtracted algebraic fractions such as

$$\frac{2}{x+3} + \frac{4}{x-3} = \frac{6(x+1)}{x^2-9}$$

In this chapter we learn how to go from right to left in similar equations. This process is sometimes called **partial fraction decomposition**. Another example is

$$\frac{4x^2+2x+6}{(x^2+3)(x-3)} = \frac{2}{x^2+3} + \frac{4}{x-3}$$

This is a useful tool in integral calculus, and partial fractions are applied this way in Specialist Mathematics Units 3 & 4.

This chapter also includes further study of quadratic functions: solving quadratic equations, using the discriminant, applying quadratic functions to problems involving rates and using quadratic equations to find the intersection of straight lines with parabolas, circles and rectangular hyperbolas.

4A Polynomial identities

Polynomials are introduced in Mathematical Methods Units 1 & 2.

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Note: The constant function $P(x) = 0$ is called the **zero polynomial**; its degree is undefined.

Two polynomials are **equal** if they give the same value for all x. It can be proved that, if two polynomials are equal, then they have the same degree and corresponding coefficients are equal. For example:

- If $ax + b = cx^2 + dx + e$ for all x, then $c = 0$, $d = a$ and $e = b$.
- If $ax^2 + bx + c = dx^2 + ex + f$ for all x, then $a = d$, $b = e$ and $c = f$.
- If $x^2 - x - 12 = x^2 + (a + b)x + ab$ for all x, then $a + b = -1$ and $ab = -12$.

This process is called **equating coefficients**.

Example 1

If $(a + 2b)x^2 - (a - b)x + 8 = 3x^2 - 6x + 8$ for all x, find the values of a and b.

Solution

Assume that

$$(a + 2b)x^2 - (a - b)x + 8 = 3x^2 - 6x + 8 \quad \text{for all } x$$

Then by equating coefficients:

$$a + 2b = 3 \qquad (1)$$

$$-(a - b) = -6 \qquad (2)$$

Solve as simultaneous equations.

Add (1) and (2):

$$3b = -3$$

$$\therefore \quad b = -1$$

Substitute into (1):

$$a - 2 = 3$$

$$\therefore \quad a = 5$$

Example 2

Express x^2 in the form $c(x-3)^2 + a(x-3) + d$.

Solution

Let $x^2 = c(x-3)^2 + a(x-3) + d$

$$= c(x^2 - 6x + 9) + a(x-3) + d$$

$$= cx^2 + (a - 6c)x + 9c - 3a + d$$

This implies that

$$c = 1 \quad (1)$$

$$a - 6c = 0 \quad (2)$$

$$9c - 3a + d = 0 \quad (3)$$

From (2): $a = 6$

From (3): $9 - 18 + d = 0$

i.e. $d = 9$

Hence $x^2 = (x-3)^2 + 6(x-3) + 9$.

Example 3

Find the values of a, b, c and d such that

$$x^3 = a(x+2)^3 + b(x+1)^2 + cx + d \quad \text{for all } x$$

Solution

Expand the right-hand side and collect like terms:

$$x^3 = a(x^3 + 6x^2 + 12x + 8) + b(x^2 + 2x + 1) + cx + d$$

$$= ax^3 + (6a + b)x^2 + (12a + 2b + c)x + (8a + b + d)$$

Equate coefficients:

$$a = 1 \quad (1)$$

$$6a + b = 0 \quad (2)$$

$$12a + 2b + c = 0 \quad (3)$$

$$8a + b + d = 0 \quad (4)$$

Substituting $a = 1$ into (2) gives

$$6 + b = 0$$

$\therefore$ $b = -6$

Substituting $a = 1$ and $b = -6$ into (3) gives

$$12 - 12 + c = 0$$

$\therefore$ $c = 0$

Substituting $a = 1$ and $b = -6$ into (4) gives

$$8 - 6 + d = 0$$

$$\therefore \quad d = -2$$

Hence $x^3 = (x+2)^3 - 6(x+1)^2 - 2$.

Example 4

Show that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x+b)^3 + c$.

Solution

Suppose that

$$2x^3 - 5x^2 + 4x + 1 = a(x+b)^3 + c$$

for some constants a, b and c.

Then expanding the right-hand side gives

$$\begin{aligned} 2x^3 - 5x^2 + 4x + 1 &= a(x^3 + 3bx^2 + 3b^2x + b^3) + c \\ &= ax^3 + 3abx^2 + 3ab^2x + ab^3 + c \end{aligned}$$

Equating coefficients:

$$a = 2 \qquad (1)$$

$$3ab = -5 \qquad (2)$$

$$3ab^2 = 4 \qquad (3)$$

$$ab^3 + c = 1 \qquad (4)$$

From (1) and (2), we have $a = 2$ and $b = -\frac{5}{6}$. But then $3ab^2 = \frac{25}{6}$, which contradicts (3).

We have obtained a contradiction. Therefore we have shown that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x+b)^3 + c$.

Summary 4A

- A **polynomial function** can be written in the form

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree** of a polynomial is the index n of the leading term.
- Equating coefficients

  Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

  For example: if $x^2 - x - 12 = x^2 + (a+b)x + ab$, then $a + b = -1$ and $ab = -12$.

Exercise 4A

1 If $ax^2 + bx + c = 10x^2 - 7$ for all x, find the values of a, b and c.

Example 1 **2** If $(2a - b)x^2 + (a + 2b)x + 8 = 4x^2 - 3x + 8$ for all x, find the values of a and b.

3 If $(2a - 3b)x^2 + (3a + b)x + c = 7x^2 + 5x + 7$ for all x, find the values of a, b and c.

4 If $2x^2 + 4x + 5 = a(x + b)^2 + c$ for all x, find the values of a, b and c.

Example 2 **5** Express x^2 in the form $c(x + 2)^2 + a(x + 2) + d$.

6 Express x^3 in the form $(x + 1)^3 + a(x + 1)^2 + b(x + 1) + c$.

Example 3 **7** Find the values of a, b and c such that $x^2 = a(x + 1)^2 + bx + c$.

Example 4 **8** **a** Show that $3x^3 - 9x^2 + 8x + 2$ cannot be expressed in the form $a(x + b)^3 + c$.

b If $3x^3 - 9x^2 + 9x + 2$ can be expressed in the form $a(x + b)^3 + c$, then find the values of a, b and c.

9 Show that constants a, b, c and d can be found such that

$$n^3 = a(n + 1)(n + 2)(n + 3) + b(n + 1)(n + 2) + c(n + 1) + d$$

10 **a** Show that no constants a and b can be found such that

$$n^2 = a(n + 1)(n + 2) + b(n + 2)(n + 3)$$

b Express n^2 in the form $a(n + 1)(n + 2) + b(n + 1) + c$.

11 **a** Express $a(x + b)^2 + c$ in expanded form.

b Express $ax^2 + bx + c$ in completed-square form.

12 Prove that, if $ax^3 + bx^2 + cx + d = (x - 1)^2(px + q)$, then $b = d - 2a$ and $c = a - 2d$.

13 If $3x^2 + 10x + 3 = c(x - a)(x - b)$ for all values of x, find the values of a, b and c.

14 For any number n, show that n^2 can be expressed as $a(n - 1)^2 + b(n - 2)^2 + c(n - 3)^2$, and find the values of a, b and c.

15 If $x^3 + 3x^2 - 9x + c$ can be expressed in the form $(x - a)^2(x - b)$, show that either $c = 5$ or $c = -27$, and find a and b for each of these cases.

16 A polynomial P is said to be **even** if $P(-x) = P(x)$ for all x. A polynomial P is said to be **odd** if $P(-x) = -P(x)$ for all x.

a Show that, if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, then $b = d = 0$.

b Show that, if $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, then $b = d = f = 0$.

4B Quadratic equations

A polynomial function of degree 2 is called a **quadratic function**. The general quadratic function can be written as $P(x) = ax^2 + bx + c$, where $a \neq 0$.

Quadratic functions are studied extensively in Mathematical Methods Units 1 & 2. In this section we provide further practice exercises.

A quadratic equation $ax^2 + bx + c = 0$ may be solved by factorising, by completing the square or by using the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The following example demonstrates each method.

Example 5

Solve the following quadratic equations for x:

a $2x^2 + 5x = 12$ **b** $3x^2 + 4x = 2$ **c** $9x^2 + 6x + 1 = 0$

Solution

a

$$2x^2 + 5x - 12 = 0$$

Rearrange the quadratic equation.

$$(2x - 3)(x + 4) = 0$$

Factorise.

$$2x - 3 = 0 \text{ or } x + 4 = 0$$

Use the null factor theorem.

Therefore $x = \frac{3}{2}$ or $x = -4$.

b

$$3x^2 + 4x - 2 = 0$$

Rearrange the quadratic equation.

$$x^2 + \frac{4}{3}x - \frac{2}{3} = 0$$

Divide both sides by 3.

$$x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 - \frac{2}{3} = 0$$

Add and subtract $\left(\frac{b}{2}\right)^2$ to 'complete the square'.

$$\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{2}{3} = 0$$

$$\left(x + \frac{2}{3}\right)^2 = \frac{10}{9}$$

$$x + \frac{2}{3} = \pm\frac{\sqrt{10}}{3}$$

$$x = -\frac{2}{3} \pm \frac{\sqrt{10}}{3}$$

Therefore $x = \frac{-2 + \sqrt{10}}{3}$ or $x = \frac{-2 - \sqrt{10}}{3}$.

c If $9x^2 + 6x + 1 = 0$, then

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 9 \times 1}}{2 \times 9}$$

$$= \frac{-6 \pm \sqrt{0}}{18}$$

$$= -\frac{1}{3}$$

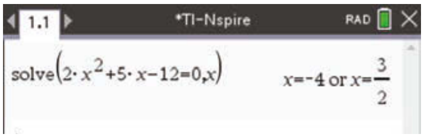
Use the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Alternatively, the equation can be solved by noting that $9x^2 + 6x + 1 = (3x + 1)^2$.

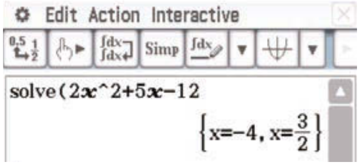
Using the TI-Nspire

Use menu > **Algebra** > **Solve** as shown.



Using the Casio ClassPad

- Select solve(from the Math1 or Math3 keyboard.
- Enter the expression $2x^2 + 5x - 12$.
- Tap EXE.



The discriminant: real solutions

The number of solutions to a quadratic equation $ax^2 + bx + c = 0$ can be determined by the **discriminant** Δ, where $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, then the equation has two real solutions.
- If $\Delta = 0$, then the equation has one real solution.
- If $\Delta < 0$, then the equation has no real solutions.

Note: In parts **a** and **b** of Example 5, we have $\Delta > 0$ and so there are two real solutions. In part **c**, we have $\Delta = 6^2 - 4 \times 9 \times 1 = 0$ and so there is only one real solution.

The discriminant: rational solutions

For a quadratic equation $ax^2 + bx + c = 0$ such that a, b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Note: In part **a** of Example 5, we have $\Delta = 121$, which is a perfect square.

Example 6

Consider the quadratic equation $x^2 - 4x = t$. Make x the subject and give the values of t for which real solution(s) to the equation can be found.

Solution

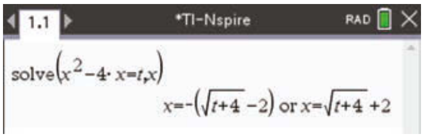
$$\begin{aligned} x^2 - 4x &= t \\ x^2 - 4x + 4 &= t + 4 \qquad \text{(completing the square)} \\ (x-2)^2 &= t + 4 \\ x - 2 &= \pm\sqrt{t+4} \\ x &= 2 \pm \sqrt{t+4} \end{aligned}$$

For real solutions to exist, we must have $t + 4 \geq 0$, i.e. $t \geq -4$.

Note: In this case the discriminant is $\Delta = 16 + 4t$. There are real solutions when $\Delta \geq 0$.

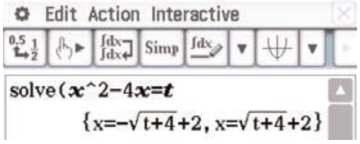
Using the TI-Nspire

Use (menu) > **Algebra** > **Solve** as shown.



Using the Casio ClassPad

- Select [solve(] from the [Math1] or [Math3] keyboard.
- Enter the equation $x^2 - 4x = t$, using the [Var] keyboard to enter the variable t.
- Tap [EXE].



Example 7

a Find the discriminant of the quadratic $x^2 + px - \dfrac{25}{4}$ in terms of p.

b Solve the quadratic equation $x^2 + px - \dfrac{25}{4} = 0$ in terms of p.

c Prove that there are two solutions for all values of p.

d Find the values of p, where p is a non-negative integer, for which the quadratic equation has rational solutions.

Solution

Here we have $a = 1$, $b = p$ and $c = -\dfrac{25}{4}$.

a $\Delta = b^2 - 4ac = p^2 + 25$

b The quadratic formula gives $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-p \pm \sqrt{p^2 + 25}}{2}$.

c We have $\Delta = p^2 + 25 > 0$, for all values of p. Thus there are always two solutions.

d If there are rational solutions, then $\Delta = p^2 + 25$ is a perfect square. Since p is an integer, we can write $p^2 + 25 = k^2$, where k is an integer with $k \geq 0$.

Rearranging, we have

$$k^2 - p^2 = 25$$

$$\therefore \quad (k - p)(k + p) = 25$$

We can factorise 25 as 5×5 or 1×25.

Note: We do not need to consider negative factors of 25, as p and k are non-negative, and so $k + p \geq 0$. Since p is non-negative, we also know that $k - p \leq k + p$.

The table on the right shows the values of k and p in each of the two cases.

Hence $p = 0$ and $p = 12$ are the only values for which the solutions are rational.

$k - p$	$k + p$	k	p
5	5	5	0
1	25	13	12

Example 8

A rectangle has an area of 288 cm^2. If the width is decreased by 1 cm and the length increased by 1 cm, the area would be decreased by 3 cm^2. Find the original dimensions of the rectangle.

Solution

Let w and ℓ be the width and length, in centimetres, of the original rectangle.

Then $\quad w\ell = 288 \quad (1)$

The dimensions of the new rectangle are $w - 1$ and $\ell + 1$, and the area is 285 cm^2.

Thus $\quad (w - 1)(\ell + 1) = 285 \quad (2)$

Rearranging (1) to make ℓ the subject and substituting in (2) gives

$$(w - 1)\left(\frac{288}{w} + 1\right) = 285$$

$$288 - \frac{288}{w} + w - 1 = 285$$

$$w - \frac{288}{w} + 2 = 0$$

$$w^2 + 2w - 288 = 0$$

Using the general quadratic formula gives

$$w = \frac{-2 \pm \sqrt{2^2 - 4 \times (-288)}}{2}$$

$$= -18 \text{ or } 16$$

But $w > 0$, and so $w = 16$. The original dimensions of the rectangle are 16 cm by 18 cm.

Example 9

Solve the equation $x - 4\sqrt{x} - 12 = 0$ for x.

Solution

For $\sqrt{x}$ to be defined, we must have $x \geq 0$.

Let $x = a^2$, where $a \geq 0$.

The equation becomes

$$\begin{aligned} a^2 - 4\sqrt{a^2} - 12 &= 0 \\ a^2 - 4a - 12 &= 0 \\ (a-6)(a+2) &= 0 \\ \therefore \quad a = 6 \text{ or } a &= -2 \end{aligned}$$

But $a \geq 0$. Hence $a = 6$ and so $x = 36$.

Summary 4B

- Quadratic equations can be solved by completing the square. This method allows us to deal with all quadratic equations, even though some have no solutions.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the brackets.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

  $$\Delta = b^2 - 4ac$$

  For the equation $ax^2 + bx + c = 0$:
  - If $\Delta > 0$, there are two solutions.
  - If $\Delta = 0$, there is one solution.
  - If $\Delta < 0$, there are no real solutions.
- For the equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers:
  - If Δ is a perfect square and $\Delta \neq 0$, there are two rational solutions.
  - If $\Delta = 0$, there is one rational solution.
  - If Δ is not a perfect square and $\Delta > 0$, there are two irrational solutions.

Skill-sheet

Exercise 4B

Example 5

1 Solve the following quadratic equations for x:

a $x^2 - 2x = -1$ **b** $x^2 - 6x + 9 = 0$ **c** $5x^2 - 10x = 1$

d $-2x^2 + 4x = 1$ **e** $2x^2 + 4x = 7$ **f** $6x^2 + 13x + 1 = 0$

2 The following equations have the number of solutions shown in brackets. Find the possible values of m.

a $x^2 + 3x + m = 0$ (0) **b** $x^2 - 5x + m = 0$ (2) **c** $mx^2 + 5x - 8 = 0$ (1)

d $x^2 + mx + 9 = 0$ (2) **e** $x^2 - mx + 4 = 0$ (0) **f** $4x^2 - mx - m = 0$ (1)

Example 6

3 Make x the subject in each of the following and give the values of t for which real solution(s) to the equation can be found:

a $2x^2 - 4t = x$ **b** $4x^2 + 4x - 4 = t - 2$

c $5x^2 + 4x + 10 = t$ **d** $tx^2 + 4tx + 10 = t$

Example 7

4 **a** Solve the quadratic equation $x^2 + px - 16 = 0$ in terms of p.

b Find the values of p, where p is an integer with $0 \le p \le 10$, for which the quadratic equation in **a** has rational solutions.

5 **a** Show that the solutions of the equation $2x^2 - 3px + (3p - 2) = 0$ are rational for all integer values of p.

b Find the value of p for which there is only one solution.

c Solve the equation when:

i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

6 **a** Show that the solutions of the equation $4(4p - 3)x^2 - 8px + 3 = 0$ are rational for all integer values of p.

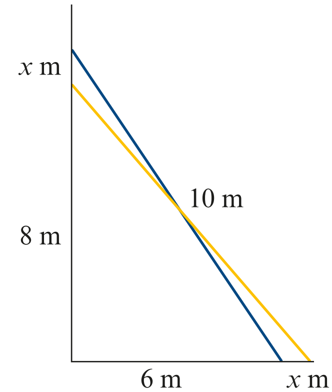
b Find the value of p for which there is only one solution.

c Solve the equation when:

i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

Example 8

7 A pole 10 m long leans against a wall. The bottom of the pole is 6 m from the wall. If the bottom of the pole is pulled away x m so that the top slides down by the same amount, find x.



8 A wire of length 200 cm is cut into two parts and each part is bent to form a square. If the area of the larger square is 9 times the area of the smaller square, find the length of the sides of the larger square.

Example 9

9 Solve each of the following equations for x:

a $x - 8\sqrt{x} + 12 = 0$

b $x - 8 = 2\sqrt{x}$

c $x - 5\sqrt{x} - 14 = 0$

d $\sqrt[3]{x^2} - 9\sqrt[3]{x} + 8 = 0$

e $\sqrt[3]{x^2} - \sqrt[3]{x} - 6 = 0$

f $x - 29\sqrt{x} + 100 = 0$

10 Find constants a, b and c such that

$$3x^2 - 5x + 1 = a(x + b)^2 + c$$

for all values of x. Hence find the minimum value of $3x^2 - 5x + 1$.

11 Show that the graphs of $y = 2 - 4x - x^2$ and $y = 24 + 8x + x^2$ do not intersect.

12 Solve the quadratic equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ for x.

13 Given that the two solutions of the equation $2x^2 - 6x - m = 0$ differ by 5, find the value of m.

14 For the equation $(b^2 - 2ac)x^2 + 4(a + c)x = 8$:

a Prove that this equation always has real solutions.

b Find the conditions for which there is only one solution.

15 The equation $\dfrac{1}{2} + \dfrac{1}{x + k} = \dfrac{1}{x}$ has no solutions. Find the possible values of k.

16 Find the smallest positive integer p for which the equation $3x^2 + px + 7 = 0$ has solutions.

4C Applying quadratic equations to rate problems

A **rate** describes how a certain quantity changes with respect to the change in another quantity (often time). An example of a rate is 'speed'. A speed of 60 km/h gives us a measure of how fast an object is travelling. A further example is 'flow', where a rate of 20 L/min is going to fill an empty swimming pool faster than a rate of 6 L/min.

Many problems are solved using rates, which can be expressed as fractions. For example, if you travel 9 km in 2 hours, then your speed can be expressed in fraction form as:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{9 \text{ km}}{2 \text{ hours}} = \frac{9}{2} \text{ km/h}$$

When solving rate problems, it is often necessary to add two or more fractions with different denominators, as shown in the following examples.

Example 10

a Express $\frac{6}{x} + \frac{6}{x+8}$ as a single fraction.

b Solve the equation $\frac{6}{x} + \frac{6}{x+8} = 2$ for x.

Solution

a
$$\frac{6}{x} + \frac{6}{x+8} = \frac{6(x+8)}{x(x+8)} + \frac{6x}{x(x+8)}$$
$$= \frac{6x + 48 + 6x}{x(x+8)}$$
$$= \frac{12(x+4)}{x(x+8)}$$

b Since $\frac{6}{x} + \frac{6}{x+8} = \frac{12(x+4)}{x(x+8)}$, we have

$$\frac{12(x+4)}{x(x+8)} = 2$$
$$12(x+4) = 2x(x+8)$$
$$6(x+4) = x(x+8)$$
$$6x + 24 = x^2 + 8x$$
$$x^2 + 2x - 24 = 0$$
$$(x+6)(x-4) = 0$$

Therefore $x = -6$ or $x = 4$.

Example 11

A car travels 500 km at a constant speed. If it had travelled at a speed of 10 km/h less, it would have taken 1 hour more to travel the distance. Find the speed of the car.

Solution

Let x km/h be the speed of the car.

The journey takes $\frac{500}{x}$ hours.

If the car's speed were 10 km/h less, then the speed would be $x - 10$ km/h.

The journey would take $\frac{500}{x-10}$ hours.

We can now write:

$$\frac{500}{x-10} = \frac{500}{x} + 1$$
$$\frac{500}{x-10} = \frac{500+x}{x}$$
$$500x = (500+x)(x-10)$$
$$500x = 500x - 5000 + x^2 - 10x$$
$$0 = x^2 - 10x - 5000$$

Thus

$$x = \frac{10 \pm \sqrt{100 + 4 \times 5000}}{2}$$
$$= 5(1 \pm \sqrt{201})$$

The speed is $5(1 + \sqrt{201}) \approx 75.887$ km/h.

Explanation

For an object travelling at a constant speed in one direction:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

and so

$$\text{time taken} = \frac{\text{distance travelled}}{\text{speed}}$$

We can summarise the two possible car journeys in the following table:

Speed (km/h)	Distance (km)	Time (h)
x	500	$\frac{500}{x}$
$x - 10$	500	$\frac{500}{x-10}$

Example 12

A tank is filled by two pipes. The smaller pipe alone will take 24 minutes longer than the larger pipe alone, and 32 minutes longer than when both pipes are used. How long will each pipe take to fill the tank alone? How long will it take for both pipes used together to fill the tank?

Solution

Let C cubic units be the capacity of the tank, and let x minutes be the time it takes for the larger pipe alone to fill the tank.

Then the average rate of flow for the larger pipe is $\frac{C}{x}$ cubic units per minute.

Since the smaller pipe alone takes $x + 24$ minutes to fill the tank, the average rate of flow for the smaller pipe is $\frac{C}{x+24}$ cubic units per minute.

The average rate of flow when both pipes are used together is the sum of these two rates:

$$\frac{C}{x} + \frac{C}{x+24} \text{ cubic units per minute}$$

Expressed as a single fraction:

$$\begin{aligned}\frac{C}{x} + \frac{C}{x+24} &= \frac{C(x+24)+Cx}{x(x+24)}\\ &= \frac{2C(x+12)}{x(x+24)}\end{aligned}$$

The time taken to fill the tank using both pipes is

$$\begin{aligned}C \div \frac{2C(x+12)}{x(x+24)} &= C \times \frac{x(x+24)}{2C(x+12)}\\ &= \frac{x(x+24)}{2(x+12)}\end{aligned}$$

Therefore the time taken for the smaller pipe alone to fill the tank can be also be expressed as $\frac{x(x+24)}{2(x+12)} + 32$ minutes.

Thus

$$\begin{aligned}\frac{x(x+24)}{2(x+12)} + 32 &= x + 24\\ \frac{x(x+24)}{2(x+12)} &= x - 8\\ x(x+24) &= 2(x+12)(x-8)\\ x^2 + 24x &= 2x^2 + 8x - 192\\ x^2 - 16x - 192 &= 0\\ (x-24)(x+8) &= 0\end{aligned}$$

But $x > 0$, and hence $x = 24$.

It takes 24 minutes for the larger pipe alone to fill the tank, 48 minutes for the smaller pipe alone to fill the tank, and 16 minutes for both pipes together to fill the tank.

Skill-sheet

Exercise 4C

Example 10

1 **a** Express $\frac{6}{x} - \frac{6}{x+3}$ as a single fraction.

b Solve the equation $\frac{6}{x} - \frac{6}{x+3} = 1$ for x.

2 Solve the equation $\frac{300}{x+5} = \frac{300}{x} - 2$ for x.

3 The sum of the reciprocals of two consecutive odd numbers is $\frac{36}{323}$. Form a quadratic equation and hence determine the two numbers.

Example 11

4 A cyclist travels 40 km at a speed of x km/h.

a Find the time taken in terms of x.

b Find the time taken when his speed is reduced by 2 km/h.

c If the difference between the times is 1 hour, find his original speed.

5 A car travels from town A to town B, a distance of 600 km, in x hours. A plane, travelling 220 km/h faster than the car, takes $5\frac{1}{2}$ hours less to cover the same distance.

a Express, in terms of x, the average speed of the car and the average speed of the plane.

b Find the actual average speed of each of them.

6 A car covers a distance of 200 km at a speed of x km/h. A train covers the same distance at a speed of $x + 5$ km/h. If the time taken by the car is 2 hours more than that taken by the train, find x.

7 A man travels 108 km, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled at an average speed 2 km/h faster. What was the man's average speed when he made the trip?

8 A bus is due to reach its destination 75 km away at a certain time. The bus usually travels with an average speed of x km/h. Its start is delayed by 18 minutes but, by increasing its average speed by 12.5 km/h, the driver arrives on time.

a Find x. **b** How long did the journey actually take?

9 Ten minutes after the departure of an express train, a slow train starts, travelling at an average speed of 20 km/h less. The slow train reaches a station 250 km away 3.5 hours after the arrival of the express. Find the average speed of each of the trains.

10 When the average speed of a car is increased by 10 km/h, the time taken for the car to make a journey of 105 km is reduced by 15 minutes. Find the original average speed.

11 A tank can be filled with water by two pipes running together in $11\frac{1}{9}$ minutes. If the larger pipe alone takes 5 minutes less to fill the tank than the smaller pipe, find the time that each pipe will take to fill the tank.

Example 12

12 At first two different pipes running together will fill a tank in $\frac{20}{3}$ minutes. The rate that water runs through each of the pipes is then adjusted. If one pipe, running alone, takes 1 minute less to fill the tank at its new rate, and the other pipe, running alone, takes 2 minutes more to fill the tank at its new rate, then the two running together will fill the tank in 7 minutes. Find in what time the tank will be filled by each pipe running alone at the new rates.

13 The journey between two towns by one route consists of 233 km by rail followed by 126 km by sea. By a second route the journey consists of 405 km by rail followed by 39 km by sea. If the time taken for the first route is 50 minutes longer than for the second route, and travelling by rail is 25 km/h faster than travelling by sea, find the average speed by rail and the average speed by sea.

14 A sea freighter travelling due north at 12 km/h sights a cruiser straight ahead at an unknown distance and travelling due east at unknown speed. After 15 minutes the vessels are 10 km apart and then, 15 minutes later, they are 13 km apart. (Assume that both travel at constant speeds.) How far apart are the vessels when the cruiser is due east of the freighter?

15 Cask A, which has a capacity of 20 litres, is filled with wine. A certain quantity of wine from cask A is poured into cask B, which also has a capacity of 20 litres. Cask B is then filled with water. After this, cask A is filled with some of the mixture from cask B. A further $\frac{20}{3}$ litres of the mixture now in A is poured back into B, and the two casks now have the same amount of wine. How much wine was first taken out of cask A?

16 Two trains travel between two stations 80 km apart. If train A travels at an average speed of 5 km/h faster than train B and completes the journey 20 minutes faster, find the average speeds of the two trains, giving your answers correct to two decimal places.

17 A tank is filled by two pipes. The smaller pipe running alone will take 24 minutes longer than the larger pipe alone, and a minutes longer than when both pipes are running together.

a Find, in terms of a, how long each pipe takes to fill the tank.

b Find how long each pipe takes to fill the tank when:

i $a = 49$ **ii** $a = 32$ **iii** $a = 27$ **iv** $a = 25$

18 Train A leaves Armadale and travels at constant speed to Bundong, which is a town 300 km from Armadale. At the same time, train B leaves Bundong and travels at constant speed to Armadale. They meet at a town Yunga, which is between the two towns. Nine hours after leaving Yunga, train A reaches Bundong, and four hours after leaving Yunga, train B reaches Armadale.

a Find the distance of Yunga from Armadale.

b Find the speed of each of the trains.

4D Partial fractions

A **rational function** is the quotient of two polynomials. If $P(x)$ and $Q(x)$ are polynomials, then $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function; e.g. $f(x) = \dfrac{4x+2}{x^2-1}$.

- If the degree of $P(x)$ is less than the degree of $Q(x)$, then $f(x)$ is a **proper fraction**.
- If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then $f(x)$ is an **improper fraction**.

By convention, we consider a rational function for its maximal domain. For example, the function $f(x) = \dfrac{4x+2}{x^2-1}$ is defined for $x \in \mathbb{R} \setminus \{-1, 1\}$.

A rational function may be expressed as a sum of simpler functions by resolving it into what are called **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

This technique can help when sketching the graphs of rational functions or when performing other mathematical procedures such as integration.

Proper fractions

For proper fractions, the technique used for obtaining partial fractions depends on the type of factors in the denominator of the original algebraic fraction. We only consider examples where the denominators have factors that are either degree 1 (linear) or degree 2 (quadratic).

- For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax+b}$.
- For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx+d}$ and $\dfrac{C}{(cx+d)^2}$.
- For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx+E}{ax^2+bx+c}$.

Note: A quadratic expression is said to be **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, both $x^2 + 1$ and $x^2 + 4x + 10$ are irreducible. You can use the discriminant to test whether a quadratic expression is irreducible.

To resolve an algebraic fraction into its partial fractions:

Step 1 Write a statement of identity between the original fraction and a sum of the appropriate number of partial fractions.

Step 2 Express the sum of the partial fractions as a single fraction, and note that the numerators of both sides are equivalent.

Step 3 Find the values of the introduced constants $A, B, C, \ldots$ by substituting appropriate values for x or by equating coefficients.

Example 13

Resolve $\dfrac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution

Method 1

Let

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \qquad (1)$$

for all $x \in \mathbb{R} \setminus \{1, -3\}$.

Express the right-hand side as a single fraction:

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$$\therefore \quad \frac{3x+5}{(x-1)(x+3)} = \frac{(A+B)x+3A-B}{(x-1)(x+3)}$$

$$\therefore \quad 3x+5 = (A+B)x+3A-B$$

Equate coefficients:

$$A + B = 3$$

$$3A - B = 5$$

Solving these equations simultaneously gives

$$4A = 8$$

and so $A = 2$ and $B = 1$.

Therefore

$$\frac{3x+5}{(x-1)(x+3)} = \frac{2}{x-1} + \frac{1}{x+3}$$

Method 2

From equation (1) we can write:

$$3x+5 = A(x+3)+B(x-1) \qquad (2)$$

Substitute $x = 1$ in equation (2):

$$8 = 4A$$

$$\therefore \quad A = 2$$

Substitute $x = -3$ in equation (2):

$$-4 = -4B$$

$$\therefore \quad B = 1$$

Explanation

Since the denominator has two linear factors, there will be two partial fractions of the form $\dfrac{A}{x-1}$ and $\dfrac{B}{x+3}$.

We know that equation (2) is true for all $x \in \mathbb{R} \setminus \{1, -3\}$.

But if this is the case, then it also has to be true for $x = 1$ and $x = -3$.

Note: You could substitute any values of x to find A and B in this way, but these values simplify the calculations.

Example 14

Resolve $\dfrac{2x + 10}{(x + 1)(x - 1)^2}$ into partial fractions.

Solution

Since the denominator has a repeated linear factor and a single linear factor, there are three partial fractions:

$$\frac{2x + 10}{(x + 1)(x - 1)^2} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}$$

$$\therefore \quad \frac{2x + 10}{(x + 1)(x - 1)^2} = \frac{A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1)}{(x + 1)(x - 1)^2}$$

This gives the equation

$$2x + 10 = A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1)$$

Let $x = 1$:
$$2(1) + 10 = C(1 + 1)$$
$$12 = 2C$$
$$\therefore \quad C = 6$$

Let $x = -1$:
$$2(-1) + 10 = A(-1 - 1)^2$$
$$8 = 4A$$
$$\therefore \quad A = 2$$

Let $x = 0$:
$$10 = A - B + C$$
$$10 = 2 - B + 6$$
$$\therefore \quad B = -2$$

Hence

$$\frac{2x + 10}{(x + 1)(x - 1)^2} = \frac{2}{x + 1} - \frac{2}{x - 1} + \frac{6}{(x - 1)^2}$$

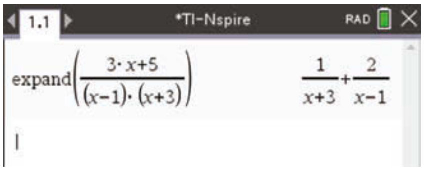
Note: In Exercise 4D, you will show that it is impossible to find A and C such that

$$\frac{2x + 10}{(x + 1)(x - 1)^2} = \frac{A}{x + 1} + \frac{C}{(x - 1)^2}$$

Using the TI-Nspire

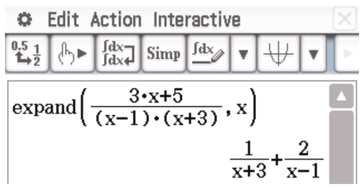
Use menu > **Algebra** > **Expand** as shown.

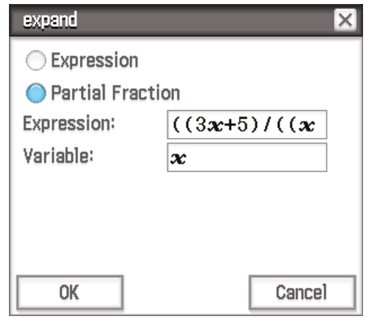
Note: You can access the fraction template using ctrl ÷.



Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, enter and highlight $\dfrac{3x+5}{(x-1)(x+3)}$.
- Go to **Interactive** > **Transformation** > **expand** and select the **Partial Fraction** option.
- Enter the variable and tap OK.





Example 15

Resolve $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)}$ into partial fractions.

Solution

Since the denominator has a single linear factor and an irreducible quadratic factor (i.e. cannot be reduced to linear factors), there are two partial fractions:

$$\frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$

$$\therefore \quad \frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{A(x^2+x+1)+(Bx+C)(x-2)}{(x-2)(x^2+x+1)}$$

This gives the equation

$$x^2+6x+5 = A(x^2+x+1)+(Bx+C)(x-2) \qquad (1)$$

Subsituting $x = 2$:

$$2^2+6(2)+5 = A(2^2+2+1)$$
$$21 = 7A$$
$$\therefore \quad A = 3$$

Note: The values of B and C could now be found by substituting $x = 0$ and $x = 1$ in equation (1). Instead we will show the method of equating coefficients.

We can rewrite equation (1) as

$$\begin{aligned} x^2+6x+5 &= A(x^2+x+1)+(Bx+C)(x-2) \\ &= A(x^2+x+1)+Bx^2-2Bx+Cx-2C \\ &= (A+B)x^2+(A-2B+C)x+A-2C \end{aligned}$$

Since $A = 3$, this gives

$$x^2 + 6x + 5 = (3 + B)x^2 + (3 - 2B + C)x + 3 - 2C$$

Equate coefficients:

$$3 + B = 1 \qquad \text{and} \qquad 3 - 2C = 5$$

$$\therefore \quad B = -2 \qquad\qquad \therefore \quad C = -1$$

Check: $3 - 2B + C = 3 - 2(-2) + (-1) = 6$

Therefore

$$\frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} = \frac{3}{x - 2} + \frac{-2x - 1}{x^2 + x + 1}$$

$$= \frac{3}{x - 2} - \frac{2x + 1}{x^2 + x + 1}$$

Improper fractions

Improper algebraic fractions can be expressed as a sum of partial fractions by first dividing the denominator into the numerator to produce a quotient and a proper fraction. This proper fraction can then be resolved into its partial fractions using the techniques just introduced.

Example 16

Express $\dfrac{x^5 + 2}{x^2 - 1}$ as partial fractions.

Solution

Dividing through:

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2 - 1 & x^5 + 2 \\ & x^5 - x^3 \\ \hline & \quad x^3 + 2 \\ & \quad x^3 - x \\ \hline & \qquad x + 2 \end{array}$$

Therefore

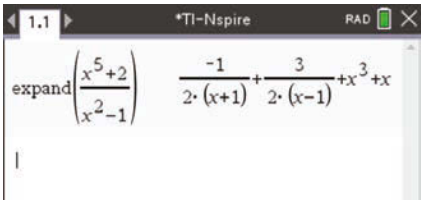
$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x + \frac{x + 2}{x^2 - 1}$$

By expressing $\dfrac{x + 2}{x^2 - 1} = \dfrac{x + 2}{(x - 1)(x + 1)}$ as partial fractions, we obtain

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x - \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

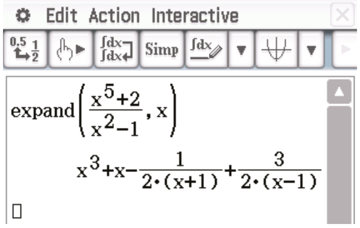
Using the TI-Nspire

Use menu > **Algebra** > **Expand** as shown.



Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, enter and highlight $\dfrac{x^5+2}{x^2-1}$.
- Go to **Interactive** > **Transformation** > **expand** and choose the **Partial Fraction** option.
- Enter the variable and tap OK.



Summary 4D

- A rational function may be expressed as a sum of simpler functions by resolving it into **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

- Examples of resolving a proper fraction into partial fractions:
  - Single linear factors

$$\frac{3x-4}{(2x-3)(x+5)} = \frac{A}{2x-3} + \frac{B}{x+5}$$

  - Repeated linear factor

$$\frac{3x-4}{(2x-3)(x+5)^2} = \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$

  - Irreducible quadratic factor

$$\frac{3x-4}{(2x-3)(x^2+5)} = \frac{A}{2x-3} + \frac{Bx+C}{x^2+5}$$

- A quadratic polynomial is **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, the quadratics x^2+5 and $x^2+4x+10$ are irreducible.
- If $f(x) = \dfrac{P(x)}{Q(x)}$ is an improper fraction, i.e. if the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then the division must be performed first.

Skill-sheet

Exercise 4D

Example 13

1 Resolve the following rational expressions into partial fractions:

a $\dfrac{5x+1}{(x-1)(x+2)}$ **b** $\dfrac{-1}{(x+1)(2x+1)}$ **c** $\dfrac{3x-2}{x^2-4}$

d $\dfrac{4x+7}{x^2+x-6}$ **e** $\dfrac{7-x}{(x-4)(x+1)}$

Example 14

2 Resolve the following rational expressions into partial fractions:

a $\dfrac{2x+3}{(x-3)^2}$ **b** $\dfrac{9}{(1+2x)(1-x)^2}$ **c** $\dfrac{2x-2}{(x+1)(x-2)^2}$

Example 15

3 Resolve the following rational expressions into partial fractions:

a $\dfrac{3x+1}{(x+1)(x^2+x+1)}$ **b** $\dfrac{3x^2+2x+5}{(x^2+2)(x+1)}$ **c** $\dfrac{x^2+2x-13}{2x^3+6x^2+2x+6}$

Example 16

4 Resolve $\dfrac{3x^2-4x-2}{(x-1)(x-2)}$ into partial fractions.

5 Show that it is not possible to find values of A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

6 Express each of the following as partial fractions:

a $\dfrac{1}{(x-1)(x+1)}$ **b** $\dfrac{x}{(x-2)(x+3)}$ **c** $\dfrac{3x+1}{(x-2)(x+5)}$

d $\dfrac{1}{(2x-1)(x+2)}$ **e** $\dfrac{3x+5}{(3x-2)(2x+1)}$ **f** $\dfrac{2}{x^2-x}$

g $\dfrac{3x+1}{x^3+x}$ **h** $\dfrac{3x^2+8}{x(x^2+4)}$ **i** $\dfrac{1}{x^2-4x}$

j $\dfrac{x+3}{x^2-4x}$ **k** $\dfrac{x^3-x^2-1}{x^2-x}$ **l** $\dfrac{x^3-x^2-6}{2x-x^2}$

m $\dfrac{x^2-x}{(x+1)(x^2+2)}$ **n** $\dfrac{x^2+2}{x^3-3x-2}$ **o** $\dfrac{2x^2+x+8}{x(x^2+4)}$

p $\dfrac{1-2x}{2x^2+7x+6}$ **q** $\dfrac{3x^2-6x+2}{(x-1)^2(x+2)}$ **r** $\dfrac{4}{(x-1)^2(2x+1)}$

s $\dfrac{x^3-2x^2-3x+9}{x^2-4}$ **t** $\dfrac{x^3+3}{(x+1)(x-1)}$ **u** $\dfrac{2x-1}{(x+1)(3x+2)}$

4E Simultaneous equations

In this section, we look at methods for finding the coordinates of the points of intersection of a linear graph with different non-linear graphs: parabolas, circles and rectangular hyperbolas. We also consider the intersections of two parabolas. These types of graphs are studied further in Mathematical Methods Units 1 & 2.

Example 17

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

Solution

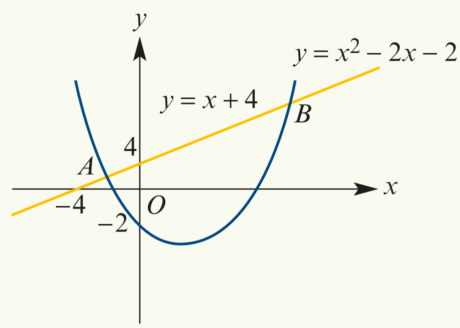
Equate the two expressions for y:

$$x^2 - 2x - 2 = x + 4$$

$$x^2 - 3x - 6 = 0$$

$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2}$$

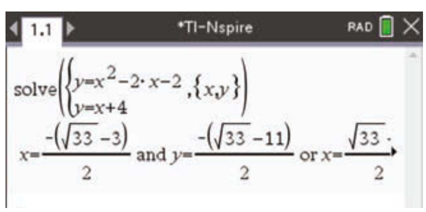
$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection are $A\left(\dfrac{3 - \sqrt{33}}{2}, \dfrac{11 - \sqrt{33}}{2}\right)$ and $B\left(\dfrac{3 + \sqrt{33}}{2}, \dfrac{11 + \sqrt{33}}{2}\right)$.

Using the TI-Nspire

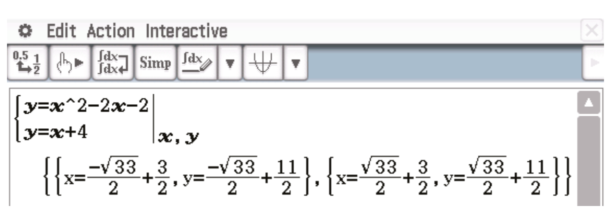
- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations** as shown.
- Use the touchpad to move the cursor up to the solution and see all the solutions.



Using the Casio ClassPad

The exact coordinates of the points of intersection can be obtained in the Main $\sqrt{\alpha}$ application.

- To select the simultaneous equations template, tap {■ from the (Math1) keyboard.
- Enter the two equations and the variables x, y in the spaces provided. Then tap (EXE).
- Tap Rotate from the icon panel and ► on the touch screen to view the entire solution.



Example 18

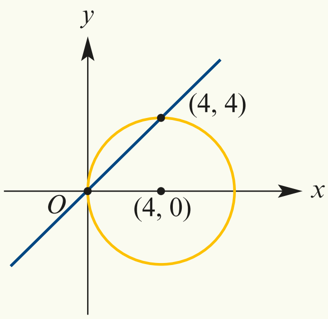
Find the points of intersection of the circle with equation $(x-4)^2 + y^2 = 16$ and the line with equation $x - y = 0$.

Solution

Rearrange $x - y = 0$ to make y the subject.

Substitute $y = x$ into the equation of the circle:

$$\begin{aligned}(x-4)^2 + x^2 &= 16\\ x^2 - 8x + 16 + x^2 &= 16\\ 2x^2 - 8x &= 0\\ 2x(x-4) &= 0\\ \therefore\quad x = 0 \text{ or } x &= 4\end{aligned}$$



The points of intersection are $(0, 0)$ and $(4, 4)$.

Example 19

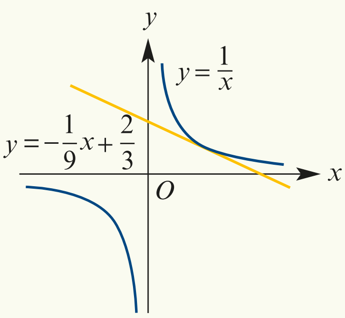
Find the point of contact of the straight line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$.

Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$.

Equate the expressions for y:

$$\begin{aligned}-\frac{1}{9}x + \frac{2}{3} &= \frac{1}{x}\\ -x^2 + 6x &= 9\\ x^2 - 6x + 9 &= 0\\ (x-3)^2 &= 0\\ \therefore\quad x &= 3\end{aligned}$$

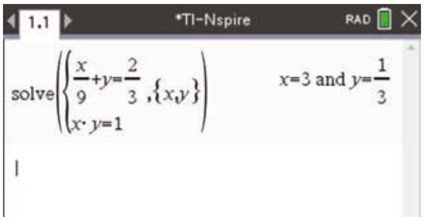


The point of intersection is $\left(3, \frac{1}{3}\right)$.

Using the TI-Nspire

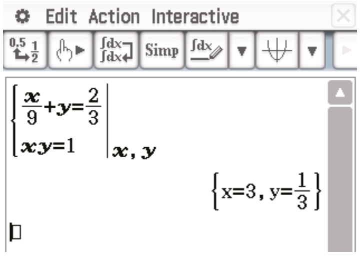
Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations** as shown.

Note: The multiplication sign between x and y is required, as the calculator will consider xy to be a single variable.



Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, select the simultaneous equations template by tapping [{☐] from the [Math1] keyboard.
- Enter the two equations and the variables x, y in the spaces provided; tap [EXE].

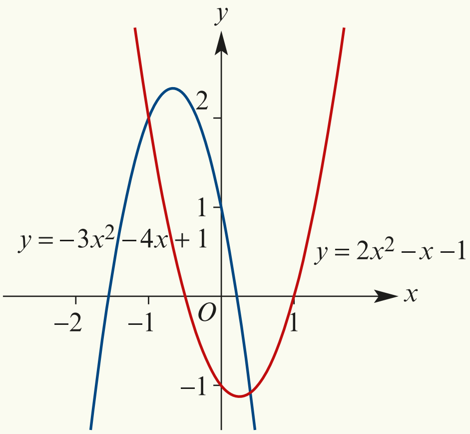


Example 20

Find the coordinates of the points of intersection of the graphs of $y = -3x^2 - 4x + 1$ and $y = 2x^2 - x - 1$.

Solution

$$-3x^2 - 4x + 1 = 2x^2 - x - 1$$
$$-5x^2 - 3x + 2 = 0$$
$$5x^2 + 3x - 2 = 0$$
$$(5x - 2)(x + 1) = 0$$
$$\therefore \quad x = \frac{2}{5} \text{ or } x = -1$$



Substitute in $y = 2x^2 - x - 1$:

When $x = -1$, $y = 2$.

When $x = \frac{2}{5}$, $y = 2 \times \frac{4}{25} - \frac{2}{5} - 1 = -\frac{27}{25}$.

The points of intersection are $(-1, 2)$ and $\left(\frac{2}{5}, -\frac{27}{25}\right)$.

Skill-sheet

Exercise 4E

Example 17

1 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$

b $y - 2x^2 = 0$, $y - x = 0$

c $y = x^2 - x$, $y = 2x + 1$

Example 18

2 Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$, $x + y = 16$

b $x^2 + y^2 = 125$, $x + y = 15$

c $x^2 + y^2 = 185$, $x - y = 3$

d $x^2 + y^2 = 97$, $x + y = 13$

e $x^2 + y^2 = 106$, $x - y = 4$

Example 19

3 Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$
$xy = 187$

b $x + y = 51$
$xy = 518$

c $x - y = 5$
$xy = 126$

4 Find the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$.

5 Find the coordinates of the points of intersection of the curves with equations $y = \dfrac{1}{x - 2} + 3$ and $y = x$.

6 Find the coordinates of the points A and B where the line with equation $x - 3y = 0$ meets the circle with equation $x^2 + y^2 - 10x - 5y + 25 = 0$.

7 Find the coordinates of the points of intersection of the line with equation $\dfrac{y}{4} - \dfrac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$.

8 Find the coordinates of the points of intersection of the curve with equation $y = \dfrac{1}{x + 2} - 3$ and the line with equation $y = -x$.

9 Find the point where the line $4y = 9x + 4$ touches the parabola $y^2 = 9x$.

10 Find the coordinates of the point where the line with equation $y = 2x + 3\sqrt{5}$ touches the circle with equation $x^2 + y^2 = 9$.

11 Find the coordinates of the point where the straight line with equation $y = \dfrac{1}{4}x + 1$ touches the curve with equation $y = -\dfrac{1}{x}$.

12 Find points of intersection of the curve $y = \dfrac{2}{x - 2}$ and the line $y = x - 1$.

Example 20

13 Find the coordinates of the points of intersection of the graphs of the following pairs of quadratic functions:

a $y = 2x^2 - 4x + 1$
$y = 2x^2 - x - 1$

b $y = -2x^2 + x + 1$
$y = 2x^2 - x - 1$

c $y = x^2 + x + 1$
$y = x^2 - x - 2$

d $y = 3x^2 + x + 2$
$y = x^2 - x + 2$

14 One solution to the simultaneous equations $5x + 4y = 11$ and $2x^2 + axy + 4y^2 = 24$ is $x = 1$ and $y = b$. Find the values of a and b, and then find the other solution to the simultaneous equations.

Review

Chapter summary

Assign-ment

Nrich

Polynomials

- A **polynomial function** can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The **degree** of a polynomial is the index n of the leading term (the term of highest index among those terms with a non-zero coefficient).
- Equating coefficients

Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.
For example: if $x^2 - x - 12 = x^2 + (a+b)x + ab$, then $a + b = -1$ and $ab = -12$.

Quadratics

- A quadratic function can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$.
- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
  - Factorising
  - Completing the square
  - Using the **general quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$, then the equation has two real solutions.
  - If $\Delta = 0$, then the equation has one real solution.
  - If $\Delta < 0$, then the equation has no real solutions.

Partial fractions

- A **rational function** has the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

For example: $f(x) = \dfrac{2x + 10}{x^3 - x^2 - x + 1}$
- Some rational functions may be expressed as a sum of **partial fractions**:
  - For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax + b}$.
  - For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx + d}$ and $\dfrac{C}{(cx + d)^2}$.
  - For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx + E}{ax^2 + bx + c}$.

For example: $\dfrac{2x + 10}{(x + 1)(x - 1)^2} = \dfrac{A}{x + 1} + \dfrac{B}{x - 1} + \dfrac{C}{(x - 1)^2}$, where $A = 2$, $B = -2$ and $C = 6$

Technology-free questions

1 If $(3a + b)x^2 + (a - 2b)x + b + 2c = 11x^2 - x + 4$ for all x, find the values of a, b and c.

2 Express x^3 in the form $(x - 1)^3 + a(x - 1)^2 + b(x - 1) + c$.

3 Prove that, if $ax^3 + bx^2 + cx + d = (x + 1)^2(px + q)$, then $b = 2a + d$ and $c = a + 2d$.

4 Prove that, if $ax^3 + bx^2 + cx + d = (x - 2)^2(px + q)$, then $b = -4a + \frac{1}{4}d$ and $c = 4a - d$.

5 Solve the following quadratic equations for x:

a $x^2 + x = 12$ **b** $x^2 - 2 = x$ **c** $-x^2 + 3x + 11 = 1$

d $2x^2 - 4x + 1 = 0$ **e** $3x^2 - 2x + 5 = t$ **f** $tx^2 + 4 = tx$

6 Solve the equation $\dfrac{2}{x-1} - \dfrac{3}{x+2} = \dfrac{1}{2}$ for x.

7 Express each of the following as partial fractions:

a $\dfrac{-3x+4}{(x-3)(x+2)}$ **b** $\dfrac{7x+2}{x^2-4}$ **c** $\dfrac{7-x}{x^2+2x-15}$

d $\dfrac{3x-9}{x^2-4x-5}$ **e** $\dfrac{3x-4}{(x+3)(x+2)^2}$ **f** $\dfrac{6x^2-5x-16}{(x-1)^2(x+4)}$

g $\dfrac{x^2-6x-4}{(x^2+2)(x+1)}$ **h** $\dfrac{-x+4}{(x-1)(x^2+x+1)}$ **i** $\dfrac{-4x+5}{(x+4)(x-3)}$

j $\dfrac{-2x+8}{(x+4)(x-3)}$

8 Express each of the following as partial fractions:

a $\dfrac{14(x-2)}{(x-3)(x^2+x+2)}$ **b** $\dfrac{1}{(x+1)(x^2-x+2)}$ **c** $\dfrac{3x^3}{x^2-5x+4}$

9 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = -x$ **b** $x^2 + y^2 = 16$, $x + y = 4$ **c** $x + y = 5$, $xy = 4$

10 Find the coordinates of the points of intersection of the line with equation $3y - x = 1$ and the circle with equation $x^2 + 2x + y^2 = 9$.

11 A motorist makes a journey of 135 km at an average speed of x km/h.

a Write an expression for the number of hours taken for the journey.

b Owing to road works, on a certain day his average speed for the journey is reduced by 15 km/h. Write an expression for the number of hours taken on that day.

c If the second journey takes 45 minutes longer than the first, form an equation in x and solve it.

d Find his average speed for each journey.

Review

Multiple-choice questions

1 If x^2 is written in the form $(x+1)^2 + b(x+1) + c$, then the values of b and c are

A $b = 0,\ c = 0$ **B** $b = -2,\ c = 0$ **C** $b = -2,\ c = 1$
D $b = 1,\ c = 2$ **E** $b = 1,\ c = -2$

2 If $x^3 = a(x+2)^3 + b(x+2)^2 + c(x+2) + d$, then the values of a, b, c and d are

A $a = 0,\ b = -8,\ c = 10,\ d = -6$ **B** $a = 0,\ b = -6,\ c = 10,\ d = -8$
C $a = 1,\ b = -8,\ c = 10,\ d = -6$ **D** $a = 1,\ b = -6,\ c = 12,\ d = -8$
E $a = 1,\ b = -8,\ c = 12,\ d = -6$

3 The quadratic equation $3x^2 - 6x + 3 = 0$ has

A two real solutions, $x = \pm 1$ **B** one real solution, $x = -1$
C no real solutions **D** one real solution, $x = 1$
E two real solutions, $x = 1$ and $x = 2$

4 The quadratic equation whose solutions are 4 and −6 is

A $(x+4)(x-6) = 0$ **B** $x^2 - 2x - 24 = 0$ **C** $2x^2 + 4x = 48$
D $-x^2 + 2x - 24 = 0$ **E** $x^2 + 2x + 24 = 0$

5 If $\dfrac{7x^2 + 13}{(x-1)(x^2+x+2)}$ is expressed in the form $\dfrac{a}{x-1} + \dfrac{bx+c}{x^2+x+2}$, then

A $a = 5,\ b = 0,\ c = -13$ **B** $a = 5,\ b = 0,\ c = -10$ **C** $a = 5,\ b = 2,\ c = -3$
D $a = 7,\ b = 2,\ c = 3$ **E** $a = 7,\ b = 3,\ c = 13$

6 $\dfrac{4x-3}{(x-3)^2}$ is equal to

A $\dfrac{3}{x-3} + \dfrac{1}{x-3}$ **B** $\dfrac{4x}{x-3} - \dfrac{3}{x-3}$ **C** $\dfrac{9}{x-3} + \dfrac{4}{(x-3)^2}$

D $\dfrac{4}{x-3} + \dfrac{9}{(x-3)^2}$ **E** $\dfrac{4}{x-3} - \dfrac{15}{(x-3)^2}$

7 $\dfrac{8x+7}{2x^2+5x+2}$ is equal to

A $\dfrac{2}{2x+1} - \dfrac{3}{x+2}$ **B** $\dfrac{2}{2x+1} + \dfrac{3}{x+2}$ **C** $\dfrac{-4}{2x+2} - \dfrac{1}{x+1}$

D $\dfrac{-4}{2x+2} + \dfrac{1}{x+1}$ **E** $\dfrac{4}{2x+2} - \dfrac{1}{x+1}$

8 $\dfrac{-3x^2 + 2x - 1}{(x^2+1)(x+1)}$ is equal to

A $\dfrac{2}{x^2+1} + \dfrac{3}{x+1}$ **B** $\dfrac{2}{x^2+1} - \dfrac{3}{x+1}$ **C** $\dfrac{5}{x^2+1} + \dfrac{2}{x+1}$

D $\dfrac{3}{x^2+1} - \dfrac{2}{x+1}$ **E** $\dfrac{3}{x^2+1} + \dfrac{2}{x+1}$

9 The line $x + y = 2k$ touches the circle $x^2 + y^2 = k$, where $k > 0$. The value of k is

A $\frac{1}{\sqrt{2}}$ **B** $\sqrt{2}$ **C** $\frac{1}{2}$ **D** 2 **E** $2\sqrt{2}$

10 The simultaneous equations $y = x^2 + x$ and $y = bx - 1$ have exactly one real solution, where $b > 0$. The value of b is

A $\frac{5}{2}$ **B** 3 **C** $\frac{7}{2}$ **D** 2 **E** $\sqrt{2}$

11 If $(bx + c)(2x - 5) = 12x^2 + kx - 10$ for all values of x, then $k =$

A -10 **B** -26 **C** 24 **D** 32 **E** 36

Extended-response questions

1 A quadratic equation with integer coefficients $x^2 + bx + c = 0$ has a solution $x = 2 - \sqrt{3}$.

a Find the values of b and c.

Hint: Use the result that, for m, n rational, if $m + n\sqrt{3} = 0$, then $m = 0$ and $n = 0$.

b Find the other solution to this quadratic equation.

c Now consider a quadratic equation with integer coefficients $x^2 + bx + c = 0$ that has a solution $x = m - n\sqrt{q}$, where q is not a perfect square. Show that:

i $b = -2m$

ii $c = m^2 - n^2q$

Hence show that:

iii $x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))$

2 A train completes a journey of 240 km at a constant speed.

a If the train had travelled 4 km/h slower, it would have taken two hours more for the journey. Find the actual speed of the train.

b If the train had travelled a km/h slower and still taken two hours more for the journey of 240 km, what would have been the actual speed? (Answer in terms of a.) Discuss the practical possible values of a and also the possible values for the speed of the train.

c If the train had travelled a km/h slower and taken a hours more for the journey of 240 km, and if a is an integer and the speed is an integer, find the possible values for a and the speed of the train.

3 Two trains are travelling at constant speeds. The slower train takes a hours longer to cover b km. It travels 1 km less than the faster train in c hours.

a What is the speed of the faster train, in terms of a, b and c?

b If a, b, c and the speeds of the trains are all rational numbers, find five sets of values for a, b and c. Choose and discuss two sensible sets of values.

4 A tank can be filled using two pipes. The smaller pipe alone will take a minutes longer than the larger pipe alone to fill the tank. Also, the smaller pipe will take b minutes longer to fill the tank than when both pipes are used.

a In terms of a and b, how long will each of the pipes take to fill the tank?

b If $a = 24$ and $b = 32$, how long will each of the pipes take to fill the tank?

c If a and b are consecutive positive integers, find five pairs of values of a and b such that $b^2 - ab$ is a perfect square. Interpret these results in the context of this problem.

5 In each of the following, use the discriminant of the resulting quadratic equation:

a Find the possible values of k for which the straight line $y = k(1 - 2x)$ touches but does not cross the parabola $y = x^2 + 2$.

b Find the possible values of c for which the line $y = 2x + c$ intersects the circle $x^2 + y^2 = 20$ in two distinct points.

c Find the value of p for which the line $y = 6$ meets the parabola $y = x^2 + (1 - p)x + 2p$ at only one point.

6 Assume that $(x - \alpha)(x - \beta) = x^2 - px + 3$ for all x, where α, β and p are real numbers.

a Find in terms of p:

i $(\alpha - 2p) + (\beta - 2p)$

ii $(\alpha - 2p)(\beta - 2p)$

b The quadratic equation $x^2 + mx + n = 0$ has solutions $x = \alpha - 2p$ and $x = \beta - 2p$. Express m and n in terms of p.

7 The line $px - qy = 1$ touches the parabola $y = ax^2$, where p, q and a are positive real numbers.

a Show that $a = \dfrac{p^2}{4q}$.

b Determine the coordinates of the point P where the line touches the parabola in terms of p and q.

c Denote the x-axis intercept of the line by X and the y-axis intercept of the line by Y. Prove that $PX^2 = XY^2 = \dfrac{p^2 + q^2}{p^2q^2}$.

d Now assume that $p = q = 1$. The line has equation $x - y = 1$.

i Give the equation of the parabola.

ii Give the coordinates of the point P where the line touches the parabola.

iii Find the distance PX.

iv The line $-x - y = 1$ also touches the parabola. Find the coordinates of the point Q where this line touches the parabola. Sketch the graphs of the parabola and the two lines on the one set of axes.

8 The line $px + qy = 1$ touches the circle $x^2 + y^2 = a^2$, where p, q and a are positive real numbers.

a Show that $a^2 = \dfrac{1}{p^2 + q^2}$.

b Determine the coordinates of the point P where the line touches the circle in terms of p and q.

c Now assume that $p = q = 1$. The line has equation $x + y = 1$.

i Give the equation of the circle.

ii Give the coordinates of the point P where the line touches the circle.

iii The line $-x + y = 1$ also touches the circle. Find the coordinates of the point Q where this line touches the circle. Sketch the graphs of the circle and the two lines on the one set of axes.

9 **a** The circle, C_1, with equation $x^2 - 6x + y^2 - 8y + 24 = 0$ is touched by the lines with equations $y = m_1x$ and $y = m_2x$, where $m_1 < m_2$.

i Write the equation of circle C_1 in the form $(x - h)^2 + (y - k)^2 = r^2$. State the coordinates of the centre of the circle and the radius of the circle.

ii Sketch the graph of C_1 and show the lines $y = m_1x$ and $y = m_2x$ touching C_1 at the points P_1 and P_2 respectively.

iii Find the values of m_1 and m_2.

iv Find the coordinates of the points P_1 and P_2.

b The circle, C_2, with centre $(3, 4)$ and radius a is touched by the line $y = 2x$.

i Find a^2 and state the equation of circle C_2.

ii Find the coordinates of the point where the line $y = 2x$ touches circle C_2.

iii The line $y = m_3x$ also touches circle C_2. Find the value of m_3.

c Circle C_3 has centre (h, k) and radius 1, where h and k are positive real numbers. The lines $y = 2x$ and $y = \frac{1}{2}x$ touch C_3 at the points Q_1 and Q_2 respectively.

i Determine the values of h and k.

ii Determine the coordinates of the points Q_1 and Q_2.

5 Revision of Chapters 1–4

5A Technology-free questions

1 Express each of the following as a product of powers of prime numbers:

a 2002 **b** 555 **c** 7007 **d** 10 000

2 Simplify $\dfrac{5m-2p}{4m^2+mp-3p^2}-\dfrac{1}{4m-3p}$.

3 Expand each of the following and collect like terms:

a $(\sqrt{3}+\sqrt{2})(\sqrt{3}-1)$ **b** $(5\sqrt{3}-\sqrt{6})(2\sqrt{6}+3\sqrt{3})$

c $(2\sqrt{x}-3)^2$ **d** $(\sqrt{x-2}-3)^2$

4 Rewrite each fraction with an integer denominator:

a $\dfrac{1}{\sqrt{2}-3}$ **b** $\dfrac{3}{\sqrt{5}-1}$ **c** $\dfrac{2}{2\sqrt{2}-1}$

d $\dfrac{3}{\sqrt{5}-\sqrt{3}}$ **e** $\dfrac{1}{\sqrt{7}-\sqrt{2}}$ **f** $\dfrac{1}{2\sqrt{5}-\sqrt{3}}$

5 For each of the following, state the condition under which the geometric series is convergent and find the sum to infinity in this case:

a $a^4-a^3+a^2-\cdots$ **b** $\dfrac{1}{a}-\dfrac{b}{a^2}+\dfrac{b^2}{a^3}-\cdots$

c $\dfrac{2x+1}{x}-1+\dfrac{x}{2x+1}-\cdots$ **d** $1-\dfrac{1}{4x-2}+\dfrac{1}{(4x-2)^2}-\cdots$

6 **a** If the equations $x^2+x-1=0$ and $x^2+bx+1=0$ have a common solution, show that $b=\pm\sqrt{5}$.

b Find the common solution when:

i $b=\sqrt{5}$ **ii** $b=-\sqrt{5}$

7 Find constants a, b and c such that $(n+1)(n-7) = a + bn + cn(n-1)$ for all n.

8 Prove that, if $n = \text{HCF}(a, b)$, then n divides $a - b$.

9 Write down the prime factorisation of each of the following numbers, and hence determine the square root of each number:

a 576 **b** 1225 **c** 1936 **d** 1296

10 Solve the equation $\dfrac{x+b}{x-c} = 1 - \dfrac{x}{x-c}$ for x.

11 Solve the equation $\dfrac{1}{x-a} + \dfrac{1}{x-b} = \dfrac{2}{x}$ for x.

12 Of the first 1000 natural numbers, find the sum of those that:

a are not divisible by 3

b are divisible by neither 2 nor 3.

13 Find two sets of values of λ, a, b such that, for all values of x,

$$x^2 - 4x - 8 + \lambda(x^2 - 2x - 5) = a(x-b)^2$$

14 If the sum of the first k terms of the geometric sequence $3, 6, 12, 24, \dots$ is equal to 189, find the value of k.

15 For each of the following recurrence relations, determine a formula for the nth term of the sequence in terms of n:

a $t_n = \dfrac{1}{2}t_{n-1}, \quad t_1 = 2$ **b** $t_n = t_{n-1} - \dfrac{5}{2}, \quad t_1 = 2$ **c** $t_n = \dfrac{1}{2}t_{n-1} - \dfrac{5}{2}, \quad t_1 = 2$

16 A frog's first jump is 4 m, the second is 2 m, the third is 1 m, and so on. If the frog continues to jump indefinitely, how far will it get?

17 A triangle is such that the lengths of its sides form the first three terms in a geometric sequence. Given that the length of the longest side is 36 cm and the perimeter is 76 cm, find the length of the shortest side.

18 Three consecutive terms $a - d,\ a,\ a + d$ of an arithmetic sequence have a sum of 36. If the first term is increased by 1, the second by 4 and the third by 43, then the three new terms are in geometric sequence. Find the values of a and d.

19 Find the points of intersection of the graphs of $y = 2x^2 - 4x - 2$ and $y = -2x^2 - 4x + 2$.

20 Solve the following equation for x:

$$\frac{4}{x^2 - x - 2} + \frac{3}{x^2 - 4} = \frac{2}{x^2 + 3x + 2}$$

21 A train travels at a constant speed of 55 km/h for 2 hours and then at a constant speed of 70 km/h for 3 hours. Find the train's average speed over the 5-hour journey.

22 Simplify $\dfrac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

23 Resolve each of the following into partial fractions:

a $\dfrac{2x}{3(x-2)(x+2)}$ **b** $\dfrac{2x+5}{(x+2)(x+3)}$ **c** $\dfrac{5x^2+4x+4}{(x+2)(x^2+4)}$

d $\dfrac{2(x^2-2x-1)}{(x+1)(x-1)^2}$ **e** $\dfrac{2x^2-3x+1}{x^3-3x^2+x-3}$ **f** $\dfrac{3x^2-x+6}{(x^2+4)(x-2)}$

5B Multiple-choice questions

1 Five is seven less than three times one more than x. Written in algebraic form, this sentence becomes

A $5 = 7 - 3(x+1)$ **B** $3x + 1 = 5 - 7$ **C** $(x+1) - 7 = 5$
D $5 = 7 - 3x + 1$ **E** $5 = 3x - 4$

2 $\dfrac{3}{x-3} - \dfrac{2}{x+3}$ is equal to

A 1 **B** $\dfrac{x+15}{x^2-9}$ **C** $\dfrac{15}{x-9}$ **D** $\dfrac{x-3}{x^2-9}$ **E** $-\dfrac{1}{6}$

3 The sum of the odd numbers from 1 to m inclusive is 100. The value of m is

A 13 **B** 15 **C** 17 **D** 19 **E** 21

4 If the sum of the first n terms of a geometric sequence is $2^{n+1} - 2$, then the nth term is

A 2^{n-1} **B** 2^n **C** $2^n - 1$ **D** $2^{n-1} + 1$ **E** $2^n + 1$

5 If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5, 6\}$ and $C = \{3, 4, 5, 6, 7\}$, then $A \cap (B \cup C)$ is equal to

A $\{1, 2, 3, 4, 5, 6, 7\}$ **B** $\{1, 2, 3, 4, 5, 6\}$ **C** $\{2, 3, 4\}$
D $\{3, 4\}$ **E** $\{2, 3, 4, 5, 6, 7\}$

6 The recurring decimal $0.\dot{7}\dot{2}$ is equal to

A $\dfrac{72}{101}$ **B** $\dfrac{72}{100}$ **C** $\dfrac{72}{99}$ **D** $\dfrac{72}{90}$ **E** $\dfrac{73}{90}$

7 $\dfrac{-4}{x-1} - \dfrac{3}{1-x} + \dfrac{x}{x-1}$ is equal to

A 1 **B** -1 **C** $\dfrac{7x}{x-1}$ **D** $\dfrac{1}{1-x}$ **E** none of these

8 $\dfrac{x+2}{3} - \dfrac{5}{6}$ is equal to

A $\dfrac{x-3}{6}$ **B** $\dfrac{2x+4}{6}$ **C** $\dfrac{2x-1}{6}$ **D** $\dfrac{2x-5}{6}$ **E** $\dfrac{x-3}{3}$

9 If $a = 1 + \frac{1}{1+b}$, then b equals

A $1 - \frac{1}{a-1}$ **B** $1 + \frac{1}{a-1}$ **C** $\frac{1}{a-1} - 1$ **D** $\frac{1}{a+1} + 1$ **E** $\frac{1}{a+1} - 1$

10 When the repeating decimal $0.\dot{3}\dot{6}$ is written in simplest fractional form, the sum of the numerator and denominator is

A 15 **B** 45 **C** 114 **D** 135 **E** 150

11 If $\frac{2x-y}{2x+y} = \frac{3}{4}$, then $\frac{x}{y}$ equals

A $\frac{2}{7}$ **B** $\frac{7}{2}$ **C** $\frac{3}{4}$ **D** $\frac{4}{3}$ **E** cannot be determined

12 The sum to infinity of the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$ is

A 2 **B** 1 **C** $\frac{1}{2}$ **D** $\frac{1}{3}$ **E** $\frac{2}{3}$

13 If $\frac{3}{3+y} = 4$, then y equals

A $\frac{1}{4}$ **B** $-\frac{9}{4}$ **C** $\frac{9}{4}$ **D** 0 **E** $-\frac{4}{9}$

14 The coordinates of the point where the lines with equations $3x + y = -7$ and $2x + 5y = 4$ intersect are

A $(3, -16)$ **B** $(-3, 2)$ **C** $(3, -2)$ **D** $(-2, 3)$ **E** no solution

15 If $\frac{m+2}{4} - \frac{2-m}{4} = \frac{1}{2}$, then m is equal to

A 1 **B** -1 **C** $\frac{1}{2}$ **D** 0 **E** $-\frac{1}{2}$

16 The number 46 200 can be written as

A $2 \times 3 \times 5 \times 7 \times 11$ **B** $2^2 \times 3^2 \times 5^2 \times 7 \times 11$ **C** $2 \times 3^2 \times 5 \times 7^2 \times 11$
D $2^3 \times 3 \times 5^2 \times 7 \times 11$ **E** $2^2 \times 3 \times 5^3 \times 7 \times 11$

17 If the three numbers y, $y - 1$ and $2y - 1$ are consecutive terms of an arithmetic sequence, then y equals

A -1 **B** 1 **C** 0 **D** 2 **E** -2

18 If the positive integers $n + 1,\ n - 1,\ n - 6,\ n - 5,\ n + 4$ are arranged in increasing order of magnitude, then the middle number is

A $n + 1$ **B** $n - 1$ **C** $n - 6$ **D** $n - 5$ **E** $n + 4$

19 An arithmetic sequence has 3 as its first term and 9 as its fourth term. The eleventh term is

A 23 **B** 11 **C** 63 **D** 21 **E** none of these

20 The expression $\dfrac{4}{n+1} + \dfrac{3}{n-1}$ is equal to

A $\dfrac{7n-1}{1-n^2}$ **B** $\dfrac{1-7n}{1-n^2}$ **C** $\dfrac{7n-1}{n^2+1}$ **D** $\dfrac{7}{n^2-1}$ **E** $\dfrac{7}{n}$

21 $(\sqrt{7}+3)(\sqrt{7}-3)$ is equal to

A -2 **B** 10 **C** $\sqrt{14}-19$ **D** $2\sqrt{7}-9$ **E** 45

22 If $\dfrac{13x-10}{2x^2-9x+4} = \dfrac{P}{x-4} + \dfrac{Q}{2x-1}$, then the values of P and Q are

A $P = 1$ and $Q = 1$ **B** $P = -1$ and $Q = 1$ **C** $P = 6$ and $Q = 1$
D $P = -6$ and $Q = 1$ **E** $P = 1$ and $Q = -6$

23 The first term of a geometric sequence is a and the infinite sum of the geometric sequence is $4a$. The common ratio of the geometric sequence is

A 3 **B** 4 **C** $\dfrac{3}{4}$ **D** $-\dfrac{3}{4}$ **E** $-\dfrac{4}{3}$

24 If $\dfrac{5x}{(x+2)(x-3)} = \dfrac{P}{x+2} + \dfrac{Q}{x-3}$, then

A $P = 2$ and $Q = 3$ **B** $P = 2$ and $Q = -3$ **C** $P = -2$ and $Q = 3$
D $P = -2$ and $Q = -3$ **E** $P = 1$ and $Q = 1$

25 If the natural number n is a perfect square, then the next perfect square is

A $n+1$ **B** n^2+1 **C** n^2+2n+1 **D** n^2+n **E** $n+2\sqrt{n}+1$

26 Which of the following is *not* a rational number?

A 0.4 **B** $\dfrac{3}{8}$ **C** $\sqrt{5}$ **D** $\sqrt{16}$ **E** 4.125

27 If $\dfrac{1}{x} = \dfrac{a}{b}$ and $\dfrac{1}{y} = a - b$, then $x + y$ equals

A $\dfrac{2}{a}$ **B** $\dfrac{a^2-b^2}{a}$ **C** $\dfrac{ba-b^2+a}{a(a-b)}$ **D** $\dfrac{2a}{a^2-b^2}$ **E** $\dfrac{-2b}{a^2-b^2}$

28 $9x^2 - 4mx + 4$ is a perfect square when m equals

A 5 **B** ± 12 **C** 2 **D** ± 1 **E** ± 3

29 If $x = (n+1)(n+2)(n+3)$, for some positive integer n, then x is not always divisible by

A 1 **B** 2 **C** 3 **D** 5 **E** 6

30 The numbers $-4, a, b, c, d, e, f, 10$ are consecutive terms of an arithmetic sequence. The sum $a + b + c + d + e + f$ is equal to

A 6 **B** 10 **C** 18 **D** 24 **E** 48

31 If both n and p are odd numbers, which one of the following numbers must be even?

A $n+p$ **B** np **C** $np+2$ **D** $n+p+1$ **E** $2n+p$

32 The sum of the first 10 terms of the sequence 4, 9, 14, 19, . . . is

A 61 **B** 250 **C** 265 **D** 290 **E** 520

33 A geometric sequence has first term 3 and common ratio 4. The twentieth term is

A $\frac{3}{4^{20}}$ **B** 3×4^{20} **C** 3×4^{19} **D** $\frac{3}{4^{19}}$ **E** $3 + 4^{20}$

34 In an arithmetic sequence, the first term is 9 and the sixteenth term is 144. The common difference is

A 2 **B** 4 **C** 6 **D** 9 **E** 10

35 Which of the following generates the sequence $-5, 13, -23, 49, -95, 193, \ldots$?

A $t_n = 3 - 2t_{n-1}, \quad t_1 = -5$ **B** $t_n = 3 + 2t_{n-1}, \quad t_1 = -5$

C $t_n = 2 - 3t_{n-1}, \quad t_1 = -5$ **D** $t_n = 2 - 2t_{n-1}, \quad t_1 = -5$

E $t_n = 2 + 3t_{n-1}, \quad t_1 = -5$

36 A study is conducted on a particular species of small marsupial. The number of animals at the start of the study is 12 500. Each year on average, there are 15 offspring per 100 animals, and 11% of the animals die. The number, P_n, of animals after n years can be modelled by the formula

A $P_n = 12\,500 \times 1.04^n$ **B** $P_n = 12\,500 \times 0.15^n - 0.11$

C $P_n = 12\,500 \times 0.11^n + 0.15$ **D** $P_n = 12\,500 \times 0.04^{n+1} - 0.11$

E $P_n = 12\,500 - 0.04n$

37 If LCM(12, n) = 60 and HCF(12, n) = 6, then n =

A 10 **B** 15 **C** 20 **D** 30 **E** 60

38 If x^2 is written in the form $(x-2)^2 + b(x-2) + c$, then the values of b and c are

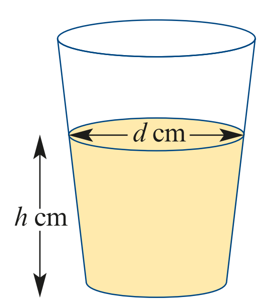
A $b = 2,\ c = 0$ **B** $b = -4,\ c = -4$ **C** $b = 4,\ c = 4$

D $b = 2,\ c = 2$ **E** $b = 0,\ c = 2$

5C Extended-response questions

1 The diagram represents a glass containing milk. When the height of the milk in the glass is h cm, the diameter, d cm, of the surface of the milk is given by the formula

$$d = \frac{h}{5} + 6$$



a Find d when $h = 10$.

b Find d when $h = 8.5$.

c What is the diameter of the bottom of the glass?

d The diameter of the top of the glass is 9 cm. What is the height of the glass?

2 At the beginning of 2012, Andrew and John bought a small catering business. The profit, $\$P$, in a particular year is given by

$$P = an + b$$

where n is the number of years of operation and a and b are constants.

a Given the table, find the values of a and b.

Year	2012	2016
Number of years of operation (n)	1	5
Profit (P)	−9000	15 000

b Find the profit when $n = 12$.

c In which year was the profit \$45 000?

3 The formula $A = 180 - \dfrac{360}{n}$ gives the size of each interior angle, $A°$, of a regular polygon with n sides.

a Find the value of A when n equals:

i 180 **ii** 360 **iii** 720 **iv** 7200

b As n becomes very large:

i What value does A approach?

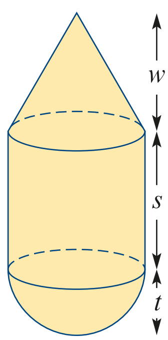
ii What shape does the polygon approach?

c Find the value of n when $A = 162$.

d Make n the subject of the formula.

e Three regular polygons, two of which are octagons, meet at a point so that they fit together without any gaps. Describe the third polygon.

4 The figure shows a solid consisting of three parts – a cone, a cylinder and a hemisphere – all of the same base radius.



a Find, in terms of w, s, t and π, the volume of each part.

b **i** If the volume of each of the three parts is the same, find the ratio $w : s : t$.

ii If also $w + s + t = 11$, find the total volume in terms of π.

5 The following information is given about a universal set ξ and subsets A, B and C of ξ.

- $|\xi| = 200$
- $|A| = 70$
- $|B| = 120$
- $|C| = 90$
- $|A \cap B| = 50$
- $|A \cap C| = 30$
- $|B \cap C| = 40$
- $|A \cap B \cap C| = 20$

Use this information to determine each of the following:

a $|A \cup B|$

b $|A \cup B \cup C|$

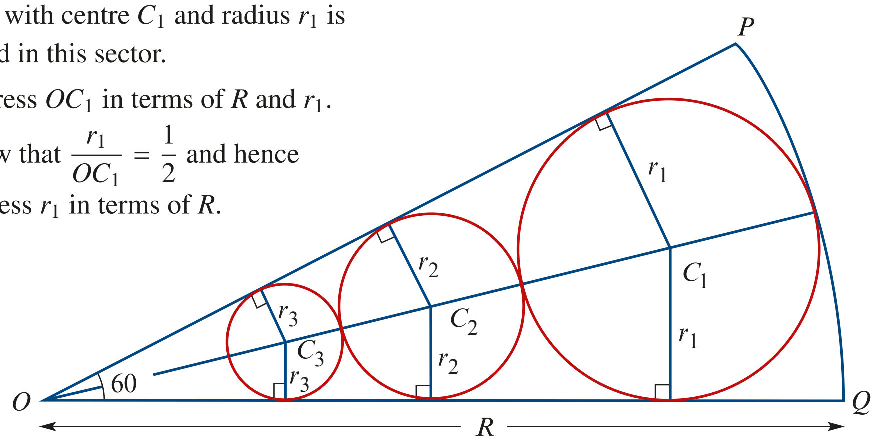
c $|A' \cap B \cap C|$

d $|A \cap B' \cap C'|$

6 **a** In the diagram, OPQ is a sector of radius R. A circle with centre C_1 and radius r_1 is inscribed in this sector.

i Express OC_1 in terms of R and r_1.

ii Show that $\frac{r_1}{OC_1} = \frac{1}{2}$ and hence express r_1 in terms of R.



b Another circle, centre C_2, is inscribed in the sector as shown.

i Express OC_2 in terms of r_2 and R.

ii Express r_2 in terms of R.

c Circles with centres at $C_3, C_4, C_5, \ldots$ are constructed in a similar way. Their radii are $r_3, r_4, r_5, \ldots$ respectively. It is known that $r_1, r_2, r_3, \ldots$ is a geometric sequence.

i Find the common ratio.

ii Find r_n.

iii Find the sum to infinity of the sequence, and interpret the result geometrically.

iv Find in terms of R and π, the sum to infinity of the areas of the circles with radii $r_1, r_2, r_3, \ldots$.

7 Two companies produce the same chemical.

- For Company A, the number of tonnes produced increases by 80 tonnes per month.
- For Company B, production increases by 4% per month.

Each company produced 1000 tonnes in January 2018.
(Let n be the number of months of production. Use $n = 1$ for January 2018.)

a Find, to the nearest tonne where appropriate:

i the production of Company A in the nth month

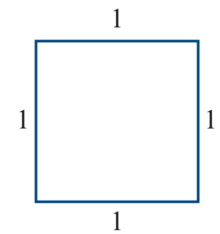
ii the production of each company in December 2019 (i.e. for $n = 24$)

iii the total production of Company A over n months (starting with $n = 1$ for January 2018)

iv the total production of each company for the period from January 2018 to December 2019 inclusive.

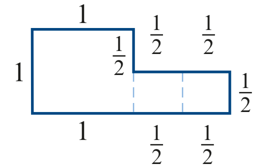
b Find in which month of which year the total production of Company A passed 100 000 tonnes.

8 The square shown has each side of length 1 unit.

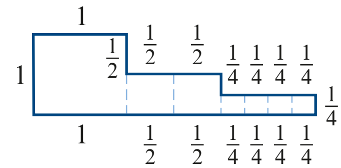


a The perimeter of the square is denoted by P_1. What is the value of P_1?

b A new figure is formed by joining two squares of side length $\frac{1}{2}$ to this square, as shown. The new perimeter is denoted by P_2. What is the value of P_2?



c What is the perimeter, P_3, of this figure?



d It is known that $P_1, P_2, P_3, \dots$ are the terms of an arithmetic sequence with first term P_1. What is the common difference?

e **i** Find P_4.

ii Find P_n in terms of P_{n-1}.

iii Find P_n in terms of n.

iv Draw the diagram of the figure corresponding to P_4.

9 The number 15 can be expressed as a sum of consecutive positive integers in four ways:

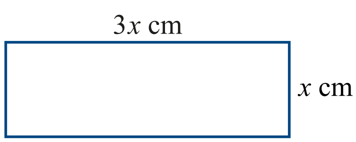
$$15 = 15, \quad 15 = 7 + 8, \quad 15 = 4 + 5 + 6, \quad 15 = 1 + 2 + 3 + 4 + 5$$

a Show that 10 can only be expressed as a sum of consecutive positive integers in two ways.

b How many ways can 100 be expressed as a sum of consecutive positive integers?

c How many ways can 15 be expressed as the sum of any sequence of consecutive integers?

10 A piece of wire 28 cm long is cut into two parts: one to make a rectangle three times as long as it is wide, and the other to make a square.



a What is the perimeter of the rectangle in terms of x?

b What is the perimeter of the square in terms of x?

c What is the length of each side of the square in terms of x?

Let A be the sum of the areas of the two figures.

d Show that $A = 7(x^2 - 4x + 7)$.

e Sketch the graph of $A = 7(x^2 - 4x + 7)$ for $0 \le x \le \frac{7}{2}$.

f Find the minimum value that A can take and the corresponding value of x.

11 **a** **i** For the equation $\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$, square both sides to show that this equation implies

$$8x - 10 = \sqrt{14x^2 - 10x}$$

ii Square both sides of this new equation and simplify to form the equation

$$x^2 - 3x + 2 = 0 \qquad (1)$$

iii The solutions to equation (1) are $x = 1$ and $x = 2$. Test these solutions for the equation

$$\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$$

and hence show that $x = 2$ is the only solution to the original equation.

b Use the techniques of part **a** to solve the equations:

i $\sqrt{x+2} - 2\sqrt{x} = \sqrt{x+1}$ **ii** $2\sqrt{x+1} + \sqrt{x-1} = 3\sqrt{x}$

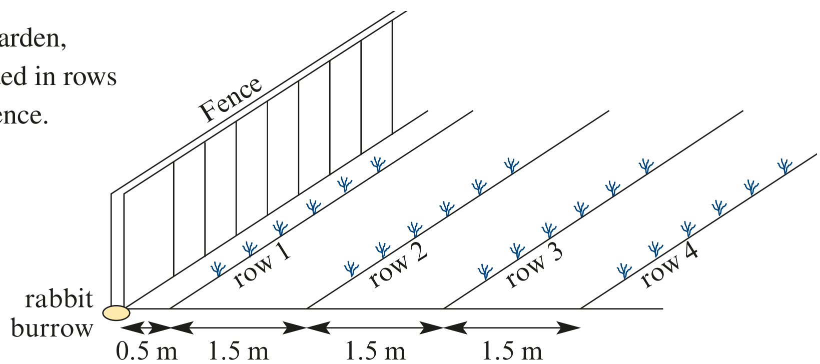
12 Let n be a natural number less than 50 such that $n + 25$ is a perfect square.

a Show that there exists an integer a such that $n = a(a + 10)$.

b Any natural number less than 100 can be written in the form $10p + q$, where p and q are digits. For this representation of n, show that $q = p^2$.

c Give all possible values of n.

13 In a vegetable garden, carrots are planted in rows parallel to the fence.

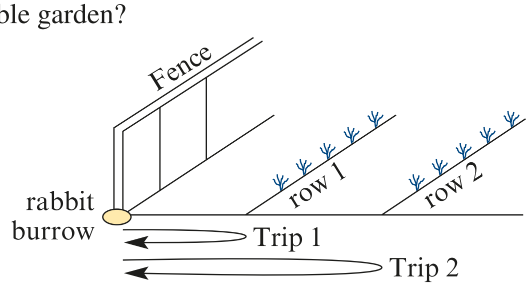


a Calculate the distance between the fence and the 10th row of carrots.

b If t_n represents the distance between the fence and the nth row, find a formula for t_n in terms of n.

c Given that the last row of carrots is less than 80 m from the fence, what is the largest number of rows possible in this vegetable garden?

d A systematic rabbit has its burrow under the fence as shown in the diagram. The rabbit runs to the first row, takes a carrot and returns it to the burrow. The rabbit then runs to the second row, takes a carrot and returns it to the burrow.



The rabbit continues in this way until it has 15 carrots. Calculate the shortest distance the rabbit has to run to accomplish this.

14 In its first month of operation, a soft-drink manufacturer produces 50 000 litres of soft drink. In each successive month, the production rises by 5000 litres per month.

a **i** The quantity of soft drink, t_n litres, produced in the nth month can be determined by a rule of the form

$$t_n = a + (n - 1)d$$

Find the values of a and d.

ii In which month will the factory double its original production?

iii How many litres in total will be produced in the first 36 months of operation?

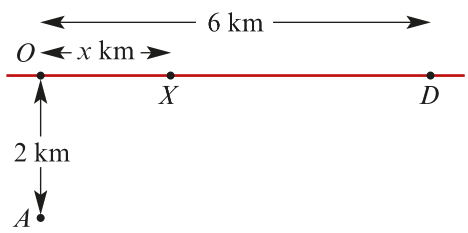
b Another soft-drink manufacturer sets up a factory at the same time as the first. In the first month, the production is 12 000 litres. The production of this factory increases by 10% every month.

i Find a rule for q_n, the quantity of soft drink produced in the nth month.

ii Find the total amount of soft drink produced in the first 12 months.

c If the two factories start production in the same month, in which month will the production of the second factory become faster than the production of the first factory?

15 The diagram shows a straight road OD, where $OD = 6$ km.



A hiker is at A, which is 2 km from O. The hiker walks directly to X and then walks along the road to D. The hiker can walk at 3 km/h off-road, but at 8 km/h along the road.

a If $OX = 3$ km, calculate the total time taken for the hiker to walk from A to D via X in hours and minutes, correct to the nearest minute.

b If the total time taken was $1\frac{1}{2}$ hours, calculate the distance OX in kilometres, correct to one decimal place.

16 A car leaves town A at 10 a.m. and arrives in town B at 11:15 a.m. During the first hour of the journey, the car travels at a constant speed of 80 km/h. The average speed of the car between 10:15 a.m. and 11:15 a.m. is 2 km/h less than the average speed for the whole journey.

a Find the distance travelled by the car from town A to town B.

b Find the average speed of the car between 10:15 a.m. and 11:15 a.m.

17 **a** There is a two-digit number □□ such that, if you add the digit 1 at the front and the back to obtain a four-digit number 1□□1, then the new four-digit number is 21 times larger than the original two-digit number. Find this two-digit number.

b Find a five-digit number □□ □□□ such that the six-digit number obtained by adding a 1 at the back, □□□ □□1, is three times larger than the six-digit number obtained by adding a 1 at the front, 1□□ □□□.

18 Find the possible integer values of k if:

a the quadratic equation $x^2 + kx - 16 = 0$ has integer solutions

b the quadratic equation $x^2 + kx + 20 = 0$ has integer solutions

c the quadratic equation $x^2 + 12x + k = 0$ has integer solutions (k positive).

19 **a** Show that $x^2 + (1 - x)^2 = 2\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right]$.

b Hence show that, if $0 \le x \le 1$, then

$$\frac{1}{2} \le x^2 + (1 - x)^2 \le 1$$

c A quadrilateral has one vertex on each side of a unit square (that is, a square of side length 1). Show that the side lengths a, b, c and d of the quadrilateral satisfy

$$2 \le a^2 + b^2 + c^2 + d^2 \le 4$$

20 Consider the quadratic expression $x^2 + bx + c$, where b and c are real numbers.

a Given that the equation $x^2 + bx + c + 1 = 0$ has only one solution, find c in terms of b.

b Given that the expression $x^2 + bx + c - 3$ can be factorised as $(x - k)(x - 2k)$, for some non-zero real number k, find c in terms of b.

c If the conditions of both parts **a** and **b** are satisfied, find the possible values of b and c.

21 **a** Find positive integers m and n such that $\sqrt{9 - 4\sqrt{5}} = \sqrt{m} - n$.

b Hence find integers b and c such that $x = \sqrt{9 - 4\sqrt{5}}$ is a solution of the quadratic equation $x^2 + bx + c = 0$.

22 Seventy-six photographers submitted work for a photographic exhibition in which they were permitted to enter not more than one photograph in each of three categories: black and white (B), colour prints (C), transparencies (T). Eighteen entrants had all their work rejected, while 30 B, 30 T and 20 C were accepted.

- From the exhibitors, as many showed T only as showed T and C.
- There were three times as many exhibitors showing B only as showing C only.
- Four exhibitors showed B and T but not C.

a Write the last three sentences in symbolic form.

b Draw a Venn diagram representing the information.

c **i** Find $|B \cap C \cap T|$. **ii** Find $|B \cap C \cap T'|$.

23 It is estimated that there are 600 black swans in a particular wildlife sanctuary. Assume that the swan population, left untouched, would increase by 5% per annum. However, at the end of each year 24 swans are removed from the wildlife sanctuary and transferred to a nearby national park.

a How many swans will there be in the wildlife sanctuary after:

i 1 year **ii** 2 years **iii** 3 years?

b Write a recurrence relation that gives the number of swans in the sanctuary after n years in terms of the number of swans in the sanctuary after $n - 1$ years.

c Write a formula for the number of swans in the sanctuary after n years in terms of n.

d Find the number of swans in the sanctuary after 12 years.

It is estimated that there are also 600 black swans in the national park and that, if left untouched, their population would decrease by 15% per annum. To help compensate, each year 24 swans are brought into the national park.

e How many swans will there be in the national park after:

i 1 year **ii** 2 years **iii** 12 years?

f How long will it take for the swan population in the national park to stabilise? What is this stable population size?

24 Each year for the past 10 years, the population of the city of Alpha has been growing at a steady rate of 2.3% per annum. The current population of Alpha is 1.35 million.

a What was the population of Alpha 10 years ago?

Over the next 10 years, the population of Alpha is predicted to grow at a steady rate of 2.8% per annum.

b What will be the population of Alpha in 10 years' time?

A neighbouring city, Beta, had a population of 1.25 million 5 years ago. Its population has been growing at a steady rate of 1.9% per annum and is predicted to maintain this growth rate for the next 10 years.

c Which city currently has the greater population?

d Which city will have the greater population in 10 years' time?

e In how many years from now will the populations of the two cities be equal?

25 **a** Prove that

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$$

Note: This is known as Sophie Germain's identity.

b Use this identity to prove that, if n is an odd number greater than 1, then $n^4 + 4^n$ is not prime.

c Hence show that the number $4^{545} + 545^4$ is not prime.

26 **a** Consider the geometric sequence $1, 4, 16, \dots$.

i Find the 10th term of this sequence.

ii Find the value of n if the sum of the first n terms of this sequence is 349 525.

b Consider the geometric sequence $1, \frac{1}{4}, \frac{1}{16}, \dots$.

i Find the 10th term of this sequence.

ii Find the sum of the first 10 terms of this sequence, correct to three decimal places.

c Now consider the sequence $2, 4\frac{1}{4}, 16\frac{1}{16}, \dots, 2^{2n-2} + \frac{1}{2^{2n-2}}, \dots$.

i Write down the sum of the first 10 terms of this sequence, correct to three decimal places.

ii Find a formula for the sum of the first n terms of this sequence.

5D Investigations

1 Arithmetic and geometric means

For positive numbers a and b, their arithmetic mean is greater than or equal to their geometric mean:

$$\frac{1}{2}(a+b) \geq \sqrt{ab}$$

Furthermore, the two means are equal if and only if $a = b$.

This result is called the **AM–GM inequality** and is easy to prove as follows:

$$\begin{aligned} \frac{1}{2}(a+b) \geq \sqrt{ab} &\Leftrightarrow \frac{1}{2}(a+b) - \sqrt{ab} \geq 0 \\ &\Leftrightarrow a + b - 2\sqrt{ab} \geq 0 \\ &\Leftrightarrow (\sqrt{a} - \sqrt{b})^2 \geq 0 \end{aligned}$$

The final inequality is true for all $a, b > 0$.

a Consider all rectangles of a given constant perimeter. Using the AM–GM inequality, find the maximum area of such rectangles in terms of this perimeter.

b Consider all rectangles of a given constant area. Using the AM–GM inequality, find the minimum perimeter of such rectangles in terms of this area.

c The AM–GM inequality can be extended to more than two numbers. For three positive numbers a, b and c, we have

$$\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$$

and the two means are equal if and only if $a = b = c$. Prove this result.

d Consider all rectangular prisms of a given constant surface area. Using the AM–GM inequality for three numbers, find the maximum volume of such prisms in terms of this surface area.

2 The arithmetic–geometric mean

Start with two positive numbers a_1 and b_1 such that $a_1 \geq b_1$. By computing arithmetic and geometric means, we can obtain a pair of sequences:

$$a_{n+1} = \frac{1}{2}(a_n + b_n) \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

a Use a spreadsheet to investigate such sequences. Try different values for a_1 and b_1.

Note that $a_n \geq b_n$ for all $n \in \mathbb{N}$, by the AM–GM inequality from Investigation 1.

b Prove that $a_{n+1} \leq a_n$ and $b_{n+1} \geq b_n$ for all $n \in \mathbb{N}$.

c Prove that $b_1 \leq a_n$ and $a_1 \geq b_n$ for all $n \in \mathbb{N}$.

d Prove that $a_{n+1} - b_{n+1} \leq \frac{1}{2}(a_n - b_n)$ for all $n \in \mathbb{N}$.

These results can be used to prove that the sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ converge to the same number (called the **arithmetic–geometric mean** of a and b). The final steps of the proof are beyond the scope of this course.

3 Modelling markets

a The shampoo market in a certain country is supplied by two distributors, X and Y. Let x_n and y_n represent the number of unit sales in week n by distributors X and Y, respectively. The market fluctuates according to the following pair of equations:

$$x_{n+1} = 30\,000 - 0.6y_n, \qquad x_1 = 30\,000$$
$$y_{n+1} = 15\,000 - 0.3x_n, \qquad y_1 = 5000$$

Find the equilibrium values by considering $x_{n+1} = x_n = x_{n-1}$.

b A third distributor, Z, joins the shampoo market. Let z_n be the number of unit sales in week n by distributor Z. In the following system of equations, week 1 is now taken to be the first week that Z enters the market:

$$x_{n+1} = 30\,000 - 0.5y_n, \qquad x_1 = 25\,610$$
$$y_{n+1} = 15\,000 - 0.2x_n - 0.2z_n, \qquad y_1 = 7317$$
$$z_{n+1} = 30\,000 - 0.5y_n, \qquad z_1 = 1000$$

Find the equilibrium values. Consider other systems of equations to obtain different equilibrium values.

4 Reciprocals of natural numbers

For a fixed natural number n, consider the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

a Let $n = 3$. Find all ordered pairs of natural numbers (x, y) that satisfy the equation.

b Let $n = 11$. Find all ordered pairs of natural numbers (x, y) that satisfy the equation.

c Let $n = p$, where p is a prime number. Find all ordered pairs of natural numbers (x, y) that satisfy the equation. Give your answers with x and y in terms of p.

d For any natural number n, is it always possible to find an ordered pair of natural numbers (x, y) that is a solution to the equation? How can we determine the number of solutions?

5 Applications of partial fractions

a **i** Show that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

ii Hence show that

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{99\times 100} = \frac{99}{100}$$

iii Hence evaluate

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+99}$$

b **i** Find A and B such that

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

ii Hence evaluate

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{99\times 101}$$

iii By using partial fractions in a similar way, evaluate

$$\frac{1}{1\times 6} + \frac{1}{6\times 11} + \frac{1}{11\times 16} + \cdots + \frac{1}{96\times 101}$$

c **i** Show that

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

ii Hence evaluate

$$\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots + \frac{1}{98\times 99\times 100}$$

d **i** By rationalising the denominator of the left-hand side, show that

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1} - \sqrt{n}$$

ii Hence show that

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{100}+\sqrt{99}} = 9$$

iii By using a similar approach, evaluate

$$\frac{1}{\sqrt{3}+\sqrt{1}} + \frac{1}{\sqrt{5}+\sqrt{3}} + \frac{1}{\sqrt{7}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{121}+\sqrt{119}}$$

6 Proof

Objectives

- To understand and use various methods of proof, including:
  - **direct proof**
  - **proof by contrapositive**
  - **proof by contradiction**.
- To write down the **negation** of a statement.
- To write and prove **converse** statements.
- To understand when mathematical statements are **equivalent**.
- To use the symbols for **implication** ($\Rightarrow$) and **equivalence** ($\Leftrightarrow$).
- To understand and use the quantifiers **'for all'** and **'there exists'**.
- To disprove statements using **counterexamples**.
- To understand and use the **principle of mathematical induction**.

A **mathematical proof** is an argument that demonstrates the absolute truth of a statement.

It is certainty that makes mathematics different from other sciences. In science, a theory is never proved true. Instead, one aims to prove that a theory is not true. And if such evidence is hard to come by, then this increases the likelihood that a theory is correct, but never provides a guarantee. The possibility of absolute certainty is reserved for mathematics alone.

When writing a proof you should always aim for three things:

- correctness
- clarity
- simplicity.

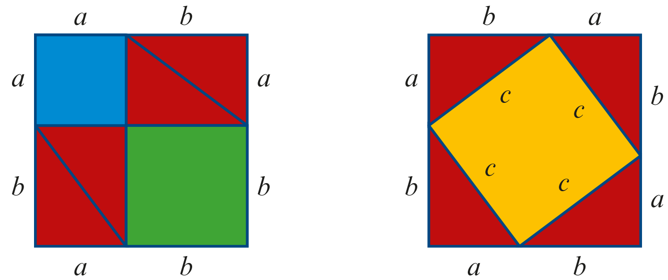
Perhaps the following proof of Pythagoras' theorem exemplifies these three aims.

Pythagoras' theorem

Take any triangle with side lengths a, b and c. If the angle between a and b is 90°, then

$$a^2 + b^2 = c^2$$

Proof Consider the two squares shown below.



The two squares each have the same total area. So subtracting four red triangles from each figure will leave the same area. Therefore $a^2 + b^2 = c^2$.

The ideas introduced in this chapter will be used in proofs throughout the rest of this book.

6A Direct proof

Conditional statements

Consider the following sentence:

Statement	If	it is raining	then	the grass is wet.

This is called a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

This can be abbreviated as

$$P \Rightarrow Q$$

which is read 'P **implies** Q'. We call P the **hypothesis** and Q the **conclusion**.

Not all conditional statements will be true. For example, switching the hypothesis and the conclusion above gives:

Statement	If	the grass is wet	then	it is raining.

Anyone who has seen dewy grass on a cloudless day knows this to be false. In this chapter we will learn how to prove (and disprove) mathematical statements.

Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 1

Prove the following statements:

a If a is odd and b is even, then $a + b$ is odd.

b If a is odd and b is odd, then ab is odd.

Solution

a Assume that a is odd and b is even.

Since a is odd, we have $a = 2m + 1$ for some $m \in \mathbb{Z}$. Since b is even, we have $b = 2n$ for some $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} a + b &= (2m + 1) + 2n \\ &= 2m + 2n + 1 \\ &= 2(m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = m + n \in \mathbb{Z} \end{aligned}$$

Hence $a + b$ is odd.

Note: We must use two different pronumerals m and n here, because these two numbers may be different.

b Assume that both a and b are odd. Then $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Therefore

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2mn + m + n \in \mathbb{Z} \end{aligned}$$

Hence ab is odd.

Example 2

Let $p, q \in \mathbb{Z}$ such that p is divisible by 5 and q is divisible by 3. Prove that pq is divisible by 15.

Solution

Since p is divisible by 5, we have $p = 5m$ for some $m \in \mathbb{Z}$. Since q is divisible by 3, we have $q = 3n$ for some $n \in \mathbb{Z}$. Thus

$$\begin{aligned} pq &= (5m)(3n) \\ &= 15mn \end{aligned}$$

and so pq is divisible by 15.

Example 3

Let x and y be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

Solution

Assume that $x > y$. Then $x - y > 0$.

Since x and y are positive, we also know that $x + y > 0$.

Therefore

$$x^2 - y^2 = \underbrace{(x - y)}_{\text{positive}}\underbrace{(x + y)}_{\text{positive}} > 0$$

Hence $x^2 > y^2$.

Explanation

When trying to prove that $x^2 > y^2$, it is easier to first prove that $x^2 - y^2 > 0$.

Also, note that the product of two positive numbers is positive.

Example 4

Let x and y be any two positive real numbers. Prove that

$$\frac{x + y}{2} \geq \sqrt{xy}$$

Solution

A **false proof** might begin with the statement that we are trying to prove.

$$
\begin{aligned}
& & \frac{x + y}{2} &\geq \sqrt{xy} \\
&\Rightarrow & x + y &\geq 2\sqrt{xy} \\
&\Rightarrow & (x + y)^2 &\geq 4xy \qquad \text{(using Example 3)} \\
&\Rightarrow & x^2 + 2xy + y^2 &\geq 4xy \\
&\Rightarrow & x^2 - 2xy + y^2 &\geq 0 \\
&\Rightarrow & (x - y)^2 &\geq 0
\end{aligned}
$$

Although it is true that $(x - y)^2 \geq 0$, the argument is faulty. We cannot prove that the result is true by assuming that the result is true! However, the above work is not a waste of time. We can correct the proof by reversing the order of the steps shown above.

Note: In the corrected proof, we need to use the fact that $a > b$ implies $\sqrt{a} > \sqrt{b}$ for all positive numbers a and b. This is shown in Question 8 of Exercise 6B.

Breaking a proof into cases

Sometimes it helps to break a problem up into different cases.

Example 5

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'We are both knaves.' What are Alice and Bob?

Solution

We will prove that Alice is a knave and Bob is a knight.

Case 1

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice and Bob are both knaves.

$\Rightarrow$ Alice is a knave and a knight.

This is impossible.

Case 2

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice and Bob are not both knaves.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knave and Bob must be a knight.

Summary 6A

- A **mathematical proof** establishes the truth of a statement.
- A **conditional statement** has the form: If P is true, then Q is true.
  This can be abbreviated as $P \Rightarrow Q$, which is read 'P **implies** Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Skill-sheet

Exercise 6A

Example 1

1 Assume that m is even and n is even. Prove that:

a $m + n$ is even **b** mn is even.

2 Assume that m is odd and n is odd. Prove that $m + n$ is even.

3 Assume that m is even and n is odd. Prove that mn is even.

Example 2

4 Suppose that m is divisible by 3 and n is divisible by 7. Prove that:

a mn is divisible by 21 **b** m^2n is divisible by 63.

5 Suppose that m and n are perfect squares. Show that mn is a perfect square.

6 Let m and n be integers. Prove that $(m + n)^2 - (m - n)^2$ is divisible by 4.

7 Suppose that n is an even integer. Prove that $n^2 - 6n + 5$ is odd.

8 Suppose that n is an odd integer. Prove that $n^2 + 8n + 3$ is even.

9 Let $n \in \mathbb{Z}$. Prove that $5n^2 + 3n + 7$ is odd.
Hint: Consider the cases when n is odd and n is even.

Example 3

10 Let x and y be positive real numbers. Show that if $x > y$, then $x^4 > y^4$.

Example 4

11 Let $x, y \in \mathbb{R}$. Show that $x^2 + y^2 \geq 2xy$.

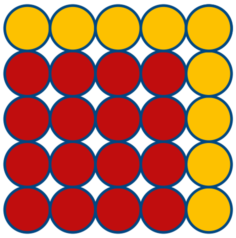
Example 5 **12** Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Determine whether Alice and Bob are knights or knaves in each of the following separate instances:

a Alice says: 'We are both knaves.'

b Alice says: 'We are both of the same kind.' Bob says: 'We are of a different kind.'

c Alice says: 'Bob is a knave.' Bob says: 'Neither of us is a knave.'

13 The diagram shows that 9 can be written as the difference of two squares: $9 = 5^2 - 4^2$.



a Draw another diagram to show that 11 can be written as the difference of two squares.

b Prove that every odd number can be written as the difference of two squares.

c Hence, express 101 as the difference of two squares.

14 **a** Consider the numbers $\frac{9}{10}$ and $\frac{10}{11}$. Which is larger?

b Let n be a natural number. Prove that $\frac{n}{n+1} > \frac{n-1}{n}$.

15 **a** Prove that

$$\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$$

b Let $n > 0$. Prove that

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

16 Let $a, b \in \mathbb{R}$. Prove that $\frac{a^2 + b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$.

17 **a** Expand $(x - y)(x^2 + xy + y^2)$.

b Prove that $x^2 + yx + y^2 \geq 0$ for all $x, y \in \mathbb{R}$.

Hint: Complete the square by thinking of y as a constant.

c Hence, prove that if $x \geq y$, then $x^3 \geq y^3$.

18 Sally travels from home to work at a speed of 12 km/h and immediately returns home at a speed of 24 km/h.

a Show that her average speed is 16 km/h.

b Now suppose that Sally travels to work at a speed of a km/h and immediately returns home at a speed of b km/h. Show that her average speed is $\frac{2ab}{a+b}$ km/h.

c Let a and b be any two positive real numbers. Prove that

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

Note: This proves that Sally's average speed for the whole journey can be no greater than the average of her speeds for the two individual legs of the journey.

6B Proof by contrapositive

The negation of a statement

To **negate** a statement P we write its very opposite, which we call '**not** P'. For example, consider the following four statements and their negations.

P		not P	
The sky is green.	(false)	The sky is not green.	(true)
$1 + 1 = 2$	(true)	$1 + 1 \neq 2$	(false)
All prime numbers are odd.	(false)	There exists an even prime number.	(true)
All triangles have three sides.	(true)	Some triangle does not have three sides.	(false)

Notice that negation turns a true statement into a false statement, and a false statement into a true statement.

Example 6

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $2 > 1$

b 5 is divisible by 3

c The sum of any two odd numbers is even.

d There are two primes whose product is 12.

Solution

a P: $2 > 1$ (true)

not P: $2 \leq 1$ (false)

b P: 5 is divisible by 3 (false)

not P: 5 is not divisible by 3 (true)

c P: The sum of any two odd numbers is even. (true)

not P: There are two odd numbers whose sum is odd. (false)

d P: There are two primes whose product is 12. (false)

not P: There are no two primes whose product is 12. (true)

De Morgan's laws

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

De Morgan's laws

not (P *and* Q) is the same as (not P) *or* (not Q)

not (P *or* Q) is the same as (not P) *and* (not Q)

Example 7

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 6 is divisible by 2 and 3

b 10 is divisible by 2 or 7

Solution

a P: 6 is divisible by 2 and 6 is divisible by 3 (true)

not P: 6 is not divisible by 2 or 6 is not divisible by 3 (false)

b P: 10 is divisible by 2 or 10 is divisible by 7 (true)

not P: 10 is not divisible by 2 and 10 is not divisible by 7 (false)

Example 8

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave or Bob is a knight.' What are Alice and Bob?

Solution

We will prove that Alice is a knight and Bob is a knight.

Case 1

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice is a knight AND Bob is a knave.

$\Rightarrow$ Alice is a knight and a knave.

This is impossible.

Case 2

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice is a knave OR Bob is a knight.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knight and Bob must be a knight.

Proof by contrapositive

Consider this statement:

Statement	If	it is the end of term	then	the students are happy.

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

Contrapositive	If	the students are *not* happy	then	it is *not* the end of term.

Note that the original statement and its contrapositive are logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.

- The **contrapositive** of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P).
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

Example 9

Let $n \in \mathbb{Z}$ and consider this statement: If n^2 is even, then n is even.

a Write down the contrapositive.

b Prove the contrapositive.

Solution

a If n is odd, then n^2 is odd.

b Assume that n is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$. Squaring n gives

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 2m \in \mathbb{Z} \end{aligned}$$

Therefore n^2 is odd.

Note: Although we proved the contrapositive, remember that we have actually proved that if n^2 is even, then n is even.

Example 10

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then n is odd.

a Write down the contrapositive.

b Prove the contrapositive.

Solution

a If n is even, then $n^2 + 4n + 1$ is odd.

b Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 4n + 1 &= (2m)^2 + 4(2m) + 1 \\ &= 4m^2 + 8m + 1 \\ &= 2(2m^2 + 4m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 4m \in \mathbb{Z} \end{aligned}$$

Hence $n^2 + 4n + 1$ is odd.

Example 11

Let x and y be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

a Write down the contrapositive.

b Prove the contrapositive.

Solution

a If $\sqrt{x} \geq \sqrt{y}$, then $x \geq y$.

b Assume that $\sqrt{x} \geq \sqrt{y}$. Then $x \geq y$ by Example 3, since $\sqrt{x}$ and $\sqrt{y}$ are positive.

Summary 6B

- To **negate** a statement we write its opposite.
- For a statement $P \Rightarrow Q$, the **contrapositive** is the statement (not Q) $\Rightarrow$ (not P). That is, we switch the hypothesis and the conclusion and negate both.
- A statement and its contrapositive are logically equivalent.
- Proving the contrapositive of a statement may be easier than giving a direct proof.

Skill-sheet

Exercise 6B

Example 6

1 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $1 > 0$

b 4 is divisible by 8

c Each pair of primes has an even sum.

d Some rectangle has four sides of equal length.

Example 7

2 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 14 is divisible by 7 and 2

b 12 is divisible by 3 or 4

c 15 is divisible by 3 and 6

d 10 is divisible by 2 or 3

Example 8

3 Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave and Bob is a knight.' What are Alice and Bob?

Example 9

4 Write down the contrapositive version of each of these statements:

a If it is raining, then there are clouds in the sky.

b If you are smiling, then you are happy.

c If $x = 1$, then $2x = 2$.

d If $x > y$, then $x^5 > y^5$.

e Let $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

f Let $m, n \in \mathbb{Z}$. If m and n are odd, then mn is odd.

g Let $m, n \in \mathbb{Z}$. If $m + n$ is even, then m and n are either both even or both odd.

Example 10

5 Let $m, n \in \mathbb{Z}$. For each of the following statements, write down and prove the contrapositive statement:

a If $3n + 5$ is even, then n is odd.

b If n^2 is odd, then n is odd.

c If $n^2 - 8n + 3$ is even, then n is odd.

d If n^2 is not divisible by 3, then n is not divisible by 3.

e If $n^3 + 1$ is even, then n is odd.

f If mn is not divisible by 3, then m is not divisible by 3 and n is not divisible by 3.

g If $m + n$ is odd, then $m \neq n$.

6 Let $x, y \in \mathbb{R}$. For each of the following statements, write down and prove the contrapositive statement:

a If $x^2 + 3x < 0$, then $x < 0$.

b If $x^3 - x > 0$, then $x > -1$.

c If $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

d If $2x + 3y \geq 12$, then $x \geq 3$ or $y \geq 2$.

7 Let $m, n \in \mathbb{Z}$ and consider this statement: If mn and $m + n$ are even, then m and n are even.

a Write down the contrapositive.

b Prove the contrapositive. You will have to consider cases.

Example 11

8 Let x and y be positive real numbers.

a Prove that

$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

b Hence, prove that if $x > y$, then $\sqrt{x} > \sqrt{y}$.

c Give a simpler proof by considering the contrapositive.

6C Proof by contradiction

There are various instances when we want to prove mathematically that something cannot be done. To do this, we assume that it can be done, and then show that something goes horribly wrong. Let's first look at a familiar example from geometry.

Example 12

An angle is called **reflex** if it exceeds 180°. Prove that no quadrilateral has more than one reflex angle.

Solution

If there is more than one reflex angle, then the angle sum must exceed $2 \times 180° = 360°$. This contradicts the fact that the angle sum of any quadrilateral is exactly 360°. Therefore there cannot be more than one reflex angle.

The example above is a demonstration of a **proof by contradiction**. The basic outline of a proof by contradiction is:

1. Assume that the statement we want to prove is false.
2. Show that this assumption leads to mathematical nonsense.
3. Conclude that we were wrong to assume that the statement is false.
4. Conclude that the statement must be true.

Example 13

A **Pythagorean triple** consists of three natural numbers (a, b, c) satisfying

$$a^2 + b^2 = c^2$$

Show that if (a, b, c) is a Pythagorean triple, then a, b and c cannot all be odd numbers.

Solution

This will be a proof by contradiction.

Let (a, b, c) be a Pythagorean triple. Then $a^2 + b^2 = c^2$.

Suppose that a, b and c are all odd numbers.
$\Rightarrow$ a^2, b^2 and c^2 are all odd numbers.
$\Rightarrow$ $a^2 + b^2$ is even and c^2 is odd.

Since $a^2 + b^2 = c^2$, this gives a contradiction.

Therefore a, b and c cannot all be odd numbers.

Possibly the most well-known proof by contradiction is the following.

Theorem

$\sqrt{2}$ is irrational.

Proof This will be a proof by contradiction.

Suppose that $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}$$

We can assume that p and q have no common factors (or else they could be cancelled). Then, squaring both sides and rearranging gives

$$p^2 = 2q^2 \qquad (1)$$

$\Rightarrow$ p^2 is divisible by 2

$\Rightarrow$ p is divisible by 2 (by Example 9)

$\Rightarrow$ $p = 2n$ for some $n \in \mathbb{Z}$

$\Rightarrow$ $(2n)^2 = 2q^2$ (substituting into (1))

$\Rightarrow$ $q^2 = 2n^2$

$\Rightarrow$ q^2 is divisible by 2

$\Rightarrow$ q is divisible by 2 (by Example 9)

Therefore both p and q are divisible by 2, which contradicts the fact that they have no common factors.

Hence $\sqrt{2}$ is irrational.

Example 14

Suppose x satisfies $5^x = 2$. Show that x is irrational.

Solution

Suppose that x is rational. Since x must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned} 5^x = 2 \quad &\Rightarrow \quad 5^{\frac{m}{n}} = 2 \\ &\Rightarrow \quad \left(5^{\frac{m}{n}}\right)^n = 2^n \qquad \text{(raise both sides to the power } n\text{)} \\ &\Rightarrow \quad 5^m = 2^n \end{aligned}$$

The left-hand side of this equation is odd and the right-hand side is even. This gives a contradiction, and so x is not rational.

We finish on a remarkable result, which is attributed to Euclid some 2300 years ago.

Theorem

There are infinitely many prime numbers.

Proof This is a proof by contradiction, so we will suppose that there are only finitely many primes. This means that we can create a list that contains *every* prime number:

$$2,\ 3,\ 5,\ 7,\ \dots,\ p$$

where p is the largest prime number.

Now for the trick. We a create a new number N by multiplying each number in the list and then adding 1:

$$N = 2 \times 3 \times 5 \times 7 \times \cdots \times p + 1$$

The number N is not divisible by any of the primes $2, 3, 5, 7, \dots, p$, since it leaves a remainder of 1 when divided by any of these numbers.

However, every natural number greater than 1 is divisible by a prime number. (This is proved in Question 13 of Exercise 6F.) Therefore N is divisible by some prime number q. But this prime number q is not in the list $2, 3, 5, 7, \dots, p$, contradicting the fact that our list contains every prime number.

Hence there are infinitely many prime numbers.

Summary 6C

- A **proof by contradiction** is used to prove that something cannot be done.
- These proofs always follow the same basic structure:
  1. Assume that the statement we want to prove is false.
  2. Show that this assumption leads to mathematical nonsense.
  3. Conclude that we were wrong to assume that the statement is false.
  4. Conclude that the statement must be true.

Skill-sheet

Exercise 6C

Example 12

1 Prove that every triangle has some interior angle with a magnitude of at least 60°.

2 Prove that there is no smallest positive rational number.

3 Let p be a prime number. Show that $\sqrt{p}$ is not an integer.

Example 14

4 Suppose that $3^x = 2$. Prove that x is irrational.

5 Prove that $\log_2 5$ is irrational.

6 Suppose that $x > 0$ is irrational. Prove that $\sqrt{x}$ is also irrational.

7 Suppose that a is rational and b is irrational. Prove that $a + b$ is irrational.

8 Suppose that $c^2 - b^2 = 4$. Prove that b and c cannot both be natural numbers.

9 Let a, b and c be real numbers with $a \neq 0$. Prove by contradiction that there is only one solution to the equation $ax + b = c$.

10 **a** Prove that all primes $p > 2$ are odd.

b Hence, prove that there are no two primes whose sum is 1001.

11 **a** Prove that there are no integers a and b for which $42a + 7b = 1$.
Hint: The left-hand side is divisible by 7.

b Prove that there are no integers a and b for which $15a + 21b = 2$.

12 **a** Prove that if n^2 is divisible by 3, then n is divisible by 3.
Hint: Prove the contrapositive by considering two cases.

b Hence, prove that $\sqrt{3}$ is irrational.

13 **a** Prove that if n^3 is divisible by 2, then n is divisible by 2.
Hint: Prove the contrapositive.

b Hence, prove that $\sqrt[3]{2}$ is irrational.

14 Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

15 **a** Let $a, b, n \in \mathbb{N}$. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

b Hence, show that 97 is a prime number.

16 **a** Let m be an integer. Prove that m^2 is divisible by 4 or leaves a remainder of 1.
Hint: Suppose that $m = 4n + r$ and consider m^2 for $r = 0, 1, 2, 3$.

b Let $a, b, c \in \mathbb{Z}$. Prove by contradiction: If $a^2 + b^2 = c^2$, then a is even or b is even.

17 **a** Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.

b Hence, find $c, d \in \mathbb{Z}$ if $\sqrt{3 + 2\sqrt{2}} = c + d\sqrt{2}$. Hint: Square both sides.

18 Let $a, b, c \in \mathbb{Z}$. Prove that if a, b and c are all odd, then the equation $ax^2 + bx + c = 0$ cannot have a rational solution.

6D Equivalent statements

The converse of a statement

At the beginning of this chapter, we proved Pythagoras' theorem. Consider any triangle with side lengths a, b and c.

Statement	If	the angle between a and b is 90°	then	$a^2 + b^2 = c^2$.

By switching the hypothesis and the conclusion, we obtain the **converse** statement:

Converse	If	$a^2 + b^2 = c^2$	then	the angle between a and b is 90°.

For this example, the converse is also a true statement.

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

Note: The converse of a true statement may not be true. For example:

Statement	If it is raining, then there are clouds in the sky.	(true)
Converse	If there are clouds in the sky, then it is raining.	(false)

Example 15

Let x and y be positive real numbers. Consider the statement: If $x < y$, then $x^2 < y^2$.

a Write down the converse of this statement.

b Prove the converse.

Solution

a If $x^2 < y^2$, then $x < y$.

b Assume that $x^2 < y^2$. Then, since both x and y are positive,

$$\begin{aligned} & x^2 - y^2 < 0 && \text{(subtract } y^2\text{)} \\ \Rightarrow\quad & (x - y)(x + y) < 0 && \text{(factorising)} \\ \Rightarrow\quad & x - y < 0 && \text{(divide both sides by } x + y > 0\text{)} \\ \Rightarrow\quad & x < y \end{aligned}$$

as required.

Example 16

Let m and n be integers. Consider the statement: If m and n are even, then $m + n$ is even.

a Write down the converse of this statement.

b Show that the converse is not true.

Solution

a If $m + n$ is even, then m is even and n is even.

b Clearly $1 + 3 = 4$ is even, although 1 and 3 are not.

Equivalent statements

Now consider the following two statements:

P: your heart is beating

Q: you are alive

Notice that both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that P and Q are **equivalent** statements and we write

$$P \Leftrightarrow Q$$

We will also say that P is true **if and only if** Q is true. So in the above example, we can say

Your heart is beating	if and only if	you are alive.

To prove that two statements P and Q are equivalent, you have to prove two things:

$$P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P$$

Example 17

Let $n \in \mathbb{Z}$. Prove that n is even if and only if $n + 1$ is odd.

Solution

($\Rightarrow$) Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$.
Therefore $n + 1 = 2m + 1$, and so $n + 1$ is odd.

($\Leftarrow$) Assume that $n + 1$ is odd. Then $n + 1 = 2m + 1$ for some $m \in \mathbb{Z}$.
Subtracting 1 from both sides gives $n = 2m$. Therefore n is even.

Note: To prove that $P \Leftrightarrow Q$, we have to show that $P \Rightarrow Q$ and $P \Leftarrow Q$. When we are about to prove $P \Rightarrow Q$, we write ($\Rightarrow$). When we are about to prove $P \Leftarrow Q$, we write ($\Leftarrow$).

Summary 6D

- For a statement $P \Rightarrow Q$, the **converse** is the statement $Q \Rightarrow P$.
  That is, we switch the hypothesis and the conclusion.
- If $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is true, then we say that P is **equivalent** to Q, or that P is true **if and only if** Q is true.
- If P and Q are equivalent, we write $P \Leftrightarrow Q$.

Skill-sheet

Exercise 6D

Example 15

1 Write down and prove the converse of each of these statements:

a Let $x \in \mathbb{R}$. If $2x + 3 = 5$, then $x = 1$.

b Let $n \in \mathbb{Z}$. If n is odd, then $n - 3$ is even.

c Let $m \in \mathbb{Z}$. If $m^2 + 2m + 1$ is even, then m is odd.

d Let $n \in \mathbb{Z}$. If n^2 is divisible by 5, then n is divisible by 5.

Example 16

2 Let m and n be integers. Consider the statement: If m and n are even, then mn is a multiple of 4.

a Write down the converse of this statement.

b Show that the converse is not true.

3 Which of these pairs of statements are equivalent?

a P: Vivian is in China.
Q: Vivian is in Asia.

b P: $2x = 4$
Q: $x = 2$

c P: $x > 0$ and $y > 0$
Q: $xy > 0$

d P: m is even or n is even, where $m, n \in \mathbb{Z}$
Q: mn is even, where $m, n \in \mathbb{Z}$

Example 17

4 Let n be an integer. Prove that $n + 1$ is odd if and only if $n + 2$ is even.

5 Let $n \in \mathbb{N}$. Prove that $n^2 - 4$ is a prime number if and only if $n = 3$.

6 Let n be an integer. Prove that n^3 is even if and only if n is even.

7 Let n be an integer. Prove that n is odd if and only if $n = 4k \pm 1$ for some $k \in \mathbb{Z}$.

8 Let $x, y \in \mathbb{R}$. Prove that $(x + y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

9 Let m and n be integers.

a By expanding the right-hand side, prove that $m^3 - n^3 = (m - n)(m^2 + mn + n^2)$.

b Hence, prove that $m - n$ is even if and only if $m^3 - n^3$ is even.

10 Prove that an integer is divisible by 4 if and only if the number formed by its last two digits is divisible by 4. Hint: 100 is divisible by 4.

6E Disproving statements

Quantification using 'for all' and 'there exists'

For all

Universal quantification claims that a property holds for *all* members of a given set. For example, consider this statement:

Statement	For all natural numbers n, we have $2n \geq n + 1$.

To prove that this statement is true, we need to give a general argument that applies to every natural number n.

There exists

Existential quantification claims that a property holds for *at least one* member of a given set. For example, consider this statement:

Statement	There exists an integer m such that $m^2 = 25$.

To prove that this statement is true, we just need to give an example: $5 \in \mathbb{Z}$ with $5^2 = 25$.

Example 18

Rewrite each statement using either 'for all' or 'there exists':

a Some real numbers are irrational.

b Every integer that is divisible by 4 is also divisible by 2.

Solution

a There exists $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$.

b For all $m \in \mathbb{Z}$, if m is divisible by 4, then m is divisible by 2.

Negating 'for all' and 'there exists'

To negate a statement involving a quantifier, we interchange 'for all' with 'there exists' and then negate the rest of the statement.

Example 19

Write down the negation of each of the following statements:

a For all natural numbers n, we have $2n \geq n + 1$.

b There exists an integer m such that $m^2 = 4$ and $m^3 = -8$.

Solution

a There exists a natural number n such that $2n < n + 1$.

b For all integers m, we have $m^2 \neq 4$ or $m^3 \neq -8$.

Note: For part **b**, we used one of De Morgan's laws.

Notation

The words ‘for all’ can be abbreviated using the *turned A* symbol, $\forall$. The words ‘there exists’ can be abbreviated using the *turned E* symbol, $\exists$. For example:

- ‘For all natural numbers n, we have $2n \geq n + 1$’ can be written as $(\forall n \in \mathbb{N})\ 2n \geq n + 1$.
- ‘There exists an integer m such that $m^2 = 25$’ can be written as $(\exists m \in \mathbb{Z})\ m^2 = 25$.

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.

Counterexamples

Consider the quadratic function $f(n) = n^2 - n + 11$. Notice how $f(n)$ is a prime number for small natural numbers n:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	11	13	17	23	31	41	53	67	83	101

From this, we might be led to believe that the following statement is true:

Statement	For all natural numbers n, the number $f(n)$ is prime.

We call this a **universal statement**, because it asserts the truth of a statement without exception. So to disprove a universal statement, we need only show that it is not true in some particular instance. For our example, we need to find $n \in \mathbb{N}$ such that $f(n)$ is not prime. Luckily, we do not have to look very hard.

Example 20

Let $f(n) = n^2 - n + 11$. Disprove this statement: For all $n \in \mathbb{N}$, the number $f(n)$ is prime.

Solution

When $n = 11$, we obtain

$$f(11) = 11^2 - 11 + 11 = 11^2$$

Therefore $f(11)$ is not prime.

To disprove a statement of the form $P \Rightarrow Q$, we simply need to give one example for which P is true and Q is not true. Such an example is called a **counterexample**.

Example 21

Find a counterexample to disprove this statement: For all $x, y \in \mathbb{R}$, if $x > y$, then $x^2 > y^2$.

Solution

Let $x = 1$ and $y = -2$. Clearly $1 > -2$, but $1^2 = 1 \leq 4 = (-2)^2$.

Disproving existence statements

Consider this statement:

Statement	There exists $n \in \mathbb{N}$ such that $n^2 + 3n + 2$ is a prime number.

We call this an **existence statement**, because it claims the existence of an object possessing a particular property. To show that such a statement is false, we prove that its negation is true:

Negation	For all $n \in \mathbb{N}$, the number $n^2 + 3n + 2$ is not a prime number.

This is easy to prove, as

$$n^2 + 3n + 2 = (n + 1)(n + 2)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 22

Disprove this statement: There exists $n \in \mathbb{N}$ such that $n^2 + 13n + 42$ is a prime number.

Solution

We need to prove that, for all $n \in \mathbb{N}$, the number $n^2 + 13n + 42$ is not prime.

This is true, since

$$n^2 + 13n + 42 = (n + 6)(n + 7)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 23

Show that this statement is false: There exists some real number x such that $x^2 = -1$.

Solution

We have to prove that the negation is true: For *all* real numbers x, we have $x^2 \neq -1$.

This is easy to prove since, for any real number x, we have $x^2 \geq 0$ and so $x^2 \neq -1$.

Summary 6E

- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- A universal statement of the form $P \Rightarrow Q$ can be disproved by giving one example of an instance when P is true but Q is not.
- Such an example is called a **counterexample**.
- To disprove an existence statement, we prove that its negation is true.

Skill-sheet

Exercise 6E

Example 18

1 Which of the following are universal statements ('for all') and which are existence statements ('there exists')?

- **a** For each $n \in \mathbb{N}$, the number $5n^2 + 3n + 7$ is odd.
- **b** There is an even prime number.
- **c** Every natural number greater than 1 has a prime factorisation.
- **d** All triangles have three sides.
- **e** Some natural numbers are primes.
- **f** At least one real number x satisfies the equation $x^2 - x - 1 = 0$.
- **g** Any positive real number has a square root.
- **h** The angle sum of a triangle is 180°.

2 Which of the following statements are true and which are false?

- **a** There exists a real number x such that $x^2 = 2$.
- **b** There exists a real number x such that $x^2 < 0$.
- **c** For all natural numbers n, the number $2n - 1$ is odd.
- **d** There exists $n \in \mathbb{N}$ such that $2n$ is odd.
- **e** For all $x \in \mathbb{R}$, we have $x^3 \geq 0$.

Example 19

3 Write down the negation of each of the following statements:

- **a** For every natural number n, the number $2n^2 - 4n + 31$ is prime.
- **b** For all $x \in \mathbb{R}$, we have $x^2 > x$.
- **c** There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.
- **d** For all $x, y \in \mathbb{R}$, we have $(x + y)^2 = x^2 + y^2$.
- **e** There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 > y^2$.

4 Prove that each of the following statements is false by finding a counterexample:

Example 20

- **a** For every natural number n, the number $2n^2 - 4n + 31$ is prime.
- **b** If $x, y \in \mathbb{R}$, then $(x + y)^2 = x^2 + y^2$.

Example 21

- **c** For all $x \in \mathbb{R}$, we have $x^2 > x$.
- **d** Let $n \in \mathbb{Z}$. If $n^3 - n$ is even, then n is even.
- **e** If $m, n \in \mathbb{N}$, then $m + n \leq mn$.
- **f** Let $m, n \in \mathbb{Z}$. If 6 divides mn, then 6 divides m or 6 divides n.

5 Show that each of the following existence statements is false:

Example 22

- **a** There exists $n \in \mathbb{N}$ such that $9n^2 - 1$ is a prime number.
- **b** There exists $n \in \mathbb{N}$ such that $n^2 + 5n + 6$ is a prime number.

Example 23

- **c** There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.

6 Provide a counterexample to disprove each of the following statements.
Hint: $\sqrt{2}$ might come in handy.

a If a is irrational and b is irrational, then ab is irrational.

b If a is irrational and b is irrational, then $a + b$ is irrational.

c If a is irrational and b is irrational, then $\frac{a}{b}$ is irrational.

7 Let $a \in \mathbb{Z}$.

a Prove that if a is divisible by 4, then a^2 is divisible by 4.

b Prove that the converse is not true.

8 Let $a, b \in \mathbb{Z}$.

a Prove that if $a - b$ is divisible by 3, then $a^2 - b^2$ is divisible by 3.

b Prove that the converse is not true.

9 Prove that each of the following statements is false:

a There exist real numbers a and b such that $a^2 - 2ab + b^2 = -1$.

b There exists some real number x such that $x^2 - 4x + 5 = \frac{3}{4}$.

10 The numbers $\{1, 2, \dots, 8\}$ can be paired so that the sum of each pair is a square number:

$$1 + 8 = 9, \quad 2 + 7 = 9, \quad 3 + 6 = 9, \quad 4 + 5 = 9$$

a Prove that you can also do this with the numbers $\{1, 2, \dots, 16\}$.

b Prove that you cannot do this with the numbers $\{1, 2, \dots, 12\}$.

11 Let $f(n) = an^2 + bn + c$ be a quadratic function, where a, b, c are natural numbers and $c \geq 2$. Show that there is an $n \in \mathbb{N}$ such that $f(n)$ is not a prime number.

6F Mathematical induction

Consider the sum of the first n odd numbers:

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1 + 3 &= 4 = 2^2 \\ 1 + 3 + 5 &= 9 = 3^2 \\ 1 + 3 + 5 + 7 &= 16 = 4^2 \end{aligned}$$

From this limited number of examples, we could make the following proposition $P(n)$ about the number n: the sum of the first n odd numbers is n^2. Since the nth odd number is $2n - 1$, we can write this proposition as

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

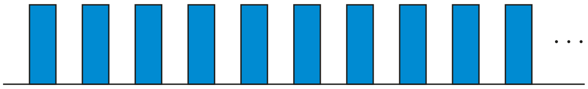
However, we have to be careful here: Just because something looks true does not mean that it is true. In this section, we will learn how to prove statements like the one above.

The principle of mathematical induction

Imagine a row of dominoes extending infinitely to the right. Each of these dominoes can be knocked over provided two conditions are met:

1 The first domino is knocked over.

2 Each domino is sufficiently close to the next domino.



This scenario provides an accurate physical model of the following proof technique.

Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number n.

We can prove that $P(n)$ is true for every natural number n as follows:

a Show that $P(1)$ is true.

b Show that, for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1) \text{ is true} \quad \Rightarrow \quad P(2) \text{ is true} \quad \Rightarrow \quad P(3) \text{ is true} \quad \Rightarrow \quad \cdots$$

Let's see how mathematical induction is used in practice.

Example 24

Prove that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Step 1 $P(1)$ is the proposition $1 = 1^2$, that is, $1 = 1$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$1 + 3 + 5 + \cdots + (2k-1) = k^2$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = (k+1)^2$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.

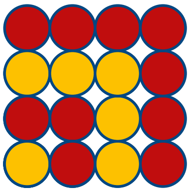
We have

$$\begin{aligned} \text{LHS of } P(k+1) &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\ &= k^2 + (2k+1) \qquad \text{(using } P(k)\text{)} \\ &= (k+1)^2 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n.

While mathematical induction is good for proving that formulas are true, it rarely indicates why they should be true in the first place. The formula $1 + 3 + 5 + \cdots + (2n-1) = n^2$ can be discovered in the diagram shown on the right.



Example 25

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$7^n - 4$ is divisible by 3

Step 1 $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. Clearly, $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3. We have

$$\begin{aligned} 7^{k+1} - 4 &= 7 \times 7^k - 4 \\ &= 7(3m+4) - 4 \qquad \text{(using } P(k)\text{)} \\ &= 21m + 28 - 4 \\ &= 21m + 24 \\ &= 3(7m+8) \end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

Proving inequalities

Induction can be used to prove certain inequalities.

For example, consider this table of values:

n	1	2	3	4	5
3^n	3	9	27	81	243
3×2^n	6	12	24	48	96

From the table, it certainly looks as though

$$3^n > 3 \times 2^n \quad \text{for all } n \geq 3$$

We will prove this formally using induction; this time starting with $P(3)$ instead of $P(1)$.

Example 26

Prove that $3^n > 3 \times 2^n$ for every natural number $n \geq 3$.

Solution

For each natural number $n \geq 3$, let $P(n)$ be the proposition:

$$3^n > 3 \times 2^n$$

Step 1 $P(3)$ is the proposition $3^3 > 3 \times 2^3$, that is, $27 > 24$. Therefore $P(3)$ is true.

Step 2 Let k be a natural number with $k \geq 3$, and assume $P(k)$ is true. That is,

$$3^k > 3 \times 2^k$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$3^{k+1} > 3 \times 2^{k+1}$$

We have

$$\begin{aligned}\text{LHS of } P(k+1) &= 3^{k+1}\\ &= 3 \times 3^k\\ &> 3 \times 3 \times 2^k \qquad \text{(using } P(k)\text{)}\\ &> 3 \times 2 \times 2^k\\ &= 3 \times 2^{k+1}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k \geq 3$.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number $n \geq 3$.

Applications to sequences

Induction proofs are also frequently used in the study of sequences.

Consider the sequence defined by the recurrence relation

$$t_{n+1} = 10t_n - 9, \quad t_1 = 11$$

The first five terms of this sequence are listed in the following table.

n	1	2	3	4	5
t_n	11	101	1001	10 001	100 001

Notice that each of these terms is one more than a power of 10. Let's see if we can prove that this is true for *every* term in the sequence.

Example 27

Given $t_1 = 11$ and $t_{n+1} = 10t_n - 9$, prove that $t_n = 10^n + 1$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$$t_n = 10^n + 1$$

Step 1 Since $t_1 = 11$ and $10^1 + 1 = 11$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$t_k = 10^k + 1$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$t_{k+1} = 10^{k+1} + 1$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= t_{k+1} \\ &= 10t_k - 9 \\ &= 10 \times (10^k + 1) - 9 \qquad \text{(using } P(k)\text{)} \\ &= 10^{k+1} + 10 - 9 \\ &= 10^{k+1} + 1 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k + 1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n.

Tower of Hanoi

You have three pegs and a collection of n discs of different sizes. Initially, all the discs are stacked in size order on the left-hand peg. Discs can be moved one at a time from one peg to any other peg, provided that a larger disc never rests on a smaller one. The aim of the puzzle is to transfer all the discs to another peg using the smallest possible number of moves.

Example 28

Let a_n be the minimum number of moves needed to solve the Tower of Hanoi with n discs.

a Find a formula for a_{n+1} in terms of a_n.

b Evaluate a_n for $n = 1, 2, 3, 4, 5$. Guess a formula for a_n in terms of n.

c Confirm your formula for a_n using mathematical induction.

d If $n = 20$, how many days are needed to transfer all the discs to another peg, assuming that one disc can be moved per second?

Solution

a Suppose there are $n + 1$ discs on the left-hand peg.

If we want to be able to move the largest disc to the right-hand peg, then first we must transfer the other n discs to the centre peg. This takes a minimum of a_n moves.

It takes 1 move to transfer the largest disc to the right-hand peg. Now we can complete the puzzle by transferring the n discs on the centre peg to the right-hand peg. This takes a minimum of a_n moves.

Hence the minimum number of moves required to transfer all the discs is

$$\begin{aligned} a_{n+1} &= a_n + 1 + a_n \\ &= 2a_n + 1 \end{aligned}$$

b We have $a_1 = 1$, since one disc can be moved in one move. Using the recurrence relation from part **a**, we find that

$$a_2 = 2a_1 + 1 = 2 \times 1 + 1 = 3$$
$$a_3 = 2a_2 + 1 = 2 \times 3 + 1 = 7$$

Continuing in this way, we obtain the following table.

n	1	2	3	4	5
a_n	1	3	7	15	31

It seems as though every term is one less than a power of 2. We guess that

$$a_n = 2^n - 1$$

c For each natural number n, let $P(n)$ be the proposition:

$$a_n = 2^n - 1$$

Step 1 The minimum number of moves required to solve the Tower of Hanoi puzzle with one disc is 1. Since $2^1 - 1 = 1$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$a_k = 2^k - 1$$

Step 3 We now wish to prove that $P(k+1)$ is true, that is,

$$a_{k+1} = 2^{k+1} - 1$$

We have

$$\begin{aligned}
\text{LHS of } P(k+1) &= a_{k+1} \\
&= 2a_k + 1 && \text{(using part } \mathbf{a}\text{)} \\
&= 2 \times (2^k - 1) + 1 && \text{(using } P(k)\text{)} \\
&= 2^{k+1} - 2 + 1 \\
&= 2^{k+1} - 1 \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$. Hence we have shown that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

d A puzzle with 20 discs requires a minimum of $2^{20} - 1$ seconds.

Since there are $60 \times 60 \times 24 = 86\,400$ seconds in a day, it will take

$$\frac{2^{20} - 1}{86\,400} \approx 12.14 \text{ days}$$

to complete the puzzle.

Summary 6F

The basic outline of a proof by mathematical induction is:

0 Define the proposition $P(n)$ for $n \in \mathbb{N}$.

1 Show that $P(1)$ is true.

2 Assume that $P(k)$ is true for some $k \in \mathbb{N}$.

3 Show that $P(k+1)$ is true.

4 Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Skill-sheet

Exercise 6F

Example 24

1 Prove each of the following by mathematical induction:

a $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

b $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$, where $x \neq 1$

c $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

d $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$

e $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

f $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\cdots\left(1 - \dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$, for $n \geq 2$

Example 25

2 Prove each of the following divisibility statements by mathematical induction:

a $11^n - 1$ is divisible by 10 for all $n \in \mathbb{N}$

b $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$

c $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$

d $5^n + 6 \times 7^n + 1$ is divisible by 4 for all $n \in \mathbb{N}$

Example 26

3 Prove each of the following inequalities by mathematical induction:

a $4^n > 10 \times 2^n$ for all integers $n \geq 4$

b $3^n > 5 \times 2^n$ for all integers $n \geq 5$

c $2^n > 2n$ for all integers $n \geq 3$

d $2^n \geq n^2$ for all integers $n \geq 4$

Example 27

4 Prove each of the following statements by mathematical induction:

a If $a_{n+1} = 2a_n - 1$ and $a_1 = 3$, then $a_n = 2^n + 1$.

b If $a_{n+1} = 5a_n + 4$ and $a_1 = 4$, then $a_n = 5^n - 1$.

c If $a_{n+1} = 2a_n - n + 1$ and $a_1 = 3$, then $a_n = 2^n + n$.

5 Prove that 3^n is odd for every $n \in \mathbb{N}$.

6 **a** Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^2 - n$.

7 **a** Prove by mathematical induction that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^3 - n$.

8 Consider the sequence defined by $a_{n+1} = 10a_n + 9$, where $a_1 = 9$.

a Find a_n for $n = 1, 2, 3, 4, 5$.

b Guess a formula for a_n in terms of n.

c Confirm that your formula is valid by using mathematical induction.

9 The Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$.

a Find f_n for $n = 1, 2, \dots, 10$.

b Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

c Evaluate $f_1 + f_3 + \cdots + f_{2n-1}$ for $n = 1, 2, 3, 4$.

d Try to find a formula for the above expression.

e Confirm that your formula works using mathematical induction.

f Using induction, prove that every third Fibonacci number, f_{3n}, is even.

10 Prove that $4^n + 5^n$ is divisible by 9 for all odd integers n.

11 Prove by induction that, for all $n \in \mathbb{N}$, every set of numbers S with exactly n elements has a largest element.

12 Standing around a circle, there are n friends and n thieves. You begin with no money, but as you go around the circle clockwise, each friend will give you \$1 and each thief will steal \$1. Prove that no matter where the friends and thieves are placed, it is possible to walk once around the circle without going into debt, provided you start at the correct point.

13 Prove by induction that every natural number $n \geq 2$ is divisible by some prime number.
Hint: Let $P(n)$ be the statement that every integer j such that $2 \leq j \leq n$ is divisible by some prime number.

14 If n straight lines are drawn across a sheet of paper, they will divide the paper into regions. Show that it is always possible to colour each region black or white, so that no two adjacent regions have the same colour.

Review

Chapter summary

Assign-ment

Nrich

- A **conditional statement** has the form: If P is true, then Q is true. This can be abbreviated as $P \Rightarrow Q$, which is read 'P implies Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements P and Q are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- The **contrapositive** of $P \Rightarrow Q$ is (not Q) $\Rightarrow$ (not P).
- Proving the contrapositive of a statement may be easier than giving a direct proof.
- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- **Counterexamples** can be used to demonstrate that a universal statement is false.
- **Mathematical induction** is used to prove that a statement is true for all natural numbers.

Technology-free questions

1 For each of the following statements, if the statement is true, then prove it, and otherwise give a counterexample to show that it is false:

a The sum of any three consecutive integers is divisible by 3.

b The sum of any four consecutive integers is divisible by 4.

2 Assume that n is even. Prove that $n^2 - 3n + 1$ is odd.

3 Let $n \in \mathbb{Z}$. Consider the statement: If n^3 is even, then n is even.

a Write down the contrapositive of this statement.

b Prove the contrapositive.

c Hence, prove by contradiction that $\sqrt[3]{6}$ is irrational.

4 **a** Show that one of three consecutive integers is always divisible by 3.

b Hence, prove that $n^3 + 3n^2 + 2n$ is divisible by 3 for all $n \in \mathbb{Z}$.

5 **a** Suppose that both m and n are divisible by d. Prove that $m - n$ is divisible by d.

b Hence, prove that the highest common factor of two consecutive integers is 1.

c Find the highest common factor of 1002 and 999.

6 A student claims that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, for all $x \geq 0$ and $y \geq 0$.

a Using a counterexample, prove that the equation is not always true.

b Prove that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ if and only if $x = 0$ or $y = 0$.

7 Let $n \in \mathbb{Z}$. Prove that $n^2 + 3n + 4$ is even.
Hint: Consider the cases when n is odd and n is even.

8 Suppose that a, b, c and d are positive integers.

a Provide a counterexample to disprove the equation
$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

b Now suppose that $\frac{c}{d} > \frac{a}{b}$. Prove that
$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

9 Prove by mathematical induction that:

a $6^n + 4$ is divisible by 10 for all $n \in \mathbb{N}$

b $1^2 + 3^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \in \mathbb{N}$

Multiple-choice questions

1 If m is even and n is odd, then which of these statements is true?

A $m + 2n$ is odd **B** $m + n$ is even **C** $m \times n$ is odd
D $m^2 - n^2$ is even **E** $m - 3n$ is odd

2 If m is divisible by 6 and n is divisible by 15, then which of these statements might be false?

A $m \times n$ is divisible by 90 **B** $m \times n$ is divisible by 30 **C** $m \times n$ is divisible by 15
D $m + n$ is divisible by 3 **E** $m + n$ is divisible by 15

3 The contrapositive of $P \Rightarrow Q$ is

A $Q \Rightarrow P$ **B** (not P) $\Rightarrow$ (not Q) **C** (not Q) $\Rightarrow$ (not P)
D $Q \Leftrightarrow P$ **E** (not P) $\Leftrightarrow$ (not Q)

4 The converse of $P \Rightarrow Q$ is

A (not Q) $\Rightarrow$ (not P) **B** $Q \Rightarrow P$ **C** $Q \Leftrightarrow Q$
D (not P) $\Leftrightarrow$ (not Q) **E** (not Q) $\Leftrightarrow$ (not P)

5 If a, b and c are any real numbers with $a > b$, the statement that must be true is

A $\frac{1}{a} > \frac{1}{b}$ **B** $\frac{1}{a} < \frac{1}{b}$ **C** $ac > bc$ **D** $a + c > b + c$ **E** $a^2 > b^2$

6 If $n = (m-1)(m-2)(m-3)$ where m is an integer, then n will not always be divisible by

A 1 **B** 2 **C** 3 **D** 5 **E** 6

7 Let $m, n \in \mathbb{Z}$. Which of the following statements is false?

A n is even if and only if $n + 1$ is odd

B $m + n$ is odd if and only if $m - n$ is odd

C $m + n$ is even if and only if m and n are even

D m and n are odd if and only if mn is odd

E mn is even if and only if m is even or n is even

8 Consider the statement: For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y^2 = x$. Which of the following provides a counterexample to this statement?

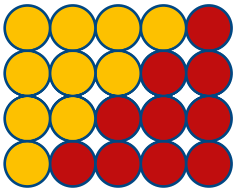
A $x = 2,\ y = \sqrt{2}$ **B** $x = \sqrt{2},\ y = 2$ **C** $x = -1$

D $x = 0$ **E** $y = -1$

Extended-response questions

1 **a** Use the diagram on the right to deduce the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad (1)$$



b Using equation (1), prove that the sum $1 + 2 + \cdots + 99$ is divisible by 99.

c Using equation (1), prove that if n is odd, then the sum of any n consecutive odd natural numbers is divisible by n.

d With the help of equation (1) and mathematical induction, prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 \quad \text{for all } n \in \mathbb{N}$$

2 For each natural number n, define $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

a Prove that $10! + 2, 10! + 3, \ldots, 10! + 10$ are nine consecutive composite numbers.
Hint: The first number is divisible by 2.

b Find a sequence of ten consecutive composite numbers.

3 We call (a, b, c) a Pythagorean triple if a, b, c are natural numbers such that $a^2 + b^2 = c^2$.

a Let $n \in \mathbb{N}$. Prove that if (a, b, c) is a Pythagorean triple, then so is (na, nb, nc).

b Prove that there is only one Pythagorean triple (a, b, c) of consecutive natural numbers.

c Prove that there is no Pythagorean triple (a, b, c) containing the numbers 1 or 2.

4 Let a be an integer that is not divisible by 3. We know that $a = 3k + 1$ or $a = 3k + 2$, for some $k \in \mathbb{Z}$.

a Show that a^2 must leave a remainder of 1 when divided by 3.

b Hence, prove that if (a, b, c) is any Pythagorean triple, then a or b is divisible by 3.

5 **a** Prove by mathematical induction that $n^2 + n$ is even for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^2 + n$.

c Hence, prove that if n is odd, then there exists an integer k such that $n^2 = 8k + 1$.

6 Let $n \in \mathbb{Z}$ and consider the statement: If n is divisible by 8, then n^2 is divisible by 8.

a Prove the statement.

b Write down the converse of the statement.

c If the converse is true, prove it. Otherwise, give a counterexample.

7 Goldbach's conjecture is that every even integer greater than 2 can be expressed as the sum of two primes. To date, no one has been able to prove this, although it has been verified for all integers less than 4×10^{18}.

a Express 100 and 102 as the sum of two prime numbers.

b Prove that 101 cannot be written as the sum of two prime numbers.

c Express 101 as the sum of three prime numbers.

d Assuming that Goldbach's conjecture is true, prove that every odd integer greater than 5 can be written as the sum of three prime numbers.

8 **a** Simplify the expression $\dfrac{1}{n-1} - \dfrac{1}{n}$.

b Hence, show that

$$\frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \frac{1}{4 \times 3} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$$

c Give another proof of the above equation using mathematical induction.

d Using the above equation, prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 \quad \text{for all } n \in \mathbb{N}$$

9 **a** Let $x \geq 0$ and $y \geq 0$. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

by substituting $x = a^2$ and $y = b^2$ into $\dfrac{x+y}{2} - \sqrt{xy}$.

b Using the above inequality, or otherwise, prove each of the following:

i If $a > 0$, then $a + \dfrac{1}{a} \geq 2$.

ii If a, b and c are positive real numbers, then $(a+b)(b+c)(c+a) \geq 8abc$.

iii If a, b and c are positive real numbers, then $a^2 + b^2 + c^2 \geq ab + bc + ca$.

c Take any rectangle of length x and width y. Prove that a square with the same perimeter has an area greater than or equal to that of the rectangle.

10 Exactly one of the following three people is lying. Who is the liar?

- Jay says: 'Kaye is lying.'
- Kaye say: 'Elle is lying.'
- Elle says: 'I am not lying.'

11 There are four sentences written below. Which of them is true?

- Exactly one of these statements is false.
- Exactly two of these statements are false.
- Exactly three of these statements are false.
- Exactly four of these statements are false.

12 We will say that a set of numbers can be **split** if it can be divided into two groups so that no two numbers appear in the same group as their sum. For example, the set $\{1, 2, 3, 4, 5, 6\}$ can be split into the two groups $\{1, 2, 4\}$ and $\{3, 5, 6\}$.

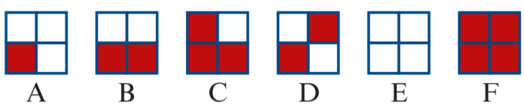
a Prove that the set $\{1, 2, \ldots, 8\}$ can be split.

b Hence, explain why the set $\{1, 2, \ldots, n\}$ can be split, where $1 \leq n \leq 8$.

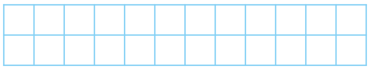
c Prove that it is impossible to split the set $\{1, 2, \ldots, 9\}$.

d Hence, prove that it is impossible to split the set $\{1, 2, \ldots, n\}$, where $n \geq 9$.

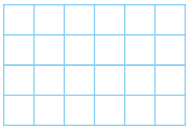
13 Consider the set of six 2×2 square tiles shown below.



a Tile the 2×12 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



b Prove that there are only four ways to tile the 4×6 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



7 Logic

Objectives

- To understand the Boolean operations ∨, ∧, ′ and the axioms of Boolean algebra.
- To understand the concept of a **statement** and its **truth value**.
- To use **logical connectives** to form compound statements.
- To construct **truth tables** for compound statements.
- To use truth tables to check the **validity** of arguments.
- To represent circuits using **logic gates**.
- To use **Karnaugh maps** to simplify Boolean expressions.

The words ‘or’, ‘and’, ‘not’, ‘true’ and ‘false’ are central to this chapter. You may have already met these words in your studies of sets, probability and proofs. In this chapter, these ideas are brought together in a formal way as **Boolean algebra**.

In Chapter 2, we looked at sets and operations on sets, including union ∪, intersection ∩ and complementation ′. This chapter begins by studying these operations on the set of all subsets of a given set. Such structures are examples of Boolean algebras.

We will see that the ideas of Boolean algebra lead to the formal study of logic. Some of the concepts introduced in Chapter 6 will reappear in this chapter, including negation, De Morgan’s laws, implication, converse and contrapositive.

Early work in logic was carried out by George Boole (1815–1864) in his book *An Investigation of the Laws of Thought*, published in 1854. His aims in the book were to

> . . . investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method.

The ideas of Boolean algebra were first applied to electrical circuits in the 1930s, notably by Claude Shannon (1916–2001). Today, Boolean algebra forms the foundation of computer science, and is central to electronics and programming.

7A The algebra of sets

In this section, we prove general statements involving the set operations of union, intersection and complementation. Throughout this chapter, you will see analogous statements arising in different contexts.

Basic set notation

We begin by revising set notation, which was introduced in Section 2A.

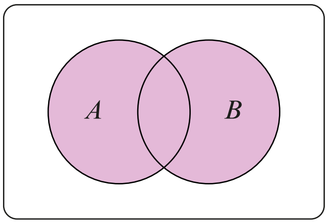
A set is any collection of objects where order is not important.

- The set of all the elements being considered in a given context is called the **universal set** and is denoted by ξ.
- The set with no elements is called the **empty set** and is denoted by $\emptyset$.
- We say that a set B is a **subset** of a set A if each element of B is also in A. In this case, we write $B \subseteq A$. Note that $\emptyset \subseteq A$ and $A \subseteq A$.

The following three operations on sets play an important role in this chapter.

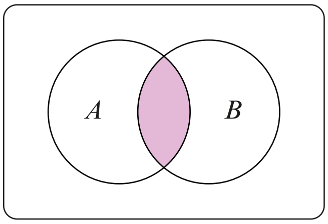
The **union** of sets A and B is denoted by $A \cup B$ and consists of all elements that belong to A or B:

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$



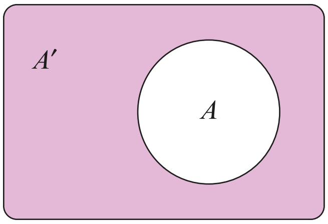
The **intersection** of sets A and B is denoted by $A \cap B$ and consists of all elements that belong to both A and B:

$$A \cap B = \{ x : x \in A \text{ and } x \in B \}$$



The **complement** of a set A is denoted by A' and consists of all elements of the universal set ξ that are not in A:

$$A' = \{ x \in \xi : x \notin A \}$$



The set of all subsets of a set

In this section, we consider the operations $\cup$, $\cap$ and $'$ on the set of all subsets of a set.

We will prove in Section 9G that a set with n elements has 2^n subsets.

For example, consider the universal set $\xi = \{a, b, c, d\}$. This set has 16 subsets:

- the empty set: $\emptyset$
- the subsets of size 1: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$
- the subsets of size 2: $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$
- the subsets of size 3: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$
- the universal set: ξ.

We can make the following observations for these sets and, more generally, for any such collection of subsets. We will illustrate how to prove these laws in Example 2. You will see similar laws reoccurring throughout this chapter.

Laws of the algebra of sets

For $A, B, C \subseteq \xi$, the following are true:

Primary
- $A \cup A = A$
- $A \cap A = A$
- $A \cup \emptyset = A$
- $A \cap \xi = A$
- $A \cup \xi = \xi$
- $A \cap \emptyset = \emptyset$

Associativity
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity
- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Distributivity
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Absorption
- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

Complements
- $A \cup A' = \xi$
- $A \cap A' = \emptyset$
- $\emptyset' = \xi$
- $\xi' = \emptyset$
- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$
- $(A')' = A$

You may notice that some of these laws are similar to laws of arithmetic involving the operations + and $\times$ and the numbers 0 and 1. For example:

Algebra of sets	$A \cup \emptyset = A$	$A \cap \xi = A$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Algebra of numbers	$a + 0 = a$	$a \times 1 = a$	$a \times (b + c) = a \times b + a \times c$

You may also notice that all but one of these laws occurs as a member of a pair. The two laws in each pair are called **dual statements**.

Dual statements

For a given statement about sets, the dual statement is obtained by interchanging:

$\cup$ with $\cap$, $\varnothing$ with ξ, $\subseteq$ with $\supseteq$

Example 1

Write the dual of $(A \cap B') \cap B = \varnothing$.

Solution

$(A \cup B') \cup B = \xi$

In the next example, we show a method for proving that two sets are equal. We also illustrate the result with a Venn diagram. However, note that we cannot prove a result using a Venn diagram, just as in geometry we cannot prove a result by simply drawing a diagram.

Two sets A and B are **equal** if they have exactly the same elements. That is, each element of set A also belongs to set B, and each element of set B also belongs to set A.

Equality for sets

When proving that two sets are equal, we can use the following equivalence:

$$X \subseteq Y \text{ and } Y \subseteq X \iff X = Y$$

Example 2

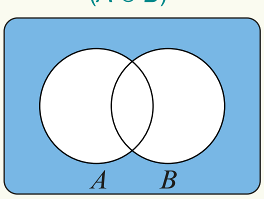
a Illustrate $(A \cup B)' = A' \cap B'$ with Venn diagrams.

b Prove that $(A \cup B)' = A' \cap B'$.

Solution

a We first draw the diagram for $(A \cup B)'$.

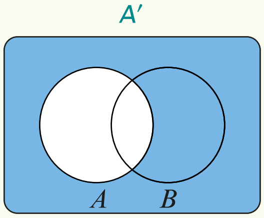
$(A \cup B)'$



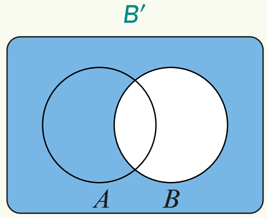
Note: Required regions are shaded.

To help draw the diagram for $A' \cap B'$, we draw diagrams for A' and B'.

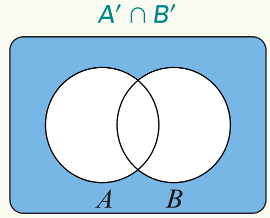
A'



B'



$A' \cap B'$



The diagrams for $(A \cup B)'$ and $A' \cap B'$ are the same.

b We must show that $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$.

Let $x \in \xi$.

i $x \in (A \cup B)' \Rightarrow x \notin A \cup B$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \in A' \cap B'$

Hence $(A \cup B)' \subseteq A' \cap B'$.

ii $x \in A' \cap B' \Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \notin A \cup B$
$\Rightarrow x \in (A \cup B)'$

Hence $A' \cap B' \subseteq (A \cup B)'$.

Note: The proof of **ii** can be obtained from the proof of **i** by reversing the steps. This can be shown by using the equivalence symbol, $\Leftrightarrow$, at each step in the proof of **i**.

The next example shows how we can use the algebra of sets to simplify expressions involving $\cup$, $\cap$ and $'$. In Section 7C, we will see similar calculations in a more general context.

Example 3

Simplify each of the following expressions:

a $X \cap (Y \cap X')$

b $X' \cup (Y \cap X)$

c $[X' \cup (Y \cap X)]'$

d $[(X \cap Y') \cup (X \cap Y)]'$

Solution

We use the laws of the algebra of sets, which are listed earlier in this section. The justification of each step is given on the right-hand side.

a

$X \cap (Y \cap X') = X \cap (X' \cap Y)$	by commutativity
$= (X \cap X') \cap Y$	by associativity
$= \emptyset \cap Y$	as $A \cap A' = \emptyset$
$= Y \cap \emptyset$	by commutativity
$= \emptyset$	as $A \cap \emptyset = \emptyset$

b

$X' \cup (Y \cap X) = (X' \cup Y) \cap (X' \cup X)$	by distributivity
$= (X' \cup Y) \cap (X \cup X')$	by commutativity
$= (X' \cup Y) \cap \xi$	as $A \cup A' = \xi$
$= X' \cup Y$	as $A \cap \xi = A$

c

$[X' \cup (Y \cap X)]' = X'' \cap (Y \cap X)'$	as $(A \cup B)' = A' \cap B'$
$= X \cap (Y' \cup X')$	as $A'' = A$ and $(A \cap B)' = A' \cup B'$
$= (X \cap Y') \cup (X \cap X')$	by distributivity
$= (X \cap Y') \cup \emptyset$	as $A \cap A' = \emptyset$
$= X \cap Y'$	as $A \cup \emptyset = A$

d $[(X \cap Y') \cup (X \cap Y)]' = (X \cap Y')' \cap (X \cap Y)'$ as $(A \cup B)' = A' \cap B'$

$= (X' \cup Y'') \cap (X' \cup Y')$ as $(A \cap B)' = A' \cup B'$

$= (X' \cup Y) \cap (X' \cup Y')$ as $A'' = A$

$= X' \cup (Y \cap Y')$ by distributivity

$= X' \cup \varnothing$ as $A \cap A' = \varnothing$

$= X'$ as $A \cup \varnothing = A$

Summary 7A

- In the **algebra of sets**, we consider general statements involving the operations $\cup$, $\cap$ and $'$ on the set of all subsets of a given set ξ. For example:
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup A' = \xi$
  - $A \cap A' = \varnothing$
- For a given statement about sets, the **dual statement** is obtained by interchanging:

  $\cup$ with $\cap$, $\varnothing$ with ξ, $\subseteq$ with $\supseteq$
- When proving that two sets are **equal**, we can use the following equivalence:

  $X \subseteq Y$ and $Y \subseteq X \Leftrightarrow X = Y$

Exercise 7A

1 Let $\xi = \{7, 8, 9, 10, 11\}$, $A = \{7, 9, 10, 11\}$ and $B = \{8, 9\}$.
Show these sets on a Venn diagram and use the diagram to find:

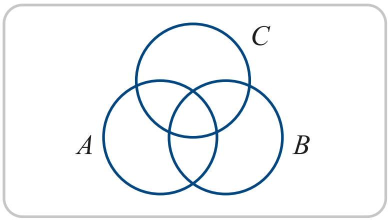
a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

2 Define two subsets of $\mathbb{N}$:
- X is the set of all natural numbers less than or equal to 5
- Y is the set of all prime numbers.

For each of the following, describe the given subset of $\mathbb{N}$ in terms of X and Y by using suitable set operations:

a $\{2, 3, 5\}$ **b** $\{1, 4\}$ **c** the set of all composite numbers greater than 5

3 Draw this diagram six times. Use shading to illustrate each of the following sets:



a $A \cap B \cap C$ **b** $A \cap B \cap C'$
c $A \cap B' \cap C'$ **d** $A' \cap B' \cap C'$
e $A \cup (B \cap C)$ **f** $(A \cup B) \cap (A \cup C)$

4 Define three subsets of $\mathbb{N}$:

- A is the set of all natural numbers less than or equal to 6
- B is the set of all even numbers
- C is the set of all multiples of 3.

For each of the following, describe the given subset of $\mathbb{N}$ in terms of A, B and C by using suitable set operations:

a $\{3, 6\}$

b $\{1, 3, 5\}$

c the set of all multiples of 6

d the set of all even numbers greater than 6

e the set containing all multiples of 3 and all odd numbers

Example 1

5 Write the dual of each of the following statements:

a $(A \cup \xi) \cap (A \cap \varnothing) = \varnothing$

b If $A \cap B = \varnothing$, then $A' \cup B = A'$.

c $A \cap B \subseteq A \cup B$

Example 2b

6 Prove each of the following results:

a $A \cup B = B \cup A$

b $A \cap B = B \cap A$

c $(A \cap B)' = A' \cup B'$

d $(A \cup B) \cap (A \cup B') = A$

e $A = (A \cap B) \cup (A \cap B')$

f $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

g $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example 3

7 Use the laws of the algebra of sets to simplify each of the following expressions:

a $X \cup (Y \cup X)$

b $(Y \cup Y') \cap Y$

c $X \cap (X' \cap Y)$

d $X \cap (Y \cup X)$

e $X \cup (Y' \cap X)$

f $[X' \cup (Y \cap Z)]'$

g $(X' \cup Y')'$

h $(X' \cap Y')'$

i $(X \cap Y') \cap (X' \cap Y')$

j $(X \cap Y) \cup (X \cap Y')$

k $[(X \cup Y) \cap (X \cup Y')]'$

l $(X \cup Y') \cap [(X \cap Z) \cup (X \cap Z')]'$

8 Prove each of the following:

a If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

b If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

c $A \subseteq B \Leftrightarrow B' \subseteq A'$

9 **Set difference** is defined as

$$A \setminus B = A \cap B' = \{\, x : x \in A \text{ and } x \notin B \,\}$$

Prove each of the following results:

a $P \setminus (Q \setminus R) = (P \setminus Q) \cup (P \cap R)$

b $P \cap (Q \setminus R) = (P \cap Q) \setminus (P \cap R)$

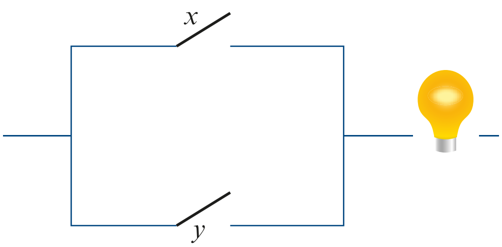
7B Switching circuits

Claude Shannon applied the ideas of logic to electrical circuits. In this chapter, we follow the opposite path. We first introduce switching circuits, and then in the next section we see how they can be interpreted using Boolean algebra.

Switches in parallel

The following diagram shows two switches x and y in **parallel**. If at least one of the switches is closed, then current flows and the light is on. In this case, we say that the system is closed.

The four possible situations for two switches in parallel are summarised in the table.



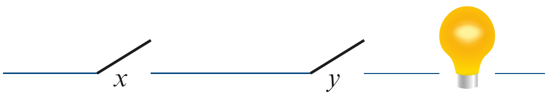
Switches x and y in parallel

x	y	State of system
open	open	open
open	closed	closed
closed	open	closed
closed	closed	closed

Switches in series

The following diagram shows two switches x and y in **series**. If both switches are closed, then current flows and the light is on. In this case, we say that the system is closed.

The four possible situations for two switches in series are summarised in the table.



Switches x and y in series

x	y	State of system
open	open	open
open	closed	open
closed	open	open
closed	closed	closed

Complementary switches

The **complement switch** x' is always in the opposite state to x.

x	x'
open	closed
closed	open

New notation

We introduce a new notation that will be used throughout this chapter in different contexts.

- Use 0 for open.
- Use 1 for closed.
- Use the notation $x \vee y$, read as 'x or y', for two switches x and y connected in parallel.
- Use the notation $x \wedge y$, read as 'x and y', for two switches x and y connected in series.
- Use x' for the complement of x.

Using this new notation, we now reproduce the tables for switches in parallel, switches in series and complementary switches.

Or (parallel)

x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

And (series)

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

Complement

x	x'
0	1
1	0

We now have three operations $\vee$, $\wedge$ and $'$ acting on the set $\{0, 1\}$.

Example 4

Evaluate $(1 \vee 0) \wedge 1'$.

Solution

$$\begin{aligned}(1 \vee 0) \wedge 1' &= 1 \wedge 1' \\ &= 1 \wedge 0 \\ &= 0\end{aligned}$$

This new notation allows us to represent more complicated switching circuits. You will see the notation used again in the next section in a more general context.

Example 5

Consider the expression

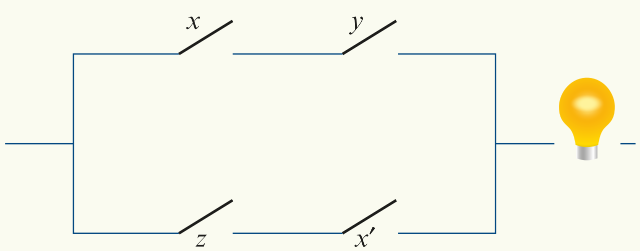
$$(x \wedge y) \vee (z \wedge x')$$

a Draw the switching circuit that is represented by this expression.

b Give a table that describes the operation of this circuit for all possible combinations of switches x, y and z being open (0) and closed (1).

Solution

a Switches x and y are connected in series, as are switches z and x'. These two pairs of switches are then connected in parallel.

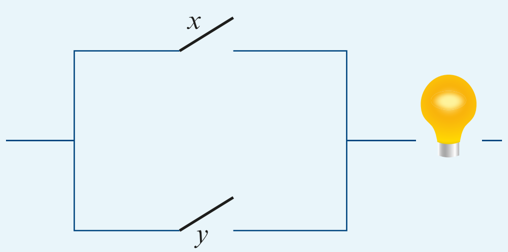


b For three variables x, y and z, there are $2^3 = 8$ possible combinations of 0s and 1s. This gives the first three columns of the table. To find the value of $(x \wedge y) \vee (z \wedge x')$ in each case, we start by finding the values of the simpler expressions $x \wedge y$ and $z \wedge x'$.

x	y	z	x'	$x \wedge y$	$z \wedge x'$	$(x \wedge y) \vee (z \wedge x')$
0	0	0	1	0	0	0
0	0	1	1	0	1	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	1	0	1
1	1	1	0	1	0	1

Summary 7B

- Use 0 to represent a switch being open (i.e. off).
- Use 1 to represent a switch being closed (i.e. on).
- 'Or' (switches in parallel)



x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

- 'And' (switches in series)

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

- 'Not' (complementary switches)
  The complement switch x' is always in the opposite state to x.

x	x'
0	1
1	0

Exercise 7B

Example 4

1 Evaluate each of the following:

a $1 \vee 0'$ **b** $1' \wedge 0$ **c** $1' \vee 0'$ **d** $(1 \wedge 0)'$

e $(1 \vee 0) \vee 1'$ **f** $0 \wedge (1' \vee 0)$ **g** $(1' \vee 1) \wedge (1 \vee 0)$ **h** $(1 \vee 0) \wedge (1' \vee 0)'$

2 Each of the following tables describes the operation of a switching circuit. Complete each table using 0s and 1s:

a

x	y	y'	$x \vee y'$
0	0		
0	1		
1	0		
1	1		

b

x	y	y'	$x \wedge y'$
0	0		
0	1		
1	0		
1	1		

c

x	y	x'	y'	$x' \wedge y'$
0	0			
0	1			
1	0			
1	1			

d

x	y	x'	y'	$x' \vee y'$
0	0			
0	1			
1	0			
1	1			

Example 5

3 For each of the following expressions:

i draw the switching circuit that is represented by the expression

ii give the table with entries 0s and 1s describing the operation of this circuit.

a $x \vee (y \wedge z)$ **b** $(x \wedge y) \vee z$ **c** $x \wedge (y \vee z)$

d $(x \vee y) \wedge (x \wedge y)$ **e** $(x \vee y') \wedge (y \vee z)$ **f** $(x \wedge y) \vee ((z \vee x) \wedge y')$

4 Draw the switching circuits and give the tables for $x \wedge (x \vee y)$ and $x \vee (x \wedge y)$. Hence show that each of these circuits is equivalent to the circuit with one switch x.

5 **a** Draw the switching circuits and give the tables for $x \wedge (y \vee z)$ and $(x \wedge y) \vee (x \wedge z)$. Hence show that these circuits are equivalent to each other.

b Draw the switching circuits and give the tables for $x \vee (y \wedge z)$ and $(x \vee y) \wedge (x \vee z)$. Hence show that these circuits are equivalent to each other.

c Draw the switching circuits and give the tables for $(x \wedge y) \vee z$ and $(x \vee y) \wedge z$. Hence show that these circuits are not equivalent to each other.

6 Draw the switching circuit for each of the following:

a $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x' \wedge y \wedge z)$

b $(x \wedge y \wedge z) \vee (x \wedge y \wedge (z \vee w \vee (u \wedge v))$

c $x \wedge ((y \wedge (z \vee w)) \vee (z \wedge (u \vee v)))$

7C Boolean algebra

We have now seen two different examples of a Boolean algebra:

- the set of all subsets of a set together with the operations $\cup$, $\cap$ and $'$
- the set $\{0, 1\}$ together with the operations $\vee$, $\wedge$ and $'$.

In general, a **Boolean algebra** is a set B with operations $\vee$, $\wedge$, $'$ and distinguished elements 0, 1 such that the following axioms are satisfied, for all $x, y, z \in B$:

Axiom 1	$x \vee y = y \vee x$	($\vee$ is commutative)
	$x \wedge y = y \wedge x$	($\wedge$ is commutative)
Axiom 2	$(x \vee y) \vee z = x \vee (y \vee z)$	($\vee$ is associative)
	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	($\wedge$ is associative)
Axiom 3	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	($\vee$ distributes over $\wedge$)
	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	($\wedge$ distributes over $\vee$)
Axiom 4	$x \vee 0 = x$	(0 is the identity for $\vee$)
	$x \wedge 1 = x$	(1 is the identity for $\wedge$)
Axiom 5	$x \vee x' = 1$	($'$ is complementation)
	$x \wedge x' = 0$	

Note: As in Section 7B, the symbols $\vee$ and $\wedge$ can be read as 'or' and 'and' respectively. Since the operations $\vee$ and $\wedge$ are also analogous to the operations of union $\cup$ and intersection $\cap$, the symbols $\vee$ and $\wedge$ are also read as 'join' and 'meet'.

Given these axioms, we can prove general results that are true for any Boolean algebra.

Example 6

Prove that, for each Boolean algebra B and all $x, y, a, b \in B$:

a $x \vee 1 = 1$

b $x \wedge 0 = 0$

c If $a \vee b = 1$ and $a \wedge b = 0$, then $a' = b$.

d $(x \vee y)' = x' \wedge y'$

Solution

a

$x \vee 1 = (x \vee 1) \wedge 1$	(axiom 4)
$= (x \vee 1) \wedge (x \vee x')$	(axiom 5)
$= x \vee (1 \wedge x')$	(axiom 3)
$= x \vee (x' \wedge 1)$	(axiom 1)
$= x \vee x'$	(axiom 4)
$= 1$	(axiom 5)

b $x \wedge 0 = (x \wedge 0) \vee 0$ (axiom 4)

$= (x \wedge 0) \vee (x \wedge x')$ (axiom 5)

$= x \wedge (0 \vee x')$ (axiom 3)

$= x \wedge (x' \vee 0)$ (axiom 1)

$= x \wedge x'$ (axiom 4)

$= 0$ (axiom 5)

Note: Part **b** is the dual of part **a**.

c Let $a \in B$. We know that $a \vee a' = 1$ and $a \wedge a' = 0$, by axiom 5. Assume that there is another element b which satisfies $a \vee b = 1$ and $a \wedge b = 0$. Then

$a' = a' \wedge 1$ (axiom 4)

$= a' \wedge (a \vee b)$ since $a \vee b = 1$

$= (a' \wedge a) \vee (a' \wedge b)$ (axiom 3)

$= (a \wedge a') \vee (a' \wedge b)$ (axiom 1)

$= 0 \vee (a' \wedge b)$ (axiom 5)

$= (a \wedge b) \vee (a' \wedge b)$ since $a \wedge b = 0$

$= (b \wedge a) \vee (b \wedge a')$ (axiom 1)

$= b \wedge (a \vee a')$ (axiom 3)

$= b \wedge 1$ (axiom 5)

$= b$ (axiom 4)

d We want to apply part **c** with $a = x \vee y$ and $b = x' \wedge y'$. To do this, we need to show that $a \vee b = 1$ and $a \wedge b = 0$. We have

$a \vee b = (x \vee y) \vee (x' \wedge y')$

$= ((x \vee y) \vee x') \wedge ((x \vee y) \vee y')$ (axiom 3)

$= (y \vee (x \vee x')) \wedge (x \vee (y \vee y'))$ (axioms 1 and 2)

$= (y \vee 1) \wedge (x \vee 1)$ (axiom 5)

$= 1 \wedge 1$ by part **a**

$= 1$ (axiom 4)

and $a \wedge b = (x \vee y) \wedge (x' \wedge y')$

$= (x' \wedge y') \wedge (x \vee y)$ (axiom 1)

$= ((x' \wedge y') \wedge x) \vee ((x' \wedge y') \wedge y)$ (axiom 3)

$= (y' \wedge (x \wedge x')) \vee (x' \wedge (y \wedge y'))$ (axioms 1 and 2)

$= (y' \wedge 0) \vee (x' \wedge 0)$ (axiom 5)

$= 0 \vee 0$ by part **b**

$= 0$ (axiom 4)

Hence $(x \vee y)' = x' \wedge y'$ by part **c**.

The set of all subsets of a set is a Boolean algebra with the operations $\cup$, $\cap$, $'$ and the identity elements $\varnothing$, ξ. All the laws for sets given in Section 7A have parallel results for Boolean algebras in general. Some of them are axioms, and the others can be proved from the axioms (see Example 6 and Question 2 in Exercise 7C).

Properties of Boolean algebras

Primary	$x \vee x = x$	$x \wedge x = x$
	$x \vee 0 = x$ (A4)	$x \wedge 1 = x$ (A4)
	$x \vee 1 = 1$	$x \wedge 0 = 0$
Associativity (A2)	$(x \vee y) \vee z = x \vee (y \vee z)$	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
Commutativity (A1)	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
Distributivity (A3)	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
Absorption	$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$
Complements	$x \vee x' = 1$ (A5)	$x \wedge x' = 0$ (A5)
	$0' = 1$	$1' = 0$
	$(x \vee y)' = x' \wedge y'$	$(x \wedge y)' = x' \vee y'$
	$(x')' = x$	

Note: The properties $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$ are called **De Morgan's laws**.

Boolean expressions and Boolean functions

A **Boolean expression** is an expression formed using $\vee$, $\wedge$, $'$, 0 and 1, such as $x \wedge (y \vee x)'$.

You are familiar with defining functions on the set $\mathbb{R}$ of real numbers. We can use Boolean expressions to define functions on the set $\{0, 1\}$. A simple example of a Boolean function is

$$f\colon \{0,1\} \to \{0,1\}, \quad f(x) = x \vee 1$$

In this case, we have $f(0) = 0 \vee 1 = 1$ and $f(1) = 1 \vee 1 = 1$.

In general, a **Boolean function** has one or more inputs from $\{0, 1\}$ and outputs in $\{0, 1\}$.

Example 7

Give the table of values for the Boolean function $f(x, y) = (x \wedge y) \vee y'$.

Solution

x	y	y'	$x \wedge y$	$f(x,y) = (x \wedge y) \vee y'$
0	0	1	0	1
0	1	0	0	0
1	0	1	0	1
1	1	0	1	1

The next example shows how to find a Boolean expression for a Boolean function given in table form. We don't give a formal proof, but you can easily verify the result by forming the table of values. In Section 7F, we will see the importance of this process in electronics.

Example 8

Find a Boolean expression for the Boolean function given by the following table.

	x	y	z	$f(x, y, z)$
1	0	0	0	0
2	0	0	1	0
3	0	1	0	1
4	0	1	1	0
5	1	0	0	1
6	1	0	1	0
7	1	1	0	1
8	1	1	1	1

Solution

We look at the rows of the table in which the output is 1. These are rows 3, 5, 7 and 8.

Each row corresponds to some combination using $\wedge$ of either x or x', either y or y', and either z or z'.

For example, row 3 corresponds to $x' \wedge y \wedge z'$. We use x for a 1-entry in the x-column and use x' for a 0-entry. The same rule is followed for the y- and z-columns.

- Row 3 $\quad x' \wedge y \wedge z'$
- Row 5 $\quad x \wedge y' \wedge z'$
- Row 7 $\quad x \wedge y \wedge z'$
- Row 8 $\quad x \wedge y \wedge z$

Now combine these terms using $\vee$. We obtain the Boolean expression

$$(x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$$

We can write the Boolean function as

$$f(x, y, z) = (x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$$

Note: You are already used to writing expressions such as $a + b + c$ without brackets. The brackets are not needed because

$$(a + b) + c = a + (b + c) \qquad \text{for all } a, b, c \in \mathbb{R}$$

That is, the operation $+$ is associative. The operations $\vee$ and $\wedge$ are also associative, and so we can write expressions such as $x \wedge y \wedge z$ and $w \vee x \vee y \vee z$ unambiguously without brackets.

Equivalent Boolean expressions

Two Boolean expressions are **equivalent** if they represent the same Boolean function.

If two Boolean expressions represent the same Boolean function, then it is possible to derive one expression from the other using the axioms of Boolean algebras.

Example 9

Consider the two Boolean expressions

$$((x \vee y) \wedge (x' \vee y)) \vee x \quad \text{and} \quad x \vee y$$

Show that these two expressions are equivalent by:

a showing that they represent the same Boolean function

b using the axioms and properties of Boolean algebras.

Solution

a

x	y	$x \vee y$	x'	$x' \vee y$	$(x \vee y) \wedge (x' \vee y)$	$((x \vee y) \wedge (x' \vee y)) \vee x$
0	0	0	1	1	0	0
0	1	1	1	1	1	1
1	0	1	0	0	0	1
1	1	1	0	1	1	1

Columns 3 and 7 of the table are the same, and therefore the two expressions determine the same Boolean function.

b

$$\begin{aligned} ((x \vee y) \wedge (x' \vee y)) \vee x &= ((y \vee x) \wedge (y \vee x')) \vee x && \text{(axiom 1)} \\ &= (y \vee (x \wedge x')) \vee x && \text{(axiom 3)} \\ &= (y \vee 0) \vee x && \text{(axiom 5)} \\ &= y \vee x && \text{(axiom 4)} \\ &= x \vee y && \text{(axiom 1)} \end{aligned}$$

Summary 7C

- A **Boolean algebra** is a set B with operations $\vee$, $\wedge$, $'$ and distinguished elements 0, 1 such that the following axioms are satisfied, for all $x, y, z \in B$:
  1. Commutativity $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$
  2. Associativity $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
  3. Distributivity $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
  4. Identities $x \vee 0 = x$ and $x \wedge 1 = x$
  5. Complementation $x \vee x' = 1$ and $x \wedge x' = 0$
- A **Boolean function** has one or more inputs from $\{0, 1\}$ and outputs in $\{0, 1\}$.
- Two Boolean expressions are **equivalent** if they represent the same Boolean function.

Exercise 7C

1 Use the properties of Boolean algebras to simplify each of the following expressions:

a $a \wedge (b \wedge a')$ **b** $(a \wedge b') \wedge a'$ **c** $a \vee (b \vee a')$

d $(a \vee b') \vee a'$ **e** $(a \vee b) \wedge a'$ **f** $a \vee (b \wedge a')$

g $a \wedge (b \vee a')$ **h** $(a \wedge b) \vee (a' \wedge b)$ **i** $(a \vee b) \vee (a' \vee b)$

Example 6

2 Each of the following gives a template for proving a property of Boolean algebras, where x and y are elements of a Boolean algebra B. Copy and complete each proof.

a *Proof that* $x \vee x = x$.

LHS $= x \vee x$
$= (x \vee x) \wedge 1$ (axiom 4)
$= (x \vee x) \wedge (x \vee x')$ (axiom 5)
$= x \vee (\square \wedge \square)$ (axiom 3)
$= x \vee \square$ (axiom $\square$)
$= x$ (axiom $\square$)
$=$ RHS

b *Proof that* $x \wedge x = x$.

LHS $= x \wedge x$
$= (x \wedge x) \vee 0$ (axiom $\square$)
$= (x \wedge x) \vee (x \wedge x')$ (axiom $\square$)
$= x \wedge (\square \vee \square)$ (axiom 3)
$= x \wedge \square$ (axiom $\square$)
$= x$ (axiom $\square$)
$=$ RHS

c *Proof that* $x \vee (x \wedge y) = x$.

LHS $= x \vee (x \wedge y)$
$= (x \wedge 1) \vee (x \wedge y)$ (axiom $\square$)
$= x \wedge (\square \vee y)$ (axiom $\square$)
$= x \wedge (y \vee \square)$ (axiom 1)
$= x \wedge \square$ (Example 6 **a**)
$= x$ (axiom $\square$)
$=$ RHS

d *Proof that* $x \wedge (x \vee y) = x$.

LHS $= x \wedge (x \vee y)$
$= (x \vee 0) \wedge (x \vee y)$ (axiom $\square$)
$= x \vee (\square \wedge y)$ (axiom $\square$)
$= x \vee (y \wedge \square)$ (axiom 1)
$= \square\ \square\ \square$ (Example 6 **b**)
$= \square$ (axiom $\square$)
$=$ RHS

e *Proof that* $0' = 1$.
We use the result from Example 6 **c** with $a = 0$ and $b = 1$. We have

$a \vee b = 0 \vee 1 = 1$ (axioms 1 and $\square$)
$a \wedge b = 0 \wedge 1 = 0$ (axiom $\square$)

Hence $0' = 1$.

f *Proof that* $1' = 0$.
We use the result from Example 6 **c** with $a = 1$ and $b = 0$. We have

$a \vee b = \square \vee \square = \square$ (axiom $\square$)
$a \wedge b = \square \wedge \square = \square$ (axioms 1 and $\square$)

Hence $1' = 0$.

g *Proof that* $(x')' = x$.
We use the result from Example 6 **c** with $a = x'$ and $b = x$. We have

$a \vee b = \square\ \square\ \square = \square$ (axioms $\square$ and $\square$)
$a \wedge b = \square\ \square\ \square = \square$ (axioms $\square$ and $\square$)

Hence $(x')' = x$.

3 Simplify $(b \wedge c') \wedge (d \wedge b')$ by using the commutativity and associativity axioms. Hence show that $a \vee ((b \wedge c') \wedge (d \wedge b')) = a$.

Example 7

4 For each of the following Boolean functions, produce a table of values:

a $f(x, y) = (x \vee y) \wedge x'$

b $f(x, y) = (x \vee y') \wedge (x' \vee y')$

c $f(x, y) = (x \wedge y') \wedge (x' \wedge y')$

d $f(x, y, z) = (x \wedge y') \vee z$

e $f(x, y, z) = (x \vee y) \wedge z$

f $f(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$

5 Draw a switching circuit for each of the following expressions. Try to simplify your circuit by first simplifying the expression using the properties of Boolean algebras.

a $(x \wedge y) \vee x$

b $(x \vee y) \wedge x$

c $(x \wedge y') \vee (x \wedge y')$

d $(x \wedge y') \vee (x' \wedge y')$

e $(x' \wedge y') \vee (x' \wedge z)$

f $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y)$

Example 8

6 For each of the following, find a Boolean expression for the given Boolean function:

a

x	y	$f(x, y)$
0	0	1
0	1	1
1	0	0
1	1	1

b

x	y	$f(x, y)$
0	0	1
0	1	1
1	0	0
1	1	0

c

x	y	z	$f(x, y, z)$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

d

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Example 9

7 Show that the expressions $(x' \vee y) \wedge (x \vee y')$ and $(x \wedge y) \vee (x' \wedge y')$ are equivalent by:

a showing that they represent the same Boolean function

b using the axioms and properties of Boolean algebras.

7D Logical connectives and truth tables

A **statement** is a sentence that is either true or false. Examples of statements are:

- The boy plays tennis.
- $5 + 7 = 12$
- $5 + 7 = 10$

Note that '5 + 7' is not a statement.

A statement can be assigned a **truth value**: T if it is true, or F if it is false. For example, the statement '$5 + 7 = 12$' is true (T), while the statement '$5 + 7 = 10$' is false (F).

A statement is often denoted by a capital letter such as A, B or C.

The logical connectives 'or', 'and', 'not'

Logical connectives enable statements to be combined together to form new statements.

In English, two sentences may be combined by a grammatical connective to form a new compound sentence. For example, consider the following sentences:

A Gary went to the cinema.

B Gary did **not** go to the cinema.

C Kay went to the cinema.

D Gary **and** Kay went to the cinema.

Statement B is the negation (not) of statement A. Statement D is the conjunction (and) of statements A and C. The same can be done using logical connectives.

We will consider two statements about an integer n.

- Let G be the statement 'n is odd'.
- Let H be the statement '$n > 10$'.

Given two statements, there are four possible combinations of truth values, as shown in the table on the right.

	G	H
1	T	T
2	T	F
3	F	T
4	F	F

Or

The symbol $\vee$ is used for 'or'.

- The statement $G \vee H$ is 'n is odd or $n > 10$'.

The statement $G \vee H$ is known as the **disjunction** of G and H. If either or both of the statements are true, then the compound statement is true. If both are false, then the compound statement is false. This is shown in the table on the right, which is called a **truth table**.

Truth table for 'or'

G	H	$G \vee H$
T	T	T
T	F	T
F	T	T
F	F	F

And

The symbol $\wedge$ is used for 'and'.

- The statement $G \wedge H$ is 'n is odd and $n > 10$'.

The statement $G \wedge H$ is known as the **conjunction** of G and H. If both statements are true, then the compound statement is true. If either or both are false, then the compound statement is false. This can be shown conveniently in a truth table.

Truth table for 'and'

G	H	$G \wedge H$
T	T	T
T	F	F
F	T	F
F	F	F

Not

The symbol ¬ is used for ‘not’.

- The statement $\neg G$ is ‘n is even’.
- The statement $\neg H$ is ‘$n \leq 10$’.

In general, the statement $\neg A$ is called the **negation** of A and has the opposite truth value to A.

Truth table for ‘not’

A	$\neg A$
T	F
F	T

Note: The negation operation ¬ corresponds to complementation in Boolean algebra. But in logic, it is common to use the notation $\neg A$ instead of A'.

Truth tables for compound statements

More complicated compound statements can be built up using these three connectives. For example:

- The statement $\neg G \wedge \neg H$ is ‘n is even and $n \leq 10$’.

We can use truth tables to find the truth values of compound statements.

To construct the truth table for $\neg G \wedge \neg H$, we first find the truth values for the simpler statements $\neg G$ and $\neg H$, and then use the truth table for $\wedge$. Note that $\neg G \wedge \neg H$ is true if and only if both $\neg G$ and $\neg H$ are true.

G	H	$\neg G$	$\neg H$	$\neg G \wedge \neg H$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example 10

Write the truth table for $\neg(A \vee B)$.

Solution

The truth values for the simpler statement $A \vee B$ are found first.

A	B	$A \vee B$	$\neg(A \vee B)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Example 11

Write the truth table for $(A \wedge B) \wedge (\neg A)$.

Solution

A	B	$A \wedge B$	$\neg A$	$(A \wedge B) \wedge (\neg A)$
T	T	T	F	F
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

Equivalent statements, tautologies and contradictions

Two statements that have the same truth values are **logically equivalent**.

Example 12

Show that $\neg(A \wedge B)$ is logically equivalent to $\neg A \vee \neg B$.

Solution

A	B	$\neg A$	$\neg B$	$A \wedge B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The truth values for $\neg(A \wedge B)$ and $\neg A \vee \neg B$ are the same, so the statements are equivalent.

- A **tautology** is a statement which is true under all circumstances.
- A **contradiction** is a statement which is false under all circumstances.

Example 13

Show that $(\neg A) \vee (A \vee B)$ is a tautology.

Solution

A	B	$\neg A$	$A \vee B$	$(\neg A) \vee (A \vee B)$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

Example 14

Show that $(A \vee B) \wedge (\neg A \wedge \neg B)$ is a contradiction.

Solution

A	B	$\neg A$	$\neg B$	$A \vee B$	$\neg A \wedge \neg B$	$(A \vee B) \wedge (\neg A \wedge \neg B)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Implication

We now consider another logical connective, called **implication**.

For statements A and B, we can form the compound statement $A \Rightarrow B$, which is read as 'A implies B' or as 'If A, then B'.

The truth table for $\Rightarrow$ is shown on the right.

Truth table for 'implies'

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

It is useful to consider whether this truth table is consistent with a 'common sense' view of implication.

- Let A be the statement 'I am elected'.
- Let B be the statement 'I will make public transport free'.

Therefore $A \Rightarrow B$ is the statement 'If I am elected, then I will make public transport free'.

Now consider each row of the truth table:

Row 1 I am elected (A is true) and public transport is made free (B is true). I have kept my election promise. The statement $A \Rightarrow B$ is true.

Row 2 I am elected (A is true) but public transport is not made free (B is false). I have broken my election promise. The statement $A \Rightarrow B$ is false.

Rows 3 & 4 I am not elected (A is false). Whether or not public transport is made free, I have not broken my promise, as I was not elected. The statement $A \Rightarrow B$ is true.

Note that the only possible way that $A \Rightarrow B$ could be false is if I am elected but do not make public transport free. Otherwise, the statement is not false, and therefore must be true.

In general, the statement $A \Rightarrow B$ is true, except when A is true and B is false.

The statement $A \Rightarrow B$ is logically equivalent to $\neg A \vee B$, as shown in the truth table on the right.

A	B	$\neg A$	$\neg A \vee B$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Example 15

Give the truth table for $B \Rightarrow (A \vee \neg B)$.

Solution

A	B	$\neg B$	$A \vee \neg B$	$B \Rightarrow (A \vee \neg B)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	T

Note: To complete the last column of the table, we look for the rows in which B is true and $A \vee \neg B$ is false. This is only row 3. In this case, the statement $B \Rightarrow (A \vee \neg B)$ is false. Otherwise, it is true.

Equivalence

Another logical connective is **equivalence**, which is represented by the symbol $\Leftrightarrow$. It has the truth table shown.

You can check using truth tables that the statement $A \Leftrightarrow B$ is logically equivalent to $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

Truth table for equivalence

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Example 16

Give the truth table for $(\neg A \vee \neg B) \Leftrightarrow \neg(A \wedge B)$.

Solution

A	B	$\neg A$	$\neg B$	$A \wedge B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$	$(\neg A \vee \neg B) \Leftrightarrow \neg(A \wedge B)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Note: This statement is a tautology. The two statements $\neg A \vee \neg B$ and $\neg(A \wedge B)$ are logically equivalent.

Converse and contrapositive

Let A be the statement 'You study Mathematics' and let B be the statement 'You study Physics'. The following two statements form a pair of **converse statements**:

- 'If you study Mathematics, then you study Physics.' ($A \Rightarrow B$)
- 'If you study Physics, then you study Mathematics.' ($B \Rightarrow A$)

The following two statements form a pair of **contrapositive statements**:

- 'If you study Mathematics, then you study Physics.' ($A \Rightarrow B$)
- 'If you do not study Physics, then you do not study Mathematics.' ($\neg B \Rightarrow \neg A$)

In general, for a conditional statement $A \Rightarrow B$:

- the **converse** statement is $B \Rightarrow A$
- the **contrapositive** statement is $\neg B \Rightarrow \neg A$.

Note: Using truth tables, you can check that a statement $A \Rightarrow B$ is equivalent to its contrapositive $\neg B \Rightarrow \neg A$, but is not equivalent to its converse $B \Rightarrow A$.

Negation of an implication

Again consider the conditional statement 'If you study Mathematics, then you study Physics'.

The only way this can be false is if you are studying Mathematics but not Physics. So the negation of the statement is 'You study Mathematics and you do not study Physics'.

For a conditional statement $A \Rightarrow B$, the negation of the statement is $A \wedge \neg B$.

Note: Using a truth table, you can check that $\neg(A \Rightarrow B)$ and $A \wedge \neg B$ are equivalent.

Example 17

For each of the following conditional statements:

i write the converse **ii** write the contrapositive **iii** write the negation.

a If you know the password, then you can get in.

b Let $n, a, b \in \mathbb{N}$. If n does not divide ab, then n does not divide a and n does not divide b.

Solution

a **i** Converse If you can get in, then you know the password.

ii Contrapositive If you cannot get in, then you do not know the password.

iii Negation You know the password and you cannot get in.

b **i** Converse If n does not divide a and n does not divide b, then n does not divide ab.

ii Contrapositive If n divides a or n divides b, then n divides ab.

iii Negation n does not divide ab and n divides a or b.

Note that statement **b** and its contrapositive are true, but its converse is false. (As a counterexample to the converse, we can take $n = 6$, $a = 2$ and $b = 3$.)

The Boolean algebra of statements

Consider the set S of all statements together with the operations $\vee$, $\wedge$ and $\neg$ and the special statements F (always false) and T (always true). We can view S as a Boolean algebra, provided we use equivalence ($\equiv$) instead of equality ($=$). For example, we have

$$A \vee \neg A \equiv \mathrm{T} \quad \text{and} \quad A \wedge \neg A \equiv \mathrm{F}$$

for each statement A.

This means that the techniques we used to simplify Boolean expressions in Section 7C can be used to help simplify compound statements down to simpler equivalent statements.

Summary 7D

- **Logical connectives**
  - The symbol $\vee$ is used for 'or'.
  - The symbol $\wedge$ is used for 'and'.
  - The symbol $\neg$ is used for 'not'.
  - The symbol $\Rightarrow$ is used for 'implies'.
  - The symbol $\Leftrightarrow$ is used for 'is equivalent to'.
- **Truth tables**
  - Or

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

  - And

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

  - Not

A	$\neg A$
T	F
F	T

  - Implies

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

  - Equivalence

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

- Two statements are **logically equivalent** if they have the same truth values.
- A **tautology** is a statement which is true under all circumstances.
- A **contradiction** is a statement which is false under all circumstances.
- For a conditional statement $A \Rightarrow B$:
  - the **converse** statement is $B \Rightarrow A$
  - the **contrapositive** statement is $\neg B \Rightarrow \neg A$.

Skill-sheet

Exercise 7D

1 For each statement (P), write down its negation ($\neg P$):

a Your eyes are blue.
b The sky is grey.
c This integer is odd.
d I live in Switzerland.
e $x > 2$
f This number is less than 100.

2 Consider the following four statements.

- A: It is dark.
- B: It is cold.
- C: It is good.
- D: It is soft.

Using the given table of opposites, write each of the following statements in English:

dark	light
cold	hot
good	bad
soft	hard

a $A \vee B$
b $A \wedge B$
c $\neg A \wedge B$
d $\neg(A \wedge B)$
e $C \vee \neg A$
f $\neg(A \vee D)$
g $A \vee \neg D$

3 Consider the following four statements.

- A: It is wet.
- B: It is rough.
- C: It is difficult.
- D: It is expensive.

Write each of the following in symbols:

a It is rough and wet.
b It is expensive or difficult.
c It is not difficult but is expensive.
d It is not wet and not rough.
e It is not expensive and not difficult.
f It is rough or wet.

4 Consider the following four statements.

- A: It is wet.
- B: It is rough.
- C: It is difficult.
- D: It is expensive.

Using the given table of opposites, write each of the following statements in English:

wet	dry
rough	smooth
difficult	easy
expensive	inexpensive

a $A \vee B$
b $A \wedge B$
c $\neg A \wedge B$
d $\neg(A \wedge B)$
e $C \vee \neg A$
f $\neg(A \vee D)$
g $A \vee \neg D$

5 Consider the following three statements for n a natural number.

- A: n is a prime number.
- B: n is an even number.
- C: n is divisible by 6.

Write each of the following in English as concisely as possible:

a $A \vee B$
b $B \wedge C$
c $A \wedge B$
d $\neg A \wedge B$
e $\neg(A \wedge B)$
f $C \vee \neg A$
g $\neg(A \vee C)$
h $A \vee \neg C$

Example 10
6 Give the truth table for $\neg(A \wedge B)$.

Example 11
7 Give the truth table for $(A \vee B) \wedge (\neg B)$.

Example 12 **8** Show that each of the following pairs of statements are equivalent:

a $\neg(A \vee B)$ $\neg A \wedge \neg B$ **b** $\neg(\neg A)$ A

c $A \vee A$ A **d** $A \vee B$ $\neg(\neg A \wedge \neg B)$

e $A \wedge B$ $\neg(\neg A \vee \neg B)$ **f** $A \wedge \neg B$ $\neg(\neg A \vee B)$

Example 13 **9** Show that $(\neg A \wedge \neg B) \vee (B \vee A)$ is a tautology.

Example 14 **10** Show that $(A \wedge B) \wedge (\neg B)$ is a contradiction.

11 Show that $(\neg A \wedge B) \wedge A$ is a contradiction.

Example 15 **12** Give a truth table for each of the following statements:

a $(A \wedge B) \Rightarrow A$ **b** $(A \vee B) \Rightarrow A$

c $(\neg B \vee \neg A) \Rightarrow A$ **d** $(\neg B \wedge A) \Rightarrow A$

e $(B \vee \neg A) \Rightarrow \neg A$ **f** $(\neg B \vee \neg A) \Rightarrow (\neg B \wedge A)$

g $(\neg B \vee A) \Rightarrow \neg(B \wedge A)$ **h** $\neg B \wedge (\neg B \Rightarrow A)$

13 Show that each of the following pairs of statements are equivalent:

a $A \wedge B$ $\neg(A \Rightarrow \neg B)$

b $A \vee B$ $\neg A \Rightarrow B$

c $A \Leftrightarrow B$ $\neg[(A \Rightarrow B) \Rightarrow \neg(B \Rightarrow A)]$

14 Show that each of the following is a tautology:

a $(A \wedge B) \Rightarrow (A \vee B)$ **b** $[A \wedge (A \Rightarrow B)] \Rightarrow B$ **c** $[(A \vee B) \wedge (\neg A)] \Rightarrow B$

Example 16 **15** Show that $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ is a tautology. Give the corresponding result for sets A and B, and illustrate with a Venn diagram.

16 The logical connective **nor** is written symbolically as $\downarrow$. The statement $A \downarrow B$ is true if and only if neither A nor B is true. Thus $A \downarrow B$ is equivalent to $\neg(A \vee B)$.

a Give the truth table for $A \downarrow B$ and $B \downarrow A$.

b Show that $A \downarrow A$ is equivalent to $\neg A$.

c Show that $[(A \downarrow A) \downarrow (B \downarrow B)] \Leftrightarrow (A \wedge B)$ is a tautology.

d Show that $\neg(A \downarrow B) \Leftrightarrow (A \vee B)$ is a tautology.

17 Using truth tables, show that the conditional statement $A \Rightarrow B$ is equivalent to its contrapositive $\neg B \Rightarrow \neg A$, but is not equivalent to its converse $B \Rightarrow A$.

18 Use a truth table to show that the statement $\neg(A \Rightarrow B)$ is equivalent to $A \wedge \neg B$.

Example 17 **19** For each of the following conditional statements:

i write the converse **ii** write the contrapositive **iii** write the negation.

a If $x = 6$, then x is an even integer.

b If I am elected, then public transport will improve.

c If I pass this exam, then I will be qualified as an actuary.

7E Valid arguments

One important use of the formal approach to logic in Section 7D is checking arguments.

- By an **argument**, we mean that one statement (called the **conclusion**) is claimed to follow from other statements (called the **premises**).
- An argument is said to be **valid** if whenever all the premises are true, the conclusion is also true.

We illustrate the procedure for checking an argument in the next example.

Example 18

By constructing a truth table, decide whether or not the following argument is valid.

Premise 1	If Australia is a democracy, then Australians have the right to vote.
Premise 2	Australia is a democracy.
Conclusion	Australians have the right to vote.

Solution

Let A be the statement 'Australia is a democracy'.
Let B be the statement 'Australians have the right to vote'.

Then the argument can be presented symbolically as follows:

Premise 1	$A \Rightarrow B$
Premise 2	A
Conclusion	B

We check this argument using a truth table.

Premise 2	Conclusion	Premise 1
A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

We look for the rows in which both premises are true. This is only row 1. In this row, the conclusion is also true. Therefore the argument is valid.

In general, we can use the following procedure to check the validity of an argument using a truth table:

- Look for the rows of the truth table in which all the premises are true.
- If the conclusion is also true in each of these rows, then argument is valid. Otherwise, the argument is not valid.

Example 19

By constructing a truth table, decide whether or not the following argument is valid.

Premise 1	If you invest in Company W, then you get rich.
Premise 2	You did not invest in Company W.
Conclusion	You did not get rich.

Solution

Let A be the statement 'You invest in Company W'.
Let B be the statement 'You get rich'.

Then the argument can be presented symbolically as follows:

Premise 1	$A \Rightarrow B$
Premise 2	$\neg A$
Conclusion	$\neg B$

We check this argument using a truth table.

		Premise 1	Premise 2	Conclusion
A	B	$A \Rightarrow B$	$\neg A$	$\neg B$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

We look for the rows in which both premises are true. These are rows 3 and 4. In row 3, the conclusion is false. Therefore the argument is not valid.

Validity versus truth

The validity of an argument depends only on whether or not the conclusion follows logically from the premises; it does not depend on whether or not the conclusion is actually true.

Valid arguments with false conclusions A valid argument is guaranteed to have a true conclusion only if all of the premises are true.

For example, consider the argument:

Premise 1	If 2 is odd, then 3 is even.	$A \Rightarrow B$
Premise 2	The number 2 is odd.	A
Conclusion	The number 3 is even.	B

This argument is *valid*, as it follows the pattern of Example 18. But the conclusion is *false*. Note that, in this case, the second premise is false. Therefore, even though the argument is valid, we are not guaranteed that the conclusion is true.

Invalid arguments with true conclusions Next consider the argument:

Premise 1	If 2 is odd, then 3 is even.	$A \Rightarrow B$
Premise 2	The number 2 is even.	$\neg A$
Conclusion	The number 3 is odd.	$\neg B$

This argument is *invalid*, as it follows the pattern of Example 19. But the conclusion is *true*.

Checking for a tautology

The following example illustrates an alternative way to check the validity of an argument. We consider the entire argument as a single compound statement, and then check whether this statement is a tautology.

Example 20

Investigate the validity of each of the following arguments by checking whether or not an appropriate compound statement is a tautology:

a In March, there are strong winds every day. The wind is not strong today. Therefore it is not March.

b On Mondays I go swimming. Today is not Monday. Therefore I do not swim today.

Solution

a Let M be the statement 'It is March'.
Let S be the statement 'There are strong winds'.

The compound statement to consider is $[(M \Rightarrow S) \wedge (\neg S)] \Rightarrow \neg M$.

M	S	$M \Rightarrow S$	$\neg S$	$(M \Rightarrow S) \wedge (\neg S)$	$\neg M$	$[(M \Rightarrow S) \wedge (\neg S)] \Rightarrow \neg M$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Since $[(M \Rightarrow S) \wedge (\neg S)] \Rightarrow \neg M$ is a tautology, the argument is valid.

b Let M be the statement 'It is Monday'.
Let S be the statement 'I swim today'.

The compound statement to consider is $[(M \Rightarrow S) \wedge (\neg M)] \Rightarrow \neg S$.

M	S	$M \Rightarrow S$	$\neg M$	$(M \Rightarrow S) \wedge (\neg M)$	$\neg S$	$[(M \Rightarrow S) \wedge (\neg M)] \Rightarrow \neg S$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

The argument is not valid. It fails to be true when M is false and S is true.

Here is a list of some useful tautologies that can be used to form valid arguments:

1. $[(A \Rightarrow B) \wedge (B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$
2. $[(A \Rightarrow B) \wedge A] \Rightarrow B$
3. $[(A \Rightarrow B) \wedge (\neg B)] \Rightarrow \neg A$
4. $[(A \vee B) \wedge (\neg A)] \Rightarrow B$
5. $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ (contrapositive)
6. $\neg(A \vee B) \Leftrightarrow (\neg A \wedge \neg B)$ (De Morgan's law)
7. $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$ (De Morgan's law)

The third of these tautologies is used in part **a** of Example 20.

Summary 7E

Checking the validity of an argument

Step 1 Represent each of the premises and the conclusion as compound statements.

Step 2 Construct a truth table that includes a column for each of the premises and the conclusion.

Step 3 Look for the rows of the truth table in which all the premises are true.

Step 4 If the conclusion is also true in each of these rows, then argument is valid. Otherwise, the argument is not valid.

Skill-sheet

Exercise 7E

Example 18

1 Consider the following argument, which is presented symbolically. Complete the truth table and state whether or not the argument is valid.

Premise 1	$A \vee B$
Premise 2	$\neg A$
Conclusion	B

A	B	$A \vee B$	$\neg A$
T	T		
T	F		
F	T		
F	F		

Example 19

2 Consider the following argument, which is presented symbolically. Complete the truth table and state whether or not the argument is valid.

Premise 1	$A \vee B$
Premise 2	$\neg A$
Conclusion	$\neg B$

A	B	$A \vee B$	$\neg A$	$\neg B$
T	T			
T	F			
F	T			
F	F			

3 Consider the following argument, which is presented symbolically. Complete the truth table and state whether or not the argument is valid.

Premise 1	A
Premise 2	$A \Rightarrow B$
Premise 3	$B \Rightarrow C$
Conclusion	C

A	B	C	$A \Rightarrow B$	$B \Rightarrow C$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Example 18

4 By constructing a truth table, decide whether or not the following argument is valid.

Premise 1	You eat lots of garlic.
Premise 2	If you eat lots of garlic, then you don't have many friends.
Conclusion	You don't have many friends.

Example 19

5 By constructing a truth table, decide whether or not the following argument is valid.

Premise 1	The number 5 is odd.
Premise 2	If 4 is even, then 5 is odd.
Conclusion	The number 4 is even.

6 By constructing a truth table, decide whether or not the following argument is valid.

Premise 1	I will buy a car or a motorcycle.
Premise 2	If I buy a car and a motorcycle, then I will need a loan.
Premise 3	I bought a motorcycle and I don't need a loan.
Conclusion	I did not buy a car.

7 For each of the following, use a truth table to decide if the argument is valid or not:

a

Premise 1	$A \Leftrightarrow B$
Premise 2	A
Conclusion	B

b

Premise 1	$A \vee B$
Premise 2	$A \Rightarrow B$
Conclusion	B

c

Premise 1	$A \wedge B$
Premise 2	$\neg A \Rightarrow B$
Conclusion	$\neg B$

d

Premise 1	$A \Rightarrow \neg B$
Premise 2	$\neg B$
Conclusion	A

8 For each of the following, use a truth table to decide if the argument is valid or not:

a

Premise 1	$A \Leftrightarrow B$
Premise 2	$A \vee C$
Premise 3	$\neg C$
Conclusion	B

b

Premise 1	A
Premise 2	$B \Leftrightarrow \neg C$
Premise 3	$C \Rightarrow A$
Conclusion	B

Example 20

9 Investigate the validity of each of the following arguments by checking whether an appropriate compound statement is a tautology:

a In January, it is warm every day. It is warm today. Therefore it is January.

b If it is not sunny, then I do not go running. I am going running today. Therefore today is sunny.

c All kangaroos jump. Jumping needs strength. So kangaroos need strength.

7F Logic circuits

In Section 7B, we introduced circuits composed of switches. In this section, we consider designing circuits composed of logic gates.

We use the Boolean operations $\vee$, $\wedge$ and $\neg$ on the set $\{0, 1\}$.

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

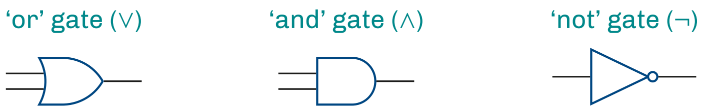
A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

A	$\neg A$
0	1
1	0

These correspond to the logical operations 'or', 'and' and 'not' if we interpret 0 as 'false' and 1 as 'true'. In a circuit, we represent 0 as 'low voltage' and 1 as 'high voltage'.

Logic gates

We will create circuits using the following three **logic gates**, which carry out the operations of 'or' ($\vee$), 'and' ($\wedge$) and 'not' ($\neg$).



Each gate is shown with the **inputs** on the left and the **output** on the right. For example:

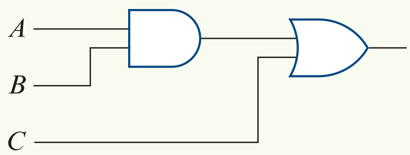
- If an 'or' gate has inputs 0 and 1, then the output will be 1.
- If an 'and' gate has inputs 0 and 1, then the output will be 0.
- If a 'not' gate has input 0, then the output will be 1.

We can build a logic circuit to represent any Boolean expression.

Example 21

Give the gate representation of the Boolean expression $(A \wedge B) \vee C$.

Solution



Note: The inputs to this circuit are labelled A, B and C. The output of the circuit will reflect the state of these inputs according to the logic statement $(A \wedge B) \vee C$.

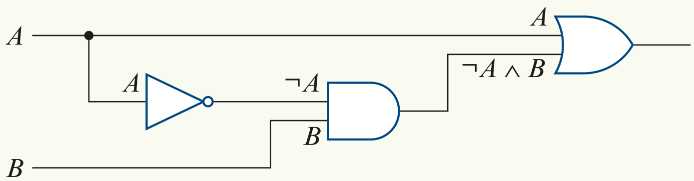
Example 22

a Give the gate representation of the Boolean expression $A \vee (\neg A \wedge B)$.

b Describe the operation of this circuit through a truth table.

Solution

a



b

A	B	$\neg A$	$\neg A \wedge B$	$A \vee (\neg A \wedge B)$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	0	0	1

In Example 8, we saw a technique for constructing a Boolean expression to match a given truth table of 0s and 1s. Given any truth table, we can construct a matching Boolean expression and therefore build a logic circuit that will operate according to the given table. This is what makes Boolean algebra so central to electronics.

Given any truth table that specifies the required operation of a circuit, it is possible to build an appropriate circuit using 'or', 'and' and 'not' gates.

Example 23

Consider the truth table shown on the right.

a Using the technique from Example 8, construct a Boolean expression to match this truth table.

b Draw a circuit for this expression.

c Use the properties of Boolean algebras to simplify the expression, and hence draw a simpler circuit that is equivalent to the circuit from **b**.

	A	B	Output
1	0	0	1
2	0	1	1
3	1	0	1
4	1	1	0

Solution

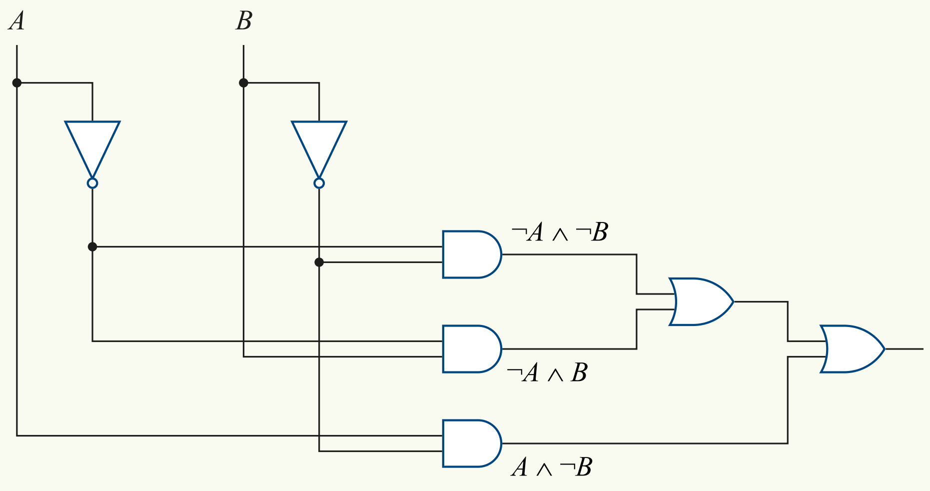
a Look at the rows of the truth table in which the output is 1:

- Row 1 $\neg A \wedge \neg B$
- Row 2 $\neg A \wedge B$
- Row 3 $A \wedge \neg B$

Therefore a Boolean expression for this truth table is

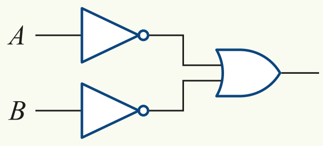
$$(\neg A \wedge \neg B) \vee (\neg A \wedge B) \vee (A \wedge \neg B)$$

b



c

$$\begin{aligned}
&(\neg A \wedge \neg B) \vee (\neg A \wedge B) \vee (A \wedge \neg B)\\
&= (\neg A \wedge (\neg B \vee B)) \vee (A \wedge \neg B)\\
&= (\neg A \wedge 1) \vee (A \wedge \neg B)\\
&= \neg A \vee (A \wedge \neg B)\\
&= (\neg A \vee A) \wedge (\neg A \vee \neg B)\\
&= 1 \wedge (\neg A \vee \neg B)\\
&= \neg A \vee \neg B
\end{aligned}$$

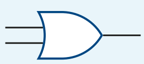


Note: This simplified circuit uses only three logic gates. In fact, we could rewrite the expression as $\neg(A \wedge B)$ and obtain an equivalent circuit that uses only two gates.

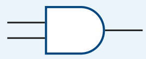
Summary 7F

- In a logic circuit, we interpret 0 as false and 1 as true.
- We will use the following three **logic gates** to create circuits (where the inputs are shown on the left and the output on the right):

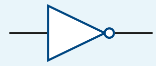
'or' gate (∨)



'and' gate (∧)



'not' gate (¬)



Exercise 7F

Example 21
Example 22

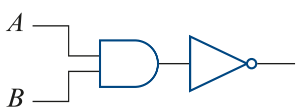
1 Draw a circuit using logic gates for each of the following:

a $A \land (\neg B)$ **b** $\neg(A \lor B)$

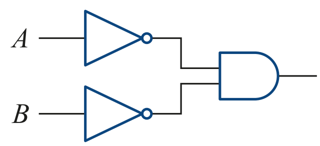
c $A \land (\neg B \lor A)$ **d** $(\neg A \lor B) \land (\neg B \lor A)$

2 For each of the following, give a Boolean expression that represents the circuit and give a truth table describing the operation of the circuit:

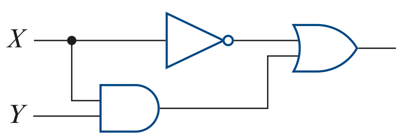
a



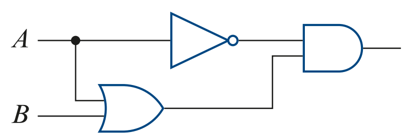
b



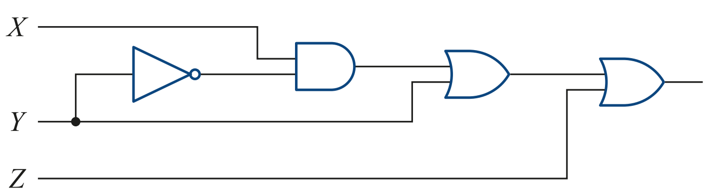
c



d



3 Write a Boolean expression for the following circuit:



4 Draw a circuit using logic gates for each of the following:

a $\neg(\neg P \land Q) \lor R$ **b** $(\neg A \land B) \lor (B \land \neg C) \lor A$

Example 23

5 Consider the truth table shown on the right.

a Using the technique from Example 8, construct a Boolean expression to match this truth table.

b Draw a circuit for this expression.

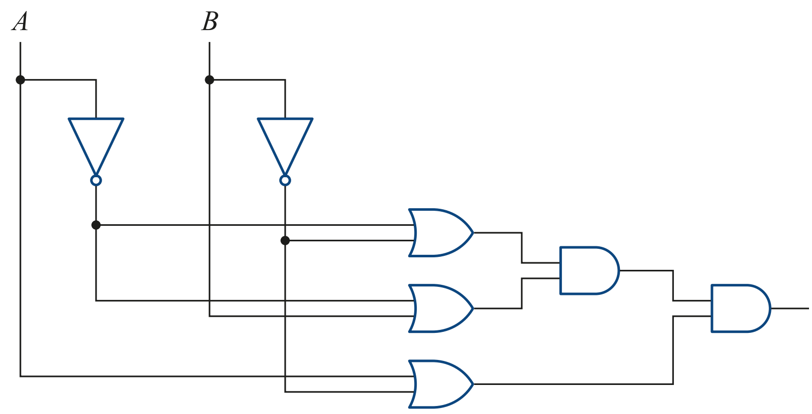
c Use the properties of Boolean algebras to simplify the expression, and hence draw a simpler circuit that is equivalent to the circuit from **b**.

A	B	Output
0	0	0
0	1	1
1	0	1
1	1	1

6 The table on the right specifies the required operation of a logic circuit. Draw an appropriate circuit.

A	B	Output
0	0	0
0	1	1
1	0	1
1	1	0

7 Consider the logic circuit shown below.



a Write a Boolean expression corresponding to the circuit.

b Simplify this expression using Boolean algebra.

c Draw an equivalent logic circuit that uses only three gates.

d Draw an equivalent logic circuit that uses only two gates.

7G Karnaugh maps

We want to be able to simplify Boolean expressions so that the circuits we build from them are simpler. These simpler circuits will be more compact and cheaper to produce.

In Section 7C, we showed how to simplify Boolean expressions using properties of Boolean algebras. In this section, we introduce a more pictorial approach.

Minimal representation

We start by formalising what it means for a Boolean expression to be 'simplified'.

Let f be a non-constant Boolean function. Then a **minimal representation** of f is a Boolean expression E which represents f and satisfies the following:

- The expression E has the form $E_1 \vee E_2 \vee \cdots \vee E_n$, where each E_i is an expression such as $x \wedge y$ or $x' \wedge y' \wedge z$ or $y \wedge z'$.
- If F is any other expression of this form which also represents f, then the number of terms F_i is greater than or equal to the number of terms E_i.
- If F and E have the same number of terms, then the number of variables in F is greater than or equal to the number of variables in E.

Karnaugh maps involving two variables

We demonstrate how a different representation of a truth table can be used to find a minimal expression. The truth table for $f(x, y) = (x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$ is shown below.

x	y	$f(x, y)$	
0	0	1	$x' \wedge y'$
0	1	1	$x' \wedge y$
1	0	0	$x \wedge y'$
1	1	1	$x \wedge y$

As in Example 8, each row of the truth table corresponds to some combination using $\wedge$ of either x or x', and either y or y'. This is shown to the right of the truth table above. Using this correspondence, we fill the values of $f(x, y)$ into the following 2×2 table.

	y	y'
x	1	0
x'	1	1

The next step is to shade the 1s which occur in pairs as 1×2 or 2×1 blocks.

	y	y'
x	1	0
x'	1	1

The table above is called a **Karnaugh map**.

To find a minimal expression, we read off the label of each coloured block:

- **Red** The two 1s in the red block have labels $x'y$ and $x'y'$. The common label for the 1s in the red block is x'. So the red block has label x'.
- **Green** The two 1s in the green block have labels xy and $x'y$. The common label for the 1s in the green block is y. So the green block has label y.
- **Together** Combine the block labels using $\vee$. This gives $x' \vee y$.

The minimal expression is $f(x, y) = x' \vee y$.

The following calculation illustrates why this process works:

$$\begin{aligned}(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y) &= \overbrace{[(x' \wedge y') \vee (x' \wedge y)]}^{\text{red block}} \vee \overbrace{[(x' \wedge y) \vee (x \wedge y)]}^{\text{green block}} \\ &= [x' \wedge (y' \vee y)] \vee [(x' \vee x) \wedge y] \\ &= [x' \wedge 1] \vee [1 \wedge y] \\ &= x' \vee y\end{aligned}$$

We could have used the original expression

$$(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$$

to fill in the Karnaugh map directly, by putting a 1 for each of the terms $x' \wedge y'$, $x' \wedge y$ and $x \wedge y$ in the expression. A Boolean expression must be of a form like this to enter directly into a Karnaugh map.

Example 24

Simplify $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y)$.

Solution

Step 1

	y	y'
x	1	1
x'	1	

Explanation

For the expression $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y)$, we fill a 1 into the cells xy, xy' and $x'y$.

It is not necessary to fill in the 0s.

Step 2

	y	y'
x	1	1
x'	1	

Complete the shading of the 1s.

Step 3 Block labels:

- Red x
- Green y
- Together $x \vee y$

The simplified expression is $x \vee y$.

Now read off the labels of the coloured blocks:

- The common label for the 1s in the red block is x.
- The common label for the 1s in the green block is y.
- Combine the block labels using $\vee$.

Karnaugh maps involving three variables

A Karnaugh map for three variables x, y and z can be labelled as shown.

	yz	$y'z$	$y'z'$	yz'
x				
x'				

Notes:

- The order of the labels yz, $y'z$, $y'z'$, yz' along the top is important. There is only one change from one label to the next.
- You need to imagine that this Karnaugh map is wrapped around a *cylinder* so that the xyz and xyz' cells are adjacent and the $x'yz$ and $x'yz'$ cells are adjacent.

The technique for three variables is similar to that for two variables. We first use either the truth table or the expression to fill the 1s into the Karnaugh map. Then we cover the 1s using blocks.

Blocks in Karnaugh maps

- You may use an $m \times n$ block in a Karnaugh map if both m and n are powers of 2. (So you may use 1×1, 1×2, 1×4, 2×1, 2×2 and 2×4 blocks.)
- You always try to form the *biggest* blocks that you can, and to use the *least number* of blocks that you can.

Examples of correct shading

	yz	$y'z$	$y'z'$	yz'
x	1			1
x'	1			1

	yz	$y'z$	$y'z'$	yz'
x		1	1	1
x'		1	1	1

	yz	$y'z$	$y'z'$	yz'
x		1	1	
x'			1	1

Examples of incorrect shading

Blocks too small

	yz	$y'z$	$y'z'$	yz'
x	1			1
x'	1			1

Blocks too small

	yz	$y'z$	$y'z'$	yz'
x		1	1	1
x'		1	1	1

Too many blocks

	yz	$y'z$	$y'z'$	yz'
x		1	1	
x'			1	1

Example 25

Simplify $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z')$.

Solution

	yz	$y'z$	$y'z'$	yz'
x	1		1	1
x'				1

Labels of the coloured blocks:

- Red $x \wedge z'$
- Green $y \wedge z'$
- Blue $x \wedge y$

The simplified expression is

$$(x \wedge y) \vee (x \wedge z') \vee (y \wedge z')$$

Explanation

In this example, the blue shading shows a 1×2 block that wraps around the back of the 'cylinder'.

The common labels for the 1s in the red block are x and z'. So the label of the red block is $x \wedge z'$.

We find the other labels similarly, and combine the block labels using $\vee$.

Example 26

Write a minimal Boolean expression for the following truth table.

x	y	z	$f(x, y, z)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Solution

	yz	$y'z$	$y'z'$	yz'
x		1	1	
x'		1	1	1

Labels of the coloured blocks:

- Red y'
- Green $x' \wedge z'$

A minimal expression is
$f(x, y, z) = (x' \wedge z') \vee y'$.

Explanation

We first fill the 1s from the rightmost column of the truth table into the Karnaugh map.

For example, row 1 of the truth table corresponds to $x' \wedge y' \wedge z'$ and thus to the $x'y'z'$ cell.

Note: A Boolean function can have more than one minimal expression, since there can be more than one correct way to choose the blocks on a Karnaugh map.

Exercise 7G

Example 24

1 Simplify each of the following using a Karnaugh map:

a $(x \wedge y) \vee (x' \wedge y)$

b $(x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$

c $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$

2 Write down the minimal Boolean expression represented by each Karnaugh map:

a

	yz	$y'z$	$y'z'$	yz'
x	1			1
x'	1			1

b

	yz	$y'z$	$y'z'$	yz'
x		1	1	1
x'		1	1	1

c

	yz	$y'z$	$y'z'$	yz'
x		1	1	
x'			1	1

Example 25

3 Simplify each of the following using a Karnaugh map:

a $(x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z')$

b $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z')$

c $(x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z')$

Example 26

4 Write a minimal Boolean expression for each of the following Boolean functions:

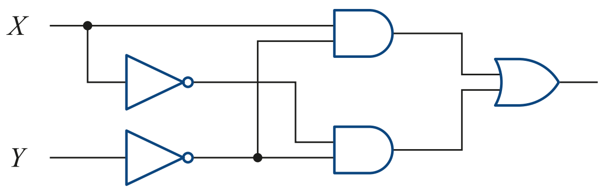
a

x	y	$f(x, y)$
0	0	1
0	1	1
1	0	1
1	1	0

b

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

5 Consider the circuit shown below.

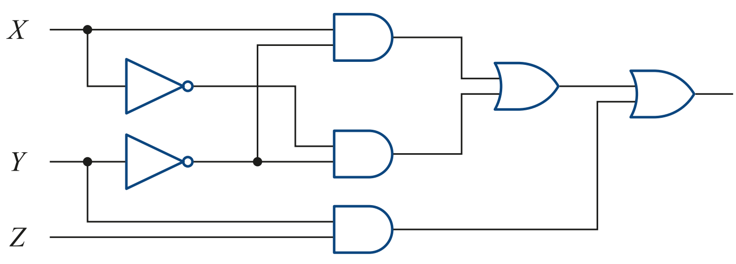


a Construct the truth table that describes the operation of this circuit.

b Find a minimal Boolean expression for the truth table.

c Draw the simplified circuit corresponding to the minimal Boolean expression.

6 Consider the circuit shown below.



a Construct the truth table that describes the operation of this circuit.

b Find a minimal Boolean expression for the truth table.

c Draw the simplified circuit corresponding to the minimal Boolean expression.

Chapter summary

Assignment

Nrich

Boolean algebra

- Basic examples of Boolean algebras:
  - the set of all subsets of a set together with the operations $\cup$, $\cap$ and $'$
  - the set $\{0, 1\}$ together with the operations $\vee$, $\wedge$ and $'$

x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

x	x'
0	1
1	0

- A **Boolean expression** is an expression formed using $\vee$, $\wedge$, $'$, 0 and 1, such as $x \wedge (y \vee x)'$.
- Two Boolean expressions are **equivalent** if they give the same Boolean function on $\{0, 1\}$.

Logical connectives

- Or

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

- And

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

- Not

A	$\neg A$
T	F
F	T

- Implies

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

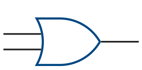
- Equivalence

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

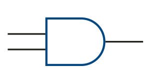
- Two statements are **logically equivalent** if they have the same truth values.
- A **tautology** is a statement which is true under all circumstances.
- A **contradiction** is a statement which is false under all circumstances.
- The **converse** of $A \Rightarrow B$ is the statement $B \Rightarrow A$.
- The **contrapositive** of $A \Rightarrow B$ is the statement $\neg B \Rightarrow \neg A$.

Logic circuits

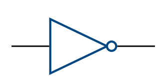
- 'Or' gate ($\vee$)



- 'And' gate ($\wedge$)



- 'Not' gate ($\neg$)



Technology-free questions

1 Which of the following statements are true?

a 2 is even.
b 3 is not even.
c 3 is even and 2 is even.
d 3 is even or 2 is even.
e If 2 is even, then 3 is odd.
f If 2 is odd, then 3 is even.

2 For each statement (P), write down the negation ($\neg P$):

a It is raining.
b It is not raining.
c $x = 5$ and $y = 5$
d $x = 3$ or $x = 5$
e It is not raining and it is not windy.
f If it is snowing, then it is cold.

3 The logical connective 'exclusive or' is denoted by $\oplus$. The statement $A \oplus B$ is true when either A or B is true, but not both. The truth table for $\oplus$ is shown on the right.

Truth table for 'exclusive or'

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
F	F	F

Construct a truth table for each of the following compound statements:

a $A \oplus (A \oplus B)$
b $A \oplus (A \vee B)$

4 Construct a truth table to show that the statement $\neg A \Rightarrow (A \Rightarrow B)$ is a tautology.

5 For each pair of Boolean expressions, prove that the two expressions are equivalent by:

i showing that they represent the same Boolean function on $\{0, 1\}$
ii using the axioms and properties of Boolean algebras.

a $x \vee (x' \wedge y)$ and $x \vee y$
b $(x \vee y) \wedge (x' \vee y)$ and y

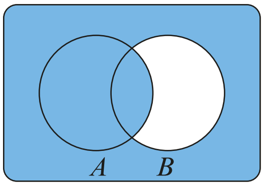
6 Draw a circuit using logic gates for each of the following:

a $\neg A \vee B$
b $A \wedge (B \vee C)$
c $(A \wedge \neg B) \vee (B \wedge \neg A)$

Multiple-choice questions

1 The blue region of the Venn diagram is

A B'
B $A \cup B'$
C $A \cap B'$
D $A' \cap B'$
E $A' \cup B'$



2 The dual of $A \cap (A \cup B)' = \varnothing$ is

A $B \cap (B \cup A)' = \varnothing$
B $A \cup (A \cap B)' = \varnothing$
C $A \cup (A \cap B)' = \xi$
D $A' \cap (A' \cup B')' = \xi$
E $A \cap (A \cup B)' = \xi$

3 Which of the following is *not* an identity of Boolean algebra?

A $x \wedge x = x$
B $x \wedge y = y \wedge x$
C $(x \wedge y)' = x' \wedge y'$
D $x \wedge (x \wedge y) = x \wedge y$
E $0 \wedge x = 0$

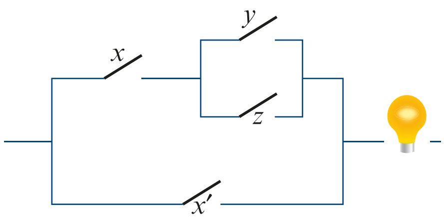
4 Consider the statement 'If n is divisible by 12, then n is divisible by 2 and 3'. Which of the following is equivalent to this statement?

A If n is not divisible by 12, then n is divisible by 2 or divisible by 3.
B If n is not divisible by 12, then n is not divisible by 2 or not divisible by 3.
C If n is divisible by 2 and divisible by 3, then n is divisible by 12.
D If n is not divisible by 2 or not divisible by 3, then n is not divisible by 12.
E If n is divisible by 2 or divisible by 3, then n is divisible by 12.

5 Let P be the statement 'If Tom is Jane's father, then Jane is Bill's niece'.
Let Q be the statement 'Bill is Tom's brother'.
Which of the following is equivalent to the statement $P \Rightarrow Q$?

A If Bill is Tom's brother, then Tom is Jane's father and Jane is not Bill's niece.
B If Bill is not Tom's brother, then Tom is Jane's father and Jane is not Bill's niece.
C If Bill is not Tom's brother, then Tom is Jane's father or Jane is Bill's niece.
D If Bill is Tom's brother, then Tom is Jane's father and Jane is Bill's niece.
E If Bill is not Tom's brother, then Tom is not Jane's father and Jane is Bill's niece.

6 Which of the following Boolean expressions represents the switching circuit shown?



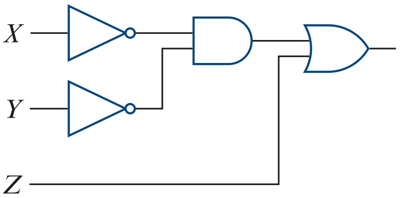
A $(x \wedge y \wedge z) \vee x'$ **B** $x \wedge (y \vee z \vee x')$
C $(x \vee x') \wedge (y \vee z)$ **D** $(x \wedge (y \vee z)) \vee x'$
E $x \wedge (y \vee z) \wedge x'$

7 Which of the following Boolean expressions corresponds to the truth table shown on the right?

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

A $(x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z)$
B $(x \wedge y \wedge z') \vee (x' \wedge y \wedge z')$
C $(x \wedge y \wedge z) \vee (x' \wedge y' \wedge z')$
D $(x \wedge y' \wedge z') \vee (x' \wedge y \wedge z)$
E $(x' \wedge y \wedge z') \vee (x \wedge y' \wedge z)$

8 Which of the following Boolean expressions represents the logic circuit shown?



A $(X \wedge Y) \vee Z$ **B** $(\neg X \wedge \neg Y) \vee Z$
C $\neg(X \wedge Y) \vee Z$ **D** $(\neg X \vee \neg Y) \wedge Z$
E $\neg X \wedge (\neg Y \vee Z)$

For Questions 9 and 10, let P be the statement 'I will pass Specialist Mathematics' and let S be the statement 'I study hard'.

9 Which of the following corresponds to the statement 'If I study hard, then I will pass Specialist Mathematics'?

A $S \Leftrightarrow P$ **B** $S \vee P$ **C** $P \Rightarrow S$ **D** $S \Rightarrow P$ **E** $S \wedge P$

10 Which of the following is *not equivalent* to the statement 'I will not pass Specialist Mathematics unless I study hard'?

A $P \Rightarrow S$ **B** $\neg P \vee S$ **C** $\neg S \Rightarrow \neg P$ **D** $S \vee \neg P$ **E** $P \wedge S$

Extended-response questions

1 Recall that set difference is defined as

$$A \setminus B = A \cap B' = \{ x : x \in A \text{ and } x \notin B \}$$

The **symmetric difference** of two sets A and B is defined as

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

a Draw a Venn diagram showing $A \oplus B$ for two sets A and B with $A \cap B \neq \varnothing$.

b Prove that $A \oplus B = (A \cup B) \setminus (A \cap B)$.

c Prove that $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

2 A light in a stairwell is controlled by two switches: one at the bottom of the stairs and one at the top. If both switches are off, then the light should be off. If either of the switches changes state, then the light should change state. In this question, we will create a switching circuit to represent such a two-way switch.

a Use x and y to denote the two switches. Use 0 for 'off' and 1 for 'on'. Complete the table on the right so that it describes the operation of the circuit.

b Based on your table for part **a**, write down a Boolean expression that represents the circuit.

c Draw the switching circuit that is represented by your Boolean expression from part **b**.

x	y	Light
0	0	0
0	1	
1	0	
1	1	

3 A committee with three members reaches its decisions by using a voting machine. The machine has three switches (x, y, z); one for each member of the committee. If at least two of the three members vote 'yes' (1), then the machine's light goes on (1). Otherwise, the light is off (0).

a Construct a table with entries 0s and 1s that describes the operation of the voting machine.

b Give a Boolean expression for the voting machine, based on your table for part **a**.

c Use a Karnaugh map to simplify the Boolean expression obtained in part **b**.

d Draw a circuit for the voting machine using logic gates, based on your simplified Boolean expression from part **c**.

4 Let B be the set of all factors of 30. Thus

$$B = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

It can be shown that axioms 1, 2 and 3 of Boolean algebras hold for B, with the operation $\vee$ as LCM (lowest common multiple) and the operation $\wedge$ as HCF (highest common factor). In this question, we will show that axioms 4 and 5 hold, in order to complete the proof that B is a Boolean algebra.

a **i** The identity element for LCM must be a number ℓ in B such that LCM$(x, \ell) = x$ for all $x \in B$. What is ℓ?

ii The identity element for HCF must be a number h in B such that HCF$(x, h) = x$ for all $x \in B$. What is h?

b For each $x \in B$, define $x' = 30 \div x$. By completing the following table, show that this operation $'$ is complementation in B.

x	1	2	3	5	6	10	15	30
x'								
LCM(x, x')								
HCF(x, x')								

Note: Can you see what is special about the number 30 here? Consider its prime factorisation. What goes wrong if you try using the number 12 instead?

5 **Ternary logic** In Boolean logic, there are two truth values: true (1) and false (0). In ternary logic, there are three truth values: true (1), false (0) and *don't know* (d). The basic example of a ternary algebra is the set $\{0, d, 1\}$ with the operations $\vee$, $\wedge$ and $'$ given by the following tables.

■ Or ($\vee$)

$\vee$	0	d	1
0	0	d	1
d	d	d	1
1	1	1	1

■ And ($\wedge$)

$\wedge$	0	d	1
0	0	0	0
d	0	d	d
1	0	d	1

■ Not ($'$)

x	x'
0	1
d	d
1	0

Note that $d' = d$, since if we don't know whether a statement is true, then we don't know whether its negation is true. Ternary logic has applications in electronic engineering and database query languages.

a Evaluate each of the following:

i $d \vee 0$ **ii** $(d \wedge 0)'$ **iii** $(d \vee 0) \wedge (d' \vee 1)'$

b Give counterexamples to show that the laws $x \vee x' = 1$ and $x \wedge x' = 0$ for Boolean algebras do not hold for the ternary algebra $\{0, d, 1\}$.

c Prove that the De Morgan law $(x \vee y)' = x' \wedge y'$ holds for the ternary algebra $\{0, d, 1\}$.

Hint: Construct a truth table to show that $(x \vee y)'$ and $x' \wedge y'$ represent the same function on $\{0, d, 1\}$. Since there are two variables, each with three possible values, the truth table will have $3 \times 3 = 9$ rows.

8

Algorithms

Objectives

- To understand the concept of an **algorithm**.
- To understand and utilise the basic constructs that are used to build algorithms, including **iteration** and **selection**.
- To understand the use of **pseudocode** to describe algorithms.
- To describe algorithms using pseudocode by applying:
  - **if–then** blocks
  - **for** loops
  - **while** loops.
- To describe algorithms using pseudocode by applying functions, lists and nested loops.

We define an **algorithm** to be a finite, unambiguous sequence of instructions for performing a specific task.

You have already used many algorithms in your study of mathematics. For example, you have used an algorithm for completing the square for any quadratic polynomial. In the previous chapter, you encountered an algorithm for finding a Boolean expression for a Boolean function given in table form. You will also see examples of algorithms used in graph theory in Chapter 12.

In recent decades, the study of algorithms has become an important area of research within mathematics. This is partly because of their obvious importance in computing. Mathematicians work to obtain new algorithms and to improve the efficiency of existing algorithms.

Note: The Interactive Textbook includes online appendices that provide an introduction to coding using the language *Python* and also to coding using the TI-Nspire and the Casio ClassPad.

8A Introduction to algorithms

In this section, we introduce the idea of an algorithm by exploring three specific examples.

Karatsuba's algorithm for multiplication

In 1956, the Russian mathematician Andrey Kolmogorov claimed that our standard long-multiplication algorithm was the most efficient possible in terms of the number of individual multiplications required. However, a more efficient algorithm was discovered in 1960 by another Russian mathematician, Anatoly Karatsuba.

The standard long-multiplication algorithm

Consider a pair of two-digit numbers. They can be written as $m = 10a + b$ and $n = 10c + d$. Using the standard long-multiplication algorithm, we find their product as follows:

$$mn = (10a + b)(10c + d) = 100ac + 10(ad + bc) + bd \qquad (*)$$

You can see that we need to perform *four* individual multiplications: ac, ad, bc and bd.

For example, let $m = 23$ and $n = 31$. Then $a = 2$, $b = 3$, $c = 3$ and $d = 1$.

We demonstrate the algorithm by substituting these values into $(*)$. We also show the algorithm using the standard layout seen in primary school.

$$\begin{aligned} 23 \times 31 &= 100(2 \times 3) + 10(2 \times 1 + 3 \times 3) + (3 \times 1) \\ &= 600 + 110 + 3 \\ &= 713 \end{aligned}$$

$$\begin{array}{r} 2\;3 \\ \times\quad 3\;1 \\ \hline 2\;3 \\ {}^{1}6\;9\;0 \\ \hline 7\;1\;3 \end{array}$$

Karatsuba's multiplication algorithm

We now describe Karatsuba's more efficient algorithm in the special case of a pair of two-digit numbers.

Karatsuba's multiplication algorithm

To find the product of a pair of two-digit numbers $m = 10a + b$ and $n = 10c + d$:

Step 1 Calculate ac. Call the result F.

Step 2 Calculate bd. Call the result G.

Step 3 Calculate $(a + b)(c + d)$. Call the result H.

Step 4 Calculate $H - F - G$. Call the result K.

Step 5 Calculate $100F + 10K + G$. The result is mn.

Note that this method requires only *three* individual multiplications. An important aim of computing with large numbers is to reduce the number of multiplications required. Each time you perform a web search, for example, your device performs a huge number of multiplications, involving numbers with hundreds or even thousands of digits.

The following calculation shows why Karatsuba's method works:

$$\begin{aligned}100F + 10K + G &= 100ac + 10(H - F - G) + bd\\ &= 100ac + 10(ac + ad + bc + bd - ac - bd) + bd\\ &= 100ac + 10(ad + bc) + bd\\ &= (10a + b)(10c + d)\\ &= mn\end{aligned}$$

Example 1

Use Karatsuba's multiplication algorithm to calculate 23×31.

Solution	Explanation
Here $m = 23$ and $n = 31$.	
So $a = 2$, $b = 3$, $c = 3$ and $d = 1$.	Follow the steps of the algorithm:
Step 1 $F = 2 \times 3 = 6$	Calculate ac. Call the result F.
Step 2 $G = 3 \times 1 = 3$	Calculate bd. Call the result G.
Step 3 $H = 5 \times 4 = 20$	Calculate $(a + b)(c + d)$. Call the result H.
Step 4 $K = 20 - 6 - 3 = 11$	Calculate $H - F - G$. Call the result K.
Step 5 $mn = 600 + 110 + 3 = 713$	Calculate $100F + 10K + G$. The result is mn.

The general form of Karatsuba's multiplication algorithm applies to pairs of numbers with any number of digits. There are now even more efficient multiplication algorithms, and mathematicians continue to work on improving them.

The binary number system

The **decimal number system** uses strings of the digits 0 to 9 to represent numbers. The positions of the digits correspond to different powers of 10. For example:

$$352 = (3 \times 10^2) + (5 \times 10^1) + (2 \times 10^0)$$

When using the decimal number system, we say that we are writing numbers in **base 10**.

The **binary number system** uses only the digits 0 and 1 to represent numbers. The positions of the digits correspond to different powers of 2.

For example, we write 53 in binary form as

$$110101 = (1 \times 2^5) + (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$$

When using the binary number system, we say that we are writing numbers in **base 2**.

Converting a decimal number to a binary number

We first remind ourselves of the meaning of the words 'quotient' and 'remainder'. For example, when dividing 53 by 2, we obtain

$$53 = 26 \times 2 + 1$$

Here 26 is the quotient and 1 is the remainder.

We now describe a method for converting 53 into binary form. The calculations are shown in the table on the right.

- Divide 53 by 2. Record the quotient and remainder.
- Divide the previous quotient by 2. Record the new quotient and remainder.
- Continue until the quotient is zero.
- Write the remainders in reverse order.

Hence 53 in binary is 110101.

Division	Quotient	Remainder
$53 \div 2$	26	1
$26 \div 2$	13	0
$13 \div 2$	6	1
$6 \div 2$	3	0
$3 \div 2$	1	1
$1 \div 2$	0	1

The general method is described more formally by the following algorithm.

Algorithm for converting a decimal number to a binary number

To convert a natural number n into binary notation:

Step 1 Input n.

Step 2 Let q be the quotient when n is divided by 2.

Step 3 Let r be the remainder when n is divided by 2.

Step 4 Record r.

Step 5 Let n have the value of q.

Step 6 If $n > 0$, then repeat from Step 2.

Step 7 Write the recorded values of r in reverse order.

Example 2

Convert the decimal number 237 into binary form.

Solution

We follow the algorithm step by step, recording the values of n, q and r after each step.

n	q	r
237	118	1
118	59	0
59	29	1
29	14	1
14	7	0
7	3	1
3	1	1
1	0	1
0		

$237 = 118 \times 2 + 1$

$118 = 59 \times 2 + 0$

$59 = 29 \times 2 + 1$

$29 = 14 \times 2 + 1$

$14 = 7 \times 2 + 0$

$7 = 3 \times 2 + 1$

$3 = 1 \times 2 + 1$

$1 = 0 \times 2 + 1$

We write the values of r in reverse order. Hence 237 in binary is 11101101.

The Euclidean algorithm

In Section 2D, we found the highest common factor of two natural numbers using prime decomposition. Here we describe a more efficient method for finding highest common factors, called the **Euclidean algorithm**. This is one of the earliest recorded algorithms.

We used division by 2 in the previous algorithm. More generally, you are familiar with dividing one natural number by another. For example:

- $65 \div 7$ gives $65 = 9 \times 7 + 2$
- $91 \div 3$ gives $91 = 30 \times 3 + 1$

We can formalise this process as follows.

Euclidean division

If a and b are integers with $b > 0$, then there are unique integers q and r such that

$$a = qb + r \quad \text{where} \quad 0 \le r < b$$

Note: Here q is the **quotient** and r is the **remainder** when a is divided by b.

The following theorem is useful for finding the highest common factor of any two given integers. We use HCF(a, b) to denote the highest common factor of two integers a and b.

Theorem

Let a and b be two integers with $b \neq 0$. If $a = qb + r$, where q and r are integers, then HCF(a, b) = HCF(b, r).

We show how to use this theorem in the next example, which will motivate our description of the Euclidean algorithm.

Example 3

Find the highest common factor of 72 and 42.

Solution

At each step, we use Euclidean division and the previous theorem:

$$\underline{72} = 1 \times \underline{42} + \underline{30} \quad \text{and so} \quad \text{HCF}(72, 42) = \text{HCF}(42, 30)$$

$$\underline{42} = 1 \times \underline{30} + \underline{12} \quad \text{and so} \quad \text{HCF}(42, 30) = \text{HCF}(30, 12)$$

$$\underline{30} = 2 \times \underline{12} + \underline{6} \quad \text{and so} \quad \text{HCF}(30, 12) = \text{HCF}(12, 6)$$

$$\underline{12} = 2 \times \underline{6} + \underline{0} \quad \text{and so} \quad \text{HCF}(12, 6) = \text{HCF}(6, 0) = 6$$

Hence it follows that HCF$(72, 42) = 6$.

Note: In general, we keep using Euclidean division until we get remainder zero. Then the HCF is the last non-zero remainder.

This method can be formalised into the Euclidean algorithm as follows.

Euclidean algorithm

To find the highest common factor of two natural numbers a and b, where $a > b$:

Step 1 Input a and b.

Step 2 Let r be the remainder when a is divided by b.

Step 3 If $r = 0$, then go to Step 7.

Step 4 Let a have the value of b.

Step 5 Let b have the value of r.

Step 6 Repeat from Step 2.

Step 7 The required value is b.

Example 4

Find the highest common factor of 72 and 42 using the Euclidean algorithm.

Solution

Here we start with $a = 72$ and $b = 42$. We follow the algorithm step by step, recording the values of a, b and r after each step.

a	b	r
72	42	30
42	30	12
30	12	6
12	6	0

$\underline{72} = 1 \times \underline{42} + \underline{30}$

$\underline{42} = 1 \times \underline{30} + \underline{12}$

$\underline{30} = 2 \times \underline{12} + \underline{6}$

$\underline{12} = 2 \times \underline{6} + \underline{0}$

The highest common factor is the final value of b. Hence $\text{HCF}(72, 42) = 6$.

Summary 8A

- An **algorithm** is a finite, unambiguous sequence of instructions for performing a specific task.
- In this section, we have seen three examples of algorithms:
  - Karatsuba's multiplication algorithm
  - an algorithm for converting a decimal number to a binary number
  - the Euclidean algorithm.

Exercise 8A

Example 1

1 Use Karatsuba's multiplication algorithm to calculate each of the following products. Give the values of F, G, H and K in each case.

a 92×37 **b** 43×26 **c** 27×19 **d** 57×23

Example 2

2 Write each of the following decimal numbers in binary form:

a 342 **b** 127 **c** 1777 **d** 2468

3 **a** Describe an algorithm for converting a decimal number into base 8.

b Use your algorithm to convert each of the following decimal numbers into base 8:

i 342 **ii** 5678 **iii** 453 **iv** 9647

Example 4

4 Use the Euclidean algorithm to find:

a HCF(9284, 4361) **b** HCF(2160, 999)

c HCF(762, 372) **d** HCF(716 485, 5255)

5 The following algorithm can be applied to a polynomial $P(x)$:

Step 1 Write the polynomial in order of decreasing powers of x.

Step 2 Factor x out of every non-constant term.

Step 3 Factor x out of every non-constant term in the innermost brackets.

Step 4 Repeat Step 3 until only a constant remains in the innermost brackets.

The resulting expression is called the **nested form** of the polynomial. For example, we express the polynomial $P(x) = 3x^3 - 4x^2 + 7x + 4$ in nested form as follows:

$$\begin{aligned} P(x) &= 3x^3 - 4x^2 + 7x + 4 \\ &= (3x^2 - 4x + 7)x + 4 \\ &= ((3x - 4)x + 7)x + 4 \\ &= (((3)x - 4)x + 7)x + 4 \end{aligned}$$

Use this algorithm to write each of the following polynomials in nested form:

a $2x^2 + 3x + 4$ **b** $x^3 + 3x^2 - 4x + 5$ **c** $4x^3 + 6x^2 - 5x - 4$

6 Prove the theorem used to justify the Euclidean algorithm: *Let a and b be two integers with b ≠ 0. If a = qb + r, where q and r are integers, then* HCF(a, b) = HCF(b, r).

7 The Babylonians (1500 BC) had a method for determining the square root of a natural number N by using a recurrence relation. The recurrence relation is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \qquad \text{where } x_1 \text{ is some initial guess for } \sqrt{N}$$

You can think of this as finding the average of x_n and $\frac{N}{x_n}$ repeatedly.

a Express this method as an algorithm for calculating a square root to a given accuracy.

Hint: To decide when to stop generating the sequence, check whether the square of the current term is sufficiently close to N; e.g. $-0.01 < (x_n)^2 - N < 0.01$.

b Use the algorithm to calculate the square root of each of the following numbers, correct to one decimal place:

i 5 **ii** 345 **iii** 1563 **iv** 7856

8 Sieve of Eratosthenes This is a simple ancient algorithm for finding all the prime numbers up to a particular natural number n.

Step 1 Create a list of all natural numbers from 1 to n.

Step 2 Cross out 1 and let p be 2 (the first prime).

Step 3 Circle p and then cross out all the other multiples of p.

Step 4 Let p be the smallest number in the list that has not been marked.

Step 5 If $p^2 \leq n$, then repeat from Step 3.

The primes are all the numbers in the list that are not crossed out.

Carry out this algorithm for $n = 100$. Hint: Write the numbers in a 10×10 grid.

8B Iteration and selection

In this section, we start to become slightly more formal in the language we use to talk about algorithms and in the notation we use to describe them.

Assigning values to variables

The concept of a variable used in algorithms is different from that used in pure mathematics.

A **variable** is a string of one or more letters that acts as a placeholder that can be assigned different values.

We will use an arrow pointing from right to left to denote the assignment of a value to a variable. For example, the notation $x \leftarrow 3$ means 'assign the value 3 to the variable x'.

Consider the following instructions:

Step 1 $x \leftarrow 3$

Step 2 $y \leftarrow x$

Step 3 $x \leftarrow 4$

After following these steps, we have $x = 4$ and $y = 3$.

Often the new value assigned to a variable will depend on its old value. The following table gives two such examples.

Notation	Meaning	Initial value	New value
$A \leftarrow A + 5$	Replace the value of A with $A + 5$	$A = 3$	$A = 8$
$A \leftarrow 2A$	Replace the value of A with $2A$	$A = 3$	$A = 6$

Controlling the flow of steps

When describing an algorithm, the order of the steps is very important. The steps are typically carried out one after the other. However, there are two fundamental constructs that allow us to change the flow of steps: *iteration* and *selection*.

Iteration

Looping constructs allow us to repeat steps in a controlled way; this is called **iteration**.

The following example illustrates iteration by generating terms of an arithmetic sequence.

Example 5

a Write an algorithm to find the first six terms of the arithmetic sequence with first term 10 and common difference 7.

b Illustrate the algorithm with a flowchart.

c Demonstrate the algorithm with a table of values.

Solution

a We use the variable T for the current term.

We use the variable n for the index of the current term (i.e. its position in the sequence).

Step 1 $T \leftarrow 10$ and $n \leftarrow 1$

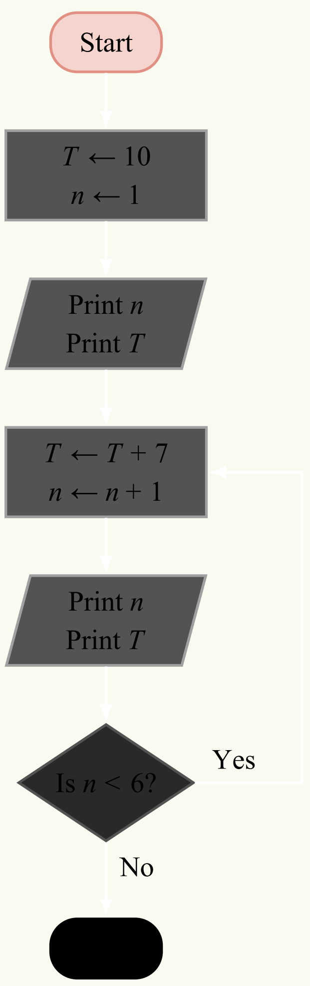
Step 2 Print n and print T

Step 3 $T \leftarrow T + 7$ and $n \leftarrow n + 1$

Step 4 Print n and print T

Step 5 Repeat from Step 3 while $n < 6$

b



c

n	T
1	10
2	17
3	24
4	31
5	38
6	45

Note: There are five iterations: you follow the instructions in the loop five times.

Example 6

An initial amount of \$100 000 is invested at an interest rate of 5% p.a. compounded annually.

a Write an algorithm to find the value of the investment at the end of each year for the first five years.

b Illustrate the algorithm with a flowchart.

c Demonstrate the algorithm with a table of values.

Solution

The yearly interest rate is $5\% = 0.05$. Therefore the value of the investment increases by a factor of 1.05 each year.

a We use the variable A for the current value of the investment.

We use the variable i to keep track of the number of iterations.

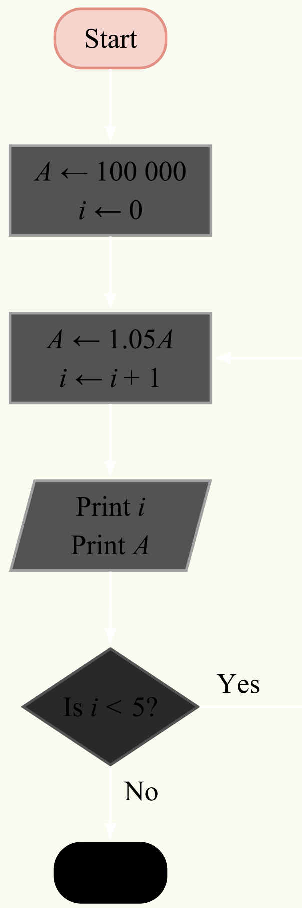
Step 1 $A \leftarrow 100\,000$ and $i \leftarrow 0$

Step 2 $A \leftarrow 1.05A$ and $i \leftarrow i + 1$

Step 3 Print i and print A

Step 4 Repeat from Step 2 while $i < 5$

b



c

i	A
0	100 000.00
1	105 000.00
2	110 250.00
3	115 762.50
4	121 550.63
5	127 628.16

Note: There are five iterations: you follow the instructions in the loop five times.

Selection

Decision-making constructs allow us to specify whether certain steps should be followed based on some condition. For example, we can use an instruction such as 'If ... then ...'. This is called **selection**.

We illustrate selection in the next example by considering a piecewise-defined sequence.

Example 7

For $n \in \mathbb{N}$, define $t_n = \begin{cases} 2n+4 & \text{if } n \text{ is even} \\ n+3 & \text{if } n \text{ is odd} \end{cases}$

a Write an algorithm to generate the first N terms of this sequence.

b Demonstrate the algorithm for $N = 6$ with a table of values.

Solution

a We use T for the current term of the sequence.
We use n for the index of the current term.

Step 1 $n \leftarrow 1$

Step 2 If n is even, then $T \leftarrow 2n + 4$
Otherwise $T \leftarrow n + 3$

Step 3 Print T

Step 4 $n \leftarrow n + 1$

Step 5 Repeat from Step 2 while $n \leq N$

b

n	T
1	4
2	8
3	6
4	12
5	8
6	16
7	

Summary 8B

- **Variables**
  - A **variable** is a string of one or more letters that acts as a placeholder that can be assigned different values.
  - The notation $x \leftarrow 3$ means 'assign the value 3 to the variable x'.
- **Iteration** Loops are used in algorithms to enable repetition of the same instructions.
- **Selection** 'If ... then ...' instructions are used in algorithms to enable logical decisions to be made within the algorithm.

Exercise 8B

Example 5

1 **a** Write an algorithm to find the first six terms of the arithmetic sequence with first term 6 and common difference 3.

b Illustrate the algorithm with a flowchart.

c Demonstrate the algorithm with a table of values.

Example 6

2 An initial amount of \$100 000 is invested at an interest rate of 2.5% p.a. compounded annually.

a Write an algorithm to find the value of the investment at the end of each of the first five years.

b Illustrate the algorithm with a flowchart.

c Demonstrate the algorithm with a table of values. (Give values to the nearest dollar.)

3 For each of the following algorithms, give a table showing the values of A and n after each step:

a
Step 1 $A \leftarrow 10$ and $n \leftarrow 1$
Step 2 $A \leftarrow A + 5$ and $n \leftarrow n + 1$
Step 3 Repeat Step 2 while $n \le 5$

b
Step 1 $A \leftarrow 2$ and $n \leftarrow 1$
Step 2 $A \leftarrow 3A$ and $n \leftarrow n + 1$
Step 3 Repeat Step 2 while $n \le 4$

4 Approximations of infinite sums Here is an example of an infinite sum:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

This formula is attributed to the Indian mathematician Madhava in the fourteenth century. The following algorithm can be used to evaluate the sum of the first N terms of this infinite sum.

Step 1 $sum \leftarrow 0$ and $n \leftarrow 1$

Step 2 $sum \leftarrow sum + \dfrac{(-1)^{n+1}}{2n-1}$

Step 3 $n \leftarrow n + 1$

Step 4 Repeat from Step 2 while $n \le N$

a In 1735 Euler proved the following remarkable formula involving π:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

Write an algorithm to evaluate the sum of the first N terms of this infinite sum.

b It can be proved that the infinite sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

grows without bound. Write an algorithm to evaluate the sum of the first N terms of this infinite sum.

Example 7

5 For $n \in \mathbb{N}$, define

$$t_n = \begin{cases} 5 - 2n & \text{if } n \text{ is even} \\ n^2 + 1 & \text{if } n \text{ is odd} \end{cases}$$

a Write an algorithm to generate the first N terms of this sequence.

b Demonstrate the algorithm for $N = 6$ with a table of values.

6 **Horner's algorithm** This algorithm was introduced in the nineteenth century to speed up the evaluation of polynomials by hand. Here we restrict to a cubic polynomial $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. To see why the algorithm works, we write the polynomial in nested form. Then we can see that the algorithm evaluates $P(x_0)$ by starting at the innermost brackets and working outwards.

Step 1 $p \leftarrow 0$ and $i \leftarrow 3$

Step 2 $p \leftarrow p \times x_0 + a_i$

Step 3 $i \leftarrow i - 1$

Step 4 Repeat from Step 2 while $i \geq 0$

Step 5 Print p

$$\begin{aligned} P(x) &= a_3x^3 + a_2x^2 + a_1x + a_0 \\ &= (a_3x^2 + a_2x + a_1)x + a_0 \\ &= ((a_3x + a_2)x + a_1)x + a_0 \\ &= (((a_3)x + a_2)x + a_1)x + a_0 \end{aligned}$$

The table of values on the right demonstrates this algorithm for $P(x) = 3x^3 - 4x^2 + 7x + 4$ and $x_0 = 5$.

Here $a_3 = 3$, $a_2 = -4$, $a_1 = 7$ and $a_0 = 4$.

The algorithm gives $P(5) = 314$.

p	i	
0	3	$p \leftarrow 0$
3	2	$p \leftarrow 0 \times 5 + 3$
11	1	$p \leftarrow 3 \times 5 - 4$
62	0	$p \leftarrow 11 \times 5 + 7$
314	−1	$p \leftarrow 62 \times 5 + 4$

Use Horner's algorithm to evaluate $P(3)$ for each of the following:

a $P(x) = x^3 + 2x^2 - 3x + 1$

b $P(x) = 2x^3 - x^2 + 4x - 2$

c $P(x) = -4x^3 + 2x^2 - x - 1$

7 **a** The following is an algorithm to draw an equilateral triangle in the plane. Verify that it does indeed produce such a triangle.

Step 1 $n \leftarrow 1$

Step 2 Draw forwards for 3 cm

Step 3 Turn through 120° anticlockwise

Step 4 $n \leftarrow n + 1$

Step 5 Repeat from Step 2 while $n \leq 3$

b Write a similar algorithm to draw a square.

c Write a similar algorithm to draw a regular hexagon.

d Draw the shape described by the following algorithm:

Step 1 $n \leftarrow 1$

Step 2 Draw forwards for 3 cm

Step 3 Turn through 144° anticlockwise

Step 4 $n \leftarrow n + 1$

Step 5 Repeat from Step 2 while $n \leq 5$

8 Collatz conjecture Starting with any natural number n, define a sequence as follows:

- If the current number is even, then obtain a new number by halving it.
- If the current number is odd and greater than 1, then obtain a new number by multiplying it by 3 and adding 1.

For example, starting with 6 we obtain the sequence 6, 3, 10, 5, 16, 8, 4, 2, 1.
The conjecture asserts that, for any natural number n, the sequence will always reach 1.

a Write an algorithm to generate the sequence for a given starting number n.

b Use your algorithm with the following starting numbers:

i $n = 5$ **ii** $n = 7$ **iii** $n = 8$

8C Introduction to pseudocode

In this section, we take a step further in formalising the language we use to write algorithms, so that it is closer to the code necessary to instruct a computer to perform an algorithm.

Terminology

A **computer** can be defined as a machine that carries out the instructions of algorithms. Computers consist of both **hardware** (the physical machine) and **software** (the instructions they follow). Since computers cannot directly understand natural language, we have to provide them with instructions in a **programming language**. The process of taking an algorithm and writing it in a programming language is called **coding**.

The algorithms in this section are written in **pseudocode**. This is an informal notation for writing instructions that is closer to natural language. It makes no reference to any particular programming language. In order to actually implement these algorithms on a computer, they must be translated into a specific programming language.

If–then blocks

In the previous section, we used 'If . . . then . . .' instructions for making decisions within an algorithm. In pseudocode, we use `if-then` blocks.

- The basic template for an `if-then` block is shown on the right.

```
if condition then
    follow these instructions
end if
```

- We can strengthen this construct by specifying alternative instructions to be followed when the given condition is not satisfied.

```
if condition then
    follow these instructions
else
    follow these instructions
end if
```

- We can iterate this construct as shown in the template on the right.

```
if first condition then
    follow these instructions
else if second condition then
    follow these instructions
else
    follow these instructions
end if
```

Example 8

Using pseudocode, write an algorithm to find the maximum of two numbers a and b.

Solution

```
input a, b
if a ≥ b then
    print a
else
    print b
end if
```

Example 9

Write an algorithm that assigns a letter grade based on a mark out of 100.

Solution

```
input mark
if mark ≥ 90 then
    print 'A'
else if mark ≥ 75 then
    print 'B'
else if mark ≥ 60 then
    print 'C'
else if mark ≥ 50 then
    print 'D'
else
    print 'E'
end if
```

For loops

A for loop provides a means of repeatedly executing the same set of instructions in a controlled way. This is achieved by performing one iteration for each term in a specified finite sequence.

We will use for loops based on the following template:

```
for i from 1 to n
    follow these instructions
end for
```

This for loop uses the sequence $1, 2, 3, \dots, n$. There are n iterations. You follow the instructions in the loop n times, with the variable i taking the values $1, 2, 3, \dots, n$ in turn.

Example 10

Consider the sequence $1^2, 2^2, 3^2, \dots, n^2$.

Using pseudocode, write an algorithm to calculate:

a the sum of the terms in this sequence

b the product of the terms in this sequence.

Provide a table of values to demonstrate each algorithm when $n = 4$.

Solution

a We use the variable *sum* for the current sum at each step of the algorithm. The initial value of *sum* must be 0.

```
input n
sum ← 0
for i from 1 to n
    sum ← sum + i^2
end for
print sum
```

i	*sum*
	0
1	$0 + 1^2 = 1$
2	$1 + 2^2 = 5$
3	$5 + 3^2 = 14$
4	$14 + 4^2 = 30$

b We use the variable *product* for the current product at each step of the algorithm. The initial value of *product* must be 1.

```
input n
product ← 1
for i from 1 to n
    product ← product × i^2
end for
print product
```

i	*product*
	1
1	$1 \times 1^2 = 1$
2	$1 \times 2^2 = 4$
3	$4 \times 3^2 = 36$
4	$36 \times 4^2 = 576$

In the previous example, we constructed a table of values to demonstrate each algorithm. This is called a **desk check**. In general, we carry out a desk check of an algorithm by carefully following the algorithm step by step, and constructing a table of the values of all the variables after each step.

While loops

A `while` loop provides another means of repeatedly executing the same set of instructions in a controlled way. This construct is useful when the number of iterations required to perform a task is unknown. For example, this may happen when we want to achieve a certain accuracy. A `while` loop will perform iterations indefinitely, as long as some condition remains true.

Every `while` loop is based on the following template:

```
while condition
    follow these instructions
end while
```

We have used Euclidean division in Section 8A. In the following example, we construct a division algorithm. The algorithm works by counting the number of times that the divisor can be subtracted from the dividend.

Example 11

Write an algorithm that divides 72 by 14 and returns the quotient and remainder. Show a desk check to test the operation of the algorithm.

Solution

We use a `while` loop, since we don't know how many iterations will be required.

The variable *count* keeps track of the number of times that 14 can be subtracted from 72. The variable *remainder* keeps track of the remainder after each subtraction.

```
count ← 0
remainder ← 72
while remainder ≥ 14
    count ← count + 1
    remainder ← remainder − 14
end while
print count, remainder
```

count	*remainder*
0	72
1	58
2	44
3	30
4	16
5	2

Note: The first output is 5 (the quotient) and the second output is 2 (the remainder). We can check that $72 = 5 \times 14 + 2$.

Example 12

Consider the sequence defined by the rule

$$x_{n+1} = 5x_n + 4, \qquad \text{where } x_1 = 3$$

Write an algorithm that will determine the smallest value of n for which $x_n > 10\,000$. Show a desk check to test the operation of the algorithm.

Solution

We use a `while` loop, since we don't know how many iterations will be required.

The variable x is used for the current term of the sequence, and the variable n is used to keep track of the number of iterations.

```
n ← 1
x ← 3
while  x ≤ 10 000
      n ← n + 1
      x ← 5x + 4
end while
print  n
```

n	x
1	3
2	19
3	99
4	499
5	2499
6	12 499

Note: The output is 6.

Summary 8C

- **If–then blocks** This construct provides a means of making decisions within an algorithm. Certain instructions are only followed if a condition is satisfied.

```
if condition then
      follow these instructions
end if
```

- **For loops** This construct provides a means of repeatedly executing the same set of instructions in a controlled way. In the template on the right, this is achieved by performing one iteration for each value of i in the sequence $1, 2, 3, \ldots, n$.

```
for i from 1 to n
      follow these instructions
end for
```

- **While loops** This construct provides another means of repeatedly executing the same set of instructions in a controlled way. This is achieved by performing iterations indefinitely, as long as some condition remains true.

```
while condition
      follow these instructions
end while
```

Skill-sheet

Exercise 8C

In this exercise, you will execute various algorithms. You may like to use a device to help carry out the steps; see the coding appendices in the Interactive Textbook for instructions.

Example 8

1 Using pseudocode, write an algorithm to find the minimum of two numbers a and b.

Example 9

2 Using this table, write an algorithm that assigns a letter grade based on a mark out of 100.

$95 \leq \text{mark} \leq 100$	A
$85 \leq \text{mark} < 95$	B
$65 \leq \text{mark} < 85$	C
$55 \leq \text{mark} < 65$	D
$0 \leq \text{mark} < 55$	E

Example 10

3 For the block of code shown on the right, determine the printed output when:

a $x = 0$

b $x = 1$

c $x = 5$

```
total ← x
for i from 1 to 5
    total ← total + i
end for
print total
```

4 For the block of code shown on the right, determine the printed output when:

a $x = 0$

b $x = 1$

c $x = 5$

```
total ← 0
for i from 1 to 5
    total ← total + x
end for
print total
```

5 For the block of code shown on the right, determine the printed output when:

a $x = 0$

b $x = 1$

c $x = 5$

```
total ← 0
for i from 1 to 5
    total ← x + i
end for
print total
```

Example 11

6 For the block of code shown on the right, determine the printed output for each of the following inputs:

a $a = 2,\ b = 3$

b $a = 2,\ b = 5$

c $a = 3,\ b = 2$

```
input a, b
while a^2 − b < 20
    b ← b + 2a
    a ← a + 2
end while
print a, b
```

7 Consider the sequence $2^1, 2^2, 2^3, \ldots, 2^n$. Using pseudocode, write an algorithm to find:

a the sum of the terms in this sequence

b the product of the terms in this sequence.

8 Using pseudocode, write an algorithm to calculate the sum $1^3 + 2^3 + 3^3 + \cdots + n^3$.

9 For the algorithm shown on the right, perform a desk check by constructing a table of the values of the variables a and b after each pass of the `while` loop.

```
a ← 8
b ← 6
while a − b < 20
    b ← ab − 30
    a ← b − 2a
end while
print a, b
```

Example 12

10 Consider the sequence defined by the rule $x_{n+1} = 3x_n + 2$, where $x_1 = 4$. Using pseudocode, write an algorithm to determine the smallest value of n for which $x_n > 1000$. Perform a desk check for your algorithm.

11 Using pseudocode, write an algorithm to find the smallest natural number n such that

$$1^1 + 2^2 + 3^3 + \cdots + n^n > 1\,000\,000$$

Perform a desk check for your algorithm.

12 Using pseudocode, write an algorithm to determine the smallest natural number n such that $2^n > 10n^2$.

13 The sequence defined by the rule $x_n = 2n + 3$ is increasing, while the sequence defined by the rule $y_n = 0.9^n \times 1000$ is decreasing. Using pseudocode, write an algorithm to determine the smallest value of n for which $x_n > y_n$.

14 Consider the block of code shown on the right.

a Perform a desk check for each of the following input values of a and b:

i $a = 64, \ b = 120$

ii $a = 360, \ b = 100$

iii $a = 144, \ b = 896$

b Describe what this algorithm does when you input two positive integers a and b.

```
input a, b
while a ≠ b
    if a < b then
        c ← a
        a ← b − a
        b ← c
    else
        c ← b
        b ← a − b
        a ← c
    end if
end while
print a
```

8D Further pseudocode

The real power of algorithms comes when we combine the basic constructs in clever ways to perform tasks that are not possible without the aid of computers.

In this section, we combine loops and `if-then` blocks. We also introduce three other important tools for the construction of algorithms: functions, lists and nested loops.

Functions

A block of code that performs a clearly defined task and can be separated out from the main algorithm is called a **function**. Functions must be defined before they are used. Once they are defined, they can be used again and again in different algorithms.

A function can have one or more inputs and return an output. Here are two simple examples of functions:

- Consider the linear function $f(x) = 3x + 2$. We can define this function for use in an algorithm as shown on the right.

  We can then call this function in an algorithm by writing $f(5)$, for example.

```
define f(x):
   y ← 3x + 2
   return y
```

- The function defined on the right has two inputs; it determines the distance from a point (x, y) to the origin.

  We can call this function in an algorithm by writing $dist(3, 4)$, for example.

```
define dist(x, y):
   dist ← √(x² + y²)
   return dist
```

The quotient and remainder functions

The following two functions can be used if we require Euclidean division of natural numbers in an algorithm. The first function returns the quotient when *number* is divided by *divisor*; the second function returns the remainder.

```
define quotient(number, divisor):
   count ← 0
   remainder ← number
   while remainder ≥ divisor
         count ← count + 1
         remainder ← remainder − divisor
   end while
   return count
```

```
define remainder(number, divisor):
   count ← 0
   remainder ← number
   while remainder ≥ divisor
         count ← count + 1
         remainder ← remainder − divisor
   end while
   return remainder
```

For example, we get $quotient(7, 3) = 2$ and $remainder(7, 3) = 1$. These two functions are built into most programming languages. The next two examples illustrate how they can be used.

Example 13

Write an algorithm that computes the sum of all the integers from 1 to 1000 that are divisible by 2 or 3.

Solution

We use the variable *sum* to keep a running total of the numbers that are divisible by 2 or 3. The initial value of *sum* must be 0.

```
sum ← 0
for i from 1 to 1000
    if remainder(i, 2) = 0 or remainder(i, 3) = 0 then
        sum ← sum + i
    end if
end for
print sum
```

Example 14

Write an algorithm that determines the number of digits in the decimal representation of a given natural number. Show a desk check for the number 7564.

Solution

We use a `while` loop, since we don't know how many iterations will be required.

The variable *count* keeps track of how many times we can divide the number by 10 before the quotient is zero.

```
input number
count ← 0
while number > 0
    number ← quotient(number, 10)
    count ← count + 1
end while
print count
```

number	*count*
7564	0
756	1
75	2
7	3
0	4

Using separately defined functions helps to make the structure of the main algorithm clearer and easier to understand.

Lists

In programming languages, a finite sequence is often called a **list**. We will write lists using square brackets. For example, we can define a list A by

$$A \leftarrow [2, 3, 5, 7, 11]$$

The notation $A[n]$ refers to the nth entry of the list. So $A[1] = 2$ and $A[5] = 11$.

We can add an entry to the end of a list using append. For example, the instruction

```
append 9 to A
```

would result in $A = [2, 3, 5, 7, 11, 9]$.

Note: The position of an entry in a list is called its **index**. In this book, we use 1 as the index of the first entry. But most programming languages use 0 as the index of the first entry.

Example 15

Perform a desk check for the algorithm shown on the right by giving a table of values for the variables i and A after each step.

```
A ← [5, 7]
for i from 1 to 3
    append 2i to A
end for
```

Solution

i	A
	$[5, 7]$
1	$[5, 7, 2]$
2	$[5, 7, 2, 4]$
3	$[5, 7, 2, 4, 6]$

Example 16

Write a function that returns a list of all the factors of a given natural number.

Solution

```
define factors(N):
  A ← []
  for i from 1 to N
      if remainder(N, i) = 0 then
          append i to A
      end if
  end for
  return A
```

Explanation

We start with an empty list A.

We use a for loop to go through all the integers from 1 to N.

Each time we find a factor of N, we append it to list A.

For example, this function will give $factors(10) = [1, 2, 5, 10]$.

We can find the length of a list using the function *length*. For example, if $A = [2, 3, 5, 7, 11]$, then the instruction

$$\ell \leftarrow length(A)$$

would result in $\ell = 5$.

Nested loops

In the previous section, we introduced `for` loops and `while` loops. In this section, we consider problems that require loops within loops. These are called **nested loops**.

A nested loop consists of an outer loop and an inner loop.

- The first pass of the outer loop starts the inner loop, which executes to completion.
- Then the second pass of the outer loop starts the inner loop again.
- This repeats until the outer loop finishes.

We begin with a simple example.

Example 17

The algorithm on the right has a nested loop. Perform a desk check for this algorithm that keeps track of the values of a, b and c after each step.

```
c ← 0
for a from 1 to 2
    for b from 1 to 4
        c ← c + 1
    end for
end for
```

Solution

a	b	c
		0
1	1	1
1	2	2
1	3	3
1	4	4
2	1	5
2	2	6
2	3	7
2	4	8

Explanation

The initial value of a in the outer loop is 1.

This value of a is taken into the inner loop, where b takes the values 1, 2, 3 and 4.

We exit the inner loop. Then a takes its next value in the outer loop, which is 2.

We enter the inner loop with this new value of a. Again b takes the values 1, 2, 3 and 4.

The next example illustrates how we can use nested loops to solve mathematical problems.

Example 18

Using pseudocode, write an algorithm to find the positive integer solutions of the equation

$$43x + 17y + 7z = 200$$

Solution

We use three loops to run through all the possible positive integer values of x, y and z. We first note that

$$200 \div 43 \approx 4.7, \qquad 200 \div 17 \approx 11.8, \qquad 200 \div 7 \approx 28.6$$

Therefore we know that we will find all the solutions from the following nest of three loops.

```
for x from 1 to 4
    for y from 1 to 11
        for z from 1 to 28
            if 43x + 17y + 7z = 200 then
                print (x, y, z)
            end if
        end for
    end for
end for
```

This algorithm prints the three solutions (1, 1, 20), (1, 8, 3) and (2, 3, 9).

Summary 8D

- **Functions** A **function** takes one or more input values and returns an output value. Functions can be defined and then used in other algorithms.

```
define function(input):
    follow these instructions
    return output
```

- **Lists** A **list** is a finite sequence. We write lists using square brackets. For example, we can assign $A \leftarrow [2, 3, 5, 7, 11]$. The notation $A[n]$ refers to the nth entry of the list. So $A[1] = 2$ and $A[5] = 11$.
  - To add an entry x to the end of list A, use the instruction `append x to A`.
  - To find the length of list A, use *length*(A).
- **Nested loops** A **nested loop** is a loop within a loop. The first pass of the outer loop starts the inner loop, which executes to completion. Then the second pass of the outer loop starts the inner loop again. This repeats until the outer loop finishes.

Skill-sheet

Exercise 8D

In this exercise, you will execute various algorithms. You may like to use a device to help carry out the steps; see the coding appendices in the Interactive Textbook for instructions.

Example 13
Example 14

1 The function defined on the right will find the sum of the squares of the first n natural numbers, that is, $1^2 + 2^2 + 3^2 + \cdots + n^2$. Modify this function so that it will find each of the following:

a $1^3 + 2^3 + 3^3 + \cdots + n^3$

b $2^1 + 2^2 + 2^3 + \cdots + 2^n$

c $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1)$

```
define sum(n):
  sum ← 0
  for i from 1 to n
      sum ← sum + i^2
  end for
  return sum
```

Example 15
Example 16

2 The function defined on the right will output the list $[1^2, 2^2, 3^2, \ldots, n^2]$.

a The second line of code is $A \leftarrow [\,]$. What is the purpose of this line?

b Rewrite the function so that it outputs the list $[2^1, 2^2, 2^3, \ldots, 2^n]$.

c Rewrite the function so that it outputs the previous list in reverse.

```
define squares(n):
  A ← [ ]
  for i from 1 to n
      append i^2 to A
  end for
  return A
```

3 The function defined on the right will find the maximum number in a given list of numbers. Rewrite this function so that it finds the minimum number.

```
define max(A):
  max ← A[1]
  for i from 1 to length(A)
      if A[i] > max then
          max ← A[i]
      end if
  end for
  return max
```

4 The factorial function can be defined as shown on the right. For a given natural number n, this function will return the value of

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

Using this function:

a write another function that will return the value of $1! + 2! + 3! + \cdots + n!$

b write an algorithm that will find the smallest natural number n for which $n! > 10^n$.

```
define factorial(n):
  product ← 1
  for i from 1 to n
      product ← product × i
  end for
  return product
```

Example 17

5 Each of the following algorithms has a nested loop. Perform a desk check for each algorithm by completing a table of values for the variables a, b and c after each step.

a

```
c ← 0
for a from 1 to 3
    for b from 1 to 3
        c ← c + 1
    end for
end for
```

b

```
c ← 0
for a from 2 to 3
    for b from 3 to 4
        c ← c + ab
    end for
end for
```

6 Consider the algorithm shown on the right.

a Perform a desk check for this algorithm by completing a table of values for the variables i and *tally* after each step.

b Describe in words what this algorithm evaluates.

```
A ← [1, 3, −5, 8]
tally ← 0
for i from 1 to 4
    tally ← tally + A[i]^2
end for
print tally
```

7 The algorithm shown on the right will print a list of the first seven terms of a well-known sequence of numbers.

```
A ← [1, 1]
for i from 1 to 5
    append A[i] + A[i + 1] to A
end for
print A
```

a Describe the result of this algorithm by completing a desk check that tabulates the values of the variables i and A.

b Modify the algorithm so that it will print the first term of the sequence that is greater than 1000.

c Modify the algorithm into a function that returns the nth term of the sequence.

d The **tribonacci sequence** is a modification of the Fibonacci sequence, where each term is the sum of the three preceding terms. The first five terms are 0, 1, 1, 2, 3. Change the algorithm so that it will print a list of the first 10 tribonacci numbers.

8 The algorithm on the right prints a list of the perfect squares between 1 and 1000.

```
A ← [ ]
n ← 1
while n^2 ≤ 1000
    append n^2 to A
    n ← n + 1
end while
print A
```

a The first entry of the printed list is $A[1] = 1$. Give the value of $A[5]$.

b Modify the algorithm so that it prints the perfect cubes between 1 and 100 000.

c How many entries will be in the printed list of your modified algorithm?

Example 18

9 Using pseudocode, write an algorithm to find all the positive integer solutions of the equation $3x + 5y + 7z = 30$.

10 **a** Using pseudocode, write an algorithm to find all the positive integer solutions of the equation $x^2 + y^2 + 10z = 30$.

b Determine the four solutions.

c Modify your algorithm from part **a** to find the positive integer solution of the simultaneous equations $x^2 + y^2 + 10z = 30$ and $x + y + z = 7$.

11 Let n be a positive integer. Write a function that will return a list of every solution of the equation $x^2 + y^2 = n$, where x and y are non-negative integers.

12 We know that a quadratic equation $ax^2 + bx + c = 0$ has no real solutions if $b^2 - 4ac < 0$. By considering integer values of the coefficients a, b and c from -10 to 10, describe what the following algorithm does.

```
count ← 0
number ← 0
for a from −10 to 10
    if a ≠ 0 then
        for b from −10 to 10
            for c from −10 to 10
                count ← count + 1
                if b² − 4ac < 0 then
                    number ← number + 1
                end if
            end for
        end for
    end if
end for
print count, number, number/count
```

13 In Example 16, we defined a function *factors*(N) that returns a list of all the factors of a given natural number N.

a Using this function, write an algorithm that determines whether a given natural number N is prime.

b Write a function *prime*(n) that returns the nth prime number.

14 In this section, we have defined the function *remainder*(*number*, *divisor*), which returns the remainder when *number* is divided by *divisor*. Making use of this function, write a function that inputs a natural number n and returns:

a the highest power of 2 that is a factor of n

b the smallest natural number that is divisible by all the numbers $1, 2, 3, \ldots, n$.

15 The sequence of Pell numbers is defined by the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$, where $P_1 = 1$ and $P_2 = 2$.

a Write a function that returns the nth Pell number.

b Using this function, write an algorithm to find the sum of the first n Pell numbers.

c Write an algorithm to find the first Pell number with at least 1000 decimal digits.

16 Here we outline a process that can be described neatly with an algorithm:

Step 1 Initially, bag A contains a counters, and bag B contains b counters.

Step 2 If one bag has more counters than the other, then we double the number of counters in the bag with fewer counters by taking the required number from the bag with more.

Step 3 Repeat Step 2 until bags A and B have the same number of counters.

Note that Step 2 can be described using an `if-then` block as shown on the right.

```
if a < b then
    b ← b − a
    a ← 2a
else if b < a then
    a ← a − b
    b ← 2b
end if
```

a Write the complete process as an algorithm in pseudocode.

b Use the algorithm to see what happens when:

i $a = 21,\ b = 28$

ii $a = 21,\ b = 49$

iii $a = 35,\ b = 105$

iv $a = 19,\ b = 133$

v $a = 37,\ b = 259$

c Find another two sets of initial values for a and b such that the algorithm will eventually stop.

17 The Euclidean algorithm for finding the highest common factor of two natural numbers is introduced in Section 8A. Describe this algorithm using pseudocode.

Chapter summary

Assign-ment

Nrich

Algorithms

- An **algorithm** is a finite, unambiguous sequence of instructions for performing a specific task.
- An algorithm can be described using step-by-step instructions, illustrated by a flowchart, or written out in pseudocode.

Pseudocode

- Variables
  - A **variable** is a string of one or more letters that acts as a placeholder that can be assigned different values.
  - The notation $x \leftarrow 3$ means 'assign the value 3 to the variable x'.
  - A **desk check** of an algorithm is achieved by following the algorithm step by step, and constructing a table of the values of all the variables after each step.
- If–then blocks This construct provides a means of making decisions within an algorithm. Certain instructions are only followed if a condition is satisfied.

```
if condition then
    follow these instructions
end if
```

- For loops This construct provides a means of repeatedly executing the same set of instructions in a controlled way. In the template on the right, this is achieved by performing one iteration for each value of i in the sequence $1, 2, 3, \dots, n$.

```
for i from 1 to n
    follow these instructions
end for
```

- While loops This construct provides another means of repeatedly executing the same set of instructions in a controlled way. This is achieved by performing iterations indefinitely, as long as some condition remains true.

```
while condition
    follow these instructions
end while
```

- Functions A **function** takes one or more input values and returns an output value. Functions can be defined and then used in other algorithms.

```
define function(input):
  follow these instructions
  return output
```

- Lists
  - A **list** is a finite sequence.
  - We write lists using square brackets. For example, we can assign $A \leftarrow [2, 3, 5, 7, 11]$. The notation $A[n]$ refers to the nth entry of the list. So $A[1] = 2$ and $A[5] = 11$.
  - To add an entry x to the end of list A, use the instruction `append x to A`.
  - To find the length of list A, use $length(A)$.

Review

Technology-free questions

1 Find the printed output for each of the following blocks of code:

a
```
x ← 5
for n from 1 to 3
    x ← x + 1
end for
print x
```

b
```
A ← [2, 1]
for i from 1 to 5
    append A[i] + A[i + 1] to A
end for
print A[7]
```

c
```
a ← 3
i ← 1
while i < 5
    a ← 2a + 3
    i ← i + 1
end while
print a
```

d
```
a ← 2
b ← 40
while a² < b − 2
    b ← b + a
    a ← a + 1
end while
print a, b
```

2 The block of code shown on the right evaluates the sum $1^2 + 2^2 + 3^2 + 4^2 + 5^2$.

Change the code so that it evaluates:

a $1^1 + 2^2 + 3^3 + 4^4 + 5^5 + 6^6$

b $1 \cdot 6 - 2 \cdot 5 + 3 \cdot 4 - 4 \cdot 3 + 5 \cdot 2 - 6 \cdot 1$

```
sum ← 0
for n from 1 to 5
    sum ← sum + n²
end for
print sum
```

3 Show a desk check for the following algorithm by completing the table of values.

```
a ← 1
b ← 2
for n from 1 to 5
    c ← n × b + 2a
    a ← b
    b ← c
    print n, a, b, c
end for
```

n	a	b	c
1			
2			
3			
4			
5			

4 Define a sequence recursively by $a_{n+1} = 3a_n + 2$, where $a_1 = 2$.

a Write down the first three terms of this sequence.

b Write pseudocode that would print the 50th term of this sequence, that is, a_{50}.

c Now adjust your pseudocode so that it would print the sum of the first 50 terms, that is, $a_1 + a_2 + a_3 + \cdots + a_{50}$.

5 For $n \in \mathbb{N}$, define $t_n = \begin{cases} 6 - 2n & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$

a Write an algorithm in pseudocode to print the first N terms of this sequence.

b Demonstrate the algorithm for $N = 5$ with a table of values.

c Modify your algorithm so that it prints the sum of the first N terms.

6 Write an algorithm in pseudocode to find all the pairs of integers (a, b) such that $9 \le a^2 + b^2 \le 36$.

7 **Bisection method** This method is used for finding approximate solutions of equations of the form $f(x) = 0$. Here we illustrate the method by finding an approximation of $\sqrt{2}$, accurate to within 0.1.

The general idea is as follows:

- Start with the interval $[a, b] = [0, 2]$.
- Find the midpoint m of the interval.
- Decide which of the two subintervals $[a, m]$ or $[m, b]$ contains $\sqrt{2}$.
- Rename the chosen subinterval as $[a, b]$ and continue the process.

The algorithm shown on the right implements this idea.

```
define f(x):
  return x^2 − 2

a ← 0
b ← 2
m ← 1
while b − a > 2 × 0.1
    if f(a) × f(m) < 0 then
        b ← m
    else
        a ← m
    end if
    m ← (a + b)/2
    print a, m, b
end while
print m
```

a Perform a desk check for the algorithm by completing the following table.

a	m	b	$f(a)$	$f(m)$	$f(b)$
0	1	2	−2	−1	2

b Modify the algorithm so that it approximates $\sqrt{3}$ to within 0.01.

Multiple-choice questions

1 When the algorithm on the right is executed, the printed value of a will be

A 8 **B** 9 **C** 10 **D** 11 **E** 16

```
a ← 1
while a < 10
    a ← 2a
end while
print a
```

Review

2 When the algorithm on the right is executed, the printed value of i will be

A 1 **B** 2 **C** 3 **D** 4 **E** 5

```
i ← 0
sum ← 0
while sum < 10
    i ← i + 1
    sum ← sum + i
end while
print i
```

3 When the algorithm on the right is executed, only one value will be printed. This value is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

```
A ← [1, 2, 3, 4, 5]
B ← [5, 4, 3, 2, 1]
for i from 1 to 5
    if A[i] = B[i] then
        print A[i]
    end if
end for
```

4 Consider the function defined on the right, which takes a natural number n as its input. The value of $f(4)$ is

A −2 **B** −1 **C** 0 **D** 1 **E** 2

```
define f(n):
  sum ← 0
  for i from 1 to n
      sum ← sum + (−1)^i × i
  end for
  return sum
```

5 The function defined on the right takes a list A as its input. If the input is $A = [1, 3, 5]$, then the output is

A [2, 5, 8] **B** [5, 3, 1]
C [2, 4, 6] **D** [2, 6, 10]
E [4, 6, 8]

```
define f(A):
  B ← [ ]
  for i from 1 to length(A)
      append A[i] + i to B
  end for
  return B
```

6 When the algorithm on the right is executed, the printed value of *sum* will be

A 7 **B** 8 **C** 9 **D** 10 **E** 11

```
sum ← 0
for i from 1 to 2
    for j from 1 to 2
        sum ← sum + i × j
    end for
end for
print sum
```

7 The function defined on the right takes a pair of positive integers (a, b) as its input. How many different inputs will return a value of 8?

A 0 **B** 1 **C** 2 **D** 4 **E** 8

```
define f(a, b):
  if a ≤ b then
      c ← a^b
  else
      c ← b^a
  end if
  return c
```

8 The function *remainder*(n, d) returns the remainder when n is divided by d.

Consider the function defined on the right, which takes a natural number n as its input. The value of $f(11)$ is

A 1 **B** 2 **C** 3
D 4 **E** 5

```
define f(n):
  count ← 0
  while n ≠ 1
      if remainder(n, 2) = 0 then
          n ← n/2
      else
          n ← n − 1
      end if
      count ← count + 1
  end while
  return count
```

9 The function *remainder*(n, d) returns the remainder when n is divided by d.

Consider the function defined on the right, which takes a natural number n as its input. If *function*$(n) = [1, 2, 4, 8, 16]$, then n must be equal to

A 1 **B** 2 **C** 4
D 8 **E** 16

```
define function(n):
  A ← [ ]
  for i from 1 to n
      if remainder(n, i) = 0 then
          append i to A
      end if
  end for
  return A
```

10 When the algorithm on the right is executed, the printed value of *sum* will be

A 10 **B** 15 **C** 20
D 25 **E** 30

```
sum ← 0
A ← [1, 2, 3, 4]
for i from 1 to 4
    sum ← sum + A[i] × A[5 − i]
end for
print sum
```

Extended-response questions

1 Converting to binary

In Section 8A, we discussed an algorithm for converting a natural number n into binary form. The following function gives a pseudocode implementation of that algorithm. We use the functions *quotient* and *remainder* from Section 8D. We also use a new function *reverse*, which reverses the order of a list.

```
define binary(n):
  A ← [ ]
  while n > 0
      r ← remainder(n, 2)
      append r to A
      n ← quotient(n, 2)
  end while
  A ← reverse(A)
  return A
```

a Carry out a desk check for:

i $n = 65$

ii $n = 4567$

iii $n = 54\,786$

b Using pseudocode, write a function to convert a natural number n into any base b. Carry out a desk check for:

i $b = 8$ and $n = 65$

ii $b = 8$ and $n = 4567$

iii $b = 8$ and $n = 54\,786$

c The following block of code illustrates how we can reverse the order of a list.

```
A ← [2, 4, 6, 8, 10]
B ← [ ]
ℓ ← length(A)
for i from 0 to ℓ − 1
    append A[ℓ − i] to B
end for
print B
```

i Carry out a desk check for this code.

ii Why do we need the second list B? What would happen if instead we used $A[i + 1] \leftarrow A[\ell - i]$ in the loop?

2 Bubble sort

Sorting lists into ascending order is an important process in both mathematics and computer science. The following algorithm illustrates a **bubble sort**. The idea is very simple: unsorted neighbours are swapped until the whole list is sorted.

This code implements a bubble sort for a list with six entries $A = [1, 9, 3, 2, 7, 6]$. The `if-then` block swaps entries $A[j]$ and $A[j+1]$ if they are in the wrong order.

```
A ← [1, 9, 3, 2, 7, 6]
for i from 1 to 6
    for j from 1 to 6 − i
        if A[j] > A[j + 1] then
            temp ← A[j]
            A[j] ← A[j + 1]
            A[j + 1] ← temp
        end if
    end for
end for
print A
```

For example, for $i = 1$, the variable j takes values from 1 to 5.

- $j = 1$: Since $A[1] \le A[2]$, these entries are *not* swapped.
- $j = 2$: Since $A[2] > A[3]$, these entries are swapped.

At the end of the $i = 1$ pass, we will have $A = [1, 3, 2, 7, 6, 9]$. The largest entry is now in the final position.

a Continue the following desk check until you reach the end of the $i = 2$ pass.

i	j	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$	
		1	9	3	2	7	6	*Initial list*
1	1	1	9	3	2	7	6	*Not swapped*
1	2	1	3	9	2	7	6	*Swapped*
1	3	1	3	2	9	7	6	*Swapped*

b Why do we need the variable *temp*? What would happen if instead we used the following code for the `if-then` block?

```
if A[j] > A[j + 1] then
    A[j] ← A[j + 1]
    A[j + 1] ← A[j]
end if
```

c Modify the pseudocode to define a function that will bubble sort an input list of any length.

Review

3 Palindromic squares

The function $R(n)$ defined on the right uses the functions *quotient* and *remainder* from Section 8D. It takes a natural number as input and returns another natural number.

```
define R(n):
  reverse ← 0
  while n > 0
      a ← remainder(n, 10)
      n ← quotient(n, 10)
      reverse ← reverse × 10 + a
  end while
  return reverse
```

a Carry out a desk check of this code with input $n = 5678$ by tabulating the values of the variables a, n and *reverse* after each pass of the `while` loop.

b Describe in words what this function does.

A natural number is **palindromic** if it remains the same when its digits are reversed. For example, the number 12321 is palindromic. Interestingly, it is also a square number, since $111^2 = 12321$.

c Making use of the function $R(n)$, write an algorithm in pseudocode that prints all the square numbers less than 1 000 000 that are palindromic.

4 Reverse-then-add algorithm

For each natural number n, let $R(n)$ be the natural number obtained when the digits of n are written in reverse order. For example, we have $R(123) = 321$. This function is implemented in pseudocode in Question 3. A natural number n is palindromic if and only if $R(n) = n$.

a Evaluate each of the following:

i $R(1234)$ **ii** $1234 + R(1234)$

iii $R(5678)$ **iv** $5678 + R(5678)$

b If each digit of n is less than 5, explain why the number $n + R(n)$ will be palindromic.

Now consider the function defined on the right. It takes a natural number, reverses its digits and adds the two numbers. It repeats this process until the result is palindromic.

```
define function(n):
  while R(n) ≠ n
      n ← n + R(n)
  end while
  return n
```

c Complete a desk check for this function with $n = 1756$. Give the values of n, $R(n)$ and $n + R(n)$ after each step.

Note: A **Lychrel number** is a natural number that does not eventually form a palindrome when repeatedly reversing its digits and adding the two numbers. So far, no natural numbers have been proven to have this property. However, the number 196 is thought to be the smallest example.

d What is the output of this function if the input is 1268? Determine how many different inputs return an output of 9889 by writing and running an appropriate program.

5 Pythagorean triples

A Pythagorean triple (a, b, c) consists of three natural numbers a, b and c such that $a^2 + b^2 = c^2$. The following algorithm will generate a Pythagorean triple for any input of a natural number.

Step 1 Input a natural number m.

Step 2 Let $n = m + 2$.

Step 3 Evaluate the sum of the reciprocals of m and n.

Step 4 Output the numerator and denominator of the result; they will be the first two entries of a Pythagorean triple.

a Find the output for $m = 3$.

b Find the output for $m = 5$.

c Prove that this algorithm always gives a Pythagorean triple for every natural number m.

6 Selection sort

In Question 2, we looked at an algorithm to sort a list into ascending order. In this question, your aim is to write your own sorting algorithm. You can use the following general method:

- Find the smallest entry in the list and swap it with the first entry.
- Find the smallest entry in the rest of the list and swap it with the second entry.
- Continue this process until the whole list is sorted.

Using pseudocode, write an algorithm to sort a list of six numbers.

7 Prime factorisation

Write an algorithm that computes the prime factorisation of an input natural number. The output should be a list with entries that give both the prime factor and its multiplicity. For example, since $200 = 2^3 \times 5^2$, the output for the number 200 should be [[2, 3], [5, 2]].

9

Combinatorics

Objectives

- ► To solve problems using the **addition** and **multiplication principles**.
- ► To solve problems involving **permutations**.
- ► To solve problems involving **combinations**.
- ► To establish and use identities associated with **Pascal's triangle**.
- ► To solve problems using the **pigeonhole principle**.
- ► To understand and apply the **inclusion–exclusion principle**.

Take a deck of 52 playing cards. This simple, familiar deck can be arranged in so many ways that if you and every other living human were to shuffle a deck once per second from the beginning of time, then by now only a tiny fraction of all possible arrangements would have been obtained. So, remarkably, every time you shuffle a deck you are likely to be the first person to have created that particular arrangement of cards!

To see this, note that we have 52 choices for the first card, and then 51 choices for the second card, and so on. This gives a total of

$$52 \times 51 \times \cdots \times 2 \times 1 \approx 8.1 \times 10^{67}$$

arrangements. This is quite an impressive number, especially in light of the fact that the universe is estimated to be merely 1.4×10^{10} years old.

Combinatorics is concerned with counting the number of ways of doing something. Our goal is to find clever ways of doing this without explicitly listing all the possibilities. This is particularly important in the study of probability. For instance, we can use combinatorics to explain why certain poker hands are more likely to occur than others without considering all 2 598 960 possible hands.

9A Basic counting methods

Tree diagrams

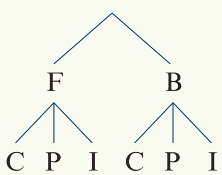
In most combinatorial problems, we are interested in the *number of solutions* to a given problem, rather than the solutions themselves. Nonetheless, for simple counting problems it is sometimes practical to list and then count all the solutions. Tree diagrams provide a systematic way of doing this, especially when the problem involves a small number of steps.

Example 1

A restaurant has a fixed menu, offering a choice of fish or beef for the main meal, and cake, pudding or ice-cream for dessert. How many different meals can be chosen?

Solution

We illustrate the possibilities on a tree diagram:



This gives six different meals, which we can write as

FC, FP, FI, BC, BP, BI

The multiplication principle

In the above example, for each of the two ways of selecting the main meal, there were three ways of selecting the dessert. This gives a total of $2 \times 3 = 6$ ways of choosing a meal. This is an example of the **multiplication principle**, which will be used extensively throughout this chapter.

Multiplication principle

If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

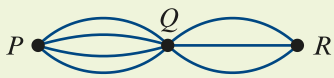
Example 2

Sandra has three different skirts, four different tops and five different pairs of shoes. How many choices does she have for a complete outfit?

Solution	Explanation
$3 \times 4 \times 5 = 60$	Using the multiplication principle, we multiply the number of ways of making each choice.

Example 3

How many paths are there from point P to point R travelling from left to right?



Solution

$4 \times 3 = 12$

Explanation

For each of the four paths from P to Q, there are three paths from Q to R.

The addition principle

In some instances, we have to count the number of ways of choosing between two alternative tasks. In this case, we use the **addition principle**.

Addition principle

Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Example 4

To travel from Melbourne to Sydney tomorrow, Kara has a choice between three different flights and two different trains. How many choices does she have?

Solution

$3 + 2 = 5$

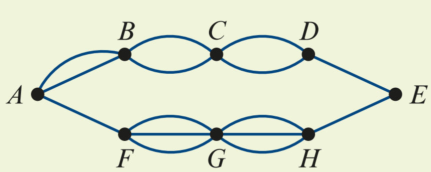
Explanation

The addition principle applies because Kara cannot travel by both plane and train. Therefore, we add the number of ways of making each choice.

Some problems will require use of both the multiplication and the addition principles.

Example 5

How many paths are there from point A to point E travelling from left to right?



Solution

We can take *either* an upper path *or* a lower path:

- Going from A to B to C to D to E there are $2 \times 2 \times 2 \times 1 = 8$ paths.
- Going from A to F to G to H to E there are $1 \times 3 \times 3 \times 1 = 9$ paths.

Using the addition principle, there is a total of $8 + 9 = 17$ paths from A to E.

Harder problems involving tree diagrams

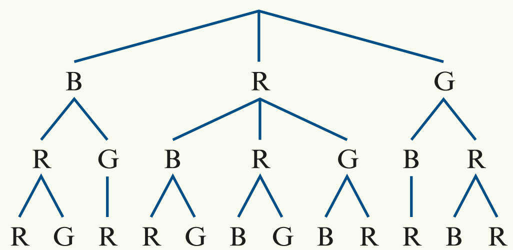
For some problems, a straightforward application of the multiplication and addition principles is not possible.

Example 6

A bag contains one blue token, two red tokens and one green token. Three tokens are removed from the bag and placed in a row. How many arrangements are possible?

Solution

The three tokens are selected without replacement. So once a blue or green token is taken, these cannot appear again. We use a tree diagram to systematically find every arrangement.



The complete set of possible arrangements can be read by tracing out each path from top to bottom of the diagram. This gives 12 different arrangements:

BRR, BRG, BGR, RBR, RBG, RRB, RRG, RGB, RGR, GBR, GRB, GRR

Summary 9A

Three useful approaches to solving simple counting problems:

- **Tree diagrams**
  These can be used to systematically list all solutions to a problem.
- **Multiplication principle**
  If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.
- **Addition principle**
  Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, there are $m + n$ ways to perform one of the tasks.

Some problems require use of both the addition and the multiplication principles.

Skill-sheet

Exercise 9A

Example 2

1 Sam has five T-shirts, three pairs of pants and three pairs of shoes. How many different outfits can he assemble using these clothes?

Example 4

2 A restaurant offers five beef dishes and three chicken dishes. How many selections of one main meal does a customer have?

3 Each of the 10 boys at a party shakes hands with each of the 12 girls. How many handshakes take place?

4 Draw a tree diagram showing all the two-digit numbers that can be formed using the digits 7, 8 and 9 if each digit:

a cannot be repeated **b** can be repeated.

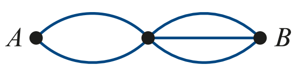
5 How many different three-digit numbers can be formed using the digits 2, 4 and 6 if each digit can be used:

a as many times as you would like **b** at most once?

6 Jack wants to travel from Sydney to Perth via Adelaide. There are four flights and two trains from Sydney to Adelaide. There are two flights and three trains from Adelaide to Perth. How many ways can Jack travel from Sydney to Perth?

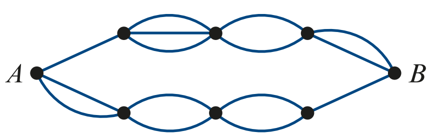
7 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

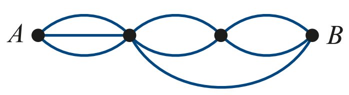
Example 3

a 

b 

Example 5

c 

d 

Example 6

8 A bag contains two blue, one red and two green tokens. Two tokens are removed from the bag and placed in a row. With the help of a tree diagram, list all the different arrangements.

9 How many ways can you make change for 50 cents using 5, 10 and 20 cent pieces?

10 Four teachers decide to swap desks at work. How many ways can this be done if no teacher is to sit at their previous desk?

11 Three runners compete in a race. In how many ways can the runners complete the race assuming:

a there are no tied places **b** the runners can tie places?

12 A six-sided die has faces labelled with the numbers 0, 2, 3, 5, 7 and 11. If the die is rolled twice and the two results are multiplied, how many different answers can be obtained?

9B Factorial notation and permutations

Factorial notation

Factorial notation provides a convenient way of expressing products of consecutive natural numbers. For each natural number n, we define

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$$

where the notation $n!$ is read as 'n factorial'.

We also define $0! = 1$. Although it might seem strange at first, this definition will turn out to be very convenient, as it is compatible with formulas that we will establish shortly.

Another very useful identity is

$$n! = n \cdot (n-1)!$$

Example 7

Evaluate:

a $3!$ **b** $\dfrac{50!}{49!}$ **c** $\dfrac{10!}{2!\,8!}$

Solution

a $3! = 3 \cdot 2 \cdot 1$
$\quad = 6$

b $\dfrac{50!}{49!} = \dfrac{50 \cdot 49!}{49!}$
$\quad = 50$

c $\dfrac{10!}{2!\,8!} = \dfrac{10 \cdot 9 \cdot 8!}{2! \cdot 8!}$
$\quad = \dfrac{10 \cdot 9}{2 \cdot 1}$
$\quad = 45$

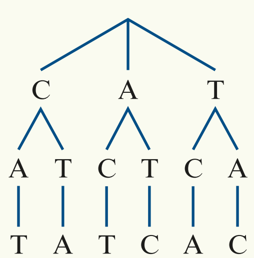
Permutations of *n* objects

A **permutation** is an ordered arrangement of a collection of objects.

Example 8

Using a tree diagram, list all the permutations of the letters in the word CAT.

Solution



There are six permutations:

CAT, CTA, ACT, ATC, TCA, TAC

Explanation

There are three choices for the first letter. This leaves only two choices for the second letter, and then one for the third.

Another way to find the number of permutations for the previous example is to draw three boxes, corresponding to the three positions. In each box, we write the number of choices we have for that position.

- We have 3 choices for the first letter (C, A or T).
- We have 2 choices for the second letter (because we have already used one letter).
- We have 1 choice for the third letter (because we have already used two letters).

3	2	1

By the multiplication principle, the total number of arrangements is

$$3 \times 2 \times 1 = 3!$$

So three objects can be arranged in 3! ways. More generally:

> The number of permutations of n objects is $n!$.

Proof The reason for this is simple:

- The first item can be chosen in n ways.
- The second item can be chosen in $n - 1$ ways, since only $n - 1$ objects remain.
- The third item can be chosen in $n - 2$ ways, since only $n - 2$ objects remain.

⋮

- The last item can be chosen in 1 way, since only 1 object remains.

Therefore, by the multiplication principle, there are

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$$

permutations of n objects.

Example 9

How many ways can six different books be arranged on a shelf?

Solution

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 720$$

Explanation

Six books can be arranged in 6! ways.

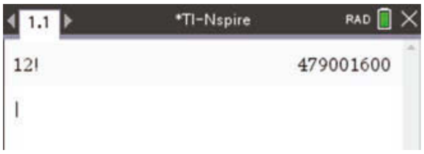
Example 10

Using your calculator, find how many ways 12 students can be lined up in a row.

Using the TI-Nspire

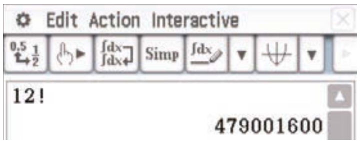
Evaluate 12! as shown.

Note: The factorial symbol (!) can be accessed using ?!▸, the Symbols palette (ctrl 📖) or menu > **Probability** > **Factorial**.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, open the keyboard.
- Enter the number 12, followed by the factorial symbol. Tap EXE.



Note: The factorial symbol (!) is found in the Advance keyboard; you need to scroll down to see this keyboard.

Example 11

How many four-digit numbers can be formed using the digits 1, 2, 3 and 4 if:

a they cannot be repeated

b they can be repeated?

Solution	Explanation
a $4! = 4 \times 3 \times 2 \times 1 = 24$	Four numbers can be arranged in 4! ways.
b $4^4 = 4 \times 4 \times 4 \times 4 = 256$	Using the multiplication principle, there are 4 choices for each of the 4 digits.

Permutations of *n* objects taken *r* at a time

Imagine a very small country with very few cars. Licence plates consist of a sequence of four digits, and repetitions of the digits are not allowed. How many such licence plates are there?

Here, we are asking for the number of permutations of 10 digits taken four at a time. We will denote this number by ${}^{10}P_4$.

To solve this problem, we draw four boxes. In each box, we write the number of choices we have for that position. For the first digit, we have a choice of 10 digits. Once chosen, we have only 9 choices for the second digit, then 8 choices for the third and 7 choices for the fourth.

10	9	8	7

By the multiplication principle, the total number of licence plates is

$$10 \times 9 \times 8 \times 7$$

There is a clever way of writing this product as a fraction involving factorials:

$$\begin{aligned} {}^{10}P_4 &= 10 \cdot 9 \cdot 8 \cdot 7 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{10!}{6!} \\ &= \frac{10!}{(10-4)!} \end{aligned}$$

We can easily generalise this procedure to give the following result.

Number of permutations

The number of permutations of n objects taken r at a time is denoted by ${}^nP_r$ and is given by the formula

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

Proof To establish this formula we note that:

- The 1st item can be chosen in n ways.
- The 2nd item can be chosen in $n - 1$ ways.

⋮

- The rth item can be chosen in $n - r + 1$ ways.

Therefore, by the multiplication principle, the number of permutations of n objects taken r at a time is

$$ \begin{aligned} {}^nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\ &= \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \cdot (n-r) \cdot \cdots \cdot 2 \cdot 1}{(n-r) \cdot \cdots \cdot 2 \cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned} $$

Notes:

- If $r = n$, then we have ${}^nP_n$, which is simply the number of permutations of n objects and so must equal $n!$. The formula still works in this instance, since

$$ {}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! $$

  Note that this calculation depends crucially on our decision to define $0! = 1$.

- If $r = 1$, then we obtain ${}^nP_1 = n$. Given n objects, there are n choices of one object, and each of these can be arranged in just one way.

Example 12

a Using the letters A, B, C, D and E without repetition, how many different two-letter arrangements are there?

b Six runners compete in a race. In how many ways can the gold, silver and bronze medals be awarded?

Solution

a There are five letters to arrange in two positions:

$$ \begin{aligned} {}^5P_2 = \frac{5!}{(5-2)!} &= \frac{5!}{3!} \\ &= \frac{5 \cdot 4 \cdot 3!}{3!} \\ &= 20 \end{aligned} $$

b There are six runners to arrange in three positions:

$$ \begin{aligned} {}^6P_3 = \frac{6!}{(6-3)!} &= \frac{6!}{3!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 120 \end{aligned} $$

Although the formula developed for ${}^nP_r$ will have an important application later in this chapter, you do not actually have to use it when solving problems. It is often more convenient to simply draw boxes corresponding to the positions, and to write in each box the number of choices for that position.

Example 13

How many ways can seven friends sit along a park bench with space for only four people?

Solution

7	6	5	4

By the multiplication principle, the total number of arrangements is

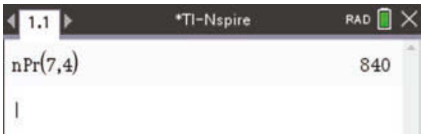
$$7 \times 6 \times 5 \times 4 = 840$$

Explanation

We draw four boxes, representing the positions to be filled. In each box we write the number of ways we can fill that position.

Using the TI-Nspire

- To evaluate ${}^7P_4$, use (menu) > **Probability** > **Permutations** as shown.

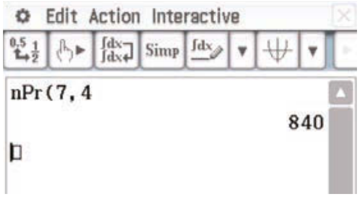


Note: Alternatively, you can simply type npr(7, 4). The command is not case sensitive.

Using the Casio ClassPad

To evaluate ${}^7P_4$:

- In Main $\sqrt{\alpha}$, select [nPr] from the (Advance) keyboard. (You need to scroll down to find this keyboard.)
- After the bracket, enter the numbers 7 and 4, separated by a comma. Then tap (EXE).



Summary 9B

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$
- $n! = n \cdot (n-1)!$
- A **permutation** is an ordered arrangement of objects.
- The number of permutations of n objects is $n!$.
- The number of permutations of n objects taken r at a time is given by

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

Exercise 9B

1 Evaluate $n!$ for $n = 0, 1, 2, \dots, 10$.

Example 7

2 Evaluate each of the following:

a $\dfrac{5!}{4!}$ **b** $\dfrac{10!}{8!}$ **c** $\dfrac{12!}{10!\,2!}$ **d** $\dfrac{100!}{97!\,3!}$

3 Simplify the following expressions:

a $\dfrac{(n+1)!}{n!}$ **b** $\dfrac{(n+2)!}{(n+1)!}$ **c** $\dfrac{n!}{(n-2)!}$ **d** $\dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

4 Evaluate ${}^4P_r$ for $r = 0, 1, 2, 3, 4$.

Example 8

5 Use a tree diagram to find all the permutations of the letters in the word DOG.

Example 9

6 How many ways can five books on a bookshelf be arranged?

7 How many ways can the letters in the word HYPERBOLA be arranged?

Example 12

8 Write down all the two-letter permutations of the letters in the word FROG.

Example 13

9 How many ways can six students be arranged along a park bench if the bench has:

a six seats **b** five seats **c** four seats?

10 Using the digits 1, 2, 5, 7 and 9 without repetition, how many numbers can you form that have:

a five digits **b** four digits **c** three digits?

11 How many ways can six students be allocated to eight vacant desks?

12 How many ways can three letters be posted in five mailboxes if each mailbox can receive:

a more than one letter **b** at most one letter?

13 Using six differently coloured flags without repetition, how many signals can you make using:

a three flags in a row **b** four flags in a row **c** five flags in a row?

14 You are in possession of four flags, each coloured differently. How many signals can you make using at least two flags arranged in a row?

15 Some car licence plates consist of a sequence of three letters followed by a sequence of three digits.

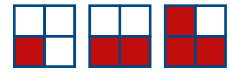
a How many different car licence plates have letters and numbers arranged this way?

b How many of these have no repeated letters or numbers?

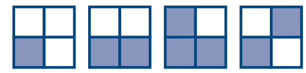
16 Find all possible values of m and n if $m! \cdot n! = 720$ and $m > n$.

17 Show that $n! = (n^2 - n) \cdot (n-2)!$ for $n \geq 2$.

18 **a** The three tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



b The four tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



19 Given six different colours, how many ways can you paint a cube so that all the faces have different colours? Two colourings are considered to be the same when one can be obtained from the other by rotating the cube.

9C Permutations with restrictions

Suppose we want to know how many three-digit numbers have no repeated digits. The answer is *not* simply ${}^{10}P_3$, the number of permutations of 10 digits taken three at a time. This is because the digit 0 cannot be used in the hundreds place.

- There are 9 choices for the first digit (1, 2, 3, . . . , 9).
- There are 9 choices for the second digit (0 and the eight remaining non-zero digits).
- This leaves 8 choices for the third digit.

100s	10s	units
9	9	8

By the multiplication principle, there are $9 \times 9 \times 8 = 648$ different three-digit numbers.

When considering permutations with restrictions, we deal with the restrictions first.

Example 14

a How many arrangements of the word DARWIN begin and end with a vowel?

b Using the digits 0, 1, 2, 3, 4 and 5 without repetition, how many odd four-digit numbers can you form?

Solution

a We draw six boxes. In each box, we write the number of choices we have for that position. We first consider restrictions. There are two choices of vowel (A or I) for the first letter, leaving only one choice for the last letter.

2					1

This leaves four choices for the second letter, three for the next, and so on.

2	4	3	2	1	1

By the multiplication principle, the number of arrangements is

$$2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$$

b We draw four boxes. Again, we first consider restrictions. The last digit must be odd (1, 3 or 5), giving three choices. We cannot use 0 in the first position, so this leaves four choices for that position.

4			3

Once these two digits have been chosen, this leaves four choices and then three choices for the remaining two positions.

4	4	3	3

Thus the number of arrangements is

$$4 \times 4 \times 3 \times 3 = 144$$

Permutations with items grouped together

For some arrangements, we may want certain items to be grouped together. In this case, the trick is to initially treat each group of items as a single object. We then multiply by the numbers of arrangements within each group.

Example 15

a How many arrangements of the word EQUALS are there if the vowels are kept together?

b How many ways can two chemistry, four physics and five biology books be arranged on a shelf if the books of each subject are kept together?

Solution	Explanation
a $4! \times 3!$ $= 144$	We group the three vowels together so that we have four items to arrange: (E, U, A), Q, L, S. They can be arranged in 4! ways. Then the three vowels can be arranged among themselves in 3! ways. We use the multiplication principle.
b $3! \times 2! \times 4! \times 5!$ $= 34\,560$	There are three groups and so they can be arranged in 3! ways. The two chemistry books can be arranged among themselves in 2! ways, the four physics books in 4! ways and the five biology books in 5! ways. We use the multiplication principle.

Summary 9C

- To count permutations that are subject to restrictions, we draw a series of boxes. In each box, we write the number of choices we have for that position. We always consider the restrictions first.
- When items are to be grouped together, we initially treat each group as a single object. We find the number of arrangements of the groups, and then multiply by the numbers of arrangements within each group.

Skill-sheet

Exercise 9C

Example 14

1 Using the digits 1, 2, 3, 4 and 5 without repetition, how many five-digit numbers can you form:

a without restriction
b that are odd
c that begin with 5
d that do not begin with 5?

Example 15

2 In how many ways can three girls and two boys be arranged in a row:

a without restriction
b if the two boys sit together
c if the two boys do not sit together
d if girls and boys alternate?

3 How many permutations of the word QUEASY:

a begin with a vowel
b begin and end with a vowel
c keep the vowels together
d keep the vowels and consonants together?

4 How many ways can four boys and four girls be arranged in a row if:

a boys and girls sit in alternate positions
b boys sit together and girls sit together?

5 The digits 0, 1, 2, 3, 4 and 5 can be combined without repetition to form new numbers. In how many ways can you form:

a a six-digit number
b a four-digit number divisible by 5
c a number less than 6000
d an even three-digit number?

6 Two parents and four children are seated in a cinema along six consecutive seats. How many ways can this be done:

a without restriction
b if the two parents sit at either end
c if the children sit together
d if the parents sit together and the children sit together
e if the youngest child must sit between and next to both parents?

7 12321 is a **palindromic number** because it reads the same backwards as forwards. How many palindromic numbers have:

a five digits
b six digits?

8 How many arrangements of the letters in VALUE do not begin and end with a vowel?

9 Using each of the digits 1, 2, 3 and 4 at most once, how many even numbers can you form?

10 How many ways can six girls be arranged in a row so that two of the girls, A and B:

a do not sit together
b have one person between them?

11 How many ways can three girls and three boys be arranged in a row if no two girls sit next to each other?

9D Permutations of like objects

The name for the Sydney suburb of WOOLLOOMOOLOO has the unusual distinction of having 13 letters in total, of which only four are different. Finding the number of permutations of the letters in this word is not as simple as evaluating 13!. This is because switching like letters does not result in a new permutation.

Our aim is to find an expression for P, where P is the number of permutations of the letters in the word WOOLLOOMOOLOO. First notice that the word has

1 letter W, 1 letter M, 3 letter Ls, 8 letter Os

Replace the three identical Ls with L_1, L_2 and L_3. These three letters can be arranged in 3! different ways. Therefore, by the multiplication principle, there are now

$$P \cdot 3!$$

permutations. Likewise, replace the eight identical Os with $O_1, O_2, \dots, O_8$. These eight letters can be arranged in 8! different ways. Therefore there are now

$$P \cdot 3! \cdot 8!$$

permutations.

On the other hand, notice that the 13 letters are now distinct, so there are 13! permutations of these letters. Therefore

$$P \cdot 3! \cdot 8! = 13! \quad \text{and so} \quad P = \frac{13!}{3!\,8!}$$

We can easily generalise this procedure to give the following result.

Permutations of like objects

The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Example 16

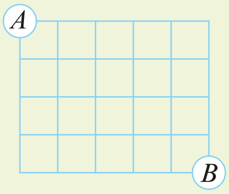
a Find the number of permutations of the letters in the word RIFFRAFF.

b There are four identical knives, three identical forks and two identical spoons in a drawer. They are taken out of the drawer and lined up in a row. How many ways can this be done?

Solution	Explanation
a $\dfrac{8!}{4!\,2!} = 840$	There are 8 letters of which 4 are alike and 2 are alike.
b $\dfrac{9!}{4!\,3!\,2!} = 1260$	There are 9 items of which 4 are alike, 3 are alike and 2 are alike.

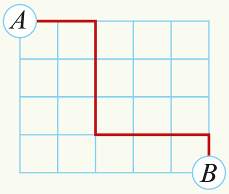
Example 17

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of four Ds and five Rs in some order. For example, the path shown can be described by the sequence RRDDDRRRD.



Since there are 9 letters of which 4 are alike and 5 are alike, the number of permutations of these letters is

$$\frac{9!}{4!\,5!} = 126$$

Summary 9D

- Switching like objects does not give a new arrangement.
- The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

Skill-sheet

Exercise 9D

Example 16

1 Ying has four identical 20 cent pieces and three identical 10 cent pieces. How many ways can she arrange these coins in a row?

2 How many ways can the letters in the word MISSISSIPPI be arranged?

3 Find the number of permutations of the letters in the word WARRNAMBOOL.

4 Using five 9s and three 7s, how many eight-digit numbers can be made?

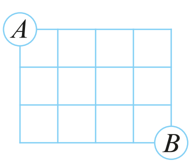
5 Using three As, four Bs and five Cs, how many sequences of 12 letters can be made?

6 How many ways can two red, two black and four blue flags be arranged in a row:

a without restriction
b if the first flag is red
c if the first and last flags are blue
d if every alternate flag is blue
e if the two red flags are adjacent?

Example 17

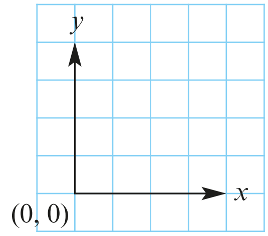
7 The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



8 The grid shown consists of unit squares. By travelling only along the grid lines, how many paths are there:

a of length 6 from $(0, 0)$ to the point $(2, 4)$

b of length $m + n$ from $(0, 0)$ to the point (m, n), where m and n are natural numbers?



9 Consider a deck of 52 playing cards.

a How many ways can the deck be arranged? Express your answer in the form $a!$.

b If two identical decks are combined, how many ways can the cards be arranged? Express your answer in the form $\dfrac{a!}{(b!)^c}$.

c If n identical decks are combined, find an expression for the number of ways that the cards can be arranged.

10 An ant starts at position $(0, 0)$ and walks north, east, south or west, one unit at a time. How many different paths of length 8 units finish at $(0, 0)$?

11 Jessica is about to walk up a flight of 10 stairs. She can take either one or two stairs at a time. How many different ways can she walk up the flight of stairs?

9E Combinations

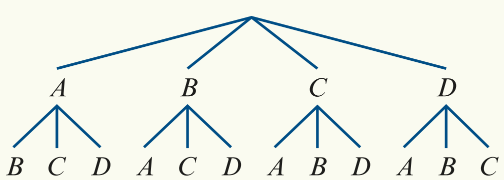
We have seen that a permutation is an ordered arrangement of objects. In contrast, a **combination** is a selection made regardless of order. We use the notation ^nP_r to denote the number of permutations of n distinct objects taken r at a time. Similarly, we use the notation ^nC_r to denote the number of combinations of n distinct objects taken r at a time.

Example 18

How many ways can two letters be chosen from the set $\{A, B, C, D\}$?

Solution

The tree diagram below shows the ways that the first and second choices can be made.



This gives 12 arrangements. But there are only six selections, since

$\{A, B\}$ is the same as $\{B, A\}$, $\{A, C\}$ is the same as $\{C, A\}$, $\{A, D\}$ is the same as $\{D, A\}$,
$\{B, C\}$ is the same as $\{C, B\}$, $\{B, D\}$ is the same as $\{D, B\}$ $\{C, D\}$ is the same as $\{D, C\}$

Suppose we want to count the number of ways that three students can be chosen from a group of seven. Let's label the students with the letters $\{A, B, C, D, E, F, G\}$. One such combination might be BDE. Note that this combination corresponds to 3! permutations:

$$BDE, \quad BED, \quad DBE, \quad DEB, \quad EBD, \quad EDB$$

In fact, each combination of three items corresponds to 3! permutations, and so there are 3! times as many permutations as combinations. Therefore

$$^7P_3 = 3! \times {}^7C_3$$

and so

$$^7C_3 = \frac{^7P_3}{3!}$$

Since we have already established that $^7P_3 = \dfrac{7!}{(7-3)!}$, we obtain

$$^7C_3 = \frac{7!}{3!\,(7-3)!}$$

This argument generalises easily so that we can establish a formula for ^nC_r.

Number of combinations

The number of combinations of n objects taken r at a time is given by the formula

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

Example 19

a A pizza can have three toppings chosen from nine options. How many different pizzas can be made?

b How many subsets of $\{1, 2, 3, \ldots, 20\}$ have exactly two elements?

Solution

a Three objects are to be chosen from nine options. This can be done in $^9C_3$ ways, and

$$\begin{aligned} ^9C_3 &= \frac{9!}{3!\,(9-3)!} \\ &= \frac{9!}{3!\,6!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! \cdot 6!} \\ &= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} \\ &= 84 \end{aligned}$$

b Two objects are to be chosen from 20 options. This can be done in $^{20}C_2$ ways, and

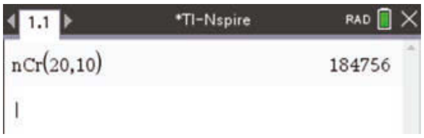
$$\begin{aligned} ^{20}C_2 &= \frac{20!}{2!\,(20-2)!} \\ &= \frac{20!}{2!\,18!} \\ &= \frac{20 \cdot 19 \cdot 18!}{2! \cdot 18!} \\ &= \frac{20 \cdot 19}{2 \cdot 1} \\ &= 190 \end{aligned}$$

Example 20

Using your calculator, find how many ways 10 students can be selected from a class of 20 students.

Using the TI-Nspire

- To evaluate ${}^{20}C_{10}$, use menu > **Probability** > **Combinations** as shown.

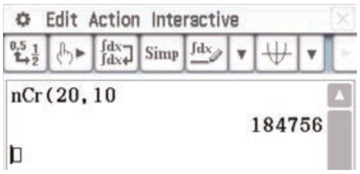


Note: Alternatively, you can simply type ncr(20, 10). The command is not case sensitive.

Using the Casio ClassPad

To evaluate ${}^{20}C_{10}$:

- In Main $\sqrt{\alpha}$, select nCr from the Advance keyboard.
- After the bracket, enter the numbers 20 and 10, separated by a comma. Then tap EXE.



Example 21

Consider a group of six students. In how many ways can a group of:

a two students be selected

b four students be selected?

Solution

a $$\begin{aligned} {}^{6}C_{2} &= \frac{6!}{2!\,(6-2)!} \\ &= \frac{6!}{2!\,4!} \\ &= \frac{6\cdot 5\cdot 4!}{2!\cdot 4!} \\ &= \frac{6\cdot 5}{2\cdot 1} \\ &= 15 \end{aligned}$$

b $$\begin{aligned} {}^{6}C_{4} &= \frac{6!}{4!\,(6-4)!} \\ &= \frac{6!}{4!\,2!} \\ &= \frac{6\cdot 5\cdot 4!}{4!\cdot 2!} \\ &= \frac{6\cdot 5}{2\cdot 1} \\ &= 15 \end{aligned}$$

Parts **a** and **b** of the previous example have the same answer. This is not a coincidence. Choosing two students out of six is the same as *not choosing* the other four students out of six. Therefore ${}^{6}C_{2} = {}^{6}C_{4}$.

More generally:

$${}^{n}C_{r} = {}^{n}C_{n-r}$$

Quick calculations

In some instances, you can avoid unnecessary calculations by noting that:

- ${}^nC_0 = 1$, since there is only one way to select no objects from n objects
- ${}^nC_n = 1$, since there is only one way to select n objects from n objects
- ${}^nC_1 = n$, since there are n ways to select one object from n objects
- ${}^nC_{n-1} = n$, since this corresponds to the number of ways of not selecting one object from n objects.

Example 22

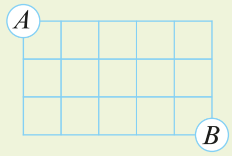
a Six points lie on a circle. How many triangles can you make using these points as the vertices?

b Each of the 20 people at a party shakes hands with every other person. How many handshakes take place?

Solution	Explanation
a ${}^6C_3 = 20$	This is the same as asking how many ways three vertices can be chosen out of six.
b ${}^{20}C_2 = 190$	This is the same as asking how many ways two people can be chosen to shake hands out of 20 people.

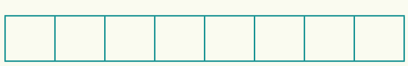
Example 23

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of three Ds and five Rs in some order. Therefore, the number of paths is equal to the number of ways of selecting three of the eight boxes below to be filled with the three Ds. (The rest will be Rs.) This can be done in ${}^8C_3 = 56$ ways.



Alternative notation

We will consistently use the notation ${}^nC_r$ to denote the number of ways of selecting r objects from n objects, regardless of order. However, it is also common to denote this number by $\binom{n}{r}$.

For example:

$$\binom{6}{4} = \frac{6!}{4!\,2!} = 15$$

Summary 9E

- A **combination** is a selection made regardless of order.
- The number of combinations of n objects taken r at a time is given by

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

Exercise 9E

1 Evaluate $^{5}C_r$ for $r = 0, 1, 2, 3, 4, 5$.

2 Evaluate each of the following without the use of your calculator:

a $^{7}C_1$ **b** $^{6}C_5$ **c** $^{12}C_{10}$ **d** $^{8}C_5$ **e** $^{100}C_{99}$ **f** $^{1000}C_{998}$

3 Simplify each of the following:

a $^{n}C_1$ **b** $^{n}C_2$ **c** $^{n}C_{n-1}$ **d** $^{n+1}C_1$ **e** $^{n+2}C_n$ **f** $^{n+1}C_{n-1}$

Example 19

4 A playlist contains ten of Nandi's favourite songs. How many ways can he:

a arrange three songs in a list **b** select three songs for a list?

5 How many ways can five cards be selected from a deck of 52 playing cards?

6 How many subsets of $\{1, 2, 3, \ldots, 10\}$ contain exactly:

a 1 element **b** 2 elements **c** 8 elements **d** 9 elements?

7 A lottery consists of drawing seven balls out of a barrel of balls numbered from 1 to 45. How many ways can this be done if their order does not matter?

Example 22

8 Eight points lie on a circle. How many triangles can you make using these points as the vertices?

9 **a** In a hockey tournament, each of the 10 teams plays every other team once. How many games take place?

b In another tournament, each team plays every other team once and 120 games take place. How many teams competed?

10 At a party, every person shakes hands with every other person. Altogether there are 105 handshakes. How many people are at the party?

11 Prove that $^{n}C_r = {}^{n}C_{n-r}$.

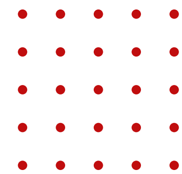
12 Explain why the number of diagonals in a regular polygon with n sides is $^{n}C_2 - n$.

13 Ten students are divided into two teams of five. Explain why the number of ways of doing this is $\dfrac{^{10}C_5}{2}$.

14 Twelve students are to be divided into two teams of six. In how many ways can this be done? (Hint: First complete the previous question.)

15 Using the formula for ${}^nC_r$, prove that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 \le r < n$.

16 Consider the 5×5 grid shown.



a How many ways can three dots be chosen?

b How many ways can three dots be chosen so that they lie on a straight line?

c How many ways can three dots be chosen so that they are the vertices of a triangle? (Hint: Use parts **a** and **b**.)

9F Combinations with restrictions

Combinations including specific items

In some problems, we want to find the number of combinations that include specific items. This reduces both the number of items we have to select and the number of items from which we are selecting.

Example 24

a Grace belongs to a group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team?

b A hand of cards consists of five cards drawn from a deck of 52 playing cards. How many hands contain both the queen and the king of hearts?

Solution	Explanation
a ${}^7C_3 = 35$	Grace must be in the selection. Therefore three more workers are to be selected from the remaining seven workers.
b ${}^{50}C_3 = 19\,600$	The queen and king of hearts must be in the selection. So three more cards are to be selected from the remaining 50 cards.

In some other problems, it can be more efficient to count the selections that we don't want.

Example 25

Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group, Sam and Tess, do not get along and cannot both be on the team. How many ways can the team be selected?

Solution

There are ${}^8C_4$ ways of selecting four students from eight. We then subtract the number of combinations that include both Sam and Tess. If Sam and Tess are on the team, then we can select two more students from the six that remain in ${}^6C_2$ ways. This gives

$${}^8C_4 - {}^6C_2 = 55$$

Combinations from multiple groups

If we are required to make multiple selections from separate groups, then the multiplication principle dictates that we simply multiply the number of ways of performing each task.

Example 26

From seven women and four men in a workplace, how many groups of five can be chosen:

a without restriction

b containing three women and two men

c containing at least one man

d containing at most one man?

Solution

a There are 11 people in total, from which we must select five. This gives

$$^{11}C_5 = 462$$

b There are $^7C_3$ ways of selecting three women from seven. There are $^4C_2$ ways of selecting two men from four. We then use the multiplication principle to give

$$^7C_3 \cdot {}^4C_2 = 210$$

c Method 1

If you select at least one man, then you select 1, 2, 3 or 4 men and fill the remaining positions with women. We use the multiplication and addition principles to give

$$^4C_1 \cdot {}^7C_4 + {}^4C_2 \cdot {}^7C_3 + {}^4C_3 \cdot {}^7C_2 + {}^4C_4 \cdot {}^7C_1 = 441$$

Method 2

It is more efficient to consider all selections of 5 people from 11 and then subtract the number of combinations containing all women. This gives

$$^{11}C_5 - {}^7C_5 = 441$$

d If there is at most one man, then either there are no men or there is one man. If there are no men, then there are $^7C_5$ ways of selecting all women. If there is one man, then there are $^4C_1$ ways of selecting one man and $^7C_4$ ways of selecting four women. This gives

$$^7C_5 + {}^4C_1 \cdot {}^7C_4 = 161$$

Permutations and combinations combined

In the following example, we first select the items and then arrange them.

Example 27

a How many arrangements of the letters in the word DUPLICATE can be made that have two vowels and three consonants?

b A president, vice-president, secretary and treasurer are to be chosen from a group containing seven women and six men. How many ways can this be done if exactly two women are chosen?

Solution	Explanation
a ${}^4C_2 \cdot {}^5C_3 \cdot 5! = 7200$	There are ${}^4C_2$ ways of selecting 2 of 4 vowels and ${}^5C_3$ ways of selecting 3 of 5 consonants. Once chosen, the 5 letters can be arranged in 5! ways.
b ${}^7C_2 \cdot {}^6C_2 \cdot 4! = 7560$	There are ${}^7C_2$ ways of selecting 2 of 7 women and ${}^6C_2$ ways of selecting 2 of 6 men. Once chosen, the 4 people can be arranged into the positions in 4! ways.

Summary 9F

- If a selection must include specific items, then this reduces both the number of items that we have to select and the number of items that we select from.
- If we are required to make multiple selections from separate groups, then we multiply the number of ways of performing each task.
- Some problems will require us to select and then arrange objects.

Skill-sheet

Exercise 9F

Example 24

1 Jane and Jenny belong to a class of 20 students. How many ways can you select a group of four students from the class if both Jane and Jenny are to be included?

2 How many subsets of $\{1, 2, 3, \ldots, 10\}$ have exactly five elements and contain the number 5?

3 Five cards are dealt from a deck of 52 playing cards. How many hands contain the jack, queen and king of hearts?

Example 25

4 Six students are to be chosen from a group of 10 students for the school basketball team. Two members of the group, Rachel and Nethra, do not get along and cannot both be on the team. How many ways can the team be selected?

Example 26

5 From eight girls and five boys, a team of seven is selected for a mixed netball team. How many ways can this be done if:

a there are no restrictions

b there are four girls and three boys on the team

c there must be at least three boys and three girls on the team

d there are at least two boys on the team?

6 There are 10 student leaders at a secondary school. Four are needed for a fundraising committee and three are needed for a social committee. How many ways can the students be selected if they can serve on:

a both committees

b at most one committee?

7 There are 18 students in a class. Seven are required for a basketball team and eight are required for a netball team. How many ways can the teams be selected if students can play in:

a both teams **b** at most one team?

8 From 10 Labor senators and 10 Liberal senators, a committee of five is formed. How many ways can this be done if:

a there are no restrictions

b there are at least two senators from each political party

c there is at least one Labor senator?

9 Consider the set of numbers $\{1, 2, 3, 4, 5, 6, 7\}$.

a How many subsets have exactly five elements?

b How many five-element subsets contain the numbers 2 and 3?

c How many five-element subsets do not contain both 2 and 3?

10 Four letters are selected from the English alphabet. How many of these selections will contain exactly two vowels?

11 A seven-card hand is dealt from a deck of 52 playing cards. How many distinct hands contain:

a four hearts and three spades **b** exactly two hearts and three spades?

12 A committee of five people is chosen from four doctors, four dentists and three physiotherapists. How many ways can this be done if the committee contains:

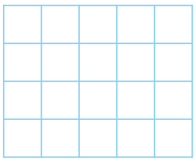
a exactly three doctors and one dentist **b** exactly two doctors?

Example 27 **13** There are four girls and five boys. Two of each are chosen and then arranged on a bench. How many ways can this be done?

14 A president, vice-president, secretary and treasurer are to be chosen from a group containing six women and five men. How many ways can this be done if exactly two women must be chosen?

15 Using five letters from the word TRAMPOLINE, how many arrangements contain two vowels and three consonants?

16 How many rectangles are there in the grid shown on the right?
Hint: Every rectangle is determined by a choice of two vertical and two horizontal lines.



17 Five cards are dealt from a deck of 52 playing cards. A full house is a hand that contains 3 cards of one rank and 2 cards of another rank (for example, 3 kings and 2 sevens). How many ways can a full house be dealt?

9G Pascal's triangle

The diagram below consists of the values of ${}^nC_r$ for $0 \le n \le 5$. They form the first 6 rows of **Pascal's triangle**, named after the seventeenth century French mathematician Blaise Pascal, one of the founders of probability theory.

Interestingly, the triangle was well known to Chinese and Indian mathematicians many centuries earlier.

$n = 0$:	${}^0C_0$	1
$n = 1$:	${}^1C_0 \quad {}^1C_1$	1 1
$n = 2$:	${}^2C_0 \quad {}^2C_1 \quad {}^2C_2$	1 2 1
$n = 3$:	${}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$	1 3 3 1
$n = 4$:	${}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4$	1 4 6 4 1
$n = 5$:	${}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$	1 5 10 10 5 1

Pascal's rule

Pascal's triangle has many remarkable properties. Most importantly:

Each entry in Pascal's triangle is the sum of the two entries immediately above.

Pascal's triangle has this property because of the following identity.

Pascal's rule

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \qquad \text{where } 1 \le r < n$$

Proof In Question 15 of Exercise 9E, you are asked to prove Pascal's rule using the formula for ${}^nC_r$. However, there is a much nicer argument.

The number of subsets of $\{1, 2, \dots, n\}$ containing exactly r elements is ${}^nC_r$. Each of these subsets can be put into one of two groups:

1. those that contain n
2. those that do not contain n.

If the subset contains n, then each of the remaining $r - 1$ elements must be chosen from $\{1, 2, \dots, n - 1\}$. Therefore the first group contains ${}^{n-1}C_{r-1}$ subsets.

If the subset does not contain n, then we still have to choose r elements from $\{1, 2, \dots, n - 1\}$. Therefore the second group contains ${}^{n-1}C_r$ subsets.

The two groups together contain all ${}^nC_r$ subsets and so

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

which establishes Pascal's rule.

Example 28

Given that ${}^{17}C_2 = 136$ and ${}^{17}C_3 = 680$, evaluate ${}^{18}C_3$.

Solution

$$\begin{aligned} {}^{18}C_3 &= {}^{17}C_2 + {}^{17}C_3 \\ &= 136 + 680 \\ &= 816 \end{aligned}$$

Explanation

We let $n = 18$ and $r = 3$ in Pascal's rule:

$${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Example 29

Write down the $n = 6$ row of Pascal's triangle and then write down the value of ${}^{6}C_3$.

Solution

$n = 6$: 1 6 15 [20] 15 6 1

${}^{6}C_3 = 20$

Explanation

Each entry in the $n = 6$ row is the sum of the two entries immediately above.

Note that ${}^{6}C_3$ is the fourth entry in the row, since the first entry corresponds to ${}^{6}C_0$.

Subsets of a set

Suppose your friend says to you: 'I have five books that I no longer need, take any that you want.' How many different selections are possible?

We will look at two solutions to this problem.

Solution 1

You could select none of the books (${}^{5}C_0$ ways), or one out of five (${}^{5}C_1$ ways), or two out of five (${}^{5}C_2$ ways), and so on. This gives the answer

$${}^{5}C_0 + {}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 + {}^{5}C_4 + {}^{5}C_5 = 32$$

Note that this is simply the sum of the entries in the $n = 5$ row of Pascal's triangle.

Solution 2

For each of the five books we have two options: either accept or reject the book. Using the multiplication principle, we obtain the answer

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

There are two important conclusions that we can draw from this example.

1. The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$${}^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$$

2. A set of size n has 2^n subsets, including the empty set and the set itself.

Example 30

a Your friend offers you any of six books that she no longer wants. How many selections are possible assuming that you take at least one book?

b How many subsets of $\{1, 2, 3, \ldots, 10\}$ have at least two elements?

Solution	Explanation
a $2^6 - 1 = 63$	There are 2^6 subsets of a set of size 6. We subtract 1 because we discard the empty set of no books.
b $2^{10} - {}^{10}C_1 - {}^{10}C_0$ $= 2^{10} - 10 - 1$ $= 1013$	There are 2^{10} subsets of a set of size 10. There are ${}^{10}C_1$ subsets containing 1 element and ${}^{10}C_0$ subsets containing 0 elements.

Summary 9G

- The values of ${}^nC_r$ can be arranged to give Pascal's triangle.
- Each entry in Pascal's triangle is the sum of the two entries immediately above.
- Pascal's rule ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$ {}^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n $$

- A set of size n has 2^n subsets, including the empty set and the set itself.

Exercise 9G

Example 28

1 Evaluate ${}^7C_2$, ${}^6C_2$ and ${}^6C_1$, and verify that the first is the sum of the other two.

Example 29

2 Write down the $n = 7$ row of Pascal's triangle. Use your answer to write down the values of ${}^7C_2$ and ${}^7C_4$.

3 Write down the $n = 8$ row of Pascal's triangle. Use your answer to write down the values of ${}^8C_4$ and ${}^8C_6$.

Example 30

4 Your friend offers you any of six different T-shirts that he no longer wants. How many different selections are possible?

5 How many subsets does the set $\{A, B, C, D, E\}$ have?

6 How many subsets does the set $\{1, 2, 3, \ldots, 10\}$ have?

7 How many subsets of $\{1, 2, 3, 4, 5, 6\}$ have at least one element?

8 How many subsets of $\{1, 2, 3, \ldots, 8\}$ have at least two elements?

9 How many subsets of $\{1, 2, 3, \ldots, 10\}$ contain the numbers 9 and 10?

10 You have one 5 cent, one 10 cent, one 20 cent and one 50 cent piece. How many different sums of money can you make assuming that at least one coin is used?

11 Let's call a set **selfish** if it contains its size as an element. For example, the set $\{1, 2, 3\}$ is selfish because the set has size 3 and the number 3 belongs to the set.

a How many subsets of $\{1, 2, 3, \ldots, 8\}$ are selfish?

b How many subsets of $\{1, 2, 3, \ldots, 8\}$ have the property that both the subset and its complement are selfish?

9H The pigeonhole principle

The pigeonhole principle is an intuitively obvious counting technique which can be used to prove some remarkably counterintuitive results. It gets its name from the following simple observation: If $n + 1$ pigeons are placed into n holes, then some hole contains at least two pigeons. Obviously, in most instances we will not be working with pigeons, so we will recast the principle as follows.

Pigeonhole principle

If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

Proof Suppose that each of the n holes contains at most one object. Then the total number of objects is at most n, which is a contradiction.

We are now in a position to prove a remarkable fact: There are at least two people in Australia with the same number of hairs on their head. The explanation is simple. No one has more than 1 million hairs on their head, so let's make 1 million holes labelled with the numbers from 1 to 1 million. We now put each of the 26 million Australians into the hole corresponding to the number of hairs on their head. Clearly, some hole contains at least two people, and all the people in that hole will have the same number of hairs on their head.

Example 31

You have thirteen red, ten blue and eight green socks. How many socks need to be selected at random to ensure that you have a matching pair?

Solution

Label three holes with the colours red, blue and green.

R	B	G

Selecting just three socks is clearly not sufficient, as you might pick one sock of each colour. Select four socks and place each sock into the hole corresponding to the colour of the sock. As there are four socks and three holes, the pigeonhole principle guarantees that some hole contains at least two socks. This is the required pair.

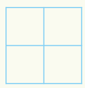
Example 32

a Show that for any five points chosen inside a 2×2 square, at least two of them will be no more than $\sqrt{2}$ units apart.

b Seven football teams play 22 games of football. Show that some pair of teams play each other at least twice.

Solution

a Split the 2×2 square into four unit squares.



Now we have four squares and five points. By the pigeonhole principle, some square contains at least two points. The distance between any two of these points cannot exceed the length of the square's diagonal, $\sqrt{1^2 + 1^2} = \sqrt{2}$.

b There are ${}^7C_2 = 21$ ways that two teams can be chosen to compete from seven. There are 22 games of football, and so some pair of teams play each other at least twice.

The generalised pigeonhole principle

Suppose that 13 pigeons are placed into four holes. By the pigeonhole principle, there is some hole with at least two pigeons. In fact, some hole must contain at least four pigeons. The reason is simple: If each of the four holes contained no more than three pigeons, then there would be no more than 12 pigeons.

This observation generalises as follows.

Generalised pigeonhole principle

If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Proof Again, let's suppose that the statement is false. Then each of the n holes contains no more than m objects. However, this means that there are no more than mn objects, which is a contradiction.

Example 33

Sixteen natural numbers are written on a whiteboard. Prove that at least four numbers will leave the same remainder when divided by 5.

Solution

We label five holes with each of the possible remainders on division by 5.

There are 16 numbers to be placed into five holes. Since $16 = 3 \times 5 + 1$, there is some hole with at least four numbers, each of which leaves the same remainder when divided by 5.

Pigeons in multiple holes

In some instances, objects can be placed into more than one hole.

Example 34

Seven people sit at a round table with 10 chairs. Show that there are three consecutive chairs that are occupied.

Solution

Number the chairs from 1 to 10. There are 10 groups of three consecutive chairs:

$$\{1,2,3\},\quad \{2,3,4\},\quad \{3,4,5\},\quad \{4,5,6\},\quad \{5,6,7\},$$
$$\{6,7,8\},\quad \{7,8,9\},\quad \{8,9,10\},\quad \{9,10,1\},\quad \{10,1,2\}$$

Each of the seven people will belong to three of these groups, and so 21 people have to be allocated to 10 groups. Since $21 = 2 \times 10 + 1$, the generalised pigeonhole principle guarantees that some group must contain three people.

Summary 9H

- **Pigeonhole principle**
  If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.
- **Generalised pigeonhole principle**
  If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Exercise 9H

Example 31

1 You have twelve red, eight blue and seven green socks. How many socks need to be selected at random to ensure that you have a matching pair?

2 A sentence contains 27 English words. Show that there are at least two words that begin with the same letter.

3 Show that in any collection of five natural numbers, at least two will leave the same remainder when divided by 4.

4 How many cards need to be dealt from a deck of 52 playing cards to be certain that you will obtain at least two cards of the same:

a colour **b** suit **c** rank?

5 Eleven points on the number line are located somewhere between 0 and 1. Show that there are at least two points no more than 0.1 apart.

Example 32

6 An equilateral triangle has side length 2 units. Choose any five points inside the triangle. Prove that there are at least two points that are no more than 1 unit apart.

7 Thirteen points are located inside a rectangle of length 6 and width 8. Show that there are at least two points that are no more than $2\sqrt{2}$ units apart.

8 The **digital sum** of a natural number is defined to be the sum of its digits. For example, the digital sum of 123 is $1 + 2 + 3 = 6$.

a Nineteen two-digit numbers are selected. Prove that at least two of them have the same digital sum.

b Suppose that 82 three-digit numbers are selected. Prove that at least four of them have the same digital sum.

Example 33

9 Whenever Eva writes down 13 integers, she notices that at least four of them leave the same remainder when divided by 4. Explain why this is always the case.

10 Twenty-nine games of football are played among eight teams. Prove that there is some pair of teams who play each other more than once.

11 A teacher instructs each member of her class to write down a different whole number between 1 and 49. She says that there will be at least one pair of students such that the sum of their two numbers is 50. How many students must be in her class?

Example 34

12 There are 10 students seated at a round table with 14 chairs. Show that there are three consecutive chairs that are occupied.

13 There are four points on a circle. Show that three of these points lie on a half-circle.
Hint: Pick any one of the four points and draw a diameter through that point.

14 There are 35 players on a football team and each player has a different number chosen from 1 to 99. Prove that there are at least four pairs of players whose numbers have the same sum.

15 Seven boys and five girls sit evenly spaced at a round table. Prove that some pair of boys are sitting opposite each other.

16 There are n guests at a party and some of these guests shake hands when they meet. Use the pigeonhole principle to show that there is a pair of guests who shake hands with the same number of people.
Hint: Place the n guests into holes labelled from 0 to $n - 1$, corresponding to the number of hands that they shake. Why must either the first or the last hole be empty?

9I The inclusion–exclusion principle

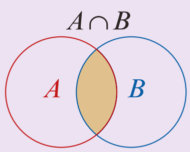
Basic set theory

A set is any collection of objects where order is not important. The set with no elements is called the **empty set** and is denoted by $\varnothing$. We say that set B is a **subset** of set A if each element of B is also in A. In this case, we can write $B \subseteq A$. Note that $\varnothing \subseteq A$ and $A \subseteq A$.

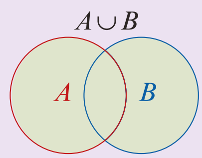
If A is a finite set, then the number of elements in A will be denoted by $|A|$.

Given any two sets A and B we define two important sets:

1 The **intersection** of sets A and B is denoted by $A \cap B$ and consists of elements belonging to A *and* B.



2 The **union** of sets A and B is denoted by $A \cup B$ and consists of elements belonging to A *or* B.



Note: It is important to realise that $A \cup B$ includes elements belonging to A and B.

Example 35

Consider the three sets of numbers $A = \{2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{3, 4, 5\}$.

a Find $B \cap C$.
b Find $A \cup C$.
c Find $A \cap B \cap C$.
d Find $A \cup B \cup C$.
e Find $|A|$.
f List all the subsets of C.

Solution

a $B \cap C = \{3, 4\}$
b $A \cup C = \{2, 3, 4, 5\}$
c $A \cap B \cap C = \{3\}$
d $A \cup B \cup C = \{1, 2, 3, 4, 5\}$
e $|A| = 2$
f $\varnothing, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}$

Earlier in the chapter we encountered the addition principle. This principle can be concisely expressed using set notation.

Addition principle

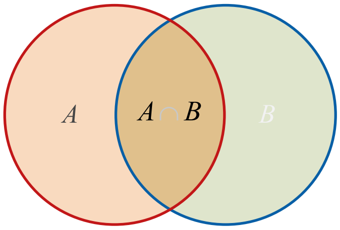
If A and B are two finite sets of objects such that $A \cap B = \varnothing$, then

$$|A \cup B| = |A| + |B|$$

Our aim is to extend this rule for instances where $A \cap B \neq \varnothing$.

Two sets

To count the number of elements in the set $A \cup B$, we first add (include) $|A|$ and $|B|$. However, this counts the elements in $A \cap B$ twice, and so we subtract (exclude) $|A \cap B|$.



Inclusion–exclusion principle for two sets

If A and B are two finite sets of objects, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 36

Each of the 25 students in a Year 11 class studies Physics or Chemistry. Of these students, 15 study Physics and 18 study Chemistry. How many students study both subjects?

Solution

$$|P \cup C| = |P| + |C| - |P \cap C|$$
$$25 = 15 + 18 - |P \cap C|$$
$$25 = 33 - |P \cap C|$$
$$\therefore \quad |P \cap C| = 8$$

Explanation

Let P and C be the sets of students who study Physics and Chemistry respectively.

Since each student studies Physics or Chemistry, we know that $|P \cup C| = 25$.

Example 37

A bag contains 100 balls labelled with the numbers from 1 to 100. How many ways can a ball be chosen that is a multiple of 2 or 5?

Solution

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 50 + 20 - 10$$
$$= 60$$

Explanation

Within the set of numbers $\{1, 2, 3, \dots, 100\}$, let A be the set of multiples of 2 and let B be the set of multiples of 5.

Then $A \cap B$ consists of numbers that are multiples of both 2 and 5, that is, multiples of 10.

Therefore $|A| = 50$, $|B| = 20$ and $|A \cap B| = 10$. We then use the inclusion–exclusion principle.

Example 38

A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly:

a two clubs

b three spades

c two clubs and three spades

d two clubs or three spades?

Solution

a ${}^{13}C_2 \cdot {}^{39}C_3 = 712\,842$

There are ${}^{13}C_2$ ways of choosing 2 clubs from 13 and ${}^{39}C_3$ ways of choosing 3 more cards from the 39 non-clubs.

b ${}^{13}C_3 \cdot {}^{39}C_2 = 211\,926$

There are ${}^{13}C_3$ ways of choosing 3 spades from 13 and ${}^{39}C_2$ ways of choosing 2 more cards from the 39 non-spades.

c ${}^{13}C_2 \cdot {}^{13}C_3 = 22\,308$

There are ${}^{13}C_2$ ways of choosing 2 clubs from 13 and ${}^{13}C_3$ ways of choosing 3 spades from 13.

d $|A \cup B|$

$= |A| + |B| - |A \cap B|$

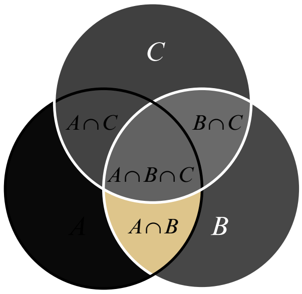
$= 712\,842 + 211\,926 - 22\,308$

$= 902\,460$

We let A be the set of all hands with 2 clubs and let B be the set of all hands with 3 spades. Then $A \cap B$ is the set of all hands with 2 clubs and 3 spades. We use the inclusion–exclusion principle to find $|A \cup B|$.

Three sets

For three sets A, B and C, the formula for $|A \cup B \cup C|$ is slightly harder to establish.



We first add $|A|$, $|B|$ and $|C|$. However, we have counted the elements in $A \cap B$, $A \cap C$ and $B \cap C$ twice, and the elements in $A \cap B \cap C$ three times.

Therefore we subtract $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ to compensate. But then the elements in $A \cap B \cap C$ will have been excluded once too often, and so we add $|A \cap B \cap C|$.

Inclusion–exclusion principle for three sets

If A, B and C are three finite sets of objects, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example 39

How many integers from 1 to 140 inclusive are not divisible by 2, 5 or 7?

Solution

Let A, B and C be the sets of all integers from 1 to 140 that are divisible by 2, 5 and 7 respectively. We then have

A	multiples of 2	$\lvert A\rvert = 140 \div 2 = 70$
B	multiples of 5	$\lvert B\rvert = 140 \div 5 = 28$
C	multiples of 7	$\lvert C\rvert = 140 \div 7 = 20$
$A \cap B$	multiples of 10	$\lvert A \cap B\rvert = 140 \div 10 = 14$
$A \cap C$	multiples of 14	$\lvert A \cap C\rvert = 140 \div 14 = 10$
$B \cap C$	multiples of 35	$\lvert B \cap C\rvert = 140 \div 35 = 4$
$A \cap B \cap C$	multiples of 70	$\lvert A \cap B \cap C\rvert = 140 \div 70 = 2$

We use the inclusion–exclusion principle to give

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 70 + 28 + 20 - 14 - 10 - 4 + 2 \\ &= 92\end{aligned}$$

Therefore the number of integers not divisible by 2, 5 or 7 is $140 - 92 = 48$.

Summary 9I

- The inclusion–exclusion principle extends the addition principle to instances where the two sets have objects in common.
- The principle works by ensuring that objects belonging to multiple sets are not counted more than once.
- The inclusion–exclusion principles for two sets and three sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Exercise 9I

Example 35

1 Consider the three sets of numbers $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 4, 6\}$.

a Find $B \cap C$.
b Find $A \cup C$.
c Find $A \cap B \cap C$.
d Find $A \cup B \cup C$.
e Find $|A|$.
f List all the subsets of A.

Example 36

2 In an athletics team, each athlete competes in track or field events. There are 25 athletes who compete in track events, 23 who compete in field events and 12 who compete in both track and field events. How many athletes are in the team?

3 Fifty patients at a medical clinic are being treated for a disease using two types of medication, A and B. There are 25 patients using medication A and 29 patients using medication B. How many patients are using both types of medication?

Example 37

4 How many integers from 1 to 630 inclusive are multiples of 7 or 9?

5 Consider the integers from 1 to 96 inclusive. How many of these are:

a divisible by 2 or 3
b not divisible by 2 or 3?

Example 38

6 How many five-letter arrangements of the word COMET:

a begin with a vowel
b end with a vowel
c begin and end with a vowel
d begin or end with a vowel?

7 **a** How many of the integers from 1 to 100 inclusive are perfect squares or perfect cubes?

b How many of the integers from 1 to 1000 inclusive are perfect squares or perfect cubes?

Example 39

8 How many of the integers from 1 to 120 inclusive are multiples of 2, 3 or 5?

9 How many of the integers from 1 to 220 inclusive are not divisible by 2, 5 or 11?

10 There are 98 Year 11 students at a secondary school and each of them must study at least one of Biology, Physics or Chemistry. There are 36 students who study Biology, 42 who study Physics and 40 who study Chemistry. Moreover, 9 study Biology and Physics, 8 study Biology and Chemistry and 7 study Physics and Chemistry. How many students study all three subjects?

11 A group of six students is selected from four students in Year 10, five in Year 11 and four in Year 12. How many selections have exactly:

a two Year 10 students
b two Year 11 students
c two Year 10 and two Year 11 students
d two Year 10 or two Year 11 students?

12 A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly one heart or exactly two diamonds?

13 Find the sum of all the integers from 1 to 100 inclusive that are divisible by 2 or 3.

14 There are seventy Year 11 students at a school and each of them must study at least one of three languages. Thirty are studying French, forty-five are studying Chinese, thirty are studying German and fifteen are studying all three languages. How many students are studying exactly two languages?

Chapter summary

Assign-ment

Nrich

- The addition and multiplication principles provide efficient methods for counting the number of ways of performing multiple tasks.
- The number of **permutations** (or arrangements) of n objects taken r at a time is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

- The number of **combinations** (or selections) of n objects taken r at a time is given by

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

- When permutations or combinations involve restrictions, we deal with them first.
- The values of ^nC_r can be arranged to give **Pascal's triangle**, where each entry is the sum of the two entries immediately above.
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

- A set of size n has 2^n subsets.
- The **pigeonhole principle** is used to show that some pair or group of objects have the same property.
- The **inclusion–exclusion principle** allows us to count the number of elements in a union of sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Technology-free questions

1 Evaluate:

a $^6C_3$ **b** $^{20}C_2$ **c** $^{300}C_1$ **d** $^{100}C_{98}$

2 Find the value of n if $^nC_2 = 55$.

3 How many three-digit numbers can be formed using the digits 1, 2 and 3 if the digits:

a can be repeated **b** cannot be repeated?

4 How many ways can six students be arranged on a bench seat with space for three?

5 How many ways can three students be allocated to five vacant desks?

6 There are four Year 11 and three Year 12 students in a school debating club. How many ways can a team of four be selected if two are chosen from each year level?

7 There are three boys and four girls in a group. How many ways can three children be selected if at least one of them is a boy?

8 On a ship's mast are two identical red flags and three identical black flags that can be arranged to send messages to nearby ships. How many different arrangements using all five flags are possible?

9 There are 53 English words written on a page. How many are guaranteed to share the same first letter?

10 Each of the twenty students in a class plays netball or basketball. Twelve play basketball and four play both sports. How many students play netball?

11 Six people are to be seated in a row. Calculate the number of ways this can be done so that two particular people, A and B, always have exactly one person between them.

Multiple-choice questions

1 Bao plans to study six subjects in Year 12. He has already chosen three subjects and for the remaining three he plans to choose one of four languages, one of three mathematics subjects and one of four science subjects. How many ways can he select his remaining subjects?

A 6 **B** 11 **C** 48 **D** 165 **E** 990

2 There are three flights directly from Melbourne to Brisbane. There are also two flights from Melbourne to Sydney and then four choices of connecting flight from Sydney to Brisbane. How many different paths are there from Melbourne to Brisbane?

A 9 **B** 11 **C** 18 **D** 20 **E** 24

3 In how many ways can 10 people be arranged in a queue at the bank?

A $10!$ **B** 10^{10} **C** 2^{10} **D** ${}^{10}C_2$ **E** ${}^{10}C_1$

4 How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5 and 6 at most once?

A ${}^{6}C_3$ **B** $3!$ **C** $6!$ **D** $6 \times 5 \times 4$ **E** $6 + 5 + 4$

5 How many permutations of the word UTOPIA begin and end with a vowel?

A 90 **B** 288 **C** 384 **D** 720 **E** 4320

6 How many ways can four identical red flags and three identical blue flags be arranged in a row?

A 4×3 **B** $\dfrac{7!}{4! \times 3!}$ **C** $7! \times 3! \times 4!$ **D** $4! \times 3!$ **E** $2 \times 3! \times 4!$

7 How many ways can three books be chosen from a collection of nine different books?

A $3!$ **B** $9 \times 8 \times 7$ **C** ${}^{9}C_3$ **D** $\dfrac{9!}{3!}$ **E** 3×9

8 The number of subsets of $\{A, B, C, D, E, F\}$ with at least one element is

A ${}^{6}C_2$ **B** ${}^{6}C_2 - 1$ **C** $2^5 - 1$ **D** $2^6 - 1$ **E** 2^6

9 A class consists of nine girls and eight boys. How many ways can a group of two boys and two girls be chosen?

A $\frac{17!}{2!\,2!}$ **B** ${}^{17}C_4$ **C** ${}^{9}C_2 \cdot {}^{8}C_2$ **D** $\frac{17!}{9!\,8!}$ **E** $9 \times 8 \times 8 \times 7$

10 There are six blue balls and five red balls in a bag. How many balls need to be selected at random before you are certain that three will have the same colour?

A 3 **B** 4 **C** 5 **D** 7 **E** 11

11 Each of the 30 students in a class studies French, German or Chinese. Of these students, 15 study French, 17 study German and 15 study Chinese. There are 15 students that study more than one subject. How many students study all three subjects?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Extended-response questions

1 A six-digit number is formed using the digits 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done if:

a the first digit is 5
b the first digit is even
c even and odd digits alternate
d the even digits are kept together?

2 Three letters from the word AUNTIE are arranged in a row. How many ways can this be done if:

a the first letter is E
b the first letter is a vowel
c the letter E is used?

3 A student leadership team consists of four boys and six girls. A group of four students is required to organise a social function. How many ways can the group be selected:

a without restriction
b if the school captain is included
c if there are two boys
d if there is at least one boy?

4 Consider the eight letters N, N, J, J, T, T, T, T. How many ways can all eight letters be arranged if:

a there is no restriction
b the first and last letters are both N
c the two Js are adjacent
d no two Ts are adjacent?

5 A pizza restaurant offers the following toppings: onion, capsicum, mushroom, olives, ham and pineapple.

a How many different kinds of pizza can be ordered with:

i three different toppings
ii three different toppings including ham
iii any number of toppings (between none and all six)?

b Another pizza restaurant boasts that they can make more than 200 varieties of pizza. What is the smallest number of toppings that they could use?

6 In how many ways can a group of four people be chosen from five married couples if:

a there is no restriction

b any two married couples are chosen

c the selected group cannot contain a married couple?

7 The name David Smith has initials DS.

a How many different two-letter initials are possible?

b How many different two-letter initials contain at least one vowel?

c Given 50 000 people, how many of them can be guaranteed to share the same two-letter initials?

8 Consider the integers from 1 to 96 inclusive. Let sets A and B consist of those integers that are multiples of 6 and 8 respectively.

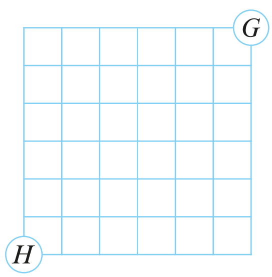
a What is the lowest common multiple of 6 and 8?

b How many integers belong to $A \cap B$?

c How many integers from 1 to 96 are divisible by 6 or 8?

d An integer from 1 to 96 is chosen at random. What is the probability that it is not divisible by 6 or 8?

9 Every morning, Milly walks from her home $H(0, 0)$ to the gym $G(6, 6)$ along city streets that are laid out in a square grid as shown. She always takes a path of shortest distance.



a How many paths are there from H to G?

b Show that there is some path that she takes at least twice in the course of three years.

c On her way to the gym, she often purchases a coffee at a cafe located at point $C(2, 2)$. How many paths are there from:

i H to C **ii** C to G **iii** H to C to G?

d A new cafe opens up at point $B(4, 4)$. How many paths can Milly take, assuming that she buys coffee at either cafe?

Hint: You will need to use the inclusion–exclusion principle here.

10 A box contains 400 balls, each of which is blue, red, green, yellow or orange. The ratio of blue to red to green balls is 1 : 4 : 2. The ratio of green to yellow to orange balls is 1 : 3 : 6. What is the smallest number of balls that must be drawn to ensure that at least 50 balls of one colour are selected?

10

Revision of Chapters 6–9

10A Technology-free questions

1 Suppose that n is odd. Prove that $n^2 + n$ is even.

2 Prove that if m and n are consecutive integers, then $n^2 - m^2 = n + m$.

3 Let $n \in \mathbb{Z}$. Consider the statement: If $5n + 3$ is even, then n is odd.

a Write down the converse statement.
b Prove the converse.
c Write down the contrapositive statement.
d Prove the contrapositive.

4 Suppose the number x is irrational. Prove by contradiction that $x + 1$ is also irrational.

5 Prove by contradiction that 6 cannot be written as the difference of two perfect squares.

6 Let $n \in \mathbb{Z}$. Prove that $3n + 1$ is even if and only if n is odd.

7 Prove that each of the following statements is false by finding a counterexample:

a The sum of two prime numbers cannot be a prime number.
b For all $x \in \mathbb{R}$, we have $x^3 > x^2$.

8 Show that this statement is false: There exists $n \in \mathbb{N}$ such that $25n^2 - 9$ is a prime number.

9 Prove by mathematical induction that:

a $2 + 4 + \cdots + 2n = n(n + 1)$
b $11^n - 6$ is divisible by 5, for all $n \in \mathbb{N}$

10 Let A, B and C be subsets of ξ.

a Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
b Illustrate this equality using a Venn diagram.

11 Let A and B be subsets of ξ. Simplify each of the following expressions:

a $A \cap \varnothing$ **b** $A \cup \xi$ **c** $(A \cup B) \cup A$

d $(A \cap B) \cup \varnothing$ **e** $A \cap A'$ **f** $A \cup A'$

g $(A \cap B) \cap B'$ **h** $(A \cup B') \cup B$ **i** $A \cup (B \cap A)$

j $A \cap (A' \cup B)$ **k** $B \cap (A \cup B)'$ **l** $A \cap (A \cap B)'$

12 Let $x, y \in B$, where B is a Boolean algebra. Simplify each of the following expressions:

a $x \wedge 1$ **b** $x \vee 0$ **c** $x' \wedge x$

d $x' \vee x$ **e** $(x \vee x') \vee x$ **f** $(x \wedge y) \wedge x$

g $(x \wedge x') \wedge y$ **h** $x \vee (x' \vee y)$ **i** $(x \wedge 0) \wedge y$

j $(x \vee 1) \wedge x'$ **k** $y \wedge (x \vee y')$ **l** $x \wedge (x \vee y)'$

13 Find a Boolean expression for the Boolean function defined by the table on the right, and simplify this expression.

x	y	$f(x, y)$
0	0	1
0	1	0
1	0	1
1	1	0

14 Consider the following two statements.

- A: Amina is in Year 11.
- B: Bao is in Year 11.

Write each of the following statements in symbolic form:

a Amina is not in Year 11.

b Amina and Bao are in Year 11.

c If Amina is in Year 11, then Bao is not.

d Amina is in Year 11 or, if she is not, then Bao is in Year 11.

e Amina and Bao are in Year 11, or else neither Amina nor Bao is in Year 11.

15 Consider the following two statements.

- P: Yasmin plays the violin.
- Q: Yasmin is in the school orchestra.

a Write each of the following statements in symbolic form:

  i Yasmin plays the violin and is in the school orchestra.

  ii If Yasmin plays the violin, then she is in the school orchestra.

b Write a statement in English corresponding to $\neg Q \Leftrightarrow \neg P$.

16 Prove each of the following by using a truth table:

a $\neg A \vee B$ is equivalent to $\neg(A \wedge \neg B)$

b $(A \vee B) \wedge (\neg A \wedge \neg B)$ is a contradiction

c $(A \wedge B) \Rightarrow (A \vee B)$ is a tautology

d $A \wedge (B \vee C)$ is equivalent to $(A \wedge B) \vee (A \wedge C)$

Hint: The truth table for part **d** will require 8 rows.

17 Consider the following two statements.

- P: I completed the task.
- Q: I was paid.

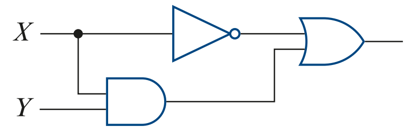
a Write the statement $\neg Q$ in words.

b Write the following statement in symbols: If I was paid, then I completed the task.

c Using a truth table, show that $((P \vee \neg Q) \wedge Q) \Rightarrow P$ is a tautology.

18 Prove that $\neg(P \vee \neg Q) \vee (\neg P \wedge \neg Q)$ is logically equivalent to $\neg P$ by constructing a truth table.

19 **a** Write the Boolean expression corresponding to the circuit shown on the right.



b Simplify this expression by using the properties of Boolean algebras.

c Draw the circuit corresponding to the simplified expression.

20 For each of the following, use a truth table to determine whether the argument is valid:

a

Premise 1	$(A \wedge B) \Rightarrow C$
Premise 2	A
Premise 3	$\neg B$
Conclusion	$\neg C$

b

Premise 1	$A \vee B$
Premise 2	$A \Rightarrow C$
Premise 3	$B \Rightarrow C$
Conclusion	C

21 Use a truth table to determine whether the following argument is valid:

Premise 1	If I am Sam's father, then Sam is Will's brother.
Premise 2	Sam is Will's brother.
Conclusion	I am not Sam's father.

22 The following algorithm begins with an empty list called A. Numbers are added to the list according to a given rule.

```
A ← [ ]
for i from 1 to 16
    if i is not the sum of two different entries in A then
        append i to A
    end if
end for
print A
```

a Give the final value of A after this algorithm is executed.

b Suppose that '`two different entries`' is changed to '`two or more different entries`'. Give the final value of A after the modified algorithm is executed.

c Describe the entries in A from part **b**.

23 Consider the function defined on the right, which takes a natural number n as input.

a Describe what this function does.

b Determine the output of the function when the input is $n = 4$.

c Determine the input if the output is 55.

d Rewrite the function so that it evaluates the product of the cubes of the first n natural numbers.

```
define function(n):
  sum ← 0
  for i from 1 to n
     sum ← sum + i^2
  end for
  return sum
```

24 The function $remainder(n, d)$ returns the remainder when n is divided by d.
Consider the function defined on the right, which takes a natural number n as input.

a Determine the output of the function when the input is $n = 120$. A desk check will help you to do this.

b Describe what this function does.

c Describe the inputs for which the output is 3.

```
define function(n):
  total ← 0
  while remainder(n, 2) = 0
     n ← n/2
     total ← total + 1
  end while
  return total
```

25 How many ways can four different books be arranged on a shelf?

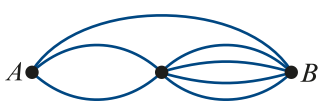
26 How many ways can three teachers and three students be arranged in a row if a teacher must be at the start of the row?

27 How many different three-digit numbers can be formed using the digits 1, 3, 5, 7 and 9:

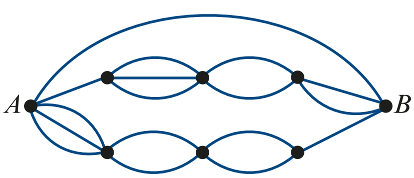
a as many times as you would like **b** at most once?

28 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

a



b



29 Evaluate each of the following:

a $4!$ **b** $\dfrac{6!}{4!}$ **c** $\dfrac{8!}{6!\,2!}$ **d** ${}^{10}C_2$

30 How many ways can five children be arranged on a bench with space for:

a four children **b** five children?

31 A bookshelf has three different mathematics books and two different physics books. How many ways can these books be arranged:

a without restriction **b** if the mathematics books are kept together?

32 Using the digits 0, 1, 2, 3 and 4 without repetition, how many five-digit numbers can you form:

a without restriction

b that are divisible by 10

c that are greater than 20 000

d that are even?

33 Asha has three identical 20 cent pieces and two identical 10 cent pieces. How many ways can she arrange these coins in a row?

34 How many ways can you select:

a three children from a group of six

b two letters from the alphabet

c four numbers from the set $\{1, 2, \dots, 10\}$

d three sides of an octagon?

35 Consider the set of numbers $X = \{1, 2, \dots, 8\}$.

a How many subsets of X have exactly two elements?

b How many subsets of X have exactly three elements, one of which is the number 8?

c Find the total number of subsets of X.

36 How many ways can you select three boys and two girls from a group of five boys and four girls?

37 There are four Labor and five Liberal parliamentarians, from which four are to be selected to form a committee. If the committee must include at least one member from each party, how many ways can this be done?

38 There are 10 blue, 11 green and 12 red balls in a bag. How many balls must be chosen at random to be sure that at least three will have the same colour?

39 How many different natural numbers from 1 to 99 inclusive must be chosen at random to be sure there will be at least one pair of numbers that sums to 100?

40 How many integers from 1 to 120 inclusive are divisible by 2 or 3?

10B Multiple-choice questions

1 Suppose that both m and n are odd. Which of the following statements is false?

A $m + n$ is even
B $m - n$ is even
C $3m + 5n$ is even
D $2m + n$ is odd
E $mn + 1$ is odd

2 Suppose that m is divisible by 4 and n is divisible by 12. Which of the following statements might be false?

A $m \times n$ is divisible by 3
B $m \times n$ is divisible by 48
C $m + n$ is divisible by 4
D $m^2 n$ is divisible by 48
E n is divisible by m

3 Let m and n be integers. Which of the following statements is always true?

A If mn is even, then m is even.
B The number $m + n$ is even if and only if both m and n are even.
C If $m + n$ is odd, then mn is odd.
D If mn is odd, then $m + n$ is even.
E If $m + n$ is even, then $m - n$ is odd.

4 Consider the statement: If n is even, then $n + 3$ is odd. The converse of this statement is

A If $n + 3$ is even, then n is even.
B If n is odd, then $n + 3$ is even.
C If $n + 3$ is odd, then n is even.
D If $n + 3$ is odd, then n is odd.
E If $n + 3$ is even, then n is odd.

5 Assume that a and b are positive real numbers with $a > b$. Which of the following might be false?

A $\frac{1}{a - b} > 0$ **B** $\frac{a}{b} - \frac{b}{a} > 0$ **C** $a + b > 2b$ **D** $a + 3 > b + 2$ **E** $2a > 3b$

6 The number of pairs of integers (m, n) that satisfy $mn - n = 12$ is

A 2 **B** 3 **C** 4 **D** 6 **E** 12

7 Suppose that n is a positive integer. For how many values of n is the number $9n^2 - 4$ a prime?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

8 If a, b, c and d are consecutive integers, then which of the following statements may be false?

A $a + b + c + d$ is divisible by 2
B $a + b + c + d$ is divisible by 4
C $a \times b \times c \times d$ is divisible by 3
D $a \times b \times c \times d$ is divisible by 8
E $a \times b \times c \times d$ is divisible by 24

9 Consider the statement:

- If you know the rules and you are not overconfident, then you win the game.

The contrapositive of this statement is

A If you lose the game, then you don't know the rules or you are overconfident.
B If you lose the game, then you don't know the rules and you are overconfident.
C If you win the game, then you know the rules and you are not overconfident.
D If you know the rules or you are not overconfident, then you win the game.
E If you don't know the rules or you are overconfident, then you lose the game.

10 Let A and B be subsets of ξ. The expression $A \cap (A' \cup B)$ simplifies to

A $A \cup B$ **B** $A \cap B$ **C** $A' \cap B$ **D** $A' \cup B$ **E** $A \cap B'$

11 Let $x, y \in B$, where B is a Boolean algebra. An expression equivalent to $((x \wedge y)' \vee z)'$ is

A $(x \wedge y) \vee z$ **B** $x \wedge y \wedge z$ **C** $x \vee y \vee z$ **D** $(x \wedge z) \vee y'$ **E** $x \wedge y \wedge z'$

12 Which of the following Boolean functions has the table of values shown on the right?

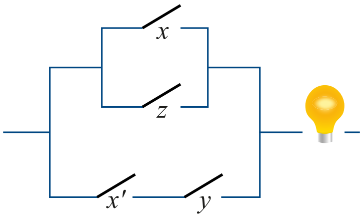
x	y	$f(x, y)$
0	0	0
0	1	0
1	0	1
1	1	0

A $f(x, y) = x' \wedge y$ **B** $f(x, y) = x \wedge y$
C $f(x, y) = x \wedge y'$ **D** $f(x, y) = x \vee y'$
E $f(x, y) = x \vee y$

13 Consider the statement: If it flies, then it is not an elephant. Which of the following is equivalent to this statement?

A If it flies, then it is an elephant.
B If it does not fly, then it is an elephant.
C If it does not fly, then it is not an elephant.
D If it is an elephant, then it does not fly.
E If it is not an elephant, then it does not fly.

14 Which of the following Boolean expressions represents the switching circuit shown?



A $(x \vee z) \vee (x' \wedge y)$ **B** $(x \wedge z) \wedge (x' \vee y)$
C $(x \vee z) \wedge (x' \wedge y)$ **D** $((x \vee z) \wedge x') \wedge y$
E $((x \wedge z) \vee x') \vee y$

15 The Boolean expression $x \vee (x \vee y)'$ simplifies to

A $x \wedge y'$ **B** $x \vee y'$ **C** $x' \wedge y'$ **D** $x' \vee y$ **E** $x' \wedge y$

16 Which of the following statements is a contradiction?

A $P \Rightarrow Q$ **B** $(\neg P) \wedge P$ **C** $(\neg P) \vee P$ **D** $P \vee Q$ **E** $P \wedge Q$

17 This statement is a tautology:

$$[(A \vee B) \wedge (\neg A)] \Rightarrow B$$

Which of the following arguments is an example of this tautology?

A The cat is grey or black. The cat is not grey. Therefore the cat is black.
B The cat is grey or black. The cat is grey. Therefore the cat is not black.
C The cat is grey. Therefore the cat is not black.
D The cat is not black. Therefore the cat is grey.
E The cat is not grey. Therefore the cat is black.

18 Consider the following argument:

- If I am 18, then I am eligible to vote. I am 18. Therefore I am eligible to vote.

Which of the following compound statements represents this argument?

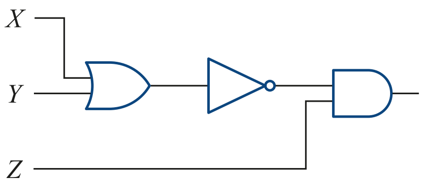
A $[(A \vee B) \wedge B] \Rightarrow A$ **B** $[(A \wedge B) \wedge A] \Rightarrow B$ **C** $[(A \wedge B) \vee A] \Rightarrow B$
D $[(A \Rightarrow B) \wedge B] \Rightarrow B$ **E** $[(A \Rightarrow B) \wedge A] \Rightarrow B$

19 Which of the following statements is *not* a tautology?

A $A \vee \neg A$ **B** $A \Rightarrow A$ **C** $(A \vee B) \Leftrightarrow (B \vee A)$

D $(A \wedge B) \Rightarrow B$ **E** $(A \vee B) \Rightarrow (A \wedge B)$

20 Which of the following Boolean expressions represents the logic circuit shown?



A $\neg(X \wedge Y) \vee Z$ **B** $\neg(X \vee Y) \wedge Z$

C $(X \wedge \neg Y) \vee Z$ **D** $(X \vee \neg Y) \wedge Z$

E $(X \vee Y) \wedge \neg Z$

21 The minimal Boolean expression represented by the Karnaugh map shown is

	yz	$y'z$	$y'z'$	yz'
x	1	1	1	1
x'			1	

A $x \vee (y' \wedge z')$ **B** $x \wedge (y' \vee z')$

C $x' \vee (y \wedge z)$ **D** $x' \wedge (y \vee z)$

E $x \vee (x' \wedge y' \wedge z')$

22 This code will print

```
product ← 1
for n from 1 to 6
    product ← product × (2n − 1)
end for
print product
```

A the value of $2 \times 4 \times 6$

B the value of $1 \times 2 \times 3 \times 4 \times 5 \times 6$

C the value of $1 \times 3 \times 5$

D the value of $1 \times 3 \times 5 \times 7 \times 9$

E the value of $1 \times 3 \times 5 \times 7 \times 9 \times 11$

23 The function *remainder*(n, d) returns the remainder when n is divided by d.

Consider the function defined on the right, which takes a natural number n as input.

The value of $f(12)$ is

```
define f(n):
  sum ← 0
  for i from 1 to n
      if remainder(n, i) = 0 then
          sum ← sum + 1
      end if
  end for
  return sum
```

A 1 **B** 2 **C** 3

D 4 **E** 6

24 The algorithm shown on the right will print the value

```
sum ← 0
for x from 1 to 2
    for y from 1 to 3
        sum ← sum + x + y
    end for
end for
print sum
```

A 18 **B** 19 **C** 20

D 21 **E** 22

25 How many ways can five people be arranged in a line?

A $5!$ **B** 2^5 **C** 5^5 **D** ${}^5C_1$ **E** $5 + 4 + 3 + 2 + 1$

26 How many arrangements of the word HOBART begin with a vowel?

A 24 **B** 48 **C** 240 **D** 120 **E** 720

27 How many four-digit numbers can be formed using the digits 1, 2, 3, 4 and 5 at most once?

A ${}^5C_4$ **B** $5 + 4 + 3 + 2$ **C** $5 \times 4 \times 3 \times 2$ **D** $4!$ **E** 5^4

28 The number of arrangements of the digits in the number 111222 is

A $\dfrac{6!}{3! \times 3!}$ **B** $6!$ **C** $\dfrac{6!}{3!}$ **D** $3! \times 3! \times 3!$ **E** $3! \times 3!$

29 Sam has n identical 10 cent pieces and n identical 20 cent pieces. How many ways can these coins be arranged in a row?

A $n! \times n!$ **B** $\dfrac{(2n)!}{(n!)^2}$ **C** $\dfrac{(n!)^2}{(2n)!}$ **D** $(2n)!$ **E** $2n!$

30 There are 10 flavours of ice-cream at a shop. Mark will select three flavours for his cone, one of which must be chocolate. The total number of different selections is

A ${}^{10}C_3$ **B** ${}^{10}C_2$ **C** ${}^9C_3$ **D** ${}^9C_2$ **E** ${}^8C_2$

31 There are four Labor and five Liberal parliamentarians, from which two of each are to be selected to form a committee. How many ways can this be done?

A ${}^9C_2$ **B** ${}^9C_4$ **C** ${}^9C_2 \times {}^9C_2$ **D** ${}^4C_2 \times {}^5C_2$ **E** ${}^4C_2 + {}^5C_2$

32 From 10 friends, you can invite any number of them to the movies. Assuming that you invite at least one friend, how many different selections can you make?

A 2^9 **B** $2^9 - 1$ **C** 2^{10} **D** $2^{10} - 1$ **E** ${}^{10}C_1$

33 An untidy kitchen drawer has a jumbled collection of eight knives, six forks and ten spoons. What is the smallest number of items that must be randomly chosen to ensure that at least four items of the same type are selected?

A 10 **B** 11 **C** 12 **D** 13 **E** 14

34 Whenever n integers are written on a whiteboard, at least six of them leave the same remainder when divided by 3. What is the smallest possible value of n?

A 3 **B** 4 **C** 7 **D** 15 **E** 16

35 How many integers from 1 to 60 inclusive are multiples of 2 or 5?

A 32 **B** 36 **C** 40 **D** 44 **E** 50

10C Extended-response questions

1 Let a, b and c be integers. Suppose you know that $a + b$ is even and $b + c$ is odd.

a Is it possible to work out whether a, b and c are even or odd?

b What if you also know that $a + b + c$ is even?

2 **a** Find all positive integer values of a, b and c such that $a < b < c$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

b Find all positive integer values of a, b, c and d such that $a < b < c < d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 2$$

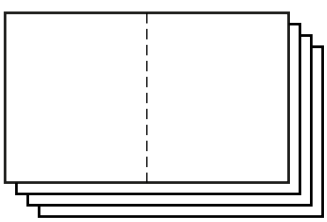
3 Let a, b and c be positive real numbers. Prove that if $b > a$, then $\dfrac{a+c}{b+c} > \dfrac{a}{b}$.

4 **a** Find the smallest value of $n \in \mathbb{N}$ such that $2^n > 10^3$.

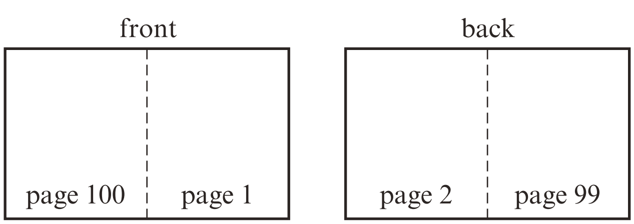
b Hence, prove that 2^{100} has at least 31 digits.

c Hence, explain why some digit in the decimal expansion of 2^{100} occurs at least four times. (Hint: There are 10 different digits: $0, 1, \ldots, 9$.)

5 A stack of paper, printed on both sides, is folded in the middle to make a newspaper.



Each sheet contains four pages. The page numbers on the top sheet of Monday's newspaper are 1, 2, 99 and 100.



a What are the page numbers on the bottom sheet of Monday's stack?

b One of the sheets in Monday's newspaper has page numbers 7 and 8. What are its other two page numbers?

c Suppose that a newspaper is made from n sheets of paper. Prove that the sum of the four page numbers on each sheet is a constant.

d Tuesday's newspaper has a sheet whose pages are numbered 11, 12, 33 and 34. How many pages does this newspaper have?

6 Sam has 20 one-dollar coins and seven pockets. He wants to put coins into his pockets so that each pocket contains a different number of coins. (The number 0 is allowed.)

a Prove that this is impossible.

b What is the minimum number of coins Sam would need to do this?

c If Sam had 50 one-dollar coins, find the maximum number of pockets that he could fill, each with a different number of coins.

7 Take a close look at the following square numbers:

$$15^2 = 225, \quad 25^2 = 625, \quad 35^2 = 1225, \quad 45^2 = 2025, \quad 55^2 = 3025, \quad 65^2 = 4225$$

a Find and describe the pattern that you see in these square numbers.

b Confirm that your pattern works for the number 75.

c Prove that your pattern actually works. (Hint: Each number is of the form $10n + 5$.)

8 Heidi has 10 wooden cubes, with edges of length 1 cm through to 10 cm.

a Using all the cubes, can she build two towers of the same height?

b Now Heidi has n wooden cubes, with edges of length 1 through to n. For what values of n can Heidi use all the cubes to build two towers of the same height?

9 **a** Suppose that a is odd and b is odd. Prove that ab is odd.

b Suppose that a is odd and $n \in \mathbb{N}$. Prove by induction that a^n is odd.

c Hence, prove that if x satisfies $3^x = 2$, then x is irrational.

10 **a** If $n^4 + 6n^3 + 11n^2 + 6n + 1 = (an^2 + bn + c)^2$, find the positive values of a, b and c.

b Hence, prove that when 1 is added to the product of four consecutive integers, the result is always a perfect square.

c Hence, write the number $5 \times 6 \times 7 \times 8 + 1$ as a product of prime numbers.

11 Consider the following list of numbers:

$$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$$

We proceed through the list entry by entry, calculating a running tally as follows. The tally is initially zero. Starting with the first entry, each entry is subtracted from the running tally if the result is non-negative, otherwise it is added to the tally. The final result is called the **net tally**.

a Complete the following table by giving the value of *tally* as each entry n in the list is either subtracted or added.

n		1	2	3	4	5	6	7	8	9	10
tally	0	1	3	0	4						

b Describe this general process by writing an algorithm in pseudocode.

c Change the order of the numbers in the list so that the net tally is 19.

d Prove that, for every natural number n, the list of numbers $[1, 2, 3, \ldots, 4n]$ can be reordered so that the net tally is zero.

12 In Chapter 7, we constructed logic circuits using 'or' gates, 'and' gates and 'not' gates. Two other commonly used logic gates are the 'nor' gate (*not or*) and the 'nand' gate (*not and*). These two gates are important because any logic circuit can be constructed using only 'nor' gates and also constructed using only 'nand' gates.

a The 'nor' gate corresponds to the Boolean expression $\neg(A \vee B)$. Describe the operation of the 'nor' gate using a truth table.

b The 'nand' gate corresponds to the Boolean expression $\neg(A \wedge B)$. Describe the operation of the 'nand' gate using a truth table.

c Show an implementation of each of the following using only 'nand' gates:

i $\neg A$ Hint: Input A twice into the same 'nand' gate.

ii $A \wedge B$ Hint: Input A and B into a 'nand' gate to produce $\neg(A \wedge B)$. Then input $\neg(A \wedge B)$ twice into a second 'nand' gate.

iii $A \vee B$

d Show an implementation of each of the following using only 'nor' gates:

i $\neg A$ **ii** $A \vee B$ **iii** $A \wedge B$

13 A five-digit number is formed using the digits 0, 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done:

a without restriction **b** if the number is divisible by 10

c if the number is odd **d** if the number is even?

14 Nic and Lucy belong to a group of eight coworkers. There are three men and five women in this group. A team of four workers is required to complete a project. How many ways can the team be selected:

a without restriction **b** if it must contain two men and two women

c if it must contain both Nic and Lucy **d** if it must not contain both Nic and Lucy?

15 A sailing boat has three identical black flags and three identical red flags. The boat can send signals to nearby boats by arranging flags along its mast.

a How many ways can all six flags be arranged in a row?

b How many ways can all six flags be arranged in a row if no two black flags are adjacent?

c Using at least one flag, how many arrangements in a row are possible?

16 Consider the letters in the word BAGGAGE.

a How many arrangements of these letters are there?

b How many arrangements begin and end with a vowel?

c How many arrangements begin and end with a consonant?

d How many arrangements have all vowels together and all consonants together?

17 There are 25 people at a party.

a If every person shakes hands with every other person, what is the total number of handshakes?

b In fact, there are two rival groups at the party, so everyone only shakes hands with every other person in their group. If there are 150 handshakes, how many people are in each of the rival groups?

c At another party, there are 23 guests. Explain why it is not possible for each person to shake hands with exactly three other guests.

18 Jacobi's method This is a method for finding approximate solutions to systems of equations. For example, consider the simultaneous equations

$$9x + 3y = 1 \quad (1) \qquad \text{and} \qquad 3x + 11y = 2 \quad (2)$$

We rewrite the equations by solving the first for x and the second for y:

$$x = \frac{1}{9}(1 - 3y) \quad (3) \qquad \text{and} \qquad y = \frac{1}{11}(2 - 3x) \quad (4)$$

We then make an initial guess at a solution of the equations, say $(x_0, y_0) = (0, 0)$. This guess can be improved by substituting these values into (3) and (4) to obtain

$$x_1 = \frac{1}{9}(1 - 3y_0) \approx 0.111 \qquad \text{and} \qquad y_1 = \frac{1}{11}(2 - 3x_0) \approx 0.182$$

In general, we substitute the values of x_n and y_n into (3) and (4) to obtain

$$x_{n+1} = \frac{1}{9}(1 - 3y_n) \qquad \text{and} \qquad y_{n+1} = \frac{1}{11}(2 - 3x_n)$$

a Complete the table on the right by continuing this process. Record your answers to three decimal places.

n	x_n	y_n
0	0	0
1	0.111	0.182
2		
3		
4		
5		

b Find the exact solution of the simultaneous equations algebraically. Compare the exact solution (x, y) with the approximate solution (x_5, y_5).

c Now use Jacobi's method to find an approximate solution of the system of equations

$$8x + y = 5 \quad \text{and} \quad 2x + 13y = 4$$

Start with the initial guess $(x_0, y_0) = (0, 0)$ and complete five iterations.

d We next look at an example where Jacobi's method does not work. Consider the system of equations

$$2x - 3y = 1 \quad \text{and} \quad 3x - 2y = -1$$

i Find the exact solution of this system algebraically.

ii Apply Jacobi's method to this system. Start with the initial guess $(x_0, y_0) = (0, 0)$ and complete five iterations. Summarise your answers in a table as in part **a**.

iii What do you notice about the values of x_n and y_n as n increases?

iv Confirm your observation by proving that $x_n \geq \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$ for all $n \geq 1$.

19 On a clock's face, twelve points are evenly spaced around a circle.

a How many ways can you select four of these points?

b How many ways can you select two points that are not diametrically opposite?

c For every selection of two points that are not diametrically opposite, you can draw one rectangle on the face that has these two points as vertices. What are the other two vertices?

d How many ways can you select four points that are the vertices of a rectangle?
Hint: Why must you divide the answer to part **b** by 4?

e Four points are randomly selected. What is the probability that the four points are the vertices of a rectangle?

20 Let $\mathbb{B} = \{0, 1\}$. We have seen that $\mathbb{B}$ forms a Boolean algebra with the operations $\vee$, $\wedge$ and $'$. Now define $\mathbb{B}^2 = \{ (a, b) : a, b \in \mathbb{B} \}$. Then $\mathbb{B}^2$ forms a Boolean algebra with the operations $\vee$, $\wedge$ and $'$ given by

- $(a, b) \vee (c, d) = (a \vee c, b \vee d)$
- $(a, b) \wedge (c, d) = (a \wedge c, b \wedge d)$
- $(a, b)' = (a', b')$

The set $\mathbb{B}^2$ has four elements in total: $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. Two of these are the distinguished elements $0 = (0, 0)$ and $1 = (1, 1)$.

a Evaluate each of the following in $\mathbb{B}^2$:

i $(1, 0) \wedge (0, 1)$ **ii** $(1, 0) \vee (0, 1)$ **iii** $(1, 1) \wedge (0, 1)$
iv $(0, 0) \vee (0, 1)$ **v** $(1, 0)'$ **vi** $(1, 1)'$

b Similarly, we can define a Boolean algebra $\mathbb{B}^3$, where $\mathbb{B}^3 = \{ (a, b, c) : a, b, c \in \mathbb{B} \}$.

i List the elements of $\mathbb{B}^3$.

ii Evaluate $(1, 0, 0) \wedge (0, 1, 0)$ and $(1, 0, 0) \vee (0, 0, 1)$.

iii Show how every element of $\mathbb{B}^3$ can be formed from joins ($\vee$) and meets ($\wedge$) of the elements $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

c More generally, we can define a Boolean algebra $\mathbb{B}^n$, for each natural number n. How many elements does $\mathbb{B}^n$ have?

In Extended-response question 4 in Chapter 7, we considered the Boolean algebra B of all factors of 30, with the operation $\vee$ as LCM and the operation $\wedge$ as HCF.

d Find a correspondence between the elements of B and the elements of $\mathbb{B}^3$ that shows that these two Boolean algebras have the same structure.
Hint: Start with the correspondences $2 \leftrightarrow (1, 0, 0)$, $3 \leftrightarrow (0, 1, 0)$ and $5 \leftrightarrow (0, 0, 1)$. This gives $6 \leftrightarrow (1, 1, 0)$. Consider the operations on both Boolean algebras.

e The number 6 has four factors. Discuss the correspondence between $\mathbb{B}^2$ and the Boolean algebra of factors of 6.

f The number 210 has 16 factors. Discuss the correspondence between $\mathbb{B}^4$ and the Boolean algebra of factors of 210.

10D Investigations

1 Sums of consecutive natural numbers

The numbers 9 and 12 can each be expressed as the sum of at least two consecutive natural numbers:

$$4 + 5 = 9 \quad \text{and} \quad 3 + 4 + 5 = 12$$

a Investigate which natural numbers from 1 to 20 can be expressed as the sum of at least two consecutive natural numbers.

b Based on your answer to part **a**, make a conjecture as to which numbers can be expressed as the sum of at least two consecutive natural numbers.

c Try to prove your conjecture from part **b**.

2 Natural numbers written on a page

There are n natural numbers written on a page, where $n \geq 2$.

- Katia chooses two of the numbers on the page; call them a and b.
- She then erases these two numbers and writes the single number $ab + a + b$.

She repeats this process until there is only one number left on the page, which is 71.

a Suppose that $n = 2$. Determine all the possibilities for the two numbers that could have been written on the page at the start.

b Suppose that $n = 3$. Determine all the possibilities for the three numbers that could have been written on the page at the start.

c What is the largest possible value of n?

d Investigate why the question would make no sense if Katia was left with a final number of 70.

3 Properties of the Fibonacci sequence

The Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

has many interesting properties. In Question 7 of Exercise 8D, you worked with an algorithm in pseudocode that generates the Fibonacci sequence. Adapt that code in order to investigate this sequence. State results and develop proofs where you can.

- What happens when you add six consecutive Fibonacci numbers and divide by four?
- Which Fibonacci numbers are even?
- Which Fibonacci numbers are divisible by 3?
- Describe the behaviour of the sequence of quotients $\dfrac{F_{n+1}}{F_n}$ as $n \to \infty$.
- Determine $F_{n-1}F_{n+1} - F_n^2$ for all $n \geq 2$.

There are many other properties. Use your programs to investigate.

4 Stars and bars

The technique of using stars and bars for combinatorics problems was introduced by William Feller (1906–1970).

a Suppose that 10 identical chocolates are to be distributed among three children, Amy, Ben and Clara. We will investigate the number of ways that this can be done.

Let a, b and c denote the number of chocolates given to Amy, Ben and Clara respectively. We can represent allocations of the 10 chocolates using *stars and bars*. For example:

- The allocation $(a, b, c) = (2, 3, 5)$ is represented as

  $$* * \mid * * * \mid * * * * *$$

- The allocation $(a, b, c) = (3, 0, 7)$ is represented as

  $$* * * \mid \mid * * * * * * *$$

i Using stars and bars, represent each of the following allocations:

- $(a, b, c) = (4, 5, 1)$
- $(a, b, c) = (0, 6, 4)$
- $(a, b, c) = (0, 0, 10)$

ii Notice that each allocation is represented by some arrangement of 10 stars and 2 bars. How many different ways can you allocate the 10 chocolates to the three children?

iii Using a similar technique, find the number of ways that eight chocolates can be distributed among four children.

iv How many ways can you distribute n chocolates among k children?

v If Amy, Ben and Clara are each to receive at least one of the 10 chocolates, how many ways can this be done?

vi How many ways can you distribute n chocolates among k children if each child is to receive at least one chocolate?

All of the following problems can be solved using this technique:

b How many ways can you distribute three identical balls into three different boxes?

c How many sequences of four non-negative integers are there that sum to 10?

d How many sequences of three odd positive integers are there that sum to 17?

e How many paths are there from the top-left corner to the bottom-right corner of an $m \times n$ grid if you can only travel right or down along the grid lines?

f List some other situations that can be considered in this way and analyse them using the technique of stars and bars.

11

Matrices

Objectives

- To identify when two matrices are **equal**.
- To **add** and **subtract** matrices of the same size.
- To multiply a matrix by a real number.
- To identify when the multiplication of two given matrices is possible.
- To perform **multiplication** of two suitable matrices.
- To find the **inverse** of a 2×2 matrix.
- To find the **determinant** of a 2×2 matrix.
- To solve **simultaneous linear equations** by using an inverse matrix.
- To use technology to find inverses and determinants of $n \times n$ matrices, where $n \geq 3$.

A **matrix** is a rectangular array of numbers. An example of a matrix is

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

Matrix algebra was first studied in England in the middle of the nineteenth century. Matrices are now used in many areas of science: for example, in physics, medical research, encryption and internet search engines.

In this chapter we will show how addition and multiplication of matrices can be defined and how matrices can be used to solve simultaneous linear equations.

In Chapter 12 we will see how matrices can be used to represent graphs, and in Chapter 20 we will see how they can be used to study transformations of the plane.

11A Matrix notation

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The following are examples of matrices:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

The size of a matrix

Matrices vary in size. The **size** of the matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines) that occur in the matrix.

The sizes of the above matrices are, in order:

$$3 \times 2, \quad 1 \times 4, \quad 3 \times 3, \quad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

An $m \times n$ matrix has m rows and n columns.

Example 1

Write down the sizes of the following matrices:

a $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$

Solution

a 2×3 **b** 4×1 **c** 1×3

Storing information in matrices

The use of matrices to store information is demonstrated by the following example.

Four exporters A, B, C and D sell refrigerators (r), dishwashers (d), microwave ovens (m) and televisions (t). The sales in a particular month can be represented by a 4×4 array of numbers. This array of numbers is called a matrix.

	r	d	m	t	
A	120	95	370	250	row 1
B	430	380	950	900	row 2
C	60	50	150	100	row 3
D	200	100	470	50	row 4
	column 1	column 2	column 3	column 4	

From this matrix it can be seen that:

- Exporter A sold 120 refrigerators, 95 dishwashers, 370 microwave ovens, 250 televisions.
- Exporter B sold 430 refrigerators, 380 dishwashers, 950 microwave ovens, 900 televisions.

The entries for the sales of refrigerators are in column 1.

The entries for the sales of exporter A are in row 1.

Example 2

A minibus has four rows of seats, with three seats in each row. If 0 indicates that a seat is vacant and 1 indicates that a seat is occupied, write down a matrix to represent:

a the 1st and 3rd rows are occupied, but the 2nd and 4th rows are vacant

b only the seat at the front-left corner of the minibus is occupied.

Solution

a $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example 3

There are four clubs in a local football league:

- Club A has 2 senior teams and 3 junior teams.
- Club B has 2 senior teams and 4 junior teams.
- Club C has 1 senior team and 2 junior teams.
- Club D has 3 senior teams and 3 junior teams.

Represent this information in a matrix.

Solution

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$$

Explanation

The rows represent clubs A, B, C, D and the columns represent the number of senior and junior teams.

Entries and equality

We will use uppercase letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ to denote matrices.

If $\mathbf{A}$ is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 3×4 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Two matrices **A** and **B** are **equal**, and we can write **A** = **B**, when:

- they have the same size, and
- they have the same entry at corresponding positions.

For example:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1 & -1 \\ 1-1 & 1 & \frac{6}{2} \end{bmatrix}$$

Example 4

If matrices **A** and **B** are equal, find the values of x and y.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ x & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -3 & y \end{bmatrix}$$

Solution

$x = -3$ and $y = 4$

Although a matrix is made from a set of numbers, it is important to think of a matrix as a single entity, somewhat like a 'super number'.

Summary 11A

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Two matrices **A** and **B** are equal when:
  - they have the same size, and
  - they have the same entry at corresponding positions.

Exercise 11A

Example 1

1 Write down the sizes of the following matrices:

a $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} a & b & c & d \end{bmatrix}$ **d** $\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$

Example 2

2 There are 25 seats arranged in five rows and five columns. Using 0 to indicate that a seat is vacant and 1 to indicate that a seat is occupied, write down a matrix to represent the situation when:

a only the seats on the two diagonals are occupied

b all seats are occupied.

3 Seating arrangements are again represented by matrices, as in Question 2. Describe the seating arrangement represented by each of the following matrices:

a the entry a_{ij} is 1 if $i = j$, but 0 if $i \neq j$

b the entry a_{ij} is 1 if $i > j$, but 0 if $i \leq j$

c the entry a_{ij} is 1 if $i = j + 1$, but 0 otherwise.

Example 3

4 At a certain school there are 200 girls and 110 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 117 in Year 8, 135 and 98 in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11, and 28 and 33 in Year 12. Summarise this information in a matrix.

Example 4

5 From the following, select those pairs of matrices which could be equal, and write down the values of x and y which would make them equal:

a $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ x \end{bmatrix}$, $\begin{bmatrix} 0 & x \end{bmatrix}$, $\begin{bmatrix} 0 & 4 \end{bmatrix}$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$, $\begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 4 & x & 1 & -2 \end{bmatrix}$

c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$

6 Find the values of the pronumerals so that matrices **A** and **B** are equal:

a $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} x & 1 & -1 \\ 0 & 1 & y \end{bmatrix}$

b $\mathbf{A} = \begin{bmatrix} x \\ 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 \\ y \end{bmatrix}$

c $\mathbf{A} = \begin{bmatrix} -3 & x \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} y & 4 \end{bmatrix}$

d $\mathbf{A} = \begin{bmatrix} 1 & y \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$

7 The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5
Player B points 8, rebounds 2, assists 3
Player C points 4, rebounds 1, assists 1
Player D points 14, rebounds 8, assists 60
Player E points 0, rebounds 1, assists 2

Express this information in a 5×3 matrix.

11B Addition, subtraction and multiplication by a real number

Addition of matrices

If **A** and **B** are two matrices of the same size, then the sum **A** + **B** is the matrix obtained by adding together the corresponding entries of the two matrices.

For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

Multiplication of a matrix by a real number

If **A** is any matrix and k is a real number, then the product k**A** is the matrix obtained by multiplying each entry of **A** by k.

For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Note: If a matrix is added to itself, then the result is twice the matrix, i.e. **A** + **A** = 2**A**. Similarly, for any natural number n, the sum of n matrices each equal to **A** is n**A**.

If **B** is any matrix, then −**B** denotes the product (−1)**B**.

Subtraction of matrices

If **A** and **B** are matrices of the same size, then **A** − **B** is defined to be the sum

A + (−**B**) = **A** + (−1)**B**

For two matrices **A** and **B** of the same size, the difference **A** − **B** can be found by subtracting the entries of **B** from the corresponding entries of **A**.

Example 5

Find:

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & -1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Zero matrix

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**, and will be denoted by **O**.

For any $m \times n$ matrix **A** and the $m \times n$ zero matrix **O**, we have

$$\mathbf{A} + \mathbf{O} = \mathbf{A} \qquad \text{and} \qquad \mathbf{A} + (-\mathbf{A}) = \mathbf{O}$$

Example 6

Let $\mathbf{X} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $3\mathbf{A} + \mathbf{B}$.

Solution

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$2\mathbf{X} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4\mathbf{Y} + \mathbf{X} = 4\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$-3\mathbf{A} = -3\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix}$$

$$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

Example 7

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find the matrix **X** such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

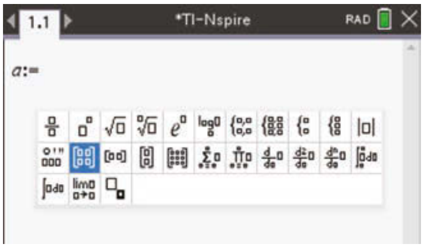
Solution

If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

Using the TI-Nspire

Entering matrices



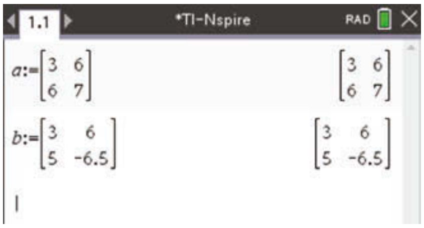
Assign the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as follows:

- In a **Calculator** page, type a := and then enter the matrix. (The assign symbol := is accessed using ctrl ⊞.)
- The simplest way to enter a 2×2 matrix is by using the 2×2 matrix template as shown. (Access the templates using either ⊞ or ctrl menu > **Math Templates**.)

Note: There is also a template for entering $m \times n$ matrices.

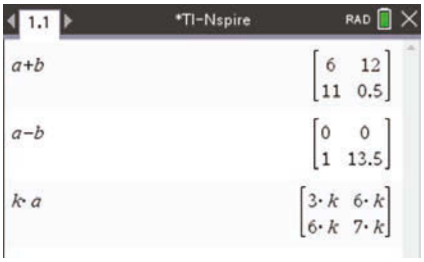
- Use the touchpad arrows (or tab) to move between the entries of the matrix.

Assign the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$ similarly.



Operations on matrices

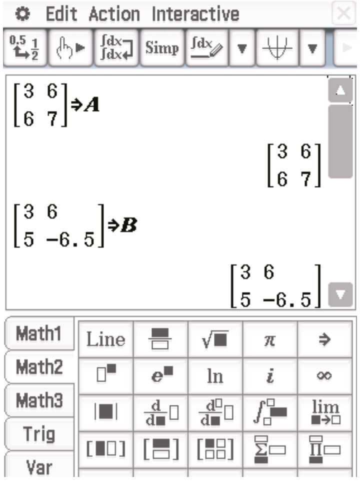
Once **A** and **B** are assigned as above, the matrices **A** + **B**, **A** − **B** and k**A** can easily be determined.



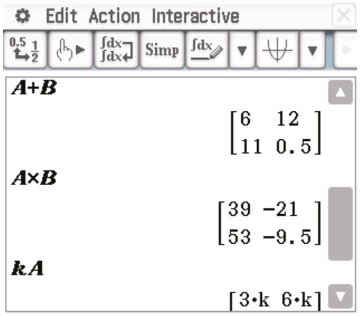
Using the Casio ClassPad

Entering matrices

- In Main $\sqrt{\alpha}$, select the Math2 keyboard.
- To enter a 2×2 matrix, tap ⊞.
- Tap on each of the entry boxes to enter the values into the matrix template.
- Move the cursor to the right-hand side of the matrix. Select the store symbol ⇒ and then the variable name A from the Var keyboard.
- Tap EXE.
- Enter the second matrix and assign it the variable name B as shown.



Note: The variables A and B will represent these two matrices until they are reassigned or you select **Edit > Clear All Variables**.



Operations on matrices

Calculate the matrices **A** + **B**, **AB** and k**A** as shown. (Use the Var keyboard to enter the variable names.)

Summary 11B

- If **A** and **B** are matrices of the same size, then:
  - the matrix **A** + **B** is obtained by adding the corresponding entries of **A** and **B**
  - the matrix **A** − **B** is obtained by subtracting the entries of **B** from the corresponding entries of **A**.
- If **A** is any matrix and k is a real number, then the matrix k**A** is obtained by multiplying each entry of **A** by k.

Exercise 11B

Example 6

1 Let $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

2 Let $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Find $2\mathbf{A}$, $-3\mathbf{A}$ and $-6\mathbf{A}$.

3 For $m \times n$ matrices **A**, **B** and **C**, is it always true that:

a $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ **b** $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$?

4 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix}$. Calculate:

a $2\mathbf{A}$ **b** $3\mathbf{B}$ **c** $2\mathbf{A} + 3\mathbf{B}$ **d** $3\mathbf{B} - 2\mathbf{A}$

5 Let $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$. Calculate:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} + 3\mathbf{Q}$ **c** $2\mathbf{P} - \mathbf{Q} + \mathbf{R}$

Example 7

6 If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -10 \\ -2 & 17 \end{bmatrix}$, find matrices **X** and **Y** such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

7 Matrices **X** and **Y** show the production of four models of cars a, b, c, d at two factories P, Q in successive weeks. Find **X** + **Y** and describe what this sum represents.

$$\text{Week 1: } \mathbf{X} = \begin{matrix} & a & b & c & d \\ P & 150 & 90 & 100 & 50 \\ Q & 100 & 0 & 75 & 0 \end{matrix} \qquad \text{Week 2: } \mathbf{Y} = \begin{matrix} & a & b & c & d \\ P & 160 & 90 & 120 & 40 \\ Q & 100 & 0 & 50 & 0 \end{matrix}$$

11C Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous section. The definition for multiplication of matrices is less straightforward. The procedure for multiplying two 2×2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

$$\begin{aligned} \text{Then} \quad \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.

Example 8

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, find:

a $\mathbf{AB}$ **b** $\mathbf{A}^2 = \mathbf{AA}$

Solution

a $\mathbf{A}$ is a 2×2 matrix and $\mathbf{B}$ is a 2×1 matrix. Therefore the product $\mathbf{AB}$ is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

b $\mathbf{A}$ is a 2×2 matrix. Therefore the product $\mathbf{AA}$ is defined and will be a 2×2 matrix.

$$\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 4 \times 3 & 2 \times 4 + 4 \times 6 \\ 3 \times 2 + 6 \times 3 & 3 \times 4 + 6 \times 6 \end{bmatrix} = \begin{bmatrix} 16 & 32 \\ 24 & 48 \end{bmatrix}$$

Note: A matrix with the same number of rows and columns is called a **square matrix**.
For any square matrix $\mathbf{A}$ and natural number n, we can define the matrix $\mathbf{A}^n$.
We can do this inductively by defining $\mathbf{A}^1 = \mathbf{A}$ and $\mathbf{A}^n = \mathbf{A}\mathbf{A}^{n-1}$ for $n \geq 2$.

Example 9

Matrix $\mathbf{X}$ shows the number of cars of models a and b bought by four dealers A, B, C, D. Matrix $\mathbf{Y}$ shows the cost in dollars of cars a and b. Find $\mathbf{XY}$ and explain what it represents.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \mathbf{Y} = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

Solution

$\mathbf{X}$ is a 4×2 matrix and $\mathbf{Y}$ is a 2×1 matrix. Therefore $\mathbf{XY}$ is a 4×1 matrix.

$$\mathbf{XY} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array}\begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

$$= \begin{bmatrix} 3 \times 26\,000 + 1 \times 32\,000 \\ 2 \times 26\,000 + 2 \times 32\,000 \\ 1 \times 26\,000 + 4 \times 32\,000 \\ 1 \times 26\,000 + 1 \times 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$$

The matrix $\mathbf{XY}$ shows that dealer A spent \$110 000, dealer B spent \$116 000, dealer C spent \$154 000 and dealer D spent \$58 000.

Example 10

For $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, find **AB**.

Solution

A is a 2×3 matrix and **B** is a 3×2 matrix. Therefore **AB** is a 2×2 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 + 4 \times 0 & 2 \times 0 + 3 \times 2 + 4 \times 3 \\ 5 \times 4 + 6 \times 1 + 7 \times 0 & 5 \times 0 + 6 \times 2 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix}$$

Summary 11C

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.
- The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.

Skill-sheet

Exercise 11C

Example 8
Example 10

1 Let $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find the products **AX**, **BX**, **AY**, **IX**, **AC**, **CA**, (**AC**)**X**, **C**(**BX**), **AI**, **IB**, **AB**, **BA**, $\mathbf{A}^2$, $\mathbf{B}^2$, **A**(**CA**) and $\mathbf{A}^2\mathbf{C}$.

2 Which of the following products of matrices from Question 1 are defined?

AY, **YA**, **XY**, $\mathbf{X}^2$, **CI**, **XI**

3 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$, find **AB**.

4 Let **A** and **B** be 2×2 matrices and let **O** be the 2×2 zero matrix. Is the following argument correct?

'If **AB** = **O** and **A** ≠ **O**, then **B** = **O**.'

5 Find a matrix **A** such that **A** ≠ **O** but $\mathbf{A}^2 = \mathbf{O}$.

6 If $\mathbf{L} = \begin{bmatrix} 2 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find **LX** and **XL**.

7 Assume that both **A** and **B** are $m \times n$ matrices. Are **AB** and **BA** defined and, if so, how many rows and columns do they have?

8 Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a Show that $ad - bc = 1$.

b Evaluate the product on the left-hand side if the order of multiplication is reversed.

9 Using the result of Question 8, write down a pair of matrices **A** and **B** such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 Choose any three 2×2 matrices **A**, **B** and **C**. Find $\mathbf{A}(\mathbf{B} + \mathbf{C})$, $\mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A}$.

11 Find matrices **A** and **B** such that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Example 9 **12** It takes John 5 minutes to drink a milk shake which costs \$2.50, and 12 minutes to eat a banana split which costs \$3.00.

a Find the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in fast-food economics.

b Two friends join John. Find $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

13 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$ and use your answer to find $\mathbf{A}^4$ and $\mathbf{A}^8$.

14 Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Write down a formula for $\mathbf{A}^n$.

11D Identities, inverses and determinants for 2 × 2 matrices

Identities

Recall that a matrix with the same number of rows and columns is called a square matrix. For square matrices of any given size (e.g. 2×2), a multiplicative identity **I** exists.

For 2×2 matrices, the **identity matrix** is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, and this result holds for any square matrix multiplied by the appropriate multiplicative identity.

For 3×3 matrices, the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Inverses

We know that, for any non-zero real number x, there is a real number x^{-1} such that $xx^{-1} = 1$. We now investigate the analogous question for matrices.

Given a 2×2 matrix $\mathbf{A}$, is there a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$?

For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2x+3u & 2y+3v \\ x+4u & y+4v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad 2x + 3u = 1 \quad$ and $\quad 2y + 3v = 0$

$\quad\quad x + 4u = 0 \quad\quad\quad y + 4v = 1$

These simultaneous equations can be solved to find x, y, u, v and hence $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

In general:

If $\mathbf{A}$ is a square matrix and if a matrix $\mathbf{B}$ can be found such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}$$

then $\mathbf{A}$ is said to be **invertible** and $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$.

We will denote the inverse of $\mathbf{A}$ by $\mathbf{A}^{-1}$. You will prove in Exercise 11D that the inverse of an invertible matrix is unique.

For an invertible matrix $\mathbf{A}$, we have

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

The inverse of a general 2 × 2 matrix

Now consider $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} ax+bu & ay+bv \\ cx+du & cy+dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad ax + bu = 1 \quad$ and $\quad ay + bv = 0$

$\quad\quad cx + du = 0 \quad\quad\quad cy + dv = 1$

These form two pairs of simultaneous equations, the first for x, u and the second for y, v.

The first pair of equations gives

$$(ad - bc)x = d \qquad \text{(eliminating } u\text{)}$$

$$(bc - ad)u = c \qquad \text{(eliminating } x\text{)}$$

These two equations can be solved for x and u provided $ad - bc \neq 0$:

$$x = \frac{d}{ad - bc} \qquad \text{and} \qquad u = \frac{c}{bc - ad} = \frac{-c}{ad - bc}$$

In a similar way, we obtain

$$y = \frac{-b}{ad - bc} \qquad \text{and} \qquad v = \frac{-a}{bc - ad} = \frac{a}{ad - bc}$$

We have established the following result.

Inverse of a 2 × 2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{(provided } ad - bc \neq 0\text{)}$$

The determinant

The quantity $ad - bc$ that appears in the formula for $\mathbf{A}^{-1}$ has a name: the **determinant** of $\mathbf{A}$. This is denoted $\det(\mathbf{A})$.

Determinant of a 2 × 2 matrix

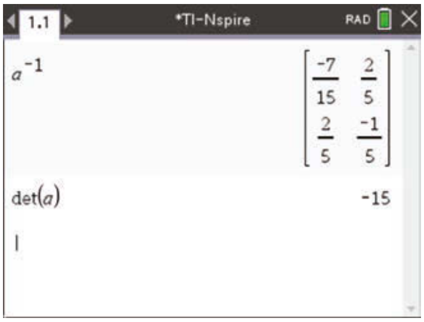
If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

Using the TI-Nspire

- The inverse of a matrix is obtained by raising the matrix to the power of -1.
- The determinant command (menu > **Matrix & Vector** > **Determinant**) is used as shown.

Hint: You can also type in det(a).



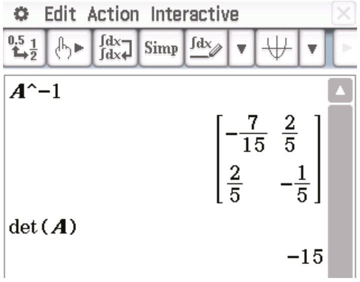
(Here a is the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ defined in Section 11B.)

Using the Casio ClassPad

- To find the inverse matrix, type A^−1 and tap EXE.

Note: If the matrix has no inverse, then the calculator will give the message **Undefined**.

- To find the determinant, enter and highlight A. Select **Interactive** > **Matrix** > **Calculation** > **det**.



Example 11

For the matrix $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, find:

a $\det(\mathbf{A})$

b $\mathbf{A}^{-1}$

Solution

a $\det(\mathbf{A}) = 5 \times 1 - 2 \times 3 = -1$

b $\mathbf{A}^{-1} = \dfrac{1}{-1}\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

Example 12

For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, find:

a $\det(\mathbf{A})$

b $\mathbf{A}^{-1}$

c $\mathbf{X}$, if $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

d $\mathbf{Y}$, if $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Solution

a $\det(\mathbf{A}) = 3 \times 6 - 2 = 16$

b $\mathbf{A}^{-1} = \dfrac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$

c $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the left) by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{IX} = \mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

d $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the right) by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y} = \frac{1}{16}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$$

Summary 11D

- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad - bc \neq 0)$$

  - the determinant of $\mathbf{A}$ is given by

$$\det(\mathbf{A}) = ad - bc$$

- A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

Skill-sheet

Exercise 11D

Example 11

1 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, find:

a $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\det(\mathbf{B})$ **d** $\mathbf{B}^{-1}$

2 Find the inverse of each of the following invertible matrices (where k is any non-zero real number):

a $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3 Let $\mathbf{A}$ and $\mathbf{B}$ be the invertible matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

a Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

b Find $\mathbf{AB}$ and hence find, if possible, $(\mathbf{AB})^{-1}$.

c From $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, find the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

Example 12

4 Let $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

a Find $\mathbf{A}^{-1}$. **b** If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{X}$. **c** If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{Y}$.

5 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$.

a Find $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$. **b** Find $\mathbf{Y}$ such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$.

6 Let $\mathbf{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, where a and b are not both zero. Prove that $\mathbf{A}^{-1}$ will always exist.

7 If the matrix $\mathbf{A}$ is invertible, show that the inverse is unique.

8 Assume that $\mathbf{A}$ is a 2×2 matrix such that $a_{12} = a_{21} = 0$, $a_{11} \neq 0$ and $a_{22} \neq 0$. Show that $\mathbf{A}$ is invertible and find $\mathbf{A}^{-1}$.

9 Let $\mathbf{A}$ be an invertible 2×2 matrix, let $\mathbf{B}$ be a 2×2 matrix and assume that $\mathbf{AB} = \mathbf{O}$. Show that $\mathbf{B} = \mathbf{O}$.

10 For what values of x does the matrix $\mathbf{A} = \begin{bmatrix} x & 2x+1 \\ 2x+1 & x \end{bmatrix}$ have an inverse?

11 **a** Let $\mathbf{A} = \begin{bmatrix} a & 4 \\ -2 & -3 \end{bmatrix}$. Find the value of a if $\mathbf{A}^{-1} = \mathbf{A}$.

b Find all 2×2 matrices such that $\mathbf{A}^{-1} = \mathbf{A}$.

12 For what values of a does the matrix $\mathbf{A} = \begin{bmatrix} a & 1 \\ 2 & a \end{bmatrix}$ not have an inverse?

13 Let n be an integer and consider the matrix $\mathbf{A} = \begin{bmatrix} n & n+1 \\ n+1 & n+2 \end{bmatrix}$.

Show that all the entries of $\mathbf{A}^{-1}$ are integers.

14 Let n be an integer and consider the matrix $\mathbf{A} = \begin{bmatrix} n & n+1 \\ n+2 & n+3 \end{bmatrix}$.

Show that the entries of $\mathbf{A}^{-1}$ cannot all be integers.

15 Let n be a natural number and consider the matrix

$$\mathbf{A} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+2} \end{bmatrix}$$

Show that all the entries of $\mathbf{A}^{-1}$ are integers.

11E Solution of simultaneous equations using matrices

Inverse matrices can be used to solve some systems of simultaneous linear equations.

Simultaneous equations with a unique solution

For example, consider the pair of simultaneous equations

$$3x - 2y = 5$$
$$5x - 3y = 9$$

This can be written as a matrix equation:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$. The determinant of $\mathbf{A}$ is $3(-3) - (-2)5 = 1$.

Since the determinant is non-zero, the inverse matrix exists:

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$$

Now multiply both sides of the original matrix equation on the left by $\mathbf{A}^{-1}$:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{I}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix} \qquad \text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is the solution to the simultaneous equations. Check by substituting $x = 3$ and $y = 2$ into the two equations.

Simultaneous equations without a unique solution

If a pair of simultaneous linear equations in two variables corresponds to two parallel lines, then a non-invertible matrix results.

For example, the following pair of simultaneous equations has no solution:

$$x + 2y = 3$$

$$-2x - 4y = 6$$

The associated matrix equation is

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The determinant of the matrix $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is $1(-4) - 2(-2) = 0$, so the matrix has no inverse.

Example 13

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution

If $\mathbf{AX} = \mathbf{K}$, then

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 14

Solve the following simultaneous equations:

$$3x - 2y = 6$$
$$7x + 4y = 7$$

Solution

The matrix equation is

$$\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$. Then $\mathbf{A}^{-1} = \frac{1}{26} \begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$.

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 38 \\ -21 \end{bmatrix}$$

Skill-sheet

Exercise 11E

Example 13

1 Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where:

a $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **b** $\mathbf{K} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Example 14

2 Use matrices to solve each of the following pairs of simultaneous equations:

a $-2x + 4y = 6$
$3x + y = 1$

b $-x + 2y = -1$
$-x + 4y = 2$

c $\frac{1}{2}x + \frac{1}{3}y = 1$
$\frac{1}{3}x + \frac{1}{4}y = 1$

d $\frac{1}{20}x + \frac{1}{21}y = \frac{1}{2}$
$\frac{1}{21}x + \frac{1}{22}y = \frac{1}{2}$

3 Use matrices to find the point of intersection of the lines given by the equations $2x - 3y = 7$ and $3x + y = 5$.

4 Two children spend their pocket money buying some books and some games. One child spends \$120 and buys four books and four games. The other child spends \$114 and buys five books and three games. Set up a system of simultaneous equations and use matrices to find the cost of a single book and a single game.

5 Consider the system

$$2x - 3y = 3$$
$$4x - 6y = 6$$

a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

b Is $\mathbf{A}$ an invertible matrix?

c Can any solutions be found for this system of equations?

d How many pairs does the solution set contain?

6 Suppose that **A**, **B**, **C** and **X** are 2×2 matrices and that both **A** and **B** are invertible. Solve the following for **X**:

a $\mathbf{AX} = \mathbf{C}$ **b** $\mathbf{ABX} = \mathbf{C}$ **c** $\mathbf{AXB} = \mathbf{C}$

d $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{C}$ **e** $\mathbf{AX} + \mathbf{B} = \mathbf{C}$ **f** $\mathbf{XA} + \mathbf{B} = \mathbf{A}$

11F Inverses and determinants for $n \times n$ matrices

In the next two sections, we see how the theory that has been developed for 2×2 matrices can be extended to $n \times n$ matrices, where $n \geq 3$. Much of the work in these two sections will be completed with the use of technology.

An $n \times n$ matrix **A** can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

Here a_{ij} is the entry in row i and column j of **A**.

We will focus on 3×3 matrices, but the techniques used for larger square matrices are similar.

Identities

For 3×3 matrices, the identity matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each 3×3 matrix **A**, we have $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$.

Similarly, for 4×4 matrices, the identity matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In general, the $n \times n$ identity matrix has 1s along the main diagonal (top-left to bottom-right) and 0s everywhere else.

Inverses

Recall that, if **A** is a square matrix and there exists a matrix **B** such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **B** is called the inverse of **A**. When it exists, the inverse of a square matrix **A** is unique and is denoted by $\mathbf{A}^{-1}$.

We can use the following useful fact to help find inverses.

Let **A** and **B** be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.

You can use technology to find the inverse of a 3×3 matrix. However, the inverse can also be found by hand, as shown in the next example. (There are more efficient methods for finding inverses, but they are beyond the scope of this course.)

Example 15

Without using a calculator, find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

Solution

We want to find a matrix $\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ such that $\mathbf{AB} = \mathbf{I}$.

That is:

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a - 3d + 2g & b - 3e + 2h & c - 3f + 2i \\ -3a + 3d - g & -3b + 3e - h & -3c + 3f - i \\ 2a - d & 2b - e & 2c - f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We first solve the three equations from the left-hand columns for a, d and g:

$$a - 3d + 2g = 1 \quad (1)$$

$$-3a + 3d - g = 0 \quad (2)$$

$$2a - d = 0 \quad (3)$$

From (3), we have $d = 2a$. Substitute into (1) and (2):

$$-5a + 2g = 1 \quad (1')$$

$$3a - g = 0 \quad (2')$$

We obtain $a = 1$, $d = 2$ and $g = 3$.

Solving the three equations from the middle columns gives $b = 2$, $e = 4$ and $h = 5$.
Solving the three equations from the right-hand columns gives $c = 3$, $f = 5$ and $i = 6$.

We obtain $\mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Check: $\mathbf{AA}^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$

Not every 3×3 matrix has an inverse. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is non-invertible.

Example 16

Let $\mathbf{A} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix}$.

Find the product **AB**, and hence find $\mathbf{A}^{-1}$.

Solution

$$\mathbf{AB} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix}\begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

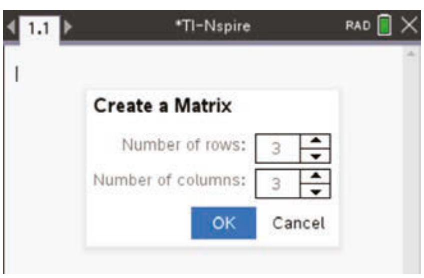
Hence $\mathbf{A}^{-1} = \mathbf{B}$.

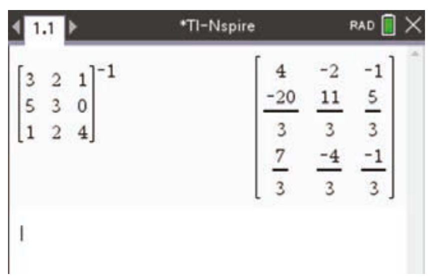
Example 17

Using your calculator, find the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 5 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$.

Using the TI-Nspire

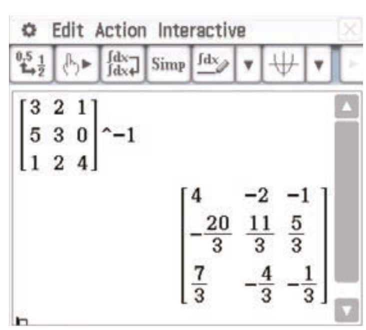
- To enter a 3×3 matrix, select the m-by-n matrix template. (The templates can be accessed using .) Complete the pop-up screen as shown below.
- The inverse of a matrix is obtained by raising the matrix to the power of -1.





Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, select the Math2 keyboard.
- To enter a 3×3 matrix, tap twice.
- Type the values into the matrix template.
- Type ^-1 and tap EXE.



The determinant

In Section 11D, we defined the determinant of a 2×2 matrix. The definition was motivated by the formula for the inverse of a 2×2 matrix. We saw that a 2×2 matrix has an inverse if and only if its determinant is non-zero.

In fact, the determinant is defined for all square matrices. You can use technology to find the determinant of an $n \times n$ matrix when $n \geq 3$. However, we will also see how to find the determinant of a 3×3 matrix by hand.

The determinant of a 3 × 3 matrix

Consider a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of **A** can be defined as follows:

$$\begin{aligned} \det(\mathbf{A}) &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

This formula comes from working through the first row of **A**:

- The first 2×2 matrix is obtained by deleting the row and column containing a_{11}.
- The second 2×2 matrix is obtained by deleting the row and column containing a_{12}.
- The third 2×2 matrix is obtained by deleting the row and column containing a_{13}.

Example 18

Find the determinant of $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

Solution

$$\begin{aligned} \det(\mathbf{A}) &= 3 \times \det \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} - 2 \times \det \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} + 0 \times \det \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \\ &= 3(4 \times 2 - 1 \times 1) - 2(3 \times 2 - 1 \times 2) + 0 \\ &= 3 \times 7 - 2 \times 4 \\ &= 13 \end{aligned}$$

We can obtain equivalent formulas for the determinant by using any row or column of **A** in a similar way. For example, working through the first column of **A**:

$$\det(\mathbf{A}) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Check that this formula gives the same result.

The sign of the a_{ij} term in a formula for the determinant is determined by $(-1)^{i+j}$.
For example:

- For a_{11}, the sign is given by $(-1)^{1+1} = 1$.
- For a_{12}, the sign is given by $(-1)^{1+2} = -1$.

These signs can also be determined using the following array:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example, working through the second row of **A**:

$$\det(\mathbf{A}) = -a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{22} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Note: When finding the determinant of a 3×3 matrix **A** by hand, it helps to work through the row or column of **A** that has the most 0 entries.

The determinant of an $n \times n$ matrix

We have seen that the determinant of a 3×3 matrix is defined using 2×2 matrices. Similarly, the determinant of a 4×4 matrix is defined using 3×3 matrices, and so on. You can use your calculator to find the determinant of large square matrices.

The determinant has the following important property.

Determinant of an $n \times n$ matrix

An $n \times n$ matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

Example 19

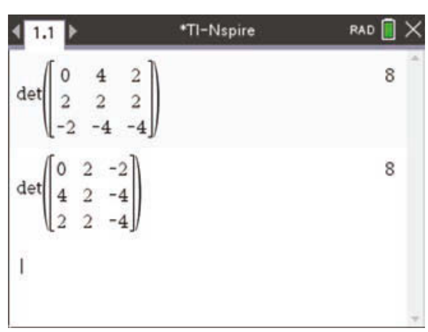
Using a calculator, find the determinant of:

a $\begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$

b $\begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix}$

Using the TI-Nspire

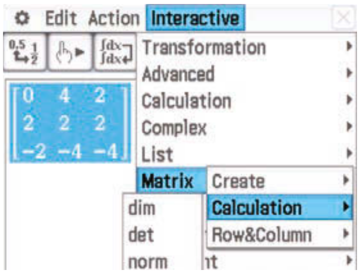
Use the determinant command (menu > **Matrix & Vector** > **Determinant**) as shown.
Alternatively, type det(.

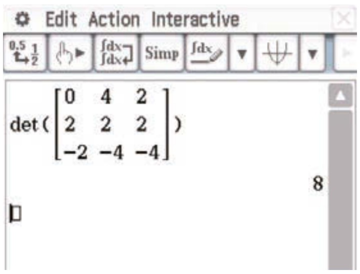


Using the Casio ClassPad

a ■ Enter the 3×3 matrix.

■ Highlight the matrix and select **Interactive** > **Matrix** > **Calculation** > **det**.





b Similarly, we can find $\det \begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix} = 8$.

Summary 11F

Identity matrix

■ For each natural number n, there is an $n \times n$ identity matrix $\mathbf{I}$. This matrix satisfies $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for all $n \times n$ matrices $\mathbf{A}$.

■ The 3×3 identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Inverse matrices

■ If $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$.

■ When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

■ Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.

Determinant

■ The determinant is defined for all square matrices.

■ A square matrix has an inverse if and only if its determinant is non-zero.

■ For a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant of $\mathbf{A}$ can be defined by

$$\det(\mathbf{A}) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Exercise 11F

Example 15

1 Without using a calculator, find the inverse matrix of each of the following:

a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

2 Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 5 \\ 1 & 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 5 & -2 & 3 \\ -7 & 3 & -5 \end{bmatrix}$.

Show that the inverse of matrix **A** is the matrix **B**. (Hint: Calculate **AB**.)

Example 16

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -4 \\ -2 & 5 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 19 & -17 & -11 \\ 6 & -5 & -2 \\ 8 & -9 & -5 \end{bmatrix}$.

Find the product **AB**, and hence find $\mathbf{A}^{-1}$.

4 Let $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$. Find $\mathbf{A}^2$, and hence find $\mathbf{A}^{-1}$.

5 Let $\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$. Find $\mathbf{A}^2$, and hence find $\mathbf{A}^{-1}$.

Example 17

6 Use your calculator to find the inverse of each of the following matrices:

a $\begin{bmatrix} 5 & 2 & 3 \\ 1 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & 8 & 3 \\ 3 & 6 & 4 \\ 2 & 1 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & 1 & 2 & 5 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **d** $\begin{bmatrix} 9 & 1 & 3 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ 5 & 1 & 0 & 2 \end{bmatrix}$

Example 18

7 Without using a calculator, find the determinant of:

a $\begin{bmatrix} 9 & 1 & 3 \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 5 \\ 1 & 0 & 0 \end{bmatrix}$

Example 19

8 Using a calculator, find the determinant of:

a **i** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

b **i** $\begin{bmatrix} 2 & 4 & 8 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 4 & 4 \\ 6 & 4 & 2 \end{bmatrix}$

9 **a** Without using a calculator, find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & p \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ in terms of p.

b Hence find the value of p for which $\mathbf{A}$ does not have an inverse.

10 **a** Without using a calculator, find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & p \\ 2 & 2 & 2 \\ p & 2 & p \end{bmatrix}$ in terms of p.

b Hence find the values of p for which $\mathbf{A}$ does not have an inverse.

11G Simultaneous linear equations with more than two variables

In this section, we use inverse matrices to solve some systems of simultaneous linear equations.

Linear equations in three variables

Consider the general system of three linear equations in three variables:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

This can be written as a matrix equation:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Define the matrices

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then the matrix equation becomes

$$\mathbf{AX} = \mathbf{B}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, we can multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$$
$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B} \qquad \text{(where } \mathbf{I} \text{ is the } 3 \times 3 \text{ identity matrix)}$$
$$\therefore \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Hence, if the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. You can use your calculator to find the inverse matrix $\mathbf{A}^{-1}$.

Example 20

Use matrix methods to solve the following system of three equations in three variables:

$$\begin{aligned} 2x + y + z &= -1 \\ 3y + 4z &= -7 \\ 6x + z &= 8 \end{aligned}$$

Solution

Define the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 6 & 0 & 1 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 \\ -7 \\ 8 \end{bmatrix}$$

Then the system of equations can be written as a matrix equation:

$$\mathbf{AX} = \mathbf{B}$$

Multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore \quad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

Use your calculator to find $\mathbf{A}^{-1}\mathbf{B}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$$

The solution is $x = 1$, $y = -5$ and $z = 2$.

You can also use your calculator to solve a system of three linear equations directly, without finding an inverse matrix.

Example 21

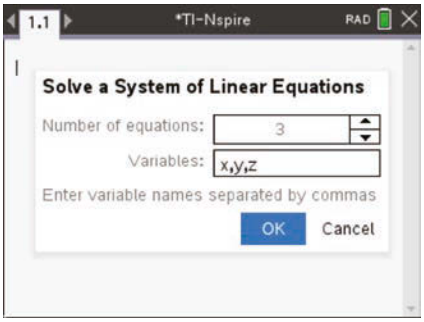
Solve the following simultaneous linear equations for x, y and z:

$$x - y + z = 6, \qquad 2x + z = 4, \qquad 3x + 2y - z = 6$$

Using the TI-Nspire

Simultaneous linear equations can be solved in a **Calculator** application.

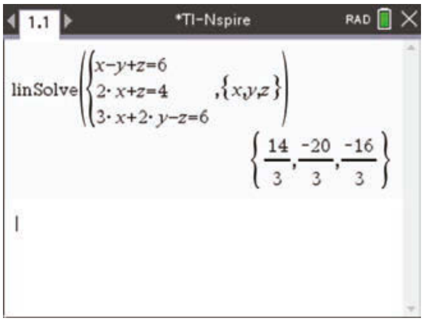
- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- Complete the pop-up screen as shown.



- Enter the three equations:

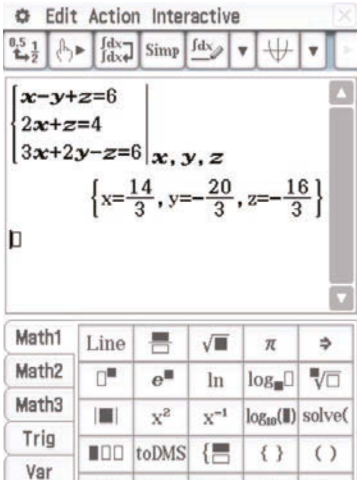
$$x - y + z = 6$$
$$2x + z = 4$$
$$3x + 2y - z = 6$$

- Hence $x = \frac{14}{3}$, $y = -\frac{20}{3}$ and $z = -\frac{16}{3}$.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, select the Math1 keyboard.
- For three simultaneous equations, tap the simultaneous equations icon twice.
- Enter the three equations into the three lines and enter the variables x, y, z in the bottom right separated by commas. Tap EXE.
- The solution is $x = \frac{14}{3}$, $y = -\frac{20}{3}$ and $z = -\frac{16}{3}$.



Simultaneous equations without a unique solution

Just as for two linear equations in two variables, there is a geometric interpretation for three linear equations in three variables. There is only a unique solution if the three equations represent three planes intersecting at a point.

There are three possible cases for a system of three linear equations in three variables:

- a unique solution
- no solution
- infinitely many solutions.

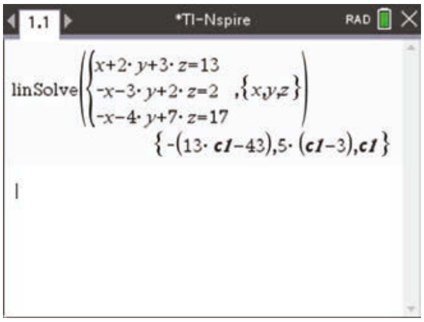
Example 22

Use your calculator to solve the simultaneous equations

$$x + 2y + 3z = 13, \qquad -x - 3y + 2z = 2, \qquad -x - 4y + 7z = 17$$

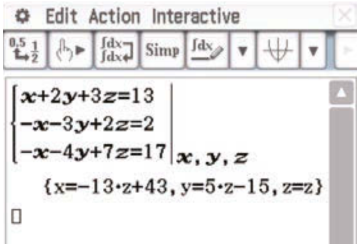
Using the TI-Nspire

- In a **Calculator** application, use menu > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- Complete the pop-up screen and then enter the three equations as shown.
- The solutions are described in terms of a parameter $c1$. Using λ for the parameter, we can write the solutions as $x = 43 - 13\lambda$, $y = 5\lambda - 15$ and $z = \lambda$, for $\lambda \in \mathbb{R}$.



Using the Casio ClassPad

- In the Math1 keyboard, tap the simultaneous equations icon twice.
- Enter the equations and variables as shown.
- The solutions are described in terms of z. We can use a parameter λ and write the solutions as $x = 43 - 13\lambda$, $y = 5\lambda - 15$ and $z = \lambda$, for $\lambda \in \mathbb{R}$.



Linear equations in more than three variables

More generally, we can consider a system of n linear equations in n variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

Such a system of equations can be written as a matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If the $n \times n$ matrix has an inverse, then the system of equations has a unique solution.

Summary 11G

Matrix method for solving simultaneous linear equations

- A system of three linear equations in three variables has the form:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

- This can be written as a matrix equation **AX** = **B**, where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- If the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

Exercise 11G

Example 20

1 Use matrix methods to solve each of the following systems of simultaneous equations:

a $2x + 3y - z = 12$
$2y + z = 7$
$2y - z = 5$

b $x + 2y + 3z = 13$
$-x - y + 2z = 2$
$-x + 3y + 4z = 26$

c $x + y = 5$
$y + z = 7$
$z + x = 12$

d $x - y - z = 0$
$5x + 20z = 50$
$10y - 20z = 30$

e $x + y - z = 3$
$x - z + w = 0$
$2x - y - z + 3w = 1$
$-4x + 2y + 3z - 4w = 0$

2 Consider the following system of simultaneous equations:

$x + 2y + 3z = 13$ (1)
$-x - 3y + 2z = 2$ (2)
$-x - 4y + 7z = 17$ (3)

a Write this system as a matrix equation **AX** = **B**.

b Find det(**A**). Is the matrix **A** invertible?

c This system of simultaneous equations has infinitely many solutions. Express the solutions in terms of a parameter λ by following these steps:

i Add equation (2) to equation (1) and subtract equation (2) from equation (3).

ii Comment on the equations obtained in part **i**.

iii Let $z = \lambda$ and find y in terms of λ.

iv Substitute for z and y in terms of λ in equation (1) to find x in terms of λ.

Chapter summary

Assign-ment

Nrich

- A **matrix** is a rectangular array of numbers.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Two matrices **A** and **B** are equal when:
  - they have the same size, and
  - they have the same entry at corresponding positions.
- Addition is defined for two matrices only when they have the same size. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

  Subtraction is performed in a similar way.
- If **A** is any matrix and k is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of **A** by k.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

  Note that the product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.
- If **A** is a square matrix and if a matrix **B** can be found such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **A** is said to be **invertible** and **B** is called the **inverse** of **A**.
- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad - bc \neq 0)$$

  - the determinant of **A** is given by

$$\det(\mathbf{A}) = ad - bc$$

- A square matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.
- Simultaneous equations can sometimes be solved using inverse matrices. For example, the system of equations

$$ax + by = c$$
$$dx + ey = f$$

  can be written as $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ and solved using $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} c \\ f \end{bmatrix}$.

Review

Technology-free questions

1 If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, find:

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **b** $\mathbf{A}^2 - \mathbf{B}^2$

2 Find all possible matrices **A** which satisfy the equation $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$.

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

a State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**.

b Find **DA** and $\mathbf{A}^{-1}$.

4 If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find **AB** and $\mathbf{C}^{-1}$.

5 Find the 2×2 matrix **A** such that $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.

6 If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, find $\mathbf{A}^2$ and hence find $\mathbf{A}^{-1}$.

7 If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ does not have an inverse, find the value of x.

8 **a** If $\mathbf{M} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, find:

i $\mathbf{MM} = \mathbf{M}^2$ **ii** $\mathbf{MMM} = \mathbf{M}^3$ **iii** $\mathbf{M}^{-1}$

b Find x and y, given that $\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Multiple-choice questions

1 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -2 & 3 \\ 3 & 0 \end{bmatrix}$ has size

A 8 **B** 4×2 **C** 2×4 **D** 1×4 **E** 3×4

2 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & -3 & -1 \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} =$

A $\begin{bmatrix} 3 & -3 \\ -2 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ **E** undefined

3 If $\mathbf{C} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix}$, then $\mathbf{D} - \mathbf{C} =$

A $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -6 & 4 \\ -2 & 0 & -4 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

D $\begin{bmatrix} 1 & -6 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ **E** undefined

4 If $\mathbf{M} = \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$, then $-\mathbf{M} =$

A $\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -4 \\ -6 & -2 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 0 \\ -2 & -6 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 6 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$

5 If $\mathbf{M} = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, then $2\mathbf{M} - 2\mathbf{N} =$

A $\begin{bmatrix} 0 & 0 \\ -9 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ -6 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 12 & -2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ 6 & -1 \end{bmatrix}$

6 If both **A** and **B** are $m \times n$ matrices, where $m \neq n$, then $\mathbf{A} + \mathbf{B}$ is

A an $m \times n$ matrix **B** an $m \times m$ matrix **C** an $n \times n$ matrix

D a $2m \times 2n$ matrix **E** not defined

7 If **P** is an $m \times n$ matrix and **Q** is an $n \times p$ matrix, where $m \neq p$, then **QP** is

A an $n \times n$ matrix **B** an $m \times p$ matrix **C** an $n \times p$ matrix

D an $m \times n$ matrix **E** not defined

8 The determinant of the matrix $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ is

A 4 **B** 0 **C** −4 **D** 1 **E** 2

9 The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ is

A −1 **B** $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

10 If $\mathbf{M} = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, then $\mathbf{NM} =$

A $\begin{bmatrix} 0 & -4 \\ -9 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} -4 & -2 \\ 2 & -8 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 4 \\ 9 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}$ **E** $\begin{bmatrix} 6 & -2 \\ -3 & -5 \end{bmatrix}$

Extended-response questions

1 **a** Consider the system of equations

$$2x - 3y = 3$$
$$4x + y = 5$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Find $\det(\mathbf{A})$ and $\mathbf{A}^{-1}$.

iii Solve the system of equations.

iv Interpret your solution geometrically.

b Consider the system of equations

$$2x + y = 3$$
$$4x + 2y = 8$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Find $\det(\mathbf{A})$ and explain why $\mathbf{A}^{-1}$ does not exist.

c Interpret your findings in part **b** geometrically.

2 The final grades for Physics and Chemistry are made up of three components: tests, practical work and exams. Each semester, a mark out of 100 is awarded for each component. Wendy scored the following marks in the three components for Physics:

Semester 1 tests 79, practical work 78, exam 80
Semester 2 tests 80, practical work 78, exam 82

a Represent this information in a 2×3 matrix.

To calculate the final grade for each semester, the three components are weighted: tests are worth 20%, practical work is worth 30% and the exam is worth 50%.

b Represent this information in a 3×1 matrix.

c Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in the three components for Chemistry:

Semester 1 tests 86, practical work 82, exam 84
Semester 2 tests 81, practical work 80, exam 70

d Calculate Wendy's final grade for Chemistry in each semester.

Students who gain a total score of 320 or more for Physics and Chemistry over the two semesters are awarded a Certificate of Merit in Science.

e Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to re-mark her Semester 2 Chemistry exam, hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

f How many extra marks on the exam does she need?

3 A company runs computing classes and employs full-time and part-time teaching staff, as well as technical staff, catering staff and cleaners. The number of staff employed depends on demand from term to term.
In one year the company employed the following teaching staff:

Term 1 full-time 10, part-time 2
Term 2 full-time 8, part-time 4
Term 3 full-time 8, part-time 8
Term 4 full-time 6, part-time 10

a Represent this information in a 4×2 matrix.

Full-time teachers are paid \$70 per hour and part-time teachers are paid \$60 per hour.

b Represent this information in a 2×1 matrix.

c Calculate the cost per hour to the company for teaching staff for each term.

In the same year the company also employed the following support staff:

Term 1 technical 2, catering 2, cleaning 1
Term 2 technical 2, catering 2, cleaning 1
Term 3 technical 3, catering 4, cleaning 2
Term 4 technical 3, catering 4, cleaning 2

d Represent this information in a 4×3 matrix.

Technical staff are paid \$60 per hour, catering staff are paid \$55 per hour and cleaners are paid \$40 per hour.

e Represent this information in a 3×1 matrix.

f Calculate the cost per hour to the company for support staff for each term.

g Calculate the total cost per hour to the company for teaching and support staff for each term.

4 Bronwyn and Noel have a clothing warehouse in Summerville. They are supplied by three contractors: Brad, Flynn and Lina.
The matrix shows the number of dresses, pants and shirts that one worker, for each of the contractors, can produce in a week.

	Brad	Flynn	Lina
Dresses	5	6	10
Pants	3	4	5
Shirts	2	6	5

The number produced varies because of the different equipment used by the contractors.
The warehouse requires 310 dresses, 175 pants and 175 shirts in a week. How many workers should each contractor employ to meet the requirement exactly?

5 Suppose that $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices.

a Prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

b Hence prove that if both $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $\mathbf{AB}$ is invertible.

6 Let $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. Prove by induction that $\mathbf{A}^n = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ for all $n \in \mathbb{N}$.

12

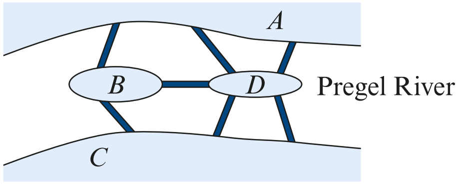
Graph theory

Objectives

- To define a graph by specifying a set of **vertices** and a set of **edges**.
- To represent graphs using diagrams and matrices.
- To define the **degree** of a vertex, and to use the fact that the sum of the degrees of all the vertices of a graph is equal to twice the number of edges.
- To understand what it means for two graphs to be **isomorphic**, and what it means for one graph to be a **subgraph** of another graph.
- To define **simple graphs**, **connected graphs**, **complete graphs**, **bipartite graphs** and **trees**.
- To introduce **Euler circuits**, **Euler trails**, **Hamiltonian cycles** and **Hamiltonian paths**.
- To count the number of **walks** of a given length between two vertices of a graph.
- To introduce **planar graphs**, and to prove and apply **Euler's formula** $v - e + f = 2$ for connected planar graphs.

Graph theory is useful for analysing 'things that are connected to other things', and so has applications in a variety of areas, including genetics, linguistics, engineering and sociology. There are many useful techniques in graph theory that are suitable for solving real-world problems – particularly optimisation problems.

The *Seven Bridges of Königsberg* is a famous historical problem in graph theory. The question is whether or not you can walk around the city of Königsberg crossing each of the seven bridges exactly once and returning to your starting point. We investigate this problem in Section 12B.

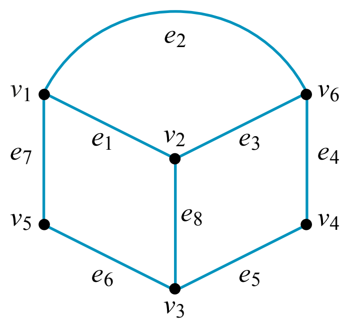


12A Graphs and adjacency matrices

Suppose that there are six teams in a hockey tournament. Label the teams as v_1, v_2, v_3, v_4, v_5 and v_6. A few weeks into the tournament, the six teams have played one another as follows:

- v_1 has played v_2, v_5 and v_6
- v_2 has played v_1, v_3 and v_6
- v_3 has played v_2, v_4 and v_5
- v_4 has played v_3 and v_6
- v_5 has played v_1 and v_3
- v_6 has played v_1, v_2 and v_4.

This situation can be represented by the diagram below. Each team is represented by a point. Two points are joined by a line if the teams they represent have played each other. The points are called **vertices**, and the lines connecting the vertices are called **edges**. The table shows the **edge-endpoint function**, which indicates the endpoints of each edge.



Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_6\}$
e_3	$\{v_2, v_6\}$
e_4	$\{v_4, v_6\}$
e_5	$\{v_3, v_4\}$
e_6	$\{v_3, v_5\}$
e_7	$\{v_1, v_5\}$
e_8	$\{v_2, v_3\}$

This is a diagram for a graph. In this diagram, it does not matter how we arrange the vertices, or whether the edges are drawn as straight, curved or intersecting lines.

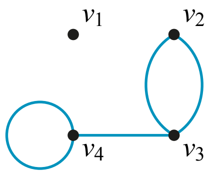
A **graph** G consists of:

1. a finite non-empty set of elements called vertices
2. a finite set of elements called edges
3. an edge-endpoint function that indicates the endpoints of each edge – this function maps each edge to a set of either one or two vertices.

Two vertices of a graph are **adjacent** if they are joined by an edge. The **adjacency matrix** of a graph with vertices $v_1, v_2, \dots, v_n$ is an $n \times n$ matrix such that the entry in row i and column j is the number of edges joining vertices v_i and v_j. For example, the adjacency matrix of the graph shown above is

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Another example of a graph is shown below.



- This graph has **multiple edges**, as there are two edges joining vertices v_2 and v_3.
- There is an edge joining v_4 to itself; this edge is called a **loop**.
- Vertex v_1 is not an endpoint of any edge; it is called an **isolated vertex**.

The edge-endpoint function and the adjacency matrix for this graph are shown below.

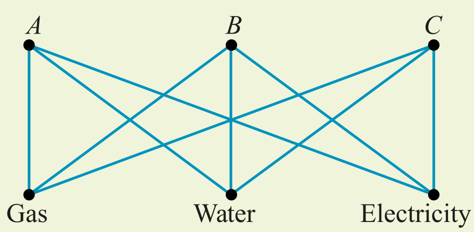
Edge	Endpoints
e_1	$\{v_2, v_3\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_3, v_4\}$
e_4	$\{v_4\}$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Note: There is only one edge joining v_4 to itself. So this loop only contributes 1 to the corresponding entry of the adjacency matrix.

Example 1

The following graph represents three houses, A, B and C, that are each connected to three utilities, gas (G), water (W) and electricity (E). Construct the adjacency matrix for this graph.

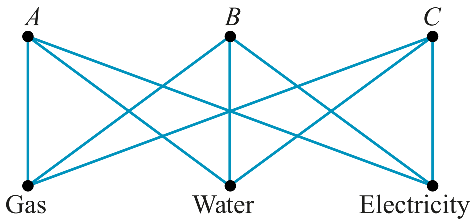


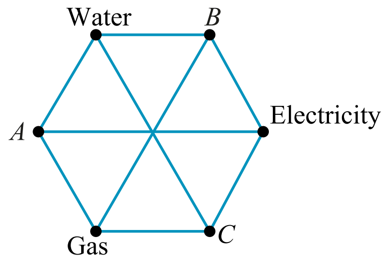
Solution

$$\begin{array}{c} \\ A \\ B \\ C \\ G \\ W \\ E \end{array} \begin{array}{c} \begin{matrix} A & B & C & G & W & E \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Graph drawing

The same graph can be drawn in very different ways. For example, the graph in Example 1 can also be represented by a different looking diagram, as shown below. The information that we get from the two diagrams about the vertices and edges is exactly the same. These two diagrams represent the same graph.





Isomorphism

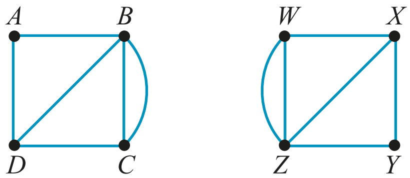
Two graphs that have exactly the same 'structure' are said to be **isomorphic**. If you ignore the labels on the vertices, then isomorphic graphs can be represented by the same diagram.

Isomorphic graphs

Two graphs are isomorphic if there is a one-to-one correspondence between their vertices that preserves the ways the vertices are connected by edges.

That is, two graphs G and H are isomorphic if graph H can be obtained from graph G by simply relabelling its vertices.

For example, the graphs G and H represented by the following two diagrams are isomorphic.



Graph G Graph H

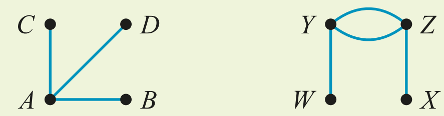
We can obtain graph H from graph G by relabelling its vertices as follows:

$$A \leftrightarrow Y, \quad B \leftrightarrow Z, \quad C \leftrightarrow W, \quad D \leftrightarrow X$$

Note: Two graphs that are isomorphic are regarded as being essentially the same. If two graphs are isomorphic, then they have exactly the same features.

Example 2

Give three reasons why the two graphs shown on the right could not possibly be isomorphic.



Graph G Graph H

Solution

There are many possible answers to this question. Here are three possible reasons:

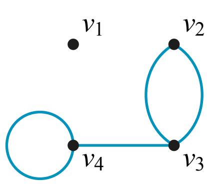
- Graph G does not have multiple edges, while graph H has multiple edges.
- Graph G has three edges, while graph H has four edges.
- Graph G has one vertex where three edges meet, while graph H has two such vertices.

For the small graphs in the previous example, it is easy to see that they are not isomorphic. But it can sometimes be very difficult to tell whether or not two large graphs are isomorphic.

Degree of a vertex

- Let v be a vertex of a graph G. The **degree** of v is equal to the number of edges that have vertex v as an endpoint, with each edge that is a loop counted twice.
- We denote the degree of v by $\deg(v)$.
- The **total degree** of the graph G is the sum of the degrees of all the vertices.

For the graph G shown below, the degrees of the vertices are given in the table.



Vertex	Degree
v_1	0
v_2	2
v_3	3
v_4	3

The total degree of the graph G is

$$\deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4) = 0 + 2 + 3 + 3 = 8$$

Note that this sum is equal to twice the number of edges of G. This is not a coincidence.

Handshaking lemma

The total degree of any graph is equal to twice the number of edges of the graph.

Proof Each edge of the graph has two ends, and so each edge contributes exactly 2 to the sum of the vertex degrees.

The handshaking lemma has two important consequences:

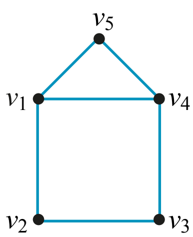
- The total degree of any graph is even.
- Every graph has an even number of vertices of odd degree.

You will prove these in Exercise 12A.

Simple graphs

A **simple graph** is a graph with no loops or multiple edges.

An example of a simple graph is shown below, together with its adjacency matrix.



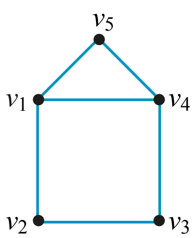
	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	1	1
v_2	1	0	1	0	0
v_3	0	1	0	1	0
v_4	1	0	1	0	1
v_5	1	0	0	1	0

In the adjacency matrix of a simple graph, every entry is either 0 or 1, and the entries on the main diagonal (top-left to bottom-right) are all 0.

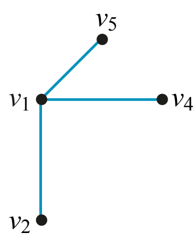
Subgraphs

A **subgraph** is a graph whose vertices and edges are subsets of another graph.

A graph G and one of its subgraphs are shown below.



Graph G

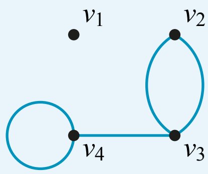


A subgraph of G

Summary 12A

- A **graph** G consists of:
  1. a finite non-empty set of elements called **vertices**
  2. a finite set of elements called **edges**
  3. an **edge-endpoint function** that indicates the endpoints of each edge – this function maps each edge to a set of either one or two vertices.

- A graph can be represented by a diagram or an **adjacency matrix**, which stores the number of edges between each pair of vertices. For example:



$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

- Two graphs are **isomorphic** if there is a one-to-one correspondence between their vertices that preserves the ways the vertices are connected by edges.
- The **degree** of a vertex v, denoted by $\deg(v)$, is equal to the number of edges that have vertex v as an endpoint, with each edge that is a loop counted twice.
- The **total degree** of a graph is the sum of the degrees of all the vertices.
- Handshaking lemma The total degree of any graph is equal to twice the number of edges of the graph.
- A **simple graph** is a graph with no loops or multiple edges.
- A **subgraph** is a graph whose vertices and edges are subsets of another graph.

Exercise 12A

Example 1

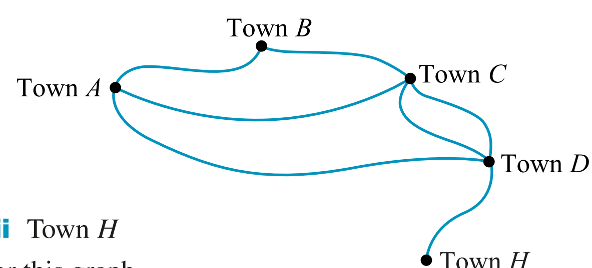
1 Houses X and Y are each connected to four utilities: gas (G), water (W), electricity (E) and broadband (B).

a Draw a graph to depict these connections.

b Redraw your graph so that the edges do not cross.

c Construct the adjacency matrix for this graph.

2 This section of a road map may be considered as a graph, with towns as vertices and the roads connecting the towns as edges.



a Give the degree of:

i Town A **ii** Town B **iii** Town H

b Construct the adjacency matrix for this graph.

c Is this graph simple? Why?

3 There are five football teams in a conference: Alphington (A), Burlington (B), Carlington (C), Darrington (D) and Eddington (E).

- A has played: B, C, C, D
- B has played: A, D, E, E
- C has played: A, A, D
- D has played: A, B, C
- E has played: B, B

a Draw a graph that models this situation.

b Construct the adjacency matrix for this graph.

c Give a reason why this is not a simple graph.

4 Six people are seated at a round table. For each of the following, draw a graph that models the situation:

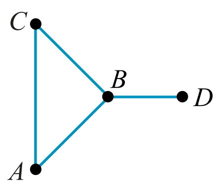
a Each person shakes hands with the two people they are sitting next to.

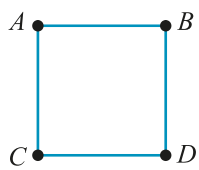
b Each person shakes hands with the person they are sitting opposite.

c Each person shakes hands with every other person.

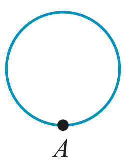
5 Draw a graph such that each state and territory of Australia is represented as a vertex. Connect two vertices by an edge if the states or territories that they represent share a land border.

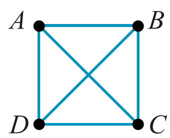
6 For each of the following graphs, give the adjacency matrix:

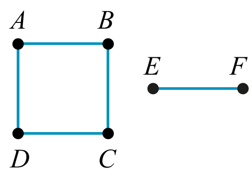
a


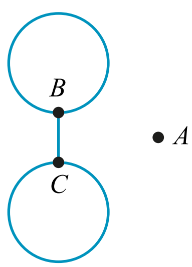
b


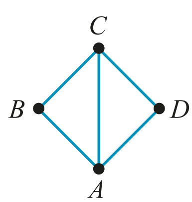
c


d


e


f


g


h


7 Which of the graphs in Question 6 are simple?

8 For each of the following adjacency matrices, give a reason why the corresponding graph is not simple:

a
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

9 Describe how you can use an adjacency matrix to determine:

a the degree of each vertex of a simple graph

b the total degree of a simple graph

c the number of edges of a simple graph.

10 For each of the following, draw the graph corresponding to the given adjacency matrix:

a
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

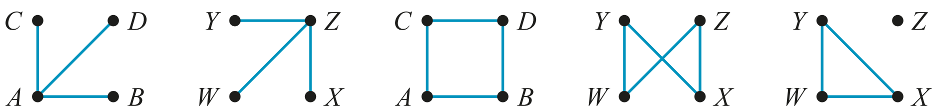
b
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

d
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$$

Example 2

11 Consider the following five graphs:



Graph *G* Graph *H* Graph *I* Graph *J* Graph *K*

a Show that graphs G and H are isomorphic by finding a one-to-one correspondence between their vertices.

b In fact, explain why there are six possible answers to the previous question.

c Show that graphs I and J are isomorphic by finding a one-to-one correspondence between their vertices.

d Give three reasons why graph G is not isomorphic to graph I.

e Give three reasons why graph I is not isomorphic to graph K.

12 Draw the simple graph with four vertices, each of which has degree 3.

13 Draw all simple graphs with vertices v_1, v_2, v_3 such that one of the edges is $\{v_1, v_2\}$.

14 Use the handshaking lemma to prove that:

a the total degree of any graph is even

b every graph has an even number of vertices of odd degree.

15 **a** Draw two different graphs with two vertices, each of which has degree 2.

b Explain why there is no graph with three vertices, each of which has degree 3.

c Draw all non-isomorphic graphs with three vertices, each of which has degree 2.

16 **a** Construct a graph with four vertices v_1, v_2, v_3, v_4 such that $\deg(v_1) = 1$, $\deg(v_2) = 2$, $\deg(v_3) = 3$ and $\deg(v_4) = 4$.

b Prove that there is no graph with four vertices v_1, v_2, v_3, v_4 such that $\deg(v_1) = 2$, $\deg(v_2) = 1$, $\deg(v_3) = 2$ and $\deg(v_4) = 2$.

17 How many different simple graphs with three vertices are there?

18 There are 11 different simple graphs with four vertices. Draw them.

19 Let G be a graph with n vertices and exactly $n - 1$ edges. Prove that G has either a vertex of degree 1 or an isolated vertex. (Hint: Prove this result by contradiction.)

20 **a** Draw a graph with three vertices whose total degree is 10.

b Consider all graphs with three vertices whose total degree is 10. Prove that each of these graphs has some vertex with degree at least 4.

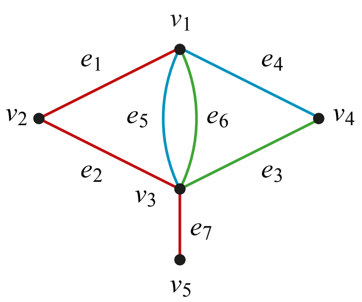
12B Euler circuits

Many applications of graphs involve travelling along the edges from one vertex to another.

Walks in graphs

Consider the graph shown on the right.

- The walk shown in red from v_1 to v_5 can be described by the sequence of vertices v_1, v_2, v_3, v_5.
- For the walk shown in green from v_4 to v_1, we must also list the edges taken, as there is more than one edge joining v_3 and v_1. This walk can be described by the sequence v_4, e_3, v_3, e_6, v_1.
- A walk can include vertices and edges more than once. For instance, the walk v_2, v_1, v_4, v_1 visits vertex v_1 twice and also uses edge e_4 twice.



Walks in graphs

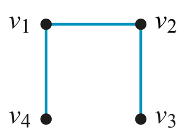
A **walk** in a graph is an alternating sequence of vertices and edges

$$v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n$$

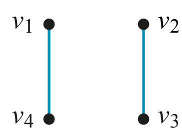
where the edge e_i joins the vertices v_i and v_{i+1}.

If each pair of adjacent vertices in a walk is joined by only one edge, then the walk can be described by the sequence of vertices $v_1, v_2, \dots, v_n$.

A graph is said to be **connected** if there is a walk between each pair of distinct vertices. Otherwise, the graph is said to be **disconnected**. The graph G shown below is connected, while the graph H is disconnected.



A connected graph G



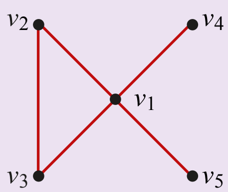
A disconnected graph H

Euler circuits

A **trail** is a walk in a graph that does not use the same edge more than once. In this section, we are interested in trails that use all the edges.

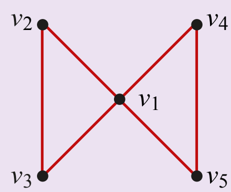
Euler trails and circuits

- An **Euler trail** is a walk in a graph that uses every edge exactly once.



The walk $v_4, v_1, v_3, v_2, v_1, v_5$ in the graph above is an Euler trail.

- An **Euler circuit** is a walk in a graph that uses every edge exactly once and that starts and ends at the same vertex.



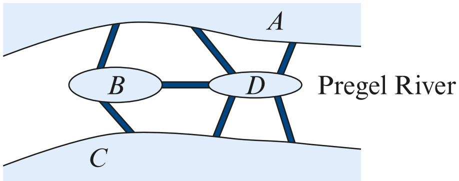
The walk $v_4, v_1, v_3, v_2, v_1, v_5, v_4$ in the graph above is an Euler circuit.

Note: In an Euler trail, some vertices may be visited more than once.

Euler circuits get their name from the Swiss mathematician Leonhard Euler, who considered the following famous problem in a paper published in 1736.

The seven bridges of Königsberg

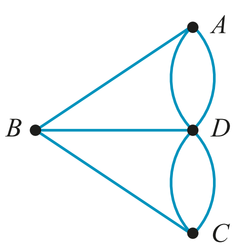
The Pregel River flows through the city of Königsberg. In the middle of the river there are two large islands. These are connected to each other and the two river banks by seven bridges, as shown in the diagram.



The problem that Euler considered was posed as follows:

> 'Is it possible to walk around the city crossing each of the seven bridges exactly once and returning to your starting point?'

We can represent this diagram using the graph shown on the right. Translated into the language of graph theory, we would like to know whether this graph has an Euler circuit.



We will show that, no matter which vertex is chosen as the starting point, it is impossible to traverse the graph and come back to the starting vertex, while using every edge exactly once.

Euler's solution of the problem

In fact, the solution follows at once from a much more general theorem that characterises when a connected graph has an Euler circuit. Half of this theorem is proved below. The other half is proved in Section 12H.

Theorem (Euler circuits)

A connected graph has an Euler circuit if and only if the degree of every vertex is even.

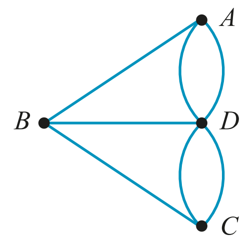
Proof Let G be a graph with an Euler circuit. We need to prove that every vertex has even degree. Suppose that the Euler circuit starts and ends at the vertex v_0.

First consider a vertex v of the graph other than v_0. Each time an edge of the Euler circuit enters v, there must be a corresponding exit edge. Since the Euler circuit includes every edge exactly once, it follows that the degree of v is even.

Now consider the starting vertex v_0. The initial edge that leads out of v_0 can be paired with the final edge that leads into v_0. If the vertex v_0 is revisited in the middle of the Euler circuit, then there must be a different entry edge and a corresponding exit edge. Hence, the degree of v_0 is also even.

We are still to prove that, if every vertex has even degree, then the graph has an Euler circuit. We will complete this half of the proof in Section 12H.

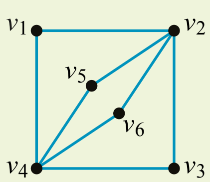
Returning to the seven bridges problem, we see that the graph has no Euler circuit, since the degree of each vertex is odd.



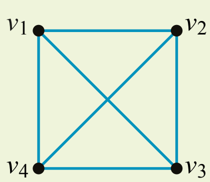
Example 3

For each of the following graphs, name an Euler circuit if one exists:

a



b



Solution

a The degree of every vertex is even. Therefore the graph has an Euler circuit. One such circuit is $v_4, v_1, v_2, v_5, v_4, v_3, v_2, v_6, v_4$. There are many others.

b This graph has vertices of odd degree. Therefore it does not have an Euler circuit.

Euler trails

We recall that an Euler trail in a graph includes every edge exactly once, but does not have to start and end at the same vertex.

We can easily identify whether an Euler trail exists in a graph by using the next theorem.

Theorem (Euler trails)

A connected graph has an Euler trail if and only if one of the following holds:

- every vertex has even degree
- exactly two vertices have odd degree.

Proof (⇒) Let G be a graph that has an Euler trail. Using the theorem for Euler circuits, we can assume that the Euler trail starts and ends at different vertices v_1 and v_2. Add an edge joining v_1 and v_2. The new graph has an Euler circuit, and so every vertex has even degree. Remove the new edge, and we see that v_1 and v_2 have odd degree in G. Therefore the graph G has exactly two vertices with odd degree.

(⇐) We know that, if every vertex has even degree, then there is an Euler circuit. So consider a connected graph G with exactly two vertices of odd degree. Let these be v_1 and v_2. Add an edge joining v_1 and v_2. We obtain a connected graph such that every vertex has even degree, and so it has an Euler circuit. Remove the new edge, and we have an Euler trail in the graph G from v_1 to v_2.

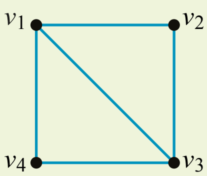
In Exercise 12B, you will prove the following result.

If a connected graph has exactly two vertices of odd degree, then every Euler trail in the graph must start at one of these vertices and end at the other.

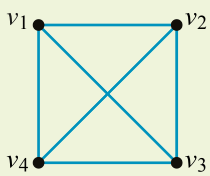
Example 4

For each of the following graphs, name an Euler trail if one exists:

a



b



Solution

a Vertices v_1 and v_3 have odd degree. Any Euler trail must start and end at these vertices. The walk $v_1, v_3, v_2, v_1, v_4, v_3$ is an Euler trail. There are many others.

b This graph has more than two vertices of odd degree. Therefore it does not have an Euler trail.

Fleury's algorithm

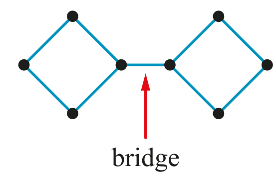
For small graphs, it is easy to find an Euler trail. But graphs used in real-world problems can have millions of vertices, and so a trial-and-error approach will not always work.

One systematic approach to finding an Euler trail is called **Fleury's algorithm**. The basic idea behind this algorithm can be recalled with the help a familiar motto:

Don't burn your bridges (unless you have to).

In a connected graph, a **bridge** is any edge whose deletion would cause the resulting graph to become disconnected.

An example of a bridge is shown in the graph on the right. Deleting this edge would disconnect the graph.



Fleury's algorithm

To find an Euler trail in a connected graph such that every vertex has even degree or exactly two vertices have odd degree:

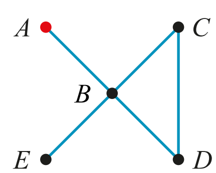
Step 1 If there are two vertices of odd degree, then start from one of them. Otherwise, start from any vertex.

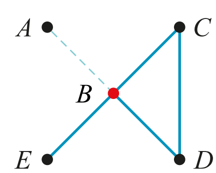
Step 2 Move from the current vertex across an edge to an adjacent vertex. Always choose a non-bridge edge unless there is no alternative.

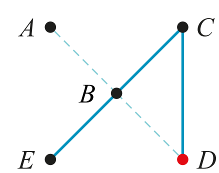
Step 3 Delete the edge that you have just traversed.

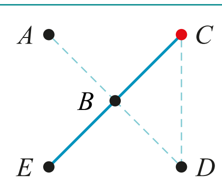
Step 4 Repeat from Step 2 until there are no edges left.

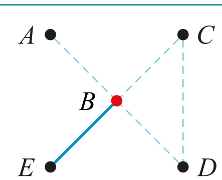
We will apply Fleury's algorithm to the graph shown in the top-left of the following table. The vertices A and E have odd degree. We choose to start at vertex A.

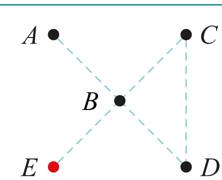
<table>
<tr>
<td>

There is only one edge at A. We must pick A-B. Delete this edge.</td>
<td>

There are three edges at B. However B-E is a bridge. We pick B-D. Delete this edge.</td>
<td>

There is only one edge at D. We must pick D-C. Delete this edge.</td>
</tr>
<tr>
<td>

There is only one edge at C. We must pick C-B. Delete this edge.</td>
<td>

There is only one edge at B. We must pick B-E. Delete this edge.</td>
<td>

All edges are deleted. An Euler trail is A-B-D-C-B-E.</td>
</tr>
</table>

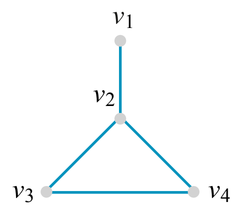
Summary 12B

- A **walk** in a graph is an alternating sequence of vertices and edges.
- An **Euler trail** is a walk in a graph that uses every edge exactly once.
- An **Euler circuit** is a walk in a graph that uses every edge exactly once and that starts and ends at the same vertex.
- A connected graph has an Euler circuit if and only if the degree of every vertex is even.
- A connected graph has an Euler trail if and only if every vertex has even degree or exactly two vertices have odd degree.

Skill-sheet

Exercise 12B

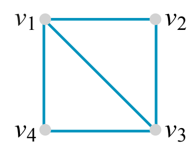
1 Consider the graph shown on the right.



a Recall that an Euler trail must start and end at the two vertices of odd degree. List all four Euler trails in this graph.

b This graph does not have an Euler circuit. Why not?

2 Consider the graph shown on the right.



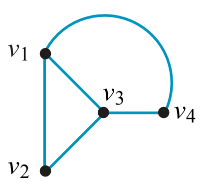
a Give a reason why this graph does not have an Euler circuit.

b Give a reason why this graph has an Euler trail.

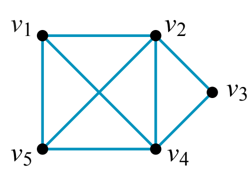
Example 3
Example 4

3 For each of the following, find an Euler circuit or trail:

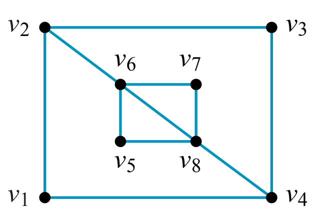
a



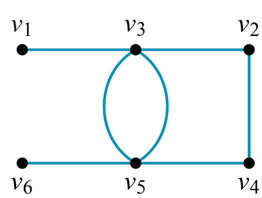
b



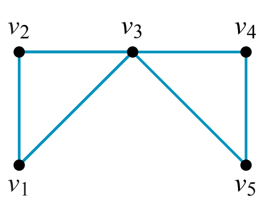
c



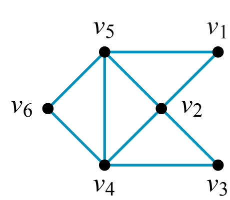
d



e

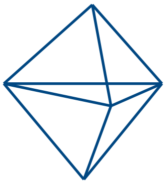


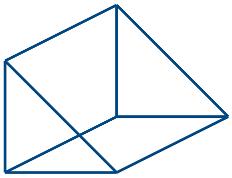
f

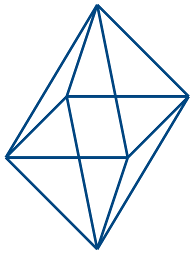


4 Is it possible for a graph with a vertex of degree 1 to have an Euler circuit? If so, draw one. If not, explain why not. What about an Euler trail?

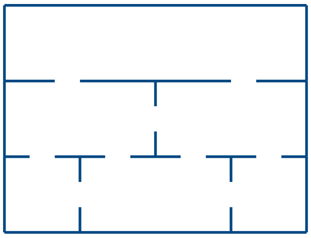
5 Can you trace the edges of the following polyhedra without lifting your pencil or tracing over the same edge twice? (Hint: First find the degree of each vertex.)

a 

b 

c 

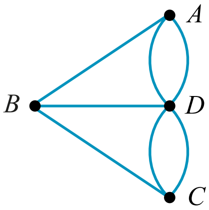
6 Suppose that six rooms in a house are laid out as shown. The doors are represented by open sections along the walls. Can you walk around the house going through each door exactly once and finishing in the same room where you started?



7 A bridge builder has come to Königsberg and would like to add bridges to the city so that it is possible to travel over every bridge exactly once. What is the least number of bridges that must be built to guarantee that there is:

a an Euler trail

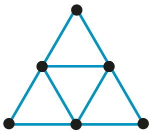
b an Euler circuit?



8 The **triangular grid graph** T_3 is shown on the right. The number of vertices of this graph is $1 + 2 + 3 = 6$.

a Draw the triangular grid graph T_4.

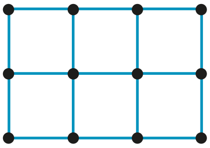
b Briefly explain why T_n has an Euler circuit for all $n \in \mathbb{N}$.



9 The 3×4 **grid graph** is shown on the right. For what values of m and n will the $m \times n$ grid graph have:

a an Euler trail

b an Euler circuit?

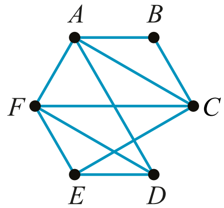


10 Prove that, if a connected graph has exactly two vertices of odd degree, then every Euler trail in the graph must start at one of these vertices and end at the other.

11 Prove that, if all the vertices of a connected graph have even degree, then every Euler trail in the graph is an Euler circuit. (Hint: Try a proof by contradiction.)

12C Hamiltonian cycles

Consider a group of six students, which we represent by six vertices A, B, C, D, E and F. An edge is drawn between two vertices if the students that they represent are friends.



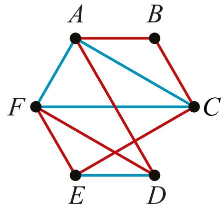
Question

Is it possible for these students to be seated at a round table so that every student is sitting next to a friend on both sides?

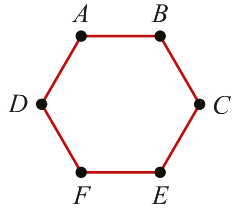
Answer

We are looking for a walk in this graph that starts and ends at the same vertex and visits every other vertex exactly once.

An example of such a walk is A, B, C, E, F, D, A.



This walk gives the order in which we can seat the students at the round table.

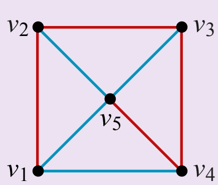


Hamiltonian paths and cycles

A **path** is a walk in a graph that does not repeat any vertices (and therefore does not repeat any edges). A **cycle** is a walk that starts and ends at the same vertex and otherwise does not repeat any vertices or edges. In this section, we are interested in paths and cycles that visit all the vertices.

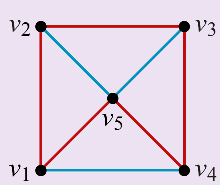
Hamiltonian paths and cycles

- A **Hamiltonian path** is a walk in a graph that visits every vertex exactly once.



The walk v_1, v_2, v_3, v_4, v_5 in the graph above is a Hamiltonian path.

- A **Hamiltonian cycle** is a walk that starts and ends at the same vertex and visits every other vertex exactly once (without repeating any edges).



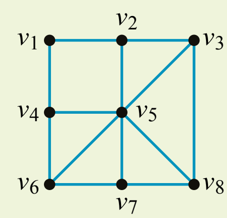
The walk $v_1, v_2, v_3, v_4, v_5, v_1$ in the graph above is a Hamiltonian cycle.

Note: In a Hamiltonian path or cycle, some edges may not be used.

Hamiltonian cycles are named after Sir William Rowan Hamilton, who devised a puzzle to find such a walk along the edges of a dodecahedron. In general, there is no simple necessary and sufficient condition that enables us to identify whether there is a Hamiltonian cycle.

Example 5

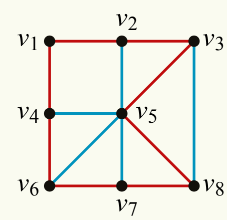
Eight towns are represented by the vertices $v_1, v_2, \ldots, v_8$. The roads that connect these towns are represented as edges. Starting and ending at v_1, how can a salesperson visit every town exactly once?



Solution

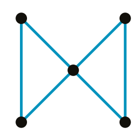
We seek a Hamiltonian cycle starting and ending at v_1.

One such cycle is $v_1, v_2, v_3, v_5, v_8, v_7, v_6, v_4, v_1$. This is shown in red. There are many others.

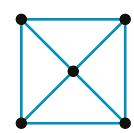


Notes:

- It is easy to remember the difference between Euler circuits and Hamiltonian cycles: EULER circuits are defined in terms of EDGES.
- A graph can have an Euler circuit but no Hamiltonian cycle. This can be seen in the graph on the right.



- A graph can have a Hamiltonian cycle but no Euler circuit. This can be seen in the graph on the right. It has vertices of odd degree.



Summary 12C

- A **Hamiltonian path** is a walk in a graph that visits every vertex exactly once.
- A **Hamiltonian cycle** is a walk in a graph that starts and ends at the same vertex and visits every other vertex exactly once (without repeating any edges).

Skill-sheet

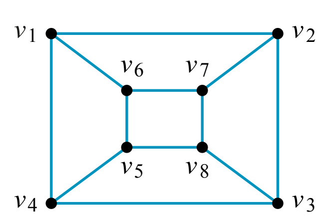
Exercise 12C

Example 5

1 For the graph shown, list a Hamiltonian path that:

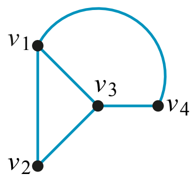
a starts at v_1 and ends at v_4

b starts at v_6 and ends at v_7.

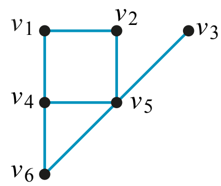


2 List a Hamiltonian path for each of the following graphs:

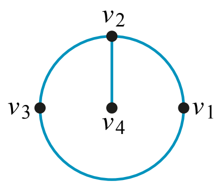
a



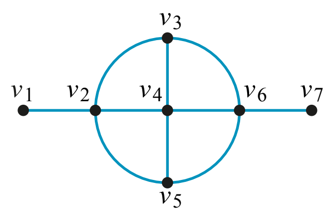
b



c

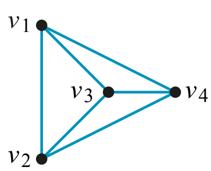


d

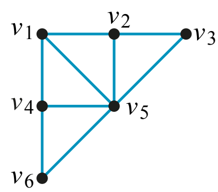


3 List a Hamiltonian cycle starting from v_1 for each of the following graphs:

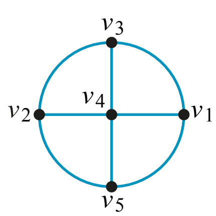
a



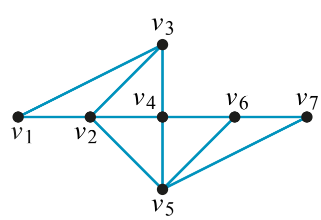
b



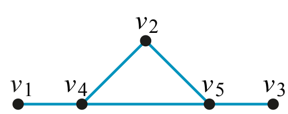
c



d



4 Consider the graph shown on the right.

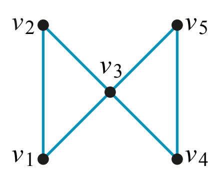


a Briefly explain why this graph does not have a Hamiltonian cycle.

b How many Hamiltonian paths does it have?

c What edge should be added to this graph so that it will have a Hamiltonian cycle?

5 Consider the **butterfly graph** shown on the right.



a Briefly explain why this graph does not have a Hamiltonian cycle.

b How many Hamiltonian paths does it have?

6 Draw a connected graph with four vertices that has:

a a Hamiltonian cycle and an Euler circuit

b a Hamiltonian cycle but no Euler circuit

c an Euler circuit but no Hamiltonian cycle

d no Euler circuit and no Hamiltonian cycle.

7 Answer *true* or *false* for each of the following statements:

a A graph with a vertex of degree 1 can have a Hamiltonian cycle.

b A graph with exactly one vertex of degree 1 can have a Hamiltonian path.

c A graph with exactly two vertices of degree 1 can have a Hamiltonian path.

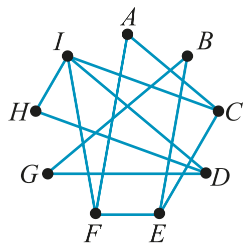
d A graph with three vertices of degree 1 can have a Hamiltonian path.

e If a vertex of a simple graph has degree 2, then both edges at this vertex must be part of any Hamiltonian cycle.

8 For the graphs in this question, each vertex represents a student. Two vertices are joined by an edge if the two students that they represent are friends.

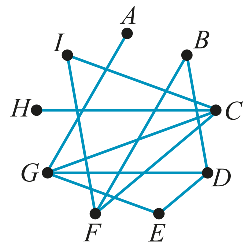
a Consider the group of students represented by the graph on the right. Can these students be seated at a round table so that students are only sitting next to their friends?

Hint: It will help to consider the vertices of degree 2.



b Consider the group of students represented by the graph on the right. Can these students be seated in a line so that students are only sitting next to their friends?

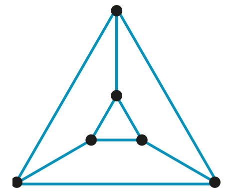
Hint: Consider the vertices of degree 1 and 2.



9 In the graph shown on the right, every vertex has degree 3.

a Colour the edges of this graph using three colours so that every vertex has three edges with three different colours.

b Find a Hamiltonian cycle in the coloured graph. What do you notice about the colours of the edges in this cycle?



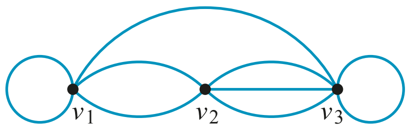
Now consider any graph such that every vertex has degree 3. Suppose that the graph also has a Hamiltonian cycle.

c Use the handshaking lemma to prove that the graph has an even number of vertices.

d Hence, explain why it is possible to colour the edges of the graph using three colours so that every vertex has three edges with three different colours.

12D Using matrix powers to count walks in graphs

The graph shown below represents three towns and the routes connecting these towns. The adjacency matrix of this graph is also shown.



$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix} \end{array}$$

There is one direct route from v_1 to v_3. This is called a **walk of length 1**. There are also several walks from v_1 to v_3 via v_2. These are called **walks of length 2**.

In general, the **length of a walk** in a graph is the number of edges in the walk. (If you use the same edge more than once, then you must count that edge more than once.)

In the graph shown above, how many different walks of length 2 are there from v_1 to v_3? There are three cases to consider.

- **Case 1:** $v_1 \to v_1 \to v_3$
  There is 1 edge from v_1 to v_1 and then 1 edge from v_1 to v_3. This gives $1 \times 1 = 1$ walk.
- **Case 2:** $v_1 \to v_2 \to v_3$
  There are 2 edges from v_1 to v_2 and then 3 edges from v_2 to v_3. This gives $2 \times 3 = 6$ walks.
- **Case 3:** $v_1 \to v_3 \to v_3$
  There is 1 edge from v_1 to v_3 and then 1 edge from v_3 to v_3. This gives $1 \times 1 = 1$ walk.

Therefore the total number of walks of length 2 from v_1 to v_3 is given by:

$$(v_1 \to v_1 \text{ then } v_1 \to v_3) \text{ or } (v_1 \to v_2 \text{ then } v_2 \to v_3) \text{ or } (v_1 \to v_3 \text{ then } v_3 \to v_3)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$(1 \times 1) + (2 \times 3) + (1 \times 1) = 8$$

Note that this calculation can also be obtained by multiplying the first row of **A** by the third column of **A**. The entries corresponding to this product are shown in red below.

$$\mathbf{A}^2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 6 & 5 & 8 \\ 5 & 13 & 5 \\ 8 & 5 & 11 \end{bmatrix} \end{array}$$

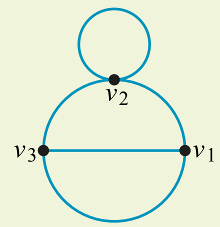
In general, the entries of $\mathbf{A}^2$ give the number of walks of length 2 between each pair of vertices. Likewise, the entries of $\mathbf{A}^3$ give the number of walks of length 3. In fact, a simple induction argument will prove the following.

If **A** is the adjacency matrix of a graph G, then the number of walks of length n from vertex v_i to vertex v_j is equal to the entry of $\mathbf{A}^n$ in row i and column j.

Note: It is helpful to define $\mathbf{A}^0 = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. We can then also consider walks of length 0. We say that there is one walk of length 0 from any vertex to itself.

Example 6

Find the number of walks of length 3 from vertex v_1 to vertex v_3 in the graph shown.



Solution

We construct the adjacency matrix $\mathbf{A}$ of the graph and then calculate $\mathbf{A}^3$:

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array} \implies \mathbf{A}^3 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 5 & 9 & 13 \\ 9 & 9 & 9 \\ 13 & 9 & 5 \end{bmatrix} \end{array}$$

The number of walks of length 3 from v_1 to v_3 is the entry of $\mathbf{A}^3$ in row 1 and column 3. Therefore there are 13 walks.

Summary 12D

- The **length of a walk** in a graph is the number of edges in the walk. (If you use the same edge more than once, then you must count that edge more than once.)
- If $\mathbf{A}$ is the adjacency matrix of a graph G, then the number of walks of length n from vertex v_i to vertex v_j is equal to the entry of $\mathbf{A}^n$ in row i and column j.

Skill-sheet

Exercise 12D

Example 6

1 A graph with four vertices has the following adjacency matrix.

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

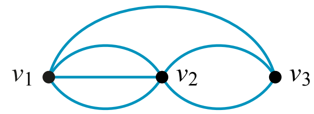
a Find the number of walks of length 2 between:

i v_1 and v_2 **ii** v_3 and v_4 **iii** v_1 and v_1 **iv** v_4 and v_2

b Find the number of walks of length 3 between:

i v_1 and v_2 **ii** v_3 and v_4 **iii** v_1 and v_1 **iv** v_4 and v_2

2 **a** Construct the adjacency matrix of the graph shown on the right.



b Find the number of walks of length 2 between:

i v_1 and v_2 **ii** v_2 and v_2 **iii** v_3 and v_1

c Find the number of walks of length 2 that start at v_1.

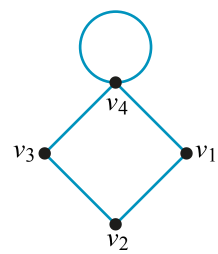
d Find the number of walks of length 2 that end at v_3.

e Find the number of walks of length 3 between:

i v_1 and v_2 **ii** v_2 and v_2 **iii** v_3 and v_1

f Find the number of walks of length 3 that start and end at the same vertex.

3 Consider the graph shown on the right.

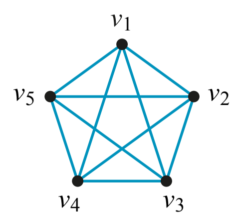


a Write down the adjacency matrix **A** of this graph.

b By finding $\mathbf{A}^3$, show that there is only one pair of vertices that is not connected by a walk of length 3.

c By finding $\mathbf{A}^4$, show that any every pair of vertices is connected by a walk of length 4.

4 In the graph on the right, each of the five vertices is adjacent to every other vertex.

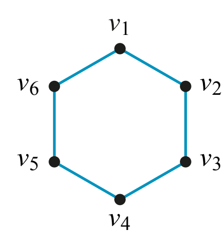


a Write down the adjacency matrix **A** of this graph.

b Without computing the matrix directly, find $\mathbf{A}^2$.

c Without computing the matrix directly, find $\mathbf{A}^3$.

5 In the graph on the right, each of the six vertices is adjacent to exactly two other vertices.



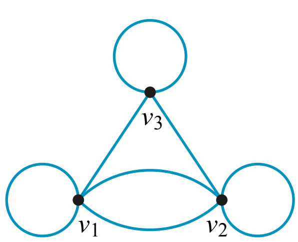
a Write down the adjacency matrix **A** of this graph.

b Without computing $\mathbf{A}^3$, find the entry in row 1 and column 1.

c Without computing $\mathbf{A}^4$, find the entry in row 1 and column 2.

d Let n be an odd natural number. Without computing the matrix, describe the entries on the main diagonal of $\mathbf{A}^n$ (top-left to bottom-right).

6 Consider the graph shown on the right.



a Write down the adjacency matrix **A** of this graph.

b Let **I** be the 3×3 identity matrix. Evaluate the matrix $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4$.

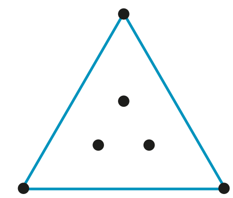
c Hence, find the number of walks of length at most 4 from vertex v_1 to vertex v_2.

12E Regular, cycle, complete and bipartite graphs

We now consider various types of graphs that frequently arise in applications, and investigate some of their properties.

Regular graphs

A graph is said to be **regular** if all its vertices have the same degree.



The graph on the right is regular, as every vertex has degree 3.

Theorem (Edges of a regular graph)

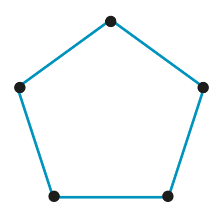
If G is a regular graph with n vertices of degree r, then G has $\frac{nr}{2}$ edges.

Proof Each of the n vertices has degree r, so the sum of the degrees of all the vertices is nr. Therefore, by the handshaking lemma, there are $\frac{nr}{2}$ edges.

We now look at two special examples of regular graphs.

Cycle graphs

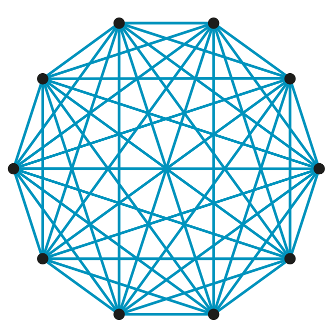
A **cycle graph** is a graph consisting of a single cycle of vertices and edges. For $n \geq 3$, the cycle graph with n vertices is denoted by C_n.



Every cycle graph C_n is regular, since each vertex has degree 2. The cycle graph C_5 is shown on the right.

Complete graphs

A **complete graph** is a simple graph with one edge joining each pair of distinct vertices. The complete graph with n vertices is denoted by K_n.



The complete graph K_{10} is shown on the right. This graph models the situation where there is a group of 10 people, and each person shakes hands with the nine other people in the group.

The graph K_n is regular, since each vertex has degree $n - 1$. The adjacency matrix of K_n has 1s in all positions, except on the main diagonal. The main diagonal has 0s.

Theorem (Edges of a complete graph)

The complete graph K_n has $\dfrac{n(n-1)}{2}$ edges.

Proof Each of the n vertices has degree $n - 1$, so the sum of the degrees of all the vertices is $n(n-1)$. Therefore, by the handshaking lemma, there are $\dfrac{n(n-1)}{2}$ edges.

The complement of a simple graph

If G is a simple graph, then its **complement** is the simple graph $\overline{G}$ defined as follows:

1. $\overline{G}$ and G have the same set of vertices
2. two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Note: To generate the complement of a simple graph, fill in all the missing edges required to form a complete graph and then remove all the edges of the original graph.

Example 7

a How many edges does the complete graph K_5 have?

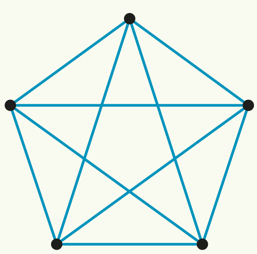
b Draw K_5.

c Draw the cycle graph C_5 and draw the complement of C_5.

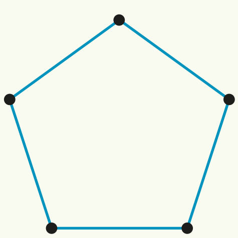
Solution

a The graph K_5 has $\dfrac{5(5-1)}{2} = 10$ edges.

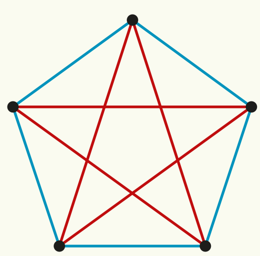
b K_5



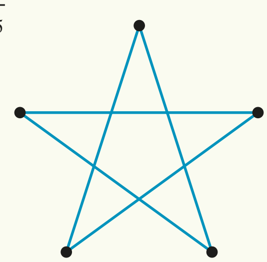
c C_5



K_5

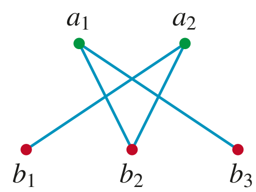


$\overline{C_5}$



Bipartite graphs

A **bipartite graph** is a graph whose vertices can be divided into two disjoint subsets A and B such that every edge of the graph joins a vertex in A to a vertex in B.



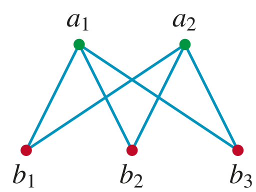
The graph shown on the right is a bipartite graph, as every edge of the graph joins a vertex in $A = \{a_1, a_2\}$ to a vertex in $B = \{b_1, b_2, b_3\}$.

A **complete bipartite graph** is a simple graph whose vertices can be divided into two disjoint subsets A and B such that:

1 every edge joins a vertex in A to a vertex in B

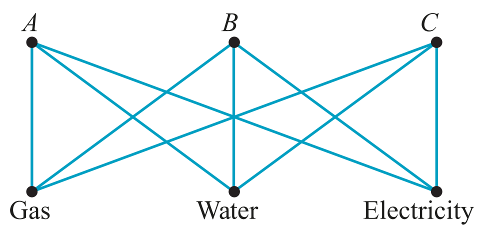
2 every vertex in A is joined to every vertex in B.

The complete bipartite graph where A contains m vertices and B contains n vertices is denoted by $K_{m,n}$.



The complete bipartite graph $K_{m,n}$ has $m + n$ vertices and mn edges. The graph above is $K_{2,3}$.

The utility graph from Example 1 corresponds to the complete bipartite graph $K_{3,3}$.

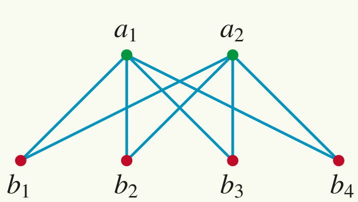


Example 8

Draw the complete bipartite graph $K_{2,4}$ and give its adjacency matrix.

Solution

Each of the two vertices in the set $A = \{a_1, a_2\}$ is joined to each of the four vertices in the set $B = \{b_1, b_2, b_3, b_4\}$.



$$\mathbf{A} = \begin{array}{c} \\ a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Every cycle graph with an even number of vertices is a bipartite graph. This is illustrated in the next example.

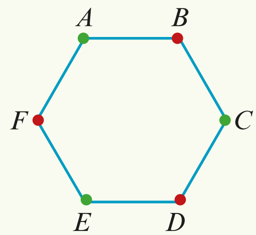
Example 9

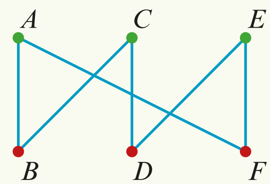
a Show that the cycle graph C_6 is a bipartite graph by dividing the set of vertices into two suitable disjoint subsets.

b Hence show that C_6 is a subgraph of $K_{3,3}$.

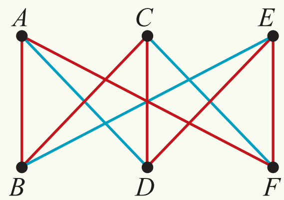
Solution

a It helps to colour the vertices with alternating colours. The vertices are then split into two disjoint subsets according to their colour.





b The graph C_6 is shown in red as a subgraph of $K_{3,3}$.



Summary 12E

- A graph is **regular** if all its vertices have the same degree.
- If G is a regular graph with n vertices of degree r, then G has $\frac{nr}{2}$ edges.
- The **cycle graph** C_n is a graph consisting of a single cycle of n vertices and n edges.
- The **complete graph** K_n is a simple graph with n vertices such that every pair of distinct vertices is joined by an edge.
- A simple graph G and its **complement** $\overline{G}$ have the same set of vertices. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.
- The vertices of a **bipartite graph** can be divided into two disjoint subsets A and B such that every edge joins a vertex in A to a vertex in B.

Skill-sheet

Exercise 12E

1 Draw a regular graph with four vertices and six edges.

2 Draw the cycle graph C_4 and its complement $\overline{C_4}$.

3 There are six teams in a competition. Each team plays every other team exactly once.

a How many matches are there?

b Represent this competition with the complete graph K_6.

4 There are five teams in a competition.

a Show that it is possible for each of the five teams to play exactly two other teams. Illustrate this competition as a regular graph.

b Using the handshaking lemma, show that it is not possible for each of the five teams to play exactly three other teams.

Example 7

5 **a** How many edges does the complete graph K_7 have?

b Draw K_7.

c Draw the cycle graph C_7 and its complement $\overline{C_7}$.

Example 8

6 **a** Draw the complete bipartite graph $K_{3,3}$ and its complement.

b Is it true that the complement of a bipartite graph is also bipartite?

7 Find the maximum number of handshakes that can take place between eight people. Represent this with the complete graph K_8.

8 Draw a regular graph with eight vertices where each vertex has degree:

a 3 **b** 4 **c** 5

9 For each of the following, draw the complete bipartite graph and give its adjacency matrix:

a $K_{1,3}$ **b** $K_{2,3}$

10 Show that the complete bipartite graph $K_{2,2}$ is isomorphic to the cycle graph C_4.

11 **a** Find a Hamiltonian cycle in $K_{3,3}$.

b Find a Hamiltonian cycle in $K_{4,4}$.

c Explain why $K_{2,3}$ cannot have a Hamiltonian cycle.

12 Prove that the complete bipartite graph $K_{m,n}$ has a Hamiltonian cycle if and only if $m = n$.

13 **a** Prove that the complete bipartite graph $K_{m,n}$ has an Euler circuit if and only if both m and n are even.

b Find an Euler circuit in $K_{2,4}$.

Example 9

14 **a** Show that the cycle graph C_8 is bipartite by dividing the set of vertices into two suitable disjoint subsets.

b Hence show that C_8 is a subgraph of $K_{4,4}$.

15 A simple graph is said to be **self-complementary** if it is isomorphic to its complement.

a Show that the cycle graph C_5 is self-complementary.

b Draw another simple graph with five vertices that is self-complementary.

c Show that the cycle graph C_4 is not self-complementary.

d There is just one simple graph with four vertices that is self-complementary. Find it.

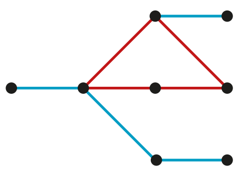
e Prove that a self-complementary graph with n vertices has $\dfrac{n(n-1)}{4}$ edges.

f Prove that, if a simple graph with n vertices is self-complementary, then $n = 4k$ or $n = 4k + 1$, where k is a non-negative integer.

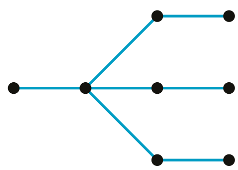
16 Recall that a graph is connected if there is a walk between each pair of distinct vertices. Let G be a simple graph. Show that either G or its complement $\overline{G}$ is connected.

12F Trees

Recall that a cycle is a walk in a graph that starts and ends at the same vertex and otherwise does not repeat any vertices or edges. (By convention, a cycle must include at least one edge.) The cycle in graph G below is indicated by red edges. Graph H has no cycles.

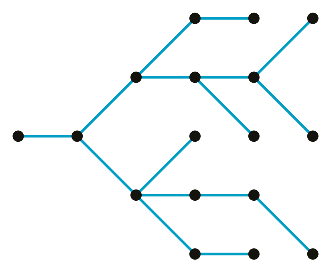


Graph *G* has a cycle



Graph *H* has no cycles

A **tree** is a connected graph that contains no cycles.

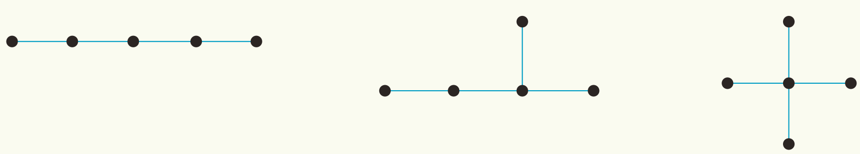


Trees get their name from the fact that they can often be drawn in a way that resembles a branching tree. A tree has no loops or multiple edges. The smallest tree is a single vertex.

Example 10

Draw the three trees with five vertices.

Solution



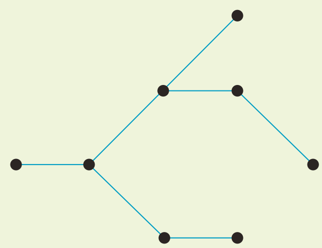
Trees have many important properties, some of which you will prove in Exercise 12F.

Properties of trees

- A tree with n vertices has $n - 1$ edges.
- In any tree, there is exactly one path between each pair of distinct vertices.
- Every tree is a bipartite graph.

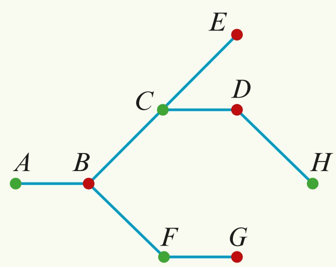
Example 11

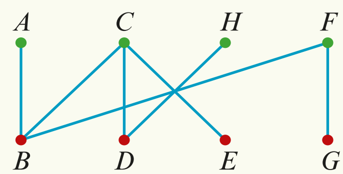
Show that this tree is a bipartite graph by dividing its vertices into two suitable disjoint subsets.



Solution

It helps to colour the vertices with alternating colours. The vertices are then split into two disjoint subsets according to their colour.

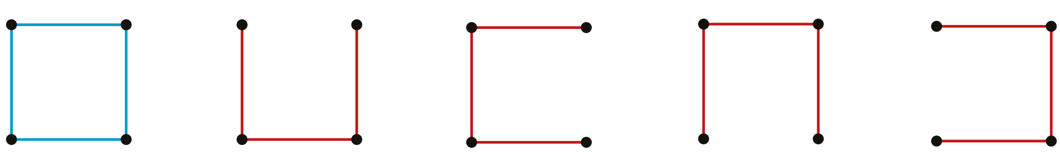




Spanning trees

Let G be a connected graph. A **spanning tree** of G is a subgraph of G that is a tree with the same set of vertices as G.

A graph may have many spanning trees. For example, the cycle graph C_4 has four spanning trees. These are shown in red below.



Trees are important largely due to the following theorem.

Theorem (Spanning trees)

Every connected graph has a spanning tree.

The proof of this theorem is suggested by the following algorithm for finding a spanning tree of any connected graph. You will complete the proof in Exercise 12F (Question 11).

Algorithm for finding a spanning tree

To find a spanning tree of a given connected graph:

Step 1 If the graph has no cycles, then stop.

Step 2 Choose any edge that belongs to a cycle, and delete the chosen edge.

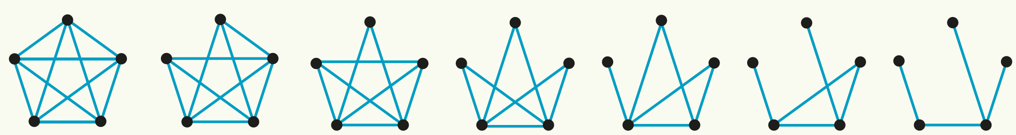
Step 3 Repeat from Step 1.

Example 12

Find a spanning tree for the complete graph K_5.

Solution

Start with K_5. Then successively delete edges belonging to cycles, until the graph has no cycles. An example is shown below. There are many other possibilities.



Summary 12F

- A **tree** is a connected graph that contains no cycles.
- A **spanning tree** of a connected graph G is a subgraph of G that is a tree with the same set of vertices as G.
- Every connected graph has a spanning tree.

Exercise 12F

Example 10

1 There are two trees with four vertices. Draw them.

2 **a** There are eight trees with five or fewer vertices. Draw them.

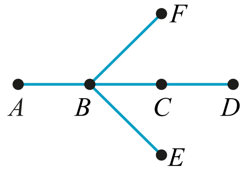
b There are six trees with six vertices. Draw them.

c By using the six trees with six vertices, draw the eleven trees with seven vertices.

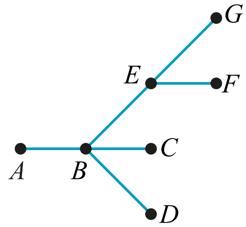
Example 11

3 Show that each of the following trees is a bipartite graph by dividing the set of vertices into two suitable disjoint subsets:

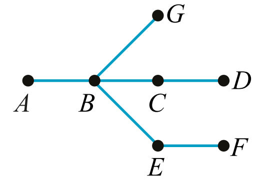
a



b



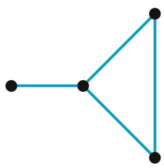
c



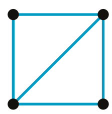
Example 12

4 By successively deleting edges belonging to cycles, find a spanning tree for each of the following graphs:

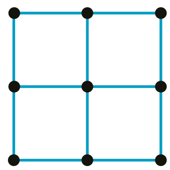
a



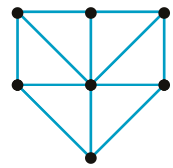
b



c



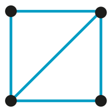
d



5 Usually a connected graph will have more than one spanning tree.

a Show that all the spanning trees of C_3 are isomorphic.

b Draw the eight spanning trees of the graph shown on the right. Put these spanning trees into two groups of isomorphic graphs.



6 Use a proof by contradiction to show that the addition of an edge to a tree cannot form more than one cycle.

7 **a** Consider a tree with at least two vertices. Prove that at least two of its vertices have degree 1. (Hint: Consider the endpoints of a path of maximal length.)

b Prove that every tree with n vertices has $n - 1$ edges. (Hint: Use part **a** and mathematical induction.)

8 Recall that a path is a walk in a graph that does not visit the same vertex more than once. Prove that, in a connected graph, there is a path between each pair of distinct vertices. (Hint: Consider a walk between the vertices of minimal length.)

9 Use a proof by contradiction for each of the following:

a Consider a connected graph G such that there is only one path between any two vertices. Prove that the graph G is a tree.

b Let G be a tree. Prove that there is only one path between any two vertices.

c Consider a connected graph G such that, if any edge is removed, then the resulting graph is disconnected. Prove that the graph G is a tree.

d Let G be a tree. Prove that deleting any edge of G will disconnect the graph.

10 **a** Give an example of a bipartite graph that is not a tree.

b Prove that every tree is a bipartite graph. (Hint: Start with a fixed vertex v. There is a unique path from v to any other vertex u. Consider the length of this path.)

11 In a connected graph, there is a path between each pair of distinct vertices.

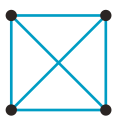
a Consider a connected graph that has a cycle. Prove that if an edge from the cycle is deleted, then the resulting graph is still connected.

b Prove that every connected graph has a spanning tree. (Hint: Use part **a**.)

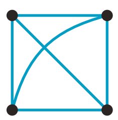
12G Euler's formula and the Platonic solids

Planar graphs

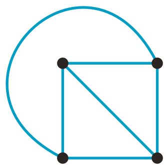
The complete graph K_4 shown below can be redrawn so that its edges do not cross. To do this, we can simply stretch one of the diagonal edges so that it lies outside the square.



K_4 with edges crossing



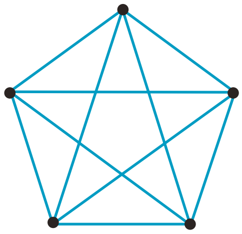
Moving an edge



K_4 without edges crossing

A graph G is called a **planar graph** if it can be drawn in the plane so that its edges only intersect at their endpoints. Any such drawing is called a **plane drawing** of G.

Not all graphs are planar. The complete graph K_5 is not planar, as it is not possible to draw this graph so that its edges do not cross. You will prove this in Exercise 12G.



Theorem (Subgraphs of planar graphs)

Any subgraph of a planar graph is also planar.

Proof The original planar graph can be drawn so that the edges do not cross. As the edges of a subgraph form a subset of the edges of the original graph, the edges of the subgraph also do not cross.

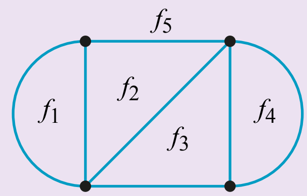
This theorem seems rather trivial, but it is surprisingly useful. To show that a graph is not planar, it is enough to find a subgraph that is not planar.

Euler’s formula

Faces of a planar graph

If G is a planar graph, then any plane drawing of G divides the plane into regions, called **faces**. The **bounded faces** are those bordered by edges. The region that is not bordered by edges is called the **unbounded face**.

For the graph shown on the right, the faces f_1, f_2, f_3 and f_4 are bounded, while the face f_5 is unbounded.



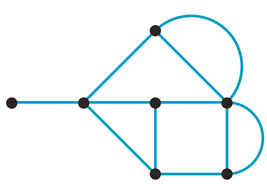
Euler discovered a remarkable result about planar graphs that relates the number of vertices, edges and faces.

Theorem (Euler’s formula)

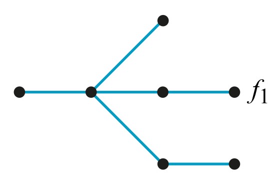
If G is a connected planar graph with v vertices, e edges and f faces, then

$$v - e + f = 2$$

Proof We start with a plane drawing of the graph G. Since G is connected, we can successively delete edges belonging to cycles until we obtain a spanning tree of G.



A connected planar graph



A spanning tree of the graph

Each time we delete an edge, the number of faces decreases by 1, the number of edges decreases by 1 and the number of vertices is unchanged. Therefore the value of $v - e + f$ is unchanged each time we delete an edge.

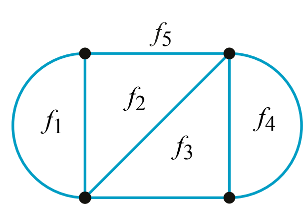
The spanning tree has v vertices. As it is a tree, it therefore has $v - 1$ edges. This tree has just one face, the unbounded face f_1. Therefore, the spanning tree satisfies

$$v - e + f = v - (v - 1) + 1 = 2$$

Moreover, since the value of $v - e + f$ did not change as each edge was deleted, this formula is also satisfied by the original graph G.

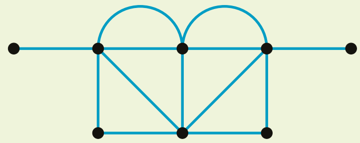
The connected planar graph on the right has four vertices, seven edges and five faces. Therefore $v = 4$, $e = 7$ and $f = 5$. We can verify that

$$\begin{aligned} v - e + f &= 4 - 7 + 5 \\ &= 2 \end{aligned}$$



Example 13

Verify Euler's formula for the planar graph shown.



Solution

There are 8 vertices, 13 edges and 7 faces (including the unbounded face).

Therefore $v = 8$, $e = 13$ and $f = 7$, giving

$$\begin{aligned} v - e + f &= 8 - 13 + 7 \\ &= 2 \end{aligned}$$

The next theorem says that, if a planar graph has many edges, then it must also have many vertices.

Theorem (Edges of planar graphs)

Let G be a connected simple graph with v vertices and e edges, where $v \geq 3$. If the graph G is planar, then $e \leq 3v - 6$.

Proof Let f be the number of faces in a plane drawing of G. As the graph is simple, it has no loops or multiple edges, so every face is bordered by at least three edges. (If the face lies on both sides of an edge, then the edge is counted twice.)

Therefore

$$\begin{aligned} 2e &= \text{sum of edges around each face} && \text{(each edge is double counted)} \\ &\geq 3f && \text{(each face has at least three edges)} \\ &= 3(2 - v + e) && \text{(by Euler's formula)} \\ &= 6 - 3v + 3e \end{aligned}$$

Overall, we find that $2e \geq 6 - 3v + 3e$. By rearranging the terms in this inequality, we obtain $e \leq 3v - 6$.

This theorem provides a simple test to show that a connected simple graph is not planar.

Example 14

A connected simple graph has 6 vertices and 14 edges. Show that this graph is not planar.

Solution

If the graph were planar, then $e \leq 3v - 6$. However, for this graph we have

$$3v - 6 = 12 < 14 = e$$

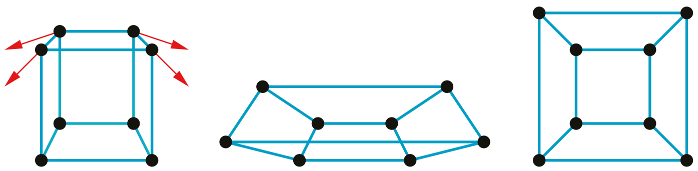
Note: This theorem can be used to show that a graph is not planar, but it cannot be used to show that a graph is planar.

Polyhedral graphs

A **polyhedron** is a three-dimensional solid formed from a collection of polygons joined along their edges.

Every convex polyhedron can be drawn as a connected planar graph.

Here we show the process for obtaining a planar graph from the cube:



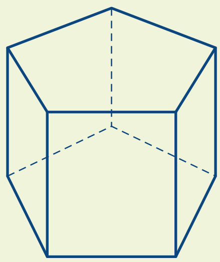
In general, start by resting a convex polyhedron on a flat surface and selecting one of the upturned faces. Create an open box by removing this face. Then flatten the box by pulling on the edges, leaving a network of points and edges on the flat plane. This network is a planar graph of the polyhedron. Graphs so obtained are called **polyhedral graphs**. The face removed in this process can be thought of as the unbounded face that surrounds the graph.

Example 15

A pentagonal prism is shown on the right.

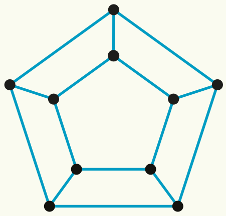
a Give a plane drawing of the graph that represents the pentagonal prism.

b Verify Euler's formula for this graph.



Solution

a



b Here $v = 10$, $e = 15$ and $f = 7$. Thus

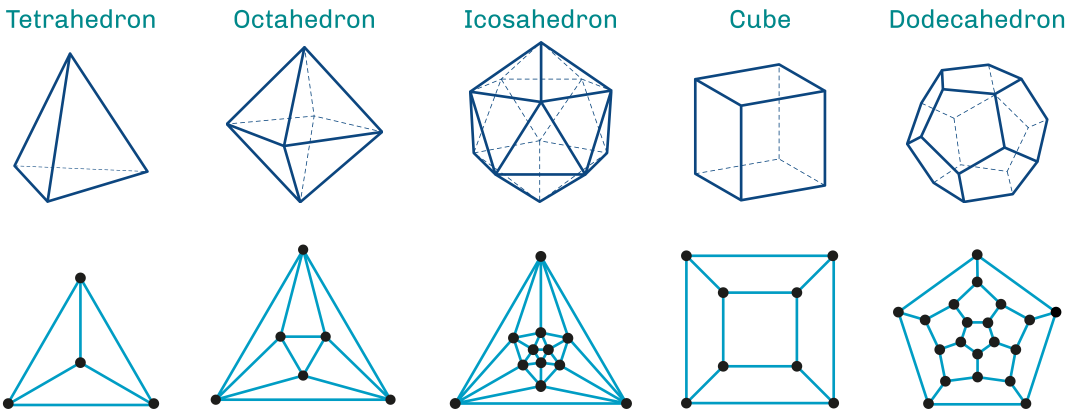
$$\begin{aligned} v - e + f &= 10 - 15 + 7 \\ &= 2 \end{aligned}$$

The Platonic solids

A **Platonic solid** is a convex polyhedron such that:

- the polygonal faces are all congruent (identical in shape and size) and regular (all angles are equal and all sides are equal), and
- the same number of faces meet at each vertex.

There are five Platonic solids. These are shown below, along with their polyhedral graphs.



We can use Euler's formula to prove that there are no other Platonic solids. You should seek out other proofs of this beautiful result.

Theorem

There are only five Platonic solids.

Proof Listed above are five Platonic solids. We will show that there are no more. Consider a Platonic solid with m edges around each face and n edges meeting at each vertex. Let v, e and f be the number of vertices, edges and faces in a planar representation of this polyhedron. Then

$$mf = 2e = nv$$

Substituting in Euler's formula gives

$$\frac{2e}{n} - e + \frac{2e}{m} = 2$$

Hence

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{e}$$

and therefore

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$$

Rearranging gives

$$\begin{aligned} \frac{2(m+n)}{mn} &> 1 \\ 2(m+n) &> mn \\ mn - 2(m+n) + 4 &< 4 \\ (m-2)(n-2) &< 4 \end{aligned}$$

Since both m and n must be at least 3, there are only five possibilities for (m, n). These are $(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$ and $(5, 3)$. Hence there are at most five Platonic solids.

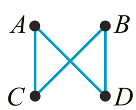
Summary 12G

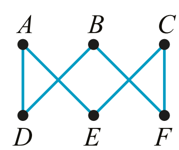
- A graph G is called a **planar graph** if it can be drawn in the plane so that its edges only intersect at their endpoints.
- Euler's formula If G is a connected planar graph with v vertices, e edges and f faces, then $v - e + f = 2$.
- Let G be a connected simple graph with v vertices and e edges, where $v \geq 3$. If the graph G is planar, then $e \leq 3v - 6$.
- Every convex polyhedron can be drawn as a connected planar graph.
- Euler's formula can be used to show that there are only five Platonic solids.

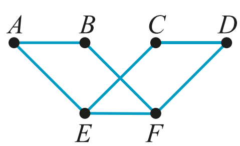
Skill-sheet

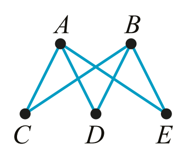
Exercise 12G

1 Show that each of the following graphs is planar by redrawing it so that the edges do not cross:

a


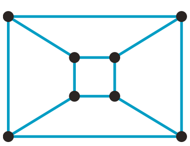
b


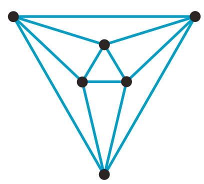
c


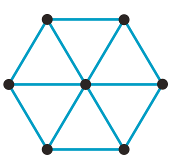
d


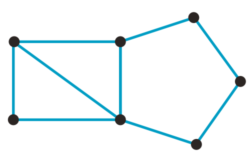
Example 13

2 Verify Euler's formula for each of the following planar graphs:

a


b


c


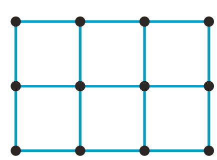
d


3 Use Euler's formula to explain why there is no connected planar graph with the following properties:

a the numbers of vertices, edges and faces are all odd

b the numbers of vertices, edges and faces are all multiples of 4.

4 The 3×4 grid graph is shown on the right.



a Verify Euler's formula for the 3×4 grid graph.

b Verify Euler's formula for the $m \times n$ grid graph.

Example 14

5 **a** Draw a simple, connected and planar graph with 4 vertices and 6 edges.

b Show that there is no simple, connected and planar graph with 4 vertices and 7 edges.

6 **a** Draw a simple, connected and planar graph with 5 vertices and 9 edges.

b Show that there is no simple, connected and planar graph with 5 vertices and 10 edges.

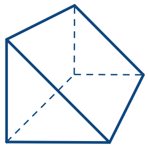
7 **a** Verify Euler's formula for the cube and the tetrahedron.

b A dodecahedron has 12 faces and 30 edges. How many vertices does it have?

c An icosahedron has 12 vertices and 20 faces. How many edges does it have?

Example 15

8 A triangular prism is shown on the right.



a Draw the polyhedral graph that represents the triangular prism.

b Verify Euler's formula for this graph.

9 By drawing a suitable diagram, show that the complete bipartite graph $K_{2,n}$ is planar for all $n \in \mathbb{N}$.

10 **a** Explain why the graph $K_{3,3}$ is not planar.

Hint: First explain why the cycle graph C_6 is a subgraph of $K_{3,3}$. Draw this subgraph as a planar graph. What goes wrong when you try to add the missing edges?

b Interpret this result in terms of the houses and utility outlets from Example 1.

c Why does part **a** imply that $K_{m,n}$ is not planar for all $m, n \geq 3$?

Hint: Use the theorem about subgraphs of planar graphs.

11 **a** Prove by contradiction that K_5 is not planar.

Hint: Use the test $e \leq 3v - 6$.

b Why does part **a** imply that K_n is not planar for all $n \geq 5$?

Hint: Use the theorem about subgraphs of planar graphs.

12 A polyhedron has v vertices. Meeting at each vertex there are three square faces and one triangular face.

a Explain why the total number of square faces is $\frac{3v}{4}$.

b Explain why the total number of triangular faces is $\frac{v}{3}$.

c Explain why the total number of edges of this polyhedron is $2v$.

d Hence, using Euler's formula, find the number of vertices, edges and faces.

e A traditional soccer ball is a polyhedron called a **truncated icosahedron**. Meeting at each of the v vertices there are two hexagonal faces and one pentagonal face. Determine the number of vertices, edges and faces of a truncated icosahedron.

12H Appendix: When every vertex has even degree

In Section 12B, we stated the condition for a connected graph to have an Euler circuit.

Theorem (Euler circuits)

A connected graph has an Euler circuit if and only if the degree of every vertex is even.

In Section 12B we proved only half of this result. We are still to prove that if the degree of every vertex is even, then the connected graph has an Euler circuit. Our proof of this result requires the following theorem.

Theorem

Let G be a graph in which every vertex has even degree. Then G can be split into cycles, no two of which have an edge in common.

Proof Begin at any non-isolated vertex and start 'moving along edges', without using the same edge twice. Every vertex has even degree, so if we arrive at a new vertex along an edge, there must be an unused edge by which we can leave the vertex. There are finitely many vertices, so at some stage we will return to a vertex that we have visited before. Therefore we have a cycle, A_1. Remove the edges of A_1 from the graph.

Removing this cycle eliminates edges in pairs at each vertex. Therefore every vertex of the new graph has even degree. Repeat this process until there are no edges left. In this way, we can split the graph into cycles $A_1, A_2, \dots, A_n$ with no edges in common.

We can now complete the proof of the theorem on Euler circuits.

Theorem

A connected graph has an Euler circuit if every vertex has even degree.

Proof Let G be a connected graph such that every vertex has even degree. By the previous theorem, the graph G can be split into cycles, no two of which have an edge in common. We want to combine these cycles to create an Euler circuit.

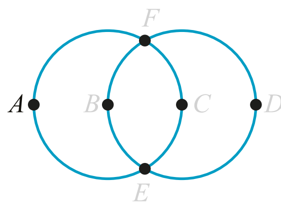
We start going around a cycle, A_1, until we arrive at the vertex of another cycle, A_2. We go all the way around cycle A_2 and then go around the rest of A_1 to return to our original starting point. We have created a circuit from cycles A_1 and A_2.

If we have not yet used all the edges, then we go around our circuit until we reach another cycle, A_3. (This works as G is connected.) We create a new circuit from the previous circuit and the cycle A_3. This process is repeated until we have a circuit that uses all the edges of the graph.

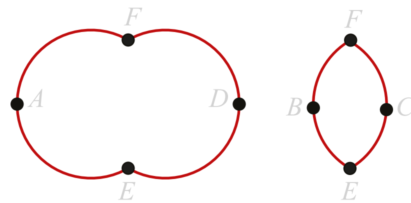
Notice that the proofs of these two theorems give an algorithm for finding an Euler circuit in a connected graph if all its vertices have even degree. The first proof describes a method for splitting the graph into cycles, and then the second proof describes a method for combining these cycles into an Euler circuit.

Exercise 12H

1 Each vertex of the graph G shown below has even degree. We have shown one way to split this graph into cycles with no edges in common.



Graph G

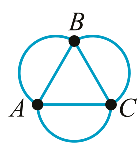


A cycle splitting for G

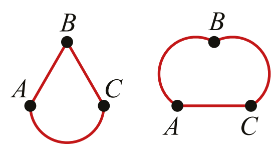
a Find an Euler circuit in G by using the given cycle splitting.

b There are two other ways to split the graph G into cycles with no edges in common. Draw them.

2 Each vertex of the graph H shown below has even degree. We have shown one way to split this graph into cycles with no edges in common.



Graph H



A cycle splitting for H

a Find an Euler circuit in H by using the given cycle splitting.

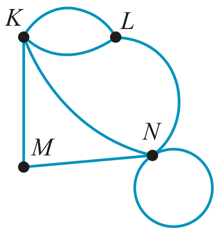
b There are four other ways to split the graph H into cycles with no edges in common. Draw them.

Chapter summary

Assignment

Nrich

- A **graph** consists of a finite non-empty set of **vertices**, a finite set of **edges** and an **edge-endpoint function** that maps each edge to a set of either one or two vertices.
- A graph can be represented by a diagram or an **adjacency matrix**.



$$\begin{array}{c c} & \begin{matrix} K & L & M & N \end{matrix} \\ \begin{matrix} K \\ L \\ M \\ N \end{matrix} & \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- The **degree** of a vertex v is the number of edges that have v as an endpoint, with each edge that is a loop counted twice.
- The sum of the degrees of all the vertices of a graph is equal to twice the number of edges.
- Two graphs are **isomorphic** if one can be obtained from the other by relabelling vertices.
- A **subgraph** is a graph whose vertices and edges are subsets of another graph.

Types of graphs

simple graph	a graph with no loops or multiple edges
connected graph	there is a walk between each pair of distinct vertices
complete graph	a simple graph with each pair of distinct vertices joined by an edge
bipartite graph	the vertex set can be divided into two disjoint subsets A and B such that each edge joins a vertex in A and a vertex in B
tree	a connected graph with no cycles
planar graph	a graph that can be drawn in the plane so that its edges do not cross

Notation for graphs

C_n	the cycle graph with n vertices
K_n	the complete graph with n vertices
$K_{m,n}$	the complete bipartite graph where A has m vertices and B has n vertices

Walks in graphs

- An **Euler circuit** is a walk that starts and ends at the same vertex and uses every edge of the graph exactly once. A connected graph has an Euler circuit if and only if every vertex has even degree.
- A **Hamiltonian cycle** is a walk that starts and ends at the same vertex and visits every other vertex exactly once (without repeating any edges).
- If $\mathbf{A}$ is the adjacency matrix of a graph G, then the number of walks of length n from vertex v_i to vertex v_j is equal to the entry of $\mathbf{A}^n$ in row i and column j.

Euler's formula

- Let G be a connected planar graph, and let v, e and f denote the number of vertices, edges and faces in a plane drawing of G. Then $v - e + f = 2$.

Technology-free questions

1 **a** Six people are seated at a round table. Draw a graph to represent each of the following situations:

  i Each person shakes hands with the two people sitting on either side.

  ii Each person shakes hands with the person sitting opposite.

  iii Each person shakes hands with exactly three other people.

  b Explain why it is not possible to draw a graph representing a group of seven people such that each person shakes hands with exactly three other people.

2 **a** Write down the definition of a simple graph.

  b Draw all non-isomorphic simple graphs with four vertices and three edges.

  c Draw all non-isomorphic simple graphs with four vertices and four edges.

  d Prove that there is no simple graph with four vertices and seven edges.

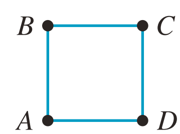
3 **a** Draw the complete graph K_6 with its vertices labelled as A, B, C, D, E, F.

  b Explain why there are 24 different Hamiltonian paths from vertex A to vertex B.

  c The subgraph of K_6 with the three vertices A, B and C and the three edges $\{A, B\}$, $\{B, C\}$ and $\{C, A\}$ is called a **triangle graph**. Show this triangle graph on your drawing of K_6 by using a different colour.

  d How many triangle subgraphs does K_6 have?

4 The cycle graph C_4 is shown on the right.



  a How many walks of length 2 are there from A to C?

  b How many walks of length 3 are there from A to C?

  c Write down the adjacency matrix $\mathbf{A}$ of C_4.

  d Without performing a calculation, explain why the entries of $\mathbf{A}^{99}$ along the main diagonal (top-left to bottom-right) are all zero.

5 **a** Draw the complete bipartite graph $K_{2,3}$.

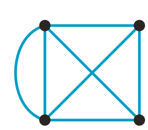
  b Draw the complement of $K_{2,3}$.

  c Show that the complement of $K_{m,n}$ has $\dfrac{m(m-1)+n(n-1)}{2}$ edges.

6 **a** Consider a connected planar graph that has twice as many edges as it has vertices and one more edge than it has faces. Determine the number of vertices, edges and faces of such a graph.

  b Draw two different (i.e. non-isomorphic) graphs that satisfy the description in part **a**.

7 **a** Show that the graph on the right is planar by drawing it in the plane without any edges crossing.



  b Verify Euler's formula for this graph.

8 **a** Draw two different connected planar graphs that have 4 vertices, 6 edges and 4 faces.

  b Give three reasons why your two graphs cannot be isomorphic.

Multiple-choice questions

1 Which one of the following statements is *false*?

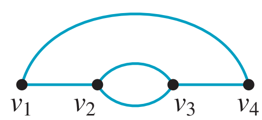
A The total degree of a graph is equal to twice the number of edges of the graph.
B A graph must have an even number of vertices with odd degree.
C A graph can have an odd number of vertices with odd degree.
D The total degree of a graph is always even.
E A graph can have an even number of vertices with even degree.

2 The adjacency matrix of a graph G is shown on the right. The graph G does not have an Euler circuit. Which one of the following edges should be added to G so that the new graph has an Euler circuit?

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A $\{v_1, v_2\}$ **B** $\{v_1, v_3\}$ **C** $\{v_1, v_4\}$
D $\{v_2, v_3\}$ **E** $\{v_2, v_4\}$

3 The graph on the right has an Euler trail that starts and ends at which one of the following pairs of vertices?

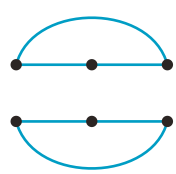


A v_1 and v_2 **B** v_1 and v_3 **C** v_1 and v_4
D v_2 and v_3 **E** v_2 and v_4

4 For which one of the following adjacency matrices does the corresponding graph *not* have an Euler circuit?

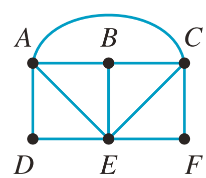
A $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

5 The complement of the graph shown on the right is



A the complete graph K_3 **B** the complete graph K_6
C the cycle graph C_3 **D** the cycle graph C_6
E the complete bipartite graph $K_{3,3}$

6 The graph G shown on the right does not have an Euler circuit. Which one of the following edges should be added to G so that the new graph has an Euler circuit?



A $\{A, D\}$ **B** $\{B, E\}$ **C** $\{C, F\}$ **D** $\{A, F\}$ **E** $\{C, D\}$

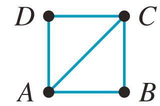
7 Which one of the following graphs has an Euler circuit?

A the complete graph K_4 **B** the complete graph K_5
C the complete graph K_6 **D** the complete bipartite graph $K_{1,3}$
E the complete bipartite graph $K_{3,3}$

8 The complete bipartite graph $K_{m,n}$ has a Hamiltonian cycle if and only if

A $m = n$ **B** $m \geq n$ **C** $m \leq n$ **D** $m < n$ **E** $m > n$

9 For the graph shown on the right, the number of walks of length 6 from vertex A to vertex A is



A 90 **B** 91 **C** 92 **D** 93 **E** 94

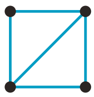
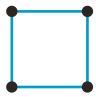
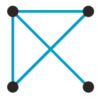
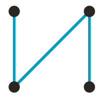
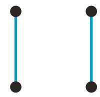
10 A simple graph G has 7 vertices and 11 edges. Its complement $\overline{G}$ has

A 10 edges **B** 11 edges **C** 12 edges **D** 22 edges **E** 42 edges

11 Which one of the following complete bipartite graphs has 10 vertices and 24 edges?

A $K_{3,8}$ **B** $K_{2,8}$ **C** $K_{2,12}$ **D** $K_{4,6}$ **E** $K_{5,5}$

12 Which one of the following graphs is a tree?

A  **B**  **C**  **D**  **E** 

13 Which one of the following statements is *false*?

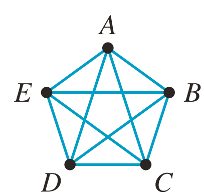
A Every tree is a connected graph. **B** No tree has a cycle.
C A tree has more edges than vertices. **D** Every tree is a bipartite graph.
E Adding an edge to a tree will always create a cycle.

14 A connected simple graph G has 6 vertices and 13 edges. We can find a spanning tree for G by deleting edges belonging to cycles. How many edges will be deleted?

A 5 **B** 6 **C** 7 **D** 8 **E** 9

Extended-response questions

1 **a** The complete graph K_5 is shown on the right. Write down the adjacency matrix $\mathbf{A}$ of this graph.



From Section 12D, we know that the entries of the matrix $\mathbf{A}^n$ give the number of walks of length n in the graph.

b The matrix $\mathbf{A}^3$ is given on the right. Without referring to this matrix, explain why there are 12 walks of length 3 in the graph K_5 that start and end at vertex A.

$$\mathbf{A}^3 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 12 & 13 & 13 & 13 & 13 \\ 13 & 12 & 13 & 13 & 13 \\ 13 & 13 & 12 & 13 & 13 \\ 13 & 13 & 13 & 12 & 13 \\ 13 & 13 & 13 & 13 & 12 \end{bmatrix} \end{array}$$

The **trace** of a square matrix $\mathbf{A}$, denoted by $\text{tr}(\mathbf{A})$, is the sum of the entries of $\mathbf{A}$ on the main diagonal (top-left to bottom-right).

c If $\mathbf{A}$ is the adjacency matrix of any simple graph, explain why:

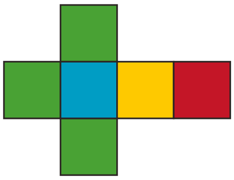
i $\text{tr}(\mathbf{A}) = 0$

ii $\text{tr}(\mathbf{A}^2) = 2 \times$ the number of edges of the graph

iii $\text{tr}(\mathbf{A}^3) = 6 \times$ the number of triangles of the graph

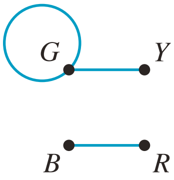
d Let $\mathbf{A}$ be the adjacency matrix of any simple graph G. Prove that if the trace of any odd power of $\mathbf{A}$ is non-zero, then the graph G is not bipartite.

2 Samira has four cubes with faces coloured red, green, blue or yellow. Cube 1 has three green faces and one each of red, blue and yellow.



- The blue face is opposite the red face.
- Two of the green faces are opposite one another.
- The other green face is opposite the yellow face.

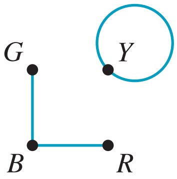
This information is illustrated by the coloured net on the right. We can also summarise the information in the graph with four vertices *R*, *G*, *B* and *Y* shown on the right.



Cube 1

a Cube 2 has a green face opposite a blue face, another green face opposite a red face and a second red face opposite a yellow face. Draw a graph to represent this information.

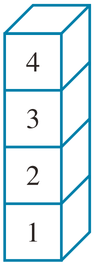
b Cube 3 is represented by the graph on the right, which indicates its opposite faces. Draw an example of a coloured net for this cube. (Note: There is more than one possibility.)



Cube 3

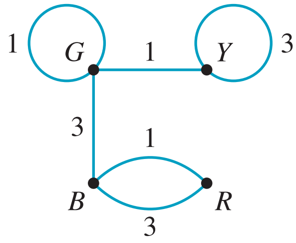
c Cube 4 has one green face, two yellow faces, one blue face and two red faces. The green face is opposite a yellow face, and the blue face is opposite a red face. Fill in the missing information and then draw a graph to represent the opposite faces of cube 4.

Samira wants to stack her four cubes into a tower as shown on the right, in such a way that all four colours are used on each of the four vertical faces of the tower. We can use a graph to help with this.



First we combine the four graphs representing the opposite faces of the four cubes into a single graph. Each edge of this graph is labelled 1, 2, 3 or 4 to indicate which cube it comes from.

d The labelled graph on the right shows cubes 1 and 3 combined together. Copy and complete this graph so that it represents all four cubes.



Cubes 1 and 3

From the graph representing all four cubes, we will find a subgraph that represents the front and back faces of the tower. Each face of the tower uses each colour once. This means that the graph representing the front and back faces must be a subgraph of the answer to part **d** with four edges labelled 1, 2, 3 and 4 and four vertices of degree 2.

e If we include the loop labelled 1 on vertex *G*, it is not possible to form a subgraph with four edges labelled 1, 2, 3 and 4 and four vertices of degree 2. Explain why.

f Suppose that we include the edge labelled 1 that joins *B* and *R*. Draw a subgraph that has the required properties.

g Using part **f**, show the two possible colourings of the front face of the tower.

Challenge: Can you find the colourings of the two side faces of the tower?

3 Consider the graph G with the following adjacency matrix:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

a Show that G is a planar graph by drawing it without any crossings.

b Verify Euler's formula for this graph.

c Explain why this graph cannot have a Hamiltonian cycle that includes the edge $\{A, B\}$.

Now consider the following algorithm for colouring the vertices of a connected graph:

Step 1 Choose any vertex. Colour this vertex red.

Step 2 Identify all the vertices that are not already coloured and that are adjacent to a red vertex. Colour each of these vertices green.

Step 3 Identify all the vertices that are not already coloured and that are adjacent to a green vertex. Colour each of these vertices red.

Step 4 Repeat from Step 2 while there are still vertices that are not coloured.

d Apply this algorithm to the graph G, starting with vertex E.

e Using your answer to part **d**, explain why the graph G is not bipartite.

f By removing one edge from the graph G, it is possible to make a bipartite graph.

i Identify which edge needs to be removed.

ii Write down the two disjoint sets of vertices that show that the new graph is bipartite.

13

Revision of Chapters 11–12

13A Technology-free questions

1 Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & -1 & -2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$.

a State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**.

b Evaluate **DA** and $\mathbf{A}^{-1}$.

2 If $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$, find:

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$

b $\mathbf{A}^2 - \mathbf{B}^2$

3 If the matrix $\begin{bmatrix} 2 & 4 \\ 8 & x \end{bmatrix}$ is non-invertible, find the value of x.

4 Find all possible matrices **A** which satisfy the equation $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.

5 If $\mathbf{A} = \begin{bmatrix} -1 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -4 \\ -1 & -6 \\ -3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, evaluate **AB** and $\mathbf{C}^{-1}$.

6 Suppose that **A**, **B**, **C** and **X** are $n \times n$ matrices and that both **A** and **B** are invertible. Denote the $n \times n$ zero matrix by **O** and the $n \times n$ identity matrix by **I**. Solve each of the following matrix equations for **X**:

a $\mathbf{B} + \mathbf{XA} = \mathbf{C}$

b $\mathbf{B}(\mathbf{X} + \mathbf{A}) = \mathbf{C}$

c $\mathbf{AX} + \mathbf{BA} = \mathbf{A}$

d $\mathbf{X} + \mathbf{A} = \mathbf{O}$

e $2\mathbf{X} - \mathbf{B} = \mathbf{O}$

f $\mathbf{AX} + \mathbf{I} = \mathbf{A}$

7 Express each of the following as a single matrix:

a $\begin{bmatrix} 3 \\ -2 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

c $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$

8 Given that $\mathbf{A} = \begin{bmatrix} w & 2w+5 \\ -1 & w+1 \end{bmatrix}$ and $\det(\mathbf{A}) = 15$, find the possible values of w.

9 Determine the value of x if

$$\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ 5 \end{bmatrix} = \begin{bmatrix} x & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

10 Determine the values of a and b if

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

11 Assume that $\mathbf{A}$ and $\mathbf{B}$ are invertible $n \times n$ matrices. Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

12 Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & a \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 \\ b & 2 \end{bmatrix}$.

a Find $\mathbf{AB}$. **b** If $\mathbf{B} = \mathbf{A}^{-1}$, determine the values of a and b.

13 Find all triples of real numbers (a, b, c) such that $\mathbf{AB} = \mathbf{O}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

14 In a group of five people, A, B, C, D and E, the following pairs of people are acquainted with each other:

- A and C
- A and D
- B and C
- C and D
- C and E

a Draw a graph G to represent this situation.

b Construct the adjacency matrix of G.

15 A tree has six vertices, with exactly three vertices of degree 1. Determine the number of vertices of degree 2 and the number of vertices of degree 3.

16 Draw three non-isomorphic trees such that each tree has seven vertices, with exactly three vertices of degree 1.

17 Let G be a graph such that every vertex has degree 4 and there are exactly 12 edges. How many vertices does the graph G have?

18 **a** Construct the adjacency matrix for the complete graph K_4.

b Determine the number of walks of length 3 between two different vertices of K_4.

19 Let n be a natural number with $n \geq 3$.

a Prove that the complete graph K_n has a Hamiltonian cycle.

b Determine the number of Hamiltonian cycles in K_n.

20 Let G_1 and G_2 be two planar graphs each with v vertices, e edges and f faces. Must these two graphs be isomorphic?

21 There are nine line segments drawn in a plane. Is it possible that each line segment intersects with exactly three others?

22 Give a plane drawing of the complete bipartite graph $K_{2,3}$.

23 A connected planar graph G has six vertices, each of degree 4. How many faces are there in a plane drawing of G?

24 Let G be a simple, connected and planar graph such that the degree of every vertex is at least 5. Show that the graph G must have at least 12 vertices.

25 **a** Draw the graph $\overline{C_5}$ with its vertices in a circle, and label the vertices clockwise as $v_1, v_2, \dots, v_5$. Write down a Hamiltonian cycle in $\overline{C_5}$ starting at v_1.

b Draw the graph $\overline{C_6}$ with its vertices in a circle, and label the vertices clockwise as $v_1, v_2, \dots, v_6$. Write down a Hamiltonian cycle in $\overline{C_6}$ starting at v_1.

c Prove that the graph $\overline{C_n}$ has a Hamiltonian cycle, for all $n \geq 5$.

26 **a** Draw a simple, connected and planar graph with 4 vertices and $2 \times 4 - 4 = 4$ faces.

b Draw a simple, connected and planar graph with 5 vertices and $2 \times 5 - 4 = 6$ faces.

c Now let G be a simple, connected and planar graph with n vertices, where $n \geq 3$.

i Show that the graph G has at most $2n - 4$ faces.

ii Show that it is possible to add edges to G so as to obtain a planar simple graph with exactly $2n - 4$ faces.

27 Consider a polyhedral graph such that every vertex has degree 3 and every face has exactly five edges. Determine the number of vertices, edges and faces of this graph.

28 **a** Draw the complete bipartite graph $K_{3,4}$.

b Explain why this graph does not have an Euler trail.

c What is the smallest number of edges that can be added to the graph $K_{3,4}$ so that the resulting graph has an Euler trail?

d What is the smallest number of edges that can be added to the graph $K_{3,4}$ so that the resulting graph has an Euler circuit?

13B Multiple-choice questions

1 If $\mathbf{P}^2 = 4\mathbf{I}$, then $\mathbf{P}^{-1}$ equals

A $\frac{1}{4}\mathbf{P}$ **B** $\frac{1}{2}\mathbf{P}$ **C** $\frac{1}{2}\mathbf{I}$ **D** $2\mathbf{P}$ **E** $4\mathbf{P}$

2 If $\mathbf{R} = \begin{bmatrix} 5 & 3 & 1 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, then $\mathbf{RS}$ is

A undefined **B** $\begin{bmatrix} -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 \\ -5 & -3 & -1 \\ 10 & 6 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -3 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$

3 If $\mathbf{A} = \begin{bmatrix} 9 & 8 \\ -11 & 5 \end{bmatrix}$, then det(**A**) equals

A -43 **B** $-\frac{1}{43}$ **C** $\frac{1}{333}$ **D** 17 **E** 133

4 If $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 6 & 4 \end{bmatrix}$, then **BA** has size

A 1×1 **B** 3×1 **C** 1×3 **D** 3×3 **E** 3×2

5 Let $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$. If $\mathbf{AX} + \mathbf{B} = \mathbf{C}$, then **X** equals

A $\frac{1}{20}\begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$ **B** $\begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$

D $\begin{bmatrix} 3 & -10 \\ -4 & 10 \end{bmatrix}$ **E** $\frac{1}{20}\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$

6 Let $\mathbf{P} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$.

If $\mathbf{X} = \mathbf{PQR}$, then the number of zero entries of **X** is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

7 If $\mathbf{X} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$, then $\mathbf{X}^{-1}$ is

A $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} \frac{1}{3} & \frac{1}{5} \\ -1 & -\frac{1}{2} \end{bmatrix}$ **D** $\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & -1 \\ -5 & -2 \end{bmatrix}$

8 The determinant of the matrix $\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$ is

A 16 **B** 4 **C** -16 **D** $\frac{1}{4}$ **E** -4

9 If $\mathbf{S} = \begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$, then $\mathbf{S}^{-1}$ is

A $-\begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & -7 \\ -2 & 5 \end{bmatrix}$ **C** $-\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$

D $\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$ **E** $\frac{1}{4}\begin{bmatrix} -2 & -7 \\ -2 & -5 \end{bmatrix}$

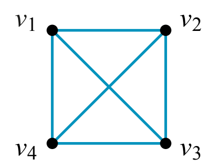
10 A connected planar graph has 8 vertices and 13 edges. The number of faces is

A 5 **B** 6 **C** 7 **D** 8 **E** 10

11 A tree with 12 vertices has

A 8 edges **B** 9 edges **C** 10 edges **D** 11 edges **E** 12 edges

12 The graph G shown on the right does not have an Euler circuit. However, by adding one or more edges to G we can produce a new graph that does have an Euler circuit. What is the minimum number of edges that we need to add?

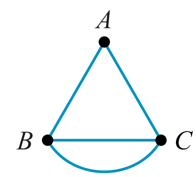


A 1 **B** 2 **C** 3
D 4 **E** 5

13 A connected planar graph G has 17 faces (including the unbounded face). Half the vertices of G have degree 4, and half have degree 5. How many vertices does G have?

A 6 **B** 12 **C** 24 **D** 36 **E** 48

14 For the graph shown, the number of walks of length 3 from vertex A to vertex B is

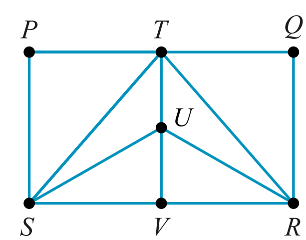


A 0 **B** 2 **C** 4
D 6 **E** 8

15 A simple graph G has 10 vertices and 24 edges. Its complement $\overline{G}$ has

A 14 edges **B** 17 edges **C** 30 edges **D** 21 edges **E** 45 edges

16 For the graph shown opposite, which of the following is a Hamiltonian cycle?



A T, P, S, V, R, U, Q, T **B** $T, Q, R, T, P, S, V, U, T$
C R, V, S, P, T, Q, R, U **D** U, V, S, P, T, Q, R
E U, V, S, P, T, Q, R, U

17 The number of edges of the graph $\overline{C_n}$ is

A $\dfrac{n^2 - 3n}{2}$ **B** n **C** $\dfrac{n^2 + n}{2}$ **D** $\dfrac{n^2 + 3n}{2}$ **E** n^2

18 The adjacency matrix of a graph with six vertices is shown on the right. This is the adjacency matrix of the graph

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A K_6 **B** C_6 **C** $K_{3,3}$
D $\overline{C_6}$ **E** $K_{4,2}$

Revision

19 The adjacency matrix of the graph $\overline{C_5}$ is shown on the right. In this graph, the number of walks of length 4 from a given vertex back to itself is

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

A 2 **B** 3 **C** 4 **D** 5 **E** 6

13C Extended-response questions

1 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $b \neq 0$ and $c \neq 0$.

a **i** Find $\mathbf{A}^2$. **ii** Find $3\mathbf{A}$.

b If $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$, show that:

i $a + d = 3$ **ii** $\det(\mathbf{A}) = 1$

c Assume that **A** has the properties:

- $a + d = 3$
- $\det(\mathbf{A}) = 1$

Show that $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$.

2 The **trace** of a square matrix **A** is defined to be the sum of the entries along the main diagonal of **A** (from top-left to bottom-right) and is denoted by tr(**A**).

For example, if $\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 2 & 2 \end{bmatrix}$, then $\text{tr}(\mathbf{A}) = 6 + 2 = 8$.

a Prove each of the following for all 2×2 matrices **X** and **Y**:

i $\text{tr}(\mathbf{X} + \mathbf{Y}) = \text{tr}(\mathbf{X}) + \text{tr}(\mathbf{Y})$

ii $\text{tr}(-\mathbf{X}) = -\text{tr}(\mathbf{X})$

iii $\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$

b Use the results of **a** to show that $\mathbf{XY} - \mathbf{YX} \neq \mathbf{I}$ for all 2×2 matrices **X** and **Y**.

3 A square matrix **A** is said to be **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.

a Show that each of the following 2×2 matrices is idempotent:

i $\begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$ **iv** $\frac{1}{2}\begin{bmatrix} 1 - \cos\theta & \sin\theta \\ \sin\theta & 1 + \cos\theta \end{bmatrix}$

b Show that each of the following 3×3 matrices is idempotent:

i $\frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$ **ii** $\frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

c Show that a product of two idempotent matrices is not necessarily idempotent.

d Show that for every idempotent matrix **A**, either $\det(\mathbf{A}) = 0$ or $\det(\mathbf{A}) = 1$.

Note: You can use the result that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

e Show that if an idempotent matrix **A** has an inverse, then $\mathbf{A} = \mathbf{I}$.

f Show that for every idempotent matrix **A**, the matrix $\mathbf{I} - \mathbf{A}$ is also idempotent.

g Describe all 2×2 idempotent matrices.

4 A square matrix $\mathbf{A}$ is said to be **involutory** if $\mathbf{A}^2 = \mathbf{I}$, that is, if it is its own inverse.

a Show that each of the following matrices is involutory:

i $\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$ **ii** $\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix}$

b Show that for every involutory matrix $\mathbf{A}$, either $\det(\mathbf{A}) = 1$ or $\det(\mathbf{A}) = -1$.

c Describe all 2×2 involutory matrices.

d Let $\mathbf{A}$ be a square matrix. Prove that $\mathbf{A}$ is involutory if and only if the matrix $\frac{1}{2}(\mathbf{A} + \mathbf{I})$ is idempotent. (See Question 3 for the definition of idempotent.)

5 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Consider the matrix $\mathbf{A} - m\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.

a Find $\det(\mathbf{A} - m\mathbf{I})$, writing your answer as a quadratic polynomial in m.

b If $m = \lambda_1$ and $m = \lambda_2$ are the solutions of the quadratic equation $\det(\mathbf{A} - m\mathbf{I}) = 0$, show that $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = \det(\mathbf{A})$.

c Suppose that $a + b = c + d = 1$. Show that $m = 1$ is a solution of the quadratic equation, and find the other solution in terms of a and c.

d **i** Suppose that $\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}$. Solve the equation $\det(\mathbf{A} - m\mathbf{I}) = 0$ for m.

ii The equation $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ has infinitely many solutions. Describe them.

iii If $m = 1$ and $m = \lambda_2$ are the two solutions from part **i**, describe the solutions of the equation $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_2\begin{bmatrix} x \\ y \end{bmatrix}$.

e Now consider examples of matrices $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + b = c + d = 6$.

More generally, consider examples with $a + b = c + d = k$, where k is an integer.

6 **a** Can you draw a graph with exactly five vertices and vertex degrees 1, 2, 3, 4 and 5? Give reasons.

b A connected graph has exactly six vertices and vertex degrees 2, 2, 2, 2, 4 and 6.

i Explain why the graph is not simple.

ii Determine if the graph has an Euler circuit. Give reasons.

c A connected simple graph has exactly six vertices.

i State the minimum possible value of the sum of the vertex degrees of the graph.

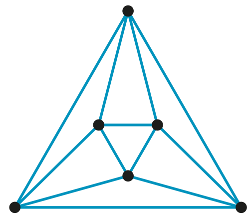
ii If the graph has an Euler circuit, what are the possible values of the degree of a vertex of this graph?

d A connected simple graph with six vertices and 10 edges has an Euler circuit.

i What can you deduce about the vertex degrees of this graph?

ii Draw a graph that satisfies these properties.

7 For the planar graph shown on the right, each of the eight faces is triangular (including the unbounded face). Given any planar graph G, we let v, e and f denote the number of vertices, edges and faces in a plane drawing of G.



a For any planar graph with only triangular faces, explain why $2e = 3f$.

b Hence, prove that if a connected planar graph has only triangular faces, then $e = 3v - 6$.

You will now try to prove that the converse of the result in part **b** is true for simple graphs. Consider any simple, connected and planar graph.

c Show that if $e = 3v - 6$, then $2e = 3f$.

d Hence, prove that if $e = 3v - 6$, then the graph has only triangular faces.

e Deduce that a convex polyhedron with 12 vertices and 20 faces is composed entirely of triangles.

13D Investigations

1 Fibonacci-style sequences

The Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ is defined by the recurrence relation

$$f_{n+2} = f_n + f_{n+1} \quad \text{and} \quad f_1 = f_2 = 1$$

a Let $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Find $\mathbf{Q}^2$, $\mathbf{Q}^3$ and $\mathbf{Q}^4$. Deduce the entries of $\mathbf{Q}^n$, for $n \geq 2$. Prove your claim using mathematical induction.

b Find $\det(\mathbf{Q})$, $\det(\mathbf{Q}^2)$, $\det(\mathbf{Q}^3)$ and $\det(\mathbf{Q}^4)$. Use these to deduce the result

$$f_{n+1}f_{n-1} - (f_n)^2 = (-1)^n$$

Note: You can use the result that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

c From the equation $\mathbf{Q}^{n+1}\mathbf{Q}^n = \mathbf{Q}^{2n+1}$, prove that $(f_{n+1})^2 + (f_n)^2 = f_{2n+1}$.

d From the equation $\mathbf{Q}^m\mathbf{Q}^{n-1} = \mathbf{Q}^{m+n-1}$, prove that $f_{m+n} = f_{m+1}f_n + f_m f_{n-1}$.

e Solve the equation $\det(\mathbf{Q} - x\mathbf{I}) = 0$ for x.

f Now start with any two positive integers and generate a sequence of 10 numbers by adding in the 'Fibonacci manner'. For example: 5, 1, 6, 7, 13, 20, 33, 53, 86, 139. Find the relationship between the sum of these numbers and the 7th number. State the result for a sequence of 10 terms generated in this way and prove your result.

g The sequence of Lucas numbers is $2, 1, 3, 4, 7, 11, 18, \ldots$. Investigate identities satisfied by the Lucas numbers and the relationship between Lucas numbers and Fibonacci numbers.

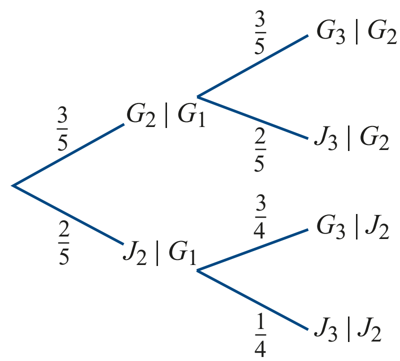
2 Transition matrices

Olivia drinks either green tea or jasmine tea every day. If she drinks green tea one day, then she drinks jasmine tea the next day with probability $\frac{2}{5}$. If she drinks jasmine tea one day, she drinks green tea the next day with probability $\frac{3}{4}$.

a Olivia drinks green tea on Monday (day 1).

Let G_n be the event 'Green tea on day n'.
Let J_n be the event 'Jasmine tea on day n'.

Using the tree diagram, find $\Pr(G_3)$. That is, find the probability that Olivia will drink green tea on Wednesday.



We can represent the probabilities in this example using a **transition matrix**, $\mathbf{T}$, and a sequence of **state vectors**, $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \dots$, defined as follows:

$$\mathbf{T} = \begin{bmatrix} \Pr(G_n \mid G_{n-1}) & \Pr(G_n \mid J_{n-1}) \\ \Pr(J_n \mid G_{n-1}) & \Pr(J_n \mid J_{n-1}) \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad \mathbf{S}_n = \begin{bmatrix} \Pr(G_n) \\ \Pr(J_n) \end{bmatrix}$$

b **i** Explain why $\mathbf{S}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

ii Explain why $\mathbf{S}_n = \mathbf{T}\mathbf{S}_{n-1}$ for all $n \geq 2$.

c Use part **b** to find $\mathbf{S}_2$ and $\mathbf{S}_3$. Check against your answer to part **a**.

d Explain why $\mathbf{S}_n = \mathbf{T}^{n-1}\mathbf{S}_1$ for all $n \geq 2$.

e Use your calculator to find $\mathbf{S}_{20}$. Hence find the probability that Olivia will drink green tea on day 20.

f Use your calculator to find $\mathbf{S}_{200}$. Hence find the probability that Olivia will drink green tea on day 200.

g Find $\mathbf{S} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that $a + b = 1$ and $\mathbf{S} = \mathbf{T}\mathbf{S}$. Compare with your answer for part **f**.

Now consider another example. A computer system operates in two different modes. Every hour, it remains in the same mode or switches to the other mode, according to the following transition matrix:

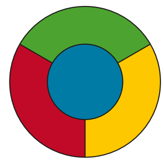
$$\mathbf{T} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

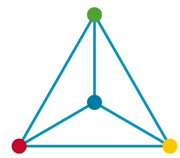
h If the system is in mode 1 at 5:30 p.m., what is the probability that it will be in mode 1 at 8:30 p.m. on the same day?

i Construct similar examples based on switching between two alternatives. Investigate what happens to the state vector $\mathbf{S}_n$ as n gets larger and larger.

3 Graph colourings

How many different colours are needed to colour the regions on a map in such a way that no two adjacent regions have the same colour? The figure below shows a typical arrangement of coloured regions. Each region can be represented by a vertex. If two regions share a border, then we represent this by drawing an edge between their corresponding vertices. As the edges cannot cross each other, we obtain a planar graph.

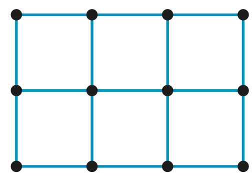




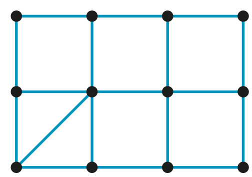
A graph is said to be ***n*-colourable** if it is possible to assign one of n colours to each vertex in such a way that no two adjacent vertices have the same colour.

a Explain why the graph shown above is not 3-colourable.

b Some planar graphs require fewer than four colours. Explain why every $m \times n$ grid graph is 2-colourable.



c What feature of the graph shown on the right means that it is not 2-colourable? Show that it is 3-colourable.



d Show that if a graph is 2-colourable, then every cycle in the graph has even length.

The **four colour theorem** asserts that every planar simple graph is 4-colourable (and therefore so is every map, no matter how large or complex). This theorem is notoriously difficult to prove. Instead, your aim is to prove a simpler result: Every planar simple graph is 6-colourable.

e In Section 12G, we proved that every simple, connected and planar graph with v vertices ($v \geq 3$) and e edges satisfies the inequality $e \leq 3v - 6$. Using this fact, prove that every planar simple graph has at least one vertex of degree less than or equal to 5.

Hint: Assume that the degree of every vertex is greater than or equal to 6, and then use the handshaking lemma.

f Using mathematical induction, prove that every planar simple graph with n vertices is 6-colourable, for all $n \in \mathbb{N}$.

Hint: At the inductive step, delete a vertex of degree less than or equal to 5.

14

Simulation, sampling and sampling distributions

Objectives

- ▶ To introduce the **mean**, **variance** and **standard deviation** of a random variable.
- ▶ To investigate the distribution of a **sum** of independent random variables, and the distribution of a **multiple** of a random variable.
- ▶ To understand the difference between a **population** and a **sample**.
- ▶ To understand **random samples** and how they may be obtained.
- ▶ To define **population parameters** and **sample statistics**.
- ▶ To introduce the concept of sample statistics as random variables which can be described by **sampling distributions**.
- ▶ To investigate the sampling distribution of the **sample mean**.
- ▶ To investigate the effect of sample size on a sampling distribution.
- ▶ To use **simulation** to generate random samples.
- ▶ To introduce the concept of the sample mean as an **estimate** of the population mean.

Statistics is concerned with the collection and analysis of data. For example, you may be interested in the amount of pocket money that Year 11 students receive each week. You could collect this information from all the Year 11 students at your school, and then analyse the data using statistical techniques such as histograms, dotplots, means and standard deviations.

Probability is concerned with the likelihood that a particular outcome may occur, and assigning a numerical value (between 0 and 1) to that likelihood. For example, if an urn contains six black balls and five white balls, then you could determine the probability of obtaining three black balls in a sample of five balls drawn from the urn.

In this chapter we link these two topics in a new area of study called **statistical inference**, which you will study further in Year 12.

14A Expected value and variance for discrete random variables

In this section, we start by reviewing discrete random variables from Mathematical Methods Units 1 & 2. We then introduce expected value and variance for discrete random variables.

Random variables

Consider the sample space obtained when a coin is tossed three times:

$$\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Suppose we are particularly interested in the number of heads associated with each outcome. We let X represent the number of heads observed when a coin is tossed three times. Then each outcome in the sample space can be associated with a value of X. The possible values of X are 0, 1, 2 and 3. Since the actual value that X will take is the result of a random experiment, we call X a random variable.

A **random variable** is a function that assigns a number to each outcome in the sample space ε.

A random variable can be discrete or continuous:

- A **discrete random variable** is one which can take only a countable number of distinct values, such as 0, 1, 2, 3, 4. Discrete random variables are usually (but not necessarily) generated by counting. The number of children in a family, the number of brown eggs in a carton of a dozen eggs, and the number times we roll a die before we observe a 'six' are all examples of discrete random variables.
- A **continuous random variable** is one which can take any value in an interval of the real number line, and is usually (but not always) generated by measuring. Height, weight, and the time taken to complete a puzzle are all examples of continuous random variables.

Discrete probability distributions

Because the values of a random variable are associated with outcomes in the sample space, we can determine the probability of each value of the random variable occurring.

Let's look again at the results obtained when a coin is tossed three times. Assuming that the coin is fair, we can summarise the probability distribution associated with the number of heads, X, observed when a fair coin is tossed three times in a table as follows.

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Note that, since every possible value of the random variable is included, the probabilities must add to 1.

The **probability distribution** of a discrete random variable X is a function

$$p(x) = \Pr(X = x)$$

that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

For any discrete probability distribution, the following two conditions must hold:

1 Each value of $p(x)$ belongs to the interval $[0, 1]$. That is, $0 \leq p(x) \leq 1$ for all x.

2 The sum of all the values of $p(x)$ is 1.

Example 1

Consider the probability distribution:

x	1	2	3	4	5	6
$\Pr(X = x)$	0.2	0.3	0.1	0.2	0.15	0.05

Use the table to find:

a $\Pr(X = 3)$ **b** $\Pr(X < 3)$ **c** $\Pr(X \geq 4)$

Solution

a $\Pr(X = 3) = 0.1$

b $\Pr(X < 3) = 0.2 + 0.3$
$= 0.5$

c $\Pr(X \geq 4) = 0.2 + 0.15 + 0.05$
$= 0.4$

Explanation

If X is less than 3, then from the table we see that X can take the value 1 or 2.

If X is greater than or equal to 4, then X can take the value 4, 5 or 6.

Example 2

Consider the function:

x	1	2	3	4	5
$\Pr(X = x)$	$2c$	$3c$	$4c$	$5c$	$6c$

For what value of c is this a probability distribution?

Solution

$$2c + 3c + 4c + 5c + 6c = 1$$
$$20c = 1$$
$$c = \frac{1}{20}$$

Explanation

For a probability distribution, we require that the probabilities add to 1.

Measures of centre and spread

From your studies of statistics, you may already be familiar with the mean as a measure of centre and with the variance and the standard deviation as measures of spread. When they are calculated from a set of data, they are called **sample statistics**. It is also possible to use the probability distribution to determine theoretically the values of the mean, variance and standard deviation. When they are calculated from the probability distribution, they are called **population parameters**.

A measure of centre: expected value

When the mean of a random variable is determined from the probability distribution, it is generally called the **expected value** of the random variable.

Expected value

The **expected value** of a discrete random variable X is determined by summing the products of each value of X and the probability that X takes that value.

We can write this in symbols as

$$\begin{aligned} \mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x) \\ &= \sum_x x \cdot p(x) \end{aligned}$$

where the notation $\sum_x$ means 'sum over all possible values of x'.

The expected value $\mathrm{E}(X)$ may be considered as the long-run average value of X. It is denoted by the Greek letter μ (*mu*), and is also called the **mean** of X.

Suppose that we are interested in the number, X, observed when a fair die is rolled. The probability distribution of X is given in the following table.

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

We can easily use this table to find the mean of X. This is demonstrated in the next example.

Example 3

Let X be the number observed when a fair die is rolled. Find the expected value of X.

Solution

$$\begin{aligned} \mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x) \\ &= \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) \\ &= \frac{21}{6} = 3.5 \end{aligned}$$

The expected value in the previous example means that, if the die was rolled many times, then in the long run an average of 3.5 would be observed. Note that you cannot actually obtain a value of 3.5 from a single roll of the die. This is the long-run average value.

Example 4

Suppose that the amount, $\$X$, that you win when you play a certain game of chance has the following probability distribution.

x	-5	2	20
$\Pr(X = x)$	0.7	0.2	0.1

Find the expected value of the amount that you win in this game.

Solution

$$\begin{aligned} E(X) &= \sum_x x \cdot \Pr(X = x) \\ &= (-5 \times 0.7) + (2 \times 0.2) + (20 \times 0.1) \\ &= -1.10 \end{aligned}$$

In the long run, you would expect to lose an average of \$1.10 per game.

Measures of spread: variance and standard deviation

As well as knowing the long-run average value of a random variable (the mean), it is also useful to have a measure of how close the possible values of the random variable are to the mean – that is, a measure of the spread of the probability distribution. The most useful measures of spread for a discrete random variable are the variance and the standard deviation.

Variance

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as

$$\text{Var}(X) = E[(X - \mu)^2]$$

and may be considered as the long-run average value of the square of the distance from X to μ. The variance is denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

From the definition,

$$\begin{aligned} \text{Var}(X) &= E[(X - \mu)^2] \\ &= \sum_x (x - \mu)^2 \cdot \Pr(X = x) \end{aligned}$$

Since the variance is determined by squaring the distance from X to μ, it is no longer in the units of measurement of the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

Standard deviation

The **standard deviation** of X is defined as

$$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

The standard deviation is denoted by σ.

Using the definition is not always the easiest way to calculate the variance. Instead we can use the following formula, which will be proved in Mathematical Methods Units 3 & 4.

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

Example 5

Let X be the number observed when a fair die is rolled. Find the variance and standard deviation of X.

Solution

We will use the formula

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

From Example 3, we know that $\mathrm{E}(X) = 3.5$. We have

$$\begin{aligned}\mathrm{E}(X^2) &= \sum_x x^2 \cdot \Pr(X = x) \\ &= \left(1^2 \times \frac{1}{6}\right) + \left(2^2 \times \frac{1}{6}\right) + \left(3^2 \times \frac{1}{6}\right) + \left(4^2 \times \frac{1}{6}\right) + \left(5^2 \times \frac{1}{6}\right) + \left(6^2 \times \frac{1}{6}\right) \\ &= \frac{91}{6}\end{aligned}$$

Hence

$$\mathrm{Var}(X) = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$

and therefore

$$\mathrm{sd}(X) = \sqrt{\frac{35}{12}} \approx 1.7078$$

Example 6

Consider again the game of chance from Example 4. The amount, $\$X$, that you win in this game has the following probability distribution.

x	-5	2	20
$\Pr(X = x)$	0.7	0.2	0.1

Find the variance and standard deviation of the amount that you win in this game.

Solution

We know from Example 4 that $E(X) = -1.10$.

Now $E(X^2) = (-5)^2 \times 0.7 + 2^2 \times 0.2 + 20^2 \times 0.1$
$= 58.3$

Hence $\text{Var}(X) = E(X^2) - [E(X)]^2$
$= 58.3 - (-1.10)^2$
$= 57.09$

and $\text{sd}(X) = \sqrt{57.09}$
≈ 7.5558

Summary 14A

- A **random variable** associates a number with each outcome of a random experiment. A **discrete** random variable is one which can take only a countable number of values.
- The **probability distribution** of a discrete random variable X is a function

$$p(x) = \Pr(X = x)$$

that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.
- For any discrete probability distribution, the following two conditions must hold:
  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is, $0 \le p(x) \le 1$ for all x.
  2. The sum of all the values of $p(x)$ is 1.
- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\mu = E(X) = \sum_x x \cdot \Pr(X = x) = \sum_x x \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \text{Var}(X) = E[(X - \mu)^2]$$

An alternative (computational) formula for variance is

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \text{sd}(X) = \sqrt{\text{Var}(X)}$$

Exercise 14A

Example 1
Example 2

1 Consider the following function:

x	1	2	3	4	5
$\Pr(X = x)$	k	$2k$	$3k$	$4k$	$5k$

a For what value of k is this a probability distribution?

b Find $\Pr(X \geq 3)$.

Example 3
Example 4

2 For each of the following probability distributions, find the mean (expected value):

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.3	0.3

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0	1	2	3	4
$p(x)$	0.18	0.22	0.26	0.21	0.13

d

x	−3	−2	−1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

3 A business consultant evaluates a proposed venture as follows. A company stands to make a profit of \$20 000 with probability 0.1, to make a profit of \$10 000 with probability 0.5, to break even with probability 0.3, and to lose \$10 000 with probability 0.1. Find the company's expected profit.

4 A spinner is numbered from 0 to 5, and each of the six numbers has an equal chance of coming up. A player who bets \$2 on any number wins \$10 if that number comes up; otherwise the \$2 is lost. What is the player's expected profit on the game?

Example 5
Example 6

5 For each of the following probability distributions, find the variance of X:

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.3	0.3

b

x	0	1	2	3
$p(x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

6 A discrete random variable X takes values 0, 1, 2, 3, 4 with probabilities as shown in the table.

x	0	1	2	3	4
$\Pr(X = x)$	p	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

a Find p. **b** Find $E(X)$. **c** Find $\text{Var}(X)$. **d** Find $\text{sd}(X)$.

7 A random variable X has the probability distribution shown on the right.

x	0	1	2	3
$\Pr(X = x)$	$4k$	$3k$	$2k$	k

a Find k. **b** Find $E(X)$. **c** Find $Var(X)$. **d** Find $sd(X)$.

14B Distribution of sums of random variables

In this section, we will consider the probability distribution of the sum of independent identical discrete random variables, and how to determine the expected value and variance of that sum. We will compare this with the distribution of a discrete random variable which has been multiplied by a positive real number.

Sum of two independent random variables

Suppose that we are interested in the sum of the two numbers observed when two fair dice are rolled. Let X_1 represent the number on the first die, and let X_2 represent the number on the second die. We can construct the following table to determine the possible values of the sum $X_1 + X_2$.

X_1 \ X_2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

From this table, we see that the sum $X_1 + X_2$ can take the values $2, 3, 4, \dots, 11, 12$.

There are 36 equally likely outcomes for the pair of numbers on the dice. So we can determine the probability of a value of $X_1 + X_2$ by counting the number of associated pairs. For example:

$$\Pr(X_1 + X_2 = 2) = \Pr(X_1 = 1, X_2 = 1) = \frac{1}{36}$$

$$\Pr(X_1 + X_2 = 3) = \Pr(X_1 = 1, X_2 = 2) + \Pr(X_1 = 2, X_2 = 1) = \frac{2}{36}$$

Continuing in this way, we can obtain the probability distribution of the sum $X_1 + X_2$.

z	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X_1 + X_2 = z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Because the outcome from rolling one die is not affected by the outcome from rolling the other die, we say that X_1 and X_2 are **independent random variables**. This means that we can determine the probability of a pair of values of X_1 and X_2 by multiplication:

$$\Pr(X_1 = a, X_2 = b) = \Pr(X_1 = a) \times \Pr(X_2 = b)$$

For example:

$$\begin{aligned}\Pr(X_1 + X_2 = 3) &= \Pr(X_1 = 1, X_2 = 2) + \Pr(X_1 = 2, X_2 = 1)\\ &= \Pr(X_1 = 1) \times \Pr(X_2 = 2) + \Pr(X_1 = 2) \times \Pr(X_2 = 1)\\ &= \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{2}{36}\end{aligned}$$

Example 7

Suppose that the amount, $\$X$, that you win when you play a certain game of chance has the following probability distribution.

x	-5	2	20
$\Pr(X = x)$	0.7	0.2	0.1

Determine the probability distribution of the total amount that you win when you play the game twice.

Solution

Let $\$X_1$ be the amount won on the first game, and let $\$X_2$ be the amount won on the second game. We can use a table to determine the possible values of $X_1 + X_2$.

		X_2: -5	2	20
	-5	-10	-3	15
X_1	2	-3	4	22
	20	15	22	40

The sum $X_1 + X_2$ can take the values -10, -3, 4, 15, 22 and 40.

In this example, the outcomes are not equally likely, so we need to use the probability distribution of X. Since X_1 and X_2 are independent, we have

$$\begin{aligned}\Pr(X_1 + X_2 = -10) &= \Pr(X_1 = -5, X_2 = -5)\\ &= \Pr(X_1 = -5) \times \Pr(X_2 = -5)\\ &= 0.7 \times 0.7 = 0.49\end{aligned}$$

$$\begin{aligned}\Pr(X_1 + X_2 = -3) &= \Pr(X_1 = -5, X_2 = 2) + \Pr(X_1 = 2, X_2 = -5)\\ &= 0.7 \times 0.2 + 0.2 \times 0.7 = 0.28\end{aligned}$$

$$\Pr(X_1 + X_2 = 4) = \Pr(X_1 = 2, X_2 = 2)$$
$$= 0.2 \times 0.2 = 0.04$$

Continuing in this way, we can obtain the probability distribution of the sum $X_1 + X_2$.

z	−10	−3	4	15	22	40
$\Pr(X_1 + X_2 = z)$	0.49	0.28	0.04	0.14	0.04	0.01

The mean of $X_1 + X_2$

We will continue with our example of the sum, $X_1 + X_2$, of the two numbers observed when two fair dice are rolled. We have already determined the following probability distribution of $X_1 + X_2$.

z	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X_1 + X_2 = z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

We can use this probability distribution to find the mean of $X_1 + X_2$ as follows:

$$\mathrm{E}(X_1 + X_2) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + \cdots + 11 \times \frac{2}{36} + 12 \times \frac{1}{36} = 7$$

How does this value relate to the means of X_1 and X_2?

From Example 3, we know that $\mathrm{E}(X_1) = \mathrm{E}(X_2) = 3.5$. So for this example, the mean of the sum is equal to the sum of the means:

$$\mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2)$$

The variance of $X_1 + X_2$

Now we use the probability distribution of $X_1 + X_2$ to calculate

$$\mathrm{E}[(X_1 + X_2)^2] = 2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + 4^2 \times \frac{3}{36} + \cdots + 11^2 \times \frac{2}{36} + 12^2 \times \frac{1}{36} = \frac{329}{6}$$

Therefore

$$\mathrm{Var}(X_1 + X_2) = \mathrm{E}[(X_1 + X_2)^2] - [\mathrm{E}(X_1 + X_2)]^2$$
$$= \frac{329}{6} - 7^2 = \frac{35}{6}$$

How does this value relate to the variances of X_1 and X_2?

From Example 5, we know that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \frac{35}{12}$. So for this example, the variance of the sum is equal to the sum of the variances:

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$$

The standard deviation of $X_1 + X_2$

From the variance of $X_1 + X_2$, we find

$$\mathrm{sd}(X_1 + X_2) = \sqrt{\mathrm{Var}(X_1 + X_2)} = \sqrt{\frac{35}{6}} \approx 2.4152$$

By Example 5, we have $\mathrm{sd}(X_1) = \mathrm{sd}(X_2) \approx 1.7078$. Hence we see that the standard deviation of the sum is *not* equal to the sum of the standard deviations:

$$\mathrm{sd}(X_1 + X_2) \neq \mathrm{sd}(X_1) + \mathrm{sd}(X_2)$$

Example 8

Consider again the game of chance from Example 7.

a Find the mean and variance of $X_1 + X_2$, the total amount won when you play the game twice.

b Compare the values found in part **a** with the mean and variance of X, the amount won when you play the game once.

Solution

a Using the probability distribution of $X_1 + X_2$ found in Example 7, we have

$$\mathrm{E}(X_1 + X_2) = (-10) \times 0.49 + (-3) \times 0.28 + 4 \times 0.04 + \cdots = -2.20$$

$$\mathrm{E}[(X_1 + X_2)^2] = (-10)^2 \times 0.49 + (-3)^2 \times 0.28 + 4^2 \times 0.04 + \cdots = 119.02$$

$$\therefore \quad \mathrm{Var}(X_1 + X_2) = \mathrm{E}[(X_1 + X_2)^2] - [\mathrm{E}(X_1 + X_2)]^2$$
$$= 119.02 - (-2.20)^2 = 114.18$$

b From Examples 4 and 6, we have $\mathrm{E}(X) = -1.10$ and $\mathrm{Var}(X) = 57.09$. We see that

$$\mathrm{E}(X_1 + X_2) = -2.20 = 2 \times (-1.10) = 2\,\mathrm{E}(X)$$

$$\mathrm{Var}(X_1 + X_2) = 114.18 = 2 \times 57.09 = 2\,\mathrm{Var}(X)$$

We can generalise our findings as follows.

Let X be a random variable with mean μ and variance σ^2. Then if X_1 and X_2 are independent random variables with identical distributions to X, we have

$$\mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2) = 2\mu$$

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 2\sigma^2$$

$$\mathrm{sd}(X_1 + X_2) = \sqrt{\mathrm{Var}(X_1 + X_2)} = \sqrt{2}\sigma$$

Note: Since $\mathrm{sd}(X_1) + \mathrm{sd}(X_2) = 2\sigma$, we see that $\mathrm{sd}(X_1 + X_2) \neq \mathrm{sd}(X_1) + \mathrm{sd}(X_2)$ for $\sigma \neq 0$.

Sum of n independent random variables

So far we have looked at the sum of two independent identically distributed random variables. In the next example, we consider a sum of three random variables.

Example 9

Consider a random variable X which has a probability distribution as follows:

x	0	1	2
$\Pr(X = x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Let X_1, X_2 and X_3 be independent random variables with identical distributions to X.

a Find the probability distribution of $X_1 + X_2 + X_3$.

b Hence find the mean, variance and standard deviation of $X_1 + X_2 + X_3$.

Solution

a Using a tree diagram or a similar strategy, we can list all the possible combinations of values of X_1, X_2 and X_3 as follows:

$$(0,0,0),\ (0,0,1),\ (0,0,2),\ (0,1,0),\ (0,1,1),\ (0,1,2),\ (0,2,0),\ (0,2,1),\ (0,2,2)$$
$$(1,0,0),\ (1,0,1),\ (1,0,2),\ (1,1,0),\ (1,1,1),\ (1,1,2),\ (1,2,0),\ (1,2,1),\ (1,2,2)$$
$$(2,0,0),\ (2,0,1),\ (2,0,2),\ (2,1,0),\ (2,1,1),\ (2,1,2),\ (2,2,0),\ (2,2,1),\ (2,2,2)$$

The value of $X_1 + X_2 + X_3$ can be determined for each of the 27 outcomes. Since all outcomes are equally likely, we can construct the probability distribution:

z	0	1	2	3	4	5	6
$\Pr(X_1 + X_2 + X_3 = z)$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{7}{27}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{27}$

b Using the probability distribution from part **a**, we have

$$\mathrm{E}(X_1 + X_2 + X_3) = 0 \times \frac{1}{27} + 1 \times \frac{1}{9} + 2 \times \frac{2}{9} + \cdots + 5 \times \frac{1}{9} + 6 \times \frac{1}{27} = 3$$

$$\mathrm{E}[(X_1 + X_2 + X_3)^2] = 0^2 \times \frac{1}{27} + 1^2 \times \frac{1}{9} + 2^2 \times \frac{2}{9} + \cdots + 5^2 \times \frac{1}{9} + 6^2 \times \frac{1}{27} = 11$$

Thus $\mathrm{Var}(X_1 + X_2 + X_3) = 11 - 3^2 = 2$

and $\mathrm{sd}(X_1 + X_2 + X_3) = \sqrt{2} \approx 1.4142$

It is easy to verify in the previous example that

$$\mathrm{E}(X_1 + X_2 + X_3) = 3\,\mathrm{E}(X)$$
$$\mathrm{Var}(X_1 + X_2 + X_3) = 3\,\mathrm{Var}(X)$$

We can extend our findings in this section to the sum of n independent identically distributed random variables.

Sums of independent random variables

Let X be a random variable with mean μ and variance σ^2. Then if $X_1, X_2, \dots, X_n$ are independent random variables with identical distributions to X, we have

$$\mathrm{E}(X_1 + X_2 + \cdots + X_n) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \cdots + \mathrm{E}(X_n) = n\mu$$

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

$$\mathrm{sd}(X_1 + X_2 + \cdots + X_n) = \sqrt{\mathrm{Var}(X_1 + X_2 + \cdots + X_n)} = \sqrt{n}\sigma$$

Example 10

Let X be a random variable with mean $\mu = 10$ and variance $\sigma^2 = 9$. If X_1, X_2, X_3, X_4 are independent random variables with identical distributions to X, find:

a $\mathrm{E}(X_1 + X_2 + X_3 + X_4)$ **b** $\mathrm{Var}(X_1 + X_2 + X_3 + X_4)$ **c** $\mathrm{sd}(X_1 + X_2 + X_3 + X_4)$

Solution

a $\mathrm{E}(X_1 + X_2 + X_3 + X_4)$
$= 4\mu = 40$

b $\mathrm{Var}(X_1 + X_2 + X_3 + X_4)$
$= 4\sigma^2 = 36$

c $\mathrm{sd}(X_1 + X_2 + X_3 + X_4)$
$= \sqrt{4}\sigma = 2\sigma = 6$

Multiples of random variables

Once again, let X represent the number observed when a fair die is rolled. We now consider the random variable $2X$, which is obtained by doubling each value of X. The probability distribution of $2X$ is given in the following table.

z	2	4	6	8	10	12
$\Pr(2X = z)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

This is very different from the distribution of $X_1 + X_2$, which takes values $2, 3, 4, \dots, 11, 12$.

We found the mean and variance of X in Examples 3 and 5:

$$\mathrm{E}(X) = 3.5 \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{35}{12}$$

What can we say about the mean and variance of $2X$?

We have

$$\mathrm{E}(2X) = 2 \times \frac{1}{6} + 4 \times \frac{1}{6} + 6 \times \frac{1}{6} + 8 \times \frac{1}{6} + 10 \times \frac{1}{6} + 12 \times \frac{1}{6} = 7$$

Thus we can see that

$$\mathrm{E}(2X) = 2\,\mathrm{E}(X)$$

To find the variance of $2X$, we first calculate

$$\mathrm{E}[(2X)^2] = 2^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} + 8^2 \times \frac{1}{6} + 10^2 \times \frac{1}{6} + 12^2 \times \frac{1}{6} = \frac{182}{3}$$

Thus

$$\begin{aligned} \mathrm{Var}(2X) &= \mathrm{E}[(2X)^2] - [\mathrm{E}(2X)]^2 \\ &= \frac{182}{3} - 7^2 = \frac{35}{3} \end{aligned}$$

We observe that

$$\mathrm{Var}(2X) = 4\,\mathrm{Var}(X) = 2^2\,\mathrm{Var}(X)$$

More generally, we can consider random variables of the form kX, which have been obtained by multiplying the values of a random variable X by a positive real number k.

Multiples of random variables

Let X be a random variable with mean μ and variance σ^2. Then if k is a positive number, we have

$$\begin{aligned} \mathrm{E}(kX) &= k\,\mathrm{E}(X) = k\mu \\ \mathrm{Var}(kX) &= k^2\,\mathrm{Var}(X) = k^2\sigma^2 \\ \mathrm{sd}(kX) &= \sqrt{\mathrm{Var}(kX)} = k\sigma \end{aligned}$$

Example 11

Let X be a random variable with mean $\mu = 10$ and variance $\sigma^2 = 9$. Find:

a $\mathrm{E}(4X)$ **b** $\mathrm{Var}(4X)$ **c** $\mathrm{sd}(4X)$

Solution

a $\mathrm{E}(4X) = 4\mu = 40$ **b** $\mathrm{Var}(4X) = 16\sigma^2 = 144$ **c** $\mathrm{sd}(4X) = 4\sigma = 12$

Comparing the distributions of $X_1 + X_2$ and $2X$

We can use the results of this section to point out some importance differences between the random variables $X_1 + X_2$ and $2X$. We have seen from the dice example that their distributions are quite different, both in the possible values that the random variables can take and in the probabilities associated with those values. We have also seen that, while $X_1 + X_2$ and $2X$ have the same mean, they have different variances and standard deviations.

Let X be a random variable with mean μ and non-zero variance σ^2. Then if X_1 and X_2 are independent random variables with identical distributions to X, we have

$$\begin{array}{lll} \mathrm{E}(X_1 + X_2) = 2\mu & \mathrm{E}(2X) = 2\mu & \therefore\ \mathrm{E}(X_1 + X_2) = \mathrm{E}(2X) \\ \mathrm{Var}(X_1 + X_2) = 2\sigma^2 & \mathrm{Var}(2X) = 4\sigma^2 & \therefore\ \mathrm{Var}(X_1 + X_2) \neq \mathrm{Var}(2X) \\ \mathrm{sd}(X_1 + X_2) = \sqrt{2}\sigma & \mathrm{sd}(2X) = 2\sigma & \therefore\ \mathrm{sd}(X_1 + X_2) \neq \mathrm{sd}(2X) \end{array}$$

Summary 14B

- **Sums of independent random variables**

  Let X be a random variable with mean μ and variance σ^2. Then if $X_1, X_2, \dots, X_n$ are independent random variables with identical distributions to X, we have
  - $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) = n\mu$
  - $Var(X_1 + X_2 + \cdots + X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n) = n\sigma^2$
  - $sd(X_1 + X_2 + \cdots + X_n) = \sqrt{Var(X_1 + X_2 + \cdots + X_n)} = \sqrt{n}\sigma$

- **Multiples of random variables**

  Let X be a random variable with mean μ and variance σ^2. Then if k is a positive number, we have
  - $E(kX) = k\,E(X) = k\mu$
  - $Var(kX) = k^2\,Var(X) = k^2\sigma^2$
  - $sd(kX) = \sqrt{Var(kX)} = k\sigma$

Exercise 14B

Example 7

1 The probability distribution of the amount won, \$$X$, when playing a game of chance is given in the following table.

x	−2	0	50
$\Pr(X = x)$	0.8	0.15	0.05

Suppose that you play the game twice.

a Determine the probability distribution of the total amount won.

b Find the probability that the total amount won is at least \$50.

2 The following table gives the probability distribution of X, the number observed when a biased die is rolled.

x	1	2	3	4	5	6
$\Pr(X = x)$	0.2	0.2	0.2	0.2	0.1	0.1

Suppose that this die is rolled twice.

a Determine the probability distribution of the sum of the two numbers observed.

b Find the probability that the sum of the two numbers observed is more than 10.

Example 8

3 Consider a spinner with five equally likely sections numbered 1, 2, 3, 4 and 5.

a Find the mean and variance of X, the number obtained from one spin.

b Find the probability distribution of $X_1 + X_2$, the sum of the two numbers obtained from two spins.

c Find the probability that the sum of the two numbers is even.

d Use the probability distribution to find the mean and variance of $X_1 + X_2$.

4 Suppose that the amount won, $\$X$, when playing a game of chance has the following probability distribution.

x	−5	0	5	50
$\Pr(X = x)$	0.9	0.06	0.03	0.01

a Find the expected value, variance and standard deviation of X.

b Find the probability distribution of $X_1 + X_2$, the total amount won when playing the game twice.

c Use the probability distribution to find the expected value, variance and standard deviation of $X_1 + X_2$.

Example 10

5 Let X be a random variable with mean $\mu = 100$ and variance $\sigma^2 = 16$. If X_1, X_2, X_3, X_4 are independent random variables with identical distributions to X, find:

a $\mathrm{E}(X_1 + X_2 + X_3 + X_4)$ **b** $\mathrm{Var}(X_1 + X_2 + X_3 + X_4)$ **c** $\mathrm{sd}(X_1 + X_2 + X_3 + X_4)$

6 Let X be a random variable with mean $\mu = 30$ and variance $\sigma^2 = 7$. If X_1, X_2, X_3 are independent random variables with identical distributions to X, find:

a $\mathrm{E}(X_1 + X_2 + X_3)$ **b** $\mathrm{Var}(X_1 + X_2 + X_3)$ **c** $\mathrm{sd}(X_1 + X_2 + X_3)$

7 Consider again the game of chance from Question 4. Find the expected value, variance and standard deviation of the total amount won when the game is played three times.

Example 11

8 **a** Let X be a random variable with mean $\mu = 100$ and variance $\sigma^2 = 16$. Find:

i $\mathrm{E}(4X)$ **ii** $\mathrm{Var}(4X)$ **iii** $\mathrm{sd}(4X)$

b Compare your answers to part **a** with your answers to Question 5.

9 Let X be a random variable with mean $\mu = 3.4$ and variance $\sigma^2 = 1.2$. Find correct to three decimal places:

a $\mathrm{E}(10X)$ **b** $\mathrm{Var}(10X)$ **c** $\mathrm{sd}(10X)$

10 Data from a recent census was used to determine the following probability distribution for the number of dogs, X, belonging to a household in a certain town.

x	0	1	2	3
$\Pr(X = x)$	0.50	0.38	0.11	0.01

a Find the mean, variance and standard deviation of X.

b Assume that the number of dogs in a household is independent of the number in any other household. Find the mean, variance and standard deviation of the total number of dogs in a street with 10 households.

c Each household in the town is required to pay \$40 per dog in registration fees. Find the mean, variance and standard deviation of the total amount paid by a household.

14C Populations and samples

In the next two sections, we apply our knowledge of probability distributions to a special type of distribution that arises from sampling.

The set of all eligible members of a group which we intend to study is called a **population**. For example, if we are interested in the Intelligence Quotient (IQ) scores of the Year 12 students at ABC Secondary College, then this group of students could be considered a population; we could collect and analyse all the IQ scores for these students. However, if we are interested in the IQ scores of all Year 12 students across Australia, then this becomes the population.

Often, dealing with an entire population is not practical:

- The population may be too large – for example, all Year 12 students in Australia.
- The population may be hard to access – for example, all blue whales in the Pacific Ocean.
- The data collection process may be destructive – for example, testing every battery to see how long it lasts would mean that there were no batteries left to sell.

Nevertheless, we often wish to make statements about a property of a population when data about the entire population is unavailable.

The solution is to select a subset of the population – called a **sample** – in the hope that what we find out about the sample is also true about the population it comes from. Dealing with a sample is generally quicker and cheaper than dealing with the whole population, and a well-chosen sample will give much useful information about this population. How to select the sample then becomes a very important issue.

Random samples

Suppose we are interested in investigating the effect of sustained computer use on the eyesight of a group of university students. To do this we go into a lecture theatre containing the students and select all the students sitting in the front two rows as our sample. This sample may be quite inappropriate, as students who already have problems with their eyesight are more likely to be sitting at the front, and so the sample may not be typical of the population. To make valid conclusions about the population from the sample, we would like the sample to have a similar nature to the population.

While there are many sophisticated methods of selecting samples, the general principle of sample selection is that the method of choosing the sample should not favour or disfavour any subgroup of the population. Since it is not always obvious if the method of selection will favour a subgroup or not, we try to choose the sample so that every member of the population has an equal chance of being in the sample. In this way, all subgroups have a chance of being represented. The way we do this is to choose the sample at random.

> The simplest way to obtain a valid sample is to choose a **random sample**, where every member of the population has an equal chance of being included in the sample.

To choose a sample from the group of university students, we could put the name of every student in a hat and then draw out, one at a time, the names of the students who will be in the sample.

Choosing a sample in an appropriate manner is critical in order to obtain results from which we can make meaningful conclusions.

Example 12

A researcher wishes to evaluate how well the local library caters to the needs of a town's residents. To do this, she hands out a questionnaire to each person entering the library over the course of a week. Will this method result in a random sample?

Solution

Since the members of the sample are already using the library, they are possibly satisfied with the service available. Additional valuable information might well be obtained by finding out the opinion of those who do not use the library.

A better sample would be obtained by selecting at random from the town's entire population, so the sample contains both people who use the library and people who do not.

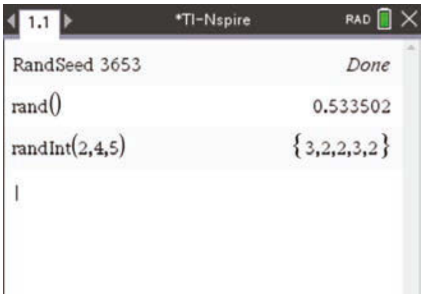
Thus, we have a very important consideration when sampling if we wish to generalise from the results of the sample.

In order to make valid conclusions about a population from a sample, we would like the sample chosen to be representative of the population as a whole. This means that all the different subgroups present in the population appear in the sample in similar proportions as they do in the population.

One very useful method for drawing random samples is to generate random numbers using a calculator or a computer.

Using the TI-Nspire

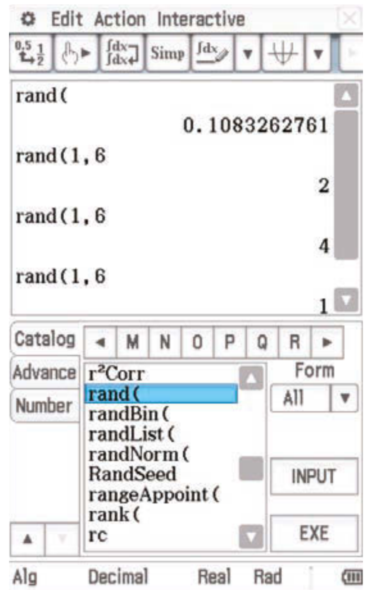
- In a **Calculator** page, go to menu > **Probability** > **Random** > **Seed** and enter the last 4 digits of your phone number. This ensures that your random-number starting point differs from the calculator default.
- For a random number between 0 and 1, use menu > **Probability** > **Random** > **Number**.



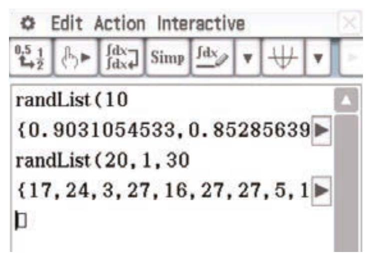
- For a random integer, use menu > **Probability** > **Random** > **Integer**.
  To obtain five random integers between 2 and 4 inclusive, use the command randInt(2, 4, 5) as shown.

Using the Casio ClassPad

- In $\sqrt{\alpha}$ (Main), press the [Keyboard] button.
- Find and then select [Catalog] by first tapping ▼ at the bottom of the left sidebar.
- Scroll across the alphabet to the letter R.



- To generate a random number between 0 and 1:
  - In [Catalog], select **rand(**.
  - Tap [EXE].
- To generate three random integers between 1 and 6 inclusive:
  - In [Catalog], select **rand(**.
  - Type: 1, 6
  - Tap [EXE] three times.
- To generate a list of 10 random numbers between 0 and 1:
  - In [Catalog], select **randList(**. Type: 10
  - Tap [EXE] and then tap ► to view all the numbers.
- To generate a list of 20 random integers between 1 and 30 inclusive:
  - In [Catalog], select **randList(**. Type: 20, 1, 30
  - Tap [EXE] and then tap ► to view all the integers.



Example 13

The table gives the IQ score for each student in the population of Year 12 students at ABC Secondary College. Each student has been given an identity number (Id). Use a random number generator to select a random sample of size 4 from this population.

Id	IQ	Id	IQ	Id	IQ	Id	IQ	Id	IQ	Id	IQ	Id	IQ	Id	IQ	Id	IQ	Id	IQ
1	101	11	94	21	102	31	85	41	113	51	92	61	103	71	107	81	86	91	113
2	116	12	116	22	84	32	122	42	109	52	85	62	78	72	104	82	87	92	108
3	107	13	98	23	128	33	125	43	76	53	111	63	128	73	99	83	92	93	90
4	76	14	104	24	81	34	96	44	101	54	106	64	98	74	96	84	94	94	103
5	104	15	87	25	91	35	104	45	137	55	97	65	114	75	82	85	95	95	83
6	101	16	130	26	91	36	89	46	106	56	133	66	87	76	117	86	99	96	106
7	103	17	88	27	111	37	99	47	106	57	112	67	75	77	104	87	106	97	73
8	112	18	105	28	94	38	94	48	97	58	69	68	126	78	92	88	63	98	80
9	72	19	88	29	89	39	120	49	124	59	92	69	114	79	100	89	105	99	99
10	89	20	92	30	121	40	107	50	84	60	117	70	105	80	84	90	113	100	109

Solution

Generating four random integers from 1 to 100 gives on this occasion:

18, 51, 92, 41

Thus the sample chosen consists of the four students listed in the table on the right.

Id	IQ
18	105
51	92
92	108
41	113

Population parameters and sample statistics

Consider the IQ scores of the population of 100 Year 12 students at ABC Secondary College from Example 13. The mean IQ for this whole group is called the **population mean** and is denoted by the Greek letter μ (*mu*).

$$\text{Population mean } \mu = \frac{\text{sum of the data values in the population}}{\text{population size}}$$

By summing the IQ scores of all students and dividing by 100, we find that the population mean is $\mu = 100.0$.

The mean IQ for the sample chosen in Example 13 is

$$\frac{105 + 92 + 108 + 113}{4} = 104.50$$

This value is called the **sample mean** and is denoted by $\bar{x}$. (We say 'x bar'.)

$$\text{Sample mean } \bar{x} = \frac{\text{sum of the data values in the sample}}{\text{sample size}}$$

In this particular case, the value of the sample mean $\bar{x}$ (104.50) is not the same as the value of the population mean μ (100.0).

We can select another three random samples, each of size 4, and calculate the sample means:

Sample				Sample mean
101	102	101	94	$\bar{x} = 99.50$
116	84	103	116	$\bar{x} = 104.75$
107	128	112	98	$\bar{x} = 111.25$

Clearly, the sample mean is going to vary from sample to sample, depending on which members of the population are selected in the sample. While the sample means are quite close to the population mean, they are not often exactly equal to the population mean.

We will look at the behaviour of the sample mean in more detail in the next section.

- The population mean μ is a **population parameter**; its value is constant for a given population.
- The sample mean $\bar{x}$ is a **sample statistic**; its value is not constant, but varies from sample to sample.

As another example, we can consider the proportion of females in the population of Year 12 students at ABC Secondary College. This is a population parameter; its value is constant. The proportion of females in a random sample of four students is a sample statistic; its value varies from sample to sample.

Summary 14C

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be meaningful unless the sample is representative of the population.
- The simplest way to obtain a valid sample is to choose a **random sample**, where every member of the population has an equal chance of being included in the sample.
- The **population mean** μ is the mean of all values of a measure in the entire population, and is constant for a given population.
- The **sample mean** $\bar{x}$ is the mean of the values of this measure in a particular sample, and varies from sample to sample.

Exercise 14C

Example 12

1 In order to estimate the amount of time that students spend playing computer games, a researcher conducted an email survey. She found that the average time was 1.5 hours per week for those who responded. Do you think that this is an appropriate way of selecting a random sample of students? Explain your answer.

2 A market researcher wishes to determine the age profile of the customers of a popular fast-food chain. She positions herself outside one of the restaurants between 4 p.m. and 8 p.m. one weekend, and asks customers to fill out a short questionnaire. Do you think this sample will be representative of the population? Explain your answer.

3 To estimate the number of days per week that students bring their lunch from home, the principal selected a group of 10 students by using a list of all enrolled students and a random number generator.

a Is this an appropriate method of choosing a sample of students? Give reasons for your answer.

b For the 10 students in the sample, the numbers of days that they brought their lunch from home last week were 1, 0, 4, 4, 5, 5, 5, 0, 0, 3. What is the value of the sample mean $\bar{x}$ for this sample?

Example 13

4 **a** Use a random number generator to select a random sample of size 5 from the population of Year 12 students at ABC Secondary College given in Example 13.
b Determine the mean IQ of the students in your sample.

5 Recent research has established that Australian adults spend on average four hours per day on sedentary leisure activities such as watching television. A group of 100 people were selected at random and found to spend an average of 3.5 hours per day on sedentary leisure activities. In this example:
a What is the population?
b What is the value of the population mean μ?
c What is the value of the sample mean $\bar{x}$?

14D Investigating the distribution of the sample mean using simulation

In Section 14C, we saw that while the population mean μ is constant for a given population, the sample mean $\bar{x}$ is not constant, but varies from sample to sample. In this section we will further investigate this variation in the sample mean.

The sampling distribution of the sample mean

Since $\bar{x}$ varies according to the contents of the random samples, we can consider the sample means $\bar{x}$ as being the values of a random variable, which we will denote by $\bar{X}$.

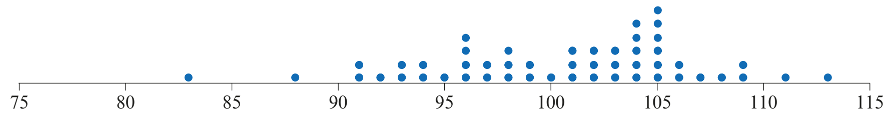
As $\bar{X}$ is a random variable, it can be described by a probability distribution. The distribution of a statistic which is calculated from a sample (such as the sample mean) has a special name – it is called a **sampling distribution**.

Consider again the IQ scores of the population of 100 Year 12 students at ABC Secondary College. The population mean is $\mu = 100.0$ and the population standard deviation is $\sigma = 15.0$.

In the previous section, we selected four random samples from this population, each of size 4, and found the sample means to be

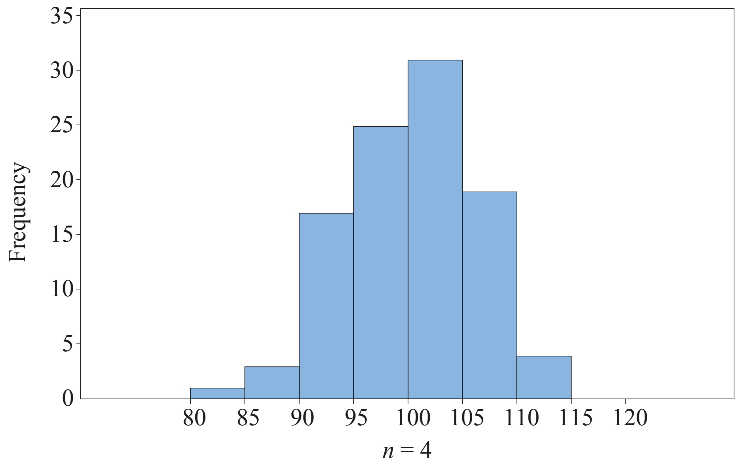
104.50, 99.50, 104.75, 111.25

Suppose that we continue this process until we have selected a total of 50 random samples (each of size 4). The values of $\bar{x}$ obtained might look like those in the following dotplot.



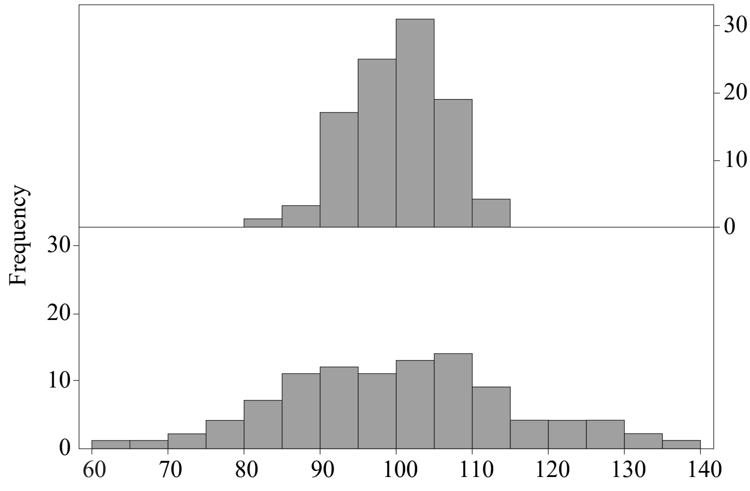
The values look to be centred around 100, ranging from about 83 to 113. In fact, the mean of these 50 values is 100.3 and the standard deviation is 6.13.

To get a better picture of the sampling distribution of $\bar{X}$, we require more sample means. The following histogram summarises the values of $\bar{x}$ observed for 100 samples (each of size 4).



Here we can see that the sampling distribution of $\bar{X}$ appears to be symmetric and centred around 100. In fact, the mean of these 100 values is 100.04 and the standard deviation is 6.27.

How does the sampling distribution of $\bar{X}$ compare with the distribution of X? In the following diagram, the lower plot shows the distribution of IQ scores for the population of 100 Year 12 students at ABC Secondary College, and the upper plot shows the sampling distribution of $\bar{X}$ based on 100 samples of size 4 from this population.

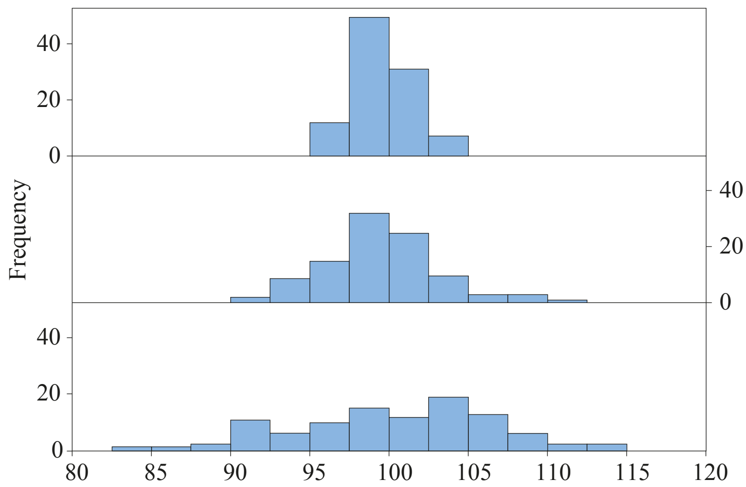


Comparing these two plots, we can see that, while both are centred around 100, the spread of the sampling distribution of $\bar{X}$ is much less than that of the population. For example, we see that several students have IQ scores of 120 or more, but none of the samples of size 4 has an average score that high.

The effect of sample size on the distribution of the sample mean

We have seen that the distribution of the sample mean $\bar{X}$ is centred at the value of the population mean, with a smaller standard deviation than the population. We next explore how the distribution of the sample mean is affected by the size of the sample chosen.

The following histograms show the sample means $\bar{x}$ obtained when 100 samples of size 4 (lower plot), then size 16 (middle plot) and then size 64 (upper plot) were chosen from the population of Year 12 students at ABC Secondary College. Here it is important not to confuse the *size* of each sample with the *number* of samples, which is quite arbitrary.



We can see from the histograms that all three sampling distributions appear to be centred at 100, the value of the population mean μ. Furthermore, as the sample size increases, the values of the sample mean $\bar{x}$ are more tightly clustered around that value.

These observations are confirmed in the following table, which gives the mean and standard deviation for each of the three sampling distributions shown in the histograms.

Sample size	4	16	64
Population mean μ	100.0	100.0	100.0
Population standard deviation σ	15.0	15.0	15.0
Mean of the values of $\bar{x}$	100.04	99.56	99.71
Standard deviation of the values of $\bar{x}$	6.27	3.78	1.84

Based on this investigation, we can make the following generalisations:

- The sampling distribution of $\bar{X}$ is centred at the value of the population mean μ.
- The variation in the sampling distribution decreases as the size of the sample increases.

Estimating the population mean

In practice, it is highly unlikely that we will know the value of the population mean μ. However, we have seen that the distribution of the sample mean $\bar{X}$ is centred at the value of the population mean. Moreover, the larger the sample we select, the closer the sample mean is likely to be to the population mean.

Thus, when the population mean μ is not known, the sample mean $\bar{x}$ can be used as an **estimate** of this parameter. The larger the sample used to calculate the sample mean, the more confident we can be that this is a good estimate of the population mean.

Example 14

The mean length of the population of adult fish in a very large lake is unknown.

A random sample of 100 fish was found to have a mean length of $\bar{x} = 34.6$ cm.
A random sample of 200 fish was found to have a mean length of $\bar{x} = 35.7$ cm.

a Which sample gives a better estimate of μ, the population mean?

b Find an even better estimate of μ by combining the information from both samples. (Assume that the two samples contain different fish.)

Solution

a We expect a larger sample to give a better estimate. So, based on the sample of 200 fish, we estimate μ as 35.7 cm.

b We can view the two samples together as a random sample of 300 fish. To find the sample mean $\bar{x}$ for this sample, we calculate the weighted mean of the two separate sample means. Thus we can estimate μ as

$$\frac{(100 \times 34.6) + (200 \times 35.7)}{300} = 35.3 \text{ cm}$$

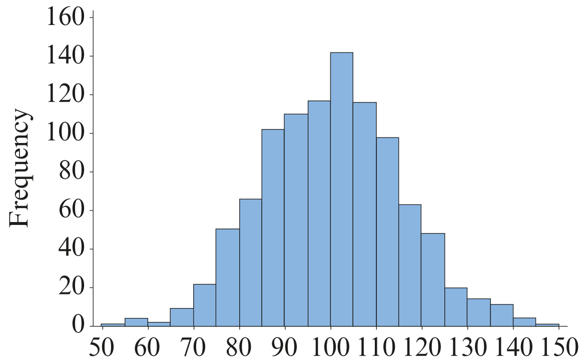
Using simulation to investigate the distribution of the sample mean

So far we have investigated the sampling distribution of the sample mean $\bar{X}$ based on data. To make our investigations easier, we can use technology (calculators or computers) to repeat a random sampling process many times. This is known as **simulation**.

Normal distributions

Consider the random variable IQ, which has a mean of 100 and a standard deviation of 15 in the population. In order to use technology to investigate this random variable, we use a distribution that you may not have met before, called the **normal distribution**. You will study this distribution in Year 12, but for now it is enough to know that many commonly occurring continuous random variables – such as height, weight and IQ – follow this distribution.

This histogram shows the IQ scores of 1000 people randomly drawn from the population.



A normal distribution is symmetric and bell-shaped, with its centre of symmetry at the population mean.

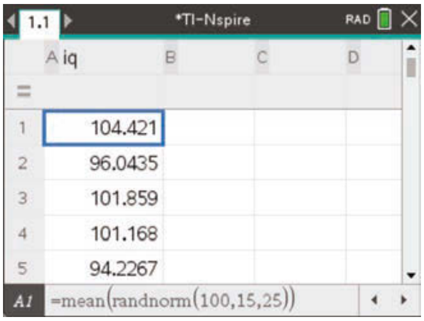
Simulating samples from a normal distribution

A normal distribution is fully defined by its mean and standard deviation. If we know these values, then we can use technology to generate sample means for random samples drawn from this population.

Using the TI-Nspire

To generate the sample means for 10 random samples of size 25 from a normal population with mean 100 and standard deviation 15:

- Start from a **Lists & Spreadsheet** page.
- Name the list 'iq' in Column A.
- In cell A1, enter the formula using (menu) > **Data** > **Random** > **Normal** and complete as: = mean(randnorm(100, 15, 25))
- Use (menu) > **Data** > **Fill** to fill down to obtain the sample means for 10 random samples.

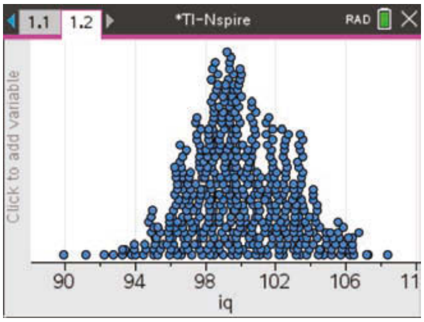


For a large number of simulations, an alternative method is easier.

To generate the sample means for 500 random samples of size 25, enter the following formula in the formula cell of Column A:

$= \text{seq}(\text{mean}(\text{randnorm}(100, 15, 25)), k, 1, 500)$

The dotplot on the right was created this way.



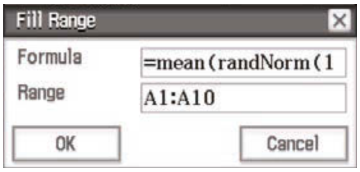
Using the Casio ClassPad

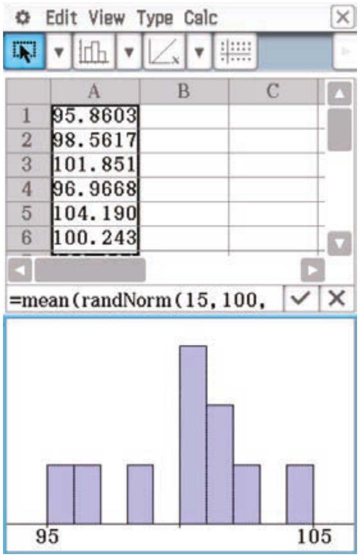
To generate the sample means for 10 random samples of size 25 from a normal population with mean 100 and standard deviation 15:

- Open the **Spreadsheet** application.
- Tap in cell A1.
- Type: = mean(randNorm(15, 100, 25))

Note: The commands **mean(** and **randNorm(** can be selected from Catalog.

- Go to **Edit** > **Fill** > **Fill Range**.
- Enter A1:A10 for the range, using the symbols A and : from the toolbar. Tap OK.



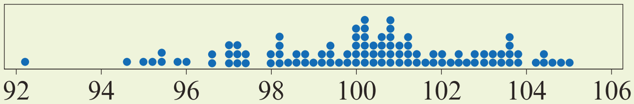


To sketch a histogram of these sample means:

- Go to **Edit** > **Select** > **Select Range**.
- Enter A1:A10 for the range and tap OK.
- Select **Graph** and tap **Histogram**.

Example 15

The IQ scores for a population have mean $\mu = 100$ and standard deviation $\sigma = 15$. The following dotplot summarises the sample means $\bar{x}$ for 100 random samples of 25 people drawn from this population.



Use the dotplot to estimate the probability that, for a random sample of 25 people drawn from this population, the sample mean $\bar{x}$ is 104 or more.

Solution

From the dotplot we can count 6 out of 100 samples where the sample mean is 104 or more. Thus we can estimate

$$\Pr(\bar{X} \geq 104) \approx \frac{6}{100} = 0.06$$

Example 16

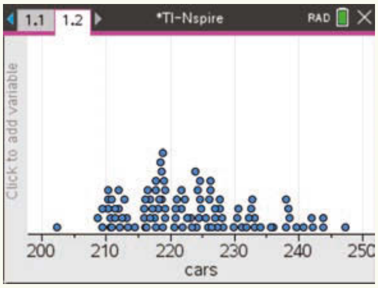
Suppose it is known that parking times in a large city car park are normally distributed, with mean $\mu = 223$ minutes and standard deviation $\sigma = 48$ minutes.

a Use your calculator to generate the sample means for 100 samples, each of size 25, drawn at random from this population. Summarise these values in a dotplot.

b Use your dotplot to estimate the probability that, in a random sample of 25 cars, the mean parking time is greater than or equal to 240 minutes.

Solution

a



b In the dotplot, there are 6 out of 100 samples where the sample mean is 240 or more. This gives

$$\Pr(\bar{X} \geq 240) \approx \frac{6}{100} = 0.06$$

Expected value and variance of the sample mean

So far we have used data and simulations to give us an insight into the sampling distribution of $\bar{X}$. Now we can confirm these insights theoretically, using the results from Section 14B.

Let X be a random variable with mean μ and variance σ^2. Consider the sample mean, $\bar{X}$, for random samples of size n. We can write

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

where $X_1, X_2, \ldots, X_n$ are independent random variables with identical distributions to X. Using our results on multiples and sums of random variables from Section 14B, we obtain

$$\mathrm{E}(\bar{X}) = \mathrm{E}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{\mathrm{E}(X_1 + X_2 + \cdots + X_n)}{n} = \frac{n\mu}{n} = \mu$$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{\mathrm{Var}(X_1 + X_2 + \cdots + X_n)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

These two new results confirm our observations:

- The sampling distribution of $\bar{X}$ is centred at the value of the population mean μ.
- The variation in the sampling distribution decreases as the size n of the sample increases.

Distribution of the sample mean

If we select samples of size n from a population with mean μ and standard deviation σ, then the sample mean $\bar{X}$ satisfies:

- $\mathrm{E}(\bar{X}) = \mu$
- $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$
- $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$

Example 17

Consider again the game of chance from Example 4. The amount, $\$X$, that you win in this game has a mean of $\mu = -1.10$ and a variance of $\sigma^2 = 57.09$.

a Find the mean and variance of $X_1 + X_2 + X_3$, the total amount won when you play the game three times.

b Find the mean and variance of $\bar{X}$, the average amount won per game when you play the game three times.

Solution

a $$\mathrm{E}(X_1 + X_2 + X_3) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \mathrm{E}(X_3) = 3\mu = -3.30$$

$$\mathrm{Var}(X_1 + X_2 + X_3) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) = 3\sigma^2 = 171.27$$

b $$\mathrm{E}(\bar{X}) = \mu = -1.10$$

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{3} = \frac{57.09}{3} = 19.03$$

Example 18

Let X be a random variable with mean $\mu = 100$ and standard deviation $\sigma = 15$. Find the mean and standard deviation of the sample mean $\bar{X}$ for each of the following sample sizes:

a $n = 4$ **b** $n = 16$ **c** $n = 64$

Solution

a $\mathrm{E}(\bar{X}) = \mu = 100, \quad \mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}} = \dfrac{15}{2} = 7.5$

b $\mathrm{E}(\bar{X}) = \mu = 100, \quad \mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}} = \dfrac{15}{4} = 3.75$

c $\mathrm{E}(\bar{X}) = \mu = 100, \quad \mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}} = \dfrac{15}{8} = 1.875$

Summary 14D

- The **sample mean** $\bar{X}$ is a random variable and so can be described by a probability distribution, called the sampling distribution of the sample mean.
- The sampling distribution of $\bar{X}$ is centred at the value of the population mean μ.
- The variation in the sampling distribution decreases as the size of the sample increases.
- When the population mean μ is not known, the sample mean $\bar{x}$ can be used as an estimate of this parameter. The larger the sample size, the more confident we can be that $\bar{x}$ is a good estimate of the population mean μ.

Exercise 14D

Example 14a

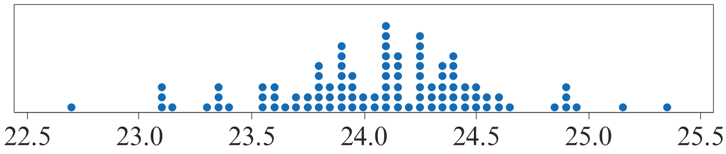
1 The mean time spent on social media each day by teenagers in a certain country is unknown. A random sample of 100 teenagers was found to spend an average of $\bar{x} = 45.6$ minutes per day on social media. Give an estimate of μ, the population mean.

Example 14b

2 The mean salary of cybersecurity engineers is unknown. A survey of 50 cybersecurity engineers found that they earned an average of $\bar{x} = \$3250$ per week. A second survey of 100 different cybersecurity engineers found that they earned an average of $\bar{x} = \$3070$ per week. Use the data from both samples to find an estimate of μ, the population mean.

Example 15

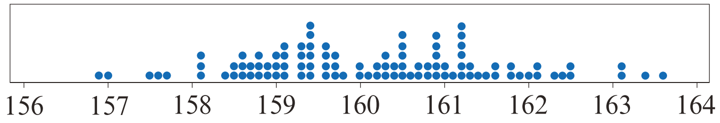
3 In a certain city, the average size of a kindergarten class is $\mu = 24$ children, with a standard deviation of $\sigma = 2$. The following dotplot shows the sample means $\bar{x}$ for 100 random samples of 20 classes.



Use the dotplot to estimate:

a $\Pr(\bar{X} \geq 25)$ **b** $\Pr(\bar{X} \leq 23)$

4 The mean height of women in a certain country is $\mu = 160$ cm, with a standard deviation of $\sigma = 8$ cm. The following dotplot shows the sample means $\bar{x}$ for 100 random samples of 30 women.



Use the dotplot to estimate:

a $\Pr(\bar{X} \geq 163)$ **b** $\Pr(\bar{X} \leq 158)$

Example 16

5 The lengths of a species of fish are normally distributed with mean length $\mu = 40$ cm and standard deviation $\sigma = 4$ cm.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 50 drawn from this population of fish, and summarise the values obtained in a dotplot.

b Use your dotplot to estimate:

i $\Pr(\bar{X} \geq 41)$ **ii** $\Pr(\bar{X} \leq 39)$

6 The marks in a statistics examination in a certain university are normally distributed with a mean of $\mu = 48$ marks and a standard deviation of $\sigma = 15$ marks.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 20 drawn from the students at this university, and summarise the values obtained in a dotplot.

b Use your dotplot to estimate:

i $\Pr(\bar{X} \geq 55)$ **ii** $\Pr(\bar{X} \leq 40)$

Example 17

7 Consider again the game of chance from Example 4. The amount, $\$X$, that you win in this game has a mean of $\mu = -1.10$ and a variance of $\sigma^2 = 57.09$.

a Find the mean and variance of the total amount won when you play the game 25 times.

b Find the mean and variance of the average amount won per game when you play the game 25 times.

8 Data from a recent census was used to determine that the number of dogs belonging to each household in a certain town has a mean of $\mu = 0.63$ and a variance of $\sigma^2 = 0.5131$.

a Find the mean and variance of the total number of dogs in a random sample of 10 households.

b Find the mean and variance of the average number of dogs per household in a random sample of 10 households.

Example 18

9 Let X be a random variable with mean $\mu = 30$ and standard deviation $\sigma = 7$. Find the mean and standard deviation of the sample mean $\bar{X}$ for each of the following sample sizes:

a $n = 25$ **b** $n = 2500$ **c** $n = 250\,000$

10 Let X be a random variable with mean $\mu = 16.77$ and standard deviation $\sigma = 2.45$. Find the mean and standard deviation of the sample mean $\bar{X}$ for each of the following sample sizes:

a $n = 10$ **b** $n = 100$ **c** $n = 1000$

11 Tickets in a game of chance can be purchased for \$5. Each ticket has a 20% chance of winning \$5, a 5% chance of winning \$40, and otherwise loses.

a Find the mean and standard deviation of the profit if you buy one ticket.

b Find the mean and standard deviation of the average profit per ticket if you buy:

i 10 tickets **ii** 100 tickets **iii** 1000 tickets

Chapter summary

Assign-ment

Nrich

Random variables

- A **random variable** associates a number with each outcome of a random experiment. A **discrete** random variable is one which can take only a countable number of values.
- The **probability distribution** of a discrete random variable X is a function

$$p(x) = \Pr(X = x)$$

that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

- For any discrete probability distribution, the following two conditions must hold:
  - Each value of $p(x)$ belongs to the interval $[0, 1]$. That is, $0 \le p(x) \le 1$ for all x.
  - The sum of all the values of $p(x)$ is 1.
- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\mu = \mathrm{E}(X) = \sum_x x \cdot \Pr(X = x)$$
$$= \sum_x x \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

Sums and multiples of random variables

- Let X be a random variable with mean μ and variance σ^2. Then if $X_1, X_2, \dots, X_n$ are independent random variables with identical distributions to X, we have
  - $\mathrm{E}(X_1 + X_2 + \dots + X_n) = n\mu$
  - $\mathrm{Var}(X_1 + X_2 + \dots + X_n) = n\sigma^2$
  - $\mathrm{sd}(X_1 + X_2 + \dots + X_n) = \sqrt{n}\sigma$
- Let X be a random variable with mean μ and variance σ^2. Then if k is a positive number, we have
  - $\mathrm{E}(kX) = k\,\mathrm{E}(X) = k\mu$
  - $\mathrm{Var}(kX) = k^2\,\mathrm{Var}(X) = k^2\sigma^2$
  - $\mathrm{sd}(kX) = \sqrt{\mathrm{Var}(kX)} = k\sigma$

Sampling

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be meaningful unless the sample is representative of the population.
- The simplest way to obtain a valid sample is to choose a **random sample**, where every member of the population has an equal chance of being included in the sample.
- The **population mean** μ is the mean of all values of a measure in the entire population. The **sample mean** $\bar{x}$ is the mean of these values in a particular sample.
- The population mean μ is a **population parameter**; its value is constant for a given population.
- The sample mean $\bar{x}$ is a **sample statistic**; its value varies from sample to sample.

Sampling distributions

- The **sample mean** $\bar{X}$ is a random variable and so can be described by a probability distribution, called the **sampling distribution** of the sample mean.
- The sampling distribution of $\bar{X}$ is centred at the value of the population mean μ.
- The variation in the sampling distribution decreases as the size of the sample increases.
- When the population mean μ is not known, the sample mean $\bar{x}$ can be used as an estimate of this parameter. The larger the sample size, the more confident we can be that $\bar{x}$ is a good estimate of the population mean μ.
- If we select samples of size n from a population with mean μ and standard deviation σ, then the sample mean $\bar{X}$ satisfies:
  - $\mathrm{E}(\bar{X}) = \mu$
  - $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$
  - $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$

Technology-free questions

1 A discrete random variable X takes values 0, 1, 2, 3 with probabilities as shown in the following table.

x	1	2	3	4
$\Pr(X = x)$	p	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

a Find p. **b** Find $\mathrm{E}(X)$. **c** Find $\mathrm{Var}(X)$.

2 A random variable X has the following probability distribution:

x	-1	0	1	2	3
$\Pr(X = x)$	k	$2k$	$3k$	$2k$	$2k$

a Find k. **b** Find $\mathrm{E}(X)$. **c** Find $\mathrm{Var}(X)$.

3 Let X be a random variable with mean $\mu = 50$ and variance $\sigma^2 = 25$. If X_1, X_2, X_3, X_4 are independent random variables with identical distributions to X, find:

a $E(X_1 + X_2 + X_3 + X_4)$ **b** $Var(X_1 + X_2 + X_3 + X_4)$ **c** $sd(X_1 + X_2 + X_3 + X_4)$

4 Let X be a random variable with mean $\mu = 30$ and variance $\sigma^2 = 16$. Find:

a $E(10X)$ **b** $Var(10X)$ **c** $sd(10X)$

5 To study the popularity of yoga, a researcher sent out a survey to students at her university and asked respondents to rate their interest in undertaking a yoga course on a scale of 1 (not at all interested) to 10 (sign me up today). Do you think that this sample will be representative of all the students at that university? Explain your answer.

6 Medical researchers were interested in the amount of water consumed by people with Type II diabetes, which they suspect may be more than the 1 litre per day average observed in the general population. They randomly selected a sample of 50 people with Type II diabetes and found their average daily water consumption was 1.5 litres per day.

a What is the population of interest here?

b Why did the researchers select a sample rather than studying the entire population?

c What is the value of the population mean μ?

d What is the value of the sample mean $\bar{x}$?

7 The mean height of females in a certain large population is unknown. A sample of 100 randomly chosen females was found to have a mean height of $\bar{x} = 1.62$ m. A second sample of 100 randomly chosen females (different from the first sample) was found to have a mean height of $\bar{x} = 1.58$ m. Find an estimate of μ, the population mean.

8 A random variable X has a mean of 10 and a standard deviation of 2. Find the mean and standard deviation of the sample mean $\bar{X}$ for each of the following sample sizes:

a $n = 9$ **b** $n = 25$ **c** $n = 100$

Multiple-choice questions

1 The random variable X has the probability distribution shown, where $0 < p < \frac{1}{2}$.

The mean of X is

x	-1	0	1
$\Pr(X = x)$	p	p	$1 - 2p$

A 1 **B** 0 **C** $1 - 3p$ **D** $1 - p$ **E** $1 + 2p$

2 A random variable X is such that $E(X) = 1.5$ and $E(X^2) = 2.89$. The standard deviation of X is equal to

A 1.7 **B** $\sqrt{1.39}$ **C** 0.64 **D** 0.7 **E** 0.8

The following information relates to Questions 3–4.

Suppose that the amount, \$$X$, that you win when you play a certain game of chance has the probability distribution shown on the right.

x	−1	0	25
$\Pr(X = x)$	0.7	0.2	0.1

3 If you play the game twice, the possible values for the total amount won (in dollars) are

A −2, 0, 50 **B** −2, −1, 0, 50 **C** −1, 0, 25
D −2, −1, 0, 24, 25, 50 **E** −2, −1, 25, 50

4 If you play the game twice, the probability of winning a total of \$25 is

A 0 **B** 0.1 **C** 0.2 **D** 0.02 **E** 0.04

The following information relates to Questions 5–7.

Consider the discrete random variable X with the probability distribution shown on the right.

x	−1	0	1	2
$\Pr(X = x)$	0.2	0.3	0.3	0.2

5 If X_1, X_2, X_3, X_4 are independent random variables with identical distributions to X, then the expected value of the sum $X_1 + X_2 + X_3 + X_4$ is equal to

A 0 **B** 0.5 **C** 0.125 **D** 2.0 **E** 2.5

6 The expected value of $5X$ is equal to

A 0 **B** 0.1 **C** 2.5 **D** 0.5 **E** 3.0

7 The variance of $5X$ is equal to

A 2.5 **B** 1.05 **C** 5.123 **D** 5.25 **E** 26.25

8 In order to estimate the ratio of males to females at a school, a teacher determines the number of males and the number of females in a particular class. The ratio that he then calculates is called a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

9 In a complete census of the population of a particular community, it is found that 59% of families have two or more children. Here '59%' represents the value of a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

10 Which of the following statements is true?

A We use sample statistics to estimate population parameters.
B We use sample parameters to estimate population statistics.
C We use population parameters to estimate sample statistics.
D We use population statistics to estimate sample parameters.
E None of the above.

11 A sampling distribution can best be described as a distribution which

A gives the most likely value of the sample statistic
B describes how a statistic's value will change from sample to sample
C describes how samples do not give reliable estimates
D gives the distribution of the values observed in a particular sample
E None of the above.

12 A market research company has decided to increase the size of the random sample of Australians that it will select for a survey, from about 1000 people to about 1500 people. What is the effect of this increase in sample size?

A The increase will ensure that the sampling distribution is symmetric.
B The effect cannot be predicted without knowing the population size.
C There will be no effect as the population size is the same.
D The variability of the sample estimate will increase, as more people are involved.
E The variability of the sample estimate will decrease.

13 Suppose a random variable X has mean $\mu = 8$ and standard deviation $\sigma = 2.5$. The mean and standard deviation of the sample mean $\bar{X}$ for a sample size of 100 are

A $E(\bar{X}) = 8,\ sd(\bar{X}) = 2.5$
B $E(\bar{X}) = 0.08,\ sd(\bar{X}) = 0.025$
C $E(\bar{X}) = 8,\ sd(\bar{X}) = 0.25$
D $E(\bar{X}) = 8,\ sd(\bar{X}) = 0.025$
E $E(\bar{X}) = 0.8,\ sd(\bar{X}) = 0.25$

Extended-response questions

1 At the Fizzy Drinks Company, the volume of soft drink in a 1 litre bottle is normally distributed with mean $\mu = 1$ litre and standard deviation $\sigma = 0.01$ litres.

a Use your calculator to simulate 100 values of the sample mean calculated from a random sample of 25 bottles from this company. Summarise the values obtained in a dotplot.

b Use your dotplot from part **a** to estimate:

i $\Pr(\bar{X} \geq 1.003)$
ii $\Pr(\bar{X} \leq 0.995)$

c Repeat the simulation carried out in part **a** but this time using samples of 50 bottles. Summarise the values obtained in a dotplot.

d Use your dotplot from part **c** to estimate:

i $\Pr(\bar{X} \geq 1.003)$
ii $\Pr(\bar{X} \leq 0.995)$

e Compare your answers to parts **b** and **d**.

2 For a certain type of mobile phone, the length of time between charges of the battery is normally distributed with a mean of 50 hours and a standard deviation of 5 hours.

a **i** Use your calculator to simulate 100 values of the sample mean calculated from a sample of 20 phones.

ii Summarise the values obtained in part **i** in a dotplot.

iii Determine the mean and standard deviation of this sampling distribution.

b **i** Use your calculator to simulate 100 values of the sample mean calculated from a sample of 50 phones.

ii Summarise the values obtained in part **i** in a dotplot.

iii Determine the mean and standard deviation of this sampling distribution.

c **i** Use your calculator to simulate 100 values of the sample mean calculated from a sample of 100 phones.

ii Summarise the values obtained in part **i** in a dotplot.

iii Determine the mean and standard deviation of this sampling distribution.

d It can be shown theoretically that the standard deviation of the sampling distribution is inversely proportional to $\sqrt{n}$, where n is the sample size. Use your answers to parts **a**–**c** to demonstrate this relationship.

3 Samar has determined the following probability distribution for the number of cups of coffee, X, that he drinks in a day.

x	0	1	2
$\Pr(X = x)$	0.1	0.6	0.3

a Find the mean, variance and standard deviation of X.

Suppose that the number of cups of coffee that Samar drinks on one day is independent of the number he drinks on any other day.

b **i** Find the probability distribution of the total number of cups of coffee that Samar drinks over a weekend (that is, over two days).

ii Find the probability that he drinks more than three cups of coffee over a weekend.

iii Find the mean and variance of the total number of cups of coffee he drinks over a weekend.

iv Find the mean and variance of the average number of cups of coffee he drinks per day over a weekend.

c **i** Find the mean and variance of the total number of cups of coffee that Samar drinks over a week (that is, over seven days).

ii Find the mean and variance of the average number of cups of coffee he drinks per day over a week.

15

Trigonometric ratios and applications

Objectives

- To solve practical problems using the trigonometric ratios.
- To use the **sine rule** and the **cosine rule** to solve problems.
- To find the **area of a triangle** given two sides and the included angle.
- To find the **length of an arc** and the **length of a chord** of a circle.
- To find the **area of a sector** and the **area of a segment** of a circle.
- To solve problems involving **angles of elevation** and **angles of depression**.
- To identify the **line of greatest slope of a plane**.
- To solve problems in **three dimensions**, including determining the angle between two planes.

Trigonometry deals with the side lengths and angles of a triangle: the word *trigonometry* comes from the Greek words for triangle and measurement.

You have studied the four standard congruence tests for triangles in earlier years. If you have the information about a triangle given in one of the congruence tests, then the triangle is uniquely determined (up to congruence). You can find the unknown side lengths and angles of the triangle using the **sine rule** or the **cosine rule**. In this chapter, we establish these rules, and apply them in two- and three-dimensional problems.

We also apply trigonometry to circles. We find the lengths and angles associated with arcs and chords of circles, and we find the areas of sectors and segments of circles.

Note: An introduction to sine, cosine and tangent as functions is given in Mathematical Methods Units 1 & 2.

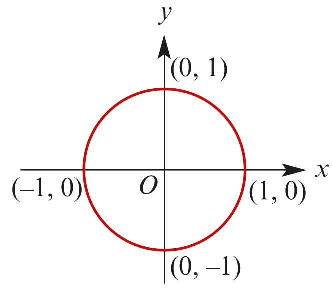
15A Reviewing trigonometry

In this section we review sine, cosine and tangent for angles between 0° and 180°.

Defining sine and cosine

The unit circle is a circle of radius 1 with centre at the origin.

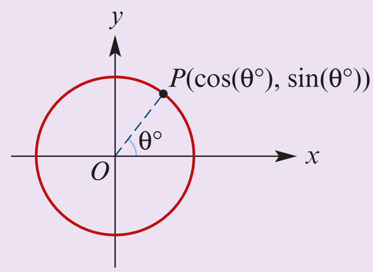
We can define the sine and cosine of any angle by using the unit circle.



Unit-circle definition of sine and cosine

For each angle θ°, there is a point P on the unit circle as shown. The angle is measured anticlockwise from the positive direction of the x-axis.

- $\cos(\theta°)$ is defined as the x-coordinate of the point P
- $\sin(\theta°)$ is defined as the y-coordinate of the point P



The trigonometric ratios

For acute angles, the unit-circle definition of sine and cosine given above is equivalent to the ratio definition.

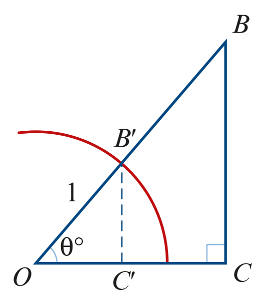
For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:

$$B'C' = \sin(\theta°) \quad \text{and} \quad OC' = \cos(\theta°)$$

As triangles OBC and $OB'C'$ are similar, we have

$$\frac{BC}{OB} = \frac{B'C'}{1} \quad \text{and} \quad \frac{OC}{OB} = \frac{OC'}{1}$$

$$\therefore \quad \frac{BC}{OB} = \sin(\theta°) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta°)$$

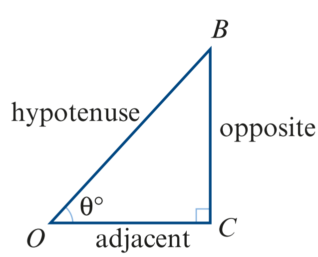


This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta°) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta°) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

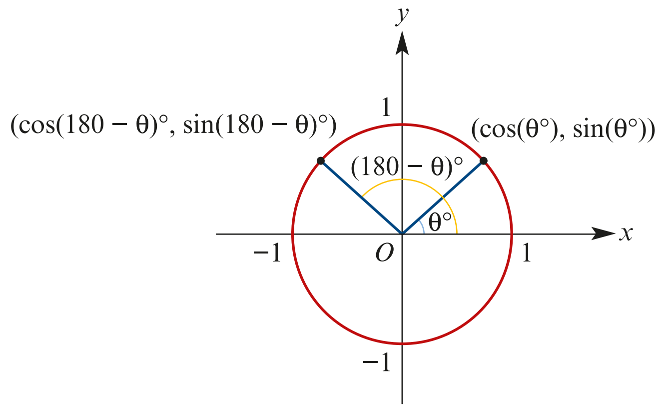
$$\tan(\theta°) = \frac{\text{opposite}}{\text{adjacent}}$$



Obtuse angles

From the unit circle, we see that

$$\sin(180 - \theta)^\circ = \sin(\theta^\circ)$$
$$\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$$



For example:

$$\sin 135^\circ = \sin 45^\circ$$
$$\cos 135^\circ = -\cos 45^\circ$$

In this chapter, we will generally use the ratio definition of tangent for acute angles. But we can also find the tangent of an obtuse angle by defining

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

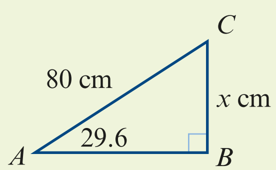
We will not consider angles greater than 180° or less than 0° in this chapter, since we are dealing with triangles.

Solving right-angled triangles

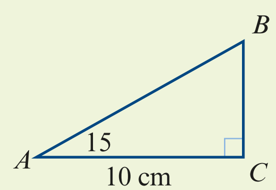
Here we provide some examples of using the trigonometric ratios.

Example 1

a Find the value of x correct to two decimal places.



b Find the length of the hypotenuse correct to two decimal places.



Solution

a
$$\frac{x}{80} = \sin 29.6^\circ$$
$$\therefore\ x = 80\sin 29.6^\circ$$
$$= 39.5153\ldots$$

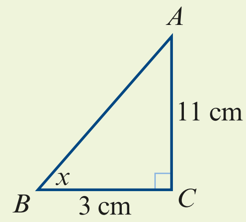
Hence $x = 39.52$, correct to two decimal places.

b
$$\frac{10}{AB} = \cos 15^\circ$$
$$10 = AB\cos 15^\circ$$
$$\therefore\ AB = \frac{10}{\cos 15^\circ}$$
$$= 10.3527\ldots$$

The length of the hypotenuse is 10.35 cm, correct to two decimal places.

Example 2

Find the magnitude of $\angle ABC$.



Solution

$$\tan x = \frac{11}{3}$$

$$\therefore \quad x = \tan^{-1}\left(\frac{11}{3}\right)$$

$$= (74.7448\ldots)^\circ$$

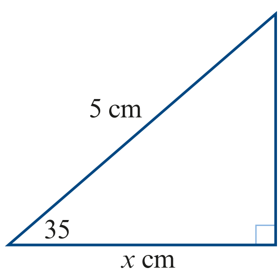
Hence $x = 74.74^\circ$, correct to two decimal places.

Exercise 15A

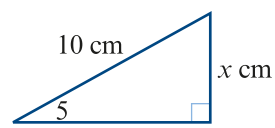
Example 1
Example 2

1 Find the value of x in each of the following:

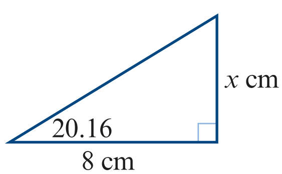
a



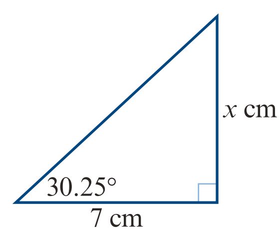
b



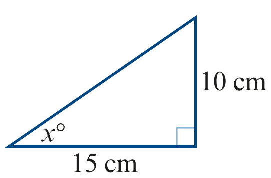
c



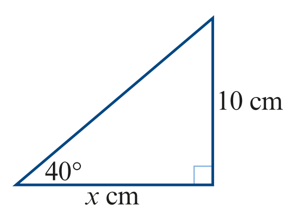
d



e



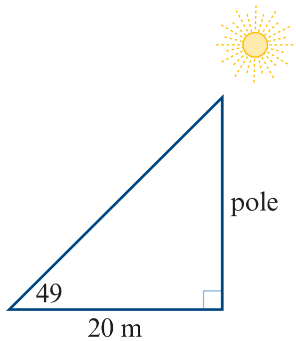
f



2 An equilateral triangle has altitudes of length 20 cm. Find the length of one side.

3 The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Find the magnitude of each of the three angles of the triangle.

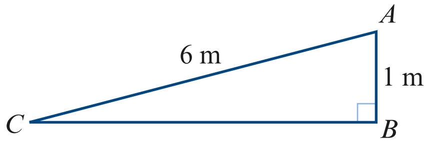
4 A pole casts a shadow 20 m long when the altitude of the sun is 49°. Calculate the height of the pole.



5 This figure represents a ramp.

a Find the magnitude of angle ACB.

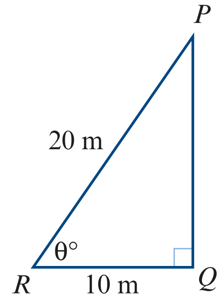
b Find the distance BC.



6 This figure shows a vertical mast PQ, which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground, which is 10 m from the foot of the mast.

a Calculate the angle of inclination, $\theta°$, of the wire to the ground.

b Calculate the height of the mast.



7 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

a the length of the ladder

b the height it reaches above the ground.

8 An engineer is designing a straight concrete entry ramp, 60 m long, for a car park that is 13 m above street level. Calculate the angle of the ramp to the horizontal.

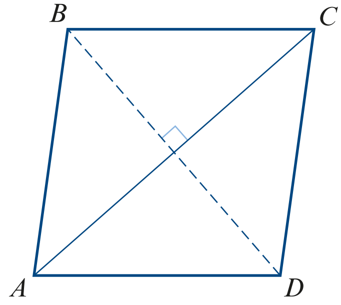
9 A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?

10 A mountain railway rises 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?

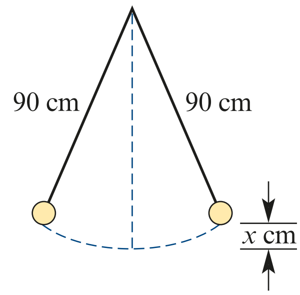
11 The diagonals of a rhombus bisect each other at right angles. If $BD = AC = 10$ cm, find:

a the length of the sides of the rhombus

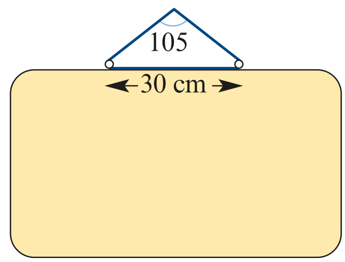
b the magnitude of angle ABC.



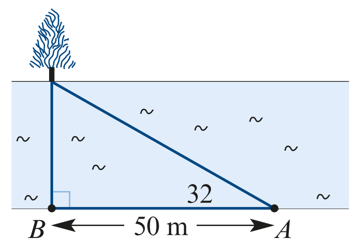
12 A pendulum swings from the vertical through an angle of 15° on each side of the vertical. If the pendulum is 90 cm long, what is the distance, x cm, between its highest and lowest points?



13 A picture is hung symmetrically by means of a string passing over a nail, with the ends of the string attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm, and the string makes an angle of 105° at the nail. Find the length of the string.

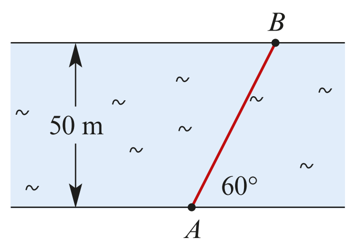


14 The distance AB is 50 m. If the line of sight to the tree of a person standing at A makes an angle of 32° with the bank, how wide is the river?



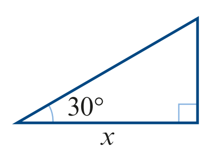
15 A ladder 4.7 m long is placed against a wall. The foot of the ladder must not be placed in a flower bed, which extends a distance of 1.7 m out from the base of the wall. How high up the wall can the ladder reach?

16 A river is known to be 50 m wide. A swimmer sets off from A to cross the river, and the path of the swimmer AB is as shown. How far does the person swim?



17 A rope is tied to the top of a flagpole. When it hangs straight down, it is 2 m longer than the pole. When the rope is pulled tight with the lower end on the ground, it makes an angle of 60° to the horizontal. How tall is the flagpole?

18 The triangle shown has perimeter 10. Find the value of x.



19 Consider the circle with equation $x^2 + y^2 - 4y = 0$ and the point $P(5, 2)$. Draw a diagram to show the circle and the two lines from P that are tangent to the circle. Find the angle between the two tangent lines, $\angle APB$, where A and B are the two points of contact.

15B The sine rule

In the previous section, we focused on right-angled triangles. In this section and the next, we consider non-right-angled triangles.

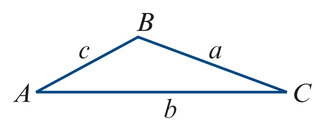
The **sine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1. one side and two angles are given
2. two sides and a non-included angle are given (that is, the given angle is not 'between' the two given sides).

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

Labelling triangles

The following convention is used in the remainder of this chapter:



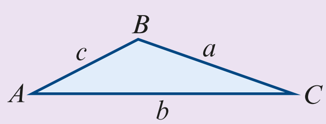
- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

For example, the magnitude of angle BAC is denoted by A, and the length of side BC is denoted by a.

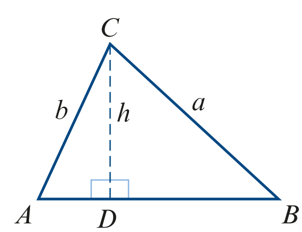
Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.



In triangle ACD:

$$\sin A = \frac{h}{b}$$

$$\therefore \quad h = b \sin A$$

In triangle BCD:

$$\sin B = \frac{h}{a}$$

$$\therefore \quad a \sin B = b \sin A$$

$$\text{i.e.} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, starting with a perpendicular from A to BC would give

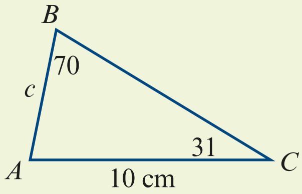
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

One side and two angles

When one side and two angles are given, this corresponds to the AAS congruence test. The triangle is uniquely defined up to congruence.

Example 3

Use the sine rule to find the length of AB.



Solution

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

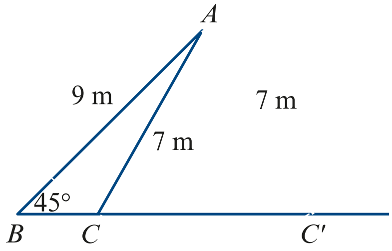
$$\therefore \quad c = \frac{10 \sin 31^\circ}{\sin 70^\circ}$$

$$= 5.4809\ldots$$

The length of AB is 5.48 cm, correct to two decimal places.

Two sides and a non-included angle

Suppose that we are given the two side lengths 7 m and 9 m and a non-included angle of 45°. There are two triangles that satisfy these conditions, as shown in the diagram.



Warning

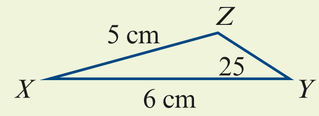
- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles.
- An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180°.

Note: If the given angle is obtuse or a right angle, then there is only one such triangle.

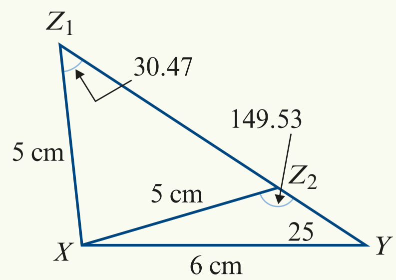
The following example illustrates the case where there are two possible triangles.

Example 4

Use the sine rule to find the magnitude of angle XZY in the triangle, given that $Y = 25°$, $y = 5$ cm and $z = 6$ cm.



Solution



$$\frac{5}{\sin 25°} = \frac{6}{\sin Z}$$

$$\frac{\sin Z}{6} = \frac{\sin 25°}{5}$$

$$\sin Z = \frac{6 \sin 25°}{5}$$

$$= 0.5071\ldots$$

$$\therefore \quad Z = (30.473\ldots)° \quad \text{or} \quad Z = (180 - 30.473\ldots)°$$

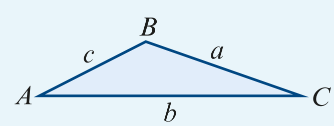
Hence $Z = 30.47°$ or $Z = 149.53°$, correct to two decimal places.

Note: Remember that $\sin(180 - \theta)° = \sin(\theta°)$.

Summary 15B

- **Sine rule** For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- When to use the sine rule:
  - one side and two angles are given (AAS)
  - two sides and a non-included angle are given.

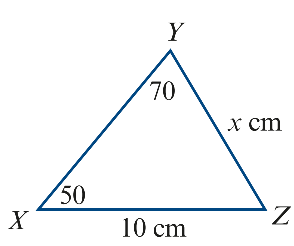
Skill-sheet

Exercise 15B

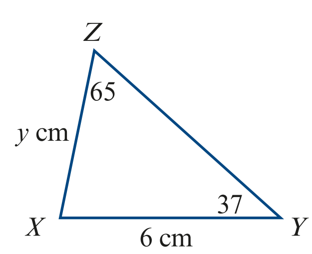
Example 3

1 Find the value of the pronumeral for each of the following triangles:

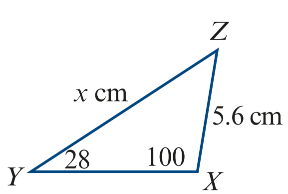
a



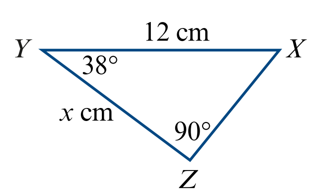
b



c



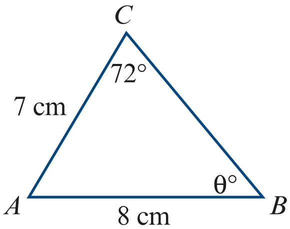
d



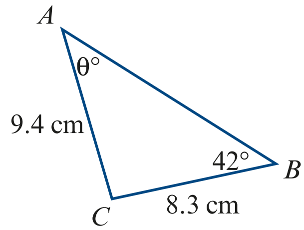
Example 4

2 Find the value of θ for each of the following triangles:

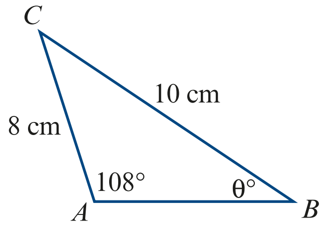
a



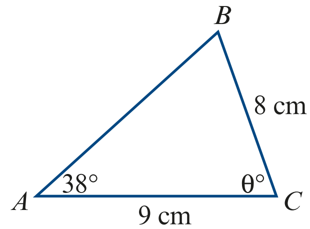
b



c



d



3 Solve the following triangles (i.e. find all sides and angles):

a $a = 12,\ B = 59°,\ C = 73°$

b $A = 75.3°,\ b = 5.6,\ B = 48.25°$

c $A = 123.2°,\ a = 11.5,\ C = 37°$

d $A = 23°,\ a = 15,\ B = 40°$

e $B = 140°,\ b = 20,\ A = 10°$

4 Solve the following triangles (i.e. find all sides and angles):

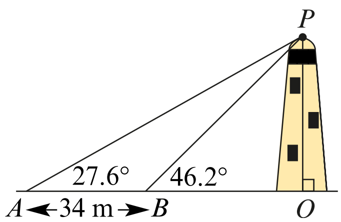
a $b = 17.6,\ C = 48.25°,\ c = 15.3$

b $B = 129°,\ b = 7.89,\ c = 4.56$

c $A = 28.35°,\ a = 8.5,\ b = 14.8$

5 A landmark A is observed from two points B and C, which are 400 m apart. The magnitude of angle ABC is measured as 68° and the magnitude of angle ACB as 70°. Find the distance of A from C.

6 P is a point at the top of a lighthouse. Measurements of the length AB and angles PBO and PAO are as shown in the diagram. Find the height of the lighthouse.

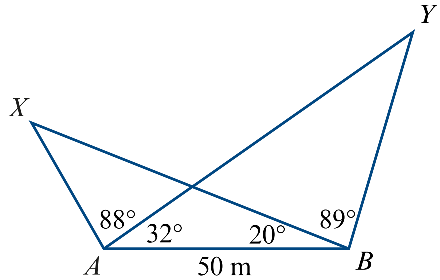


7 A and B are two points on a coastline, and C is a point at sea. The points A and B are 1070 m apart. The angles CAB and CBA have magnitudes of 74° and 69° respectively. Find the distance of C from A.

8 Find:

a AX

b AY



9 Use the sine rule to establish the following identities for triangles:

a $\dfrac{a+b}{c} = \dfrac{\sin A + \sin B}{\sin C}$ **b** $\dfrac{a-b}{c} = \dfrac{\sin A - \sin B}{\sin C}$

15C The cosine rule

The **cosine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1 two sides and the included angle are given

2 three sides are given.

In each case, the triangle is uniquely defined up to congruence.

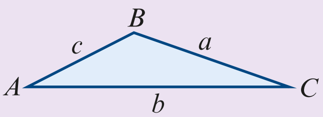
Cosine rule

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

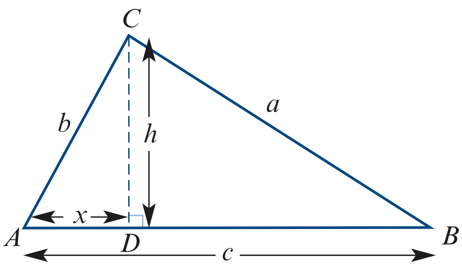


Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD:

$$\cos A = \frac{x}{b}$$

$$\therefore \quad x = b\cos A$$



Using Pythagoras' theorem in triangles ACD and BCD:

$$b^2 = x^2 + h^2$$
$$a^2 = (c-x)^2 + h^2$$

Expanding gives

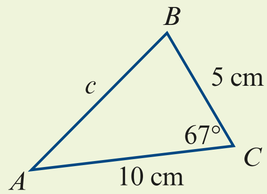
$$a^2 = c^2 - 2cx + x^2 + h^2$$
$$= c^2 - 2cx + b^2 \qquad (\text{as } b^2 = x^2 + h^2)$$
$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A \qquad (\text{as } x = b\cos A)$$

Two sides and the included angle

When two sides and the included angle are given, this corresponds to the SAS congruence test. The triangle is uniquely defined up to congruence.

Example 5

For triangle ABC, find the length of AB in centimetres correct to two decimal places.



Solution

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore \quad c = 9.2696\ldots$$

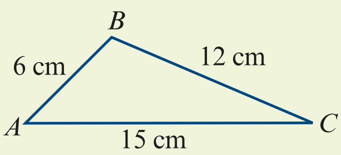
The length of AB is 9.27 cm, correct to two decimal places.

Three sides

When three sides are given, this corresponds to the SSS congruence test. The triangle is uniquely defined up to congruence.

Example 6

Find the magnitude of angle ABC.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$

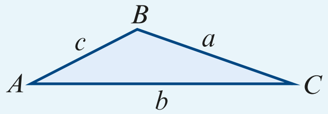
The magnitude of angle ABC is 108.21°, correct to two decimal places.

Summary 15C

- **Cosine rule** For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- When to use the cosine rule:
  - two sides and the included angle are given (SAS)
  - three sides are given (SSS).

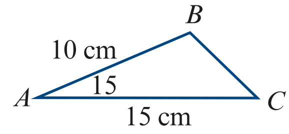


Skill-sheet

Exercise 15C

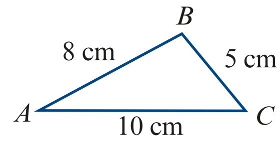
Example 5

1 Find the length of BC.



Example 6

2 Find the magnitudes of angles ABC and ACB.



3 For triangle ABC with:

a $A = 60°$ $b = 16$ $c = 30$, find a

b $a = 14$ $B = 53°$ $c = 12$, find b

c $a = 27$ $b = 35$ $c = 46$, find the magnitude of angle ABC

d $a = 17$ $B = 120°$ $c = 63$, find b

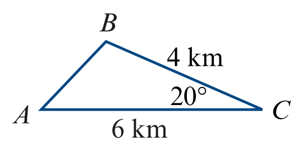
e $a = 31$ $b = 42$ $C = 140°$, find c

f $a = 10$ $b = 12$ $c = 9$, find the magnitude of angle BCA

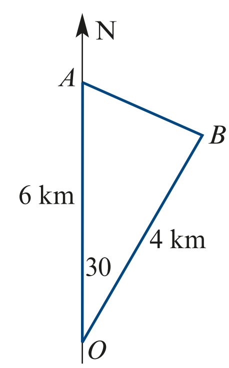
g $a = 11$ $b = 9$ $C = 43.2°$, find c

h $a = 8$ $b = 10$ $c = 15$, find the magnitude of angle CBA.

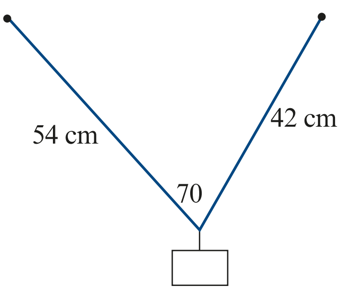
4 A section of an orienteering course is as shown. Find the length of leg AB.



5 Two ships sail in different directions from a point O. At a particular time, their positions A and B are as shown. Find the distance between the ships at this time.



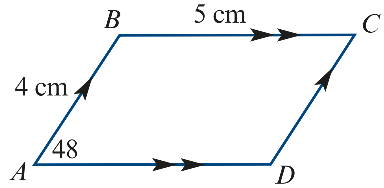
6 A weight is hung from two hooks in a ceiling by strings of length 54 cm and 42 cm, which are inclined at 70° to each other. Find the distance between the hooks.



7 *ABCD* is a parallelogram. Find the lengths of the diagonals:

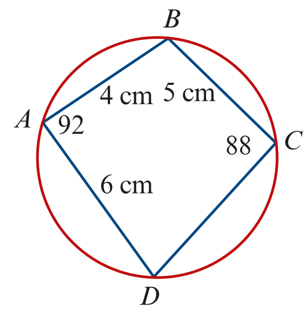
a *AC*

b *BD*



8 **a** Find the length of diagonal *BD*.

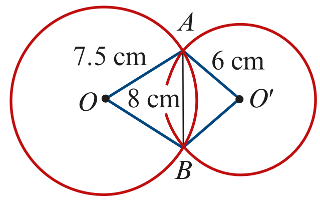
b Use the sine rule to find the length of *CD*.



9 Two circles of radius 7.5 cm and 6 cm have a common chord of length 8 cm.

a Find the magnitude of angle $AO'B$.

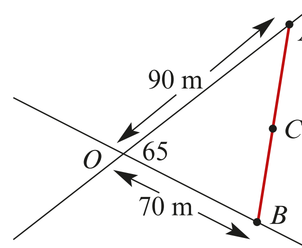
b Find the magnitude of angle AOB.



10 Two straight roads intersect at an angle of 65°. A point *A* on one road is 90 m from the intersection and a point *B* on the other road is 70 m from the intersection, as shown.

a Find the distance of *A* from *B*.

b If *C* is the midpoint of *AB*, find the distance of *C* from the intersection.

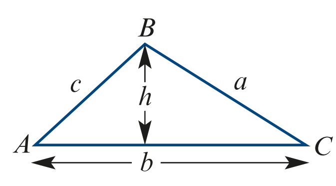


15D The area of a triangle

The area of a triangle is given by

$$\text{Area} = \frac{1}{2} \times \text{base length} \times \text{height}$$

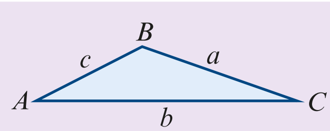
$$= \frac{1}{2}bh$$



By observing that $h = c \sin A$, we obtain the following useful formula.

For triangle *ABC*:

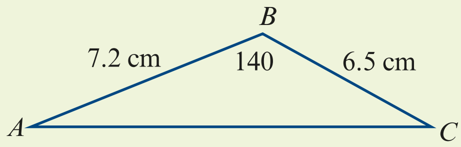
$$\text{Area} = \frac{1}{2}bc \sin A$$



That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

Example 7

Find the area of triangle ABC shown in the diagram.



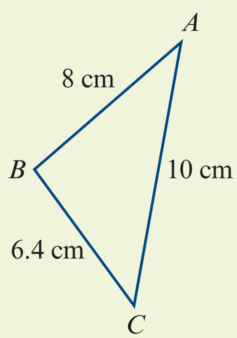
Solution

$$\text{Area} = \frac{1}{2} \times 7.2 \times 6.5 \sin 140^\circ$$
$$= 15.04 \text{ cm}^2 \qquad \text{(correct to two decimal places)}$$

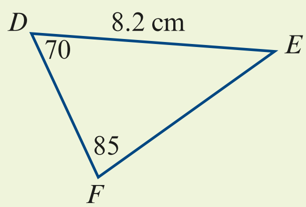
Example 8

Find the area of each of the following triangles, correct to three decimal places:

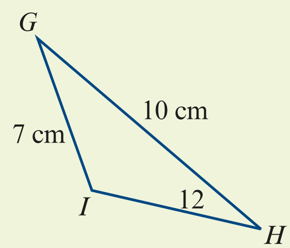
a



b



c



Solution

a Using the cosine rule:

$$8^2 = 6.4^2 + 10^2 - 2 \times 6.4 \times 10 \cos C$$
$$64 = 140.96 - 128 \cos C$$
$$\cos C = 0.60125$$
$$\therefore \quad C^\circ = (53.0405\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle ABC = \frac{1}{2} \times 6.4 \times 10 \times \sin C$$
$$= 25.570 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

b Note that $E^\circ = (180 - (70 + 85))^\circ = 25^\circ$.

Using the sine rule:

$$DF = \sin 25^\circ \times \frac{8.2}{\sin 85^\circ}$$
$$= 3.4787\ldots \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle DEF = \frac{1}{2} \times 8.2 \times DF \times \sin 70^\circ$$
$$= 13.403 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

c Using the sine rule:

$$\sin I = 10 \times \frac{\sin 12^\circ}{7}$$
$$= 0.2970\ldots$$

$$\therefore \quad I^\circ = (180 - 17.27\ldots)^\circ \qquad \text{(since } I \text{ is an obtuse angle)}$$
$$= (162.72\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

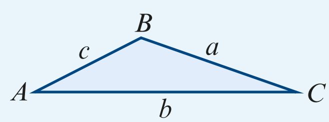
$$\therefore \quad G^\circ = (180 - (12 + I))^\circ$$
$$= (5.27\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle GHI = \frac{1}{2} \times 10 \times 7 \times \sin G$$
$$= 3.220 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

Summary 15D

For triangle ABC:

$$\text{Area} = \frac{1}{2} bc \sin A$$



That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

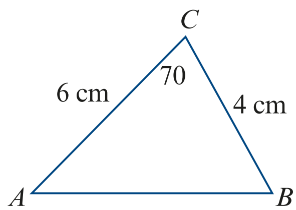
Skill-sheet

Exercise 15D

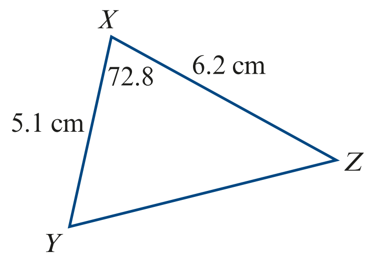
Example 7

1 Find the area of each of the following triangles:

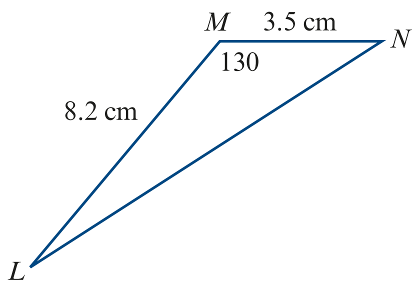
a



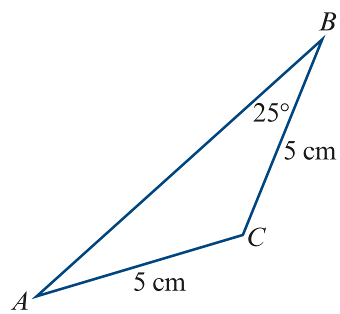
b



c



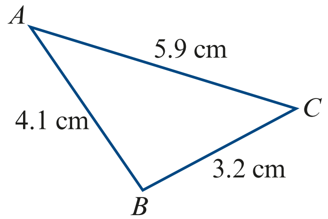
d



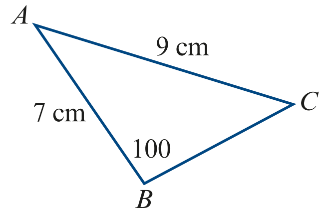
Example 8

2 Find the area of each of the following triangles, correct to three decimal places:

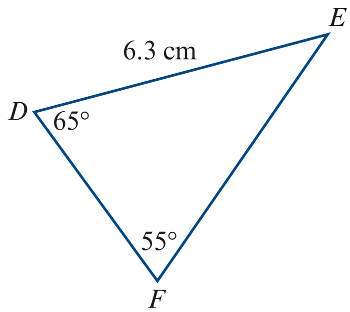
a



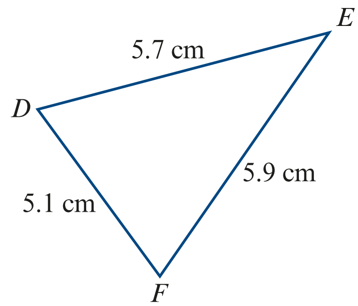
b



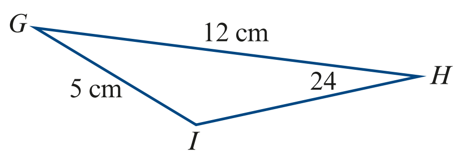
c



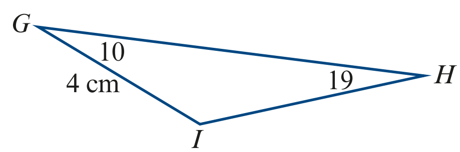
d



e



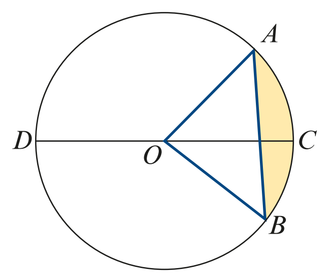
f



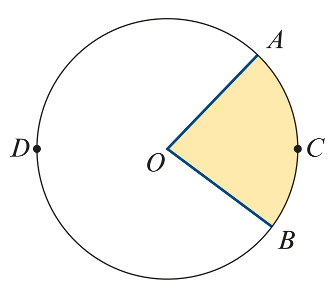
15E Circle mensuration

Terminology

In the diagram, the circle has centre O.

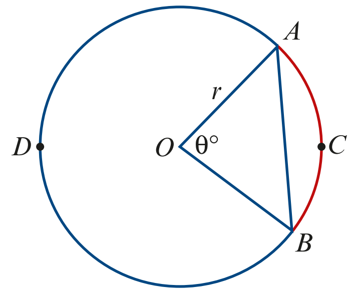


- **Chords** A **chord** of a circle is a line segment with endpoints on the circle; e.g. line segment AB in the diagram. A chord passing through the centre of the circle is called a **diameter**; e.g. line segment CD in the diagram.
- **Arcs** Any two points on a circle divide the circle into arcs. The shorter arc is called the **minor arc** and the longer is the **major arc**. In the diagram, arc ACB is a minor arc and arc ADB is a major arc. The arcs DAC and DBC are called **semicircular arcs**.
- **Segments** Every chord divides the interior of a circle into two regions called segments. The smaller is called the **minor segment** and the larger is the **major segment**. In the above diagram, the minor segment has been shaded.
- **Sectors** Two radii and an arc define a region called a sector. In this diagram, with circle centre O, the shaded region is a **minor sector** and the unshaded region is a **major sector**.



Arc length

The circle in the diagram has centre O and radius r. The arc ACB and the corresponding chord AB are said to **subtend** the angle $\angle AOB$ at the centre of the circle.



The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The length ℓ of arc ACB will be the same fraction of the circumference of the circle, $2\pi r$.

Length of an arc using degrees

$$\ell = \frac{\theta}{360} \times 2\pi r$$

$$= \frac{\pi r \theta}{180} \qquad \text{(where } \theta \text{ is measured in degrees)}$$

Radian measure of angles is introduced in Mathematical Methods Units 1 & 2.

We recall that, in the unit circle, an arc of length θ units subtends an angle of θ radians at the centre. A circle of radius r is similar to the unit circle, with similarity factor r, and therefore an arc of length $r\theta$ units subtends an angle of θ radians at the centre.

Length of an arc using radians

$$\ell = r\theta \qquad \text{(where } \theta \text{ is measured in radians)}$$

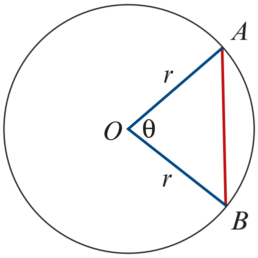
Note: As there are 2π radians in a circle, the arc length is $\ell = \frac{\theta}{2\pi} \times 2\pi r = r\theta$.

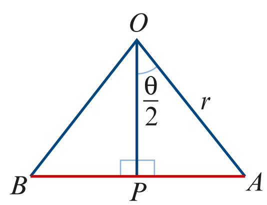
Chord length

In triangle OAP:

$$AP = r\sin\left(\frac{\theta}{2}\right)$$

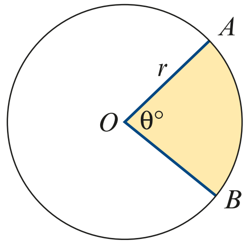
$$\therefore \quad AB = 2r\sin\left(\frac{\theta}{2}\right)$$





Area of a sector

The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The area of the sector will be the same fraction of the area of the circle, πr^2.

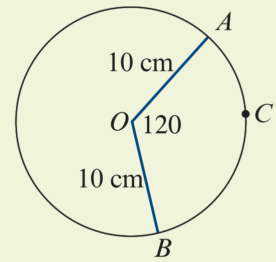


Using degrees: Area of sector $= \dfrac{\pi r^2 \theta}{360}$

Using radians: Area of sector $= \dfrac{1}{2} r^2 \theta$

Example 9

The circle shown has centre O and radius length 10 cm. The angle subtended at O by arc ACB has magnitude 120°. Find:



a **i** the exact length of the chord AB

ii the exact length of the arc ACB

b the exact area of the minor sector AOB

c the magnitude of angle AOC, in degrees, if the minor arc AC has length 4 cm.

Solution

a **i** Chord length $= 2r\sin\left(\dfrac{\theta}{2}\right)$

$= 20\sin 60°$ since $r = 10$ and $\theta = 120°$

$= 20 \times \dfrac{\sqrt{3}}{2}$

$= 10\sqrt{3}$

Length of chord is $10\sqrt{3}$ cm.

ii Arc length $\ell = r\theta$ using radians

$= 10 \times \dfrac{2\pi}{3}$ since $r = 10$ and $\theta = \dfrac{2\pi}{3}$

$= \dfrac{20\pi}{3}$

Length of arc is $\dfrac{20\pi}{3}$ cm.

Check: Verify that length of arc is greater than length of chord.

b Area of sector $= \dfrac{1}{2}r^2\theta$ using radians

$= \dfrac{1}{2} \times 10^2 \times \dfrac{2\pi}{3}$ since $r = 10$ and $\theta = \dfrac{2\pi}{3}$

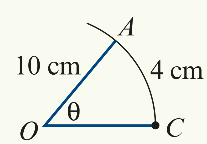
$= \dfrac{100\pi}{3}$

Area of minor sector AOB is $\dfrac{100\pi}{3}$ cm^2.

c Using radians: $\ell = r\theta$

$4 = 10\theta$

$\therefore \quad \theta = \dfrac{4}{10}$



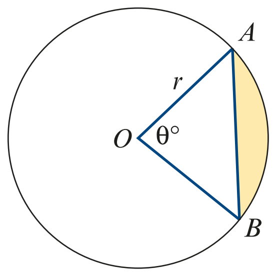
Convert to degrees: $\angle AOC = 0.4 \times \dfrac{180}{\pi}$

$= (22.9183\ldots)°$

$= 22.92°$ (correct to two decimal places)

Area of a segment

The area of the shaded segment is found by subtracting the area of $\triangle AOB$ from the area of the minor sector OAB.



Using degrees: Area of segment $= \dfrac{\pi r^2\theta}{360} - \dfrac{1}{2}r^2 \sin\theta$

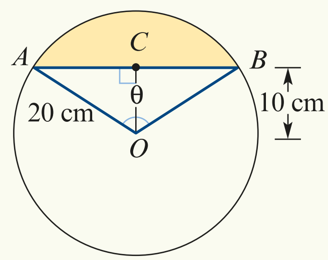
Using radians: Area of segment $= \dfrac{1}{2}r^2\theta - \dfrac{1}{2}r^2 \sin\theta$

Example 10

A circle, with centre O and radius length 20 cm, has a chord AB that is 10 cm from the centre of the circle. Calculate the area of the minor segment formed by this chord.

Solution

The area of the segment is $\dfrac{1}{2}r^2(\theta - \sin\theta)$. We know $r = 20$, but we need to find θ.



In $\triangle OCB$:

$$\cos\left(\frac{\theta}{2}\right) = \frac{10}{20}$$

$$\frac{\theta}{2} = \frac{\pi}{3}$$

$$\therefore \quad \theta = \frac{2\pi}{3}$$

$$\text{Area of segment} = \frac{1}{2} \times 20^2\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$$

$$= 200\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 200\left(\frac{4\pi - 3\sqrt{3}}{6}\right)$$

$$= \frac{100(4\pi - 3\sqrt{3})}{3}\ \text{cm}^2$$

Summary 15E

- **Circle mensuration formulas with θ in radians**
  - Arc length $= r\theta$
  - Chord length $= 2r\sin\left(\dfrac{\theta}{2}\right)$
  - Area of sector $= \dfrac{1}{2}r^2\theta$
  - Area of segment $= \dfrac{1}{2}r^2(\theta - \sin\theta)$
- **Circle mensuration formulas with θ in degrees**
  - Arc length $= \dfrac{\pi r\theta}{180}$
  - Chord length $= 2r\sin\left(\dfrac{\theta}{2}\right)$
  - Area of sector $= \dfrac{\pi r^2\theta}{360}$
  - Area of segment $= \dfrac{\pi r^2\theta}{360} - \dfrac{1}{2}r^2\sin\theta$

Skill-sheet

Exercise 15E

Example 9

1 Find the length of an arc which subtends an angle of magnitude 105° at the centre of a circle of radius length 25 cm.

2 Find the magnitude, in degrees, of the angle subtended at the centre of a circle of radius length 30 cm by:

a an arc of length 50 cm

b a chord of length 50 cm.

Example 10

3 A chord of length 6 cm is drawn in a circle of radius 7 cm. Find:

a the length of the minor arc cut off by the chord

b the area of the smaller region inside the circle cut off by the chord.

4 Sketch, on the same set of axes, the graphs of $A = \{(x, y) : x^2 + y^2 \leq 16\}$ and $B = \{(x, y) : y \geq 2\}$. Find the area measure of the region $A \cap B$.

5 Find the area of the region between an equilateral triangle of side length 10 cm and the circumcircle of the triangle (the circle that passes through the three vertices of the triangle).

6 A person stands on level ground 60 m from the nearest point of a cylindrical tank of radius length 20 m. Calculate:

a the circumference of the tank

b the percentage of the circumference that is visible to the person.

7 The minute hand of a large clock is 4 m long.

a How far does the tip of the minute hand move between 12:10 p.m. and 12:35 p.m.?

b What is the area covered by the minute hand between 12:10 p.m. and 12:35 p.m.?

8 Two circles of radii 3 cm and 4 cm have their centres 5 cm apart. Calculate the area of the region common to both circles.

9 A sector of a circle has perimeter 32 cm and area 63 cm^2. Find the radius length and the magnitude of the angle subtended at the centre of the two possible sectors.

10 Two wheels (pulleys) have radii of length 15 cm and 25 cm and have their centres 60 cm apart. What is the length of the belt required to pass tightly around the pulleys without crossing?

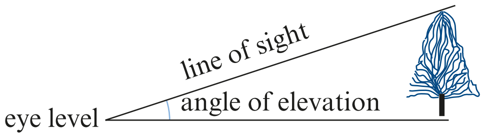
11 A frame in the shape of an equilateral triangle encloses three circular discs of radius length 5 cm so that the discs touch each other. Find:

a the perimeter of the smallest frame which can enclose the discs

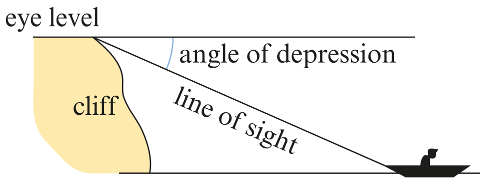
b the area enclosed between the discs.

15F Angles of elevation, angles of depression and bearings

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.



The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



Example 11

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Calculate the horizontal distance of the boat to the helicopter.

Solution

Note that $\angle ABH = 1.2°$, using alternate angles.

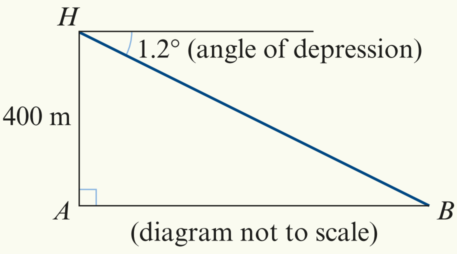
Thus

$$\frac{AH}{AB} = \tan 1.2°$$

$$\frac{400}{AB} = \tan 1.2°$$

$$\therefore \quad AB = \frac{400}{\tan 1.2°}$$

$$= 19\,095.800\ldots$$



The horizontal distance is 19 100 m, correct to the nearest 10 m.

Example 12

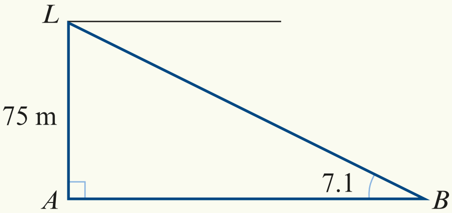
The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Calculate the distance of the boat from the lighthouse.

Solution

$$\frac{75}{AB} = \tan 7.1°$$

$$\therefore \quad AB = \frac{75}{\tan 7.1°}$$

$$= 602.135\ldots$$



The distance of the boat from the lighthouse is 602 m, correct to the nearest metre.

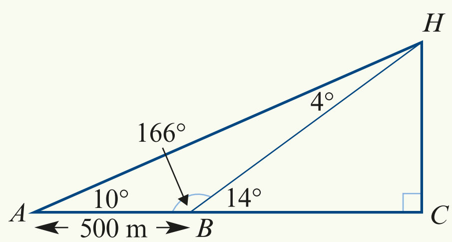
Example 13

From the point A, a man observes that the angle of elevation of the summit of a hill is 10°. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of 14°. Find the height of the hill above the level of A.

Solution

Magnitude of $\angle HBA = (180 - 14)^\circ = 166^\circ$

Magnitude of $\angle AHB = (180 - (166 + 10))^\circ = 4^\circ$



Using the sine rule in triangle ABH:

$$\frac{500}{\sin 4^\circ} = \frac{HB}{\sin 10^\circ}$$

$$\therefore \quad HB = \frac{500 \sin 10^\circ}{\sin 4^\circ}$$

$$= 1244.67\ldots$$

In triangle BCH:

$$\frac{HC}{HB} = \sin 14^\circ$$

$$\therefore \quad HC = HB \sin 14^\circ$$

$$= 301.11\ldots$$

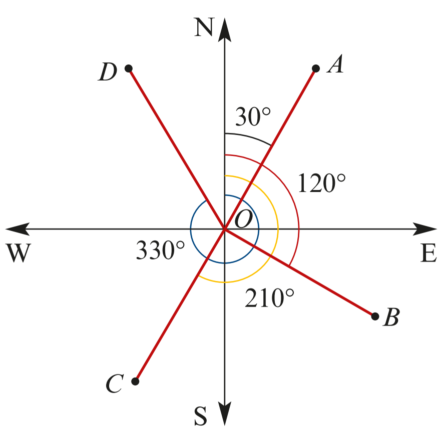
The height of the hill is 301 m, correct to the nearest metre.

Bearings

The **bearing** (or compass bearing) is the direction measured from north clockwise.

For example:

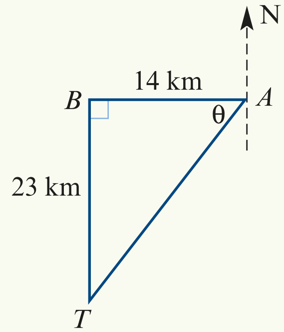
- The bearing of A from O is 030°.
- The bearing of B from O is 120°.
- The bearing of C from O is 210°.
- The bearing of D from O is 330°.



Example 14

The road from town A runs due west for 14 km to town B. A television mast is located due south of B at a distance of 23 km. Calculate the distance and bearing of the mast from the centre of town A.

Solution



$$\tan\theta = \frac{23}{14}$$

$\therefore \quad \theta = 58.67^\circ \quad$ (to two decimal places)

Thus the bearing is

$$180^\circ + (90 - 58.67)^\circ = 211.33^\circ$$

To find the distance, use Pythagoras' theorem:

$$\begin{aligned} AT^2 &= AB^2 + BT^2 \\ &= 14^2 + 23^2 \\ &= 725 \end{aligned}$$

$\therefore \quad AT = 26.925\ldots$

The mast is 27 km from the centre of town A (to the nearest kilometre) and on a bearing of 211.33°.

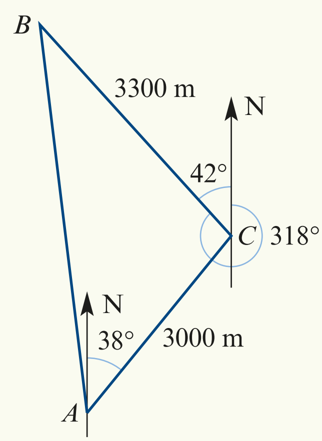
Example 15

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318° and after sailing for a further 3300 m reaches a point B. Find:

a the distance AB

b the bearing of B from A.

Solution



a The magnitude of angle ACB needs to be found so that the cosine rule can be applied in triangle ABC:

$$\angle ACB = (180 - (38 + 42))^\circ = 100^\circ$$

In triangle ABC:

$$\begin{aligned} AB^2 &= 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \cos 100^\circ \\ &= 23\,328\,233.917\ldots \\ \therefore \quad AB &= 4829.931\ldots \end{aligned}$$

The distance of B from A is 4830 m (to the nearest metre).

b To find the bearing of B from A, the magnitude of angle BAC must first be found. Using the sine rule:

$$\frac{3300}{\sin A} = \frac{AB}{\sin 100°}$$

$$\therefore \quad \sin A = \frac{3300 \sin 100°}{AB}$$

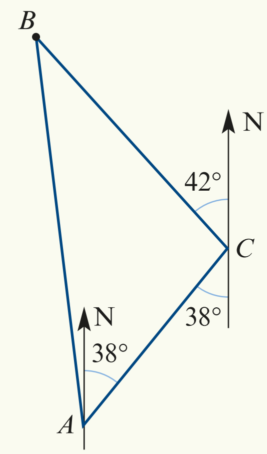
$$= 0.6728\ldots$$

$$\therefore \quad A = (42.288\ldots)°$$

The bearing of B from A = 360° − (42.29° − 38°)

= 355.71°

The bearing of B from A is 356° to the nearest degree.



Exercise 15F

Example 11

1 From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

Example 12

2 The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Find the height of the chimney.

3 A hiker standing on top of a mountain observes that the angle of depression to the base of a building is 41°. If the height of the hiker above the base of the building is 500 m, find the horizontal distance from the hiker to the building.

4 A person lying down on top of a cliff 40 m high observes the angle of depression to a buoy in the sea below to be 20°. If the person is in line with the buoy, find the distance between the buoy and the base of the cliff, which may be assumed to be vertical.

Example 13

5 A person standing on top of a cliff 50 m high is in line with two buoys whose angles of depression are 18° and 20°. Calculate the distance between the buoys.

Example 14

6 A ship sails 10 km north and then sails 15 km east. What is its bearing from the starting point?

7 A ship leaves port A and travels 15 km due east. It then turns and travels 22 km due north.

a What is the bearing of the ship from port A?

b What is the bearing of port A from the ship?

Example 15

8 A yacht sails from point A on a bearing of 035° for 2000 m. It then alters course to a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

a Find the distance AB.

b Find the bearing of B from A.

9 The bearing of a point A from a point B is 207°. What is the bearing of B from A?

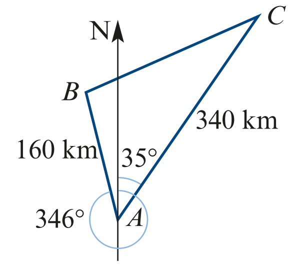
10 The bearing of a ship S from a lighthouse A is 055°. A second lighthouse B is due east of A. The bearing of S from B is 302°. Find the magnitude of angle ASB.

11 A yacht starts from L and sails 12 km due east to M. It then sails 9 km on a bearing of 142° to K. Find the magnitude of angle MLK.

12 The bearing of C from A is 035°. The bearing of B from A is 346°. The distance of C from A is 340 km. The distance of B from A is 160 km.

a Find the magnitude of angle BAC.

b Use the cosine rule to find the distance from B to C.



13 From a ship S, two other ships P and Q are on bearings 320° and 075° respectively. The distance PS is 7.5 km and the distance QS is 5 km. Find the distance PQ.

15G Problems in three dimensions

Some problems in three dimensions can be solved by picking out triangles from a main figure and finding lengths and angles through these triangles.

Example 16

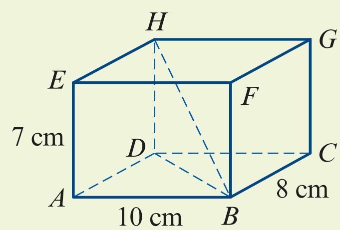
$ABCDEFGH$ is a cuboid. Find:

a the distance DB

b the distance HB

c the magnitude of angle HBD

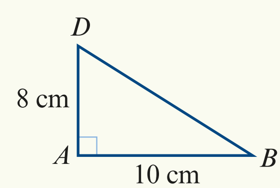
d the magnitude of angle HBA.



Solution

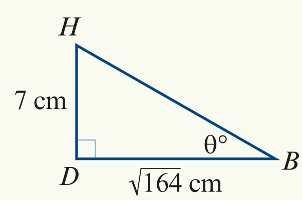
a

$$DB^2 = 8^2 + 10^2$$
$$= 164$$
$$\therefore DB = \sqrt{164}$$
$$= 12.81 \text{ cm} \quad \text{(correct to two decimal places)}$$



b

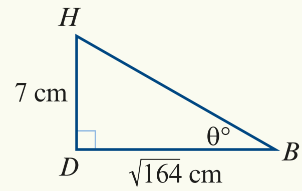
$$HB^2 = HD^2 + DB^2$$
$$= 7^2 + 164$$
$$= 213$$
$$\therefore HB = \sqrt{213}$$
$$= 14.59 \text{ cm} \quad \text{(correct to two decimal places)}$$



c $\tan\theta = \dfrac{HD}{BD}$

$= \dfrac{7}{\sqrt{164}}$

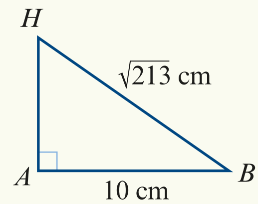
$\therefore\ \theta = 28.66^\circ$ (correct to two decimal places)



d From triangle HBA:

$\cos B = \dfrac{10}{\sqrt{213}}$

$\therefore\ B = 46.75^\circ$ (correct to two decimal places)



Example 17

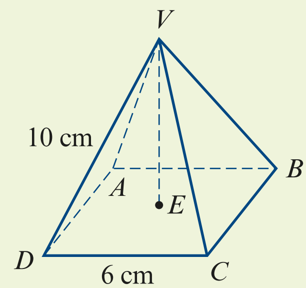
The figure shows a pyramid with a square base. The base has sides 6 cm long and the edges VA, VB, VC and VD are each 10 cm long.

a Find the length of DB.

b Find the length of BE.

c Find the length of VE.

d Find the magnitude of angle VBE.



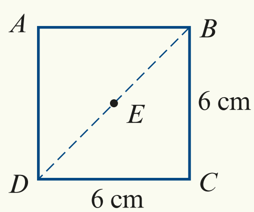
Solution

a $DB^2 = 6^2 + 6^2$

$= 72$

$\therefore\ DB = 6\sqrt{2}$

$= 8.4852\ldots$



The length of DB is 8.49 cm, correct to two decimal places.

b $BE = \dfrac{1}{2}DB$

$= 3\sqrt{2}$

$= 4.2426\ldots$

The length of BE is 4.24 cm, correct to two decimal places.

c $VE^2 = VB^2 - BE^2$

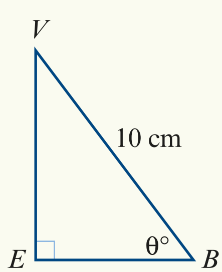
$= 10^2 - (3\sqrt{2})^2$

$= 100 - 18$

$= 82$

$\therefore\ VE = \sqrt{82}$

$= 9.0553\ldots$



The length of VE is 9.06 cm, correct to two decimal places.

d $\sin\theta = \dfrac{VE}{VB}$

$= \dfrac{\sqrt{82}}{10}$

$= 0.9055\ldots$

$\therefore\ \theta = (64.8959\ldots)^\circ$

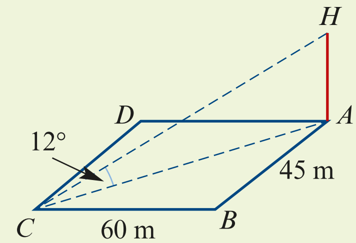
The magnitude of $\angle VBE$ is 64.90°, correct to two decimal places.

Example 18

A communications mast is erected at corner A of a rectangular courtyard $ABCD$ with side lengths 60 m and 45 m as shown. If the angle of elevation of the top of the mast from C is 12°, find:

a the height of the mast

b the angle of elevation of the top of the mast from B.

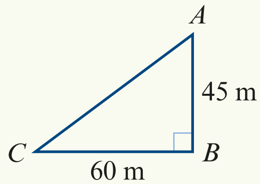


Solution

a $AC^2 = 45^2 + 60^2$

$= 5625$

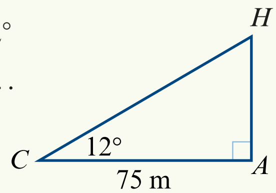
$\therefore \quad AC = 75$



$\dfrac{HA}{75} = \tan 12°$

$\therefore \quad HA = 75 \tan 12°$

$= 15.941\ldots$

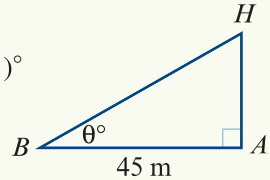


The height of the mast is 15.94 m, correct to two decimal places.

b $\tan \theta = \dfrac{HA}{45}$

$= 0.3542\ldots$

$\therefore \quad \theta = (19.507\ldots)°$



The angle of elevation of the top of the mast, H, from B is 19.51°, correct to two decimal places.

Exercise 15G

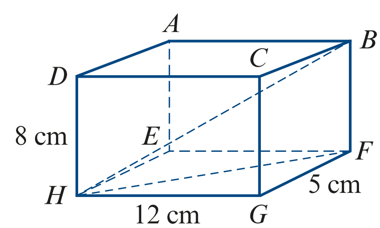
Example 16

1 $ABCDEFGH$ is a cuboid with dimensions as shown. Find:

a the length of FH **b** the length of BH

c the magnitude of angle BHF

d the magnitude of angle BHG.



Example 17

2 $VABCD$ is a right pyramid with a square base. The sides of the base are 8 cm in length. The height, VF, of the pyramid is 12 cm. If E is the midpoint of AD, find:

a the length of EF

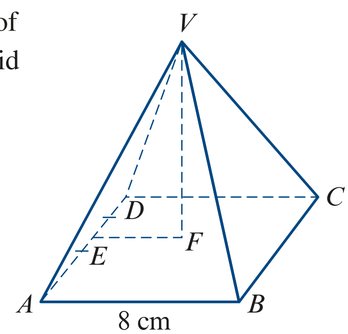
b the magnitude of angle VEF

c the length of VE

d the length of a sloping edge

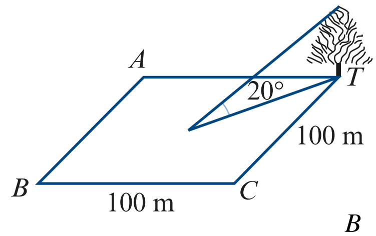
e the magnitude of angle VAD

f the surface area of the pyramid.



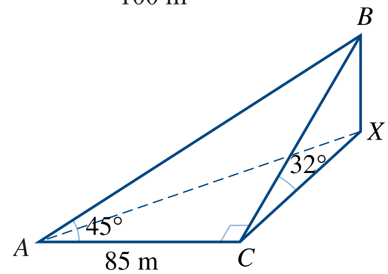
Example 18

3 A tree stands at a corner of a square playing field. Each side of the square is 100 m long. At the centre of the field, the tree subtends an angle of 20°. What angle does it subtend at each of the other three corners of the field?



4 Suppose that A, C and X are three points in a horizontal plane and that B is a point vertically above X. The length of AC is 85 m and the magnitudes of angles BAC, ACB and BCX are 45°, 90° and 32° respectively. Find:

a the distance CB **b** the height XB.

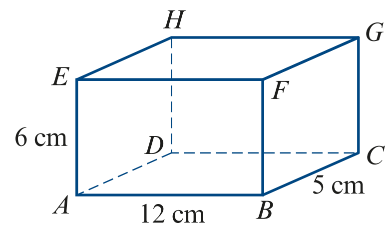


5 Standing due south of a tower 50 m high, the angle of elevation of the top is 26°. What is the angle of elevation after walking a distance 120 m due east?

6 From the top of a cliff 160 m high, two buoys are observed. Their bearings are 337° and 308°. Their respective angles of depression are 3° and 5°. Calculate the distance between the buoys.

7 Find the magnitude of each of the following angles for the cuboid shown:

a ACE **b** HDF **c** ECH



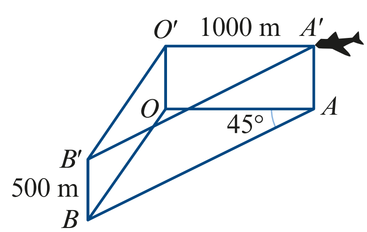
8 From a point A due north of a tower, the angle of elevation to the top of the tower is 45°. From point B, which is 100 m from A on a bearing of 120°, the angle of elevation is 26°. Find the height of the tower.

9 A and B are two positions on level ground. From an advertising balloon at a vertical height of 750 m, point A is observed in an easterly direction and point B at a bearing of 160°. The angles of depression of A and B, as viewed from the balloon, are 40° and 20° respectively. Find the distance between A and B.

10 A right pyramid, height 6 cm, stands on a square base of side length 5 cm. Find:

a the length of a sloping edge **b** the area of a triangular face.

11 A light aircraft flying at a height of 500 m above the ground is sighted at a point A' due east of an observer at a point O on the ground, measured horizontally to be 1 km from the plane. The aircraft is flying south-west (along $A'B'$) at 300 km/h.



a How far will it travel in one minute?

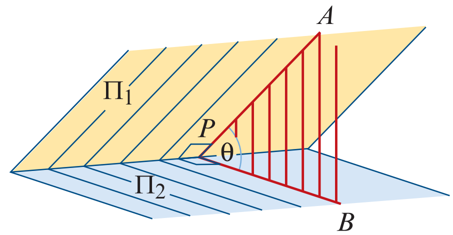
b Find its bearing from O (O') at this time.

c What will be its angle of elevation from O at this time?

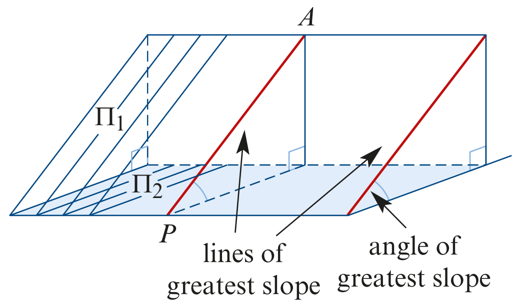
15H Angles between planes and more difficult 3D problems

Angles between planes

Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between planes Π_1 and Π_2.



Note: If the plane Π_2 is horizontal, then PA is called a **line of greatest slope** in the plane Π_1.

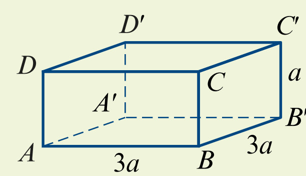


Example 19

For the cuboid shown in the diagram, find:

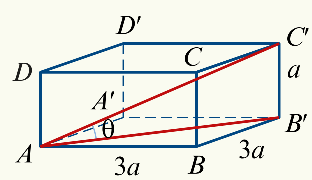
a the angle between AC' and the plane $ABB'A'$

b the angle between the planes ACD' and DCD'.



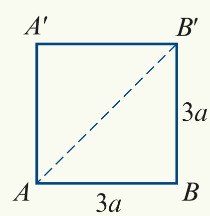
Solution

a To find the angle θ between AC' and the plane $ABB'A'$, we need the projection of AC' in the plane.



We drop a perpendicular from C' to the plane (line $C'B'$), and join the foot of the perpendicular to A (line $B'A$).

The required angle, θ, lies between $C'A$ and $B'A$.

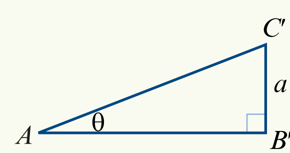


Draw separate diagrams showing the base and the section through A, C' and B'. Then we see that

$$AB' = \sqrt{(3a)^2 + (3a)^2} = 3a\sqrt{2}$$

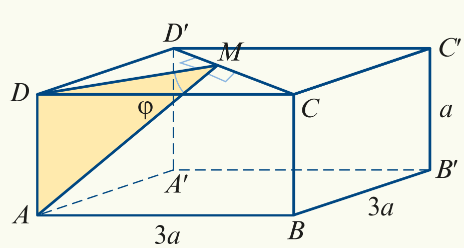
and $\quad \tan\theta = \dfrac{a}{3a\sqrt{2}} = \dfrac{1}{3\sqrt{2}}$

Hence the required angle, θ, is 13.26°.

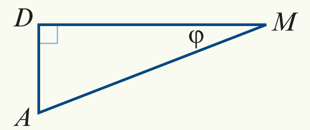


b The line common to the planes ACD' and DCD' is CD'. Let M be the midpoint of the line segment CD'.

Then MD is perpendicular to CD' in the plane DCD', and MA is perpendicular to CD' in the plane ACD'.



Thus φ is the angle between the planes DCD' and ACD'. We have



$$DM = \frac{1}{2}DC' = \frac{1}{2}(3a\sqrt{2})$$

$$\therefore \quad \tan\varphi = a \div \left(\frac{3a\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{3}$$

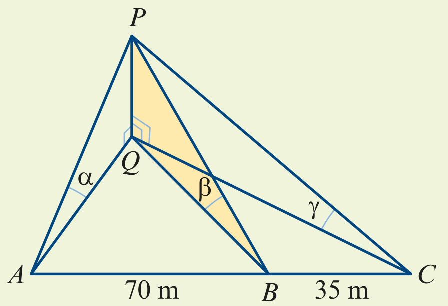
Hence the required angle is $\varphi = 25.24°$.

Example 20

Three points A, B and C are on a horizontal line such that $AB = 70$ m and $BC = 35$ m. The angles of elevation of the top of a tower are α, β and γ, where

$$\tan\alpha = \frac{1}{13}, \quad \tan\beta = \frac{1}{15}, \quad \tan\gamma = \frac{1}{20}$$

as shown in the diagram.



The base of the tower is at the same level as A, B and C. Find the height of the tower.

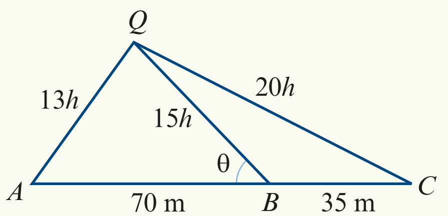
Solution

Let the height of the tower, PQ, be h m. Then

$$h = QA\tan\alpha = QB\tan\beta = QC\tan\gamma$$

which implies that

$$QA = 13h, \quad QB = 15h, \quad QC = 20h$$



Now consider the base triangle ACQ.

Using the cosine rule in $\triangle AQB$:

$$\cos\theta = \frac{(70)^2 + (15h)^2 - (13h)^2}{2(70)(15h)}$$

Using the cosine rule in $\triangle CQB$:

$$-\cos\theta = \cos(180° - \theta) = \frac{(35)^2 + (15h)^2 - (20h)^2}{2(35)(15h)}$$

Hence

$$\frac{(70)^2 + (15h)^2 - (13h)^2}{2(70)(15h)} = \frac{(20h)^2 - (15h)^2 - (35)^2}{2(35)(15h)}$$

$$4900 + 56h^2 = 2(175h^2 - 1225)$$

$$7350 = 294h^2$$

$$\therefore \quad h = 5$$

The height of the tower is 5 m.

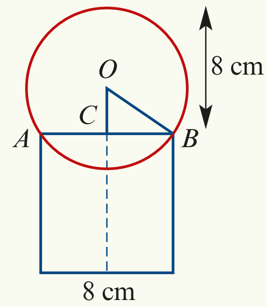
Example 21

A sphere rests on the top of a vertical cylinder which is open at the top. The inside diameter of the cylinder is 8 cm. The sphere projects 8 cm above the top of the cylinder. Find the radius length of the sphere.

Solution

This 3D problem can be represented by a 2D diagram without loss of information.

Let the radius length of the sphere be r cm. Then, in $\triangle OBC$, we have



$$OC = (8 - r) \text{ cm}, \quad BC = 4 \text{ cm}, \quad OB = r \text{ cm}$$

Using Pythagoras' theorem:

$$(8 - r)^2 + 4^2 = r^2$$
$$64 - 16r + r^2 + 16 = r^2$$
$$-16r + 80 = 0$$
$$\therefore \quad r = 5$$

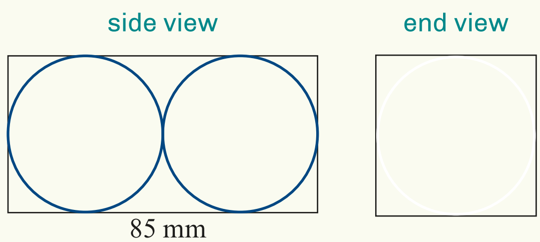
The radius length of the sphere is 5 cm.

Example 22

A box contains two standard golf balls that fit snugly inside. The box is 85 mm long. What percentage of the space inside the box is air?

Solution

Two 2D diagrams may be used to represent the 3D situation.



Let r mm be the radius length of a golf ball.

Length of box = 85 mm = $4r$ mm

Thus $r = \dfrac{85}{4}$, i.e. $r = 21.25$

So the box has dimensions 85 mm by 42.5 mm by 42.5 mm.

Now volume of box $= 42.5^2 \times 85$ using $V = Ah$

volume of two golf balls $= 2 \times \dfrac{4}{3} \times \pi \times 21.25^3$ using $V = \dfrac{4}{3}\pi r^3$

$$= \frac{8}{3}\pi \times 21.25^3$$

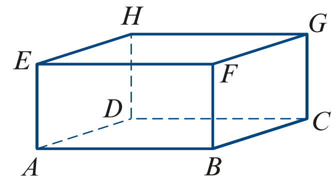
Hence percentage air $= \dfrac{100(42.5^2 \times 85 - \frac{8}{3}\pi \times 21.25^3)}{42.5^2 \times 85}$

$= 47.6\%$ to one decimal place

Exercise 15H

Example 19

1 The diagram shows a rectangular prism. Assume that $AB = 4a$ units, $BC = 3a$ units, $GC = a$ units.



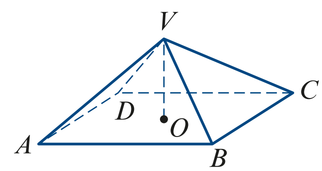
a Calculate the areas of the faces $ABFE$, $BCGF$ and $ABCD$.

b Calculate the magnitude of the angle which plane $GFAD$ makes with the base.

c Calculate the magnitude of the angle which plane $HGBA$ makes with the base.

d Calculate the magnitude of the angle which AG makes with the base.

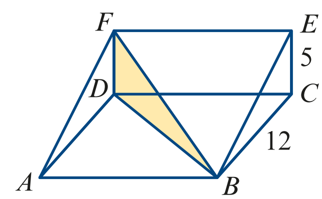
2 $VABCD$ is a right pyramid with square base $ABCD$, and with $AB = 2a$ and $OV = a$.



a Find the slope of the edge VA. That is, find the magnitude of $\angle VAO$.

b Find the slope of the face VBC.

3 A hill has gradient $\frac{5}{12}$. If BF makes an angle of 45° with the line of greatest slope, find:



a the gradient of BF

b the magnitude of $\angle FBD$.

4 The cross-section of a right prism is an isosceles triangle ABC with $AB = BC = 16$ cm and $\angle ABC = 58°$. The equal edges AD, BE and CF are parallel and of length 12 cm. Calculate:

a the length of AC

b the length of AE

c the magnitude of the angle between AE and EC.

Example 20

5 A vertical tower, AT, of height 50 m, stands at a point A on a horizontal plane. The points A, B and C lie on the same horizontal plane, where B is due west of A and C is due south of A. The angles of elevation of the top of the tower, T, from B and C are 25° and 30° respectively.

a Giving answers to the nearest metre, calculate the distances:

i AB **ii** AC **iii** BC

b Calculate the angle of elevation of T from the midpoint, M, of AB.

6 A right square pyramid, vertex O, stands on a square base $ABCD$. The height is 15 cm and the base side length is 10 cm. Find:

a the length of the slant edge

b the inclination of a slant edge to the base

c the inclination of a sloping face to the base

d the magnitude of the angle between two adjacent sloping faces.

7 A post stands at one corner of a rectangular courtyard. The elevations of the top of the post from the nearest corners are 30° and 45°. Find the elevation from the diagonally opposite corner.

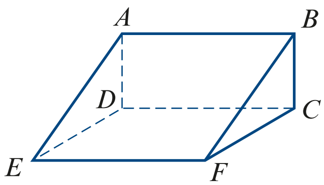
8 $VABC$ is a regular tetrahedron with base $\triangle ABC$. (All faces are equilateral triangles.) Find the magnitude of the angle between:

a a sloping edge and the base

b adjacent sloping faces.

9 An observer at a point A at sea level notes an aircraft due east at an elevation of 35°. At the same time an observer at B, which is 2 km due south of A, reports the aircraft on a bearing of 50°. Calculate the altitude of the aircraft.

10 $ABFE$ represents a section of a ski run which has a uniform inclination of 30° to the horizontal, with $AE = 100$ m and $AB = 100$ m. A skier traverses the slope from A to F. Calculate:



a the distance that the skier has traversed

b the inclination of the skier's path to the horizontal.

Example 21 **11** A sphere of radius length 8 cm rests on the top of a hollow inverted cone of height 15 cm whose vertical angle is 60°. Find the height of the centre of the sphere above the vertex of the cone.

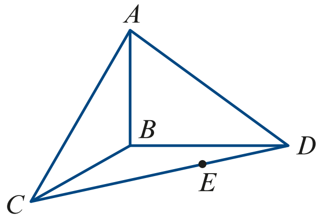
Example 22 **12** Four congruent spheres, radius length 10 cm, are placed on a horizontal table so that each touches two others and their centres form a square. A fifth congruent sphere rests on top of them. Find the height of the top of this fifth sphere above the table.

13 A cube has edge length a cm. What is the radius length, in terms of a, of:

a the sphere that just contains the cube

b the sphere that just fits inside the cube?

14 In the diagram, the edge AB is vertical, $\triangle BCD$ is horizontal, $\angle CBD$ is a right angle and $AB = 20$ m, $BD = 40$ m, $BC = 30$ m. Calculate the inclination to the horizontal of:



a AD

b AE, where AE is the line of greatest slope

c AE, where E is the midpoint of CD.

Chapter summary

Assign-
ment

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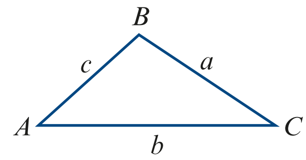
Triangles

■ **Labelling triangles**

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

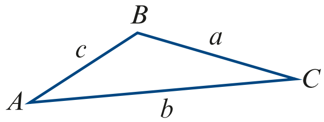
For example, the magnitude of angle BAC is denoted by A, and the length of side BC by a.



■ **Sine rule**

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle in the following cases:

- one side and two angles are given
- two sides and a non-included angle are given.

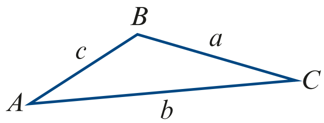
In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

■ **Cosine rule**

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



The symmetrical results also hold:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

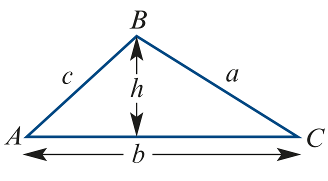
The cosine rule is used to find unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

■ **Area of a triangle**

$$\text{Area} = \frac{1}{2}bh$$

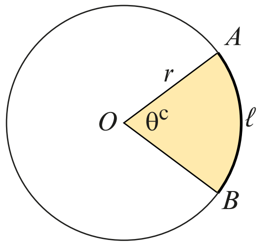
$$\text{Area} = \frac{1}{2}bc\sin A$$



That is, the area of a triangle is half the product of the lengths of two sides and the sine of the angle included between them.

Review

Circles

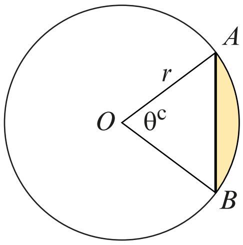


- Length of minor arc AB (red curve) is given by

$$\ell = r\theta$$

- Area of sector AOB (shaded) is given by

$$\text{Area} = \frac{1}{2}r^2\theta$$



- Length of chord AB (red line) is given by

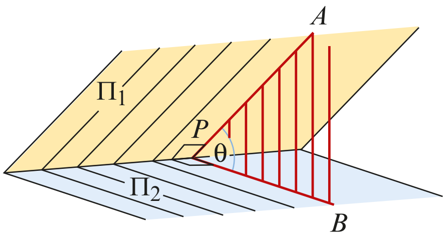
$$\ell = 2r\sin\left(\frac{\theta}{2}\right)$$

- Area of segment (shaded) is given by

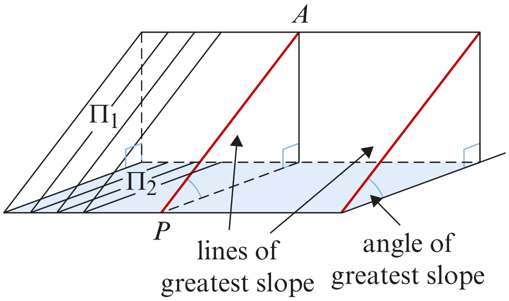
$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

Angle between planes

- Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between Π_1 and Π_2.



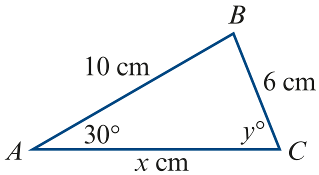
- If plane Π_2 is horizontal, then PA is called a **line of greatest slope** in plane Π_1.



Technology-free questions

1 **a** Find x.

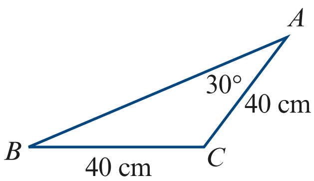
b Find y.



2 **a** Find $\angle ABC$ and $\angle ACB$.

b Find the length of side AB.

c Find the distance CM, where M is the midpoint of side AB.



3 From a port P, a ship Q is 20 km away on a bearing of 112°, and a ship R is 12 km away on a bearing of 052°. Find the distance between the two ships.

4 In a quadrilateral $ABCD$, $AB = 5$ cm, $BC = 5$ cm, $CD = 7$ cm, $B = 120°$ and $C = 90°$. Find:

a the length of the diagonal AC

b the area of triangle ABC

c the area of triangle ADC

d the area of the quadrilateral.

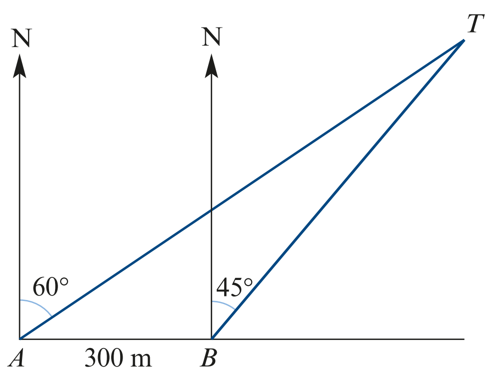
5 If $\sin x = \sin 37°$ and x is obtuse, find x.

6 A point T is 10 km due north of a point S. A point R, which is east of the straight line joining T and S, is 8 km from T and 7 km from S. Calculate the cosine of the bearing of R from S.

7 In $\triangle ABC$, $AB = 5$ cm, $\angle BAC = 60°$ and $AC = 6$ cm. Calculate the sine of $\angle ABC$.

8 The area of a sector of a circle with radius 6 cm is 33 cm^2. Calculate the angle of the sector.

9 The diagram shows two survey points, A and B, which are on an east–west line on level ground. From point A, the bearing of a tower T is 060°, while from point B, the bearing of the tower is 045°.



a **i** Find the magnitude of $\angle TAB$.

ii Find the magnitude of $\angle ATB$.

b Given that $\sin 15° = \dfrac{\sqrt{6} - \sqrt{2}}{4}$, find the distances AT and BT.

10 A boat sails 11 km from a harbour on a bearing of 220°. It then sails 15 km on a bearing of 340°. How far is the boat from the harbour?

11 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 150° to a checkpoint B. It then flies due east to its base C.

a If the bearing of C from A is 120°, find the distances AC and BC.

b The helicopter flies at a constant speed throughout and takes five minutes to fly from A to C. Find its speed.

12 A sector of a circle has an arc length of 30 cm. If the radius of the circle is 12 cm, find the area of the sector.

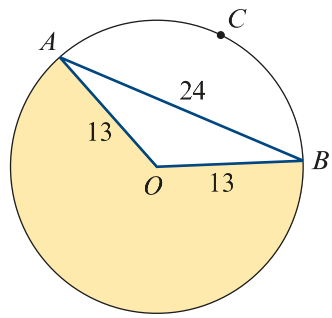
13 A chord PQ of a circle, radius 5 cm, subtends an angle of 2 radians at the centre of the circle. Taking π to be 3.14, calculate the length of the major arc PQ, correct to one decimal place.

Review

14 The diagram shows a circle of radius length 13 cm and a chord AB of length 24 cm. Calculate:

a the length of arc ACB

b the area of the shaded region.



15 From a cliff top 11 m above sea level, two boats are observed. One has an angle of depression of 45° and is due east, the other an angle of depression of 30° on a bearing of 120°. Calculate the distance between the boats.

Multiple-choice questions

1 In a triangle XYZ, $x = 21$ cm, $y = 18$ cm and $\angle YXZ = 62°$. The magnitude of $\angle XYZ$, correct to one decimal place, is

A 0.4° **B** 0.8° **C** 1.0° **D** 49.2° **E** 53.1°

2 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of c, to the nearest whole number, is

A 9 **B** 10 **C** 11 **D** 81 **E** 129

3 In a triangle ABC, $a = 5.2$ cm, $b = 6.8$ cm and $c = 7.3$ cm. The magnitude of $\angle ACB$, correct to the nearest degree, is

A 43° **B** 63° **C** 74° **D** 82° **E** 98°

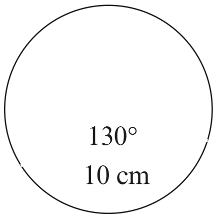
4 The area of the triangle ABC, where $b = 5$ cm, $c = 3$ cm, $\angle A = 30°$ and $\angle B = 70°$, is

A 2.75 cm^2 **B** 3.75 cm^2 **C** 6.5 cm^2 **D** 7.5 cm^2 **E** 8 cm^2

5 The length of the radius of the circle shown, correct to two decimal places, is

A 5.52 cm **B** 8.36 cm **C** 9.01 cm

D 12.18 cm **E** 18.13 cm



6 A chord of length 5 cm is drawn in a circle of radius 6 cm. The area of the smaller region inside the circle cut off by the chord, correct to one decimal place, is

A 1.8 cm^2 **B** 2.3 cm^2 **C** 3.9 cm^2 **D** 13.6 cm^2 **E** 15.5 cm^2

7 From a point on a cliff 500 m above sea level, the angle of depression to a boat is 20°. The distance from the foot of the cliff to the boat, to the nearest metre, is

A 182 m **B** 193 m **C** 210 m **D** 1374 m **E** 1834 m

8 A tower 80 m high is 1.3 km away from a point on the ground. The angle of elevation to the top of the tower from this point, correct to the nearest degree, is

A 1° **B** 4° **C** 53° **D** 86° **E** 89°

9 A man walks 5 km due east followed by 7 km due south. The bearing he must take to return to the start is

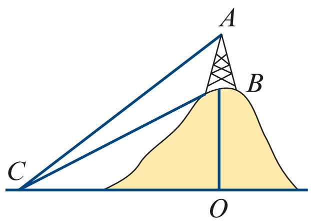
A 036° **B** 306° **C** 324° **D** 332° **E** 348°

10 A boat sails at a bearing of 215° from A to B. The bearing it must take from B to return to A is

A 035° **B** 055° **C** 090° **D** 215° **E** 250°

Extended-response questions

1 AB is a tower 60 m high on top of a hill. The magnitude of $\angle ACO$ is 49° and the magnitude of $\angle BCO$ is 37°.

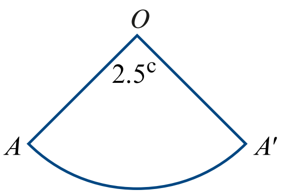


a Find the magnitudes of $\angle ACB$, $\angle CBO$ and $\angle CBA$.

b Find the length of BC.

c Find the height of the hill, i.e. the length of OB.

2 The angle of a sector of a circle, centre O and radius length 12 cm, has magnitude 2.5 radians. The sector is folded so that OA and OA' are joined to form a cone. Calculate:

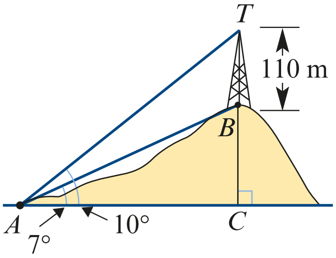


a the base radius length of the cone

b the curved surface area of the cone

c the shortest distance between two points diametrically opposite on the edge of the base.

3 A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill, the angle of elevation of the bottom of the tower is 7° and that of the top is 10°.



a Find the magnitudes of angles TAB, ABT and ATB.

b Use the sine rule to find the length of AB.

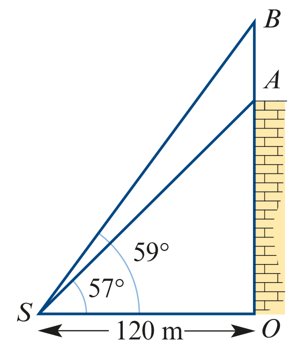
c Find CB, the height of the hill.

4 Point S is a distance of 120 m from the base of a building. On the building is an aerial, AB. The angle of elevation from S to A is 57°. The angle of elevation from S to B is 59°. Find:

a the distance OA

b the distance OB

c the distance AB.

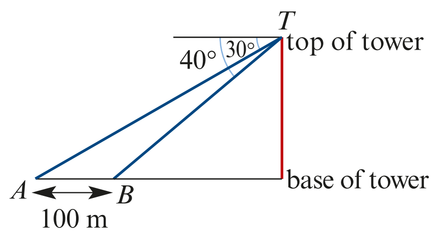


5 From the top of a communications tower, T, the angles of depression of two points A and B on a horizontal line through the base of the tower are 30° and 40°. The distance between the points is 100 m. Find:

a the distance AT

b the distance BT

c the height of the tower.



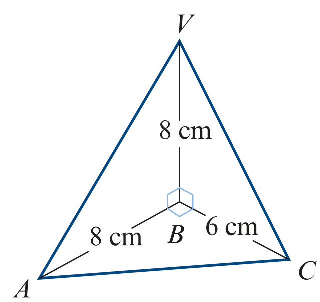
6 Angles VBA, VBC and ABC are right angles. Find:

a the distance VA

b the distance VC

c the distance AC

d the magnitude of angle VCA.



7 The perimeter of a triangle ABC is L metres. Find the area of the triangle in terms of L and the triangle's angles α, β and γ.

Hint: Let $AB = x$. Using the sine rule, first find the other side lengths in terms of x.

16

Trigonometric identities

Objectives

- To introduce the **reciprocal circular functions** and use them to obtain alternative forms of the **Pythagorean identity**.
- To evaluate simple trigonometric expressions using **trigonometric identities**.
- To prove simple trigonometric identities.
- To apply the **compound angle formulas** for circular functions.
- To apply the **double angle formulas** for circular functions.
- To simplify expressions of the form $a \cos x + b \sin x$.
- To sketch graphs of functions of the form $f(x) = a \cos x + b \sin x$.
- To solve equations of the form $a \cos x + b \sin x = c$.
- To apply the trigonometric identities for products of sines and cosines expressed as sums or differences, and vice versa.

There are many interesting and useful relationships between the circular functions. The most fundamental is the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

Some of these identities were discovered a very long time ago. For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

They are of great importance in many areas of mathematics, including calculus.

Note: An introduction to sine, cosine and tangent as functions is given in Mathematical Methods Units 1 & 2.

16A Reciprocal circular functions and the Pythagorean identity

In this section we introduce the reciprocals of the basic circular functions. The graphs of these functions appear in Chapter 17, where reciprocal functions are studied in general. Here we use these functions in alternative forms of the Pythagorean identity.

Reciprocal circular functions

The circular functions sine, cosine and tangent can be used to form three new functions, called the reciprocal circular functions.

Secant, cosecant and cotangent

- $\sec\theta = \dfrac{1}{\cos\theta}$ (for $\cos\theta \neq 0$)
- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ (for $\sin\theta \neq 0$)
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (for $\sin\theta \neq 0$)

Note: For $\cos\theta \neq 0$ and $\sin\theta \neq 0$, we have

$$\cot\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}$$

Example 1

Find the exact value of each of the following:

a $\sec\left(\dfrac{2\pi}{3}\right)$ **b** $\cot\left(\dfrac{5\pi}{4}\right)$ **c** $\operatorname{cosec}\left(\dfrac{7\pi}{4}\right)$

Solution

a
$$\begin{aligned}\sec\left(\frac{2\pi}{3}\right) &= \frac{1}{\cos\left(\frac{2\pi}{3}\right)}\\ &= \frac{1}{\cos\left(\pi - \frac{\pi}{3}\right)}\\ &= \frac{1}{-\cos\left(\frac{\pi}{3}\right)}\\ &= 1 \div \left(-\frac{1}{2}\right)\\ &= -2\end{aligned}$$

b
$$\begin{aligned}\cot\left(\frac{5\pi}{4}\right) &= \frac{\cos\left(\frac{5\pi}{4}\right)}{\sin\left(\frac{5\pi}{4}\right)}\\ &= \frac{\cos\left(\pi + \frac{\pi}{4}\right)}{\sin\left(\pi + \frac{\pi}{4}\right)}\\ &= \frac{-1}{\sqrt{2}} \div \left(\frac{-1}{\sqrt{2}}\right)\\ &= 1\end{aligned}$$

c
$$\begin{aligned}&\operatorname{cosec}\left(\frac{7\pi}{4}\right)\\ &= \frac{1}{\sin\left(2\pi - \frac{\pi}{4}\right)}\\ &= \frac{1}{-\sin\left(\frac{\pi}{4}\right)}\\ &= 1 \div \left(-\frac{1}{\sqrt{2}}\right)\\ &= -\sqrt{2}\end{aligned}$$

Note: In this example, we are using symmetry properties and exact values of circular functions, which are covered in Mathematical Methods Units 1 & 2.

Example 2

Find the values of x between 0 and 2π for which:

a $\sec x = -2$

b $\cot x = -1$

Solution

a $\sec x = -2$

$$\frac{1}{\cos x} = -2$$

$$\cos x = -\frac{1}{2}$$

$$\therefore \quad x = \pi - \frac{\pi}{3} \quad \text{or} \quad x = \pi + \frac{\pi}{3}$$

$$\therefore \quad x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}$$

b $\cot x = -1$

$$\tan x = -1$$

$$\therefore \quad x = \pi - \frac{\pi}{4} \quad \text{or} \quad x = 2\pi - \frac{\pi}{4}$$

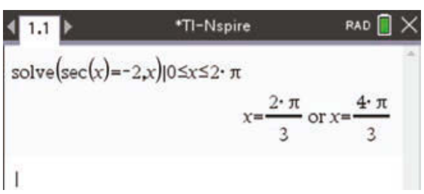
$$\therefore \quad x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4}$$

Using the TI-Nspire

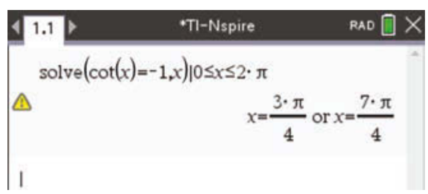
Check that your calculator is in radian mode. Use menu > **Algebra** > **Solve** as shown.

Note: Access sec and cot using trig. Access ≤ using ctrl =.

a



b



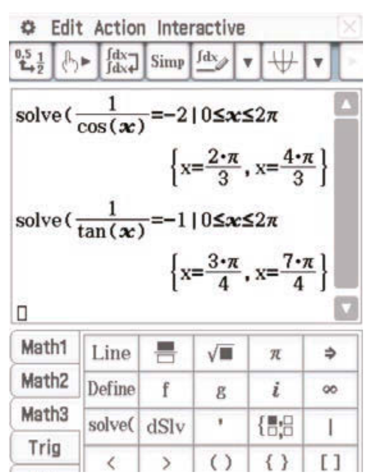
Using the Casio ClassPad

The ClassPad does not recognise $\sec x$, $\operatorname{cosec} x$ and $\cot x$. These functions must be entered as reciprocals of $\cos x$, $\sin x$ and $\tan x$ respectively.

a
- Select solve(from the Math1 or Math3 keyboard.
- Enter $\dfrac{1}{\cos(x)} = -2 \,\big|\, 0 \le x \le 2\pi$ and tap EXE.

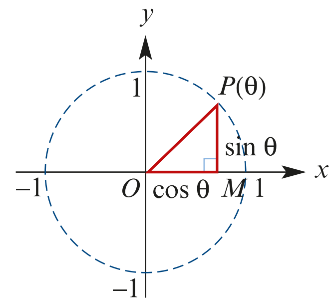
b
- Select solve(from the Math1 or Math3 keyboard.
- Enter $\dfrac{1}{\tan(x)} = -1 \,\big|\, 0 \le x \le 2\pi$ and tap EXE.

Note: The 'for' operator | is found in the Math3 keyboard and is used to specify a condition. In this case, the condition is the domain restriction.



The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.



By Pythagoras' theorem:

$$OP^2 = OM^2 + MP^2$$

$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Since this is true for all values of θ, it is called an identity.

We write $(\cos\theta)^n$ as $\cos^n\theta$, and similarly for other circular functions. Therefore we have:

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

We can derive other forms of this identity:

- Dividing both sides by $\cos^2\theta$ gives

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\therefore \quad 1 + \tan^2\theta = \sec^2\theta$$

- Dividing both sides by $\sin^2\theta$ gives

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\therefore \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Example 3

a If $\operatorname{cosec} x = \frac{7}{4}$, find $\cos x$.

b If $\sec x = -\frac{3}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find $\sin x$.

Solution

a Since $\operatorname{cosec} x = \frac{7}{4}$, we have $\sin x = \frac{4}{7}$.

Now $\cos^2 x + \sin^2 x = 1$

$$\cos^2 x + \frac{16}{49} = 1$$

$$\cos^2 x = \frac{33}{49}$$

$$\therefore \quad \cos x = \pm\frac{\sqrt{33}}{7}$$

b Since $\sec x = -\frac{3}{2}$, we have $\cos x = -\frac{2}{3}$.

Now $\cos^2 x + \sin^2 x = 1$

$$\frac{4}{9} + \sin^2 x = 1$$

$$\therefore \quad \sin x = \pm\frac{\sqrt{5}}{3}$$

But $\sin x$ is positive for $P(x)$ in the 2nd quadrant, and so $\sin x = \frac{\sqrt{5}}{3}$.

Example 4

If $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the values of $\cos\theta$ and $\tan\theta$.

Solution

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta + \frac{9}{25} = 1$$

$$\therefore \quad \cos^2\theta = \frac{16}{25}$$

Thus $\cos\theta = -\frac{4}{5}$, since $\frac{\pi}{2} < \theta < \pi$, and therefore $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4}$.

Example 5

Prove that $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2\operatorname{cosec}^2\theta$.

Solution

$$\begin{aligned} \text{LHS} &= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \\ &= \frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta} \\ &= \frac{2}{1-\cos^2\theta} \\ &= \frac{2}{\sin^2\theta} \\ &= 2\operatorname{cosec}^2\theta \\ &= \text{RHS} \end{aligned}$$

Summary 16A

- **Reciprocal circular functions**

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{for } \cos\theta \neq 0)$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Exercise 16A

Example 1

1 Find the exact value of each of the following:

a $\cot\left(\frac{3\pi}{4}\right)$ **b** $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$

d $\operatorname{cosec}\left(\frac{\pi}{2}\right)$ **e** $\sec\left(\frac{4\pi}{3}\right)$ **f** $\operatorname{cosec}\left(\frac{13\pi}{6}\right)$

g $\cot\left(\frac{7\pi}{3}\right)$ **h** $\sec\left(\frac{5\pi}{3}\right)$

2 Without using a calculator, write down the exact value of each of the following:

a $\cot 135°$ **b** $\sec 150°$ **c** $\operatorname{cosec} 90°$

d $\cot 240°$ **e** $\operatorname{cosec} 225°$ **f** $\sec 330°$

g $\cot 315°$ **h** $\operatorname{cosec} 300°$ **i** $\cot 420°$

Example 2

3 Find the values of x between 0 and 2π for which:

a $\operatorname{cosec} x = 2$ **b** $\cot x = \sqrt{3}$

c $\sec x + \sqrt{2} = 0$ **d** $\operatorname{cosec} x = \sec x$

Example 3
Example 4

4 If $\sec\theta = -\frac{17}{8}$ and $\frac{\pi}{2} < \theta < \pi$, find:

a $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$

5 If $\tan\theta = -\frac{7}{24}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\cos\theta$ and $\sin\theta$.

6 Find the value of $\sec\theta$ if $\tan\theta = \frac{2}{5}$ and θ is not in the 1st quadrant.

7 If $\tan\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, evaluate $\frac{\sin\theta - 2\cos\theta}{\cot\theta - \sin\theta}$.

8 If $\cos\theta = \frac{2}{3}$ and θ is in the 4th quadrant, express $\frac{\tan\theta - 3\sin\theta}{\cos\theta - 2\cot\theta}$ in simplest surd form.

Example 5

9 Prove each of the following identities for suitable values of θ and φ:

a $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$ **b** $\cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta = 1$

c $\frac{\tan\theta}{\tan\varphi} = \frac{\tan\theta + \cot\varphi}{\cot\theta + \tan\varphi}$ **d** $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

e $\frac{1 + \cot^2\theta}{\cot\theta\operatorname{cosec}\theta} = \sec\theta$ **f** $\sec\theta + \tan\theta = \frac{\cos\theta}{1 - \sin\theta}$

16B Compound and double angle formulas

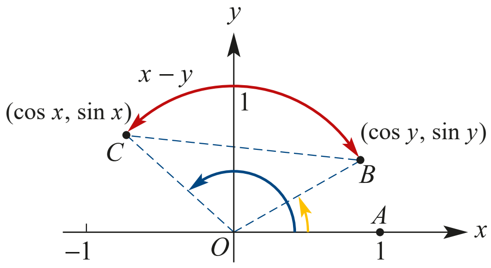
The compound angle formulas

Compound angle formulas for cosine

1 $\cos(x + y) = \cos x \cos y - \sin x \sin y$

2 $\cos(x - y) = \cos x \cos y + \sin x \sin y$

Proof Consider a unit circle as shown:



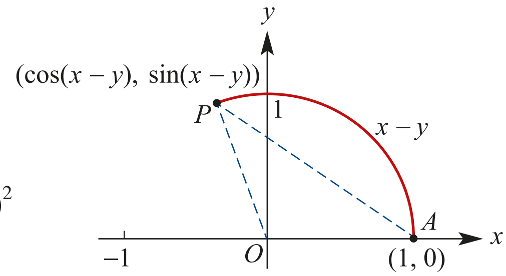
arc length $AB = y$ units

arc length $AC = x$ units

arc length $BC = x - y$ units

Rotate ΔOCB so that B is coincident with A. Then C is moved to

$P(\cos(x - y), \sin(x - y))$



Since the triangles CBO and PAO are congruent, we have $CB = PA$.

Using the coordinate distance formula:

$$\begin{aligned} CB^2 &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\ &= 2 - 2(\cos x \cos y + \sin x \sin y) \end{aligned}$$

$$\begin{aligned} PA^2 &= (\cos(x - y) - 1)^2 + (\sin(x - y) - 0)^2 \\ &= 2 - 2\cos(x - y) \end{aligned}$$

Since $CB = PA$, this gives

$$2 - 2\cos(x - y) = 2 - 2(\cos x \cos y + \sin x \sin y)$$

$$\therefore \quad \cos(x - y) = \cos x \cos y + \sin x \sin y$$

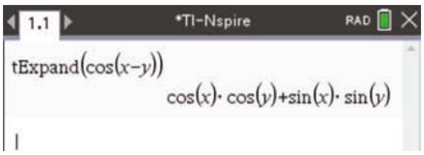
We can now obtain the first formula from the second by replacing y with $-y$:

$$\begin{aligned} \cos(x + y) &= \cos(x - (-y)) \\ &= \cos x \cos(-y) + \sin x \sin(-y) \\ &= \cos x \cos y - \sin x \sin y \end{aligned}$$

Note: Here we used $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

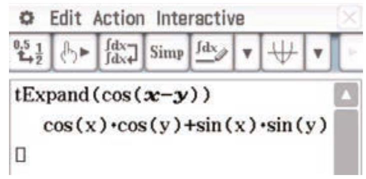
Using the TI-Nspire

Access the **tExpand( )** command from menu > **Algebra** > **Trigonometry** > **Expand** and complete as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight $\cos(x - y)$.
- Go to **Interactive** > **Transformation** > **tExpand** and tap OK.



Example 6

Evaluate $\cos 75°$.

Solution

$$
\begin{aligned}
\cos 75° &= \cos(45° + 30°)\\
&= \cos 45° \cos 30° - \sin 45° \sin 30°\\
&= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\
&= \frac{\sqrt{3}-1}{2\sqrt{2}}\\
&= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}
$$

Compound angle formulas for sine

1 $\sin(x + y) = \sin x \cos y + \cos x \sin y$

2 $\sin(x - y) = \sin x \cos y - \cos x \sin y$

Proof We use the symmetry properties $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$:

$$
\begin{aligned}
\sin(x + y) &= \cos\left(\frac{\pi}{2} - (x + y)\right)\\
&= \cos\left(\left(\frac{\pi}{2} - x\right) - y\right)\\
&= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y\\
&= \sin x \cos y + \cos x \sin y
\end{aligned}
$$

We can now obtain the second formula from the first by replacing y with $-y$:

$$
\begin{aligned}
\sin(x - y) &= \sin x \cos(-y) + \cos x \sin(-y)\\
&= \sin x \cos y - \cos x \sin y
\end{aligned}
$$

Example 7

Evaluate:

a $\sin 75^\circ$

b $\sin 15^\circ$

Solution

a
$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ)\\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ\\ &= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

b
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Compound angle formulas for tangent

1 $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

2 $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Proof To obtain the first formula, we write

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Now divide the numerator and denominator by $\cos x \cos y$. The second formula can be obtained from the first by using $\tan(-\theta) = -\tan\theta$.

Example 8

If x and y are acute angles such that $\tan x = 4$ and $\tan y = \frac{3}{5}$, show that $x - y = \frac{\pi}{4}$.

Solution

$$\begin{aligned}\tan(x - y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y}\\ &= \frac{4 - \frac{3}{5}}{1 + 4 \times \frac{3}{5}}\\ &= \frac{20 - 3}{5 + 4 \times 3}\\ &= 1\end{aligned}$$

$$\therefore \quad x - y = \frac{\pi}{4}$$

Note: The function $\tan\theta$ is one-to-one for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

The double angle formulas

Using the compound angle formulas, we can easily derive useful expressions for $\sin(2x)$, $\cos(2x)$ and $\tan(2x)$.

Double angle formulas for cosine

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1 && (\text{since } \sin^2 x = 1 - \cos^2 x)\\ &= 1 - 2\sin^2 x && (\text{since } \cos^2 x = 1 - \sin^2 x)\end{aligned}$$

Proof $\cos(x + x) = \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x$

Double angle formula for sine

$$\sin(2x) = 2\sin x \cos x$$

Proof $\sin(x + x) = \sin x \cos x + \cos x \sin x$

$= 2\sin x \cos x$

Double angle formula for tangent

$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

Proof $\tan(x + x) = \dfrac{\tan x + \tan x}{1 - \tan x \tan x}$

$= \dfrac{2\tan x}{1 - \tan^2 x}$

Example 9

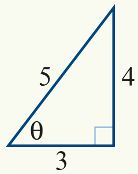
If $\tan\theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\tan(2\theta)$

Solution

a $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

$$\begin{aligned}\therefore \sin(2\theta) &= 2\sin\theta\cos\theta\\ &= 2 \times \frac{4}{5} \times \frac{3}{5}\\ &= \frac{24}{25}\end{aligned}$$



b $$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}\\ &= \frac{2 \times \frac{4}{3}}{1 - \frac{16}{9}}\\ &= \frac{2 \times 4 \times 3}{9 - 16}\\ &= -\frac{24}{7}\end{aligned}$$

Example 10

Prove each of the following identities:

a $\dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tan(2\theta)$

b $\dfrac{\sin\theta}{\sin\varphi} + \dfrac{\cos\theta}{\cos\varphi} = \dfrac{2\sin(\theta+\varphi)}{\sin(2\varphi)}$

c $\dfrac{1}{\cos\theta + \sin\theta} + \dfrac{1}{\cos\theta - \sin\theta} = \tan(2\theta)\operatorname{cosec}\theta$

Solution

a
$$\begin{aligned} \text{LHS} &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \\ &= \frac{\sin(2\theta)}{\cos(2\theta)} \\ &= \tan(2\theta) \\ &= \text{RHS} \end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$.

b
$$\begin{aligned} \text{LHS} &= \frac{\sin\theta}{\sin\varphi} + \frac{\cos\theta}{\cos\varphi} \\ &= \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\sin\varphi\cos\varphi} \\ &= \frac{\sin(\theta+\varphi)}{\frac{1}{2}\sin(2\varphi)} \\ &= \frac{2\sin(\theta+\varphi)}{\sin(2\varphi)} \\ &= \text{RHS} \end{aligned}$$

Note: Identity holds when $\sin(2\varphi) \neq 0$.

c
$$\begin{aligned} \text{LHS} &= \frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta} \\ &= \frac{\cos\theta - \sin\theta + \cos\theta + \sin\theta}{\cos^2\theta - \sin^2\theta} \\ &= \frac{2\cos\theta}{\cos(2\theta)} \\ &= \frac{2\cos\theta}{\cos(2\theta)} \times \frac{\sin\theta}{\sin\theta} \\ &= \frac{\sin(2\theta)}{\cos(2\theta)\sin\theta} \\ &= \frac{\tan(2\theta)}{\sin\theta} \\ &= \tan(2\theta)\operatorname{cosec}\theta \\ &= \text{RHS} \end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$ and $\sin\theta \neq 0$.

Sometimes the easiest way to prove that two expressions are equal is to simplify each of them separately. This is demonstrated in the following example.

Example 11

Prove that $(\sec A - \cos A)(\operatorname{cosec} A - \sin A) = \dfrac{1}{\tan A + \cot A}$.

Solution

$$\begin{aligned}\text{LHS} &= (\sec A - \cos A)(\operatorname{cosec} A - \sin A)\\ &= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{1}{\sin A} - \sin A\right)\\ &= \frac{1-\cos^2 A}{\cos A} \times \frac{1-\sin^2 A}{\sin A}\\ &= \frac{\cos^2 A \sin^2 A}{\cos A \sin A}\\ &= \cos A \sin A\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{\tan A + \cot A}\\ &= \frac{1}{\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}}\\ &= \frac{\cos A \sin A}{\sin^2 A + \cos^2 A}\\ &= \cos A \sin A\end{aligned}$$

We have shown that LHS = RHS.

Summary 16B

- Compound angle formulas

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- Double angle formulas

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$
$$\sin(2x) = 2 \sin x \cos x$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

Skill-sheet

Exercise 16B

Example 6

1 By using the appropriate compound angle formulas, find exact values for the following:

a $\cos 15^\circ$ **b** $\cos 105^\circ$

Example 7

2 By using the appropriate compound angle formulas, find exact values for the following:

a $\sin 165^\circ$ **b** $\tan 75^\circ$

3 Find the exact value of:

a $\cos\left(\dfrac{5\pi}{12}\right)$ **b** $\sin\left(\dfrac{11\pi}{12}\right)$ **c** $\tan\left(-\dfrac{\pi}{12}\right)$

Example 8

4 If $\sin x = \dfrac{12}{13}$ and $\sin y = \dfrac{3}{5}$, evaluate $\sin(x+y)$. (Note: There is more than one answer.)

5 Simplify the following:

a $\sin\left(\theta + \frac{\pi}{6}\right)$ **b** $\cos\left(\varphi - \frac{\pi}{4}\right)$ **c** $\tan\left(\theta + \frac{\pi}{3}\right)$ **d** $\sin\left(\theta - \frac{\pi}{4}\right)$

6 Simplify:

a $\cos(u - v)\sin v + \sin(u - v)\cos v$ **b** $\sin(u + v)\sin v + \cos(u + v)\cos v$

Example 9

7 If $\sin\theta = -\frac{3}{5}$, with θ in the 3rd quadrant, and $\cos\varphi = -\frac{5}{13}$, with φ in the 2nd quadrant, evaluate each of the following without using a calculator:

a $\cos(2\varphi)$ **b** $\sin(2\theta)$ **c** $\tan(2\theta)$ **d** $\sec(2\varphi)$
e $\sin(\theta + \varphi)$ **f** $\cos(\theta - \varphi)$ **g** $\operatorname{cosec}(\theta + \varphi)$ **h** $\cot(2\theta)$

8 For acute angles u and v such that $\tan u = \frac{4}{3}$ and $\tan v = \frac{5}{12}$, evaluate:

a $\tan(u + v)$ **b** $\tan(2u)$ **c** $\cos(u - v)$ **d** $\sin(2u)$

9 If $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{24}{25}$, with $\frac{\pi}{2} < \beta < \alpha < \pi$, evaluate:

a $\cos(2\alpha)$ **b** $\sin(\alpha - \beta)$ **c** $\tan(\alpha + \beta)$ **d** $\sin(2\beta)$

10 If $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\cos(2\theta)$

11 Simplify each of the following expressions:

a $(\sin\theta - \cos\theta)^2$ **b** $\cos^4\theta - \sin^4\theta$

Example 10
Example 11

12 Prove the following identities:

a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta - \cos\theta$ **b** $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

c $\tan\left(\theta + \frac{\pi}{4}\right)\tan\left(\theta - \frac{\pi}{4}\right) = -1$ **d** $\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$

e $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$ **f** $\frac{\sin(u + v)}{\cos u \cos v} = \tan v + \tan u$

g $\frac{\tan u + \tan v}{\tan u - \tan v} = \frac{\sin(u + v)}{\sin(u - v)}$ **h** $\cos(2\theta) + 2\sin^2\theta = 1$

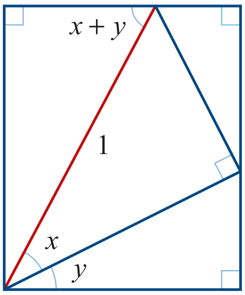
i $\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$ **j** $\frac{1 - \sin(2\theta)}{\sin\theta - \cos\theta} = \sin\theta - \cos\theta$

13 **a** Suppose that angles x, y and $x + y$ are all acute. Use the diagram to show that:

i $\sin(x + y) = \sin x \cos y + \cos x \sin y$
ii $\cos(x + y) = \cos x \cos y - \sin x \sin y$

b Suppose that angles x, y and $x - y$ are all acute. Adapt the diagram to show that:

i $\sin(x - y) = \sin x \cos y - \cos x \sin y$
ii $\cos(x - y) = \cos x \cos y + \sin x \sin y$



16C Simplifying $a\cos x + b\sin x$

In this section, we see how to rewrite the rule of a function $f(x) = a\cos x + b\sin x$ in terms of a single circular function.

$$a\cos x + b\sin x = r\cos(x - \alpha) \quad \text{where } r = \sqrt{a^2 + b^2},\ \cos\alpha = \frac{a}{r} \text{ and } \sin\alpha = \frac{b}{r}$$

Proof Let $r = \sqrt{a^2 + b^2}$. Consider the point $P\left(\frac{a}{r}, \frac{b}{r}\right)$ and its distance from the origin O:

$$OP^2 = \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

The point P is on the unit circle, and so $\frac{a}{r} = \cos\alpha$ and $\frac{b}{r} = \sin\alpha$, for some angle α.

We can now write

$$\begin{aligned} a\cos x + b\sin x &= r\left(\frac{a}{r}\cos x + \frac{b}{r}\sin x\right) \\ &= r(\cos\alpha\,\cos x + \sin\alpha\,\sin x) \\ &= r\cos(x - \alpha) \end{aligned}$$

Similarly, it may be shown that

$$a\cos x + b\sin x = r\sin(x + \beta) \quad \text{where } r = \sqrt{a^2 + b^2},\ \sin\beta = \frac{a}{r},\ \cos\beta = \frac{b}{r}$$

Example 12

Express $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x - \alpha)$. Hence find the range of the function f with rule $f(x) = \cos x - \sqrt{3}\sin x$ and find the maximum and minimum values of f.

Solution

Here $a = 1$ and $b = -\sqrt{3}$. Therefore

$$r = \sqrt{1 + 3} = 2, \quad \cos\alpha = \frac{a}{r} = \frac{1}{2} \quad \text{and} \quad \sin\alpha = \frac{b}{r} = -\frac{\sqrt{3}}{2}$$

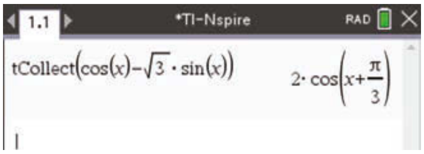
We see that $\alpha = -\frac{\pi}{3}$ and so

$$f(x) = \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

Thus the range of f is $[-2, 2]$, the maximum value is 2 and the minimum value is -2.

Using the TI-Nspire

Access the **tCollect( )** command from menu > **Algebra** > **Trigonometry** > **Collect** and complete as shown.



Example 13

Solve $\cos x - \sqrt{3} \sin x = 1$ for $x \in [0, 2\pi]$.

Solution

From Example 12, we have

$$\cos x - \sqrt{3} \sin x = 2 \cos\left(x + \frac{\pi}{3}\right)$$

$$\therefore \quad 2 \cos\left(x + \frac{\pi}{3}\right) = 1$$

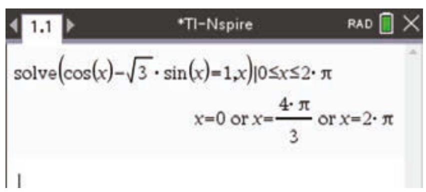
$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3} \text{ or } \frac{7\pi}{3}$$

$$x = 0, \frac{4\pi}{3} \text{ or } 2\pi$$

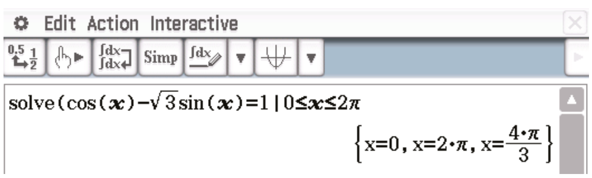
Using the TI-Nspire

Use **solve( )** from the **Algebra** menu as shown.



Using the Casio ClassPad

Use solve(and complete as shown.



Example 14

Express $\sqrt{3} \sin(2x) - \cos(2x)$ in the form $r \sin(2x + \alpha)$.

Solution

A slightly different technique is used. Assume that

$$\sqrt{3} \sin(2x) - \cos(2x) = r \sin(2x + \alpha)$$

$$= r\left(\sin(2x) \cos \alpha + \cos(2x) \sin \alpha\right)$$

This is to hold for all x.

For $x = \frac{\pi}{4}$: $\quad \sqrt{3} = r \cos \alpha \qquad (1)$

For $x = 0$: $\quad -1 = r \sin \alpha \qquad (2)$

Squaring and adding (1) and (2) gives

$$r^2\cos^2\alpha + r^2\sin^2\alpha = 4$$
$$r^2 = 4$$
$$\therefore\quad r = \pm 2$$

We take the positive solution. Substituting in (1) and (2) gives

$$\frac{\sqrt{3}}{2} = \cos\alpha \quad\text{and}\quad -\frac{1}{2} = \sin\alpha$$

Thus $\alpha = -\frac{\pi}{6}$ and hence

$$\sqrt{3}\sin(2x) - \cos(2x) = 2\sin\left(2x - \frac{\pi}{6}\right)$$

Check: Expand the right-hand side of the equation using a compound angle formula.

Summary 16C

- $a\cos x + b\sin x = r\cos(x - \alpha)$ where $r = \sqrt{a^2 + b^2}$, $\cos\alpha = \frac{a}{r}$, $\sin\alpha = \frac{b}{r}$
- $a\cos x + b\sin x = r\sin(x + \beta)$ where $r = \sqrt{a^2 + b^2}$, $\sin\beta = \frac{a}{r}$, $\cos\beta = \frac{b}{r}$

Skill-sheet

Exercise 16C

Example 12

1 Find the maximum and minimum values of the following:

a $4\cos x + 3\sin x$
b $\sqrt{3}\cos x + \sin x$
c $\cos x - \sin x$
d $\cos x + \sin x$
e $3\cos x + \sqrt{3}\sin x$
f $\sin x - \sqrt{3}\cos x$
g $\cos x - \sqrt{3}\sin x + 2$
h $5 + 3\sin x - 2\cos x$

Example 13

2 Solve each of the following for $x \in [0, 2\pi]$ or for $\theta \in [0, 360]$:

a $\sin x - \cos x = 1$
b $\sqrt{3}\sin x + \cos x = 1$
c $\sin x - \sqrt{3}\cos x = -1$
d $3\cos x - \sqrt{3}\sin x = 3$
e $4\sin\theta^\circ + 3\cos\theta^\circ = 5$
f $2\sqrt{2}\sin\theta^\circ - 2\cos\theta^\circ = 3$

3 Write $\sqrt{3}\cos(2x) - \sin(2x)$ in the form $r\cos(2x + \alpha)$.

Example 14

4 Write $\cos(3x) - \sin(3x)$ in the form $r\sin(3x - \alpha)$.

5 Sketch the graph of each of the following, showing one cycle:

a $f(x) = \sin x - \cos x$
b $f(x) = \sqrt{3}\sin x + \cos x$
c $f(x) = \sin x + \cos x$
d $f(x) = \sin x - \sqrt{3}\cos x$

16D Sums and products of sines and cosines

In Section 16B, we derived the compound angle formulas for sine and cosine. We use them in this section to obtain new identities which allow us to rewrite products of sines and cosines as sums or differences, and vice versa.

Expressing products as sums or differences

Product-to-sum identities

$$2\cos x\,\cos y = \cos(x-y) + \cos(x+y)$$
$$2\sin x\,\sin y = \cos(x-y) - \cos(x+y)$$
$$2\sin x\,\cos y = \sin(x+y) + \sin(x-y)$$

Proof We use the compound angle formulas for sine and cosine:

$$\cos(x+y) = \cos x\,\cos y - \sin x\,\sin y \qquad (1)$$
$$\cos(x-y) = \cos x\,\cos y + \sin x\,\sin y \qquad (2)$$
$$\sin(x+y) = \sin x\,\cos y + \cos x\,\sin y \qquad (3)$$
$$\sin(x-y) = \sin x\,\cos y - \cos x\,\sin y \qquad (4)$$

The first product-to-sum identity is obtained by adding (2) and (1), the second identity is obtained by subtracting (1) from (2), and the third by adding (3) and (4).

Example 15

Express each of the following products as sums or differences:

a $2\sin(3\theta)\cos(\theta)$

b $2\sin 50^\circ \cos 60^\circ$

c $2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right)$

Solution

a Use the third product-to-sum identity:

$$\begin{aligned}2\sin(3\theta)\cos(\theta) &= \sin(3\theta+\theta) + \sin(3\theta-\theta)\\ &= \sin(4\theta) + \sin(2\theta)\end{aligned}$$

b Use the third product-to-sum identity:

$$\begin{aligned}2\sin 50^\circ \cos 60^\circ &= \sin 110^\circ + \sin(-10)^\circ\\ &= \sin 110^\circ - \sin 10^\circ\end{aligned}$$

c Use the first product-to-sum identity:

$$\begin{aligned}2\cos\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2}\right) + \cos(2\theta)\\ &= \cos(2\theta)\end{aligned}$$

Expressing sums and differences as products

Sum-to-product identities

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

Proof Using the first product-to-sum identity, we have

$$\begin{aligned} 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) &= \cos\left(\frac{x+y}{2} - \frac{x-y}{2}\right) + \cos\left(\frac{x+y}{2} + \frac{x-y}{2}\right) \\ &= \cos y + \cos x \\ &= \cos x + \cos y \end{aligned}$$

The other three sum-to-product identities can be obtained similarly.

Example 16

Express each of the following as products:

a $\sin 36° + \sin 10°$

b $\cos 36° + \cos 10°$

c $\sin 36° - \sin 10°$

d $\cos 36° - \cos 10°$

Solution

a $\sin 36° + \sin 10° = 2\sin 23° \cos 13°$

b $\cos 36° + \cos 10° = 2\cos 23° \cos 13°$

c $\sin 36° - \sin 10° = 2\cos 23° \sin 13°$

d $\cos 36° - \cos 10° = -2\sin 23° \sin 13°$

Example 17

Prove that

$$\frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} = \tan(2\theta)$$

Solution

$$\begin{aligned} \text{LHS} &= \frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} \\ &= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \frac{2\sin(2\theta)\sin(\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \tan(2\theta) \\ &= \text{RHS} \end{aligned}$$

Example 18

Solve the equation $\sin(3x) + \sin(11x) = 0$ for $x \in [0, \pi]$.

Solution

$$\begin{aligned}
& \sin(3x) + \sin(11x) = 0 \\
\Leftrightarrow\ & 2\sin(7x)\cos(4x) = 0 \\
\Leftrightarrow\ & \sin(7x) = 0 \quad \text{or} \quad \cos(4x) = 0 \\
\Leftrightarrow\ & 7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi \quad \text{or} \quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\
\Leftrightarrow\ & x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8} \text{ or } \frac{7\pi}{8}
\end{aligned}$$

Summary 16D

- Product-to-sum identities

$$2\cos x \cos y = \cos(x - y) + \cos(x + y)$$
$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$

- Sum-to-product identities

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

Exercise 16D

Example 15

1 Express each of the following products as sums or differences:

a $2\sin(3\pi t)\cos(2\pi t)$

b $\sin 20° \cos 30°$

c $2\cos\left(\frac{\pi x}{4}\right)\sin\left(\frac{3\pi x}{4}\right)$

d $2\sin\left(\frac{A+B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)$

2 Express $2\sin(3\theta)\sin(2\theta)$ as a difference of cosines.

3 Use a product-to-sum identity to derive the expression for $2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$ as a difference of sines.

4 Show that $\sin 75° \sin 15° = \frac{1}{4}$.

Example 16 **5** Express each of the following as products:

a $\sin 56° + \sin 22°$ **b** $\cos 56° + \cos 22°$

c $\sin 56° - \sin 22°$ **d** $\cos 56° - \cos 22°$

6 Express each of the following as products:

a $\sin(6A) + \sin(2A)$ **b** $\cos(x) + \cos(2x)$

c $\sin(4x) - \sin(3x)$ **d** $\cos(3A) - \cos(A)$

Example 17 **7** Show that $\sin(A) + 2\sin(3A) + \sin(5A) = 4\cos^2(A)\sin(3A)$.

8 For any three angles α, β and γ, show that

$$\sin(\alpha + \beta)\sin(\alpha - \beta) + \sin(\beta + \gamma)\sin(\beta - \gamma) + \sin(\gamma + \alpha)\sin(\gamma - \alpha) = 0$$

9 Show that $\cos 70° + \sin 40° = \cos 10°$.

10 Show that $\cos 20° + \cos 100° + \cos 140° = 0$.

Example 18 **11** Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\cos(5x) + \cos(x) = 0$ **b** $\cos(5x) - \cos(x) = 0$

c $\sin(5x) + \sin(x) = 0$ **d** $\sin(5x) - \sin(x) = 0$

12 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\cos(2\theta) - \sin(\theta) = 0$ **b** $\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$

c $\sin(7\theta) - \sin(\theta) = \sin(3\theta)$ **d** $\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$

13 Prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A + B}{2}\right)$.

14 Prove the identity:

$$4\sin(A + B)\sin(B + C)\sin(C + A) = \sin(2A) + \sin(2B) + \sin(2C) - \sin(2A + 2B + 2C)$$

15 Prove that $\dfrac{\cos(2A) - \cos(2B)}{\sin(2A - 2B)} = -\dfrac{\sin(A + B)}{\cos(A - B)}$.

16 Prove each of the following identities:

a $\dfrac{\sin(A) + \sin(3A) + \sin(5A)}{\cos(A) + \cos(3A) + \cos(5A)} = \tan(3A)$

b $\cos^2(A) + \cos^2(B) - 1 = \cos(A + B)\cos(A - B)$

c $\cos^2(A - B) - \cos^2(A + B) = \sin(2A)\sin(2B)$

d $\cos^2(A - B) - \sin^2(A + B) = \cos(2A)\cos(2B)$

17 Find the sum

$$\sin(x) + \sin(3x) + \sin(5x) + \cdots + \sin(99x)$$

Hint: First multiply this sum by $2\sin(x)$.

Chapter summary

Assignment

Nrich

- **Reciprocal circular functions**

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \quad (\text{if } \cos\theta \neq 0)$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

- **Compound angle formulas**

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- **Double angle formulas**

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin(2x) = 2\sin x \cos x$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

- **Linear combinations**

$$a\cos x + b\sin x = r\cos(x - \alpha) \qquad \text{where } r = \sqrt{a^2 + b^2}, \quad \cos\alpha = \frac{a}{r}, \quad \sin\alpha = \frac{b}{r}$$
$$a\cos x + b\sin x = r\sin(x + \beta) \qquad \text{where } r = \sqrt{a^2 + b^2}, \quad \sin\beta = \frac{a}{r}, \quad \cos\beta = \frac{b}{r}$$

- **Product-to-sum identities**

$$2\cos x \cos y = \cos(x - y) + \cos(x + y)$$
$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$

- **Sum-to-product identities**

$$\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$
$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$
$$\sin x - \sin y = 2\sin\left(\frac{x - y}{2}\right)\cos\left(\frac{x + y}{2}\right)$$

Review

Technology-free questions

1 Prove each of the following identities:

a $\sec\theta + \operatorname{cosec}\theta\cot\theta = \sec\theta\operatorname{cosec}^2\theta$ **b** $\sec\theta - \sin\theta = \dfrac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

2 Find the values of $\theta \in [0, 2\pi]$ for which:

a $\operatorname{cosec}^2\theta = 4$ **b** $\operatorname{cosec}(2\theta) = 2$ **c** $\sec(3\theta) = \dfrac{2\sqrt{3}}{3}$ **d** $\operatorname{cosec}^2(2\theta) = 1$

e $\cot^2\theta = 3$ **f** $\cot(2\theta) = -1$ **g** $\operatorname{cosec}(3\theta) = -1$ **h** $\sec(2\theta) = \sqrt{2}$

3 Solve the equation $\tan(\theta°) = 2\sin(\theta°)$ for values of $\theta°$ from $0°$ to $360°$.

4 If $\sin A = \dfrac{5}{13}$ and $\sin B = \dfrac{8}{17}$, where A and B are acute, find:

a $\cos(A + B)$ **b** $\sin(A - B)$ **c** $\tan(A + B)$

5 Find:

a $\cos 80° \cos 20° + \sin 80° \sin 20°$ **b** $\dfrac{\tan 15° + \tan 30°}{1 - \tan 15° \tan 30°}$

6 If $A + B = \dfrac{\pi}{2}$, find the value of:

a $\sin A \cos B + \cos A \sin B$ **b** $\cos A \cos B - \sin A \sin B$

7 Prove each of the following:

a $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B$

b $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} = \dfrac{2}{\sin\theta}$ **c** $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

8 Given that $\sin A = \dfrac{\sqrt{5}}{3}$ and that A is obtuse, find the value of:

a $\cos(2A)$ **b** $\sin(2A)$ **c** $\sin(4A)$

9 Prove:

a $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \cos(2A)$ **b** $\sqrt{2r^2(1 - \cos\theta)} = 2r\sin\left(\dfrac{\theta}{2}\right)$ for $r > 0$ and θ acute

10 Find $\tan 15°$ in simplest surd form.

11 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sin x + \cos x = 1$ **b** $\sin\left(\dfrac{1}{2}x\right)\cos\left(\dfrac{1}{2}x\right) = -\dfrac{1}{4}$

c $3\tan(2x) = 2\tan x$ **d** $\sin^2 x = \cos^2 x + 1$

e $\sin(3x)\cos x - \cos(3x)\sin x = \dfrac{\sqrt{3}}{2}$ **f** $2\cos\left(2x - \dfrac{\pi}{3}\right) = -\sqrt{3}$

12 Sketch the graph of:

a $y = 2\cos^2 x$ **b** $y = 2\sin^2 x$

13 If $\tan A = 2$ and $\tan(\theta + A) = 4$, find the exact value of $\tan\theta$.

14 **a** Express $2\cos\theta + 9\sin\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b **i** Give the maximum value of $2\cos\theta + 9\sin\theta$.

ii Give the cosine of θ for which this maximum occurs.

iii Find the smallest positive solution of the equation $2\cos\theta + 9\sin\theta = 1$.

15 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\sin(4\theta) + \sin(2\theta) = 0$ **b** $\sin(2\theta) - \sin(\theta) = 0$

16 Prove that $\dfrac{\cos A - \cos B}{\sin A + \sin B} = \tan\left(\dfrac{B - A}{2}\right)$.

Multiple-choice questions

1 $\operatorname{cosec} x - \sin x$ is equal to

A $\cos x \cot x$ **B** $\operatorname{cosec} x \tan x$ **C** $1 - \sin^2 x$

D $\sin x \operatorname{cosec} x$ **E** $\dfrac{1 - \sin x}{\sin x}$

2 If $\cos x = -\frac{1}{3}$, then the possible values of $\sin x$ are

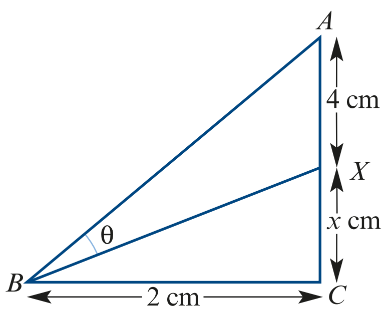
A $-\dfrac{2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}$ **B** $-\dfrac{2}{3}, \dfrac{2}{3}$ **C** $-\dfrac{8}{9}, \dfrac{8}{9}$

D $-\dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}$ **E** $-\dfrac{1}{2}, \dfrac{1}{2}$

3 If $\cos\theta = \frac{a}{b}$ both a and b are positive, and $0 < \theta < \frac{\pi}{2}$, then $\tan\theta$ is equal to

A $\dfrac{\sqrt{a^2 + b^2}}{b}$ **B** $\dfrac{\sqrt{b^2 - a^2}}{a}$ **C** $\dfrac{a}{\sqrt{b^2 - a^2}}$ **D** $\dfrac{a}{\sqrt{b^2 + a^2}}$ **E** $\dfrac{a}{b\sqrt{b^2 + a^2}}$

4 In the diagram, the magnitude of $\angle ABX$ is θ, $AX = 4$ cm, $XC = x$ cm and $BC = 2$ cm. Therefore $\tan\theta$ is equal to



A $\dfrac{8}{(x + 2)^2}$ **B** $\dfrac{4}{x}$ **C** $8 - x$

D $8 + x$ **E** $\dfrac{8}{\sqrt{x^2 + 4}}$

5 If $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$ with $\cos A = t$ and $\sin B = t$, then $\sin(B + A)$ equals

A 0 **B** 1 **C** $2t^2 - 1$ **D** $1 - 2t^2$ **E** -1

6 $\dfrac{\sin(2A)}{\cos(2A) - 1}$ is equal to

A $\cot(2A) - 1$ **B** $\sin(2A) + \sec(2A)$ **C** $\dfrac{\sin A}{\cos A - 1}$

D $\sin(2A) - \tan(2A)$ **E** $-\cot A$

7 $(1 + \cot x)^2 + (1 - \cot x)^2$ is equal to

A $2 + \cot(x) + 2\cot(2x)$ **B** 2 **C** $-4\cot x$

D $2 + \cot(2x)$ **E** $2\operatorname{cosec}^2 x$

8 If $\sin(2A) = m$ and $\cos A = n$, then $\tan A$ is equal to

A $\dfrac{m}{2n^2}$ **B** $\dfrac{n}{m}$ **C** $\dfrac{2n}{m^2}$ **D** $\dfrac{2n}{m}$ **E** $\dfrac{2n^2}{m}$

9 Expressing $-\cos x + \sin x$ in the form $r\sin(x + \alpha)$, where $r > 0$, gives

A $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ **B** $-\sin\left(x + \frac{\pi}{4}\right)$ **C** $\sqrt{2}\sin\left(x + \frac{5\pi}{4}\right)$

D $\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$ **E** $\sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$

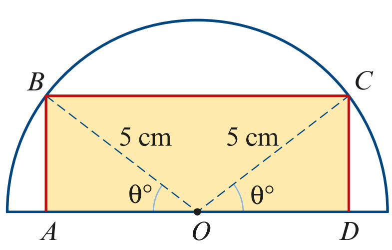
10 The product $\sin 25° \cos 75°$ can be rewritten as

A $\sin 100° - \sin 50°$ **B** $2(\sin 100° + \sin 50°)$ **C** $2(\sin 100° - \sin 50°)$

D $\frac{1}{2}(\sin 100° + \sin 50°)$ **E** $\frac{1}{2}(\sin 100° - \sin 50°)$

Extended-response questions

1 The diagram shows a rectangle $ABCD$ inside a semicircle, centre O and radius 5 cm, with $\angle BOA = \angle COD = \theta°$.



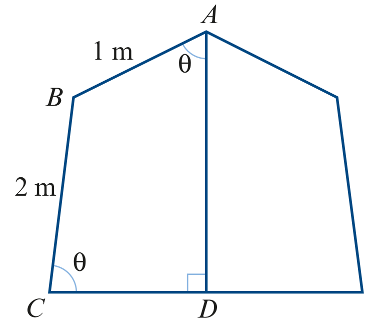
a Show that the perimeter, P cm, of the rectangle is given by $P = 20\cos\theta + 10\sin\theta$.

b Express P in the form $r\cos(\theta - \alpha)$ and hence find the value of θ for which $P = 16$.

c Find the value of k for which the area of the rectangle is $k\sin(2\theta)$ cm^2.

d Find the value of θ for which the area is a maximum.

2 The diagram shows a vertical section through a tent in which $AB = 1$ m, $BC = 2$ m and $\angle BAD = \angle BCD = \theta$. The line CD is horizontal, and the diagram is symmetrical about the vertical AD.



a Obtain an expression for AD in terms of θ.

b Express AD in the form $r\cos(\theta - \alpha)$, where r is positive.

c State the maximum length of AD and the corresponding value of θ.

d Given that $AD = 2.15$ m, find the value of θ for which $\theta > \alpha$.

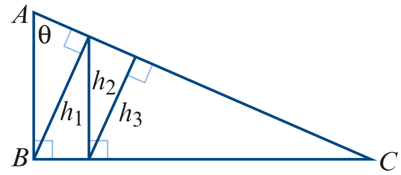
3 **a** Prove the identity $\cos(2\theta) = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$.

b **i** Use the result of **a** to show that $1 + x^2 = \sqrt{2}x^2 - \sqrt{2}$, where $x = \tan(67\tfrac{1}{2})^\circ$.

ii Hence find the values of integers a and b such that $\tan(67\tfrac{1}{2})^\circ = a + b\sqrt{2}$.

c Find the value of $\tan(7\tfrac{1}{2})^\circ$.

4 In the diagram, $\triangle ABC$ has a right angle at B, the length of BC is 1 unit and $\angle BAC = \theta$.



a Find in terms of θ:

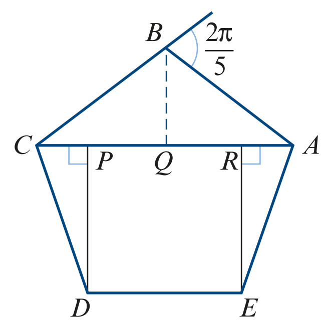
i h_1 **ii** h_2 **iii** h_3 **iv** h_n

b Show that the infinite sum is given by

$$h_1 + h_2 + h_3 + \cdots = \frac{\cos\theta}{1 - \sin\theta}$$

c If the value of the infinite sum is $\sqrt{2}$, find θ.

5 $ABCDE$ is a regular pentagon with side length one unit. The exterior angles of a regular pentagon each have magnitude $\dfrac{2\pi}{5}$.



a **i** Show that the magnitude of $\angle BCA$ is $\dfrac{\pi}{5}$.

ii Find the length of CA.

b **i** Show the magnitude of $\angle DCP$ is $\dfrac{2\pi}{5}$.

ii Use the fact that $AC = 2CQ = 2CP + PR$ to show that $2\cos\left(\dfrac{\pi}{5}\right) = 2\cos\left(\dfrac{2\pi}{5}\right) + 1$.

iii Use the identity $\cos(2\theta) = 2\cos^2\theta - 1$ to form a quadratic equation in terms of $\cos\left(\dfrac{\pi}{5}\right)$.

iv Find the exact value of $\cos\left(\dfrac{\pi}{5}\right)$.

6 **a** Prove each of the identities:

i $\cos\theta = \dfrac{1 - \tan^2(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$ **ii** $\sin\theta = \dfrac{2\tan(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$

b Use the results of **a** to find the value of $\tan(\tfrac{1}{2}\theta)$, given that $8\cos\theta - \sin\theta = 4$.

17

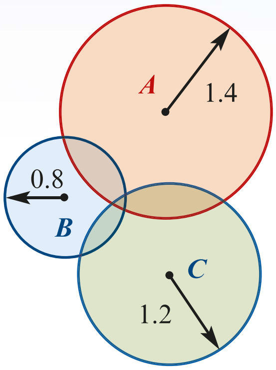
Graphing functions and relations

Objectives

- To define and sketch the graphs of the **inverse circular functions**.
- To sketch graphs of **reciprocal functions**, including those of polynomial functions and circular functions.
- To solve equations and sketch graphs involving the **modulus function**.
- To give **locus definitions** of lines, circles, parabolas, ellipses and hyperbolas, and to find the **Cartesian equations** of these curves.
- To use **parametric equations** to describe curves in the plane.
- To understand **polar coordinates** and their relationship to **Cartesian coordinates**.
- To sketch graphs in polar form.

The extensive use of mobile phones has led to an increased awareness of potential threats to the privacy of their users. For example, a little basic mathematics can be employed to track the movements of someone in possession of a mobile phone.

Suppose that there are three transmission towers within range of your mobile phone. By measuring the time taken for signals to travel between your phone and each tower, it is possible to estimate your distance from each tower.



In the diagram, there are towers at points A, B and C. If it is estimated that a person is no more than 1.4 km from A, no more than 0.8 km from B and no more than 1.2 km from C, then the person can be located in the intersection of the three circles.

In this chapter, we will look at different ways of describing circles and various other interesting figures.

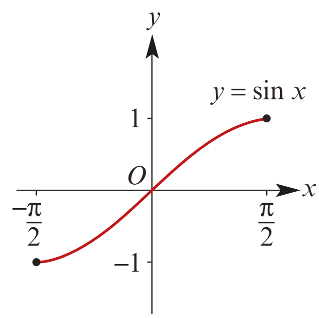
17A The inverse circular functions

As the circular functions sine, cosine and tangent are periodic, they are not one-to-one and therefore they do not have inverse functions. However, by restricting their domains to form one-to-one functions, we can define the inverse circular functions.

The inverse sine function: $y = \sin^{-1} x$

Restricting the sine function

When the domain of the sine function is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the resulting function is one-to-one and therefore has an inverse function.



Note: Other intervals (defined through consecutive turning points of the graph) could have been used for the restricted domain, but this is the convention.

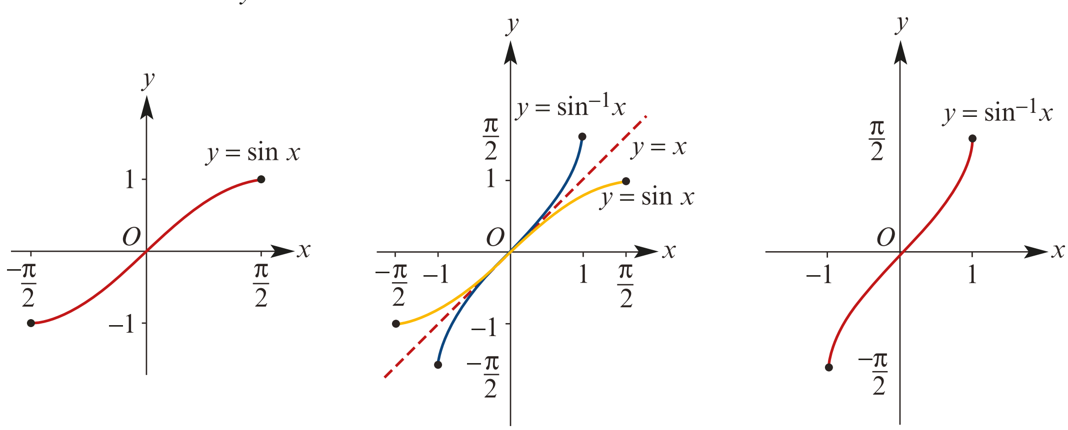
Defining the inverse function

The inverse of the restricted sine function is usually denoted by $\sin^{-1}$ or arcsin.

Inverse sine function

$$\sin^{-1} x = y \quad \text{if} \quad \sin y = x, \quad \text{for } x \in [-1, 1] \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The graph of $y = \sin^{-1} x$ is obtained from the graph of $y = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, through a reflection in the line $y = x$.



- **Domain** Domain of $\sin^{-1}$ = range of restricted sine function = $[-1, 1]$
- **Range** Range of $\sin^{-1}$ = domain of restricted sine function = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- **Inverse relationship**
  - $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$
  - $\sin^{-1}(\sin x) = x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The inverse cosine function: $y = \cos^{-1} x$

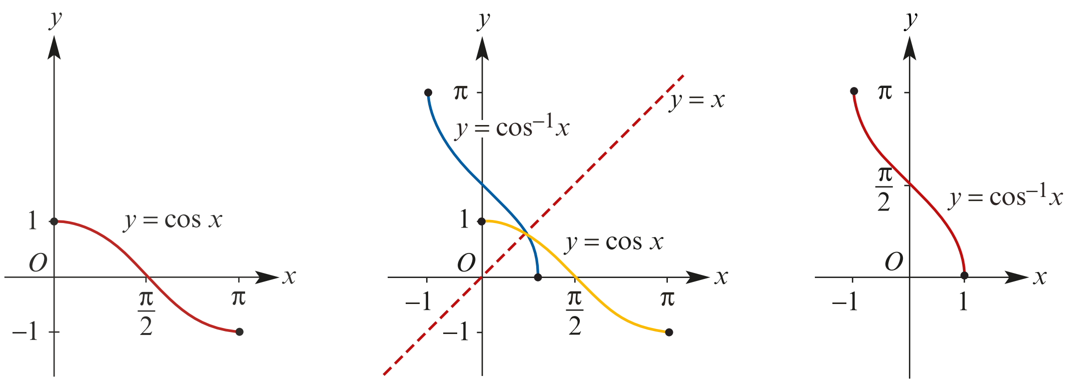
The standard domain for the restricted cosine function is $[0, \pi]$.

The restricted cosine function is one-to-one, and its inverse is denoted by $\cos^{-1}$ or arccos.

Inverse cosine function

$$\cos^{-1} x = y \quad \text{if} \quad \cos y = x, \quad \text{for } x \in [-1, 1] \text{ and } y \in [0, \pi]$$

The graph of $y = \cos^{-1} x$ is obtained from the graph of $y = \cos x$, $x \in [0, \pi]$, through a reflection in the line $y = x$.



- **Domain** Domain of $\cos^{-1}$ = range of restricted cosine function = $[-1, 1]$
- **Range** Range of $\cos^{-1}$ = domain of restricted cosine function = $[0, \pi]$
- **Inverse relationship**
  - $\cos(\cos^{-1} x) = x$ for all $x \in [-1, 1]$
  - $\cos^{-1}(\cos x) = x$ for all $x \in [0, \pi]$

The inverse tangent function: $y = \tan^{-1} x$

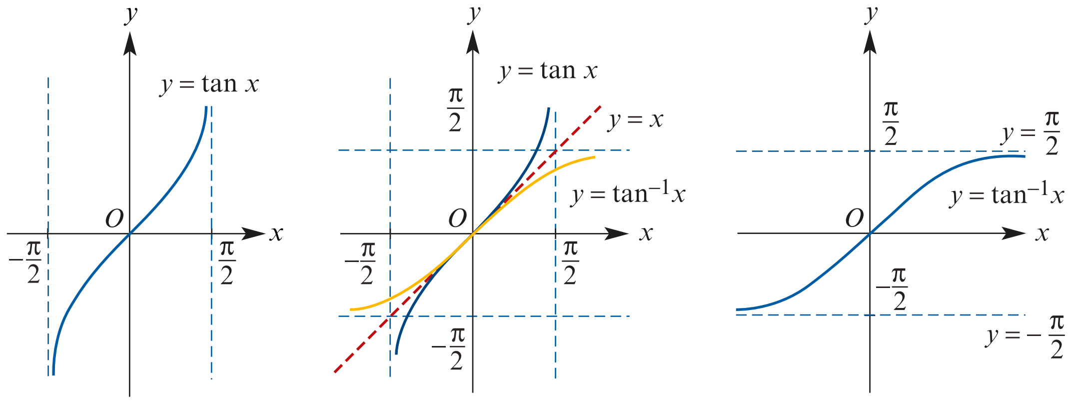
The domain of the restricted tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The restricted tangent function is one-to-one, and its inverse is denoted by $\tan^{-1}$ or arctan.

Inverse tangent function

$$\tan^{-1} x = y \quad \text{if} \quad \tan y = x, \quad \text{for } x \in \mathbb{R} \text{ and } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graph of $y = \tan^{-1} x$ is obtained from the graph of $y = \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, through a reflection in the line $y = x$.



- **Domain** Domain of $\tan^{-1}$ = range of restricted tangent function = $\mathbb{R}$
- **Range** Range of $\tan^{-1}$ = domain of restricted tangent function = $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
- **Inverse relationship**
  - $\tan(\tan^{-1} x) = x$ for all $x \in \mathbb{R}$
  - $\tan^{-1}(\tan x) = x$ for all $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Example 1

Sketch the graph of each of the following functions for the maximal domain:

a $y = \cos^{-1}(2 - 3x)$ **b** $y = \tan^{-1}(x + 2) + \dfrac{\pi}{2}$

Solution

a $\cos^{-1}(2 - 3x)$ is defined $\Leftrightarrow \quad -1 \le 2 - 3x \le 1$

$$\Leftrightarrow \quad -3 \le -3x \le -1$$

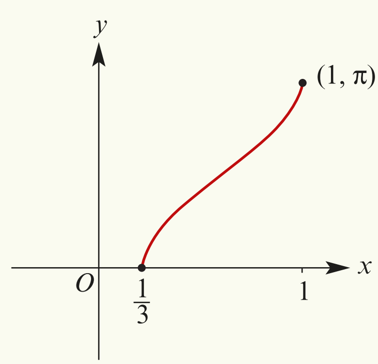
$$\Leftrightarrow \quad \frac{1}{3} \le x \le 1$$

The implied domain is $\left[\dfrac{1}{3}, 1\right]$.

It helps to write $y = \cos^{-1}\left(-3\left(x - \dfrac{2}{3}\right)\right)$.

The graph is obtained from the graph of $y = \cos^{-1} x$ by the following sequence of transformations:

- a dilation of factor $\frac{1}{3}$ from the y-axis
- a reflection in the y-axis
- a translation of $\frac{2}{3}$ units in the positive direction of the x-axis.

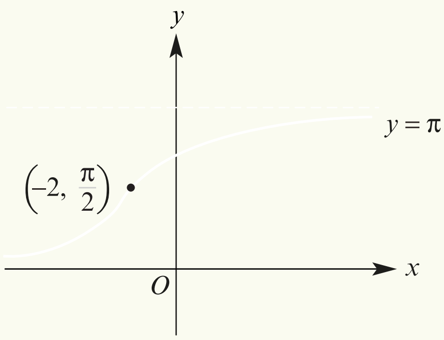


b The domain of $\tan^{-1}$ is $\mathbb{R}$.

The graph of

$$y = \tan^{-1}(x + 2) + \frac{\pi}{2}$$

is obtained from the graph of $y = \tan^{-1} x$ by a translation of 2 units in the negative direction of the x-axis and $\frac{\pi}{2}$ units in the positive direction of the y-axis.



Example 2

a Evaluate $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

b Simplify:

i $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$

ii $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

iii $\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$

iv $\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$

Solution

a Evaluating $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is equivalent to solving $\sin y = -\frac{\sqrt{3}}{2}$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

b **i** Since $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by definition we have

$$\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$$

ii

$$\begin{aligned}\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) &= \sin^{-1}\left(\sin\left(\pi - \frac{5\pi}{6}\right)\right)\\ &= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)\\ &= \frac{\pi}{6}\end{aligned}$$

iii

$$\begin{aligned}\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) &= \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right)\\ &= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)\\ &= \frac{\pi}{6}\end{aligned}$$

iv

$$\begin{aligned}\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) &= \sin\left(\frac{\pi}{4}\right)\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Example 3

Find the implied domain and range of:

a $y = \sin^{-1}(2x - 1)$

b $y = 3\cos^{-1}(2 - 2x)$

Solution

a For $\sin^{-1}(2x - 1)$ to be defined:

$$-1 \le 2x - 1 \le 1$$
$$\Leftrightarrow \quad 0 \le 2x \le 2$$
$$\Leftrightarrow \quad 0 \le x \le 1$$

Thus the implied domain is $[0, 1]$.

The range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b For $3\cos^{-1}(2 - 2x)$ to be defined:

$$-1 \le 2 - 2x \le 1$$
$$\Leftrightarrow \quad -3 \le -2x \le -1$$
$$\Leftrightarrow \quad \frac{1}{2} \le x \le \frac{3}{2}$$

Thus the implied domain is $\left[\frac{1}{2}, \frac{3}{2}\right]$.

The range is $[0, 3\pi]$.

Summary 17A

Inverse circular functions

- $\sin^{-1}\colon [-1, 1] \to \mathbb{R}$, $\sin^{-1} x = y$, where $\sin y = x$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos^{-1}\colon [-1, 1] \to \mathbb{R}$, $\cos^{-1} x = y$, where $\cos y = x$ and $y \in [0, \pi]$
- $\tan^{-1}\colon \mathbb{R} \to \mathbb{R}$, $\tan^{-1} x = y$, where $\tan y = x$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Skill-sheet

Exercise 17A

Example 1

1 Sketch the graphs of the following functions, stating clearly the implied domain and the range of each:

a $y = \tan^{-1}(x - 1)$ **b** $y = \cos^{-1}(x + 1)$ **c** $y = 2\sin^{-1}\left(x + \frac{1}{2}\right)$

d $y = 2\tan^{-1}(x) + \frac{\pi}{2}$ **e** $y = \cos^{-1}(2x)$ **f** $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$

Example 2a

2 Evaluate each of the following:

a $\sin^{-1} 1$ **b** $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **c** $\sin^{-1} 0.5$

d $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **e** $\cos^{-1} 0.5$ **f** $\tan^{-1} 1$

g $\tan^{-1}(-\sqrt{3})$ **h** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **i** $\cos^{-1}(-1)$

Example 2b

3 Simplify:

a $\sin(\cos^{-1} 0.5)$ **b** $\sin^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right)$ **c** $\tan\left(\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$ **d** $\cos(\tan^{-1} 1)$

e $\tan^{-1}\left(\sin\left(\frac{5\pi}{2}\right)\right)$ **f** $\tan(\cos^{-1} 0.5)$ **g** $\cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)$ **h** $\sin^{-1}\left(\sin\left(-\frac{2\pi}{3}\right)\right)$

i $\tan^{-1}\left(\tan\left(\frac{11\pi}{4}\right)\right)$ **j** $\cos^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$ **k** $\cos^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$ **l** $\sin^{-1}\left(\cos\left(-\frac{3\pi}{4}\right)\right)$

4 Let $f: \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}$, $f(x) = \sin x$.

a Define f^{-1}, clearly stating its domain and its range.

b Evaluate:

i $f\left(\frac{\pi}{2}\right)$ **ii** $f\left(\frac{3\pi}{4}\right)$ **iii** $f\left(\frac{7\pi}{6}\right)$ **iv** $f^{-1}(-1)$ **v** $f^{-1}(0)$ **vi** $f^{-1}(0.5)$

Example 3

5 Given that the domains of sin, cos and tan are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $[0, \pi]$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ respectively, give the implied domain and range of each of the following:

a $y = \sin^{-1}(2 - x)$ **b** $y = \sin\left(x + \frac{\pi}{4}\right)$ **c** $y = \sin^{-1}(2x + 4)$

d $y = \sin\left(3x - \frac{\pi}{3}\right)$ **e** $y = \cos\left(x - \frac{\pi}{6}\right)$ **f** $y = \cos^{-1}(x + 1)$

g $y = \cos^{-1}(x^2)$ **h** $y = \cos\left(2x + \frac{2\pi}{3}\right)$ **i** $y = \tan^{-1}(x^2)$

j $y = \tan\left(2x - \frac{\pi}{2}\right)$ **k** $y = \tan^{-1}(2x + 1)$ **l** $y = \tan(x^2)$

6 Simplify each of the following expressions, in an exact form:

a $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$ **b** $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$ **c** $\cos\left(\tan^{-1}\left(\frac{7}{24}\right)\right)$ **d** $\tan\left(\sin^{-1}\left(\frac{40}{41}\right)\right)$

e $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$ **f** $\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$ **g** $\sin(\tan^{-1}(-2))$ **h** $\cos\left(\sin^{-1}\left(\frac{3}{7}\right)\right)$

7 Let $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{5}{13}$, where $\alpha \in \left[0, \frac{\pi}{2}\right]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$.

a Find:

i $\cos\alpha$ **ii** $\cos\beta$

b Use a compound angle formula to show that:

i $\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) = \sin^{-1}\left(\frac{16}{65}\right)$

ii $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

8 Given that the domains of sin and cos are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively, explain why each expression cannot be evaluated:

a $\cos(\sin^{-1}(-0.5))$ **b** $\sin(\cos^{-1}(-0.2))$ **c** $\cos(\tan^{-1}(-1))$

17B Reciprocal functions

Reciprocals of polynomials

You have learned in previous years that the **reciprocal** of a non-zero number a is $\frac{1}{a}$. Likewise, we have the following definition.

> If $y = f(x)$ is a polynomial function, then its **reciprocal function** is defined by the rule
>
> $$y = \frac{1}{f(x)}$$

For example, the reciprocal of the function $y = x^3$ is $y = \frac{1}{x^3}$.

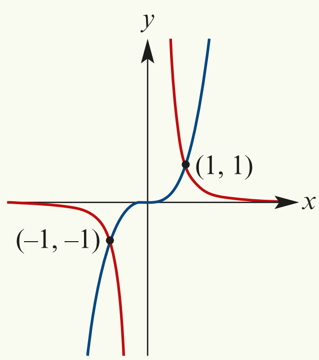
In this section, we will find relationships between the graph of a function and the graph of its reciprocal. Let's consider some specific examples, from which we will draw general conclusions.

Example 4

Sketch the graphs of $y = x^3$ and $y = \frac{1}{x^3}$ on the same set of axes.

Solution

We first sketch the graph of $y = x^3$. This is shown in blue.



Horizontal asymptotes

If $x \to \pm\infty$, then $\frac{1}{x^3} \to 0$. Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes

Notice that $x^3 = 0$ when $x = 0$.

If x is a small positive number, then $\frac{1}{x^3}$ is a large positive number.

If x is a small negative number, then $\frac{1}{x^3}$ is a large negative number.

Therefore the line $x = 0$ is a vertical asymptote of the reciprocal function.

Observations from the example

This example highlights behaviour typical of reciprocal functions:

- If $y = f(x)$ is a non-zero polynomial function, then the graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes where $f(x) = 0$.
- The graphs of a function and its reciprocal are always on the same side of the x-axis.
- If the graphs of a function and its reciprocal intersect, then it must be where $f(x) = \pm 1$.

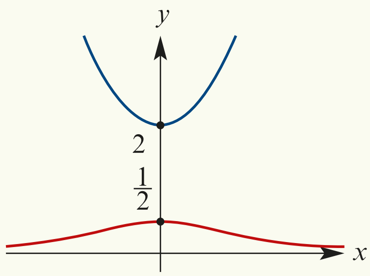
The following example is perhaps easier, because the reciprocal graph has no vertical asymptotes. This time we are interested in turning points.

Example 5

Consider the function $f(x) = x^2 + 2$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = x^2 + 2$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\dfrac{1}{f(x)} \to 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes There are no vertical asymptotes, as there is no solution to the equation $f(x) = 0$.

Turning points Notice that the graph of $y = x^2 + 2$ has a minimum at $(0, 2)$. The reciprocal function therefore has a maximum at $(0, \frac{1}{2})$.

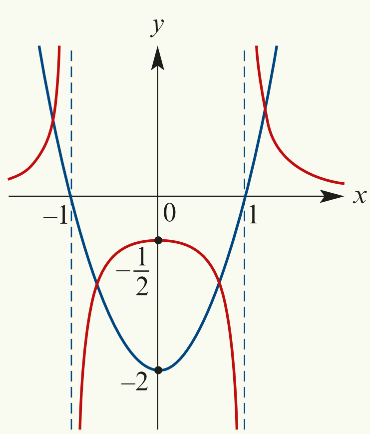
- If the graph of $y = f(x)$ has a local minimum at $x = a$, then the graph of $y = \dfrac{1}{f(x)}$ will have a local maximum at $x = a$.
- If the graph of $y = f(x)$ has a local maximum at $x = a$, then the graph of $y = \dfrac{1}{f(x)}$ will have a local minimum at $x = a$.

Example 6

Consider the function $f(x) = 2(x - 1)(x + 1)$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2(x - 1)(x + 1)$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\dfrac{1}{f(x)} \to 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes We have $f(x) = 0$ when $x = -1$ or $x = 1$. Therefore the lines $x = -1$ and $x = 1$ are vertical asymptotes of the reciprocal function.

Turning points The graph of $y = f(x)$ has a minimum at $(0, -2)$. Therefore the reciprocal has a local maximum at $(0, -\frac{1}{2})$.

Reciprocals of further functions

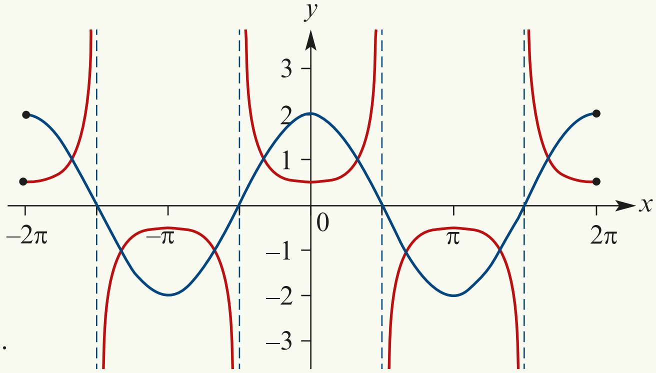
The techniques used to sketch graphs of the reciprocals of polynomial functions can also be used for the reciprocals of other functions. We give two examples here.

Example 7

Let $f(x) = 2\cos x$ for $-2\pi \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2\cos x$ for $x \in [-2\pi, 2\pi]$. This is shown in blue.



Vertical asymptotes

Vertical asymptotes of the reciprocal function will occur when $f(x) = 0$.

These are given by $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}$.

Turning points

The points $(0, 2)$ and $(\pm 2\pi, 2)$ are local maximums of $y = f(x)$. Therefore the points $(0, \frac{1}{2})$ and $(\pm 2\pi, \frac{1}{2})$ are local minimums of the reciprocal.

The points $(\pm\pi, -2)$ are local minimums of $y = f(x)$. Therefore the points $(\pm\pi, -\frac{1}{2})$ are local maximums of the reciprocal.

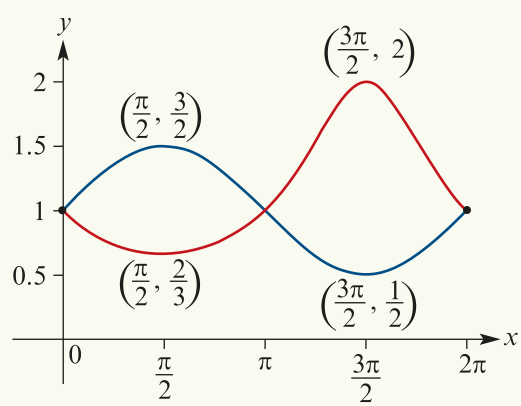
The graph of the next function has no x-axis intercepts, and so its reciprocal has no vertical asymptotes.

Example 8

Let $f(x) = 0.5\sin x + 1$ for $0 \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 0.5\sin x + 1$ for $x \in [0, 2\pi]$. This is shown in blue.



Turning points

The point $(\frac{\pi}{2}, \frac{3}{2})$ is a local maximum of $y = f(x)$. Therefore the point $(\frac{\pi}{2}, \frac{2}{3})$ is a local minimum of the reciprocal.

The point $(\frac{3\pi}{2}, \frac{1}{2})$ is a local minimum of $y = f(x)$. Therefore the point $(\frac{3\pi}{2}, 2)$ is a local maximum of the reciprocal.

Summary 17B

Given the graph of a continuous function $y = f(x)$, we can sketch the graph of $y = \frac{1}{f(x)}$ with the help of the following observations:

Function $y = f(x)$	Reciprocal function $y = \frac{1}{f(x)}$
x-axis intercept at $x = a$	vertical asymptote $x = a$
local maximum at $x = a$	local minimum at $x = a$
local minimum at $x = a$	local maximum at $x = a$
above the x-axis	above the x-axis
below the x-axis	below the x-axis
increasing over an interval	decreasing over the interval
decreasing over an interval	increasing over the interval
values approach ∞	values approach 0 from above
values approach $-\infty$	values approach 0 from below
values approach 0 from above	values approach ∞
values approach 0 from below	values approach $-\infty$

Skill-sheet

Exercise 17B

Example 4 Example 5 Example 6

1 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes:

a $f(x) = x + 3$ **b** $f(x) = x^2$ **c** $f(x) = x^2 + 4$
d $f(x) = (x - 1)(x + 1)$ **e** $f(x) = 4 - x^2$ **f** $f(x) = (x - 1)^2 - 1$
g $f(x) = x^2 - 2x - 3$ **h** $f(x) = -x^2 - 2x + 3$ **i** $f(x) = x^3 + 1$

Example 7 Example 8

2 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes. Label asymptotes, turning points and endpoints.

a $f(x) = \sin x$ for $0 \le x \le 2\pi$ **b** $f(x) = \cos x$ for $0 \le x \le 2\pi$
c $f(x) = -2\cos x$ for $-\pi \le x \le \pi$ **d** $f(x) = \cos x + 1$ for $0 \le x \le 4\pi$
e $f(x) = -\sin x - 1$ for $-2\pi \le x \le 2\pi$ **f** $f(x) = \cos x - 2$ for $0 \le x \le 2\pi$
g $f(x) = -\sin x + 2$ for $0 \le x \le 2\pi$ **h** $f(x) = -2\cos x + 3$ for $-\pi \le x \le \pi$

3 Consider the quadratic function $f(x) = x^2 + 2x + 2$.

a By completing the square, find the turning point of the graph of $y = f(x)$.

b Hence sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

4 Consider the quadratic function $f(x) = 5x(1 - x)$.

a Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

b Find the points of intersection of the two graphs by solving $f(x) = 1$ and $f(x) = -1$.

5 Sketch the graphs of $y = 2\sin^2 x$ and $y = \dfrac{1}{2\sin^2 x}$ on the same set of axes, over the interval $0 \le x \le 2\pi$.

6 Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = 2^x - 1$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

7 Let $k \in \mathbb{R}$ and consider the function $f(x) = x^2 + 2kx + 1$.

a By completing the square, show that the graph of $y = f(x)$ has a minimum turning point at $(-k, 1 - k^2)$.

b For what values of k does the graph of $y = f(x)$ have:

i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts?

c Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when the graph of $y = f(x)$ has:

i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts.

Hint: It helps to ignore the y-axis.

17C Graphing the reciprocal circular functions

The reciprocal circular functions were introduced in Chapter 16. In this section, we consider the graphs of these functions and basic transformations of these functions.

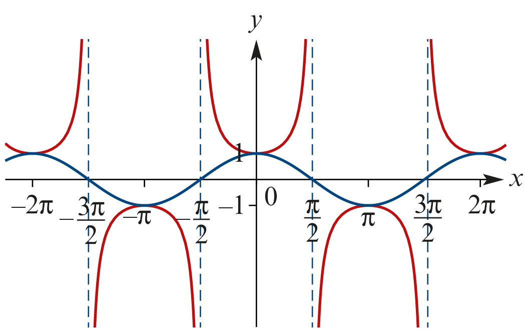
The secant function

The secant function is defined by

$$\sec x = \frac{1}{\cos x}$$

provided $\cos x \neq 0$.

The graphs of $y = \cos x$ and $y = \sec x$ are shown here on the same axes.



The significant features of the two graphs are listed in the following table.

Function $y = \cos x$	Reciprocal function $y = \sec x$
x-axis intercepts at $x = \dfrac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$	vertical asymptotes at $x = \dfrac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$
domain $= \mathbb{R}$	domain $= \mathbb{R} \setminus \left\{ \dfrac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$
local maximums at $(2n\pi, 1)$, $n \in \mathbb{Z}$	local minimums at $(2n\pi, 1)$, $n \in \mathbb{Z}$
local minimums at $((2n+1)\pi, -1)$, $n \in \mathbb{Z}$	local maximums at $((2n+1)\pi, -1)$, $n \in \mathbb{Z}$
range $= [-1, 1]$	range $= (-\infty, -1] \cup [1, \infty)$

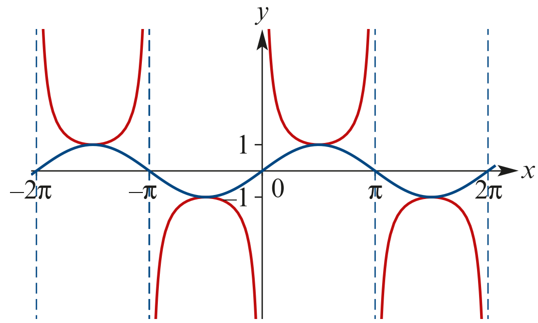
The cosecant function

The cosecant function is defined by

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

provided $\sin x \neq 0$.

The graphs of $y = \sin x$ and $y = \operatorname{cosec} x$ are shown here on the same axes.



Function $y = \sin x$	Reciprocal function $y = \operatorname{cosec} x$
x-axis intercepts at $x = n\pi,\ n \in \mathbb{Z}$	vertical asymptotes at $x = n\pi,\ n \in \mathbb{Z}$
domain $= \mathbb{R}$	domain $= \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \}$
local maximums at $\left(2n\pi + \frac{\pi}{2}, 1\right),\ n \in \mathbb{Z}$	local minimums at $\left(2n\pi + \frac{\pi}{2}, 1\right),\ n \in \mathbb{Z}$
local minimums at $\left(2n\pi - \frac{\pi}{2}, -1\right),\ n \in \mathbb{Z}$	local maximums at $\left(2n\pi - \frac{\pi}{2}, -1\right),\ n \in \mathbb{Z}$
range $= [-1, 1]$	range $= (-\infty, -1] \cup [1, \infty)$

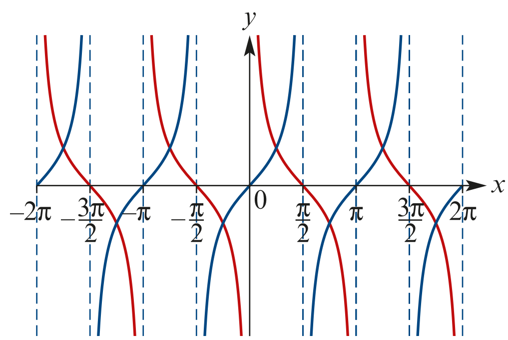
The cotangent function

The cotangent function is defined by

$$\cot x = \frac{\cos x}{\sin x}$$

provided $\sin x \neq 0$.

This diagram shows the graph of $y = \tan x$ in blue and the graph of $y = \cot x$ in red.



Function $y = \tan x$	Function $y = \cot x$
x-axis intercepts at $x = n\pi,\ n \in \mathbb{Z}$	vertical asymptotes at $x = n\pi,\ n \in \mathbb{Z}$
vertical asymptotes at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$	x-axis intercepts at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$
domain $= \mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$	domain $= \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \}$
range $= \mathbb{R}$	range $= \mathbb{R}$

Note the similarity between the graphs of $y = \cot x$ and $y = \tan x$. Using the complementary relationship between sine and cosine, we have

$$\cot x = \frac{\cos x}{\sin x} = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \tan\left(\frac{\pi}{2} - x\right) = \tan\left(-\left(x - \frac{\pi}{2}\right)\right) = -\tan\left(x - \frac{\pi}{2}\right)$$

Therefore the graph of $y = \cot x$ can be obtained from the graph of $y = \tan x$ by a reflection in the x-axis followed by a translation of $\frac{\pi}{2}$ units in the positive direction of the x-axis.

Transformations of the reciprocal circular functions

We now look at dilations, reflections and translations of the reciprocal circular functions.

Example 9

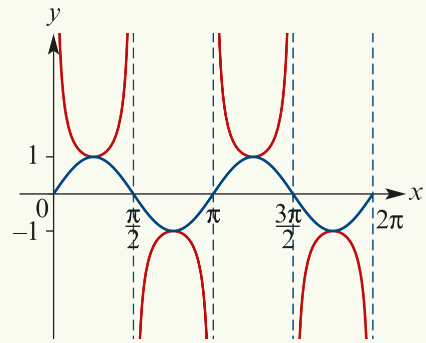
Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}(2x)$ **b** $y = 2\sec x$ **c** $y = -\cot x$

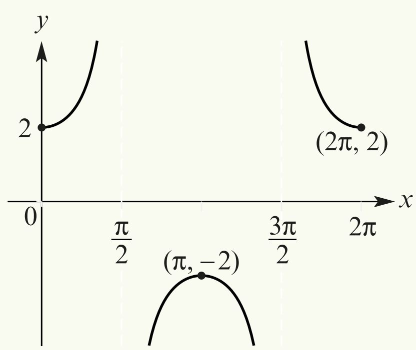
Solution

a The graph of $y = \operatorname{cosec}(2x)$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

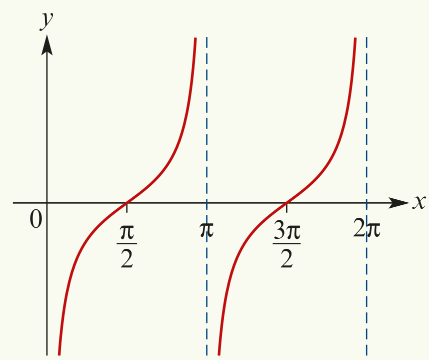
It is helpful to draw the graph of $y = \sin(2x)$ on the same axes.



b The graph of $y = 2\sec x$ can be obtained from the graph of $y = \sec x$ by a dilation of factor 2 from the x-axis.



c The graph of $y = -\cot x$ can be obtained from the graph of $y = \cot x$ by a reflection in the x-axis.



Example 10

Sketch the graph of each of the following over the interval $[0, 2\pi]$:

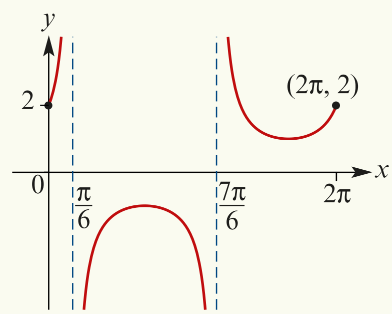
a $y = \sec\left(x + \frac{\pi}{3}\right)$ **b** $y = \operatorname{cosec}(x) - 2$ **c** $y = \cot\left(x - \frac{\pi}{4}\right)$

Solution

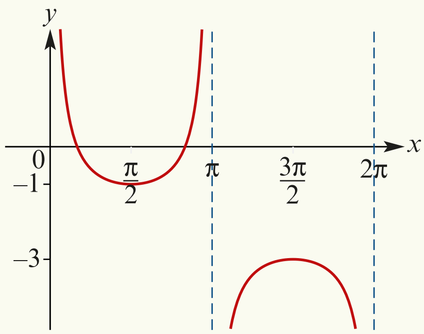
a The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ can be obtained from the graph of $y = \sec x$ by a translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.

The y-axis intercept is $\sec\left(\frac{\pi}{3}\right) = 2$.

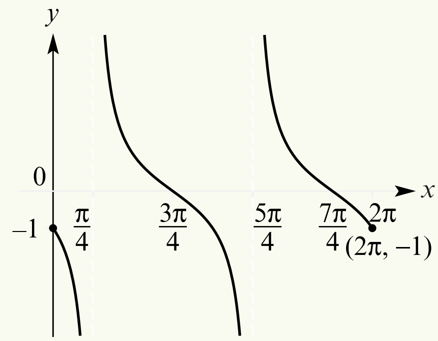
The asymptotes are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.



b The graph of $y = \operatorname{cosec}(x) - 2$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a translation of 2 units in the negative direction of the y-axis.



c The graph of $y = \cot\left(x - \frac{\pi}{4}\right)$ can be obtained from the graph of $y = \cot x$ by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis.



Example 11

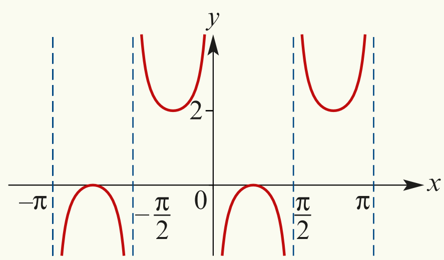
Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -\sec\left(2x - \frac{\pi}{2}\right) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

Solution

It helps to write the equation of the transformed graph as $y = -\sec\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

An appropriate sequence is:

1. reflection in the x-axis
2. dilation of factor $\frac{1}{2}$ from the y-axis
3. translation of $\frac{\pi}{4}$ units to the right and 1 unit up.



Summary 17C

Reciprocal circular functions

- $\sec x = \dfrac{1}{\cos x}$ (provided $\cos x \neq 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (provided $\sin x \neq 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$ (provided $\sin x \neq 0$)

Exercise 17C

Example 9

1 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec(2x)$ **b** $y = \cot(2x)$ **c** $y = 3\sec x$

d $y = 2\operatorname{cosec} x$ **e** $y = -\operatorname{cosec} x$ **f** $y = -2\sec x$

Example 10

2 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec\left(x - \frac{\pi}{2}\right)$ **b** $y = \cot\left(x + \frac{\pi}{4}\right)$ **c** $y = -\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$

d $y = 1 + \sec x$ **e** $y = 2 - \operatorname{cosec} x$ **f** $y = 1 + \cot\left(x + \frac{\pi}{4}\right)$

Example 11

3 Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -2\sec\left(x - \frac{\pi}{2}\right)$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

4 Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = \operatorname{cosec}(-2x) + 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

5 Describe a sequence of transformations that will take the graph of $y = \cot x$ to the graph of $y = -\cot\left(2x - \frac{\pi}{2}\right) - 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

6 On the one set of axes, sketch the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ over the interval $[0, 2\pi]$. Find and label the points of intersection.

17D The modulus function

The **modulus** or **absolute value** of a real number x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example:

$$|5| = 5 \quad \text{and} \quad |-5| = 5$$

The modulus function may also be defined as $|x| = \sqrt{x^2}$.

Example 12

Evaluate each of the following:

a i $|-3 \times 2|$ **ii** $|-3| \times |2|$

b i $\left|\frac{-4}{2}\right|$ **ii** $\frac{|-4|}{|2|}$

c i $|-6 + 2|$ **ii** $|-6| + |2|$

Solution

a i $|-3 \times 2| = |-6| = 6$ **ii** $|-3| \times |2| = 3 \times 2 = 6$ Note: $|-3 \times 2| = |-3| \times |2|$

b i $\left|\frac{-4}{2}\right| = |-2| = 2$ **ii** $\frac{|-4|}{|2|} = \frac{4}{2} = 2$ Note: $\left|\frac{-4}{2}\right| = \frac{|-4|}{|2|}$

c i $|-6 + 2| = |-4| = 4$ **ii** $|-6| + |2| = 6 + 2 = 8$ Note: $|-6 + 2| \neq |-6| + |2|$

Properties of the modulus function

- $|ab| = |a|\,|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \le |a| + |b|$
- If a and b are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \ge 0$, then $|x| \le a$ is equivalent to $-a \le x \le a$.
- If $a \ge 0$, then $|x - k| \le a$ is equivalent to $k - a \le x \le k + a$.

The following example uses the second property listed above to solve simple equations.

Example 13

Solve each of the following equations for x:

a $|x - 4| = 6$ **b** $|2x - 4| = 16$

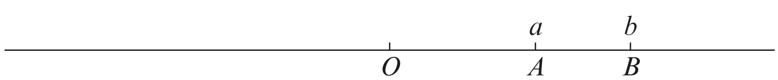
Solution

a
$$|x - 4| = 6$$
$$\Rightarrow \quad x - 4 = 6 \quad \text{or} \quad x - 4 = -6$$
$$\Rightarrow \quad x = 10 \quad \text{or} \quad x = -2$$

b
$$|2x - 4| = 16$$
$$\Rightarrow \quad 2x - 4 = 16 \quad \text{or} \quad 2x - 4 = -16$$
$$\Rightarrow \quad x = 10 \quad \text{or} \quad x = -6$$

The modulus function as a measure of distance

Consider two points A and B on a number line:



On a number line, the distance between points A and B is $|a - b| = |b - a|$.

For example:

- $|x - 2| \le 3$ can be read as 'the distance of x from 2 is less than or equal to 3'
- $|x| \le 3$ can be read as 'the distance of x from the origin is less than or equal to 3'.

Note that $|x| \le 3$ is equivalent to $-3 \le x \le 3$ or $x \in [-3, 3]$.

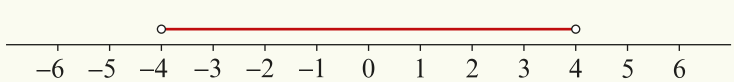
Example 14

Illustrate each of the following sets on a number line and represent the sets using interval notation:

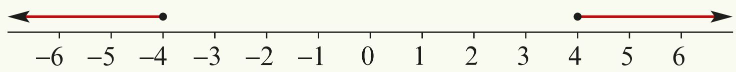
a $\{ x : |x| < 4 \}$ **b** $\{ x : |x| \ge 4 \}$ **c** $\{ x : |x - 1| \le 4 \}$

Solution

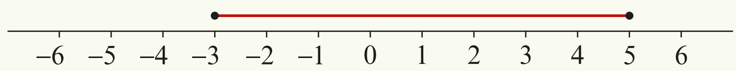
a $\{ x : |x| < 4 \}$
$= (-4, 4)$



b $\{ x : |x| \geq 4 \}$
$= (-\infty, -4] \cup [4, \infty)$



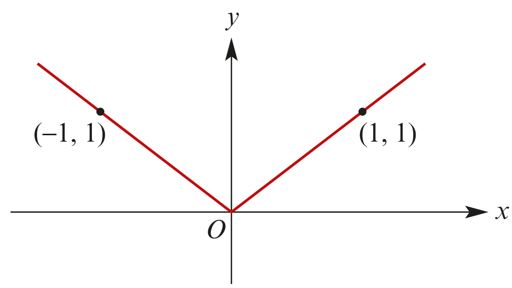
c $\{ x : |x - 1| \leq 4 \}$
$= [-3, 5]$



The graph of $y = |x|$

The graph of the function $y = |x|$ is shown on the right.

This graph is symmetric about the y-axis, since $|x| = |-x|$.



Example 15

For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 3| + 1$

b $f(x) = -|x - 3| + 1$

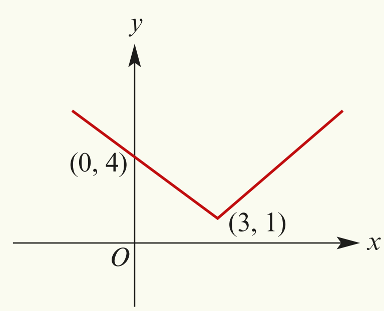
Solution

Note that $|a - b| = a - b$ if $a \geq b$, and $|a - b| = b - a$ if $b \geq a$.

a $f(x) = |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases}$

$= \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases}$

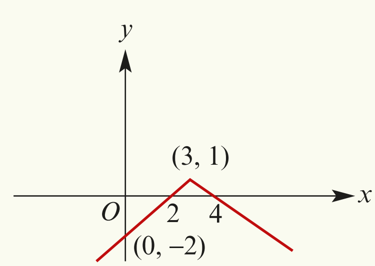
Range $= [1, \infty)$



b $f(x) = -|x - 3| + 1 = \begin{cases} -(x - 3) + 1 & \text{if } x \geq 3 \\ -(3 - x) + 1 & \text{if } x < 3 \end{cases}$

$= \begin{cases} -x + 4 & \text{if } x \geq 3 \\ -2 + x & \text{if } x < 3 \end{cases}$

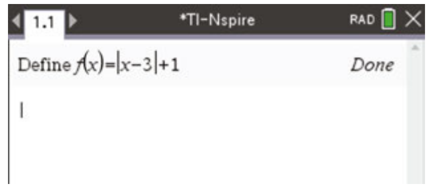
Range $= (-\infty, 1]$



Using the TI-Nspire

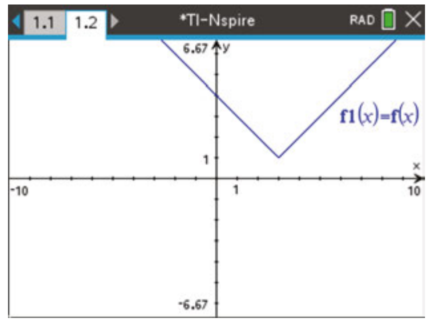
- Use (menu) > **Actions** > **Define** to define the function $f(x) = \text{abs}(x - 3) + 1$.

Note: The absolute value function can be obtained by typing **abs( )** or using the 2D-template palette.



- Open a **Graphs** application ((ctrl) (I) > **Graphs**) and let $f1(x) = f(x)$.
- Press (enter) to obtain the graph.

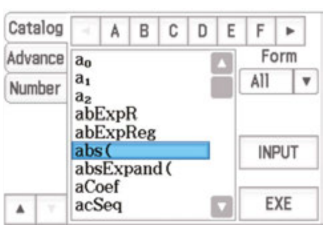
Note: The expression $\text{abs}(x - 3) + 1$ could have been entered directly for $f1(x)$.

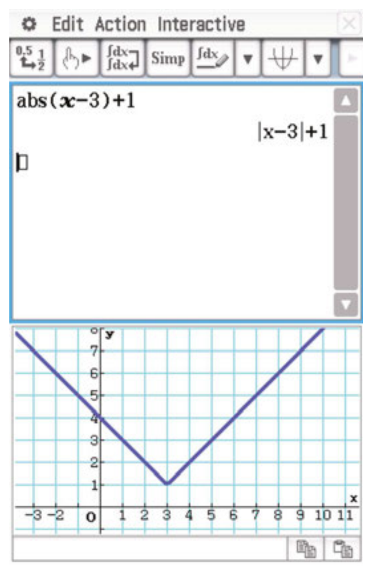


Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the expression $|x - 3| + 1$.

Note: To obtain the absolute value function, either choose **abs(** from the catalog (as shown below) or select the absolute value template from the (Math1) keyboard.

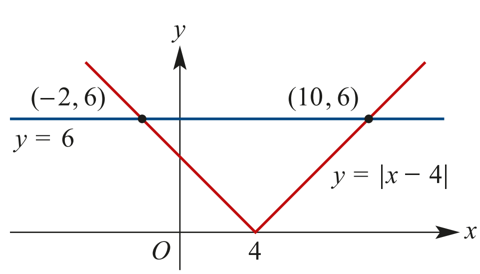




- Tap the graph icon to open the graph window.
- Highlight $|x - 3| + 1$ and drag into the graph window.
- Use the window icon to adjust the window manually.

Note: Alternatively, the function can be graphed using the **Graph & Table** application.

Note: The solution of equations can be shown graphically. For example, this graph shows the solutions of the equation $|x - 4| = 6$ from Example 13 **a**.



Functions with rules of the form $y = |f(x)|$ and $y = f(|x|)$

If the graph of $y = f(x)$ is known, then we can sketch the graph of $y = |f(x)|$ using the following observation:

$$|f(x)| = f(x) \text{ if } f(x) \geq 0 \quad \text{and} \quad |f(x)| = -f(x) \text{ if } f(x) < 0$$

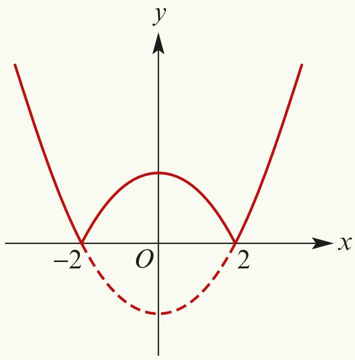
Example 16

Sketch the graph of each of the following:

a $y = |x^2 - 4|$ **b** $y = |2^x - 1|$

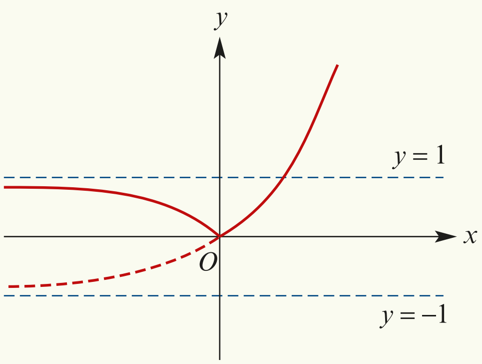
Solution

a



The graph of $y = x^2 - 4$ is drawn and the negative part reflected in the x-axis.

b



The graph of $y = 2^x - 1$ is drawn and the negative part reflected in the x-axis.

A graph with a rule of the form $y = f(|x|)$ is symmetric about the y-axis, since $f(|-x|) = f(|x|)$ for all x. Thus we can sketch the graph of $y = f(|x|)$, for $x \in \mathbb{R}$, by reflecting the graph of $y = f(x)$, for $x \geq 0$, in the y-axis.

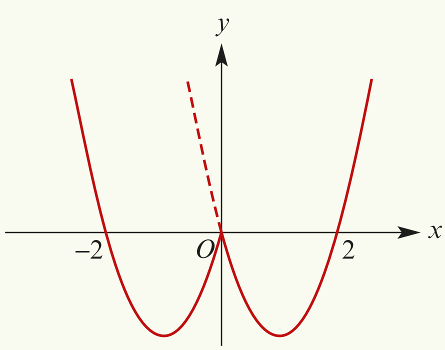
Example 17

Sketch the graph of each of the following:

a $y = |x|^2 - 2|x|$ **b** $y = 2^{|x|}$

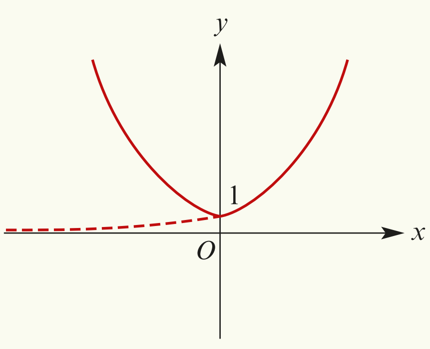
Solution

a



The graph of $y = x^2 - 2x$, $x \geq 0$, is reflected in the y-axis.

b



The graph of $y = 2^x$, $x \geq 0$, is reflected in the y-axis.

Example 18

a Solve the equation $|x^2 - 4x| = 3$ for x.

b Illustrate the solutions by graphing $y = |x^2 - 4x|$ and $y = 3$ on the same set of axes.

Solution

a

$$|x^2 - 4x| = 3$$

$$\Rightarrow \quad x^2 - 4x = 3 \quad \text{or} \quad x^2 - 4x = -3$$

$$\Rightarrow \quad x^2 - 4x - 3 = 0 \quad \text{or} \quad x^2 - 4x + 3 = 0$$

$$\Rightarrow \quad x = 2 \pm \sqrt{7} \quad \text{or} \quad x = 1 \text{ or } x = 3$$

Therefore $x = 2 + \sqrt{7}$, $x = 2 - \sqrt{7}$, $x = 1$ or $x = 3$.

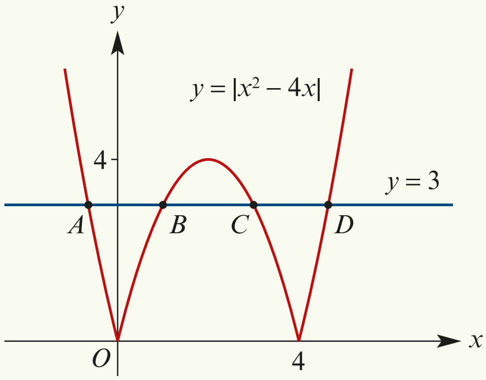
b The solutions correspond to the points of intersection of the two graphs:

$A(2 - \sqrt{7}, 3)$

$B(1, 3)$

$C(3, 3)$

$D(2 + \sqrt{7}, 3)$



Note: We can see from the graph of $y = |x^2 - 4x|$ that the equation $|x^2 - 4x| = 4$ has three solutions and the equation $|x^2 - 4x| = 5$ has two solutions.

Summary 17D

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- On the number line, the distance between a and b is given by $|a - b| = |b - a|$.
  For example: $|x - 2| < 5$ can be read as 'the distance between x and 2 is less than 5'.
- To sketch the graph of $y = |f(x)|$, first draw the graph of $y = f(x)$. Then reflect the sections of the graph that are below the x-axis so that they are above the x-axis.
- To sketch the graph of $y = f(|x|)$, first draw the graph of $y = f(x)$ for $x \geq 0$. Then reflect the graph across the y-axis to obtain the graph for $x \leq 0$.

Skill-sheet

Exercise 17D

Example 12

1 Evaluate each of the following:

a $|-5| + 3$ **b** $|-5| + |-3|$ **c** $|-5| - |-3|$

d $|-5| - |-3| - 4$ **e** $|-5| - |-3| - |-4|$ **f** $|-5| + |-3| - |-4|$

Example 13

2 Solve each of the following equations for x:

a $|x - 1| = 2$ **b** $|2x - 3| = 4$ **c** $|5x - 3| = 9$ **d** $|x - 3| - 9 = 0$

e $|3 - x| = 4$ **f** $|3x + 4| = 8$ **g** $|5x + 11| = 9$

Example 14

3 For each of the following, illustrate the set on a number line and represent the set using interval notation:

a $\{ x : |x| < 3 \}$ **b** $\{ x : |x| \geq 5 \}$ **c** $\{ x : |x - 2| \leq 1 \}$

d $\{ x : |x - 2| < 3 \}$ **e** $\{ x : |x + 3| \geq 5 \}$ **f** $\{ x : |x + 2| \leq 1 \}$

Example 15

4 For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 4| + 1$ **b** $f(x) = -|x + 3| + 2$

c $f(x) = |x + 4| - 1$ **d** $f(x) = 2 - |x - 1|$

5 Solve each of the following inequalities for x:

a $|x| \leq 5$ **b** $|x| \geq 2$ **c** $|2x - 3| \leq 1$

d $|5x - 2| < 3$ **e** $|-x + 3| \geq 7$ **f** $|-x + 2| \leq 1$

6 Let $f(x) = 4 - x$.

a Sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.

b State the set of values of x for which $|f(x)| = f(|x|)$.

Example 16

7 Sketch the graph of each of the following:

a $y = |x^2 - 9|$ **b** $y = |3^x - 3|$ **c** $y = |x^2 - x - 12|$

d $y = |x^2 - 3x - 40|$ **e** $y = |x^2 - 2x - 8|$ **f** $y = |2^x - 4|$

Example 17

8 Sketch the graph of each of the following:

a $y = |x|^2 - 4|x|$ **b** $y = 3^{|x|}$ **c** $y = |x|^2 - 7|x| + 12$

d $y = |x|^2 - |x| - 12$ **e** $y = |x|^2 + |x| - 12$ **f** $y = -3^{|x|} + 1$

Example 18

9 Solve each of the following equations for x:

a $|x^2 - 2x| = \dfrac{1}{2}$ **b** $|x^2 - 2x| = 1$ **c** $|x^2 - 2x| = 8$

d $|x^2 - 6x| = 8$ **e** $|x^2 - 6x| = 16$ **f** $|x^2 - 6x| = 9$

10 Solve each of the following equations for x:

a $|x - 4| - |x + 2| = 6$ **b** $|2x - 5| - |4 - x| = 10$ **c** $|2x - 1| + |4 - 2x| = 10$

11 If $f(x) = |x - a| + b$ with $f(3) = 3$ and $f(-1) = 3$, find the values of a and b.

12 Prove the following properties of the modulus function:

a $|ab| = |a|\,|b|$ Hint: Start by writing $|ab|^2 = (ab)^2$.

b $|a + b| \leq |a| + |b|$ Hint: Start by writing $|a + b|^2 = (a + b)^2$.

13 Prove each of the following for all $x, y, z \in \mathbb{R}$:

a $|x - y| \leq |x| + |y|$ **b** $|x| - |y| \leq |x - y|$ **c** $|x + y + z| \leq |x| + |y| + |z|$

17E Locus of points

Until now, all the curves we have studied have been described by an algebraic relationship between the x- and y-coordinates, such as $y = x^2 + 1$. In this section, we are interested in sets of points described by a geometric condition. A set described in this way is often called a **locus**. Many of these descriptions will give curves that are already familiar.

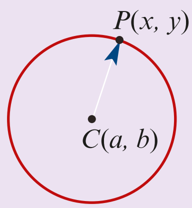
Circles

Circles have a very simple geometric description.

Locus definition of a circle

A **circle** is the locus of a point $P(x, y)$ that moves so that its distance from a fixed point $C(a, b)$ is constant.

Note: The constant distance is called the **radius** and the fixed point $C(a, b)$ is called the **centre** of the circle.



This definition can be used to find the Cartesian equation of a circle.

Recall that the distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let r be the radius of the circle. Then

$$CP = r$$
$$\sqrt{(x-a)^2 + (y-b)^2} = r$$
$$(x-a)^2 + (y-b)^2 = r^2$$

The Cartesian equation of the circle with centre $C(a, b)$ and radius r is

$$(x-a)^2 + (y-b)^2 = r^2$$

Example 19

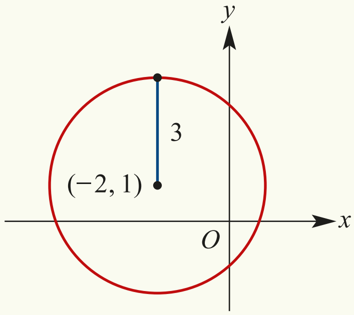
Find the locus of a point $P(x, y)$ that moves so that its distance from $C(-2, 1)$ is 3.

Solution

We know that the point $P(x, y)$ satisfies

$$CP = 3$$
$$\sqrt{(x+2)^2 + (y-1)^2} = 3$$
$$(x+2)^2 + (y-1)^2 = 3^2$$

This is a circle with centre $(-2, 1)$ and radius 3.



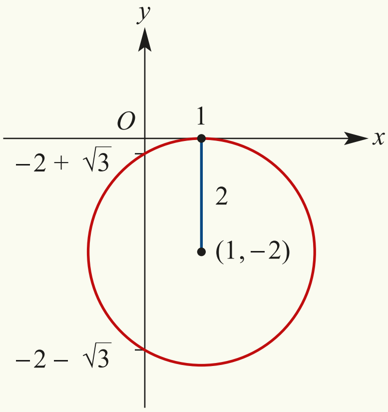
Example 20

Sketch the circle with equation $x^2 - 2x + y^2 + 4y = -1$. Label the axis intercepts.

Solution

We first complete the square in both variables:

$$x^2 - 2x + y^2 + 4y = -1$$
$$(x^2 - 2x + 1) - 1 + (y^2 + 4y + 4) - 4 = -1$$
$$(x - 1)^2 + (y + 2)^2 = 4$$



This is a circle with centre $(1, -2)$ and radius 2.

The x-axis intercept is 1.

To find the y-axis intercepts, we let $x = 0$, giving

$$1 + (y + 2)^2 = 4$$
$$(y + 2)^2 = 3$$
$$y = -2 \pm \sqrt{3}$$

The y-axis intercepts are $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$.

Straight lines

You have learned in previous years that a straight line is the set of points (x, y) satisfying

$$ax + by = c$$

for some constants a, b, c with $a \neq 0$ or $b \neq 0$.

Lines can also be described geometrically as follows.

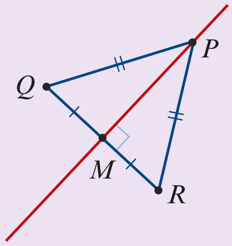
Locus definition of a straight line

Suppose that points Q and R are fixed.

A **straight line** is the locus of a point P that moves so that its distance from Q is the same as its distance from R. That is,

$$QP = RP$$

We can say that point P is **equidistant** from points Q and R.



Note: This straight line is the **perpendicular bisector** of line segment QR. To see this, we note that the midpoint M of QR is on the line. If P is any other point on the line, then

$$QP = RP, \quad QM = RM \quad \text{and} \quad MP = MP$$

and so $\triangle QMP$ is congruent to $\triangle RMP$. Therefore $\angle QMP = \angle RMP = 90°$.

Example 21

a Find the locus of a point $P(x, y)$ that moves so that it is equidistant from the two points $Q(1, 1)$ and $R(3, 5)$.

b Show that this is the perpendicular bisector of line segment QR.

Solution

a We know that the point $P(x, y)$ satisfies

$$QP = RP$$
$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-3)^2 + (y-5)^2}$$
$$(x-1)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$$
$$x + 2y = 8$$
$$y = -\tfrac{1}{2}x + 4$$

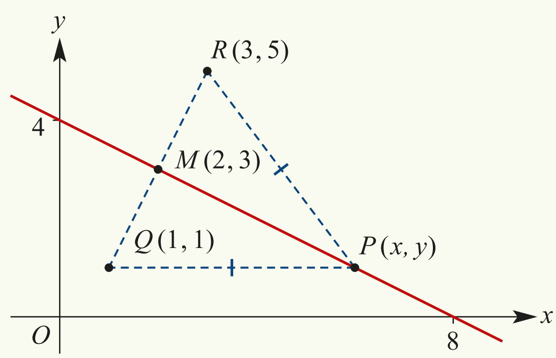
The locus is the straight line with equation $y = -\frac{1}{2}x + 4$.

b This line has gradient $-\frac{1}{2}$.

The line through the two points $Q(1, 1)$ and $R(3, 5)$ has gradient

$$\frac{5-1}{3-1} = 2$$

Because the product of the two gradients is -1, the two lines are perpendicular.



We also need to check that the line $y = -\frac{1}{2}x + 4$ passes through the midpoint of QR, which is $M(2, 3)$. When $x = 2$, $y = -\frac{1}{2} \times 2 + 4 = 3$. Thus $M(2, 3)$ is on the line.

Summary 17E

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- The Cartesian equation of the circle with centre $C(a, b)$ and radius r is

  $$(x-a)^2 + (y-b)^2 = r^2$$

- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.

Skill-sheet

Exercise 17E

Example 19

1 Find the locus of a point $P(x, y)$ that moves so that its distance from $Q(1, -2)$ is 4.

2 Find the locus of a point $P(x, y)$ that moves so that its distance from $Q(-4, 3)$ is 5.

3 Sketch the circles defined by the following equations:

a $x^2 + y^2 = 2^2$

b $x^2 + y^2 = 3^2$

c $(x - 1)^2 + y^2 = 4^2$

d $x^2 + (y + 1)^2 = 3^2$

e $(x - 1)^2 + (y - 2)^2 = 2^2$

f $(x + 2)^2 + (y - 1)^2 = 3^2$

g $(x - 3)^2 + (y - 2)^2 = 2^2$

h $(x + 3)^2 + (y + 2)^2 = 2^2$

Example 20

4 Sketch the circle with equation $x^2 + 4x + y^2 - 2y = -1$. Label the axis intercepts.

5 **a** Find the Cartesian equation of the circle with centre $C(1, 0)$ that passes through the point $Q(0, -2)$.

b Find the Cartesian equation of the circle that touches the x-axis at the point $Q(2, 0)$ and passes through the point $R(-2, 2)$.

Example 21

6 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(-1, -1)$ and $R(1, 1)$.

b Show that this is the perpendicular bisector of line segment QR.

7 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(0, 2)$ and $R(1, 0)$.

b Show that this is the perpendicular bisector of line segment QR.

8 Point P is equidistant from points $Q(0, 1)$ and $R(2, 3)$. Moreover, its distance from point $S(3, 3)$ is 3. Find the possible coordinates of P.

9 Point P is equidistant from points $Q(0, 1)$ and $R(2, 0)$. Moreover, it is also equidistant from points $S(-1, 0)$ and $T(0, 2)$. Find the coordinates of P.

10 A valuable item is buried in a forest. It is 10 metres from a tree stump located at coordinates $T(0, 0)$ and 2 metres from a rock at coordinates $R(6, 10)$. Find the possible coordinates of the buried item.

11 Consider the three points $R(4, 5)$, $S(6, 1)$ and $T(1, -4)$.

a Find the locus of points $P(x, y)$ that are equidistant from the points R and S.

b Find the locus of points $P(x, y)$ that are equidistant from the points S and T.

c Hence find the point that is equidistant from the points R, S and T.

d Hence find the equation of the circle through the points R, S and T.

12 Given two fixed points $A(0, 1)$ and $B(2, 5)$, find the locus of P if the gradient of AB equals that of BP.

13 A triangle OAP has vertices $O(0, 0)$, $A(4, 0)$ and $P(x, y)$, where $y > 0$. The triangle has area 12 square units. Find the locus of P.

14 **a** Determine the locus of a point $P(x, y)$ that moves so that its distance from the origin is equal to the sum of its x- and y-coordinates.

b Determine the locus of a point $P(x, y)$ that moves so that the *square* of its distance from the origin is equal to the sum of its x- and y-coordinates.

15 $A(0, 0)$ and $B(3, 0)$ are two vertices of a triangle ABP. The third vertex P is such that $AP : BP = 2$. Find the locus of P.

16 Find the locus of a point P that moves so that its distance from the line $y = 3$ is always 2 units.

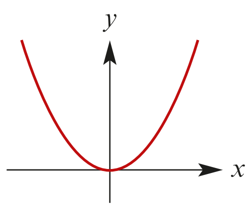
17 A steel pipe is too heavy to drag, but can be lifted at one end and rotated about its opposite end. How many moves are required to rotate the pipe into the parallel position indicated by the dotted line? The distance between the parallel lines is less than the length of the pipe.

17F Parabolas

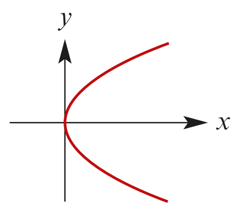
The parabola has been studied since antiquity and is admired for its range of applications, one of which we will explore at the end of this section.

The standard form of a parabola is $y = ax^2$.

Rotating the figure by 90° gives a parabola with equation $x = ay^2$.



$y = ax^2$



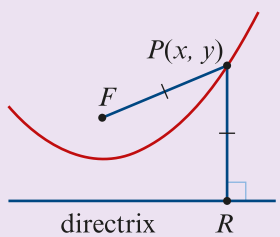
$x = ay^2$

The parabola can also be defined geometrically.

Locus definition of a parabola

A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.

Note: The fixed point is called the **focus** and the fixed line is called the **directrix**.

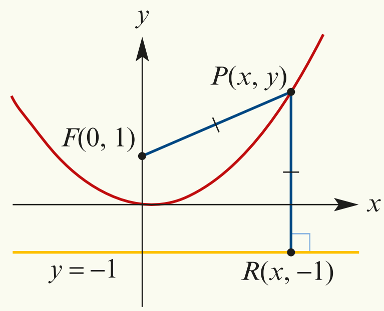


Example 22

Find the locus of a point $P(x, y)$ that moves so that its distance from the fixed point $F(0, 1)$ is equal to its perpendicular distance from the fixed line $y = -1$.

Solution

We know that the point $P(x, y)$ satisfies



$$
\begin{aligned}
FP &= RP \\
\sqrt{x^2 + (y-1)^2} &= \sqrt{(y-(-1))^2} \\
x^2 + (y-1)^2 &= (y+1)^2 \\
x^2 + y^2 - 2y + 1 &= y^2 + 2y + 1 \\
x^2 - 2y &= 2y \\
x^2 &= 4y \\
y &= \frac{x^2}{4}
\end{aligned}
$$

Therefore the set of points is the parabola with equation $y = \dfrac{x^2}{4}$.

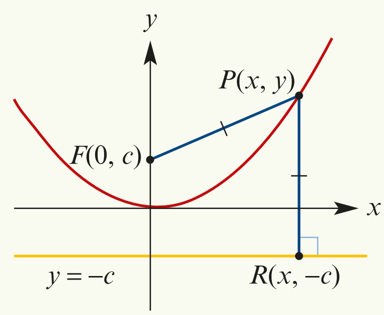
Example 23

a Find the equation of the parabola with focus $F(0, c)$ and directrix $y = -c$.

b Hence find the focus of the parabola with equation $y = 2x^2$.

Solution

a A point $P(x, y)$ on the parabola satisfies



$$
\begin{aligned}
FP &= RP \\
\sqrt{x^2 + (y-c)^2} &= \sqrt{(y-(-c))^2} \\
x^2 + (y-c)^2 &= (y+c)^2 \\
x^2 + y^2 - 2cy + c^2 &= y^2 + 2cy + c^2 \\
x^2 - 2cy &= 2cy \\
x^2 &= 4cy
\end{aligned}
$$

The parabola has equation $4cy = x^2$.

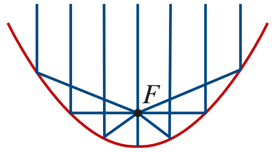
b Since $\dfrac{y}{2} = x^2$, we solve $\dfrac{1}{2} = 4c$, giving $c = \dfrac{1}{8}$.

Hence the focus is $F\left(0, \dfrac{1}{8}\right)$.

In the previous example, we proved the following result:

The parabola with focus $F(0, c)$ and directrix $y = -c$ has equation $4cy = x^2$.

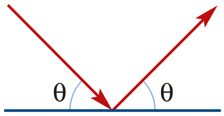
A remarkable application



Parabolas have a remarkable property that makes them extremely useful. Light travelling parallel to the axis of symmetry of a reflective parabola is always reflected to its focus.

Parabolas can therefore be used to make reflective telescopes. Low intensity signals from outer space will reflect off the dish and converge at a receiver located at the focus.

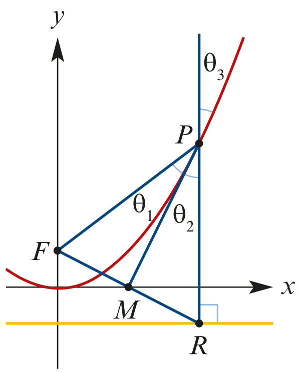
To see how this works, we require a simple law of physics:



- When light is reflected off a surface, the angle between the ray and the tangent to the surface is preserved after reflection.

Reflective property of the parabola

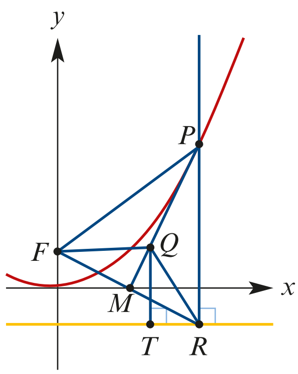
Any ray of light parallel to the axis of symmetry of the parabola that reflects off the parabola at point P will pass through the focus at F.



Proof Since point P is on the parabola, the distance to the focus F is the same as the distance to the directrix. Therefore $FP = RP$, and so $\triangle FPR$ is isosceles.

Let M be the midpoint of FR. Then $\triangle FMP$ is congruent to $\triangle RMP$ (by SSS). Therefore MP is the perpendicular bisector of FR and

$$\begin{aligned} \theta_1 &= \theta_2 && (\text{as } \triangle FMP \equiv \triangle RMP) \\ &= \theta_3 && (\text{vertically opposite angles}) \end{aligned}$$



However, we also need to ensure that line MP is tangent to the parabola. To see this, we will show that point P is the only point common to the parabola and line MP.

Take any other point Q on line MP. Suppose that point T is the point on the directrix closest to Q. Then

$$FQ = RQ > TQ$$

and so point Q is not on the parabola.

Summary 17F

- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- The fixed point is called the **focus** and the fixed line is called the **directrix**.
- The parabola with equation $4cy = x^2$ has focus $F(0, c)$ and directrix $y = -c$.

Exercise 17F

Example 22

1 Find the locus of a point $P(x, y)$ that moves so that its distance from the point $F(0, 3)$ is equal to its perpendicular distance from the line with equation $y = -3$.

2 Find the locus of a point $P(x, y)$ that moves so that its distance from the point $F(0, -4)$ is equal to its perpendicular distance from the line with equation $y = 2$.

3 Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(2, 0)$ is equal to the perpendicular distance to the line with equation $x = -4$.

Example 23

4 **a** Find the equation of the parabola with focus $F(c, 0)$ and directrix $x = -c$.
b Hence find the focus of the parabola with equation $x = 3y^2$.

5 **a** Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(a, b)$ is equal to the perpendicular distance to the line with equation $y = c$.
b Hence find the equation of the parabola with focus $(1, 2)$ and directrix $y = 3$.

6 A parabola goes through the point $P(7, 9)$ and its focus is $F(1, 1)$. The axis of symmetry of the parabola is $x = 1$. Find the equation of its directrix.
Hint: The directrix will be a horizontal line, $y = c$. Expect to find two answers.

7 A parabola goes through the point $(1, 1)$, its axis of symmetry is the line $x = 2$ and its directrix is the line $y = 3$. Find the coordinates of its focus.
Hint: The focus must lie on the axis of symmetry.

17G Ellipses

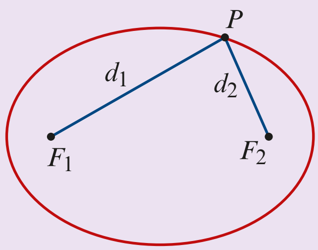
A ball casts a shadow that looks like a squashed circle. This figure – called an ellipse – is of considerable geometric significance. For instance, the planets in our solar system have elliptic orbits.

Locus definition of an ellipse

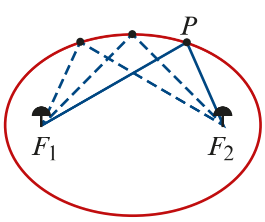
An **ellipse** is the locus of a point P that moves so that the sum of its distances from two fixed points F_1 and F_2 is a constant. That is,

$$F_1P + F_2P = k$$

Note: Points F_1 and F_2 are called the **foci** of the ellipse.

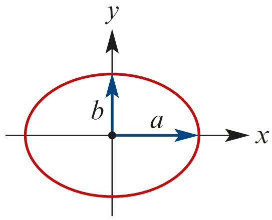


Drawing an ellipse An ellipse can be drawn by pushing two pins into paper. These will be the foci. A string of length k is tied to each of the two pins and the tip of a pen is used to pull the string taut and form a triangle. The pen will trace an ellipse if it is moved around the pins while keeping the string taut.



Cartesian equations of ellipses

The standard form of the Cartesian equation of an ellipse centred at the origin is



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

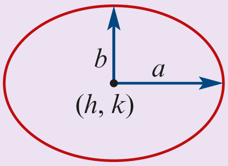
This ellipse has x-axis intercepts $\pm a$ and y-axis intercepts $\pm b$.

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).



Example 24

For each of the following equations, sketch the graph of the corresponding ellipse. Give the coordinates of the centre and the axis intercepts.

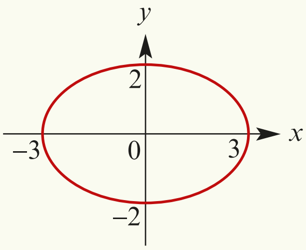
a $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **b** $4x^2 + 9y^2 = 1$ **c** $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$

Solution

a The equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

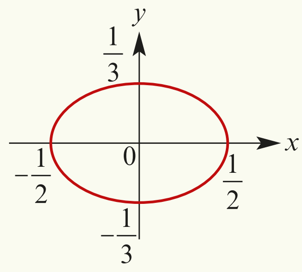
This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm 3$ and $y = \pm 2$.



b The equation can be written as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$$

This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm\frac{1}{2}$ and $y = \pm\frac{1}{3}$.



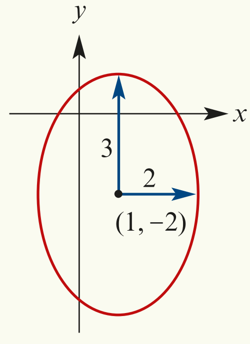
c This is an ellipse with centre $(1, -2)$.

To find the x-axis intercepts, let $y = 0$. Then solving for x gives

$$x = \frac{3 \pm 2\sqrt{5}}{3}$$

Likewise, to find the y-axis intercepts, let $x = 0$. This gives

$$y = \frac{-4 \pm 3\sqrt{3}}{2}$$



Using the locus definition

Example 25

Consider points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP + BP = 8$.

Solution

Let (x, y) be the coordinates of point P. If $AP + BP = 8$, then

$$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 8$$

and so $$\sqrt{(x+2)^2 + y^2} = 8 - \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$(x+2)^2 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$$

$$x^2 + 4x + 4 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$$

$$x - 8 = -2\sqrt{(x-2)^2 + y^2}$$

Square both sides again:

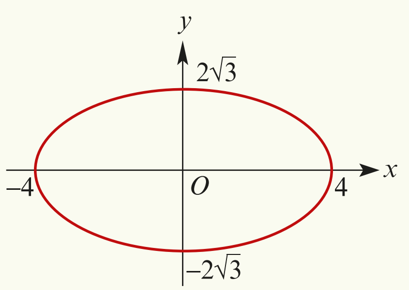
$$x^2 - 16x + 64 = 4(x^2 - 4x + 4 + y^2)$$

Simplifying yields

$$3x^2 + 4y^2 = 48$$

i.e. $$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

This is an ellipse with centre the origin and axis intercepts at $x = \pm 4$ and $y = \pm 2\sqrt{3}$.



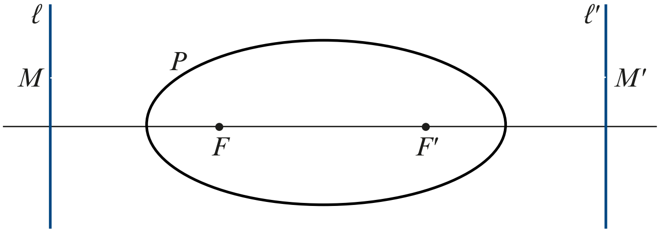
Every point P on the ellipse satisfies $AP + BP = 8$.

Eccentricity of an ellipse

It can also be shown that an ellipse is the locus of points $P(x, y)$ satisfying

$$FP = eMP$$

where F is a fixed point, $0 < e < 1$ and MP is the perpendicular distance from P to a fixed line ℓ. The number e is called the **eccentricity** of the ellipse.



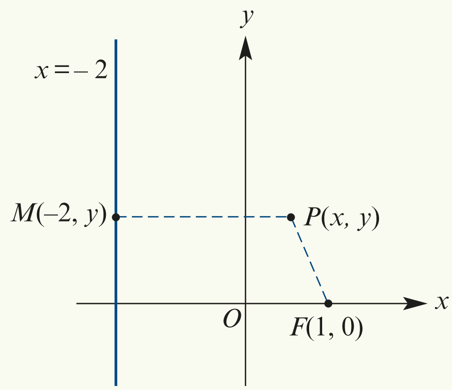
From the symmetry of the ellipse, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eM'P$ defines the same locus, where $M'P$ is the perpendicular distance from P to ℓ'.

Example 26

Find the equation of the locus of points $P(x, y)$ if the distance from P to the point $F(1, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = \frac{1}{2}MP$.

Solution

Let (x, y) be the coordinates of point P.



If $FP = \frac{1}{2}MP$, then

$$\sqrt{(x-1)^2 + y^2} = \frac{1}{2}\sqrt{(x+2)^2}$$

Square both sides:

$$(x-1)^2 + y^2 = \frac{1}{4}(x+2)^2$$

$$4(x^2 - 2x + 1) + 4y^2 = x^2 + 4x + 4$$

$$3x^2 - 12x + 4y^2 = 0$$

Complete the square:

$$3(x^2 - 4x + 4) + 4y^2 = 12$$

$$3(x-2)^2 + 4y^2 = 12 \quad \text{or equivalently} \quad \frac{(x-2)^2}{4} + \frac{y^2}{3} = 1$$

This is an ellipse with centre $(2, 0)$.

Summary 17G

- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 (called the **foci**) is equal to a fixed positive constant.
- The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).

Skill-sheet

Exercise 17G

Example 24

1 Sketch the graph of each ellipse, labelling the axis intercepts:

a $\frac{x^2}{9} + \frac{y^2}{64} = 1$ **b** $\frac{x^2}{100} + \frac{y^2}{25} = 1$ **c** $\frac{y^2}{9} + \frac{x^2}{64} = 1$ **d** $25x^2 + 9y^2 = 225$

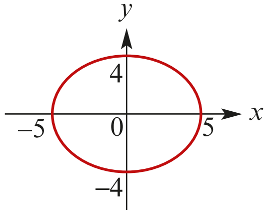
2 Sketch the graph of each ellipse, labelling the centre and the axis intercepts:

a $\frac{(x-3)^2}{9} + \frac{(y-4)^2}{16} = 1$ **b** $\frac{(x+3)^2}{9} + \frac{(y+4)^2}{25} = 1$

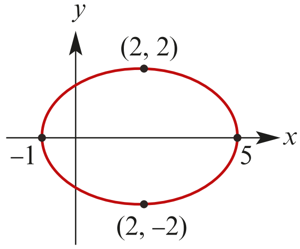
c $\frac{(y-3)^2}{16} + \frac{(x-2)^2}{4} = 1$ **d** $25(x-5)^2 + 9y^2 = 225$

3 Find the Cartesian equations of the following ellipses:

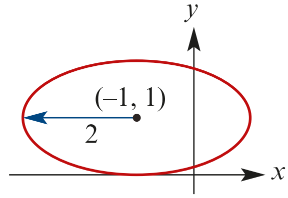
a



b



c



Example 25

4 Find the locus of a point P that moves such that the sum of its distances from two fixed points $A(1, 0)$ and $B(-1, 0)$ is 4 units.

5 Find the locus of a point P that moves such that the sum of its distances from two fixed points $A(0, 2)$ and $B(0, -2)$ is 6 units.

Example 26

6 Find the equation of the locus of points $P(x, y)$ such that the distance from P to the point $F(2, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -4$. That is, $FP = \frac{1}{2}MP$.

7 A circle has equation $x^2 + y^2 = 1$. It is then dilated by a factor of 3 from the x-axis and by a factor of 5 from the y-axis. Find the equation of the image and sketch its graph.

17H Hyperbolas

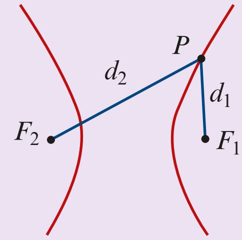
Hyperbolas are defined analogously to ellipses, but using the difference instead of the sum.

Locus definition of a hyperbola

A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant. That is,

$$|F_2P - F_1P| = k$$

Note: Points F_1 and F_2 are called the **foci** of the hyperbola.



The standard form of the Cartesian equation of a hyperbola centred at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k).

Note: Interchanging x and y in this equation produces another hyperbola (rotated by 90°).

Asymptotes of the hyperbola

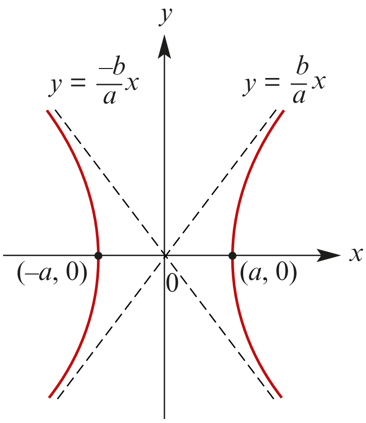
We now investigate the behaviour of the hyperbola as $x \to \pm\infty$. We first show that the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

To see why this should be the case, we rearrange the equation of the hyperbola as follows:



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$y^2 = \frac{b^2x^2}{a^2} - b^2$$

$$= \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

If $x \to \pm\infty$, then $\frac{a^2}{x^2} \to 0$. This suggests that $y^2 \to \frac{b^2x^2}{a^2}$ as $x \to \pm\infty$. That is,

$$y \to \pm\frac{bx}{a} \quad \text{as} \quad x \to \pm\infty$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we obtain the following result:

The hyperbola with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm\frac{b}{a}(x - h)$$

Example 27

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

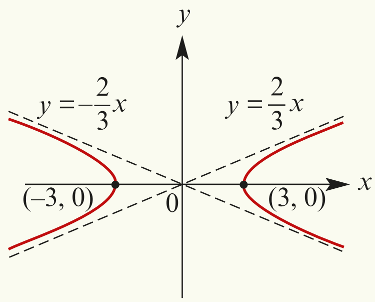
c $(x-1)^2 - (y+2)^2 = 1$

d $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

Solution

a Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$$

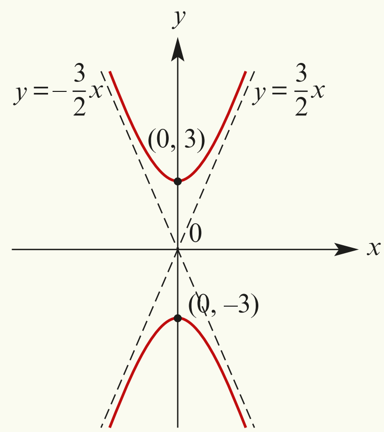


Thus the equations of the asymptotes are $y = \pm\frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The x-axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.

b Since $\frac{y^2}{9} - \frac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4}\left(1 + \frac{4}{x^2}\right)$$



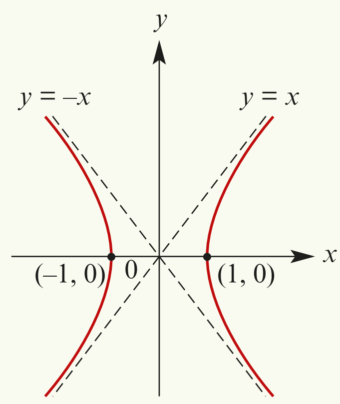
Thus the equations of the asymptotes are $y = \pm\frac{3}{2}x$.

The y-axis intercepts are $(0, 3)$ and $(0, -3)$.

The centre is $(0, 0)$.

c First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This hyperbola is called a **rectangular hyperbola**, as its asymptotes are perpendicular.

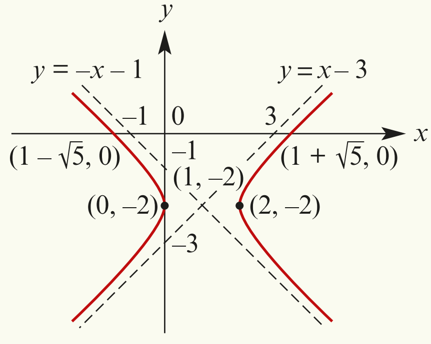


Now to sketch the graph of

$$(x-1)^2 - (y+2)^2 = 1$$

we apply the translation $(x, y) \to (x + 1, y - 2)$.

The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x - 1)$. That is, $y = x - 3$ and $y = -x - 1$.



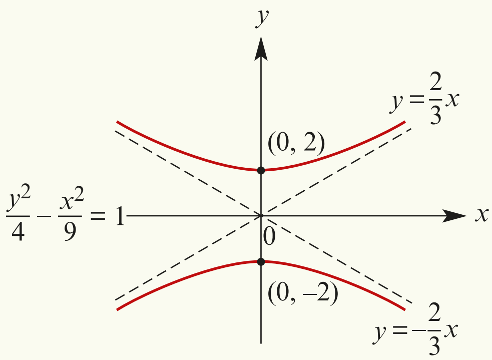
Axis intercepts

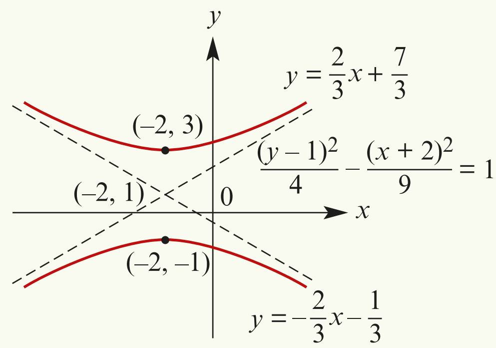
If $x = 0$, then $y = -2$.

If $y = 0$, then $(x-1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.

d The graph of $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$ through the translation $(x, y) \to (x - 2, y + 1)$. Its centre will be $(-2, 1)$.





Using the locus definition

Example 28

Consider the points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 3$.

Solution

Let (x, y) be the coordinates of point P.

If $AP - BP = 3$, then

$$\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2} = 3$$

and so

$$\sqrt{(x+2)^2 + y^2} = 3 + \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$(x+2)^2 + y^2 = 9 + 6\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$$

$$x^2 + 4x + 4 + y^2 = 9 + 6\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$$

$$8x - 9 = 6\sqrt{(x-2)^2 + y^2}$$

Note that this only holds if $x \geq \frac{9}{8}$. Squaring both sides again gives

$$64x^2 - 144x + 81 = 36(x^2 - 4x + 4 + y^2)$$

$$28x^2 - 36y^2 = 63$$

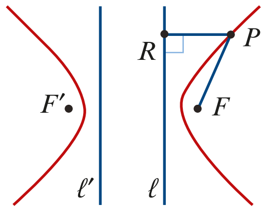
$$\frac{4x^2}{9} - \frac{4y^2}{7} = 1 \qquad \text{for } x \geq \frac{3}{2}$$

This is the right branch of a hyperbola with centre the origin and x-axis intercept $\frac{3}{2}$.

It can also be shown that a hyperbola is the locus of points $P(x, y)$ satisfying

$$FP = eRP$$

where F is a fixed point, $e > 1$ and RP is the perpendicular distance from P to a fixed line ℓ.



From the symmetry of the hyperbola, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eR'P$ defines the same locus, where $R'P$ is the perpendicular distance from P to ℓ'.

Example 29

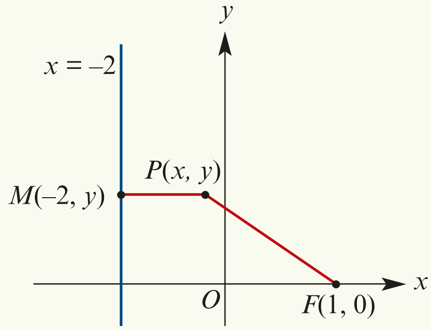
Find the equation of the locus of points $P(x, y)$ for which the distance from P to the point $F(1, 0)$ is twice the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = 2MP$.

Solution

Let (x, y) be the coordinates of point P.

If $FP = 2MP$, then

$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{(x+2)^2}$$



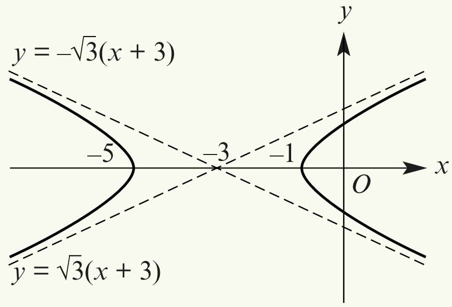
Squaring both sides gives

$$(x-1)^2 + y^2 = 4(x+2)^2$$
$$x^2 - 2x + 1 + y^2 = 4(x^2 + 4x + 4)$$
$$3x^2 + 18x - y^2 + 15 = 0$$

By completing the square, we obtain

$$3(x^2 + 6x + 9) - 27 - y^2 + 15 = 0$$
$$3(x+3)^2 - y^2 = 12$$
$$\frac{(x+3)^2}{4} - \frac{y^2}{12} = 1$$

This is a hyperbola with centre $(-3, 0)$.



Summary 17H

- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 (called the **foci**) is a constant. That is, $|F_2P - F_1P| = k$.
- The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k). Its asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

Skill-sheet

Exercise 17H

Example 27

1 Sketch the graph of each of the following hyperbolas. Label axis intercepts and give the equations of the asymptotes.

a $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ **b** $x^2 - \dfrac{y^2}{4} = 1$

c $\dfrac{y^2}{25} - \dfrac{x^2}{100} = 1$ **d** $25x^2 - 9y^2 = 225$

2 Sketch the graph of each of the following hyperbolas. State the centre and label axis intercepts and asymptotes.

a $(x-1)^2 - (y+2)^2 = 1$ **b** $\dfrac{(x+1)^2}{4} - \dfrac{(y-2)^2}{16} = 1$

c $\dfrac{(y-3)^2}{9} - (x-2)^2 = 1$ **d** $25(x-4)^2 - 9y^2 = 225$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$ **f** $9x^2 - 25y^2 - 90x + 150y = 225$

Example 28

3 Consider the points $A(4, 0)$ and $B(-4, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 6$.

4 Find the equation of the locus of points $P(x, y)$ satisfying $AP - BP = 4$, given coordinates $A(-3, 0)$ and $B(3, 0)$.

Example 29

5 Find the equation of the locus of points $P(x, y)$ for which the distance to P from the point $F(5, 0)$ is twice the distance MP, the perpendicular distance to P from the line with equation $x = -1$. That is, $FP = 2MP$.

6 Find the equation of the locus of points $P(x, y)$ for which the distance to P from the point $F(0, -1)$ is twice the distance MP, the perpendicular distance to P from the line with equation $y = -4$. That is, $FP = 2MP$.

17I Parametric equations

A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable t is called the **parameter**, and for each choice of t we get a point in the plane $(f(t), g(t))$. The set of all such points will be a curve in the plane.

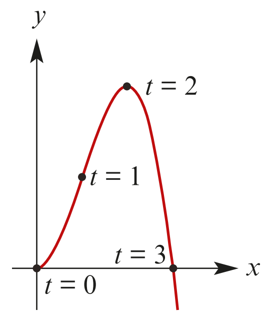
It is sometimes useful to think of t as being *time*, so that the equations $x = f(t)$ and $y = g(t)$ give the position of an object at time t. Points on the curve can be plotted by substituting various values of t into the two equations.

For instance, we can plot points on the curve defined by the parametric equations

$$x = t \quad \text{and} \quad y = 3t^2 - t^3$$

by letting $t = 0, 1, 2, 3$.

t	0	1	2	3
x	0	1	2	3
y	0	2	4	0



In this instance, it is possible to eliminate the parameter t to obtain a Cartesian equation in x and y alone. Substituting $t = x$ into the second equation gives $y = 3x^2 - x^3$.

Lines

Example 30

a Find the Cartesian equation for the curve defined by the parametric equations

$$x = t + 2 \quad \text{and} \quad y = 2t - 3$$

b Find parametric equations for the line through the points $A(2, 3)$ and $B(4, 7)$.

Solution

a Substitute $t = x - 2$ into the second equation to give

$$\begin{aligned} y &= 2(x - 2) - 3 \\ &= 2x - 7 \end{aligned}$$

Thus every point lies on the straight line with equation $y = 2x - 7$.

b The gradient of the straight line through points $A(2, 3)$ and $B(4, 7)$ is

$$m = \frac{7 - 3}{4 - 2} = 2$$

Therefore the line has equation

$$\begin{aligned} y - 3 &= 2(x - 2) \\ y &= 2x - 1 \end{aligned}$$

We can simply let $x = t$ and so $y = 2t - 1$.

Note: There are infinitely many pairs of parametric equations that describe the same curve. In part **b**, we could also let $x = 2t$ and $y = 4t - 1$. These parametric equations describe exactly the same set of points. As t increases, the point moves along the same line twice as fast.

Parabolas

Example 31

Find the Cartesian equation of the parabola defined by the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 + 1$$

Solution

Substitute $t = x + 1$ into the second equation to give $y = (x + 1)^2 + 1$.

Circles

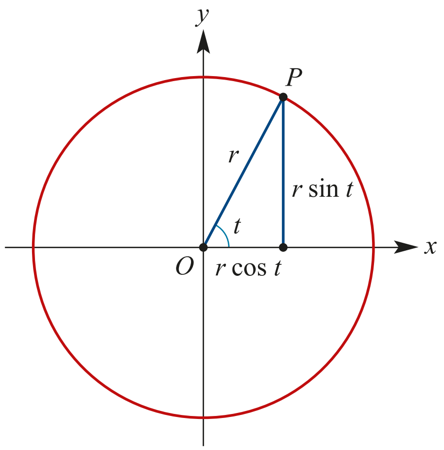
We have seen that the circle with radius r and centre at the origin can be written in Cartesian form as

$$x^2 + y^2 = r^2$$

We now introduce the parameter t and let

$$x = r\cos t \quad \text{and} \quad y = r\sin t$$

As t increases from 0 to 2π, the point $P(x, y)$ travels from $(r, 0)$ anticlockwise around the circle and returns to its original position.



To demonstrate that this parameterises the circle, we evaluate

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2 t + r^2\sin^2 t \\ &= r^2(\cos^2 t + \sin^2 t) \\ &= r^2 \end{aligned}$$

where we have used the Pythagorean identity $\cos^2 t + \sin^2 t = 1$.

Example 32

a Find the Cartesian equation of the circle defined by the parametric equations

$$x = \cos t + 1 \quad \text{and} \quad y = \sin t - 2$$

b Find parametric equations for the circle with Cartesian equation

$$(x + 1)^2 + (y + 3)^2 = 4$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$x - 1 = \cos t \quad \text{and} \quad y + 2 = \sin t$$

Using the Pythagorean identity:

$$(x - 1)^2 + (y + 2)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the circle with equation $(x - 1)^2 + (y + 2)^2 = 1$.

b We let

$$\cos t = \frac{x + 1}{2} \quad \text{and} \quad \sin t = \frac{y + 3}{2}$$

giving

$$x = 2\cos t - 1 \quad \text{and} \quad y = 2\sin t - 3$$

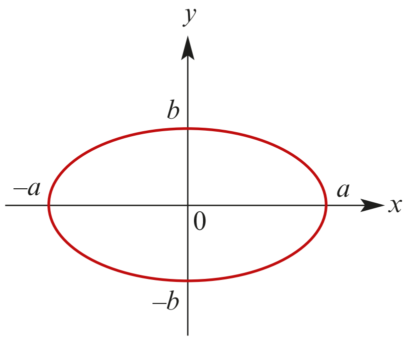
We can easily check that these equations parameterise the given circle.

Ellipses

An ellipse can be thought of as a squashed circle.

This is made apparent from the parametric equations for an ellipse:

$$x = a\cos t \quad \text{and} \quad y = b\sin t$$



As with the circle, we see the sine and cosine functions, but these are now scaled by different constants, giving different dilations from the x- and y-axes.

We can turn this pair of parametric equations into one Cartesian equation as follows:

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

giving

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

which is the standard form of an ellipse centred at the origin with axis intercepts at $x = \pm a$ and $y = \pm b$.

Example 33

a Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t + 1 \quad \text{and} \quad y = 2\sin t - 1$$

b Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$\frac{x-1}{3} = \cos t \quad \text{and} \quad \frac{y+1}{2} = \sin t$$

Using the Pythagorean identity:

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the ellipse with equation $\dfrac{(x-1)^2}{3^2} + \dfrac{(y+1)^2}{2^2} = 1$.

b We let

$$\cos t = \frac{x-1}{2} \quad \text{and} \quad \sin t = \frac{y+2}{4}$$

giving

$$x = 2\cos t + 1 \quad \text{and} \quad y = 4\sin t - 2$$

Hyperbolas

We can parameterise a hyperbola using the equations

$$x = a\sec t \quad \text{and} \quad y = b\tan t$$

From these two equations, we can find the more familiar Cartesian equation:

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

giving

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

which is the standard form of a hyperbola centred at the origin.

Example 34

a Find the Cartesian equation of the hyperbola defined by the parametric equations

$$x = 3\sec t - 1 \quad \text{and} \quad y = 2\tan t + 2$$

b Find parametric equations for the hyperbola with Cartesian equation

$$\frac{(x+2)^2}{4} - \frac{(y-3)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\sec t$ and $\tan t$ respectively. This gives

$$\frac{x+1}{3} = \sec t \quad \text{and} \quad \frac{y-2}{2} = \tan t$$

and therefore

$$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-2}{2}\right)^2 = \sec^2 t - \tan^2 t = 1$$

So each point on the graph lies on the hyperbola with equation $\dfrac{(x+1)^2}{3^2} - \dfrac{(y-2)^2}{2^2} = 1$.

b We let

$$\sec t = \frac{x+2}{2} \quad \text{and} \quad \tan t = \frac{y-3}{4}$$

giving

$$x = 2\sec t - 2 \quad \text{and} \quad y = 4\tan t + 3$$

Parametric equations with restricted domains

Example 35

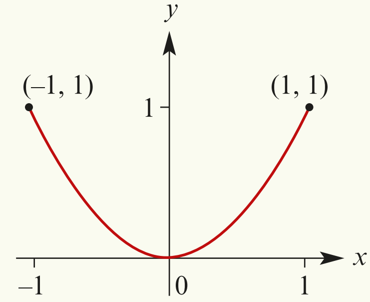
Eliminate the parameter to determine the graph of the parameterised curve

$$x = t - 1, \quad y = t^2 - 2t + 1 \qquad \text{for } 0 \le t \le 2$$

Solution

Substitute $t = x + 1$ from the first equation into the second equation, giving

$$\begin{aligned} y &= (x+1)^2 - 2(x+1) + 1 \\ &= x^2 + 2x + 1 - 2x - 2 + 1 \\ &= x^2 \end{aligned}$$



Since $0 \le t \le 2$, it follows that $-1 \le x \le 1$.

Therefore, as t increases from 0 to 2, the point travels along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.

Intersections of curves defined parametrically

It is often difficult to find the intersection of two curves defined parametrically. This is because, although the curves may intersect, they might do so for different values of the parameter t.

In many instances, it is easiest to find the points of intersection using the Cartesian equations for the two curves.

Example 36

Find the points of intersection of the circle and line defined by the parametric equations:

circle $\quad x = 5\cos t \quad$ and $\quad y = 5\sin t$

line $\quad x = t - 3 \quad$ and $\quad y = 2t - 8$

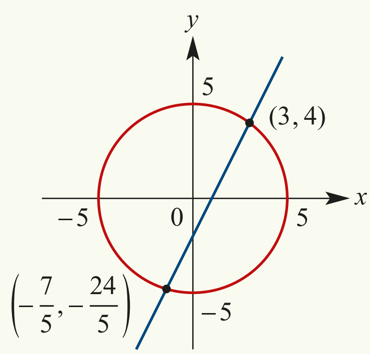
Solution

The Cartesian equation of the circle is $x^2 + y^2 = 25$.

The Cartesian equation of the line is $y = 2x - 2$.

Substituting the second equation into the first gives

$$\begin{aligned} x^2 + (2x-2)^2 &= 25 \\ x^2 + 4x^2 - 8x + 4 &= 25 \\ 5x^2 - 8x - 21 &= 0 \\ (x-3)(5x+7) &= 0 \end{aligned}$$



This gives solutions $x = 3$ and $x = -\frac{7}{5}$.

Substituting these into the equation $y = 2x - 2$ gives $y = 4$ and $y = -\frac{24}{5}$ respectively.

The points of intersection are $(3, 4)$ and $\left(-\frac{7}{5}, -\frac{24}{5}\right)$.

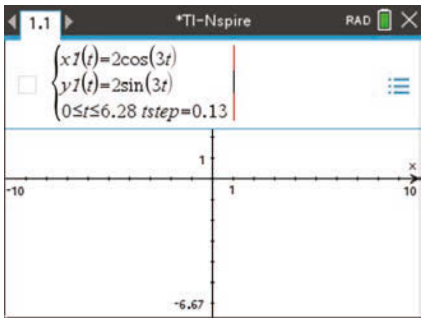
Using a CAS calculator with parametric equations

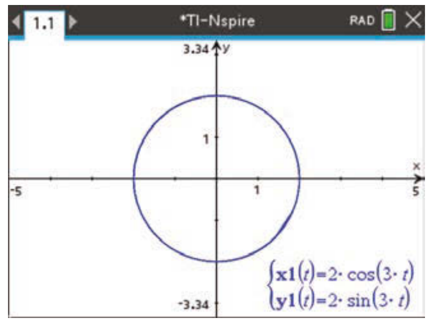
Example 37

Plot the graph of the parametric curve given by $x = 2\cos(3t)$ and $y = 2\sin(3t)$.

Using the TI-Nspire

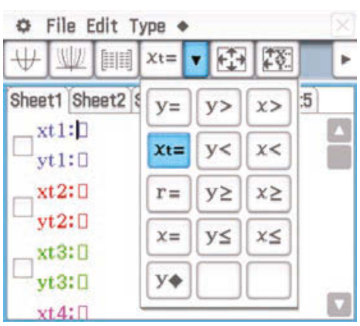
- Open a **Graphs** application ((on) > **New Document** > **Add Graphs**).
- Use (menu) > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown.



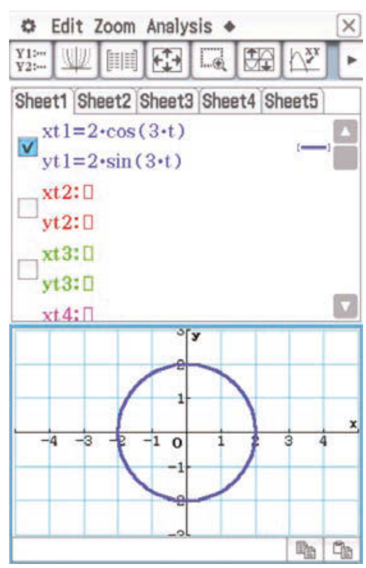


Using the Casio ClassPad

- Open the **Graph & Table** application.
- Clear all equations and graphs.
- Tap on ▼ next to [y=] in the toolbar and select [xt=].



- Enter the equations in $xt1$ and $yt1$ as shown.
- Tick the box and tap [graph icon]. Adjust the window as required.



Summary 17I

- A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is called the **parameter** of the curve. For example:

	Cartesian equation	Parametric equations
Circle	$x^2 + y^2 = r^2$	$x = r\cos t$ and $y = r\sin t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t$ and $y = b\sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec t$ and $y = b\tan t$

- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

Skill-sheet

Exercise 17I

1 Consider the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 - 1$$

a Find the Cartesian equation of the curve described by these equations.

b Sketch the curve and label the points on the curve corresponding to $t = 0, 1, 2$.

Example 30
Example 31

2 For each of the following pairs of parametric equations, find the Cartesian equation and sketch the curve:

a $x = t + 1$ and $y = 2t + 1$

b $x = t - 1$ and $y = 2t^2 + 1$

c $x = t^2$ and $y = t^6$

d $x = t + 2$ and $y = \frac{1}{t+1}$

Example 32

3 **a** Find the Cartesian equation of the circle defined by the parametric equations

$$x = 2\cos t \quad \text{and} \quad y = 2\sin t$$

Example 33

b Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t - 1 \quad \text{and} \quad y = 2\sin t + 2$$

c Find parametric equations for the circle with Cartesian equation

$$(x + 3)^2 + (y - 2)^2 = 9$$

d Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

4 Find parametric equations for the line through the points $A(-1, -2)$ and $B(1, 4)$.

Example 34

5 **a** Find the Cartesian equation of the hyperbola defined by the parametric equations

$$x = 2\sec t + 1 \quad \text{and} \quad y = 3\tan t - 2$$

b Find parametric equations for the hyperbola with Cartesian equation

$$(x-2)^2 - \frac{(y+1)^2}{4} = 1$$

Example 35

6 **a** Eliminate the parameter t to determine the equation of the parameterised curve

$$x = t - 1 \quad \text{and} \quad y = -2t^2 + 4t - 2 \qquad \text{for } 0 \le t \le 2$$

b Sketch the graph of this curve over an appropriate domain.

Example 36

7 Find the points of intersection of the circle and line defined by the parametric equations:

circle $\quad x = \cos t \quad$ and $\quad y = \sin t$

line $\quad x = 3t + 6 \quad$ and $\quad y = 4t + 8$

8 A curve is parameterised by the equations $x = \sin t$ and $y = 2\sin^2 t + 1$ for $0 \le t \le 2\pi$.

a Find the curve's Cartesian equation. **b** What is the domain of the curve?

c What is the range of the curve? **d** Sketch the graph of the curve.

9 A curve is parameterised by the equations $x = 2^t$ and $y = 2^{2t} + 1$ for $t \in \mathbb{R}$.

a Find the curve's Cartesian equation. **b** What is the domain of the curve?

c What is the range of the curve? **d** Sketch the graph of the curve.

10 Eliminate the parameter to determine the graph of the parameterised curve

$$x = \cos t \quad \text{and} \quad y = 1 - 2\sin^2 t \qquad \text{for } 0 \le t \le 2\pi$$

11 Consider the parametric equations

$$x = 2^t + 2^{-t} \quad \text{and} \quad y = 2^t - 2^{-t}$$

a Show that the Cartesian equation of the curve is $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$ for $x \ge 2$.

b Sketch the graph of the curve.

12 Consider the circle with Cartesian equation $x^2 + (y-1)^2 = 1$.

a Sketch the graph of the circle.

b Show that the parametric equations $x = \cos t$ and $y = \sin t + 1$ define the same circle.

c A different parameterisation of the circle can be found without the use of the cosine and sine functions. Suppose that t is any real number and let $P(x, y)$ be the point of intersection of the line $y = 2 - tx$ with the circle. Solve for x and y in terms of t, assuming that $x \ne 0$.

d Verify that the equations found in part **c** parameterise the same circle.

Example 37

13 The curve with parametric equations $x = \dfrac{t}{2\pi}\cos t$ and $y = \dfrac{t}{2\pi}\sin t$ is called an **Archimedean spiral**.

a With the help of your calculator, sketch the curve over the interval $0 \le t \le 6\pi$.

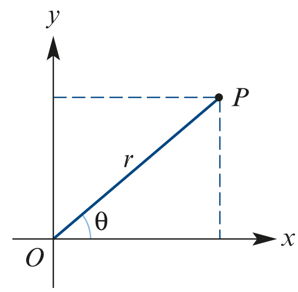
b Label the points on the curve corresponding $t = 0, 1, 2, 3, 4, 5, 6$.

17J Polar coordinates

Until now, we have described each point in the plane by a pair of numbers (x, y). These are called Cartesian coordinates, and take their name from the French intellectual René Descartes (1596–1650) who introduced them. However, they are not the only way to describe points in the plane. In fact, for many situations it is more convenient to use **polar coordinates**.

Using polar coordinates, every point P in the plane is described by a pair of numbers $[r, \theta]$, where:

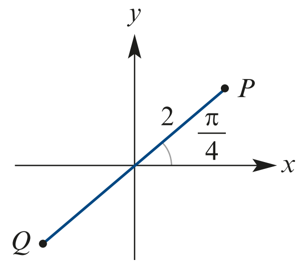
- the number r is the distance from the origin O to P
- the number θ measures the angle between the positive direction of the x-axis and the ray OP, as shown.



Note: To distinguish polar coordinates from Cartesian coordinates, we write the numbers in square brackets.

For example, the diagram on the right shows the point P with polar coordinates $\left[2, \frac{\pi}{4}\right]$.

We can even make sense of polar coordinates such as $Q\left[-2, \frac{\pi}{4}\right]$: go to the direction $\frac{\pi}{4}$ and then move a distance of 2 in the opposite direction.



Converting between the two coordinate systems requires little more than basic trigonometry.

- If a point P has polar coordinates $[r, \theta]$, then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- If a point P has Cartesian coordinates (x, y), then its polar coordinates $[r, \theta]$ satisfy

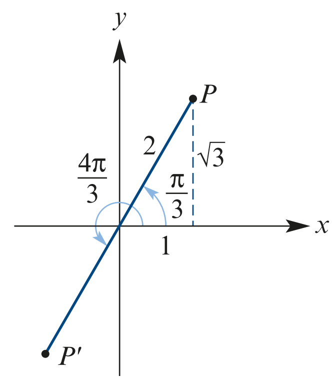
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \qquad (\text{if } x \neq 0)$$

Non-uniqueness of polar coordinates

Polar coordinates differ from Cartesian coordinates in that each point in the plane has more than one representation in polar coordinates.

For example, the following polar coordinates all represent the same point:

$$\left[2, \frac{\pi}{3}\right], \quad \left[-2, \frac{4\pi}{3}\right] \quad \text{and} \quad \left[2, \frac{7\pi}{3}\right]$$



The point $P[r, \theta]$ can be described in infinitely many ways:

$$[r, \theta + 2n\pi] \quad \text{and} \quad [-r, \theta + (2n+1)\pi] \qquad \text{for all } n \in \mathbb{Z}$$

Example 38

Convert polar coordinates $\left[2, \frac{5\pi}{6}\right]$ into Cartesian coordinates.

Solution

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 2\cos\left(\frac{5\pi}{6}\right) \qquad = 2\sin\left(\frac{5\pi}{6}\right)$$
$$= -\sqrt{3} \qquad = 1$$

The Cartesian coordinates are $(-\sqrt{3}, 1)$.

Example 39

For each pair of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$.

a $(3, 3)$ **b** $(1, -\sqrt{3})$ **c** $(-5, 0)$ **d** $(0, 3)$

Solution

a $r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

The point has polar coordinates $\left[3\sqrt{2}, \frac{\pi}{4}\right]$.

We could also let $r = -3\sqrt{2}$ and add π to the angle, giving $\left[-3\sqrt{2}, \frac{5\pi}{4}\right]$.

b $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

The point has polar coordinates $\left[2, -\frac{\pi}{3}\right]$.

We could also let $r = -2$ and add π to the angle, giving $\left[-2, \frac{2\pi}{3}\right]$.

c $r = 5$ and $\theta = \pi$

The point has polar coordinates $[5, \pi]$.

We could also let $r = -5$ and subtract π from the angle, giving $[-5, 0]$.

d $r = 3$ and $\theta = \frac{\pi}{2}$

The point has polar coordinates $\left[3, \frac{\pi}{2}\right]$.

We could also let $r = -3$ and subtract π from the angle, giving $\left[-3, -\frac{\pi}{2}\right]$.

Summary 17J

- Each point P in the plane can be represented using polar coordinates $[r, \theta]$, where:
  - r is the distance from the origin O to P
  - θ is the angle between the positive direction of the x-axis and the ray OP.
- If a point P has polar coordinates $[r, \theta]$, then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- If a point P has Cartesian coordinates (x, y), then all its polar coordinates $[r, \theta]$ satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \qquad (\text{if } x \neq 0)$$

- Each point in the plane has more than one representation in polar coordinates. For example, the coordinates $\left[2, \frac{\pi}{4}\right]$, $\left[2, \frac{9\pi}{4}\right]$ and $\left[-2, \frac{5\pi}{4}\right]$ all represent the same point.

Exercise 17J

Example 38

1 Plot the points with the following polar coordinates and then find their Cartesian coordinates:

a $A\left[1, \frac{\pi}{2}\right]$ **b** $B\left[2, \frac{3\pi}{4}\right]$ **c** $C\left[3, -\frac{\pi}{2}\right]$ **d** $D\left[-2, \frac{\pi}{4}\right]$ **e** $E[-1, \pi]$

f $F\left[0, \frac{\pi}{4}\right]$ **g** $G\left[4, -\frac{5\pi}{6}\right]$ **h** $H\left[-2, \frac{2\pi}{3}\right]$ **i** $I\left[-2, -\frac{\pi}{4}\right]$

Example 39

2 For each of the following pairs of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$:

a $(1, -1)$ **b** $(1, \sqrt{3})$ **c** $(2, -2)$

d $(-\sqrt{2}, -\sqrt{2})$ **e** $(3, 0)$ **f** $(0, -2)$

3 Two points have polar coordinates $P\left[2, \frac{\pi}{6}\right]$ and $Q\left[3, \frac{\pi}{2}\right]$ respectively. Find the exact length of line segment PQ.

4 Two points have polar coordinates $P[r_1, \theta_1]$ and $Q[r_2, \theta_2]$. Find a formula for the length of PQ.

17K Graphing using polar coordinates

Polar coordinates are useful for describing and sketching curves in the plane, especially in situations that involve symmetry with respect to the origin. Suppose that f is a function. The graph of f in polar coordinates is simply the set of all points $[r, \theta]$ such that $r = f(\theta)$.

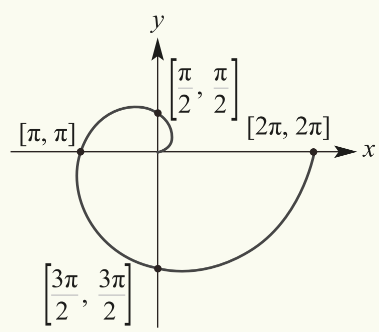
Example 40

Sketch the spiral with polar equation $r = \theta$, for $0 \le \theta \le 2\pi$.

Solution

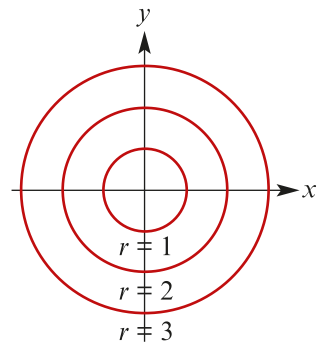
The distance r from the origin exactly matches the angle θ. So as the angle increases, so too does the distance from the origin.

Note that the coordinates on the graph are in polar form.



Circles

If a circle is centred at the origin, then its polar equation could not be simpler.



A circle of radius a centred at the origin has polar equation

$$r = a$$

That is, the distance r from the origin is constant, having no dependence on the angle θ. This illustrates rather forcefully the utility of polar coordinates: they simplify situations that involve symmetry with respect to the origin.

For circles not centred at the origin, the polar equations are less obvious.

Example 41

A curve has polar equation $r = 2\sin\theta$. Show that its Cartesian equation is $x^2 + (y-1)^2 = 1$.

Solution

The trick here is to first multiply both sides of the polar equation by r to get

$$r^2 = 2r\sin\theta$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, this equation becomes

$$\begin{aligned} x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y &= 0 \\ x^2 + (y^2 - 2y + 1) - 1 &= 0 \qquad \text{(completing the square)} \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

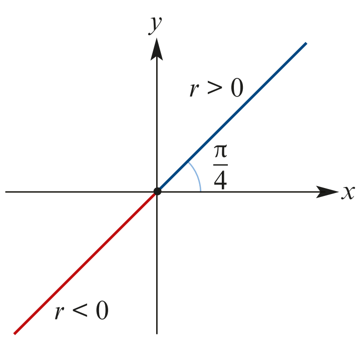
This is a circle with centre (0, 1) and radius 1.

Lines

For a straight line through the origin, the angle θ is fixed and the distance r varies. Because we have allowed negative values of r, the straight line goes in both directions.

The straight line shown has equation

$$\theta = \frac{\pi}{4}$$



For a straight line that does not go through the origin, the equation is more complicated. A line in Cartesian form $ax + by = c$ can be converted into polar form by substituting $x = r\cos\theta$ and $y = r\sin\theta$.

Example 42

a Express $x + y = 1$ in polar form. **b** Express $r = \dfrac{2}{3\cos\theta - 4\sin\theta}$ in Cartesian form.

Solution

a Since $x = r\cos\theta$ and $y = r\sin\theta$, the equation $x + y = 1$ becomes

$$r\cos\theta + r\sin\theta = 1$$

$$r(\cos\theta + \sin\theta) = 1$$

Therefore the straight line has polar equation

$$r = \frac{1}{\cos\theta + \sin\theta}$$

b Since $\dfrac{x}{r} = \cos\theta$ and $\dfrac{y}{r} = \sin\theta$, the equation becomes

$$r = \frac{2}{\dfrac{3x}{r} - \dfrac{4y}{r}}$$

$$r = \frac{2r}{3x - 4y}$$

$$1 = \frac{2}{3x - 4y}$$

Therefore the Cartesian equation is

$$3x - 4y = 2$$

Further graphs

Various geometrically significant figures are best described using polar coordinates.

Cardioids

The name **cardioid** comes from the Greek word for heart. A cardioid is the curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius.

Example 43

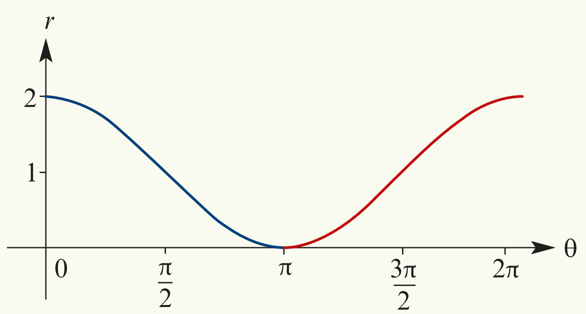
Graph the cardioid with equation $r = 1 + \cos\theta$, for $\theta \in [0, 2\pi]$.

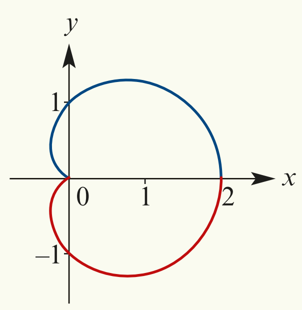
Solution

To help sketch this curve, we first graph the function $r = 1 + \cos\theta$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases.

- As the angle θ increases from 0 to π, the distance r decreases from 2 to 0.
- As the angle θ increases from π to 2π, the distance r increases from 0 to 2.

This gives the graph of the cardioid shown on the right.



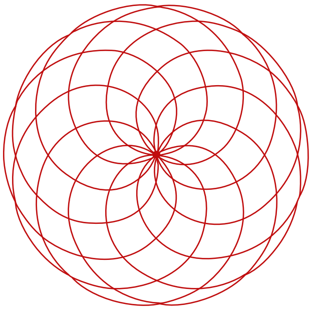


Roses

This impressive curve is fittingly called a **rose**. It belongs to the family of curves with polar equations of the form

$$r = \cos(n\theta)$$

For the example shown, $n = \frac{5}{8}$.



Example 44

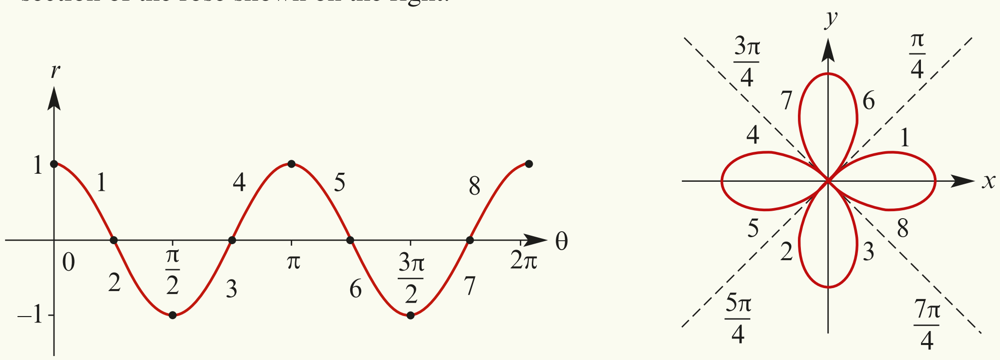
A curve has polar equation $r = \cos(2\theta)$.

a Sketch the graph of the curve.

b Show that its Cartesian equation is $(x^2 + y^2)^3 = (x^2 - y^2)^2$.

Solution

a To help sketch this curve, we first graph the function $r = \cos(2\theta)$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases. Using numbers, we have labelled how each section of this graph corresponds to a section of the rose shown on the right.



b Using the double angle formula $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, we have

$$\begin{aligned} r &= \cos(2\theta) \\ r &= \cos^2\theta - \sin^2\theta \\ r &= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \\ r^3 &= x^2 - y^2 \\ (r^2)^3 &= (x^2 - y^2)^2 \\ (x^2 + y^2)^3 &= (x^2 - y^2)^2 \end{aligned}$$

Note: This example further illustrates how polar coordinates can give more pleasing equations than their Cartesian counterparts.

The curve in this example is a **four-leaf rose**. More generally, the equations $r = \cos(n\theta)$ and $r = \sin(n\theta)$ give $2n$-leaf roses if n is even, and give n-leaf roses if n is odd.

Using a CAS calculator with polar coordinates

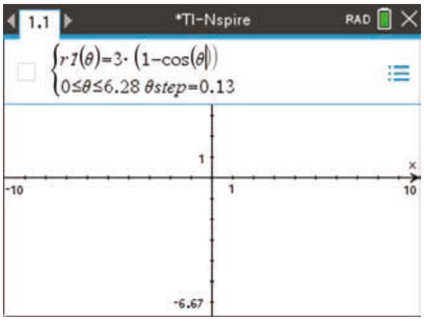
Example 45

Plot the graph of $r = 3(1 - \cos\theta)$.

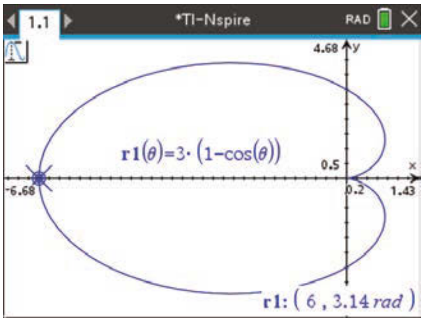
Using the TI-Nspire

- Open a **Graphs** application (on > **New Document** > **Add Graphs**) and set to polar using menu > **Graph Entry/Edit** > **Polar**.
- Enter $r1(\theta) = 3(1 - \cos(\theta))$ as shown. The variable θ is entered using π▸ or the Symbols palette (ctrl).

Note: The domain and the step size can be adjusted in this window.

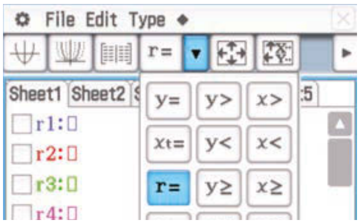


- Set the scale using menu > **Window/Zoom** > **Zoom – Fit**.
- You can see the polar coordinates $[r, \theta]$ of points on the graph using menu > **Trace** > **Graph Trace**.
- To go to the point where $\theta = \pi$, simply type π and then press enter. The cursor will move to the point $[r, \theta] = [6, \pi]$ as shown.



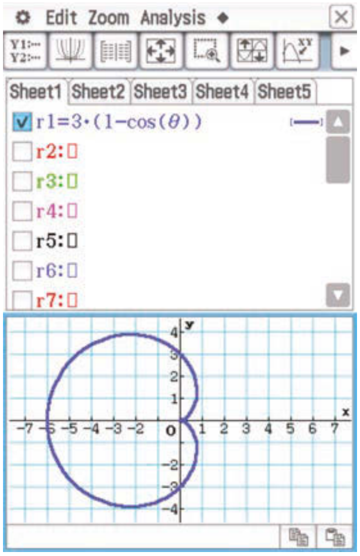
Using the Casio ClassPad

- Open the **Graph & Table** application.
- Clear all equations and graphs.
- Tap on ▼ next to y= in the toolbar and select r=.



- Enter $3(1 - \cos(\theta))$ in $r1$.
- Tick the box and tap .
- Select **Zoom** > **Initialize** to adjust the window.

Note: The variable θ is found in the Trig keyboard.



Summary 17K

- For a function f, the graph of f in polar coordinates is the set of all points $[r, \theta]$ such that $r = f(\theta)$.
- To convert between the polar form and the Cartesian form of an equation, substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Skill-sheet

Exercise 17K

Example 40

1 Sketch the spiral with polar equation $r = \dfrac{\theta}{2\pi}$, for $0 \le \theta \le 4\pi$.

2 Express each of the following Cartesian equations in polar form:

a $x = 4$ **b** $y = x^2$ **c** $x^2 + y^2 = 9$ **d** $x^2 - y^2 = 1$ **e** $2x - 3y = 5$

3 Express each of the following polar equations in Cartesian form:

a $r = \dfrac{2}{\cos\theta}$ **b** $r = 2$ **c** $\theta = \dfrac{\pi}{4}$ **d** $r = \dfrac{4}{3\cos\theta - 2\sin\theta}$

Example 41

4 By finding the Cartesian equation, show that each of the following polar equations describes a circle:

a $r = 6\cos\theta$ **b** $r = 4\sin\theta$ **c** $r = -6\cos\theta$ **d** $r = -8\sin\theta$

5 Show that the graph of $r = 2a\cos\theta$ is a circle of radius a centred at $(a, 0)$.

Example 42

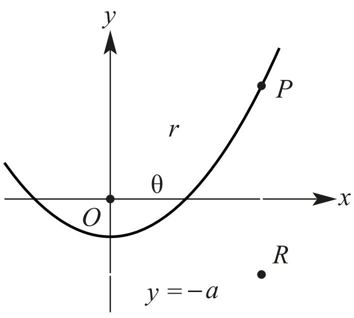
6 a Show that the graph of $r = \dfrac{a}{\cos\theta}$ is a vertical line.

b Find the polar form of the horizontal line $y = a$.

7 Consider the set of points $P[r, \theta]$ such that the distance from P to the origin O is equal to the perpendicular distance from P to the line $y = -a$, where $a > 0$. This set of points is a parabola.

a Show that the distance from P to the line is $a + r\sin\theta$.

b Conclude that the equation for the parabola can be written as $r = \dfrac{a}{1 - \sin\theta}$.



Example 43

8 The curve with polar equation $r = 1 - \sin\theta$ is a cardioid.

a Sketch the graph of the cardioid.

b Show that its Cartesian equation is $(x^2 + y^2 + y)^2 = x^2 + y^2$.

Example 44

9 Sketch the graphs of the roses with the following polar equations:

a $r = \cos(3\theta)$ **b** $r = \sin(3\theta)$

10 The polar equation $r = \sin(2\theta)$ defines a four-leaf rose.

a Sketch the graph of the rose.

b Using a double angle formula, show that its Cartesian equation is $(x^2 + y^2)^3 = 4x^2y^2$.

Chapter summary

Assign-ment

Nrich

Inverse circular functions

- $\sin^{-1}: [-1, 1] \to \mathbb{R}$, $\sin^{-1} x = y$, where $\sin y = x$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos^{-1}: [-1, 1] \to \mathbb{R}$, $\cos^{-1} x = y$, where $\cos y = x$ and $y \in [0, \pi]$
- $\tan^{-1}: \mathbb{R} \to \mathbb{R}$, $\tan^{-1} x = y$, where $\tan y = x$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Reciprocal functions

- If $y = f(x)$ is a function, then the **reciprocal function** is defined by the rule $y = \frac{1}{f(x)}$.
- To sketch the graph of $y = \frac{1}{f(x)}$, we first sketch the graph of $y = f(x)$.
- The x-axis intercepts of $y = f(x)$ will become vertical asymptotes of $y = \frac{1}{f(x)}$.
- Local maximums of $y = f(x)$ will become local minimums of $y = \frac{1}{f(x)}$, and vice versa.

Reciprocal circular functions

- $\sec x = \frac{1}{\cos x}$ (provided $\cos x \neq 0$)
- $\operatorname{cosec} x = \frac{1}{\sin x}$ (provided $\sin x \neq 0$)
- $\cot x = \frac{\cos x}{\sin x}$ (provided $\sin x \neq 0$)

The modulus function

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

  For example: $|5| = 5$ and $|-5| = 5$.
- On the number line, the distance between two numbers a and b is given by $|a - b| = |b - a|$.
  For example: $|x - 2| < 5$ can be read as 'the distance between x and 2 is less than 5'.
- To sketch the graph of $y = |f(x)|$, first draw the graph of $y = f(x)$. Then reflect the sections of the graph that are below the x-axis so that they are above the x-axis.
- To sketch the graph of $y = f(|x|)$, first draw the graph of $y = f(x)$ for $x \geq 0$. Then reflect the graph across the y-axis to obtain the graph for $x \leq 0$.

Circles and straight lines

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- The Cartesian equation of the circle with centre $C(a, b)$ and radius r is

$$(x - a)^2 + (y - b)^2 = r^2$$

- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.

Review

Parabolas, ellipses and hyperbolas

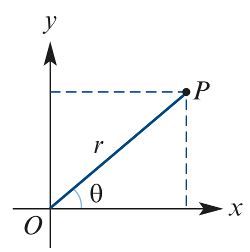
- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- An **ellipse** is the locus of a point P that moves so that the sum of its distances from two fixed points F_1 and F_2 is a constant.
- The graph of
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
is an ellipse centred at the point (h, k).
- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant.
- The graph of
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
is a hyperbola centred at the point (h, k). Its asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

Parametric curves

- A **parametric curve** in the plane is a pair of functions
$$x = f(t) \quad \text{and} \quad y = g(t)$$
where t is called the **parameter** of the curve.
- It can be helpful to think of the parameter t as describing time. Parametric curves are then useful for describing the motion of an object.
- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

Polar coordinates

- Each point P in the plane can be represented using polar coordinates $[r, \theta]$, where:
  - r is the distance from the origin O to P
  - θ is the angle between the positive direction of the x-axis and the ray OP.



- If a point P has polar coordinates $[r, \theta]$, then its Cartesian coordinates (x, y) satisfy
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \qquad (1)$$
- If a point P has Cartesian coordinates (x, y), then all its polar coordinates $[r, \theta]$ satisfy
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad (\text{if } x \neq 0) \qquad (2)$$
- If f is a function, then the graph of f in polar coordinates is the set of all points $[r, \theta]$ such that $r = f(\theta)$.
- To convert between the polar form and the Cartesian form of an equation, use formulas (1) and (2) above.

Technology-free questions

1 Evaluate each of the following:

a $\sin^{-1}(1)$ **b** $\tan^{-1}(\sqrt{3})$ **c** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

d $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **e** $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **f** $\tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right)$

2 Sketch the graphs of the following functions, stating clearly the implied domain and the range of each:

a $y = \sin^{-1}(x + 1)$ **b** $y = 2\cos^{-1}\left(x + \frac{1}{2}\right) - \pi$ **c** $y = -2\tan^{-1}(x) + \frac{\pi}{4}$

3 State the value of each of the following without using the absolute value function in your answer:

a $|-9|$ **b** $\left|-\frac{1}{400}\right|$ **c** $|9 - 5|$ **d** $|5 - 9|$ **e** $|\pi - 3|$ **f** $|\pi - 4|$

4 Let $f(x) = |x^2 - 3x|$. Solve the equation $f(x) = x$.

5 For each of the following, sketch the graph of $y = f(x)$ and state the range of f:

a $f(x) = |x^2 - 4x|$ **b** $f(x) = |x^2 - 4x| - 3$ **c** $f(x) = 3 - |x^2 - 4x|$

6 **a** Find the four integer values of n such that $|n^2 - 9|$ is a prime number.

b Solve each equation for x:

i $x^2 + 5|x| - 6 = 0$ **ii** $x + |x| = 0$

c Solve the inequality $5 - |x| < 4$ for x.

7 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes:

a $f(x) = \frac{1}{2}(x^2 - 4)$ **b** $f(x) = (x + 1)^2 + 1$

c $f(x) = \cos(x) + 1, \ x \in [0, 2\pi]$ **d** $f(x) = \sin(x) + 2, \ x \in [0, 2\pi]$

8 Sketch the graph of each of the following functions over the interval $[0, 2\pi]$:

a $y = 2\sec(x)$ **b** $y = -\operatorname{cosec}(x - \pi) + 1$ **c** $y = -\cot(2x)$

9 Find the locus of a point $P(x, y)$ that moves so that its distance to point $A(3, 2)$ is 6.

10 Sketch the graph of each circle and label its axis intercepts:

a $x^2 + (y - 2)^2 = 4$ **b** $(x - 3)^2 + (y - 1)^2 = 5$

11 A circle has equation $x^2 + 4x + y^2 - 8y = 0$. Find the coordinates of the centre and the radius of the circle.

12 Sketch the graph of each ellipse and find the coordinates of its axis intercepts:

a $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **b** $\frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{9} = 1$

13 An ellipse has equation $x^2 + 4x + 2y^2 = 0$. Find the coordinates of the centre and the axis intercepts of the ellipse.

Review

14 Sketch the graph of each hyperbola and write down the equations of its asymptotes:

a $x^2 - \frac{y^2}{4} = 1$ **b** $\frac{(y-1)^2}{16} - \frac{(x+2)^2}{4} = 1$

15 A point $P(x, y)$ moves so that its distance from the point $K(-2, 5)$ is twice its distance from the line $x = 1$. Find its locus.

16 For each of the following pairs of parametric equations, find the corresponding Cartesian equation:

a $x = 2t - 1$ and $y = 6 - 4t$ **b** $x = 2\cos t$ and $y = 2\sin t$

c $x = 3\cos t + 1$ and $y = 5\sin t - 1$ **d** $x = \cos t$ and $y = 3\sin^2 t - 2$

17 A curve has parametric equations $x = t - 1$ and $y = 2t^2 - 1$ for $0 \le t \le 2$.

a Find the curve's Cartesian equation. **b** What is the domain of the curve?

c What is the range of the curve? **d** Sketch the graph of the curve.

18 Convert the polar coordinates $\left[2, \frac{3\pi}{4}\right]$ into Cartesian coordinates.

19 The point P has Cartesian coordinates $(2, -2\sqrt{3})$. Find two representations of P using polar coordinates, one with $r > 0$ and the other with $r < 0$.

20 Find an equation for the straight line $2x + 3y = 5$ using polar coordinates.

21 Show that the graph of $r = 6\sin\theta$ is a circle of radius 3 centred at $(0, 3)$.
Hint: First multiply both sides of the equation by r.

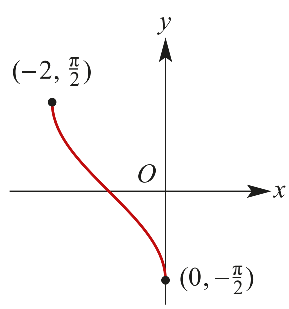
Multiple-choice questions

1 The rule for the graph shown is

A $y = \cos^{-1}(x+1) - \frac{\pi}{2}$ **B** $y = \cos^{-1}(x-1) + \frac{\pi}{2}$

C $y = \sin^{-1}(x+1) + \frac{\pi}{2}$ **D** $y = \sin^{-1}(x-1) - \frac{\pi}{2}$

E $y = \sin^{-1}(x-1) + \frac{\pi}{2}$

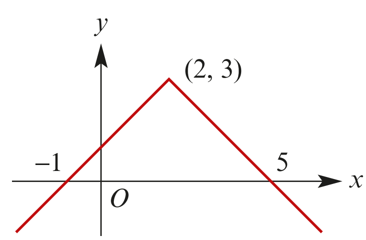


2 The rule for the graph shown is

A $y = |x-2| + 3$ **B** $y = |-x+2| + 3$

C $y = |x+2| + 3$ **D** $y = -|x-2| + 3$

E $y = -|x+2| + 3$



3 The graph of $y = \frac{1}{x^2 + 8x + k}$ will have two vertical asymptotes provided

A $k = 16$ **B** $k < 16$ **C** $k > 16$ **D** $k < -4$ or $k > 4$ **E** $-4 < k < 4$

4 An equation for the graph shown is

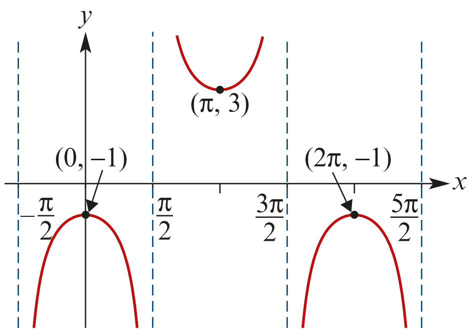
A $y = -2\operatorname{cosec}(x) + 1$

B $y = -2\sec(x) + 1$

C $y = 2\operatorname{cosec}(x) + 1$

D $y = 2\sec(x) + 1$

E $y = 2\operatorname{cosec}(x) - 1$



5 The locus of points $P(x, y)$ which satisfy the property that $AP = BP$, given points $A(2, -5)$ and $B(-4, 1)$, is described by the equation

A $y = x - 1$ **B** $y = x - 6$ **C** $y = -x - 3$ **D** $y = x + 1$ **E** $y = 3 - x$

6 A parabola has focus $(0, 2)$ and directrix $y = -2$. Which of the following is not true about the parabola?

A Its axis of symmetry is the line $x = 0$.

B It passes through the origin.

C It contains no point below the x-axis.

D The point $(2, 1)$ lies on the parabola.

E The point $(4, 2)$ lies on the parabola.

7 The graph of the ellipse with equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ has x-axis intercepts at

A $(-5, 0)$ and $(-3, 0)$ **B** $(-3, 0)$ and $(3, 0)$ **C** $(0, -5)$ and $(0, 5)$

D $(-5, 0)$ and $(5, 0)$ **E** $(5, 0)$ and $(3, 0)$

8 The equation of the graph shown is

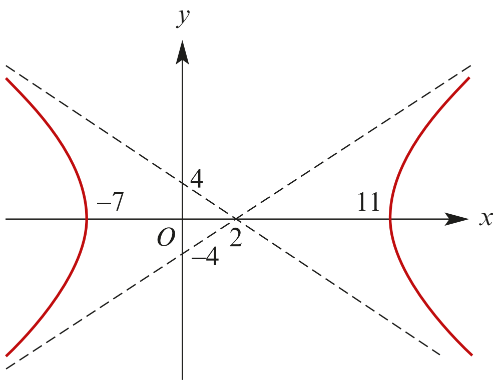
A $\dfrac{(x+2)^2}{27} - \dfrac{y^2}{108} = 1$

B $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{34} = 1$

C $\dfrac{(x+2)^2}{81} - \dfrac{y^2}{324} = 1$

D $\dfrac{(x-2)^2}{81} - \dfrac{y^2}{324} = 1$

E $\dfrac{(x+2)^2}{9} - \dfrac{y^2}{36} = 1$



9 The asymptotes of the hyperbola with equation $\dfrac{(y-2)^2}{9} - \dfrac{(x+3)^2}{4} = 1$ intersect at the point

A $(3, 2)$ **B** $(3, -2)$ **C** $(-3, 2)$ **D** $(2, -3)$ **E** $(-2, 3)$

10 An ellipse is parameterised by the equations $x = 4\cos t + 1$ and $y = 2\sin t - 1$. The coordinates of its x-axis intercepts are

A $(1 - 3\sqrt{2}, 0),\ (1 + 3\sqrt{2}, 0)$

B $(-3, 0),\ (5, 0)$

C $(1 - 2\sqrt{3}, 0),\ (1 + 2\sqrt{3}, 0)$

D $(0, -3),\ (0, 5)$

E $(0, 1 - 2\sqrt{3}),\ (0, 1 + 2\sqrt{3})$

11 Which of the following pairs of polar coordinates represent the same point?

A $\left[2, \frac{\pi}{4}\right]$ and $\left[2, \frac{3\pi}{4}\right]$ **B** $\left[3, \frac{\pi}{2}\right]$ and $\left[-3, \frac{\pi}{2}\right]$ **C** $\left[2, \frac{\pi}{3}\right]$ and $\left[-2, \frac{2\pi}{3}\right]$

D $\left[3, \frac{\pi}{4}\right]$ and $\left[3, \frac{5\pi}{4}\right]$ **E** $\left[1, \frac{\pi}{6}\right]$ and $\left[-1, \frac{7\pi}{6}\right]$

12 A curve has polar equation $r = 1 + \cos\theta$. Its equation in Cartesian coordinates is

A $xy = x^2 + y^2$ **B** $(x^2 + y^2 - x)^2 = x^2 + y^2$ **C** $x = x^2 + y^2$

D $(x^2 + y^2 - y)^2 = x^2 + y^2$ **E** $y = x^2 + y^2$

Extended-response questions

1 **a** Use a compound angle formula to prove that

$$\tan\left(\tan^{-1}(x) + \tan^{-1}(y)\right) = \frac{x + y}{1 - xy}$$

b Use the identity from part **a** to show that $\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}) = \frac{\pi}{4}$.

2 Let $f(x) = |mx + 2|$, where $m > 0$.

a Find the x-axis intercept of the graph of f in terms of m.

b For what values of m is the x-axis intercept less than -2?

c **i** Find the equation of the line ℓ that is perpendicular to the graph of f at the point with coordinates $(0, 2)$.

ii For $m > 1$, find the coordinates of the other point of intersection of the line ℓ with the graph of f.

iii What happens for $m = 1$?

iv For what value of m does the line ℓ meet the graph of f where $x = -\frac{3}{2}$?

3 **a** Consider the function with rule $f(x) = |x^2 - ax|$, where a is a constant.

i Sketch the graph of $y = f(x)$ for $a = 2$.

ii For $a \neq 0$, find the x-axis intercepts of the graph of $y = f(x)$.

iii For $a \neq 0$, find the coordinates of the local maximum on the graph of $y = f(x)$.

iv Find the values of a for which the point $(-1, 4)$ lies on the graph of $y = f(x)$.

b Consider the function with rule $g(x) = |x|^2 - a|x|$, where a is a constant.

i Sketch the graph of $y = g(x)$ for $a = 2$.

ii For $a > 0$, find the x-axis intercepts of the graph of $y = g(x)$.

iii For $a > 0$, find the coordinates of the local minimums on the graph of $y = g(x)$.

iv Find the value of a for which the point $(-1, 4)$ lies on the graph of $y = g(x)$.

c For $a > 0$, find the values of x such that $f(x) = g(x)$.

d For $a < 0$, find the values of x such that $f(x) = g(x)$.

4 Consider points $A(0, 3)$ and $B(6, 0)$. Find the locus of the point $P(x, y)$ given that:

a $AP = BP$ **b** $AP = 2BP$

5 Find the equation of the locus of points $P(x, y)$ which satisfy the property that the distance to P from the point $F(0, 4)$ is equal to:

a MP, the perpendicular distance from the line with equation $y = -2$

b half the distance MP, the perpendicular distance from the line $y = -2$

c twice the distance MP, the perpendicular distance from the line $y = -2$.

6 A ball is thrown into the air. The position of the ball at time $t \geq 0$ is given by the parametric equations $x = 10t$ and $y = 20t - 5t^2$.

a Find the Cartesian equation of the ball's flight.

b Sketch the graph of the ball's path.

c What is the maximum height reached by the ball?

A second ball is thrown into the air. Its position at time $t \geq 0$ is given by the parametric equations $x = 60 - 10t$ and $y = 20t - 5t^2$.

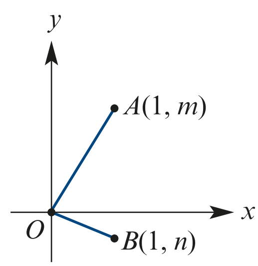
d Find the Cartesian equation of the second ball's flight.

e Sketch the graph of the second ball's path on the same set of axes.

f Find the points of intersection of the two paths.

g Do the balls collide?

7 Consider the lines $y = mx$ and $y = nx$ shown in the diagram.

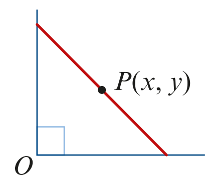


a Use the diagram and Pythagoras' theorem to prove that if $\angle AOB = 90°$, then $mn = -1$.

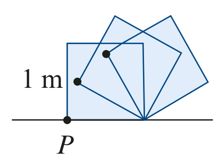
b Use the diagram and the cosine rule to prove that if $mn = -1$, then $\angle AOB = 90°$.

c Consider points $A(0, 4)$ and $B(8, 10)$. Find the equation of the locus of points $P(x, y)$ where $AP \perp BP$.

8 A ladder of length 6 metres stands against a vertical wall. The ladder then slides along on the floor until it lies flat. Show that the midpoint $P(x, y)$ of the ladder moves along a circular path.



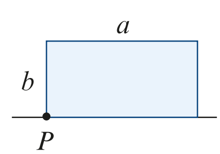
9 A square box of side length 1 metre is too heavy to lift, but can be rolled along the flat ground, using each edge as a pivot. The box is rolled one full revolution.



a Sketch the full path of the point P.

b Find the total distance travelled by the point P.

c A second rectangular box has length a metres and width b metres. Sketch the path taken by the point P when the box is rolled one full revolution, and find the total distance travelled by this point.



d For the second box, find the area between the path taken by P and the ground.

18

Complex numbers

Objectives

- To understand the **imaginary number** i and the set of **complex numbers** $\mathbb{C}$.
- To find the **real part** and the **imaginary part** of a complex number.
- To perform **addition**, **subtraction**, **multiplication** and **division** of complex numbers.
- To find the **conjugate** of a complex number.
- To represent complex numbers graphically on an **Argand diagram**.
- To work with complex numbers in **polar form**, and to understand the geometric interpretation of multiplication and division of complex numbers in this form.
- To solve polynomial equations over the complex numbers.
- To sketch subsets of the **complex plane**, including lines, rays and circles.

In this chapter we introduce a new set of numbers, called *complex numbers*. These numbers first arose in the search for solutions to polynomial equations.

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as ‘impossible’, they arose in calculations to find real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano’s formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which you can show equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.

18A Starting to build the complex numbers

Mathematicians in the eighteenth century introduced the imaginary number i with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely i and $-i$.

By declaring that $i = \sqrt{-1}$, we can find square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)}\\ &= \sqrt{4} \times \sqrt{-1}\\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers a and b, but does not hold when both a and b are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

Now consider the equation $x^2 + 2x + 3 = 0$. Using the quadratic formula gives

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 12}}{2}\\ &= \frac{-2 \pm \sqrt{-8}}{2}\\ &= -1 \pm \sqrt{-2}\end{aligned}$$

This equation has no real solutions. However, using complex numbers we obtain solutions

$$x = -1 \pm \sqrt{2}i$$

The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$$

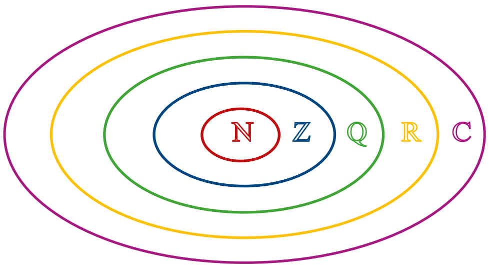
The letter z is often used to denote a complex number.

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

We can now extend the diagram from Chapter 2 to include the complex numbers.



Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\operatorname{Re}(z) = a \quad \text{and} \quad \operatorname{Im}(z) = b$$

where $\operatorname{Re}(z)$ is called the **real part** of z and $\operatorname{Im}(z)$ is called the **imaginary part** of z.

For example, for the complex number $z = 2 + 5i$, we have $\operatorname{Re}(z) = 2$ and $\operatorname{Im}(z) = 5$.

Note: Both $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ are real numbers. That is, $\operatorname{Re}\colon \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im}\colon \mathbb{C} \to \mathbb{R}$.

Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Example 1

If $4 - 3i = 2a + bi$, find the real values of a and b.

Solution

$$2a = 4 \quad \text{and} \quad b = -3$$

$$\Rightarrow \quad a = 2 \quad \text{and} \quad b = -3$$

Example 2

Find the real values of a and b such that $(2a + 3b) + (a - 2b)i = -1 + 3i$.

Solution

$$2a + 3b = -1 \qquad (1)$$

$$a - 2b = 3 \qquad (2)$$

Multiply (2) by 2:

$$2a - 4b = 6 \qquad (3)$$

Subtract (3) from (1):

$$7b = -7$$

Therefore $b = -1$ and $a = 1$.

Operations on complex numbers

Addition and subtraction

Addition of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.

If $z = a + bi$, then we define $-z = -a - bi$.

Subtraction of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$$

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z + 0 = z$
- $z + (-z) = 0$

Example 3

If $z_1 = 2 - 3i$ and $z_2 = -4 + 5i$, find:

a $z_1 + z_2$

b $z_1 - z_2$

Solution

a $z_1 + z_2 = (2 - 3i) + (-4 + 5i)$
$= (2 + (-4)) + (-3 + 5)i$
$= -2 + 2i$

b $z_1 - z_2 = (2 - 3i) - (-4 + 5i)$
$= (2 - (-4)) + (-3 - 5)i$
$= 6 - 8i$

Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = k(a + bi) = ka + kbi$$

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

Powers of i

Successive multiplication by i gives the following:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \ldots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Example 4

Simplify:

a i^{13}

b $3i^4 \times (-2i)^3$

Solution

a $i^{13} = i^{4\times3+1}$
$= i$

b $3i^4 \times (-2i)^3 = 3 \times (-2)^3 \times i^4 \times i^3$
$= -24i^7$
$= 24i$

Summary 18A

- The imaginary number i satisfies $i^2 = -1$.
- If a is a positive real number, then $\sqrt{-a} = \sqrt{a} \cdot i$.
- The set of **complex numbers** is $\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\mathrm{Re}(z) = a$
  - the **imaginary part** of z is $\mathrm{Im}(z) = b$.
- Equality of complex numbers:

  $$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

- If $z_1 = a + bi$ and $z_2 = c + di$, then

  $$z_1 + z_2 = (a + c) + (b + d)i \quad \text{and} \quad z_1 - z_2 = (a - c) + (b - d)i$$

- When simplifying powers of i, remember that $i^4 = 1$.

Exercise 18A

1 State the values of Re(z) and Im(z) for each of the following:

a $z = 2 + 3i$ **b** $z = 4 + 5i$ **c** $z = \frac{1}{2} - \frac{3}{2}i$

d $z = -4$ **e** $z = 3i$ **f** $z = \sqrt{2} - 2\sqrt{2}i$

Example 1
Example 2

2 Find the real values of a and b in each of the following:

a $2a - 3bi = 4 + 6i$ **b** $a + b - 2abi = 5 - 12i$

c $2a + bi = 10$ **d** $3a + (a - b)i = 2 + i$

Example 3

3 Simplify:

a $(2 - 3i) + (4 - 5i)$ **b** $(4 + i) + (2 - 2i)$

c $(-3 - i) - (3 + i)$ **d** $(2 - \sqrt{2}i) + (5 - \sqrt{8}i)$

e $(1 - i) - (2i + 3)$ **f** $(2 + i) - (-2 - i)$

g $4(2 - 3i) - (2 - 8i)$ **h** $-(5 - 4i) + (1 + 2i)$

i $5(i + 4) + 3(2i - 7)$ **j** $\frac{1}{2}(4 - 3i) - \frac{3}{2}(2 - i)$

Example 4

4 Simplify:

a $\sqrt{-16}$ **b** $2\sqrt{-9}$ **c** $\sqrt{-2}$

d i^3 **e** i^{14} **f** i^{20}

g $-2i \times i^3$ **h** $4i^4 \times 3i^2$ **i** $\sqrt{8}i^5 \times \sqrt{-2}$

5 Simplify:

a $i(2 - i)$ **b** $i^2(3 - 4i)$

c $\sqrt{2}i\,(i - \sqrt{2})$ **d** $-\sqrt{3}\,(\sqrt{-3} + \sqrt{2})$

18B Multiplication and division of complex numbers

In the previous section, we defined addition and subtraction of complex numbers. We begin this section by defining multiplication.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \qquad (\text{since } i^2 = -1) \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that the following is a *definition* of multiplication for $\mathbb{C}$.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$.

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z \times 1 = z$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 5

If $w = 3 - 2i$ and $z = 1 + i$, find wz.

Solution

$$\begin{aligned} wz &= (3 - 2i)(1 + i) \\ &= 3 + 3i - 2i - 2i^2 \\ &= 5 + i \end{aligned}$$

Explanation

Expand the brackets in the usual way.

Remember that $i^2 = -1$.

The conjugate of a complex number

Let $z = a + bi$. The **conjugate** of z is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi$$

For example, the conjugate of $-4 + 3i$ is $-4 - 3i$, and vice versa.

Example 6

If $w = 2 - 3i$ and $z = -1 + 2i$, find:

a $\overline{w + z}$ and $\overline{w} + \overline{z}$

b $\overline{w \cdot z}$ and $\overline{w} \cdot \overline{z}$

Solution

We have $\overline{w} = 2 + 3i$ and $\overline{z} = -1 - 2i$.

a
$$\begin{aligned} w + z &= (2 - 3i) + (-1 + 2i) \\ &= 1 - i \end{aligned}$$
$$\overline{w + z} = 1 + i$$
$$\begin{aligned} \overline{w} + \overline{z} &= (2 + 3i) + (-1 - 2i) \\ &= 1 + i \end{aligned}$$

b
$$\begin{aligned} w \cdot z &= (2 - 3i)(-1 + 2i) \\ &= -2 + 4i + 3i - 6i^2 \\ &= 4 + 7i \end{aligned}$$
$$\overline{w \cdot z} = 4 - 7i$$
$$\begin{aligned} \overline{w} \cdot \overline{z} &= (2 + 3i)(-1 - 2i) \\ &= -2 - 4i - 3i - 6i^2 \\ &= 4 - 7i \end{aligned}$$

- The conjugate of a sum is equal to the sum of the conjugates:
$$\overline{w + z} = \overline{w} + \overline{z}$$
- The conjugate of a product is equal to the product of the conjugates:
$$\overline{w \cdot z} = \overline{w} \cdot \overline{z}$$

The modulus of a complex number

For a complex number $z = a + bi$, we have

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 + b^2 \qquad \text{where } a^2 + b^2 \text{ is a real number} \end{aligned}$$

The **modulus** of the complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$

The calculation above shows that

$$z\overline{z} = |z|^2$$

Note: In the case that z is a real number, this definition of $|z|$ agrees with the definition of the modulus of a real number given in Chapter 17.

In Section 18F, we will see the geometric interpretation of the modulus of a complex number.

Example 7

If $w = 2 + 3i$ and $z = 1 - 2i$, find:

a $|wz|$

b $|w|\,|z|$

Solution

a We first find that

$$\begin{aligned} wz &= (2 + 3i)(1 - 2i) \\ &= 2 - 4i + 3i - 6i^2 \\ &= 8 - i \end{aligned}$$

Therefore

$$|wz| = \sqrt{8^2 + (-1)^2} = \sqrt{65}$$

b We first find that

$$|w| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$|z| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Therefore

$$|w|\,|z| = \sqrt{13}\sqrt{5} = \sqrt{65}$$

The modulus of a product is equal to the product of the moduli:

$$|wz| = |w|\,|z|$$

Division of complex numbers

Multiplicative inverse

We begin with some familiar algebra that will motivate the definition:

$$\begin{aligned} \frac{1}{a + bi} &= \frac{1}{a + bi} \times \frac{a - bi}{a - bi} \\ &= \frac{a - bi}{(a + bi)(a - bi)} \\ &= \frac{a - bi}{a^2 + b^2} \end{aligned}$$

We can see that

$$(a + bi) \times \frac{a - bi}{a^2 + b^2} = 1$$

Although we have carried out this arithmetic, we have not yet defined what $\dfrac{1}{a + bi}$ means.

Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

Note: We can check that $(wz)^{-1} = w^{-1}z^{-1}$.

Division

The formal definition of division in the complex numbers is via the multiplicative inverse:

Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2} \qquad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di}$$

Multiply the numerator and denominator by the conjugate of z_2:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2}\end{aligned}$$

We complete the division by simplifying.

This procedure is demonstrated in the next example.

Example 8

a Find $\dfrac{2 - i}{3 + 2i}$.

b Find z if $(2 + 3i)z = -1 - 2i$.

Solution

a
$$\begin{aligned}\frac{2 - i}{3 + 2i} &= \frac{2 - i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i} \\ &= \frac{6 - 4i - 3i + 2i^2}{3^2 + 2^2} \\ &= \frac{4 - 7i}{13} \\ &= \frac{1}{13}(4 - 7i)\end{aligned}$$

b $(2 + 3i)z = -1 - 2i$

$$\begin{aligned}\Rightarrow \quad z &= \frac{-1 - 2i}{2 + 3i} \\ &= \frac{-1 - 2i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i} \\ &= \frac{-2 + 3i - 4i + 6i^2}{2^2 + 3^2} \\ &= \frac{-8 - i}{13} \\ &= -\frac{1}{13}(8 + i)\end{aligned}$$

There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

Example 9

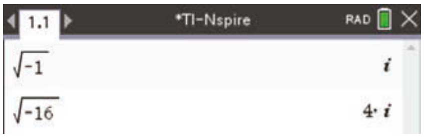
If $z = 2 - 5i$, find z^{-1} and express with a real denominator.

Solution

$$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{2-5i} \\ &= \frac{1}{2-5i} \times \frac{2+5i}{2+5i} \\ &= \frac{2+5i}{29} \\ &= \frac{1}{29}(2+5i) \end{aligned}$$

Using the TI-Nspire

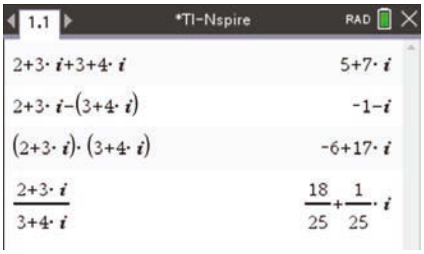
Set to complex mode using (on) > **Settings** > **Document Settings**. Select **Rectangular** from the **Real or Complex** field.



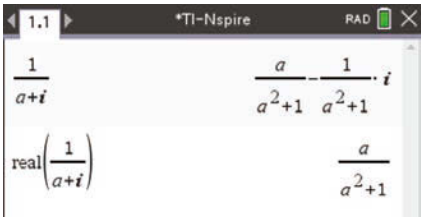
Note: The square root of a negative number can be found only in complex mode. But most computations with complex numbers can also be performed in real mode.

- The results of the arithmetic operations +, −, × and ÷ are illustrated using the two complex numbers $2 + 3i$ and $3 + 4i$.

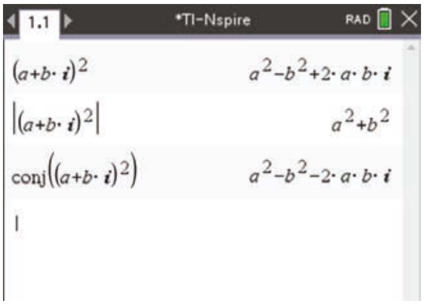
Note: Do not use the text i for the imaginary constant. The symbol i is found using (π▸) or the Symbols palette ((ctrl)(📖)).



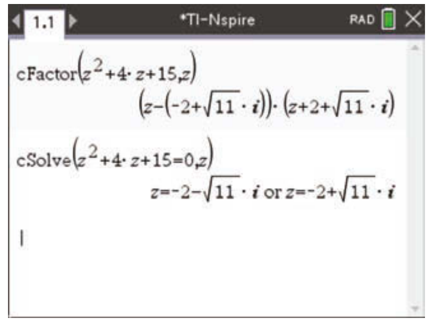
- To find the real part of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Real Part**. Alternatively, type real(.



- To find the modulus of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Magnitude**. Alternatively, use |□| from the 2D-template palette (template key) or type abs(.
- To find the conjugate of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Complex Conjugate**. Alternatively, type conj(.



There are also commands for factorising polynomials over the complex numbers and for solving polynomial equations over the complex numbers. These are available from (menu) > **Algebra** > **Complex**.

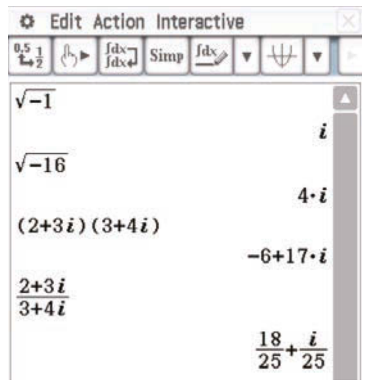


Note: You must use this menu even if the calculator is in complex mode. When using **cFactor**, you must include the variable as shown.

Using the Casio ClassPad

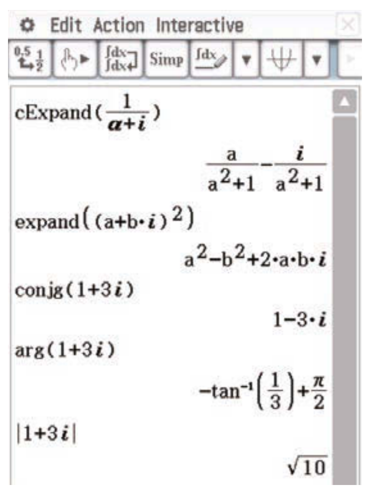
In Main $\sqrt{\alpha}$, tap **Real** in the status bar at the bottom of the screen to change to **Cplx** mode.

- Enter $\sqrt{-1}$ and tap (EXE) to obtain the answer i.
- Enter $\sqrt{-16}$ to obtain the answer $4i$.
- The arithmetic operations $+$, $-$, $\times$ and $\div$ can be applied to complex numbers as shown.

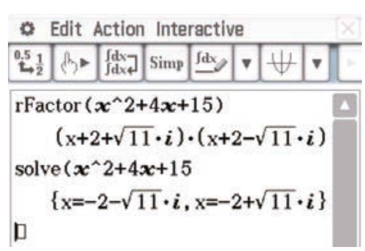


Note: The symbol i is found in both the (Math2) and the (Math3) keyboards.

With the calculator set to complex mode, various operations on complex numbers can be carried out using options from **Interactive** > **Complex**.



With the calculator set to complex mode, you can factorise polynomials and solve equations in the usual way, and the answers will be given over the complex numbers.



Summary 18B

- **Multiplication** To find a product $(a + bi)(c + di)$, expand the brackets in the usual way, remembering that $i^2 = -1$.
- **Conjugate** If $z = a + bi$, then $\overline{z} = a - bi$.
- **Modulus** If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.
- **Division** To perform a division, start with

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2}$$

and then simplify.
- **Multiplicative inverse** To find z^{-1}, calculate $\frac{1}{z}$.

Exercise 18B

Example 5

1 Expand and simplify:

a $(4 + i)^2$ **b** $(2 - 2i)^2$ **c** $(3 + 2i)(2 + 4i)$
d $(-1 - i)^2$ **e** $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i)$ **f** $(5 - 2i)(-2 + 3i)$

2 **a** If $w = 3 + 2i$ and $z = 2 + 4i$, find $\text{Re}(wz)$.
b If $w = 4 + 5i$ and $z = 3 - 2i$, find $\text{Im}(wz)$.

3 Write down the conjugate of each of the following complex numbers:

a $2 - 5i$ **b** $-1 + 3i$ **c** $\sqrt{5} - 2i$ **d** $-5i$

4 Evaluate $z \cdot \overline{z}$ for each of the following:

a $z = 3 + 4i$ **b** $z = 1 + i$ **c** $z = 2 - 3i$ **d** $z = \sqrt{2} + \sqrt{3}i$

Example 6

5 If $z_1 = 2 - i$ and $z_2 = -3 + 2i$, find:

a $\overline{z_1}$ **b** $\overline{z_2}$ **c** $z_1 \cdot z_2$ **d** $\overline{z_1 \cdot z_2}$
e $\overline{z_1} \cdot \overline{z_2}$ **f** $z_1 + z_2$ **g** $\overline{z_1 + z_2}$ **h** $\overline{z_1} + \overline{z_2}$

Example 7

6 If $w = 1 + i$ and $z = 3 - 4i$, find:

a $|wz|$ **b** $|w|\,|z|$ **c** $|w + z|$ **d** $|3w - 2z|$

7 If $z = 2 - 4i$, express each of the following in the form $x + yi$:

a $\overline{z}$ **b** $z\overline{z}$ **c** $z + \overline{z}$ **d** $z(z + \overline{z})$

Example 9

e $z - \overline{z}$ **f** $i(z - \overline{z})$ **g** z^{-1} **h** $\frac{z}{i}$

8 Find the real values of a and b such that $(a + bi)(2 + 5i) = 3 - i$.

Example 8a

9 Find each of the following, expressing your answer in the form $x + yi$:

a $\dfrac{2-i}{4+i}$ **b** $\dfrac{3+2i}{2-3i}$ **c** $\dfrac{4+3i}{1+i}$ **d** $\dfrac{2-2i}{4i}$ **e** $\dfrac{1}{2-3i}$ **f** $\dfrac{i}{2+6i}$

10 Find the real values of a and b if $(3 - i)(a + bi) = 6 - 7i$.

Example 8b

11 Solve each of the following for z:

a $(2 - i)z = 42i$ **b** $(1 + 3i)z = -2 - i$ **c** $(3i + 5)z = 1 + i$

d $2(4 - 7i)z = 5 + 2i$ **e** $z(1 + i) = 4$

12 If $a, b \in \mathbb{R}$ and $(a + bi)^2 = -5 + 12i$, find a and b.

13 If $a \in \mathbb{R}$ and $\dfrac{1}{a+3i} + \dfrac{1}{a-3i} = \dfrac{4}{13}$, find a.

14 Let $z = a + bi$ be a complex number.

a If $z = \bar{z}$, prove that z is a real number. That is, prove that $\mathrm{Im}(z) = 0$.

b Prove that $z + \bar{z}$ is a real number.

c For $z \neq 0$, prove that $\dfrac{1}{z} + \dfrac{1}{\bar{z}}$ is a real number.

15 Let $z = \dfrac{a+bi}{a-bi}$, where $a, b \in \mathbb{R}$. Prove that $\dfrac{z^2+1}{2z}$ is a real number.

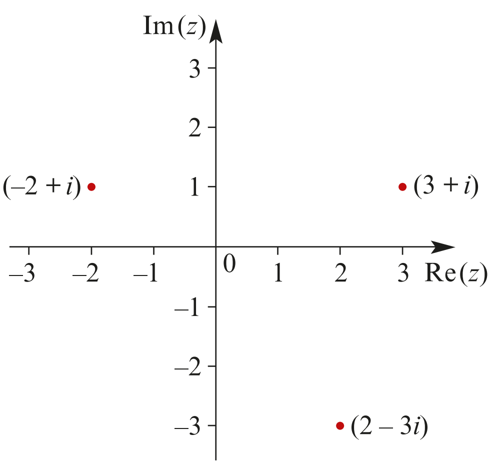
18C Argand diagrams

An **Argand diagram** is a geometric representation of the set of complex numbers. A complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\mathrm{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\mathrm{Im}(z)$, for $z \in \mathbb{C}$.

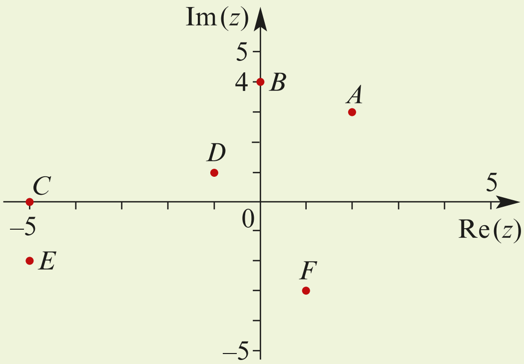
Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent Cartesian axes, as shown by the examples in this figure.

A complex number written as $a + bi$ is said to be in **Cartesian form**.



Example 10

Write down the complex number represented by each of the points shown on this Argand diagram.



Solution

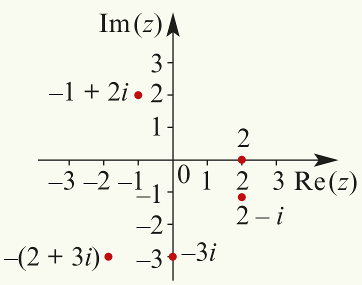
A $2 + 3i$ **B** $4i$ **C** -5 **D** $-1 + i$ **E** $-5 - 2i$ **F** $1 - 3i$

Example 11

Represent the following complex numbers as points on an Argand diagram:

a 2 **b** $-3i$ **c** $2 - i$ **d** $-(2 + 3i)$ **e** $-1 + 2i$

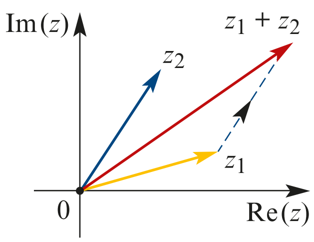
Solution

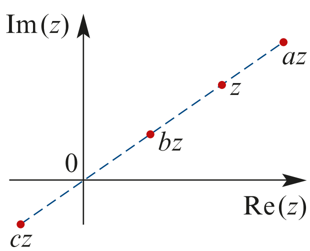


Geometric representation of the basic operations on complex numbers

In an Argand diagram, the sum of two complex numbers z_1 and z_2 can be found geometrically by placing the 'tail' of z_2 on the 'tip' of z_1, as shown in the diagram on the left.

When a complex number is multiplied by a real constant, it maintains the same 'direction', but its distance from the origin is scaled. This is shown in the diagram on the right.





$a > 1$

$0 < b < 1$

$c < 0$

The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

Example 12

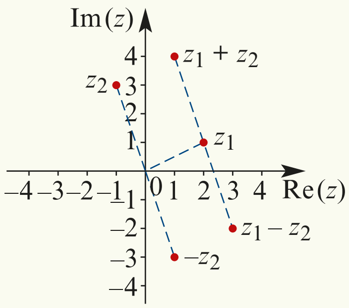
Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

Represent the complex numbers z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

Solution

$$z_1 + z_2 = (2 + i) + (-1 + 3i)$$
$$= 1 + 4i$$

$$z_1 - z_2 = (2 + i) - (-1 + 3i)$$
$$= 3 - 2i$$

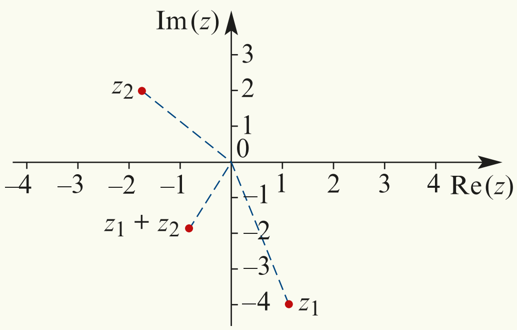


Example 13

Let $z_1 = 1 - 4i$ and $z_2 = -2 + 2i$. Find $z_1 + z_2$ algebraically and illustrate $z_1 + z_2$ on an Argand diagram.

Solution

$$z_1 + z_2 = (1 - 4i) + (-2 + 2i)$$
$$= -1 - 2i$$

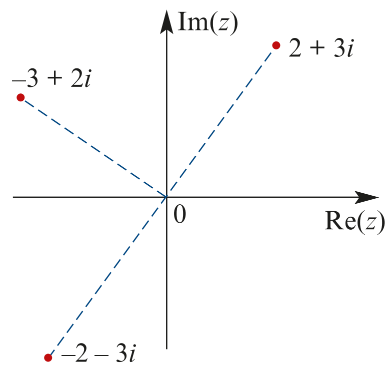


Rotation about the origin

When the complex number $2 + 3i$ is multiplied by -1, the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.

When the complex number $2 + 3i$ is multiplied by i, we obtain

$$i(2 + 3i) = 2i + 3i^2$$
$$= 2i - 3$$
$$= -3 + 2i$$

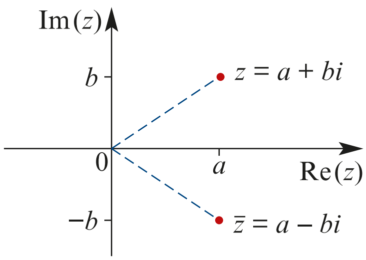


The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by i, the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.

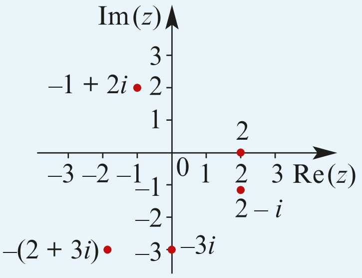
Reflection in the horizontal axis

The conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$. Therefore $\bar{z}$ is the reflection of z in the horizontal axis of an Argand diagram.



Summary 18C

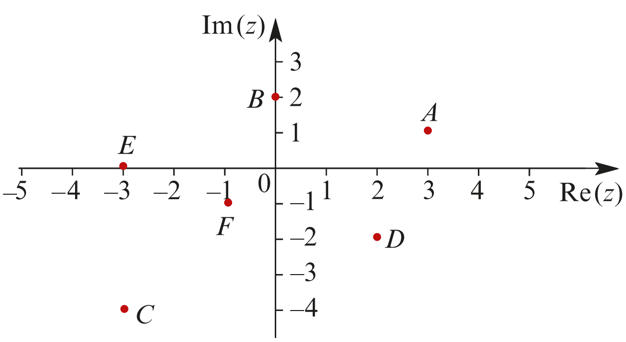
- An **Argand diagram** is a geometric representation of the set of complex numbers.
- The horizontal axis represents Re(z) and the vertical axis represents Im(z), for $z \in \mathbb{C}$.
- The operations of addition, subtraction and multiplication by a real constant all have geometric interpretations on an Argand diagram.
- Multiplication of a complex number by i corresponds to a rotation of 90° anticlockwise about the origin.
- Complex conjugate corresponds to reflection in the horizontal axis.



Exercise 18C

Example 10

1 Write down the complex numbers represented on this Argand diagram.



Example 11

2 Represent each of the following complex numbers as points on an Argand diagram:

a $3 - 4i$ **b** $-4 + i$ **c** $4 + i$ **d** -3 **e** $-2i$ **f** $-5 - 2i$

Example 12
Example 13

3 If $z_1 = 6 - 5i$ and $z_2 = -3 + 4i$, represent each of the following on an Argand diagram:

a $z_1 + z_2$ **b** $z_1 - z_2$

4 If $z = 1 + 3i$, represent each of the following on an Argand diagram:

a z **b** $\bar{z}$ **c** z^2 **d** $-z$ **e** $\frac{1}{z}$

5 If $z = 2 - 5i$, represent each of the following on an Argand diagram:

a z **b** zi **c** zi^2 **d** zi^3 **e** zi^4

18D Solving quadratic equations over the complex numbers

Quadratic equations with a negative discriminant have no real solutions. The introduction of complex numbers enables us to solve such quadratic equations.

Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

This allows us to solve equations of the form $z^2 + a^2 = 0$.

Example 14

Solve the following equations over $\mathbb{C}$:

a $z^2 + 16 = 0$ **b** $2z^2 + 6 = 0$

Solution

a

$$\begin{aligned} z^2 + 16 &= 0 \\ z^2 - 16i^2 &= 0 \\ (z + 4i)(z - 4i) &= 0 \\ \therefore \quad z &= \pm 4i \end{aligned}$$

b

$$\begin{aligned} 2z^2 + 6 &= 0 \\ z^2 + 3 &= 0 \\ z^2 - 3i^2 &= 0 \\ (z + \sqrt{3}i)(z - \sqrt{3}i) &= 0 \\ \therefore \quad z &= \pm\sqrt{3}i \end{aligned}$$

Solution of quadratic equations

To solve a quadratic equation with a negative discriminant, we can either complete the square or use the quadratic formula.

Example 15

a Solve $z^2 + 6z + 11 = 0$ over $\mathbb{C}$ by completing the square.

b Solve $3z^2 + 5z + 3 = 0$ over $\mathbb{C}$ by using the quadratic formula.

Solution

a

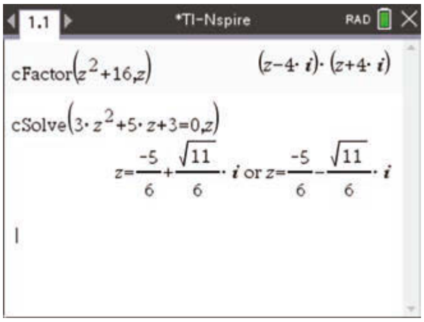
$$\begin{aligned} z^2 + 6z + 11 &= 0 \\ (z^2 + 6z + 9) - 9 + 11 &= 0 \\ (z + 3)^2 + 2 &= 0 \\ (z + 3)^2 - 2i^2 &= 0 \\ (z + 3 + \sqrt{2}i)(z + 3 - \sqrt{2}i) &= 0 \\ \therefore \quad z &= -3 \pm \sqrt{2}i \end{aligned}$$

b Using the quadratic formula:

$$\begin{aligned} z &= \frac{-5 \pm \sqrt{25 - 36}}{6} \\ &= \frac{-5 \pm \sqrt{-11}}{6} \\ &= \frac{1}{6}(-5 \pm \sqrt{11}i) \end{aligned}$$

Using the TI-Nspire

- To factorise polynomials over the complex numbers, use menu > **Algebra** > **Complex** > **Factor** as shown.
- To solve polynomial equations over the complex numbers, use menu > **Algebra** > **Complex** > **Solve** as shown.



Using the Casio ClassPad

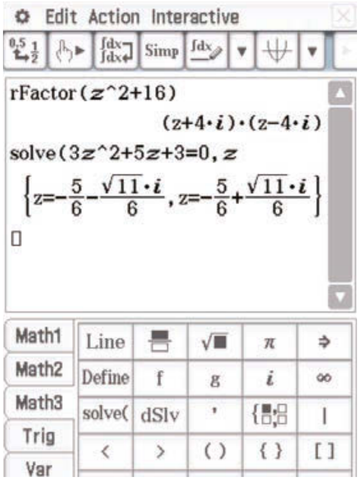
To factorise:

- Ensure the mode is set to **Cplx**.
- Enter and highlight the expression $z^2 + 16$.
- Select **Interactive** > **Transformation** > **factor** > **rFactor**.

To solve:

- Ensure the mode is set to **Cplx**.
- Select solve(from the Math1 or Math3 keyboard.
- Enter $3z^2 + 5z + 3 = 0, z$ and tap EXE.

Note: Recall that '= 0' could be omitted here, but ', z' is required as the default variable is x.



Example 16

Let $b, c \in \mathbb{R}$. If the quadratic equation $z^2 + bz + c = 0$ has solutions $z = 2 - 3i$ and $z = 2 + 3i$, find the values of b and c.

Solution

The quadratic has factors $z - 2 + 3i$ and $z - 2 - 3i$. Multiplying them together gives

$$\begin{aligned}(z - 2 + 3i)(z - 2 - 3i) &= ((z - 2) + 3i)((z - 2) - 3i)\\ &= (z - 2)^2 - (3i)^2\\ &= z^2 - 4z + 4 + 9\\ &= z^2 - 4z + 13\end{aligned}$$

Therefore $b = -4$ and $c = 13$.

Summary 18D

- Quadratic equations can be solved over the complex numbers by completing the square or by using the quadratic formula.
- Two properties of complex numbers that are useful when solving equations:
  - $z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$
  - $\sqrt{-a} = \sqrt{a} \cdot i$, where a is a positive real number.

Skill-sheet

Exercise 18D

Example 14

1 Solve the following equations over $\mathbb{C}$:

a $z^2 + 1 = 0$ **b** $z^2 + 9 = 0$ **c** $z^2 = -16$
d $4z^2 = -25$ **e** $z^2 = -2$ **f** $2z^2 + 8 = 0$
g $3z^2 + 75 = 0$ **h** $4z^2 + 1 = 0$ **i** $16z^2 + 9 = 0$
j $z^2 + 3 = 0$ **k** $2z^2 + 10 = 0$ **l** $(z + 1)^2 + 1 = 0$
m $(z - 2)^2 + 5 = 0$ **n** $(z + 3)^2 + 3 = 0$ **o** $(z - 2)^2 = -4$

Example 15a

2 Solve the following quadratic equations over $\mathbb{C}$ by completing the square:

a $z^2 + 2z + 3 = 0$ **b** $z^2 - 4z + 5 = 0$
c $z^2 + 6z + 12 = 0$ **d** $2z^2 - 8z + 10 = 0$
e $3z^2 + 2z + 1 = 0$ **f** $2z^2 + 2z + 1 = 0$

Example 15b

3 Solve the following quadratic equations over $\mathbb{C}$ by using the quadratic formula:

a $z^2 + 3z + 3 = 0$ **b** $z^2 - 4z + 5 = 0$
c $z^2 + 6z + 12 = 0$ **d** $z^2 - 4z + 8 = 0$
e $3z^2 + 2z + 1 = 0$ **f** $2z^2 - \sqrt{2}z + 1 = 0$

4 Solve the following equations over $\mathbb{C}$ using any method:

a $z^2 + 4 = 0$ **b** $2z^2 + 18 = 0$
c $3z^2 = -15$ **d** $(z - 2)^2 + 16 = 0$
e $(z + 1)^2 = -49$ **f** $z^2 - 2z + 3 = 0$
g $z^2 + 3z + 3 = 0$ **h** $2z^2 + 5z + 4 = 0$
i $3z^2 = z - 2$ **j** $2z = z^2 + 5$
k $2z^2 - 6z = -10$ **l** $z^2 - 6z = -14$

Example 16

5 Find the values of $b, c \in \mathbb{R}$ if the quadratic equation $z^2 + bz + c = 0$ has solutions:

a $z = 1 + i$ and $z = 1 - i$ **b** $z = -2 - 5i$ and $z = -2 + 5i$

6 Consider the quadratic equation $az^2 + bz + c = 0$, where a, b and c are consecutive positive integers. Show that the solutions of this equation are not real numbers.

18E Solving polynomial equations over the complex numbers

In Mathematical Methods Units 1 & 2, you have seen the correspondence between the linear factors of a polynomial $P(x)$ and the solutions of the equation $P(x) = 0$. This correspondence extends to the complex numbers.

Factor theorem

Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of a polynomial $P(z)$ if and only if $P(\alpha) = 0$.

Every quadratic equation has two solutions over the complex numbers, if we count repeated solutions twice. For example, the equation $(z-3)^2 = 0$ has a repeated solution $z = 3$. We say that this solution has a **multiplicity** of 2.

Likewise, every cubic equation has three solutions over the complex numbers, counting multiplicity. More generally, we have the following important theorem.

Fundamental theorem of algebra

For $n \geq 1$, every polynomial of degree n can be expressed as a product of n linear factors over the complex numbers. Therefore every polynomial equation of degree n has n solutions (counting multiplicity).

Note: This theorem applies to polynomials with real or complex coefficients. The proof of the theorem is surprisingly difficult and beyond the scope of this book.

Example 17

Show that $z = 1$ is a solution of $z^3 + z^2 + 3z - 5 = 0$, and then find the other two solutions.

Solution

Let $P(z) = z^3 + z^2 + 3z - 5$. Since

$$P(1) = 1^3 + 1^2 + 3 - 5 = 0$$

we see that $z = 1$ is a solution of $P(z) = 0$. Therefore $z - 1$ is a factor of $P(z)$.

By inspection (or polynomial division), we can find the other factors:

$$\begin{aligned} z^3 + z^2 + 3z - 5 &= (z-1)(z^2 + 2z + 5) \\ &= (z-1)((z^2 + 2z + 1) - 1 + 5) \\ &= (z-1)((z+1)^2 + 4) \\ &= (z-1)((z+1)^2 - (2i)^2) \\ &= (z-1)(z+1+2i)(z+1-2i) \end{aligned}$$

Therefore the remaining two solutions are $z = -1 - 2i$ and $z = -1 + 2i$.

Notice in the previous example that the solutions $z = -1 - 2i$ and $z = -1 + 2i$ are conjugates of each other. This is not a coincidence.

Conjugate root theorem

Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of the equation $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

You will prove this theorem in Exercise 18E. Note that the theorem does not hold without the assumption that $P(z)$ has real coefficients. For example, the linear equation $z - i = 0$ has just one solution, $z = i$, and its conjugate is clearly not a solution.

Example 18

Let $P(z) = z^3 + 4z^2 + 6z + 4$. Given that $z = -1 + i$ is a solution of the equation $P(z) = 0$, find all three solutions.

Solution

Note that the polynomial $P(z)$ has real coefficients. Since $-1 + i$ is a solution of $P(z) = 0$, its conjugate $-1 - i$ is also a solution.

We now have two monic linear factors $z + 1 - i$ and $z + 1 + i$ of $P(z)$. Their product is also a factor:

$$\begin{aligned}(z + 1 - i)(z + 1 + i) &= ((z + 1) - i)((z + 1) + i)\\ &= (z + 1)^2 - i^2\\ &= z^2 + 2z + 1 + 1\\ &= z^2 + 2z + 2\end{aligned}$$

The remaining factor can be found by inspection or polynomial division. This gives

$$P(z) = z^3 + 4z^2 + 6z + 4 = (z + 2)(z^2 + 2z + 2)$$

Therefore the three solutions are $z = -2$, $z = -1 + i$ and $z = -1 - i$.

Summary 18E

The following three theorems are useful when solving polynomial equations over $\mathbb{C}$:

- **Factor theorem**
  Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of a polynomial $P(z)$ if and only if $P(\alpha) = 0$.
- **Fundamental theorem of algebra**
  For $n \geq 1$, every polynomial of degree n can be expressed as a product of n linear factors over the complex numbers. Therefore every polynomial equation of degree n has n solutions (counting multiplicity).
- **Conjugate root theorem**
  Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

Exercise 18E

Example 17

1 Show that $z = 2$ is a solution of the cubic equation $z^3 + 2z^2 - 3z - 10 = 0$, and then find the other two solutions.

2 Find a real solution of $z^3 + 3z^2 + 4z + 2 = 0$, and then find the other two solutions.

Example 18

3 Given that $z = 3 - 2i$ is a solution of $z^3 - 9z^2 + 31z - 39 = 0$, find all three solutions.

4 Given that $z = 1 - \sqrt{2}i$ is a solution of $z^3 - 4z^2 + 7z - 6 = 0$, find all three solutions.

5 Show that $z = 2i$ is a solution of the cubic equation $z^3 - 3z^2 + 4z - 12 = 0$, and then find the other two solutions.

6 Show that $z = 3i$ is a solution of the quartic equation $z^4 + z^3 + 7z^2 + 9z - 18 = 0$, and then find the other three solutions.

7 Solve each of the following cubic equations over $\mathbb{C}$:

a $z^3 - z^2 + z - 1 = 0$
b $z^3 - z^2 + 3z + 5 = 0$
c $z^3 - 2z + 4 = 0$
d $z^3 + 3z^2 - 6z - 36 = 0$

8 Let $a, b, c \in \mathbb{R}$. If $z = 1 + i$ and $z = 3$ are solutions of the equation $z^3 + az^2 + bz + c = 0$, find the values of a, b and c.

9 Let $c \in \mathbb{R}$. If $z = 1 - 2i$ is a solution of $2z^3 - 5z^2 + cz - 5 = 0$, find the value of c.

10 Give an example of a quartic polynomial $P(z)$ with real coefficients such that the equation $P(z) = 0$ has:

a four distinct complex solutions
b two imaginary and two real solutions.

11 For a quadratic polynomial $P(z)$ with real coefficients, there are three possible cases for the equation $P(z) = 0$: two distinct real solutions, one real solution of multiplicity 2, or two conjugate complex solutions.

a What are the possible cases for a cubic polynomial $P(z)$ with real coefficients?
b Give an example for each case.

12 Let $P(z)$ be a quartic polynomial with real coefficients. Explain why it is not possible for the equation $P(z) = 0$ to have:

a exactly one real solution
b exactly three real solutions.

13 In this question, you will prove the conjugate root theorem.

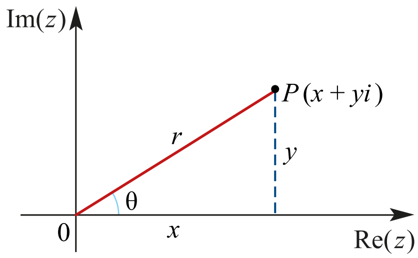
a If $z = a + bi$ and $w = c + di$ are complex numbers, prove that $\overline{z + w} = \overline{z} + \overline{w}$.
b If $z = a + bi$ and $w = c + di$ are complex numbers, prove that $\overline{zw} = \overline{z}\,\overline{w}$.
c If $z = a + bi$ is a complex number and c is a real number, prove that $\overline{cz} = c\,\overline{z}$.
d Let z be a complex number. Using mathematical induction, prove that $\overline{z^n} = \overline{z}^{\,n}$.
e Consider a polynomial equation $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, where all the coefficients are real. Let z be a solution of this equation. Show that $\overline{z}$ is also a solution. (Hint: Take the complex conjugate of both sides of the equation.)

18F Polar form of a complex number

Polar coordinates for points in the plane were introduced in Chapter 17. Similarly, each complex number may be described by an angle and a distance from the origin. In this section, we will see that this is a very useful way to describe complex numbers.

The diagram shows the point P corresponding to the complex number $z = x + yi$. We see that $x = r\cos\theta$ and $y = r\sin\theta$, and so we can write

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)\,i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$



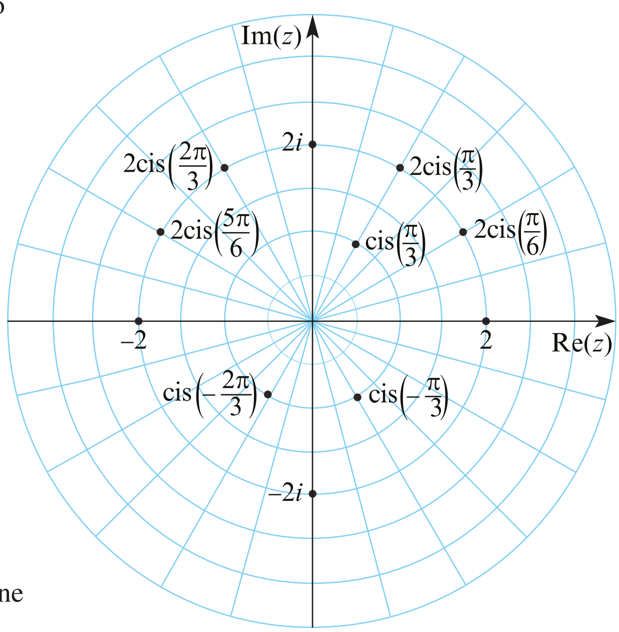
This is called the **polar form** of the complex number. The polar form is abbreviated to

$$z = r\operatorname{cis}\theta$$

- The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z.

Polar form for complex numbers is also called **modulus–argument form**.

This Argand diagram uses a polar grid with rays at intervals of $\frac{\pi}{12} = 15°$.



Non-uniqueness of polar form

Each complex number has more than one representation in polar form.

Since $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$, for all $n \in \mathbb{Z}$, we can write

$$z = r\operatorname{cis}\theta = r\operatorname{cis}(\theta + 2n\pi) \quad \text{for all } n \in \mathbb{Z}$$

The convention is to use the angle θ such that $-\pi < \theta \le \pi$. This value of θ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$. That is,

$$-\pi < \operatorname{Arg} z \le \pi$$

Note: The principal value of the argument is not defined for the complex number 0, since it can be written in polar form as $0\operatorname{cis}\theta$ for any angle θ.

Example 19

Express each of the following complex numbers in polar form:

a $z = 1 + \sqrt{3}i$

b $z = 2 - 2i$

Solution

a We have $x = 1$ and $y = \sqrt{3}$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1+3} \\ &= 2 \end{aligned}$$

The point $z = 1 + \sqrt{3}i$ is in the 1st quadrant, and so $0 < \theta < \dfrac{\pi}{2}$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$

and $\quad \sin\theta = \dfrac{y}{r} = \dfrac{\sqrt{3}}{2}$

Hence $\theta = \dfrac{\pi}{3}$ and therefore

$$\begin{aligned} z &= 1 + \sqrt{3}i \\ &= 2\operatorname{cis}\left(\frac{\pi}{3}\right) \end{aligned}$$

b We have $x = 2$ and $y = -2$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4+4} \\ &= 2\sqrt{2} \end{aligned}$$

The point $z = 2 - 2i$ is in the 4th quadrant, and so $-\dfrac{\pi}{2} < \theta < 0$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

and $\quad \sin\theta = \dfrac{y}{r} = -\dfrac{1}{\sqrt{2}}$

Hence $\theta = -\dfrac{\pi}{4}$ and therefore

$$\begin{aligned} z &= 2 - 2i \\ &= 2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right) \end{aligned}$$

Example 20

Express $z = 2\operatorname{cis}\left(-\dfrac{2\pi}{3}\right)$ in Cartesian form.

Solution

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\left(-\frac{2\pi}{3}\right) \\ &= 2 \times \left(-\frac{1}{2}\right) \\ &= -1 \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 2\sin\left(-\frac{2\pi}{3}\right) \\ &= 2 \times \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

Hence $z = 2\operatorname{cis}\left(-\dfrac{2\pi}{3}\right) = -1 - \sqrt{3}i$.

Multiplication and division in polar form

We can give a simple geometric interpretation of multiplication and division of complex numbers in polar form.

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

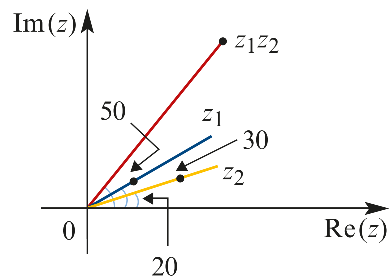
$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{(multiply the moduli and add the angles)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \qquad \text{(divide the moduli and subtract the angles)}$$

For example, if $z_1 = 2 \operatorname{cis} 30°$ and $z_2 = 4 \operatorname{cis} 20°$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= (2 \cdot 4) \operatorname{cis}(30° + 20°) \\ &= 8 \operatorname{cis} 50° \end{aligned}$$

You will prove this result in Exercise 18F.



When we multiply a complex number z by $r \operatorname{cis} \theta$, the effect on the point representing z in an Argand diagram is a dilation of factor r from the origin followed by a rotation about the origin by angle θ anticlockwise.

Example 21

Let $z_1 = 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ and $z_2 = 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$. Find the product $z_1 z_2$ and express in Cartesian form.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 6 \operatorname{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) \\ &= 6 \operatorname{cis}\left(\frac{4\pi}{3}\right) \end{aligned}$$

$$\therefore \quad z_1 z_2 = 6 \operatorname{cis}\left(-\frac{2\pi}{3}\right) \qquad \text{since } -\pi < \operatorname{Arg} z \leq \pi$$

Expressing this in Cartesian form, we find that

$$\begin{aligned} z_1 z_2 &= 6 \operatorname{cis}\left(-\frac{2\pi}{3}\right) \\ &= 6 \cos\left(-\frac{2\pi}{3}\right) + 6 \sin\left(-\frac{2\pi}{3}\right) i \\ &= 6\left(-\frac{1}{2}\right) + 6\left(-\frac{\sqrt{3}}{2}\right) i \\ &= -3 - 3\sqrt{3} i \end{aligned}$$

Example 22

Let $z_1 = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$ and $z_2 = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$. Find the quotient $\frac{z_1}{z_2}$ and express in Cartesian form.

Solution

We find that

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)\\ &= \frac{2}{4}\operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)\\ &= \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{3}\right)\end{aligned}$$

Expressing this in Cartesian form, we find that

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{1}{2}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)i\\ &= \frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)i\\ &= -\frac{1}{4}(1 - \sqrt{3}i)\end{aligned}$$

Example 23

The point $(2, 3)$ is rotated about the origin by angle $\frac{\pi}{6}$ anticlockwise. By multiplying two complex numbers, find the image of the point.

Solution

The point $(2, 3)$ corresponds to the complex number $z = 2 + 3i$.

To rotate z about the origin by $\frac{\pi}{6}$ anticlockwise, we multiply z by $\operatorname{cis}\left(\frac{\pi}{6}\right)$.

This gives

$$\begin{aligned}(2 + 3i)\operatorname{cis}\left(\frac{\pi}{6}\right) &= (2 + 3i)\left(\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)i\right)\\ &= (2 + 3i)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\\ &= \sqrt{3} + i + \frac{3\sqrt{3}}{2}i + \frac{3}{2}i^2\\ &= \frac{2\sqrt{3} - 3}{2} + \left(\frac{3\sqrt{3} + 2}{2}\right)i\end{aligned}$$

Therefore the image is the point $\left(\frac{2\sqrt{3} - 3}{2}, \frac{3\sqrt{3} + 2}{2}\right)$.

Summary 18F

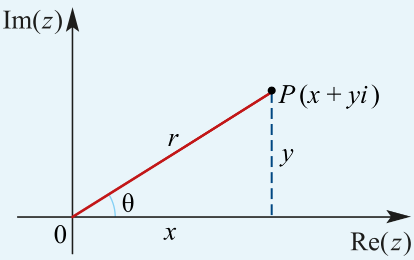
- **Polar form**

  A complex number in Cartesian form

  $$z = x + yi$$

  can be written in polar form as

  $$z = r(\cos\theta + i\sin\theta)$$
  $$= r\operatorname{cis}\theta$$



  - The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
  - The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z.

- The polar form of a complex number is not unique. For a non-zero complex number z, the argument θ of z such that $-\pi < \theta \le \pi$ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$.

- **Multiplication and division in polar form**

  If $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$, then

  $$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

Skill-sheet

Exercise 18F

Example 19

1 Express each of the following in polar form $r\operatorname{cis}\theta$ with $-\pi < \theta \le \pi$:

a $1 + \sqrt{3}i$ **b** $1 - i$ **c** $-2\sqrt{3} + 2i$

d $-4 - 4i$ **e** $12 - 12\sqrt{3}i$ **f** $-\frac{1}{2} + \frac{1}{2}i$

Example 20

2 Express each of the following in the form $x + yi$:

a $3\operatorname{cis}\left(\frac{\pi}{2}\right)$ **b** $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{3}\right)$ **c** $2\operatorname{cis}\left(\frac{\pi}{6}\right)$ **d** $5\operatorname{cis}\left(\frac{3\pi}{4}\right)$

e $12\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **f** $3\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **g** $5\operatorname{cis}\left(\frac{4\pi}{3}\right)$ **h** $5\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

3 Simplify the following and express the answers in Cartesian form:

Example 21

a $2\operatorname{cis}\left(\frac{\pi}{6}\right)\cdot 3\operatorname{cis}\left(\frac{\pi}{12}\right)$ **b** $4\operatorname{cis}\left(\frac{\pi}{12}\right)\cdot 3\operatorname{cis}\left(\frac{\pi}{4}\right)$ **c** $\operatorname{cis}\left(\frac{\pi}{4}\right)\cdot 5\operatorname{cis}\left(\frac{5\pi}{12}\right)$

d $12\operatorname{cis}\left(-\frac{\pi}{3}\right)\cdot 3\operatorname{cis}\left(\frac{2\pi}{3}\right)$ **e** $12\operatorname{cis}\left(\frac{5\pi}{6}\right)\cdot 3\operatorname{cis}\left(\frac{\pi}{2}\right)$ **f** $(\sqrt{2}\operatorname{cis}\pi)\cdot\sqrt{3}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

Example 22

g $\dfrac{10\operatorname{cis}\left(\frac{\pi}{4}\right)}{5\operatorname{cis}\left(\frac{\pi}{12}\right)}$ **h** $\dfrac{12\operatorname{cis}\left(-\frac{\pi}{3}\right)}{3\operatorname{cis}\left(\frac{2\pi}{3}\right)}$ **i** $\dfrac{12\sqrt{8}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{3\sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)}$ **j** $\dfrac{20\operatorname{cis}\left(-\frac{\pi}{6}\right)}{8\operatorname{cis}\left(\frac{5\pi}{6}\right)}$

Example 23

4 For each of the following, multiply two complex numbers to find the image of the point under the rotation about the origin:

a $(5, 2)$ is rotated by $\frac{\pi}{3}$ anticlockwise

b $(3, 2)$ is rotated by $\frac{\pi}{4}$ clockwise

c (x, y) is rotated by θ anticlockwise

5 Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$. Use the compound angle formulas for sine and cosine to prove that $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$.

18G Sketching subsets of the complex plane

We have already seen how complex numbers can be plotted on an Argand diagram (also called the **complex plane**). In this section, we treat complex numbers as points in the complex plane, and therefore we can illustrate sets of complex numbers.

Distance in the complex plane

Recall that, if $z = x + yi$ is a complex number, then its modulus

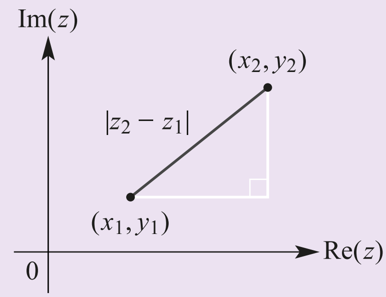
$$|z| = \sqrt{x^2 + y^2}$$

is equal to its distance from the origin in the complex plane. More generally:

Distance between two complex numbers

For complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, the distance between z_1 and z_2 in the complex plane is equal to

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 24

Find the distance between $z_1 = -1 + 4i$ and $z_2 = 3 + 2i$ in the complex plane.

Solution

The distance can be found by evaluating

$$\begin{aligned}|z_2 - z_1| &= |(3 + 2i) - (-1 + 4i)| \\ &= |4 - 2i| \\ &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5}\end{aligned}$$

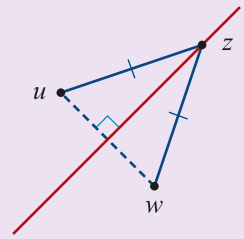
Lines in the complex plane

Equation of a line in the complex plane

Let u and w be fixed complex numbers. Then the equation

$$|z - u| = |z - w|$$

defines the set of all points z that are equal distance from u and w. This set is a straight line.



Example 25

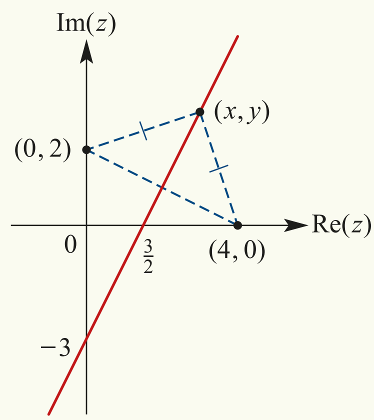
Find the Cartesian equation for the set of points z such that $|z - 2i| = |z - 4|$.

Solution

Note that the point z is equidistant from $2i$ and 4. Therefore the set of points is the straight line that is the perpendicular bisector of the line segment between (0, 2) and (4, 0).

Letting $z = x + yi$, we can find the Cartesian equation algebraically as follows:

$$\begin{aligned}
|z - 2i| &= |z - 4| \\
|x + yi - 2i| &= |x + yi - 4| \\
|x + (y - 2)i| &= |(x - 4) + yi| \\
\sqrt{x^2 + (y-2)^2} &= \sqrt{(x-4)^2 + y^2} \\
x^2 + y^2 - 4y + 4 &= x^2 - 8x + 16 + y^2 \\
-4y + 4 &= -8x + 16 \\
y &= 2x - 3
\end{aligned}$$



The set of points is the straight line with Cartesian equation $y = 2x - 3$.

Rays in the complex plane

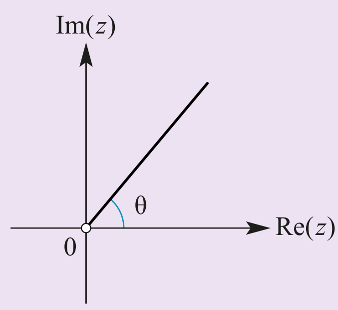
Equation of a ray starting at the origin

Let θ be a fixed angle. Then the equation

$$\operatorname{Arg} z = \theta$$

defines a ray extending from the origin at an angle of θ measured anticlockwise from the horizontal axis.

Note: The origin is not included in the set of points, as the principal argument is not defined for the complex number 0.

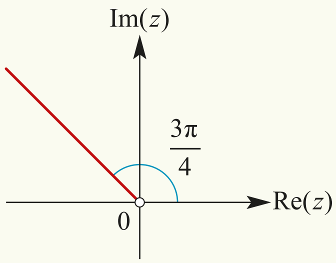


Example 26

Sketch the subset of the complex plane defined by $\operatorname{Arg} z = \frac{3\pi}{4}$.

Solution

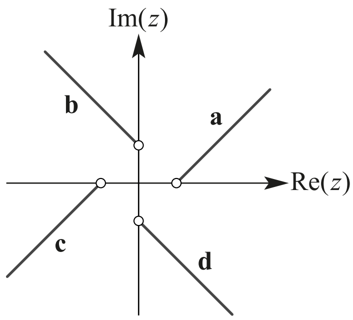
The equation defines the set of complex numbers with a principal argument of $\frac{3\pi}{4}$.



By applying a translation, we can describe rays that do not start at the origin. The diagram on the right shows the rays with equations:

a $\operatorname{Arg}(z - 1) = \frac{\pi}{4}$ **b** $\operatorname{Arg}(z - i) = \frac{3\pi}{4}$

c $\operatorname{Arg}(z + 1) = -\frac{3\pi}{4}$ **d** $\operatorname{Arg}(z + i) = -\frac{\pi}{4}$



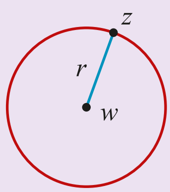
Circles in the complex plane

Equation of a circle in the complex plane

Let w be a fixed complex number and let $r > 0$. Then the equation

$$|z - w| = r$$

defines a circle with centre w and radius r.



Example 27

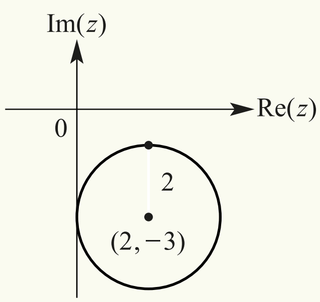
Find the Cartesian equation for the set of points z such that $|z - (2 - 3i)| = 2$.

Solution

Note that the point z is a distance of 2 units from $2 - 3i$. Therefore the set of points is the circle of radius 2 centred at $(2, -3)$.

Letting $z = x + yi$, we can find the Cartesian equation algebraically as follows:

$$\begin{aligned} |z - (2 - 3i)| &= 2 \\ |x + yi - (2 - 3i)| &= 2 \\ |(x - 2) + (y + 3)i| &= 2 \\ \sqrt{(x - 2)^2 + (y + 3)^2} &= 2 \\ (x - 2)^2 + (y + 3)^2 &= 4 \end{aligned}$$



Example 28

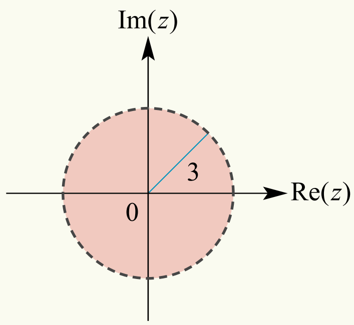
Sketch the set of complex numbers z described by each rule:

a $|z| < 3$

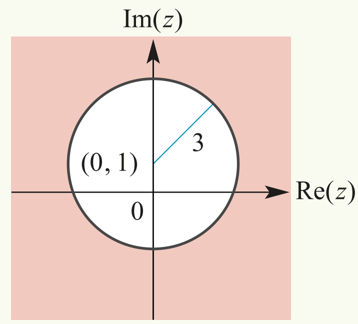
b $|z - i| \geq 3$

Solution

a If $|z| < 3$, then z is less than 3 units from the origin. Therefore z is inside the circle of radius 3 centred at the origin.



b If $|z - i| \geq 3$, then z is at least 3 units from i. Therefore z lies on or outside the circle of radius 3 centred at $(0, 1)$.



Other subsets of the complex plane

Sometimes the subset of the complex plane defined by a particular rule is not obvious until we find a Cartesian description for the set.

Example 29

Consider the set of points z in the complex plane such that

$$2|z - 2| = |z - \bar{z} + 2i|$$

Find the Cartesian equation that describes this set.

Solution

Let $z = x + yi$. Then

$$\begin{aligned} 2|z - 2| &= |z - \bar{z} + 2i| \\ 2|x + yi - 2| &= |x + yi - (x - yi) + 2i| \\ 2|(x - 2) + yi| &= |(2y + 2)i| \\ |(x - 2) + yi| &= |(y + 1)i| \\ \sqrt{(x - 2)^2 + y^2} &= \sqrt{(y + 1)^2} \\ (x - 2)^2 + y^2 &= y^2 + 2y + 1 \\ y &= \tfrac{1}{2}(x - 2)^2 - \tfrac{1}{2} \end{aligned}$$

This set of points is a parabola in the complex plane.

In the next example, we look at combining regions of the complex plane using union and intersection.

Example 30

Define sets S and T of complex numbers by

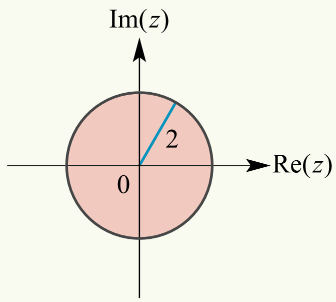
$$S = \{ z : |z| \leq 2 \} \quad \text{and} \quad T = \left\{ z : -\frac{\pi}{4} \leq \operatorname{Arg} z \leq \frac{\pi}{4} \right\}$$

Sketch the following regions of the complex plane:

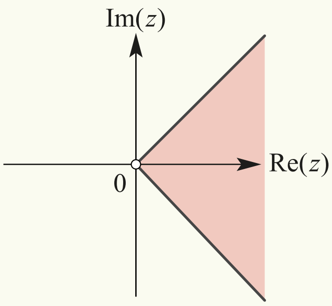
a S **b** T **c** $S \cap T$ **d** $S \cup T$

Solution

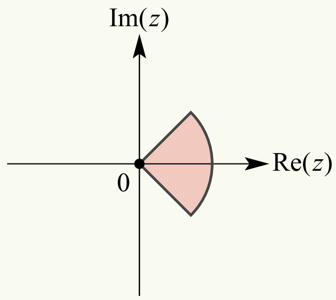
a Region S is the set of points at most 2 units from the origin. This is a disc of radius 2 that includes its boundary.



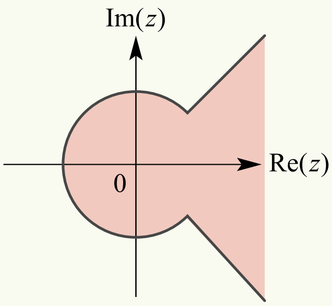
b Region T is the set of points with principal argument between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. So T is the wedge shown below.



c The intersection consists of the points in common to S and T.



d The union consists of all points in S or in T (or both).



Summary 18G

■ **Distance in the complex plane**

- For $z \in \mathbb{C}$, the distance of z from the origin is equal to $|z|$.
- For $z_1, z_2 \in \mathbb{C}$, the distance between z_1 and z_2 is equal to $|z_2 - z_1|$.

■ **Subsets of the complex plane**

- The equation $|z - w| = r$ defines the circle with centre w and radius r.
- The equation $|z - u| = |z - w|$ defines a line.
- The equation $\operatorname{Arg} z = \theta$ defines the ray extending from the origin at angle θ. (The origin is not included.)

Skill-sheet

Exercise 18G

Example 24

1 For each of the following, find the distance between z and w in the complex plane:

a $z = 1 + i,\ w = 4 + 5i$ **b** $z = 3 - 4i,\ w = 2 - 3i$

c $z = 4 - 6i,\ w = -1 + 6i$ **d** $z = 2,\ w = -2i$

e $z = 10i,\ w = -3i$ **f** $z = \sqrt{2} + i,\ w = 2i$

Example 25

2 For each of the following, find the Cartesian equation of the line described by the rule and sketch the line on an Argand diagram:

a $\text{Re}(z) = 2$ **b** $\text{Im}(z) = -1$

c $\text{Im}(z) = 3\,\text{Re}(z)$ **d** $3\,\text{Re}(z) + 4\,\text{Im}(z) = 12$

e $|z - 1| = |z - i|$ **f** $|z - (1 + i)| = |z + 1|$

g $z + \bar{z} = 6$ **h** $z - \bar{z} = 4i$

Example 26

3 Sketch the subsets of the complex plane described by the following rules:

a $\text{Arg}\, z = \dfrac{\pi}{4}$ **b** $\text{Arg}\, z = -\dfrac{5\pi}{6}$ **c** $0 \le \text{Arg}\, z \le \dfrac{\pi}{2}$

d $\text{Arg}(z - 1) = \dfrac{3\pi}{4}$ **e** $\text{Arg}(z + i) = -\dfrac{\pi}{4}$ **f** $\text{Arg}(z - 1 + i) = \pi$

Example 27

4 Consider the set of points $z \in \mathbb{C}$ for which $|z - 2| = 1$. By letting $z = x + yi$, show algebraically that this corresponds to the circle with equation $(x - 2)^2 + y^2 = 1$.

5 Consider the set of points $z \in \mathbb{C}$ for which $|z| = |z - 2 - 2i|$. By letting $z = x + yi$, show algebraically that this corresponds to the straight line with equation $y = 2 - x$.

6 Sketch the set of points $z \in \mathbb{C}$ that are distance 2 from the point $w = 2 + 2i$.

Example 28

7 By interpreting $|z - w|$ as the distance from z to w, sketch the set of complex numbers z described by each rule. (You do *not* have to find the Cartesian equation algebraically.)

a $|z| = 3$ **b** $|z| \le 2$ **c** $|z| > 2$

d $|z - 1| = 2$ **e** $|z - i| < 2$ **f** $|z + 2| \ge 3$

g $|z + 2i| < 2$ **h** $|z - (1 + i)| > 3$ **i** $|z + 1 - 2i| \le 3$

Example 30

8 Define three sets of complex numbers by

$$R = \{\, z : |z| \le 3 \,\}, \quad S = \{\, z : \text{Re}(z) \ge 0 \,\} \quad \text{and} \quad T = \left\{ z : \frac{\pi}{4} < \text{Arg}\, z \le \frac{3\pi}{4} \right\}$$

Sketch each of the following regions of the complex plane:

a R **b** S **c** T **d** $R \cap S$

e $R \cap T$ **f** $S \cap T$ **g** $R \cup S$ **h** $R \cap S \cap T$

9 Define sets S and T of complex numbers by

$$S = \{ z : z + \bar{z} \le |z|^2 \} \quad \text{and} \quad T = \left\{ z : \frac{\pi}{4} \le \operatorname{Arg} z \le \frac{\pi}{2} \right\}$$

a By letting $z = x + yi$, find a Cartesian description for the set S.

b Sketch S in the complex plane.

c Sketch T in the complex plane.

d Sketch $S \cap T$.

10 Show that the equation $|z + 2i| = |2iz - 1|$ defines a circle in the complex plane. Find its centre and radius.

Example 29 **11** Consider the set of points z in the complex plane such that

$$2|z - i| = |z + \bar{z} + 2|$$

Find the Cartesian equation that describes this set.

12 Define the set $S = \{ z \in \mathbb{C} : |z + 16| = 4|z + 1| \}$.

a Prove that $S = \{ z \in \mathbb{C} : |z| = 4 \}$.

b Hence, sketch the set S in the complex plane.

13 Show that the equation $|z| = 3|z + 8|$ defines a circle in the complex plane. Find its centre and radius.

14 Let $S = \{ z \in \mathbb{C} : |z - 1| = 1 \}$.

a Sketch the set S in the complex plane.

b Hence, sketch each of the following subsets of the complex plane:

i $T = \{ z + 1 : z \in S \}$

ii $U = \{ z + i : z \in S \}$

iii $V = \{ 2z : z \in S \}$

iv $W = \{ iz : z \in S \}$

Hint: Multiplication by i corresponds to a rotation.

Review

Chapter summary

Assign-ment

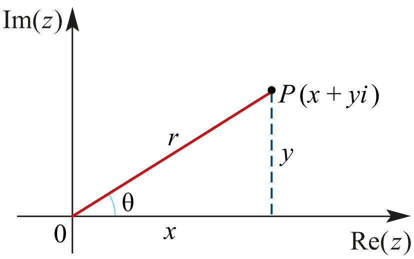
Nrich

Complex numbers

- The imaginary number i has the property $i^2 = -1$.
- The set of **complex numbers** is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\text{Re}(z) = a$
  - the **imaginary part** of z is $\text{Im}(z) = b$.
- Complex numbers z_1 and z_2 are equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of z, denoted by $|z|$, is the distance from the origin to the point representing z in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The complex number $z = x + yi$ can be expressed in **polar form** as

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\operatorname{cis}\theta$$

where $r = |z| = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, $y = r\sin\theta$.
This is also called modulus–argument form.



- The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z.
- For a non-zero complex number z, the argument θ of z such that $-\pi < \theta \le \pi$ is called the **principal value** of the argument of z and is denoted by $\text{Arg}\, z$.

Operations on complex numbers

- The **complex conjugate** of $z = a + bi$ is given by $\bar{z} = a - bi$. Note that $z\bar{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

- Multiplication and division in polar form:
  Let $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$. Then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

Polynomial equations over the complex numbers

- Factor theorem A polynomial $P(z)$ has $z - \alpha$ as a factor if and only if $P(\alpha) = 0$.
- Fundamental theorem of algebra For $n \ge 1$, every polynomial of degree n can be expressed as a product of n linear factors over the complex numbers. Therefore every polynomial equation of degree n has n solutions (counting multiplicity).
- Conjugate root theorem Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

Subsets of the complex plane

- For $z_1, z_2 \in \mathbb{C}$, the distance between z_1 and z_2 is equal to $|z_2 - z_1|$.
- The equation $|z - w| = r$ defines the circle with centre w and radius r.
- The equation $|z - u| = |z - w|$ defines a line.
- The equation $\operatorname{Arg} z = \theta$ defines the ray extending from the origin at angle θ. (The origin is not included.)

Technology-free questions

1 For $z_1 = m + ni$ and $z_2 = p + qi$, express each of the following in the form $a + bi$:

a $2z_1 + 3z_2$ **b** $\overline{z_2}$ **c** $z_1\overline{z_2}$

d $\dfrac{z_1}{z_2}$ **e** $z_1 + \overline{z_1}$ **f** $(z_1 + z_2)(z_1 - z_2)$

g $\dfrac{1}{z_1}$ **h** $\dfrac{z_2}{z_1}$ **i** $\dfrac{3z_1}{z_2}$

2 Let $z = 1 - \sqrt{3}i$. For each of the following, express in the form $a + bi$ and mark on an Argand diagram:

a z **b** z^2 **c** z^3 **d** $\dfrac{1}{z}$ **e** $\overline{z}$ **f** $\dfrac{1}{\overline{z}}$

3 Write each of the following in polar form:

a $1 + i$ **b** $1 - \sqrt{3}i$ **c** $2\sqrt{3} + i$

d $3\sqrt{2} + 3\sqrt{2}i$ **e** $-3\sqrt{2} - 3\sqrt{2}i$ **f** $\sqrt{3} - i$

4 Write each of the following in Cartesian form:

a $-2\operatorname{cis}\left(\dfrac{\pi}{3}\right)$ **b** $3\operatorname{cis}\left(\dfrac{\pi}{4}\right)$ **c** $3\operatorname{cis}\left(\dfrac{3\pi}{4}\right)$

d $-3\operatorname{cis}\left(-\dfrac{3\pi}{4}\right)$ **e** $3\operatorname{cis}\left(-\dfrac{5\pi}{6}\right)$ **f** $\sqrt{2}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

5 Let $z = \operatorname{cis}\left(\dfrac{\pi}{3}\right)$. On an Argand diagram, carefully plot:

a z^2 **b** $\overline{z}$ **c** $\dfrac{1}{z}$ **d** $\operatorname{cis}\left(\dfrac{2\pi}{3}\right)$

6 Let $z = \operatorname{cis}\left(\dfrac{\pi}{4}\right)$. On an Argand diagram, carefully plot:

a iz **b** $\overline{z}$ **c** $\dfrac{1}{z}$ **d** $-iz$

7 Solve each of the following quadratic equations over $\mathbb{C}$:

a $z^2 + 4 = 0$ **b** $3z^2 + 9 = 0$

c $z^2 + 4z + 5 = 0$ **d** $2z^2 - 3z + 4 = 0$

Review

8 Show that $z = 2$ is a solution of the equation $z^3 - 2z^2 + 4z - 8 = 0$, and then find the other two solutions.

9 **a** Show that $z = i$ is a solution of the equation $12z^3 - 11z^2 + 12z - 11 = 0$.

b Hence, find the other two solutions.

c Now consider the equation $nz^3 - (n-1)z^2 + nz - (n-1) = 0$, where n is an integer. Show that there is only one value of n such that the equation has an integer solution.

10 Sketch the following subsets of the complex plane:

a $\{ z : |z - 1| \le 3 \}$ **b** $\{ z : z + \bar{z} = 4 \}$ **c** $\left\{ z : \operatorname{Arg} z = -\frac{3\pi}{4} \right\}$

11 **a** Let $z = x + yi$. Express z^2 in Cartesian form.

b Sketch the subset of the complex plane defined by $\operatorname{Re}(z^2) = 1$.

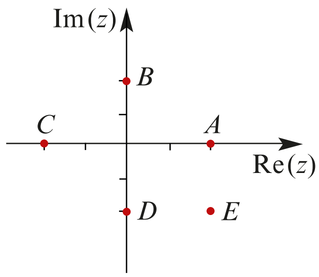
c Sketch the subset of the complex plane defined by $\operatorname{Im}(z^2) = 1$.

Multiple-choice questions

1 If $u = 1 + i$, then $\dfrac{1}{2-u}$ is equal to

A $-\frac{1}{2} - \frac{1}{2}i$ **B** $\frac{1}{5} + \frac{2}{5}i$ **C** $\frac{1}{2} + \frac{1}{2}i$ **D** $-\frac{1}{2} + \frac{1}{5}i$ **E** $1 + 5i$

2 The point C on the Argand diagram represents the complex number z. Which point represents the complex number $i \times z$?



A A **B** B **C** C

D D **E** E

3 If $|z| = 5$, then $\left|\dfrac{1}{z}\right| =$

A $\frac{1}{\sqrt{5}}$ **B** $-\frac{1}{\sqrt{5}}$ **C** $\frac{1}{5}$ **D** $-\frac{1}{5}$ **E** $\sqrt{5}$

4 If $(x + yi)^2 = -32i$ for real values of x and y, then

A $x = 4, y = 4$ **B** $x = -4, y = 4$

C $x = 4, y = -4$ **D** $x = 4, y = -4$ or $x = -4, y = 4$

E $x = 4, y = 4$ or $x = -4, y = -4$

5 The quadratic polynomial $z^2 + 6z + 10$ can be factorised over $\mathbb{C}$ as

A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$ **C** $(z + 3 + i)(z - 3 + i)$

D $(z + 3 - i)(z + 3 + i)$ **E** $(z + 3 + i)(z - 3 - i)$

6 Let $z = \frac{1}{1-i}$. If $r = |z|$ and $\theta = \text{Arg}\, z$, then

A $r = 2$ and $\theta = \frac{\pi}{4}$ **B** $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ **C** $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$

D $r = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$ **E** $r = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$

7 The solution of the equation $\frac{z-2i}{z-(3-2i)} = 2$, where $z \in \mathbb{C}$, is

A $z = 6 + 2i$ **B** $z = 6 - 2i$ **C** $z = -6 - 6i$ **D** $z = 6 - 6i$ **E** $z = -6 + 2i$

8 Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z^2(1+i) = 2 - 2i$, then z could be equal to

A $\sqrt{2}i$ **B** $-\sqrt{2}i$ **C** $-1 - i$ **D** $-1 + i$ **E** $\sqrt{-2}$

9 The value of the discriminant for the quadratic expression $(2+2i)z^2 + 8iz - 4(1-i)$ is

A -32 **B** 0 **C** 64 **D** 32 **E** -64

10 If $\text{Arg}(ai + 1) = \frac{\pi}{6}$, then the real number a is

A $\sqrt{3}$ **B** $-\sqrt{3}$ **C** 1 **D** $\frac{1}{\sqrt{3}}$ **E** $-\frac{1}{\sqrt{3}}$

11 Let $b, c \in \mathbb{R}$. If $z = 3 + 4i$ is a solution of the equation $z^2 + bz + c = 0$, then

A $b = -6,\ c = 25$ **B** $b = 6,\ c = 25$ **C** $b = -6,\ c = -25$

D $b = 6,\ c = -25$ **E** $b = 25,\ c = -6$

Extended-response questions

1 **a** Find the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$.

b **i** Plot the two solutions from part **a** on an Argand diagram.

ii Find the equation of the circle, with centre the origin, which passes through these two points.

iii Find the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.

2 Let z be a complex number with $|z| = 6$. Let A be the point representing z and let B be the point representing $(1+i)z$.

a Find:

i $|(1+i)z|$ **ii** $|(1+i)z - z|$

b Prove that OAB is a right-angled isosceles triangle.

3 Let $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

a On an Argand diagram, the points O, A, Z, P and Q represent the complex numbers 0, 1, z, $1 + z$ and $1 - z$ respectively. Show these points on a diagram.

b Prove that the magnitude of $\angle POQ$ is $\frac{\pi}{2}$. Find the ratio $\frac{OP}{OQ}$.

4 Let z_1 and z_2 be two complex numbers. Prove the following:

a $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$

b $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$

c $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

State a geometric theorem from the result of **c**.

5 Let z_1 and z_2 be two complex numbers.

a Prove the following:

i $\overline{\overline{z_1}z_2} = z_1\overline{z_2}$

ii $z_1\overline{z_2} + \overline{z_1}z_2$ is a real number

iii $z_1\overline{z_2} - \overline{z_1}z_2$ is an imaginary number

iv $(z_1\overline{z_2} + \overline{z_1}z_2)^2 - (z_1\overline{z_2} - \overline{z_1}z_2)^2 = 4|z_1z_2|^2$

b Use the results from part **a** and Question 4 to prove that $|z_1 + z_2| \le |z_1| + |z_2|$.

Hint: Show that $(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 \ge 0$.

c Hence prove that $|z_1 - z_2| \ge |z_1| - |z_2|$.

6 Assume that $|z| = 1$ and that the argument of z is θ, where $0 < \theta < \pi$. Find the modulus and argument of:

a $z + 1$ **b** $z - 1$ **c** $\dfrac{z-1}{z+1}$

7 The quadratic expression $ax^2 + bx + c$ has real coefficients.

a Find the discriminant of $ax^2 + bx + c$.

b Find the condition in terms of a, b and c for which the equation $ax^2 + bx + c = 0$ has no real solutions.

c If this condition is fulfilled, let z_1 and z_2 be the complex solutions of the equation and let P_1 and P_2 the corresponding points on an Argand diagram.

i Find $z_1 + z_2$ and $|z_1|$ in terms of a, b and c.

ii Find $\cos(\angle P_1OP_2)$ in terms of a, b and c.

8 Let z_1 and z_2 be the solutions of the quadratic equation $z^2 + z + 1 = 0$.

a Find z_1 and z_2.

b Prove that $z_1 = z_2^2$ and $z_2 = z_1^2$.

c Find the modulus and the principal value of the argument of z_1 and z_2.

d Let P_1 and P_2 be the points on an Argand diagram corresponding to z_1 and z_2. Find the area of triangle P_1OP_2.

19

Revision of Chapters 15–18

19A Technology-free questions

1 Given that $\sin A = \frac{3}{5}$, where A is acute, and that $\cos B = -\frac{1}{2}$, where B is obtuse, find the exact values of:

a $\sec A$ **b** $\cot A$ **c** $\cot B$ **d** $\operatorname{cosec} B$

2 Given that $\cos A = \frac{1}{3}$, find the possible values of $\cos\left(\frac{A}{2}\right)$.

3 Prove the identity $\dfrac{1}{1 + \sin A} + \dfrac{1}{1 - \sin A} = 2\sec^2 A$.

4 If $w = 3 + 2i$ and $z = 3 - 2i$, express each of the following in the form $a + bi$, where a and b are real numbers:

a $w + z$ **b** $w - z$ **c** wz **d** $w^2 + z^2$
e $(w + z)^2$ **f** $(w - z)^2$ **g** $w^2 - z^2$ **h** $(w - z)(w + z)$

5 If $w = 1 - 2i$ and $z = 2 - 3i$, express each of the following in the form $a + bi$, where a and b are real numbers:

a $w + z$ **b** $w - z$ **c** wz **d** $\dfrac{w}{z}$ **e** iw **f** $\dfrac{i}{w}$
g $\dfrac{w}{i}$ **h** $\dfrac{z}{w}$ **i** $\dfrac{w}{w + z}$ **j** $(1 + i)w$ **k** $\dfrac{w}{1 + i}$ **l** w^2

6 Write each polynomial as a product of linear factors:

a $z^2 + 49$ **b** $z^2 - 2z + 10$ **c** $9z^2 - 6z + 5$ **d** $4z^2 + 12z + 13$

7 **a** Find the two square roots of $3 - 4i$ by solving $(x + yi)^2 = 3 - 4i$ for $x, y \in \mathbb{R}$.
b Use the quadratic formula to solve $(2 - i)z^2 + (4 + 3i)z + (-1 + 3i) = 0$ for z.

8 Let $a, b, c \in \mathbb{R}$ and consider the equation $z^3 + az^2 + bz + c = 0$. Given that $z = -1 + i$ is a solution and that the sum of the solutions is 4, find the values of a, b and c.

9 Let $S = \{ z \in \mathbb{C} : |z - (1 + i)| \le 1 \}$ and $T = \left\{ z \in \mathbb{C} : 0 \le \operatorname{Arg} z \le \frac{\pi}{3} \right\}$.

a Sketch S in the complex plane.

b Sketch T in the complex plane.

c Sketch the region $S \cap T$.

10 Sketch the graphs of the following functions over their implied domains:

a $y = \cos^{-1}(x - 1) - \pi$ **b** $y = -\sin^{-1}(2x)$ **c** $y = \tan^{-1}(-x) + \frac{\pi}{4}$

11 **a** Sketch the graphs of $y = x^2 - 4$ and $y = |x| + 2$ on the same set of axes.

b Find the coordinates of the points of intersection.

12 Let $f(x) = x(x - 1)$. Sketch the graph of each of the following functions:

a $y = f(x)$ **b** $y = |f(x)|$ **c** $y = f(|x|)$ **d** $y = |f(|x|)|$

13 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes:

a $f(x) = x^2 + 3x + 2$ **b** $f(x) = (x - 1)^2 + 1$

c $f(x) = \sin(x) + 1,\ x \in [0, 2\pi]$ **d** $f(x) = \cos(x) + 2,\ x \in [0, 2\pi]$

14 Sketch the graphs of the following functions over the domain $[-\pi, \pi]$:

a $f(x) = 2\sec(x) + 1$ **b** $f(x) = -\operatorname{cosec}(2x)$ **c** $f(x) = 3\cot(x + \pi)$

15 Sketch the graph of each ellipse and find the coordinates of its axis intercepts:

a $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$ **b** $\frac{(x+1)^2}{2^2} + \frac{(y-2)^2}{3^2} = 1$

16 Sketch the graph of each hyperbola and write down the equations of its asymptotes:

a $x^2 - \frac{y^2}{3^2} = 1$ **b** $\frac{(y+1)^2}{4^2} - \frac{(x-2)^2}{2^2} = 1$

17 A point $P(x, y)$ moves so that it is equidistant from points $A(2, 2)$ and $B(3, 4)$. Find the locus of the point P.

18 A point $P(x, y)$ moves so that its distance from the point $K(0, 1)$ is half its distance from the line $x = -3$. Find its locus.

19 Each point $P(x, y)$ on a curve has the following property: the distance to $P(x, y)$ from the point $F(0, 1)$ is the same as the shortest distance to $P(x, y)$ from the line $y = -3$. Find the equation of the curve.

20 Find the Cartesian equation corresponding to each pair of parametric equations:

a $x = 2t + 1$ and $y = 2 - 3t$ **b** $x = \cos(2t)$ and $y = \sin(2t)$

c $x = 2\cos t + 2$ and $y = 3\sin t + 3$ **d** $x = 2\tan t$ and $y = 3\sec t$

21 A curve has parametric equations

$$x = t - 1 \quad \text{and} \quad y = 1 - 2t^2 \qquad \text{for } 0 \le t \le 2$$

a Find the curve's Cartesian equation.
b What is the domain of the curve?
c What is the range of the curve?
d Sketch the graph of the curve.

22 Convert the polar coordinates $\left[2, \frac{7\pi}{6}\right]$ into Cartesian coordinates.

23 A point P has Cartesian coordinates $(2, -2)$. Find two representations of P using polar coordinates, one with $r > 0$ and the other with $r < 0$.

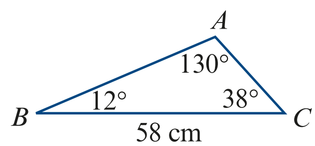
24 Convert the following polar equations into Cartesian equations:

a $r = 5$ **b** $\theta = \frac{\pi}{3}$ **c** $r = \frac{3}{\sin\theta}$ **d** $r = \frac{2}{3\sin\theta + 4\cos\theta}$ **e** $r^2 = \frac{1}{\sin(2\theta)}$

25 **a** Sketch the circle with equation $x^2 + (y-2)^2 = 2^2$.
b Show that this circle has polar equation $r = 4\sin\theta$.

19B Multiple-choice questions

1 Which one of the following gives the correct value for c?



A $\frac{58\cos 38°}{\cos 130°}$ **B** $\frac{58\sin 38°}{\sin 130°}$ **C** $58\sin 38°$
D $\frac{58\cos 130°}{\cos 38°}$ **E** $\frac{58\sin 130°}{\sin 38°}$

2 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, then $\sin(A - B)$ is given by

A $\frac{140}{221}$ **B** $-\frac{21}{221}$ **C** $\frac{34\,209}{23\,560}$ **D** $-\frac{107}{140}$ **E** $\frac{107}{140}$

3 In triangle ABC, $c = 5$, $b = 9$ and $A = 43°$. Which of the following statements are correct?

I With the information given, we can find the area of triangle ABC.
II With the information given, we can find angle B.
III With the information given, we can find side a.

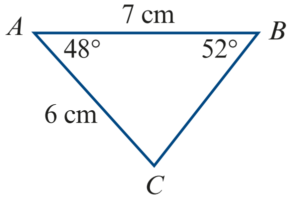
A I and II only **B** I and III only **C** II and III only
D I, II and III **E** none of these

4 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, then $\tan(A + B)$ is given by

A $\frac{140}{221}$ **B** $-\frac{21}{221}$ **C** $\frac{34\,209}{23\,560}$ **D** $-\frac{171}{140}$ **E** $\frac{171}{140}$

Revision

5 Which one of the following expressions will give the area of triangle ABC?



A $\frac{1}{2} \times 6 \times 7 \sin 48°$ **B** $\frac{1}{2} \times 6 \times 7 \cos 48°$

C $\frac{1}{2} \times 6 \times 7 \sin 52°$ **D** $\frac{1}{2} \times 6 \times 7 \cos 52°$

E $\frac{1}{2} \times 6 \times 7 \tan 48°$

6 If $\cos\theta = c$ and θ is acute, then $\cot\theta$ can be expressed in terms of c as

A $c\sqrt{1-c^2}$ **B** $\sqrt{1-c^2}$ **C** $\frac{1}{\sqrt{1-c^2}}$ **D** $\frac{c}{\sqrt{1-c^2}}$ **E** $2c\sqrt{1-c^2}$

7 A child on a swing travels through an arc of length 3 m. If the ropes of the swing are 4 m in length, then the angle which the arc makes at the top of the swing (where the swing is attached to the support) is best approximated by

A 135° **B** 75° **C** 12° **D** 75^c **E** 43°

8 If $A + B = \frac{\pi}{2}$, then the value of $\cos A \, \cos B - \sin A \, \sin B$ is

A −2 **B** 1 **C** −1 **D** 0 **E** 2

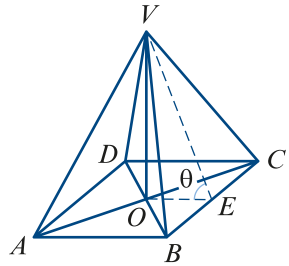
9 Given that $\sin A = \frac{\sqrt{5}}{3}$ and that A is obtuse, the value of $\sin(2A)$ is

A $\frac{16\sqrt{5}}{243}$ **B** $-\frac{1}{9}$ **C** $-\frac{8\sqrt{5}}{27}$ **D** $\frac{5}{9}$ **E** $-\frac{4\sqrt{5}}{9}$

10 Correct to two decimal places, the area of a sector with an included angle of 60° in a circle of diameter 10 cm is

A 104.72 cm^2 **B** 52.36 cm^2 **C** 13.09 cm^2 **D** 26.16 cm^2 **E** 750 cm^2

11 $VABCD$ is a right square pyramid with base length 80 mm and perpendicular height 100 mm. The angle θ between a sloping face and the base $ABCD$, to the nearest degree, is



A 22° **B** 29° **C** 51°

D 61° **E** 68°

12 If $\cos\theta = c$ and θ is acute, then $\sin(2\theta)$ can be expressed in terms of c as

A $c\sqrt{1-c^2}$ **B** $\sqrt{1-c^2}$ **C** $\frac{1}{\sqrt{1-c^2}}$ **D** $\frac{c}{\sqrt{1-c^2}}$ **E** $2c\sqrt{1-c^2}$

13 The expression $\cos(3x) + \cos(x)$ is equivalent to

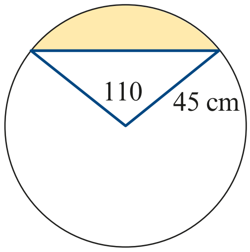
A $\cos(4x)$ **B** $2\cos(4x)$ **C** $2\cos(3x)\cos(x)$

D $2\cos(2x)\cos(x)$ **E** $2\sin(2x)\cos(x)$

14 The angles between 0° and 360° which satisfy the equation $4\cos x - 3\sin x = 1$, given correct to two decimal places, are

A 53.13° and 126.87° **B** 48.41° and 205.33° **C** 41.59° and 244.67°
D 131.59° and 334.67° **E** 154.67° and 311.59°

15 The area of the shaded region in the diagram is closest to



A 951 cm^2 **B** 992 cm^2
C 1944 cm^2 **D** 2895 cm^2
E 110 424 cm^2

16 The expression $8\sin\theta\cos^3\theta - 8\sin^3\theta\cos\theta$ is equal to

A $8\sin\theta\cos\theta$ **B** $\sin(8\theta)$ **C** $2\sin(4\theta)$
D $4\cos(2\theta)$ **E** $2\sin(2\theta)\cos(2\theta)$

17 If v, w and z are complex numbers such that $v = 4\operatorname{cis}(-0.3\pi)$, $w = 5\operatorname{cis}(0.6\pi)$ and $z = v\overline{w}$, then $\operatorname{Arg} z$ is equal to

A 0.9π **B** -0.9π **C** 0.3π **D** -0.3π **E** 1.8π

18 The complex number $2\operatorname{cis}\left(\frac{2\pi}{3}\right)$ is written in Cartesian form as

A $\sqrt{3} - i$ **B** $-\sqrt{3} + i$ **C** $1 - \sqrt{3}i$ **D** $-1 + \sqrt{3}i$ **E** $\frac{1}{3} - \frac{\sqrt{3}}{2}i$

19 If $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, then $\operatorname{Arg} z$ is equal to

A $\frac{4\pi}{3}$ **B** $\frac{7\pi}{6}$ **C** $-\frac{\pi}{6}$ **D** $-\frac{2\pi}{3}$ **E** $-\frac{5\pi}{6}$

20 If $u = 3\operatorname{cis}\left(\frac{\pi}{2}\right)$ and $v = 5\operatorname{cis}\left(\frac{2\pi}{3}\right)$, then uv is equal to

A $15\operatorname{cis}\left(\frac{\pi}{3}\right)$ **B** $15\operatorname{cis}\left(\frac{\pi^2}{3}\right)$ **C** $15\operatorname{cis}\left(-\frac{5\pi}{6}\right)$ **D** $8\operatorname{cis}\left(\frac{\pi^2}{3}\right)$ **E** $8\operatorname{cis}\left(\frac{7\pi}{6}\right)$

21 The modulus of $12 - 5i$ is

A 169 **B** 7 **C** 13 **D** $\sqrt{119}$ **E** $\sqrt{7}$

22 Let $z = x + yi$, where x and y are real numbers which are not both zero. Which one of the following expressions does not necessarily represent a real number?

A z^2 **B** $z\overline{z}$ **C** $z^{-1}z$ **D** $\operatorname{Im}(z)$ **E** $z + \overline{z}$

23 If $z = -14 - 7i$, then the complex conjugate of z is equal to

A $7 - 14i$ **B** $14 + 7i$ **C** $-14 + 7i$ **D** $14 - 7i$ **E** $-7 + 14i$

24 The expression $3z^2 + 9$ is factorised over $\mathbb{C}$. Which one of the following is a factor?

A $3z$ **B** $z + 3$ **C** $z + 3i$ **D** $z - 3i$ **E** $z + \sqrt{3}i$

25 $(1 + 2i)^2$ is equal to

A -3 **B** $-3 + 2i$ **C** $-3 + 4i$ **D** $-1 + 4i$ **E** $5 + 4i$

26 Let $r > 0$ and consider the set $S = \{z \in \mathbb{C} : |z - 1 + 2i| = r\}$. If $4 + 2i$ belongs to S, then the value of r is

A 2 **B** 3 **C** 4 **D** 5 **E** 6

27 Which of the following equations has $z = 2i$ as a solution?

A $z^2 - 2 = 0$ **B** $z^2 + 2 = 0$ **C** $z^2 - 4 = 0$

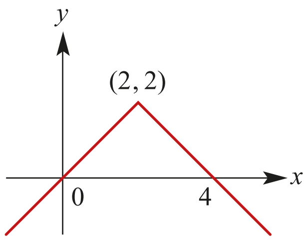
D $z^3 - 3z^2 + 4z - 11 = 0$ **E** $z^3 - 3z^2 + 4z - 12 = 0$

28 Which of the following equations has the graph shown?

A $y = |x - 2| + 2$ **B** $y = |x + 2| - 2$

C $y = -|x - 2| + 2$ **D** $y = -|x + 2| + 2$

E $y = -|x - 2| - 2$



29 Let $f(x) = \sin^{-1}(ax + 2) + b$, where $a, b \in \mathbb{R}$. If the implied domain of f is $[2, 6]$ and the range of f is $[0, \pi]$, then the values of a and b are

A $a = -\frac{1}{2},\ b = -\frac{\pi}{2}$ **B** $a = -\frac{1}{2},\ b = \frac{\pi}{2}$ **C** $a = \frac{1}{2},\ b = \frac{\pi}{2}$

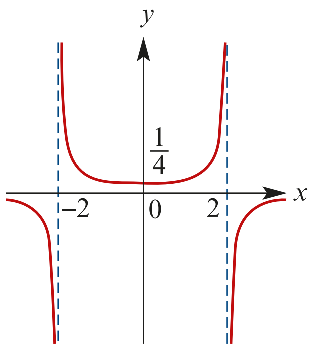
D $a = 2,\ b = \frac{\pi}{2}$ **E** $a = -2,\ b = -\frac{\pi}{2}$

30 Which of the following equations has the graph shown?

A $y = \dfrac{1}{4 - x^2}$ **B** $y = \dfrac{1}{x^2 - 4}$

C $y = \dfrac{1}{2 - x^2}$ **D** $y = \dfrac{1}{x^2 - 2}$

E $y = \dfrac{1}{(x - 2)^2}$



31 If a and b are positive real numbers, then the graph of the reciprocal of $y = a\sin(x) + b$, where $0 \le x \le 2\pi$, will have two vertical asymptotes provided

A $b > a$ **B** $a > b$ **C** $b > -a$ **D** $a > -b$ **E** $a > 0$

32 The graph of $f(x) = \sec(2x)$, for $-\pi \le x \le \pi$, has its local minimum points at

A $x = 0$ **B** $x = -\pi, \pi$ **C** $x = -\pi, 0, \pi$

D $x = -\frac{\pi}{2}, \frac{\pi}{2}$ **E** $x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

33 Given point $A(1, -2)$, a set of points $P(x, y)$ satisfy $AP = 3$. This set of points is a

A line **B** circle **C** parabola **D** ellipse **E** hyperbola

34 A line has equation $y = x + 1$. For some pair of points A and B, each point $P(x, y)$ on the line satisfies $AP = BP$. The coordinates of A and B could be

A $A(0, 0)$ and $B(0, 1)$ **B** $A(0, 0)$ and $B(-1, 1)$ **C** $A(-1, 0)$ and $B(0, 1)$
D $A(0, 1)$ and $B(1, 0)$ **E** $A(0, 1)$ and $B(-1, 0)$

35 A parabola has focus $F(0, 2)$ and directrix $y = -4$. Which of the following is true?

A The parabola has axis of symmetry $y = 0$.
B The parabola goes through the origin.
C The parabola goes through the point $(0, -1)$.
D The parabola goes through the point $(1, 2)$.
E The parabola has equation $y = 2x^2 - 4$.

36 Let a and b be positive real numbers. The graphs of $x^2 - y^2 = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will have four points of intersection provided

A $b > a$ **B** $a > 1$ **C** $a < 1$ **D** $b > 1$ **E** $b < 1$

37 A hyperbola has asymptotes $y = 2x + 1$ and $y = -2x + 1$ and has no x-axis intercepts. The equation of the hyperbola could be

A $x^2 - \frac{(y-1)^2}{4} = 1$ **B** $\frac{x^2}{4} - (y-1)^2 = 1$ **C** $\frac{(x-1)^2}{4} - y^2 = 1$
D $\frac{(y-1)^2}{4} - x^2 = 1$ **E** $\frac{y^2}{4} - (x-1)^2 = 1$

38 A curve is parameterised by the equations $x = 1 + t$ and $y = \frac{1-t}{1+t}$. The Cartesian equation of the curve is

A $y = \frac{2}{x} - 1$ **B** $y = \frac{1}{x} - 1$ **C** $y = \frac{1}{x} - 2$ **D** $y = \frac{1}{x} + 2$ **E** $y = \frac{2}{x} + 1$

39 The ellipse with equation $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$ can be parameterised by the pair of equations

A $x = 4\cos(t) - 1$ and $y = 9\sin(t) + 1$ **B** $x = 4\cos(t) + 1$ and $y = 9\sin(t) - 1$
C $x = 4\cos(t) - 1$ and $y = 9\sin(t) - 1$ **D** $x = 2\cos(t) - 1$ and $y = 3\sin(t) - 1$
E $x = 2\cos(t) + 1$ and $y = 3\sin(t) - 1$

40 A curve is parameterised by the equations $x = 2t - 3$ and $y = t^2 - 3t$. Which of the following points does the curve pass through?

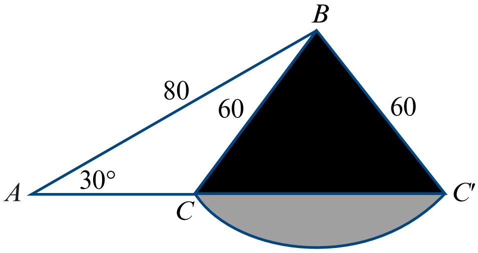
A $(5, 1)$ **B** $(5, 2)$ **C** $(5, 3)$ **D** $(5, 4)$ **E** $(5, 5)$

41 The Cartesian equation $y = x^2$ written in polar form is

A $r = \sec\theta \tan\theta$ **B** $r = \cos\theta \cot\theta$ **C** $r = \sec\theta \cot\theta$
D $r = \cos\theta \tan\theta$ **E** $r = \sin\theta \tan\theta$

19C Extended-response questions

1 For $\triangle ABC$ in the diagram, $A = 30°$, $a = 60$ and $c = 80$.



a Find the magnitudes of angles:

  i BCA and ABC

  ii $BC'A$ and ABC'

b Find the lengths of line segments:

  i AC **ii** AC' **iii** CC'

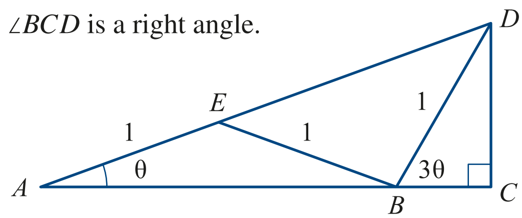
c Show that the magnitude of $\angle CBC'$ is 96.38° (correct to two decimal places). Then using this value:

  i find the area of triangle BCC'

  ii find the area of the shaded sector

  iii find the area of the shaded segment.

2 In the figure, $AE = BE = BD = 1$ unit and $\angle BCD$ is a right angle.



a Show that the magnitude of $\angle BDE$ is 2θ.

b Use the cosine rule in $\triangle BDE$ to show that $DE = 2\cos(2\theta)$.

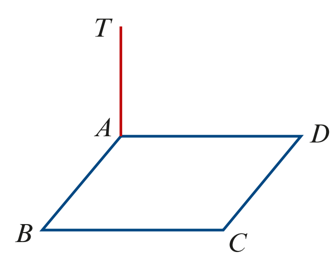
c Show that:

  i $DC = \sin(3\theta)$ **ii** $AD = \dfrac{\sin(3\theta)}{\sin\theta}$

d Use the results of **b** and **c** to show that $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$.

3 a Adam notices a distinctive tree while orienteering on a flat horizontal plane. From where he is standing, the tree is 200 m away on a bearing of 050°. Two other people, Brian and Colin, who are both standing due east of Adam, each claim that the tree is 150 m away from them. Given that their claims are true and that Brian and Colin are not standing in the same place, how far apart are they? Give your answer to the nearest metre.

b A vertical tower of height 10 m stands in one corner of a rectangular courtyard. From the top of the tower, T, the angles of depression to the nearest corners B and D are 32° and 19° respectively. Find:



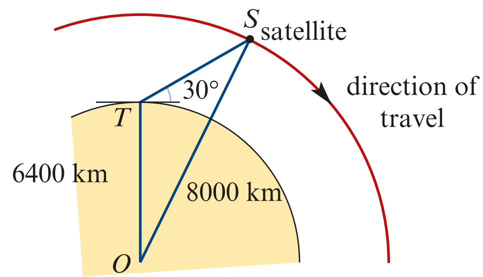
  i AB, correct to two decimal places

  ii AD, correct to two decimal places

  iii the angle of depression from T to the corner C diagonally opposite the tower, correct to the nearest degree.

c Two circles, each of radius length 10 cm, have their centres 16 cm apart. Calculate the area common to both circles, correct to one decimal place.

4 A satellite travelling in a circular orbit 1600 km above the Earth is due to pass directly over a tracking station at 12 p.m. Assume that the satellite takes two hours to make an orbit and that the radius of the Earth is 6400 km.

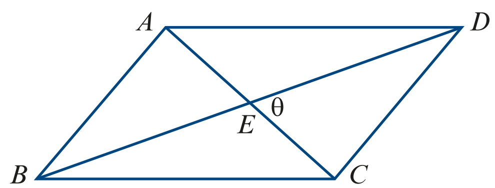


a If the tracking station antenna is aimed at 30° above the horizon, at what time will the satellite pass through the beam of the antenna?

b Find the distance between the satellite and the tracking station at 12:06 p.m.

c At what angle above the horizon should the antenna be aimed so that its beam will intercept the satellite at 12:06 p.m.?

5 The diagonals of parallelogram $ABCD$ intersect at point E, with $\angle CED = \theta°$. Let $AB = CD = x$, $AD = BC = y$, $BD = p$ and $AC = q$.



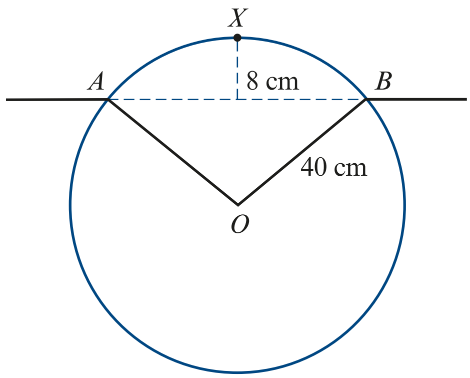
a Apply the cosine rule to triangle DEC to find x in terms of p, q and θ.

b Apply the cosine rule to triangle DEA to find y in terms of p, q and θ.

c Use the results of **a** and **b** to show that $2(x^2 + y^2) = p^2 + q^2$.

d A parallelogram has side lengths 8 cm and 6 cm and one diagonal of length 13 cm. Find the length of the other diagonal.

6 The figure shows the circular cross-section of a uniform log of radius 40 cm floating in water. The points A and B are on the surface of the water and the highest point X is 8 cm above the surface.



a Show that the magnitude of $\angle AOB$ is approximately 1.29 radians.

b **i** Find the length of arc AXB.

ii Find the area of the cross-section below the surface.

iii Find the percentage of the volume of the log below the surface.

7 Consider triples of real numbers (a, b, c) such that

$$a \le b \le c, \quad |a| + |b| + |c| = 14, \quad |a + b + c| = 2 \quad \text{and} \quad |abc| = 72$$

a Explain why such triples must satisfy

$$a \le 0 \le b \le c \quad \text{or} \quad a \le b \le 0 \le c$$

b Determine all such triples.

8 Suppose that k is a positive real number and consider the function $f\colon [0, 2\pi] \to \mathbb{R}$ given by $f(x) = 2\sin(x) + k$.

a Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when:

i $k = 1$ **ii** $k = 3$

b For what value of k does the graph of $y = \dfrac{1}{f(x)}$ have only one vertical asymptote?

c For this value of k, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

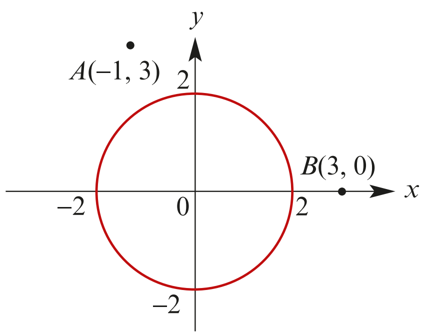
9 Two towns A and B are located on a rectangular grid with coordinates $A(1, 2)$ and $B(2, -2)$, where the units are kilometres. A straight section of road is to be constructed so that each point $P(x, y)$ on the road is equidistant from the two towns.

a Find the equation of the road.

b Show that the road is the perpendicular bisector of the line segment AB.

c Hence find the shortest distance from town A to the road.

10 The circle with equation $x^2 + y^2 = 4$ is shown. We will say that point A is **visible** to point B if the line AB does not intersect the circle.



a Consider points $A(-1, 3)$ and $B(3, 0)$. Show that the equations

$$x = 4t - 1 \quad \text{and} \quad y = 3 - 3t$$

parameterise the line AB.

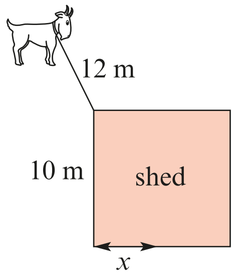
b Show that A is not visible to B by showing that there are two values of t for which the line AB intersects the circle.

c Find parametric equations for the line that goes through points $C(-1, 4)$ and $B(3, 0)$.

d Show that C is visible to B by showing that there is no value of t for which the line CB intersects the circle.

e Find the range of values k for which the point $D(-1, k)$ is visible to B.

11 A shed has a square base of side length 10 metres. A goat is tied to a corner of the shed by a rope of length 12 metres. As the goat pulls tightly on the rope and walks around the shed in both directions, a path is traced by the goat.



a Sketch the shed and the path described above.

b Find the size of the area over which the goat can walk.

c The goat is now tied to a point on the shed x metres from the corner, where $0 \le x \le 5$. Find a formula for the area A over which the goat can walk, in terms of x.

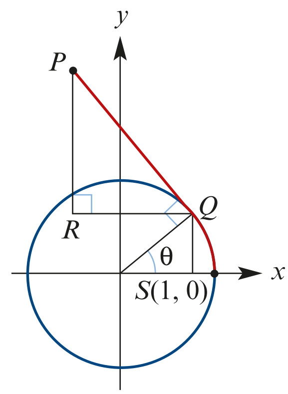
Hint: Consider the two cases $0 \le x \le 2$ and $2 < x \le 5$.

d Sketch the graph of A against x for $0 \le x \le 5$.

e Where should the goat be tied if the area is to be:

i a maximum **ii** a minimum?

12 A rope of length π is affixed to point $S(1, 0)$ on one side of a circle of radius 1 centred at the origin. The rope can be pulled tight and wrapped around the circle in both directions. The end of the rope traces out a curve.



a Explain why the rope can reach to the opposite side of the circle.

b Sketch the unit circle and the curve described above.

We now find parametric equations to describe the part of the curve obtained when some of the rope is wrapped anticlockwise around the unit circle.

c Referring to the diagram, find the following in terms of θ:

i Arc length SQ **ii** Length PQ **iii** Angle RPQ

iv Length RQ **v** Length RP

d Hence, by finding the coordinates of point P, give parametric equations for the curve in terms of θ.

13 **a** Let w and z be complex numbers such that the angles $\text{Arg}(w)$ and $\text{Arg}(z)$ are acute. Using the rule for multiplying complex numbers in polar form, show that

$$\text{Arg}(wz) = \text{Arg}(w) + \text{Arg}(z)$$

b Find the exact values of:

i $\text{Arg}(2 + i)$ **ii** $\text{Arg}(3 + i)$ **iii** $\text{Arg}(5 + 5i)$

c Show that $(2 + i)(3 + i) = 5 + 5i$.

d By taking the argument of both sides of the equation from part **c**, prove that

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

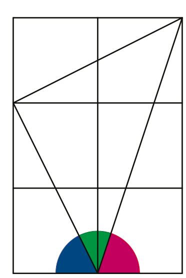
e By first evaluating $(3 + i)^2(7 + i)$, prove that

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

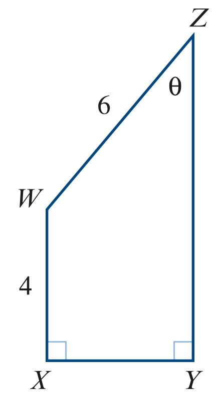
f By first evaluating $(1 + i)(1 + 2i)(1 + 3i)$, prove that

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$$

g Explain how the result from part **f** can also be shown using the diagram on the right. The grid is composed of unit-length squares.



14 The diagram shows a trapezium $XYZW$ such that $XW = 4$, $WZ = 6$, $\angle YXW = \angle XYZ = \frac{\pi}{2}$ and $\angle YZW = \theta$, where $0 < \theta < \frac{\pi}{2}$.



a Show that the area, A, of the trapezium is given by

$$A = 24\sin\theta + 9\sin(2\theta)$$

b Show that the perimeter, P, of the trapezium is given by

$$P = 14 + 6(\sin\theta + \cos\theta)$$

c Find the maximum value of P and the corresponding value of θ.

d Find the values of θ for which $P = 21$.

e Find the exact value of A when $\theta = \frac{\pi}{4}$.

f Use your calculator to find the maximum value of A (correct to two decimal places) and the value of θ for which this occurs (correct to one decimal place).

g Sketch the graph of A against θ.

h Change of variable Let $x = XY$. Show that

$$A = 4x + \frac{1}{2}x\sqrt{36 - x^2}$$

Use your calculator to find the maximum value of A by graphing A against x.

19D Investigations

1 Graphs involving the modulus function

a **i** Sketch the graphs of the functions $y = |x - 2| + |x - 3|$ and $y = |2x - 3| + |x - 4|$.

ii Investigate graphs of the form $y = |ax + b| + |cx + d|$ and $y = |ax + b| - |cx + d|$.

b **i** Sketch the graph of the relation $|2x + y| = 2$.

ii Investigate graphs of the form $|ax + by| = c$.

c **i** Sketch the region of the plane defined by $|x + y| \le 2$.

ii Investigate graphs of the form $|ax + by| \le c$.

d **i** Sketch the graph of the relation $|x| + |y| = 2$. Find the area of the enclosed region.

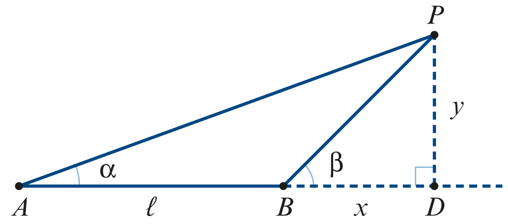
ii Investigate graphs of the form $|ax| + |by| = c$ and the corresponding areas.

e Sketch the region of the plane defined by $|x| + |y| + |x + y| \le 2$ and find the area of this region.

f Investigate other families of graphs involving the modulus function. For example, consider the graphs of $|xy| = 1$ and $|xy| \le 2$.

2 Measurement errors

A practical use of trigonometry is in determining the position of an inaccessible object. The technique used in this activity is called *triangulation*.



In this diagram: The point P is inaccessible. The distance $\ell = AB$ is known. The distances $x = BD$ and $y = PD$ are unknown. The angles α and β are measured.

a Note that $\tan\beta = \dfrac{y}{x}$ and $\tan\alpha = \dfrac{y}{x+\ell}$. Show that $x = \dfrac{\ell\tan\alpha}{\tan\beta - \tan\alpha}$.

b Hence, show that

$$x = \frac{\ell\sin\alpha\ \cos\beta}{\sin(\beta-\alpha)} \quad \text{and} \quad y = \frac{\ell\sin\alpha\ \sin\beta}{\sin(\beta-\alpha)}$$

c Investigate the effect that measurement errors in the angles α and β have on the values of x and y calculated using the formulas from part **b**. In particular, consider the effect for different magnitudes of $\beta - \alpha$. For example, what is the effect of a 1° error in α when the measured values are $\alpha = 40^\circ$ and $\beta = 45^\circ$?

d Take actual measurements in real-world situations and investigate your errors.

3 Complex quadratics

In this question, we use the quadratic formula to help solve polynomial equations over the complex numbers. Recall that the solutions of a quadratic equation $az^2 + bz + c = 0$ are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a **i** Solve the quadratic equation $z^2 + az + a^2 = 0$, where a is a positive real number.

ii Hence, solve the cubic equation $z^3 = a^3$, where a is a positive real number.
Hint: Use the factorisation $z^3 - a^3 = (z-a)(z^2 + az + a^2)$.

iii For the case $a = 1$, write the solutions of the cubic equation in polar form. Plot them on an Argand diagram with the unit circle.

iv Add the solutions for the cases $a = 2$ and $a = 3$ to your Argand diagram.

v Summarise what you have found.

b Repeat part **a** for $z^2 - az + a^2 = 0$ and $z^3 = -a^3$, where a is a positive real number.
Hint: Use the factorisation $z^3 + a^3 = (z+a)(z^2 - az + a^2)$.

c Repeat part **a** for $z^2 + aiz - a^2 = 0$ and $z^3 = -a^3 i$, where a is a positive real number.

d Repeat part **a** for $z^2 - aiz - a^2 = 0$ and $z^3 = a^3 i$, where a is a positive real number.

e There is more to consider in this way. For example, look at the equation $z^4 + 1 = 0$. Factorise the left-hand side by considering $z^4 + 2z^2 + 1 - 2z^2$.

20

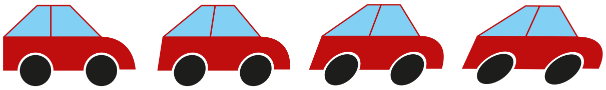
Transformations of the plane

Objectives

- To define **linear transformations**.
- To represent a linear transformation as a 2×2 matrix.
- To study the effect of important transformations, including **dilations**, **reflections**, **rotations**, **shears** and **translations**.
- To investigate **compositions** and **inverses** of transformations.
- To investigate the connection between the **determinant** of a transformation matrix and area.
- To investigate the effect of transformations on regions of the plane, including points, shapes and graphs.

Modern animations are largely created with the use of computers. Many basic visual effects can be understood in terms of simple transformations of the plane.

For example, suppose that an animator wants to give the car below a sense of movement. This can be achieved by gradually tilting the car so that it leans forwards. We will see later how this can easily be done using a transformation called a **shear**.



Aside from computer graphics, linear transformations play an important role in many diverse fields such as mathematics, physics, engineering and economics.

20A Linear transformations

Each point in the plane can be denoted by an ordered pair (x, y). The set of all ordered pairs is often called the **Cartesian plane**: $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

A **transformation** of the plane maps each point (x, y) in the plane to a new point (x', y'). We say that (x', y') is the **image** of (x, y).

We will mainly be concerned with **linear transformations**, which have rules of the form

$$(x, y) \to (ax + by, cx + dy)$$

Example 1

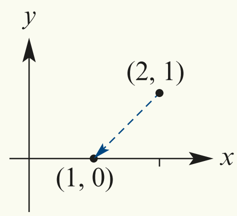
Find the image of the point (2, 1) under the transformation with rule

$$(x, y) \to (3x - 5y, 2x - 4y)$$

Solution

We let $x = 2$ and $y = 1$, giving

$$(2, 1) \to (3 \times 2 - 5 \times 1, 2 \times 2 - 4 \times 1) = (1, 0)$$



Matrices and linear transformations

Each ordered pair can also be written as a 2×1 matrix, which we will call a **column vector**:

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

This is a very useful observation, since we can now easily perform the linear transformation $(x, y) \to (ax + by, cx + dy)$ by using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Example 2

a Find the matrix of the linear transformation with rule $(x, y) \to (x - 2y, 3x + y)$.

b Use the matrix to find the image of the point (2, 3) under the transformation.

Solution

a $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 - 2 \times 3 \\ 3 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$

Therefore the image of (2, 3) is (−4, 9).

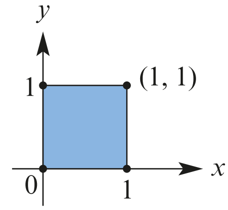
Explanation

The rows of the matrix are given by the coefficients of x and y.

We write the point (2, 3) as a column vector and multiply by the transformation matrix.

Transforming the unit square

The **unit square** is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. The effect of a linear transformation can often be demonstrated by studying its effect on the unit square.



Example 3

A linear transformation is represented by the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

a Find the image of the unit square under this transformation.

b Sketch the unit square and its image.

Solution

a We could find the images of the four vertices of the square one at a time:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

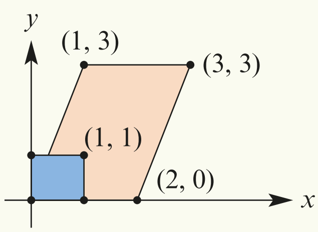
However, this can be done in a single step by multiplying the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex of the square:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

The columns of the result give the images of the vertices:

$$(0,0), \quad (2,0), \quad (1,3), \quad (3,3)$$

b The unit square is shown in blue and its image in red.



Mapping the standard unit vectors

Let's express the points $(1,0)$ and $(0,1)$ as column vectors:

$$(1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad (0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These are called the **standard unit vectors** in $\mathbb{R}^2$.

We now consider the images of these points under the transformation with matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \text{first column of the matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \text{second column of the matrix}$$

To find the matrix of a linear transformation:

- The first column is the image of $(1, 0)$, written as a column vector.
- The second column is the image of $(0, 1)$, written as a column vector.

This observation allows us to write down the matrix of a linear transformation given just two pieces of information.

Example 4

A linear transformation maps the points $(1, 0)$ and $(0, 1)$ to the points $(1, 1)$ and $(-2, 3)$ respectively.

a Find the matrix of the transformation.

b Find the image of the point $(-3, 4)$.

Solution

a $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \end{bmatrix}$

Therefore $(-3, 4) \to (-11, 9)$.

Explanation

The image of $(1, 0)$ is $(1, 1)$, and the image of $(0, 1)$ is $(-2, 3)$. We write these images as the columns of a matrix.

Write the point $(-3, 4)$ as a column vector and multiply by the transformation matrix.

Summary 20A

- A **transformation** maps each point (x, y) in the plane to a new point (x', y').
- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- The **unit square** has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can be seen by looking at the image of the unit square.
- In the matrix of a linear transformation:
  - the first column is the image of $(1, 0)$, written as a column vector
  - the second column is the image of $(0, 1)$, written as a column vector.

Skill-sheet

Exercise 20A

Example 1

1 Find the image of the point $(2, -4)$ under the transformation with rule:

a $(x, y) \to (x + y, x - y)$
b $(x, y) \to (2x + 3y, 3x - 4y)$
c $(x, y) \to (3x - 5y, x)$
d $(x, y) \to (y, -x)$

Example 2

2 Find the image of the point $(2, 3)$ under the linear transformation with matrix:

a $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
b $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$
c $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
d $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

3 Find the matrix of the linear transformation defined by the rule:

a $(x, y) \to (2x + 3y, 4x + 5y)$
b $(x, y) \to (11x - 3y, 3x - 8y)$
c $(x, y) \to (2x, x - 3y)$
d $(x, y) \to (y, -x)$

Example 3

4 Find and sketch the image of the unit square under the linear transformation represented by the matrix:

a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
b $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
c $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
d $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$

5 Find the image of the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$ under the linear transformation represented by the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

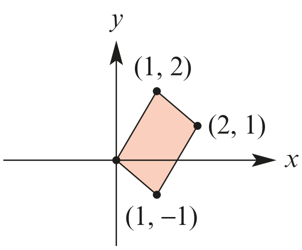
Example 4

6 Find the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(3, 4)$ and $(5, 6)$ respectively. Hence find the image of the point $(-2, 4)$.

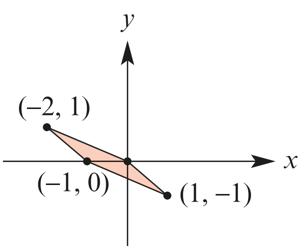
7 Find the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(-3, 2)$ and $(1, -1)$ respectively. Hence find the image of the point $(2, 3)$.

8 Find a matrix that transforms the unit square to each of the following parallelograms.
Note: There are two possible answers for each part.

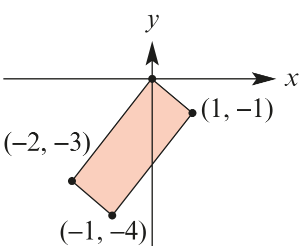
a



b



c

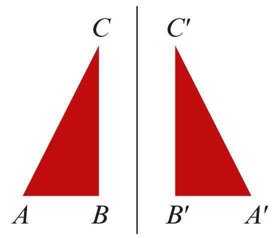


20B Geometric transformations

We now look at various important transformations that are geometric in nature.

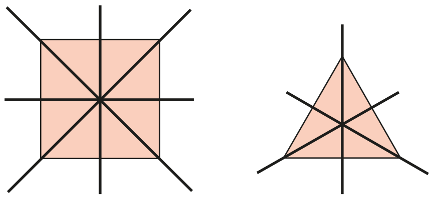
Reflections

A **reflection** in a line ℓ maps each point in the plane to its mirror image on the other side of the line. The point A and its image A' are the same distance from ℓ and the line AA' is perpendicular to ℓ.



These transformations are important for studying figures with **reflective symmetry**, that is, figures that look the same when reflected in a **line of symmetry**.

A square has four lines of symmetry, while an equilateral triangle has just three.



Note: A reflection is an example of a transformation that does not change lengths. Such a transformation is called an **isometry**.

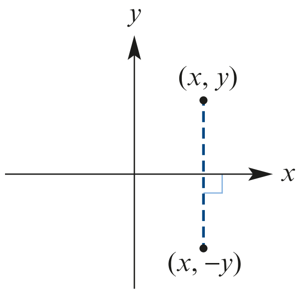
Reflection in the x-axis

A reflection in the x-axis is defined by

$$(x, y) \to (x, -y)$$

So if (x', y') is the image of the point (x, y), then

$$x' = x \quad \text{and} \quad y' = -y$$



This transformation can also be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

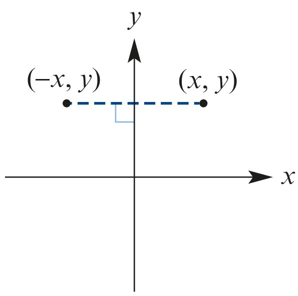
Reflection in the y-axis

A reflection in the y-axis is defined by

$$(x, y) \to (-x, y)$$

So if (x', y') is the image of the point (x, y), then

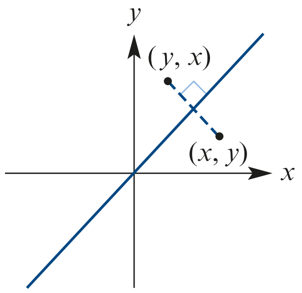
$$x' = -x \quad \text{and} \quad y' = y$$



Once again, this transformation can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection in the line $y = x$



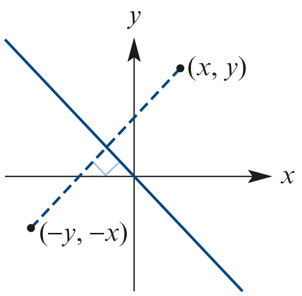
If the point (x, y) is reflected in the line $y = x$, then it is mapped to the point (y, x). So if (x', y') is the image of (x, y), then

$$x' = y \quad \text{and} \quad y' = x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection in the line $y = -x$



If the point (x, y) is reflected in the line $y = -x$, it is mapped to $(-y, -x)$. So if (x', y') is the image of (x, y), then

$$x' = -y \quad \text{and} \quad y' = -x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformation	Rule	Matrix
Reflection in the x-axis	$x' = 1x + 0y$ $y' = 0x - 1y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$x' = -1x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$x' = 0x + 1y$ $y' = 1x + 0y$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$x' = 0x - 1y$ $y' = -1x + 0y$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

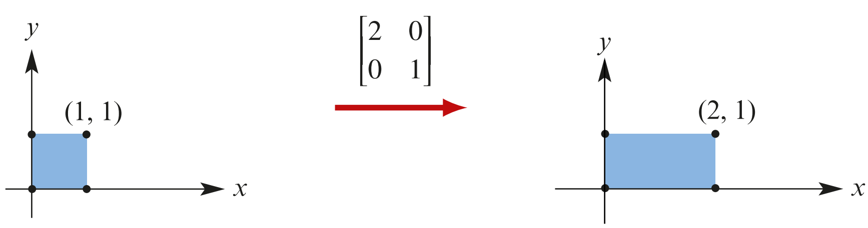
Dilations

Dilation from the y-axis

A dilation from the y-axis is a transformation of the form

$$(x, y) \to (cx, y)$$

where $c > 0$. The x-coordinate is scaled by a factor of c, but the y-coordinate is unchanged.

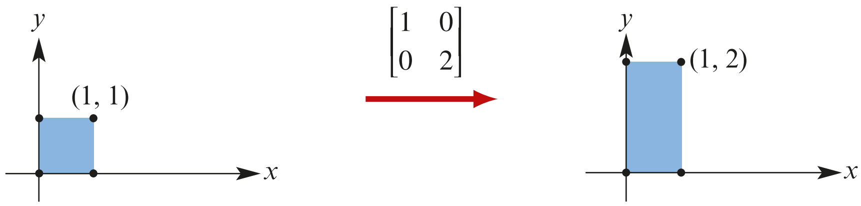


Dilation from the x-axis

Likewise, a dilation from the x-axis is a transformation of the form

$$(x, y) \to (x, cy)$$

where $c > 0$. The y-coordinate is scaled by a factor of c, but the x-coordinate is unchanged.



Dilation from the x- and y-axes

We can also simultaneously scale along the x- and y-axes using the transformation

$$(x, y) \to (cx, dy)$$

with scale factors $c > 0$ and $d > 0$.

Transformation	Rule	Matrix
Dilation from the y-axis	$x' = cx + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$
Dilation from the x-axis	$x' = 1x + 0y$ $y' = 0x + cy$	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Dilation from the x- and y-axes	$x' = cx + 0y$ $y' = 0x + dy$	$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

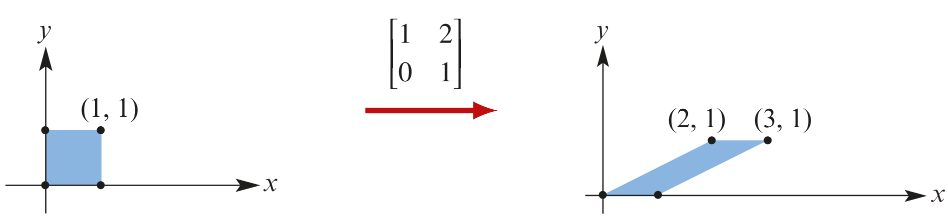
Shears

Shear parallel to the x-axis

A shear parallel to the x-axis is a transformation of the form

$$(x, y) \to (x + cy, y)$$

Notice that each point is moved in the x-direction by an amount proportional to the distance from the x-axis. This means that the unit square is tilted in the x-direction.



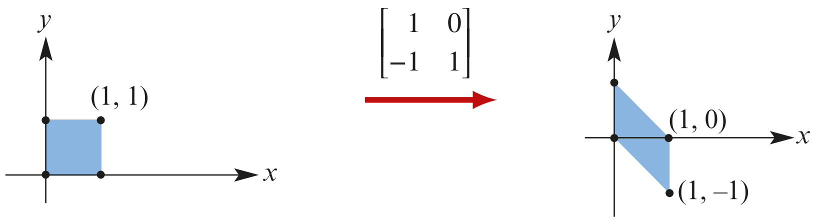
Shear parallel to the y-axis

A shear parallel to the y-axis is a transformation of the form

$$(x, y) \to (x, cx + y)$$

Here, each point is moved in the y-direction by an amount proportional to the distance from the y-axis. Now the unit square is tilted in the y-direction.

Note that if $c < 0$, then we obtain a shear in the negative direction.



Transformation	Rule	Matrix
Shear parallel to the x-axis	$x' = 1x + cy$ $y' = 0x + 1y$	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$
Shear parallel to the y-axis	$x' = 1x + 0y$ $y' = cx + 1y$	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$

Projections

The transformation defined by

$$(x, y) \to (x, 0)$$

will project the point (x, y) onto the x-axis.

Likewise, the transformation defined by

$$(x, y) \to (0, y)$$

will project the point (x, y) onto the y-axis.

Transformation	Rule	Matrix
Projection onto the x-axis	$x' = 1x + 0y$ $y' = 0x + 0y$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the y-axis	$x' = 0x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projections are an important class of transformations. For example, the image on a television screen is the projection of a three-dimensional scene onto a two-dimensional surface.

Example 5

Find the image of the point (3, 4) under each of the following transformations:

a reflection in the y-axis

b dilation of factor 2 from the y-axis

c shear of factor 4 parallel to the x-axis

d projection onto the y-axis

Solution

a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$(3, 4) \to (-3, 4)$

b $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$(3, 4) \to (6, 4)$

c $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix}$

$(3, 4) \to (19, 4)$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$(3, 4) \to (0, 4)$

Translations

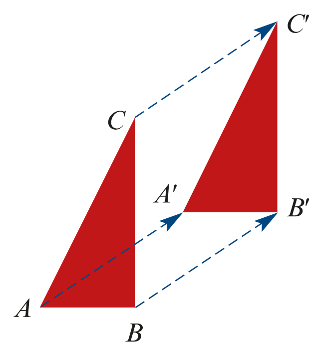
A **translation** moves a figure so that every point in the figure moves in the same direction and over the same distance.

A translation of a units in the x-direction and b units in the y-direction is defined by the rule

$$(x, y) \to (x + a, y + b)$$

This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$



Note: Translations cannot be represented using matrix multiplication. To see this, note that matrix multiplication will always map the point (0, 0) to itself. Therefore, there is no matrix that will translate the point (0, 0) to (a, b), unless $a = b = 0$.

Example 6

Find the rule for a translation of 2 units in the x-direction and −1 units in the y-direction, and sketch the image of the unit square under this translation.

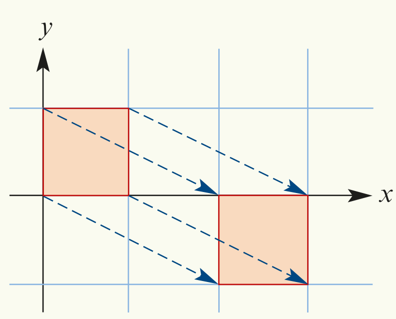
Solution

Using vector addition, this translation can be defined by the rule

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x + 2 \\ y - 1 \end{bmatrix}$$

or equivalently

$$x' = x + 2 \quad \text{and} \quad y' = y - 1$$



Summary 20B

- Important geometric transformation matrices are summarised in the table below.

Transformation	Matrix	Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y-axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x-axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- A translation of a units in the x-direction and b units in the y-direction is defined by the rule $(x, y) \to (x + a, y + b)$. This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Skill-sheet

Exercise 20B

Example 5

1 For each of the transformations described below:

i find the matrix of the transformation

ii sketch the image of the unit square under this transformation.

a dilation of factor 2 from the x-axis
b dilation of factor 3 from the y-axis
c shear of factor 3 parallel to the x-axis
d shear of factor -1 parallel to the y-axis
e reflection in the x-axis
f reflection in the line $y = -x$

Example 6

2 For each of the translations described below:

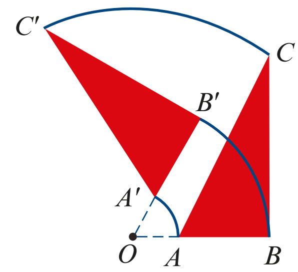
i find the rule for the translation using column vectors

ii sketch the image of the unit square under this translation.

a translation of 2 units in the x-direction
b translation of -3 units in the y-direction
c translation of -2 units in the x-direction and -4 units in the y-direction
d translation by the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$
e translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

20C Rotations and general reflections

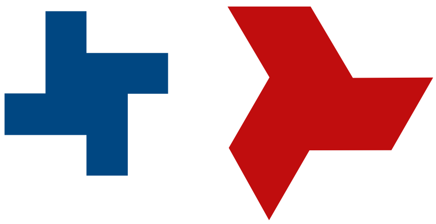
Rotations



A **rotation** turns an object about a point, but keeps its distance to the point fixed. A rotation does not change lengths, and so is another example of an isometry.

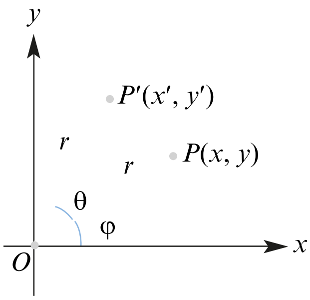
Rotations are important for studying figures with **rotational symmetry**, that is, figures that look the same when rotated through a certain angle.

These two figures have rotational symmetry, but no reflective symmetry.



Finding the rotation matrix

Consider the transformation that rotates each point in the plane about the origin by angle θ anticlockwise. We will show that this is a linear transformation and find its matrix.



Let O be the origin and let $P(x, y)$ be a point in the plane.

Then we can write

$$x = r\cos\varphi \quad \text{and} \quad y = r\sin\varphi$$

where r is the distance OP and φ is the angle between OP and the positive direction of the x-axis.

Now let $P'(x', y')$ be the image of $P(x, y)$ under a rotation about O by angle θ anticlockwise.

As $OP' = r$, we can use the compound angle formulas to show that

$$\begin{aligned} x' &= r\cos(\varphi + \theta) \\ &= r\cos\varphi\,\cos\theta - r\sin\varphi\,\sin\theta \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

and

$$\begin{aligned} y' &= r\sin(\varphi + \theta) \\ &= r\sin\varphi\,\cos\theta + r\cos\varphi\,\sin\theta \\ &= y\cos\theta + x\sin\theta \end{aligned}$$

Writing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 7

Find the matrix that represents a rotation of the plane about the origin by:

a 90° anticlockwise

b 45° clockwise.

Solution

a $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Explanation

An anticlockwise rotation means that we let $\theta = 90°$ in the formula for the rotation matrix.

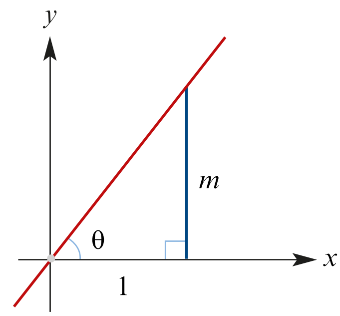
A clockwise rotation means that we let $\theta = -45°$ in the formula for the rotation matrix.

Reflection in the line $y = mx$

Reflection in a line that passes through the origin is also a linear transformation. We will find the matrix that will reflect the point (x, y) in the line $y = mx$.

Let's suppose that the angle between the positive direction of the x-axis and the line $y = mx$ is θ. Then $\tan\theta = m$ and so

$$y = mx = x\tan\theta$$



Finding the reflection matrix

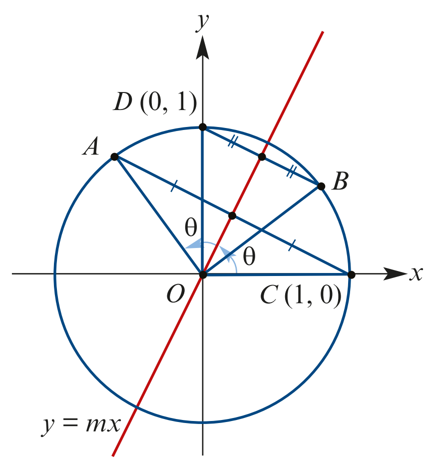
We will use the fact that the first column of the required matrix will be the image A of $C(1, 0)$, written as a column vector, and the second column will be the image B of $D(0, 1)$, written as a column vector.

Since $\angle AOC = 2\theta$, we have

$$(1, 0) \to (\cos(2\theta), \sin(2\theta))$$

Moreover, since $\angle BOC = 2\theta - 90°$, we have

$$\begin{aligned}(0, 1) &\to (\cos(2\theta - 90°), \sin(2\theta - 90°)) \\ &= (\sin(2\theta), -\cos(2\theta))\end{aligned}$$



Writing these images as column vectors gives the reflection matrix:

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Example 8

a Find the matrix that will reflect the point (x, y) in the line through the origin at an angle of 30° to the positive direction of the x-axis.

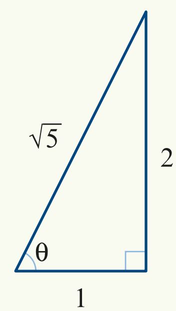
b Find the matrix that will reflect the point (x, y) in the line $y = 2x$.

Solution

a We simply let $\theta = 30°$, and so the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

b Since $\tan\theta = 2 = \frac{2}{1}$, we draw a right-angled triangle with opposite and adjacent lengths 2 and 1 respectively.



Pythagoras' theorem gives the hypotenuse as $\sqrt{5}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \sin\theta = \frac{2}{\sqrt{5}}$$

We then use the double angle formulas to show that

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$$

$$\sin(2\theta) = 2\sin\theta\,\cos\theta = 2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = \frac{4}{5}$$

Therefore the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Summary 20C

Rotation matrix

The matrix that will rotate the plane about the origin by angle θ anticlockwise is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Reflection matrix

The matrix that will reflect the plane in the line $y = mx = x\tan\theta$ is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Exercise 20C

Example 7

1 Find the matrix for each of the following rotations about the origin:

a 270° anticlockwise
b 30° anticlockwise
c 60° clockwise
d 135° clockwise

2 Find the image of the point $(2, 3)$ under a rotation about the origin by:

a $90°$ anticlockwise **b** $45°$ clockwise.

Example 8a

3 Find the matrix that will reflect the point (x, y) in the line through the origin that makes an angle with the positive direction of the x-axis of:

a $45°$ **b** $60°$ **c** $-30°$ **d** $15°$

Example 8b

4 Find the matrix that will reflect the point (x, y) in the line:

a $y = 3x$ **b** $y = 5x$ **c** $y = \dfrac{2x}{3}$ **d** $y = -3x$

5 **a** Find a formula (in terms of m) for the matrix that will reflect the point (x, y) in the line $y = mx$.

b Hence find the image of $(1, 1)$ when reflected in the line $y = 6x$.

6 The unit square is rotated about the origin by $45°$ anticlockwise.

a Find the matrix of this transformation.

b Draw the unit square and its image on the same set of axes.

c Find the area of the overlapping region.

7 The point $A(1, 0)$ is rotated to point B through angle $120°$ anticlockwise. The point $A(1, 0)$ is also rotated to point C through angle $240°$ anticlockwise.

a Find the coordinates of B and C.

b What sort of triangle is ABC?

c Triangle ABC has three lines of symmetry. Find the equations of all three lines.

20D Composition of transformations

Composition as matrix multiplication

The matrix representation of linear transformations makes it easy to find the effect of one transformation followed by another. Suppose that we would like to:

1 reflect the point (x, y) in the y-axis

2 then dilate the result by a factor of 2 from the x-axis.

To reflect the point (x, y), we would first evaluate

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

To dilate the result by a factor of 2 from the x-axis, we would then multiply

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore the matrix that will achieve both transformations is the product

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

Example 9

Find the matrix that corresponds to:

a a reflection in the x-axis and then a rotation about the origin by 90° anticlockwise

b a rotation about the origin by 90° anticlockwise and then a reflection in the x-axis.

Solution

a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Explanation

A reflection in the x-axis has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

An anticlockwise rotation by 90° has matrix

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We then multiply these two matrices together in the correct order.

Note: In this example, we get a different matrix when the same two transformations take place in reverse order. This should not be a surprise, as matrix multiplication is not commutative in general.

Compositions involving translations

Example 10

a Find the rule for the transformation that will reflect (x, y) in the x-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

b Find the rule for the transformation if the translation takes place before the reflection.

Solution

a
$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} x-3 \\ -y+4 \end{bmatrix} \end{aligned}$$

Therefore the transformation is $(x, y) \to (x - 3, -y + 4)$.

b
$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x-3 \\ y+4 \end{bmatrix} \\ &= \begin{bmatrix} x-3 \\ -y-4 \end{bmatrix} \end{aligned}$$

Therefore the transformation is $(x, y) \to (x - 3, -y - 4)$.

Summary 20D

If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

The order is important, as matrix multiplication is not commutative in general.

Skill-sheet

Exercise 20D

Example 9

1 Find the matrix that represents a reflection in the y-axis followed by a dilation of factor 3 from the x-axis.

2 Find the matrix that represents a rotation about the origin by 90° anticlockwise followed by a reflection in the x-axis.

3 **a** Find the matrix that represents a reflection in the x-axis followed by a reflection in the y-axis.

b Show that this matrix corresponds to a rotation about the origin by 180°.

4 Consider these two transformations:

- T_1: A reflection in the x-axis.
- T_2: A dilation of factor 2 from the y-axis.

a Find the matrix of T_1 followed by T_2. **b** Find the matrix of T_2 followed by T_1.

c Does the order of transformation matter in this instance?

5 Consider these two transformations:

- T_1: A rotation about the origin by 90° clockwise.
- T_2: A reflection in the line $y = x$.

a Find the matrix of T_1 followed by T_2. **b** Find the matrix of T_2 followed by T_1.

c Does the order of transformation matter in this instance?

Example 10

6 Consider these two transformations:

- T_1: A reflection in the y-axis.
- T_2: A translation of −3 units in the x-direction and 5 units in the y-direction.

a Find the rule for T_1 followed by T_2. **b** Find the rule for T_2 followed by T_1.

c Does the order of transformation matter in this instance?

7 Express each of the following transformation matrices as the product of a dilation matrix and a reflection matrix:

a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$

8 **a** Find the matrix for the transformation that is a reflection in the x-axis followed by a reflection in the line $y = x$.

b Show that these two reflections can be achieved with one rotation.

9 Suppose that matrix $\mathbf{A}$ gives a rotation about the origin by angle θ anticlockwise and that matrix $\mathbf{B}$ gives a reflection in the line $y = x$. If $\mathbf{AB} = \mathbf{BA}$, find the angle θ.

10 Suppose that matrix $\mathbf{A}$ rotates the plane about the origin by angle θ anticlockwise.

a Through what angle will the matrix $\mathbf{A}^2$ rotate the plane?

b Evaluate $\mathbf{A}^2$.

c Hence find formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

11 A transformation T consists of a reflection in the line $y = x$ followed by a translation by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

a Find the rule for the transformation T.

b Show that the transformation T can also be obtained by a translation and then a reflection in the line $y = x$. Find the translation vector.

12 **a** Find the rotation matrix for an angle of 60° anticlockwise.

b Find the rotation matrix for an angle of 45° clockwise.

c By multiplying these two matrices, find the rotation matrix for an angle of 15° anticlockwise.

d Hence write down the exact values of $\sin 15°$ and $\cos 15°$.

13 A transformation consists of a reflection in the line $y = x\tan\varphi$ and then in the line $y = x\tan\theta$. Show that this is equivalent to a single rotation.

20E Inverse transformations

If transformation T maps the point (x, y) to the point (x', y'), then the **inverse transformation** T^{-1} maps the point (x', y') to the point (x, y).

For a linear transformation T, we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{X}' = \mathbf{AX}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, then we have

$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{A}^{-1}\mathbf{AX}$$
$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{IX}$$
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}'$$

Therefore $\mathbf{A}^{-1}$ is the matrix of the inverse transformation T^{-1}.

If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse exists if and only if $\det(\mathbf{A}) = ad - bc \neq 0$.

Example 11

Find the inverse of the transformation with rule $(x, y) \to (3x + 2y, 5x + 4y)$.

Solution

Since the matrix of this linear transformation is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

the inverse transformation will have matrix

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3 \times 4 - 2 \times 5}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \end{aligned}$$

Therefore the rule of the inverse transformation is $(x, y) \to (2x - y, -\frac{5}{2}x + \frac{3}{2}y)$.

Example 12

Find the matrix of the linear transformation such that $(4, 3) \to (9, 10)$ and $(2, 1) \to (5, 6)$.

Solution

We need to find a matrix $\mathbf{A}$ such that

$$\mathbf{A}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This can be written as a single equation:

$$\mathbf{A}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}$$

Therefore

$$\mathbf{A} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

Inverses of important transformations

For important geometric transformations, it is often obvious what the inverse transformation should be.

Example 13

Let $\mathbf{R}$ be the matrix corresponding to a rotation of the plane by angle θ anticlockwise. Show that $\mathbf{R}^{-1}$ corresponds to a rotation by angle θ clockwise.

Solution

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

This matrix corresponds to a rotation of the plane by angle θ clockwise.

Explanation

We find the inverse matrix using the formula

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We also use the symmetry properties:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

The following table summarises the important geometric transformations along with their inverses. You will demonstrate some of these results in the exercises.

Transformation	Matrix $\mathbf{A}$	Inverse matrix $\mathbf{A}^{-1}$	Inverse transformation
Dilation from the y-axis	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the y-axis
Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$	Dilation from the x-axis
Shear parallel to the x-axis	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$	Shear parallel to the x-axis
Rotation by θ anticlockwise	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$	Rotation by θ clockwise
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the x-axis
Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection in the y-axis
Reflection in the line $y = mx$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	Reflection in the line $y = mx$

Summary 20E

If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Exercise 20E

1 Find the inverse matrix of each of the following transformation matrices:

a $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$

c $\begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} -1 & 3 \\ -4 & 5 \end{bmatrix}$

Example 11

2 For each of the following transformations, find the rule for their inverse:

a $(x, y) \to (5x - 2y, 2x - y)$ **b** $(x, y) \to (x - y, x)$

3 Find the point (x, y) that is mapped to $(1, 1)$ by the transformation with matrix:

a $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

Example 12

4 Find the matrix of the linear transformation such that $(1, 2) \to (2, 1)$ and $(2, 3) \to (1, 1)$.

5 Find the vertices of the rectangle that is mapped to the unit square by the transformation with matrix $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

Example 13

6 Consider a dilation of factor k from the y-axis, where $k > 0$.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a dilation of factor $\frac{1}{k}$ from the y-axis.

7 Consider a shear of factor k parallel to the x-axis.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the x-axis.

8 Consider the transformation that reflects each point in the x-axis.

a Write down the matrix $\mathbf{A}$ of this transformation.

b Show that $\mathbf{A}^{-1} = \mathbf{A}$, and explain why you should expect this result.

9 Consider the transformation that reflects each point in the line $y = mx = x\tan\theta$.

a Write down the matrix **B** of this transformation.

b Show that $\mathbf{B}^{-1} = \mathbf{B}$, and explain why you should expect this result.

20F Transformations of straight lines and other graphs

We have considered the effect of various transformations on points and figures in the plane. We will now turn our attention to graphs.

Here, we will aim to find the equations of transformed graphs. We will also investigate the effects of linear transformations on straight lines. You will study this topic in much greater detail in Mathematical Methods.

Linear transformations of straight lines

We will first investigate the effect of linear transformations on straight lines.

Example 14

Find the equation of the image of the line $y = 2x + 3$ under a reflection in the x-axis followed by a dilation of factor 2 from the y-axis.

Solution

The matrix of the combined transformation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

If (x', y') are the coordinates of the image of (x, y), then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$$

Therefore

$$x' = 2x \quad \text{and} \quad y' = -y$$

Rearranging gives

$$x = \frac{x'}{2} \quad \text{and} \quad y = -y'$$

Therefore the equation $y = 2x + 3$ becomes

$$-y' = 2\left(\frac{x'}{2}\right) + 3$$

$$-y' = x' + 3$$

$$y' = -x' - 3$$

We now ignore the dashes, and so the equation of the image is simply

$$y = -x - 3$$

Example 15

Consider the graph of $y = x + 1$. Find the equation of its image under the linear transformation $(x, y) \to (x + 2y, y)$.

Solution

Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x' - 2y' \\ y' \end{bmatrix}$$

and so $x = x' - 2y'$ and $y = y'$.

The equation $y = x + 1$ becomes

$$\begin{aligned} y' &= x' - 2y' + 1 \\ 3y' &= x' + 1 \\ y' &= \frac{x'}{3} + \frac{1}{3} \end{aligned}$$

The equation of the image is $y = \dfrac{x}{3} + \dfrac{1}{3}$.

In the previous two examples, you will have noticed that the image of each straight line was another straight line. In fact, linear transformations get their name in part from the following fact, which is proved in the exercises.

The image of any straight line under an invertible linear transformation is a straight line.

Example 16

Find a matrix that transforms the line $y = x + 2$ to the line $y = -2x + 4$.

Solution

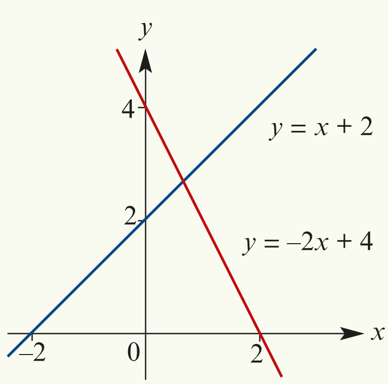
Let's find the matrix that maps the x-axis intercept of the first line to the x-axis intercept of the second line, and likewise for the y-axis intercepts.

We want

$$(-2, 0) \to (2, 0) \quad \text{and} \quad (0, 2) \to (0, 4)$$

This can be achieved by a reflection in the y-axis and then a dilation of factor 2 from the x-axis.

This transformation has the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.



Transformations of other graphs

The method for finding the image of a straight line can be used for other graphs.

Example 17

Find the image of the graph of $y = x^2 + 1$ under a translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the y-axis.

Solution

Let (x', y') be the image of (x, y). Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x+2 \\ y-1 \end{bmatrix} = \begin{bmatrix} -x-2 \\ y-1 \end{bmatrix}$$

Therefore $x' = -x - 2$ and $y' = y - 1$.

This gives $x = -x' - 2$ and $y = y' + 1$.

The equation $y = x^2 + 1$ becomes

$$y' + 1 = (-x' - 2)^2 + 1$$
$$y' = (-x' - 2)^2$$
$$= (x' + 2)^2$$

The equation of the image is $y = (x + 2)^2$.

Example 18

Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 2 from the y-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

Solution

The dilation matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

The rotation matrix is $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

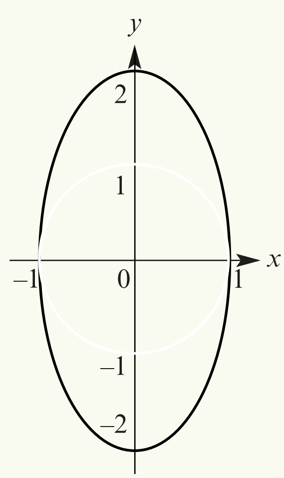
Let (x', y') be the image of (x, y). Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 2x \end{bmatrix}$$

Thus $x' = -y$ and $y' = 2x$, giving $y = -x'$ and $x = \frac{y'}{2}$.

The equation $x^2 + y^2 = 1$ becomes $\left(\frac{y'}{2}\right)^2 + (-x')^2 = 1$.

Hence the image is the ellipse with equation $x^2 + \frac{y^2}{2^2} = 1$.



Exercise 20F

Example 14

1 Find the equation of the image of the graph of $y = 3x + 1$ under:

a a reflection in the x-axis

b a dilation of factor 2 from the y-axis

c a dilation of factor 3 from the x-axis and factor 2 from the y-axis

d a reflection in the x-axis and then in the y-axis

e a reflection in the y-axis and then a dilation of factor 3 from the x-axis

f a rotation about the origin by 90° anticlockwise

g a rotation about the origin by 90° clockwise and then a reflection in the x-axis.

Example 15

2 Find the image of $y = 2 - 3x$ under each of the following transformations:

a $(x, y) \to (2x, 3y)$

b $(x, y) \to (-y, x)$

c $(x, y) \to (x - 2y, y)$

d $(x, y) \to (3x + 5y, x + 2y)$

Example 16

3 Find a matrix that transforms the line $x + y = 1$ to the line $x + y = 2$.

4 Find a matrix that transforms the line $y = x + 1$ to the line $y = 6 - 2x$.

Example 17

5 Find the equation of the image of the graph of $y = x^2 - 1$ under a translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and then a reflection in the x-axis.

6 Find the equation of the image of the graph of $y = (x - 1)^2$ under a reflection in the y-axis and then a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Example 18

7 Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 3 from the x-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

8 Consider any invertible linear transformation

$$(x, y) \to (ax + by, cx + dy)$$

Show that the image of the straight line $px + qy = r$ is a straight line.

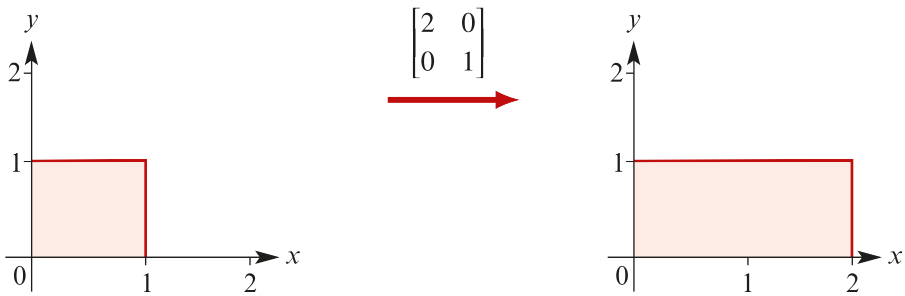
9 Rotate the graph of $y = \dfrac{1}{x}$ by 45° anticlockwise. Show that the equation of the image is $y^2 - x^2 = 2$.

Note: This shows that the two curves are congruent hyperbolas.

20G Area and determinant

If we apply a linear transformation to some region of the plane, then the area may change.

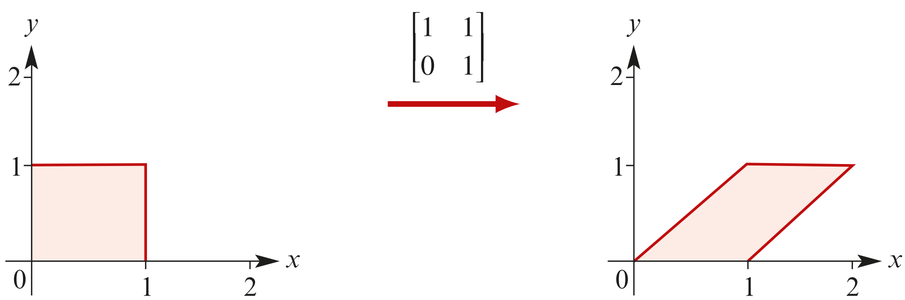
For example, if we dilate the unit square by a factor of 2 from the y-axis, then the area increases by a factor of 2.



Notice that this increase corresponds to the determinant of the transformation matrix:

$$\det\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

On the other hand, if we shear the unit square by a factor of 1 parallel to the x-axis, then the area is unchanged.



Notice that the determinant of this transformation matrix is

$$\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

More generally, we can prove the following remarkable result.

> If a region of the plane is transformed by matrix $\mathbf{B}$, then
>
> $$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

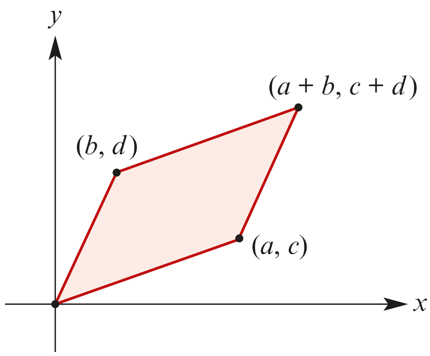
Proof We will prove the result when the unit square is transformed by matrix

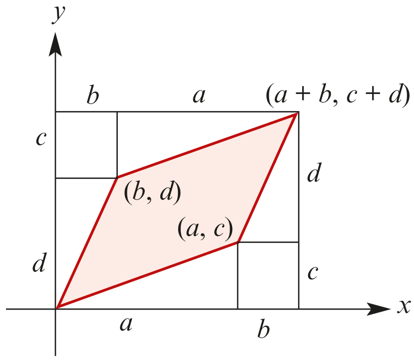
$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The result can be extended to other regions by approximating them by squares.

We will assume that a, b, c and d are all positive and that $\det(\mathbf{B}) > 0$. The proof can easily be adapted if we relax these assumptions.

The image of the unit square under transformation **B** is a parallelogram.





To find the area of the image, we draw a rectangle around it as shown, and subtract the area of the two small rectangles and four triangles from the total area:

$$\begin{aligned}\text{Area of image} &= (a+b)(c+d) - bc - bc - \frac{ac}{2} - \frac{ac}{2} - \frac{bd}{2} - \frac{bd}{2} \\ &= (a+b)(c+d) - 2bc - ac - bd \\ &= ac + ad + bc + bd - 2bc - ac - bd \\ &= ad - bc\end{aligned}$$

This is equal to the determinant of matrix **B**.

Example 19

The triangular region with vertices (1, 1), (2, 1) and (1, 2) is transformed by the rule $(x, y) \to (-x + 2y, 2x + y)$.

a Find the matrix of the linear transformation.

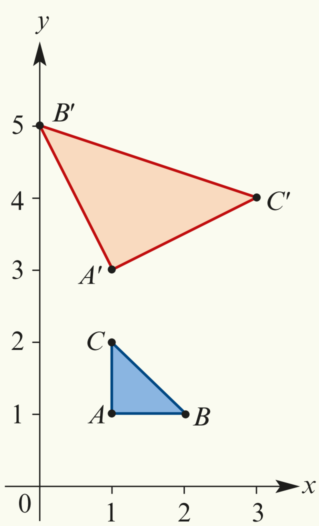
b On the same set of axes, sketch the region and its image.

c Find the area of the image.

Solution

a The matrix is given by $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

b The region is shown in blue and its image in red.



c The area of the original region is $\frac{1}{2}$.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = (-1) \times 1 - 2 \times 2 = -5$$

Therefore

$$\begin{aligned}\text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region} \\ &= |-5| \times \frac{1}{2} = \frac{5}{2}\end{aligned}$$

Example 20

The unit square is mapped to a parallelogram of area 3 by the matrix

$$\mathbf{B} = \begin{bmatrix} m & 2 \\ m & m \end{bmatrix}$$

Find the possible values of m.

Solution

The original area is 1. Therefore

$$\begin{aligned} \text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region} \\ 3 &= |ad - bc| \times 1 \\ 3 &= |m^2 - 2m| \end{aligned}$$

Therefore either $m^2 - 2m = 3$ or $m^2 - 2m = -3$.

Case 1:

$$\begin{aligned} m^2 - 2m &= 3 \\ m^2 - 2m - 3 &= 0 \\ (m+1)(m-3) &= 0 \\ m = -1 \text{ or } m &= 3 \end{aligned}$$

Case 2:

$$\begin{aligned} m^2 - 2m &= -3 \\ m^2 - 2m + 3 &= 0 \end{aligned}$$

This quadratic equation has no solutions, since the discriminant is

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(3) = 4 - 12 < 0 \end{aligned}$$

The connection between area and determinant has many important applications. In the next example, we see how it can be used to find the area of an ellipse. Alternative approaches to finding this area are much more sophisticated.

Example 21

The circle with equation $x^2 + y^2 = 1$ is mapped to an ellipse by the rule $(x, y) \to (ax, by)$, where both a and b are positive.

a Find the equation of the ellipse and sketch its graph.

b Find the area of the ellipse.

Solution

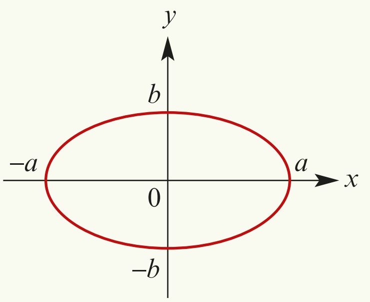
a We have $x' = ax$ and $y' = by$.

This gives $x = \dfrac{x'}{a}$ and $y = \dfrac{y'}{b}$.

The equation $x^2 + y^2 = 1$ becomes

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

Hence the equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



b The area of the original circle of radius 1 is π.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \times b - 0 \times 0 = ab$$

Therefore the area of the ellipse is πab.

Note: When $a = b = r$, this formula gives the area of a circle of radius r.

Summary 20G

If a region of the plane is transformed by matrix **B**, then

Area of image = $|\det(\mathbf{B})|$ × Area of region

Exercise 20G

1 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

a $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

Example 19

2 The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ maps the triangle with vertices (0, 1), (1, 1) and (0, 0) to a new triangle.

a Sketch the original triangle and its image.

b Find the areas of both triangles.

3 The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ maps the triangle with vertices (−1, 1), (1, 1) and (1, 0) to a new triangle.

a Sketch the original triangle and its image.

b Find the areas of both triangles.

Example 20

4 The matrix $\begin{bmatrix} m & 2 \\ -1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 6 square units. Find the value(s) of m.

5 The matrix $\begin{bmatrix} m & m \\ 1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 2 square units. Find the value(s) of m.

6 **a** By evaluating a determinant, show that each of the following transformations will not change the area of any region:

i a shear of factor k parallel to the x-axis

ii an anticlockwise rotation about the origin by angle θ

iii a reflection in any straight line through the origin

Note: We say that each of these transformations **preserves area**.

b Let $k > 0$. A linear transformation has matrix $\begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$.

i Describe the geometric effect of the transformation.

ii Show that this transformation preserves area.

7 For each $x \in \mathbb{R}$, the matrix $\begin{bmatrix} x & 1 \\ -2 & x+2 \end{bmatrix}$ maps the unit square to a parallelogram.

a Show that the area of the parallelogram is $(x+1)^2 + 1$.

b For what value of x is the area of the parallelogram a minimum?

8 For what values of m does the matrix $\begin{bmatrix} m & 2 \\ 3 & 4 \end{bmatrix}$ map the unit square to a parallelogram of area greater than 2?

9 Find all matrices that will map the unit square to a rhombus of area $\frac{1}{2}$ with one vertex at $(0,0)$ and another at $(1,0)$.

10 **a** Find a matrix that transforms the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to the triangle with vertices $(0,0)$, (a,c) and (b,d).

b Hence show that the area of the triangle with vertices $(0,0)$, (a,c) and (b,d) is given by the formula

$$A = \frac{1}{2}|ad - bc|$$

c Hence prove that if a, b, c and d are rational numbers, then the area of this triangle is rational.

d A **rational point** has coordinates (x, y) such that both x and y are rational numbers. Prove that no equilateral triangle can be drawn in the Cartesian plane so that all three of its vertices are rational points.

Hint: You can assume that the vertices of the triangle are $(0,0)$, (a,c) and (b,d). Find another expression for the area of the triangle using Pythagoras' theorem. You can also assume that $\sqrt{3}$ is irrational.

20H General transformations

Earlier in this chapter we considered rotations about the origin. But what if we want to rotate a figure about a point that is not the origin? In this section we will see how a more complicated transformation can be achieved by a sequence of simpler transformations.

Rotation about the point (a, b)

If we want to rotate the plane about the point (a, b) by angle θ anticlockwise, we can do this in a sequence of three steps:

Step 1 Translate the plane so that the centre of rotation is now the origin, by adding $\begin{bmatrix} -a \\ -b \end{bmatrix}$.

Step 2 Rotate the plane through angle θ anticlockwise, by multiplying by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} a \\ b \end{bmatrix}$.

Chaining these three transformations together gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Example 22

a Find the transformation that rotates the plane by 90° anticlockwise about the point (1, 1).

b Check your answer by showing that (0, 1) is mapped to the correct point.

Solution

a We do this in a sequence of three steps, starting with the initial point (x, y):

Initial point	Translate	Rotate 90° anticlockwise	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This gives the overall transformation

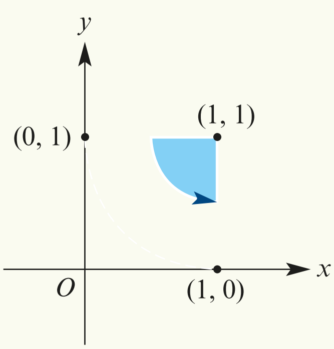
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -y + 2 \\ x \end{bmatrix}$$

b We check our answer by finding the image of (0, 1).

Let $x = 0$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 + 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore $(0, 1) \to (1, 0)$, as expected.



Reflection in the line $y = x\tan\theta + c$

To reflect the plane in a line $y = x\tan\theta + c$ that does not go through the origin, we can also do this in a sequence of three steps:

Step 1 Translate the plane so that the line passes through the origin, by adding $\begin{bmatrix} 0 \\ -c \end{bmatrix}$.

Step 2 Reflect the plane in the line $y = x\tan\theta$, by multiplying by $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} 0 \\ c \end{bmatrix}$.

Example 23

a Find the transformation that reflects the plane in the line $y = -x + 1$.

b Check your answer by finding the image of the point $(1, 1)$.

Solution

a We do this in a sequence of three steps, starting with the initial point (x, y). The first step translates the line $y = -x + 1$ so that it passes through the origin.

Initial point	Translate	Reflect in line $y = -x$	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

This gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

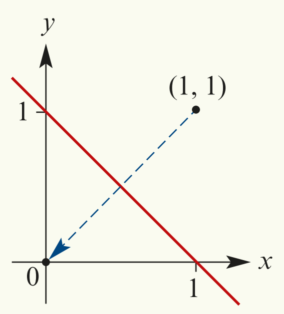
$$= \begin{bmatrix} -y+1 \\ -x+1 \end{bmatrix}$$

b We check our answer by finding the image of $(1, 1)$.

Let $x = 1$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $(1, 1) \to (0, 0)$, as expected.



Summary 20H

More difficult transformations can be achieved by combining simpler transformations.

- To rotate the plane about the point (a, b):
  1. Translate the plane so that the origin is the centre of rotation.
  2. Rotate the plane about the origin.
  3. Translate the plane back to its original position.
- To reflect the plane in the line $y = mx + c$:
  1. Translate the plane so that the line passes through the origin.
  2. Reflect the plane in the line $y = mx$.
  3. Translate the plane back to its original position.

Exercise 20H

Example 22

1 Find the transformation that rotates the plane by 90° clockwise about the point $(2, 2)$. Check your answer by showing that the point $(2, 1)$ is mapped to the correct point.

2 Find the transformation that rotates the plane by 180° anticlockwise about the point $(-1, 1)$. Check your answer by showing that the point $(-1, 0)$ is mapped to the correct point.

Example 23

3 Find the transformation that reflects the plane in each of the following lines. Check your answer by showing that the point $(0, 0)$ is mapped to the correct point.

a $y = x - 1$ **b** $y = -x - 1$

c $y = 1$ **d** $x = -2$

4 **a** Write down the matrix **A** for a rotation about the origin by angle θ clockwise.

b Write down the matrix **B** for a dilation of factor k from the x-axis.

c Write down the matrix **C** for a rotation about the origin by angle θ anticlockwise.

d Hence find the matrix that increases the perpendicular distance from the line $y = x \tan\theta$ by a factor of k.

5 Find the transformation matrix that projects the point (x, y) onto the line $y = x \tan\theta$.
Hint: First rotate the plane clockwise by angle θ.

6 Consider these two transformations:

- T_1: A reflection in the line $y = x + 1$.
- T_2: A reflection in the line $y = x$.

Show that T_1 followed by T_2 is a translation.

Chapter summary

Assign-ment

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- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The point (x', y') is called the **image** of the point (x, y).

- The matrix of a composition of two linear transformations can be found by multiplying the two transformation matrices in the correct order.
- If $\mathbf{A}$ is the matrix of a linear transformation, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.
- If a region of the plane is transformed by matrix $\mathbf{B}$, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

- Difficult transformations can be achieved by combining simpler transformations.

Transformation	Matrix	Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y-axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x-axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Rotation by θ anticlockwise	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Reflection in the line $y = x\tan\theta$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Technology-free questions

1 The rule for a transformation is $(x, y) \to (2x + y, -x + 2y)$.

a Find the image of the point $(2, 3)$.

b Find the matrix of this transformation.

c Sketch the image of the unit square and find its area.

d Find the rule for the inverse transformation.

2 Find the matrix corresponding to each of the following linear transformations:

a reflection in the y-axis
b dilation of factor 5 from the x-axis
c shear of factor -3 parallel to the x-axis
d projection onto the x-axis
e rotation by 30° anticlockwise
f reflection in the line $y = x$

3 **a** Find the matrix that will reflect the plane in the line $y = 3x$.
b Find the image of the point $(2, 4)$ under this transformation.

4 Find the transformation matrix that corresponds to:

a a reflection in the x-axis and then a reflection in the line $y = -x$
b a rotation about the origin by 90° anticlockwise and then a dilation of factor 2 from the x-axis
c a reflection in the line $y = x$ and then a shear of factor 2 parallel to the y-axis.

5 **a** Find the rule for the transformation that will reflect (x, y) in the x-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
b Find the rule for the transformation if the translation takes place before the reflection.

6 **a** Write down the matrix for a shear of factor k parallel to the y-axis.
b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the y-axis.

7 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

a $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$
b $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

8 **a** Find the rule for the transformation that rotates the plane about the point $(1, -1)$ by 90° anticlockwise. (Hint: Translate the point $(1, -1)$ to the origin, rotate the plane, and then translate the point back to its original position.)
b Find the image of the point $(2, -1)$ under this transformation.
c Sketch the unit square and its image under this transformation.

Multiple-choice questions

1 The image of the point $(2, -1)$ under the transformation $(x, y) \to (2x - 3y, -x + 4y)$ is

A $(1, -6)$ **B** $(7, -6)$ **C** $(7, 6)$ **D** $(7, 2)$ **E** $(1, 2)$

2 The matrix that will reflect the plane in the line $y = -x$ is

A $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

3 The matrix that corresponds to a dilation of factor 2 from the y-axis followed by a reflection in the x-axis is

A $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

4 The matrix $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ will

A rotate the plane by $30°$
B rotate the plane by $-30°$
C rotate the plane by $-60°$
D reflect the plane in the line $y = x \tan 30°$
E reflect the plane in the line $y = x \tan 60°$

5 The transformation that translates (x, y) by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and then rotates the result about the origin by $90°$ anticlockwise is given by

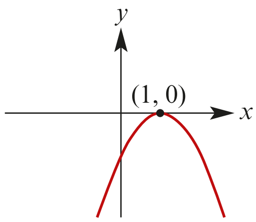
A $(x, y) \to (y - 3, x + 2)$
B $(x, y) \to (-y + 3, -x - 2)$
C $(x, y) \to (-y + 3, x + 2)$
D $(x, y) \to (-x - 2, -y + 3)$
E $(x, y) \to (x + 2, -y - 3)$

6 The matrix $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ corresponds to

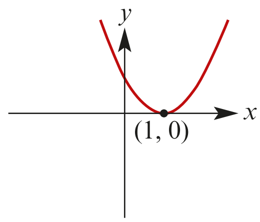
A a rotation by $180°$ and then a dilation of factor 2 from the x-axis
B a rotation by $90°$ clockwise and then a dilation of factor 2 from the x-axis
C a rotation by $180°$ and then a dilation of factor 2 from the y-axis
D a reflection in the y-axis and then a dilation of factor 2 from the y-axis
E a reflection in the x-axis and then a dilation of factor 2 from the x-axis

7 Which of the following shows the image of the graph of $y = (x - 1)^2$ under the linear transformation with matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$?

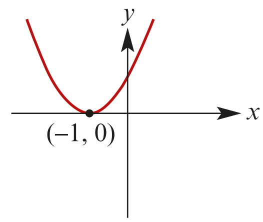
A



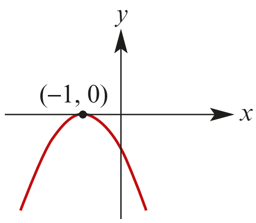
B



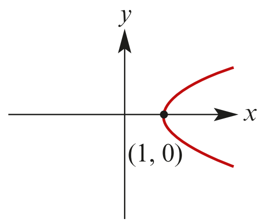
C



D



E



8 Which of these matrices maps the unit square to a parallelogram of area 2 square units?

A $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

9 The matrix $\mathbf{R}$ will rotate the plane through angle 40°. The smallest value of m such that $\mathbf{R}^m = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, is

A 6 **B** 7 **C** 8 **D** 9 **E** 10

Extended-response questions

1 **a** Find the matrix that will rotate the plane by 45° anticlockwise.

b Find the matrix that will rotate the plane by 30° anticlockwise.

c Hence find the matrix that will rotate the plane by 75° anticlockwise.

d Hence deduce exact values for cos 75° and sin 75°.

2 The triangle with vertices (0, 0), (2, 0) and (0, 2) is transformed by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

a Sketch the triangle and its image on the same set of axes.

b Find the area of the triangle and its image.

c The image of the triangle is revolved around the y-axis to create a three-dimensional solid. Find the volume of this solid.

3 Consider the transformation with rule $(x, y) \to (x + y, y)$.

a Write down the matrix of this transformation.

b What name is given to this type of transformation?

c Find the images of the points (−1, 1), (0, 0) and (1, 1) under this transformation.

d Hence sketch the graph of $y = x^2$ and its image under this transformation.

4 A square with vertices (±1, ±1) is rotated about the origin by 45° anticlockwise.

a Find the coordinates of the vertices of its image.

b Sketch the square and its image on the same set of axes.

c When these two squares are combined, the resulting figure is called a Star of Lakshmi. Find its area.

5 In this chapter we investigated two important transformation matrices. These were the rotation and reflection matrices, which we will now denote by

$$\text{Rot}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \text{Ref}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

a Using matrix multiplication and an application of trigonometric identities, prove the following four matrix equations:

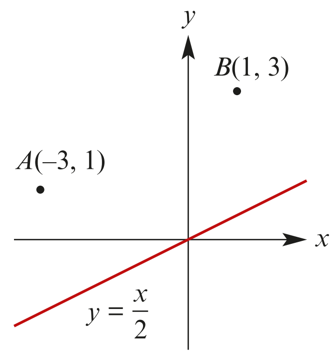
i $\text{Rot}(\theta)\,\text{Rot}(\varphi) = \text{Rot}(\theta + \varphi)$ **ii** $\text{Ref}(\theta)\,\text{Ref}(\varphi) = \text{Rot}(2\theta - 2\varphi)$

iii $\text{Rot}(\theta)\,\text{Ref}(\varphi) = \text{Ref}(\varphi + \frac{1}{2}\theta)$ **iv** $\text{Ref}(\theta)\,\text{Rot}(\varphi) = \text{Ref}(\theta - \frac{1}{2}\varphi)$

b Explain in words what each of the above four equations shows.

c Using these identities, find the matrix $\text{Rot}(60°)\,\text{Ref}(60°)\,\text{Ref}(60°)\,\text{Rot}(60°)$.

6 An ant is at point $A(-3, 1)$. His friend is at point $B(1, 3)$. The ant wants to walk from A to B, but first wants to visit the straight line $y = \frac{1}{2}x$. Being an economical ant, he wants the total length of his path to be as short as possible.



a Find the matrix that will reflect the plane in the line $y = \frac{1}{2}x$.

b Find the image A' of the point A when reflected in the line $y = \frac{1}{2}x$.

c Find the distance from point A' to point B.

d The straight line $A'B$ intersects the line $y = \frac{1}{2}x$ at the point C. What type of triangle is ACA'?

e Suppose that D is any other point on the line $y = \frac{1}{2}x$. Show that

$$AD + DB \geq AC + CB$$

f Hence find the shortest possible distance travelled by the ant.

7 A rectangle R_1 has vertices $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b), where a and b are positive real numbers.

a Sketch the rectangle R_1.

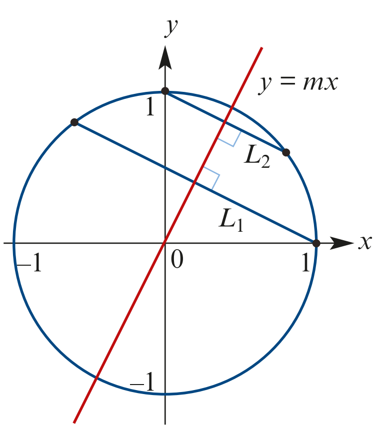
b The rectangle R_1 is rotated about the origin by angle θ anticlockwise, where $0° \leq \theta \leq 90°$. The image is another rectangle R_2. Find the coordinates of the vertices of rectangle R_2 in terms of a, b and θ.

c The vertices of R_2 lie on another rectangle R_3 that has edges parallel to the coordinate axes. Show that the area of rectangle R_3 is

$$A = \frac{1}{2}(a^2 + b^2)\sin(2\theta) + ab$$

d Hence show that the maximum area of rectangle R_3 is $\frac{1}{2}(a + b)^2$, which occurs when $\theta = 45°$.

8 The graphs of the unit circle $x^2 + y^2 = 1$ and the line $y = mx$ are shown. Lines L_1 and L_2 are perpendicular to the line $y = mx$ and go through the points $(1, 0)$ and $(0, 1)$ respectively.



a Find the equation of the line L_1, and find where it intersects the unit circle in terms of m.

b Find the equation of the line L_2, and find where it intersects the unit circle in terms of m.

c Hence deduce the formula for the matrix that reflects the point (x, y) in the line $y = mx$.

Hint: Recall that the columns of the matrix will be the images of the standard unit vectors.

21

Vectors in the plane

Objectives

- To understand the concept of a **vector** and to apply the basic operations on vectors.
- To recognise when two vectors are **parallel**.
- To use the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ to represent vectors in two dimensions.
- To find the **scalar product** of two vectors.
- To use the scalar product to find the magnitude of the angle between two vectors.
- To use the scalar product to recognise when two vectors are **perpendicular**.
- To resolve a vector into **rectangular components**.
- To apply vectors to displacement, velocity, relative velocity and equilibrium.
- To use the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to represent vectors in three dimensions.

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

length	30 cm is the length of the page of a particular book
time	10 s is the time for one athlete to run 100 m

More is required to describe displacement, velocity or force. The direction must be recorded as well as the magnitude.

displacement	30 km in the direction north
velocity	60 km/h in the direction south-east

A quantity that has both a magnitude and a direction is called a **vector**. Our study of vectors will tie together different ideas from previous chapters, including trigonometry, complex numbers and transformations.

21A Introduction to vectors

Suppose that you are asked: ‘Where is your school in relation to your house?’

It is not enough to give an answer such as ‘four kilometres’. You need to specify a direction as well as a distance. You could give the answer ‘four kilometres north-east’.

Position is an example of a vector quantity.

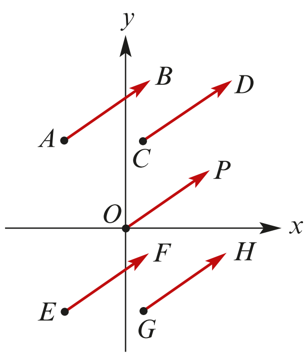
Directed line segments

A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

The five directed line segments shown all have the same length and direction, and so they are equivalent.



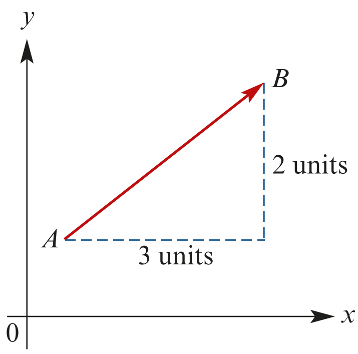
A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

For simplicity of language, this is also called vector $\overrightarrow{AB}$. That is, the set of equivalent segments can be named through one member of the set.

Note: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$

Column vectors

In Chapter 20, we introduced vectors in the context of translations of the plane. We represented each translation by a column of numbers, which was called a vector.



This is consistent with the approach here, as the column of numbers corresponds to a set of equivalent directed line segments.

For example, the column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segments which go 3 across and 2 up.

Vector notation

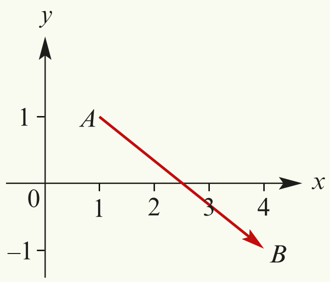
A vector is often denoted by a single bold lowercase letter. The vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single letter, such as $\boldsymbol{v}$. We can write $\boldsymbol{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underset{\sim}{v}$.

Example 1

Draw a directed line segment corresponding to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Solution



Explanation

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 across to the right and 2 down'.

Note: Here the segment starts at (1, 1) and goes to (4, −1). It can start at any point.

Example 2

The vector $\boldsymbol{u}$ is defined by the directed line segment from (2, 6) to (3, 1).

If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

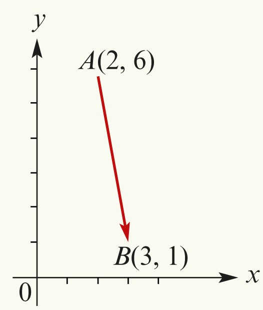
Solution

The vector is

$$\boldsymbol{u} = \begin{bmatrix} 3-2 \\ 1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

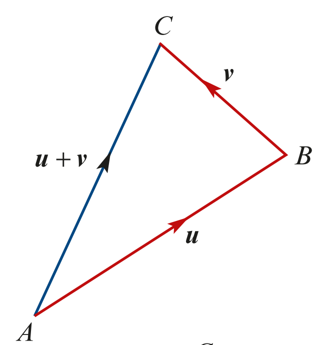
Hence $a = 1$ and $b = -5$.

Explanation



Addition of vectors

Adding vectors geometrically

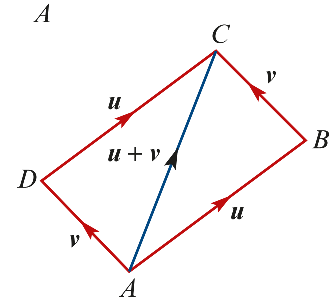


Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from A to B and then a line segment representing $\boldsymbol{v}$ from B to C.

The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from A to C. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$$

The same result is achieved if the order is reversed. This is represented in the diagram on the right:



$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{AC} \\ &= \boldsymbol{v} + \boldsymbol{u} \end{aligned}$$

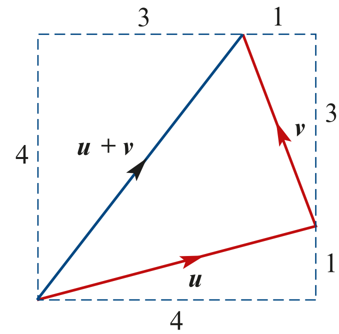
Hence addition of vectors is commutative.

Adding column vectors

Two vectors can be added using column-vector notation.

For example, if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

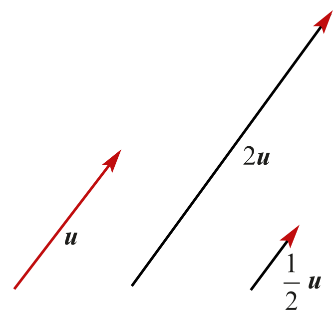
$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



Scalar multiplication

Multiplication by a real number (scalar) changes the length of the vector. For example:

- $2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$
- $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$

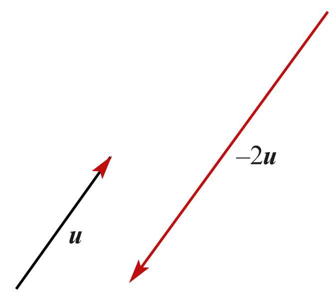


We have $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$.

In general, for $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.

When a vector is multiplied by -2, the vector's direction is reversed and the length is doubled.

When a vector is multiplied by -1, the vector's direction is reversed and the length remains the same.



If $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

If $\boldsymbol{u} = \overrightarrow{AB}$, then

$$-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$$

The directed line segment $-\overrightarrow{AB}$ goes from B to A.

Zero vector

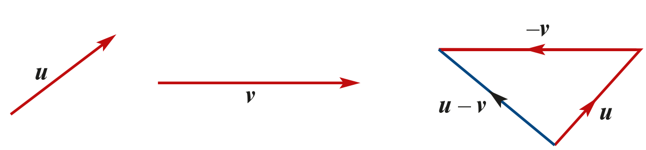
The **zero vector** is denoted by $\mathbf{0}$ and represents a line segment of zero length. The zero vector has no direction.

Subtraction of vectors

To find $\boldsymbol{u} - \boldsymbol{v}$, we add $-\boldsymbol{v}$ to $\boldsymbol{u}$.

That is, we define

$$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$$



Example 3

For the vectors $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, find:

a $2\boldsymbol{u} + 3\boldsymbol{v}$

b $2\boldsymbol{u} - 3\boldsymbol{v}$

Solution

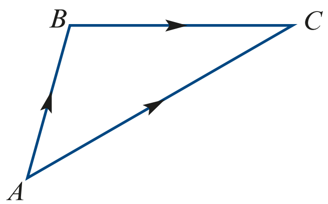
a $$2\boldsymbol{u} + 3\boldsymbol{v} = 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

b $$2\boldsymbol{u} - 3\boldsymbol{v} = 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} - \begin{bmatrix} -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

Polygons of vectors

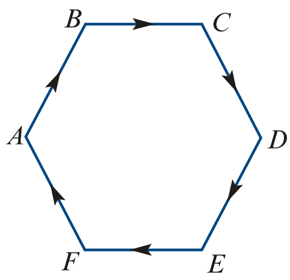
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$



Parallel vectors

Two parallel vectors have the same direction or opposite directions.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

For example, if $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel as $\boldsymbol{v} = 3\boldsymbol{u}$.

Position vectors

We can use a point O, the origin, as a starting point for a vector to indicate the position of a point A in space relative to O.

For most of this chapter, we study vectors in two dimensions and the point O is the origin of the Cartesian plane. (Vectors in three dimensions are studied in Section 21I.)

For a point A, the **position vector** is $\overrightarrow{OA}$.

Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

Proof Assume that $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$. Then

$$m\boldsymbol{a} - p\boldsymbol{a} = q\boldsymbol{b} - n\boldsymbol{b}$$

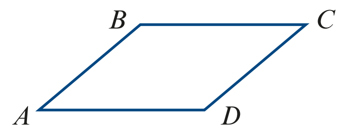
$$\therefore \quad (m-p)\boldsymbol{a} = (q-n)\boldsymbol{b}$$

If $m \neq p$ or $n \neq q$, we could therefore write

$$\boldsymbol{a} = \frac{q-n}{m-p}\boldsymbol{b} \quad \text{or} \quad \boldsymbol{b} = \frac{m-p}{q-n}\boldsymbol{a}$$

But this is not possible, as $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel. Therefore $m = p$ and $n = q$.

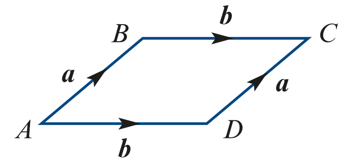
Parallelograms



A parallelogram $ABCD$ is a quadrilateral whose opposite sides are parallel. Therefore $\overrightarrow{DC} = k\overrightarrow{AB}$ and $\overrightarrow{BC} = \ell\overrightarrow{AD}$, for some $k, \ell \in \mathbb{R} \setminus \{0\}$. But we have

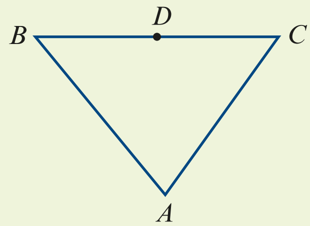
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AD} + \overrightarrow{DC}$$

$$\therefore \quad \overrightarrow{AB} + \ell\overrightarrow{AD} = \overrightarrow{AD} + k\overrightarrow{AB}$$



So the previous result gives $k = 1$ and $\ell = 1$. We have shown that every parallelogram is spanned by two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, as illustrated on the right.

Example 4



Let A, B and C be the vertices of a triangle, and let D be the midpoint of BC.

Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$.

Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BD}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{AD}$ **e** $\overrightarrow{CA}$

Solution	Explanation
a $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\boldsymbol{b}$	Same direction and half the length
b $\overrightarrow{DC} = \overrightarrow{BD} = \frac{1}{2}\boldsymbol{b}$	Equivalent vectors
c $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$	
d $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$	
e $\overrightarrow{CA} = -\overrightarrow{AC} = -(\boldsymbol{a} + \boldsymbol{b})$	

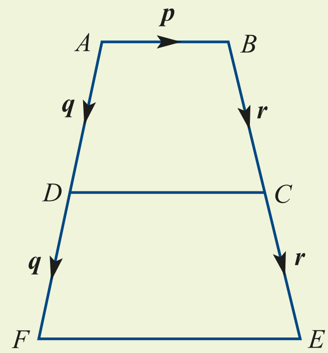
Example 5

In the figure, $\overrightarrow{DC} = k\boldsymbol{p}$ where $k \in \mathbb{R} \setminus \{0\}$.

a Express $\boldsymbol{p}$ in terms of k, $\boldsymbol{q}$ and $\boldsymbol{r}$.

b Express $\overrightarrow{FE}$ in terms of k and $\boldsymbol{p}$ to show that FE is parallel to DC.

c If $\overrightarrow{FE} = 4\overrightarrow{AB}$, find the value of k.



Solution

a $\boldsymbol{p} = \overrightarrow{AB}$

$= \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$

$= \boldsymbol{q} + k\boldsymbol{p} - \boldsymbol{r}$

Therefore

$(1-k)\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{r}$

b $\overrightarrow{FE} = -2\boldsymbol{q} + \boldsymbol{p} + 2\boldsymbol{r}$

$= 2(\boldsymbol{r} - \boldsymbol{q}) + \boldsymbol{p}$

From part **a**, we have

$\boldsymbol{r} - \boldsymbol{q} = k\boldsymbol{p} - \boldsymbol{p}$

$= (k-1)\boldsymbol{p}$

Therefore

$\overrightarrow{FE} = 2(k-1)\boldsymbol{p} + \boldsymbol{p}$

$= 2k\boldsymbol{p} - 2\boldsymbol{p} + \boldsymbol{p}$

$= (2k-1)\boldsymbol{p}$

c $\overrightarrow{FE} = 4\overrightarrow{AB}$

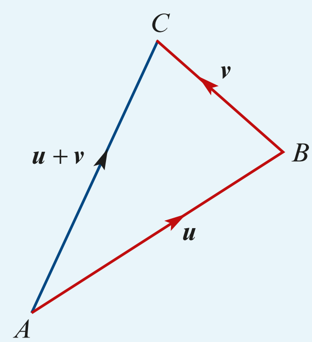
$(2k-1)\boldsymbol{p} = 4\boldsymbol{p}$

$2k - 1 = 4$

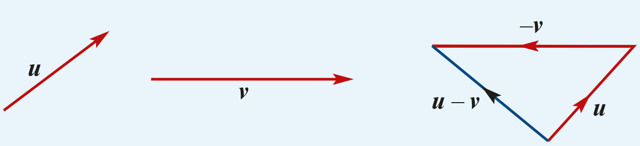
$\therefore \quad k = \dfrac{5}{2}$

Summary 21A

- A **vector** is a set of equivalent **directed line segments**.
- Addition of vectors
  If $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{BC}$, then $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.
- Zero vector
  The **zero vector**, denoted by $\mathbf{0}$, has zero length and has no direction.
- Subtraction of vectors
  $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$



- Parallel vectors
  Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

Skill-sheet

Exercise 21A

Example 1

1 On the same set of axes, draw arrows which represent the following vectors:

a $\begin{bmatrix}1\\5\end{bmatrix}$ **b** $\begin{bmatrix}0\\-2\end{bmatrix}$ **c** $\begin{bmatrix}-1\\-2\end{bmatrix}$ **d** $\begin{bmatrix}-4\\3\end{bmatrix}$

Example 2

2 The vector $\boldsymbol{u}$ is defined by the directed line segment from $(1, 5)$ to $(6, 6)$.
If $\boldsymbol{u} = \begin{bmatrix}a\\b\end{bmatrix}$, find a and b.

3 The vector $\boldsymbol{v}$ is defined by the directed line segment from $(-1, 5)$ to $(2, -10)$.
If $\boldsymbol{v} = \begin{bmatrix}a\\b\end{bmatrix}$, find a and b.

4 Let $A = (1, -2)$, $B = (3, 0)$ and $C = (2, -3)$ and let O be the origin.
Express each of the following vectors in the form $\begin{bmatrix}a\\b\end{bmatrix}$:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

Example 3

5 Let $\boldsymbol{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}1\\-3\end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix}-2\\1\end{bmatrix}$.

a Find:

i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $2\boldsymbol{c} - \boldsymbol{a}$ **iii** $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

b Show that $\boldsymbol{a} + \boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

6 If $A = (2, -3)$, $B = (4, 0)$, $C = (1, -4)$ and O is the origin, sketch the following vectors:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

7 On graph paper, sketch the vectors joining the following pairs of points in the direction indicated:

a $(0, 0) \to (2, 1)$ **b** $(3, 4) \to (0, 0)$ **c** $(1, 3) \to (3, 4)$
d $(2, 4) \to (4, 3)$ **e** $(-2, 2) \to (5, -1)$ **f** $(-1, -3) \to (3, 0)$

8 Identify vectors from Question 7 which are parallel to each other.

9 **a** Plot the points $A(-1, 0)$, $B(1, 4)$, $C(4, 3)$ and $D(2, -1)$ on a set of coordinate axes.

b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$.

c Show that:

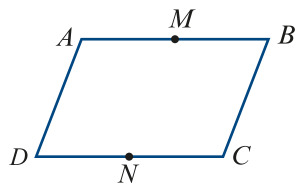
i $\overrightarrow{AB} = \overrightarrow{DC}$ **ii** $\overrightarrow{BC} = \overrightarrow{AD}$

d Describe the shape of the quadrilateral $ABCD$.

10 Find the values of m and n such that $m\begin{bmatrix}3\\-3\end{bmatrix} + n\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}-19\\61\end{bmatrix}$

Example 4

11 Points A, B, C, D are the vertices of a parallelogram, and M and N are the midpoints of AB and DC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AD}$.

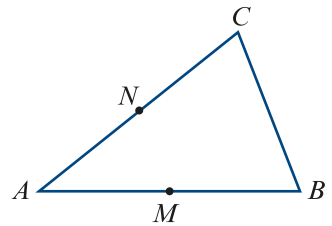


a Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{MD}$ **ii** $\overrightarrow{MN}$

b Find the relationship between $\overrightarrow{MN}$ and $\overrightarrow{AD}$.

12 The figure represents the triangle ABC, where M and N are the midpoints of AB and AC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AC}$.

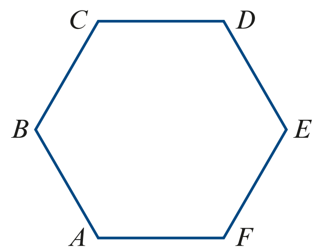


a Express $\overrightarrow{CB}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Hence describe the relation between the two vectors.

Example 5

13 The figure shows a regular hexagon $ABCDEF$. Let $\boldsymbol{a} = \overrightarrow{AF}$ and $\boldsymbol{b} = \overrightarrow{AB}$. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:



a $\overrightarrow{CD}$ **b** $\overrightarrow{ED}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{FC}$

e $\overrightarrow{FA}$ **f** $\overrightarrow{FB}$ **g** $\overrightarrow{FE}$

14 In parallelogram $ABCD$, let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{CA}$ **e** $\overrightarrow{BD}$

15 In triangle OAB, let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. The point P on AB is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$ and the point Q is such that $\overrightarrow{OP} = 3\overrightarrow{PQ}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BA}$ **b** $\overrightarrow{PB}$ **c** $\overrightarrow{OP}$ **d** $\overrightarrow{PQ}$ **e** $\overrightarrow{BQ}$

16 $PQRS$ is a quadrilateral in which $\overrightarrow{PQ} = \boldsymbol{u}$, $\overrightarrow{QR} = \boldsymbol{v}$ and $\overrightarrow{RS} = \boldsymbol{w}$. Express each of the following vectors in terms of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$:

a $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$

17 $OABC$ is a parallelogram. Let $\boldsymbol{u} = \overrightarrow{OA}$ and $\boldsymbol{v} = \overrightarrow{OC}$. Let M be the midpoint of AB.

a Express $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

b Express $\overrightarrow{CM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

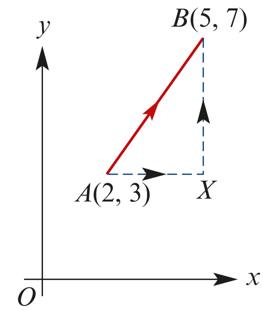
c If P is a point on CM and $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CP}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

d Find $\overrightarrow{OP}$ and hence show that P lies on the line segment OB.

e Find the ratio $OP : PB$.

21B Components of vectors

The vector $\overrightarrow{AB}$ in the diagram is described by the column vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.



From the diagram, we see that the vector $\overrightarrow{AB}$ can also be expressed as the sum

$$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$$

Using column-vector notation:

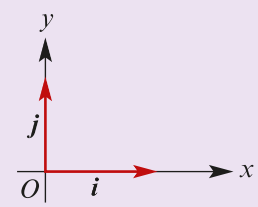
$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

This suggests the introduction of two important vectors.

Standard unit vectors in two dimensions

- Let $\boldsymbol{i}$ be the vector of unit length in the positive direction of the x-axis.
- Let $\boldsymbol{j}$ be the vector of unit length in the positive direction of the y-axis.

Using column-vector notation, we have $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Note: These two vectors also played an important role in our study of linear transformations using matrices in Chapter 20.

For the example above, we have $\overrightarrow{AX} = 3\boldsymbol{i}$ and $\overrightarrow{XB} = 4\boldsymbol{j}$. Therefore

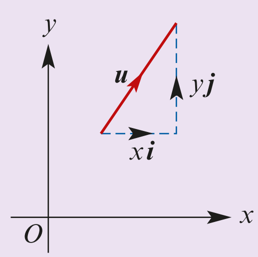
$$\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

It is possible to describe any two-dimensional vector in this way.

Component form

- We can write the vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.

  We say that $\boldsymbol{u}$ is the sum of the two **components** $x\boldsymbol{i}$ and $y\boldsymbol{j}$.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|\boldsymbol{u}|$ and is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.



Operations with vectors now look more like basic algebra:

- $(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
- $k(x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$

Two vectors are equal if and only if their components are equal:

$$x\boldsymbol{i} + y\boldsymbol{j} = m\boldsymbol{i} + n\boldsymbol{j} \quad \text{if and only if} \quad x = m \text{ and } y = n$$

Example 6

a Find $\overrightarrow{AB}$ if $\overrightarrow{OA} = 3\boldsymbol{i}$ and $\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j}$.

b Find $|2\boldsymbol{i} - 3\boldsymbol{j}|$.

Solution

a
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -3\boldsymbol{i} + (2\boldsymbol{i} - \boldsymbol{j}) \\ &= -\boldsymbol{i} - \boldsymbol{j}\end{aligned}$$

b
$$\begin{aligned}|2\boldsymbol{i} - 3\boldsymbol{j}| &= \sqrt{2^2 + (-3)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13}\end{aligned}$$

Example 7

Let A and B be points in the Cartesian plane such that $\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 3\boldsymbol{j}$.

Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

Solution

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ \therefore \quad \overrightarrow{AB} &= -(2\boldsymbol{i} + \boldsymbol{j}) + \boldsymbol{i} - 3\boldsymbol{j} \\ &= -\boldsymbol{i} - 4\boldsymbol{j}\end{aligned}$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{1 + 16} = \sqrt{17}$$

Unit vectors

A **unit vector** is a vector of length one unit. For example, both $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors.

The unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$. (We say 'a hat'.)

Since $|\hat{\boldsymbol{a}}| = 1$, we have

$$|\boldsymbol{a}|\,\hat{\boldsymbol{a}} = \boldsymbol{a}$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Example 8

Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$.

Find $|\boldsymbol{a}|$, the magnitude of $\boldsymbol{a}$, and hence find the unit vector in the direction of $\boldsymbol{a}$.

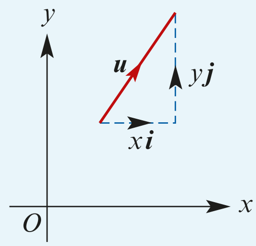
Solution

$$|\boldsymbol{a}| = \sqrt{9 + 16} = 5$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$$

Summary 21B

- A **unit vector** is a vector of length one unit.
- Each vector $\boldsymbol{u}$ in the plane can be written in **component form** as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where:
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.



- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Skill-sheet

Exercise 21B

Example 6a

1 If A and B are points in the plane such that $\overrightarrow{OA} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\overrightarrow{OB} = 3\boldsymbol{i} - 5\boldsymbol{j}$, find $\overrightarrow{AB}$.

2 $OAPB$ is a rectangle with $\overrightarrow{OA} = 5\boldsymbol{i}$ and $\overrightarrow{OB} = 6\boldsymbol{j}$. Express each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

a $\overrightarrow{OP}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

Example 6b

3 Determine the magnitude of each of the following vectors:

a $5\boldsymbol{i}$ **b** $-2\boldsymbol{j}$ **c** $3\boldsymbol{i} + 4\boldsymbol{j}$ **d** $-5\boldsymbol{i} + 12\boldsymbol{j}$

4 The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are given by $\boldsymbol{u} = 7\boldsymbol{i} + 8\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j}$.

a Find $|\boldsymbol{u} - \boldsymbol{v}|$.

b Find constants x and y such that $x\boldsymbol{u} + y\boldsymbol{v} = 44\boldsymbol{j}$.

5 Points A and B have position vectors $\overrightarrow{OA} = 10\boldsymbol{i}$ and $\overrightarrow{OB} = 4\boldsymbol{i} + 5\boldsymbol{j}$. If M is the midpoint of AB, find $\overrightarrow{OM}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

6 $OPAQ$ is a rectangle with $\overrightarrow{OP} = 2\boldsymbol{i}$ and $\overrightarrow{OQ} = \boldsymbol{j}$. Let M be the point on OP such that $OM = \frac{1}{5}OP$ and let N be the point on MQ such that $MN = \frac{1}{6}MQ$.

a Find each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

i $\overrightarrow{OM}$ **ii** $\overrightarrow{MQ}$ **iii** $\overrightarrow{MN}$ **iv** $\overrightarrow{ON}$ **v** $\overrightarrow{OA}$

b **i** Hence show that N is on the diagonal OA.

ii State the ratio of the lengths $ON : NA$.

7 The position vectors of A and B are given by $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Find the distance between A and B.

8 Find the pronumerals in the following equations:

a $\boldsymbol{i} + 3\boldsymbol{j} = 2(\ell\boldsymbol{i} + k\boldsymbol{j})$ **b** $(x - 1)\boldsymbol{i} + y\boldsymbol{j} = 5\boldsymbol{i} + (x - 4)\boldsymbol{j}$

c $(x + y)\boldsymbol{i} + (x - y)\boldsymbol{j} = 6\boldsymbol{i}$ **d** $k(\boldsymbol{i} + \boldsymbol{j}) = 3\boldsymbol{i} - 2\boldsymbol{j} + \ell(2\boldsymbol{i} - \boldsymbol{j})$

Example 7

9 Let $A = (2, 3)$ and $B = (5, 1)$. Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

10 Let $\overrightarrow{OA} = 3\boldsymbol{i}$, $\overrightarrow{OB} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{OC} = -3\boldsymbol{i} + \boldsymbol{j}$. Find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{AC}$ **c** $|\overrightarrow{BC}|$

11 Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Find:

a D such that $\overrightarrow{AB} = \overrightarrow{CD}$ **b** F such that $\overrightarrow{AF} = \overrightarrow{BC}$ **c** G such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

12 Let $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 2\boldsymbol{j}$. Points A, B and C are such that $\overrightarrow{AO} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{BC} = 2\boldsymbol{a}$, where O is the origin. Find the coordinates of A, B and C.

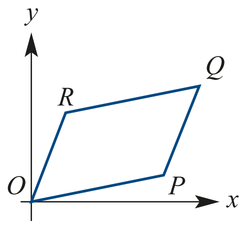
13 A, B, C and D are the vertices of a parallelogram and O is the origin. $A = (2, -1)$, $B = (-5, 4)$ and $C = (1, 7)$.

a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{BC}$ **v** $\overrightarrow{AD}$

b Hence find the coordinates of D.

14 The diagram shows a parallelogram $OPQR$. The points P and Q have coordinates $(12, 5)$ and $(18, 13)$ respectively. Find:



a $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ **b** $|\overrightarrow{RQ}|$ and $|\overrightarrow{OR}|$

15 $A(1, 6)$, $B(3, 1)$ and $C(13, 5)$ are the vertices of a triangle ABC.

a Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

b Hence show that ABC is a right-angled triangle.

16 $A(4, 4)$, $B(3, 1)$ and $C(7, 3)$ are the vertices of a triangle ABC.

a Find the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

c Hence show that triangle ABC is a right-angled isosceles triangle.

17 $A(-3, 2)$ and $B(0, 7)$ are points in the Cartesian plane, O is the origin and M is the midpoint of AB.

a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{BA}$ **iv** $\overrightarrow{BM}$

b Hence find the coordinates of M. (Hint: $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$.)

Example 8

18 Find the unit vector in the direction of each of the following vectors:

a $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ **b** $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$ **c** $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j}$

d $\boldsymbol{d} = \boldsymbol{i} - \boldsymbol{j}$ **e** $\boldsymbol{e} = \frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}$ **f** $\boldsymbol{f} = 6\boldsymbol{i} - 4\boldsymbol{j}$

19 **a** Find a vector of length 4 in the direction of the vector $\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j}$.

b Find a vector of length 10 in the direction of the vector $\boldsymbol{b} = -3\boldsymbol{i} - 4\boldsymbol{j}$.

21C Scalar product of vectors

The scalar product is an operation that takes two vectors and gives a real number.

Definition of the scalar product

We define the **scalar product** of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

For example:

$$(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} - 4\boldsymbol{j}) = 2 \times 1 + 3 \times (-4) = -10$$

The scalar product is often called the **dot product**.

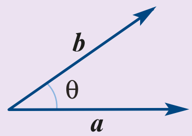
Note: If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Geometric description of the scalar product

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



Proof Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$. Then using the cosine rule in $\triangle OAB$ gives

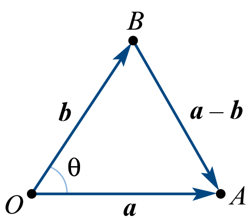
$$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = |\boldsymbol{a} - \boldsymbol{b}|^2$$

$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$2(a_1 b_1 + a_2 b_2) = 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

$$a_1 b_1 + a_2 b_2 = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

$$\therefore \quad \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$



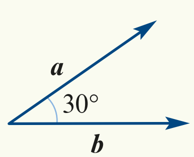
Example 9

a If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

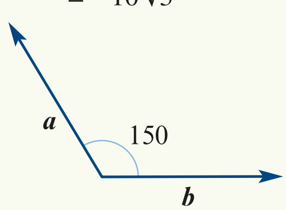
b If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $150°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Solution

a
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 30° \\ &= 20 \times \frac{\sqrt{3}}{2} \\ &= 10\sqrt{3}\end{aligned}$$



b
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 150° \\ &= 20 \times \left(-\frac{\sqrt{3}}{2}\right) \\ &= -10\sqrt{3}\end{aligned}$$



Properties of the scalar product

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

Note: In a more advanced treatment of vectors, it is useful to define two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ to be *orthogonal* if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. We will not need this definition in this chapter.

Finding the magnitude of the angle between two vectors

One important use of the scalar product is to find the angle between two vectors. Consider the two different forms of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos \theta \quad \text{and} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

Therefore

$$\cos \theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Example 10

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \quad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$$

Find the magnitude of $\angle ABC$.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \cos(\angle ABC)$$

We have

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\boldsymbol{i} + 2\boldsymbol{j}) \cdot (-\boldsymbol{i} - 3\boldsymbol{j}) = 1 - 6 = -5$$

$$|\overrightarrow{BA}| = \sqrt{1 + 4} = \sqrt{5}$$

$$|\overrightarrow{BC}| = \sqrt{1 + 9} = \sqrt{10}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{-5}{\sqrt{5}\sqrt{10}} = -\frac{1}{\sqrt{2}}$$

Hence $\angle ABC = \dfrac{3\pi}{4}$.

Summary 21C

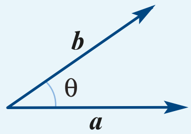
- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

- The scalar product can be described geometrically by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

  where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Skill-sheet

Exercise 21C

1 Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$
e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

2 Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{a} \cdot \boldsymbol{b}$
d $\boldsymbol{a} \cdot \boldsymbol{c}$ **e** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b})$

Example 9

3 **a** If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 45°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.
b If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 135°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

4 Expand and simplify:

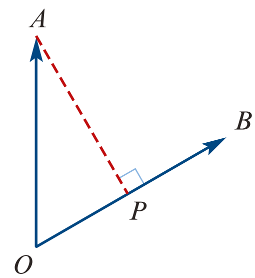
a $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 2\boldsymbol{b})$ **b** $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$
c $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} + \boldsymbol{b})$ **d** $\dfrac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

5 Let C and D be points with position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c} \cdot \boldsymbol{d} = 4$, find $|\overrightarrow{CD}|$.

6 Solve each of the following equations:

a $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (5\boldsymbol{i} + x\boldsymbol{j}) = -6$ **b** $(x\boldsymbol{i} + 7\boldsymbol{j}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j}) = 10$
c $(x\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j}) = x$ **d** $x(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} + x\boldsymbol{j}) = 6$

7 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j}$. Let P be the point on OB such that AP is perpendicular to OB. Then $\overrightarrow{OP} = q\boldsymbol{b}$, for a constant q.



a Express $\overrightarrow{AP}$ in terms of q, $\boldsymbol{a}$ and $\boldsymbol{b}$.
b Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to find the value of q.
c Find the coordinates of the point P.

Example 10

8 Find the angle, in degrees, between each of the following pairs of vectors, correct to two decimal places:

a $\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{i} - 4\boldsymbol{j}$
b $-2\boldsymbol{i} + \boldsymbol{j}$ and $-2\boldsymbol{i} - 2\boldsymbol{j}$
c $2\boldsymbol{i} - \boldsymbol{j}$ and $4\boldsymbol{i}$
d $7\boldsymbol{i} + \boldsymbol{j}$ and $-\boldsymbol{i} + \boldsymbol{j}$

9 If A and B are points defined by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ respectively, find:

a $\overrightarrow{AB}$
b $|\overrightarrow{AB}|$
c the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$.

10 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors such that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Use the geometric description of the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

For Questions 11–12, find the angles in degrees correct to two decimal places.

11 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$ respectively. Let M be the midpoint of AB. Find:

a $\overrightarrow{OM}$
b $\angle AOM$
c $\angle BMO$

12 Let A, B and C be the points defined by the position vectors $3\boldsymbol{i}$, $4\boldsymbol{j}$ and $-2\boldsymbol{i} + 6\boldsymbol{j}$ respectively. Let M and N be the midpoints of AB and AC respectively. Find:

a **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$
b $\angle MON$
c $\angle MOC$

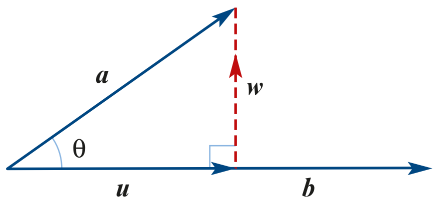
21D Vector projections

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram, it can be seen that

$$\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$$

where $\boldsymbol{u} = k\boldsymbol{b}$ and so $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$.



For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, we must have

$$\boldsymbol{w} \cdot \boldsymbol{b} = 0$$
$$(\boldsymbol{a} - k\boldsymbol{b}) \cdot \boldsymbol{b} = 0$$
$$\boldsymbol{a} \cdot \boldsymbol{b} - k(\boldsymbol{b} \cdot \boldsymbol{b}) = 0$$

Hence $k = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}$ and therefore $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

This vector $\boldsymbol{u}$ is called the **vector projection** (or **vector resolute**) of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Vector resolute

The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be expressed in any one of the following equivalent forms:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Note: The quantity $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that, from our previous calculation, we have $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

Expressing $\boldsymbol{a}$ as the sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, gives

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**, one parallel to $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

Example 11

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$. Find the vector resolute of:

a $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 3 = -2$

$\boldsymbol{b} \cdot \boldsymbol{b} = 1 + 1 = 2$

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \frac{-2}{2}(\boldsymbol{i} - \boldsymbol{j})$$
$$= -1(\boldsymbol{i} - \boldsymbol{j})$$
$$= -\boldsymbol{i} + \boldsymbol{j}$$

b $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = -2$

$\boldsymbol{a} \cdot \boldsymbol{a} = 1 + 9 = 10$

The vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$$\frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} = \frac{-2}{10}(\boldsymbol{i} + 3\boldsymbol{j})$$
$$= -\frac{1}{5}(\boldsymbol{i} + 3\boldsymbol{j})$$

Example 12

Find the scalar resolute of $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ in the direction of $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$.

Solution

$\boldsymbol{a} \cdot \boldsymbol{b} = -2 + 6 = 4$

$|\boldsymbol{b}| = \sqrt{1 + 9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{10}}{5}$$

Example 13

Resolve $\boldsymbol{i} + 3\boldsymbol{j}$ into rectangular components, one of which is parallel to $2\boldsymbol{i} - 2\boldsymbol{j}$.

Solution

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - 6 = -4$$

$$\boldsymbol{b} \cdot \boldsymbol{b} = 4 + 4 = 8$$

Therefore the vector resolute is

$$\frac{-4}{8}(2\boldsymbol{i} - 2\boldsymbol{j}) = -\frac{1}{2}(2\boldsymbol{i} - 2\boldsymbol{j}) = -\boldsymbol{i} + \boldsymbol{j}$$

The perpendicular component is

$$\boldsymbol{a} - (-\boldsymbol{i} + \boldsymbol{j}) = (\boldsymbol{i} + 3\boldsymbol{j}) - (-\boldsymbol{i} + \boldsymbol{j}) = 2\boldsymbol{i} + 2\boldsymbol{j}$$

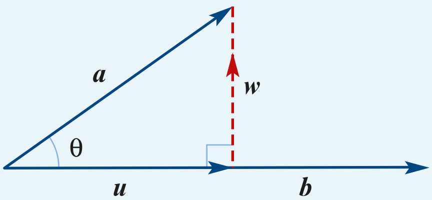
Hence we can write

$$\boldsymbol{i} + 3\boldsymbol{j} = (-\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} + 2\boldsymbol{j})$$

Check: We can check our calculation by verifying that the second component is indeed perpendicular to $\boldsymbol{b}$. We have $(2\boldsymbol{i} + 2\boldsymbol{j}) \cdot (2\boldsymbol{i} - 2\boldsymbol{j}) = 4 - 4 = 0$, as expected.

Summary 21D

- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.



Exercise 21D

1 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$.

a Find $\hat{\boldsymbol{a}}$. **b** Find $\hat{\boldsymbol{b}}$. **c** Find $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = \overrightarrow{AB}$.

2 Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $|\boldsymbol{b}|$

b Find the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

3 Points A and B are defined by the position vectors $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $\hat{\boldsymbol{b}}$

b Find the unit vector which bisects $\angle AOB$.

Example 11

4 For each pair of vectors, find the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:

a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

b $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

c $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{i}$

Example 12

5 For each of the following pairs of vectors, find the scalar resolute of the first vector in the direction of the second vector:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$

c $\boldsymbol{b} = 2\boldsymbol{j}$ and $\boldsymbol{a} = 2\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$

d $\boldsymbol{b} = \boldsymbol{i} - \sqrt{5}\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$

Example 13

6 For each of the following pairs of vectors, find the resolution of the vector $\boldsymbol{a}$ into rectangular components, one of which is parallel to $\boldsymbol{b}$:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 5\boldsymbol{i}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$

c $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$

7 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b a unit vector perpendicular to OB

8 Let A and B be the points defined by the position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$

c the shortest distance from A to line OB

9 Points A, B and C have position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

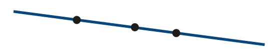
c the shortest distance from B to line AC

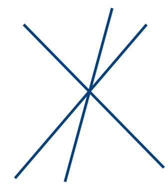
d the area of triangle ABC

21E Geometric proofs

In this section we see how vectors can be used as a tool for proving geometric results.

We require the following two definitions:





- **Collinear points** Three or more points are **collinear** if they all lie on a single line.
- **Concurrent lines** Three or more lines are **concurrent** if they all pass through a single point.

Here are some properties of vectors that will be useful:

Parallel vectors

- For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R} \setminus \{0\}$, then A, B and C are collinear.

Scalar product

- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Linear combinations of non-parallel vectors

- For two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that are not parallel, if $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$, then $m = p$ and $n = q$.

Example 14

Three points P, Q and R have position vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $k(2\boldsymbol{p} + \boldsymbol{q})$ respectively, relative to a fixed origin O. The points O, P and Q are not collinear. Find the value of k if:

a $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$ **b** $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$ **c** P, Q and R are collinear.

Solution

a
$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR}\\ &= -\boldsymbol{q} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= 2k\boldsymbol{p} + (k-1)\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$, then there is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$2k\boldsymbol{p} + (k-1)\boldsymbol{q} = \lambda\boldsymbol{p}$$

This implies that

$$2k = \lambda \quad \text{and} \quad k - 1 = 0$$

Hence $k = 1$.

b
$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PO} + \overrightarrow{OR}\\ &= -\boldsymbol{p} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= (2k-1)\boldsymbol{p} + k\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$, then there is some $m \in \mathbb{R} \setminus \{0\}$ such that

$$(2k-1)\boldsymbol{p} + k\boldsymbol{q} = m\boldsymbol{q}$$

This implies that

$$2k - 1 = 0 \quad \text{and} \quad k = m$$

Hence $k = \frac{1}{2}$.

Note: Since points O, P and Q are not collinear, the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are not parallel.

c If points P, Q and R are collinear, then there exists $n \in \mathbb{R} \setminus \{0\}$ such that

$$n\overrightarrow{PQ} = \overrightarrow{QR}$$

$$n(-\boldsymbol{p} + \boldsymbol{q}) = 2k\boldsymbol{p} + (k-1)\boldsymbol{q}$$

$$\therefore \quad -n\boldsymbol{p} + n\boldsymbol{q} = 2k\boldsymbol{p} + (k-1)\boldsymbol{q}$$

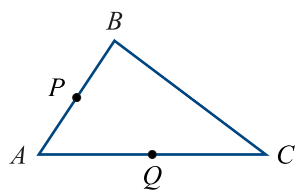
This implies that

$$-n = 2k \quad \text{and} \quad n = k - 1$$

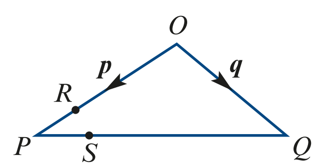
Therefore $3k - 1 = 0$ and so $k = \frac{1}{3}$.

Exercise 21E

1 Consider a triangle ABC. Let P be the midpoint of side AB and let Q be the midpoint of side AC. Prove that $PQ = \dfrac{1}{2}BC$ and that PQ is parallel to BC.



2 In the diagram, $OR = \dfrac{4}{5}OP$, $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and $PS : SQ = 1 : 4$.



a Express each of the following in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$:

i $\overrightarrow{OR}$ **ii** $\overrightarrow{RP}$ **iii** $\overrightarrow{PO}$

iv $\overrightarrow{PS}$ **v** $\overrightarrow{RS}$

b What can be said about line segments RS and OQ?

c What type of quadrilateral is $ORSQ$?

d The area of triangle PRS is 5 cm^2. What is the area of $ORSQ$?

3 The position vectors of three points A, B and C relative to an origin O are $\boldsymbol{a}$, $\boldsymbol{b}$ and $k\boldsymbol{a}$ respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC and is such that $CQ = 6QB$.

a Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i the position vector of P

ii the position vector of Q

b Given that OPQ is a straight line, find:

i the value of k

ii the ratio $\dfrac{OP}{PQ}$

c The position vector of a point R is $\dfrac{7}{3}\boldsymbol{a}$. Show that PR is parallel to BC.

Example 14

4 The position vectors of two points A and B relative to an origin O are $3\boldsymbol{i} + 3.5\boldsymbol{j}$ and $6\boldsymbol{i} - 1.5\boldsymbol{j}$ respectively.

a **i** Given that $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ and $\overrightarrow{AE} = \frac{1}{4}\overrightarrow{AB}$, write down the position vectors of D and E.

ii Hence find $|\overrightarrow{ED}|$.

b Given that OE and AD intersect at X and that $\overrightarrow{OX} = p\overrightarrow{OE}$ and $\overrightarrow{XD} = q\overrightarrow{AD}$, find the position vector of X in terms of:

i p **ii** q

c Hence determine the values of p and q.

5 Points P and Q have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, with reference to an origin O, and M is the point on PQ such that

$$\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$$

a Prove that the position vector of M is given by $\boldsymbol{m} = \dfrac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha + \beta}$.

b Write the position vectors of P and Q as $\boldsymbol{p} = k\boldsymbol{a}$ and $\boldsymbol{q} = \ell\boldsymbol{b}$, where k and ℓ are positive real numbers and $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors.

i Prove that the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\boldsymbol{a} + \boldsymbol{b})$.

ii If M is the point where the internal bisector of $\angle POQ$ meets PQ, show that

$$\frac{\alpha}{\beta} = \frac{k}{\ell}$$

6 A **rhombus** is a parallelogram with all sides of equal length. Suppose that $OABC$ is a rhombus. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$

b Find $\overrightarrow{OB} \cdot \overrightarrow{AC}$.

c Prove that the diagonals of a rhombus intersect at right angles.

7 Suppose that $ORST$ is a parallelogram, where O is the origin. Let U be the midpoint of RS and let V be the midpoint of ST. Denote the position vectors of R, S, T, U and V by $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.

a Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$.

b Express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.

c Hence, or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

8 Prove that, for any quadrilateral, the midpoints of the four sides are the vertices of a parallelogram.

9 Prove that the diagonals of a square are of equal length and bisect each other.

10 Prove that the diagonals of a parallelogram bisect each other.

11 Apollonius' theorem

For $\triangle OAB$, the point C is the midpoint of side AB. Prove that:

a $4\overrightarrow{OC} \cdot \overrightarrow{OC} = OA^2 + OB^2 + 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

b $4\overrightarrow{AC} \cdot \overrightarrow{AC} = OA^2 + OB^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

c $2OC^2 + 2AC^2 = OA^2 + OB^2$

12 If P is any point in the plane of rectangle $ABCD$, prove that $PA^2 + PC^2 = PB^2 + PD^2$.

13 A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. Prove that the medians bisecting the equal sides of an isosceles triangle are equal.

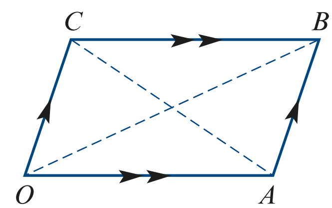
14 **a** Prove that if $(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$ and $(\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} = 0$, then $(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{c} = 0$.

b An **altitude** of a triangle is a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side. Use part **a** to prove that the altitudes of a triangle are concurrent.

15 For a parallelogram $OABC$, prove that

$$OB^2 + AC^2 = 2OA^2 + 2OC^2$$

That is, prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.



21F Applications of vectors: displacement and velocity

For the next three sections, we will be working with vector and scalar quantities:

- A **vector quantity** has both magnitude and direction. We will introduce the vector quantities displacement, velocity and force.
- A **scalar quantity** has only magnitude. We will use the scalar quantities distance, time, speed and mass.

Displacement

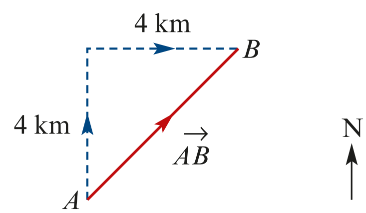
We have been describing points in the plane using position vectors. Points A and B have position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

If an object moves from point A to point B, then the **displacement** of the object is the change in position of the object; it is described by the vector $\overrightarrow{AB}$.

For example, suppose that a person walks 4 km north and then 4 km east.

The person's displacement is $4\sqrt{2}$ km north-east.

Note: The total distance that the person has walked is 8 km, which is not equal to the magnitude of the displacement vector.



Example 15

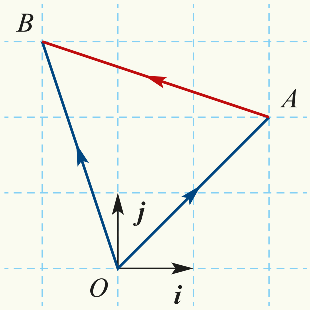
A particle moves from point $A(2, 2)$ to point $B(-1, 3)$. Express the displacement vector of the particle in component form.

Solution

We have $\overrightarrow{OA} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\overrightarrow{OB} = -\boldsymbol{i} + 3\boldsymbol{j}$.

The displacement vector is

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -(2\boldsymbol{i} + 2\boldsymbol{j}) + (-\boldsymbol{i} + 3\boldsymbol{j}) \\ &= -3\boldsymbol{i} + \boldsymbol{j}\end{aligned}$$



Displacement problems involving bearings can be solved using the trigonometric techniques demonstrated in Section 15F.

Velocity

Velocity is the rate of change of position with respect to time.

Velocity is a vector quantity; it has magnitude and direction. The units of velocity which will be used in this chapter are metres per second (m/s) and kilometres per hour (km/h).

Some examples of velocity vectors are:

- 80 km/h in the direction north
- 10 km/h on a bearing of 080°
- $3\boldsymbol{i} + 4\boldsymbol{j}$ m/s

The first two vectors have magnitudes 80 km/h and 10 km/h respectively. The third vector has magnitude $|3\boldsymbol{i} + 4\boldsymbol{j}| = \sqrt{3^2 + 4^2} = 5$ m/s. The magnitude of velocity is called **speed**.

Motion with constant velocity

In this chapter, we only deal with constant velocity (that is, the velocity does not change over a particular time interval). Consider the following two examples:

- If a car travels for 2 hours with a constant velocity of 80 km/h north, then its displacement is $2 \times 80 = 160$ km north.
- If a particle starts at the origin and moves with a velocity of $3\boldsymbol{i} + 4\boldsymbol{j}$ m/s for 2 seconds, then its position is $2(3\boldsymbol{i} + 4\boldsymbol{j}) = 6\boldsymbol{i} + 8\boldsymbol{j}$ m.

If an object moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by

$$\boldsymbol{s} = t\boldsymbol{v}$$

Note: Here $\boldsymbol{s}$ and $\boldsymbol{v}$ are vector quantities and t is a scalar quantity. So this is an example of scalar multiplication.

Example 16

A particle starts at the point A with position vector $\overrightarrow{OA} = \boldsymbol{i} + 3\boldsymbol{j}$, where the unit is metres. The particle begins moving with a constant velocity of $2\boldsymbol{i} + 4\boldsymbol{j}$ m/s. Find the position vector of the particle after:

a 5 seconds

b t seconds.

Solution

a Let P be the point that the particle reaches after 5 seconds. Then

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + 5(2\boldsymbol{i} + 4\boldsymbol{j})\\ &= \boldsymbol{i} + 3\boldsymbol{j} + 10\boldsymbol{i} + 20\boldsymbol{j}\\ &= 11\boldsymbol{i} + 23\boldsymbol{j}\end{aligned}$$

b Let Q be the point that the particle reaches after t seconds. Then

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OA} + t(2\boldsymbol{i} + 4\boldsymbol{j})\\ &= \boldsymbol{i} + 3\boldsymbol{j} + 2t\boldsymbol{i} + 4t\boldsymbol{j}\\ &= (1 + 2t)\boldsymbol{i} + (3 + 4t)\boldsymbol{j}\end{aligned}$$

Direction of motion

The velocity vector is in the direction of motion. We often use the unit vector of the velocity vector to describe the direction of motion.

For example, if $\boldsymbol{v} = 3\boldsymbol{i} + 4\boldsymbol{j}$, then the unit vector $\hat{\boldsymbol{v}} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ is in the direction of motion.

Example 17

Particle A starts moving from point O with a constant velocity of $\boldsymbol{v}_A = 3\boldsymbol{i} + 4\boldsymbol{j}$ m/s. Three seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 7 m/s. When and where will B catch up to A?

Solution

At time t seconds, particle A is at the point with position vector

$$\overrightarrow{OP}_A = t(3\boldsymbol{i} + 4\boldsymbol{j})$$

At time t seconds, for $t \geq 3$, particle B has been moving for $t - 3$ seconds and is at the point with position vector

$$\overrightarrow{OP}_B = \frac{7(t-3)}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$$

The two particles are at the same point when

$$\begin{aligned}\frac{7(t-3)}{5} &= t\\ 7(t-3) &= 5t\\ 2t &= 21\\ \therefore \quad t &= \frac{21}{2}\end{aligned}$$

Particle B catches up to particle A at time $t = 10.5$ seconds.

At this time, both particles have position vector $31.5\boldsymbol{i} + 42\boldsymbol{j}$.

Example 18

A particle starts from O with a constant velocity of $\boldsymbol{v}_1 = 3\boldsymbol{i} + 4\boldsymbol{j}$ m/s. At the same time, a second particle starts moving with constant velocity from point B, where $\overrightarrow{OB} = 25\boldsymbol{j}$.

Given that the two particles meet and their paths are at right angles, find:

a the position vector of the point where they meet

b the velocity of the second particle.

Solution

a Assume that the particles meet at the point P at time t seconds. Since their paths are at right angles, we have

$$\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$$

At time t seconds, the position vector of the first particle is

$$\overrightarrow{OP} = t(3\boldsymbol{i} + 4\boldsymbol{j}) = 3t\boldsymbol{i} + 4t\boldsymbol{j}$$

Therefore

$$\begin{aligned}\overrightarrow{BP} &= \overrightarrow{BO} + \overrightarrow{OP}\\ &= -25\boldsymbol{j} + (3t\boldsymbol{i} + 4t\boldsymbol{j})\\ &= 3t\boldsymbol{i} + (4t - 25)\boldsymbol{j}\end{aligned}$$

Since $\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$, we obtain

$$\begin{aligned}(3t\boldsymbol{i} + 4t\boldsymbol{j}) \cdot (3t\boldsymbol{i} + (4t - 25)\boldsymbol{j}) &= 0\\ 9t^2 + 4t(4t - 25) &= 0\\ 25t^2 - 100t &= 0\\ \therefore \quad t(t - 4) &= 0\end{aligned}$$

The particles do not meet at time 0 s, so they meet at time $t = 4$ s.

The position vector of the point where they meet is

$$\overrightarrow{OP} = 4(3\boldsymbol{i} + 4\boldsymbol{j}) = 12\boldsymbol{i} + 16\boldsymbol{j}$$

b Let $\boldsymbol{v}$ m/s be the velocity of the second particle. We use the formula $\boldsymbol{s} = t\boldsymbol{v}$.

At time $t = 4$, the displacement of the second particle is $\overrightarrow{BP}$. Therefore

$$\begin{aligned}\overrightarrow{BP} &= 4\boldsymbol{v}\\ \overrightarrow{BO} + \overrightarrow{OP} &= 4\boldsymbol{v}\\ -25\boldsymbol{j} + (12\boldsymbol{i} + 16\boldsymbol{j}) &= 4\boldsymbol{v}\\ 12\boldsymbol{i} - 9\boldsymbol{j} &= 4\boldsymbol{v}\end{aligned}$$

Hence the velocity of the second particle is $\boldsymbol{v} = 3\boldsymbol{i} - \frac{9}{4}\boldsymbol{j}$ m/s.

Summary 21F

- The **displacement** of a particle is the change in its position. If a particle moves from point A to point B, then its displacement is $\overrightarrow{AB}$.
- The **velocity** of a particle is the rate of change of its position with respect to time.
- Displacement and velocity are vector quantities. The magnitude of velocity is **speed**.
- Motion with constant velocity If a particle moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by $\boldsymbol{s} = t\boldsymbol{v}$.

Exercise 21F

Example 15

1 For each of the following, find the displacement vector in component form for a particle that moves from point A to point B:

a $A(3, 7)$, $B(2, -4)$ **b** $A(-2, 4)$, $B(3, -2)$ **c** $A(3, 1)$, $B(4, 6)$
d $A(3, 7)$, $B(3, -4)$ **e** $A(-2, -7)$, $B(2, -7)$ **f** $A(5, -6)$, $B(11, 5)$

2 From the point O, a hiker walks 5 km north and then 8 km on a bearing of 330°, finishing at a point A. Describe the displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing. (Hint: Use the cosine rule and the sine rule.)

3 From the point O, a yacht sails 3 km east and then 5 km on a bearing of 060°, finishing at a point A. Describe the displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing.

4 Give the corresponding speed for each of the following velocity vectors:

a $5\boldsymbol{i} + 4\boldsymbol{j}$ m/s **b** $3\boldsymbol{i} - 4\boldsymbol{j}$ m/s **c** $-\boldsymbol{i} + 4\boldsymbol{j}$ m/s
d $-2\boldsymbol{i} - 6\boldsymbol{j}$ m/s **e** $5\boldsymbol{i} - 12\boldsymbol{j}$ m/s **f** $-7\boldsymbol{i} + 11\boldsymbol{j}$ m/s

In each of the following questions, the unit of distance is metres.

Example 16

5 A particle starts from the point A with position vector $\overrightarrow{OA} = -\boldsymbol{i} + 2\boldsymbol{j}$ and moves with a constant velocity of $5\boldsymbol{i} + 12\boldsymbol{j}$ m/s. Find the position vector of the particle after:

a 5 seconds **b** t seconds.

6 An object takes 5 seconds to move with constant velocity from point A to point B, where $\overrightarrow{OA} = 5\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{OB} = -15\boldsymbol{i} + 24\boldsymbol{j}$. Find the velocity of the object.

7 A particle starts from the point B with position vector $\overrightarrow{OB} = -2\boldsymbol{i} + 3\boldsymbol{j}$ and moves with a constant velocity of $7\boldsymbol{i} + 24\boldsymbol{j}$ m/s.

a Find the position vector of the particle after:

i 4 seconds **ii** t seconds.

b Find the particle's distance from the origin after:

i 4 seconds **ii** t seconds.

8 Let O be the origin and let A and B be the points with $\overrightarrow{OA} = 5\boldsymbol{i} + 2\boldsymbol{j}$ and $\overrightarrow{OB} = -5\boldsymbol{i} - 3\boldsymbol{j}$. A particle moves with constant velocity from A to B in 10 seconds. Find:

a the velocity of the particle **b** the speed of the particle.

Example 17

9 Particle A starts moving from point O with a constant velocity of $\boldsymbol{v}_A = \boldsymbol{i} + 2\boldsymbol{j}$ m/s. Two seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 6 m/s. When and where will B catch up to A?

Example 18

10 A particle starts from O with a constant velocity of $\boldsymbol{v}_1 = 2\boldsymbol{i} + \boldsymbol{j}$ m/s. At the same time, a second particle starts moving with constant velocity from point B, where $\overrightarrow{OB} = 20\boldsymbol{j}$. Given that the two particles meet and their paths are at right angles, find:

a the position vector of the point where they meet

b the velocity of the second particle.

11 Points A and B have position vectors $\overrightarrow{OA} = 10\boldsymbol{j}$ and $\overrightarrow{OB} = 20\boldsymbol{i}$. A particle starts moving from point A with a constant velocity of $\boldsymbol{v}_1 = 2\boldsymbol{i}$ m/s. At the same time, a second particle starts moving from point B with constant velocity. Given that the two particles meet and their paths are at right angles, find:

a the position vector of the point where they meet

b the velocity of the second particle.

21G Applications of vectors: relative velocity

Resultant velocity

If two or more velocity vectors are added, then the sum is called a **resultant velocity**.

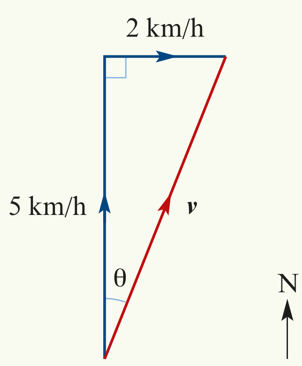
Example 19

A river is flowing north at 5 km/h. Mila can swim at 2 km/h in still water. She dives in from the west bank of the river and swims towards the opposite bank.

a In which direction does she travel? **b** What is her actual speed?

Solution

The swimmer's actual velocity, $\boldsymbol{v}$, is the vector sum of her velocity relative to the water (2 km/h east) and the water's velocity (5 km/h north).



a From the diagram, we have

$$\tan\theta = \frac{2}{5}$$

$$\therefore \quad \theta \approx 21.8°$$

She is travelling on a bearing of 022°.

b Her actual speed is

$$|\boldsymbol{v}| = \sqrt{2^2 + 5^2} \approx 5.39 \text{ km/h}$$

Relative velocity

In the previous example, the velocity of the water is given *relative to the bank* and the velocity of the swimmer is given *relative to the water*. The velocity of the swimmer *relative to the bank* is found by taking the vector sum. That is:

Velocity of swimmer relative to bank	=	Velocity of swimmer relative to water	+	Velocity of water relative to bank

The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.

Consider another example: A train is travelling north at 60 km/h, and a passenger walks at 3 km/h along the corridor towards the back of the train.

Velocity of passenger relative to Earth	=	Velocity of passenger relative to train	+	Velocity of train relative to Earth

The passenger is moving with a velocity of 57 km/h north relative to Earth.

In general, if an object A is in motion relative to another object B, then we can find the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

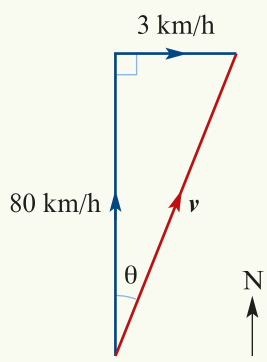
Velocities measured relative to Earth are often called **true velocities** or **actual velocities**.

Example 20

A train is moving with a constant velocity of 80 km/h north. A passenger walks straight across a carriage from the west side to the east side at 3 km/h. What is the true velocity of the passenger?

Solution

The passenger's true velocity, $\boldsymbol{v}$, is the vector sum of his velocity relative to the train (3 km/h east) and the train's velocity (80 km/h north).



Speed:
$$|\boldsymbol{v}| = \sqrt{80^2 + 3^2} = \sqrt{6409} \approx 80.06 \text{ km/h}$$

Direction:
$$\tan\theta = \frac{3}{80}$$
$$\therefore \quad \theta \approx 2.15^\circ$$

The passengers's true velocity is 80.06 km/h on a bearing of 002°.

Example 21

Car A is moving with a velocity of 50 km/h due north, while car B is moving with a velocity of 120 km/h due west. What is the velocity of car A relative to car B?

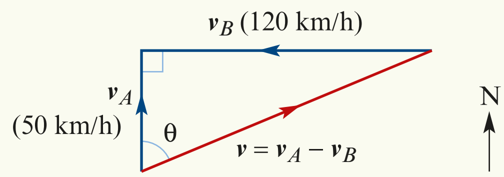
Solution

Let $\boldsymbol{v}_A$ be the velocity of car A, and let $\boldsymbol{v}_B$ be the velocity of car B.

The velocity of car A relative to car B is given by $\boldsymbol{v} = \boldsymbol{v}_A - \boldsymbol{v}_B$.

Speed: $|\boldsymbol{v}| = \sqrt{50^2 + 120^2}$
$= 130$ km/h

Direction: $\theta = \tan^{-1}\left(\frac{12}{5}\right)$
$\approx 67.38°$



The velocity of car A relative to car B is 130 km/h on a bearing of 067°.

Wind effect on flight paths

The **airspeed** of an aircraft is its speed relative to air. In the next example, we see how the wind affects the actual velocity of an aircraft.

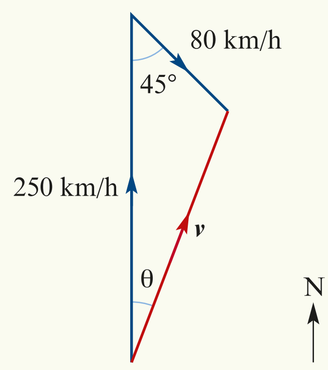
Example 22

A light aircraft has an airspeed of 250 km/h. The pilot sets a course due north. If the wind is blowing from the north-west at 80 km/h, what is the true speed and direction of the aircraft?

Solution

We can use the cosine rule to find the true speed:

$$|\boldsymbol{v}| = \sqrt{250^2 + 80^2 - 2 \times 250 \times 80 \cos 45°}$$
$$= 201.5334\ldots \text{ km/h}$$



We can now use the sine rule to find the angle θ:

$$\frac{80}{\sin\theta} = \frac{|\boldsymbol{v}|}{\sin 45°}$$

$$\sin\theta = \frac{80 \sin 45°}{|\boldsymbol{v}|}$$
$$= 0.2806\ldots$$

$\therefore \quad \theta \approx 16.30°$

The aircraft is flying at 201.53 km/h on a bearing of 016°.

To fly an aircraft in a given direction, the pilot must compensate for the effect of the wind.

Example 23

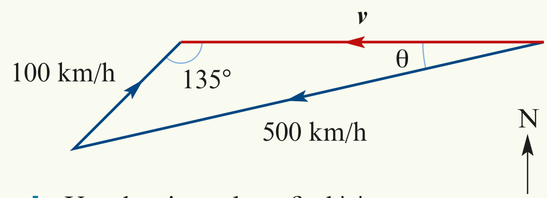
An aeroplane is scheduled to travel from a point P to a point Q, which is 1000 km due west of P. The aeroplane's airspeed is 500 km/h and the wind is blowing from the south-west at 100 km/h.

a In which direction should the pilot set the course?

b How long will the flight take?

Solution

We want to ensure that the plane's true velocity, $\boldsymbol{v}$, is due west.



a Use the sine rule to find θ:

$$\frac{500}{\sin 135^\circ} = \frac{100}{\sin\theta}$$

$$\sin\theta = \frac{100\sin 135^\circ}{500}$$

$$= 0.1414\ldots$$

$$\therefore \quad \theta = (8.130\ldots)^\circ$$

The pilot should head on a bearing of 262°.

b Use the sine rule to find $|\boldsymbol{v}|$:

$$\frac{500}{\sin 135^\circ} = \frac{|\boldsymbol{v}|}{\sin(36.869\ldots)^\circ}$$

$$\therefore \quad |\boldsymbol{v}| = \frac{500\sin(36.869\ldots)^\circ}{\sin 135^\circ}$$

$$= 424.264\ldots$$

$$\approx 424.26 \text{ km/h}$$

The plane's speed relative to the ground is approximately 424 km/h. The flight will take approximately 2.4 hours.

Summary 21G

- If two or more velocity vectors are added, then the sum is called a **resultant velocity**.
- The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.
- If an object A is in motion relative to another object B, we can find the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

Exercise 21G

Example 19

1 A river is flowing south at 4 km/h. Max can swim at 3 km/h in still water. He dives in from the west bank of the river and swims towards the opposite bank.

a In which direction does he travel? **b** What is his actual speed?

Example 20

2 A train is moving due north at 100 km/h. A passenger walks straight across a carriage from the east side to the west side at 4 km/h. What is the true velocity of the passenger?

3 Cars A and B are driving along a straight level road that runs east–west.

a If car A has a velocity of 100 km/h west and car B has a velocity of 80 km/h west, what is the velocity of car A relative to car B?

b If car A has a velocity of 100 km/h west and car B has a velocity of 80 km/h east, what is the velocity of car A relative to car B?

4 A cricketer is on a moving walkway which runs from south to north at 2 m/s. He bowls his fastest delivery, which is 45 m/s, again in a direction north. What is the velocity of the ball (relative to Earth)?

5 A ship is moving in a straight line at 15 m/s. A bird flies horizontally from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?

6 Car A is travelling north at 60 km/h along a straight level road. Car B is on the same road travelling north at 40 km/h. Find:

a the velocity of car A relative to car B **b** the velocity of car B relative to car A.

7 A plane is heading due north, its airspeed is 240 km/h and there is an 80 km/h wind blowing from west to east. What is the velocity of the plane relative to Earth?

Example 21

8 Car A is moving with a velocity of 60 km/h due north, while car B is moving with a velocity of 80 km/h due west. What is the velocity of car A relative to car B?

9 A glider P is travelling due north at 60 km/h, and another glider Q is travelling north-west at 40 km/h. Find the velocity of P relative to Q.

10 Two particles, A and B, are moving with constant velocities of $\boldsymbol{v}_A = 4\boldsymbol{i} - 3\boldsymbol{j}$ m/s and $\boldsymbol{v}_B = 5\boldsymbol{i} - 7\boldsymbol{j}$ m/s respectively.

a Find the velocity of B relative to A.

b Find the magnitude of this relative velocity.

11 A ship is moving in a straight line at 15 m/s. A bird flies at an angle of 18° to the horizontal from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?

Example 22

12 A light aircraft has an airspeed of 240 km/h. The pilot sets a course due north. The wind is blowing from the north-east at 70 km/h. What is the true speed and direction of the aircraft?

Example 23

13 An aeroplane with an airspeed of 200 km/h is flying to an airport south-west of its present position. There is a wind blowing at 70 km/h from the east.

a Find the course that the pilot must set.

b Find the speed of the aeroplane relative to the ground.

14 A canoeist can paddle at 2 m/s in still water. He wishes to go straight across a river so that his path is at right angles to the banks of the river. The river is flowing at 1.5 m/s.

a Find the direction in which he must paddle.

b If the river is 60 m wide, how long will it take him to cross the river?

21H Applications of vectors: forces and equilibrium

A **force** is a measure of the strength of a *push* or *pull*. Forces can start motion, stop motion, make objects move faster or slower, and change the direction of motion.

Force can be defined as the physical quantity that causes a change in motion.

We will focus on situations where the forces 'cancel each other out'. For example, if an inflatable raft is floating in a swimming pool, then the water is exerting an upwards force on the raft (called the buoyant force) that cancels out the downwards force of gravity.

Introduction to forces

A force has both magnitude and direction – it may be represented by a vector.

When considering the forces that act on an object, it is convenient to treat the forces as acting on a single particle. The single particle may be thought of as a point at which the entire mass of the object is concentrated.

Weight and units of force

Every object near the surface of the Earth is subject to the force of gravity. We refer to this force as the **weight** of the object. Weight is a force that acts vertically downwards on an object (actually towards the centre of the Earth).

The unit of force used in this section is the **kilogram weight** (kg wt). If an object has a **mass** of 1 kg, then the force due to gravity acting on the object is 1 kg wt.

This unit is convenient for objects near the Earth's surface. An object with a mass of 1 kg would have a different weight on the moon.

Note: The standard unit of force is the newton (N). At the Earth's surface, a mass of m kg has a force of m kg wt = mg N acting on it, where g m/s^2 is the acceleration due to gravity ($g \approx 9.8$).

Resultant force and equilibrium

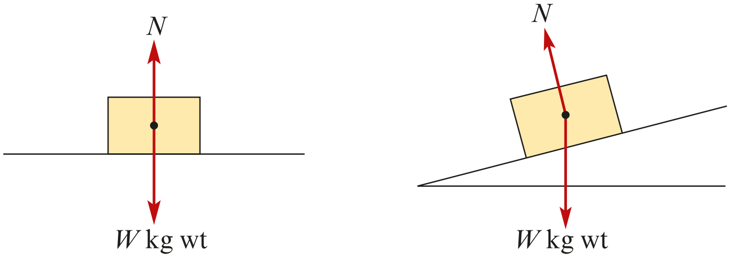
When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**. The resultant force is the vector sum of the forces acting on the particle.

If the resultant force acting on an object is zero, the object will remain at rest or continue moving with constant velocity. The object is said to be in **equilibrium**.

Note: Planet Earth is moving and our galaxy is moving, but we use Earth as our frame of reference and so our observation of an object being at rest is determined in this way.

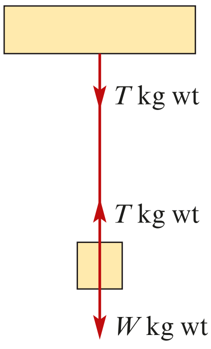
Normal force

Any mass placed on a surface, either horizontal or inclined, experiences a force perpendicular to the surface. This force is referred to as a **normal force**.



For example, a book sitting on a table is obviously being subjected to a force due to gravity. But the fact that it does not fall to the ground indicates that there must be a second force on the book. The table is exerting a force on the book equal in magnitude to gravity, but in the opposite direction. Hence the book remains at rest; it is in **equilibrium**.

Tension force



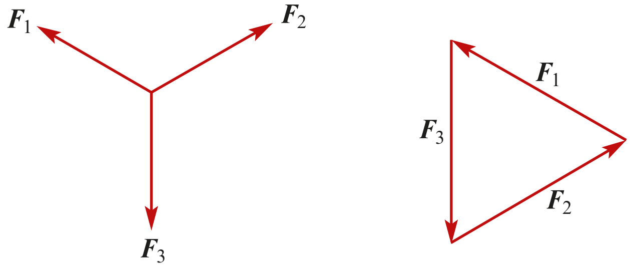
The diagram shows a string attached to the ceiling supporting a mass, which is at rest. The force of gravity, W kg wt, acts downwards on the mass and the string exerts an equal force, T kg wt, upwards on the mass. The force exerted by the string is called the **tension force**.

Note that there is a force, equal in magnitude but opposite in direction, acting on the ceiling at the point of contact.

Triangle of forces

If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

Suppose that three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on a particle in equilibrium, as shown in the diagram on the left. Since the particle is in equilibrium, we must have $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \mathbf{0}$. Therefore the three forces can be rearranged into a triangle as shown on the right.

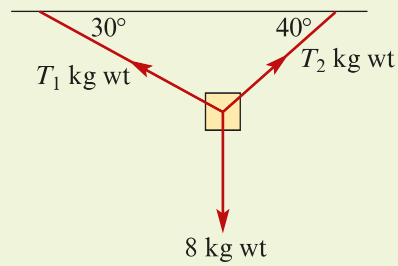


The magnitudes of the forces and the angles between the forces can now be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.

In the following examples and exercise, strings and ropes are considered to have negligible mass. A smooth light pulley is considered to have negligible mass and the friction between a rope and pulley is considered to be negligible.

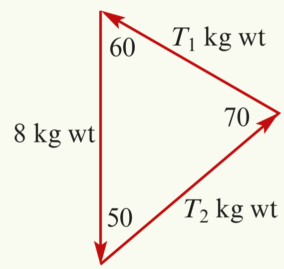
Example 24

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 40° to the horizontal, find the tension in each string.



Solution

Represent the forces in a triangle. The sine rule gives



$$\frac{T_1}{\sin 50°} = \frac{T_2}{\sin 60°} = \frac{8}{\sin 70°}$$

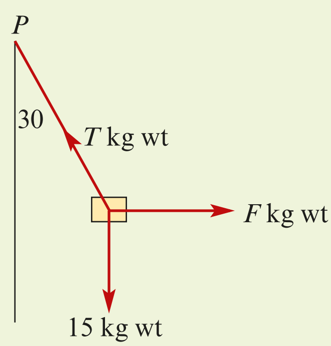
$$T_1 = \frac{8 \sin 50°}{\sin 70°} \approx 6.52 \text{ kg wt}$$

$$T_2 = \frac{8 \sin 60°}{\sin 70°} \approx 7.37 \text{ kg wt}$$

Example 25

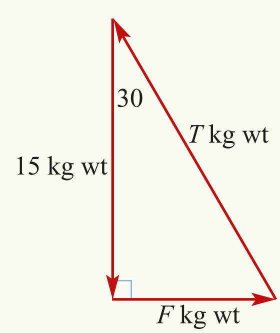
A particle of mass 15 kg is suspended vertically from a point P by a string. The particle is pulled horizontally by a force of F kg wt so that the string makes an angle of 30° with the vertical.

Find the value of F and the tension in the string.



Solution

Representing the forces in a triangle gives



$$\frac{F}{15} = \tan 30°$$

$$F = 15 \tan 30° = 5\sqrt{3}$$

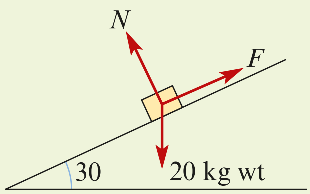
and $$\frac{15}{T} = \cos 30°$$

$$T = \frac{15}{\cos 30°} = 10\sqrt{3}$$

The tension in the string is $10\sqrt{3}$ kg wt.

Example 26

A body of mass 20 kg is placed on a smooth plane inclined at 30° to the horizontal. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.

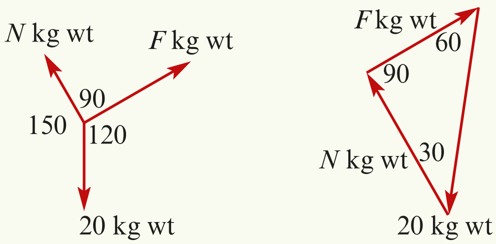


Solution

The three forces form a triangle (as the body is in equilibrium). Therefore

$$F = 20\sin 30^\circ = 10 \text{ kg wt}$$

$$N = 20\cos 30^\circ = 10\sqrt{3} \text{ kg wt}$$



Note: Force is a vector quantity, but it is often useful to employ only the magnitude of a force in calculations, and the direction is evident from the context. In this section, and in particular in diagrams, we often denote the magnitude of a force (for example, $\boldsymbol{F}$) by the same unbolded letter (in this case, F).

Resolution of forces

Obviously there are many situations where more than three forces (or in fact only two forces) will be acting on a body. An alternative method is required to solve such problems.

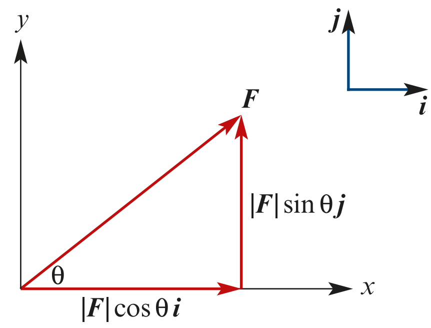
If all forces under consideration are acting in the same plane, then these forces and the resultant force can each be expressed as a sum of its $\boldsymbol{i}$- and $\boldsymbol{j}$-components.

If a force $\boldsymbol{F}$ acts at an angle of θ to the x-axis, then $\boldsymbol{F}$ can be written as the sum of two forces, one 'horizontal' and the other 'vertical':

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\, \boldsymbol{i} + |\boldsymbol{F}|\sin\theta\, \boldsymbol{j}$$

The force $\boldsymbol{F}$ is **resolved** into two components:

- the $\boldsymbol{i}$-component is parallel to the x-axis
- the $\boldsymbol{j}$-component is parallel to the y-axis.

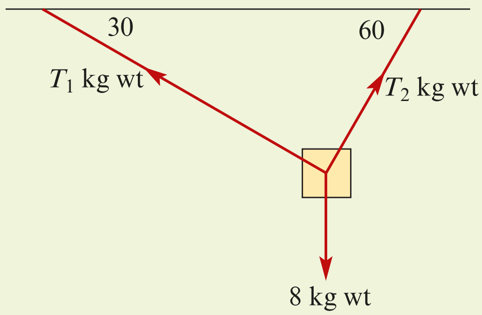


For a particle that is in equilibrium, if all the forces acting on the particle are resolved into their $\boldsymbol{i}$- and $\boldsymbol{j}$-components, then:

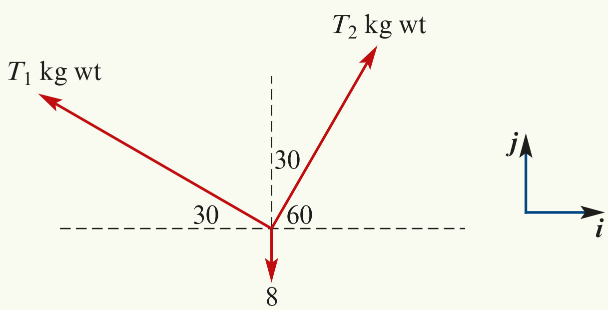
- the sum of all the $\boldsymbol{i}$-components is zero
- the sum of all the $\boldsymbol{j}$-components is zero.

Example 27

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 60° to the horizontal, find the tension in each string.



Solution



Resolution in the $\boldsymbol{j}$-direction:

$$T_1 \sin 30^\circ + T_2 \sin 60^\circ - 8 = 0$$

$$T_1\left(\frac{1}{2}\right) + T_2\left(\frac{\sqrt{3}}{2}\right) - 8 = 0 \qquad (1)$$

Resolution in the $\boldsymbol{i}$-direction:

$$-T_1 \cos 30^\circ + T_2 \cos 60^\circ = 0$$

$$-T_1\left(\frac{\sqrt{3}}{2}\right) + T_2\left(\frac{1}{2}\right) = 0 \qquad (2)$$

From (2): $\sqrt{3}\,T_1 = T_2$

Substituting in (1) gives

$$T_1\left(\frac{1}{2}\right) + \sqrt{3}\,T_1\left(\frac{\sqrt{3}}{2}\right) - 8 = 0$$

$$4T_1 = 16$$

$$\therefore \quad T_1 = 4$$

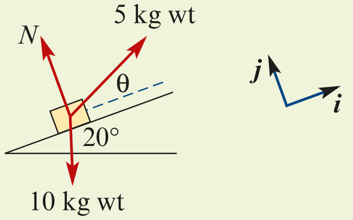
Hence $T_1 = 4$ and $T_2 = 4\sqrt{3}$.

The tensions in the strings are 4 kg wt and $4\sqrt{3}$ kg wt.

Example 28

A body of mass 10 kg is held at rest on a smooth plane inclined at 20° by a string with tension 5 kg wt as shown.

Find the angle between the string and the inclined plane.



Solution

We resolve the forces parallel and perpendicular to the plane. Then the normal force $\boldsymbol{N}$ has no parallel component, since it is perpendicular to the plane.

Resolving in the $\boldsymbol{i}$-direction:

$$5\cos\theta - 10\sin 20^\circ = 0$$

$$\cos\theta = \frac{10\sin 20^\circ}{5}$$

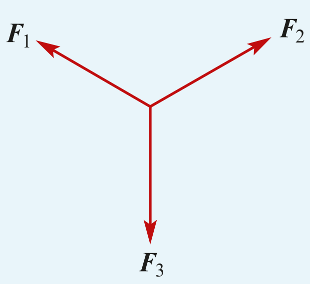
$$= 0.6840\ldots$$

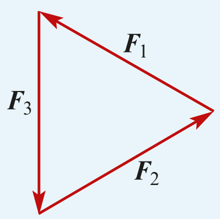
$$\therefore \quad \theta \approx 46.84^\circ$$

The angle between the string and the inclined plane is 46.84°.

Summary 21H

- **Force** is a vector quantity.
- The magnitude of a force can be measured using **kilogram weight** (kg wt).
  If an object near the surface of the Earth has a mass of 1 kg, then the force due to gravity acting on the object is 1 kg wt.
- Triangle of forces
  If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.





- Resolution of forces
  - A force $\boldsymbol{F}$ is **resolved** into components by writing it in the form $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$.
  - If forces are acting on a particle that is in equilibrium, then:
    - the sum of the $\boldsymbol{i}$-components of all the forces is zero
    - the sum of the $\boldsymbol{j}$-components of all the forces is zero.

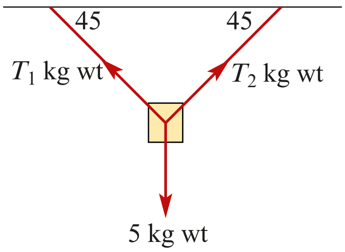
Skill-sheet

Exercise 21H

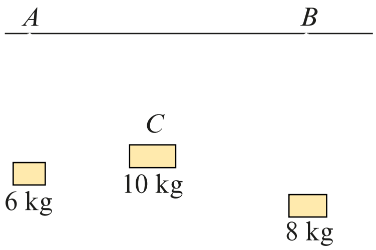
Complete Questions 1–10 using triangles of forces.

Example 24

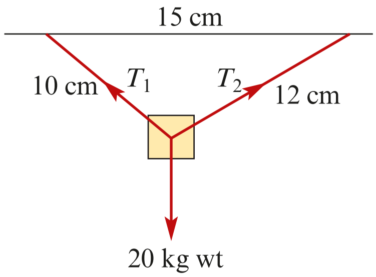
1 A particle of mass 5 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 45° with the horizontal, find the tension in each string.



2 Using strings and pulleys, three weights of mass 6 kg, 8 kg and 10 kg are suspended in equilibrium as shown. Calculate the magnitude of the angle ACB.



3 A mass of 20 kg is suspended from two strings of length 10 cm and 12 cm, the ends of the strings being attached to two points in a horizontal line, 15 cm apart. Find the tension in each string.

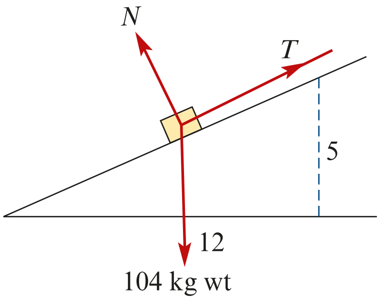


Example 25

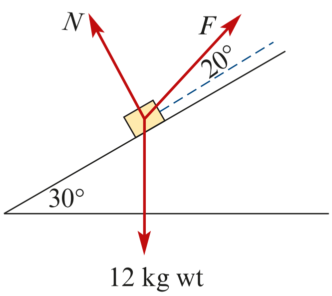
4 A boat is being pulled by a force of 40 kg wt towards the east and by a force of 30 kg wt towards the north-west. What third force must be acting on the boat if it remains stationary? Give the magnitude and direction.

Example 26

5 A body of mass 104 kg is placed on a smooth inclined plane which rises 5 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.

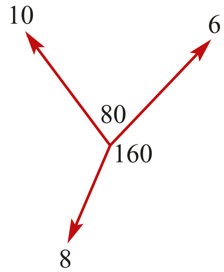


6 A body of mass 12 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 20° to the plane. Find the magnitude of the force.

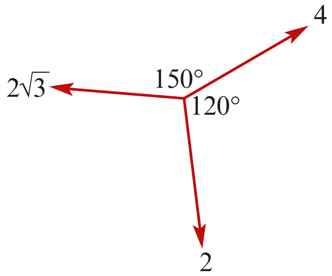


7 In each of the following cases, determine whether the particle is in equilibrium:

a



b



8 Three forces of magnitude 4 kg wt, 7 kg wt and 10 kg wt are in equilibrium. Determine the magnitudes of the angles between the forces.

9 A mass of 15 kg is maintained at rest on a smooth inclined plane by a string that is parallel to the plane. Determine the tension in the string if:

a the plane is at 30° to the horizontal

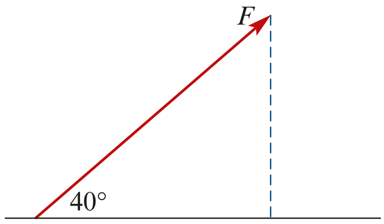
b the plane is at 40° to the horizontal

c the plane is at 30° to the horizontal, but the string is held at an angle of 10° to the plane.

10 The two ends of a string are connected to two points A and D in a horizontal line, and masses of 12 kg and W kg are attached at points B and C on the string. Given that C is lower than B and that AB, BC and CD make angles of 40°, 20° and 50° respectively with the horizontal, calculate the tensions in the string and the value of W.

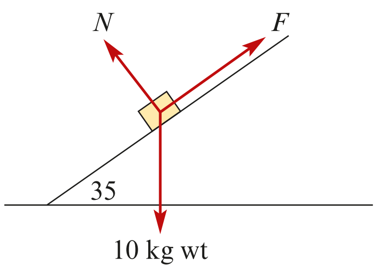
Complete Questions 11–17 using resolution of forces.

11 A force of F kg wt makes an angle of 40° with the horizontal. If its horizontal component is a force of 10 kg wt, find the value of F.

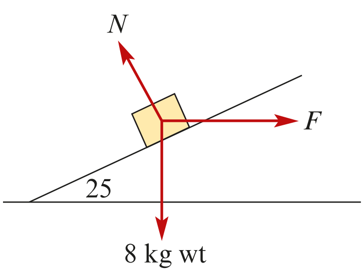


Example 27

12 Find the magnitude of the force, acting on a smooth inclined plane of angle 35°, required to support a mass of 10 kg resting on the plane.

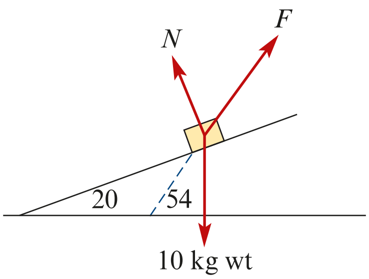


13 A body of mass 8 kg rests on a smooth inclined plane of angle 25° under the action of a horizontal force. Find the magnitude of the force and the reaction of the plane on the body.



Example 28

14 A body of mass 10 kg rests on a smooth inclined plane of angle 20°. Find the force that will keep it in equilibrium when it acts at an angle of 54° with the horizontal.

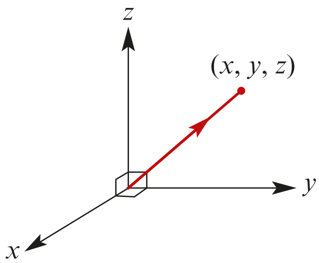


15 If a body of mass 12 kg is suspended by a string, find the horizontal force required to hold it at an angle of 30° from the vertical.

16 A force of 20 kg wt acting directly up a smooth plane inclined at an angle of 40° maintains a body in equilibrium on the plane. Calculate the mass of the body and the force it exerts on the plane.

17 Two men are supporting a block by ropes. One exerts a force of 20 kg wt, his rope making an angle of 35° with the vertical, and the other exerts a force of 30 kg wt. Determine the mass of the block and the angle of direction of the second rope.

21I Vectors in three dimensions

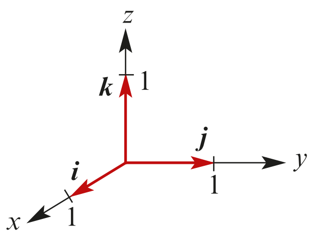


Points in three dimensions are represented using three perpendicular axes as shown.

Vectors in three dimensions are of the form

$$\boldsymbol{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

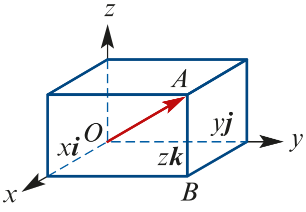
where $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



The vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the standard unit vectors for three dimensions.

The position vector for point $A(x, y, z)$ is

$$\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$



Using Pythagoras' theorem twice:

$$\begin{aligned} OA^2 &= OB^2 + BA^2 \\ &= OB^2 + z^2 \\ &= x^2 + y^2 + z^2 \end{aligned}$$

$$\therefore \quad |\overrightarrow{OA}| = \sqrt{x^2 + y^2 + z^2}$$

Example 29

Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 7\boldsymbol{k}$. Find:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{b} - 3\boldsymbol{a}$ **c** $|\boldsymbol{a}|$

Solution

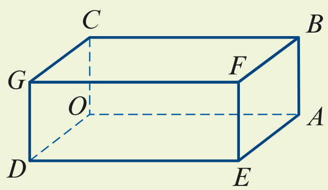
a $\boldsymbol{a} + \boldsymbol{b}$
$= \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \boldsymbol{i} + 7\boldsymbol{k}$
$= 2\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}$

b $\boldsymbol{b} - 3\boldsymbol{a}$
$= \boldsymbol{i} + 7\boldsymbol{k} - 3(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$
$= -2\boldsymbol{i} - 3\boldsymbol{j} + 10\boldsymbol{k}$

c $|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + (-1)^2}$
$= \sqrt{3}$

Example 30

$OABCDEFG$ is a cuboid such that $\overrightarrow{OA} = 3\boldsymbol{j}$, $\overrightarrow{OC} = \boldsymbol{k}$ and $\overrightarrow{OD} = \boldsymbol{i}$.



a Express each of the following in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

i $\overrightarrow{OE}$ **ii** $\overrightarrow{OF}$ **iii** $\overrightarrow{GF}$ **iv** $\overrightarrow{GB}$

b Let M and N be the midpoints of OD and GF respectively. Find MN.

Solution

a **i** $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = 3\boldsymbol{j} + \boldsymbol{i}$ (as $\overrightarrow{AE} = \overrightarrow{OD}$)

ii $\overrightarrow{OF} = \overrightarrow{OE} + \overrightarrow{EF} = 3\boldsymbol{j} + \boldsymbol{i} + \boldsymbol{k}$ (as $\overrightarrow{EF} = \overrightarrow{OC}$)

iii $\overrightarrow{GF} = \overrightarrow{OA} = 3\boldsymbol{j}$

iv $\overrightarrow{GB} = \overrightarrow{DA} = \overrightarrow{DO} + \overrightarrow{OA} = -\boldsymbol{i} + 3\boldsymbol{j}$

b $\overrightarrow{MN} = \overrightarrow{MD} + \overrightarrow{DG} + \overrightarrow{GN}$

$= \frac{1}{2}\overrightarrow{OD} + \overrightarrow{OC} + \frac{1}{2}\overrightarrow{OA}$

$= \frac{1}{2}\boldsymbol{i} + \boldsymbol{k} + \frac{3}{2}\boldsymbol{j}$

$|\overrightarrow{MN}| = \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2}$

Example 31

If $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$, find $\hat{\boldsymbol{a}}$.

Solution

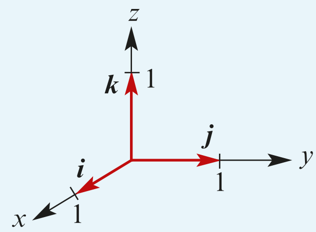
$|\boldsymbol{a}| = \sqrt{9 + 4 + 4} = \sqrt{17}$

$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{\sqrt{17}}(3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$

Summary 21I

In three dimensions:

- The standard unit vectors are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.
- Each vector can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.



Skill-sheet

Exercise 21I

Example 29

1 Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{k}$. Find:

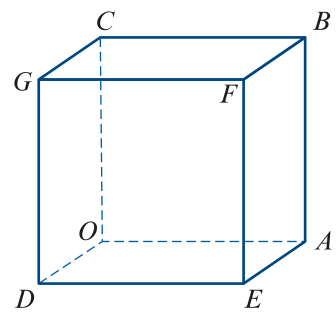
a $\boldsymbol{a} - \boldsymbol{b}$ **b** $3\boldsymbol{b} - 2\boldsymbol{a} + \boldsymbol{c}$ **c** $|\boldsymbol{b}|$

d $|\boldsymbol{b} + \boldsymbol{c}|$ **e** $3(\boldsymbol{a} - \boldsymbol{b}) + 2\boldsymbol{c}$

Example 30

2 *OABCDEFG* is a cuboid such that

$$\overrightarrow{OA} = 2\boldsymbol{j}, \quad \overrightarrow{OC} = 2\boldsymbol{k} \quad \text{and} \quad \overrightarrow{OD} = \boldsymbol{i}$$



Express the following vectors in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OE}$ **c** $\overrightarrow{OG}$

d $\overrightarrow{OF}$ **e** $\overrightarrow{ED}$ **f** $\overrightarrow{EG}$

g $\overrightarrow{CE}$ **h** $\overrightarrow{BD}$

Example 31

3 Let $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

a **i** Find $\hat{\boldsymbol{a}}$.

ii Find $-2\hat{\boldsymbol{a}}$.

b Find the vector $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ such that $|\boldsymbol{b}| = 5$.

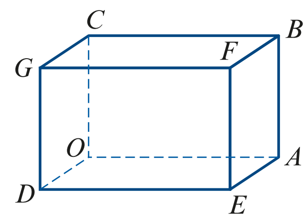
4 If $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$, find the vector $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ such that $|\boldsymbol{c}| = |\boldsymbol{b}|$.

5 Let P and Q be the points defined by the position vectors $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ respectively. Let M be the midpoint of PQ. Find:

a $\overrightarrow{PQ}$ **b** $|\overrightarrow{PQ}|$ **c** $\overrightarrow{OM}$

6 *OABCDEFG* is a cuboid such that

$$\overrightarrow{OA} = 3\boldsymbol{j}, \quad \overrightarrow{OC} = 2\boldsymbol{k} \quad \text{and} \quad \overrightarrow{OD} = \boldsymbol{i}$$



The point M is such that $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OE}$, and N is the midpoint of BF. Find:

a $\overrightarrow{MN}$ **b** $|\overrightarrow{MN}|$

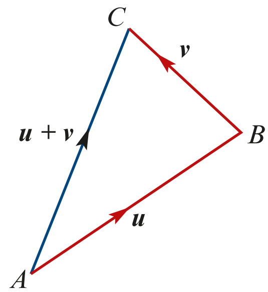
Chapter summary

Assign-ment

Nrich

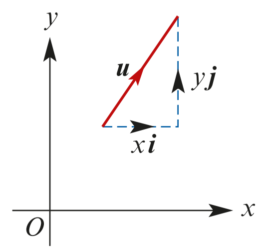
- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point A is the vector $\overrightarrow{OA}$, where O is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

Basic operations on vectors



- Addition
  - If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.
  - The sum $\boldsymbol{u} + \boldsymbol{v}$ can also be obtained geometrically as shown.
- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - The vector $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$, but the opposite direction.
  - Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- Subtraction $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

Component form

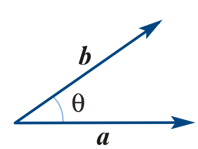


- In two dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where:
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Scalar product and vector projections

- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$



- The scalar product is described geometrically by $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

Displacement and velocity

- The **displacement** of a particle is the change in its position. If a particle moves from point A to point B, then its displacement is $\overrightarrow{AB}$.
- The **velocity** of a particle is the rate of change of its position with respect to time.
- Motion with constant velocity If a particle moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by $\boldsymbol{s} = t\boldsymbol{v}$.

Relative velocity

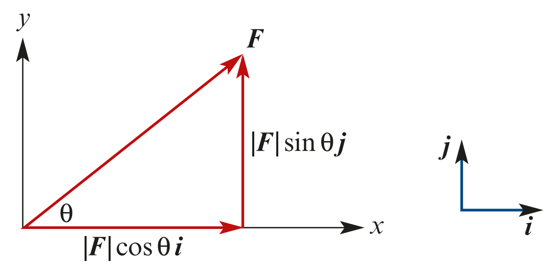
- The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.
- If an object A is in motion relative to another object B, we can find the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

Forces and equilibrium

- Resultant force When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**.
- Equilibrium If the resultant force acting on an object is zero, then the object is said to be in **equilibrium**; it will remain at rest or continue moving with constant velocity.
- Triangle of forces If three forces are acting on a particle in equilibrium, then the vectors representing the forces may be arranged to form a triangle. The magnitudes of the forces and the angles between them can be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.
- Resolution of forces
  If all forces on a particle are acting in two dimensions, then each force can be expressed in terms of its components in the $\boldsymbol{i}$- and $\boldsymbol{j}$-directions:

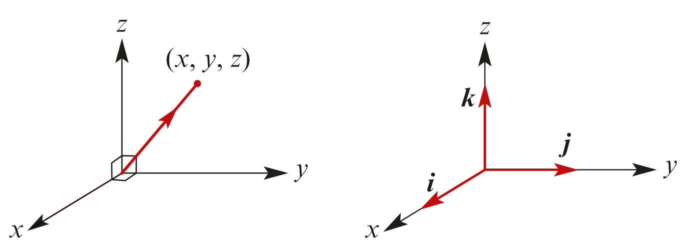
$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\,\boldsymbol{j}$$



For the particle to be in equilibrium, the sum of all the $\boldsymbol{i}$-components must be zero and the sum of all the $\boldsymbol{j}$-components must be zero.

Vectors in three dimensions

- In three dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors as shown.



- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

Technology-free questions

1 Given that $\boldsymbol{a} = 7\boldsymbol{i} + 6\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + x\boldsymbol{j}$, find the values of x for which:

a $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ **b** $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same magnitude.

2 $ABCD$ is a parallelogram where $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AD} = -2\boldsymbol{i} + 5\boldsymbol{j}$. Find the coordinates of the four vertices of the parallelogram.

3 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$. Find the values of p and q such that $\boldsymbol{a} + p\boldsymbol{b} + q\boldsymbol{c}$ is parallel to the x-axis.

4 The position vectors of P and Q are $2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $3\boldsymbol{i} - 7\boldsymbol{j} + 12\boldsymbol{k}$ respectively.

a Find $|\overrightarrow{PQ}|$. **b** Find the unit vector in the direction of $\overrightarrow{PQ}$.

5 The position vectors of A, B and C are $2\boldsymbol{j} + 2\boldsymbol{k}$, $4\boldsymbol{i} + 10\boldsymbol{j} + 18\boldsymbol{k}$ and $x\boldsymbol{i} + 14\boldsymbol{j} + 26\boldsymbol{k}$ respectively. Find x if A, B and C are collinear.

6 $\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j}$ and C is a point on OA such that $|\overrightarrow{OC}| = \dfrac{16}{5}$.

a Find the unit vector in the direction of $\overrightarrow{OA}$.

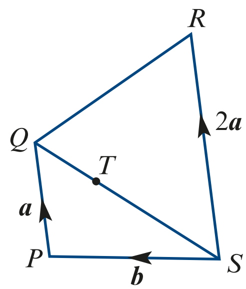
b Hence find $\overrightarrow{OC}$.

7 In the diagram, $ST = 2TQ$, $\overrightarrow{PQ} = \boldsymbol{a}$, $\overrightarrow{SR} = 2\boldsymbol{a}$ and $\overrightarrow{SP} = \boldsymbol{b}$.

a Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{SQ}$ **ii** $\overrightarrow{TQ}$ **iii** $\overrightarrow{RQ}$ **iv** $\overrightarrow{PT}$ **v** $\overrightarrow{TR}$

b Show that P, T and R are collinear.



8 If $\boldsymbol{a} = 5\boldsymbol{i} - s\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = t\boldsymbol{i} + 2\boldsymbol{j} + u\boldsymbol{k}$ are equal vectors.

a Find s, t and u. **b** Find $|\boldsymbol{a}|$.

9 The vector $\boldsymbol{p}$ has magnitude 7 units and bearing 050° and the vector $\boldsymbol{q}$ has magnitude 12 units and bearing 170°. (These are compass bearings on the horizontal plane.) Draw a diagram (not to scale) showing $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{p} + \boldsymbol{q}$. Calculate the magnitude of $\boldsymbol{p} + \boldsymbol{q}$.

10 If $\boldsymbol{a} = 5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, find:

a $\boldsymbol{a} + 2\boldsymbol{b}$ **b** $|\boldsymbol{a}|$ **c** $\hat{\boldsymbol{a}}$ **d** $\boldsymbol{a} - \boldsymbol{b}$

11 Let O, A and B be the points $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

a If C is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$, find the coordinates of C.

b If D is the point $(1, 24)$ and $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, find the values of h and k.

12 Let $\boldsymbol{p} = 3\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} - 5\boldsymbol{j}$. Find the values of m and n such that $m\boldsymbol{p} + n\boldsymbol{q} = 8\boldsymbol{i} + 9\boldsymbol{j}$.

13 The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ relative to an origin O. Write down an equation connecting $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for each of the following cases:

a $OABC$ is a parallelogram

b B divides AC in the ratio $3 : 2$. That is, $AB : BC = 3 : 2$.

14 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

15 Points A, B and C have position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{c} = -5\boldsymbol{i} + 3\boldsymbol{j}$ respectively. Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and hence show that $\triangle ABC$ is right-angled at B.

16 Given the vectors $\boldsymbol{p} = 5\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} + t\boldsymbol{j}$, find the values of t for which:

a $\boldsymbol{p} + \boldsymbol{q}$ is parallel to $\boldsymbol{p} - \boldsymbol{q}$

b $\boldsymbol{p} - 2\boldsymbol{q}$ is perpendicular to $\boldsymbol{p} + 2\boldsymbol{q}$

c $|\boldsymbol{p} - \boldsymbol{q}| = |\boldsymbol{q}|$

17 Points A, B and C have position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to the line AC.

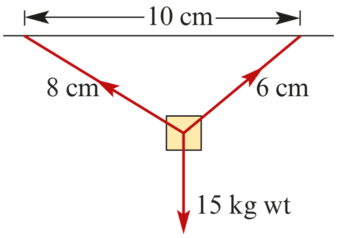
18 Priya can swim at a speed of 1.6 m/s in still water. She swims across a river that is 48 metres wide and flows at 1.2 m/s between parallel banks.

a Find the speed of the swimmer relative to the river bank.

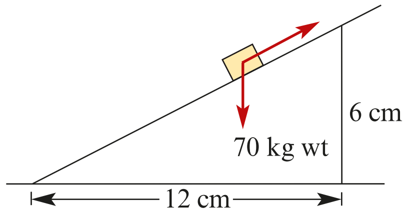
b Find the time that it takes her to cross the river.

c Describe the position at which she arrives on the opposite bank.

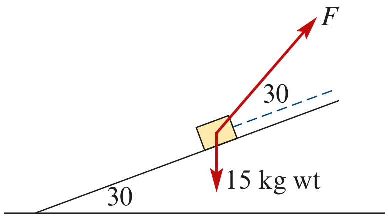
19 A mass of 15 kg is suspended from two strings of length 6 cm and 8 cm, the ends of the strings being attached to two points in a horizontal line, 10 cm apart. Find the tension in each string.



20 A body of mass 70 kg is placed on a smooth inclined plane which rises 6 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.



21 A body of mass 15 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 30° to the plane. Find the magnitude of the force.



Multiple-choice questions

1 The vector $\boldsymbol{v}$ is defined by the directed line segment from (1, 1) to (3, 5). If $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, then

A $a = 3$ and $b = 5$ **B** $a = -2$ and $b = -4$ **C** $a = 2$ and $b = 4$
D $a = 2$ and $b = 3$ **E** $a = 4$ and $b = 2$

2 If vector $\overrightarrow{AB} = \boldsymbol{u}$ and vector $\overrightarrow{AC} = \boldsymbol{v}$, then vector $\overrightarrow{CB}$ is equal to

A $\boldsymbol{u} + \boldsymbol{v}$ **B** $\boldsymbol{v} - \boldsymbol{u}$ **C** $\boldsymbol{u} - \boldsymbol{v}$ **D** $\boldsymbol{u} \times \boldsymbol{v}$ **E** $\boldsymbol{v} + \boldsymbol{u}$

3 If vector $\boldsymbol{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and vector $\boldsymbol{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $2\boldsymbol{a} - 3\boldsymbol{b} =$

A $\begin{bmatrix} 9 \\ -13 \end{bmatrix}$ **B** $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ **C** $\begin{bmatrix} 9 \\ -7 \end{bmatrix}$ **D** $\begin{bmatrix} 3 \\ -13 \end{bmatrix}$ **E** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

4 $PQRS$ is a parallelogram. If $\overrightarrow{PQ} = \boldsymbol{p}$ and $\overrightarrow{QR} = \boldsymbol{q}$, then $\overrightarrow{SQ}$ is equal to

A $\boldsymbol{p} + \boldsymbol{q}$ **B** $\boldsymbol{p} - \boldsymbol{q}$ **C** $\boldsymbol{q} - \boldsymbol{p}$ **D** $2\boldsymbol{q}$ **E** $2\boldsymbol{p}$

5 $|3\boldsymbol{i} - 5\boldsymbol{j}| =$

A 2 **B** $\sqrt{34}$ **C** 34 **D** 8 **E** -16

6 If $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 2\boldsymbol{j}$, then $\overrightarrow{AB}$ equals

A $-\boldsymbol{i} - 5\boldsymbol{j}$ **B** $-\boldsymbol{i} + 5\boldsymbol{j}$ **C** $-\boldsymbol{i} - \boldsymbol{j}$ **D** $-\boldsymbol{i} + \boldsymbol{j}$ **E** $\boldsymbol{i} + \boldsymbol{j}$

7 If $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 2\boldsymbol{j}$, then $|\overrightarrow{AB}|$ equals

A 6 **B** 26 **C** $\sqrt{26}$ **D** $\sqrt{24}$ **E** 36

8 If $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j}$, then the unit vector in the direction of $\boldsymbol{a}$ is

A $2\boldsymbol{i} + 3\boldsymbol{j}$ **B** $\frac{1}{13}(2\boldsymbol{i} + 3\boldsymbol{j})$ **C** $\frac{1}{\sqrt{5}}(2\boldsymbol{i} + 3\boldsymbol{j})$
D $\frac{1}{\sqrt{13}}(2\boldsymbol{i} + 3\boldsymbol{j})$ **E** $\sqrt{13}(2\boldsymbol{i} + 3\boldsymbol{j})$

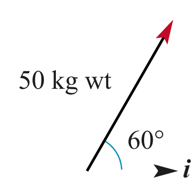
9 An aircraft has an airspeed of 100 km/h. The aircraft is heading in the direction $3\boldsymbol{i} - 4\boldsymbol{j}$ and the wind is blowing with a velocity of $-5\boldsymbol{i} + 20\boldsymbol{j}$ km/h. The velocity of the aircraft relative to the ground (in km/h) is

A $55\boldsymbol{i} - 60\boldsymbol{j}$ **B** $65\boldsymbol{i} - 60\boldsymbol{j}$ **C** $305\boldsymbol{i} - 420\boldsymbol{j}$ **D** $60\boldsymbol{i} - 40\boldsymbol{j}$ **E** $295\boldsymbol{i} - 380\boldsymbol{j}$

10 The velocity of a ship is $20\boldsymbol{i}$ km/h and the velocity of the wind is $-4\boldsymbol{i} + 3\boldsymbol{j}$ km/h. The direction of the smoke trail coming from the ship's funnel is given by

A $16\boldsymbol{i} + 3\boldsymbol{j}$ **B** $-24\boldsymbol{i} + 3\boldsymbol{j}$ **C** $-4\boldsymbol{i} + 3\boldsymbol{j}$ **D** $-16\boldsymbol{i} - 3\boldsymbol{j}$ **E** $24\boldsymbol{i} - 3\boldsymbol{j}$

11 A force $\boldsymbol{F}$ of magnitude 50 kg wt acts as shown in the diagram. The magnitude of the component of $\boldsymbol{F}$ in the $\boldsymbol{i}$-direction is



A 300 kg wt **B** 50 kg wt **C** 40 kg wt
D 20 kg wt **E** 25 kg wt

12 Two perpendicular forces have magnitudes 5 kg wt and 4 kg wt. The magnitude of the resultant force is

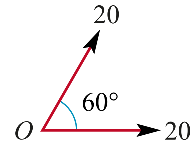
A 3 kg wt **B** $\sqrt{11}$ kg wt **C** $\sqrt{41}$ kg wt **D** 1 kg wt **E** 9 kg wt

13 A particle is acted on by a force of magnitude 7 kg wt acting on a bearing of 45° and by a force of magnitude a kg wt acting on a bearing of 135°. If the magnitude of the resultant force is 9 kg wt, then the value of a must be

A 2 **B** $4\sqrt{2}$ **C** $\sqrt{130}$ **D** 16 **E** 32

14 Two forces of magnitude 20 kg wt act on a particle at O as shown. The magnitude of the resultant force (in kg wt) is

A 40 **B** $20\sqrt{3}$ **C** 0
D 20 **E** 10



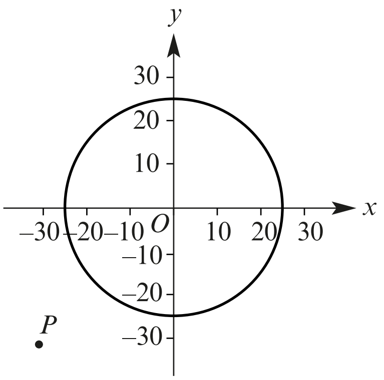
Extended-response questions

1 Let $\begin{bmatrix}1\\0\end{bmatrix}$ represent a displacement 1 km due east.

Let $\begin{bmatrix}0\\1\end{bmatrix}$ represent a displacement 1 km due north.

The diagram shows a circle of radius 25 km with centre at $O(0,0)$. A lighthouse entirely surrounded by sea is located at O. The lighthouse is not visible from points outside the circle.

A ship is initially at point P, which is 31 km west and 32 km south of the lighthouse.



a Write down the vector $\overrightarrow{OP}$.

The ship is travelling in the direction of vector $\boldsymbol{u} = \begin{bmatrix}4\\3\end{bmatrix}$ with speed 20 km/h.

An hour after leaving P, the ship is at point R.

b Show that $\overrightarrow{PR} = \begin{bmatrix}16\\12\end{bmatrix}$ and hence find the vector $\overrightarrow{OR}$.

c Show that the lighthouse first becomes visible when the ship reaches R.

2 Given that $\boldsymbol{p} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{q} = -2\boldsymbol{i} + 4\boldsymbol{j}$, find:

a $|\boldsymbol{p} - \boldsymbol{q}|$ **b** $|\boldsymbol{p}| - |\boldsymbol{q}|$ **c** $\boldsymbol{r}$ such that $\boldsymbol{p} + 2\boldsymbol{q} + \boldsymbol{r} = \boldsymbol{0}$

3 The quadrilateral $PQRS$ is a parallelogram. The point P has coordinates $(5, 8)$, the point R has coordinates $(32, 17)$ and the vector $\overrightarrow{PQ}$ is given by $\overrightarrow{PQ} = \begin{bmatrix}20\\-15\end{bmatrix}$.

a Find the coordinates of Q and write down the vector $\overrightarrow{QR}$.

b Write down the vector $\overrightarrow{RS}$ and show that the coordinates of S are $(12, 32)$.

4 Let $\boldsymbol{a} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 7 \\ 9 \\ 7 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$.

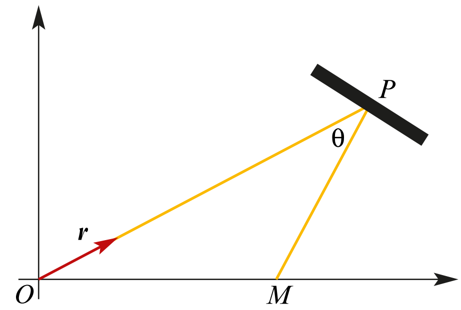
a Find the value of the scalar k such that $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = k\boldsymbol{d}$.

b Find the scalars x and y such that $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{d}$.

c Use your answers to **a** and **b** to find scalars p, q and r (not all zero) such that $p\boldsymbol{a} + q\boldsymbol{b} + r\boldsymbol{c} = \boldsymbol{0}$.

5 The diagram shows the path of a light beam from its source at O in the direction of the vector $\boldsymbol{r} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

At point P, the beam is reflected by an adjustable mirror and meets the x-axis at M. The position of M varies, depending on the adjustment of the mirror at P.



a Given that $\overrightarrow{OP} = 4\boldsymbol{r}$, find the coordinates of P.

b The point M has coordinates $(k, 0)$. Find an expression, in terms of k, for vector $\overrightarrow{PM}$.

c Find the magnitudes of vectors $\overrightarrow{OP}$, $\overrightarrow{OM}$ and $\overrightarrow{PM}$, and hence find the value of k for which θ is equal to $90°$.

d Find the value θ for which M has coordinates $(9, 0)$.

6 A helicopter can fly at 150 km/h in still air. The wind is blowing at 30 km/h from the east.

a How long in total would it take the helicopter to fly directly to a point 180 km due east and back again?

b On what bearing should the helicopter head in order to fly directly to a point 90 km due north? How long would this take?

c On what bearing should the helicopter head in order to fly due south?

7 The unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ represent 1 km east and 1 km north respectively. Two motor boats, A and B, are moving with velocities $\boldsymbol{v}_A = 12\boldsymbol{i} + 16\boldsymbol{j}$ km/h and $\boldsymbol{v}_B = 8\boldsymbol{i} + \alpha\boldsymbol{j}$ km/h, where α is a real number.

a Find an expression for the velocity of boat A relative to boat B.

b When the two boats first sight each other, boat A is 10 km due west of boat B.

i Find the value of α for which the two boats would collide if they maintained their current velocities.

ii Find the time between the boats first sighting each other and the collision.

22

Revision of Chapters 20–21

22A Technology-free questions

1 A transformation has rule $(x, y) \to (2x + y, -x - 2y)$.

a Find the image of the point $(2, 3)$.

b Find the matrix of this transformation.

c Sketch the image of the unit square and find its area.

d Find the rule for the inverse transformation.

2 Find the matrix corresponding to each of the following linear transformations:

a reflection in the x-axis

b dilation of factor 3 from the y-axis

c shear of factor 2 parallel to the y-axis

d projection onto the y-axis

e rotation by $45°$ anticlockwise

f rotation by $30°$ clockwise

g reflection in the line $y = -x$

h reflection in the line $y = x \tan 30°$

3 **a** Find the matrix that will reflect the plane in the line $y = 4x$.

b Find the image of the point $(2, 4)$ under this transformation.

4 Find the transformation matrix that corresponds to:

a a reflection in the y-axis and then a dilation of factor 2 from the x-axis

b a rotation by $90°$ anticlockwise and then a reflection in the line $y = x$

c a reflection in the line $y = -x$ and then a shear of factor 2 parallel to the x-axis

5 **a** Find the rule for the transformation that will reflect (x, y) in the y-axis then translate the result by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

b Find the rule for the transformation if the translation takes place before the reflection.

6 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

a $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

7 **a** Find the rule for the transformation that will reflect the plane in the line $y = x - 1$.

Hint: Translate the plane 1 unit in the y-direction, reflect in the line $y = x$, and then translate the plane back to its original position.

b Find the image of the point $(0, 0)$ under this transformation.

c Sketch the unit square and its image under this transformation.

8 Let $\boldsymbol{a} = 2\boldsymbol{i} + 6\boldsymbol{j}$.

a Find $|\boldsymbol{a}|$.

b Find the unit vector in the direction of $\boldsymbol{a}$.

c Write down a vector of magnitude 8 that has the same direction as $\boldsymbol{a}$.

d Write down a vector of magnitude 2 that has the opposite direction to $\boldsymbol{a}$.

9 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -3\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

10 The points A, B, C and D have position vectors $\overrightarrow{OA} = 4\boldsymbol{i} + 2\boldsymbol{j}$, $\overrightarrow{OB} = -\boldsymbol{i} + 7\boldsymbol{j}$, $\overrightarrow{OC} = 8\boldsymbol{i} + 6\boldsymbol{j}$ and $\overrightarrow{OD} = p\boldsymbol{i} - 2\boldsymbol{j}$.

a Find the values of m and n such that $m\overrightarrow{OA} + n\overrightarrow{BC} = 2\boldsymbol{i} + 10\boldsymbol{j}$.

b Find the value of p such that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CD}$.

c Find the values of p such that $|\overrightarrow{AD}| = \sqrt{17}$.

11 A motorboat heads due east at 16 m/s across a river that flows due north at 9 m/s.

a What is the resultant velocity of the boat?

b If the river is 136 m wide, how long does it take the boat to cross the river?

c How far downstream is the boat when it reaches the other side of the river?

12 Two forces of equal magnitude F kg wt act on a particle and they have a resultant force of magnitude 6 kg wt. When one of the forces is doubled in magnitude, the resultant force is 11 kg wt. Find the value of F and the cosine of the angle between the two forces.

13 A block of mass 10 kg is maintained at rest on a smooth plane inclined at 30° to the horizontal by a string. Calculate the tension in the string and the reaction of the plane if:

a the string is parallel to the plane **b** the string is horizontal.

14 A mass of 10 kg is suspended by two strings of lengths 5 cm and 12 cm that are attached to fixed points on the same horizontal level 13 cm apart. Find the tension in each string.

22B Multiple-choice questions

1 The point (a, b) is reflected in the line with equation $x = m$. The image has coordinates

A $(2m - a, b)$ **B** $(a, 2m - b)$ **C** $(a - m, b)$ **D** $(a, b - m)$ **E** $(2m + a, b)$

2 The image of the line $\{ (x, y) : x + y = 4 \}$ under a dilation of factor $\frac{1}{2}$ from the y-axis followed by a reflection in the line $x = 4$ is

A $\{ (x, y) : y = 2x \}$ **B** $\{ (x, y) : y + 2 = 0 \}$ **C** $\{ (x, y) : y + 2x - 16 = 0 \}$
D $\{ (x, y) : x + y = 0 \}$ **E** $\{ (x, y) : y = 2x - 12 \}$

3 The image of $\{ (x, y) : y = x^2 \}$ under a translation by the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ followed by a reflection in the x-axis is

A $\{ (x, y) : y = (x - 3)^2 + 2 \}$ **B** $\{ (x, y) : -(x - 3)^2 = y + 2 \}$
C $\{ (x, y) : y = (x + 3)^2 + 2 \}$ **D** $\{ (x, y) : -y + 2 = (x - 3)^2 \}$
E none of these

4 The image of the graph of $y = 2^x$ under a dilation of factor 2 from the x-axis followed by a dilation of factor $\frac{1}{3}$ from the y-axis has the equation

A $y = \frac{1}{3} \times 2^{3x}$ **B** $y = 3 \times 2^{\frac{x}{2}}$ **C** $y = 2 \times 2^{3x}$ **D** $y = 2 \times 2^{\frac{x}{3}}$ **E** none of these

5 Consider these two transformations:

- T_1: A reflection in the line with equation $x = 2$.
- T_2: A translation by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

The rule for T_1 followed by T_2 is given by

A $(x, y) \to (2 - x, y + 3)$ **B** $(x, y) \to (-x, y + 3)$ **C** $(x, y) \to (x + 2, y + 3)$
D $(x, y) \to (6 - x, y + 3)$ **E** none of these

6 A transformation has rule $(x, y) \to (4x + 3y, 5x + 4y)$. The rule for the inverse transformation is

A $(x, y) \to (3x + 4y, 5x + 4y)$ **B** $(x, y) \to (3x - 4y, 5x - 4y)$
C $(x, y) \to (4x + 3y, 5x + 4y)$ **D** $(x, y) \to (4x - 3y, -5x + 4y)$
E $(x, y) \to (-4x + 3y, 5x - 4y)$

7 Transformation T rotates the plane about the origin by 35° clockwise.
Transformation S rotates the plane about the origin by 15° anticlockwise.
The matrix of T followed by S is

A $\begin{bmatrix} \cos 50^\circ & -\sin 50^\circ \\ \sin 50^\circ & \cos 50^\circ \end{bmatrix}$ **B** $\begin{bmatrix} \cos 20^\circ & -\sin 20^\circ \\ \sin 20^\circ & \cos 20^\circ \end{bmatrix}$ **C** $\begin{bmatrix} \cos 50^\circ & \sin 50^\circ \\ -\sin 50^\circ & \cos 50^\circ \end{bmatrix}$

D $\begin{bmatrix} \cos 20^\circ & \sin 20^\circ \\ -\sin 20^\circ & \cos 20^\circ \end{bmatrix}$ **E** $\begin{bmatrix} \cos 50^\circ & \sin 50^\circ \\ \sin 50^\circ & -\cos 50^\circ \end{bmatrix}$

8 The linear transformation determined by the matrix $\begin{bmatrix} \cos 40° & -\sin 40° \\ \sin 40° & \cos 40° \end{bmatrix}$ is

A clockwise rotation by 40°
B anticlockwise rotation by 40°
C reflection in the line $y = x \tan 40°$
D reflection in the line $y = x \tan 20°$
E anticlockwise rotation by 20°

9 The unit vector in the direction of vector $\boldsymbol{a} = 3\boldsymbol{i} - 4\boldsymbol{j}$ is

A $\boldsymbol{i} - \boldsymbol{j}$ **B** $\frac{1}{5}(3\boldsymbol{i} - 4\boldsymbol{j})$ **C** $\boldsymbol{i} + \boldsymbol{j}$ **D** $\frac{1}{25}(3\boldsymbol{i} - 4\boldsymbol{j})$ **E** $\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$

10 If $\overrightarrow{OA} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{OB} = 3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$, then $\overrightarrow{AB}$ equals

A $5\boldsymbol{i} + 2\boldsymbol{k}$ **B** $-\boldsymbol{i} - 8\boldsymbol{j}$ **C** $\boldsymbol{i} + 8\boldsymbol{j} + 2\boldsymbol{k}$ **D** $\boldsymbol{i} + 8\boldsymbol{j}$ **E** $\boldsymbol{i}$

11 If $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$, then $\boldsymbol{a} - \boldsymbol{b}$ equals

A $5\boldsymbol{i} - 6\boldsymbol{j}$ **B** $-\boldsymbol{i} + 6\boldsymbol{j}$ **C** $5\boldsymbol{i} - 2\boldsymbol{j}$ **D** $5\boldsymbol{i} + 2\boldsymbol{j}$ **E** $\boldsymbol{i} - 6\boldsymbol{j}$

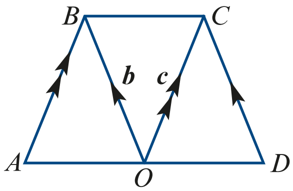
12 The magnitude of vector $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ is

A $\sqrt{21}$ **B** 21 **C** 19 **D** $\sqrt{19}$ **E** 7

13 In the diagram, AB is parallel to OC, DC is parallel to OB, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $AB = OB = OC = DC$.

Vector $\overrightarrow{AD}$ is equal to

A $\boldsymbol{b} + \boldsymbol{c}$ **B** $2(\boldsymbol{c} - \boldsymbol{b})$ **C** $2(\boldsymbol{b} - \boldsymbol{c})$
D $2\boldsymbol{b} + 2\boldsymbol{c}$ **E** $|\boldsymbol{b} + \boldsymbol{c}|$

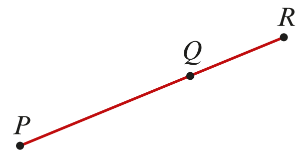


14 If $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{s} = -\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, then $2\boldsymbol{r} - \boldsymbol{s}$ equals

A $3\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ **B** $3\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ **C** $5\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ **D** $5\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ **E** $6\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$

15 PQR is a straight line and $PQ = 2QR$. If $\overrightarrow{OQ} = 2\boldsymbol{i} - 3\boldsymbol{j}$ and $\overrightarrow{OR} = \boldsymbol{i} + 2\boldsymbol{j}$, then $\overrightarrow{OP}$ could be equal to

A $4\boldsymbol{i} - 13\boldsymbol{j}$ **B** $3\boldsymbol{i} - \boldsymbol{j}$ **C** $2\boldsymbol{i} - 10\boldsymbol{j}$
D $3\boldsymbol{i} + \boldsymbol{j}$ **E** $\boldsymbol{i} - 5\boldsymbol{j}$



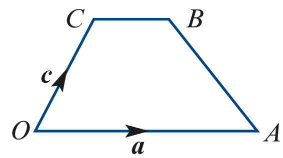
16 Let $\boldsymbol{u} = \boldsymbol{i} + a\boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{v} = b\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$. Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel when

A $a = -3$ and $b = -1$ **B** $a = \frac{5}{2}$ and $b = -\frac{6}{5}$ **C** $a = 3$ and $b = -1$
D $a = -\frac{5}{6}$ and $b = \frac{6}{5}$ **E** $a = \frac{2}{5}$ and $b = \frac{5}{6}$

17 Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{x} = \boldsymbol{i} + 5\boldsymbol{j}$. If $\boldsymbol{x} = s\boldsymbol{a} + t\boldsymbol{b}$, then the scalars s and t are given by

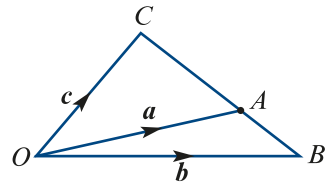
A $s = -1$ and $t = -1$ **B** $s = -1$ and $t = 1$ **C** $s = 1$ and $t = -1$
D $s = 1$ and $t = 1$ **E** $s = 5$ and $t = 5$

18 In this diagram, $OABC$ is a trapezium.
If $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\overrightarrow{OA} = 3\overrightarrow{CB}$, then $\overrightarrow{AB}$ equals



A $3\boldsymbol{c}$ **B** $\boldsymbol{c} - \frac{2}{3}\boldsymbol{a}$ **C** $3\boldsymbol{c} - 2\boldsymbol{a}$
D $\frac{2}{3}\boldsymbol{a} - \boldsymbol{c}$ **E** $\frac{4}{3}\boldsymbol{a} + \boldsymbol{c}$

19 In this diagram, $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $AC : AB = 2 : 1$. The vector $\boldsymbol{c}$ is equal to

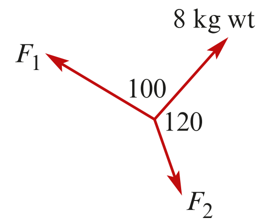


A $\boldsymbol{a} + 2\boldsymbol{b}$ **B** $3\boldsymbol{a} - 2\boldsymbol{b}$ **C** $2\boldsymbol{a} + \boldsymbol{b}$
D $2\boldsymbol{a} - \boldsymbol{b}$ **E** $3\boldsymbol{a} + \boldsymbol{b}$

20 A bus and a car are on a straight level road that runs east–west. The bus is moving east at 20 m/s and the car is moving west at 20 m/s. If a man walks from the back to the front of the bus at 2 m/s, what is the velocity of the man relative to the car?

A 38 m/s east **B** 38 m/s west **C** 42 m/s east **D** 42 m/s west **E** 22 m/s east

Questions 21–22 refer to this system of forces, which is in equilibrium.



21 The magnitude of force $\boldsymbol{F}_1$ is approximately

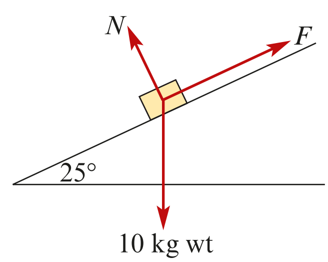
A 10.78 kg wt **B** 5.94 kg wt **C** 9.10 kg wt
D 12.26 kg wt **E** 7.04 kg wt

22 The magnitude of force $\boldsymbol{F}_2$ is approximately

A 10.78 kg wt **B** 5.94 kg wt **C** 9.10 kg wt **D** 12.26 kg wt **E** 7.04 kg wt

Questions 23–24 refer to the following information:

A 10 kg block is resting on a smooth plane inclined at 25° to the horizontal and is prevented from slipping down the plane by a string, as shown in the diagram.



23 The magnitude, N, of the normal reaction force is approximately

A 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt **E** 10 kg wt

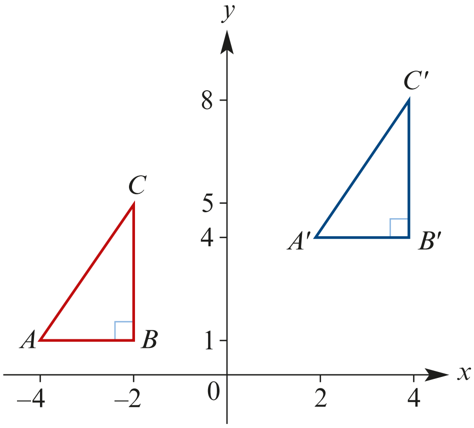
24 The magnitude, F, of the tension in the string is approximately

A 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt **E** 10 kg wt

Revision

22C Extended-response questions

1 The coordinates of A, B and C are $(-4, 1)$, $(-2, 1)$ and $(-2, 5)$ respectively.



a Find the rule of the transformation that maps triangle ABC to triangle $A'B'C'$.

b On graph paper, draw triangle ABC and its image under a reflection in the x-axis.

c On the same set of axes, draw the image of ABC under a dilation of factor 2 from the y-axis.

d Find the image of the parabola $y = x^2$ under a dilation of factor 2 from the x-axis followed by a translation by the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

e Find the rule for the transformation that maps the graph of $y = x^2$ to the graph of $y = -2(x-3)^2 + 4$.

2 A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

a Show that this transformation is a rotation.

b Let C be the circle that passes through the origin and has its centre at $(0, 1)$.

i Find the equation of C.

ii Find the equation of C', the image of C under the transformation defined by $\mathbf{M}$.

c Find the coordinates of the points of intersection of C and C'.

3 Let $\mathbf{R}$ be the transformation matrix for a rotation about the origin by $\frac{\pi}{4}$ anticlockwise.

a Give the 2×2 matrix $\mathbf{R}$.

b Find the inverse of this matrix.

c If the image of (a, b) is $(1, 1)$, find the values of a and b.

d If the image of (c, d) is $(1, 2)$, find the values of c and d.

e **i** If $(x, y) \to (x', y')$ under this transformation, use the result of **b** to find x and y in terms of x' and y'.

ii Find the image of $y = x^2$ under this transformation.

4 Consider lines $y = x$ and $y = 2x$.

a Sketch these two lines on the same set of axes.

b The acute angle between the two lines, θ radians, can be written in the form $\theta = \tan^{-1}(a) - b$. What are the values of a and b?

c Hence find a rotation matrix that will rotate the line $y = x$ to the line $y = 2x$. You will need to use the compound angle formulas for sine and cosine.

5 Let M be the transformation that reflects the plane in the line $y = x$.

a **i** Find the image of the point $A(1, 3)$ under this transformation.

ii The image of the triangle with vertices $A(1, 3)$, $B(1, 5)$ and $C(3, 3)$ is another triangle. Find the coordinates of the vertices of the image.

iii Sketch triangle ABC and its image on a set of axes, with both axes from -5 to 5.

b **i** Show that the equation of the image of the graph of $y = x^2 - 2$ under the transformation M is $x = y^2 - 2$.

ii Find the coordinates of the points of intersection of $y = x^2 - 2$ and the line $y = x$.

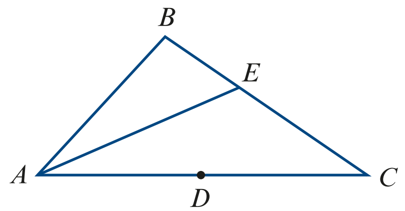
iii Show that the x-coordinates of the points of intersection of $y = x^2 - 2$ and its image may be determined by the equation $x^4 - 4x^2 - x + 2 = 0$.

iv Two solutions of the equation $x^4 - 4x^2 - x + 2 = 0$ are

$$x = \frac{1}{2}(-1 + \sqrt{5}) \quad \text{and} \quad x = \frac{1}{2}(-1 - \sqrt{5})$$

Use this result and the result of **b ii** to find the coordinates of the points of intersection of $y = x^2 - 2$ and its image under M.

6 In the diagram, D is the midpoint of AC and E is the point on BC such that $BE : EC = 1 : t$, where $t > 0$. Suppose that DE is extended to a point F such that $DE : EF = 1 : 7$.



Let $\boldsymbol{a} = \overrightarrow{AD}$ and $\boldsymbol{b} = \overrightarrow{AB}$.

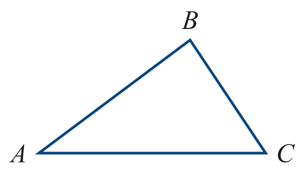
a Express $\overrightarrow{AE}$ in terms of t, $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Express $\overrightarrow{AE}$ in terms of $\boldsymbol{a}$ and $\overrightarrow{AF}$.

c Show that $\overrightarrow{AF} = \dfrac{9 - 7t}{1 + t}\boldsymbol{a} + \dfrac{8t}{1 + t}\boldsymbol{b}$.

d If A, B and F are collinear, find the value of t.

7 The vertices A, B and C of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin in the plane ABC.



a Let P be an arbitrary point on the line segment AB. Show that the position vector of P can be written in the form

$$m\boldsymbol{a} + n\boldsymbol{b}, \quad \text{where } m \geq 0,\ n \geq 0 \text{ and } m + n = 1$$

Hint: Assume that P divides AB in the ratio $x : y$.

b Find $\overrightarrow{PC}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

c Let Q be an arbitrary point on the line segment PC. Show that the position vector of Q can be written in the form

$$\lambda\boldsymbol{a} + \mu\boldsymbol{b} + \gamma\boldsymbol{c}, \quad \text{where } \lambda \geq 0,\ \mu \geq 0,\ \gamma \geq 0 \text{ and } \lambda + \mu + \gamma = 1$$

Note: The triple of numbers (λ, μ, γ) are known as the **barycentric coordinates** of the point Q in the triangle ABC.

8 **a** A man walks north at a rate of 4 km/h and notices that the wind *appears* to blow from the west. He doubles his speed and now the wind appears to blow from the north-west. What is the velocity of the wind?

Note: Both the direction and the magnitude must be given.

b A river 400 m wide flows from east to west at a steady speed of 1 km/h. A swimmer, whose speed in still water is 2 km/h, starts from the south bank and heads north across the river. Find the swimmer's speed over the river bed and how far downstream he is when he reaches the north bank.

c To a motorcyclist travelling due north at 50 km/h, the wind appears to come from the north-west at 60 km/h. What is the true velocity of the wind?

d A dinghy in distress is 6 km on a bearing of 230° from a lifeboat and is drifting in a direction of 150° at 5 km/h. In what direction should the lifeboat travel to reach the dinghy as quickly as possible if the maximum speed of the lifeboat is 35 km/h?

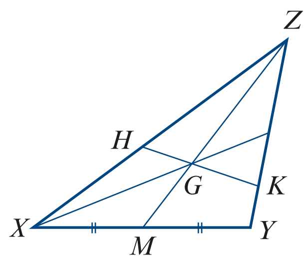
9 **a** Let points O, A, B and C be coplanar and let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel. If points A, B and C are collinear with

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} \quad \text{where } \alpha, \beta \in \mathbb{R}$$

show that $\alpha + \beta = 1$.

b In the figure, the point G is the centroid of a triangle (i.e. the point where the lines joining each vertex to the midpoint of the opposite side meet).

A line passing through G meets ZX and ZY at points H and K respectively, with $ZH = hZX$ and $ZK = kZY$.



i Prove that $\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM}$.

ii Express $\overrightarrow{ZG}$ in terms of h, k, $\overrightarrow{ZH}$ and $\overrightarrow{ZK}$.

iii Find the value of $\frac{1}{h} + \frac{1}{k}$. (Use the result from **a**.)

iv If $h = k$, find the value of h and describe geometrically what this implies.

v If the area of triangle XYZ is 1 cm^2, find the area of triangle HKZ when $h = k$.

vi If $k = 2h$, find the value of h and describe geometrically what this implies.

vii Describe the restrictions on h and k, and sketch the graph of h against k for suitable values of k.

viii Investigate the area, A cm^2, of triangle HKZ as a ratio with respect to the area of triangle XYZ, as k varies. Sketch the graph of A against k. Be careful with the domain.

23 Kinematics

Objectives

- To model **motion in a straight line**.
- To apply **differentiation** to problems involving motion in a straight line.
- To apply **antidifferentiation** to problems involving motion in a straight line.
- To use the formulas for motion with **constant acceleration**.
- To use **graphical methods** to solve problems involving motion in a straight line.

Kinematics is the study of motion without reference to the cause of the motion.

In this chapter, we will consider the motion of a particle in a straight line only. This simple model can be applied in various real-life situations. For example:

- finding the braking distance of a car travelling at 60 km/h
- finding the maximum height reached by a stone thrown into the air
- finding the time required for a train to travel between two stations.

When studying motion, it is important to make a distinction between vector quantities and scalar quantities:

Vector quantities Position, displacement, velocity and acceleration must be specified by both magnitude and direction.

Scalar quantities Distance, time and speed are specified by their magnitude only.

Since we are considering movement in a straight line, the *direction* of each vector quantity is simply specified by the *sign* of the numerical value.

This chapter uses your knowledge of differential calculus from Mathematical Methods Units 1 & 2.

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23A Position, velocity and acceleration

Position

The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.

Consider a particle which starts at O and begins to move. The position of the particle at any instant can be specified by a real number x. For example, if the unit is metres and if $x = -3$, the position is 3 m to the left of O; while if $x = 3$, the position is 3 m to the right of O.

Sometimes there is a rule that enables the position at any instant to be calculated. In this case, we can view x as being a function of t. Hence $x(t)$ is the position at time t.

For example, imagine that a stone is dropped from the top of a vertical cliff 45 metres high. Assume that the stone is a particle travelling in a straight line. Let $x(t)$ metres be the downwards position of the particle from O, the top of the cliff, t seconds after the particle is dropped. If air resistance is neglected, then an approximate model for the position is

$$x(t) = 5t^2 \quad \text{for } 0 \le t \le 3$$

Example 1

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \ge 0$.

a Find its initial position. **b** Find its position at $t = 4$.

Solution

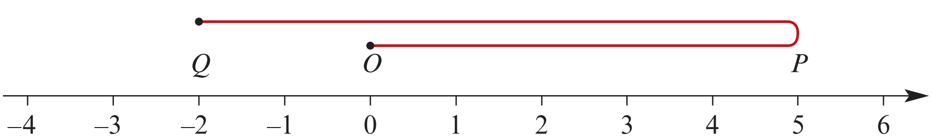
a At $t = 0$, $x = +6$, i.e. the particle is 6 cm to the right of O.

b At $t = 4$, $x = (4)^2 - 7(4) + 6 = -6$, i.e. the particle is 6 cm to the left of O.

Displacement and distance

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction). For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is -2. So the displacement of the particle is -2 units. However, the distance it has travelled is 12 units.

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Velocity and speed

You are already familiar with rates of change through your studies in Mathematical Methods.

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_1 is the position at time t_1 and x_2 is the position at time t_2.

Instantaneous velocity

The instantaneous rate of change of position with respect to time is **instantaneous velocity**. We will refer to the instantaneous velocity as simply the **velocity**.

If a particle's position, x, at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

If x is the position of a particle at time t, then

$$\text{velocity } v = \frac{dx}{dt}$$

Velocity may be positive, negative or zero. If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

Speed and average speed

- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is given by $\dfrac{\text{distance travelled}}{t_2 - t_1}$

Units of measurement

Common units for velocity (and speed) are:

$$\begin{aligned}
&\text{1 metre per second} &&= 1\text{ m/s} &&= 1\text{ m s}^{-1}\\
&\text{1 centimetre per second} &&= 1\text{ cm/s} &&= 1\text{ cm s}^{-1}\\
&\text{1 kilometre per hour} &&= 1\text{ km/h} &&= 1\text{ km h}^{-1}
\end{aligned}$$

The first and third units are connected in the following way:

$$1\text{ km/h} = 1000\text{ m/h} = \frac{1000}{60 \times 60}\text{ m/s} = \frac{5}{18}\text{ m/s}$$

$$\therefore \quad 1\text{ m/s} = \frac{18}{5}\text{ km/h}$$

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Example 2

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

Solution

a $x = t^2 - 7t + 6$

$$v = \frac{dx}{dt} = 2t - 7$$

At $t = 0$, $v = -7$. The particle is initially moving to the left at 7 cm/s.

b $\frac{dx}{dt} = 0$ implies $2t - 7 = 0$, i.e. $t = 3.5$

When $t = 3.5$, $x = (3.5)^2 - 7(3.5) + 6$

$= -6.25$

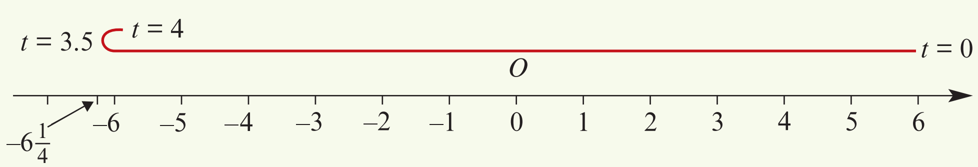
So, at $t = 3.5$ seconds, the particle is at rest 6.25 cm to the left of O.

c $\text{Average velocity} = \dfrac{\text{change in position}}{\text{change in time}}$

Position is given by $x = t^2 - 7t + 6$. So at $t = 4$, $x = -6$, and at $t = 0$, $x = 6$.

$\therefore$ Average velocity $= \dfrac{-6 - 6}{4} = -3$ cm/s

d $\text{Average speed} = \dfrac{\text{distance travelled}}{\text{change in time}}$



The particle stopped at $t = 3.5$ and began to move in the opposite direction. So we must consider the distance travelled in the first 3.5 seconds (from $x = 6$ to $x = -6.25$) and then the distance travelled in the final 0.5 seconds (from $x = -6.25$ to $x = -6$).

Total distance travelled = 12.25 + 0.25 = 12.5

$\therefore$ Average speed $= \dfrac{12.5}{4} = 3.125$ cm/s

Note: Remember that speed is the magnitude of the velocity. However, we can see from this example that average speed is *not* the magnitude of the average velocity.

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Acceleration

The acceleration of a particle is the rate of change of its velocity with respect to time.

- **Average acceleration** for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Note: The second derivative $\dfrac{d^2x}{dt^2}$ is just the derivative of the derivative.

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

Example 3

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^3 - 6t^2 + 5$, $t \geq 0$.

a Find its initial position, velocity and acceleration, and hence describe its motion.

b Find the times when it is instantaneously at rest and determine its position and acceleration at those times.

Solution

a $x = t^3 - 6t^2 + 5$

$$v = \frac{dx}{dt} = 3t^2 - 12t$$

$$a = \frac{dv}{dt} = 6t - 12$$

So when $t = 0$, we have $x = 5$, $v = 0$ and $a = -12$.

Initially, the particle is instantaneously at rest 5 cm to the right of O, with an acceleration of $-12\ \text{cm/s}^2$.

b $v = 0$ implies $3t^2 - 12t = 0$

$$3t(t-4) = 0$$

$$\therefore \quad t = 0 \text{ or } t = 4$$

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The particle is initially at rest and stops again after 4 seconds.

At $t = 0$, $x = 5$ and $a = -12$.
At $t = 4$, $x = (4)^3 - 6(4)^2 + 5 = -27$ and $a = 6(4) - 12 = 12$.

After 4 seconds, the particle's position is 27 cm to the left of O, and its acceleration is 12 cm/s^2.

Summary 23A

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is positive.
- **Average velocity** for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

  where x_2 is the position at time t_2 and x_1 is the position at time t_1.
- The instantaneous rate of change of position with respect to time is called the **instantaneous velocity**, or simply the **velocity**.
  If x is the position of the particle at time t, then its velocity is $v = \dfrac{dx}{dt}$
- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is $\dfrac{\text{distance travelled}}{t_2 - t_1}$
- **Average acceleration** for a time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Skill-sheet

Exercise 23A

Example 1
Example 2

1 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^2 - 7t + 12$. Find:

a its initial position
b its position at $t = 5$
c its initial velocity
d when and where its velocity equals zero
e its average velocity in the first 5 s
f its average speed in the first 5 s.

Example 3

2 The position, x metres, at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = t^2 - 7t + 10$. Find:

a when its velocity equals zero
b its acceleration at this time
c the distance travelled in the first 5 s
d when and where its velocity is -2 m/s.

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3 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = t^3 - 11t^2 + 24t - 3$. Find:

a its initial position and velocity
b its velocity at any time t
c at what times the particle is stationary
d where the particle is stationary
e for how long the particle's velocity is negative
f its acceleration at any time t
g when the particle's acceleration is zero and its velocity and position at that time.

4 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = 2t^3 - 5t^2 + 4t - 5$. Find:

a when its velocity is zero and its acceleration at that time
b when its acceleration is zero and its velocity at that time.

5 A particle is moving in a straight line in such a way that its position, x cm, relative to the point O at time t seconds ($t \geq 0$) satisfies $x = t^3 - 13t^2 + 46t - 48$. When does the particle pass through O, and what is its velocity and acceleration at those times?

6 Two particles are moving along a straight path so that their positions, x_1 cm and x_2 cm, relative to a fixed point P at any time t seconds are given by $x_1 = t + 2$ and $x_2 = t^2 - 2t - 2$. Find:

a the time when the particles are at the same position
b the time when they are moving with the same velocity.

23B Applications of antidifferentiation to kinematics

In the previous section, we considered examples in which we were given a rule for the position of a particle in terms of time, and from it we derived rules for the velocity and the acceleration by differentiation.

We may be given a rule for acceleration and, by using antidifferentiation and some additional information, we can deduce rules for both velocity and position.

Example 4

A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 4$.

a Find its position x in terms of t.
b Find its position after 3 seconds.
c What is the distance travelled in the first 3 seconds?
d Find its average velocity in the first 3 seconds.
e Find its average speed in the first 3 seconds.

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Solution

a We are given the velocity:

$$v = 2t - 4$$

Find the position by antidifferentiating:

$$x = t^2 - 4t + c$$

When $t = 0$, $x = 0$, and so $c = 0$.

$$\therefore \quad x = t^2 - 4t$$

b When $t = 3$, $x = -3$. The body is 3 m to the left of O.

c First find when the body is at rest: $v = 0$ implies $2t - 4 = 0$, i.e. $t = 2$.

When $t = 2$, $x = -4$.

Therefore the body goes from $x = 0$ to $x = -4$ in the first 2 seconds, and then back to $x = -3$ in the next second.

Thus it has travelled 5 m in the first 3 seconds.

d Average velocity $= \dfrac{-3 - 0}{3} = -1$ m/s

e From part **c**, the distance travelled is 5 m.

$\therefore$ Average speed $= \dfrac{5}{3}$ m/s

Example 5

A particle starts from rest 3 metres from a fixed point and moves in a straight line with an acceleration of $a = 6t + 8$. Find its position and velocity at any time t seconds.

Solution

We are given the acceleration:

$$a = \frac{dv}{dt} = 6t + 8$$

Find the velocity by antidifferentiating:

$$v = 3t^2 + 8t + c$$

At $t = 0$, $v = 0$, and so $c = 0$.

$$\therefore \quad v = 3t^2 + 8t$$

Find the position by antidifferentiating again:

$$x = t^3 + 4t^2 + d$$

At $t = 0$, $x = 3$, and so $d = 3$.

$$\therefore \quad x = t^3 + 4t^2 + 3$$

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Example 6

A stone is projected vertically upwards from the top of a 20 m high building with an initial velocity of 15 m/s.

a Find the time taken for the stone to reach its maximum height.

b Find the maximum height reached by the stone.

c What is the time taken for the stone to reach the ground?

d What is the velocity of the stone as it hits the ground?

In this case we only consider the stone's motion in a vertical direction, so we can treat it as motion in a straight line. Also we will assume that the acceleration due to gravity is approximately -10 m/s^2. (Note that downwards is considered the negative direction.)

Solution

We have

$$a = -10$$
$$v = -10t + c$$

At $t = 0$, $v = 15$, so $c = 15$.

$$\therefore \quad v = -10t + 15$$
$$x = -5t^2 + 15t + d$$

At $t = 0$, $x = 20$, so $d = 20$.

$$\therefore \quad x = -5t^2 + 15t + 20$$

a The stone will reach its maximum height when $v = 0$, i.e. when $-10t + 15 = 0$, which implies $t = 1.5$.

The stone reaches its maximum height when $t = 1.5$ seconds.

b At $t = 1.5$, $x = -5(1.5)^2 + 15(1.5) + 20$
$$= 31.25$$

The maximum height reached by the stone is 31.25 metres.

c The stone reaches the ground when $x = 0$:

$$-5t^2 + 15t + 20 = 0$$
$$-5(t^2 - 3t - 4) = 0$$
$$-5(t - 4)(t + 1) = 0$$

Thus $t = 4$. (The solution of $t = -1$ is rejected, since $t \geq 0$.)

The stone takes 4 seconds to reach the ground.

d At $t = 4$, $v = -10(4) + 15$
$$= -25$$

Thus its velocity on impact is -25 m/s.

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Summary 23B

Antidifferentiation may be used to go from acceleration to velocity, and from velocity to position.

Skill-sheet

Exercise 23B

Example 4

1 A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v cm/s, is given by $v = 4t - 6$. Find:

- **a** its position x in terms of t
- **b** its position after 3 s
- **c** the distance travelled in the first 3 s
- **d** its average velocity in the first 3 s
- **e** its average speed in the first 3 s.

2 The velocity of a particle, v m/s, at time t seconds ($t \geq 0$) is given by $v = 3t^2 - 8t + 5$. It is initially 4 m to the right of a point O. Find:

- **a** its position and acceleration at any time t
- **b** its position when the velocity is zero
- **c** its acceleration when the velocity is zero.

Example 5

3 A body moves in a straight line with an acceleration of 10 m/s^2. If after 2 s it passes through O and after 3 s it is 25 m from O, find its initial position relative to O.

4 A body moves in a straight line so that its acceleration, a m/s^2, after time t seconds ($t \geq 0$) is given by $a = 2t - 3$. If the initial position of the body is 2 m to the right of a point O and its velocity is 3 m/s, find the particle's position and velocity after 10 s.

Example 6

5 An object is projected vertically upwards with a velocity of 25 m/s. (Its acceleration due to gravity is -10 m/s^2.) Find:

- **a** the object's velocity at any time t
- **b** its height above the point of projection at any time t
- **c** the time it takes to reach its maximum height
- **d** the maximum height reached
- **e** the time taken to return to the point of projection.

6 The lift in a tall building passes the 50th floor with a velocity of -8 m/s and an acceleration of $\frac{1}{9}(t - 5)$ m/s^2. If each floor spans a height of 6 metres, find at which floor the lift will stop.

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23C Constant acceleration

If an object is moving due to a constant force (for example, gravity), then its acceleration is constant. There are several useful formulas that apply in this situation.

Formulas for constant acceleration

For a particle moving in a straight line with constant acceleration a, we can use the following formulas, where u is the initial velocity, v is the final velocity, s is the displacement and t is the time taken:

1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$

Proof 1 We can write

$$\frac{dv}{dt} = a$$

where a is a constant and v is the velocity at time t. By antidifferentiating with respect to t, we obtain

$$v = at + c$$

where the constant c is the initial velocity. We denote the initial velocity by u, and therefore $v = u + at$.

2 We now write

$$\frac{dx}{dt} = v = u + at$$

where x is the position at time t. By antidifferentiating again, we have

$$x = ut + \frac{1}{2}at^2 + d$$

where the constant d is the initial position. The particle's displacement (change in position) is given by $s = x - d$, and so we obtain the second equation.

3 Transform the first equation $v = u + at$ to make t the subject:

$$t = \frac{v - u}{a}$$

Now substitute this into the second equation:

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{u(v - u)}{a} + \frac{a(v - u)^2}{2a^2}$$

$$\begin{aligned} 2as &= 2u(v - u) + (v - u)^2 \\ &= 2uv - 2u^2 + v^2 - 2uv + u^2 \\ &= v^2 - u^2 \end{aligned}$$

4 Similarly, the fourth equation can be derived from the first and second equations.

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These four formulas are very useful, but it must be remembered that they only apply when the acceleration is constant.

When approaching problems involving constant acceleration, it is a good idea to list the quantities you are given, establish which quantity or quantities you require, and then use the appropriate formula. Ensure that all quantities are converted to compatible units.

Example 7

An object is moving in a straight line with uniform acceleration. Its initial velocity is 12 m/s and after 5 seconds its velocity is 20 m/s. Find:

a the acceleration

b the distance travelled during the first 5 seconds

c the time taken to travel a distance of 200 m.

Solution

We are given $u = 12$, $v = 20$ and $t = 5$.

a Find a using

$$\begin{aligned} v &= u + at \\ 20 &= 12 + 5a \\ a &= 1.6 \end{aligned}$$

The acceleration is $1.6\ \text{m/s}^2$.

b Find s using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= 12(5) + \frac{1}{2}(1.6)5^2 = 80 \end{aligned}$$

The distance travelled is 80 m.

Note: Since the object is moving in one direction, the distance travelled is equal to the displacement.

c We are now given $a = 1.6$, $u = 12$ and $s = 200$.

Find t using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ 200 &= 12t + \frac{1}{2} \times 1.6 \times t^2 \\ 200 &= 12t + \frac{4}{5}t^2 \\ 1000 &= 60t + 4t^2 \\ 250 &= 15t + t^2 \\ t^2 + 15t - 250 &= 0 \\ (t - 10)(t + 25) &= 0 \\ \therefore \quad t &= 10 \text{ or } t = -25 \end{aligned}$$

As $t \geq 0$, the only allowable solution is $t = 10$.

The object takes 10 s to travel a distance of 200 m.

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Summary 23C

Constant acceleration

If acceleration is constant, then the following formulas can be used (for acceleration a, initial velocity u, final velocity v, displacement s and time taken t):

1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$

Skill-sheet

Exercise 23C

1 How long does it take for an object that is initially at rest to travel a distance of 30 m if it is accelerated at 1.5 m/s^2?

2 A car is travelling at 25 m/s when the brakes are applied. It is brought to rest with uniform deceleration in 3 s. How far did it travel after the brakes were applied?

Example 7

3 A motorcycle accelerates uniformly from 3 m/s to 30 m/s in 9 seconds. Find:

a the acceleration

b the time it will take to increase in speed from 30 m/s to 50 m/s

c the distance travelled in the first 15 seconds (assuming it starts from rest)

d the time taken to reach a speed of 200 km/h (assuming it starts from rest).

4 A car accelerating uniformly from rest reaches a speed of 45 km/h in 5 seconds.

a Find its acceleration.

b Find the distance travelled in the 5 seconds.

5 A train starts from rest at a station and accelerates uniformly at 0.5 m/s^2 until it reaches a speed of 90 km/h.

a How long does the train take to reach this speed?

b How far does the train travel in reaching this speed?

6 A train travelling at 54 km/h begins to climb an incline of constant gradient that produces a deceleration of 0.25 m/s^2.

a How long will the train take to travel a distance of 250 m?

b What will the train's speed be then?

For Questions 7–11, assume that the acceleration due to gravity is −9.8 m/s^2 and ignore air resistance. Upward motion is considered to be in the positive direction.

7 A stone is projected vertically upwards from O with a speed of 20 m/s. Find:

a the velocity of the stone after 4 s

b the position of the stone relative to O after 4 s.

8 Repeat Question 7 for the stone being projected downwards from O with the same speed.

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9 An object is projected vertically upwards with a velocity of 49 m/s.

a After what time will the object return to the point of projection?

b When will the object be at a height of 102.9 m above the point of projection?

10 A man dives from a springboard where his centre of gravity is initially 3 m above the water and his initial velocity is 4.9 m/s upwards. Regarding the diver as a particle at his centre of gravity and assuming that the diver's motion is vertical, find:

a the diver's velocity after t seconds

b the diver's height above the water after t seconds

c the maximum height of the diver above the water

d the time taken for the diver to reach the water.

11 A stone is thrown vertically upwards from the top edge of a cliff 24.5 m high with a speed of 19.6 m/s. Find:

a the time taken for the stone to reach its maximum height

b the maximum height above the base of the cliff reached by the stone

c the time taken for the stone to return to the point of projection

d the time taken for the stone to reach the base of the cliff.

12 A body is travelling at 20 m/s when it passes point P and 40 m/s when it passes point Q. Find its speed when it is halfway from P to Q, assuming uniform acceleration.

23D Velocity–time graphs

Many kinematics problems can be solved using velocity–time graphs. These are particularly useful if acceleration is constant, but with a broader knowledge of integral calculus they can also be used when acceleration is variable. (Integration will not be used in this chapter.)

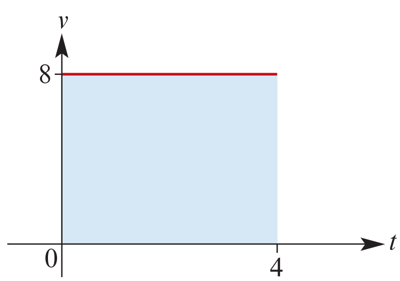
We begin with examples where the velocity is always positive.

Constant velocity

When a particle is moving with constant velocity, the corresponding velocity–time graph (v against t) is a straight line parallel to the t-axis.

The velocity–time graph for a particle moving at 8 m/s for 4 seconds is shown.

The shaded region is a rectangle of area $8 \times 4 = 32$, which is the product of the velocity and the time taken. Therefore this area is equal to the particle's displacement, 32 m, over the 4 seconds.



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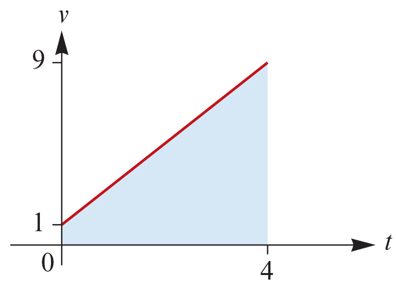
Constant acceleration

If a particle moves with constant acceleration a, its velocity v at time t is given by $v = u + at$, where u is the initial velocity. The velocity–time graph is a straight line with gradient a.

This graph shows the motion of a particle with initial velocity $u = 1$ m/s and acceleration $a = 2$ m/s^2. The equation of the straight line is $v = 1 + 2t$.



The particle's displacement over the 4 seconds is

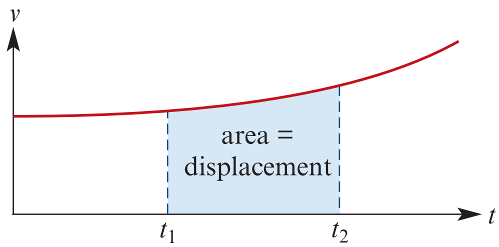
$$s = \frac{1}{2}(u + v)t = \frac{1}{2}(1 + 9)4 = 20 \text{ m}$$

This is the area of the shaded trapezium.

Variable acceleration

If the velocity is always positive, then the displacement is equal to the distance travelled.

The total area of the region(s) between the velocity–time graph and the t-axis corresponds to the distance travelled by the particle between times t_1 and t_2.



Note: You may have met the fundamental theorem of calculus in Mathematical Methods. Since $v = \dfrac{dx}{dt}$, it follows that $\int_{t_1}^{t_2} v(t)\, dt = x(t_2) - x(t_1)$.

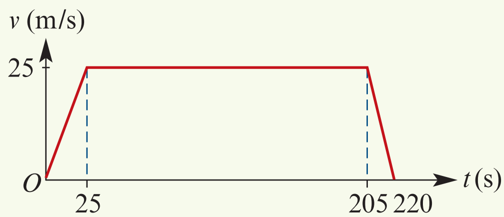
A velocity–time graph is particularly useful in situations where there are several stages to the particle's motion.

Example 8

A car starts from rest and accelerates uniformly for 25 s until it is travelling at 25 m/s. It maintains this velocity for 3 minutes, before decelerating uniformly until it stops in another 15 s. Construct a velocity–time graph and use it to determine the total distance travelled in kilometres.

Solution

From the graph we can calculate the area of the trapezium:



$$\begin{aligned} \text{Area} &= \frac{1}{2}(a + b)h \\ &= \frac{1}{2}(220 + 180)25 \\ &= 5000 \text{ m} \\ &= 5 \text{ km} \end{aligned}$$

The total distance travelled is 5 km.

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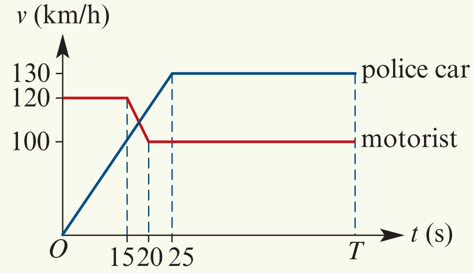
Example 9

A motorist is travelling at a constant speed of 120 km/h when he passes a stationary police car. He continues at that speed for another 15 s before uniformly decelerating to 100 km/h in 5 s. The police car takes off after the motorist the instant that he passes. It accelerates uniformly for 25 s, by which time it has reached 130 km/h. It continues at that speed until it catches up to the motorist. After how long does the police car catch up to the motorist and how far has he travelled in that time?

Solution

We start by representing the information on a velocity–time graph.



The distances travelled by the motorist and the police car will be the same, so the areas under the two velocity–time graphs will be equal. This fact can be used to find T, the time taken for the police car to catch up to the motorist.

Note: The factor $\frac{5}{18}$ changes velocities from km/h to m/s.

The distances travelled (in metres) after T seconds are given by

$$\begin{aligned}\text{Distance for motorist} &= \frac{5}{18}\left(120 \times 15 + \frac{1}{2}(120 + 100) \times 5 + 100(T - 20)\right)\\ &= \frac{5}{18}(1800 + 550 + 100T - 2000)\\ &= \frac{5}{18}(100T + 350)\end{aligned}$$

$$\begin{aligned}\text{Distance for police car} &= \frac{5}{18}\left(\frac{1}{2} \times 25 \times 130 + 130(T - 25)\right)\\ &= \frac{5}{18}(130T - 1625)\end{aligned}$$

When the police car catches up to the motorist:

$$\begin{aligned}100T + 350 &= 130T - 1625\\ 30T &= 1975\\ T &= \frac{395}{6}\end{aligned}$$

The police car catches up to the motorist after 65.83 s.

$$\begin{aligned}\therefore \quad \text{Distance for motorist} &= \frac{5}{18}(100T + 350) \qquad \text{where } T = \frac{395}{6}\\ &= \frac{52\,000}{27} \text{ m}\\ &= 1.926 \text{ km}\end{aligned}$$

The motorist has travelled 1.926 km when the police car catches up.

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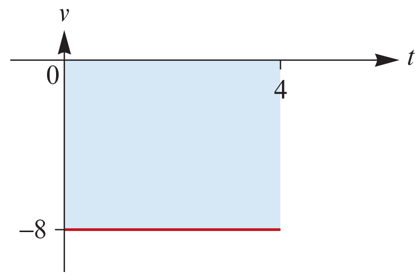
Signed area

This graph shows the motion of a particle with a velocity of −8 m/s for 4 seconds. The shaded region represents a displacement of −32 m. The region has a signed area of −32.

- A region *above* the t-axis has *positive* signed area.
- A region *below* the t-axis has *negative* signed area.



Example 10

A particle is moving in a straight line. The initial velocity of the particle is 10 m/s and it has a constant acceleration of −2 m/s^2.

a Sketch the velocity–time graph for the motion.

b Describe the motion of the particle during the first 8 seconds.

c Find the total distance travelled in the first 8 seconds of motion.

d Find the displacement of the particle after the first 8 seconds of motion.

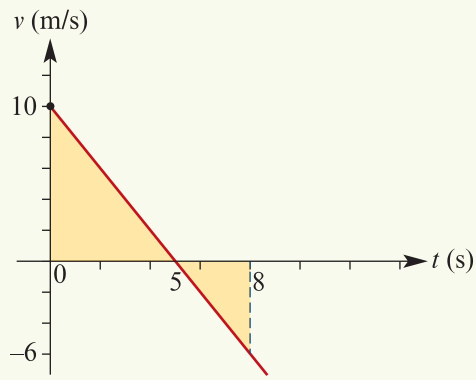
Solution

a We are given $u = 10$ and $a = -2$.

The equation of the line is

$$v = -2t + 10$$

where v m/s is the velocity at time t s.



b From $t = 0$ to $t = 5$, the particle has positive velocity; it is moving to the right.

At $t = 5$, the particle has velocity zero; it is momentarily stationary.

From $t = 5$ to $t = 8$, the particle has negative velocity; it is moving to the left.

c Distance travelled = total area $= \frac{1}{2} \times 5 \times 10 + \frac{1}{2} \times 3 \times 6 = 34$ metres

d Displacement = total signed area $= \frac{1}{2} \times 5 \times 10 - \frac{1}{2} \times 3 \times 6 = 16$ metres

Summary 23D

- **Distance travelled** is given by the sum of the **areas** of the regions between the velocity–time graph and the t-axis.
- **Displacement** is given by the sum of the **signed areas** of the regions between the velocity–time graph and the t-axis.

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Skill-sheet

Exercise 23D

It is suggested that you draw a velocity–time graph for each of the following questions.

Example 8

1 A particle starts from rest and accelerates uniformly for 5 s until it reaches a speed of 10 m/s. It immediately decelerates uniformly until it comes to rest after a further 8 s. How far did it travel?

2 A car accelerates uniformly from rest for 10 s to a speed of 15 m/s. It maintains this speed for 25 s before decelerating uniformly to rest after a further 15 s. Find:

a the total distance travelled by the car

b the distance it had travelled when it started to decelerate

c the time taken for it to reach the halfway point of its journey.

3 A particle starts from rest and travels 1 km before coming to rest again. For the first 5 s it accelerates uniformly. It next maintains a constant speed for 500 m, and then decelerates uniformly for the last 10 s. Find the maximum speed of the particle.

4 A car passes point P with a speed of 36 km/h and continues at this speed for 12 s before accelerating to a speed of 72 km/h in 6 s. How far from P is the car when it reaches a speed of 72 km/h?

5 A tram decelerates uniformly from a speed of 60 km/h to rest in 60 s. Find:

a the distance travelled by the tram

b how far it had travelled by the time it had reduced its speed by half

c the time taken for it to travel half the total distance.

Example 9

6 A car passes a point A with a speed of 15 m/s and continues travelling at that speed. A second car is stationary at point A. At the moment when the first car passes A, the second car accelerates uniformly until it reaches a speed of 25 m/s in 10 s. Both cars continue with a constant speed on to point B, which they reach at the same time.

a How long does it take for both cars to reach point B?

b How far is it from A to B?

Example 10

7 A particle is moving in a straight line. The initial velocity of the particle is 20 m/s and it has a constant acceleration of -2 m/s^2.

a Sketch the velocity–time graph for the motion.

b Describe the motion of the particle during the first 14 seconds.

c Find the total distance travelled in the first 14 seconds of motion.

d Find the displacement of the particle after the first 14 seconds of motion.

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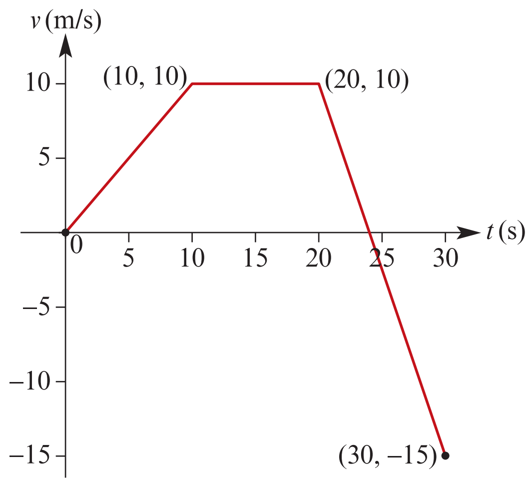
8 The velocity–time graph for the motion of a particle is shown.

a Find the acceleration for the first 10 seconds.

b Find the acceleration for the period from $t = 20$ to $t = 30$, where t is the time in seconds from the beginning of the motion.

c Find the total distance travelled in the first 30 seconds.

d Find the displacement of the particle after 30 seconds.



9 A particle moves in a straight line, starting from rest at a point O. It first moves in a positive direction with an acceleration of 2 m/s^2, until its velocity reaches 10 m/s. It then continues with a constant velocity of 10 m/s for some time, before decelerating to rest after a total time of 20 seconds. The total distance travelled is 160 m.

a Sketch the velocity–time graph.

b Find the magnitude of the deceleration.

10 Two stations A and B are 14 km apart. A train passes through station A, heading towards B, maintaining a constant speed of 60 km/h. At the instant that it passes through A, a second train on the same track leaves station B, heading towards A, and accelerates uniformly. After 5 minutes, the alarm is raised at both stations simultaneously that a collision is imminent. Both trains are radioed and instructed to brake. The first train decelerates uniformly so that it will stop in 2.5 minutes. The second train, which has reached a speed of 80 km/h, will take 4 minutes to stop. Will they collide?

11 Two tram stops are 800 m apart. A tram starts from rest at the first stop and accelerates at a constant rate of a m/s^2 for a certain time and then decelerates at a constant rate of $2a$ m/s^2, before coming to rest at the second stop. The time taken to travel between the two stops is 1 minute 40 seconds. Find:

a the maximum speed reached by the tram (in km/h)

b the time at which the brakes are applied

c the value of a.

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Chapter summary

Assign-ment

Nrich

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the origin, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.
- **Average velocity** $= \dfrac{\text{change in position}}{\text{change in time}}$
- For a particle moving in a straight line with position x at time t:
  - **velocity** (v) is the rate of change of position with respect to time
  - **acceleration** (a) is the rate of change of velocity with respect to time

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

- **Displacement** is the change in position (i.e. final position minus initial position).
- Scalar quantities:
  - **Distance travelled** means the total distance travelled.
  - **Speed** is the magnitude of the velocity.
  - **Average speed** $= \dfrac{\text{distance travelled}}{\text{change in time}}$
- Constant acceleration

  If acceleration is constant, then the following formulas can be used (for acceleration a, initial velocity u, final velocity v, displacement s and time taken t):

  1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$
- Velocity–time graphs
  - *Distance travelled* is given by the sum of the *areas* of the regions between the velocity–time graph and the t-axis.
  - *Displacement* is given by the sum of the *signed areas* of the regions between the velocity–time graph and the t-axis.

Technology-free questions

1 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^2 - 4t - 5$. Find:

a its initial position
b its position at $t = 3$
c its initial velocity
d when and where its velocity equals zero
e its average velocity in the first 3 s
f its average speed in the first 3 s.

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^3 - 2t^2 + 8$. Find:

a its initial position, velocity and acceleration and hence describe its motion
b the times when it is stationary and its position and acceleration at those times.

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3 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = -2t^3 + 3t^2 + 12t + 7$. Find:

a when the particle passes through O and its velocity and its acceleration at those times

b when the particle is at rest

c the distance travelled in the first 3 seconds.

4 Two particles A and B are moving in a straight line such that their positions, x_A cm and x_B cm, relative to the point O at time t seconds ($t \geq 0$) are given by

$$x_A(t) = t^3 - t^2 \quad \text{and} \quad x_B(t) = t^2$$

a Find:

i the position of A after $\frac{1}{2}$ s

ii the acceleration of A after $\frac{1}{2}$ s

iii the velocity of B after $\frac{1}{2}$ s.

b Find:

i the times when A and B collide (i.e. have the same position)

ii the maximum distance between A and B during the first 2 s of motion.

5 A particle moving in a straight line has an acceleration of $6t$ m/s^2 at time t seconds ($t \geq 0$). If the particle starts from rest at the origin O, find:

a the velocity after 2 s

b the position at any time t.

6 A particle moving in a straight line has an acceleration of $(3 - 2t)$ m/s^2 at time t seconds ($t \geq 0$). If the particle starts at the origin O with a velocity of 4 m/s, find:

a the time when the particle comes to rest

b the position of the particle at the instant it comes to rest

c the acceleration at this instant

d the time when the acceleration is zero

e the velocity at this time.

7 A particle moves in a straight line and, at time t seconds after it starts from point O, its velocity is $(2t^2 - 3t^3)$ m/s. Find:

a the position after 1 s

b the velocity after 1 s

c the acceleration after 1 s.

8 For a particle moving in a straight line, the velocity function is $v\colon \mathbb{R}^+ \to \mathbb{R}$, $v(t) = \dfrac{1}{2t^2}$. Find:

a the acceleration at time t

b the position at time t, given that the particle is at O when $t = 1$.

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9 The velocity, v m/s, of an object t seconds after it starts moving from O along a straight line is given by $v = t^3 - 11t^2 + 24t$, $t \geq 0$.

a Find the acceleration at time t.

b Find the acceleration at the instant when the object first changes direction.

c Find the displacement of the object from O after 5 s, and the total distance travelled in the first 5 s.

10 A car is travelling at 20 m/s when the brakes are applied. It is brought to rest with uniform deceleration in 4 s. How far did it travel after the brakes were applied?

11 A car accelerates uniformly from 0 m/s to 30 m/s in 12 seconds. Find:

a the acceleration

b the time it will take to increase in speed from 30 m/s to 50 m/s

c the distance travelled in the first 20 seconds

d the time taken to reach a speed of 100 km/h.

12 A train starts from rest at a station and accelerates uniformly at 0.4 m/s^2 until it reaches a speed of 60 km/h.

a How long does the train take to reach this speed?

b How far does the train travel in reaching this speed?

For Questions 13–14, assume that the acceleration due to gravity is −9.8 m/s^2 and ignore air resistance. Upward motion is considered to be in the positive direction.

13 An object is projected vertically upwards with a velocity of 35 m/s.

a After what time will the object return to the point of projection?

b When will the object be at a height of 60 m above the point of projection?

14 A stone is projected vertically upwards from the top of a cliff 20 m high with a speed of 19.6 m/s. Find:

a the time taken for the stone to reach its maximum height

b the maximum height reached with respect to the base of the cliff

c the time taken for the stone to return to the point of projection

d the time taken for the stone to reach the base of the cliff.

It is suggested that you draw a velocity–time graph for each of Questions 15–18.

15 A particle starts from rest and accelerates uniformly for 15 s until it reaches a speed of 25 m/s. It immediately decelerates uniformly until it comes to rest after a further 20 s. How far did it travel?

16 A car accelerates uniformly from rest for 8 s to a speed of 12 m/s. It maintains this speed for 15 s before decelerating uniformly to rest after a further 10 s. Find:

a the total distance travelled by the car

b the time taken for it to reach the halfway point of its journey.

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17 A vehicle starts from rest and travels 1 km before coming to rest again. For the first 15 s it accelerates uniformly, before maintaining a constant speed for 800 m and then finally decelerating uniformly to rest in 10 s. Find the maximum speed of the vehicle.

18 A car travels at a constant speed of 12 m/s along a straight road. It passes a second stationary car, which sets off in pursuit 3 s later. Find the constant acceleration required for the second car so that it catches the first car after a further 27 s has passed.

19 A particle moves in a straight line so that t seconds after passing a fixed point O in the line its velocity, v m/s, is given by $v = \frac{t^2}{4} - 3t + 5$. Calculate:

a the velocity after 10 s

b the acceleration when $t = 0$

c the minimum velocity

d the distance travelled in the first 2 s

e the distance travelled in the 3rd second.

20 A spot of light moves along a straight line so that its acceleration t seconds after passing a fixed point O on the line is $(2 - 2t)$ cm/s^2. Three seconds after passing O, the spot has a velocity of 5 cm/s. Find an expression, in terms of t, for:

a the velocity of the spot of light after t seconds

b the position of the spot relative to O after t seconds.

21 A particle P is moving along a straight line. It passes through a point O with a velocity of 6 m/s. At time t seconds after passing through O, its acceleration is $(4 - 4t)$ m/s^2.

a Show that, at time t seconds, the velocity of P is $(6 + 4t - 2t^2)$ m/s.

b Calculate:

- **i** the maximum velocity of P
- **ii** the value of t when the velocity of P is again 6 m/s
- **iii** the distance OP when the velocity of P is zero.

22 A particle travelling in a straight line passes a fixed point O with velocity 5 m/s. Its acceleration, a m/s^2, is given by $a = 27 - 4t^2$, where t seconds is the time after passing O. Calculate:

a the acceleration of the particle as it passes O

b its velocity when $t = 3$

c the value of t when its velocity is again 5 m/s.

23 A particle passes a fixed point O with a velocity of 2 m/s and moves in a straight line with an acceleration of $3(1 - t)$ m/s^2, where t is the time in seconds after passing O. Calculate:

a the velocity when $t = 4$

b the position of the particle at this instant.

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24 A particle P travels in a straight line starting at a fixed point O so that its velocity, v m/s, is given by $v = t^2 - 10t + 24$, where t is the time in seconds after leaving O. Find:

- **a** the values of t for which P is instantaneously at rest
- **b** the distance OP when $t = 3$
- **c** the range of values of t for which the acceleration is negative.

Multiple-choice questions

1 A particle moves in a straight line so that its position, x cm, relative to a fixed point O at time t seconds ($t \geq 0$) is given by $x = -t^3 + 7t^2 - 12t$. The initial position of the particle relative to O is

A 0 cm **B** −6 cm **C** 12 cm **D** −20 cm **E** 5 cm

2 A particle moves in a straight line so that its position, x cm, relative to a fixed point O at time t seconds ($t \geq 0$) is given by $x = -t^3 + 7t^2 - 12t$. The average velocity of the particle in the first 2 seconds, correct to two decimal places, is

A 4 cm/s **B** −4 cm/s **C** 2 cm/s **D** 4.06 cm/s **E** −2 cm/s

3 A particle moves in a straight line with an acceleration of $4 - 6t$ m/s^2 at time t seconds. The particle has an initial velocity of −1 m/s and an initial position of 4 m relative to a fixed point O. The velocity of the particle when $t = 1$ is

A −1 m/s **B** 6 m/s **C** 0 m/s **D** 4 m/s **E** −2 m/s

4 A body starts from rest with a uniform acceleration of 1.8 m/s^2. The time it will take for the body to travel 90 m is

A 5 s **B** $\sqrt{10}$ s **C** 10 s **D** $\sqrt{10}$ **E** $10\sqrt{2}$ s

5 A car accelerating uniformly from rest reaches a speed of 60 km/h in 4 s. The car's acceleration is

A 15 km/h^2 **B** 15 m/s^2 **C** 54 m/s^2 **D** $\frac{25}{6}$ km/h^2 **E** $\frac{25}{6}$ m/s^2

6 A car accelerating uniformly from rest reaches a speed of 60 km/h in 4 s. The distance travelled by the car in the 4 s is

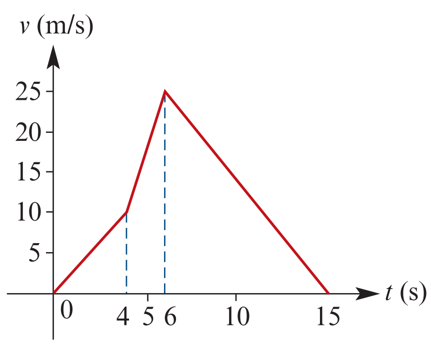
A 200 m **B** 100 km **C** $\frac{100}{3}$ m **D** 100 m **E** 360 m

7 This velocity–time graph shows the motion of a car. The total distance travelled by the car over the 15 s is

A 75 m **B** 315 m

C 182.5 m **D** 167.5 m

E 375 m



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8 A rock falls from the top of a cliff 40 m high. Assuming that the acceleration due to gravity is 9.8 m/s^2, the rock's speed just before it hits the ground is

A 20 m/s **B** 22 m/s **C** 24 m/s **D** 26 m/s **E** 28 m/s

9 A body initially travelling at 20 m/s is subject to a constant deceleration of 4 m/s^2. The time it takes to come to rest (t seconds) and the distance travelled before it comes to rest (s metres) are given by

A $t = 5,\ s = 50$ **B** $t = 5,\ s = 45$ **C** $t = 4,\ s = 20$
D $t = 5,\ s = 40$ **E** $t = 4,\ s = 35$

10 A particle moves in a straight line with an acceleration of $12t - 5$ m/s^2 at time t seconds. The particle has an initial velocity of 1 m/s and an initial position of 0 m relative to a fixed point O. The velocity of the particle at time $t = 1$ is

A 1 m/s **B** −5 m/s **C** 7 m/s **D** 2 m/s **E** 3 m/s

Extended-response questions

1 A particle moves in a straight line so that its position, x cm, relative to point O at time t seconds is given by

$$x = \frac{1}{3}t^3 - 2t^2 + 4t - 2\tfrac{1}{3}$$

a Find its initial position.
b Find its initial velocity.
c Find its acceleration after 3 seconds.
d When is its velocity zero?
e What is its position when the velocity is zero?
f When is the particle at point O?

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^4 + 2t^2 - 8t$. Show that:

a the particle moves first to the left
b the greatest distance of the particle to the left of O occurs after 1 second
c after this time, the particle always moves to the right.

3 A defective rocket rises vertically upwards into the air and then crashes back to the ground. The rocket's height above the ground, h metres, at time t seconds after take-off is given by $h = 6t^2 - t^3$. (This is an approximate model.)

a When does the rocket crash and what is its velocity at this time?
b At what time is the speed of the rocket zero, and what is its maximum height?
c When does the acceleration of the rocket become negative?

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4 An object is projected vertically upwards at 20 m/s from the top of a tower 10 m high on the edge of a vertical cliff. At time t seconds after projection, the object has position $x(t)$ metres relative to the base of the tower, where $x(t) = -4.9t^2 + 20t + 10$ for $t \geq 0$. Use a CAS calculator to evaluate the values

$$x(1) - x(0), \quad x(2) - x(1), \quad x(3) - x(2), \quad \dots, \quad x(10) - x(9)$$

Analyse your results and draw some inference about the motion of the object.

5 A particle is projected vertically upwards with an initial speed of u m/s, and the magnitude of the acceleration due to gravity is g m/s^2. Prove that:

a the time taken by the particle to reach its highest point is $\frac{u}{g}$ seconds

b the total time taken for the particle to return to the point of projection is $\frac{2u}{g}$ seconds

c the particle's speed when returning to the point of projection is u m/s.

6 A stone is projected vertically upwards with a speed of 14 m/s from a point O at the top of a mine shaft. Five seconds earlier, a lift began to descend the mine shaft from O with a constant speed of 3.5 m/s. Find the depth of the lift (to the nearest metre) at the instant when the stone falls on it. (Neglect air resistance and take the acceleration due to gravity to be 9.8 m/s^2.)

7 A car is travelling along a straight road at 90 km/h when the brakes are applied. The car comes to rest in 5 seconds and, during this time, its velocity decreases linearly with time. Find:

a the rule for the velocity function after the brakes are applied

b the distance travelled in the 5 seconds.

8 A particle moves in a straight line so that its position, x cm, relative to point O at time t seconds ($t \geq 0$) is given by $x = 3t^4 - 4t^3 + 24t^2 - 48t$. Show that the particle moves at first to the left, comes to rest at a point A and then moves always to the right. Find the position of A.

9 A particle is projected vertically upwards with a velocity of u m/s from a point O on the ground, and T seconds later a second particle is projected vertically upwards from O with the same velocity. Let g m/s^2 be the magnitude of the acceleration due to gravity.

a Prove that:

i the time taken for the two particles to collide is $\frac{u}{g} + \frac{T}{2}$ seconds after the first particle was launched

ii the height of the particles when they collide is $\frac{4u^2 - g^2T^2}{8g}$ metres above O.

b Interpret the case where $T = \frac{2u}{g}$.

c What happens if $T > \frac{2u}{g}$?

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Glossary

A

Absolute value function [p. 557] The absolute value of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

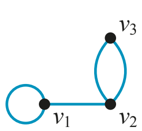
Also called the *modulus function*

Addition of complex numbers [p. 606]
If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.

Addition of vectors [p. 696]
If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$.

Addition principle [p. 290] Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Adjacency matrix [p. 383] a matrix that represents a graph. The entries of the matrix give the number of edges joining each pair of vertices. For example:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

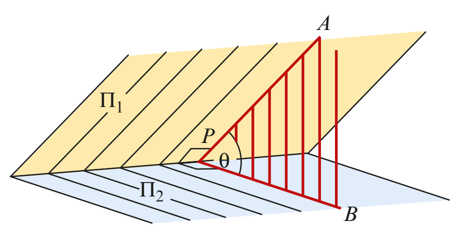
Adjacent vertices [p. 383] Two vertices of a graph are adjacent if they are joined by an edge.

Algebra of sets [p. 205] general statements involving the operations $\cup$, $\cap$ and $'$ acting on the set of all subsets of a given set ξ; e.g. $A \cup A' = \xi$

Algorithm [p. 250] a finite, unambiguous sequence of instructions for performing a specific task

Altitude of a triangle [p. 717] a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side

Angle between planes [p. 506] For any point P on the common line of two planes Π_1 and Π_2, if lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between Π_1 and Π_2.

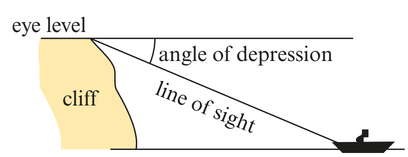


Angle between two vectors [p. 708] can be found using the scalar product:

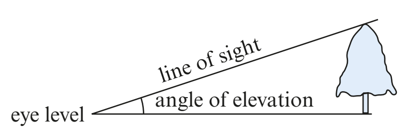
$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

Angle of depression [p. 498] the angle between the horizontal and a direction below the horizontal

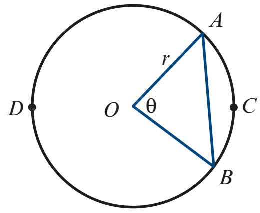


Angle of elevation [p. 498] the angle between the horizontal and a direction above the horizontal



Arc [p. 493] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Arc length [p. 494] The length of arc ACB is given by $\ell = r\theta$, where $\theta^c = \angle AOB$.



Arccos *see* inverse cosine function

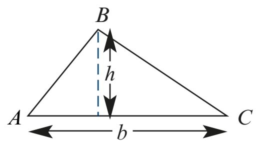
Arcsin *see* inverse sine function

Arctan *see* inverse tangent function

Area of a triangle [p. 490] given by half the product of the lengths of two sides and the sine of the angle included between them.

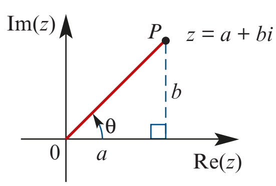
$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}bc\sin A$$



Area of image [p. 681] If a linear transformation (with matrix $\mathbf{B}$) is applied to a region of the plane, then Area of image = $|\det(\mathbf{B})| \times$ Area of region.

Argand diagram [p. 616] a geometric representation of the set of complex numbers



Argument of a complex number [p. 626]

- The argument of z is an angle θ from the positive direction of the x-axis to the line joining the origin to z.
- The *principal value* of the argument, denoted by $\operatorname{Arg} z$, is the angle in the interval $(-\pi, \pi]$.

Arithmetic sequence [p. 75] a sequence in which each successive term is found by adding a fixed amount to the previous term; e.g. $2, 5, 8, 11, \ldots$. An arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is the common difference. The nth term can be found using $t_n = a + (n-1)d$, where $a = t_1$.

Arithmetic series [p. 79] the sum of the terms in an arithmetic sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

where $a = t_1$ and d is the common difference.

Arrangement [p. 293] *see* permutation

Asymptote [p. 576] A straight line is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to the straight line. An asymptote can be horizontal, vertical or oblique.

Asymptotes of hyperbolas [p. 576] The hyperbola with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm\frac{b}{a}(x-h)$$

B

Bearing [p. 499] the compass bearing; the direction measured from north clockwise

Binary number system [p. 252] uses only the digits 0 and 1 to represent numbers. The positions of the digits correspond to different powers of 2.

Boolean algebra [p. 214] a set B equipped with operations $\vee, \wedge, '$ that are analogous to the set-theoretic operations $\cup, \cap, '$ and also to the logical connectives 'or', 'and', 'not'

Boolean expression [p. 216] an expression formed using $\vee, \wedge, '$, 0 and 1; e.g. $x \wedge (y \vee x)'$

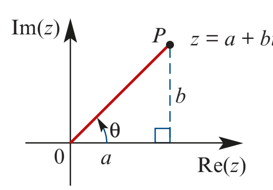
Boolean function [p. 216] a function with one or more inputs from $\{0, 1\}$ and outputs in $\{0, 1\}$; e.g. $f\colon \{0,1\}^3 \to \{0,1\}$, $f(x, y, z) = x \wedge (y \vee x)'$

C

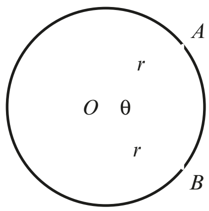
$\mathbb{C}$ [p. 605] the set of complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Cartesian equation [p. 564] an equation that describes a curve in the plane by giving the relationship between the x- and y-coordinates of the points on the curve; e.g. $y = x^2 + 1$

Cartesian form of a complex number [p. 605] A complex number is expressed in Cartesian form as $z = a + bi$, where a is the real part of z and b is the imaginary part of z.



Chord [p. 493]
a line segment with endpoints on a circle



Chord length [p. 494]

$AB = 2r\sin\left(\frac{\theta}{2}\right)$

where $\theta^c = \angle AOB$

Circle, general Cartesian equation [p. 564] The circle with radius r and centre (h, k) has equation $(x - h)^2 + (y - k)^2 = r^2$.

Circular functions [pp. 478, 479] the sine, cosine and tangent functions

Circular functions, exact values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

cis θ [p. 626] $\cos\theta + i\sin\theta$

Collinear points [p. 714] Three or more points are collinear if they all lie on a single line.

Column vector [pp. 657, 695] an $n \times 1$ matrix.
A column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be used to represent a vector in the plane, an ordered pair, a point in the Cartesian plane or a translation of the plane.

Combination [p. 304] a selection where order is not important. The number of combinations of n objects taken r at a time is given by

${}^nC_r = \frac{n!}{r!\,(n-r)!}$

An alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

Common difference, *d* [p. 75] the difference between two consecutive terms of an arithmetic sequence, i.e. $d = t_n - t_{n-1}$

Common ratio, *r* [p. 85] the quotient of two consecutive terms of a geometric sequence, i.e.
$r = \frac{t_n}{t_{n-1}}$

Compass bearing [p. 499] the direction measured from north clockwise

Complement of a set [p. 40] The complement of a set A, written A', is the set of all elements of ξ that are not elements of A.

Complement of a simple graph [p. 406]
If G is a simple graph, then its complement $\overline{G}$ is the simple graph with the same vertices as G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Complex conjugate, $\bar{z}$ [p. 609]
- If $z = a + bi$, then $\bar{z} = a - bi$.
- If $z = r\operatorname{cis}\theta$, then $\bar{z} = r\operatorname{cis}(-\theta)$.

Complex conjugate, properties [p. 610]
- $z + \bar{z} = 2\operatorname{Re}(z)$
- $z\bar{z} = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Complex number [p. 605] an expression of the form $a + bi$, where a and b are real numbers

Complex plane [p. 616] *see* Argand diagram

Composite [p. 52] A natural number m is a composite number if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

Compound angle formulas [p. 523]
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Concurrent lines [p. 714] Three or more lines are concurrent if they all pass through a single point.

Conditional statement [pp. 169, 224]
a statement of the form 'If P is true, then Q is true', which can be abbreviated to $P \Rightarrow Q$

Congruence tests Two triangles are congruent if one of the following conditions holds:
- SSS the three sides of one triangle are equal to the three sides of the other triangle
- SAS two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
- AAS two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
- RHS the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

Congruent figures have exactly the same shape and size

Conjugate root theorem [p. 624] Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of the equation $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

Constant velocity [p. 718] If a particle moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by $\boldsymbol{s} = t\boldsymbol{v}$.

Contradiction [p. 223] a statement which is false under all circumstances; *see also* proof by contradiction

Contrapositive [pp. 175, 226] The contrapositive of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P). The contrapositive is equivalent to the original statement.

Convergent series [p. 109] An infinite series $t_1 + t_2 + t_3 + \cdots$ is convergent if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. An infinite geometric series is convergent if $-1 < r < 1$, where r is the common ratio.

Converse [pp. 182, 226] The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

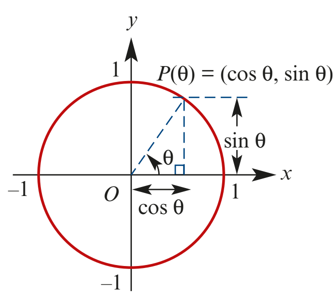
Conversion between Cartesian and polar forms [pp. 589, 626]

$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2$

Cosecant function [pp. 518, 554]

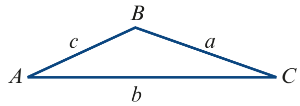
$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ for $\sin\theta \neq 0$

Cosine function [p. 478] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Cosine rule [p. 487] used to find unknown quantities in a triangle given two sides and the included angle, or given three sides. For $\triangle ABC$:

$a^2 = b^2 + c^2 - 2bc\cos A$



Cotangent function [pp. 518, 554]

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ for $\sin\theta \neq 0$

Counterexample [p. 186] an example that shows that a universal statement is false. For example, the number 2 is a counterexample to the claim 'Every prime number is odd.'

Cycle [pp. 398, 405] a walk in a graph that starts and ends at the same vertex and otherwise does not repeat any vertices or edges

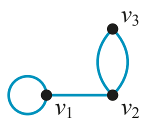
D

De Morgan's laws [pp. 174, 216]

- 'not (P and Q)' is '(not P) or (not Q)'
- 'not (P or Q)' is '(not P) and (not Q)'

Degree of a polynomial [p. 120] given by the highest power of x with a non-zero coefficient; e.g. the polynomial $2x^5 - 7x^2 + 4$ has degree 5

Degree of a vertex [p. 386] the number of edges that end at the vertex, with each edge that is a loop counted twice. For example:



Vertex	Degree
v_1	3
v_2	3
v_3	2

Desk check [p. 266] To carry out a desk check of an algorithm, you carefully follow the algorithm step by step, and construct a table of the values of all the variables after each step.

Determinant of a matrix [pp. 359, 368] Associated with each square matrix $\mathbf{A}$, there is a real number called the determinant of $\mathbf{A}$, which is denoted by $\det(\mathbf{A})$. A square matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

Diameter [p. 493] a chord of a circle that passes through the centre

Dilation [p. 662] A dilation scales the x- or y-coordinate of each point in the plane.

- Dilation from the x-axis: $(x, y) \to (x, cy)$
- Dilation from the y-axis: $(x, y) \to (cx, y)$

Direct proof [p. 170] To give a direct proof of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Discrete random variable [p. 440] a random variable X which can take only a countable number of values, usually whole numbers

Discriminant, Δ, of a quadratic [p. 125] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac > 0$, there are two real solutions.
- If $b^2 - 4ac = 0$, there is one real solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint sets [p. 39] Sets A and B are said to be disjoint if they have no elements in common, i.e. if $A \cap B = \emptyset$.

Displacement [p. 717] the change in position. If a particle moves from point A to point B, then its displacement is described by the vector $\overrightarrow{AB}$.

Distance in the complex plane [p. 631] The distance between complex numbers z_1 and z_2 is equal to $|z_2 - z_1|$.

Division of complex numbers [pp. 611, 628]

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Dot product [p. 707] *see* scalar product

Double angle formulas [p. 526]

- $\cos(2x) = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$
- $\sin(2x) = 2 \sin x \cos x$
- $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$

E

Ellipse [p. 571] The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).

Equality of complex numbers [p. 606]

$a + bi = c + di$ if and only if $a = c$ and $b = d$

Equilibrium [p. 727] A particle is said to be in equilibrium if the resultant force acting on it is zero; the particle will remain at rest or continue moving with constant velocity.

Equivalence of vectors [p. 703]
Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$.
If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Equivalent statements [pp. 183, 225] Statements P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$; this is abbreviated to $P \Leftrightarrow Q$.
For equivalent statements P and Q, we also say 'P is true *if and only if* Q is true'.

Euclidean algorithm [p. 254] a method for finding the highest common factor of two natural numbers

Euler circuit [pp. 392, 393] a walk in a graph that uses every edge exactly once and that starts and ends at the same vertex. A connected graph has an Euler circuit if and only if every vertex has even degree.

Euler trail [pp. 392, 394] a walk in a graph that uses every edge exactly once. A connected graph has an Euler trail if and only if every vertex has even degree or exactly two vertices have odd degree.

Euler's formula [p. 415] If G is a connected planar graph with v vertices, e edges and f faces, then $v - e + f = 2$.

Existence statement [pp. 185, 187] a statement claiming that a property holds for some member of a given set. Such a statement can be written using the quantifier 'there exists'.

Expected value of a random variable, E(X) [p. 442] also called the mean, μ. For a discrete random variable X:

$$\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x) = \sum_x x \cdot p(x)$$

F

Factor [p. 52] A natural number a is a factor of a natural number b if there exists a natural number k such that $b = ak$.

Factor theorem [p. 623] A polynomial $P(z)$ has $z - \alpha$ as a factor if and only if $P(\alpha) = 0$.

Factorial notation [p. 293] The notation $n!$ (read as 'n factorial') is an abbreviation for the product of all the integers from n down to 1:

$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

Force [p. 727] causes a change in motion; e.g. gravitational force, tension force, normal reaction force. Force is a vector quantity.

Formula [p. 19] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

Fundamental theorem of algebra [p. 623] For $n \geq 1$, every polynomial of degree n can be expressed as a product of n linear factors over the complex numbers. Therefore every polynomial equation of degree n has n solutions (counting multiplicity).

Fundamental theorem of arithmetic [p. 53] Every natural number greater than 1 either is a prime number or can be represented as a product of prime numbers. Furthermore, this representation is unique apart from rearrangement of the order of the prime factors.

G

Geometric mean [p. 87] For $a, b, c \in \mathbb{R}^+$, if $\frac{c}{a} = \frac{b}{c}$, then c is the geometric mean of a and b.

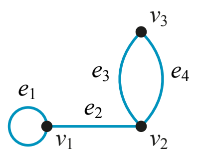
Geometric sequence [p. 85] a sequence in which each successive term is found by multiplying the previous term by a fixed amount; e.g. $2, 6, 18, 54, \dots$. A geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is the common ratio. The nth term can be found using $t_n = ar^{n-1}$, where $a = t_1$.

Geometric series [p. 90] the sum of the terms in a geometric sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Graph [p. 383] A graph consists of a finite non-empty set of *vertices*, a finite set of *edges* and an *edge-endpoint function* that maps each edge to a set of either one or two vertices. A graph can be represented by a diagram, where the vertices are shown as points and the edges as lines connecting the vertices. For example:



Edge	Endpoints
e_1	$\{v_1\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2, v_3\}$

Graph, bipartite [p. 407] The vertices of a bipartite graph can be divided into two disjoint subsets A and B such that every edge of the graph joins a vertex in A to a vertex in B.

Graph, complete [p. 405] A complete graph is a simple graph with one edge joining each pair of distinct vertices. The complete graph with n vertices is denoted by K_n.

Graph, complete bipartite [p. 407] A complete bipartite graph is a simple graph whose vertices can be divided into two disjoint subsets A and B such that:

- every edge joins a vertex in A to a vertex in B
- every vertex in A is joined to every vertex in B.

The complete bipartite graph where $|A| = m$ and $|B| = n$ is denoted by $K_{m,n}$.

Graph, connected [p. 391] A graph is said to be connected if there is a walk between each pair of distinct vertices.

Graph, disconnected [p. 391] A graph is said to be disconnected if there are two distinct vertices that are not connected by a walk.

Graph, planar [p. 414] A planar graph can be drawn in the plane so that its edges do not cross.

Graph, regular [p. 405] A graph is said to be regular if all its vertices have the same degree.

Graph, simple [p. 387] A simple graph has no loops or multiple edges.

Graphs, isomorphic [p. 385] Two graphs are isomorphic if there is a one-to-one correspondence between their vertices that preserves the ways the vertices are connected by edges.

H

Hamiltonian cycle [p. 398] a walk in a graph that starts and ends at the same vertex and visits every other vertex exactly once (without repeating any edges)

Hamiltonian path [p. 398] a walk in a graph that visits every vertex exactly once (and therefore cannot repeat any edges)

Handshaking lemma [p. 386] The sum of the degrees of all the vertices of a graph is equal to twice the number of edges.

Highest common factor [p. 54] The highest common factor of two natural numbers a and b, denoted by $\text{HCF}(a, b)$, is the largest natural number that is a factor of both a and b.

Hyperbola [p. 575] The graph of the equation

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k); the asymptotes are given by

$$y - k = \pm\frac{b}{a}(x - h)$$

I

Imaginary number i [p. 605] $i^2 = -1$

Imaginary part of a complex number [p. 606] If $z = a + bi$, then $\text{Im}(z) = b$. Note that $\text{Im}(z)$ is a real number.

Implication [pp. 169, 224] *see* conditional statement

Inclusion–exclusion principle [p. 320] allows us to count the number of elements in a union of sets. In the case of two sets: $|A \cup B| = |A| + |B| - |A \cap B|$

Index laws [p. 2]

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
- $a^{-n} = \dfrac{1}{a^n}$
- $\dfrac{1}{a^{-n}} = a^n$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Infinite geometric series [p. 109] For an infinite geometric series with $-1 < r < 1$, the sum to infinity is given by

$$S_\infty = \frac{a}{1-r}$$

where $a = t_1$ and r is the common ratio.

Integers [p. 42] $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Intersection of sets [p. 40] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 45] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Inverse cosine function (arccos) [p. 544]

$\cos^{-1} x = y$ if $\cos y = x$,

for $x \in [-1, 1]$ and $y \in [0, \pi]$

Inverse sine function (arcsin) [p. 543]

$\sin^{-1} x = y$ if $\sin y = x$,

for $x \in [-1, 1]$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Inverse tangent function (arctan) [p. 544]

$\tan^{-1} x = y$ if $\tan y = x$,

for $x \in \mathbb{R}$ and $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Irrational number [p. 42] a real number that is not rational; e.g. π and $\sqrt{2}$

Iteration [p. 258] In an algorithm, we can use looping constructs to repeat steps in a controlled way; e.g. `for` loops and `while` loops.

Iterative rule [p. 68] *see* recurrence relation

K

Karnaugh map [p. 240] a special form of truth table used for simplifying Boolean expressions

Kilogram weight, kg wt [p. 727] a unit of force. If an object on the surface of the Earth has a mass of 1 kg, then the gravitational force acting on this object is 1 kg wt.

L

Like surds [p. 48] surds with the same irrational factor; e.g. $2\sqrt{7}$ and $9\sqrt{7}$

Linear equation [p. 8] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear transformation [p. 657] a transformation of the plane with a rule of the form $(x, y) \to (ax + by, cx + dy)$. Each linear transformation can be represented by a 2×2 matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear transformation, inverse [p. 673] If $\mathbf{A}$ is the matrix of a linear transformation and $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.

Linear transformations, composition [p. 670] If $\mathbf{A}$ and $\mathbf{B}$ are the matrices of two linear transformations, then the product $\mathbf{BA}$ is the matrix of the transformation $\mathbf{A}$ followed by $\mathbf{B}$.

Literal equation [p. 25] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

Locus [p. 564] a set of points described by a geometric condition; e.g. the locus of points P that satisfy $PO = 3$, where O is the origin, is the circle of radius 3 centred at the origin

Logic circuit [p. 235] an electronic circuit built using logic gates. Every Boolean function can be realised as a logic circuit, where 0 corresponds to 'low voltage' and 1 to 'high voltage'.

Logic gates [p. 235] the components of logic circuits. Logic gates carry out logical operations such as 'or' ($\vee$), 'and' ($\wedge$) and 'not' ($\neg$).

Logical connectives [p. 221] used to combine statements together to form new statements; e.g. 'and', 'or', 'not', 'implies'

Logically equivalent [p. 223] Two compound statements (each expressed in terms of simple statements $A, B, C, \dots$) are logically equivalent if they have the same truth value for all possible combinations of the truth values of $A, B, C, \dots$.

Loop in a graph [p. 384] an edge that joins a vertex to itself

Loop in an algorithm [p. 258] a sequence of instructions that is to be repeated. Each repeat is a *pass* of the loop.

Lowest common multiple [p. 55] The lowest common multiple of two natural numbers a and b, denoted by $\text{LCM}(a, b)$, is the smallest natural number that is a multiple of both a and b.

M

Magnitude of a vector [p. 703] the length of a directed line segment corresponding to the vector.

- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

Mass [p. 727] The mass of an object is the amount of matter it contains, and can be measured in kilograms. Mass is not the same as weight.

Mathematical induction [p. 189] a proof technique for showing that a statement is true for all natural numbers; uses the *principle of mathematical induction*

Matrices, addition [p. 350] Addition is defined for two matrices of the same size. The sum is found by adding corresponding entries. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

Matrices, equal [p. 348] Two matrices **A** and **B** are equal, and we can write **A** = **B**, when:

- they have the same size, and
- they have the same entry at corresponding positions.

Matrices, multiplication [p. 354] The product of two matrices **A** and **B** is defined only if the number of columns of **A** is the same as the number of rows of **B**. If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

Matrix, identity [pp. 357, 365] For square matrices of a given size (e.g. 2×2), there is a multiplicative identity matrix **I**.

For 2×2 matrices, the identity is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ for each 2×2 matrix **A**.

Matrix, inverse [pp. 358, 365] If **A** is a square matrix and there exists a matrix **B** such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **B** is called the inverse of **A**. When it exists, the inverse of a square matrix **A** is unique and is denoted by $\mathbf{A}^{-1}$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ provided $ad - bc \neq 0$.

Matrix, invertible [p. 358] A square matrix is said to be invertible if its inverse exists.

Matrix, multiplication by a scalar [p. 350] If **A** is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are k times the corresponding entries of **A**. For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Matrix, non-invertible [p. 358] A square matrix is said to be non-invertible if it does not have an inverse.

Matrix, size [p. 346] A matrix with m rows and n columns is said to be an $m \times n$ matrix.

Matrix, square [p. 355] A matrix with the same number of rows and columns is called a square matrix; e.g. a 2×2 matrix.

Matrix, zero [p. 351] The $m \times n$ matrix with all entries equal to zero is called the zero matrix and is usually denoted by **O**.

Matrix algebra [pp. 350–357] Some properties of arithmetic operations on $n \times n$ matrices:

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ — commutative law
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ — associative law
- $\mathbf{A} + \mathbf{O} = \mathbf{A}$ — zero matrix
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$ — additive inverse
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ — associative law
- $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ — identity matrix
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ — distributive law
- $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ — distributive law

Note: Matrix multiplication is not commutative.

Mean of a random variable, μ [p. 442] *see* expected value of a random variable, $\text{E}(X)$

Median of a triangle [p. 717] a line segment from a vertex to the midpoint of the opposite side

Modulus–argument form [p. 626] *see* polar form of a complex number

Modulus function [p. 557] The modulus of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *absolute value function*

Modulus of a complex number, $|z|$ [pp. 610, 626] the distance of the complex number from the origin. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

Modulus, properties [pp. 611, 628]
For complex numbers z_1 and z_2:

- $|z_1 z_2| = |z_1|\,|z_2|$ (the modulus of a product is the product of the moduli)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)

Multiple [p. 55] A natural number a is a multiple of a natural number b if there exists a natural number k such that $a = kb$.

Multiple edges [p. 384] A graph is said to have multiple edges if it has a pair of vertices joined by more than one edge.

Multiple of a random variable [p. 452] If X is a random variable and k is a positive number, then:

- $\mathrm{E}(kX) = k\,\mathrm{E}(X)$
- $\mathrm{Var}(kX) = k^2\,\mathrm{Var}(X)$

Multiplication of a complex number by i [pp. 618, 628] corresponds to a rotation about the origin by 90° anticlockwise. If $z = a + bi$, then $iz = i(a + bi) = -b + ai$.

Multiplication of a vector by a scalar [p. 697] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $m \in \mathbb{R}$, then $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j} + ma_3\boldsymbol{k}$.

Multiplication of complex numbers [pp. 609, 628] If $z_1 = a + bi$ and $z_2 = c + di$, then

$z_1 z_2 = (ac - bd) + (ad + bc)i$

If $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$, then

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

Multiplication principle [p. 289] If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

N

$n!$ [p. 293] The notation $n!$ (read as 'n factorial') is an abbreviation for the product of all the integers from n down to 1:

$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

Natural numbers [p. 42] $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Negation [pp. 174, 222] The negation of a statement P is the opposite statement, called 'not P' and written $\neg P$. For example, if P is the statement 'n is odd', then $\neg P$ is the statement 'n is even'.

Normal distribution [p. 464] a symmetric, bell-shaped distribution that often occurs for a measure in a population (e.g. height, weight, IQ); its centre is determined by the mean, μ, and its width by the standard deviation, σ.

Normal reaction force [p. 728] A mass placed on a surface (horizontal or inclined) experiences a force perpendicular to the surface, called the normal force.

O

Ordered pair [p. 43] a pair of elements, denoted (x, y), where x is the first coordinate and y is the second coordinate

P

Parallelogram [p. 699] a quadrilateral with both pairs of opposite sides parallel

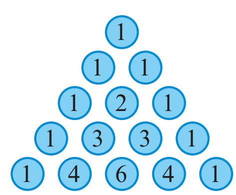
Parametric equations [p. 580] a pair of equations $x = f(t)$ and $y = g(t)$ describing a curve in the plane, where t is called the *parameter* of the curve

Partial fractions [p. 135] Some rational functions may be expressed as a sum of partial fractions; e.g.

$$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{ex^2+fx+g}$$

Particle model [p. 727] an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

Pascal's triangle [p. 313] a triangular pattern of numbers formed by the values of ${}^nC_r$. Each entry of Pascal's triangle is the sum of the two entries immediately above.



Path [p. 398] a walk in a graph that does not repeat any vertices

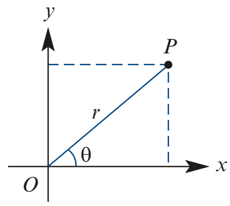
Permutation [p. 293] an ordered arrangement of objects. The number of permutations of n objects taken r at a time is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

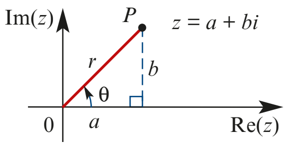
Pigeonhole principle [p. 316] If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

Polar coordinates [p. 589] A point P in the plane has polar coordinates $[r, \theta]$, where:

- r is the distance from the origin O to P
- θ is the angle between the positive direction of the x-axis and the ray OP.



Polar form of a complex number [p. 626] A complex number is expressed in polar form as $z = r \operatorname{cis} \theta$, where r is the modulus of z and θ is an argument of z. This is also called *modulus–argument form*.



Polynomial function [p. 120] A polynomial has a rule of the type

$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N} \cup \{0\}$

where $a_0, a_1, \ldots, a_n$ are numbers called coefficients.

Population [p. 456] the set of all eligible members of a group which we intend to study

Population mean, μ [p. 459] the mean of all values of a measure in the entire population

Population parameter [pp. 442, 460] a statistical measure that is based on the whole population; the value is constant for a given population

Position vector [p. 698] A position vector, $\overrightarrow{OP}$, indicates the position in space of the point P relative to the origin O.

Prime [p. 52] A natural number greater than 1 is a prime number if its only factors are itself and 1.

Prime decomposition [p. 53] expressing a composite number as a product of powers of prime numbers; e.g. $500 = 2^2 \times 5^3$

Principle of mathematical induction [p. 189] used to prove that a statement is true for all natural numbers

Probability distribution [p. 441] a function, denoted $p(x)$ or $\Pr(X = x)$, which assigns a probability to each value of a discrete random variable X

Product-to-sum identities [p. 533]

- $2 \cos x \cos y = \cos(x - y) + \cos(x + y)$
- $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$
- $2 \sin x \cos y = \sin(x + y) + \sin(x - y)$

Projection [p. 664] A projection maps each point in the plane onto an axis.

- Projection onto the x-axis: $(x, y) \to (x, 0)$
- Projection onto the y-axis: $(x, y) \to (0, y)$

Proof by contradiction [p. 178] a proof that begins by assuming the negation of what is to be proved

Pseudocode [p. 263] a notation for describing algorithms that is less formal than a programming language

Pythagorean identity [p. 520]

- $\cos^2 \theta + \sin^2 \theta = 1$
- $1 + \tan^2 \theta = \sec^2 \theta$
- $\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$

Q

Quadratic formula [p. 124] An equation of the form $az^2 + bz + c = 0$, with $a \neq 0$, may be solved quickly by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic function [p. 124] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic surd [p. 47] a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number

Quantifier [p. 185] *see* existence statement, universal statement

R

$\mathbb{R}^+$ [p. 46] $\{ x : x > 0 \}$, positive real numbers

$\mathbb{R}^-$ [p. 46] $\{ x : x < 0 \}$, negative real numbers

$\mathbb{R} \setminus \{0\}$ [p. 46] real numbers excluding 0

$\mathbb{R}^2$ [p. 43] $\{ (x, y) : x, y \in \mathbb{R} \}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 494] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180°}{\pi} \quad \text{and} \quad 1° = \frac{\pi^c}{180}$$

Random sample [p. 456] a sample chosen using a random process so that each member of the population has an equal chance of being included

Random variable [p. 440] a variable that takes its value from the outcome of a random experiment; e.g. the number of heads observed when a coin is tossed three times

Rate [p. 130] describes how a certain quantity changes with respect to the change in another quantity (often time)

Rational function [p. 135] a function with a rule of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials

Rational number [p. 42] a number that can be written as $\frac{p}{q}$, for some integers p and q with $q \neq 0$

Real part of a complex number [p. 606] If $z = a + bi$, then $\text{Re}(z) = a$.

Reciprocal circular functions [pp. 518, 553] the secant, cosecant and cotangent functions

Reciprocal function [p. 549] The reciprocal of the function $y = f(x)$ is defined by $y = \frac{1}{f(x)}$.

Recurrence relation [p. 68] a rule which enables each subsequent term of a sequence to be found from previous terms; e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$

Recurrence relation, first-order linear [p. 105] a recurrence relation $t_n = f(n)\,t_{n-1} + g(n)$. In the special case that

$t_n = rt_{n-1} + d$

where r and d are constants with $r \neq 1$, we can find an explicit formula for t_n of the form

$t_n = Ar^{n-1} + B$

for constants A and B.

Reflection [p. 661] A reflection in a line ℓ maps each point in the plane to its mirror image on the other side of the line.

- Reflection in the x-axis: $(x, y) \to (x, -y)$
- Reflection in the y-axis: $(x, y) \to (-x, y)$
- Reflection in the line $y = x$: $(x, y) \to (y, x)$
- Reflection in the line $y = -x$: $(x, y) \to (-y, -x)$

Reflection matrix [p. 668] A reflection in the line $y = mx = x \tan\theta$ is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Regular polygon a polygon in which all the angles are equal and all the sides are equal

Resultant force [p. 727] the vector sum of the forces acting at a point

Rhombus [p. 716] a parallelogram with all sides of equal length

Rotation matrix [p. 667] A rotation about the origin by angle θ anticlockwise is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

S

Sample [p. 456] a subset of the population which we select in order to make inferences about the whole population

Sample mean, $\bar{x}$ [p. 459] the mean of all values of a measure in a particular sample. The values $\bar{x}$ are the values of a random variable $\bar{X}$.

Sample statistic [pp. 442, 460] a statistical measure that is based on a sample from the population; the value varies from sample to sample

Sampling distribution [p. 461] the distribution of a statistic which is calculated from a sample

Scalar [p. 697] a real number; name used when working with vectors or matrices

Scalar product [p. 707] The scalar product of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$

Scalar product, properties [p. 708]

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Scalar quantity [p. 717] a quantity determined by its magnitude; e.g. distance, time, speed, mass

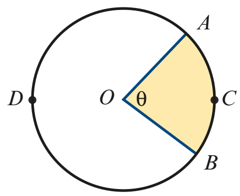
Scientific notation [p. 5] A number is in *standard form* when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}.

Secant function [pp. 518, 553] $\sec\theta = \dfrac{1}{\cos\theta}$ for $\cos\theta \neq 0$

Sector [pp. 493, 494] Two radii and an arc define a region called a sector. In this diagram, the shaded region is a *minor sector* and the unshaded region is a *major sector*.

Area of sector $= \dfrac{1}{2}r^2\theta$

where $\theta^c = \angle AOB$

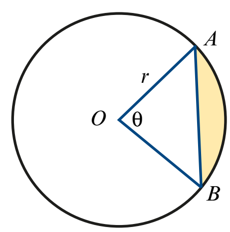


Segment [pp. 493, 496] Every chord divides the interior of a circle into two regions called segments; the smaller is the *minor segment* (shaded), and the larger is the *major segment*.

Area of segment

$= \dfrac{1}{2}r^2(\theta - \sin\theta)$

where $\theta^c = \angle AOB$



Selection [p. 260] In an algorithm, we can use decision-making constructs to specify whether certain steps should be followed based on some condition; e.g. `if-then` blocks.
see also combination

Sequence [p. 67] a list of numbers, with the order being important; e.g. 1, 1, 2, 3, 5, 8, 13, . . . The numbers of a sequence are called its *terms*, and the nth term is often denoted by t_n.

Series [p. 79] the sum of the terms in a sequence

Set notation [p. 39]

$\in$	means 'is an element of'
$\notin$	means 'is not an element of'
$\subseteq$	means 'is a subset of'
$\cup$	means 'union'
$\cap$	means 'intersection'
$\varnothing$	is the empty set, containing no elements
ξ	is the universal set, containing all elements being considered
A'	is the complement of a set A
$\lvert A\rvert$	is the number of elements in a finite set A

Sets of numbers [pp. 42, 605]

$\mathbb{N}$	is the set of natural numbers
$\mathbb{Z}$	is the set of integers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{R}$	is the set of real numbers
$\mathbb{C}$	is the set of complex numbers

Shear [p. 663] A shear moves each point in the plane by an amount proportional to its distance from an axis.

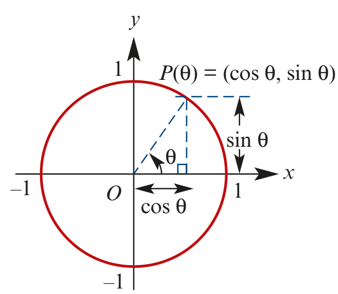
- Shear parallel to the x-axis: $(x, y) \to (x + cy, y)$
- Shear parallel to the y-axis: $(x, y) \to (x, cx + y)$

Simplest form [p. 47] A surd $\sqrt{a}$ is in simplest form if the number under the square root has no factors which are squares of a rational number.

Simulation [p. 464] using technology (calculators or computers) to repeat a random process many times; e.g. random sampling

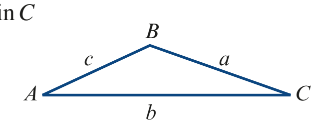
Simultaneous equations [pp. 8, 142, 362, 372] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [p. 478] sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Sine rule [p. 483] used to find unknown quantities in a triangle given one side and two angles, or given two sides and a non-included angle. For $\triangle ABC$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Speed [p. 718] the magnitude of velocity

Standard deviation of a random variable, σ [p. 444] a measure of the spread or variability, given by $\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$

Standard form [p. 5] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}. Also called *scientific notation*

Subgraph [p. 387] a graph whose vertices and edges are subsets of another graph

Subset [p. 39] A set B is called a subset of a set A if every element of B is also an element of A. We write $B \subseteq A$.

Subtraction of complex numbers [p. 607]
If $z_1 = a + bi$ and $z_2 = c + di$, then
$z_1 - z_2 = (a - c) + (b - d)i$.

Subtraction of vectors [p. 697]
If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$.

Sum of independent random variables [p. 447] If X and Y are independent random variables, then:

- $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
- $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Sum to infinity [p. 109] The sum to infinity of an infinite geometric series exists provided $-1 < r < 1$ and is given by

$$S_\infty = \frac{a}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Sum-to-product identities [p. 534]

- $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
- $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

Surd of order *n* [p. 47] a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power

Surd, quadratic [p. 47] a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number

Switching circuit [p. 210] an electrical circuit built using combinations of switches in series and in parallel. A switching circuit can be represented by a Boolean expression.

T

Tangent function [p. 479] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$

Tautology [p. 223] a statement which is true under all circumstances

Tension force [p. 728] the pulling force exerted by a string that connects two objects. The forces at each end of the string have equal magnitude.

Total degree of a graph [p. 386] the sum of the degrees of all the vertices of the graph. The total degree is equal to twice the number of edges.

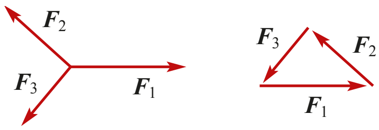
Transformation [p. 657] A transformation of the plane maps each point (x, y) in the plane to a new point (x', y'). We say that (x', y') is the *image* of (x, y).

Translation [p. 665] a transformation that moves each point in the plane in the same direction and over the same distance: $(x, y) \to (x + a, y + b)$

Tree [p. 410] a connected graph with no cycles. A tree with n vertices has $n - 1$ edges.

Tree, spanning [p. 411] If G is a connected graph, then a spanning tree of G is a subgraph of G that is a tree with the same set of vertices as G.

Triangle of forces [p. 728] If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

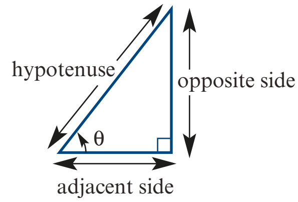


Trigonometric ratios [p. 478]

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Truth table [p. 221] gives the truth value of a compound statement for each combination of truth values of the constituent statements

U

Union of sets [p. 39] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

Unit vector [p. 704] a vector of magnitude 1. The unit vectors in the positive directions of the x-, y- and z-axes are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. The unit vector in the direction of $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Universal statement [pp. 185, 186] a statement claiming that a property holds for all members of a given set. Such a statement can be written using the quantifier 'for all'.

V

Valid argument [p. 230] An argument is said to be valid if whenever all the premises are true, the conclusion is also true.

Variance of a random variable, σ^2 [p. 443] a measure of the spread or variability, defined by

$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$

An alternative (computational) formula is

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

Vector [p. 695] a set of equivalent directed line segments

Vector quantity [p. 717] a quantity determined by its magnitude and direction; e.g. position, displacement, velocity, force

Vectors, parallel [p. 698] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = k\boldsymbol{b}$ for some $k \in \mathbb{R} \setminus \{0\}$.

Vectors, perpendicular [p. 708] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Vectors, properties [pp. 695–697]

- $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ — commutative law
- $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$ — associative law
- $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$ — zero vector
- $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$ — additive inverse
- $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$ — distributive law

Vectors, resolution [p. 710] A vector $\boldsymbol{a}$ is resolved into rectangular components by writing it as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

The *vector resolute* of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}$$

Velocity [p. 718] the rate of change of position with respect to time. Velocity is a vector quantity.

Velocity, relative [p. 723] The relative velocity of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.

Velocity, resultant [p. 722] the sum of two or more velocity vectors. For example, if a train is travelling north at 60 km/h and a passenger walks at 3 km/h towards the back of the train, then the passenger's resultant velocity is 57 km/h north.

Velocity, true [p. 723] the velocity of an object measured relative to Earth

W

Walk [pp. 391, 402] A walk in a graph is an alternating sequence of vertices and edges

$v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}, v_n$

where the edge e_i joins the vertices v_i and v_{i+1}.

- The *length* of a walk is the number of edges in the walk, counting repetitions.
- If $\mathbf{A}$ is the adjacency matrix of a graph G, then the matrix $\mathbf{A}^n$ gives the number of walks of length n between each pair of vertices of G.

Weight [p. 727] On the Earth's surface, a mass of m kg has a force of m kg wt acting on it; this force is known as the weight.

Z

Zero vector, 0 [p. 697] a line segment of zero length with no direction

Answers

Chapter 1

Exercise 1A

1 **a** x^7 **b** a^2 **c** x^3 **d** y^{-4}
e x^{12} **f** p^{-7} **g** $a^{-\frac{1}{6}}$ **h** a^{-8}
i y^{14} **j** x^{15} **k** a^{-12} **l** x^2
m n^2 **n** $8x^{\frac{7}{2}}$ **o** a **p** x^4
q $\frac{1}{2n^6}$ **r** $-8x^2$ **s** $a^{-2}b^5$ **t** 1

2 **a** 5 **b** 4 **c** $\frac{4}{3}$ **d** $\frac{1}{4}$
e $\frac{6}{7}$ **f** 3 **g** 12 **h** 16
i 27 **j** $\frac{3}{2}$ **k** 1 **l** 8

3 **a** 18.92 **b** 79.63 **c** 5.89
d 125 000 **e** 0.9 **f** 1.23
g 0.14 **h** 1.84 **i** 0.4

4 **a** a^4b^7 **b** $64a^4b^7$ **c** b
d a^6b^9 **e** $2a^4b^7$ **f** $\frac{a^2b^5}{128}$

5 2^{2n-4}

6 6^{3x}

7 **a** $\left(\frac{1}{2}\right)^{\frac{1}{6}}$ **b** $a^{\frac{11}{20}}$ **c** $2^{\frac{5}{6}}$ **d** $2^{\frac{19}{6}}$ **e** $2^{\frac{3}{5}}$

8 **a** $a^{\frac{1}{3}}b$ **b** $a^{\frac{5}{2}}b^{\frac{1}{2}}$ **c** $ab^{\frac{1}{5}}$
d $\left(\frac{b}{a}\right)^{\frac{1}{2}}$ **e** $a^{\frac{5}{2}}b^{\frac{1}{2}}c^{-4}$ **f** $a^{\frac{1}{5}}b^{\frac{3}{5}}$
g $a^{-4}b^{\frac{7}{2}}c^5$

Exercise 1B

1 **a** 4.78×10 **b** 6.728×10^3 **c** 7.923×10
d 4.358×10^4 **e** 2.3×10^{-3} **f** 5.6×10^{-7}
g $1.200\,034 \times 10$ **h** 5.0×10^7
i 2.3×10^{10} **j** 1.3×10^{-9} **k** 1.65×10^5
l 1.4567×10^{-5}

2 **a** 1.0×10^{-8} **b** 1.67×10^{-24}
c 5×10^{-5} **d** $1.853\,18 \times 10^3$
e 9.461×10^{12} **f** 2.998×10^{10}

3 **a** 81 280 000 000 000 **b** 270 000 000
c 0.000 000 000 000 28

4 **a** 4.569×10^2 **b** 3.5×10^4
c 5.6791×10^3 **d** 4.5×10^{-2}
e 9.0×10^{-2} **f** 4.5682×10^3

5 **a** 0.000 0567 **b** $\frac{262}{2625}$

6 **a** 11.8 **b** 4.76×10^7

Exercise 1C

1 **a** $x = \frac{8}{3}$ **b** $x = 48$ **c** $x = -\frac{20}{3}$
d $x = 63$ **e** $x = -0.7$ **f** $x = 2.4$
g $x = 0.3$ **h** $x = -6$ **i** $x = -\frac{15}{92}$
j $x = -\frac{21}{17}$

2 **a** $x = \frac{160}{9}$ **b** $x = 19.2$ **c** $x = -4$
d $x = \frac{80}{51}$ **e** $x = 6.75$ **f** $x = -\frac{85}{38}$
g $x = \frac{487}{13}$ **h** $x = \frac{191}{91}$

3 **a** $x = \frac{18}{13}, y = -\frac{14}{13}$ **b** $x = \frac{16}{11}, y = -\frac{18}{11}$
c $x = 12, y = 17$ **d** $x = 8, y = 2$
e $x = 0, y = 2$ **f** $x = 1, y = 6$

Exercise 1D

1 **a** $4(x - 2) = 60;\ x = 17$
b $\left(\frac{2x + 7}{4}\right)^2 = 49;\ x = 10.5$
c $x - 5 = 2(12 - x);\ x = \frac{29}{3}$
d $y = 6x - 4$ **e** $Q = np$

f $R = 1.1pS$ **g** $\frac{60n}{5} = 2400$

h $a = \frac{\pi}{3}(x + 3)$

2 \$2500

3 Eight dresses and three handbags

4 8.375 m by 25.125 m

5 \$56.50 **6** Nine

7 20, 34 and 17

8 Annie 165, Belinda 150, Cassie 189

9 15 km/h **10** 2.04×10^{-23} g

11 30 pearls

12 Oldest \$48, middle \$36, youngest \$12

13 98% **14** 25 students

15 After 20 minutes

16 a 40 minutes **b** 90 minutes **c** 20 minutes

17 200 km

18 39 km/h

Exercise 1E

1 140.625 km **2** 50 guests

3 10 000 adults **4** Men \$220, boys \$160

5 127 and 85

6 252 litres 40% and 448 litres 15%

7 120 and 100; 60 **8** \$370 588

9 500 adults, 1100 students

10 18 draws

Exercise 1F

1 a 25 **b** 330 **c** 340.47 **d** 1653.48
e 612.01 **f** 77.95 **g** 2.42 **h** 2.1
i ±9.43 **j** ±9.54

2 a $a = \frac{v - u}{t}$ **b** $\ell = \frac{2S}{n} - a$ **c** $b = \frac{2A}{h}$

d $I = \pm\sqrt{\frac{P}{R}}$ **e** $a = \frac{2(s - ut)}{t^2}$

f $v = \pm\sqrt{\frac{2E}{m}}$ **g** $h = \frac{Q^2}{2g}$ **h** $x = -\frac{z}{y}$

i $x = \frac{-b(c + y)}{a - c}$ **j** $x = \frac{-b(c + 1)}{m - c}$

3 a 82.4°F **b** $C = \frac{5(F - 32)}{9}$; 57.22°C

4 a 1080° **b** $n = \frac{S}{180} + 2$; 9 sides

5 a 115.45 cm^3 **b** 12.53 cm **c** 5.00 cm

6 a 66.5 **b** 4 **c** 11

Exercise 1G

1 a $\frac{13x}{6}$ **b** $\frac{5a}{4}$ **c** $-\frac{h}{8}$ **d** $\frac{5x - 2y}{12}$

e $\frac{3y + 2x}{xy}$ **f** $\frac{7x - 2}{x(x - 1)}$

g $\frac{5x - 1}{(x - 2)(x + 1)}$ **h** $\frac{-7x^2 - 36x + 27}{2(x + 3)(x - 3)}$

i $\frac{4x + 7}{(x + 1)^2}$ **j** $\frac{5a^2 + 8a - 16}{8a}$

k $\frac{4(x^2 + 1)}{5x}$ **l** $\frac{2x + 5}{(x + 4)^2}$

m $\frac{3x + 14}{(x - 1)(x + 4)}$ **n** $\frac{x + 14}{(x - 2)(x + 2)}$

o $\frac{7x^2 + 28x + 16}{(x - 2)(x + 2)(x + 3)}$ **p** $\frac{(x - y)^2 - 1}{x - y}$

q $\frac{4x + 3}{x - 1}$ **r** $\frac{3 - 2x}{x - 2}$

2 a $2xy^2$ **b** $\frac{xy}{8}$ **c** $\frac{2}{x}$ **d** $\frac{x}{y^2}$

e $\frac{a}{3}$ **f** $\frac{1}{2x}$ **g** $\frac{x - 1}{x + 4}$ **h** $x + 2$

i $\frac{x - 1}{x}$ **j** $\frac{a}{4b}$ **k** $\frac{2x}{x + 2}$ **l** $\frac{x - 1}{4x}$

m $\frac{x + 1}{2x}$ **n** $\frac{1}{3}x(x + 3)$

o $\frac{x - 2}{3x(3x - 2)(x + 5)}$

3 a $\frac{3}{x - 3}$ **b** $\frac{4x - 14}{x^2 - 7x + 12}$

c $\frac{5x - 1}{x^2 + x - 12}$ **d** $\frac{2x^2 + 10x - 6}{x^2 + x - 12}$

e $\frac{2x - 9}{x^2 - 10x + 25}$ **f** $\frac{5x - 8}{(x - 4)^2}$

g $\frac{1}{3 - x}$ **h** $\frac{23 - 3x}{x^2 + x - 12}$

i $\frac{5x^2 - 3x}{x^2 - 9}$ **j** $\frac{11 - 2x}{x^2 - 10x + 25}$

k $\frac{12}{(x - 6)^3}$ **l** $\frac{9x - 25}{x^2 - 7x + 12}$

4 a $\frac{3 - x}{\sqrt{1 - x}}$ **b** $\frac{2\sqrt{x - 4} + 6}{3\sqrt{x - 4}}$ **c** $\frac{5}{\sqrt{x + 4}}$

d $\frac{x + 7}{\sqrt{x + 4}}$ **e** $-\frac{12x^2}{\sqrt{x + 4}}$ **f** $\frac{9x^2(x + 2)}{2\sqrt{x + 3}}$

5 a $\frac{6x - 4}{(6x - 3)^{\frac{2}{3}}}$ **b** $\frac{3}{(2x + 3)^{\frac{2}{3}}}$ **c** $\frac{3 - 3x}{(x - 3)^{\frac{2}{3}}}$

Exercise 1H

1 a $x = \frac{m - n}{a}$ **b** $x = \frac{b}{b - a}$ **c** $x = -\frac{bc}{a}$

d $x = \frac{5}{p - q}$ **e** $x = \frac{m + n}{n - m}$ **f** $x = \frac{ab}{1 - b}$

g $x = 3a$ **h** $x = -mn$ **i** $x = \frac{a^2 - b^2}{2ab}$

j $x = \frac{p - q}{p + q}$ **k** $x = \frac{3ab}{b - a}$ **l** $x = \frac{1}{3a - b}$

m $x = \frac{p^2 + p^2t + t^2}{q(p + t)}$ **n** $x = -\frac{5a}{3}$

4 a $x = \frac{d - bc}{1 - ab}, y = \frac{c - ad}{1 - ab}$

b $x = \frac{a^2 + ab + b^2}{a + b}, y = \frac{ab}{a + b}$

c $x = \frac{t + s}{2a}, y = \frac{t - s}{2b}$ **d** $x = a + b, y = a - b$

e $x = c, y = -a$ **f** $x = a + 1, y = a - 1$

5 a $s = a(2a + 1)$ **b** $s = \frac{2a^2}{1 - a}$

c $s = \frac{a^2 + a + 1}{a(a + 1)}$ **d** $s = \frac{a}{(a - 1)^2}$

e $s = 3a^3(3a + 1)$ **f** $s = \frac{3a}{a + 2}$

g $s = 2a^2 - 1 + \frac{1}{a^2}$ **h** $s = \frac{5a^2}{a^2 + 6}$

Exercise 1I

1 a $x = a - b$ **b** $x = 7$

c $x = -\frac{a \pm \sqrt{a^2 + 4ab - 4b^2}}{2}$ **d** $x = \frac{a + c}{2}$

2 a $(x - 1)(x + 1)(y - 1)(y + 1)$

b $(x - 1)(x + 1)(x + 2)$

c $(a^2 - 12b)(a^2 + 4b)$ **d** $(a - c)(a - 2b + c)$

3 a $x = \frac{a + b + c}{a + b}, y = \frac{a + b}{c}$

b $x = \frac{-(a - b - c)}{a + b - c}, y = \frac{a - b + c}{a + b - c}$

Chapter 1 review

Technology-free questions

1 a x^{12} **b** y^{-9} **c** $-15x^{\frac{11}{2}}$ **d** x^{-1}

2 3.84×10^8

3 a $\frac{2x + y}{10}$ **b** $\frac{4y - 7x}{xy}$

c $\frac{7x - 1}{(x + 2)(x - 1)}$ **d** $\frac{7x + 20}{(x + 2)(x + 4)}$

e $\frac{13x^2 + 2x + 40}{2(x + 4)(x - 2)}$ **f** $\frac{3(x - 4)}{(x - 2)^2}$

4 a $\frac{2}{x}$ **b** $\frac{x - 4}{4x}$ **c** $\frac{x^2 - 4}{3}$ **d** $4x^2$

5 10^6 seconds or $11\frac{31}{54}$ days

6 50

7 12

8 88 crime, 80 science fiction, 252 romance

9 a 300π cm^3 **b** $h = \frac{V}{\pi r^2}$; $\frac{117}{5\pi}$ cm

c $r = \sqrt{\frac{V}{\pi h}}$; $\sqrt{\frac{128}{\pi}}$ cm $= \frac{8\sqrt{2}}{\sqrt{\pi}}$ cm

10 a $x = \frac{b}{a + y}$ **b** $x = \frac{a + b}{c}$

c $x = \frac{2ab}{b - a}$ **d** $x = \frac{ab + b^2 d - d^2}{d(a + b)}$

11 a $\frac{p^2 + q^2}{p^2 - q^2}$ **b** $\frac{x + y}{x(y - x)}$

c $(x - 2)(2x - 1)$ **d** $\frac{2}{a}$

12 A 36; B 12; C 2

13 a $a = 8, b = 18$ **b** $x = p + q, y = 2q$

14 $x = 3.5$

15 a $4n^2k^2$ **b** $\frac{40cx^2}{ab^2}$

16 $x = -1$

Multiple-choice questions

1 A **2** A **3** C **4** A **5** B **6** E
7 B **8** B **9** B **10** B **11** E

Extended-response questions

1 a $\frac{5x}{4}$ hours **b** $\frac{4x}{7}$ hours **c** $\frac{19x}{28}$ hours

d i $x = \frac{14}{19} \approx 0.737$

ii Jack $\frac{140}{19} \approx 7$ km; Benny $\frac{560}{19} \approx 29$ km

2 a 18 000 cm^3 per minute **b** $V = 18\,000t$

c $h = \frac{45t}{4\pi}$ **d** After

3 a Thomas a; George $\frac{3a}{2}$; Sally $a - 18$; Zeb $\frac{a}{3}$; Henry $\frac{5a}{6}$

b $\frac{3a}{2} + a - 18 + \frac{a}{3} = a + \frac{5a}{6} + 6$

c $a = 24$; Thomas 24; Henry 20; George 36; Sally 6; Zeb 8

4 a 1.9×10^{-8} N **b** $m_1 = \frac{Fr^2 10^{11}}{6.67m_2}$

c 9.8×10^{24} kg

5 a $V = (1.8 \times 10^7)d$ **b** 5.4×10^8 m^3

c $k = 9.81 \times 10^3$ **d** 1.325×10^{15} J

e 1202 days (to the nearest day)

6 $\frac{10\sqrt{3}}{3}$ cm **7** $-40°$ **8** $\frac{240}{11}$ km/h

9 a $h = 20 - r$

b i $V = \left(20r^2 - \frac{r^3}{3}\right)\pi$

ii $r = 5.94$ cm; $h = 14.06$ cm

10 a $\frac{2}{3}$ litre from A; $\frac{1}{3}$ litre from B

b 600 mL from A; 400 mL from B

c $\frac{(p - q)(n + m)}{2(np - qm)}$ litres from A,

$\frac{(n - m)(p + q)}{2(np - qm)}$ litres from B,

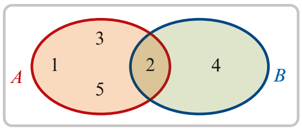
where $\frac{n}{m} \neq \frac{q}{p}$ and one of $\frac{n}{m}$ or $\frac{q}{p}$ is ≥ 1 and the other is ≤ 1

11 a $h = 2(10 - r)$ **b** $V = 2\pi r^2(10 - r)$

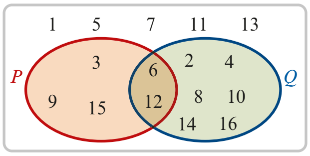
c $r = 3.4985, h = 13$ or $r = 9.022, h = 1.955$

Chapter 2

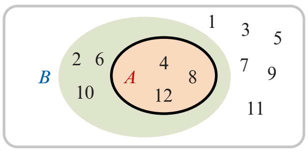
Exercise 2A

1 ξ


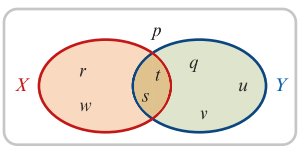
a {4} **b** {1, 3, 5} **c** {1, 2, 3, 4, 5} = ξ
d ∅ **e** ∅

2 ξ


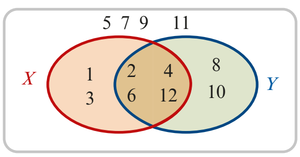
a {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16}
b {1, 3, 5, 7, 9, 11, 13, 15}
c {2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16}
d {1, 5, 7, 11, 13} **e** {1, 5, 7, 11, 13}

3 ξ


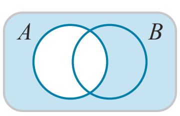
a {1, 2, 3, 5, 6, 7, 9, 10, 11}
b {1, 3, 5, 7, 9, 11} **c** {2, 4, 6, 8, 10, 12}
d {1, 3, 5, 7, 9, 11} **e** {1, 3, 5, 7, 9, 11}

4 ξ


a {*p*, *q*, *u*, *v*} **b** {*p*, *r*, *w*} **c** {*p*}
d {*p*, *q*, *r*, *u*, *v*, *w*} **e** {*q*, *r*, *s*, *t*, *u*, *v*, *w*} **f** {*p*}

5 ξ


a {5, 7, 8, 9, 10, 11} **b** {1, 3, 5, 7, 9, 11}
c {1, 3, 5, 7, 8, 9, 10, 11}
d {1, 3, 5, 7, 8, 9, 10, 11}
e {1, 2, 3, 4, 6, 8, 10, 12} **f** {5, 7, 9, 11}

6 a


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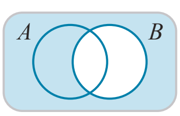

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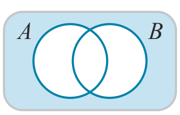

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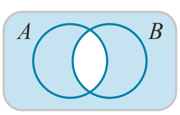

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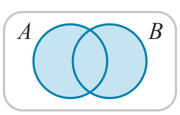

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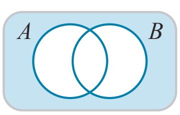
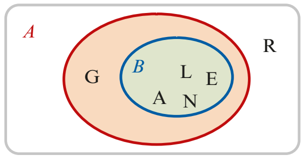
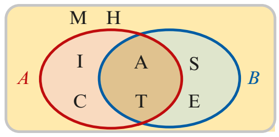


7 ξ


a {R} **b** {G, R} **c** {L, E, A, N}
d {A, N, G, E, L} **e** {R} **f** {G, R}

8 ξ


a {E, H, M, S} **b** {C, H, I, M}
c {A, T} **d** {H, M} **e** {C, E, H, I, M, S}
f {H, M}

9 a 10 **b** 10
c If you have chosen a 2-element subset of *C*, its complement is a 3-element subset of *C*.

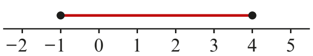
Exercise 2B

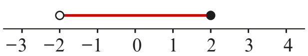
1 a Yes **b** Yes **c** Yes

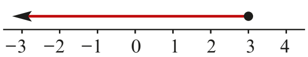
2 a No **b** No **c** No

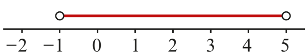
3 a $\frac{9}{20}$ **b** $\frac{2}{9}$ **c** $\frac{3}{11}$ **d** $\frac{3}{25}$ **e** $\frac{4}{11}$ **f** $\frac{2}{7}$

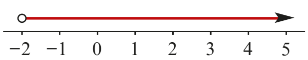
4 a $0.\dot{2}8571\dot{4}$ **b** $0.\dot{4}\dot{5}$ **c** 0.35
d $0.\dot{3}0769\dot{2}$ **e** $0.\dot{0}58823529411764\dot{7}$

5 a


b


c


d


e


6 a $(-\infty, 3)$ **b** $[-3, \infty)$ **c** $(-\infty, -3]$
d $(5, \infty)$ **e** $[-2, 3)$ **f** $[-2, 3]$
g $(-2, 3]$ **h** $(-5, 3)$

Exercise 2C

1 a $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $3\sqrt{3}$ **d** $5\sqrt{2}$
e $3\sqrt{5}$ **f** $11\sqrt{10}$ **g** $7\sqrt{2}$ **h** $6\sqrt{3}$
i 5 **j** $5\sqrt{3}$ **k** $16\sqrt{2}$

2 a $3\sqrt{2}$ **b** $6\sqrt{3}$ **c** $4\sqrt{7}$
d $5\sqrt{10}$ **e** $28\sqrt{2}$ **f** 0

3 a $11\sqrt{3} + \sqrt{14}$ **b** $5\sqrt{7}$
c 0 **d** $\sqrt{2} + \sqrt{3}$
e $5\sqrt{2} + 15\sqrt{3}$ **f** $\sqrt{2} + \sqrt{5}$

4 a $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{7}}{7}$ **c** $-\frac{\sqrt{2}}{2}$
d $\frac{2\sqrt{3}}{3}$ **e** $\frac{\sqrt{6}}{2}$ **f** $\frac{\sqrt{2}}{4}$

g $\sqrt{2}-1$ **h** $2+\sqrt{3}$ **i** $\dfrac{4+\sqrt{10}}{6}$

j $\sqrt{6}-2$ **k** $\dfrac{\sqrt{5}+\sqrt{3}}{2}$ **l** $3(\sqrt{6}+\sqrt{5})$

m $3+2\sqrt{2}$

5 a $6+4\sqrt{2}$ **b** $9+4\sqrt{5}$ **c** $-1+\sqrt{2}$

d $4-2\sqrt{3}$ **e** $\dfrac{2\sqrt{3}}{9}$ **f** $\dfrac{8+5\sqrt{3}}{11}$

g $\dfrac{3+\sqrt{5}}{2}$ **h** $\dfrac{6+5\sqrt{2}}{14}$

6 a $4a-4\sqrt{a}+1$

b $3+2x+2\sqrt{(x+1)(x+2)}$

7 $3\sqrt{5}, 4\sqrt{3}, 7, 5\sqrt{2}$

8 a $5-3\sqrt{2}$ **b** $7-2\sqrt{6}$

9 a $\dfrac{3}{\sqrt{2}}$ **b** $\dfrac{\sqrt{5}}{2}$ **c** $\dfrac{\sqrt{5}}{5}$ **d** $\dfrac{8}{\sqrt{3}}$

10 a $b=0, c=-3$ **b** $b=0, c=-12$

c $b=-2, c=-1$ **d** $b=-4, c=1$

e $b=-6, c=1$

f $b=-7+5\sqrt{5}, c=-58-13\sqrt{5}$

11 $\dfrac{3\sqrt{2}+2\sqrt{3}-\sqrt{30}}{12}$

12 b $-1-2^{\frac{1}{3}}-2^{\frac{2}{3}}$

13 3

Exercise 2D

1 a $2^2\times 3\times 5$ **b** $2^2\times 13^2$

c $2^2\times 3\times 19$ **d** $2^2\times 3^2\times 5^2$

e $2^2\times 3^2\times 7$ **f** $2^2\times 3^2\times 5^2\times 7$

g $2^5\times 3\times 5\times 11\times 13$

h $2^5\times 3\times 7\times 11\times 13$

i $2^5\times 7\times 11\times 13$

j $2^5\times 7\times 11\times 13\times 17$

2 a 1 **b** 27 **c** 5 **d** 31 **e** 6

3 a 18: 1, 2, 3, 6, 9, 18;
36 : 1, 2, 3, 4, 6, 9, 12, 18, 36

b 36 is a square number ($36=6\times 6$)

c 121 has factors 1, 11 and 121

4 5, 14 and 15 **5** $n=121$

6 15 **7** 105

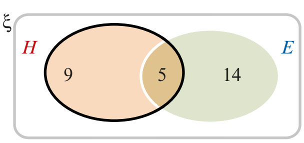
8 8 **9** 4

10 1:12 p.m.

11 600 and 108 000; 2400 and 27 000; 3000 and 21 600; 5400 and 12 000

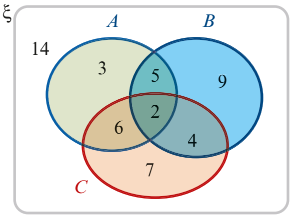
Exercise 2E

1 a ξ



b i 19 **ii** 9 **iii** 23

2 a ξ



b i 23 **ii** 37 **iii** 9

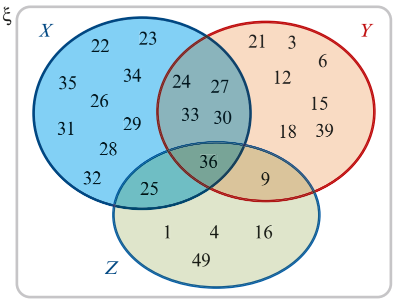
3 20%

4 a 5 **b** 10

5 45

6 a $x=5$ **b** 16 **c** 0

7 a ξ



b i $X\cap Y\cap Z=\{36\}$ **ii** $|X\cap Y|=5$

8 31 students; 15 black, 12 green, 20 red

9 $|M\cap F|=11$ **10** 1

11 $x=6$; 16 students **12** 102 students

Chapter 2 review

Technology-free questions

1 a $\dfrac{7}{90}$ **b** $\dfrac{5}{11}$ **c** $\dfrac{1}{200}$

d $\dfrac{81}{200}$ **e** $\dfrac{4}{15}$ **f** $\dfrac{6}{35}$

2 $2^3\times 3^2\times 7$

3 a $\dfrac{2\sqrt{6}-\sqrt{2}}{2}$ **b** $4\sqrt{5}+9$ **c** $2\sqrt{6}+5$

4 $-23-12\sqrt{3}$

5 a $2\sqrt{6}+6$ **b** $\dfrac{a-\sqrt{a^2-b^2}}{b}$

6 a 15 **b** 15

7 a 1 **b** 22 **c** 22

8 5 **9** 2 cm^2 **10** $-15\sqrt{7}$

11 $x=\pm 2$ **12** $\sqrt{5}-\sqrt{6}$ **13** $\dfrac{51\sqrt{3}}{5}$

14 $2\sqrt{2}+3$

15 a 57 **b** 3 **c** 32

Multiple-choice questions

1 A **2** D **3** D **4** D **5** C **6** D

7 B **8** B **9** C **10** A **11** D **12** B

Extended-response questions

1 c i $\sqrt{11}+\sqrt{3}$
ii $2\sqrt{2}-\sqrt{7}$ or $\sqrt{7}-2\sqrt{2}$
iii $3\sqrt{3}-2\sqrt{6}$ or $2\sqrt{6}-3\sqrt{3}$

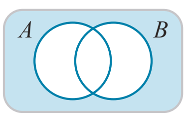
2 a $a=6,\ b=5$ **b** $p=26,\ q=16$
c $a=-1,\ b=\frac{2}{3}$
d i $\frac{12\sqrt{3}-19}{71}$ **ii** $3\pm\sqrt{3}$ **iii** $\frac{1\pm\sqrt{3}}{2}$
e $\mathbb{Q}=\{a+0\sqrt{3} : a\in\mathbb{Q}\}$

3 a (20, 21, 29)

4 a i 4 factors **ii** 8 factors
b $n+1$ factors
c i 32 factors **ii** $(n+1)(m+1)$ factors
d $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_n+1)$ factors
e 24

5 a $1080=2^3\times3^3\times5$;
$25\,200=2^4\times3^2\times5^2\times7$
b 75 600
d i 3470, 3472, 3474, 3476
ii 1735, 1736, 1737, 1738

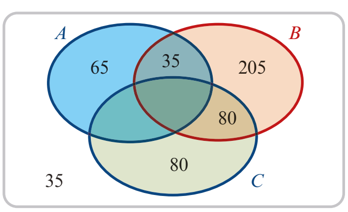
6 a i Students shorter than or equal to 180 cm
ii Students who are female or taller than 180 cm
iii Students who are not female and shorter than or equal to 180 cm
b



$(A\cup B)'=A'\cap B'$ is shaded

8 a i Region 8
ii non-female, red hair, blue eyes
iii non-female, not red hair, blue eyes
b i 5 **ii** 182

9 a



$|A\cap C|=0$

b 160 **c** 65 **d** 0

Chapter 3

Exercise 3A

1 a 3, 7, 11, 15, 19 **b** 5, 19, 61, 187, 565
c 1, 5, 25, 125, 625 **d** −1, 1, 3, 5, 7
e 1, 3, 7, 17, 41

2 a $t_n=t_{n-1}+3,\ t_1=3$ **b** $t_n=2t_{n-1},\ t_1=1$
c $t_n=-2t_{n-1},\ t_1=3$ **d** $t_n=t_{n-1}+3,\ t_1=4$
e $t_n=t_{n-1}+5,\ t_1=4$

3 a $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ **b** 2, 5, 10, 17
c 2, 4, 6, 8 **d** 2, 4, 8, 16
e 5, 8, 11, 14 **f** −1, 8, −27, 64
g 3, 5, 7, 9 **h** 2, 6, 18, 54

4 a $t_n=3n$ **b** $t_n=2^{n-1}$
c $t_n=\frac{1}{n^2}$ **d** $t_n=3(-2)^{n-1}$
e $t_n=3n+1$ **f** $t_n=5n-1$

5 $t_{n+1}=3n+4,\ t_{2n}=6n+1$

6 a $t_1=15,\ t_n=t_{n-1}+3$
b $t_n=12+3n$ **c** $t_{13}=51$

7 a $t_1=94.3,\ t_n=0.96t_{n-1}$
b $t_n=94.3(0.96)^{n-1}$ **c** $t_9=68.03$

8 a $t_0=100,\ t_n=1.8t_{n-1}+20$
b $t_1=200,\ t_2=380,\ t_3=704,\ t_4=1287,\ t_5=2336$

9 a 1st year \$2120; 2nd year \$2671.20; 3rd year \$3255.47
b $t_n=1.06(t_{n-1}+400),\ t_1=2120$
c \$8454.02

10 a 1, 4, 7, 10, 13, 16 **b** 3, 1, −1, −3, −5, −7
c $\frac{1}{2}$, 1, 2, 4, 8, 16 **d** 32, 16, 8, 4, 2, 1

11 a 1.1, 1.21, 1.4641, 2.144, 4.595, 21.114
b 27, 18, 12, 8, $\frac{16}{3}$, $\frac{32}{9}$
c −1, 3, 11, 27, 59, 123
d −3, 7, −3, 7, −3, 7

12 a $t_1=1, t_2=2, t_3=4$
b $u_1=1, u_2=2, u_3=4$
c $t_1=u_1, t_2=u_2, t_3=u_3$
d $t_4=8, u_4=7$

13 $S_1=a+b,\ S_2=4a+2b,\ S_3=9a+3b,$
$S_{n+1}-S_n=2an+a+b$

14 $t_2=\frac{3}{2}, t_3=\frac{17}{12}, t_4=\frac{577}{408}$; the number is $\sqrt{2}$

15 $F_3=2,\ F_4=3,\ F_5=5$

Exercise 3B

1 a 0, 2, 4, 6 **b** −3, 2, 7, 12
c $-\sqrt{5}, -2\sqrt{5}, -3\sqrt{5}, -4\sqrt{5}$ **d** 11, 9, 7, 5

2 a −31 **b** 24 **c** 5 **d** $6\sqrt{3}$

3 a $a=3,\ d=4,\ t_n=4n-1$
b $a=3,\ d=-4,\ t_n=7-4n$
c $a=-\frac{1}{2},\ d=2,\ t_n=2n-\frac{5}{2}$
d $a=5-\sqrt{5},\ d=\sqrt{5},\ t_n=\sqrt{5}n+5-2\sqrt{5}$

4 a 13 **b** 8 **c** 20 **d** 56

5 $a=-2,\ d=3,\ t_7=16$

6 $t_n=156n-450$

7 −2

8 54

9 $27\sqrt{3} - 60$

10 a 672 **b** 91st week

11 a 70 **b** 94 **c** Row F

12 117

13 $\frac{218}{9}$

14 7, 9, 11, 13

15 $t_n = a - \frac{a(n-1)}{m-1}$

16 a 11.5 **b** $\frac{2\sqrt{2}}{7}$

17 16

18 5

20 3

Exercise 3C

1 a 426 **b** 55 **c** $60\sqrt{2}$ **d** 108

2 112

3 680

4 2450

5 a 14 **b** 322

6 a 20 **b** −280

7 a 12 **b** 105

8 a 180 **b** $n = 9$

9 11

10 20

11 0

12 a 16.5 km **b** 45 km **c** 7 walks **d** 189 km

13 a 10 days **b** 25 per day

14 a 86 **b** 2600 **c** 224 **d** 2376
e 5 extra rows

15 $176 400

16 $a = -15,\ d = 3,\ t_6 = 0,\ S_6 = -45$

17 2160

18 266

19 a $t_n = \frac{5}{4}n + \frac{11}{4}$ **b** $t_n = \frac{46\sqrt{5}}{5} - 2\sqrt{5}n$

20 a b **b** $\frac{n}{2}(b + bn)$

21 $t_5 = -10,\ S_{25} = -1250$

22 $1575d$

23 a $S_{n-1} = 23n - 3n^2 - 20$
b $t_n = 20 - 6n$ **c** $a = 14,\ d = -6$

24 7, 12, 17

27 4860

28 Sequence of four positive integers such that $2a + 3d = 50$; therefore $a = 1 + 3t$ and $d = 16 - 2t$ for $0 \le t \le 16$. (The sequence 25, 25, 25, 25 is included.)

29 60 (the equilateral case is included with common difference 0)

Exercise 3D

1 a 3, 6, 12, 24 **b** 3, −6, 12, −24
c 10 000, 1000, 100, 10 **d** 3, 9, 27, 81

2 a $\frac{5}{567}$ **b** $\frac{1}{256}$ **c** 32 **d** a^{x+5}

3 a $t_n = 3\left(\frac{2}{3}\right)^{n-1}$ **b** $t_n = 2(-2)^{n-1}$
c $t_n = 2(\sqrt{5})^{n-1}$

4 a 3 **b** $\pm\frac{2}{5}$

5 t_9

6 a 6 **b** 9 **c** 9 **d** 6 **e** 8

7 $\frac{2}{3^5}$

8 $16\sqrt{2}$

9 a 24 **b** 12 288

10 a 21 870 m^2 **b** 9th day

11 47.46 cm

12 a $5397.31 **b** 48th year

13 a 57.4 km **b** 14th day

14 $5 369 000 **15** $t_{10} = 2048$

16 $t_6 = 729$ **17** 5 weeks

18 a 60 **b** 2.5 **c** 1 **d** x^4y^7

19 3 or 1

20 $a = \frac{1 \pm \sqrt{5}}{2}$

21 a 168.07 mL **b** 20 times

22 a Side length $\frac{a+b}{2}$ **b** Side length $\sqrt{ab}$

Exercise 3E

1 a 5115 **b** −182 **c** $-\frac{57}{64}$

2 a 1094 **b** −684 **c** 7812

3 10

4 7

5 a 1062.9 mL **b** 5692.30 mL **c** 11 days

6 a 49 minutes (to nearest minute)
b 164 minutes **c** Friday

7 $\frac{481\ 835}{6561} \approx 73.44$ m

8 a $18 232.59 **b** $82 884.47

9 a 155 **b** $\frac{15\sqrt{2}}{2} + 15$

10 a 8 **b** $\{n : n > 19\}$

11 $\frac{x^{2m+2} + 1}{x^2 + 1}$

12 a 54 976 km **b** 43 times

13 Option 1: $52 000 000;
Option 2: Approx. $45 036 000 000 000

14 $\left(\frac{1}{3}\right)^{\frac{1}{50}}$

Exercise 3F

1 a \$7092.60 **b** 12 years

2 \$3005.61

3 a $60\,000(1.15)^{n-1}$ **b** 23rd year
c $400\,000(1.15^n - 1)$

4 a \$39 918.13 **b** During the 5th year

5 11.6% p.a.

6 During the 9th year

7 \$330 169.68

8 \$1587.24

9 a \$279 432.85 **b** 8 years

10 a \$37 110.54
b At the end of the 15th year, the final repayment is \$2107.18

11 \$4497.06

12 Bianca \$3247.32; Andrew \$3000

13 a i 21 000 **ii** 22 150 **iii** 23 473
b $P_n = 1.15P_{n-1} - 2000$
c $P_n = 20\,000 \times 1.15^n - \frac{40\,000}{3}(1.15^n - 1)$
d 67 580

14 a i \$290 000 **ii** \$279 000 **iii** \$266 900
b $A_n = 1.1A_{n-1} - 40\,000$
c $A_n = 300\,000 \times 1.1^n - 400\,000(1.1^n - 1)$
d At the end of the 15th year, the final payment is \$22 275.18

Exercise 3G

1 a 6, 22, 70, 214 **b** 1, 8, 50, 302
c 6, 14, 38, 110 **d** 2, 9, 37, 149

2 a $d = -4$ **b** $r = \frac{3}{2}$ **c** $d = -100$
d $a = 5$

3 a $r = 2,\ d = 1$ **b** $r = \frac{1}{2},\ d = -64$
c $a = 1,\ d = 5$ **d** $r = 3,\ d = -100$

4 a $a_2 = 5k + 3,\ a_3 = 25k + 18$
b $S_4 = 156k + 114$

5 a $t_n = 2^{n-1} + 6$ **b** $t_n = 2 - 2^{n-1}$
c $t_n = 20$ **d** $t_n = 28 - \frac{8}{2^{n-1}}$
e $t_n = \frac{40}{2^{n-1}} - 20$ **f** $t_n = 1$

6 a $t_n = 10 - \frac{4}{2^{n-1}}$ **b** 6, 8, 9, 9.5
c As n gets larger, the value of t_n gets closer to 10 from below

7 a $t_n = \frac{1}{3}\left(10 + \frac{8}{(-2)^{n-1}}\right)$ **b** 6, 2, 4, 3
c Values of t_n oscillate about $\frac{10}{3}$, getting closer as n gets larger

8 $t_n = 12 \times 3^{n-1} - 5$

9 $t_n = 22 \times \left(\frac{1}{2}\right)^{n-1} - 6$

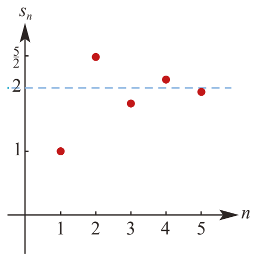
10 a $r = 2,\ d = 4$ **b** $n = 9$
c $t_{n+1} - t_n = 3 \times 2^n > 0$ for all $n \in \mathbb{N}$; hence $t_{n+1} > t_n$ for all $n \in \mathbb{N}$

11 $t_n = 5 + (n-1)(n+2) = n^2 + n + 3$

12 $t_n = 5 + (n-1)(n+3) = n^2 + 2n + 2$

13 a $a_1 = 11,\ a_3 = 171$ **b** $a_n = \frac{1}{3}(32 \times 4^{n-1} + 1)$

14 a $s_n = 2 - \left(-\frac{1}{2}\right)^{n-1}$ **b** $1, \frac{5}{2}, \frac{7}{4}, \frac{17}{8}, \frac{31}{16}$
c



d Values of s_n oscillate about 2, getting closer as n gets larger

15 a $N_n = 1.22N_{n-1} - 250,\ N_1 = 1356$
b Deer population is still increasing

16 a $t_n = 1.085t_n + 250,\ t_1 = 3000$
b $t_n = \frac{1}{17}(101\,000 \times 1.085^{n-1} - 50\,000)$
c $t_{11} \approx 10\,492$ **d** 2013

17 a $t_n = 1.27t_{n-1} - 20,\ t_1 = 200$
b $t_n = \frac{200}{27}(17 \times 1.27^{n-1} + 10)$
c 1156 goats

18 a $A_{n+1} = 1.007A_n - 400,\ A_1 = 15\,000$
b $A_n = \frac{1}{7}(400\,000 - 295\,000 \times 1.007^{n-1})$
c Paid off at the start of the 45th month

19 a $t_n = 150 - 118 \times 0.6^{n-1}$
b $t_{n+1} - t_n = \frac{236}{5} \times 0.6^{n-1} > 0$ for all $n \in \mathbb{N}$
c $n = 23$ **f** $d = 80$

Exercise 3H

1 a $\frac{5}{4}$ **b** $\frac{3}{5}$

2 Perimeter $p\left(\frac{1}{2}\right)^{n-1}$; Area $\frac{p^2\sqrt{3}}{9 \times 4^n}$;
Sum of perimeters $2p$; Sum of areas $\frac{p^2\sqrt{3}}{27}$

3 $3333\frac{1}{3}$ m

4 Yes, as the number of hours approaches infinity, but the problem becomes unrealistic after 4 to 5 hours

5 $S_\infty = 8$ m **6** 12 m **7** 75 m

8 a $\frac{4}{9}$ **b** $\frac{1}{30}$ **c** $\frac{31}{3}$ **d** $\frac{7}{198}$ **e** 1 **f** $\frac{37}{9}$

9 b $S_\infty = \frac{a^2}{b}$

10 a $-1 < x < 1,\ S_\infty = \frac{1}{1-x}$

b $0 < a < 1,\ S_\infty = \frac{a}{2(1-a)}$

c $x < \frac{1}{4}$ or $x > \frac{1}{2},\ S_\infty = \frac{(3x-1)^2}{x(2x-1)}$

d $x < -\frac{1}{\sqrt{2}}$ or $-\frac{1}{2} < x < \frac{1}{2}$ or $x > \frac{1}{\sqrt{2}}$, $S_\infty = \frac{(3x^2-1)^2}{x^2(2x^2-1)}$

11 $r = \frac{1}{2},\ t_1 = 16,\ t_2 = 8;$

$r = -\frac{1}{2},\ t_1 = 48,\ t_2 = -24$

12 $\frac{5}{8}$

13 $\frac{2}{3}$

Chapter 3 review

Technology-free questions

1 a 3, −1, −5, −9, −13, −17
b 5, 12, 26, 54, 110, 222

2 a 2, 4, 6, 8, 10, 12
b −1, −4, −7, −10, −13, −16

3 a \$5250, \$6037.50
b $t_1 = 5250,\ t_n = 1.05(t_{n-1} + 500)$

4 147

5 −0.1

6 −258.75

7 {12}

8 1

9 1000×1.035^n

10 $t_n = 2^{n-1} + 3$

11 $t_2 = 6,\ t_4 = \frac{8}{3}$ or $t_2 = -6,\ t_4 = -\frac{8}{3}$

12 96

13 −9840

14 $\frac{3}{4}$

15 $x = 8$ or $x = -2$

Multiple-choice questions

1 D 2 B 3 A 4 A 5 B
6 D 7 E 8 C 9 E 10 D
11 B 12 E 13 B

Extended-response questions

1 a 0.8, 1.5, 2.2, . . . b Yes c 8.5 m

2 a Yes b $t_n = 25n + 25$ c 650

3 $22\frac{1}{7}$ km from town A; $9\frac{6}{7}$ km from town B

4 a $D_n = 7n - 5$ b 27

5 472 mm

6 520

7 a 99.9999 mg b 100 mg

8 a $\frac{1}{729}$ m b 1.499 m
No, maximum height of water is 1.5 m

9 a 27.49 b 1680.8

10 a $7\frac{1}{9}$ m b 405 m

11 $2^{64} - 1 = 1.845 \times 10^{19}$

12 a i $t_n = 3750 + 250n$
ii $S_n = 3875n + 125n^2$
iii $n = 22$
iv $m = \frac{T - 4000}{250} + 1$
v $p = 51$
b i $S_n = 37\,500(1.08^n - 1)$
ii $Q_B - Q_A = 37\,500(1.08^n - 1) - 3875n - 125n^2;\ n = 18$

13 a Old Swamp 338; New Swamp 64.5
b $Q_0 = 30$ and $Q_n = 1.15Q_{n-1} + 30$ for $n \in \mathbb{N}$
c $P_n = 120(1.15)^n + 200,$
$Q_n = 230(1.15)^n - 200$
d i $P_5 = 441.363,\ Q_5 = 262.612$
ii 9 years

14 a 3^{n-1} b $\left(\frac{1}{2}\right)^{n-1}$ c $\left(\frac{3}{4}\right)^{n-1}$
d Area of white region approaches zero

15 a 8^{n-1} b $\left(\frac{1}{3}\right)^{n-1}$ c $\left(\frac{8}{9}\right)^{n-1}$
d Area of white region approaches zero

Chapter 4

Exercise 4A

1 $a = 10,\ b = 0,\ c = -7$

2 $a = 1,\ b = -2$

3 $a = 2,\ b = -1,\ c = 7$

4 $a = 2,\ b = 1,\ c = 3$

5 $(x+2)^2 - 4(x+2) + 4$

6 $(x+1)^3 - 3(x+1)^2 + 3(x+1) - 1$

7 $a = 1,\ b = -2,\ c = -1$

8 a It is impossible to find a, b and c such that $a = 3,\ 3ab = -9,\ 3ab^2 = 8$ and $ab^3 + c = 2$
b $a = 3,\ b = -1,\ c = 5$

9 $a = 1,\ b = -6,\ c = 7,\ d = -1$

10 a If $a = -\frac{5}{3}b$ and $a = -3b$, then both a and b are zero, but then $a + b = 1$ is not satisfied
b $(n+1)(n+2) - 3(n+1) + 1$

11 a $ax^2 + 2abx + ab^2 + c$
b $a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

13 $a = -3,\ b = -\frac{1}{3},\ c = 3$ or
$a = -\frac{1}{3},\ b = -3,\ c = 3$

14 $a = 3,\ b = -3,\ c = 1$

15 If $c = 5$, then $a = 1$ and $b = -5$;
if $c = -27$, then $a = -3$ and $b = 3$

Exercise 4B

1 a $x = 1$ **b** $x = 3$
c $x = 1 \pm \frac{\sqrt{30}}{5}$ **d** $x = 1 \pm \frac{\sqrt{2}}{2}$
e $x = -1 \pm \frac{3\sqrt{2}}{2}$ **f** $x = \frac{-13 \pm \sqrt{145}}{12}$

2 a $m > \frac{9}{4}$ **b** $m < \frac{25}{4}$ **c** $m = -\frac{25}{32}$
d $m < -6$ or $m > 6$ **e** $-4 < m < 4$
f $m = 0$ or $m = -16$

3 a $x = \frac{1 \pm \sqrt{32t + 1}}{4}$, $t \geq -\frac{1}{32}$
b $x = \frac{-1 \pm \sqrt{t + 3}}{2}$, $t \geq -3$
c $x = \frac{-2 \pm \sqrt{5t - 46}}{5}$, $t \geq \frac{46}{5}$
d $x = -2 \pm \frac{\sqrt{5t(t - 2)}}{t}$, $t < 0$ or $t \geq 2$

4 a $x = \frac{-p \pm \sqrt{p^2 + 64}}{2}$ **b** $p = 0$ or $p = 6$

5 a $\Delta = (3p - 4)^2$ **b** $p = \frac{4}{3}$
c i $x = 1$ or $x = \frac{1}{2}$ **ii** $x = 1$ or $x = 2$
iii $x = 1$ or $x = -\frac{5}{2}$

6 a $\Delta = 16(2p - 3)^2$ **b** $p = \frac{3}{2}$
c i $x = \frac{3}{2}$ or $x = \frac{1}{2}$ **ii** $x = \frac{1}{2}$ or $x = \frac{3}{10}$
iii $x = \frac{1}{2}$ or $x = -\frac{3}{14}$

7 $x = 2$

8 Side length 37.5 cm

9 a $x = 4$ or $x = 36$ **b** $x = 16$ **c** $x = 49$
d $x = 1$ or $x = 512$ **e** $x = 27$ or $x = -8$
f $x = 16$ or $x = 625$

10 $a = 3$, $b = -\frac{5}{6}$, $c = -\frac{13}{12}$; Minimum $-\frac{13}{12}$

12 $x = 1$ or $x = \frac{a - b}{b - c}$

13 $m = 8$

14 a $\Delta = (a - c)^2 + 2b^2 \geq 0$
b $a = c$ and $b = 0$

15 $-8 < k < 0$

16 $p = 10$

Exercise 4C

1 a $\frac{18}{x(x + 3)}$ **b** $x = -6$ or $x = 3$

2 $x = -30$ or $x = 25$

3 17 and 19

4 a $\frac{40}{x}$ hours **b** $\frac{40}{x - 2}$ hours **c** 10 km/h

5 a Car $\frac{600}{x}$ km/h; Plane $\left(\frac{600}{x} + 220\right)$ km/h
b Car 80 km/h; Plane 300 km/h

6 $x = 20$

7 6 km/h

8 a $x = 50$ **b** 72 minutes

9 Slow train 30 km/h; Express train 50 km/h

10 60 km/h

11 Small pipe 25 minutes; Large pipe 20 minutes

12 Each pipe running alone takes 14 minutes

13 Rail 43 km/h; Sea 18 km/h

14 22 km

15 10 litres

16 32.23 km/h, 37.23 km/h

17 a $a + \sqrt{a^2 - 24a}$ minutes,
$a - 24 + \sqrt{a^2 - 24a}$ minutes
b i 84 minutes, 60 minutes
ii 48 minutes, 24 minutes
iii 36 minutes, 12 minutes
iv 30 minutes, 6 minutes

18 a 120 km **b** 20 km/h, 30 km/h

Exercise 4D

1 a $\frac{2}{x - 1} + \frac{3}{x + 2}$ **b** $\frac{1}{x + 1} - \frac{2}{2x + 1}$
c $\frac{2}{x + 2} + \frac{1}{x - 2}$ **d** $\frac{1}{x + 3} + \frac{3}{x - 2}$
e $\frac{3}{5(x - 4)} - \frac{8}{5(x + 1)}$

2 a $\frac{2}{x - 3} + \frac{9}{(x - 3)^2}$
b $\frac{4}{1 + 2x} + \frac{2}{1 - x} + \frac{3}{(1 - x)^2}$
c $\frac{-4}{9(x + 1)} + \frac{4}{9(x - 2)} + \frac{2}{3(x - 2)^2}$

3 a $\frac{-2}{x + 1} + \frac{2x + 3}{x^2 + x + 1}$ **b** $\frac{x + 1}{x^2 + 2} + \frac{2}{x + 1}$
c $\frac{x - 2}{x^2 + 1} - \frac{1}{2(x + 3)}$

4 $3 + \frac{3}{x - 1} + \frac{2}{x - 2}$

5 It is impossible to find A and C such that $A = 0$, $C - 2A = 2$ and $A + C = 10$

6 a $\frac{1}{2(x - 1)} - \frac{1}{2(x + 1)}$ **b** $\frac{2}{5(x - 2)} + \frac{3}{5(x + 3)}$
c $\frac{1}{x - 2} + \frac{2}{x + 5}$ **d** $\frac{2}{5(2x - 1)} - \frac{1}{5(x + 2)}$
e $\frac{3}{3x - 2} - \frac{1}{2x + 1}$ **f** $\frac{2}{x - 1} - \frac{2}{x}$
g $\frac{1}{x} + \frac{3 - x}{x^2 + 1}$ **h** $\frac{2}{x} + \frac{x}{x^2 + 4}$

i $\frac{1}{4(x-4)} - \frac{1}{4x}$ **j** $\frac{7}{4(x-4)} - \frac{3}{4x}$

k $x + \frac{1}{x} - \frac{1}{x-1}$ **l** $-x - 1 - \frac{3}{x} - \frac{1}{2-x}$

m $\frac{2}{3(x+1)} + \frac{x-4}{3(x^2+2)}$

n $\frac{2}{3(x-2)} + \frac{1}{3(x+1)} - \frac{1}{(x+1)^2}$

o $\frac{2}{x} + \frac{1}{x^2+4}$ **p** $\frac{8}{2x+3} - \frac{5}{x+2}$

q $\frac{26}{9(x+2)} + \frac{1}{9(x-1)} - \frac{1}{3(x-1)^2}$

r $\frac{16}{9(2x+1)} - \frac{8}{9(x-1)} + \frac{4}{3(x-1)^2}$

s $x - 2 + \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$

t $x - \frac{1}{x+1} + \frac{2}{x-1}$ **u** $\frac{3}{x+1} - \frac{7}{3x+2}$

Exercise 4E

1 a $(1, 1), (0, 0)$ **b** $(0, 0), \left(\frac{1}{2}, \frac{1}{2}\right)$

c $\left(\frac{3+\sqrt{13}}{2}, 4+\sqrt{13}\right), \left(\frac{3-\sqrt{13}}{2}, 4-\sqrt{13}\right)$

2 a $(13, 3), (3, 13)$ **b** $(10, 5), (5, 10)$
c $(-8, -11), (11, 8)$ **d** $(9, 4), (4, 9)$
e $(9, 5), (-5, -9)$

3 a $(11, 17), (17, 11)$ **b** $(37, 14), (14, 37)$
c $(14, 9), (-9, -14)$

4 $(0, 0), (2, 4)$

5 $\left(\frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right), \left(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$

6 $\left(\frac{15}{2}, \frac{5}{2}\right), (3, 1)$

7 $\left(\frac{-130+80\sqrt{2}}{41}, \frac{60+64\sqrt{2}}{41}\right)$, $\left(\frac{-130-80\sqrt{2}}{41}, \frac{60-64\sqrt{2}}{41}\right)$

8 $\left(\frac{1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}\right), \left(\frac{1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$

9 $\left(\frac{4}{9}, 2\right)$ **10** $\left(-\frac{6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$

11 $\left(-2, \frac{1}{2}\right)$ **12** $(0, -1), (3, 2)$

13 a $\left(\frac{2}{3}, -\frac{7}{9}\right)$ **b** $\left(-\frac{1}{2}, 0\right), (1, 0)$

c $\left(-\frac{3}{2}, \frac{7}{4}\right)$ **d** $(-1, 4), (0, 2)$

14 $a = \frac{26}{3},\ b = \frac{3}{2};\ x = -\frac{75}{31},\ y = \frac{179}{31}$

Chapter 4 review

Technology-free questions

1 $a = 3,\ b = 2,\ c = 1$

2 $(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$

5 a $x = -4$ or $x = 3$ **b** $x = -1$ or $x = 2$

c $x = -2$ or $x = 5$ **d** $x = \frac{2 \pm \sqrt{2}}{2}$

e $x = \frac{1 \pm \sqrt{3t-14}}{3}$ **f** $x = \frac{t \pm \sqrt{t^2-16t}}{2t}$

6 $x = \frac{-3 \pm \sqrt{73}}{2}$

7 a $\frac{-1}{x-3} - \frac{2}{x+2}$ **b** $\frac{3}{x+2} + \frac{4}{x-2}$

c $\frac{1}{2(x-3)} - \frac{3}{2(x+5)}$ **d** $\frac{1}{x-5} + \frac{2}{x+1}$

e $\frac{13}{x+2} - \frac{13}{x+3} - \frac{10}{(x+2)^2}$

f $\frac{4}{x+4} + \frac{2}{x-1} - \frac{3}{(x-1)^2}$

g $\frac{1}{x+1} - \frac{6}{x^2+2}$ **h** $\frac{1}{x-1} - \frac{x+3}{x^2+x+1}$

i $\frac{1}{3-x} - \frac{3}{x+4}$ **j** $\frac{2}{7(x-3)} - \frac{16}{7(x+4)}$

8 a $\frac{1}{x-3} - \frac{x-10}{x^2+x+2}$

b $\frac{1}{4(x+1)} - \frac{x-2}{4(x^2-x+2)}$

c $3x + 15 + \frac{64}{x-4} - \frac{1}{x-1}$

9 a $(0, 0), (-1, 1)$ **b** $(0, 4), (4, 0)$
c $(1, 4), (4, 1)$

10 $(-4, -1), (2, 1)$

11 a $t = \frac{135}{x}$ **b** $t = \frac{135}{x-15}$ **c** $x = 60$
d 60 km/h, 45 km/h

Multiple-choice questions

1 C **2** D **3** D **4** C **5** C **6** D
7 B **8** B **9** C **10** B **11** B

Extended-response questions

1 a $b = -4,\ c = 1$ **b** $x = 2 + \sqrt{3}$

2 a 24 km/h

b Speed $= \frac{a + \sqrt{a(a+480)}}{2}$, $a > 0$;
When $a = 60$, speed = 120 km/h, which is a very fast constant speed for a train. If we choose this as the upper limit for the speed, then $0 < a < 60$ and $0 <$ speed < 120

c

a	1	8	14	22	34	43	56	77	118
speed	16	20	24	30	40	48	60	80	120

3 a $\dfrac{a+\sqrt{a^2+4abc}}{2ac}$

b e.g. $a = 3, b = 1, c = \dfrac{4}{3}$

4 a Smaller pipe $(b + \sqrt{b^2 - ab})$ minutes;
Larger pipe $(b - a + \sqrt{b^2 - ab})$ minutes

b Smaller pipe 48 minutes;
Larger pipe 24 minutes

c

a	3	8	15	24	35
b	4	9	16	25	36

5 a $k = -2$ or $k = 1$ **b** $-10 < c < 10$ **c** $p = 5$

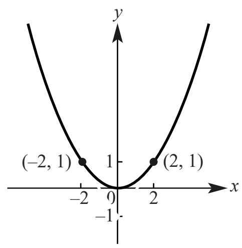
6 a i $-3p$ **ii** $3 + 2p^2$

b $m = 3p,\ n = 3 + 2p^2$

7 b $P\left(\dfrac{2}{p}, \dfrac{1}{q}\right)$

d i $y = \dfrac{1}{4}x^2$ **ii** $P(2, 1)$ **iii** $PX = \sqrt{2}$

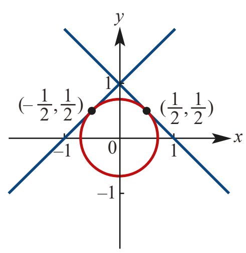
iv $Q(-2, 1)$



8 b $P\left(\dfrac{p}{p^2+q^2}, \dfrac{q}{p^2+q^2}\right)$

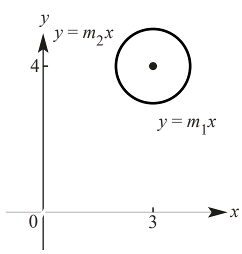
c i $x^2 + y^2 = \dfrac{1}{2}$ **ii** $P\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

iii $Q\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$



9 a i $(x - 3)^2 + (y - 4)^2 = 1$; $C(3, 4),\ r = 1$

ii



iii $m_1 = \dfrac{6-\sqrt{6}}{4},\ m_2 = \dfrac{6+\sqrt{6}}{4}$

iv $P_1\left(\dfrac{72+8\sqrt{6}}{25}, \dfrac{96-6\sqrt{6}}{25}\right)$,
$P_2\left(\dfrac{72-8\sqrt{6}}{25}, \dfrac{96+6\sqrt{6}}{25}\right)$

b i $a^2 = \dfrac{4}{5},\ (x-3)^2 + (y-4)^2 = \dfrac{4}{5}$

ii $\left(\dfrac{11}{5}, \dfrac{22}{5}\right)$ **iii** $m_3 = \dfrac{38}{41}$

c i $h = k = \sqrt{5}$

ii $Q_1\left(\dfrac{3\sqrt{5}}{5}, \dfrac{6\sqrt{5}}{5}\right),\ Q_2\left(\dfrac{6\sqrt{5}}{5}, \dfrac{3\sqrt{5}}{5}\right)$

Chapter 5

Technology-free questions

1 a $2 \times 7 \times 11 \times 13$ **b** $3 \times 5 \times 37$
c $7^2 \times 11 \times 13$ **d** $2^4 \times 5^4$

2 $\dfrac{1}{m+p}$

3 a $\sqrt{6} - \sqrt{3} - \sqrt{2} + 3$ **b** $21\sqrt{2} + 33$
c $4x - 12\sqrt{x} + 9$ **d** $-6\sqrt{x-2} + x + 7$

4 a $-\dfrac{\sqrt{2}+3}{7}$ **b** $\dfrac{3(\sqrt{5}+1)}{4}$
c $\dfrac{4\sqrt{2}+2}{7}$ **d** $\dfrac{3(\sqrt{5}+\sqrt{3})}{2}$
e $\dfrac{\sqrt{7}+\sqrt{2}}{5}$ **f** $\dfrac{2\sqrt{5}+\sqrt{3}}{17}$

5 a $a < -1$ or $a > 1$, $S_\infty = \dfrac{a^5}{a+1}$

b $-1 < \dfrac{b}{a} < 1,\ S_\infty = \dfrac{1}{a+b}$

c $x < -1$ or $x > -\dfrac{1}{3},\ S_\infty = \dfrac{(2x+1)^2}{x(3x+1)}$

d $x > \dfrac{3}{4}$ or $x < \dfrac{1}{4},\ S_\infty = \dfrac{4x-2}{4x-1}$

6 b i $x = \dfrac{-1-\sqrt{5}}{2}$ **ii** $x = \dfrac{-1+\sqrt{5}}{2}$

7 $a = -7,\ b = -5,\ c = 1$

9 a $576 = 2^6 \times 3^2,\ \sqrt{576} = 24$
b $1225 = 5^2 \times 7^2,\ \sqrt{1225} = 35$
c $1936 = 4^2 \times 11^2,\ \sqrt{1936} = 44$
d $1296 = 6^4,\ \sqrt{1296} = 36$

10 $x = -b - c$

11 $x = \dfrac{2ab}{a+b}$

12 a 333 667 **b** 166 333

13 $a = -\dfrac{1}{3},\ b = -2,\ \lambda = -\dfrac{4}{3}$;
$a = -\dfrac{1}{2},\ b = -1,\ \lambda = -\dfrac{3}{2}$

14 $k = 6$

15 a $t_n = \dfrac{2}{2^{n-1}}$ **b** $t_n = \dfrac{1}{2}(9 - 5n)$

c $t_n = \dfrac{7}{2^{n-1}} - 5$

16 8 m **17** 16 cm

18 $a = 12$; $d = -51$ or $d = 9$

19 $(1, -4), (-1, 4)$ **20** $x = -3$

21 64 km/h **22** $\sqrt{6} - 2$

23 a $\dfrac{1}{3(x-2)} + \dfrac{1}{3(x+2)}$

b $\dfrac{1}{x+2} + \dfrac{1}{x+3}$ **c** $\dfrac{2}{x+2} + \dfrac{3x-2}{x^2+4}$

d $\dfrac{1}{x+1} + \dfrac{1}{x-1} - \dfrac{2}{(x-1)^2}$

e $\dfrac{x}{x^2+1} + \dfrac{1}{x-3}$ **f** $\dfrac{x+1}{x^2+4} + \dfrac{2}{x-2}$

Multiple-choice questions

1 E	**2** B	**3** D	**4** B	**5** C
6 C	**7** A	**8** C	**9** C	**10** A
11 B	**12** D	**13** B	**14** B	**15** A
16 D	**17** A	**18** B	**19** A	**20** B
21 A	**22** C	**23** C	**24** A	**25** E
26 C	**27** C	**28** E	**29** D	**30** C
31 A	**32** C	**33** C	**34** D	**35** A
36 A	**37** D	**38** C		

Extended-response questions

1 a 8 **b** 7.7 **c** 6 cm **d** 15 cm

2 a $a = 6000$, $b = -15\,000$ **b** \$57 000
c 2021

3 a i 178 **ii** 179 **iii** 179.5 **iv** 179.95
b i 180 **ii** Circle
c 20 **d** $n = \dfrac{360}{180 - A}$ **e** Square

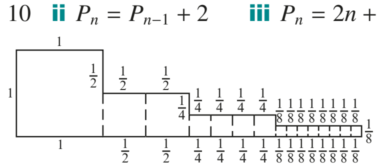
4 a Volume of hemisphere $= \dfrac{2}{3}\pi t^3$,
Volume of cylinder $= \pi t^2 s$,
Volume of cone $= \dfrac{1}{3}\pi t^2 w$
b i 6 : 2 : 3 **ii** 54π cubic units

5 a 140 **b** 180 **c** 20 **d** 10

6 a i $OC_1 = R - r_1$ **ii** $r_1 = \dfrac{R}{3}$
b i $OC_2 = \dfrac{R}{3} - r_2$ **ii** $r_2 = \dfrac{R}{9}$
c i $r = \dfrac{1}{3}$ **ii** $r_n = \dfrac{R}{3^n}$
iii $S_\infty = \dfrac{R}{2}$ **iv** $S_\infty = \dfrac{\pi R^2}{8}$

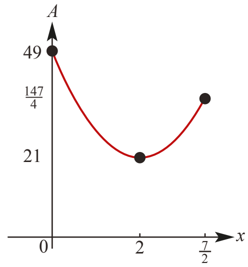
7 a i $80n + 920$
ii A: 2840 tonnes; B: 2465 tonnes
iii $40n(n + 24)$
iv A: 46 080 tonnes; B: 39 083 tonnes
b April 2021

8 a 4 **b** 6 **c** 8 **d** 2
e i 10 **ii** $P_n = P_{n-1} + 2$ **iii** $P_n = 2n + 2$
iv



9 b 3 ways **c** 8 ways

10 a $8x$ cm **b** $28 - 8x$ cm **c** $7 - 2x$ cm
e



f $A = 21$ when $x = 2$

11 b i $x = \dfrac{1}{24}$ **ii** $x = \dfrac{25}{24}$

12 c 11, 24 and 39

13 a 14 m **b** $t_n = 1.5n - 1$ **c** 53 **d** 330 m

14 a i $a = 50\,000$, $d = 5000$
ii 11th month **iii** 4 950 000 litres
b i $q_n = 12\,000(1.1)^{n-1}$ **ii** 256 611 litres
c 31st month

15 a 1 hour 35 minutes **b** 2.5 km

16 a 90 km **b** 70 km/h

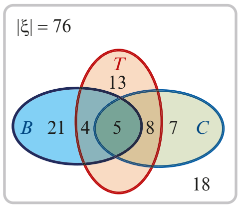
17 a 91 **b** 42 857

18 a 0, ±6, ±15 **b** ±9, ±12, ±21
c 11, 20, 27, 32, 35, 36

20 a $c = \dfrac{b^2 - 4}{4}$ **b** $c = \dfrac{2b^2}{9} + 3$
c $b = \pm 12$, $c = 35$

21 a $m = 5$, $n = 2$ **b** $b = 4$, $c = -1$

22 a $|B' \cap C' \cap T| = |C \cap T|$,
$|B \cap C' \cap T'| = 3|B' \cap C \cap T'|$,
$|B \cap C' \cap T| = 4$
b



c i 5 **ii** 0

23 a i 606 **ii** 612 **iii** 619
b $t_n = 1.05t_{n-1} - 24$
c $t_n = 600 \times 1.05^n - \dfrac{24(1.05^n - 1)}{0.05}$
d 696
e i 534 **ii** 478 **iii** 223
f The population stabilises to 160 after approximately 42 years

24 a 1.075 million **b** 1.779 million
c Beta (1.373 million)
d Alpha (1.779 million) will be greater than Beta (1.658 million)
e 2 years (1.426 million)

26 a i 262 144 **ii** $n = 10$

b i $\frac{1}{262\,144}$ **ii** 1.333

c i 349 526.333 **ii** $S_n = \frac{1}{3}(4^n - 4^{1-n}) + 1$

Investigations

See solutions supplement

Chapter 6

See solutions supplement

Chapter 7

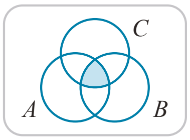
Exercise 7A

1 a {8} **b** {7, 10, 11}

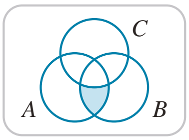
c $\{7, 8, 9, 10, 11\} = \xi$ **d** $\varnothing$ **e** $\varnothing$

2 a $X \cap Y$ **b** $X \cap Y'$ **c** $X' \cap Y'$

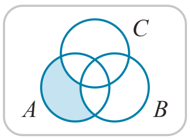
3 a



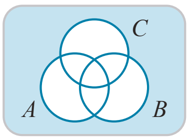
b



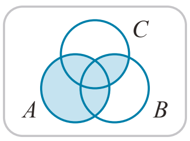
c



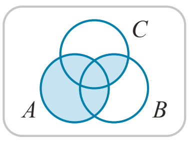
d



e



f



4 a $A \cap C$ **b** $A \cap B'$ **c** $B \cap C$ **d** $A' \cap B$

e $B' \cup C$

5 a $(A \cap \varnothing) \cup (A \cup \xi) = \xi$

b If $A \cup B = \xi$, then $A' \cap B = A'$.

c $A \cup B \supseteq A \cap B$

7 a $X \cup Y$ **b** Y **c** $\varnothing$ **d** X

e X **f** $X \cap (Y' \cup Z')$ **g** $X \cap Y$

h $X \cup Y$ **i** $\varnothing$ **j** X **k** X'

l $X' \cap Y'$

Exercise 7B

1 a 1 **b** 0 **c** 1 **d** 1 **e** 1 **f** 0 **g** 1 **h** 1

2 a

x	y	y'	$x \vee y'$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

b

x	y	y'	$x \wedge y'$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

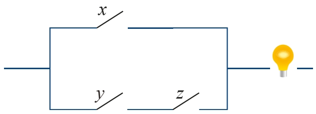
c

x	y	x'	y'	$x' \wedge y'$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

d

x	y	x'	y'	$x' \vee y'$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

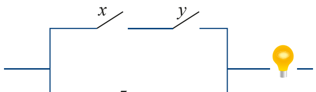
3 a i



ii

x	y	z	$y \wedge z$	$x \vee (y \wedge z)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

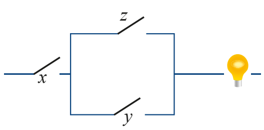
b i



ii

x	y	z	$x \wedge y$	$(x \wedge y) \vee z$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

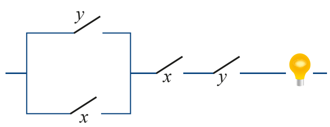
c i



ii

x	y	z	$y \vee z$	$x \wedge (y \vee z)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

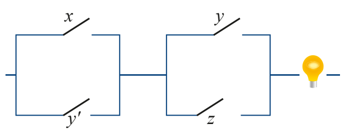
d **i**



ii

x	y	$x \vee y$	$x \wedge y$	$(x \vee y) \wedge (x \wedge y)$
0	0	0	0	0
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

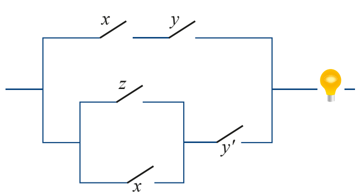
e **i**



ii

x	y	z	$a = x \vee y'$	$b = y \vee z$	$a \wedge b$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

f **i**



ii

x	y	z	$a = x \wedge y$	$b = (z \vee x) \wedge y'$	$a \vee b$
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	0	1

Exercise 7C

1 **a** 0 **b** 0 **c** 1
d 1 **e** $a' \wedge b$ **f** $a \vee b$
g $a \wedge b$ **h** b **i** 1

3 0

4 **a**

x	y	$f(x, y)$
0	0	0
0	1	1
1	0	0
1	1	0

b

x	y	$f(x, y)$
0	0	1
0	1	0
1	0	1
1	1	0

c

x	y	$f(x, y)$
0	0	0
0	1	0
1	0	0
1	1	0

d

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

e

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

f

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

5 **a** $(x \wedge y) \vee x = x$ **b** $(x \vee y) \wedge x = x$
c $(x \wedge y') \vee (x \wedge y') = x \wedge y'$
d $(x \wedge y') \vee (x' \wedge y') = y'$
e $(x' \wedge y') \vee (x' \wedge z) = x' \wedge (y' \vee z)$
f $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) = x \vee y$

6 **a** $(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$
b $(x' \wedge y') \vee (x' \wedge y)$
c $(x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y \wedge z)$
d $(x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y' \wedge z)$

Exercise 7D

1 **a** Your eyes are not blue.
b The sky is not grey. **c** This integer is even.
d I do not live in Switzerland. **e** $x \leq 2$
f This number is greater than or equal to 100.

2 **a** It is dark or it is cold.
b It is dark and cold. **c** It is light and cold.
d It is light or hot. **e** It is good or light.
f It is light and hard. **g** It is dark or hard.

3 **a** $B \wedge A$ **b** $D \vee C$
c $\neg C \wedge D$ **d** $\neg A \wedge \neg B$
e $\neg D \wedge \neg C$ **f** $B \vee A$

4 **a** It is wet or rough. **b** It is wet and rough.
c It is dry and rough. **d** It is dry or smooth.
e It is difficult or dry.
f It is dry and inexpensive.
g It is wet or inexpensive.

5 **a** n is a prime number or an even number.
b n is divisible by 6. **c** n is 2.
d n is an even number greater than 2.
e n is not 2. **f** n is not prime.
g n is neither prime nor divisible by 6.
h n is not divisible by 6.

6

A	B	$A \wedge B$	$\neg(A \wedge B)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

7

A	B	$A \vee B$	$\neg B$	$(A \vee B) \wedge (\neg B)$
T	T	T	F	F
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

12 a

A	B	$A \wedge B$	$(A \wedge B) \Rightarrow A$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

b

A	B	$A \vee B$	$(A \vee B) \Rightarrow A$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

c

A	B	$\neg A$	$\neg B$	C: $\neg B \vee \neg A$	$C \Rightarrow A$
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	F	T	F
F	F	T	T	T	F

d

A	B	$\neg B$	$\neg B \wedge A$	$(\neg B \wedge A) \Rightarrow A$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

e

A	B	$\neg A$	$B \vee \neg A$	$(B \vee \neg A) \Rightarrow \neg A$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

f

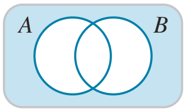
A	B	C: $\neg B \vee \neg A$	D: $\neg B \wedge A$	$C \Rightarrow D$
T	T	F	F	T
T	F	T	T	T
F	T	T	F	F
F	F	T	F	F

g

A	B	C: $\neg B \vee A$	D: $\neg(B \wedge A)$	$C \Rightarrow D$
T	T	T	F	F
T	F	T	T	T
F	T	F	T	T
F	F	T	T	T

h

A	B	$\neg B$	$\neg B \Rightarrow A$	$\neg B \wedge (\neg B \Rightarrow A)$
T	T	F	T	F
T	F	T	T	T
F	T	F	T	F
F	F	T	F	F

15 

$(A \cup B)' = A' \cap B'$

16 a

A	B	$A \downarrow B$	$B \downarrow A$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

19 a **i** If x is an even integer, then $x = 6$.
ii If x is not an even integer, then $x \neq 6$.
iii $x = 6$ and x is not an even integer.

b **i** If public transport improves, then I was elected.
ii If public transport does not improve, then I was not elected.
iii I was elected and public transport did not improve.

c **i** If I qualify as an actuary, then I passed the exam.
ii If I do not qualify as an actuary, then I failed the exam.
iii I passed the exam and I did not qualify as an actuary.

Exercise 7E

1 Valid

A	B	$A \vee B$	$\neg A$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

2 Not valid

A	B	$A \vee B$	$\neg A$	$\neg B$
T	T	T	F	F
T	F	T	F	T
F	T	T	T	F
F	F	F	T	T

3 Valid

A	B	C	$A \Rightarrow B$	$B \Rightarrow C$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

4 Prem 1 Conc Prem 2 Valid

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

5 Prem 1 Conc Prem 2 Not valid

A	B	$B \Rightarrow A$
T	T	T
T	F	T
F	T	F
F	F	T

6

			P1	P2	P3	C
C	M	L	$C \vee M$	$(C \wedge M) \Rightarrow L$	$M \wedge (\neg L)$	$\neg C$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	T	T	F	F
T	F	F	T	T	F	F
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

Argument is valid

7 a Valid **b** Valid
c Not valid **d** Not valid

8 a Valid **b** Not valid

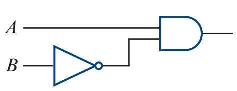
9 a $[(J \Rightarrow W) \wedge W] \Rightarrow J$ is not a tautology; argument is invalid

b $[(\neg S \Rightarrow \neg R) \wedge R] \Rightarrow S$ is a tautology; argument is valid

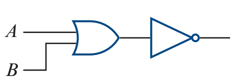
c $[(K \Rightarrow J) \wedge (J \Rightarrow S)] \Rightarrow (K \Rightarrow S)$ is a tautology; argument is valid

Exercise 7F

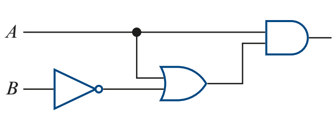
1 a



b

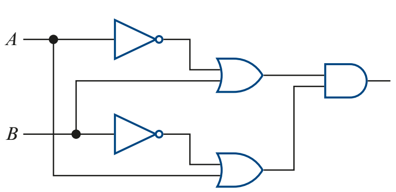


c



Note: Equivalent to A

d



2 a $\neg(A \wedge B)$

A	B	$A \wedge B$	$\neg(A \wedge B)$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

b $\neg A \wedge \neg B$

A	B	$\neg A$	$\neg B$	$\neg A \wedge \neg B$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

c $\neg X \vee (X \wedge Y) \equiv \neg X \vee Y$

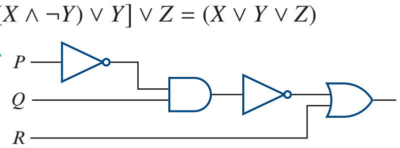
X	Y	$\neg X$	$X \wedge Y$	$\neg X \vee (X \wedge Y)$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	0
1	1	0	1	1

d $\neg A \wedge (A \vee B) \equiv \neg A \wedge B$

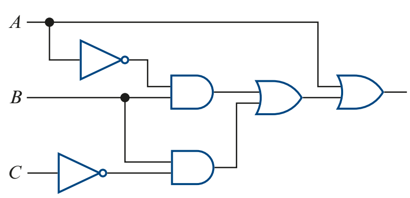
A	B	$\neg A$	$A \vee B$	$\neg A \wedge (A \vee B)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

3 $[(X \wedge \neg Y) \vee Y] \vee Z = (X \vee Y \vee Z)$

4 a



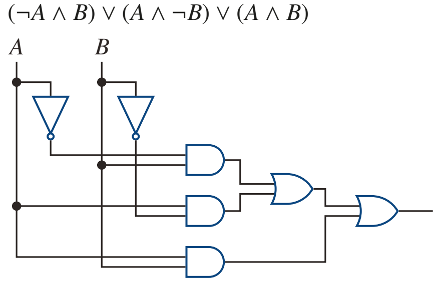
b



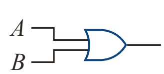
Note: Equivalent to $A \vee B$

5 a $(\neg A \wedge B) \vee (A \wedge \neg B) \vee (A \wedge B)$

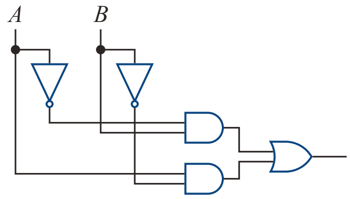
b



c $A \vee B$;



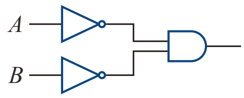
6 $(\neg A \wedge B) \vee (A \wedge \neg B)$



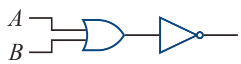
7 a $(\neg A \vee \neg B) \wedge (\neg A \vee B) \wedge (A \vee \neg B)$

b $\neg A \wedge \neg B$

c



d



Exercise 7G

1 a y **b** $x' \vee y'$ **c** $x \vee y'$

2 a y **b** $y' \vee z'$ **c** $(x \wedge y') \vee (x' \wedge z')$

3 a $(x \wedge y') \vee (x \wedge z) \vee (x' \wedge y \wedge z')$

b $(x \wedge y) \vee (x' \wedge z')$

c $(x \wedge y') \vee (x' \wedge z') \vee (y \wedge z)$
or $(x \wedge z) \vee (x' \wedge y) \vee (y' \wedge z')$

4 a $x' \vee y'$ **b** $x \vee (y \wedge z')$

5 a

X	Y	Output
0	0	1
0	1	0
1	0	1
1	1	0

b $\neg Y$

6 a

X	Y	Z	Output
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

b $\neg Y \vee Z$

Chapter 7 review

Technology-free questions

1 True: a, b, d, e, f; False: c

2 a It is not raining. **b** It is raining.

c $x \neq 5$ or $y \neq 5$

d $x \neq 3$ and $x \neq 5$ (i.e. $x \notin \{3, 5\}$)

e It is raining or it is windy.

f It is snowing and it is not cold.

3 a

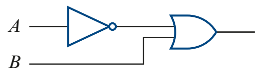
A	B	$A \oplus B$	$A \oplus (A \oplus B)$
T	T	F	T
T	F	T	F
F	T	T	T
F	F	F	F

Note: $A \oplus (A \oplus B) \equiv B$

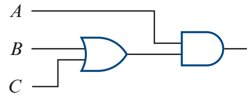
b

A	B	$A \vee B$	$A \oplus (A \vee B)$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	F

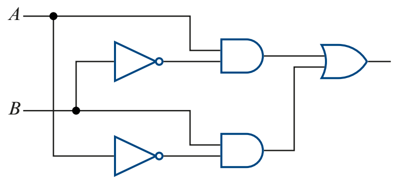
6 a



b



c

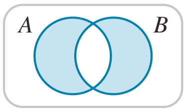


Multiple-choice questions

1 B **2** C **3** C **4** D **5** B
6 D **7** A **8** B **9** D **10** E

Extended-response questions

1 a

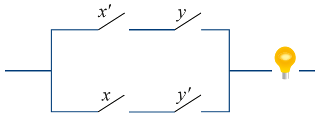


2 a

x	y	Light
0	0	0
0	1	1
1	0	1
1	1	0

b $(x' \wedge y) \vee (x \wedge y')$

c



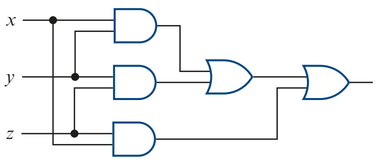
3 a

x	y	z	Light
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

b $(x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$

c $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

d



4 a i $\ell = 1$ **ii** $h = 30$

b $\text{LCM}(x, x') = 30 = h$, for all $x \in B$; $\text{HCF}(x, x') = 1 = \ell$, for all $x \in B$

5 a i d **ii** 1 **iii** 0

b $d \vee d' = d \neq 1$ and $d \wedge d' = d \neq 0$

Chapter 8

Exercise 8A

1 a 3404; $F = 27$, $G = 14$, $H = 110$, $K = 69$

b 1118; $F = 8$, $G = 18$, $H = 56$, $K = 30$

c 513; $F = 2$, $G = 63$, $H = 90$, $K = 25$

d 1311; $F = 10$, $G = 21$, $H = 60$, $K = 29$

2 a 101010110 **b** 1111111

c 11011110001 **d** 100110100100

3 a Step 1 Input n.

Step 2 Let q be the quotient when n is divided by 8.

Step 3 Let r be the remainder when n is divided by 8.

Step 4 Record r.

Step 5 Let n have the value of q.

Step 6 If $n > 0$, then repeat from Step 2.

Step 7 Write the recorded values of r in reverse order.

b i 526 **ii** 13056 **iii** 705 **iv** 22657

4 a 1 **b** 27 **c** 6 **d** 5

5 a $((2)x + 3)x + 4$

b $(((1)x + 3)x - 4)x + 5$

c $(((4)x + 6)x - 5)x - 4$

7 a Step 1 Choose an initial estimate x for $\sqrt{N}$.

Step 2 Let $x_{\text{new}} = \frac{1}{2}\left(x + \frac{N}{x}\right)$.

Step 3 Let x have the value of x_{new}.

Step 4 Repeat from Step 2 unless $-0.01 < x^2 - N < 0.01$.

Step 5 The required estimate is x.

b i 2.2 **ii** 18.6 **iii** 39.5 **iv** 88.6

8 Circled: 2, 3, 5, 7; Unmarked: 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Exercise 8B

1 a Step 1 $T \leftarrow 6$ and $n \leftarrow 1$

Step 2 Print n and print T

Step 3 $T \leftarrow T + 3$ and $n \leftarrow n + 1$

Step 4 Print n and print T

Step 5 Repeat from Step 3 while $n < 6$

c

n	1	2	3	4	5	6
T	6	9	12	15	18	21

2 a Step 1 $A \leftarrow 100\,000$ and $i \leftarrow 0$

Step 2 $A \leftarrow 1.025A$ and $i \leftarrow i + 1$

Step 3 Print i and print A

Step 4 Repeat from Step 2 while $i < 5$

c

i	A
0	100 000
1	102 500
2	105 063
3	107 689
4	110 381
5	113 141

3 a

n	1	2	3	4	5	6
A	10	15	20	25	30	35

b

n	1	2	3	4	5
A	2	6	18	54	162

4 a Step 1 $sum \leftarrow 0$ and $n \leftarrow 1$

Step 2 $sum \leftarrow sum + \frac{1}{n^2}$

Step 3 $n \leftarrow n + 1$

Step 4 Repeat from Step 2 while $n \leq N$

b Step 1 $sum \leftarrow 0$ and $n \leftarrow 1$

Step 2 $sum \leftarrow sum + \frac{1}{n}$

Step 3 $n \leftarrow n + 1$

Step 4 Repeat from Step 2 while $n \leq N$

5 a Step 1 $n \leftarrow 1$

Step 2 If n is even, then $T \leftarrow 5 - 2n$ Otherwise $T \leftarrow n^2 + 1$

Step 3 Print T

Step 4 $n \leftarrow n + 1$

Step 5 Repeat from Step 2 while $n \leq N$

b

n	1	2	3	4	5	6	7
T	2	1	10	−3	26	−7	

6 a $P(3) = 37$ **b** $P(3) = 55$ **c** $P(3) = -94$

p	i
0	3
1	2
5	1
12	0
37	−1

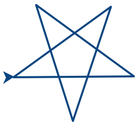
p	i
0	3
2	2
5	1
19	0
55	−1

p	i
0	3
−4	2
−10	1
−31	0
−94	−1

7 b Step 1 $n \leftarrow 1$
Step 2 Draw forwards for 3 cm
Step 3 Turn through 90° anticlockwise
Step 4 $n \leftarrow n + 1$
Step 5 Repeat from Step 2 while $n \le 4$

c Step 1 $n \leftarrow 1$
Step 2 Draw forwards for 3 cm
Step 3 Turn through 60° anticlockwise
Step 4 $n \leftarrow n + 1$
Step 5 Repeat from Step 2 while $n \le 6$

d



8 a Step 1 Input n
Step 2 Print n
Step 3 If $n = 1$, then stop
Step 4 If n is even, then $n \leftarrow n \div 2$
Otherwise $n \leftarrow 3n + 1$
Step 5 Repeat from Step 2

b **i** 5, 16, 8, 4, 2, 1
ii 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1
iii 8, 4, 2, 1

Exercise 8C

1

```
input a, b
if a ≤ b then
  print a
else
  print b
end if
```

2

```
input mark
if mark ≥ 95 then
  print 'A'
else if mark ≥ 85 then
  print 'B'
else if mark ≥ 65 then
  print 'C'
else if mark ≥ 55 then
  print 'D'
else
  print 'E'
end if
```

3 a 15 **b** 16 **c** 20
4 a 0 **b** 5 **c** 25
5 a 5 **b** 6 **c** 10
6 a $a = 6,\ b = 15$ **b** $a = 8,\ b = 29$
c $a = 7,\ b = 18$

7 a

```
input n
sum ← 0
for i from 1 to n
  sum ← sum + 2^i
end for
print sum
```

b

```
input n
product ← 1
for i from 1 to n
  product ← product × 2^i
end for
print product
```

8

```
input n
sum ← 0
for i from 1 to n
  sum ← sum + i^3
end for
print sum
```

9

a	b
8	6
2	18
2	6
−22	−18
410	366

10

```
n ← 1
x ← 4
while x ≤ 1000
  n ← n + 1
  x ← 3x + 2
end while
print n
```

n	x
1	4
2	14
3	44
4	134
5	404
6	1214

11

```
n ← 0
sum ← 0
while sum ≤ 1 000 000
  n ← n + 1
  sum ← sum + n^n
end while
print n
```

n	sum
0	0
1	1
2	5
3	32
4	288
5	3413
6	50 069
7	873 612
8	17 650 828

12

```
n ← 1
while 2^n ≤ 10n^2
  n ← n + 1
end while
print n
```

$n = 10$

13

```
n ← 0
x ← 3
y ← 1000
while x ≤ y
   n ← n + 1
   x ← 2n + 3
   y ← 0.9^n × 1000
end while
print n
```

$n = 28$

14 a i Prints 8 **ii** Prints 20 **iii** Prints 16

b Prints the highest common factor of a and b

Exercise 8D

1 Change instruction inside for loop:

a $sum \leftarrow sum + i^3$

b $sum \leftarrow sum + 2^i$

c $sum \leftarrow sum + i \times (i + 1)$

2 a Initialise the variable A as an empty list

b Change for loop:

```
for i from 1 to n
   append 2^i to A
end for
```

c Change for loop:

```
for i from 0 to n − 1
   append 2^(n−i) to A
end for
```

3

```
define min(A):
   min ← A[1]
   for i from 1 to length(A)
      if A[i] < min then
         min ← A[i]
      end if
   end for
   return min
```

4 a

```
define sum(n):
   sum ← 0
   for i from 1 to n
      sum ← sum + factorial(i)
   end for
   return sum
```

b

```
n ← 1
while factorial(n) ≤ 10^n
   n ← n + 1
end while
print n
```

5 a

a	b	c
1	1	1
1	2	2
1	3	3
2	1	4
2	2	5
2	3	6
3	1	7
3	2	8
3	3	9

b

a	b	c
2	3	6
2	4	14
3	3	23
3	4	35

6 a

i	$tally$
	0
1	1
2	10
3	35
4	99

b Finds the sum of the squares of the entries in list A

7 a

i	A	
	[1, 1]	
1	[1, 1, 2]	$A[1] + A[2] = 2$
2	[1, 1, 2, 3]	$A[2] + A[3] = 3$
3	[1, 1, 2, 3, 5]	$A[3] + A[4] = 5$
4	[1, 1, 2, 3, 5, 8]	$A[4] + A[5] = 8$
5	[1, 1, 2, 3, 5, 8, 13]	$A[5] + A[6] = 13$

b

```
A ← [1, 1]
i ← 1
while A[i] ≤ 1000
   append A[i] + A[i + 1] to A
   i ← i + 1
end while
print A[i]
```

c

```
define fibonacci(n):
   A ← [1, 1]
   for i from 1 to n − 2
      append A[i] + A[i + 1] to A
   end for
   return A[n]
```

d

```
A ← [0, 1, 1]
for i from 1 to 7
   append A[i] + A[i + 1] + A[i + 2] to A
end for
print A
```

8 a $A[5] = 25$

b Change while loop:

```
while n^3 ≤ 100 000
   append n^3 to A
   n ← n + 1
end while
```

c 46 entries

9

```
for x from 1 to 10
  for y from 1 to 6
    for z from 1 to 4
      if 3x + 5y + 7z = 30 then
        print (x, y, z)
      end if
    end for
  end for
end for
```

Printed solutions: $(1, 4, 1), (2, 2, 2), (6, 1, 1)$

10 a

```
for x from 1 to 5
  for y from 1 to 5
    for z from 1 to 3
      if x^2 + y^2 + 10z = 30 then
        print (x, y, z)
      end if
    end for
  end for
end for
```

b $(1, 3, 2), (2, 4, 1), (3, 1, 2), (4, 2, 1)$

c Change condition on if-then block:

```
if x^2 + y^2 + 10z = 30 and x + y + z = 7
then
```

Printed solutions: $(2, 4, 1), (4, 2, 1)$

11

```
define f(n):
  A ← [ ]
  for x from 0 to n
    for y from 0 to n
      if x^2 + y^2 = n then
        append [x, y] to A
      end if
    end for
  end for
  return A
```

12 Finds the number and proportion of cases in which the quadratic equation $ax^2 + bx + c = 0$ has no real solutions, where a, b, c are integers between -10 and 10 inclusive ($a \neq 0$)

Note: *There is more than one correct way to answer Questions 13–16.*

13 a

```
input N
if length(factors(N)) = 2 then
  print 'prime'
else
  print 'not prime'
end if
```

b

```
define prime(n):
  i ← 0
  count ← 0
  while count < n
    i ← i + 1
    if length(factors(i)) = 2 then
      count ← count + 1
    end if
  end while
  return i
```

14 a

```
define power(n):
  i ← 0
  while remainder(n, 2^i) = 0
    i ← i + 1
  end while
  return i − 1
```

b

```
define number(n):
  m ← n − 1
  found ← false
  while found = false
    m ← m + 1
    found ← true
    for i from 1 to n
      if remainder(m, i) ≠ 0 then
        found ← false
      end if
    end for
  end while
  return m
```

15 a

```
define pell(n):
  A ← [1, 2]
  for i from 1 to n − 2
    append A[i] + 2 × A[i + 1] to A
  end for
  return A[n]
```

b

```
sum ← 0
for i from 1 to n
  sum ← sum + pell(i)
end for
print sum
```

c

```
A ← [1, 2]
i ← 1
while A[i] < 10^999
  append A[i] + 2 × A[i + 1] to A
  i ← i + 1
end while
print A[i]
```

16 a

```
input a, b
print (a, b)
i ← 0
while a ≠ b and i < 100
   [insert given if-then block]
   print (a, b)
   i ← i + 1
end while
```

Note: The variable i is not necessary, but is used to ensure that the program stops.

b **i** Cycles indefinitely: (21, 28), (42, 7), (35, 14), (21, 28), . . .
ii Cycles indefinitely: (21, 49), (42, 28), (14, 56), (28, 42), (56, 14), (42, 28), . . .
iii (35, 105), (70, 70)
iv (19, 133), (38, 114), (76, 76)
v (37, 259), (74, 222), (148, 148)

Chapter 8 review

Technology-free questions

1 a 8 **b** 18 **c** 93 **d** 9, 75

2 Change for loop:

a

```
for n from 1 to 6
   sum ← sum + n^n
end for
```

b

```
for n from 1 to 6
   sum ← sum + (−1)^(n+1) × n × (7 − n)
end for
```

3

n	a	b	c
1	2	4	4
2	4	12	12
3	12	44	44
4	44	200	200
5	200	1088	1088

4 a 2, 8, 26

b

```
a ← 0
for i from 1 to 50
   a ← 3a + 2
end for
print a
```

c

```
a ← 0
sum ← 0
for i from 1 to 50
   a ← 3a + 2
   sum ← sum + a
end for
print sum
```

5 a

```
input N
for n from 1 to N
   if remainder(n, 2) = 0 then
      T ← 6 − 2n
   else
      T ← 3n + 1
   end if
   print T
end for
```

b

n	1	2	3	4	5
T	4	2	10	−2	16

c

```
input N
sum ← 0
for n from 1 to N
   if remainder(n, 2) = 0 then
      sum ← sum + 6 − 2n
   else
      sum ← sum + 3n + 1
   end if
end for
print sum
```

6

```
for a from −6 to 6
   for b from −6 to 6
      if 9 ≤ a^2 + b^2 ≤ 36 then
         print (a, b)
      end if
   end for
end for
```

7 a

a	m	b	$f(a)$	$f(m)$	$f(b)$
0	1	2	−2	−1	2
1	$\frac{3}{2}$	2	−1	$\frac{1}{4}$	2
1	$\frac{5}{4}$	$\frac{3}{2}$	−1	$-\frac{7}{16}$	$\frac{1}{4}$
$\frac{5}{4}$	$\frac{11}{8}$	$\frac{3}{2}$	$-\frac{7}{16}$	$-\frac{7}{64}$	$\frac{1}{4}$
$\frac{11}{8}$	$\frac{23}{16}$	$\frac{3}{2}$			

b

```
define f(x):
   return x^2 − 3

a ← 0
b ← 3
m ← 1.5
while b − a > 2 × 0.01
   [inside while loop is unchanged]
end while
print m
```

Multiple-choice questions

1 E **2** D **3** C **4** E **5** A
6 C **7** C **8** E **9** E **10** C

Extended-response questions

1 a i [1, 0, 0, 0, 0, 0, 1]
ii [1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1]
iii [1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0]
b i [1, 0, 1] **ii** [1, 0, 7, 2, 7]
iii [1, 5, 3, 0, 0, 2]
c i Output $B = [10, 8, 6, 4, 2]$
ii Output $A = [10, 8, 6, 8, 10]$

2 a

i	j	$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$	
		1	9	3	2	7	6	
1	1	1	9	3	2	7	6	*NS*
1	2	1	3	9	2	7	6	*S*
1	3	1	3	2	9	7	6	*S*
1	4	1	3	2	7	9	6	*S*
1	5	1	3	2	7	6	9	*S*
2	1	1	3	2	7	6	9	*NS*
2	2	1	2	3	7	6	9	*S*
2	3	1	2	3	7	6	9	*NS*
2	4	1	2	3	6	7	9	*S*

b Gives both entries the value of $A[j+1]$ instead of swapping their values
c Change conditions on for loops:

```
for i from 1 to length(A)
  for j from 1 to length(A) − i
```

3 a

a	n	*reverse*
	5678	0
8	567	8
7	56	87
6	5	876
5	0	8765

b Reverses the digits of a given natural number n
c

```
for n from 1 to 1000
  if R(n^2) = n^2 then
    print n^2
  end if
end for
```

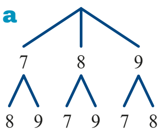
4 a i 4321 **ii** 5555 **iii** 8765 **iv** 14443
c

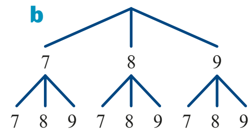
n	$R(n)$	$n + R(n)$
1756	6571	15565
15565	56551	72116
72116	61127	133243
133243	342331	475574

d Output 9889; 82 inputs
5 a 8, 15 **b** 12, 35

Chapter 9

Exercise 9A

1 45 **2** 8 **3** 120
4 a


b


5 a 27 **b** 6
6 30
7 a 6 **b** 18 **c** 20 **d** 15
8 BB, BR, BG, RB, RG, GB, GR, GG
9 12
10 9
11 a 6 **b** 13
12 16

Exercise 9B

1 1, 1, 2, 6, 24, 120, 720, 5040, 40 320, 362 880, 3 628 800
2 a 5 **b** 90 **c** 66 **d** 161 700
3 a $n+1$ **b** $n+2$ **c** $n(n-1)$ **d** $\dfrac{n+2}{(n+1)!}$
4 1, 4, 12, 24, 24
5 DOG, DGO, ODG, OGD, GOD, GDO
6 120
7 362 880
8 FR, FO, FG, RF, RO, RG, OF, OR, OG, GF, GR, GO
9 a 720 **b** 720 **c** 360
10 a 120 **b** 120 **c** 60
11 20 160
12 a 125 **b** 60
13 a 120 **b** 360 **c** 720
14 60
15 a 17 576 000 **b** 11 232 000
16 $(m, n) = (6, 0), (6, 1), (5, 3)$
17 $(n^2 - n) \cdot (n-2)! = n \cdot (n-1) \cdot (n-2)! = n!$
18 a 384 **b** 3072
19 30

Exercise 9C

1 a 120 **b** 72 **c** 24 **d** 96
2 a 120 **b** 48 **c** 72 **d** 12
3 a 360 **b** 144 **c** 144 **d** 72
4 a 1152 **b** 1152
5 a 600 **b** 108 **c** 431 **d** 52
6 a 720 **b** 48 **c** 144 **d** 96 **e** 48
7 a 900 **b** 900

8 84

9 32

10 a 480 b 192

11 144

Exercise 9D

1 35 2 34 650

3 4 989 600 4 56

5 27 720

6 a 420 b 105 c 90 d 12 e 105

7 35

8 a 15 b $\dfrac{(m+n)!}{m!\cdot n!}$

9 a 52! b $\dfrac{104!}{(2!)^{52}}$ c $\dfrac{(52n)!}{(n!)^{52}}$

10 4900

11 89

Exercise 9E

1 1, 5, 10, 10, 5, 1

2 a 7 b 6 c 66 d 56 e 100
f 499 500

3 a n b $\dfrac{n(n-1)}{2}$ c n d $n+1$
e $\dfrac{(n+2)(n+1)}{2}$ f $\dfrac{n(n+1)}{2}$

4 a 720 b 120

5 2 598 960

6 a 10 b 45 c 45 d 10

7 45 379 620

8 56

9 a 45 b 16

10 15

14 462

16 a 2300 b 152 c 2148

Exercise 9F

1 153 2 126

3 1176 4 140

5 a 1716 b 700 c 980 d 1568

6 a 25 200 b 4200

7 a 1 392 554 592 b 5 250 960

8 a 15 504 b 10 800 c 15 252

9 a 21 b 10 c 11

10 2100

11 a 204 490 b 7 250 100

12 a 48 b 210

13 1440 14 3600

15 14 400 16 150

17 3744

Exercise 9G

1 $^7C_2 = 21, {}^6C_2 = 15, {}^6C_1 = 6$

2 1, 7, 21, 35, 35, 21, 7, 1; $^7C_2 = 21, {}^7C_4 = 35$

3 1, 8, 28, 56, 70, 56, 28, 8, 1;
$^8C_4 = 70, {}^8C_6 = 28$

4 $2^6 = 64$ 5 $2^5 = 32$

6 $2^{10} = 1024$ 7 $2^6 - 1 = 63$

8 $2^8 - {}^8C_1 - {}^8C_0 = 247$

9 $2^8 = 256$ 10 $2^4 - 1 = 15$

11 a 128 b 44

Exercise 9H

1 4

4 a 3 b 5 c 14

11 At least 26 students

Exercise 9I

1 a $\{1, 3, 4\}$ b $\{1, 3, 4, 5, 6\}$ c $\{4\}$
d $\{1, 2, 3, 4, 5, 6\}$ e 3
f $\varnothing, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{4, 5, 6\}$

2 36 3 4 4 150

5 a 64 b 32

6 a 48 b 48 c 12 d 84

7 a 12 b 38

8 88 9 80 10 4

11 a 756 b 700 c 360 d 1096

12 1 452 555 13 3417 14 5

Chapter 9 review

Technology-free questions

1 a 20 b 190 c 300 d 4950

2 11

3 a 27 b 6

4 120 5 60 6 18 7 31

8 10 9 3 10 12 11 192

Multiple-choice questions

1 C 2 B 3 A 4 D 5 B 6 B

7 C 8 D 9 C 10 C 11 A

Extended-response questions

1 a 120 b 360 c 72 d 144

2 a 20 b 80 c 60

3 a 210 b 84 c 90 d 195

4 a 420 b 15 c 105 d 12

5 a i 20 ii 10 iii 64
b 8

6 a 210 b 10 c 80

7 a 676 b 235 c 74

8 a 24 b 4 c 24 d $\dfrac{3}{4}$

9 a 924

b There are at least $365 \times 3 = 1095$ days in three years and there are 924 different paths, so some path is taken at least twice.

c i 6 **ii** 70 **iii** 420

d 624

10 196

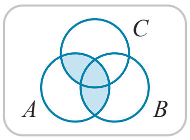
Chapter 10

Technology-free questions

3 a If n is odd, then $5n + 3$ is even.

c If n is even, then $5n + 3$ is odd.

10 b



11 a $\varnothing$ **b** ξ **c** $A \cup B$ **d** $A \cap B$

e $\varnothing$ **f** ξ **g** $\varnothing$ **h** ξ

i A **j** $A \cap B$ **k** $\varnothing$ **l** $A \cap B'$

12 a x **b** x **c** 0 **d** 1

e 1 **f** $x \wedge y$ **g** 0 **h** 1

i 0 **j** x' **k** $x \wedge y$ **l** 0

13 $(x' \wedge y') \vee (x \wedge y') = y'$

14 a $\neg A$ **b** $A \wedge B$ **c** $A \Rightarrow \neg B$

d $A \vee (\neg A \Rightarrow B)$ **e** $(A \wedge B) \vee (\neg A \wedge \neg B)$

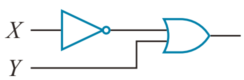
15 a i $P \wedge Q$ **ii** $P \Rightarrow Q$

b Yasmin is not in the school orchestra if and only if Yasmin does not play the violin.

17 a I was not paid. **b** $Q \Rightarrow P$

19 a $\neg X \vee (X \wedge Y)$ **b** $\neg X \vee Y$

c



20 a Not valid **b** Valid

21 Not valid

22 a $A = [1, 2, 4, 7, 10, 13, 16]$

b $A = [1, 2, 4, 8, 16]$

c Entries are consecutive powers of 2

23 a Evaluates $1^2 + 2^2 + \cdots + n^2$, i.e. the sum of the squares of the first n natural numbers

b $function(4) = 30$

c $function(5) = 55$

d

```
define function(n):
  product ← 1
  for i from 1 to n
    product ← product × i³
  end for
  return product
```

24 a 3

b Returns the index of the highest power of 2 that is a factor of n

c Inputs of the form $8m$, where m is an odd natural number

25 24

26 360

27 a 125 **b** 60

28 a 9 **b** 25

29 a 24 **b** 30 **c** 28 **d** 45

30 a 120 **b** 120

31 a 120 **b** 36

32 a 96 **b** 24 **c** 72 **d** 60

33 10

34 a 20 **b** 325 **c** 210 **d** 56

35 a 28 **b** 21 **c** $2^8 = 256$

36 60 **37** 120 **38** 7 **39** 51 **40** 80

Multiple-choice questions

1 E **2** E **3** D **4** C **5** E

6 E **7** B **8** B **9** A **10** B

11 E **12** C **13** D **14** A **15** B

16 B **17** A **18** E **19** E **20** B

21 A **22** E **23** E **24** D **25** A

26 C **27** C **28** A **29** B **30** D

31 D **32** D **33** A **34** E **35** B

Extended-response questions

1 a No

b Yes; both a and b are odd, and c is even

2 a $(a, b, c) = (2, 3, 6)$

b $(a, b, c, d) = (1, 2, 3, 4)$ or $(a, b, c, d) = (1, 2, 3, 5)$

4 a 10

5 a 49, 50, 51 and 52 **b** 93 and 94 **d** 44

6 b 21 coins **c** 10

8 a No **b** $n = 4k$ or $n = 4k - 1$

10 a $a = 1$, $b = 3$, $c = 1$ **c** 41^2

11 a

n		1	2	3	4	5	6	7	8	9	10
$tally$	0	1	3	0	4	9	3	10	2	11	1

b

```
input A
tally ← 0
for i from 1 to length(A)
  if tally − A[i] ≥ 0 then
    tally ← tally − A[i]
  else
    tally ← tally + A[i]
  end if
end for
print tally
```

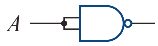
c One possibility: $[1, 4, 2, 3, 5, 8, 6, 7, 9, 10]$

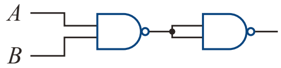
12 a

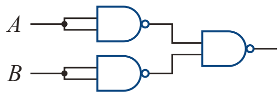
A	B	$\neg(A \vee B)$
0	0	1
0	1	0
1	0	0
1	1	0

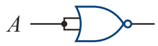
b

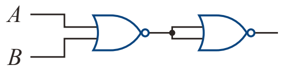
A	B	$\neg(A \wedge B)$
0	0	1
0	1	1
1	0	1
1	1	0

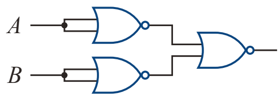
c **i**


ii


iii


d **i**


ii


iii


13 a 2160 **b** 360 **c** 900 **d** 1260

14 a 70 **b** 30 **c** 15 **d** 55

15 a 20 **b** 4 **c** 68

16 a 420 **b** 60 **c** 120 **d** 24

17 a 300 **b** 10 and 15

18 a

n	1	2	3	4	5
x_n	0.111	0.051	0.061	0.055	0.056
y_n	0.182	0.151	0.168	0.165	0.167

b $x = \frac{1}{18},\ y = \frac{1}{6}$

c

n	1	2	3	4	5
x_n	0.625	0.587	0.599	0.598	0.598
y_n	0.308	0.212	0.217	0.216	0.216

d **i** $x = -1, y = -1$

ii

n	1	2	3	4	5
x_n	0.500	1.250	2.375	4.063	6.594
y_n	0.500	1.250	2.375	4.063	6.594

19 a 495 **b** 60

c The two points diametrically opposite

d 15 **e** $\frac{1}{33}$

20 a **i** (0, 0) **ii** (1, 1) **iii** (0, 1)

iv (0, 1) **v** (0, 1) **vi** (0, 0)

b **i** (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)

ii (0, 0, 0), (1, 0, 1)

c 2^n

Investigations

See solutions supplement

Chapter 11

Exercise 11A

1 a 2×2 **b** 2×3 **c** 1×4 **d** 4×1

2 a $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

3 a $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

4 $\begin{bmatrix} 200 & 180 & 135 & 110 & 56 & 28 \\ 110 & 117 & 98 & 89 & 53 & 33 \end{bmatrix}$

5 a $\begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ if $x = 4$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$ if $x = 4$

c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$ if $x = 0, y = 2$

6 a $x = 2,\ y = 3$ **b** $x = 3,\ y = 2$

c $x = 4,\ y = -3$ **d** $x = 3,\ y = -2$

7 $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$

Exercise 11B

1 $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $2\mathbf{X} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $4\mathbf{Y} + \mathbf{X} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$

$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix}$

$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -7 & -7 \end{bmatrix}$

2 $2\mathbf{A} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ 0 & -6 \end{bmatrix}$

$-6\mathbf{A} = \begin{bmatrix} -6 & 6 \\ 0 & -12 \end{bmatrix}$

3 a Yes **b** Yes

4 a $\begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix}$

c $\begin{bmatrix} 6 & -5 \\ 8 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} -6 & -13 \\ 16 & 7 \end{bmatrix}$

5 a $\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$ **b** $\begin{bmatrix}-2 & 3\\6 & 3\end{bmatrix}$ **c** $\begin{bmatrix}3 & 3\\-1 & 7\end{bmatrix}$

6 $\mathbf{X} = \begin{bmatrix}2 & 4\\0 & -3\end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix}-\frac{9}{2} & -\frac{23}{2}\\-\frac{1}{2} & 11\end{bmatrix}$

7 $\mathbf{X}+\mathbf{Y} = \begin{bmatrix}310 & 180 & 220 & 90\\200 & 0 & 125 & 0\end{bmatrix}$
represents the total production at two factories in two successive weeks

Exercise 11C

1 $\mathbf{AX} = \begin{bmatrix}4\\-5\end{bmatrix}$ $\mathbf{BX} = \begin{bmatrix}4\\1\end{bmatrix}$ $\mathbf{AY} = \begin{bmatrix}-5\\8\end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix}2\\-1\end{bmatrix}$ $\mathbf{AC} = \begin{bmatrix}0 & -1\\1 & 2\end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$ $(\mathbf{AC})\mathbf{X} = \begin{bmatrix}1\\0\end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix}9\\5\end{bmatrix}$ $\mathbf{AI} = \begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$ $\mathbf{AB} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\mathbf{A}^2 = \begin{bmatrix}3 & -8\\-4 & 11\end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix}11 & 8\\4 & 3\end{bmatrix}$ $\mathbf{A}(\mathbf{CA}) = \begin{bmatrix}1 & -3\\-1 & 4\end{bmatrix}$

$\mathbf{A}^2\mathbf{C} = \begin{bmatrix}-2 & -5\\3 & 7\end{bmatrix}$

2 Defined: **AY**, **CI**;
Not defined: **YA**, **XY**, $\mathbf{X}^2$, **XI**

3 $\mathbf{AB} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

4 No

5 One possible answer is $\mathbf{A} = \begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}$

6 $\mathbf{LX} = \begin{bmatrix}7\end{bmatrix}$, $\mathbf{XL} = \begin{bmatrix}4 & -2\\-6 & 3\end{bmatrix}$

7 **AB** and **BA** are not defined unless $m = n$

8 b $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

9 One possible answer is
$\mathbf{A} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $\mathbf{B} = \begin{bmatrix}-2 & 1\\1.5 & -0.5\end{bmatrix}$

10 One possible answer is
$\mathbf{A} = \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}$, $\mathbf{B} = \begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$, $\mathbf{C} = \begin{bmatrix}-1 & 2\\-2 & 1\end{bmatrix}$,
$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{bmatrix}-1 & 11\\-4 & 24\end{bmatrix}$, $\mathbf{AB}+\mathbf{AC} = \begin{bmatrix}-1 & 11\\-4 & 24\end{bmatrix}$,
$(\mathbf{B}+\mathbf{C})\mathbf{A} = \begin{bmatrix}11 & 7\\16 & 12\end{bmatrix}$

11 For example: $\mathbf{A} = \begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$

12 a $\begin{bmatrix}29\\8.50\end{bmatrix}$, John took 29 minutes to eat food costing \$8.50

b $\begin{bmatrix}29 & 22 & 12\\8.50 & 8.00 & 3.00\end{bmatrix}$,
John's friends took 22 and 12 minutes to eat food costing \$8.00 and \$3.00 respectively

13 $\mathbf{A}^2 = \begin{bmatrix}-3 & 4\\-4 & -3\end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix}-7 & -24\\24 & -7\end{bmatrix}$,
$\mathbf{A}^8 = \begin{bmatrix}-527 & 336\\-336 & -527\end{bmatrix}$

14 $\mathbf{A}^2 = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}$,
$\mathbf{A}^n = \begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$

Exercise 11D

1 a 1 **b** $\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$ **c** 2 **d** $\frac{1}{2}\begin{bmatrix}2 & 2\\-3 & -2\end{bmatrix}$

2 a $\begin{bmatrix}-1 & 1\\-4 & 3\end{bmatrix}$ **b** $\begin{bmatrix}\frac{2}{7} & -\frac{1}{14}\\\frac{1}{7} & \frac{3}{14}\end{bmatrix}$

c $\begin{bmatrix}1 & 0\\0 & \frac{1}{k}\end{bmatrix}$ **d** $\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$

3 a $\mathbf{A}^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\0 & -1\end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix}5 & 1\\-3 & -1\end{bmatrix}$, $(\mathbf{AB})^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\-\frac{3}{2} & -\frac{5}{2}\end{bmatrix}$

c $\mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix}-1 & \frac{1}{2}\\3 & -1\end{bmatrix}$,
$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\-\frac{3}{2} & -\frac{5}{2}\end{bmatrix}$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

4 a $\begin{bmatrix}-\frac{1}{2} & \frac{3}{2}\\1 & -2\end{bmatrix}$ **b** $\begin{bmatrix}0 & 7\\1 & -8\end{bmatrix}$ **c** $\begin{bmatrix}\frac{5}{2} & -\frac{7}{2}\\\frac{11}{2} & -\frac{21}{2}\end{bmatrix}$

5 a $\begin{bmatrix}-\frac{3}{8} & \frac{11}{8}\\\frac{1}{16} & \frac{7}{16}\end{bmatrix}$ **b** $\begin{bmatrix}-\frac{11}{16} & \frac{17}{16}\\-\frac{1}{4} & \frac{3}{4}\end{bmatrix}$

8 $\begin{bmatrix}\frac{1}{a_{11}} & 0\\0 & \frac{1}{a_{22}}\end{bmatrix}$

10 $x \in \mathbb{R} \setminus \{-1, -\frac{1}{3}\}$

11 a $a = 3$

b $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix},$
$\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}, k \in \mathbb{R},$
$\begin{bmatrix} a & b \\ \dfrac{1-a^2}{b} & -a \end{bmatrix}, a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}$

12 $a = \pm\sqrt{2}$

Exercise 11E

1 a $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$ **b** $\begin{bmatrix} 5 \\ 17 \end{bmatrix}$

2 a $x = -\frac{1}{7},\ y = \frac{10}{7}$ **b** $x = 4,\ y = 1.5$
c $x = -6,\ y = 12$ **d** $x = -210,\ y = 231$

3 $(2, -1)$

4 Book $12, game $18

5 a $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

b $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ is non-invertible

c System has solutions (not a unique solution)

d Solution set contains infinitely many pairs

6 a $\mathbf{A}^{-1}\mathbf{C}$ **b** $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C}$ **c** $\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$
d $\mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$ **e** $\mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$
f $(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1}$

Exercise 11F

1 a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$ **b** $\begin{bmatrix} 1 & -2 & \frac{1}{5} \\ 0 & \frac{1}{2} & -\frac{3}{10} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$

3 $\mathbf{AB} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix};\quad \mathbf{A}^{-1} = \frac{1}{7}\mathbf{B}$

4 $\mathbf{A}^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix};\quad \mathbf{A}^{-1} = \frac{1}{9}\mathbf{A}$

5 $\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix};\quad \mathbf{A}^{-1} = \frac{1}{4}\mathbf{A}$

6 a $\begin{bmatrix} 2 & 1 & -10 \\ 3 & 2 & -17 \\ -5 & -3 & 28 \end{bmatrix}$

b $\frac{1}{29}\begin{bmatrix} 8 & -13 & 14 \\ 2 & 4 & -11 \\ -9 & 11 & 6 \end{bmatrix}$

c $\frac{1}{37}\begin{bmatrix} 6 & 4 & -7 & -17 \\ -13 & -21 & 46 & 43 \\ 8 & 30 & -34 & -35 \\ -4 & -15 & 17 & 36 \end{bmatrix}$

d $\frac{1}{37}\begin{bmatrix} 6 & -13 & 8 & -4 \\ 4 & -21 & 30 & -15 \\ -7 & 46 & -34 & 17 \\ -17 & 43 & -35 & 36 \end{bmatrix}$

7 a -36 **b** 1

8 a i -2 **ii** -2
b i -4 **ii** -16

9 a $\det(\mathbf{A}) = -2p + 6$ **b** $p = 3$

10 a $\det(\mathbf{A}) = -2(p-2)(p-1)$
b $p = 2$ or $p = 1$

Exercise 11G

1 a $x = 2, y = 3, z = 1$ **b** $x = -3, y = 5, z = 2$
c $x = 5, y = 0, z = 7$ **d** $x = 6, y = 5, z = 1$
e $x = 5, y = 2, z = 4, w = -1$

2 a $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \\ -1 & -4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 17 \end{bmatrix}$

b $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is non-invertible

c i $-y + 5z = 15,\ -y + 5z = 15$
ii The two equations are the same
iii $y = 5\lambda - 15$
iv $x = 43 - 13\lambda$

Chapter 11 review

Technology-free questions

1 a $\begin{bmatrix} 0 & 0 \\ 12 & 8 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 \\ 8 & 8 \end{bmatrix}$

2 $\begin{bmatrix} a \\ 2 - \frac{3}{4}a \end{bmatrix}, a \in \mathbb{R}$

3 a Exist: **AC**, **CD**, **BE**; Does not exist: **AB**

b $\mathbf{DA} = \begin{bmatrix} 14 & 0 \end{bmatrix},\quad \mathbf{A}^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

4 $\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix},\quad \mathbf{C}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5 $\begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$

6 $\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix},\quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

7 $x = 8$

8 a i $\begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$ **iii** $\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

b $x = 2,\ y = 1$

Multiple-choice questions

1 B **2** E **3** C **4** E **5** C
6 A **7** E **8** A **9** E **10** D

Extended-response questions

1 a i $\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

ii $\det(\mathbf{A}) = 14,\ \mathbf{A}^{-1} = \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

iii $\frac{1}{7}\begin{bmatrix} 9 \\ -1 \end{bmatrix}$

iv Two lines intersect at point $\left(\frac{9}{7}, -\frac{1}{7}\right)$

b i $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

ii $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is non-invertible

c Two parallel lines

2 a $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$ **b** $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

c Semester 1: 79.2; Semester 2: 80.4

d Semester 1: 83.8; Semester 2: 75.2

e No, total score is 318.6

f 3 marks

3 a $\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix}$ **b** $\begin{bmatrix} 70 \\ 60 \end{bmatrix}$

c Term 1: \$820; Term 2: \$800; Term 3: \$1040; Term 4: \$1020

d $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix}$

f Term 1: \$270; Term 2: \$270; Term 3: \$480; Term 4: \$480

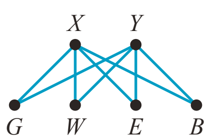
g Term 1: \$1090; Term 2: \$1070; Term 3: \$1520; Term 4: \$1500

4 Brad 20; Flynn 10; Lina 15

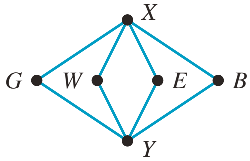
Chapter 12

Exercise 12A

1 a



b



c

$$\begin{array}{c} \\ X \\ Y \\ G \\ W \\ E \\ B \end{array}\begin{array}{c} \begin{matrix} X & Y & G & W & E & B \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

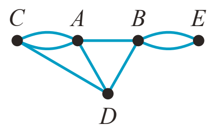
2 a i 3 **ii** 2 **iii** 1

b

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ H \end{array}\begin{array}{c} \begin{matrix} A & B & C & D & H \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

c Not simple, as two edges join C and D

3 a

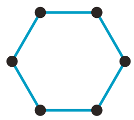


b

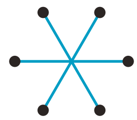
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

c Two edges join A and C, and also B and E

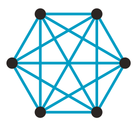
4 a



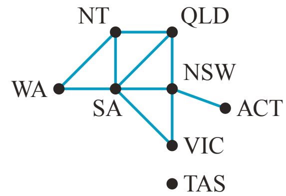
b



c



5



6 a

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

c

	A	B	C	D
A	0	1	0	0
B	1	0	0	0
C	0	0	0	1
D	0	0	1	0

d

	A
A	1

e

	A	B	C	D
A	0	1	1	1
B	1	0	1	1
C	1	1	0	1
D	1	1	1	0

f

	A	B	C	D	E	F
A	0	1	1	0	0	0
B	1	0	0	1	0	0
C	1	0	0	1	0	0
D	0	1	1	0	0	0
E	0	0	0	0	0	1
F	0	0	0	0	1	0

g

	A	B	C
A	0	0	0
B	0	1	1
C	0	1	1

h

	A	B	C	D
A	0	1	1	1
B	1	0	1	0
C	1	1	0	1
D	1	0	1	0

7 a, b, c, e, f, h

8 a Loop at vertex v_1

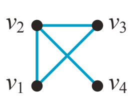
b Two edges join vertices v_1 and v_2

9 a The degree of vertex v_i is the sum of the entries in row i (or column i)

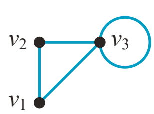
b Sum of all the entries

c Half the sum of all of the entries

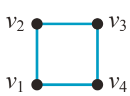
10 a



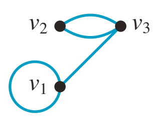
b



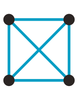
c



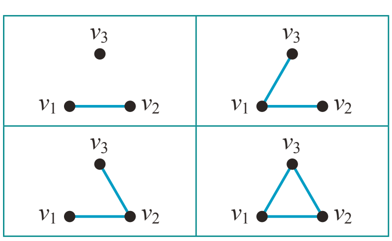
d



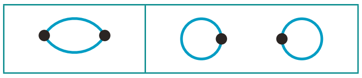
12



13

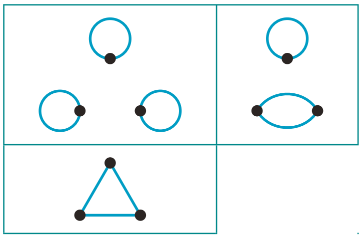


15 a

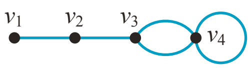


b The total degree of the graph would be 9, but the total degree must be even.

c

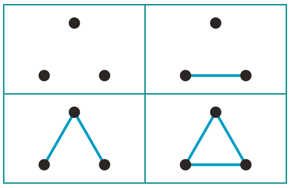


16 a

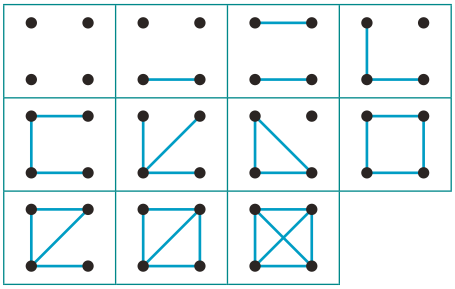


b The total degree of the graph would be 7, but the total degree must be even.

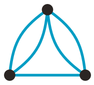
17 Four different graphs:



18



20 a For example:



Exercise 12B

1 a
- v_1, v_2, v_3, v_4, v_2
- v_1, v_2, v_4, v_3, v_2
- v_2, v_3, v_4, v_2, v_1
- v_2, v_4, v_3, v_2, v_1

b It has vertices of odd degree (v_1 and v_2).

2 a It has vertices of odd degree (v_1 and v_3).

b Exactly two vertices have odd degree.

3 Many possible answers. For example:

a $v_3, v_4, v_1, v_3, v_2, v_1$

b $v_1, v_2, v_3, v_4, v_1, v_5, v_4, v_2, v_5$

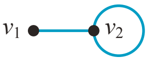
c $v_2, v_3, v_4, v_1, v_2, v_6, v_7, v_8, v_5, v_6, v_8, v_4$

d $v_1, v_3, v_5, v_4, v_2, v_3, v_5, v_6$

e $v_1, v_2, v_3, v_4, v_5, v_3, v_1$

f $v_5, v_1, v_2, v_3, v_4, v_6, v_5, v_2, v_4, v_5$

4
- Cannot have an Euler circuit, as it has a vertex of odd degree.
- Can have an Euler trail. For example, the following graph has Euler trail v_1, v_2, v_2.

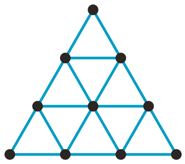


5 a Yes **b** No **c** Yes

6 Yes

7 a One edge **b** Two edges

8 a



9 a $1 \times n$ for $n \in \mathbb{N}$, $m \times 1$ for $m \in \mathbb{N}$, 2×2, 2×3, 3×2

b 1×1, 2×2

Exercise 12C

Note: *Questions 1–3 have many correct answers.*

1 a $v_1, v_2, v_3, v_8, v_7, v_6, v_5, v_4$

b $v_6, v_1, v_4, v_5, v_8, v_3, v_2, v_7$

2 a v_1, v_2, v_3, v_4 **b** $v_3, v_5, v_6, v_4, v_1, v_2$

c v_1, v_3, v_2, v_4 **d** $v_1, v_2, v_3, v_4, v_5, v_6, v_7$

3 a v_1, v_4, v_3, v_2, v_1 **b** $v_1, v_2, v_3, v_5, v_6, v_4, v_1$

c $v_1, v_5, v_2, v_3, v_4, v_1$

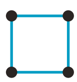
d $v_1, v_2, v_5, v_7, v_6, v_4, v_3, v_1$

4 b Two: v_1, v_4, v_2, v_5, v_3 and v_3, v_5, v_2, v_4, v_1

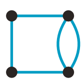
c $\{v_1, v_3\}$

5 b 8

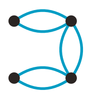
6 a



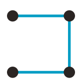
b



c



d

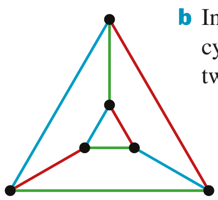


7 a False **b** True **c** True **d** False **e** True

8 a Yes, as there is a Hamiltonian cycle: $I, H, D, G, B, E, F, A, C, I$

b Yes, as there is a Hamiltonian path: $A, G, E, D, B, F, I, C, H$

9 a



b In each Hamiltonian cycle, the edges have two alternating colours.

Exercise 12D

1 a i 4 **ii** 0 **iii** 5 **iv** 2

b i 9 **ii** 9 **iii** 8 **iv** 2

2 a

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$$

b i 2 **ii** 13 **iii** 6

c 18

d 14

e i 42 **ii** 12 **iii** 14

f 36

3 a

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

4 a

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b

$$\mathbf{A}^2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 4 & 3 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 3 & 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{bmatrix} \end{array}$$

c

$$\mathbf{A}^3 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 12 & 13 & 13 & 13 & 13 \\ 13 & 12 & 13 & 13 & 13 \\ 13 & 13 & 12 & 13 & 13 \\ 13 & 13 & 13 & 12 & 13 \\ 13 & 13 & 13 & 13 & 12 \end{bmatrix} \end{array}$$

5 a

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b 0 **c** 0 **d** All 0

6 a

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

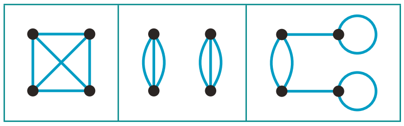
b

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 105 & 104 & 76 \\ 104 & 105 & 76 \\ 76 & 76 & 57 \end{bmatrix} \end{array}$$

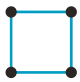
c 104

Exercise 12E

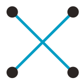
1 Many possible answers. For example:



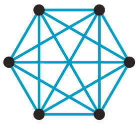
2 C_4



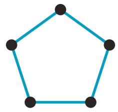
$\overline{C_4}$



3 a 15 **b**

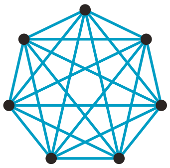


4 a

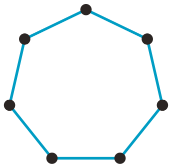


b The total degree of the corresponding graph would be $5 \times 3 = 15$, but the total degree must be even.

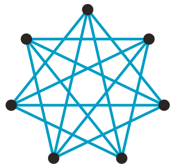
5 a 21 **b**



c

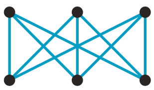


C_7

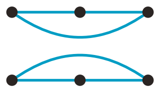


$\overline{C_7}$

6 a



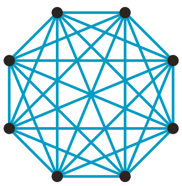
$K_{3,3}$



$\overline{K_{3,3}}$

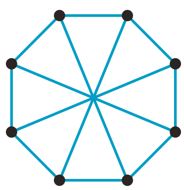
b No; the complement of $K_{3,3}$ is not bipartite.

7 28

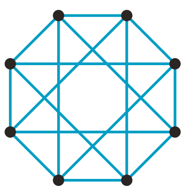


8 Many possible answers. For example:

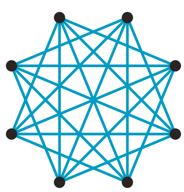
a



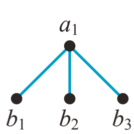
b



c

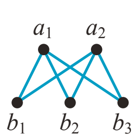


9 a



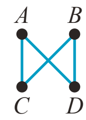
	a_1	b_1	b_2	b_3
a_1	0	1	1	1
b_1	1	0	0	0
b_2	1	0	0	0
b_3	1	0	0	0

b

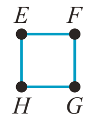


	a_1	a_2	b_1	b_2	b_3
a_1	0	0	1	1	1
a_2	0	0	1	1	1
b_1	1	1	0	0	0
b_2	1	1	0	0	0
b_3	1	1	0	0	0

10



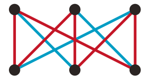
$K_{2,2}$



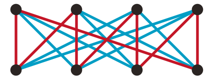
C_4

One isomorphism is $A \leftrightarrow E$, $D \leftrightarrow F$, $B \leftrightarrow G$, $C \leftrightarrow H$

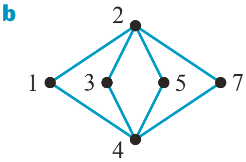
11 a



b



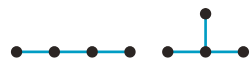
13 b



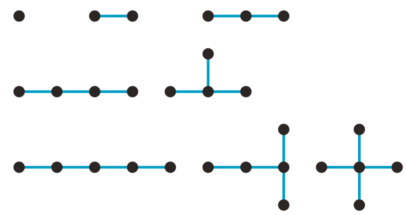
Euler circuit: 1, 2, 3, 4, 5, 2, 7, 4, 1

Exercise 12F

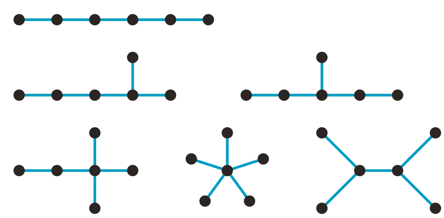
1



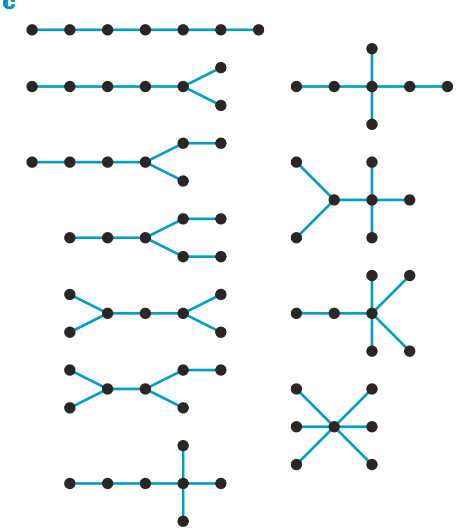
2 a



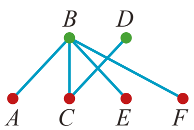
b



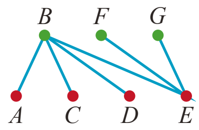
c



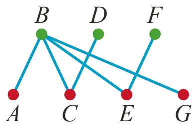
3 a



b

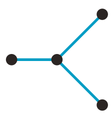


c

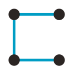


4 Many possible answers. For example:

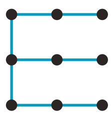
a



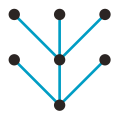
b



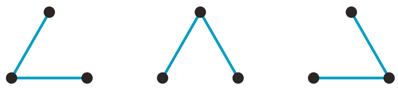
c



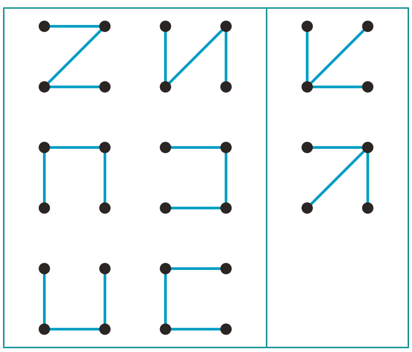
d



5 a Each spanning tree is a path of length 2:



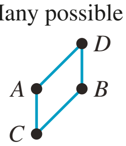
b



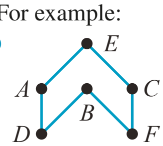
Exercise 12G

1 Many possible answers. For example:

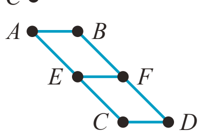
a



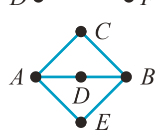
b



c



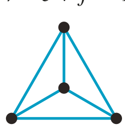
d



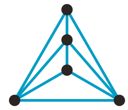
2 a $v - e + f = 8 - 12 + 6 = 2$
b $v - e + f = 6 - 12 + 8 = 2$
c $v - e + f = 7 - 12 + 7 = 2$
d $v - e + f = 7 - 9 + 4 = 2$

4 a $v - e + f = 12 - 17 + 7 = 2$

5 a



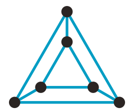
6 a



7 a ■ Cube: $v - e + f = 8 - 12 + 6 = 2$
■ Tetrahedron: $v - e + f = 4 - 6 + 4 = 2$

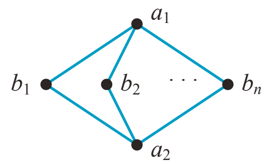
b 20 vertices **c** 30 edges

8 a



b $v - e + f = 6 - 9 + 5 = 2$

9



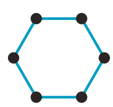
Exercise 12H

See solutions supplement

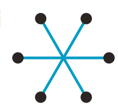
Chapter 12 review

Technology-free questions

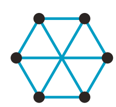
1 a i



ii



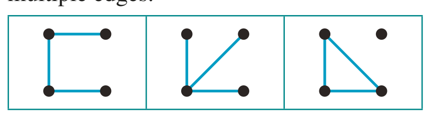
iii



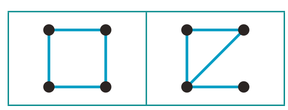
b The total degree of the corresponding graph would be $7 \times 3 = 21$, but the total degree must be even.

2 a A simple graph is a graph with no loops or multiple edges.

b



c

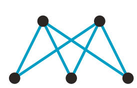


3 d 20

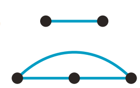
4 a 2 **b** 0 **c**

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

5 a

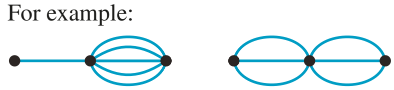


b

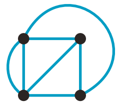


6 a $v = 3,\ e = 6,\ f = 5$

b For example:

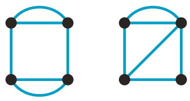


7 a For example:



b $v - e + f = 4 - 7 + 5 = 2$

8 a For example:

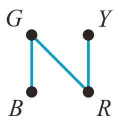


Multiple-choice questions

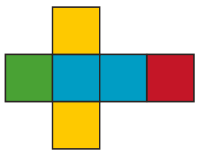
1 C **2** D **3** D **4** E **5** E
6 B **7** B **8** A **9** B **10** A
11 D **12** D **13** C **14** D

Extended-response questions

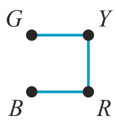
2 a



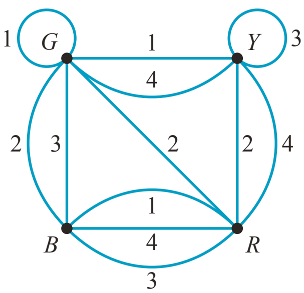
b



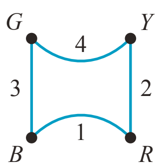
c



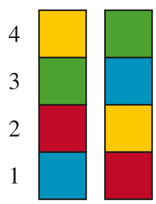
d



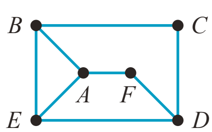
f



g

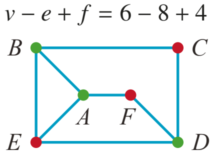


3 a



b $v - e + f = 6 - 8 + 4 = 2$

d



f i Edge joining A and B

ii Red $\{C, E, F\}$, green $\{A, B, D\}$

Chapter 13

Technology-free questions

1 a All defined except **AB**

b $\mathbf{DA} = \begin{bmatrix} 6 & -12 \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$

2 a $\begin{bmatrix} -2 & 4 \\ 18 & -24 \end{bmatrix}$ **b** $\begin{bmatrix} -10 & -19 \\ 7 & -16 \end{bmatrix}$

3 $x = 16$

4 $\mathbf{A} = \begin{bmatrix} t \\ 3t - 5 \end{bmatrix}$, $t \in \mathbb{R}$

5 $\mathbf{AB} = \begin{bmatrix} -9 & -8 \\ -15 & 10 \end{bmatrix}$, $\mathbf{C}^{-1} = \begin{bmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

6 a $(\mathbf{C} - \mathbf{B})\mathbf{A}^{-1}$ **b** $\mathbf{B}^{-1}\mathbf{C} - \mathbf{A}$
c $\mathbf{I} - \mathbf{A}^{-1}\mathbf{BA}$ **d** $-\mathbf{A}$
e $\frac{1}{2}\mathbf{B}$ **f** $\mathbf{I} - \mathbf{A}^{-1}$

7 a $\begin{bmatrix} -1 \\ -10 \end{bmatrix}$ **b** $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & 24 \\ 7 & 4 \end{bmatrix}$

8 $w = -5$ or $w = 2$

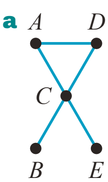
9 $x = 3$

10 $a = 0$, $b = -1$

12 a $\begin{bmatrix} 1 - 2b & 0 \\ ab & 2a \end{bmatrix}$ **b** $a = \frac{1}{2}$, $b = 0$

13 $(0, 0, 0)$, $(1, 0, -1)$

14 a

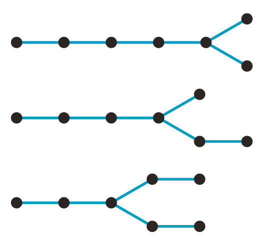


b

	A	B	C	D	E
A	0	0	1	1	0
B	0	0	1	0	0
C	1	1	0	1	1
D	1	0	1	0	0
E	0	0	1	0	0

15 Two vertices of degree 2 and one vertex of degree 3

16



17 6 vertices

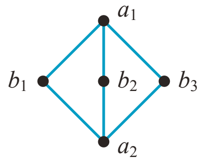
18 a $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **b** 7 walks

19 b $\frac{(n-1)!}{2}$

20 No

21 No

22



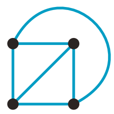
23 8 faces

25 a $v_1, v_3, v_5, v_2, v_4, v_1$

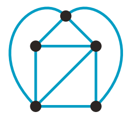
b $v_1, v_3, v_5, v_2, v_6, v_4, v_1$

26 For example:

a

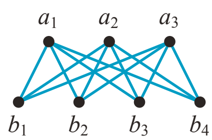


b



27 20 vertices, 30 edges, 12 faces

28 a



b Four vertices have odd degree (b_1, b_2, b_3, b_4)

c One edge (for example: $\{b_1, b_2\}$)

d Two edges (for example: $\{b_1, b_2\}$, $\{b_3, b_4\}$)

Multiple-choice questions

1 A	**2** B	**3** E	**4** A	**5** B
6 C	**7** A	**8** B	**9** D	**10** C
11 D	**12** B	**13** B	**14** D	**15** D
16 E	**17** A	**18** B	**19** E	

Extended-response questions

1 a i $\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & d^2+bc \end{bmatrix}$ **ii** $\begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$

3 g $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ k & 1 \end{bmatrix}, k \in \mathbb{R},$

$\begin{bmatrix} a & b \\ \dfrac{a-a^2}{b} & 1-a \end{bmatrix}, a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}$

4 c $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix},$

$\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}, k \in \mathbb{R},$

$\begin{bmatrix} a & b \\ \dfrac{1-a^2}{b} & -a \end{bmatrix}, a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}$

5 a $m^2 - (a+d)m + ad - bc$

c $m = a - c$

d i $m = 1$ or $m = -9$ **ii** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}, k \in \mathbb{R}$

iii $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -\dfrac{3k}{2} \end{bmatrix}, k \in \mathbb{R}$

6 a No, as the total degree of the graph would be 15, but the total degree must be even.

b i One vertex has degree 6 and there are only five other vertices, so the graph must have a loop or multiple edges.

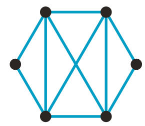
ii The graph has an Euler circuit as all vertices have even degree.

c i 10, which is the total degree of a tree with 6 vertices

ii 2, 4

d i Vertex degrees 2, 2, 4, 4, 4, 4

ii



Investigations

See solutions supplement

Chapter 14

Exercise 14A

1 a $k = \dfrac{1}{15}$ **b** $\Pr(X \geq 3) = \dfrac{4}{5}$

2 a 4.6 **b** 0.5 **c** 1.89 **d** 0

3 \$6000

4 A loss of 33c

5 a $\text{Var}(X) = 3.84$ **b** $\text{Var}(X) = 1.25$

6 a $p = \dfrac{1}{16}$ **b** $\text{E}(X) = 1.625$

c $\text{Var}(X) = 0.9844$ **d** $\text{sd}(X) = 0.9922$

7 a $k = \dfrac{1}{10}$ **b** $\text{E}(X) = 1$

c $\text{Var}(X) = 1$ **d** $\text{sd}(X) = 1$

Exercise 14B

1 a

z	−4	−2	0	48	50	100
$p(z)$	0.64	0.24	0.0225	0.08	0.015	0.0025

b $\Pr(X_1 + X_2 \geq 50) = 0.0175$

2 a

z	2	3	4	5	6	7
$p(z)$	0.04	0.08	0.12	0.16	0.16	0.16

z	8	9	10	11	12
$p(z)$	0.12	0.08	0.05	0.02	0.01

b $\Pr(X_1 + X_2 > 10) = 0.03$

3 a $\text{E}(X) = 3$, $\text{Var}(X) = 2$

b

z	2	3	4	5	6
$p(z)$	0.04	0.08	0.12	0.16	0.2

z	7	8	9	10
$p(z)$	0.16	0.12	0.08	0.04

c $\Pr(X_1 + X_2 \text{ is even}) = 0.52$

d $\text{E}(X_1 + X_2) = 6$, $\text{Var}(X_1 + X_2) = 4$

4 a $E(X) = -3.85$, $Var(X) = 33.428$, $sd(X) = 5.782$

b

z	−10	−5	0	5	10
$p(z)$	0.81	0.108	0.0576	0.0036	0.0009

z	45	50	55	100
$p(z)$	0.018	0.0012	0.0006	0.0001

c $E(X_1 + X_2) = -7.70$, $Var(X_1 + X_2) = 66.855$, $sd(X_1 + X_2) = 8.176$

5 a $E(X_1 + X_2 + X_3 + X_4) = 400$
b $Var(X_1 + X_2 + X_3 + X_4) = 64$
c $sd(X_1 + X_2 + X_3 + X_4) = 8$

6 a $E(X_1 + X_2 + X_3) = 90$
b $Var(X_1 + X_2 + X_3) = 21$
c $sd(X_1 + X_2 + X_3) = 4.583$

7 $E(X_1 + X_2 + X_3) = -11.55$, $Var(X_1 + X_2 + X_3) = 100.283$, $sd(X_1 + X_2 + X_3) = 10.014$

8 a i $E(4X) = 400$ **ii** $Var(4X) = 256$
iii $sd(4X) = 16$
b The means are the same, but $4X$ has a much higher variability than $X_1 + X_2 + X_3 + X_4$

9 a $E(10X) = 34$ **b** $Var(10X) = 120$
c $sd(10X) = 10.954$

10 a $E(X) = 0.63$, $Var(X) = 0.513$, $sd(X) = 0.716$
b $E(X_1 + \cdots + X_{10}) = 6.3$, $Var(X_1 + \cdots + X_{10}) = 5.131$, $sd(X_1 + \cdots + X_{10}) = 2.265$
c $E(40X) = 25.2$, $Var(40X) = 820.8$, $sd(40X) = 28.650$

Exercise 14C

1 No, as students who do not use email will not be included in the sample.

2 No, as customers who use the restaurant on weekdays or at other times on the weekend will not be included in the sample.

3 a Yes, as every student in the school has the same probability of being included in the sample.
b $\bar{x} = 2.7$

4 Answers will vary

5 a All Australian adults **b** $\mu = 4$ **c** $\bar{x} = 3.5$

Exercise 14D

1 45.6 minutes per day

2 $3130

3 a $\Pr(\bar{X} \geq 25) \approx 0.02$ **b** $\Pr(\bar{X} \leq 23) \approx 0.01$

4 a $\Pr(\bar{X} \geq 163) \approx 0.04$ **b** $\Pr(\bar{X} \leq 158) \approx 0.05$

5 Answers will vary

6 Answers will vary

7 a $E(X_1 + \cdots + X_{25}) = -27.5$, $Var(X_1 + \cdots + X_{25}) = 1427.25$
b $E(\bar{X}) = -1.10$, $Var(\bar{X}) = 2.284$

8 a $E(X_1 + \cdots + X_{10}) = 6.3$, $Var(X_1 + \cdots + X_{10}) = 5.131$
b $E(\bar{X}) = 0.63$, $Var(\bar{X}) = 0.0513$

9 a $E(\bar{X}) = 30$, $sd(\bar{X}) = 1.4$
b $E(\bar{X}) = 30$, $sd(\bar{X}) = 0.14$
c $E(\bar{X}) = 30$, $sd(\bar{X}) = 0.014$

10 a $E(\bar{X}) = 16.77$, $sd(\bar{X}) = 0.775$
b $E(\bar{X}) = 16.77$, $sd(\bar{X}) = 0.245$
c $E(\bar{X}) = 16.77$, $sd(\bar{X}) = 0.0775$

11 a $E(P) = -2$, $sd(P) = 8.718$
b i $E(\bar{P}) = -2$, $sd(\bar{P}) = 2.757$
ii $E(\bar{P}) = -2$, $sd(\bar{P}) = 0.872$
iii $E(\bar{P}) = -2$, $sd(\bar{P}) = 0.276$

Chapter 14 review

Technology-free questions

1 a $\frac{1}{4}$ **b** $\frac{5}{2}$ **c** $\frac{5}{4}$

2 a $\frac{1}{10}$ **b** $\frac{6}{5}$ **c** $\frac{39}{25}$

3 a $E(X_1 + X_2 + X_3 + X_4) = 200$
b $Var(X_1 + X_2 + X_3 + X_4) = 100$
c $sd(X_1 + X_2 + X_3 + X_4) = 10$

4 a $E(10X) = 300$
b $Var(10X) = 1600$
c $sd(10X) = 40$

5 No, as it is likely that students who are not interested in yoga will not respond to the survey.

6 a People with Type II diabetes
b Population is too large and dispersed
c Unknown **d** $\bar{x} = 1.5$

7 1.6 m

8 a $E(\bar{X}) = 10$, $sd(\bar{X}) = \frac{2}{3}$
b $E(\bar{X}) = 10$, $sd(\bar{X}) = \frac{2}{5}$
c $E(\bar{X}) = 10$, $sd(\bar{X}) = \frac{1}{5}$

Multiple-choice questions

1 C **2** E **3** D **4** E **5** D
6 C **7** E **8** B **9** C **10** A
11 B **12** E **13** C

Extended-response questions

1 Answers will vary

2 a iii mean ≈ 50, s.d. ≈ 1.12
b iii mean ≈ 50, s.d. ≈ 0.71
c iii mean ≈ 50, s.d. ≈ 0.50

3 **a** $E(X) = 1.2$, $Var(X) = 0.36$, $sd(X) = 0.6$

b **i**

z	0	1	2	3	4
$p(z)$	0.01	0.12	0.42	0.36	0.09

ii $Pr(X_1 + X_2 > 3) = 0.09$

iii $E(X_1 + X_2) = 2.4$, $Var(X_1 + X_2) = 0.72$

iv $E(\bar{X}) = 1.2$, $Var(\bar{X}) = 0.18$

c **i** $E(X_1 + \cdots + X_7) = 8.4$, $Var(X_1 + \cdots + X_7) = 2.52$

ii $E(\bar{X}) = 1.2$, $Var(\bar{X}) = 0.051$

Chapter 15

Exercise 15A

1 **a** 4.10 **b** 0.87 **c** 2.94 **d** 4.08 **e** 33.69° **f** 11.92

2 $\dfrac{40\sqrt{3}}{3}$ cm

3 66.42°, 66.42°, 47.16°

4 23 m

5 **a** 9.59° **b** $\sqrt{35}$ m

6 **a** 60° **b** 17.32 m

7 **a** 6.84 m **b** 6.15 m

8 12.51°

9 182.7 m

10 1451 m

11 **a** $5\sqrt{2}$ cm **b** 90°

12 3.07 cm

13 37.8 cm

14 31.24 m

15 4.38 m

16 57.74 m

17 $\dfrac{2\sqrt{3}}{2-\sqrt{3}} \approx 12.93$ m

18 $\dfrac{10}{1+\sqrt{3}} \approx 3.66$

19 $\angle APB = 47.16°$

Exercise 15B

1 **a** 8.15 **b** 3.98 **c** 11.75 **d** 9.46

2 **a** 56.32° **b** 36.22° **c** 49.54° **d** 98.16° or 5.84°

3 **a** $A = 48°$, $b = 13.84$, $c = 15.44$

b $a = 7.26$, $C = 56.45°$, $c = 6.26$

c $B = 19.8°$, $b = 4.66$, $c = 8.27$

d $C = 117°$, $b = 24.68$, $c = 34.21$

e $C = 30°$, $a = 5.40$, $c = 15.56$

4 **a** $B = 59.12°$, $A = 72.63°$, $a = 19.57$ or $B = 120.88°$, $A = 10.87°$, $a = 3.87$

b $C = 26.69°$, $A = 24.31°$, $a = 4.18$

c $B = 55.77°$, $C = 95.88°$, $c = 17.81$ or $B = 124.23°$, $C = 27.42°$, $c = 8.24$

5 554.26 m

6 35.64 m

7 1659.86 m

8 **a** 26.60 m **b** 75.12 m

Exercise 15C

1 5.93 cm

2 $\angle ABC = 97.90°$, $\angle ACB = 52.41°$

3 **a** 26 **b** 11.74 **c** 49.29° **d** 73 **e** 68.70 **f** 47.22° **g** 7.59 **h** 38.05°

4 2.626 km

5 3.23 km

6 55.93 cm

7 **a** 8.23 cm **b** 3.77 cm

8 **a** 7.326 cm **b** 5.53 cm

9 **a** 83.62° **b** 64.46°

10 **a** 87.61 m **b** 67.7 m

Exercise 15D

1 **a** 11.28 cm^2 **b** 15.10 cm^2 **c** 10.99 cm^2 **d** 9.58 cm^2

2 **a** 6.267 cm^2 **b** 15.754 cm^2 **c** 19.015 cm^2 **d** 13.274 cm^2

e 24.105 cm^2 or 29.401 cm^2

f 2.069 cm^2

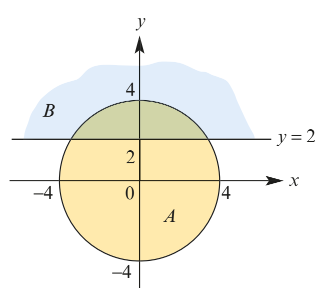
Exercise 15E

1 45.81 cm

2 **a** 95.5° **b** 112.88°

3 **a** 6.20 cm **b** 2.73 cm^2

4



Area of $A \cap B = 9.83$ square units

5 61.42 cm^2

6 **a** 125.66 m **b** 41.96%

7 **a** 10.47 m **b** 20.94 m^2

8 6.64 cm^2

9 $r = 7$ cm, $\theta = \left(\dfrac{18}{7}\right)^c$ or $r = 9$ cm, $\theta = \left(\dfrac{14}{9}\right)^c$

10 247.33 cm

11 **a** 81.96 cm **b** 4.03 cm^2

Exercise 15F

1 400.10 m

2 34.77 m

3 575.18 m

4 109.90 m

5 16.51 m

6 056°

7 **a** 034° **b** 214°

8 **a** 3583.04 m **b** 353°

9 027°

10 113°

11 22.01°

12 a $\angle BAC = 49°$ **b** 264.24 km

13 10.63 km

Exercise 15G

1 a 13 cm **b** 15.26 cm **c** 31.61° **d** 38.17°

2 a 4 cm **b** 71.57° **c** 12.65 cm **d** 13.27 cm **e** 72.45° **f** 266.39 cm^2

3 10.31° at B; 14.43° at A and C

4 a 85 m **b** 45.04 m

5 17.58°

6 1702.55 m

7 a 24.78° **b** 65.22° **c** 20.44°

8 42.40 m

9 1945.54 m

10 a 6.96 cm **b** 16.25 cm^2

11 a 5 km **b** 215.65° **c** 6.56°

Exercise 15H

1 a $4a^2$, $3a^2$ and $12a^2$ square units respectively **b** 14.04° **c** 18.43° **d** 11.31°

2 a 35.26° **b** 45°

3 a 0.28 **b** 15.78°

4 a 15.51 cm **b** 20 cm **c** 45.64°

5 a i 107 m **ii** 87 m **iii** 138 m **b** 43.00°

6 a $5\sqrt{11}$ cm **b** 64.76° **c** 71.57° **d** 95.74°

7 26.57°

8 a 54.74° **b** 70.53°

9 1.67 km

10 a 141.42 m **b** 20.70°

11 16 cm

12 34.14 cm

13 a $\dfrac{a\sqrt{3}}{2}$ cm **b** $\dfrac{a}{2}$ cm

14 a 26.57° **b** 39.81° **c** 38.66°

Chapter 15 review

Technology-free questions

1 a $5\sqrt{3} \pm \sqrt{11}$ **b** $\sin^{-1}\left(\frac{5}{6}\right)$ or $\pi - \sin^{-1}\left(\frac{5}{6}\right)$

2 a $\angle ABC = 30°$, $\angle ACB = 120°$ **b** $40\sqrt{3}$ cm **c** 20 cm

3 $4\sqrt{19}$ km

4 a $5\sqrt{3}$ cm **b** $\dfrac{25\sqrt{3}}{4}$ cm^2 **c** $\dfrac{105}{4}$ cm^2 **d** $\dfrac{5(21+5\sqrt{3})}{4}$ cm^2

5 143° **6** $\dfrac{17}{28}$

7 $\dfrac{3\sqrt{93}}{31}$ **8** $\left(\dfrac{11}{6}\right)^c$

9 a i 30° **ii** 15° **b** $AT = 300(1+\sqrt{3})$ m, $BT = 150(\sqrt{6}+\sqrt{2})$ m

10 $\sqrt{181}$ km

11 a $AC = \dfrac{12\sqrt{3}}{5}$ km, $BC = 2.4$ km **b** 57.6 km/h

12 180 cm^2

13 21.4 cm

14 a $26\tan^{-1}\left(\dfrac{12}{5}\right)$ cm **b** $169\left(\pi - \tan^{-1}\left(\dfrac{12}{5}\right)\right)$ cm^2

15 11 m

Multiple-choice questions

1 D **2** C **3** C **4** B **5** A
6 A **7** D **8** B **9** C **10** A

Extended-response questions

1 a $\angle ACB = 12°$, $\angle CBO = 53°$, $\angle CBA = 127°$ **b** 189.33 m **c** 113.94 m

2 a 4.77 cm **b** 180 cm^2 **c** 9.55 cm

3 a $\angle TAB = 3°$, $\angle ABT = 97°$, $\angle ATB = 80°$ **b** 2069.87 m **c** 252.25 m

4 a 184.78 m **b** 199.71 m **c** 14.93 m

5 a 370.17 m **b** 287.94 m **c** 185.08 m

6 a $8\sqrt{2}$ cm **b** 10 cm **c** 10 cm **d** 68.90°

7 Area $= \dfrac{L^2 \sin\alpha \sin\beta \sin\gamma}{2(\sin\alpha + \sin\beta + \sin\gamma)^2}$

Chapter 16

Exercise 16A

1 a -1 **b** $-\sqrt{2}$ **c** $-\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$ **d** 1 **e** -2 **f** 2 **g** $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ **h** 2

2 a -1 **b** $-\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$ **c** 1 **d** $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ **e** $-\sqrt{2}$ **f** $\dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ **g** -1 **h** $-\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$ **i** $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

3 a $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$ **b** $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$ **c** $\dfrac{3\pi}{4}, \dfrac{5\pi}{4}$ **d** $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

4 a $-\dfrac{8}{17}$ **b** $\dfrac{15}{17}$ **c** $-\dfrac{15}{8}$

5 $\cos\theta = \dfrac{24}{25}$, $\sin\theta = -\dfrac{7}{25}$

6 $-\frac{\sqrt{29}}{5}$

7 $\frac{8}{31}$

8 $\frac{15}{4(9+\sqrt{5})} = \frac{15(6-\sqrt{5})}{124}$

Exercise 16B

1 a $\frac{\sqrt{2}+\sqrt{6}}{4}$ **b** $\frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

2 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

3 a $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

c $\frac{1-\sqrt{3}}{1+\sqrt{3}} = -2+\sqrt{3}$

4 For $x, y \in \left(0, \frac{\pi}{2}\right)$, $\sin(x+y) = \frac{63}{65}$;

For $x, y \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(x+y) = -\frac{63}{65}$;

For $x \in \left(0, \frac{\pi}{2}\right)$, $y \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(x+y) = -\frac{33}{65}$;

For $x \in \left(\frac{\pi}{2}, \pi\right)$, $y \in \left(0, \frac{\pi}{2}\right)$, $\sin(x+y) = \frac{33}{65}$

5 a $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$ **b** $\frac{1}{\sqrt{2}}(\cos\varphi + \sin\varphi)$

c $\frac{\tan\theta + \sqrt{3}}{1-\sqrt{3}\tan\theta}$ **d** $\frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

6 a $\sin u$ **b** $\cos u$

7 a $-\frac{119}{169}$ **b** $\frac{24}{25}$ **c** $\frac{24}{7}$ **d** $-\frac{169}{119}$

e $-\frac{33}{65}$ **f** $-\frac{16}{65}$ **g** $-\frac{65}{33}$ **h** $\frac{7}{24}$

8 a $\frac{63}{16}$ **b** $-\frac{24}{7}$ **c** $\frac{56}{65}$ **d** $\frac{24}{25}$

9 a $\frac{7}{25}$ **b** $\frac{3}{5}$ **c** $\frac{117}{44}$ **d** $-\frac{336}{625}$

10 a $-\frac{\sqrt{3}}{2}$ for $\theta = \frac{5\pi}{3}$ **b** $-\frac{1}{2}$

11 a $1-\sin(2\theta)$ **b** $\cos(2\theta)$

Exercise 16C

1 a $5, -5$ **b** $2, -2$

c $\sqrt{2}, -\sqrt{2}$ **d** $\sqrt{2}, -\sqrt{2}$

e $2\sqrt{3}, -2\sqrt{3}$ **f** $2, -2$

g $4, 0$ **h** $5+\sqrt{13}, 5-\sqrt{13}$

2 a $\frac{\pi}{2}, \pi$ **b** $0, \frac{2\pi}{3}, 2\pi$

c $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $0, \frac{5\pi}{3}, 2\pi$

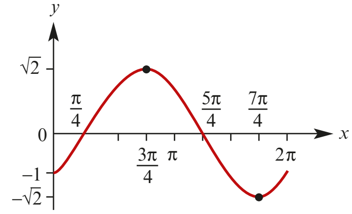
e $53.13°$ **f** $95.26°, 155.26°$

3 $2\cos\left(2x+\frac{\pi}{6}\right)$

4 $\sqrt{2}\sin\left(3x-\frac{5\pi}{4}\right)$

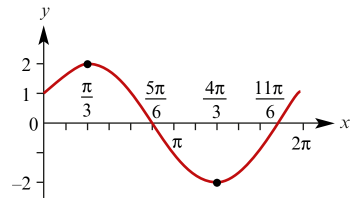
5 a $f(x) = \sin x - \cos x = \sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)$

$= \sqrt{2}\sin\left(x+\frac{7\pi}{4}\right) = \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$



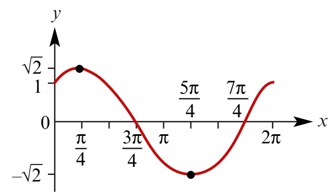
b $f(x) = \sqrt{3}\sin x + \cos x$

$= 2\cos\left(x-\frac{\pi}{3}\right) = 2\sin\left(x+\frac{\pi}{6}\right)$



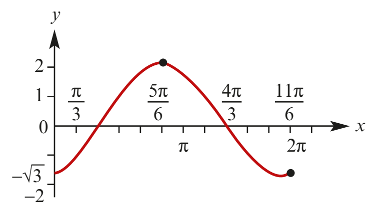
c $f(x) = \sin x + \cos x$

$= \sqrt{2}\cos\left(x-\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$



d $f(x) = \sin x - \sqrt{3}\cos x = 2\cos\left(x-\frac{5\pi}{6}\right)$

$= 2\sin\left(x+\frac{5\pi}{3}\right) = 2\sin\left(x-\frac{\pi}{3}\right)$



Exercise 16D

1 a $\sin(5\pi t) + \sin(\pi t)$ **b** $\frac{1}{2}(\sin 50° - \sin 10°)$

c $\sin(\pi x) + \sin\left(\frac{\pi x}{2}\right)$ **d** $\sin(A) + \sin(B+C)$

2 $\cos(\theta) - \cos(5\theta)$

3 $\sin x - \sin y$

5 a $2\sin 39^\circ \cos 17^\circ$ **b** $2\cos 39^\circ \cos 17^\circ$
c $2\cos 39^\circ \sin 17^\circ$ **d** $-2\sin 39^\circ \sin 17^\circ$

6 a $2\sin(4A)\cos(2A)$ **b** $2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$
c $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{7x}{2}\right)$ **d** $-2\sin(2A)\sin(A)$

11 a $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{\pi}{4}$

b $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

c $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$

d $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

12 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

c $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$

d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

17 $\frac{1-\cos(100x)}{2\sin(x)}$

Chapter 16 review

Technology-free questions

2 a $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

c $\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$

d $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **e** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

f $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$ **g** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

h $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

3 $60^\circ, 300^\circ, 0^\circ, 180^\circ, 360^\circ$

4 a $\frac{140}{221}$ **b** $-\frac{21}{221}$ **c** $\frac{171}{140}$

5 a $\frac{1}{2}$ **b** 1

6 a 1 **b** 0

8 a $-\frac{1}{9}$ **b** $-\frac{4\sqrt{5}}{9}$ **c** $\frac{8\sqrt{5}}{81}$

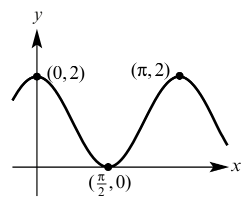
10 $2-\sqrt{3}$

11 a $0, \frac{\pi}{2}, 2\pi$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $0, \pi, 2\pi$

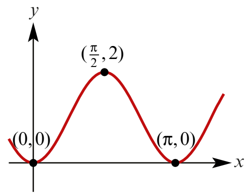
d $\frac{\pi}{2}, \frac{3\pi}{2}$ **e** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

f $\frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$

12 a



b



13 $\frac{2}{9}$

14 a $\sqrt{85}\cos(\theta-\alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$

b i $\sqrt{85}$ **ii** $\frac{2}{\sqrt{85}}$

iii $\theta = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$

15 a $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$ **b** $0, \frac{\pi}{3}, \pi$

Multiple-choice questions

1 A	**2** A	**3** B	**4** A	**5** C
6 E	**7** E	**8** A	**9** D	**10** E

Extended-response questions

1 b $P = 10\sqrt{5}\cos(\theta-\alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$;
$\theta = 70.88^\circ$

c $k = 25$ **d** $\theta = 45^\circ$

2 a $AD = \cos\theta + 2\sin\theta$

b $AD = \sqrt{5}\cos(\theta-\alpha)$ where
$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63^\circ$

c Max length of AD is $\sqrt{5}$ m when $\theta = 63^\circ$

d $\theta = 79.38^\circ$

3 b ii $a = 1,\ b = 1$

c $\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$
$= \sqrt{6}+\sqrt{2}-\sqrt{3}-2$

4 a i $h_1 = \cos\theta$
ii $h_2 = \sin\theta\cos\theta$
iii $h_3 = \sin^2\theta\cos\theta$
iv $h_n = \sin^{n-1}\theta\cos\theta$ for $n \in \mathbb{N}$

c 19.47°

5 a ii $2\cos\left(\frac{\pi}{5}\right)$

b iii $4\cos^2\left(\frac{\pi}{5}\right) - 2\cos\left(\frac{\pi}{5}\right) - 1 = 0$

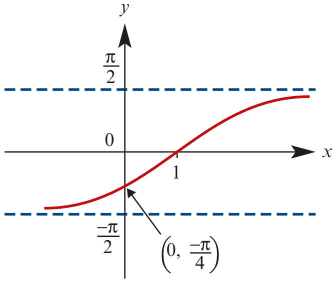
iv $\frac{1+\sqrt{5}}{4}$

6 b $-\frac{2}{3}$ or $\frac{1}{2}$

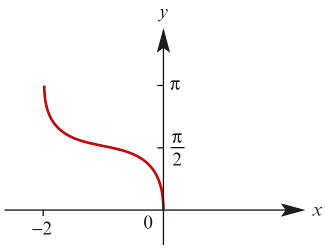
Chapter 17

Exercise 17A

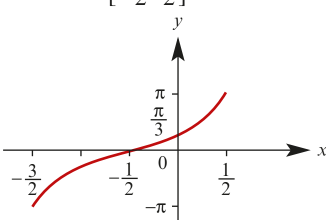
1 a Domain = $\mathbb{R}$; Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



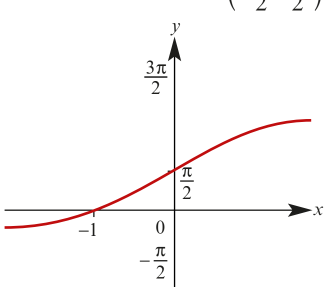
b Domain = $[-2, 0]$; Range = $[0, \pi]$



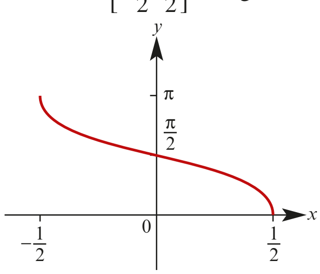
c Domain = $\left[-\frac{3}{2}, \frac{1}{2}\right]$; Range = $[-\pi, \pi]$



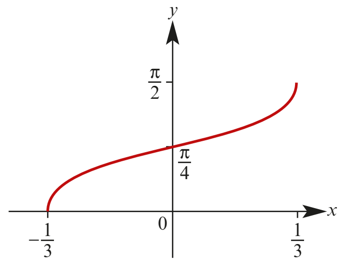
d Domain = $\mathbb{R}$; Range = $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$



e Domain = $\left[-\frac{1}{2}, \frac{1}{2}\right]$; Range = $[0, \pi]$



f Domain = $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range = $\left[0, \frac{\pi}{2}\right]$



2 a $\frac{\pi}{2}$ **b** $-\frac{\pi}{4}$ **c** $\frac{\pi}{6}$ **d** $\frac{5\pi}{6}$ **e** $\frac{\pi}{3}$
f $\frac{\pi}{4}$ **g** $-\frac{\pi}{3}$ **h** $\frac{\pi}{6}$ **i** π

3 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\pi}{3}$ **c** -1 **d** $\frac{\sqrt{2}}{2}$ **e** $\frac{\pi}{4}$
f $\sqrt{3}$ **g** $\frac{\pi}{3}$ **h** $-\frac{\pi}{3}$ **i** $-\frac{\pi}{4}$ **j** $\frac{5\pi}{6}$
k π **l** $-\frac{\pi}{4}$

4 a $f^{-1}\colon [-1, 1] \to \mathbb{R}$, $f^{-1}(x) = y$,
where $\sin y = x$ and $y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$(f^{-1}(x) = \pi - \sin^{-1}(x))$

b i 1 **ii** $\frac{\sqrt{2}}{2}$ **iii** $-\frac{1}{2}$ **iv** $\frac{3\pi}{2}$ **v** π **vi** $\frac{5\pi}{6}$

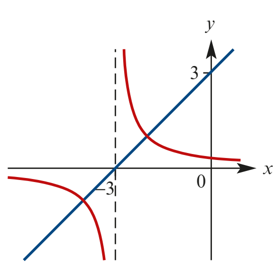
5 a $[1, 3]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **b** $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$, $[-1, 1]$
c $\left[-\frac{5}{2}, -\frac{3}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **d** $\left[-\frac{\pi}{18}, \frac{5\pi}{18}\right]$, $[-1, 1]$
e $\left[\frac{\pi}{6}, \frac{7\pi}{6}\right]$, $[-1, 1]$ **f** $[-2, 0]$, $[0, \pi]$
g $[-1, 1]$, $\left[0, \frac{\pi}{2}\right]$ **h** $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$, $[-1, 1]$
i $\mathbb{R}$, $\left[0, \frac{\pi}{2}\right)$ **j** $\left(0, \frac{\pi}{2}\right)$, $\mathbb{R}$ **k** $\mathbb{R}$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
l $\left(-\frac{\sqrt{2\pi}}{2}, \frac{\sqrt{2\pi}}{2}\right)$, $[0, \infty)$

6 a $\frac{3}{5}$ **b** $\frac{12}{5}$ **c** $\frac{24}{25}$ **d** $\frac{40}{9}$
e $\sqrt{3}$ **f** $\frac{\sqrt{5}}{3}$ **g** $-\frac{2\sqrt{5}}{5}$ **h** $\frac{2\sqrt{10}}{7}$

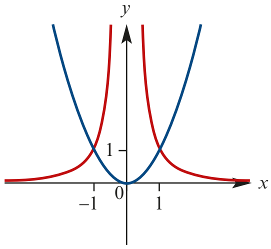
7 a i $\frac{4}{5}$ **ii** $\frac{12}{13}$

Exercise 17B

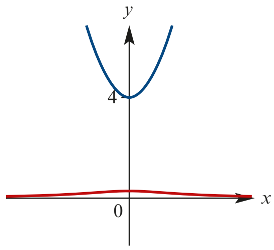
1 a



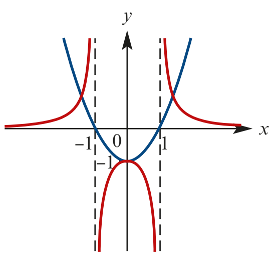
b



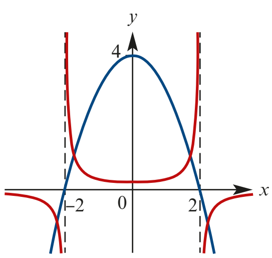
c



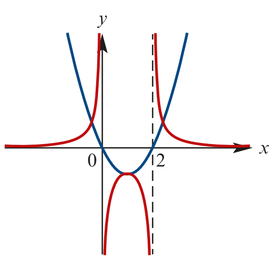
d



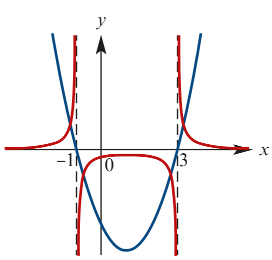
e



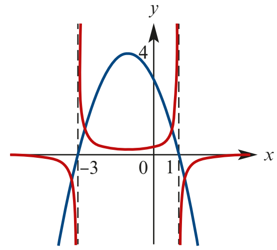
f



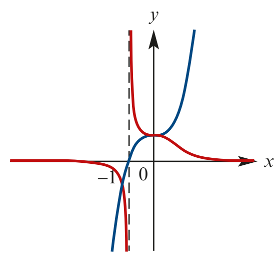
g



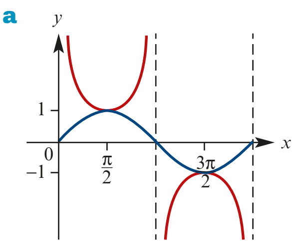
h



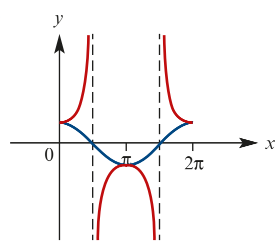
i



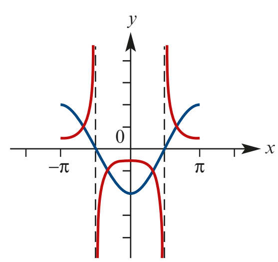
2 a



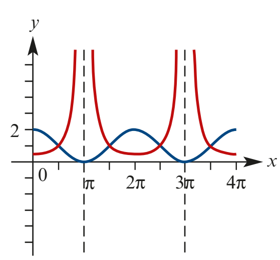
b



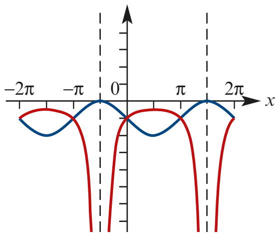
c



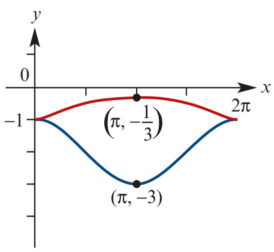
d



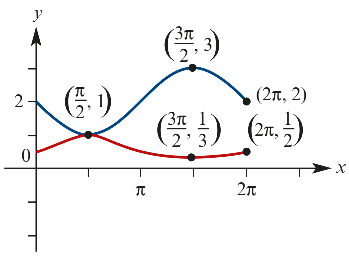
e



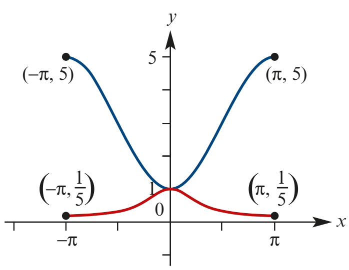
f



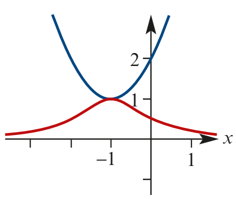
g



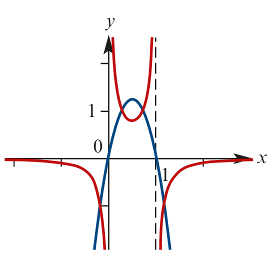
h



3 a $(-1, 1)$ **b**

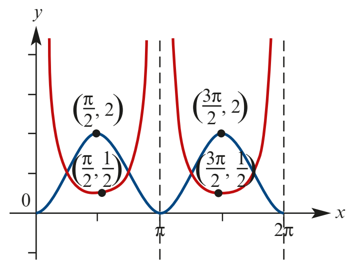


4 a

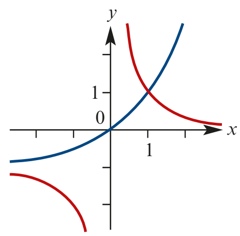


b $\left(\frac{5 \pm 3\sqrt{5}}{10}, -1\right), \left(\frac{5 \pm \sqrt{5}}{10}, 1\right)$

5



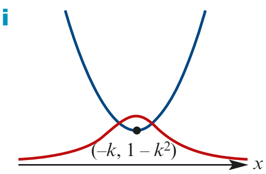
6



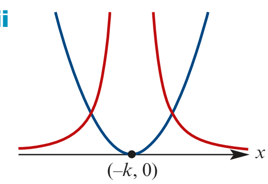
7 a $f(x) = (x + k)^2 + 1 - k^2$

b i $-1 < k < 1$ **ii** $k = \pm 1$ **iii** $k > 1$ or $k < -1$

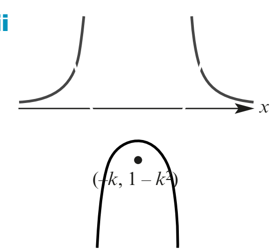
c i



ii

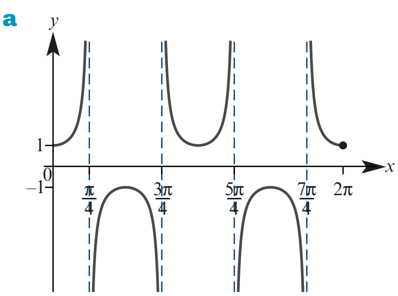


iii

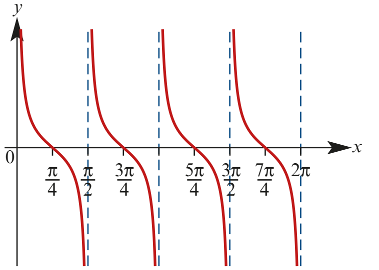


Exercise 17C

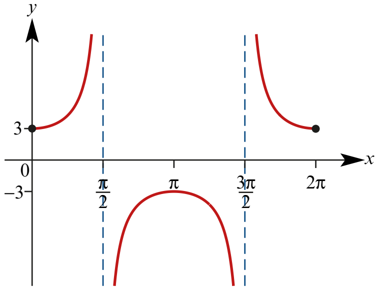
1 a



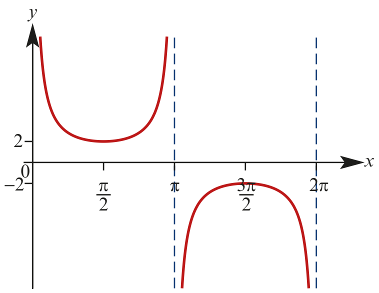
b



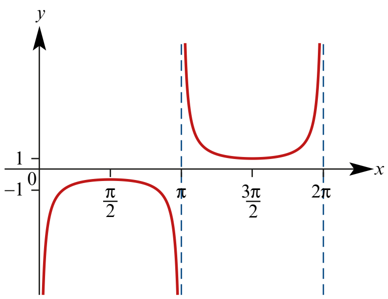
c



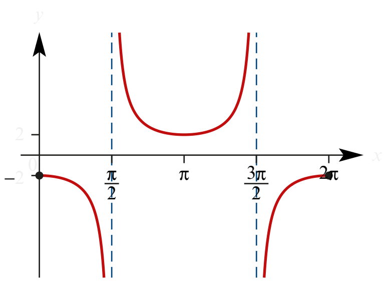
d



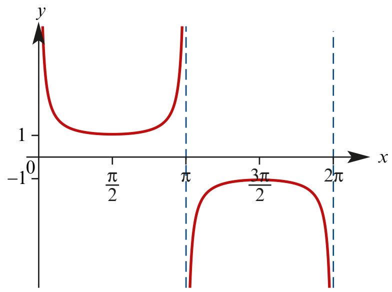
e



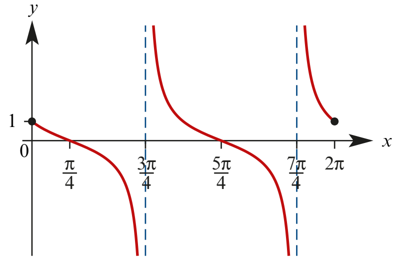
f



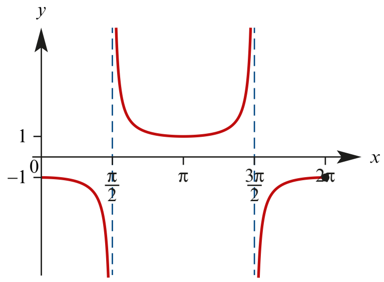
2 a



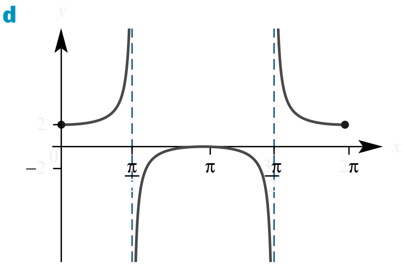
b



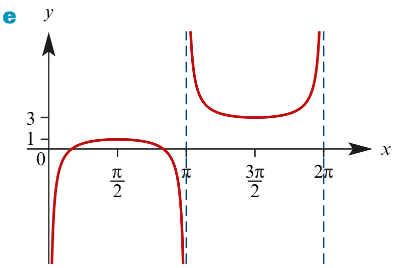
c



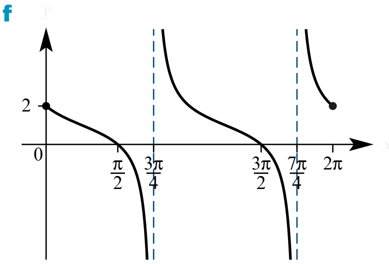
d



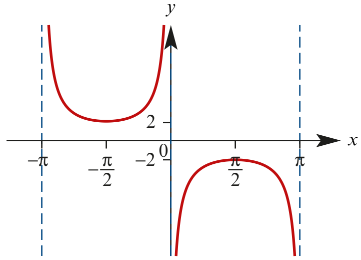
e



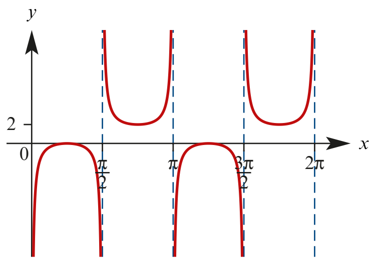
f



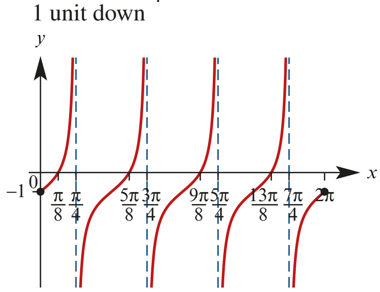
3
- Reflection in the x-axis
- Dilation of factor 2 from the x-axis
- Translation $\frac{\pi}{2}$ units to the right



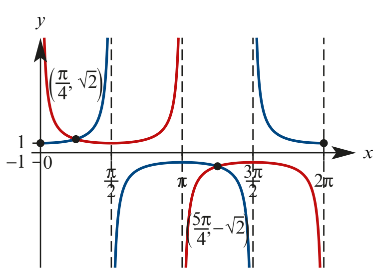
4
- Reflection in the y-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation 1 unit up



5
- Reflection in the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\frac{\pi}{4}$ units to the right and 1 unit down



6

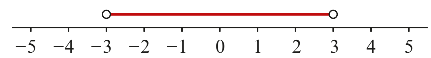


Exercise 17D

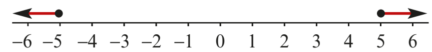
1 a 8 **b** 8 **c** 2 **d** −2
e −2 **f** 4

2 a 3, −1 **b** $\frac{7}{2}, -\frac{1}{2}$ **c** $\frac{12}{5}, -\frac{6}{5}$ **d** 12, −6
e −1, 7 **f** $\frac{4}{3}, -4$ **g** $-\frac{2}{5}, -4$

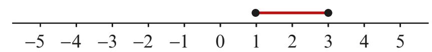
3 a $(-3, 3)$



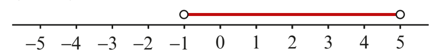
b $(-\infty, -5] \cup [5, \infty)$



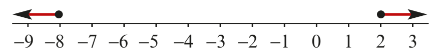
c $[1, 3]$



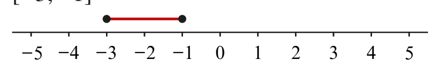
d $(-1, 5)$



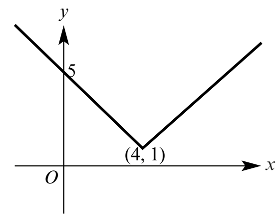
e $(-\infty, -8] \cup [2, \infty)$



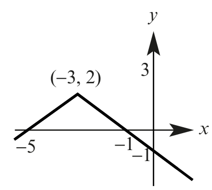
f $[-3, -1]$



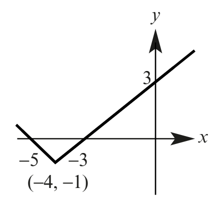
4 a Range = $[1, \infty)$



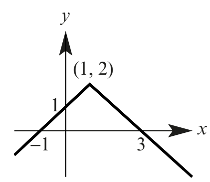
b Range = $(-\infty, 2]$



c Range = $[-1, \infty)$

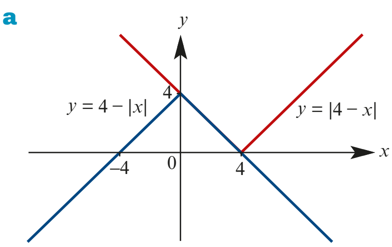


d Range = $(-\infty, 2]$



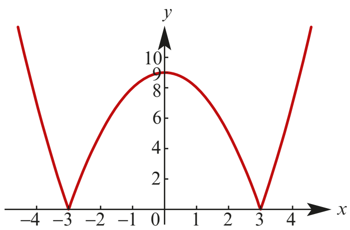
5 a $-5 \le x \le 5$ **b** $x \le -2$ or $x \ge 2$
c $1 \le x \le 2$ **d** $-\frac{1}{5} < x < 1$
e $x \le -4$ or $x \ge 10$ **f** $1 \le x \le 3$

6 a

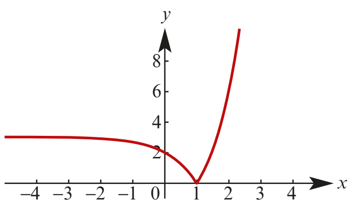


b $0 \le x \le 4$

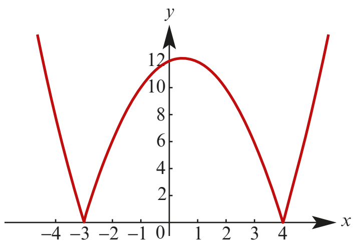
7 a



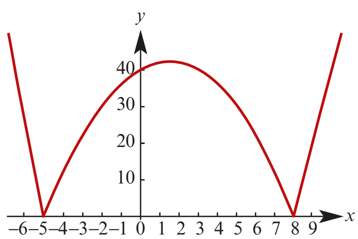
b



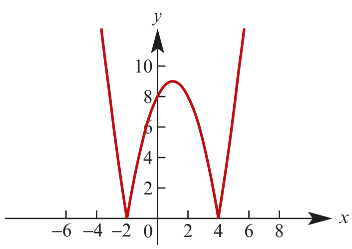
c



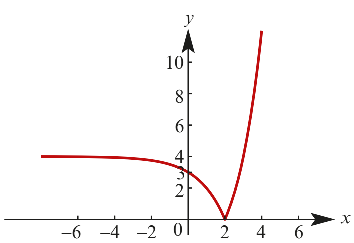
d



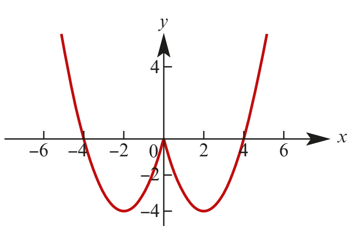
e



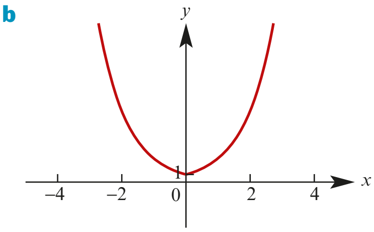
f



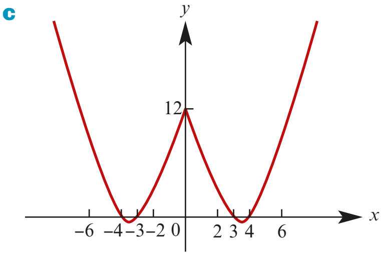
8 a



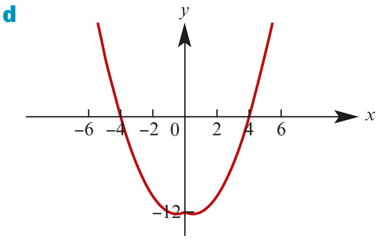
b



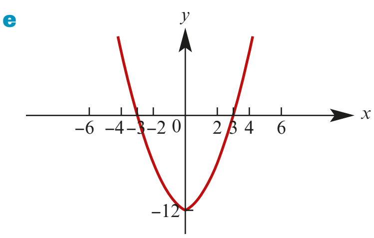
c



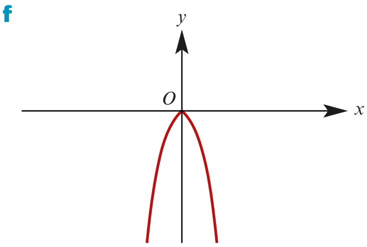
d



e



f



9 a $\dfrac{2-\sqrt{6}}{2}, \dfrac{2+\sqrt{6}}{2}, \dfrac{2-\sqrt{2}}{2}, \dfrac{2+\sqrt{2}}{2}$

b $1-\sqrt{2}, 1+\sqrt{2}, 1$ **c** $-2, 4$

d $3-\sqrt{17}, 3+\sqrt{17}, 2, 4$ **e** $-2, 8$

f $3-3\sqrt{2}, 3+3\sqrt{2}, 3$

10 a $x \le -2$ **b** $x = -9$ or $x = 11$

c $x = -\dfrac{5}{4}$ or $x = \dfrac{15}{4}$

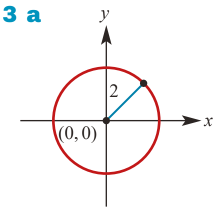
11 $a = 1,\ b = 1$

Exercise 17E

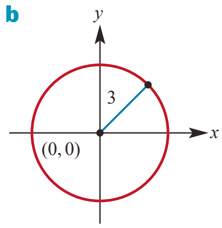
1 $(x-1)^2 + (y+2)^2 = 4^2$

2 $(x+4)^2 + (y-3)^2 = 5^2$

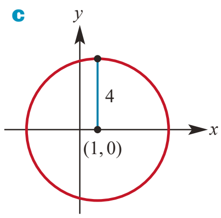
3 a



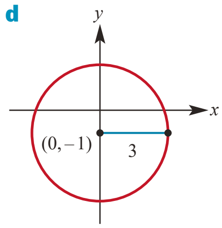
b



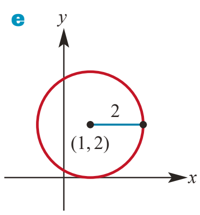
c



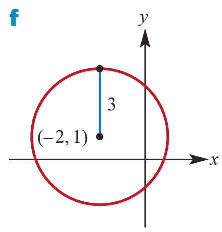
d



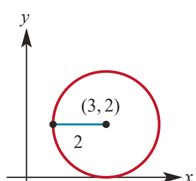
e



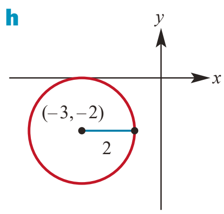
f



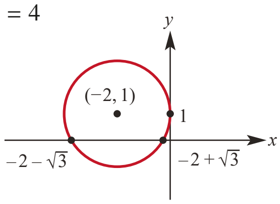
g



h



4 $(x + 2)^2 + (y - 1)^2 = 4$



5 a $(x - 1)^2 + y^2 = 5$
b $(x - 2)^2 + (y - 5)^2 = 5^2$

6 a $y = -x$

7 a $y = \frac{1}{2}x + \frac{3}{4}$

8 (0, 3) or (3, 0)

9 $\left(\frac{9}{10}, \frac{3}{10}\right)$

10 (6, 8) or $\left(\frac{72}{17}, \frac{154}{17}\right)$

11 a $2y - x = 1$ **b** $x + y = 2$ **c** (1, 1)
d $(x - 1)^2 + (y - 1)^2 = 5^2$

12 $y = 2x + 1$

13 $y = 6$

14 a The lines $x = 0$ and $y = 0$
b $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$

15 $(x - 4)^2 + y^2 = 4$

16 The lines $y = 1$ and $y = 5$

17 3 moves

Exercise 17F

1 $y = \frac{x^2}{12}$ **2** $y = -\frac{x^2}{12} - 1$ **3** $x = \frac{y^2}{12} - 1$

4 a $x = \frac{y^2}{4c}$ **b** $\left(\frac{1}{12}, 0\right)$

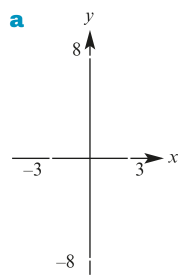
5 a $y = \frac{1}{2b - 2c}(x^2 - 2ax + a^2 + b^2 - c^2)$
b $y = -\frac{1}{2}(x^2 - 2x - 4)$

6 $y = -1$ or $y = 19$

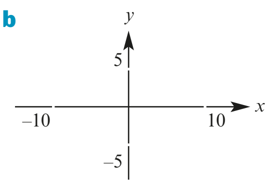
7 $(2, 1 + \sqrt{3})$ or $(2, 1 - \sqrt{3})$

Exercise 17G

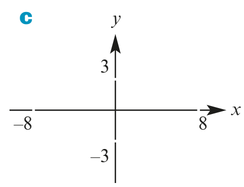
1 a



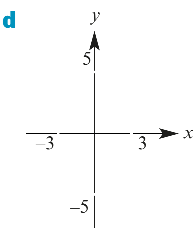
b



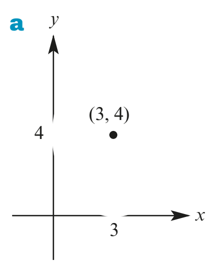
c



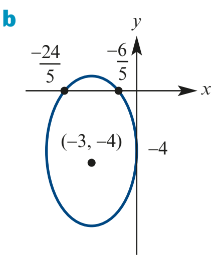
d



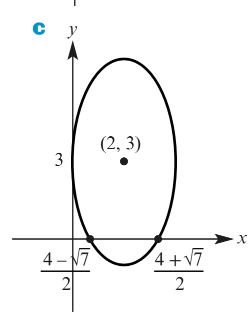
2 a



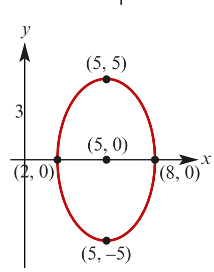
b



c



d



3 a $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **b** $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$

c $\frac{(x+1)^2}{4} + (y-1)^2 = 1$

4 $\frac{x^2}{4} + \frac{y^2}{3} = 1$

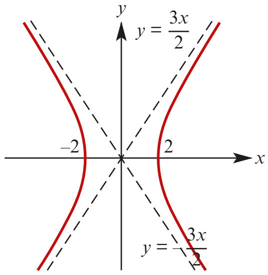
5 $\frac{x^2}{5} + \frac{y^2}{9} = 1$

6 $\frac{(x-4)^2}{16} + \frac{y^2}{12} = 1$

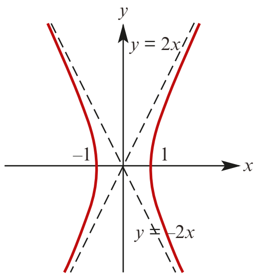
7 $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Exercise 17H

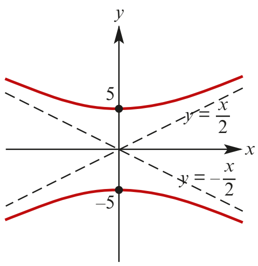
1 a



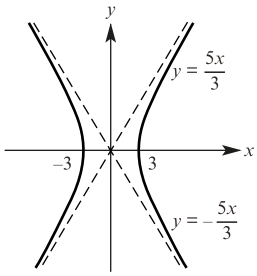
b



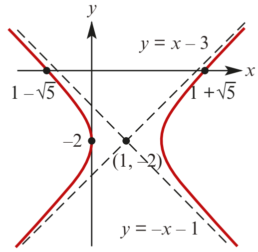
c



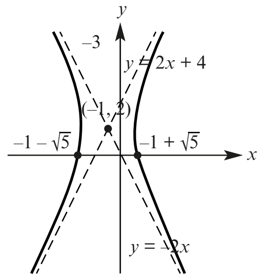
d



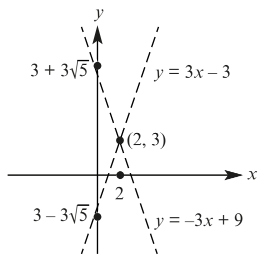
2 a



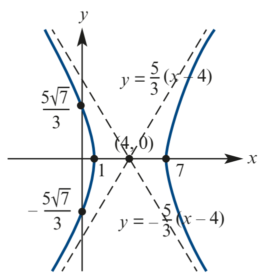
b



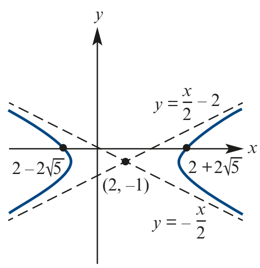
c



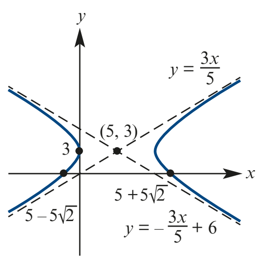
d



e



f



3 $\frac{x^2}{9} - \frac{y^2}{7} = 1$

4 $5x^2 - 4y^2 = 20$

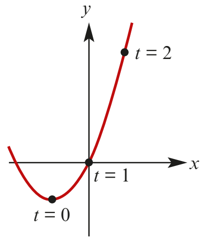
5 $\frac{(x+3)^2}{16} - \frac{y^2}{48} = 1$

6 $\frac{(y+5)^2}{4} - \frac{x^2}{12} = 1$

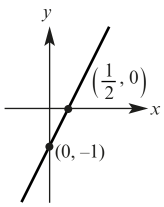
Exercise 17I

1 a $y = x^2 + 2x$

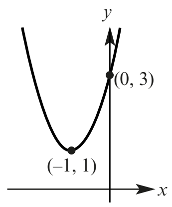
b



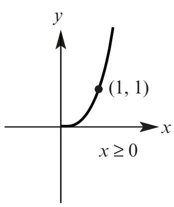
2 a $y = 2x - 1$



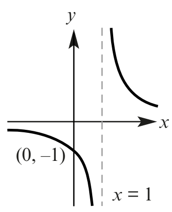
b $y = 2(x+1)^2 + 1$



c $y = x^3$



d $y = \frac{1}{x-1}$



3 a $x^2 + y^2 = 2^2$

b $\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1$

c $x = 3\cos t - 3$ and $y = 3\sin t + 2$
(other answers are possible)

d $x = 3\cos t - 2$ and $y = 2\sin t + 1$
(other answers are possible)

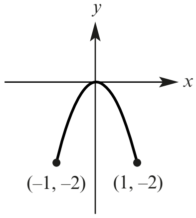
4 $x = t$ and $y = 3t + 1$
(other answers are possible)

5 a $\left(\frac{x-1}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$

b $x = \sec t + 2$ and $y = 2\tan t - 1$

6 a $y = -2x^2$ where $-1 \le x \le 1$

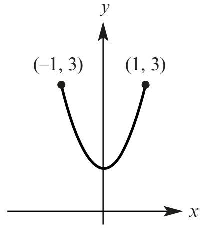
b



7 $\left(-\frac{3}{5}, -\frac{4}{5}\right), \left(\frac{3}{5}, \frac{4}{5}\right)$

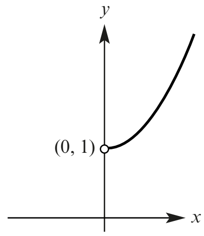
8 a $y = 2x^2 + 1$ **b** $-1 \le x \le 1$ **c** $1 \le y \le 3$

d

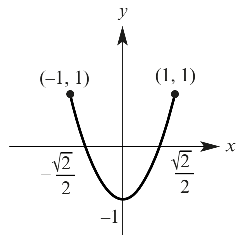


9 a $y = x^2 + 1$ **b** $x > 0$ **c** $y > 1$

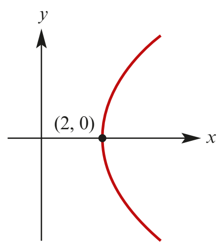
d



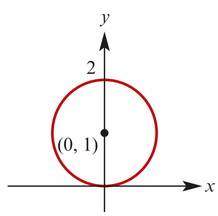
10 $y = -1 + 2x^2$ where $-1 \le x \le 1$



11 b



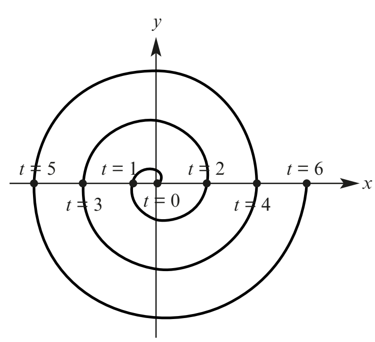
12 a



c $x = \frac{2t}{t^2+1}$

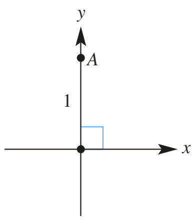
$y = \frac{2}{t^2+1}$

13

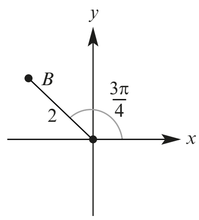


Exercise 17J

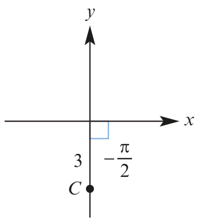
1 a $(0, 1)$



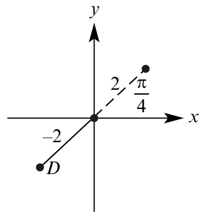
b $(-\sqrt{2}, \sqrt{2})$



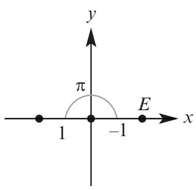
c $(0, -3)$



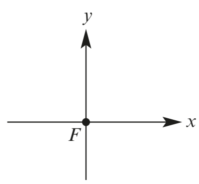
d $(-\sqrt{2}, -\sqrt{2})$



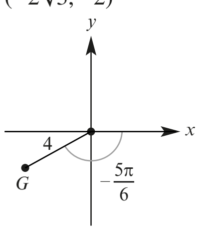
e $(1, 0)$



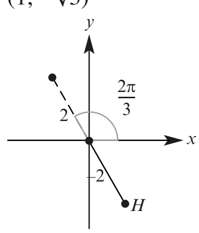
f $(0, 0)$



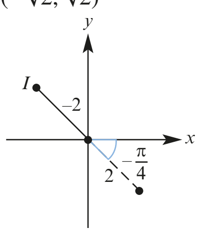
g $(-2\sqrt{3}, -2)$



h $(1, -\sqrt{3})$



i $(-\sqrt{2}, \sqrt{2})$



2 a $\left[\sqrt{2}, -\frac{\pi}{4}\right], \left[-\sqrt{2}, \frac{3\pi}{4}\right]$ **b** $\left[2, \frac{\pi}{3}\right], \left[-2, \frac{4\pi}{3}\right]$

c $\left[2\sqrt{2}, -\frac{\pi}{4}\right], \left[-2\sqrt{2}, \frac{3\pi}{4}\right]$

d $\left[2, -\frac{3\pi}{4}\right], \left[-2, \frac{\pi}{4}\right]$ **e** $[3, 0], [-3, \pi]$

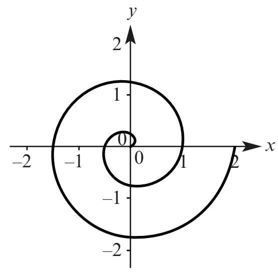
f $\left[2, -\frac{\pi}{2}\right], \left[-2, \frac{\pi}{2}\right]$

3 $\sqrt{7}$

4 $PQ = \sqrt{(r_1)^2 + (r_2)^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$

Exercise 17K

1



2 a $r = \dfrac{4}{\cos\theta}$ **b** $r = \tan\theta\sec\theta$

c $r = 3$ or $r = -3$ **d** $r^2 = \dfrac{1}{\cos(2\theta)}$

e $r = \dfrac{5}{2\cos\theta - 3\sin\theta}$

3 a $x = 2$ **b** $x^2 + y^2 = 2^2$

c $y = x$ **d** $3x - 2y = 4$

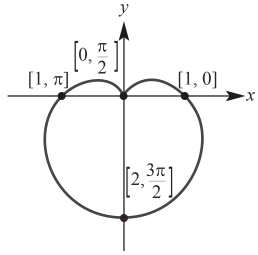
4 a $(x-3)^2 + y^2 = 9$ **b** $x^2 + (y-2)^2 = 4$

c $(x+3)^2 + y^2 = 9$ **d** $x^2 + (y+4)^2 = 16$

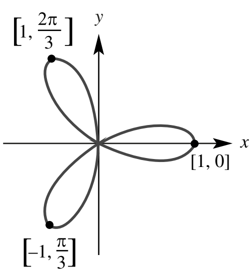
5 $(x-a)^2 + y^2 = a^2$

6 a Equation $x = a$ **b** $r = \dfrac{a}{\sin\theta}$

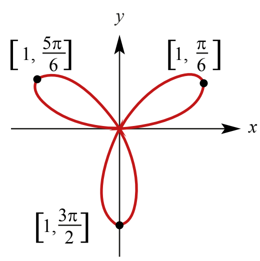
8 a



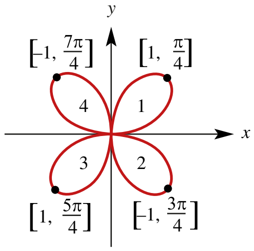
9 a



b



10 a

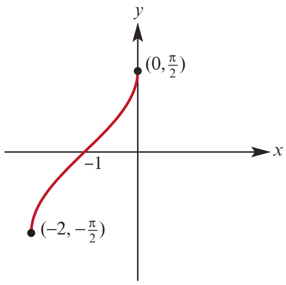


Chapter 17 review

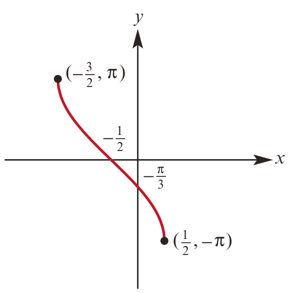
Technology-free questions

1 a $\frac{\pi}{2}$ **b** $\frac{\pi}{3}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{3}$ **e** $\frac{\sqrt{3}}{2}$ **f** $\frac{\pi}{4}$

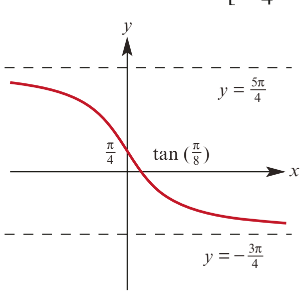
2 a Domain $= [-2, 0]$; Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



b Domain $= \left[-\frac{3}{2}, \frac{1}{2}\right]$; Range $= [-\pi, \pi]$



c Domain $= \mathbb{R}$; Range $= \left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

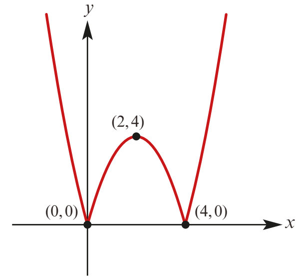


3 a 9 **b** $\frac{1}{400}$ **c** 4

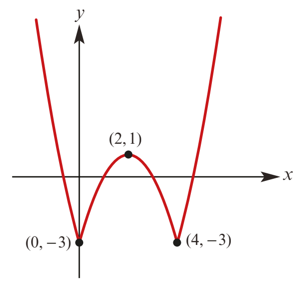
d 4 **e** $\pi - 3$ **f** $4 - \pi$

4 $x = 0$, $x = 2$ or $x = 4$

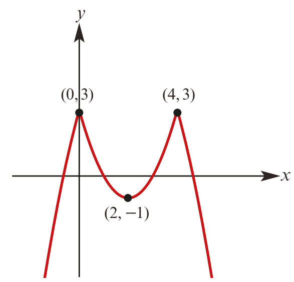
5 a Range $= [0, \infty)$



b Range $= [-3, \infty)$



c Range $= (-\infty, 3]$

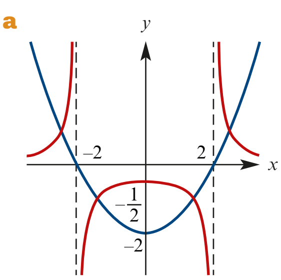


6 a $n = \pm 2, \pm 4$

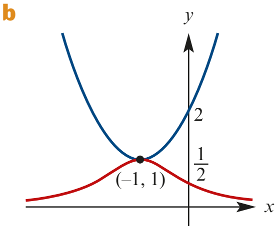
b i $x = \pm 1$ **ii** $x \leq 0$

c $x < -1$ or $x > 1$

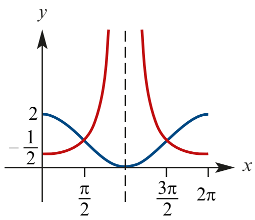
7 a



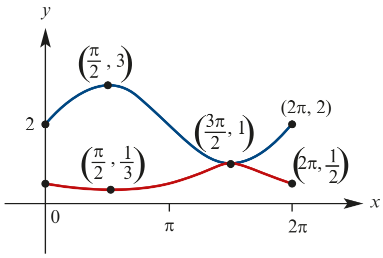
b



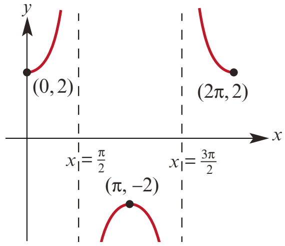
c



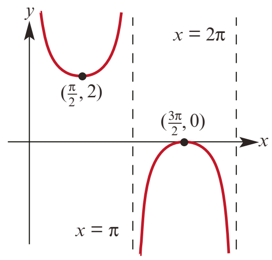
d



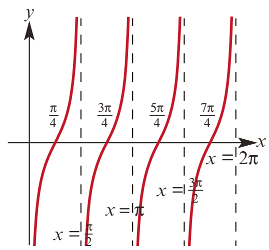
8 a



b

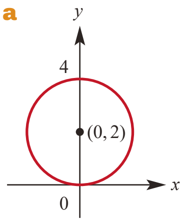


c

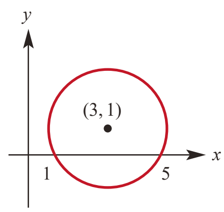


9 $(x-3)^2 + (y-2)^2 = 6^2$

10 a

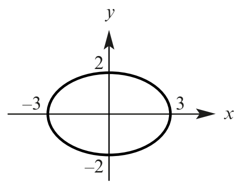


b

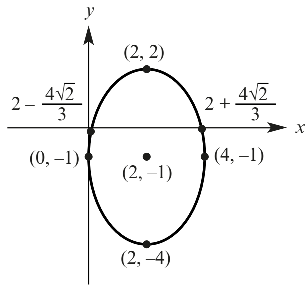


11 $C(-2, 4),\ r = \sqrt{20}$

12 a

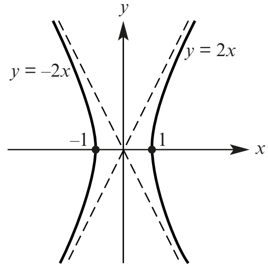


b

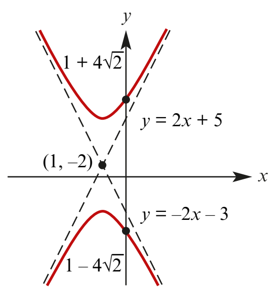


13 $C(-2, 0)$; Intercepts $(0, 0)$, $(-4, 0)$

14 a



b



15 $\dfrac{(x-2)^2}{4} - \dfrac{(y-5)^2}{12} = 1$

16 a $y = 4 - 2x$ **b** $x^2 + y^2 = 2^2$

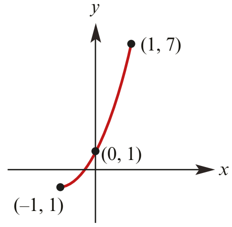
c $\dfrac{(x-1)^2}{3^2} + \dfrac{(y+1)^2}{5^2} = 1$

d $y = 1 - 3x^2$ where $-1 \le x \le 1$

17 a $y = 2(x+1)^2 - 1$ **b** $-1 \le x \le 1$

c $-1 \le y \le 7$

d



18 $(-\sqrt{2}, \sqrt{2})$

19 $\left[4, -\dfrac{\pi}{3}\right], \left[-4, \dfrac{2\pi}{3}\right]$

20 $r = \dfrac{5}{2\cos\theta + 3\sin\theta}$

21 $x^2 + (y-3)^2 = 9$

Multiple-choice questions

1 A	**2** D	**3** B	**4** B	**5** A	**6** D
7 D	**8** D	**9** C	**10** C	**11** E	**12** B

Extended-response questions

2 a $-\dfrac{2}{m}$

b $0 < m < 1$

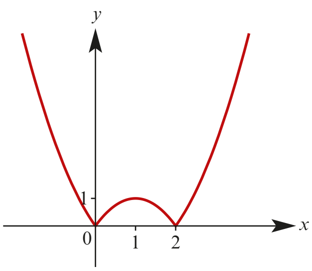
c i $y = -\dfrac{1}{m}x + 2$

ii $\left(\dfrac{4m}{1-m^2}, \dfrac{2(m^2+1)}{m^2-1}\right)$

iii The line ℓ is parallel to the graph of $y = f(x)$ for $x < -2$

iv $m = 3$

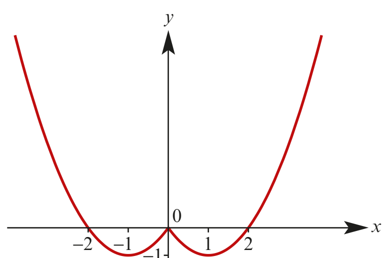
3 a i



ii $(0, 0), (a, 0)$ **iii** $\left(\dfrac{a}{2}, \dfrac{a^2}{4}\right)$

iv $a = 3$ or $a = -5$

b i



ii $(0, 0), (a, 0), (-a, 0)$

iii $\left(\dfrac{a}{2}, -\dfrac{a^2}{4}\right), \left(-\dfrac{a}{2}, -\dfrac{a^2}{4}\right)$ **iv** $a = -3$

c $x = 0$ or $x \geq a$

d $x \geq 0$

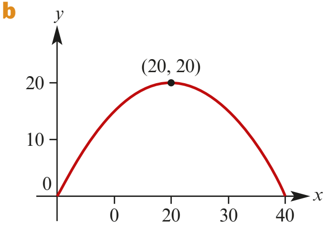
4 a $y = 2x - \dfrac{9}{2}$ **b** $(x-8)^2 + (y+1)^2 = 20$

5 a $y = \dfrac{x^2}{12} + 1$ **b** $\dfrac{x^2}{12} + \dfrac{(y-6)^2}{16} = 1$

c $\dfrac{(y+4)^2}{16} - \dfrac{x^2}{48} = 1$

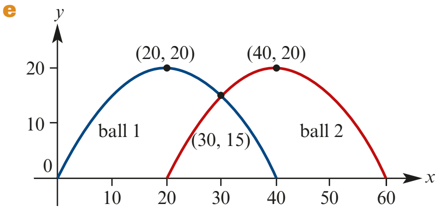
6 a $y = \dfrac{1}{20}x(40 - x)$

b



c 20 metres **d** $y = -\dfrac{1}{20}(x-20)(x-60)$

e

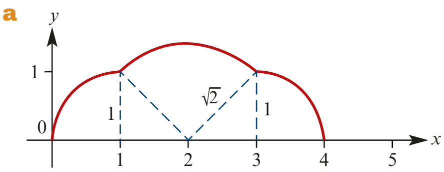


f $(30, 15)$

g Yes (same position at same time)

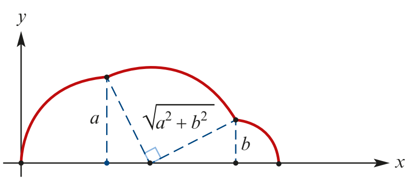
7 c $(x-4)^2 + (y-7)^2 = 25$

9 a



b Distance $= \dfrac{\pi}{2}(2 + \sqrt{2})$

c



Distance $= \dfrac{\pi}{2}(a + \sqrt{a^2 + b^2} + b)$

d Area $= \dfrac{\pi}{2}(a^2 + b^2) + ab$

Chapter 18

Exercise 18A

1

	Re(z)	Im(z)
a	2	3
c	$\frac{1}{2}$	$-\frac{3}{2}$
e	0	3

	Re(z)	Im(z)
b	4	5
d	−4	0
f	$\sqrt{2}$	$-2\sqrt{2}$

2 a $a = 2,\ b = -2$

b $a = 3,\ b = 2$ or $a = 2,\ b = 3$

c $a = 5,\ b = 0$ **d** $a = \dfrac{2}{3},\ b = -\dfrac{1}{3}$

3 a $6 - 8i$ **b** $6 - i$ **c** $-6 - 2i$
d $7 - 3\sqrt{2}i$ **e** $-2 - 3i$ **f** $4 + 2i$
g $6 - 4i$ **h** $-4 + 6i$ **i** $-1 + 11i$
j -1

4 a $4i$ **b** $6i$ **c** $\sqrt{2}i$
d $-i$ **e** -1 **f** 1
g -2 **h** -12 **i** -4

5 a $1 + 2i$ **b** $-3 + 4i$
c $-\sqrt{2} - 2i$ **d** $-\sqrt{6} - 3i$

Exercise 18B

1 a $15 + 8i$ **b** $-8i$ **c** $-2 + 16i$
d $2i$ **e** 5 **f** $-4 + 19i$

2 a -2 **b** 7

3 a $2 + 5i$ **b** $-1 - 3i$ **c** $\sqrt{5} + 2i$ **d** $5i$

4 a 25 **b** 2 **c** 13 **d** 5

5 a $2 + i$ **b** $-3 - 2i$ **c** $-4 + 7i$ **d** $-4 - 7i$
e $-4 - 7i$ **f** $-1 + i$ **g** $-1 - i$ **h** $-1 - i$

6 a $5\sqrt{2}$ **b** $5\sqrt{2}$ **c** 5 **d** $\sqrt{130}$

7 a $2 + 4i$ **b** 20 **c** 4
d $8 - 16i$ **e** $-8i$ **f** 8
g $\frac{1}{10}(1 + 2i)$ **h** $-4 - 2i$

8 $a = \frac{1}{29},\ b = -\frac{17}{29}$

9 a $\frac{7}{17} - \frac{6}{17}i$ **b** i **c** $\frac{7}{2} - \frac{1}{2}i$
d $-\frac{1}{2} - \frac{1}{2}i$ **e** $\frac{2}{13} + \frac{3}{13}i$ **f** $\frac{3}{20} + \frac{1}{20}i$

10 $a = \frac{5}{2},\ b = -\frac{3}{2}$

11 a $-\frac{42}{5}(1 - 2i)$ **b** $-\frac{1}{2}(1 - i)$ **c** $\frac{1}{17}(4 + i)$
d $\frac{1}{130}(6 + 43i)$ **e** $2 - 2i$

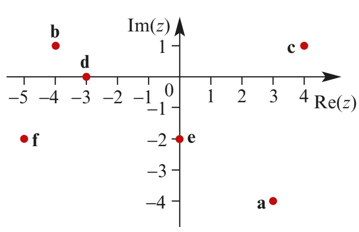
12 $a = 2,\ b = 3$ or $a = -2,\ b = -3$

13 $a = 2$ or $a = \frac{9}{2}$

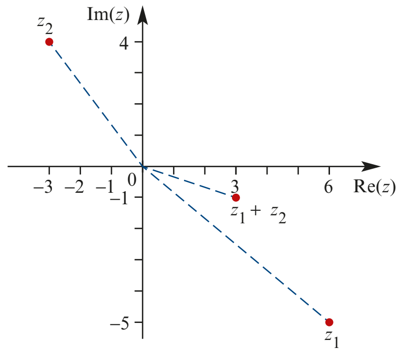
Exercise 18C

1 $A = 3 + i,\quad B = 2i,\quad C = -3 - 4i$
$D = 2 - 2i,\quad E = -3,\quad F = -1 - i$

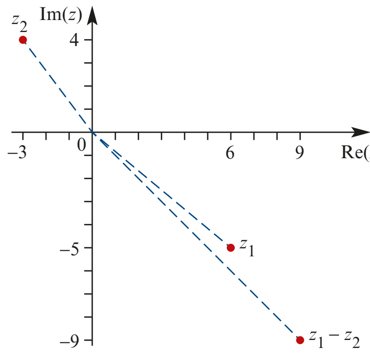
2



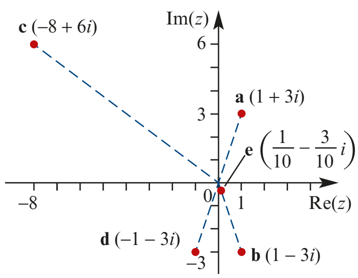
3 a $z_1 + z_2 = 3 - i$



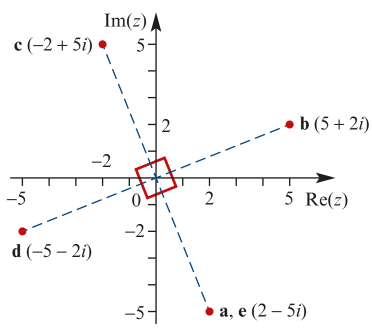
b $z_1 - z_2 = 9 - 9i$



4



5



Exercise 18D

1 a $\pm i$ **b** $\pm 3i$ **c** $\pm 4i$ **d** $\pm\frac{5}{2}i$ **e** $\pm\sqrt{2}i$
f $\pm 2i$ **g** $\pm 5i$ **h** $\pm\frac{1}{2}i$ **i** $\pm\frac{3}{4}i$ **j** $\pm\sqrt{3}i$
k $\pm\sqrt{5}i$ **l** $-1 \pm i$ **m** $2 \pm \sqrt{5}i$
n $-3 \pm \sqrt{3}i$ **o** $2 \pm 2i$

2 a $-1 \pm \sqrt{2}i$ **b** $2 \pm i$ **c** $-3 \pm \sqrt{3}i$
d $2 \pm i$ **e** $\frac{1}{3}(-1 \pm \sqrt{2}i)$ **f** $\frac{1}{2}(-1 \pm i)$

3 a $\frac{1}{2}(-3 \pm \sqrt{3}i)$ **b** $2 \pm i$ **c** $-3 \pm \sqrt{3}i$
d $2 \pm 2i$ **e** $\frac{1}{3}(-1 \pm \sqrt{2}i)$ **f** $\frac{1}{4}(\sqrt{2} \pm \sqrt{6}i)$

4 a $\pm 2i$ **b** $\pm 3i$ **c** $\pm\sqrt{5}i$ **d** $2 \pm 4i$
e $-1 \pm 7i$ **f** $1 \pm \sqrt{2}i$ **g** $\frac{1}{2}(-3 \pm \sqrt{3}i)$
h $\frac{1}{4}(-5 \pm \sqrt{7}i)$ **i** $\frac{1}{6}(1 \pm \sqrt{23}i)$ **j** $1 \pm 2i$
k $\frac{1}{2}(3 \pm \sqrt{11}i)$ **l** $3 \pm \sqrt{5}i$

5 a $b = -2,\ c = 2$ **b** $b = 4,\ c = 29$

Exercise 18E

1 $2, -2 \pm i$
2 $-1, 1 \pm i$
3 $3, 3 \pm 2i$
4 $2, 1 \pm \sqrt{2}i$
5 $3, \pm 2i$
6 $-2, 1, \pm 3i$
7 a $1, \pm i$ **b** $1, 1 \pm 2i$
c $2, 1 \pm i$ **d** $3, -3 \pm \sqrt{3}i$
8 $a = -5,\ b = 8,\ c = -6$
9 $c = 12$

Exercise 18F

1 a $2\,\text{cis}\left(\frac{\pi}{3}\right)$ **b** $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$
c $4\,\text{cis}\left(\frac{5\pi}{6}\right)$ **d** $4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$
e $24\,\text{cis}\left(-\frac{\pi}{3}\right)$ **f** $\frac{1}{\sqrt{2}}\,\text{cis}\left(\frac{3\pi}{4}\right)$

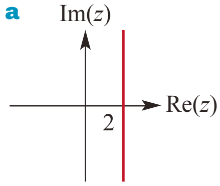
2 a $3i$ **b** $\frac{1}{\sqrt{2}}(1 + \sqrt{3}i) = \frac{\sqrt{2}}{2}(1 + \sqrt{3}i)$
c $\sqrt{3} + i$ **d** $-\frac{5}{\sqrt{2}}(1 - i) = -\frac{5\sqrt{2}}{2}(1 - i)$
e $-6(\sqrt{3} - i)$ **f** $3(1 - i)$
g $-\frac{5}{2}(1 + \sqrt{3}i)$ **h** $-\frac{5}{2}(1 + \sqrt{3}i)$

3 a $3\sqrt{2}(1 + i)$ **b** $6(1 + \sqrt{3}i)$
c $-\frac{5}{2}(1 - \sqrt{3}i)$ **d** $18(1 + \sqrt{3}i)$
e $-18(1 + \sqrt{3}i)$ **f** $\sqrt{3}(1 + i)$
g $\sqrt{3} + i$ **h** -4
i $-4(1 - \sqrt{3}i)$ **j** $-\frac{5}{2}$

4 a $\left(\frac{5 - 2\sqrt{3}}{2}, \frac{5\sqrt{3} + 2}{2}\right)$
b $\left(\frac{5}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$
c $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

Exercise 18G

1 a 5 **b** $\sqrt{2}$ **c** 13 **d** $2\sqrt{2}$ **e** 13 **f** $\sqrt{3}$

2 a


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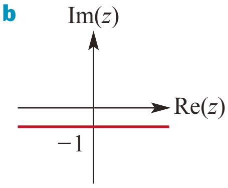

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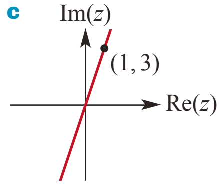

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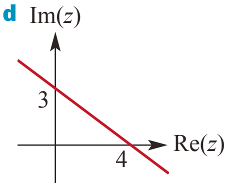

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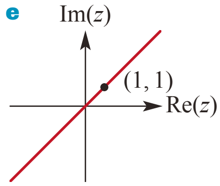

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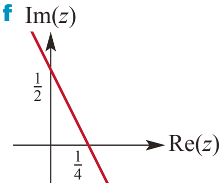

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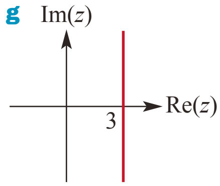

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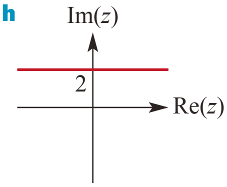

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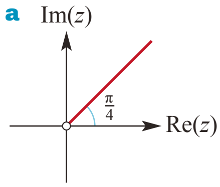

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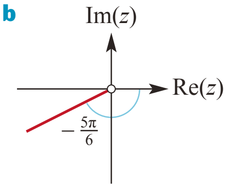

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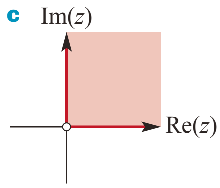

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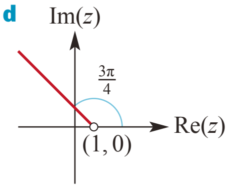

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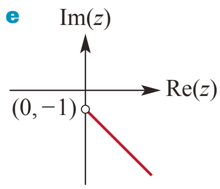

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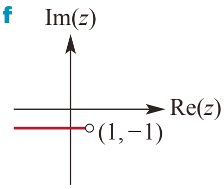

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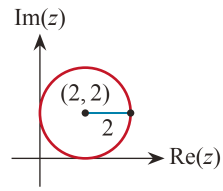
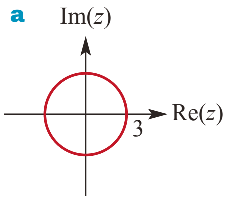


7 a



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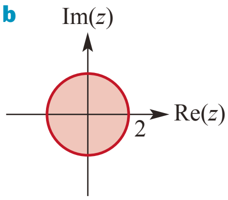

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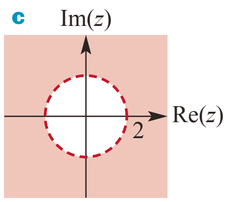

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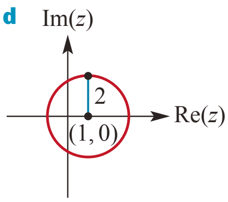

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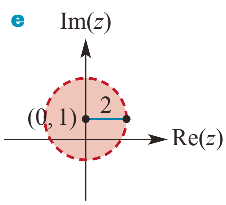

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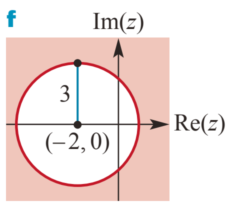

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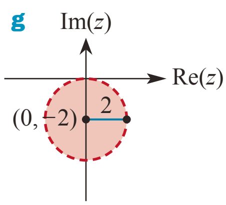

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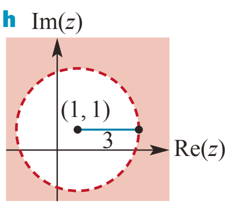
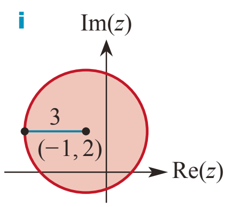
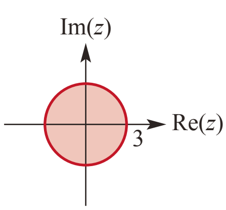


i



8 a



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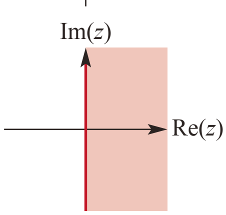

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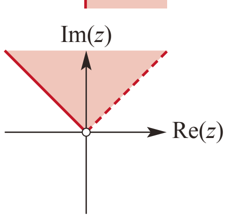

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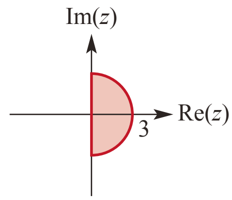

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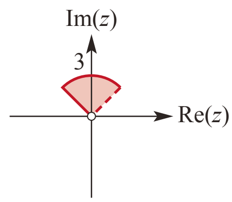
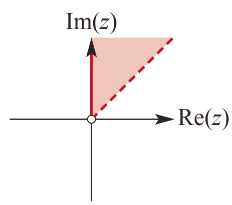


f



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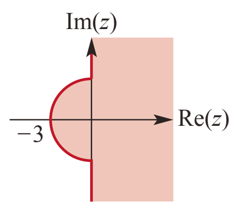

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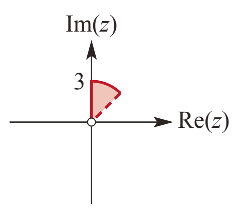
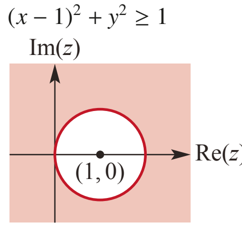


9 a $(x-1)^2 + y^2 \geq 1$

b



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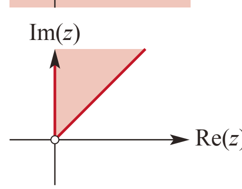
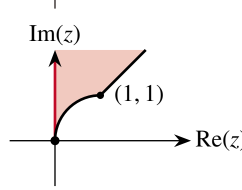


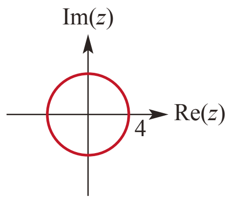
d



10 Centre $(0, 0)$, radius 1

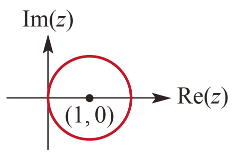
11 $x = \frac{1}{2}(y^2 - 2y)$

12 b

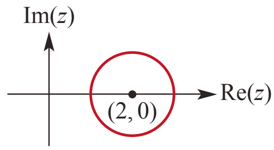


13 Centre $(-9, 0)$, radius 3

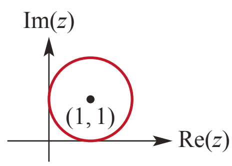
14 a



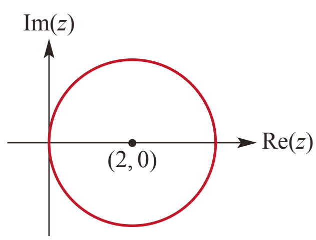
b i



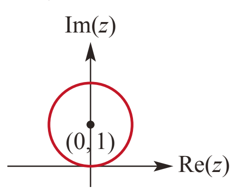
ii



iii



iv



Chapter 18 review

Technology-free questions

1 a $(2m + 3p) + (2n + 3q)i$ **b** $p - qi$

c $(mp + nq) + (np - mq)i$

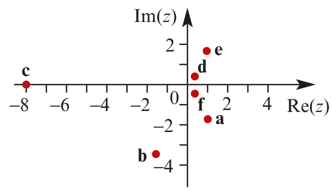
d $\dfrac{(mp + nq) + (np - mq)i}{p^2 + q^2}$ **e** $2m$

f $(m^2 - n^2 - p^2 + q^2) + (2mn - 2pq)i$

g $\dfrac{m - ni}{m^2 + n^2}$ **h** $\dfrac{(mp + nq) + (mq - np)i}{m^2 + n^2}$

i $\dfrac{3((mp + nq) + (np - mq)i)}{p^2 + q^2}$

2



a $1 - \sqrt{3}i$ **b** $-2 - 2\sqrt{3}i$

c -8 **d** $\frac{1}{4}(1 + \sqrt{3}i)$

e $1 + \sqrt{3}i$ **f** $\frac{1}{4}(1 - \sqrt{3}i)$

3 a $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$ **b** $2\operatorname{cis}\left(-\frac{\pi}{3}\right)$

c $\sqrt{13}\operatorname{cis}\left(\tan^{-1}\left(\frac{\sqrt{3}}{6}\right)\right)$ **d** $6\operatorname{cis}\left(\frac{\pi}{4}\right)$

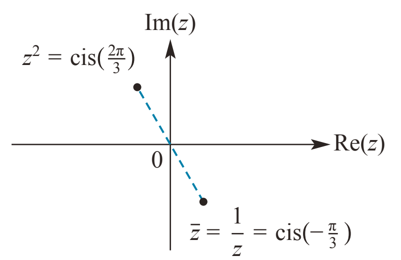
e $6\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ **f** $2\operatorname{cis}\left(-\frac{\pi}{6}\right)$

4 a $-1 - \sqrt{3}i$ **b** $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

c $-\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$ **d** $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

e $-\frac{3\sqrt{3}}{2} - \frac{3}{2}i$ **f** $1 - i$

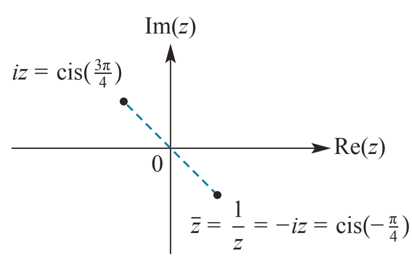
5



a $z^2 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$ **b** $\bar{z} = \operatorname{cis}\left(-\frac{\pi}{3}\right)$

c $\frac{1}{z} = \operatorname{cis}\left(-\frac{\pi}{3}\right)$ **d** $\operatorname{cis}\left(\frac{2\pi}{3}\right)$

6



a $iz = \operatorname{cis}\left(\frac{3\pi}{4}\right)$ **b** $\bar{z} = \operatorname{cis}\left(-\frac{\pi}{4}\right)$

c $\frac{1}{z} = \operatorname{cis}\left(-\frac{\pi}{4}\right)$ **d** $-iz = \operatorname{cis}\left(-\frac{\pi}{4}\right)$

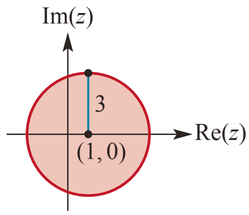
7 a $\pm 2i$ **b** $\pm\sqrt{3}i$ **c** $-2 \pm i$

d $\frac{1}{4}(3 \pm \sqrt{23}i)$

8 2, $\pm 2i$

9 b $\frac{11}{12}, \pm i$ **c** $n = 1$

10 a



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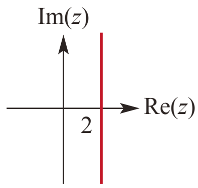
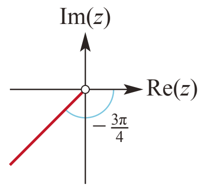
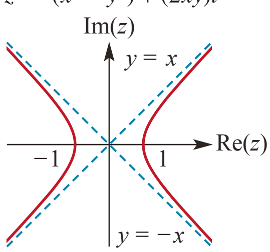


c

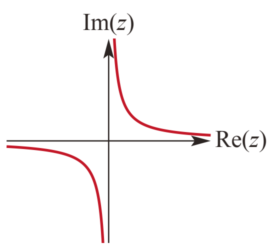


11 a $z^2 = (x^2 - y^2) + (2xy)i$

b



c



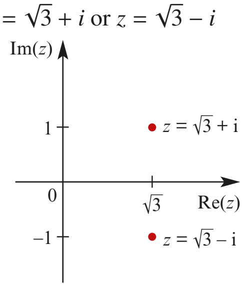
Multiple-choice questions

1 C **2** D **3** C **4** D **5** D **6** E
7 D **8** D **9** B **10** D **11** A

Extended-response questions

1 a $z = \sqrt{3} + i$ or $z = \sqrt{3} - i$

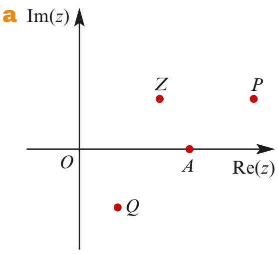
b i



ii $x^2 + y^2 = 4$ **iii** $a = 2$

2 a i $6\sqrt{2}$ **ii** 6

3 a



b $\sqrt{2} + 1$

6 a $|z + 1| = \sqrt{2 + 2\cos\theta} = 2\cos\left(\frac{\theta}{2}\right)$,

$\operatorname{Arg}(z + 1) = \frac{\theta}{2}$

b $|z - 1| = \sqrt{2 - 2\cos\theta} = 2\sin\left(\frac{\theta}{2}\right)$,

$\operatorname{Arg}(z - 1) = \frac{\pi + \theta}{2}$

c $\left|\frac{z-1}{z+1}\right| = \tan\left(\frac{\theta}{2}\right)$, $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

7 a $\Delta = b^2 - 4ac$

b $b^2 < 4ac$

c i $-\frac{b}{a}, \frac{\sqrt{ac}}{a}$ **ii** $\frac{b^2}{2ac} - 1$

8 a $z_1 = \frac{1}{2}(-1 + \sqrt{3}i)$, $z_2 = \frac{1}{2}(-1 - \sqrt{3}i)$

c $|z_1| = 1$, $\operatorname{Arg}(z_1) = \frac{2\pi}{3}$;

$|z_2| = 1$, $\operatorname{Arg}(z_2) = -\frac{2\pi}{3}$

d $\frac{\sqrt{3}}{4}$

Chapter 19

Technology-free questions

1 a $\frac{5}{4}$ **b** $\frac{4}{3}$ **c** $-\frac{\sqrt{3}}{3}$ **d** $\frac{2\sqrt{3}}{3}$

2 $\pm\frac{\sqrt{6}}{3}$

4 a 6 **b** $4i$ **c** 13 **d** 10
e 36 **f** −16 **g** $24i$ **h** $24i$

5 a $3 - 5i$ **b** $-1 + i$ **c** $-4 - 7i$ **d** $\frac{8 - i}{13}$
e $2 + i$ **f** $\frac{-2 + i}{5}$ **g** $-2 - i$ **h** $\frac{8 + i}{5}$
i $\frac{13 - i}{34}$ **j** $3 - i$ **k** $\frac{-1 - 3i}{2}$ **l** $-3 - 4i$

6 a $(z - 7i)(z + 7i)$

b $(z - 1 - 3i)(z - 1 + 3i)$

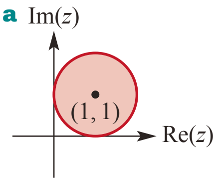
c $9\left(z - \frac{1}{3} - \frac{2}{3}i\right)\left(z - \frac{1}{3} + \frac{2}{3}i\right)$

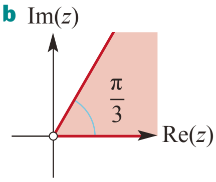
d $4\left(z + \frac{3}{2} - i\right)\left(z + \frac{3}{2} + i\right)$

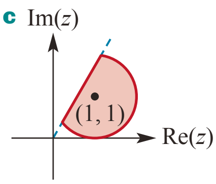
7 a $2 - i,\ -2 + i$

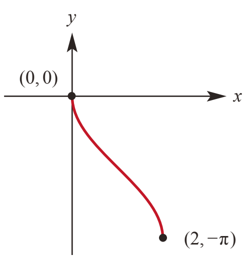
b $z = -i$ or $z = -1 - i$

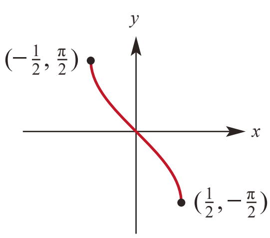
8 $a = -4,\ b = -10,\ c = -12$

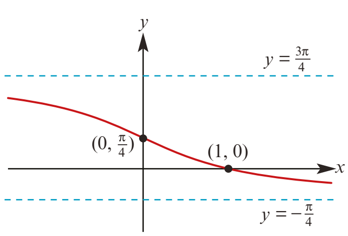
9 a


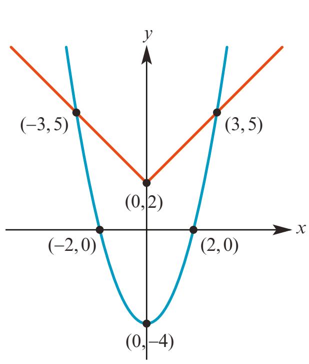
b


c


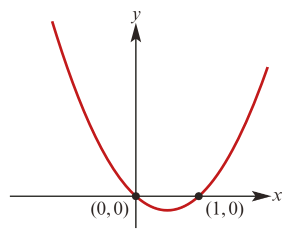
10 a


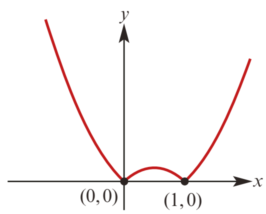
b


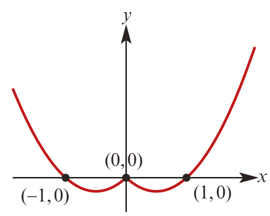
c


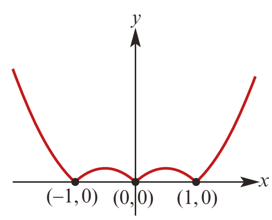
11 a


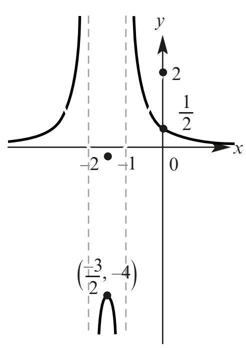
b $(-3, 5), (3, 5)$

12 a


b


c


d


13 a


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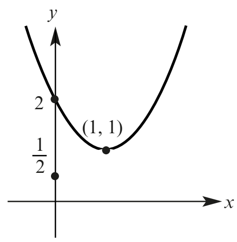
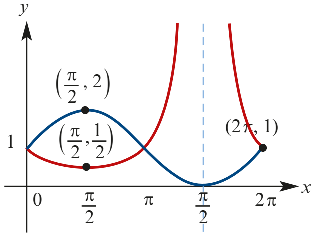
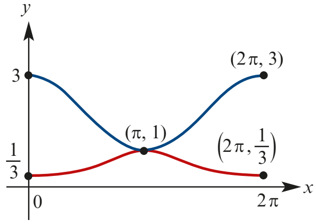


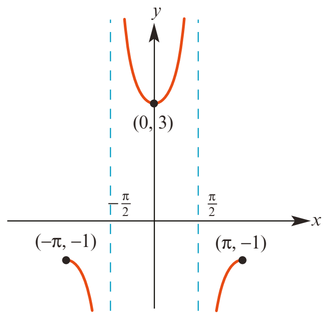
c



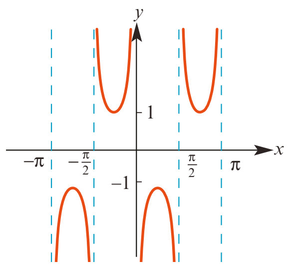
d



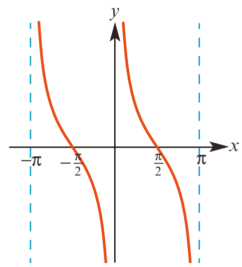
14 a



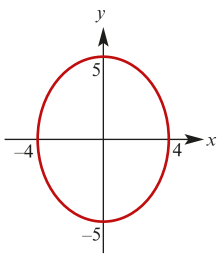
b



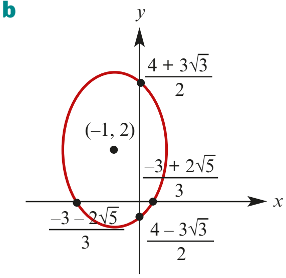
c



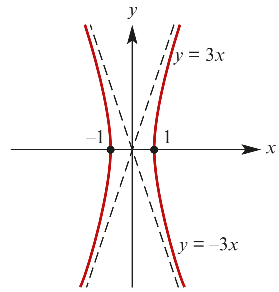
15 a



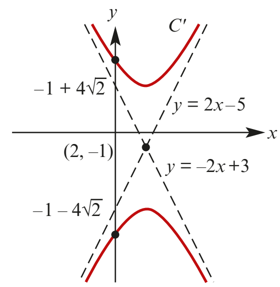
b



16 a



b



17 $2x + 4y = 17$

18 $\dfrac{(x-1)^2}{4} + \dfrac{(y-1)^2}{3} = 1$

19 $y = \dfrac{x^2}{8} - 1$

20 a $3x + 2y = 7$ **b** $x^2 + y^2 = 1$

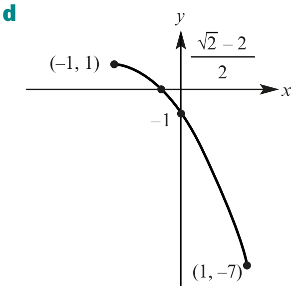
c $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{9} = 1$

d $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

21 a $y = 1 - 2(x+1)^2$ **b** $-1 \le x \le 1$

c $-7 \le y \le 1$

d



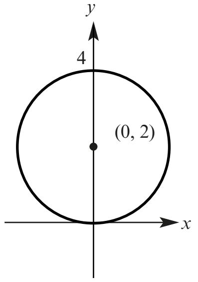
22 $(-\sqrt{3}, -1)$

23 $\left[2\sqrt{2}, -\frac{\pi}{4}\right], \left[-2\sqrt{2}, \frac{3\pi}{4}\right]$

24 a $x^2 + y^2 = 5^2$ **b** $y = \sqrt{3}x$ **c** $y = 3$

d $3y + 4x = 2$ **e** $y = \frac{1}{2x}$

25 a



Multiple-choice questions

1 B	**2** B	**3** D	**4** E	**5** A	**6** D
7 E	**8** D	**9** E	**10** C	**11** E	**12** E
13 D	**14** C	**15** B	**16** C	**17** B	**18** D
19 E	**20** C	**21** C	**22** A	**23** C	**24** E
25 C	**26** D	**27** E	**28** C	**29** B	**30** A
31 B	**32** C	**33** B	**34** B	**35** C	**36** B
37 D	**38** A	**39** E	**40** D	**41** A	

Extended-response questions

1 a i $\angle BCA = 138.19°$, $\angle ABC = 11.81°$

ii $\angle BC'A = 41.81°$, $\angle ABC' = 108.19°$

b i 24.56 **ii** 114.00 **iii** 89.44

c i 1788.85 **ii** 3027.87 **iii** 1239.01

3 a 155 m

b i 16.00 m **ii** 29.04 m **iii** 17°

c 32.7 cm^2

4 a 12:05 p.m. **b** 2752 km

c 26.1°

5 a $x = \sqrt{\frac{p^2}{4} + \frac{q^2}{4} - \frac{pq}{2}\cos\theta}$

b $y = \sqrt{\frac{p^2}{4} + \frac{p^2}{4} + \frac{pq}{2}\cos\theta}$

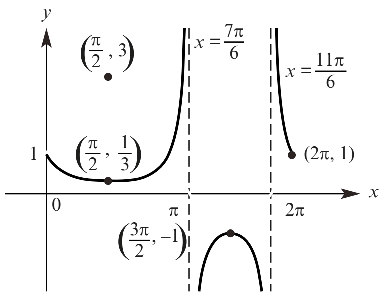
d $\sqrt{31}$ cm

6 b i 51.48 cm **ii** 4764.95 cm^2

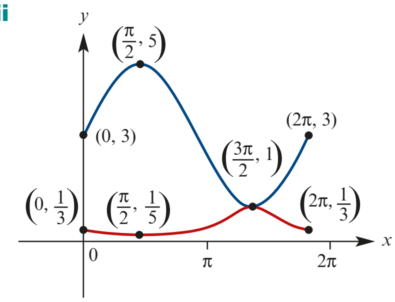
iii 94.80%

7 b $(-6, 2, 6), (-6, -2, 6), (-8, 3, 3), (-3, -3, 8)$

8 a i

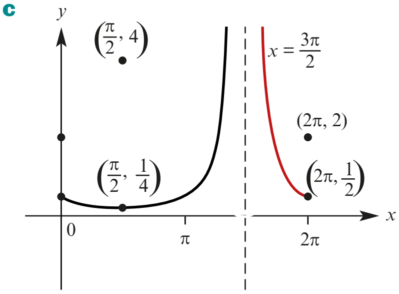


ii



b $k = 2$

c

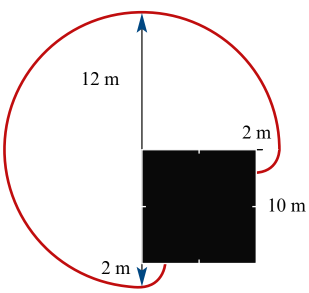


9 a $y = \frac{x}{4} - \frac{3}{8}$ **c** $\frac{\sqrt{17}}{2}$ km

10 c $x = t$ and $y = -t + 3$

e $k > \frac{8}{\sqrt{5}}$ or $k < -\frac{8}{\sqrt{5}}$

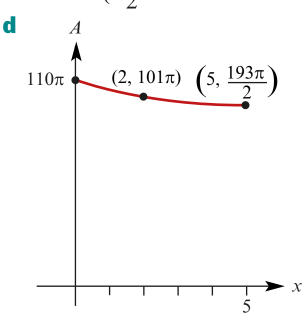
11 a



b $110\pi\ \text{m}^2$

c $A(x) = \begin{cases} \frac{3\pi x^2}{4} - 6\pi x + 110\pi, & 0 \le x \le 2 \\ \frac{\pi x^2}{2} - 5\pi x + 109\pi, & 2 < x \le 5 \end{cases}$

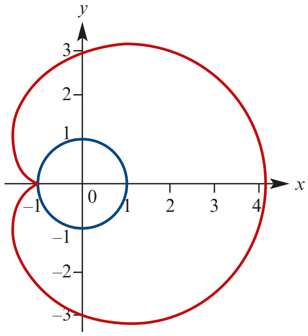
d



e i $x = 0$ **ii** $x = 5$

12 a Length of rope, π, is equal to the arc length from S to the opposite side of the circle

b



c **i** θ **ii** $\pi - \theta$ **iii** θ

iv $(\pi - \theta)\sin\theta$ **v** $(\pi - \theta)\cos\theta$

d $x = \cos\theta - (\pi - \theta)\sin\theta$ and $y = \sin\theta + (\pi - \theta)\cos\theta$

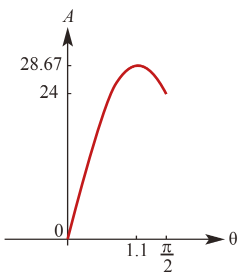
13 b i $\tan^{-1}\left(\frac{1}{2}\right)$ **ii** $\tan^{-1}\left(\frac{1}{3}\right)$ **iii** $\frac{\pi}{4}$

14 c $P = 14 + 6\sqrt{2}, \ \theta = \frac{\pi}{4}$

d 0.1845, 1.3861 **e** $A = 9 + 12\sqrt{2}$

f $A \approx 28.67, \ \theta \approx 1.1$

g



h $A \approx 28.67, \ x \approx 5.36$

Investigations

See solutions supplement

Chapter 20

Exercise 20A

1 a $(-2, 6)$ **b** $(-8, 22)$

c $(26, 2)$ **d** $(-4, -2)$

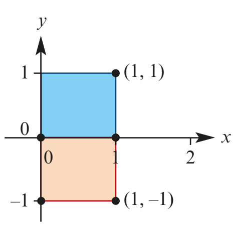
2 a $(3, 2)$ **b** $(-4, 9)$ **c** $(8, 3)$ **d** $(7, 11)$

3 a $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 11 & -3 \\ 3 & -8 \end{bmatrix}$

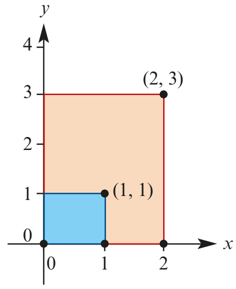
c $\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4 Unit square is blue; image is red

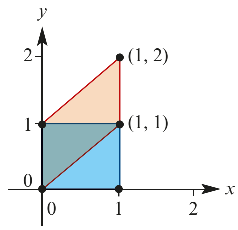
a



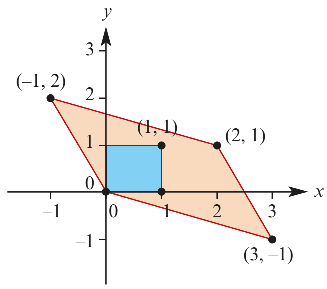
b



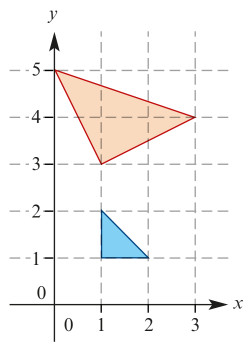
c



d



5



6 $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$

7 $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

8 a $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

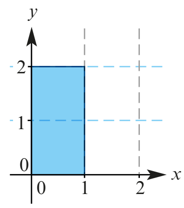
b $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

c $\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}$

Exercise 20B

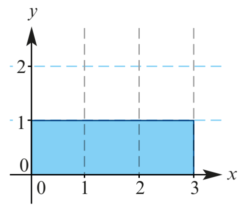
1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

ii



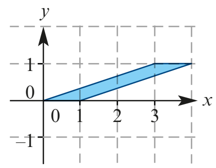
b i $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

ii



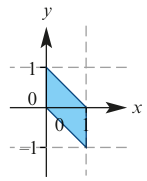
c i $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

ii



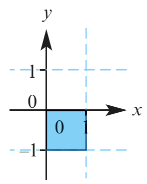
d i $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

ii



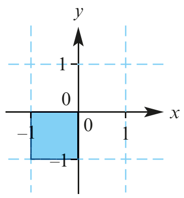
e i $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

ii



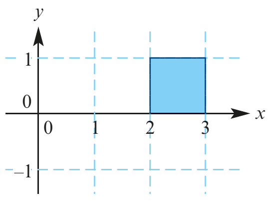
f i $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

ii



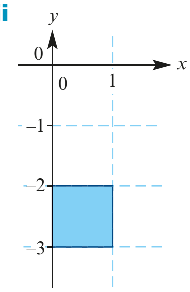
2 a i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$

ii



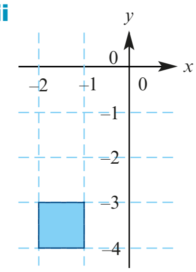
b i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y-3 \end{bmatrix}$

ii



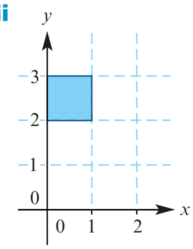
c i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x-2 \\ y-4 \end{bmatrix}$

ii



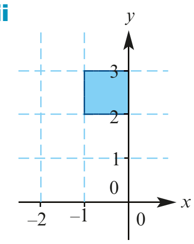
d i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y+2 \end{bmatrix}$

ii



e i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$

ii



Exercise 20C

1 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

d $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

2 a $(-3, 2)$ **b** $\left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3 a $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

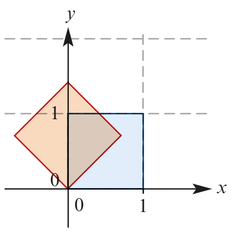
4 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$

c $\begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

5 a $\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$ **b** $\left(-\frac{23}{37}, \frac{47}{37}\right)$

6 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

b



c $\sqrt{2} - 1$

7 a $B\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

b Equilateral

c $y = -\sqrt{3}x,\ \ y = 0,\ \ y = \sqrt{3}x$

Exercise 20D

1 $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ **2** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

3 a $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

b $\begin{bmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4 a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **c** No

5 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **c** Yes

6 a $(x, y) \to (-x-3, y+5)$

b $(x, y) \to (-x+3, y+5)$ **c** Yes

7 a $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

8 a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9 $\theta = 180°k$, where $k \in \mathbb{Z}$

10 a 2θ

b $\begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$

c $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

11 a $(x, y) \to (y+1, x+2)$

b $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

12 a $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{2} \end{bmatrix}$

d $\cos 15° = \frac{\sqrt{2}+\sqrt{6}}{4}$, $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$

13 $\begin{bmatrix} \cos(2\theta - 2\varphi) & -\sin(2\theta - 2\varphi) \\ \sin(2\theta - 2\varphi) & \cos(2\theta - 2\varphi) \end{bmatrix}$,
rotation matrix for angle $2\theta - 2\varphi$

Exercise 20E

1 a $\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{14} \end{bmatrix}$

c $\begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$ **d** $\begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$

2 a $(x, y) \to (x - 2y, 2x - 5y)$

b $(x, y) \to (y, -x + y)$

3 a $(-1, 1)$ **b** $\left(-\frac{1}{2}, 1\right)$

4 $\begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}$

5 $(0, 0), (-1, -2), (1, 1), (0, -1)$

6 a $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$

7 a $\mathbf{A} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$

8 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b Reflecting twice in the same axis will return any point (x, y) to its original position

9 a $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

b Reflecting twice in the same line will return any point (x, y) to its original position

Exercise 20F

1 a $y = -3x - 1$ **b** $y = \frac{3x}{2} + 1$ **c** $y = \frac{9x}{2} + 3$

d $y = 3x - 1$ **e** $y = -9x + 3$ **f** $y = \frac{-x-1}{3}$

g $y = \frac{x-1}{3}$

2 a $y = 6 - \frac{9x}{2}$ **b** $y = \frac{x+2}{3}$

c $y = \frac{2-3x}{7}$ **d** $y = \frac{5x-2}{12}$

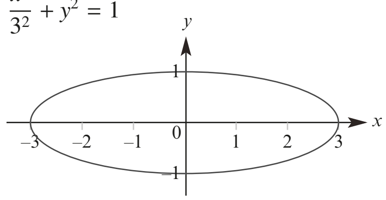
3 $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

4 $\begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}$

5 $y = -(x + 1)^2 - 1$

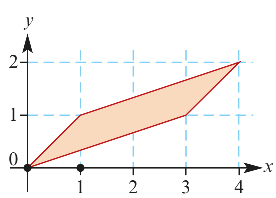
6 $y = (x - 1)^2 - 3$

7 $\frac{x^2}{3^2} + y^2 = 1$

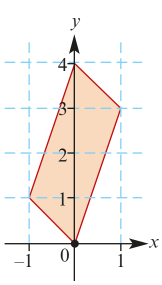


Exercise 20G

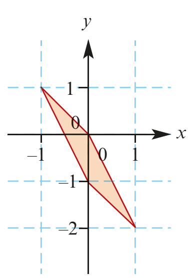
1 a Area = 2



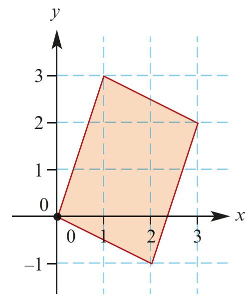
b Area = 4



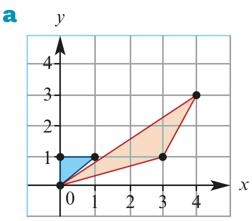
c Area = 1



d Area = 7

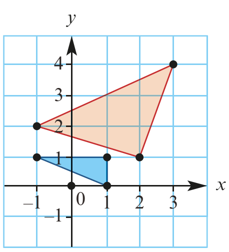


2 a



b Original area = $\frac{1}{2}$; Image area = $\frac{5}{2}$

3 a



b Original area = 1; Image area = 5

4 $m = \pm 2$

5 $m = -1, 2$

6 a i $\det \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = 1$

ii $\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1$

iii $\det \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = -1$

b i Dilation of factor k from the y-axis and dilation of factor $\frac{1}{k}$ from the x-axis

ii Determinant of matrix is 1

7 b $x = -1$

8 $m > 2$ or $m < 1$

9 $\begin{bmatrix} 1 & \pm\frac{\sqrt{3}}{2} \\ 0 & \pm\frac{1}{2} \end{bmatrix}$ or $\begin{bmatrix} \pm\frac{\sqrt{3}}{2} & 1 \\ \pm\frac{1}{2} & 0 \end{bmatrix}$

10 a $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Exercise 20H

1 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x+4 \end{bmatrix}$

2 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-2 \\ -y+2 \end{bmatrix}$

3 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}$ **b** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y-1 \\ -x-1 \end{bmatrix}$

c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ -y+2 \end{bmatrix}$ **d** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-4 \\ y \end{bmatrix}$

4 a $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

b $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

c $\mathbf{C} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

d $\mathbf{CBA} =$

$$\begin{bmatrix} \cos^2\theta + k\sin^2\theta & \cos\theta\sin\theta - k\sin\theta\cos\theta \\ \cos\theta\sin\theta - k\sin\theta\cos\theta & \sin^2\theta + k\cos^2\theta \end{bmatrix}$$

5 $\begin{bmatrix} \cos^2\theta & \cos\theta\,\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

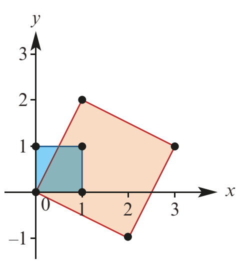
6 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$

Chapter 20 review

Technology-free questions

1 a $(7, 4)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

c Area = 5



d $(x, y) \to \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **f** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ **b** $\left(\frac{4}{5}, \frac{22}{5}\right)$

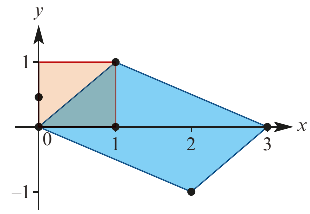
4 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

5 a $(x, y) \to (x-3, -y+4)$

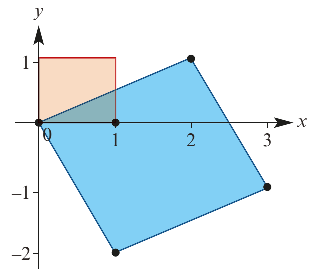
b $(x, y) \to (x-3, -y-4)$

6 a $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$

7 a Image area = 3 square units

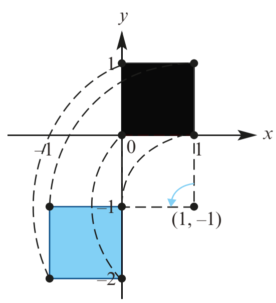


b Image area = 5 square units



8 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ x-2 \end{bmatrix}$ **b** $(1, 0)$

c



Multiple-choice questions

1 B **2** D **3** A **4** D **5** C
6 A **7** D **8** E **9** D

Extended-response questions

1 a $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

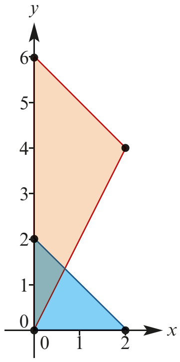
c Product of these two matrices:

$$\begin{bmatrix} \frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

d $\cos 75^\circ = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4}$

$\sin 75^\circ = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$

2 a



b Original area = 2 square units; Image area = 6 square units

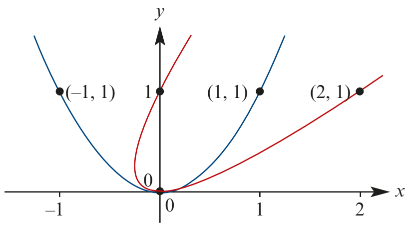
c 8π cubic units

3 a $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b Shear of factor 1 parallel to the x-axis

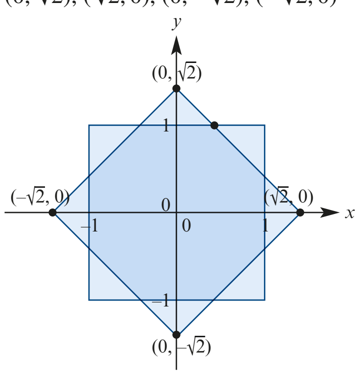
c $(0,0)$, $(2,1)$, $(0,1)$

d



4 a $(0,\sqrt{2})$, $(\sqrt{2},0)$, $(0,-\sqrt{2})$, $(-\sqrt{2},0)$

b



c $16 - 8\sqrt{2}$ square units

5 b i The composition of two rotations is a rotation

ii The composition of two reflections is a rotation

iii The composition of a reflection followed by a rotation is a reflection

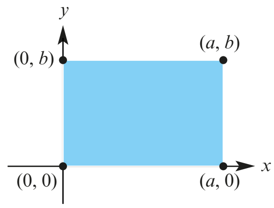
iv The composition of a rotation followed by a reflection is a reflection

c $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

6 a $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ **b** $A'(-1,-3)$ **c** $2\sqrt{10}$

d Isosceles **f** $2\sqrt{10}$

7 a



b $O(0,0)$, $A(a\cos\theta, a\sin\theta)$, $B(-b\sin\theta, b\cos\theta)$, $C(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$

8 a $y = \frac{1}{m} - \frac{x}{m}$; $(1,0)$, $\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$

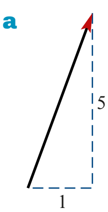
b $y = 1 - \frac{x}{m}$; $(0,1)$, $\left(\frac{2m}{1+m^2}, \frac{m^2-1}{1+m^2}\right)$

c $\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{bmatrix}$

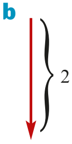
Chapter 21

Exercise 21A

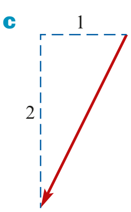
1 a



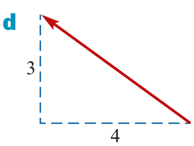
b



c



d



2 $a = 5$, $b = 1$

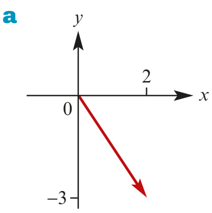
3 $a = 3$, $b = -15$

4 a $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ **d** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **e** $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

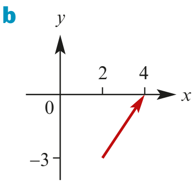
5 a i $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **ii** $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

b $\boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{c}$

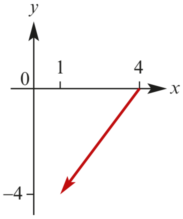
6 a



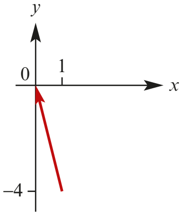
b



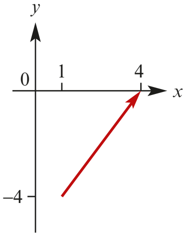
c



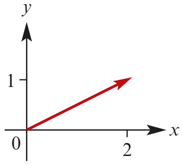
d



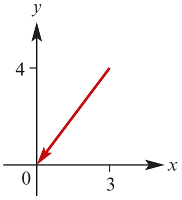
e



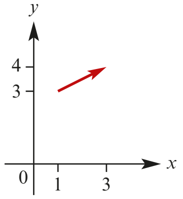
7 a



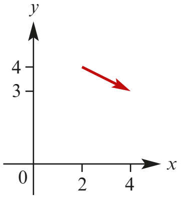
b



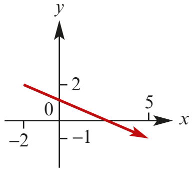
c



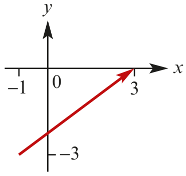
d



e

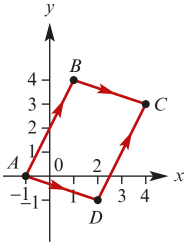


f



8 a and c

9 a b



d Parallelogram

10 $m = -11,\ n = 7$

11 a i $\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ **ii** $\boldsymbol{b}$

b $\overrightarrow{MN} = \overrightarrow{AD}$

12 a $\overrightarrow{CB} = \boldsymbol{a} - \boldsymbol{b},\ \overrightarrow{MN} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

b $\overrightarrow{CB} = -2\overrightarrow{MN}$

13 a $\boldsymbol{a}$ **b** $\boldsymbol{b}$ **c** $2\boldsymbol{a}$ **d** $2\boldsymbol{b}$

e $-\boldsymbol{a}$ **f** $\boldsymbol{b} - \boldsymbol{a}$ **g** $\boldsymbol{a} + \boldsymbol{b}$

14 a $\boldsymbol{a}$ **b** $-\boldsymbol{b}$ **c** $\boldsymbol{a} + \boldsymbol{b}$

d $-\boldsymbol{a} - \boldsymbol{b}$ **e** $\boldsymbol{b} - \boldsymbol{a}$

15 a $\boldsymbol{a} - \boldsymbol{b}$ **b** $\frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})$ **c** $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

d $\frac{1}{9}(\boldsymbol{a} + 2\boldsymbol{b})$ **e** $\frac{1}{9}(4\boldsymbol{a} - \boldsymbol{b})$

16 a $\boldsymbol{u} + \boldsymbol{v}$ **b** $\boldsymbol{v} + \boldsymbol{w}$ **c** $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$

17 a $\overrightarrow{OB} = \boldsymbol{u} + \boldsymbol{v},\ \overrightarrow{OM} = \boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$ **b** $\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$

c $\frac{2}{3}\left(\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}\right)$

d $\overrightarrow{OP} = \frac{2}{3}(\boldsymbol{u} + \boldsymbol{v}) = \frac{2}{3}\overrightarrow{OB}$ **e** 2 : 1

Exercise 21B

1 $2\boldsymbol{i} - 7\boldsymbol{j}$

2 a $5\boldsymbol{i} + 6\boldsymbol{j}$ **b** $-5\boldsymbol{i} + 6\boldsymbol{j}$ **c** $5\boldsymbol{i} - 6\boldsymbol{j}$

3 a 5 **b** 2 **c** 5 **d** 13

4 a 13 **b** $x = 2,\ y = -7$

5 $7\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

6 a i $\frac{2}{5}\boldsymbol{i}$ **ii** $-\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}$ **iii** $\frac{1}{6}\left(-\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}\right)$

iv $\frac{1}{3}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$ **v** $2\boldsymbol{i} + \boldsymbol{j}$

b i $\overrightarrow{ON} = \frac{1}{6}\overrightarrow{OA}$ **ii** 1 : 5

7 $4\sqrt{2}$ units

8 a $k = \frac{3}{2},\ \ell = \frac{1}{2}$ **b** $x = 6,\ y = 2$

c $x = 3,\ y = 3$ **d** $k = -\frac{1}{3},\ \ell = -\frac{5}{3}$

9 $3\boldsymbol{i} - 2\boldsymbol{j},\ \sqrt{13}$

10 a $-2\boldsymbol{i} + 4\boldsymbol{j}$ **b** $-6\boldsymbol{i} + \boldsymbol{j}$ **c** 5

11 a $D(-6, 3)$ **b** $F(4, -3)$ **c** $G\left(\frac{3}{2}, -\frac{3}{2}\right)$

12 $A(-1, -4),\ B(-2, 2),\ C(0, 10)$

13 a i $2\boldsymbol{i} - \boldsymbol{j}$ **ii** $-5\boldsymbol{i} + 4\boldsymbol{j}$ **iii** $\boldsymbol{i} + 7\boldsymbol{j}$

iv $6\boldsymbol{i} + 3\boldsymbol{j}$ **v** $6\boldsymbol{i} + 3\boldsymbol{j}$

b $D(8, 2)$

14 a $\overrightarrow{OP} = 12\boldsymbol{i} + 5\boldsymbol{j},\ \overrightarrow{PQ} = 6\boldsymbol{i} + 8\boldsymbol{j}$ **b** 13, 10

15 a i $\sqrt{29}$ **ii** $\sqrt{116}$ **iii** $\sqrt{145}$

b $(\sqrt{29})^2 + (\sqrt{116})^2 = (\sqrt{145})^2$

16 a i $-\boldsymbol{i} - 3\boldsymbol{j}$ **ii** $4\boldsymbol{i} + 2\boldsymbol{j}$ **iii** $-3\boldsymbol{i} + \boldsymbol{j}$

b i $\sqrt{10}$ **ii** $2\sqrt{5}$ **iii** $\sqrt{10}$

17 a i $-3\boldsymbol{i} + 2\boldsymbol{j}$ **ii** $7\boldsymbol{j}$

iii $-3\boldsymbol{i} - 5\boldsymbol{j}$ **iv** $\frac{1}{2}(-3\boldsymbol{i} - 5\boldsymbol{j})$

b $M\left(-\frac{3}{2}, \frac{9}{2}\right)$

18 a $\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ **b** $\frac{1}{\sqrt{10}}(3\boldsymbol{i} - \boldsymbol{j})$

c $\frac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$ **d** $\frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

e $\frac{6}{\sqrt{13}}\left(\frac{1}{2}\boldsymbol{i}+\frac{1}{3}\boldsymbol{j}\right)$ **f** $\frac{1}{\sqrt{13}}(3\boldsymbol{i}-2\boldsymbol{j})$

19 a $\frac{4}{5}(4\boldsymbol{i}+3\boldsymbol{j})$ **b** $-6\boldsymbol{i}-8\boldsymbol{j}$

Exercise 21C

1 a 17 **b** 13 **c** 8 **d** −10
e −4 **f** 3 **g** −58

2 a 5 **b** 13 **c** 8 **d** −5 **e** 13

3 a $15\sqrt{2}$ **b** $-15\sqrt{2}$

4 a $|\boldsymbol{a}|^2+4|\boldsymbol{b}|^2+4\boldsymbol{a}\cdot\boldsymbol{b}$ **b** $4\boldsymbol{a}\cdot\boldsymbol{b}$
c $|\boldsymbol{a}|^2-|\boldsymbol{b}|^2$ **d** $|\boldsymbol{a}|$

5 $\sqrt{66}$

6 a $-\frac{11}{2}$ **b** $\frac{10}{3}$ **c** −1 **d** $\frac{-2\pm\sqrt{76}}{6}$

7 a $-\boldsymbol{a}+q\boldsymbol{b}$ **b** $\frac{22}{29}$ **c** $\left(\frac{44}{29},\frac{110}{29}\right)$

8 a 139.40° **b** 71.57° **c** 26.57° **d** 126.87°

9 a $-3\boldsymbol{i}+\boldsymbol{j}$ **b** $\sqrt{10}$ **c** 116.57°

11 a $\frac{3}{2}\boldsymbol{i}$ **b** 45° **c** 116.57°

12 a i $\frac{3}{2}\boldsymbol{i}+2\boldsymbol{j}$ **ii** $\frac{1}{2}\boldsymbol{i}+3\boldsymbol{j}$
b 27.41°
c 55.30°

Exercise 21D

1 a $\frac{1}{\sqrt{10}}(\boldsymbol{i}+3\boldsymbol{j})$ **b** $\frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})$ **c** $\frac{1}{\sqrt{2}}(\boldsymbol{i}-\boldsymbol{j})$

2 a i $\frac{1}{5}(3\boldsymbol{i}+4\boldsymbol{j})$ **ii** $\sqrt{2}$
b $\frac{\sqrt{2}}{5}(3\boldsymbol{i}+4\boldsymbol{j})$

3 a i $\frac{1}{5}(3\boldsymbol{i}+4\boldsymbol{j})$ **ii** $\frac{1}{13}(5\boldsymbol{i}+12\boldsymbol{j})$
b $\frac{1}{\sqrt{65}}(4\boldsymbol{i}+7\boldsymbol{j})$

4 a $-\frac{11}{17}(\boldsymbol{i}-4\boldsymbol{j})$ **b** $\frac{13}{17}(\boldsymbol{i}-4\boldsymbol{j})$ **c** $4\boldsymbol{i}$

5 a 2 **b** $\frac{1}{\sqrt{5}}$ **c** $\frac{2\sqrt{3}}{\sqrt{7}}$ **d** $\frac{-1-4\sqrt{5}}{\sqrt{17}}$

6 a $\boldsymbol{a}=\boldsymbol{u}+\boldsymbol{w}$ where $\boldsymbol{u}=2\boldsymbol{i}$ and $\boldsymbol{w}=\boldsymbol{j}$
b $\boldsymbol{a}=\boldsymbol{u}+\boldsymbol{w}$ where $\boldsymbol{u}=2\boldsymbol{i}+2\boldsymbol{j}$ and $\boldsymbol{w}=\boldsymbol{i}-\boldsymbol{j}$
c $\boldsymbol{a}=\boldsymbol{u}+\boldsymbol{w}$ where $\boldsymbol{u}=\boldsymbol{0}$ and $\boldsymbol{w}=-\boldsymbol{i}+\boldsymbol{j}$

7 a $2\boldsymbol{i}+2\boldsymbol{j}$ **b** $\frac{1}{\sqrt{2}}(-\boldsymbol{i}+\boldsymbol{j})$

8 a $\frac{3}{2}(\boldsymbol{i}-\boldsymbol{j})$ **b** $\frac{5}{2}(\boldsymbol{i}+\boldsymbol{j})$ **c** $\frac{5\sqrt{2}}{2}$

9 a i $\boldsymbol{i}-\boldsymbol{j}$ **ii** $\boldsymbol{i}-5\boldsymbol{j}$
b $\frac{3}{13}(\boldsymbol{i}-5\boldsymbol{j})$ **c** $\frac{\sqrt{104}}{13}$ **d** 2

Exercise 21E

2 a i $\frac{4}{5}\boldsymbol{p}$ **ii** $\frac{1}{5}\boldsymbol{p}$ **iii** $-\boldsymbol{p}$ **iv** $\frac{1}{5}(\boldsymbol{q}-\boldsymbol{p})$ **v** $\frac{1}{5}\boldsymbol{q}$
b RS and OQ are parallel
c Trapezium
d 120 cm^2

3 a i $\frac{1}{3}\boldsymbol{a}+\frac{2}{3}\boldsymbol{b}$ **ii** $\frac{k}{7}\boldsymbol{a}+\frac{6}{7}\boldsymbol{b}$
b i 3 **ii** $\frac{7}{2}$

4 a i $\overrightarrow{OD}=2\boldsymbol{i}-0.5\boldsymbol{j}$, $\overrightarrow{OE}=\frac{15}{4}\boldsymbol{i}+\frac{9}{4}\boldsymbol{j}$
ii $\frac{\sqrt{170}}{4}$
b i $p\left(\frac{15}{4}\boldsymbol{i}+\frac{9}{4}\boldsymbol{j}\right)$
ii $(q+2)\boldsymbol{i}+(4q-0.5)\boldsymbol{j}$
c $p=\frac{2}{3}$, $q=\frac{1}{2}$

6 a i $\overrightarrow{AB}=\boldsymbol{c}$ **ii** $\overrightarrow{OB}=\boldsymbol{a}+\boldsymbol{c}$ **iii** $\overrightarrow{AC}=\boldsymbol{c}-\boldsymbol{a}$
b $|\boldsymbol{c}|^2-|\boldsymbol{a}|^2$

7 a $\boldsymbol{r}+\boldsymbol{t}$ **b** $\frac{1}{2}(\boldsymbol{s}+\boldsymbol{t})$

Exercise 21F

1 a $-\boldsymbol{i}-11\boldsymbol{j}$ **b** $5\boldsymbol{i}-6\boldsymbol{j}$ **c** $\boldsymbol{i}+5\boldsymbol{j}$
d $-11\boldsymbol{j}$ **e** $4\boldsymbol{i}$ **f** $6\boldsymbol{i}+11\boldsymbol{j}$

2 12.58 km on a bearing of 341.46°

3 7.74 km on a bearing of 071.17°

4 a $\sqrt{41}$ m/s **b** 5 m/s **c** $\sqrt{17}$ m/s
d $2\sqrt{10}$ m/s **e** 13 m/s **f** $\sqrt{170}$ m/s

5 a $24\boldsymbol{i}+62\boldsymbol{j}$ **b** $(5t-1)\boldsymbol{i}+(12t+2)\boldsymbol{j}$

6 $-4\boldsymbol{i}+4\boldsymbol{j}$ m/s

7 a i $26\boldsymbol{i}+99\boldsymbol{j}$
ii $(7t-2)\boldsymbol{i}+(24t+3)\boldsymbol{j}$
b i 102.36 m
ii $\sqrt{(7t-2)^2+(24t+3)^2}$ m

8 a $-\boldsymbol{i}-\frac{1}{2}\boldsymbol{j}$ m/s **b** $\frac{\sqrt{5}}{2}$ m/s

9 After $\frac{12(6+\sqrt{5})}{31}$ seconds;
position vector $\frac{12(6+\sqrt{5})}{31}(\boldsymbol{i}+2\boldsymbol{j})$

10 a $8\boldsymbol{i}+4\boldsymbol{j}$ **b** $2\boldsymbol{i}-4\boldsymbol{j}$ m/s

11 a $20\boldsymbol{i}+10\boldsymbol{j}$ **b** $\boldsymbol{j}$ m/s

Exercise 21G

1 a On a bearing of 143.13°
b 5 km/h

2 100.08 km/h on a bearing of 357.71°

3 a 20 km/h west **b** 180 km/h west

4 47 m/s north

5 10 m/s

6 a 20 km/h north **b** 20 km/h south

7 252.98 km/h on a bearing of 018.43°

8 100 km/h on a bearing of 053.13°

9 42.5 km/h on a bearing of 41.73°

10 a $\boldsymbol{i} - 4\boldsymbol{j}$ m/s **b** 4.12 m/s

11 10.36 m/s

12 196.83 km/h on a bearing of 345.44°

13 a Bearing 210.67°
b 243.28 km/h

14 a Upstream at an angle of 48.59° to his desired path
b 45.36 seconds

Exercise 21H

1 $T_1 = T_2 = \frac{5\sqrt{2}}{2}$ kg wt

2 90°

3 $T_1 = 14.99$ kg wt, $T_2 = 12.10$ kg wt

4 28.34 kg wt, W48.5°S

5 $T = 40$ kg wt, $N = 96$ kg wt

6 $F = 6.39$ kg wt

7 a No **b** Yes

8 146.88°, 51.32°, 161.8°

9 a 7.5 kg wt **b** 9.64 kg wt **c** 7.62 kg wt

10 32.97 kg wt, 26.88 kg wt, 39.29 kg wt, $W = 39.29$ kg

11 13.05 kg wt

12 5.74 kg wt

13 3.73 kg wt, 8.83 kg wt

14 4.13 kg wt

15 6.93 kg wt

16 31.11 kg, 23.84 kg wt

17 44.10 kg, 22.48° to the vertical

Exercise 21I

1 a $-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ **b** $3\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$ **c** $\sqrt{14}$
d $3\sqrt{2}$ **e** $-5\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$

2 a $2\boldsymbol{j} + 2\boldsymbol{k}$ **b** $\boldsymbol{i} + 2\boldsymbol{j}$ **c** $\boldsymbol{i} + 2\boldsymbol{k}$
d $\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ **e** $-2\boldsymbol{j}$ **f** $-2\boldsymbol{j} + 2\boldsymbol{k}$
g $\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$ **h** $\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$

3 a i $\frac{3}{\sqrt{11}}\boldsymbol{i} + \frac{1}{\sqrt{11}}\boldsymbol{j} - \frac{1}{\sqrt{11}}\boldsymbol{k}$
ii $-\frac{6}{\sqrt{11}}\boldsymbol{i} - \frac{2}{\sqrt{11}}\boldsymbol{j} + \frac{2}{\sqrt{11}}\boldsymbol{k}$
b $\frac{15}{\sqrt{11}}\boldsymbol{i} + \frac{5}{\sqrt{11}}\boldsymbol{j} - \frac{5}{\sqrt{11}}\boldsymbol{k}$

4 $\frac{\sqrt{14}}{3\sqrt{3}}(\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$

5 a $\boldsymbol{i} - 3\boldsymbol{j}$ **b** $\sqrt{10}$ **c** $\frac{3}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} - \boldsymbol{k}$

6 a $\frac{1}{6}\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ **b** $\frac{17}{6}$

Chapter 21 review

Technology-free questions

1 a $\frac{12}{7}$ **b** ± 9

2 $A(2, -1)$, $B(5, 3)$, $C(3, 8)$, $D(0, 4)$

3 $p = \frac{1}{6}$, $q = -\frac{11}{12}$

4 a $3\sqrt{10}$ **b** $\frac{1}{3\sqrt{10}}(\boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k})$

5 $x = 6$

6 a $\frac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$ **b** $\frac{16}{25}(4\boldsymbol{i} + 3\boldsymbol{j})$

7 a i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$ **iii** $\boldsymbol{b} - \boldsymbol{a}$
iv $\frac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$ **v** $\frac{2}{3}(2\boldsymbol{a} - \boldsymbol{b})$
b $\overrightarrow{TR} = 2\overrightarrow{PT}$, so P, T and R are collinear

8 a $s = -2$, $t = 5$, $u = 2$
b $\sqrt{33}$

9 $\sqrt{109}$ units

10 a $11\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ **b** $\sqrt{30}$
c $\frac{1}{\sqrt{30}}(5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$ **d** $2\boldsymbol{i} + 4\boldsymbol{j}$

11 a $(-1, 10)$ **b** $h = 3$, $k = -2$

12 $m = 2$, $n = 1$

13 a $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c}$ **b** $\boldsymbol{b} = \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$

14 a 13 **b** 10 **c** 8 **d** −11
e −9 **f** 0 **g** −27

16 a $\frac{6}{5}$ **b** $\pm\frac{3}{\sqrt{2}}$ **c** $\frac{7}{3}$

17 a i $\overrightarrow{AB} = -\boldsymbol{i}$ **ii** $\overrightarrow{AC} = -5\boldsymbol{j}$
b $\boldsymbol{0}$
c 1

18 a 2 m/s **b** 30 seconds
c 36 m downstream of her starting point

19 9 kg wt, 12 kg wt

20 $14\sqrt{5}$ kg wt, $28\sqrt{5}$ kg wt

21 $5\sqrt{3}$ kg wt

Multiple-choice questions

1 C	**2** C	**3** A	**4** B	**5** B
6 A	**7** C	**8** D	**9** A	**10** B
11 E	**12** C	**13** B	**14** B	

Extended-response questions

1 a $\begin{bmatrix} -31 \\ -32 \end{bmatrix}$ **b** $\begin{bmatrix} -15 \\ -20 \end{bmatrix}$ **c** $|OR| = 25$

2 a $\sqrt{34}$ **b** $\sqrt{10} - \sqrt{20}$
c $\boldsymbol{r} = \boldsymbol{i} - 9\boldsymbol{j}$

3 a $(25, -7)$, $\begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **b** $\begin{bmatrix} -20 \\ 15 \end{bmatrix}$

4 a $k = \frac{1}{2}$ **b** $x = -2,\ y = 2$
c $p = 2k,\ q = k,\ r = k$
$k \in \mathbb{R} \setminus \{0\}$

5 a $(12, 4)$ **b** $\begin{bmatrix} k - 12 \\ -4 \end{bmatrix}$
c $\sqrt{160},\ k,\ \sqrt{(k-12)^2 + 16},\ k = \frac{40}{3}$
d $34.7°$

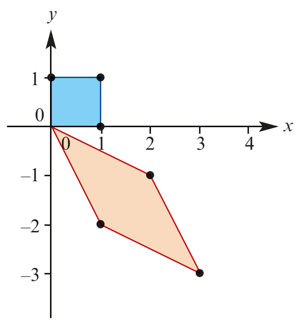
6 a 2.5 hours
b 011.54°, 36.7 minutes
c 168.46°

7 a $4\boldsymbol{i} + (16 - \alpha)\boldsymbol{j}$ km/h
b i $\alpha = 16$ **ii** 2.5 hours

Chapter 22

Technology-free questions

1 a $(7, -8)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$
c Area = 3



d $(x, y) \to \left(\frac{2}{3}x + \frac{1}{3}y, -\frac{1}{3}x - \frac{2}{3}y\right)$

2 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

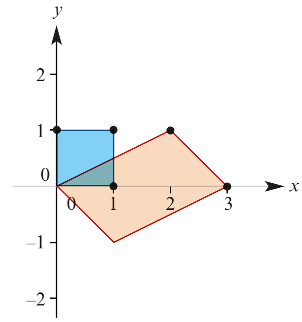
f $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **g** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

3 a $\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}$ **b** $\left(\frac{2}{17}, \frac{76}{17}\right)$

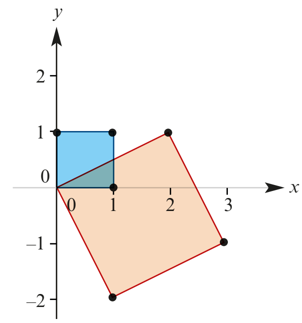
4 a $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **c** $\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$

5 a $(x, y) \to (-x + 2, y - 1)$
b $(x, y) \to (-x - 2, y - 1)$

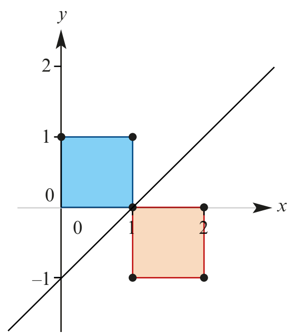
6 a Area = 3



b Area = 5



7 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y + 1 \\ x - 1 \end{bmatrix}$ **b** $(0, 0) \to (1, 1)$
c



8 a $2\sqrt{10}$ **b** $\frac{1}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$
c $\frac{8}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$ **d** $-\frac{2}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$

9 a 13 **b** 13 **c** 13 **d** −13
e −13 **f** 0 **g** −13

10 a $m = \frac{46}{11},\ n = -\frac{18}{11}$ **b** $p = -48$
c $p = 3, 5$

11 a $\sqrt{337}$ m/s on a bearing of $\tan^{-1}\left(\frac{16}{9}\right)$
b 8.5 seconds **c** 76.5 m

12 $F = 7$ kg wt, $\cos\theta = \frac{31}{49}$

13 a $T = 5$ kg wt, $N = 5\sqrt{3}$ kg wt
b $T = \frac{10\sqrt{3}}{3}$ kg wt, $N = \frac{20\sqrt{3}}{3}$ kg wt

14 $\frac{50}{13}$ kg wt, $\frac{120}{13}$ kg wt

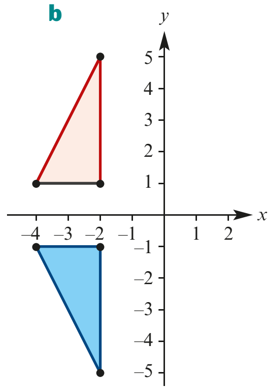
Multiple-choice questions

1 A **2** E **3** B **4** C **5** D **6** D
7 D **8** B **9** B **10** D **11** B **12** A
13 B **14** D **15** A **16** B **17** C **18** B
19 B **20** C **21** A **22** D **23** B **24** A

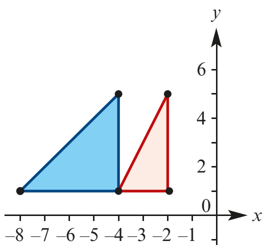
Extended-response questions

1 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+6 \\ y+3 \end{bmatrix}$

b



c



d $y = 2(x+3)^2 + 2$ **e** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+3 \\ -2y+4 \end{bmatrix}$

2 b i $x^2 + (y-1)^2 = 1$

ii $\left(x + \frac{4}{5}\right)^2 + \left(y - \frac{3}{5}\right)^2 = 1$

c $(0, 0), \left(-\frac{4}{5}, \frac{8}{5}\right)$

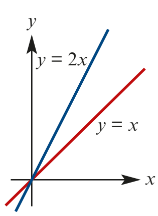
3 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

c $a = \sqrt{2},\ b = 0$ **d** $c = \frac{3\sqrt{2}}{2},\ d = \frac{\sqrt{2}}{2}$

e i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \\ -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{bmatrix}$

ii $\sqrt{2}(y - x) = (x + y)^2$

4 a

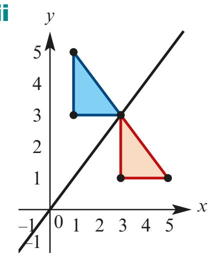


b $a = 2,\ b = \frac{\pi}{4}$

c $\begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

5 a i (3, 1) **ii** $A'(3, 1), B'(5, 1), C'(3, 3)$

iii



b ii $(-1, -1), (2, 2)$

iv $(-1, -1), (2, 2),$
$\left(\frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})\right),$
$\left(\frac{1}{2}(-1-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})\right)$

6 a $\overrightarrow{AE} = \frac{1}{t+1}(2\boldsymbol{a} + t\boldsymbol{b})$

b $\overrightarrow{AE} = \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$ **d** $t = \frac{9}{7}$

7 b $(n-1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c}$

8 a $4\sqrt{2}$ km/h blowing from the south-west
b $\sqrt{5}$ km/h; 200 m downstream
c 43.1 km/h on a bearing of 080° **d** 222°

9 b ii $\overrightarrow{ZG} = \frac{1}{3h}\overrightarrow{ZH} + \frac{1}{3k}\overrightarrow{ZK}$

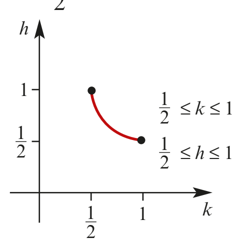
iii $\frac{1}{h} + \frac{1}{k} = 3$

iv $h = \frac{2}{3}$; similarity

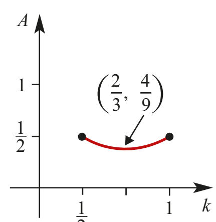
v $\frac{4}{9}$ cm^2

vi $h = \frac{1}{2}$; H is midpoint of ZX, $K = Y$

vii



viii



Chapter 23

Exercise 23A

1 **a** 12 cm to the right of O
b 2 cm to the right of O
c Moving to the left at 7 cm/s
d When $t = 3.5$ s and the particle is 0.25 cm to the left of O
e -2 cm/s
f 2.9 cm/s

2 **a** After 3.5 s **b** 2 m/s^2 **c** 14.5 m
d When $t = 2.5$ s and the particle is 1.25 m to the left of O

3 **a** 3 cm to the left of O moving to the right at 24 cm/s
b $v = 3t^2 - 22t + 24$
c After $\frac{4}{3}$ s and 6 s
d $11\frac{22}{27}$ cm to the right of O and 39 cm to the left of O
e $4\frac{2}{3}$ s
f $a = 6t - 22$
g When $t = \frac{11}{3}$ s and the particle is $13\frac{16}{27}$ cm left of O moving to the left at $16\frac{1}{3}$ cm/s

4 **a** When $t = \frac{2}{3}$ s and $a = -2$ cm/s^2; when $t = 1$ s and $a = 2$ cm/s^2
b When $t = \frac{5}{6}$ s and the particle is moving to the left at $\frac{1}{6}$ cm/s

5 When $t = 2$ s, $v = 6$ cm/s, $a = -14$ cm/s^2; when $t = 3$ s, $v = -5$ cm/s, $a = -8$ cm/s^2; when $t = 8$ s, $v = 30$ cm/s, $a = 22$ cm/s^2

6 **a** $t = 4$ s and $t = -1$ s
b $t = \frac{3}{2}$ s

Exercise 23B

1 **a** $x = 2t^2 - 6t$ **b** At the origin O
c 9 cm **d** 0 cm/s **e** 3 cm/s

2 **a** $x = t^3 - 4t^2 + 5t + 4$, $a = 6t - 8$
b When $t = 1$, $x = 6$; when $t = \frac{5}{3}$, $x = 5\frac{23}{27}$
c When $t = 1$, $a = -2$ m/s^2; when $t = \frac{5}{3}$, $a = 2$ m/s^2

3 20 m to the left of O

4 $x = 215\frac{1}{3}$ m, $v = 73$ m/s

5 **a** $v = -10t + 25$ **b** $x = -5t^2 + 25t$
c 2.5 s **d** $31\frac{1}{4}$ m **e** 5 s

6 29th floor

Exercise 23C

1 $2\sqrt{10}$ s

2 37.5 m

3 **a** 3 m/s^2 **b** $6\frac{2}{3}$ s **c** 337.5 m **d** $\frac{500}{27}$ s

4 **a** 2.5 m/s^2 **b** 31.25 m

5 **a** 50 s **b** 625 m

6 **a** 20 s **b** 10 m/s

7 **a** -19.2 m/s **b** 1.6 m

8 **a** -59.2 m/s **b** -158.4 m

9 **a** 10 s **b** After 3 s and 7 s

10 **a** $4.9(1 - 2t)$ m/s **b** $4.9t(1 - t) + 3$ m
c 4.225 m **d** $\frac{10}{7}$ s

11 **a** 2 s **b** 44.1 m **c** 4 s **d** 5 s

12 $10\sqrt{10}$ m/s

Exercise 23D

1 65 m

2 **a** 562.5 m **b** 450 m **c** 23.75 s

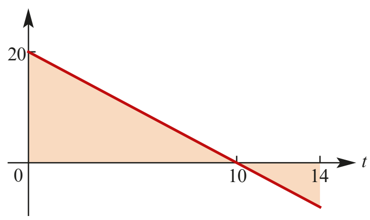
3 $\frac{200}{3}$ m/s

4 210 m

5 **a** 500 m **b** 375 m **c** 17.57 s

6 **a** 12.5 s **b** 187.5 m

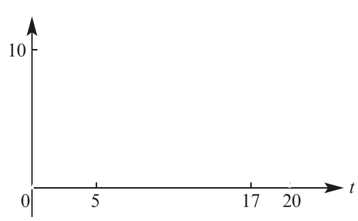
7 **a**



b From initial position O, the particle moves to the right with initial velocity 20 m/s. It slows until after 10 seconds it is 100 m from O and momentarily stops. It then moves to the left towards O, getting faster.
c 116 m
d 84 m to the right of initial position

8 **a** 1 m/s^2 **b** -2.5 m/s^2 **c** 215 m
d 125 m to the right of initial position

9 **a**



b $\frac{10}{3}$ m/s^2

10 No, the first train will stop after 6.25 km and the second train will stop after 6 km.

11 a 57.6 km/h **b** 1 minute $6\frac{2}{3}$ seconds
c $a = 0.24$

Chapter 23 review

Technology-free questions

1 a 5 cm to the left of O
b 8 cm to the left of O **c** −4 cm/s
d $t = 2$ s, 9 cm to the left of O **e** −1 cm/s
f $1\frac{2}{3}$ cm/s

2 a 8 cm to the right, 0 cm/s, −4 cm/s^2
b At $t = 0$ s, 8 cm to the right, −4 cm/s^2; at $t = \frac{4}{3}$ s, $6\frac{22}{27}$ cm to the right, 4 cm/s^2

3 a 3.5 s, −40.5 cm/s, −36 cm/s^2 **b** 2 s
c 31 cm

4 a i $\frac{1}{8}$ cm to the left **ii** 1 cm/s^2 **iii** 1 cm/s
b i 0 s, 2 s **ii** $\frac{32}{27}$ cm

5 a 12 m/s **b** $x = t^3$

6 a 4 s **b** $18\frac{2}{3}$ m to the right **c** −5 m/s^2
d 1.5 s **e** $6\frac{1}{4}$ m/s

7 a $\frac{1}{12}$ m to the left **b** −1 m/s **c** −5 m/s^2

8 a $a = -\frac{1}{t^3}$ **b** $x = \frac{1}{2} - \frac{1}{2t}$

9 a $a = 3t^2 - 22t + 24$ **b** −15 m/s^2
c $2\frac{1}{12}$ m to the left, $60\frac{7}{12}$ m

10 40 m

11 a 2.5 m/s^2 **b** 8 s **c** 500 m **d** $\frac{100}{9}$ s

12 a $41\frac{2}{3}$ s **b** $347\frac{2}{9}$ m

13 a 7.143 s **b** $2\frac{6}{7}$ s, $4\frac{2}{7}$ s

14 a 2 s **b** 39.6 m **c** 4 s **d** 4.84 s

15 437.5 m

16 a 288 m **b** 16 s

17 16 m/s

18 $\frac{80}{81}$ m/s^2

19 a 0 m/s **b** −3 m/s^2 **c** −4 m/s
d $4\frac{2}{3}$ m **e** $\frac{11}{12}$ m

20 a $2t - t^2 + 8$ **b** $t^2 - \frac{t^3}{3} + 8t$

21 b i 8 m/s **ii** 2 s **iii** 18 m

22 a 27 m/s^2 **b** 50 m/s **c** 4.5 s

23 a −10 m/s **b** 0 m

24 a 4 s, 6 s **b** 36 m **c** $0 \le t < 5$

Multiple-choice questions

1 A **2** E **3** C **4** C **5** E
6 C **7** D **8** E **9** A **10** D

Extended-response questions

1 a $2\frac{1}{3}$ cm to the left of O **b** 4 cm/s
c 2 cm/s^2 **d** At 2 s
e $\frac{1}{3}$ cm to the right of O **f** At 1 s

3 a After 6 s at −36 m/s
b When $t = 0$ or $t = 4$; when $t = 4$, the maximum height is 32 m
c After 2 s

4 $x(1) - x(0) = 15.1$, $x(2) - x(1) = 5.3$,
$x(3) - x(2) = -4.5$, $x(4) - x(3) = -14.3$,
$x(5) - x(4) = -24.1$, $x(6) - x(5) = -33.9$,
$x(7) - x(6) = -43.7$, $x(8) - x(7) = -53.5$,
$x(9) - x(8) = -63.3$, $x(10) - x(9) = -73.1$

The constant difference between successive numbers is −9.8 (acceleration due to gravity)

6 33 m

7 a $v = -5t + 25,\ 0 \le t \le 5$ **b** 62.5 m

8 25 m to the left of O

9 b The second particle is projected upwards at the instant the first particle lands.
c The second particle is projected upwards after the first particle has landed, so there is no collision.