Chapter 1 – Preliminary topics

Solutions to Exercise 1A

1 **a** **i** $720^\circ = \left(720 \times \frac{\pi}{180}\right)^c = 4\pi^c$

ii $540^\circ = \left(540 \times \frac{\pi}{180}\right)^c = 3\pi^c$

iii $-450^\circ = \left(-450 \times \frac{\pi}{180}\right)^c = \frac{-5\pi}{2}^c$

iv $15^\circ = \left(15 \times \frac{\pi}{180}\right)^c = \frac{\pi}{12}^c$

v $-10^\circ = \left(-10 \times \frac{\pi}{180}\right)^c = \frac{-\pi}{18}^c$

vi $-315^\circ = \left(-315 \times \frac{\pi}{180}\right)^c = \frac{-7\pi}{4}^c$

b **i** $\frac{5\pi}{4}^c = \left(\frac{5\pi}{4} \times \frac{180}{\pi}\right)^\circ = 225^\circ$

ii $\frac{-2\pi}{3}^c = \left(\frac{-2\pi}{3} \times \frac{180}{\pi}\right)^\circ = -120^\circ$

iii $\frac{7\pi}{12}^c = \left(\frac{7\pi}{12} \times \frac{180}{\pi}\right)^\circ = 105^\circ$

iv $\frac{-11\pi}{6}^c = \left(\frac{-11\pi}{6} \times \frac{180}{\pi}\right)^\circ = -330^\circ$

v $\frac{13\pi}{9}^c = \left(\frac{13\pi}{9} \times \frac{180}{\pi}\right)^\circ = 260^\circ$

vi $\frac{-11\pi}{12}^c = \left(\frac{-11\pi}{12} \times \frac{180}{\pi}\right)^\circ = -165^\circ$

2 **a** **i** $7^\circ = \left(7 \times \frac{\pi}{180}\right)^c \approx 0.12^c$

ii $-100^\circ = \left(-100 \times \frac{\pi}{180}\right)^c \approx -1.75^c$

iii $-25^\circ = \left(-25 \times \frac{\pi}{180}\right)^c \approx -0.44^c$

iv $51^\circ = \left(51 \times \frac{\pi}{180}\right)^c \approx 0.89^c$

v $206^\circ = \left(206 \times \frac{\pi}{180}\right)^c \approx 3.60^c$

vi $-410^\circ = \left(-410 \times \frac{\pi}{180}\right)^c \approx -7.16^c$

b **i** $1.7^c = \left(1.7 \times \frac{180}{\pi}\right)^\circ \approx 97.40^\circ$

ii $-0.87^c = \left(-0.87 \times \frac{180}{\pi}\right)^\circ \approx -49.85^\circ$

iii $2.8^c = \left(2.8 \times \frac{180}{\pi}\right)^\circ \approx 160.43^\circ$

iv $0.1^c = \left(0.1 \times \frac{180}{\pi}\right)^\circ \approx 5.73^\circ$

v $-3^c = \left(-3 \times \frac{180}{\pi}\right)^\circ \approx -171.89^\circ$

vi $-8.9^c = \left(-8.9 \times \frac{180}{\pi}\right)^\circ \approx -509.93^\circ$

3 **a**
$$\begin{aligned}\sin(135^\circ) &= \sin(180 - 45)^\circ \\ &= \sin(45^\circ) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

b
$$\begin{aligned}\cos(-300^\circ) &= \cos(300)^\circ \\ &= \cos(360 - 60)^\circ \\ &= \cos(60^\circ) \\ &= \frac{1}{2}\end{aligned}$$

c $\sin(480°) = \sin(540 - 60)°$

$= \sin(180 - 60)°$

$= \sin(60)°$

$= \dfrac{\sqrt{3}}{2}$

d $\cos(240°) = \cos(180 + 60)°$

$= -\cos(60°)$

$= \dfrac{-1}{2}$

e $\sin(-225°) = -\sin(225°)$

$= -\sin(180 + 45)°$

$= \sin(45°)$

$= \dfrac{\sqrt{2}}{2}$

f $\sin(420°) = \sin(360 + 60)°$

$= \sin(60°)$

$= \dfrac{\sqrt{3}}{2}$

4 a $\sin\left(\dfrac{2\pi}{3}\right) = \sin\left(\pi - \dfrac{2\pi}{3}\right)$

$= \sin\left(\dfrac{\pi}{3}\right)$

$= \dfrac{\sqrt{3}}{2}$

b $\cos\left(\dfrac{3\pi}{4}\right) = -\cos\left(\pi - \dfrac{3\pi}{4}\right)$

$= -\cos\left(\dfrac{\pi}{4}\right)$

$= \dfrac{-\sqrt{2}}{2}$

c $\cos\left(\dfrac{-\pi}{3}\right) = \cos\left(\dfrac{\pi}{3}\right)$

$= \dfrac{1}{2}$

d $\cos\left(\dfrac{5\pi}{4}\right) = \cos\left(\pi + \dfrac{\pi}{4}\right)$

$= -\cos\left(\dfrac{\pi}{4}\right)$

$= \dfrac{-\sqrt{2}}{2}$

e $\cos\left(\dfrac{9\pi}{4}\right) = \cos\left(2\pi + \dfrac{\pi}{4}\right)$

$= \cos\left(\dfrac{\pi}{4}\right)$

$= \dfrac{\sqrt{2}}{2}$

f $\sin\left(\dfrac{11\pi}{3}\right) = \sin\left(4\pi - \dfrac{\pi}{3}\right)$

$= -\sin\left(\dfrac{\pi}{3}\right)$

$= \dfrac{-\sqrt{3}}{2}$

g $\cos\left(\dfrac{31\pi}{6}\right) = \cos\left(5\pi + \dfrac{\pi}{6}\right)$

$= -\cos\left(\dfrac{\pi}{6}\right)$

$= \dfrac{-\sqrt{3}}{2}$

h $\cos\left(\dfrac{29\pi}{6}\right) = \cos\left(5\pi - \dfrac{\pi}{6}\right)$

$= -\cos\left(\dfrac{\pi}{6}\right)$

$= \dfrac{-\sqrt{3}}{2}$

i $\sin\left(\dfrac{-23\pi}{6}\right) = -\sin\left(\dfrac{23\pi}{6}\right)$

$= -\sin\left(4\pi - \dfrac{\pi}{6}\right)$

$= \sin\left(\dfrac{\pi}{6}\right)$

$= \dfrac{1}{2}$

5 **a** $\sin^2(x°) + \cos^2(x°) = 1$

$\therefore 0.25 + \cos^2(x°) = 1$

$\therefore \cos^2(x°) = \frac{3}{4}$

$\therefore \cos(x°) = \pm\sqrt{\frac{3}{4}}$

$\therefore \cos(x°) = \frac{-\sqrt{3}}{2}$ as $90 < x < 180$

b $\tan(x°) = \frac{\sin(x°)}{\cos(x°)}$

$= \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$

$= -\frac{1}{2} \times \frac{2}{\sqrt{3}}$

$= -\frac{1}{\sqrt{3}}$

$= -\frac{\sqrt{3}}{3}$

6 **a** $\sin^2(x°) + \cos^2(x°) = 1$

$\therefore \sin^2(x°) + 0.49 = 1$

$\therefore \sin^2(x°) = \frac{51}{100}$

$\therefore \sin(x°) = \pm\sqrt{\frac{51}{100}}$

$\therefore \sin(x°) = -\frac{\sqrt{51}}{10}$ as $180 < x < 270$

b $\tan(x°) = \frac{\sin(x°)}{\cos(x°)}$

$= \frac{-\frac{\sqrt{51}}{10}}{-\frac{7}{10}}$

$= \frac{\sqrt{51}}{10} \times \frac{10}{7}$

$= \frac{\sqrt{51}}{7}$

7 **a** $\sin^2(x) + \cos^2(x) = 1$

$\therefore 0.25 + \cos^2(x) = 1$

$\therefore \cos^2(x) = \frac{3}{4}$

$\therefore \cos(x) = \pm\sqrt{\frac{3}{4}}$

$\therefore \cos(x) = -\frac{\sqrt{3}}{2}$ as $\pi < x \leq \frac{3\pi}{2}$

b $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$

$= \frac{1}{2} \times \frac{2}{\sqrt{3}}$

$= \frac{\sqrt{3}}{3}$

8 **a** $\sin^2(x) + \cos^2(x) = 1$

$\therefore 0.09 + \cos^2(x) = 1$

$\therefore \cos^2(x) = \frac{91}{100}$

$\therefore \cos(x) = \pm\sqrt{\frac{91}{100}}$

$\therefore \cos(x) = \frac{\sqrt{91}}{10}$ as $\frac{3\pi}{2} < x \le 2\pi$

b $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$= \frac{-\frac{3}{10}}{\frac{\sqrt{91}}{10}}$

$= -\frac{3}{10} \times \frac{10}{\sqrt{91}}$

$= -\frac{3\sqrt{91}}{91}$

9 The graph of cosine is that of an even function and the period is 2π
Hence,
$f(a) = f(-a) = f(2\pi - a)$
$f(b) = f(-b) = f(2\pi - b)$
$f(c) = f(-c) = f(2\pi - c)$
$f(d) = f(-d) = f(2\pi - d)$

10 a $\sin x = \frac{-\sqrt{3}}{2}$

$\therefore x = \frac{4\pi}{3}, \frac{5\pi}{3}$ as $x \in [0, 2\pi]$

b $\sin(2x) = -\frac{\sqrt{3}}{2}, \ x \in [0, 2\pi]$

$\therefore 2x \in [0, 4\pi]$

$\therefore 2x = \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi + \frac{4\pi}{3}, 2\pi + \frac{5\pi}{3}$
as $2x \in [0, 4\pi]$

$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$ as $x \in [0, 2\pi]$

c $2\cos 2x = -1$

$\therefore \cos 2x = -\frac{1}{2}, \ x \in [0, 2\pi]$

$\therefore 2x \in [0, 4\pi]$

$\therefore 2x = \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi + \frac{2\pi}{3}, 2\pi + \frac{4\pi}{3}$
as $2x \in [0, 4\pi]$

$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
as $x \in [0, 2\pi]$

d $\sin\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}, \ x \in [0, 2\pi]$

$\therefore x + \frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$

$\therefore x + \frac{\pi}{3} = \frac{7\pi}{6}, \frac{11\pi}{6}$
as $x + \frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$

$\therefore x = \frac{5\pi}{6}, \frac{3\pi}{2}$ as $x \in [0, 2\pi]$

e $2\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -1$

$\therefore \cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -\frac{1}{2}, \ x \in [0, 2\pi]$

$\therefore x + \frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$

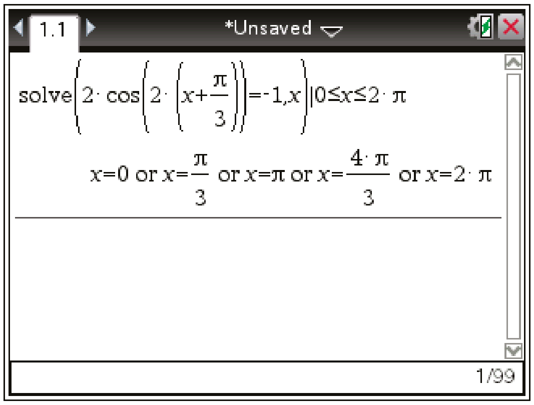
$\therefore 2\left(x + \frac{\pi}{3}\right) \in \left[\frac{2\pi}{3}, \frac{14\pi}{3}\right]$

$\therefore 2\left(x + \frac{\pi}{3}\right) = \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi + \frac{2\pi}{3}, 2\pi + \frac{4\pi}{3}, 4\pi + \frac{2\pi}{3}$
as $2\left(x + \frac{\pi}{3}\right) \in \left[\frac{2\pi}{3}, \frac{14\pi}{3}\right]$

$\therefore x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\therefore x = 0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ as $x \in [0, 2\pi]$

CAS: Type
solve $\left(\mathbf{2\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -1, x}\right) \mathbf{\Big| 0 \le x \le 2\pi}$
For part **e** we have,



f $2\sin\left(2x+\frac{\pi}{3}\right)=-\sqrt{3}$

$\therefore \sin\left(2x+\frac{\pi}{3}\right)=\frac{-\sqrt{3}}{2},\ x\in[0,\ 2\pi]$

$\therefore 2x\ \in[0,\ 4\pi]$

$\therefore\ 2x+\frac{\pi}{3}\in\left[\frac{\pi}{3},\ \frac{13\pi}{3}\right]$

$\therefore 2x+\frac{\pi}{3}=\frac{4\pi}{3},\ \frac{5\pi}{3},\ 2\pi+\frac{4\pi}{3},\ 2\pi+\frac{5\pi}{3}$

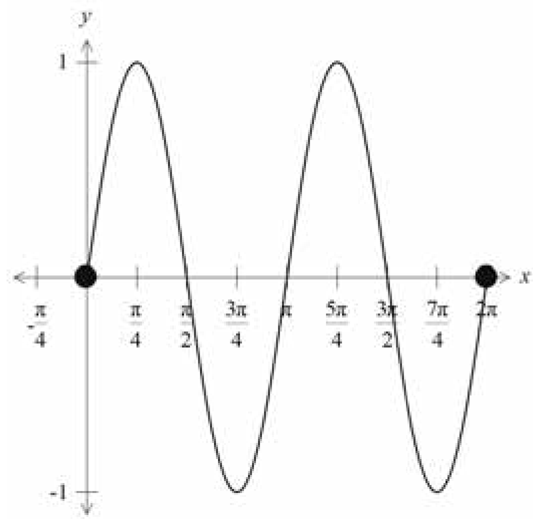
as $2x+\frac{\pi}{3}\in\left[\frac{\pi}{3},\ \frac{13\pi}{3}\right]$

$\therefore 2x=\pi,\ \frac{4\pi}{3},\ 3\pi,\ \frac{10\pi}{3}$

$\therefore x=\frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{3\pi}{2},\ \frac{5\pi}{3}$ as $x\in[0,\ 2\pi]$

11 a $f(x)=\sin 2x,\ x\in[0,2\pi]$

The transformation from the graph of $g(x)=\sin x$ is a dilation from the y axis of factor $\frac{1}{2}$.



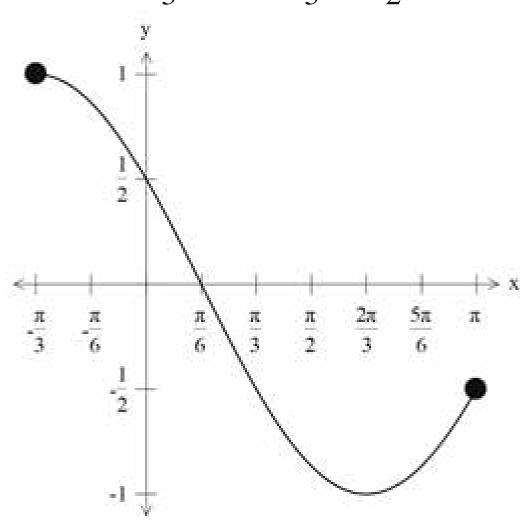
b $f(x)=\cos\left(x+\frac{\pi}{3}\right),\ x\in\left[-\frac{\pi}{3},\ \pi\right]$

The transformation from the graph of $g(x)=\cos x$ is a translation of $\frac{\pi}{3}$ to the left.

$f\left(\frac{-\pi}{3}\right)=\cos 0=1$

$f(0)=\cos\frac{\pi}{3}=\frac{1}{2}$

$f(\pi)=\cos\frac{4\pi}{3}=-\cos\frac{\pi}{3}=-\frac{1}{2}$



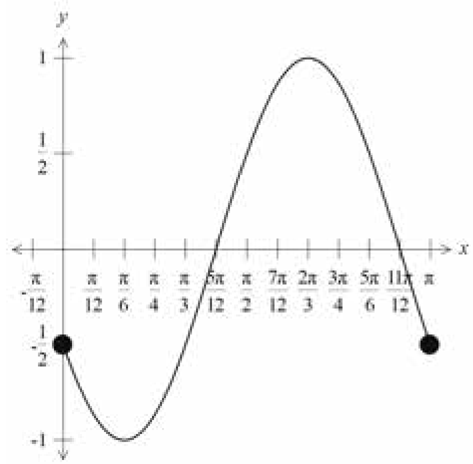
c $f(x)=\cos\left(2\left(x+\frac{\pi}{3}\right)\right),\ x\in[0,\ \pi]$

The transformations from the graph of $g(x)=\cos x$ are a dilation from the y axis of factor $\frac{1}{2}$ and a translation of

$\frac{\pi}{3}$ to the left.

$f(0) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$f(\pi) = \cos\left(\frac{8\pi}{3}\right) = -\frac{1}{2}$



d $f(x) = 2\sin(3x) + 1,\ x \in [0,\ \pi]$

The transformations from the graph of $g(x) = \sin x$ are a dilation from the y axis of factor $\frac{1}{3}$, a dilation from the x axis of factor 2 and a translation of 1 in the positive direction of the y axis.

To find x axis intercepts for $f(x)$, solve $f(x) = 0$

i.e. $2\sin(3x) + 1 = 0,\ x \in [0,\ \pi]$

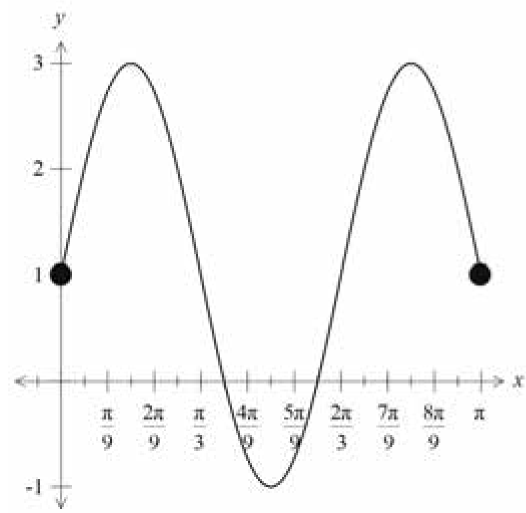
$\therefore \sin(3x) = -\frac{1}{2},\ 3x \in [0,\ 3\pi]$

$\therefore 3x = \frac{7\pi}{6},\ \frac{11\pi}{6}$

$\therefore x = \frac{7\pi}{18},\ \frac{11\pi}{18}$

$f(0) = 1,\ f(\pi) = 2\sin(3\pi) + 1$

$= 1$



e $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3},\ x \in [0,\ 2\pi]$

The transformations from the graph of $g(x) = \sin x$ are a dilation from the x axis of factor 2, a translation of $\frac{\pi}{4}$ to the right and a translation of $\sqrt{3}$ in the positive direction of the y axis.

$f(0) = 2\sin\left(-\frac{\pi}{4}\right) + \sqrt{3}$

$= -2\sin\left(\frac{\pi}{4}\right) + \sqrt{3}$

$= \sqrt{3} - \sqrt{2}$

$f(2\pi) = 2\sin\left(\frac{7\pi}{4}\right) + \sqrt{3}$

$= \sqrt{3} - \sqrt{2}$

To find x axis intercepts for $f(x)$, solve $f(x) = 0$

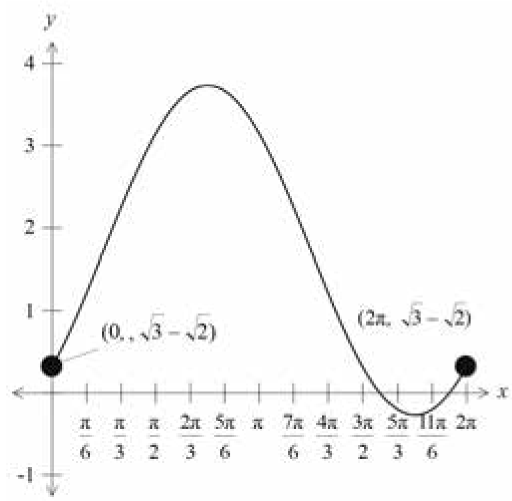
i.e. $2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3} = 0,\ x \in [0,\ 2\pi]$

$\therefore \sin\left(x - \frac{\pi}{4}\right) = \frac{-\sqrt{3}}{2},$

$x - \frac{\pi}{4} \in \left[-\frac{\pi}{4},\ \frac{7\pi}{4}\right]$

$\therefore x - \frac{\pi}{4} = \frac{4\pi}{3},\ \frac{5\pi}{3}$

$\therefore x = \frac{19\pi}{12},\ \frac{23\pi}{12}$



12 **a** $\tan\left(\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right)$

$= \tan\left(\frac{\pi}{4}\right)$

$= 1$

b $\tan\left(-\frac{2\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right)$

$= \tan\left(\frac{\pi}{3}\right)$

$= \sqrt{3}$

c $\tan\left(-\frac{29\pi}{6}\right) = \tan\left(-5\pi + \frac{\pi}{6}\right)$

$= \tan\left(\pi + \frac{\pi}{6}\right)$

$= \tan\left(\frac{\pi}{6}\right)$

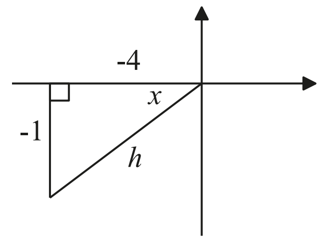
$= \frac{\sqrt{3}}{3}$

d $\tan(240°) = \tan(180 + 60)°$

$= \tan(60)°$

$= \sqrt{3}$

13



$h^2 = 1 + 16$

$\therefore h = \sqrt{17}$

a $\sin x = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$

b $\cos x = -\frac{4}{\sqrt{17}} = -\frac{4\sqrt{17}}{17}$

c Since $\pi \le x \le \frac{3\pi}{2}$

$\therefore -\frac{3\pi}{2} \le -x \le -\pi \Leftrightarrow \frac{\pi}{2} \le -x \le \pi$

$\therefore \tan(-x) = -\frac{1}{4}$ as $\frac{\pi}{2} \le -x \le \pi$

d Since $\pi \le x \le \frac{3\pi}{2}$

$\therefore \frac{3\pi}{2} \le \pi - x \le 2\pi$

$\therefore \tan(\pi - x) = -\frac{1}{4}$ as $\frac{3\pi}{2} \le \pi - x \le 2\pi$

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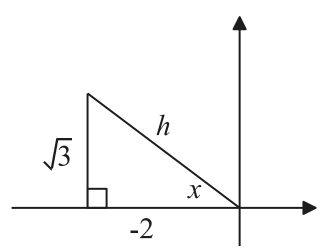


$h^2 = 3 + 4$

$\therefore h = \sqrt{7}$

a $\sin x = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{21}}{7}$

b $\cos x = \dfrac{-2}{\sqrt{7}} = \dfrac{-2\sqrt{7}}{7}$

c Since $\dfrac{\pi}{2} \le x \le \pi$

$\therefore -\pi \le -x \le -\dfrac{\pi}{2} \Leftrightarrow \pi \le -x \le \dfrac{3\pi}{2}$

$\therefore \tan(-x) = \dfrac{\sqrt{3}}{2}$ as $\pi \le -x \le \dfrac{3\pi}{2}$

d Since $\dfrac{\pi}{2} \le x \le \pi$

$\therefore -\dfrac{\pi}{2} \le x\pi \le 0$

$\therefore \tan(x - \pi) = -\dfrac{\sqrt{3}}{2}$ as

$-\dfrac{\pi}{2} \le x - \pi \le 0$

15 a $\tan x = -\sqrt{3}$

$\therefore x = \pi - \dfrac{\pi}{3}, 2\pi - \dfrac{\pi}{3}$

$\therefore x = \dfrac{2\pi}{3}, \dfrac{5\pi}{3}$ as $x \in [0, 2\pi]$

b $\tan\left(3x - \dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{3}$

as $x \in [0, 2\pi]$

$\therefore 3x \in [0, 6\pi]$

$\therefore 3x - \dfrac{\pi}{6} \in \left[-\dfrac{\pi}{6}, \dfrac{35\pi}{6}\right]$

$\therefore 3x - \dfrac{\pi}{6} = \dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{\pi}{6} + 2\pi, \dfrac{7\pi}{6} +$
$2\pi, \dfrac{\pi}{6} + 4\pi, \dfrac{7\pi}{6} + 4\pi$

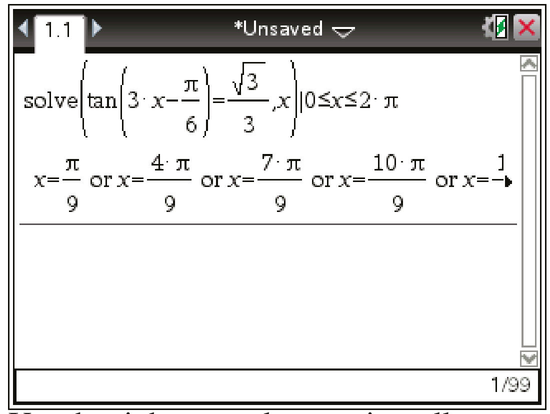
$\therefore 3x - \dfrac{\pi}{6} =$
$\dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{13\pi}{6}, \dfrac{19\pi}{6}, \dfrac{25\pi}{6}, \dfrac{31\pi}{6}$

$\therefore 3x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}, \dfrac{7\pi}{3}, \dfrac{10\pi}{3}, \dfrac{13\pi}{3}, \dfrac{16\pi}{3}$

$\therefore x = \dfrac{\pi}{9}, \dfrac{4\pi}{9}, \dfrac{7\pi}{9}, \dfrac{10\pi}{9}, \dfrac{13\pi}{9}, \dfrac{16\pi}{9}$

CAS: Type

$\mathbf{solve}\left(\mathbf{tan}\left(\mathbf{3}\boldsymbol{x} - \dfrac{\boldsymbol{\pi}}{\mathbf{6}}\right) = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{3}}, \boldsymbol{x}\right)\Big| \mathbf{0} \le \boldsymbol{x}$
$\le \mathbf{2}\boldsymbol{\pi}$



Use the right arrow key to view all solutions.

c $2\tan\left(\dfrac{x}{2}\right) + 2 = 0$

$\therefore \tan\left(\dfrac{x}{2}\right) = -1$

and $\dfrac{x}{2} \in [0, \pi]$

$\therefore \dfrac{x}{2} = \dfrac{3\pi}{4}$ as $\dfrac{x}{2} \in [0, \pi]$

$\therefore x = \dfrac{3\pi}{2}$

d $3\tan\left(\dfrac{\pi}{2} + 2x\right) = -3$

$\therefore \tan\left(\dfrac{\pi}{2} + 2x\right) = -1$

as $x \in [0, 2\pi]$

$\therefore \dfrac{\pi}{2} + 2x \in \left[\dfrac{\pi}{2}, \dfrac{9\pi}{2}\right]$

$\therefore \dfrac{\pi}{2} + 2x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{3\pi}{4} + 2\pi, \dfrac{7\pi}{4} + 2\pi$

$\therefore \dfrac{\pi}{2} + 2x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \dfrac{15\pi}{4}$

$\therefore 2x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$

$\therefore x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$

16 a $f(x) = \tan(2x)$

Period: $= \dfrac{\pi}{|n|} = \dfrac{\pi}{2}$

Asymptotes:

$x = \dfrac{(2k+1)\pi}{2n}$

$\therefore x = \dfrac{(2k+1)\pi}{4}$

$\therefore\ x = \frac{\pi}{4},\ \frac{3\pi}{4}$ as $x \in [0,\ \pi]$

x-intercepts:

as $x \in [0,\ \pi]$

$\therefore\ 2x \in [0, 2\pi]$

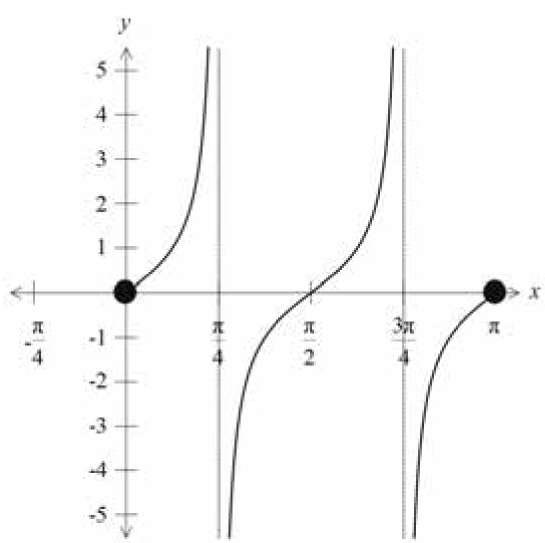
$\tan(2x) = 0$

$\therefore\ 2x = 0,\ \pi,\ 2\pi$

$\therefore\ x = 0,\ \frac{\pi}{2},\ \pi$

y-intercept:

$f(0) = \tan(0) = 0$



b $f(x) = \tan\left(x - \frac{\pi}{3}\right)$

Period: $= \frac{\pi}{|n|} = \pi$

Asymptotes:

$x = \frac{(2k+1)\pi}{2n} + \frac{\pi}{3}$

$\therefore\ x = \frac{(2k+1)\pi}{2} + \frac{\pi}{3}$

$\therefore\ x = \frac{\pi}{2} + \frac{\pi}{3}$ as $x \in [0,\ \pi]$

$\therefore\ x = \frac{5\pi}{6}$

x-intercepts:

as $x \in [0,\ \pi]$

$\therefore\ x - \frac{\pi}{3} \in \left[-\frac{\pi}{3},\ \frac{2\pi}{3}\right]$

$\tan\left(x - \frac{\pi}{3}\right) = 0$

$\therefore\ x - \frac{\pi}{3} = 0$

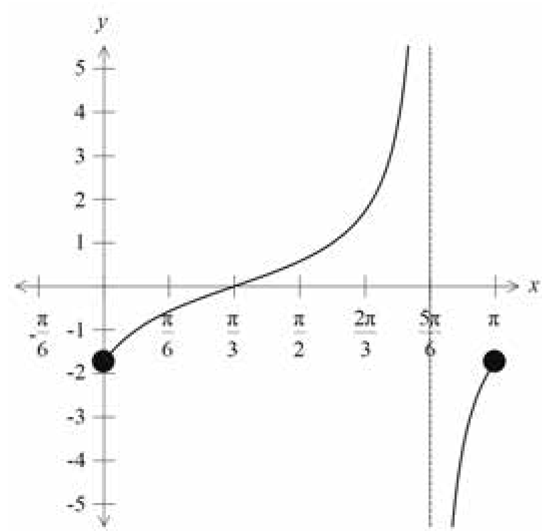
$\therefore\ x = \frac{\pi}{3}$

y-intercept:

$f(0) = \tan\left(-\frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -sqrt3$

Endpoint:

$f(\pi) = \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$



c $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) =$

$2\tan\left(2\left(x + \frac{\pi}{6}\right)\right)$

Period: $= \frac{\pi}{|n|} = \frac{\pi}{2}$

Asymptotes:

$x = \frac{(2k+1)\pi}{2n} - \frac{\pi}{6}$

$\therefore\ x = \frac{(2k+1)\pi}{4} - \frac{\pi}{6}$

$\therefore\ x = \frac{\pi}{4} - \frac{\pi}{6},\ \frac{3\pi}{4} - \frac{\pi}{6}$ as $x \in [0,\ \pi]$

$\therefore\ x = \frac{\pi}{12},\ \frac{7\pi}{12}$

x-intercepts:

as $x \in [0,\ \pi]$

$\therefore\ 2x + \frac{\pi}{3} \in \left[\frac{\pi}{3},\ \frac{7\pi}{3}\right]$

$\therefore\ \tan\left(2x + \frac{\pi}{3}\right) = 0$

$\therefore\ 2x + \frac{\pi}{3} = \pi,\ 2\pi$

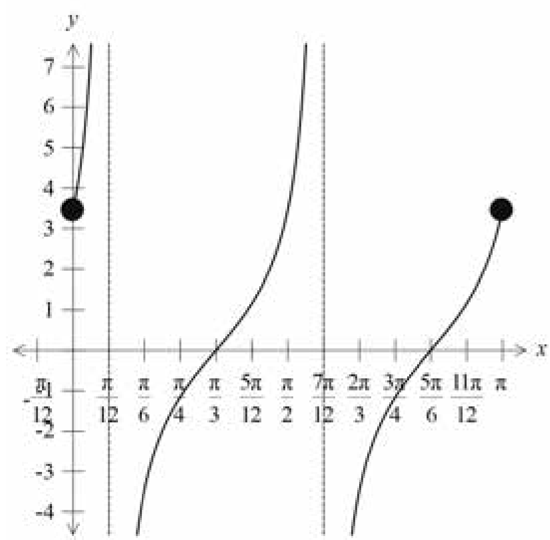
$\therefore\ x = \frac{\pi}{3},\ \frac{5\pi}{6}$

y-intercept:

$f(0) = 2\tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$

Endpoint:

$f(\pi) = 2\tan\left(\frac{7\pi}{3}\right) = 2\sqrt{3}$



d $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$

$= 2\tan\left(2\left(x + \frac{\pi}{6}\right)\right) - 2$

Period: $= \frac{\pi}{|n|} = \frac{\pi}{2}$

Asymptotes:

$x = \frac{(2k+1)\pi}{2n} - \frac{\pi}{6}$

$\therefore x = \frac{(2k+1)\pi}{4} - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{4} - \frac{\pi}{6}, \frac{3\pi}{4} - \frac{\pi}{6}$ as $x \in [0, \pi]$

$\therefore x = \frac{\pi}{12}, \frac{7\pi}{12}$

x-intercepts:

as $x \in [0, \pi]$

$\therefore 2x + \frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$

$\therefore \tan\left(2x + \frac{\pi}{3}\right) = 1$

$\therefore 2x + \frac{\pi}{3} = \frac{5\pi}{4}, \frac{9\pi}{4}$

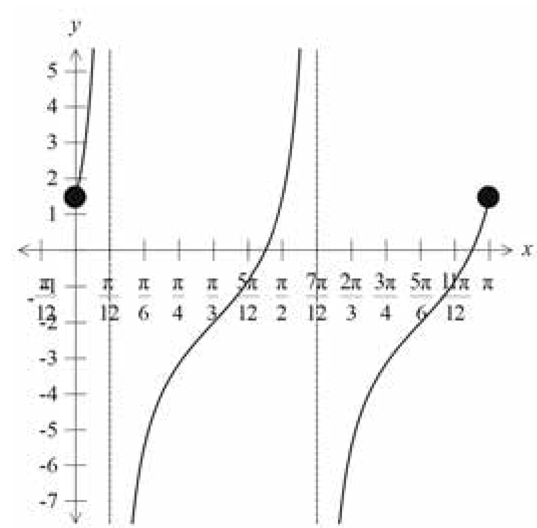
$\therefore x = \frac{11\pi}{24}, \frac{23\pi}{24}$

y-intercept:

$f(0) = 2\tan\left(\frac{\pi}{3}\right) - 2 = 2\sqrt{3} - 2$

Endpoint:

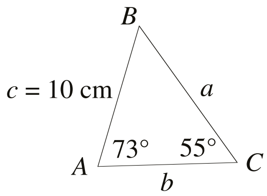
$f(\pi) = 2\tan\left(\frac{7\pi}{3}\right) - 2 = 2\sqrt{3} - 2$



Solutions to Exercise 1B

1 $A + B + C = 180°$

$\therefore B = (180 - (73 + 55))° = 52°$



a Applying the sine rule:

$$\frac{10}{\sin 55°} = \frac{a}{\sin 73°}$$

$$\therefore BC = a = \frac{10 \sin 73°}{\sin 55°} \approx 11.67$$

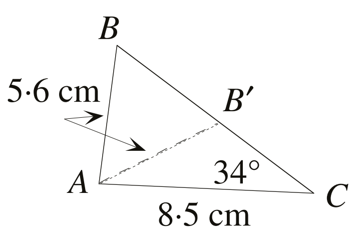
BC is 11.67 cm, correct to two decimal places.

b $$\frac{10}{\sin 55°} = \frac{b}{\sin 52°}$$

$$\therefore AC = b = \frac{10 \sin 52°}{\sin 55°} \approx 9.62$$

AC is 9.62 cm, correct to two decimal places.

2 The two possible triangles are:



a Applying the sine rule:

$$\frac{\sin 34°}{5 \cdot 6} = \frac{\sin B°}{8 \cdot 5}$$

$$\therefore B = \sin^{-1}\left(\frac{8 \cdot 5 \sin 34°}{5 \cdot 6}\right)$$

$$= (58.07867\ldots)°$$

$$\text{or } B = 180° - \sin^{-1}\left(\frac{8 \cdot 5 \sin 34°}{5 \cdot 6}\right)$$

$$= (121.92132\ldots)°$$

$\angle ABC$ is either 58.08° or 121.92°, correct to two decimal places.

b If $\angle ABC = 58.08°$, then $\angle BAC = (180 - (58.08 + 34))° = 87.92°$

Applying the cosine rule:

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos\angle BAC$$

$$= 5.6^2 + 8.5^2 - 2(5.6)(8.5)\cos 87.92°$$

$$= 100.15472\ldots$$

$\therefore BC = 10.00773\ldots$

BC is 10.01 cm, correct to two decimal places.

If $\angle ABC = 121.92°$, then

$\angle BAC = (180 - (121.92 + 34))°$

$= 24.08°$

Applying the cosine rule:

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos\angle BAC$$

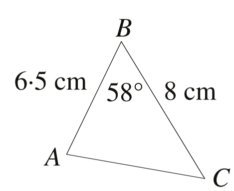
$$= 5.6^2 + 8.5^2 - 2(5.6)(8.5)\cos 24.08°$$

$$= 16.69462\ldots$$

$\therefore BC = 4.08590\ldots$

BC is 4.09 cm, correct to two decimal places.

3



a Applying the cosine rule:

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos B$$
$$= 6.5^2 + 8^2 - 2(6.5)(8)\cos 58°$$
$$= 51.13839\ldots$$
$\therefore AC = 7.15111\ldots$

AC is 7.15 cm, correct to two decimal places.

b Applying the sine rule:

$$\frac{6 \cdot 5}{\sin C°} = \frac{AC}{\sin 58°}$$
$$\therefore C = \sin^{-1}\left(\frac{6 \cdot 5 \sin 58°}{AC}\right)$$
$$= (50.42874\ldots)°$$
$$\text{or } C = 180 - \sin^{-1}\left(\frac{6 \cdot 5 \sin 58°}{AC}\right)$$
$$= (129.57125\ldots)°$$

Therefore $\angle BCA = 50.43°$, correct to two decimal places. (A triangle with two angles of 58° and 129.57° cannot be formed.)

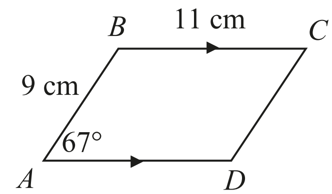
4 a Using the cosine rule

$$12^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos B$$
$$\therefore \cos B = \frac{5^2 + 10^2 - 12^2}{2 \times 5 \times 10}$$
$$\therefore B = \cos^{-1}\left(-\frac{19}{100}\right)$$
$$\therefore B \approx 100.95°$$

b

$$10^2 = 5^2 + 12^2 - 2 \times 5 \times 12 \cos B$$
$$\therefore \cos B = \frac{5^2 + 12^2 - 10^2}{2 \times 5 \times 12}$$
$$\therefore B = \cos^{-1}\left(-\frac{23}{40}\right)$$
$$\therefore B \approx 54.90°$$

5



Since $AD \parallel BC$,

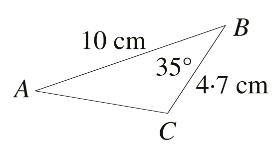
$$\angle ABC = 180° - \angle BAC$$
$$= (180 - 67)°$$
$$= 113°$$

Applying the cosine rule:

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos \angle ABC$$
$$= 9^2 + 11^2 - 2(9)(11)\cos 113°$$
$$= 279.36476\ldots$$
$\therefore AC = 16.71420\ldots$

The length of the longer diagonal is 16.71 cm, correct to two decimal places.

6



a Applying the cosine rule:

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos A$$
$$= 10^2 + 4.7^2 - 2(10)(4.7)\cos 35°$$
$$= 45.08970\ldots$$
$\therefore AC = 6.71488\ldots$

AC is 6.71 cm, correct to two decimal places.

b Applying the sine rule:

$$\frac{\sin C°}{10} = \frac{\sin 35°}{AC}$$
$$\therefore \angle ACB = C = \sin^{-1}\left(\frac{10 \sin 35°}{AC}\right)$$
$$= (58.66995\ldots)°$$

$$\text{or } C = 180° - \sin^{-1}\left(\frac{10\sin 35°}{AC}\right)$$
$$= (121.33004\ldots)°$$

If $C = 58.67$ then

$A = (180 - (58.67 + 35))° = 86.33°$

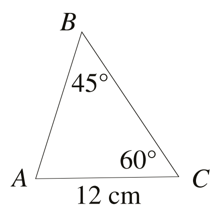
But $|AB| > |BC|$

$\therefore C > A$

$\therefore C = 121.33°$

$\angle ACB$ is 121.33°, correct to two decimal places.

7

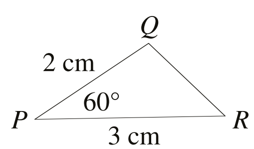


$$\frac{AB}{\sin 60°} = \frac{AC}{\sin 45°}$$

$$\therefore AB = \frac{12 \times \frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = 6\sqrt{6}$$

AB is $6\sqrt{6}$ cm.

8

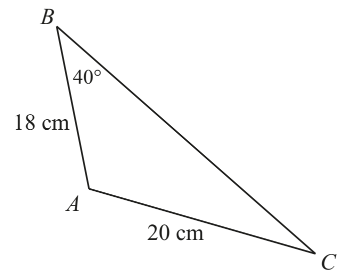


$$QR^2 = PQ^2 + PR^2 - 2(PQ)(PR)\cos P$$
$$= 2^2 + 3^2 - 2(2)(3)\cos 60°$$
$$= 4 + 9 - 12 \times \frac{1}{2}$$
$$= 7$$

$\therefore QR = \sqrt{7}$

QR is $\sqrt{7}$ cm.

9



Applying the sine rule:

$$\frac{\sin C}{18} = \frac{\sin 40°}{20}$$

$$\therefore C = \sin^{-1}\left(\frac{9\sin 40°}{10}\right)$$
$$= (35.34573\ldots)°$$

Hence, A = 180 − 40 − 35.35 = 104.65°

Applying the sine rule:

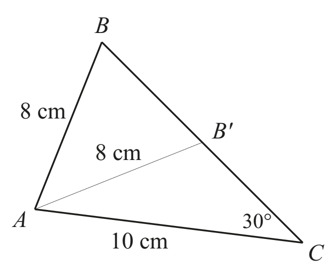
$$\frac{BC}{\sin 104.65°} = \frac{20}{\sin 40°}$$

$$\therefore BC = \frac{20\sin 104.65°}{\sin 40°}$$

$\therefore BC = 30.102322\ldots$

BC is 30.10 cm

10 The ambiguous case applies in this instance as the smaller known side is opposite the known angle.



Applying the cosine rule:

$$AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos 30°$$

$$64 = 100 + BC^2 - 2(10)(BC)\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore BC^2 - 10\sqrt{3}BC + 36 = 0$$

$$\therefore BC = \frac{10\sqrt{3} \pm \sqrt{300 - 144}}{2}$$

$$= \frac{10\sqrt{3} \pm \sqrt{156}}{2}$$

$$= \frac{10\sqrt{3} \pm 2\sqrt{39}}{2}$$

$$= 5\sqrt{3} \pm \sqrt{39}$$

BC is $5\sqrt{3} \pm \sqrt{39}$ cm

Solutions to Exercise 1C

1 $t_1 = 3, t_2 = -1, t_3 = -5, t_4 = -9$
Using $t_n = t_{n-1} - 4$

2 $t_1 = -2, t_{n+1} = -3t_n$

3 $t_n = 2n - 3$
$t_1 = -1, t_2 = 1, t_3 = 3, t_4 = 5$

4
$t = 1$ $\quad t_6 = t_5 + t_4 = 8$
$t_2 = 1$ $\quad t_7 = t_6 + t_5 = 13$
$t_3 = t_2 + t_1 = 2$ $\quad t_8 = t_7 + t_6 = 21$
$t_4 = t_3 + t_2 = 3$ $\quad t_9 = t_8 + t_7 = 34$
$t_5 = t_4 + t_3 = 5$ $\quad t_{10} = t_9 + t_8 = 55$
The first ten terms are:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55

5 $t_n = a + 9d, a = -4, d = -3$
$\therefore t_{10} = -4 + 9 \times (-3) = -31$

6 $t_10 = 2(-3)^9 = -39366$

7 $a = 3,\ d = 4,\ n = 10$
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$\begin{aligned}\therefore S_{10} &= \frac{10}{2}[2 \times 3 + (10-1) \times 4]\\ &= 5[6 + 9 \times 4]\\ &= 5 \times 42\\ &= 210\end{aligned}$$

8 $S_n = \dfrac{a(r^n - 1)}{r - 1}$,
$a = 6,\ r = -3,\ n = 8$
$$\begin{aligned}\therefore S_8 &= \frac{6((-3)^8 - 1)}{-3 - 1}\\ &= \frac{-3}{2}((-3)^8 - 1)\\ &= -9840\end{aligned}$$

9 $s_\infty = \dfrac{a}{1-r},\ a = 1,\ r = \dfrac{-1}{3}$
$$\begin{aligned}&= \frac{1}{1 - \left(\frac{-1}{3}\right)}\\ &= \frac{1}{\frac{4}{3}}\\ &= \frac{3}{4}\end{aligned}$$

10 **a** $\dfrac{x}{x+5} = \dfrac{x-4}{x}$
$$\begin{aligned}\therefore x^2 &= (x+5)(x-4)\\ &= x^2 + x - 20\\ \therefore x &= 20\end{aligned}$$

b $$\begin{aligned}r &= \frac{x}{x+5}\\ &= \frac{20}{25}\\ &= \frac{4}{5}\end{aligned}$$

c $$\begin{aligned}s_\infty - S_n &= \frac{a}{1-r} - \frac{a(1-r^n)}{1-r}\\ &= \frac{ar^n}{1-r},\end{aligned}$$
$a = x + 5 = 25,\ r = \dfrac{4}{5}, n = 10$

$$\therefore s_\infty - S_{10} = \frac{25 \times \left(\frac{4}{5}\right)^{10}}{1 - \frac{4}{5}}$$

$$= 5 \times 25 \times \left(\frac{4}{5}\right)^{10}$$

$$= \frac{4^{10}}{5^7}$$

11 $s_\infty = \frac{a}{1-r},\ a = a,\ r = \frac{a}{\sqrt{2}} \div a = \frac{1}{\sqrt{2}}$

$$= \frac{a}{1 - \frac{1}{\sqrt{2}}}$$

$$= \frac{a}{\frac{\sqrt{2}-1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}a}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{a(2+\sqrt{2})}{1}$$

$$= a(2+\sqrt{2})$$

12 a $S_n = \frac{a(r^n - 1)}{r-1},\ n = 10,\ a = 1,\ r = \frac{x}{2}$

$$\therefore S_{10} = \frac{1\left(\left(\frac{x}{2}\right)^{10} - 1\right)}{\frac{x}{2} - 1}$$

$$= \frac{2}{x-2}\left(\left(\frac{x}{2}\right)^{10} - 1\right)$$

When $x = 1.5,\ S_{10} = 4\left[1 - \left(\frac{3}{4}\right)^{10}\right]$

b i $s_\infty = \frac{a}{1-r},\ a = 1,\ r = \frac{x}{2}$

$$= \frac{1}{1 - \frac{x}{2}},\ x \neq 2$$

$$= \frac{2}{2-x}$$

Now $-1 < r < 1$

$\therefore -1 < \frac{x}{2} < 1$

$\therefore -2 < x < 2$

The infinite sum exists for $-2 < x < 2$

ii Let $S = \frac{2}{2-x},\ x \neq 2$

Given $S = 2S_{10}$,

$$\frac{2}{2-x} = \frac{4}{x-2}\left(\left(\frac{x}{2}\right)^{10} - 1\right)$$

$$\therefore \left(\frac{x}{2}\right)^{10} - 1 = -\frac{1}{2}$$

$$\therefore \left(\frac{x}{2}\right)^{10} = \frac{1}{2}$$

$$\therefore \frac{x}{2} = \pm\left(\frac{1}{2}\right)^{\frac{1}{10}}$$

$$\therefore x = \pm 2 \times 2^{\frac{-1}{10}}$$

$$= \pm 2^{\frac{9}{10}}$$

13 a $s_\infty = \frac{a}{1-r},\ a = 1,\ r = \sin\theta$

$$= \frac{1}{1 - \sin\theta}$$

b $\frac{1}{1-\sin\theta} = 2$

$\therefore 2(1 - \sin\theta) = 1$

$\therefore 1 - \sin\theta = \frac{1}{2}$

$\therefore \sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{\pi}{6} \pm 2\pi,\ \frac{5\pi}{6} \pm 2\pi,\ \frac{\pi}{6} \pm 4\pi,\ \ldots$

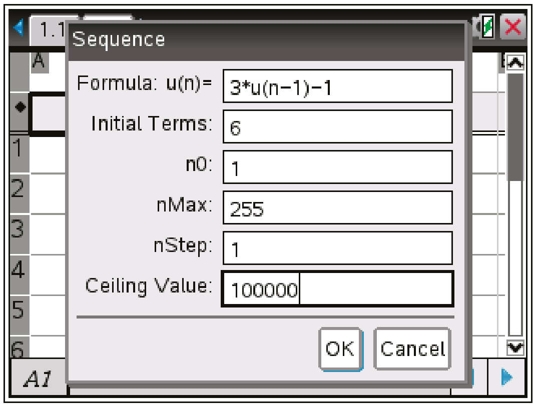
$\therefore \theta = \frac{\pi}{6} + 2k\pi,\ \frac{5\pi}{6} + 2k\pi,\ k \in Z$

14 $t_{n+1} = 3t_n - 1,\ t_1 = 6$

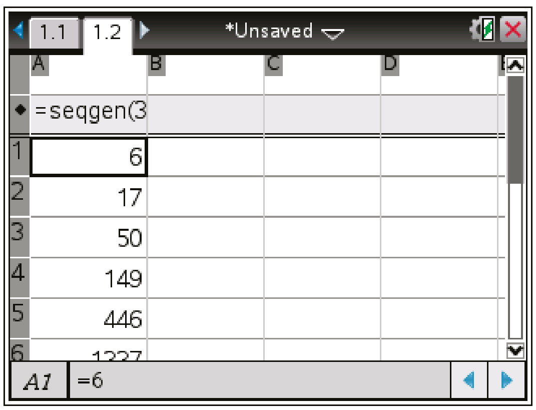
$t_2 = 3t_1 - 1 = 3 \times 6 - 1 = 17$

$t_3 = 3t_2 - 1 = 3 \times 17 - 1 = 50$

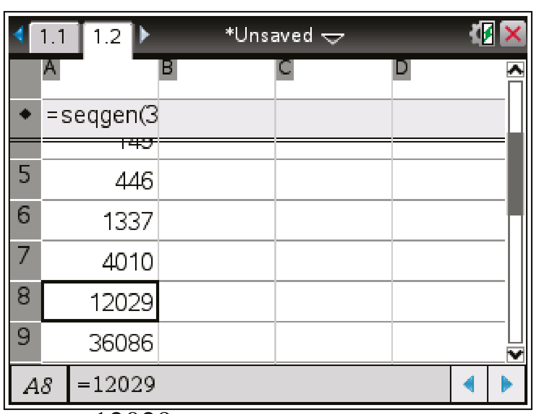
TI: Open a Lists & Spreadsheet application.
Press **Menu** → **3** : **Data** → **1** : **Generate Sequence** and input as shown below



You will now have the sequence of numbers listed in column A like shown.



Scroll down to cell A8 to find the value of t_8.



$\therefore t_8 = 12029$

CP: Open the Sequence application and input the following:

$\boldsymbol{a_{n+1} = 3a_n - 1}$

$\boldsymbol{a_0 = 6}$

Tap 8 and change the Table End value to 10. Now tap # to generate the sequence. Read the value of t_8 from the table (this occurs when n is 7)

15 $y_{n+1} = 2y_n + 6,\ y_1 = 5$

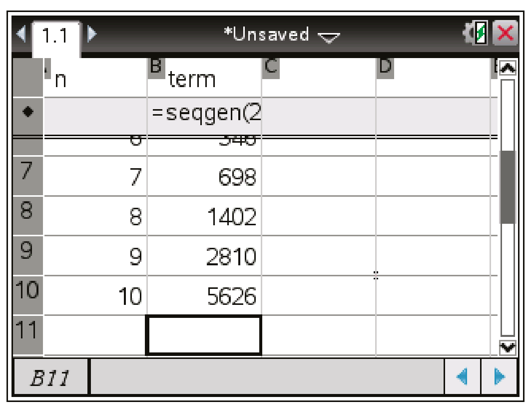
$y_2 = 2y_1 + 6 \qquad y_3 = 2y_2 + 6$

$ = 2 \times 5 + 6 \qquad = 2 \times 16 + 6$

$ = 16 \qquad\qquad = 38$

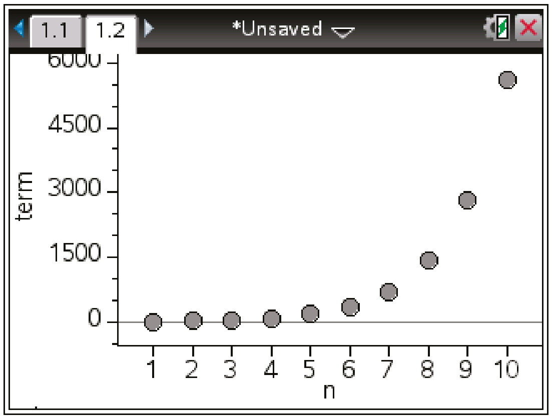
TI: In a new Lists & Spreadsheet, enter the values from 1 to 10 into column A. Give column A the name **n**. Give column B the name **term** and generate the following sequence into column B (as per question 1)

Formula: $2u(n-1)+6$
Initial Terms: 5
n0:1
nMax:10
nstep: 1
Ceiling Value (upper limit): 6000
Scroll down to cell B10 to find the value of y_{10}.

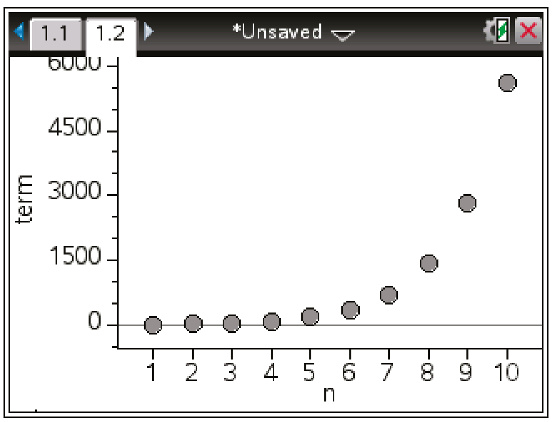


$\therefore y_{10} = 5626$

Open a Data & Statistics page. Add the variable **n** along the horizontal axis and add the variable **term** along the vertical axis.



or sketching by hand we have:



16 **a** $t_n = 2t_{n-1} - 8$

Use

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

$$t_1 = 8, r = 2, d = -8$$

$$t_n = 2^{n-1} \times 8 - \frac{8(2^{n-1} - 1)}{2 - 1}$$

$$= 8$$

b $t_n = 2t_{n-1} - 2$

Use

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

$$t_1 = 10, r = 2, d = -2$$

$$t_n = 2^{n-1} \times 10 - \frac{2(2^{n-1} - 1)}{2 - 1}$$

$$= 8 \times 2^{n-1} + 2$$

c $t_{n+1} = \frac{1}{2}t_n + 6$

We can rewrite as $t_n = \frac{1}{2}t_{n-1} + 6$ with $t_0 = 68$

Use

$$t_n = r^n t_1 + \frac{d(r^n - 1)}{r - 1}$$

$$t_1 = 68, r = \frac{1}{2}, d = 6$$

$$t_n = \left(\frac{1}{2}\right)^n \times 68 + \frac{6\left(\left(\frac{1}{2}\right)^n - 1\right)}{\frac{1}{2} - 1}$$

$$= 68 \times \left(\frac{1}{2}\right)^n - 12 \times \left(\frac{1}{2}\right)^n + 12$$

$$= 28 \times \left(\frac{1}{2}\right)^{n-1} + 12$$

17 $t_1 = 6, t_2 = 7, t_3 = 9$

$$6 = Ar^0 + B \ldots (1)$$

$$7 = Ar + B \ldots (2)$$

$$9 = Ar^2 + B \ldots (3)$$

(2) − (1)

$$1 = A(r - 1) \ldots (3)$$

(3) − (2)

$$2 = Ar(r - 1) \ldots (4)$$

(4) ÷ (3)

$$2 = r, A = 1, B = 5$$

18 $t_1 = 8, t_2 = 23, t_3 = 98$

$$8 = Ar^0 + B \ldots (1)$$

$$23 = Ar + B \ldots (2)$$

$$98 = Ar^2 + B \ldots (3)$$

$(2) - (1)$

$$15 = A(r-1) \ldots (3)$$

$(3) - (2)$

$$75 = Ar(r-1) \ldots (4)$$

$(4) \div (3)$

$$5 = r, A = \frac{15}{4}, B = \frac{17}{4}$$

Solutions to Exercise 1D

1 **a** 8

b 8

c 2

d −2

e −2

f 4

2 **a**

$$|x-1| = 2$$
$$x - 1 = \pm 2$$
$$x = 3 \text{ or } x = -1$$

b

$$|2x-3| = 4$$
$$2x - 3 = \pm 4$$
$$2x = 7 \text{ or } 2x = -1$$
$$x = \frac{7}{2} \text{ or } x = -\frac{1}{2}$$

c

$$|5x-3| = 9$$
$$5x - 3 = \pm 9$$
$$5x = 12 \text{ or } 5x = -6$$
$$x = \frac{12}{5} \text{ or } x = -\frac{6}{5}$$

d

$$|x-3| = 9$$
$$x - 3 = \pm 9$$
$$x = 12 \text{ or } x = -6$$

e

$$|x-3| = 4$$
$$x - 3 = \pm 4$$
$$x = 7 \text{ or } x = -1$$

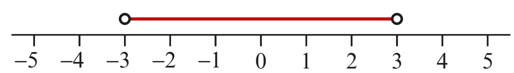
f

$$|3x+4| = 8$$
$$3x + 4 = \pm 8$$
$$3x = 4 \text{ or } 3x = -12$$
$$x = \frac{4}{3} \text{ or } x = -4$$

g

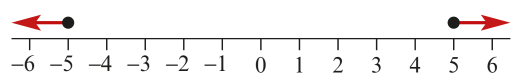
$$|5x+11| = 9$$
$$5x + 11 = \pm 9$$
$$5x = -2 \text{ or } 5x = -20$$
$$x = -\frac{2}{5} \text{ or } x = -4$$

3 **a**



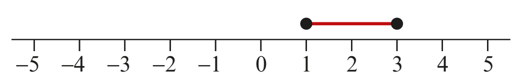
$(-3, 3)$

b



Answer:$(-\infty, -5] \cup [5, \infty)$

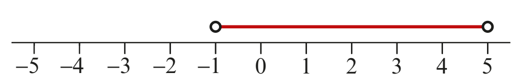
c



$$|x-2| \le 1 \Leftrightarrow -1 \le x - 2 \le 1$$
$$\Leftrightarrow 1 \le x \le 3$$

Answer:$[1, 3]$

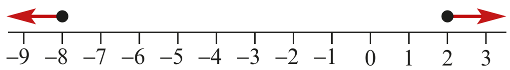
d



$$|x-2| < 3 \Leftrightarrow -3 < x - 2 < 3$$
$$\Leftrightarrow -1 < x < 5$$

Answer:$(-1, 5)$

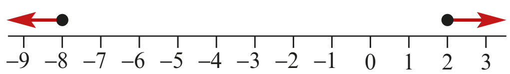
e



$$|x+3| \geq 5 \Leftrightarrow x+3 \geq 5 \text{ or } x+3 \leq -5$$
$$\Leftrightarrow x \geq 2 \text{ or } x \leq -8$$

Answer:$(-\infty, -8] \cup [2, \infty)$

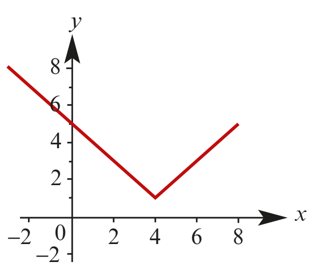
f



$$|x+2| \leq 1 \Leftrightarrow -1 < x+2 < 1$$
$$\Leftrightarrow -3 < x < -1$$

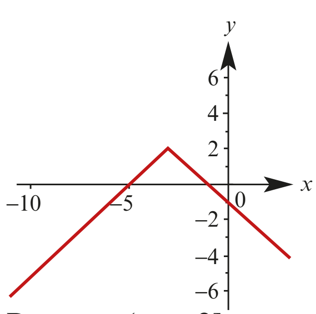
Answer: $[-3, -1]$

4 a



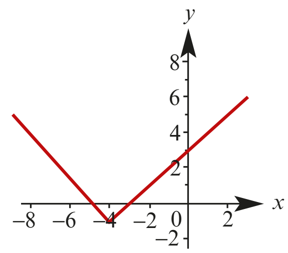
Range $= [1, \infty)$

b



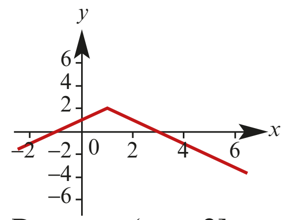
Range $= (-\infty, 2]$

c



Range $= [-1, \infty)$

d



Range $= (-\infty, 2]$

5 a $|x| \leq 5 \Leftrightarrow -5 \leq x \leq 5$

Answer: $\{x : -5 \leq x \leq 5\}$

b $|x| \geq 2 \Leftrightarrow x \geq 2 \text{ or } x \leq -2$

Answer: $\{x : x \leq -2\} \cup \{x : x \geq 2\}$

c $|2x-3| \leq 1 \Leftrightarrow -1 < 2x-3 < 1$

$$\Leftrightarrow 2 \leq 2x \leq 4$$
$$\Leftrightarrow 1 \leq x \leq 2$$

Answer: $\{x : 1 \leq x \leq 2\}$

d $|5x-2| < 31 \Leftrightarrow -3 < 5x-2 < 3$

$$\Leftrightarrow -1 \leq 5x \leq 5$$
$$\Leftrightarrow -\frac{1}{5} < x < 1$$

Answer: $\{x : -\frac{1}{5} < x < 1\}$

e

$$|-x+3| \geq 7 \Leftrightarrow -x+3 \geq 7 \text{ or } -x+3 \leq -7$$
$$\Leftrightarrow -x \geq 4 \text{ or } -x \leq -10$$
$$\Leftrightarrow x \leq -4 \text{ or } x \geq 10$$

Answer: $\{x : x \leq -4\} \cup \{x : x \geq 10\}$

f $|-x+2| \le 1 \Leftrightarrow -1 < -x+2 < 1$

$\Leftrightarrow -3 \le -x \le -1$

$\Leftrightarrow 1 \le x \le 3$

Answer: $\{x : 1 \le x \le 3\}$

6 We use an algebraic approach but using graphs to help simplifies it somewhat.

a Consider Cases:

Crucial points are -2 and 4

Case 1 :$x \ge 4$

x-4-(x+2) =6

No soln

Case 2: $-2 \le x \le 4$

$4 - x - (x + 2) = 6$

$2 - 2x = 6$

$-2x = 4$

$x = -2$

Case 3: $x \le -2$

$4 - x - (-x - 2) = 6$

$6 = 6$

Always true

Solution: $(-\infty, -2]$

b Consider Cases:

Crucial points are $\frac{5}{2}$ and 4

Case 1 :$x \ge 4$

$2x - 5 - (x - 4) = 10$

$x - 1 = 10$

$x = 11$ (Solution)

Case 2: $\frac{5}{2} \le x \le 4$

$2x - 5 - (4 - x) = 10$

$3x - 9 = 6$

$x = 5$ (No solution)

Case 3: $x \le \frac{5}{2}$

$5 - 2x - (4 - x) = 10$

$1 - x = 10$

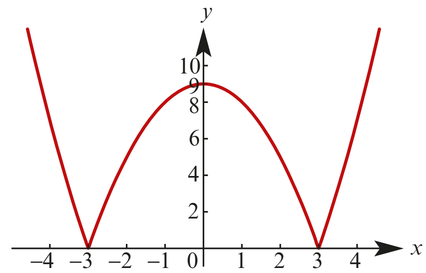
$x = -9$ (Solution)

Therefore $x = 11$ or $x = -9$

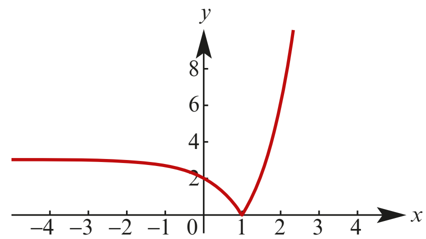
c Use a calculator

$x = \frac{5}{4}$ or $x = \frac{15}{4}$

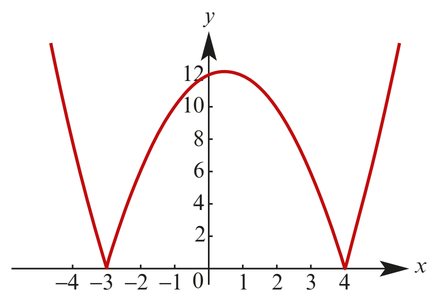
7 a



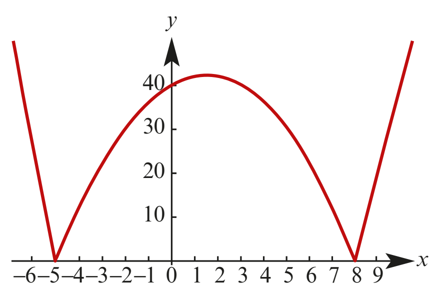
b



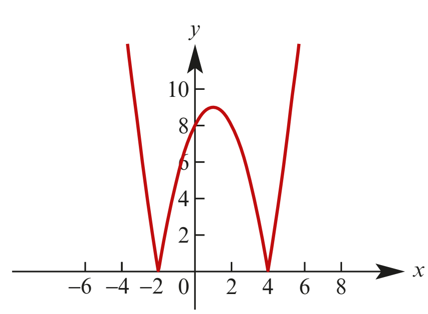
c



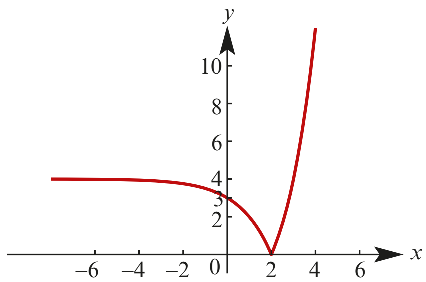
d



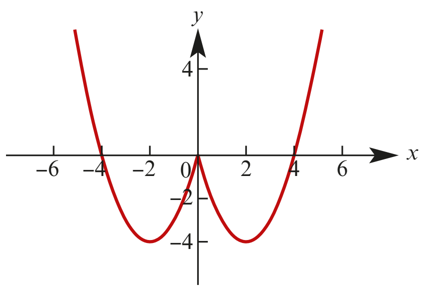
e



f

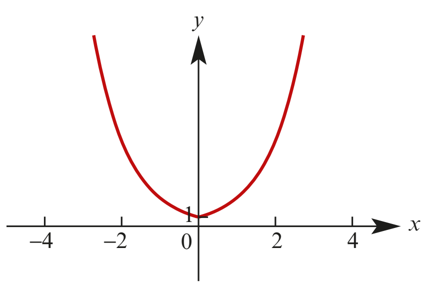


8 a



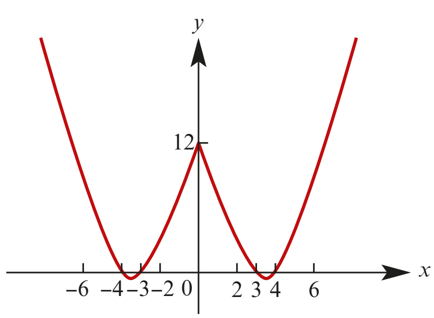
$$y = \begin{cases} x^2 - 4x & x \geq 0 \\ x^2 + 4x & x < 0 \end{cases}$$

b



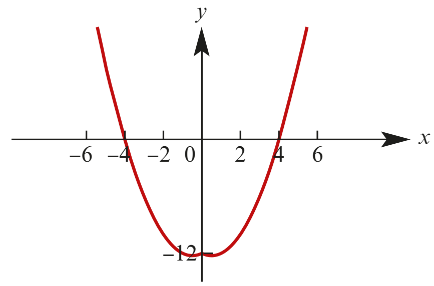
$$y = \begin{cases} 3^x & x \geq 0 \\ 3^{-x} & x < 0 \end{cases}$$

c



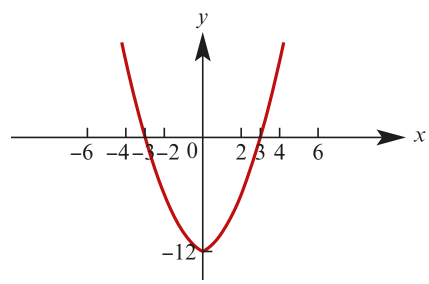
$$y = \begin{cases} x^2 - 7x + 12 & x \geq 0 \\ x^2 + 7x + 12 & x < 0 \end{cases}$$

d



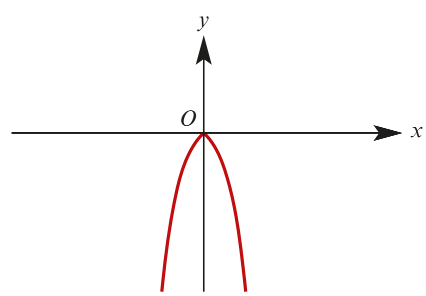
$$y = \begin{cases} x^2 - x - 12 & x \geq 0 \\ x^2 + x - 12 & x < 0 \end{cases}$$

e



$$y = \begin{cases} x^2 + x - 12 & x \geq 0 \\ x^2 - x - 12 & x < 0 \end{cases}$$

f



$$y = \begin{cases} -3^x & x \geq 0 \\ -3^{-x} & x < 0 \end{cases}$$

9 $f(x) = |x - a| + b$

Given, $f(3) = 3$ and $f(-1) = 3$

The symmetry of f gives us that $a = 1$

Hence $b = 1$

Solutions to Exercise 1E

1 **a** $(x-2)^2+(y-3)^2=1$

b $(x+3)^2+(y-4)^2=25$

c $x^2+(y+5)^2=25$

d $(x-3)^2+y^2=2$

2 **a** $x^2+y^2+4x-6y+12=0$
Completing the square in x and y gives:
$(x^2+4x+4)+(y^2-6y+9)+12=13$
$\therefore (x+2)^2+(y-3)^2=1$
A circle with centre $(-2, 3)$ and radius 1 is described.

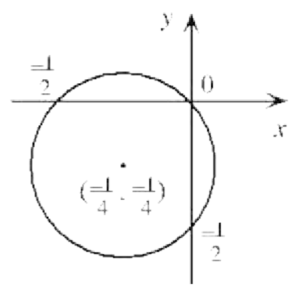
b $x^2+y^2-2x-4y+1=0$
Completing the square in x and y gives:
$(x^2-2x+1)+(y^2-4y+4)+1=5$
$\therefore (x-1)^2+(y-2)^2=4$
A circle with centre $(1, 2)$ and radius 2 is described.

c $x^2+y^2-3x=0$
$\therefore \left(x^2-3x+\frac{9}{4}\right)+y^2=\frac{9}{4}$
$\therefore \left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$
A circle with centre $\left(\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$ is described.

d $x^2+y^2+4x-10y+25=0$
$\therefore (x^2+4x+4)+(y^2-10y+25)+25=29$
$\therefore (x+2)^2+(y-5)^2=4$
A circle with centre $(-2, 5)$ and radius 2 is described.

3 **a** $2x^2+2y^2+x+y=0$
$\therefore 2\left[x^2+y^2+\frac{1}{2}x+\frac{1}{2}y\right]=0$
$\therefore \left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+\left(y^2+\frac{1}{2}y+\frac{1}{16}\right)=\frac{1}{8}$
$\therefore \left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2=\frac{1}{8}$
centre $\left(-\frac{1}{4}, -\frac{1}{4}\right)$, radius $\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$

When $x=0$, $\frac{1}{16}+\left(y+\frac{1}{4}\right)^2=\frac{1}{8}$
$\therefore \left(y+\frac{1}{4}\right)^2=\frac{1}{16}$
$\therefore y+\frac{1}{4}=\pm\frac{1}{4}$
$\therefore y=0, -\frac{1}{2}$
Similarly when $y=0$, $x=0, -\frac{1}{2}$



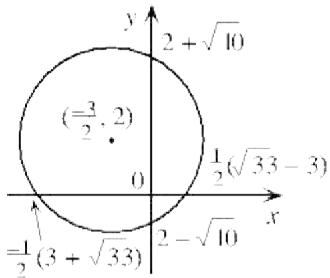
b $x^2+y^2+3x-4y=6$
$\therefore \left(x^2+3x+\frac{9}{4}\right)+(y^2-4y+4)=\frac{49}{4}$
$\therefore \left(x+\frac{3}{2}\right)^2+(y-2)^2=\frac{49}{4}$
centre $\left(-\frac{3}{2}, 2\right)$, radius $\frac{7}{2}$

When $x=0$, $\frac{9}{4}+(y-2)^2=\frac{49}{4}$
$\therefore (y-2)^2=10$
$\therefore y-2=\pm\sqrt{10}$
$\therefore y=2\pm\sqrt{10}$
When $y=0$, $\left(x+\frac{3}{2}\right)^2+4=\frac{49}{4}$

$\therefore \left(x + \frac{3}{2}\right)^2 = \frac{33}{4}$

$\therefore x + \frac{3}{2} = \pm\frac{\sqrt{33}}{2}$

$\therefore x = -\frac{1}{2}(3 \pm \sqrt{33})$



c $x^2 + y^2 + 8x - 10y + 16 = 0$

$\therefore (x^2 + 8x + 16) + (y^2 - 10y + 25) + 16 = 41$

$\therefore (x + 4)^2 + (y - 5)^2 = 25$

centre $(-4, 5)$, radius 5

When $x = 0$, $16 + (y - 5)^2 = 25$

$\therefore (y - 5)^2 = 9$

$\therefore y - 5 = \pm 3$

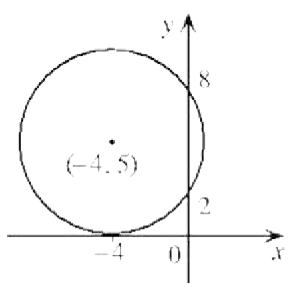
$\therefore y = 2, 8$

When $y = 0$, $(x + 4)^2 + 25 = 25$

$\therefore (x + 4)^2 = 0$

$\therefore x + 4 = 0$

$\therefore x = -4$



d $x^2 + y^2 - 8x - 10y + 16 = 0$

$\therefore (x^2 - 8x + 16) + (y^2 - 10y + 25) + 16 = 41$

$\therefore (x - 4)^2 + (y - 5)^2 = 25$

centre $(4, 5)$, radius 5

When $x = 0$, $16 + (y - 5)^2 = 25$

$\therefore (y - 5)^2 = 9$

$\therefore y - 5 = \pm 3$

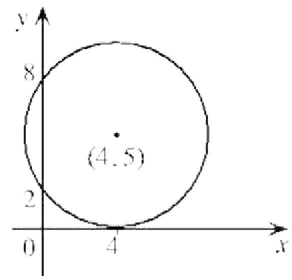
$\therefore y = 2, 8$

When $y = 0$, $(x - 4)^2 + 25 = 25$

$\therefore (x - 4)^2 = 0$

$\therefore x - 4 = 0$

$\therefore x = 4$



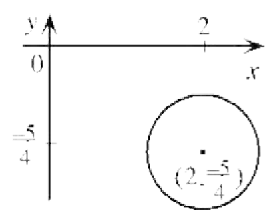
e $2x^2 + 2y^2 - 8x + 5y + 10 = 0$

$\therefore 2\left(x^2 + y^2 - 4x + \frac{5}{2}y + 5\right) = 0$

$\therefore (x^2 - 4x + 4) + \left(y^2 + \frac{5}{2}y + \frac{25}{16}\right) + 5 = \frac{89}{16}$

$\therefore (x - 2)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{9}{16}$

centre $\left(2, \frac{-5}{4}\right)$, radius $\frac{3}{4}$



f $3x^2 + 3y^2 + 6x - 9y = 100$

$\therefore 3(x^2 + 2x + 1) + 3\left(y^2 - 3y + \frac{9}{4}\right) = \frac{439}{4}$

$\therefore 3(x + 1)^2 + 3\left(y - \frac{3}{2}\right)^2 = \frac{439}{4}$

$\therefore (x + 1)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{439}{12}$

centre $\left(-1, \frac{3}{2}\right)$, radius $\frac{1}{2}\sqrt{\frac{439}{3}} = \frac{\sqrt{1317}}{6}$

When $x = 0, 1 + \left(y - \frac{3}{2}\right)^2 = \frac{439}{12}$

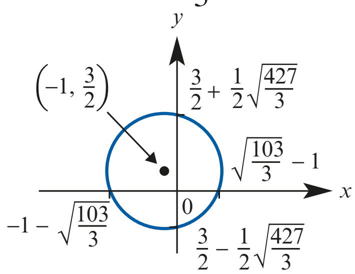
$\therefore \left(y - \frac{3}{2}\right)^2 = \frac{427}{12}$

$\therefore y = \frac{3}{2} \pm \frac{\sqrt{1281}}{6}$

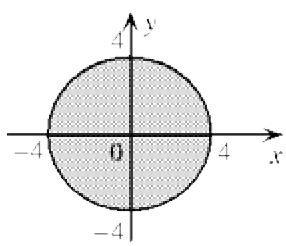
When $y = 0,\ (x + 1)^2 + \frac{9}{4} = \frac{439}{12}$

$\therefore (x + 1)^2 = \frac{412}{12}$

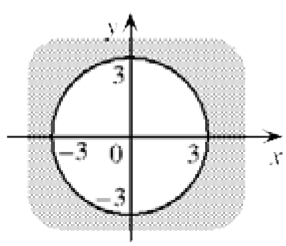
$\therefore x = -1 \pm \frac{\sqrt{309}}{3}$



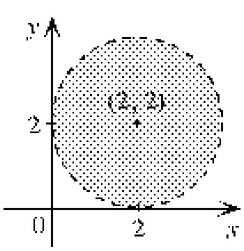
4 a $x^2 + y^2 \le 16$



b $x^2 + y^2 \ge 9$



c $(x - 2)^2 + (y - 2)^2 < 4$



d $(x - 3)^2 + (y + 2)^2 > 16$

For $(x - 3)^2 + (y + 2)^2 = 16$

When $x = 0, 9 + (y + 2)^2 = 16$

$\therefore (y + 2)^2 = 7$

$\therefore y + 2 = \pm\sqrt{7}$

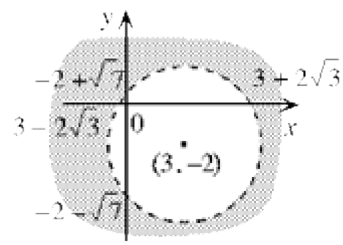
$\therefore y = -2 \pm \sqrt{7}$

When $y = 0,\ (x - 3)^2 + 4 = 16$

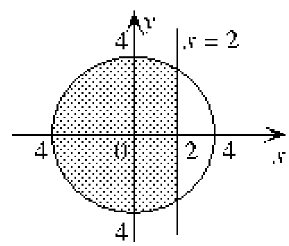
$\therefore (x - 3)^2 = 12$

$\therefore x - 3 = \pm 2\sqrt{3}$

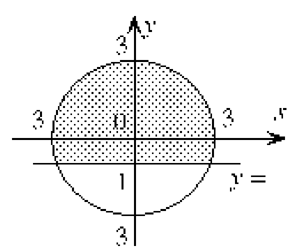
$\therefore x = 3 \pm 2\sqrt{3}$



e $x^2 + y^2 \le 16$ and $x \le 2$



f $x^2 + y^2 \le 9$ and $y \ge -1$



5 Length of diameter

$= \sqrt{(8 - 2)^2 + (4 - 2)^2}$

$= \sqrt{36 + 4}$

$= \sqrt{40}$

$= 2\sqrt{10}$

$\therefore r = \sqrt{10}$

The centre of the circle lies at the midpoint of the diameter and has coordinates $\left(\frac{8+2}{2}, \frac{4+2}{2}\right)$ i.e. (5, 3)

centre (5, 3), radius $\sqrt{10}$

6 centre (2, −3), radius 3
$\therefore (x-2)^2 + (y+3)^2 = 9$

7 $(x-h)^2 + (y-k)^2 = r^2$
At (3, 1), $(3-h)^2 + (1-k)^2 = r^2$
$\therefore 9 - 6h + h^2 + 1 - 2k + k^2 = r^2$
$\therefore 10 - 6h + h^2 - 2k + k^2 = r^2$ ①
At (8, 2), $(8-h)^2 + (2-k)^2 = r^2$
$\therefore 64 - 16h + h^2 + 4 - 4k + k^2 = r^2$
$\therefore 68 - 16h + h^2 - 4k + k^2 = r^2$ ②
At (2, 6), $(2-h)^2 + (6-k)^2 = r^2$
$\therefore 4 - 4h + h^2 + 36 - 12k + k^2 = r^2$
$\therefore 40 - 4h + h^2 - 12k + k^2 = r^2$ ③
① − ② $-58 + 10h + 2k = 0$
$\therefore k = 29 - 5h$ ④
③ − ① $30 + 2h - 10k = 0$
$\therefore 15 + h - 5k = 0$ ⑤
Substituting ④ in ⑤ yields
$15 + h - 5(29 - 5h) = 0$
$\therefore 15 + h - 145 + 25h = 0$
$\therefore 26h = 130$
$\therefore h = 5$
Substituting $h = 5$ in ④ yields
$k = 29 - 5 \times 5$
$= 29 - 25$
$= 4$
Substituting $h = 5,\ k = 4$ in ① yields
$10 - 6 \times 5 + 5^2 - 2 \times 4 + 4^2 = r^2$
$\therefore r^2 = 10 - 30 + 25 - 8 + 16$
$= 13$
$\therefore (x-5)^2 + (y-4)^2 = 13$ is the circle passing through (3, 1), (8, 2) and (2, 6)

8 $4x^2 + 4y^2 - 60x - 76y + 536 = 0$
$\therefore x^2 + y^2 - 15x - 19y + 134 = 0$
$\therefore \left(x - \frac{15}{2}\right)^2 + \left(y - \frac{19}{2}\right)^2 = \frac{25}{2}$
centre $\left(\frac{15}{2}, \frac{19}{2}\right)$, radius $\frac{5\sqrt{2}}{2}$

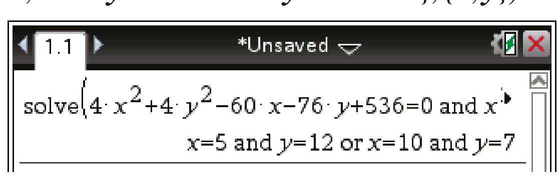
$x^2 + y^2 - 10x - 14y + 49 = 0$ ①
$\therefore (x^2 - 10x + 25) + (y^2 - 14y + 49) + 49 = 74$
$\therefore (x-5)^2 + (y-7)^2 = 25$
centre (5, 7), radius 5

To find points of intersection, let
$x^2 + y^2 - 15x - 19y + 134 =$
$x^2 + y^2 - 10x - 14y + 49$
$\therefore 5x + 5y = 85$
$\therefore x + y = 17$
$\therefore y = 17 - x$ ②
Substituting ② in ① yields
$x^2 + (17-x)^2 - 10x - 14(17-x) + 49 = 0$
$\therefore x^2 + 289 - 34x + x^2 - 10x - 238 + 14x + 49 = 0$
$\therefore 2x^2 - 30x + 100 = 0$
$\therefore x^2 - 15x + 50 = 0$
$\therefore (x-5)(x-10) = 0$
$\therefore x = 5$ or $x = 10$
When $x = 5,\ y = 17 - 5 = 12$
When $x = 10,\ y = 17 - 10 = 7$

The points of intersection of the two circles are (5, 12) and (10, 7)

TI: Type
solve(4x^2 + 4y^2 − 60x − 76y + 536 = 0 and x^2 + y^2 − 10x − 14y + 49 = 0, x)

CP: Type
solve ({4x^2 + 4y^2 − 60x − 76y + 536 = 0, x^2 + y^2 − 10x − 14y + 49 = 0}, {x, y})



9 **a** Substituting $y = x$ into $x^2 + y^2 = 25$ yields

$x^2 + x^2 = 25$

$\therefore 2x^2 = 25$

$\therefore x^2 = \frac{25}{2}$

$\therefore x = \pm\frac{5}{\sqrt{2}} = \pm\frac{5 \cdot \sqrt{2}}{2}$

Hence $y = x = \pm\frac{5\sqrt{2}}{2}$

The points of intersection are $\left(\frac{5 \cdot \sqrt{2}}{2}, \frac{5 \cdot \sqrt{2}}{2}\right)$ and $\left(\frac{-5 \cdot \sqrt{2}}{2}, \frac{-5 \cdot \sqrt{2}}{2}\right)$

b Substituting $y = 2x$ into $x^2 + y^2 = 25$ yields

$x^2 + 4x^2 = 25$

$\therefore 5x^2 = 25$

$\therefore x^2 = 5$

$y = 2x$

Hence

$= \pm 2\sqrt{5}$

The points of intersection are

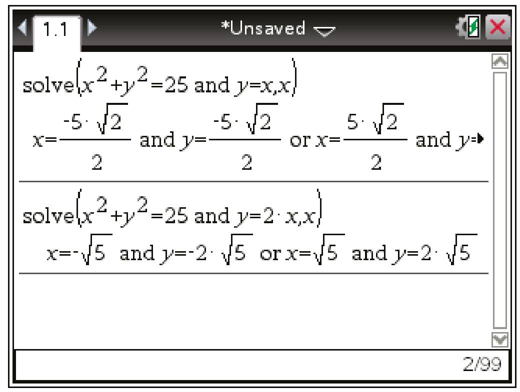
$(\sqrt{5}, 2\sqrt{5})$ and $(-\sqrt{5}\ -2\sqrt{5})$

TI: Type

solve($x^2 + y^2 = 25$ and $y = x, x$)

CP: Type

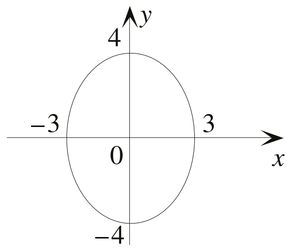
solve ($\{x^2 + y^2 = 25, y = x\}, \{x, y\}$)



Solutions to Exercise 1F

1 a $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

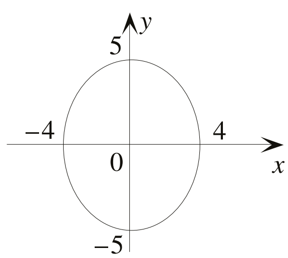
ellipse, centre $(0, 0)$



b $25x^2 + 16y^2 = 400$

$\therefore \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

ellipse, centre $(0, 0)$



c $\dfrac{(x-4)^2}{9} + \dfrac{(y-1)^2}{16} = 1$

ellipse, centre $(4, 1)$

When $x = 0$, $\dfrac{16}{9} + \dfrac{(y-1)^2}{16} = 1$

$\therefore \dfrac{(y-1)^2}{16} = \dfrac{-7}{9}$

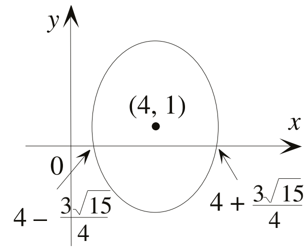
$\therefore$ no y axis intercepts

When $y = 0$, $\dfrac{(x-4)^2}{9} + \dfrac{1}{16} = 1$

$\therefore \dfrac{(x-4)^2}{9} = \dfrac{15}{16}$

$\therefore (x-4)^2 = \dfrac{9 \times 15}{16}$

$\therefore x = 4 \pm \dfrac{3\sqrt{15}}{4}$



d $x^2 + \dfrac{(y-2)^2}{9} = 1$

ellipse, centre $(0, 2)$

When $x = 0$, $\dfrac{(y-2)^2}{9} = 1$

$\therefore (y-2)^2 = 9$

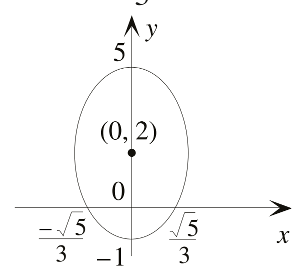
$\therefore y = 2 \pm 3$

$= -1, 5$

When $y = 0$, $x^2 + \dfrac{4}{9} = 1$

$\therefore x^2 = \dfrac{5}{9}$

$\therefore x = \dfrac{\pm\sqrt{5}}{3}$



e $9x^2 + 25y^2 - 54x - 100y = 44$

$\therefore 9(x^2 - 6x + 9)$

$+ 25(y^2 - 4y + 4) = 225$

$\therefore 9(x-3)^2 + 25(y-2)^2 = 225$

$\therefore \dfrac{(x-3)^2}{25} + \dfrac{(y-2)^2}{9} = 1$

ellipse, centre $(3, 2)$

When $x = 0$, $\dfrac{9}{25} + \dfrac{(y-2)^2}{9} = 1$

$\therefore \dfrac{(y-2)^2}{9} = \dfrac{16}{25}$

$\therefore (y-2)^2 = \dfrac{9 \times 16}{25}$

$\therefore y = 2 \pm \frac{12}{5}$

$= \frac{-2}{5}, \frac{22}{5}$

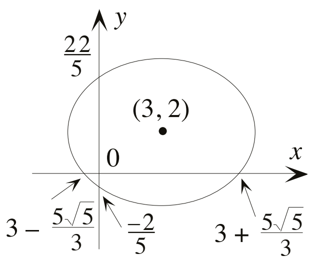
$\therefore (x + 2)^2 = \frac{36}{5}$

When $y = 0$, $\frac{(x-3)^2}{25} + \frac{4}{9} = 1$

$\therefore \frac{(x-3)^2}{25} = \frac{5}{9}$

$\therefore (x - 3)^2 = \frac{25 \times 5}{9}$

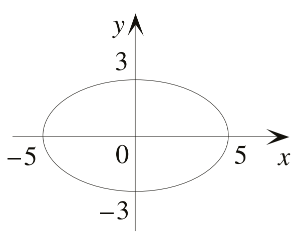
$\therefore x = 3 \pm \frac{5\sqrt{5}}{3}$



f $9x^2 + 25y^2 = 225$

$\therefore \frac{x^2}{25} + \frac{y^2}{9} = 1$

ellipse, centre (0, 0)



g $5x^2 + 9y^2 + 20x - 18y - 16 = 0$

$\therefore 5(x^2 + 4x + 4) + 9(y^2 - 2y + 1) -$
$16 - 29 = 0$

$\therefore 5(x + 2)^2 + 9(y - 1)^2 = 45$

$\therefore \frac{(x+2)^2}{9} + \frac{(y-1)^2}{5} = 1$

ellipse, centre (−2, 1)

When $x = 0$, $\frac{4}{9} + \frac{(y-1)^2}{5} = 1$

$\therefore \frac{(y-1)^2}{5} = \frac{5}{9}$

$\therefore (y - 1)^2 = \frac{25}{9}$

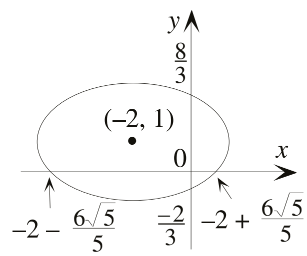
$\therefore y = 1 \pm \frac{5}{3}$

$= \frac{-2}{3}, \frac{8}{3}$

When $y = 0$, $\frac{(x+2)^2}{9} + \frac{1}{5} = 1$

$\therefore \frac{(x+2)^2}{9} = \frac{4}{5}$

$\therefore x = -2 \pm \frac{6\sqrt{5}}{5}$



h $16x^2 + 25y^2 - 32x + 100y - 284 = 0$

$\therefore 16(x^2 - 2x + 1) + 25(y^2 + 4y + 4) -$
$284 - 116 = 0$

$\therefore 16(x - 1)^2 + 25(y + 2)^2 = 400$

$\therefore \frac{(x-1)^2}{25} + \frac{(y+2)^2}{16} = 1$

ellipse, centre (1, −2)

When $x = 0$, $\frac{1}{25} + \frac{(y+2)^2}{16} = 1$

$\therefore \frac{(y+2)^2}{16} = \frac{24}{25}$

$\therefore (y + 2)^2 = \frac{16 \times 24}{25}$

$= \frac{384}{25}$

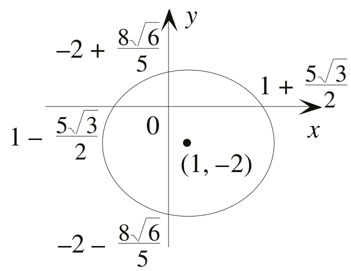
$\therefore y = -2 \pm \frac{8\sqrt{6}}{5}$

When $y = 0$, $\frac{(x-1)^2}{25} + \frac{1}{4} = 1$

$\therefore \frac{(x-1)^2}{25} = \frac{3}{4}$

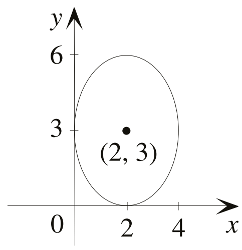
$\therefore (x - 1)^2 = \frac{75}{4}$

$\therefore x = 1 \pm \frac{5\sqrt{3}}{2}$



i $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

ellipse, centre (2, 3)



j $2(x-2)^2 + 4(y-1)^2 = 16$

$\therefore \frac{(x-2)^2}{8} + \frac{(y-1)^2}{4} = 1$

ellipse, centre (2, 1)

When $x = 0$, $\frac{1}{2} + \frac{(y-1)^2}{4} = 1$

$\therefore \frac{(y-1)^2}{4} = \frac{1}{2}$

$\therefore (y-1)^2 = 2$

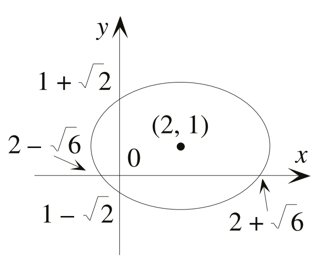
$\therefore y = 1 \pm \sqrt{2}$

When $y = 0$, $\frac{(x-2)^2}{8} + \frac{1}{4} = 1$

$\therefore \frac{(x-2)^2}{8} = \frac{3}{4}$

$\therefore (x-2)^2 = \frac{24}{4} = 6$

$\therefore x = 2 \pm \sqrt{6}$



2 a $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\therefore \frac{y^2}{9} = \frac{x^2}{16} - 1$

$\therefore y^2 = \frac{9x^2}{16} - 9$

$= \frac{9x^2}{16}\left(1 - \frac{16}{x^2}\right)$

As $x \to \pm\infty$, $\frac{16}{x^2} \to 0$

$\therefore y^2 \to \frac{9x^2}{16}$

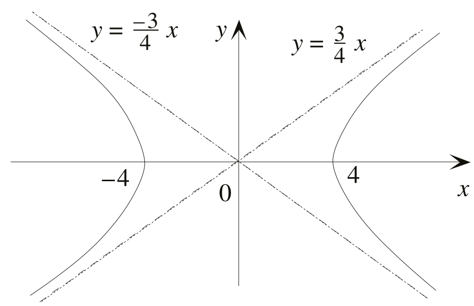
$\therefore y \to \frac{\pm 3x}{4}$

Equations of asymptotes: $y = \pm\frac{3x}{4}$

When $y = 0$, $x^2 = 16$

$\therefore x = \pm 4$

centre (0, 0)



b $\frac{y^2}{16} - \frac{x^2}{9} = 1$

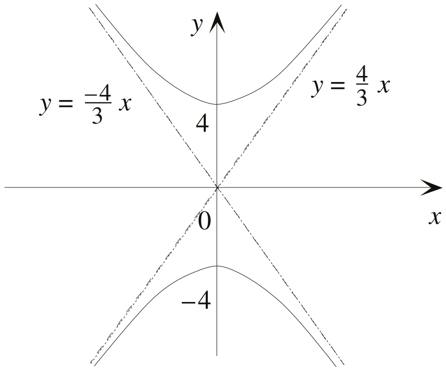
This is the reflection of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ in the line

$y = x$

Asymptotes are $x = \pm\frac{3}{4}y$

$\therefore y = \pm\frac{4}{3}x$

The y axis intercepts are (0, 4) and (0, −4)



c $x^2 - y^2 = 4$

$\therefore \dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$

$\therefore \dfrac{y^2}{4} = \dfrac{x^2}{4} - 1$

$\therefore y^2 = \dfrac{4x^2}{4} - 4$

$= x^2\left(1 - \dfrac{4}{x^2}\right)$

As $x \to \pm\infty$, $\dfrac{4}{x^2} \to 0$

$\therefore y^2 \to x^2$

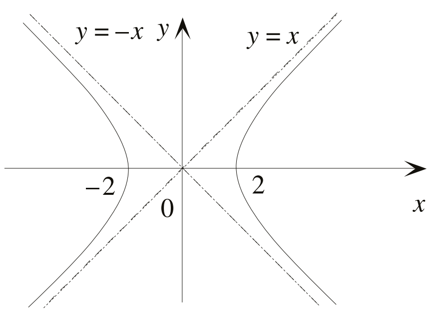
$\therefore y \to \pm x$

Equations of asymptotes: $y = \pm x$

When $y = 0$, $x^2 = 4$

$\therefore x = \pm 2$

centre $(0, 0)$



d $2x^2 - y^2 = 4$

$\therefore \dfrac{x^2}{2} - \dfrac{y^2}{4} = 1$

$\therefore \dfrac{y^2}{4} = \dfrac{x^2}{2} - 1$

$\therefore y^2 = 2x^2 - 4$

$= 2x^2\left(1 - \dfrac{2}{x^2}\right)$

As $x \to \pm\infty$, $\dfrac{2}{x^2} \to 0$

$\therefore y^2 \to 2x^2$

$\therefore y \to \pm\sqrt{2}x$

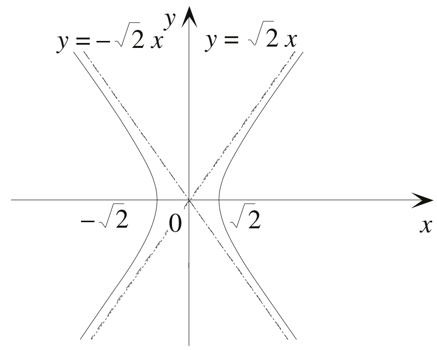
Equations of asymptotes: $y = \pm\sqrt{2}x$

When $y = 0$, $2x^2 = 4$

$\therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$

centre $(0, 0)$



e $x^2 - 4y^2 - 4x - 8y - 16 = 0$

$\therefore (x^2 - 4x + 4) - 4(y^2 + 2y + 1) - 16 = 0$

$\therefore (x - 2)^2 - 4(y + 1)^2 = 16$

$\therefore \dfrac{(x-2)^2}{16} - \dfrac{(y+1)^2}{4} = 1$

$\therefore \dfrac{(y+1)^2}{4} = \dfrac{(x-2)^2}{16} - 1$

$\therefore (y + 1)^2 = \dfrac{(x-2)^2}{4} - 4$

$= \dfrac{(x-2)^2}{4}\left[1 - \dfrac{16}{(x-2)^2}\right]$

As $x \to \pm\infty$, $\dfrac{16}{(x-2)^2} \to 0$

$\therefore (y + 1)^2 \to \dfrac{(x-2)^2}{4}$

$\therefore y + 1 \to \pm\dfrac{x-2}{2}$

$\therefore y \to -1 \pm \dfrac{x-2}{2}$

Equations of asymptotes:

$y = -1 \pm \dfrac{x-2}{2}$

i.e. $y = \frac{x-4}{2}$ and $y = \frac{-x}{2}$

$= \frac{1}{2}x - 2$ $\qquad = -\frac{1}{2}x$

When $y = -1$, $\frac{(x-2)^2}{16} = 1$

$\therefore (x-2)^2 = 16$

$\therefore x - 2 = \pm 4$

$\therefore x = -2, 6$

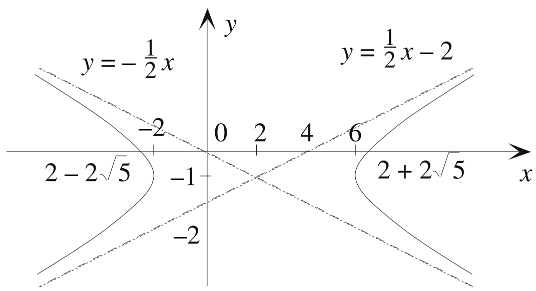
centre $(2, -1)$

When $y = 0$, $\frac{(x-2)^2}{16} - \frac{1}{4} = 1$

$\therefore \frac{(x-2)^2}{16} = \frac{5}{4}$

$\therefore (x-2)^2 = 20$

$\therefore x = 2 \pm 2\sqrt{5}$



f $9x^2 - 25y^2 - 90x + 150y = 225$

$\therefore 9(x^2 - 10x + 25) - 25(y^2 - 6y + 9) = 225$

$\therefore 9(x-5)^2 - 25(y-3)^2 = 225$

$\therefore \frac{(x-5)^2}{25} - \frac{(y-3)^2}{9} = 1$

$\therefore \frac{(y-3)^2}{9} = \frac{(x-5)^2}{25} - 1$

$\therefore (y-3)^2 = \frac{9(x-5)^2}{25} - 9$

$= \frac{9(x-5)^2}{25}\left[1 - \frac{25}{(x-5)^2}\right]$

As $x \to \pm\infty$, $\frac{25}{(x-5)^2} \to 0$

$\therefore (y-3)^2 \to \frac{9(x-5)^2}{25}$

$\therefore y - 3 \to \pm\frac{3(x-5)}{5}$

$\therefore y \to 3 \pm \frac{3(x-5)}{5}$

Equations of asymptotes:

$y = 3 + \frac{3(x-5)}{5}$

$= \frac{15 + 3x - 15}{5}$

$= \frac{3}{5}x$

and $y = 3 - \frac{3(x-5)}{5}$

$= \frac{15 - 3x + 15}{5}$

$= \frac{30 - 3x}{5}$

$= 6 - \frac{3}{5}x$

When $y = 3$, $\frac{(x-5)^2}{25} = 1$

$\therefore (x-5)^2 = 25$

$\therefore x - 5 = \pm 5$

$\therefore x = 0, 10$

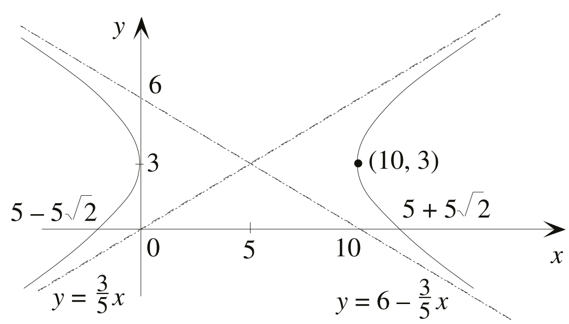
centre $(5, 3)$

When $y = 0$, $\frac{(x-5)^2}{25} - 1 = 1$

$\therefore \frac{(x-5)^2}{25} = 2$

$\therefore (x-5)^2 = 50$

$\therefore x = 5 \pm 5\sqrt{2}$



g $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{9} = 1$

$\therefore \frac{(y-3)^2}{9} = \frac{(x-2)^2}{4} - 1$

$\therefore (y-3)^2 = \frac{9(x-2)^2}{4} - 9$

$= \frac{9(x-2)^2}{4}\left[1 - \frac{4}{(x-2)^2}\right]$

As $x \to \pm\infty$, $\dfrac{4}{(x-2)^2} \to 0$

$\therefore (y-3)^2 \to \dfrac{9(x-2)^2}{4}$

$\therefore y - 3 \to \pm\dfrac{3(x-2)}{2}$

$\therefore y \to 3 \pm \dfrac{3(x-2)}{2}$

Equations of asymptotes:

$y = 3 + \dfrac{3(x-2)}{2}$

$= \dfrac{6 + 3x - 6}{2}$

$= \dfrac{3}{2}x$

and $y = 3 - \dfrac{3(x-2)}{2}$

$= \dfrac{6 - 3x + 6}{2}$

$= \dfrac{12 - 3x}{2}$

$= 6 - \dfrac{3}{2}x$

When $y = 3$, $\dfrac{(x-2)^2}{4} = 1$

$\therefore (x-2)^2 = 4$

$\therefore x = 2 \pm 2 = 0, 4$

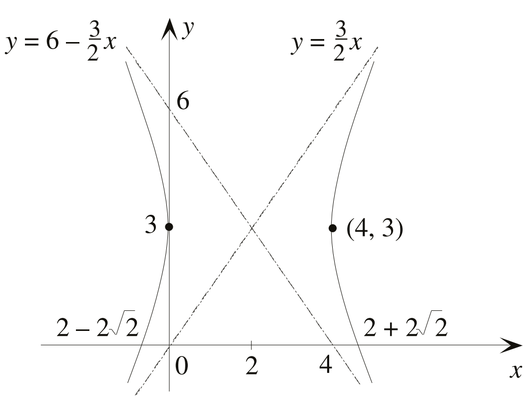
centre (2, 3)

When $y = 0$, $\dfrac{(x-2)^2}{4} - 1 = 1$

$\therefore \dfrac{(x-2)^2}{4} = 2$

$\therefore (x-2)^2 = 8$

$\therefore x = 2 \pm 2\sqrt{2}$



h $4x^2 - 8x - y^2 + 2y = 0$

$\therefore 4(x^2 - 2x + 1) - (y^2 - 2y + 1) = 3$

$\therefore 4(x-1)^2 - (y-1)^2 = 3$

$\therefore \dfrac{4(x-1)^2}{3} - \dfrac{(y-1)^2}{3} = 1$

$\therefore \dfrac{(y-1)^2}{3} = \dfrac{4(x-1)^2}{3} - 1$

$\therefore (y-1)^2 = 4(x-1)^2 - 3$

$= 4(x-1)^2\left[1 - \dfrac{3}{4(x-1)^2}\right]$

As $x \to \pm\infty$, $\dfrac{3}{4(x-1)^2} \to 0$

$\therefore (y-1)^2 \to 4(x-1)^2$

$\therefore y - 1 \to \pm 2(x-1)$

$\therefore y \to 1 \pm 2(x-1)$

Equations of asymptotes:

$y = 1 + 2(x-1)$

$= 1 + 2x - 2$

$= 2x - 1$

and $y = 1 - 2(x-1)$

$= 1 - 2x + 2$

$= 3 - 2x$

When $y = 1$, $\dfrac{4(x-1)^2}{3} = 1$

$\therefore 4(x-1)^2 = 3$

$\therefore (x-1)^2 = \dfrac{3}{4}$

$\therefore x = 1 \pm \dfrac{\sqrt{3}}{2}$

centre (1, 1)

When $y = 0$, $\dfrac{4(x-1)^2}{3} - \dfrac{1}{3} = 1$

$\therefore \dfrac{4(x-1)^2}{3} = \dfrac{4}{3}$

$\therefore (x-1)^2 = 1$

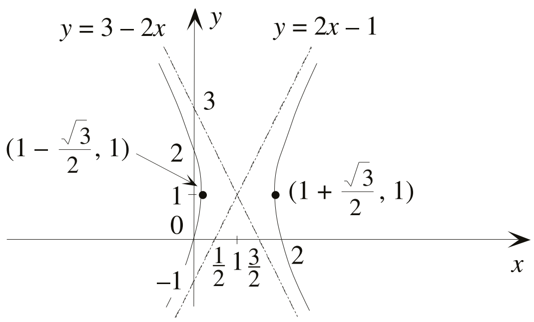
$\therefore x = 1 \pm 1 = 0, 2$

When $x = 0$, $\dfrac{4}{3} - \dfrac{(y-1)^2}{3} = 1$

$\therefore \dfrac{(y-1)^2}{3} = \dfrac{1}{3}$

$\therefore (y-1)^2 = 1$

$\therefore y = 1 \pm 1 = 0, 2$



i $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

$\therefore 9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) - 151 + 7 = 0$

$\therefore 9(x-1)^2 - 16(y-1)^2 = 144$

$\therefore \dfrac{(x-1)^2}{16} - \dfrac{(y-1)^2}{9} = 1$

$\therefore \dfrac{(y-1)^2}{9} = \dfrac{(x-1)^2}{16} - 1$

$\therefore (y-1)^2 = \dfrac{9(x-1)^2}{16} - 9$

$= \dfrac{9(x-1)^2}{16}\left[1 - \dfrac{16}{(x-1)^2}\right]$

As $x \to \pm\infty$, $\dfrac{16}{(x-1)^2} \to 0$

$\therefore (y-1)^2 \to \dfrac{9(x-1)^2}{16}$

$\therefore y - 1 \to \pm\dfrac{3(x-1)}{4}$

$\therefore y \to 1 \pm \dfrac{3(x-1)}{4}$

Equations of asymptotes:

$y = 1 + \dfrac{3(x-1)}{4}$

$= \dfrac{4 + 3x - 3}{4}$

$= \dfrac{1 + 3x}{4}$

$= \dfrac{3}{4}x + \dfrac{1}{4}$

and $y = 1 - \dfrac{3(x-1)}{4}$

$= \dfrac{4 - 3x + 3}{4}$

$= \dfrac{7 - 3x}{4}$

$= \dfrac{7}{4} - \dfrac{3}{4}x$

When $y = 1$, $\dfrac{(x-1)^2}{16} = 1$

$\therefore (x-1)^2 = 16$

$\therefore x = 1 \pm 4 = -3, 5$

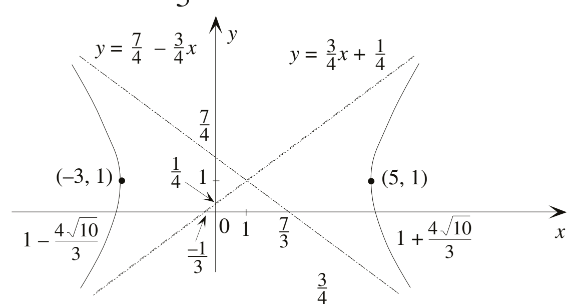
centre $(1, 1)$

When $y = 0$, $\dfrac{(x-1)^2}{16} - \dfrac{1}{9} = 1$

$\therefore \dfrac{(x-1)^2}{16} = \dfrac{10}{9}$

$\therefore (x-1)^2 = \dfrac{160}{9}$

$\therefore x = 1 \pm \dfrac{4\sqrt{10}}{3}$



j $25x^2 - 16y^2 = 400$

$\therefore \dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$

$\therefore \dfrac{y^2}{25} = \dfrac{x^2}{16} - 1$

$\therefore y^2 = \dfrac{25x^2}{16} - 25$

$= \dfrac{25x^2}{16}\left[1 - \dfrac{16}{x^2}\right]$

As $x \to \pm\infty$, $\dfrac{16}{x^2} \to 0$

$\therefore y^2 \to \dfrac{25x^2}{16}$

$\therefore y \to \pm\dfrac{5}{4}x$

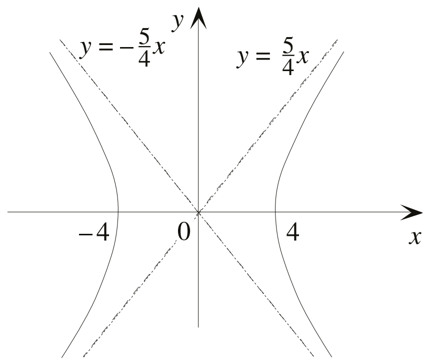
Equations of asymptotes: $y = \pm\frac{5}{4}x$

When $y = 0$, $25x^2 = 400$

$\therefore x^2 = 16$

$\therefore x = \pm 4$

centre (0, 0)



$\frac{x^2}{4} + \left(\frac{1}{2}x\right)^2 = 1$

$\therefore \frac{x^2}{4} + \frac{x^2}{4} = 1$

$\therefore \frac{x^2}{2} = 1$

$\therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$

When $x = \sqrt{2}$, $y = \frac{\sqrt{2}}{2}$

When $x = -\sqrt{2}$, $y = \frac{-\sqrt{2}}{2}$

The points of intersection are

$\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\sqrt{2}, \frac{-\sqrt{2}}{2}\right)$

3 **a** Substituting $y = \frac{1}{2}x$ into $x^2 - y^2 = 1$ gives

$x^2 - \left(\frac{1}{2}x\right)^2 = 1$

$\therefore x^2 - \frac{1}{4}x^2 = 1$

$\therefore \frac{3}{4}x^2 = 1$

$\therefore x^2 = \frac{4}{3}$

$\therefore x = \pm\frac{2\sqrt{3}}{3}$

Now $y = \frac{1}{2}x$

$\therefore y = \frac{\sqrt{3}}{3}$ when $x = \frac{2\sqrt{3}}{3}$

and $y = \frac{-\sqrt{3}}{3}$ when $x = \frac{-2\sqrt{3}}{3}$

The points of intersection are

$\left(\frac{2\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ and

$\left(\frac{-2\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$

b Substituting $y = \frac{1}{2}x$ into $\frac{x^2}{4} + y^2 = 1$ gives

4 Substituting $y = x + 5$ into $x^2 + \frac{y^2}{4} = 1$ gives

$x^2 + \frac{(x+5)^2}{4} = 1$

$\therefore 4x^2 + x^2 + 10x + 25 = 4$

$\therefore 5x^2 + 10x + 21 = 0$

$\therefore 5(x^2 + 2x + 1) + 16 = 0$

$\therefore 5(x+1)^2 = -16$

$\therefore (x+1)^2 = \frac{-16}{5}$

But $(x+1)^2 \geq 0$ $\quad\therefore$ there is no intersection point.

5 Since $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is a reflection of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in either of the lines $y = \pm x$, the points of intersection of the two ellipses occur when $y = \pm x$.

Substituting $y = \pm x$ into $\frac{x^2}{9} + \frac{y^2}{4} = 1$ gives

$\frac{x^2}{9} + \frac{x^2}{4} = 1$

$\therefore 4x^2 + 9x^2 = 36$

$\therefore 13x^2 = 36$

$\therefore x^2 = \frac{36}{13}$

$\therefore x = \pm\frac{6}{\sqrt{13}} = \pm\frac{6\sqrt{13}}{13}$

Hence the points of intersection are $\left(\frac{-6\sqrt{13}}{13}, \frac{-6\sqrt{13}}{13}\right)$, $\left(\frac{6\sqrt{13}}{13}, \frac{6\sqrt{13}}{13}\right)$, $\left(\frac{-6\sqrt{13}}{13}, \frac{-6\sqrt{13}}{13}\right)$ and $\left(\frac{6\sqrt{13}}{13}, \frac{-6\sqrt{13}}{13}\right)$.

These four points are all equidistant from the origin and hence form the vertices of a square.

6 $5x = 4y \qquad \therefore y = \frac{5}{4}x$

Substituting $y = \frac{5}{4}x$ into $\frac{x^2}{16} + \frac{y^2}{25} = 1$ gives

$\frac{x^2}{16} + \frac{\left(\frac{5}{4}x\right)^2}{25} = 1$

$\therefore \frac{x^2}{16} + \frac{25x^2}{16 \times 25} = 1$

$\therefore \frac{x^2}{16} + \frac{x^2}{16} = 1$

$\therefore \frac{x^2}{8} = 1$

$\therefore x^2 = 8$

$\therefore x = \pm 2\sqrt{2}$

When $x = \pm 2\sqrt{2}$,

$y = \frac{5}{4} \times \pm 2\sqrt{2} = \pm\frac{5\sqrt{2}}{2}$

The points of intersection are $\left(-2\sqrt{2}, \frac{-5\sqrt{2}}{2}\right)$ and $\left(2\sqrt{2}, \frac{5\sqrt{2}}{2}\right)$

7 $x^2 + y^2 = 9$

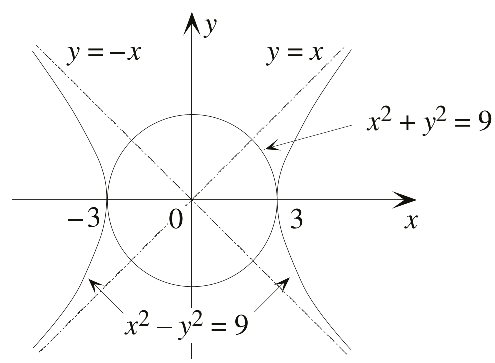
A circle with centre (0, 0) and radius 3

$x^2 - y^2 = 9 \qquad \therefore \frac{x^2}{9} - \frac{y^2}{9} = 1$

A hyperbola with centre (0, 0) and asymptotes at $y = \pm x$

When $y = 0$, $x^2 = 9$

$\therefore x = \pm 3$



Solutions to Exercise 1G

1 $x = 2\cos 3t$ and $y = 2\sin 3t$

$\text{ran}(x) = [-2, 2]$

$= \text{dom(cartesian equation)}$

$\text{ran}(y) = [-2, 2]$

$= \text{ran(cartesian equation)}$

$\therefore \frac{x}{2} = \cos 3t$ and $\frac{y}{2} = \sin 3t$

Squaring both sides of each equation gives $\frac{x}{4} = \cos^2 3t$ and $\frac{y^2}{4} = \sin^2 3t$

Adding these two equations together gives $\frac{x^2}{4} + \frac{y^2}{4} = 1$

$\therefore x^2 + y^2 = 4,\ \text{dom} = [-2,\ 2]$

$\text{ran} = [-2, 2]$

2 $x = 4t^2, y = 8t$

a $x = 4 \times \frac{y^2}{64}$ since $t = \frac{y}{8}$

$\therefore x = \frac{y^2}{16}$ or $y^2 = 16x$

b When $t = 1, x = 4, y = 8$

When $t = -1, x = 4, y = -8$

Therefore equation of line $x = 4$

c When $t = -3, x = 36, y = -24$

Length PR of chord joining $P(4, 8)$ and $R(36, -24)$ is given by

$PR = \sqrt{32^2 + 32^2}$

$= 32\sqrt{2}$

3 $x = 2 + 3\sin t$

$\therefore \sin t = \frac{x-2}{3}$

$y = 3 - 2\cos t$

$\therefore \cos t = \frac{y-3}{-2}$

$\therefore \left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{-2}\right)^2 = 1$

That is, $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1$

4 $x = 2\sec t$

$\therefore \sec t = \frac{x}{2}$

$y = 3\tan t$

$\therefore \tan t = \frac{y}{3}$

$\therefore \left(\frac{y}{3}\right)^2 + 1 = \left(\frac{x}{2}\right)^2$

That is, $\frac{x^2}{4} - \frac{y^2}{9} = 1, x \le 2, y \in \mathbb{R}$

5 a $x = 4\cos 2t, y = 4\sin 2t$

$\therefore \cos 2t = \frac{x}{4}$ and $\sin 2t = \frac{y}{4}$

$\therefore \frac{x^2}{16} + \frac{y^2}{16} = \cos^2 2t + \sin^2 2t = 1$

$\therefore x^2 + y^2 = 16$

b $x = 2\sin 2t, y = 2\cos 2t$

$\therefore \sin 2t = \frac{x}{4}$ and $\cos 2t = \frac{y}{4}$

$\therefore \frac{x^2}{4} + \frac{y^2}{4} = \cos^2 2t + \sin^2 2t = 1$

$\therefore x^2 + y^2 = 4$

c $x = 4\cos t, y = 3\sin t$

$\therefore \cos t = \frac{x}{4}$ and $\sin t = \frac{y}{3}$

$\therefore \frac{x^2}{16} + \frac{y^2}{9} = \cos^2 t + \sin^2 t = 1$

$\therefore \frac{x^2}{16} + \frac{y^2}{9} = 1$

d $x = 4\sin t, y = 3\cos t$

$\therefore \sin t = \frac{x}{4}$ and $\cos t = \frac{y}{3}$

$\therefore \frac{x^2}{16} + \frac{y^2}{9} = \cos^2 t + \sin^2 t = 1$

$\therefore \frac{x^2}{16} + \frac{y^2}{9} = 1$

e $x = 2\tan 2t$

$\therefore \tan 2t = \frac{x}{2}$

$y = 3\sec 2t$

$\therefore \sec 2t = \frac{y}{3}$

$\therefore \frac{y^2}{9} - \frac{x^2}{4} = 1$

f $x = 1 - t, \quad y = t^2 - 4$

$\therefore t = 1 - x$ and

$\therefore y = (x-1)^2 - 4 = x^2 - 2x - 3$

g $x = t + 2, \quad y = \frac{1}{t}$

$\therefore t = x - 2$ and

$\therefore y = \frac{1}{x-2}$

h $x = t^2 - 1, \quad y = t^2 + 1$

$\therefore t^2 = x + 1$ and

$\therefore y = \frac{1}{x-2}$ Note: $x \geq -1$

i $x = t - \frac{1}{t}, \quad y = 2(t + \frac{1}{t})$

$x^2 = t^2 - 2 + \frac{1}{t^2}$ and

$y^2 = 4(t^2 + 2 + \frac{1}{t^2})$

$\therefore x^2 = t^2 - 2 + \frac{1}{t^2}$ and

$\frac{y^2}{4} = t^2 + 2 + \frac{1}{t^2}$

$t^2 + \frac{1}{t^2} = x^2 + 2$ and $t^2 + \frac{1}{t^2} = \frac{y^2}{4} - 2$

$\therefore \frac{y^2}{4} - 2 = x^2 - 2$

$\frac{y^2}{16} - \frac{x^2}{4} = 1$

6 a $x = \sec t$ and $y = \tan t, \ t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Squaring both sides of each equation gives

$x = \sec^2 t$ ① and $y^2 = \tan^2 t$ ②

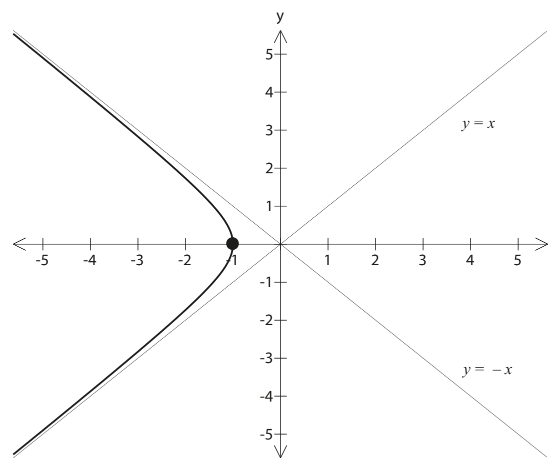
① - ②

$x^2 - y^2 = \sec^2 t - \tan^2 t$

$\therefore x^2 - y^2 = 1$

For the function $x = \sec t, t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ the range is $(-\infty, -1]$. Hence the domain of the cartesian equation is $(-\infty, -1]$ which corresponds to the left branch of the hyperbola.

Equations of asymptotes: $y = \pm x$



b $x = 3\cos 2t$ and $y = -4t\sin 2t$

$\text{ran}(x) = [-3, 3]$

$= \text{dom(cartesian equation)}$

$\text{ran}(y) = [-4, 4]$

$= \text{ran(cartesian equation)}$

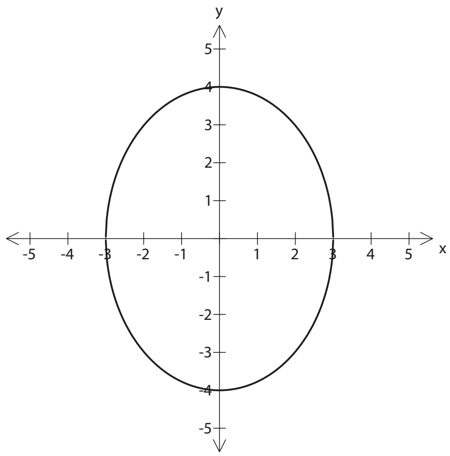
$\therefore \frac{x}{3} = \cos 2t$ and $\frac{y}{4} = -\sin 2t$

Squaring both sides of each equation

gives $\frac{x^2}{9} = \cos^2 2t$ and $\frac{y^2}{16} = \sin^2 2t$
Adding these two equations together gives $\frac{x^2}{9} + \frac{y^2}{16} = 1$

$\therefore \frac{x^2}{9} + \frac{y^2}{16} = 1$, dom $= [-3, 3]$

ran $= [-4, 4]$



c $x = 3 - 3\cos t$ and $y = 2 + 2\sin t$

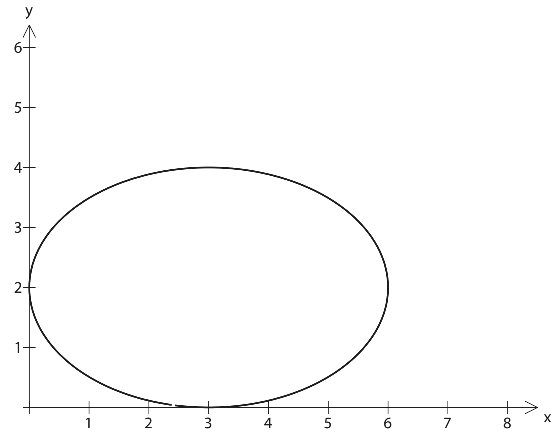
$$\begin{aligned}\text{ran}(x) &= [-3 + 3, 3 + 3]\\ &= [0, 6]\\ &= \text{dom(cartesian equation)}\\ \text{ran}(y) &= [-2 + 2, 2 + 2]\\ &= [0, 4]\\ &= \text{ran(cartesian equation)}\end{aligned}$$

$\therefore x - 3 = (-3\cos t$ and $y - 2 = 2\sin t$
Squaring both sides of each equation gives $(x - 3)^2 = 9\cos^2 t$ and $(y-2)^2 = 4\sin^2 t$

$\therefore \frac{(x-3)^2}{9} = \cos^2 t$ and $\frac{(y-2)^2}{4} = \sin^2 t$ Adding these two equations together gives $\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$

$\therefore \frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$, dom $= [0\ 6]$

ran $= [0, 4]$



A CAS calculator has the capability to sketch parametric equations.

In order to sketch a graph for part **c**:

Note: ensure your handheld unit is set to radian/Rad mode.

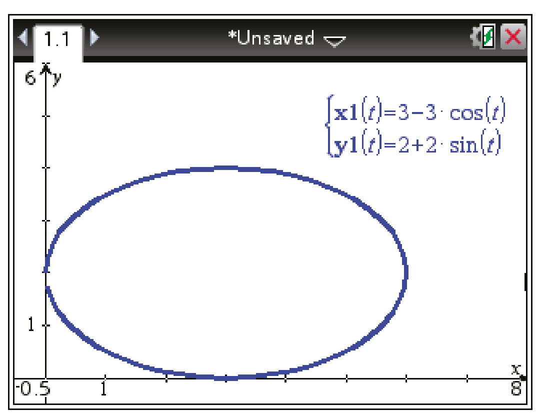
TI: Open a Graphs page. Press **Menu → 3: Graph Entry/Edit → 3: Parametric**

Now type the following information:

$x1(t) = 3 - 3\cos t$

$y1(t) = 2 + 2\sin t$

$0 < f < 2\pi$ *tstep* $= 0.13$
and press ENTER
Set the window to:
Xmin $= -0.5$
Xmax $= 8$
Ymin $= -0.5$
Ymax $= 6$



CP: In the Graph & Table application tap n and select $\boldsymbol{x_t}$**=**. Input the equations into the corresponding positions followed by EXE.
Tap $ to see the graph.

d $x = 3\sin t$ and $y = 4\cos t,\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\text{ran}(x) = [-3, 3]$ for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$= \text{dom(cartesian equation)}$

$\text{ran}(y) = [0, 4]$ for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$= \text{ran(cartesian equation)}$

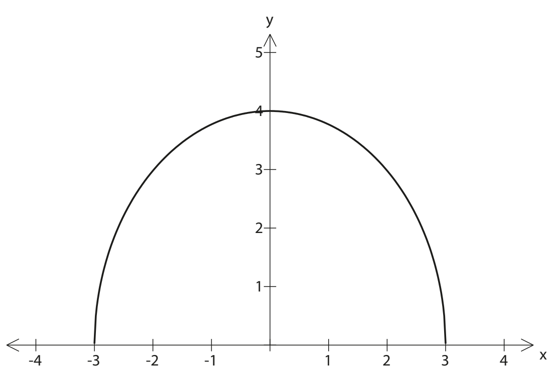
$\therefore \frac{x}{3} = \sin t$ and $\frac{y}{4} = \cos t$

Squaring both sides of each equation gives $\frac{x^2}{9} = \sin^2 t$ and $\frac{y^2}{16} = \cos^2 t$

Adding these two equations together gives $\frac{x^2}{9} + \frac{y^2}{16} = 1$

$\therefore \frac{x^2}{9} + \frac{y^2}{16} = 1,\ \text{dom} = [-3,\ 3]$

$\text{ran} = [0, 4]$



e $x = \sec t$ and $y = \tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Squaring both sides of each equation gives

$x^2 = \sec^2 t$ ① and $y^2 = \tan^2 t$ ②

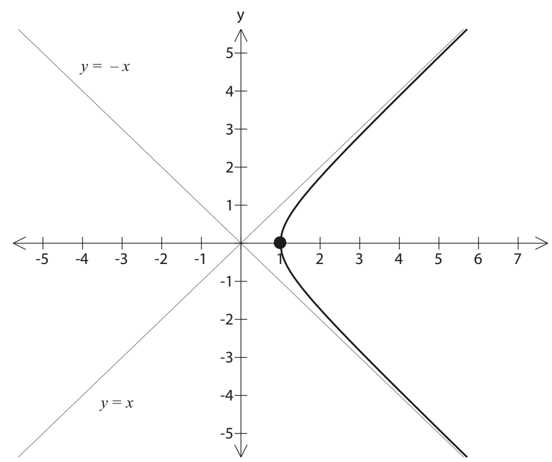
① − ②

$x^2 - y^2 = \sec^2 t - \tan^2 t$

$\therefore x^2 - y^2 = 1$

For the function $x = \sec t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the range is $[1, \infty)$. Hence the domain of the cartesian equation is $[1, \infty)$ which corresponds to the right branch of the hyperbola.

Equations of asymptotes: $y = \pm x$



f $x = 1 - \sec(2t)$ and $y = 1 + \tan(2t)$, where $t \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

$\therefore x - 1 = -\sec(2t)$ and $y - 1 = \tan(2t)$

Squaring both sides of each equation gives

$(x-1)^2 = \sec^2(2t)$ ①

and $(y-1)^2 = \tan^2(2t)$ ②

① − ②

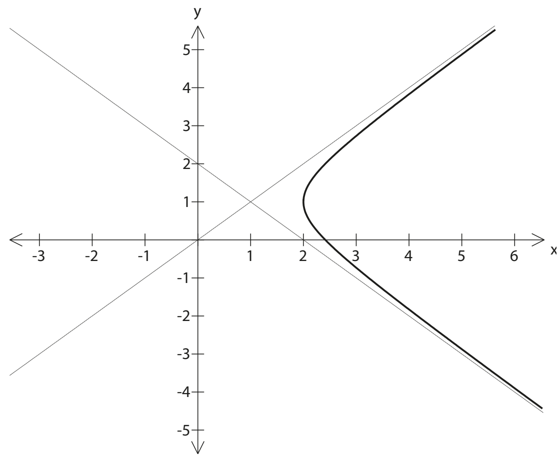
$(x-1)^2 - (y-1)^2 = 1$

For the function $x = 1 - \sec(2t)$ where $t \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ the range is $[2, \infty)$. Hence the domain of the cartesian equation is $[2, \infty)$ which corresponds to the right branch of the hyperbola.

Equations of asymptotes:

$y = 1 \pm (x - 1)$

$\therefore y = x$ and $y = 2 - x$



7 **a** $x = 2\cos\left(\frac{8\pi}{3}\right) = -1$

$y = -2\sin\left(\frac{8\pi}{3}\right) = -\sqrt{3}$

The point P has coordinates $(-1, -\sqrt{3})$

b The circle has centre $O(0, 0)$. The gradient of $OR = \sqrt{3}$

Therefore gradient of tangent $= -\frac{1}{\sqrt{3}}$

Equation of tangent:

$y + \sqrt{3} = -\frac{1}{\sqrt{3}}(x + 1)$

$\sqrt{3}y + 3 = -x - 1$

$\sqrt{3}y + x = -4$

8 **a** $x^2 + y^2 = 16$

$\therefore x^2 + y^2 = 4^2$

In general for $x^2 + y^2 = a^2$ the most basic parametric equations have the form $x = a\cos t$ and $y = a\sin t$

Hence the parametric equations are $x = 4\cos t$ and $y = 4\sin t$

b $\frac{x^2}{9} - \frac{y^2}{4} = 1$

$\therefore \frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$

In general for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the most basic parametric equations have the form $x = a\sec t$ and $y = a\tan t$

Hence the parametric equations are $x = 3\sec t$ and $y = 2\tan t$

c $(x - 1)^2 + (y + 2)^2 = 9$

$\therefore (x - 1)^2 + (y + 2)^2 = 3^2$

centre $(1, -2)$ radius is 3

In general for $(x - h)^2 + (y - k)^2 = a^2$ the parametric equations have the form

$x = h + a\cos t$ and $y = k + a\sin t$

Hence the parametric equations are $x = 1 + 3\cos t$ and $y = 3\sin t - 2$

d $\frac{(x-1)^2}{9} + \frac{(y+3)^2}{4} = 9$

$\therefore \frac{(x-1)^2}{9^2} + \frac{(y+3)^2}{6^2} = 1$

In general for $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ the parametric equations have the form $x = h + a\cos t$ and $y = k + b\sin t$

Hence the parametric equations are $x = 1 + 9\cos t$ and $y = 6\sin t - 3$

9 centre (1, 3) and radius 2

$\therefore (x-1)^2 + (y-3)^2 = 2^2$

As the parametric equations are in the form $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$

$\therefore x = 1 + 2\cos(2\pi t)$and

$y = 3 + 2\sin(2\pi t)$

$\therefore$ a=1, b=2, c=3 and d=2

10 Ellipse: x–intercepts at $(-4, 0)$ and $(4, 0)$

y –intercepts at $(0, 3)$ and $(0, -3)$

Hence a possible cartesian equation for

this ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Thus a possible pair of parametric equations for the above ellipse is $x = 4\cos t$ and $y = 3\sin t$

11 $x = 2\cos(2t)$ and $y = 2\sin(2t)$

a For a dilation of factor 3 from the x-axis the point $(x,\ y)$ is mapped onto $(x,\ 3y)$

i.e. $(x,\ y) \to (x,\ 3y)$

Thus to find the equation of the image curve under the dilation $(x,\ y) \to (x,\ 3y)$, replace y with $\frac{y}{3}$.

$\therefore \frac{y}{3} = 2\sin(2t)$

$\therefore y = 6\sin(2t)$

Hence one possible pair of parametric equations for the image curve is $x = 2\cos(2t)$ and $y = 6\sin(2t)$

b $x = 2\cos(2t)$ and $y = 6\sin(2t)$

$\therefore \frac{x}{2} = \cos(2t)$ and $\frac{y}{6} = \sin(2t)$

Squaring both sides of each equation gives $\frac{x^2}{4} = \cos^2(2t)$ and $\frac{y^2}{36} = \sin^2(2t)$

Adding these two equations together gives $\frac{x^2}{4} + \frac{y^2}{36} = 1$ as $\cos^2(kt) + \sin^2(kt) = 1$

Hence the cartesian equation is

$\frac{x^2}{4} + \frac{y^2}{36} = 1$

12 $x = 3 - 2\cos\left(\frac{t}{2}\right)$ and $y = 4 + 3\sin\left(\frac{t}{2}\right)$

a For a translation of 3 units in the negative direction of the x-axis and a translation of 2 units in the negative direction of the y-axis:

$(x, y) \to (x - 3, y - 2)$

Let (x', y') be the coordinates of the image of (x, y) so $x' = x - 3$, $y' = y - 2$

Rearranging gives $x = x' + 3$ and $y = y' + 2$

So $x = 3 - 2\cos\left(\frac{t}{2}\right)$ becomes

$x' + 3 = 3 - 2\cos\left(\frac{t}{2}\right)$

$\therefore x' = -\cos\left(\frac{t}{2}\right)$

and $y = 4 + 3\sin\left(\frac{t}{2}\right)$ becomes

$y' + 2 = 4 + 3\sin\left(\frac{t}{2}\right)$

$\therefore y' = 2 + 3\sin\left(\frac{t}{2}\right)$

Thus the parametric equations of the image curve are

$x = -2\cos\left(\frac{t}{2}\right)$ and $y = 2 + 3\sin\left(\frac{t}{2}\right)$

b $x = -2\cos\left(\frac{t}{2}\right)$ and $y = 2 + 3\sin\left(\frac{t}{2}\right)$

$\therefore \frac{x}{2} = -\cos\left(\frac{t}{2}\right)$ and $\frac{y-2}{3} = \sin\left(\frac{t}{2}\right)$

Squaring both sides of each equation gives $\frac{x^2}{4} = \cos^2\left(\frac{t}{2}\right)$ and

$\frac{(y-2)^2}{9} = \sin^2\left(\frac{t}{2}\right)$

Adding these two equations together gives $\frac{x^2}{4} + \frac{(y-2)^2}{9} = 1$

Hence the cartesian equation is

$\frac{x^2}{4} + \frac{(y-2)^2}{9} = 1$

13 $x = 2 + 3\sin(2\pi t)$ and $y = 4 + 2\cos(2\pi t)$

a ran$(x) = [2, 5]$ for $t \in \left[0, \frac{1}{4}\right]$

$= \text{dom(cartesian equation)}$

ran$(y) = [4, 6]$ for $t \in \left[0, \frac{1}{4}\right]$

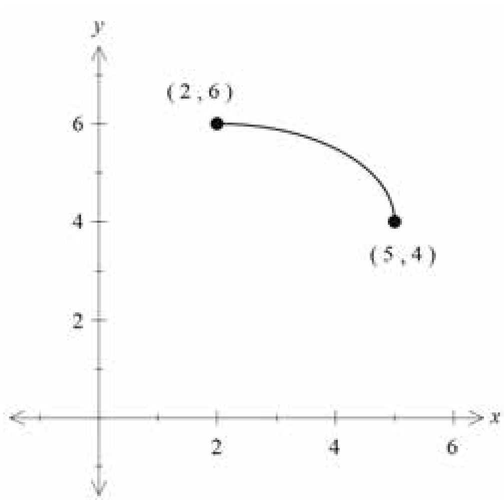
$= \text{ran(cartesian equation)}$

and the cartesian equation is

$$\frac{(x-2)^2}{9} + \frac{(y-4)^2}{4} = 1$$

$$\therefore \frac{(x-2)^2}{9} + \frac{(y-4)^2}{4} = 1, \text{ dom} = [2, 5]$$

$$\text{ran} = [4, 6]$$



b ran$(x) = [2, 5]$ for $t \in \left[0, \frac{1}{2}\right]$

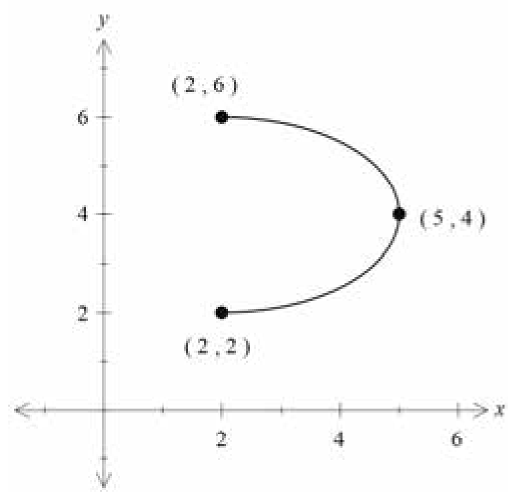
$= \text{dom(cartesian equation)}$

ran$(y) = [2, 6]$ for $t \in \left[0, \frac{1}{2}\right]$

$= \text{ran(cartesian equation)}$

$$\therefore \frac{(x-2)^2}{9} + \frac{(y-4)^2}{4} = 1, \text{ dom} = [2, 5]$$

$$\text{ran} = [2, 6]$$



c ran$(x) = [-1, 5]$ for $t \in \left[0, \frac{3}{2}\right]$

$= \text{dom(cartesian equation)}$

ran$(y) = [2, 6]$ for $t \in \left[0, \frac{3}{2}\right]$

$= \text{ran(cartesian equation)}$

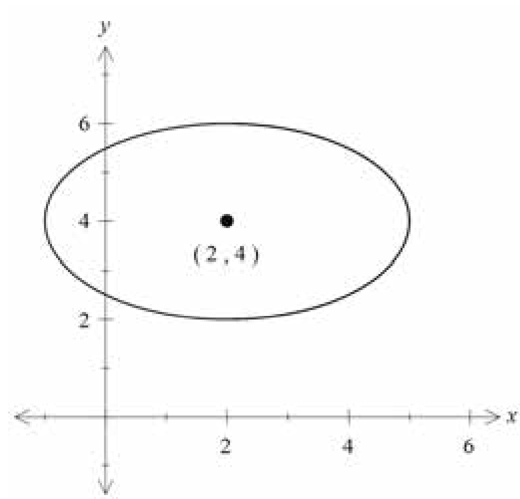
When $x = 0, \frac{4}{9} + \frac{(y-4)^2}{4} = 1$

$\therefore \frac{(y-4)^2}{4} = \frac{5}{9}$

$\therefore (y-4)^2 = \frac{20}{9}$

$\therefore y - 4 = \pm\frac{2\sqrt{5}}{3}$

$y = 4 \pm \frac{2\sqrt{5}}{3}$



Solutions to Exercise 1H

1

```
n ← 1
x ← 3
while x ≤ 100
   n ← n + 1
   x ← 2x + 3
end while
print n
```

n	x
1	3
2	9
3	21
4	45
5	93
6	189

2

```
define evenprod(n):
   product ← 1
   for i from 1 to n
      product ← product × 2i
   end for
   return product
```

3

```
define powers(n):
   A ← [ ]
   for i from 1 to n
      append 2^(i−1) to A
   end for
   return A
```

4 **a**

```
for x from 1 to 22
   for y from 1 to 22
      for z from 1 to 22
         if x^2 + y^2 + z^2 = 500 then
            print (x, y, z)
         end if
      end for
   end for
end for
```

b

```
for x from 1 to 99
   for y from 1 to 99
      for z from 1 to 99
         if x^3 + y^3 + z^3 = 1 000 000
then
            print (x, y, z)
         end if
      end for
   end for
end for
```

5 **a** **i** 0.099 833

ii 0.841 468

iii 0.907 937

6 **a**

```
define cossum(x, n):
   sum ← 0
   for k from 1 to n
      sum ← sum + ((−1)^(k+1) × x^(2k−2)) / factorial(2k − 2)
   end for
   return sum
```

b $-\dfrac{19}{45} \approx -0.422, \ \cos 2 \approx -0.416$

7 **a**

i	$strip$	sum	$left$
		0	0
1	1.5	1.5	0.5
2	1.8125	3.3125	1
3	3	6.3125	1.5
4	5.4375	11.75	2
5	9.5	21.25	2.5
6	15.5625	36.8125	3
7	24	60.8125	3.5
8	35.1875	96	4
9	49.5	145.5	4.5
10	67.3125	212.8125	5

b

```
define f(x):
    return x^3 + 2x^2 + 3

a ← 0
b ← 5
n ← 50
h ← (b − a)/n
left ← a
right ← a + h
sum ← 0
for i from 1 to n
    strip ← 0.5 × (f(left) + f(right)) × h
    sum ← sum + strip
    left ← left + h
    right ← right + h
end for
print sum
```

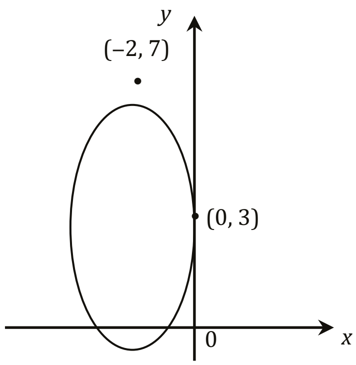
8 **a** 1.259 921

b 3.141 593

Solutions to Short-answer questions

1 $f_n = 5f_{n-1}, f_0 = 1$
Geometric sequence with $r = 5, f_1 = 5$.
$f_n = 5 \times 5^{n-1}$

2



The centre of the ellipse is $(-2, 3)$. The minor axis has length 4 and the major axis length 8. Hence using the general equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

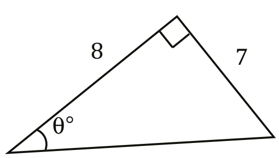
gives $\frac{(x+2)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$

$(0, 3)$ is on the ellipse. Hence $\frac{4}{a^2} = 1$ and $a^2 = 4$.

Also $(-2, 7)$ is on the ellipse. Hence $\frac{16}{b^2} = 1$ and $b^2 = 16$

Hence the equation is $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{16} = 1$

3



The triangle is right-angled and so the hypotenuse has length $\sqrt{49+64} = \sqrt{113}$.

Therefore $\sin(\theta^\circ) = \frac{7}{\sqrt{113}}$

4 $\frac{x}{9} = \sin(30^\circ)$

Therefore $x = 9\sin(30^\circ) = \frac{9}{2}$

5 **a** $\cos(315^\circ) = \cos(360 - 45)^\circ$

$$= \cos(45^\circ) = \frac{\sqrt{2}}{2}$$

b If $\tan(x^\circ) = \frac{3}{4}$ and $180 < x < 270$,

use $1 + \tan^2\theta = \sec^2\theta$

$$\sec^2(x^\circ) = 1 + \frac{9}{16}$$

$$\sec^2(x^\circ) = \frac{25}{16}$$

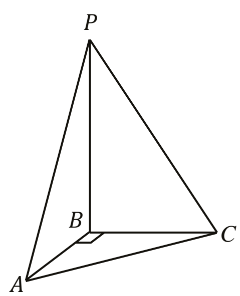
and $\sec(x^\circ) = \pm\frac{5}{4}$

Hence $\cos(x^\circ) = -\frac{4}{5}$

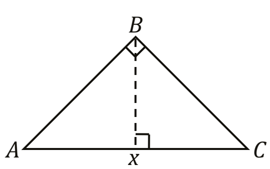
as $180 < x < 270$.

c $\sin A = \sin 330^\circ$.

6



The triangle ABC is right-angled at B, and $AB = BC = 1$ cm.
Pythagoras' theorem gives that $AC = \sqrt{2}$.
Triangle ABC is isosceles, and X is the midpoint of AC.
Using Pythagoras' theorem again gives $BX = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{2}}{2}$



Let angle BXP have magnitude θ°.

Then $\tan\theta° = 3 \div \dfrac{\sqrt{2}}{2}$

$= \dfrac{6}{\sqrt{2}}$

$= 3\sqrt{2}$

Therefore $\theta° = \tan^{-1}(3\sqrt{2})$

7 **a** $2\cos(2x + \pi) - 1 = 0$

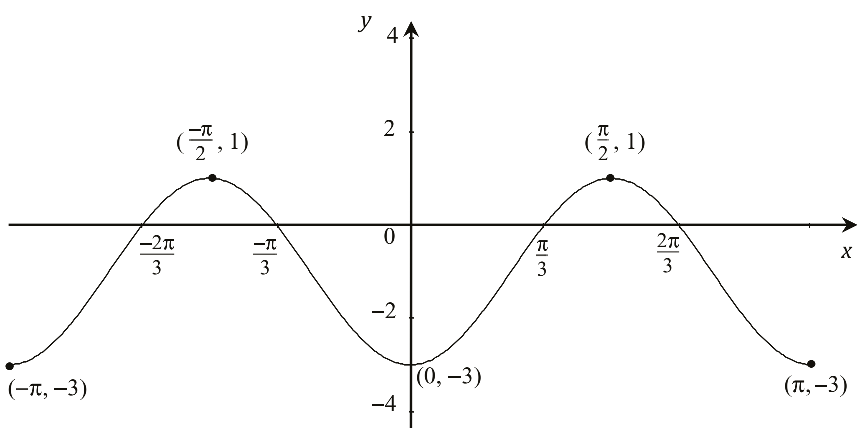
implies $-2\cos(2x) = 1$

and therefore $\cos(2x) = -\dfrac{1}{2}$

$2x = \ldots, \dfrac{-4\pi}{3}, \dfrac{-2\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$

$X = \dfrac{-2\pi}{3}, \dfrac{-\pi}{3}, \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

b



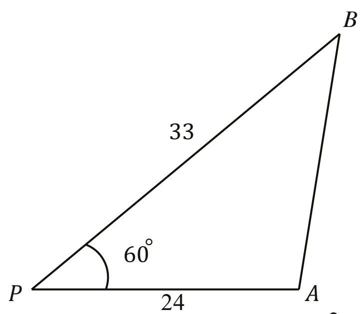
c From the graph, $2\cos(2x + \pi) < 1$ for $\left[-\pi, \dfrac{-2\pi}{3}\right) \cup \left(\dfrac{-\pi}{3}, \dfrac{\pi}{3}\right) \cup \left(\dfrac{2\pi}{3}, \pi\right]$

8 **a** $15^2 = 12^2 + 9^2$

Therefore right angle at C.

b $\tan^{-1}\dfrac{9}{9} = \tan^{-1} 1 = 45°$ the other $\tan^{-1}\dfrac{9}{12} = \tan^{-1}\dfrac{3}{4}$

9 **a**



The cosine rule gives $AB^2 = 24^2 + 33^2 - 2 \times 24 \times 33 \cos(60°) = 873$

$AB = 3\sqrt{97}$ km

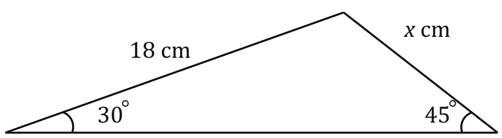
The distance apart after the hours is $3\sqrt{97}$ nautical miles.

b The speeds are 8 nautical miles per hour and 11 nautical miles per hour. Therefore the distances travelled are 40 nautical miles and 55 nautical miles respectively.

The new triangle formed is similar to the triangle of part **a**, with a scale factor of $\frac{5}{3}$.

The distance apart is $5\sqrt{97}$ nautical miles after 5 hours.

10

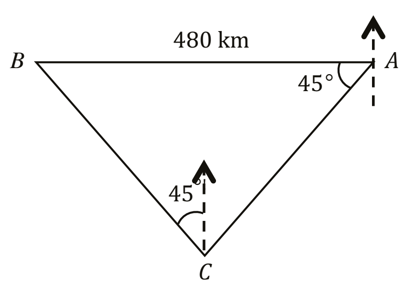


Using the sine rule gives $\frac{x}{\sin(30°)} = \frac{18}{\sin(45°)}$

Therefore $x = \frac{18}{\sin(45°)} \times \sin(30°)$

$= 18\sqrt{2} \times \frac{1}{2} = 9\sqrt{2}$

11 **a**



b The triangle ABC is right-angled at C.

$$\frac{AC}{480} = \cos(45°)$$

Therefore $AC = 240\sqrt{2}$

c The triangle is isosceles and so the total distance flown $= 480\sqrt{2}$ km.

12 For $x^2 - \frac{(y-2)^2}{9} = 15$

Rearrange to give $\frac{(y-2)^2}{9} = x^2 - 15$

and hence $y - 2 = 3\sqrt{x^2 - 15}$

and hence $y - 2 = \pm 3x\left(1 - \frac{15}{x^2}\right)^{\frac{1}{2}}$

It now can be observed that the asymptotes will have equations

$y = \pm 3x + 2$

or $y = 3x + 2$ and $y = -3x + 2$

13 For $x = 3\cos(2t) + 4$ and $y = \sin(2t) - 6$, first rearrange each of the equations.

$\cos(2t) = \frac{x-4}{3}$ and $\sin(2t) = y + 6$

Square each of these equations and add

$\cos^2(2t) + \sin^2(2t) = \frac{(x-4)^2}{9} + (y+6)^2$

Therefore the cartesian equation is $\frac{(x-4)^2}{9} + (y+6)^2 = 1$

14 For $x = 2\cos(\pi t)$ and $y = 2\sin(\pi t) + 2$,

first rearrange; $\cos(\pi t) = \frac{x}{2}$ and $\sin(\pi t) = \frac{y-2}{2}$

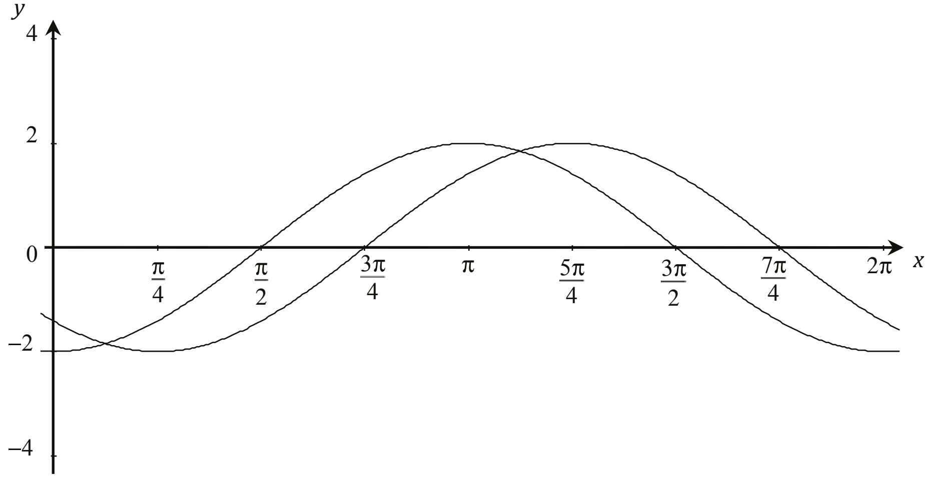
Squaring and adding gives

$\cos^2(\pi t) + \sin^2(\pi t) = \frac{x^2}{4} + \frac{(y-2)^2}{4}$

Hence the cartesian equation is

$x^2 + (y-4)^2 = 4$

15 a



b $$-2\cos\left(x-\frac{\pi}{4}\right)=0$$

implies $\cos\left(x-\frac{\pi}{4}\right)=0$

$$x-\frac{\pi}{4}=\frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or} \ldots$$

$$x=\frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

c $-2\cos x \le 0$ is equivalent to $\cos x \ge 0$

From the graph for $x \in [0, 2\pi]$, $\cos x \ge 0$ for $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

16 **a** $\sin\theta = \frac{1}{2}$

$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

b $\cos\theta = \frac{\sqrt{3}}{2}$

$$\theta = \frac{\pi}{6} \text{ or } \frac{11\pi}{6}$$

c $\tan\theta = 1$

$$\theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

17 For $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$ rearranging gives

$\frac{x-a}{b} = \cos(2\pi t)$ and $\frac{y-c}{d} = \sin(2\pi t)$

Squaring and adding gives

$$\frac{(x-a)^2}{b^2} + \frac{(y-c)^2}{d^2} = 1$$

The centre of the circle is (1, 2) and the radius is 3.

Hence $a = 1$, $c = 2$ and $b = d = 3$

18 $x^2 + 8x + y^2 - 12y + 3 = 0$

Completing the square gives

$x^2 + 8x + 16 + y^2 - 12y + 36 + 3 = 52$

$$(x+4)^2 + (y-6)^2 = 49$$

The centre of the circle is the point with coordinates $(-4, 6)$ and the radius is 7.

19 $\frac{x^2}{81} + \frac{y^2}{9} = 1$

When $x = 0,\ y^2 = 9$

and $y = 3$ or -3

When $y = 0,\ x^2 = 81$

and $x = 9$ or -9

20 **a** **i** Use $t_n = a + (n - 1)d$

$17p + 17 = 3p + 5 + 2(n - 1)$

$14p + 12 = 2(n - 1)$

Therefore $n = 7p + 7$

ii The sum of the sequence,

$S_n = \frac{7p + 7}{2}(3p + 5 + 17p + 17)$

$= 7(p + 1)(10p + 11)$

$= 7(10p^2 + 21p + 11)$

$= 70p^2 + 147p + 77$

b sum = $7(p + 1)(10p + 11)$

If p is even, $p + 1$ is odd and $10p + 1$ is odd. Therefore the sum is not divisible by 14.

If p is odd, $p + 1$ is even and hence the sum is divisible by 14.

21 **a** The n^{th} term is 3^{n-1}

b $3^0 \times 3^1 \times 3^2 \times \ldots \times 3^{n-1} = 3^{0+1+2+\ldots+(n-1)}$

$= 3^{1+2+3+\ldots+(n-1)}$

$1 + 2 + 3 + \ldots + 19 = \frac{19(19 + 1)}{2}$

$= 190$

Therefore the product of the first 20 terms is 3^{190}.

22 **a** 9

b $\frac{1}{400}$

c 4

d 4

e $\pi - 3$

f $4 - \pi$

23 **a** $(0, \frac{1}{10^4})$

b $(100, \infty)$

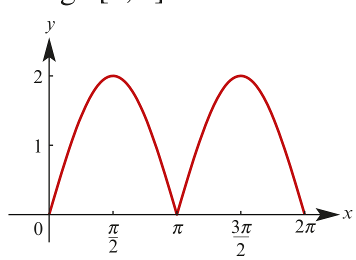
24 $|x^2 - 3x| = x$

$$\begin{cases} x^2 - 3x = x & \text{if } x^2 - 3x \geq 0 \\ -(x^2 - 3x) = x & \text{if } x^2 - 3x < 0 \end{cases}$$

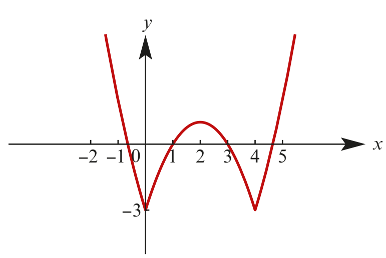
$$\begin{cases} x^2 - 4x = 0 & \text{if } x \geq 3 \text{ or } x \leq 0 \\ -x^2 + 2x = 0 & \text{if } 0 < x < 3 < 0 \end{cases}$$

$\therefore x = 0, 2, 4$

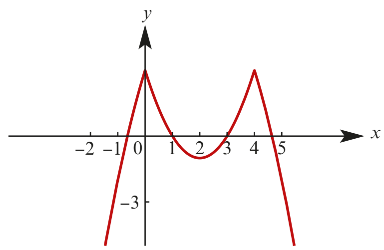
25 **a** Range $[0, 2]$



b Range $[-3, \infty)$



c Range $(-\infty, 3]$



y
x
−2
−1
0
1
2
3
4
5
−3

Solutions to multiple-choice questions

1 B $t_3 = 4$ and $t_8 = 128$

For a geometric sequence $t_n = ar^{n-1}$

Using $t_3 : 4 = ar^2$ ①

Using $t_8 : 128 = ar^7$ ②

② ÷ ① gives

$r^5 = 32$

$\therefore r = 2$

Hence $t_n = a(2)^{n-1}$

As $t_3 = 4$ then $4 = a(2^2)$

$\therefore a = 1$

Thus the first term of the sequence is 1

2 D The first term of the arithmetic sequence is not known thus the following sequence should be used.

$t_n = t_{n-1} + d$

If 5, x and y are in arithmetic sequence then

$x = 5 + d$ ①

$y = x + d$ ②

Rearranging ① for d gives:

$d = x - 5$ ③

Substituting ③ into ② gives

$y = x + (x - 5)$

$\therefore y = 2x - 5$

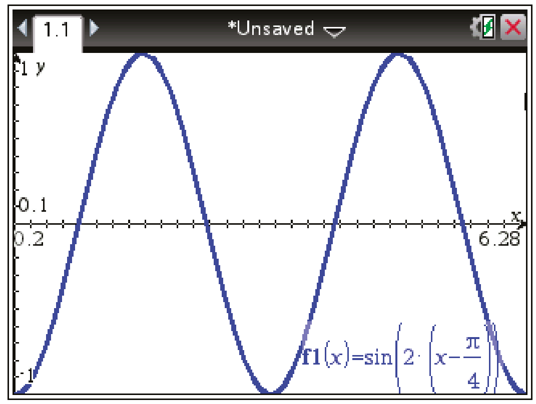
3 C $2\cos x° - \sqrt{2} = 0$

$\therefore \cos x° = \dfrac{\sqrt{2}}{2}$

$\therefore x° = 45°$ for $x° \in [0°, 90°]$

4 A As the range of the given graph is $[-1, 1]$, response D is incorrect. Clearly, the given graph has period π. Thus response B and E are also incorrect.

The graph also has a y-intercept of -1. Response C clearly does not pass through the point $(0, -1)$ while response A does Hence the given graph is $y = \sin 2\left(x - \dfrac{\pi}{4}\right)$ A quick sketch of response A on your CAS calculator will alleviate all doubt.



5 D

$6 = 4 \times 2 - 2$

$22 = 4 \times 6 - 2$

$86 = 4 \times 22 - 2$

$342 = 4 \times 86 - 2$

6 B $ar = 24 \ldots (1)$

$ar^3 = 54 \ldots (2)$

$(2) \div (1)$

$r^2 = \dfrac{9}{4}$

$r = \dfrac{3}{2}$

$\therefore a = 16$

$16 + 24 + 36 + 54 + 81 = 211$

7 C Using the cosine rule,

$$c^2 = 30^2 + 21^2 - 2(30)(21)\cos C$$
$$= 1341 - 260 \times \left(\frac{51}{53}\right)$$
$$= 1341 - \frac{64260}{53}$$
$$= \frac{6813}{53}$$
$\therefore c = \sqrt{\frac{6813}{53}}$ as $c > 0$

$\therefore c = 11.33786\ldots$

Thus $c = 11$ rounded to the nearest whole number.

8 D $x^2 - 8x + y^2 - 2y = 8$

$\therefore (x^2 - 8x + 16) + (y^2 - 2y + 1) = 25$

$\therefore (x - 4)^2 + (y - 1)^2 = 25$

centre (4, 1)

9 D From the graph:

① The centre occurs at $(2, 0)$

$\therefore$ Responses A, C and E are incorrect

② The vertices occur at $(-7, 0)$ and $(11, 0)$

Generally the vertices of a hyperbola occur at

$(\pm a + h,\ k)$

For response B: $a = 3,\ h = 2$ and $k = 0$ So the vertices are $(\pm 3 + 2, 0)$

i.e. $(-1, 0)$ and $(5, 0)$

$\therefore$ Response B is incorrect

For response D: $a = 9,\ h = 2$ and $k = 0$ So the vertices are $(\pm 9 + 2, 0)$

i.e. $(-7, 0)$ and $(11, 0)$

Thus response D is correct.

Solutions to extended-response questions

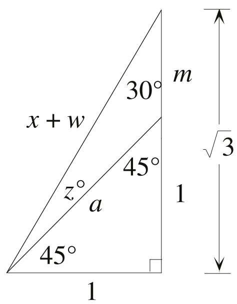
1 a $\sin 45° = \frac{1}{a}$

$\therefore \frac{1}{\sqrt{2}} = \frac{1}{a}$

$\therefore a = \sqrt{2}$

$45 + z = 60$

$\therefore z = 15$



$\sin 30° = \frac{1}{2}$

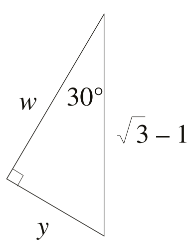
$\therefore \frac{1}{x + w} = \frac{1}{2}$

$\therefore x + w = 2$

$\cos 30° = \frac{\sqrt{3}}{2}$

$\therefore \frac{1 + m}{2} = \frac{\sqrt{3}}{2}$

$\therefore 1 + m = \sqrt{3}$, so $m = \sqrt{3} - 1$



$\cos 30° = \frac{\sqrt{3}}{2}$

$\therefore \frac{w}{\sqrt{3} - 1} = \frac{\sqrt{3}}{2}$

$\therefore w = \frac{\sqrt{3}(\sqrt{3} - 1)}{2} = \frac{3 - \sqrt{3}}{2}$

Now $x + w = 2$

$\therefore x = 2 - w$

$$= 2 - \left(\frac{3 - \sqrt{3}}{2}\right)$$

$$= \frac{4 - (3 - \sqrt{3})}{2}$$

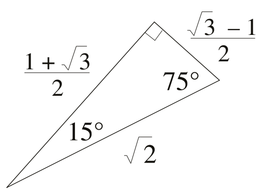
$$\therefore x = \frac{1 + \sqrt{3}}{2}$$

$$\sin 30° = \frac{1}{2}$$

$$\therefore \frac{y}{\sqrt{3} - 1} = \frac{1}{2}$$

$$\therefore y = \frac{\sqrt{3} - 1}{2}$$

b



$$\sin(15°) = \frac{\sqrt{3} - 1}{2} \div \sqrt{2}$$

$$= \frac{\sqrt{3} - 1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\cos(15°) = \frac{1 + \sqrt{3}}{2} \div \sqrt{2}$$

$$= \frac{1 + \sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\tan(15°) = \frac{\sqrt{3} - 1}{2} \div \frac{1 + \sqrt{3}}{2}$$

$$= \frac{\sqrt{3} - 1}{2} \times \frac{2}{\sqrt{3} + 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

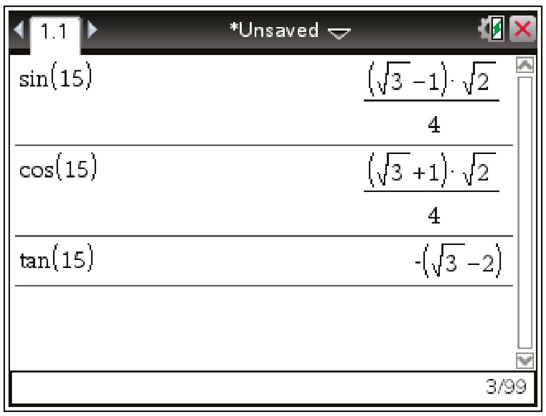
$$= \frac{(\sqrt{3} - 1)^2}{2}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= 2 - \sqrt{3}$$

CAS:

Change to **Degree**/**Deg** mode



c $$\sin(75°) = \frac{1 + \sqrt{3}}{2} \div \sqrt{2}$$

$$= \frac{1 + \sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

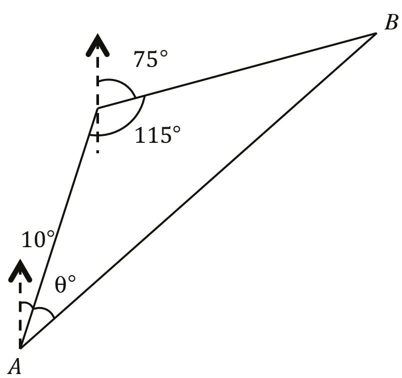
$$\cos(75°) = \frac{\sqrt{3} - 1}{2} \div \sqrt{2}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\tan(75°) = \frac{1+\sqrt{3}}{2} \div \frac{\sqrt{3}-1}{2}$$
$$= \frac{1+\sqrt{3}}{2} \times \frac{2}{\sqrt{3}-1}$$
$$= \frac{1+\sqrt{3}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$
$$= \frac{(1+\sqrt{3})^2}{2}$$
$$= \frac{4+2\sqrt{3}}{2}$$
$$= 2+\sqrt{3}$$

2



a $AB^2 = 25 + 49 - 70\cos(115°)$

Therefore $AB = 10.2$ km, correct to two decimal places.

b Then using the sine rule, $\frac{7}{\sin\theta} = \frac{AB}{\sin(115°)}$

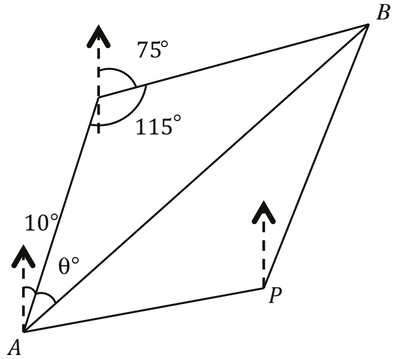
Therefore $\sin\theta = \frac{7\sin(115°)}{AB}$

which gives $\theta = 38.56\ldots$

and the bearing of B from A is given by $10 + 38.56\ldots$

The bearing is 049°.

c



i The magnitude of angle $BAP = (80 - (\theta + 10))^\circ = (31.43\ \ldots)^\circ$

Using the cosine in triangle APB gives

$$BP^2 = AB^2 + 4^2 - 8\,AB\cos(31.43\ \ldots)^\circ$$

Therefore $BP = 7.079\ \ldots$

The total distance travelled by the second hiker

$= 4 + 7.079\ \ldots$

$= 11.08$ km, correct to two decimal places.

ii Use the cosine rule to find the size of angle APB.

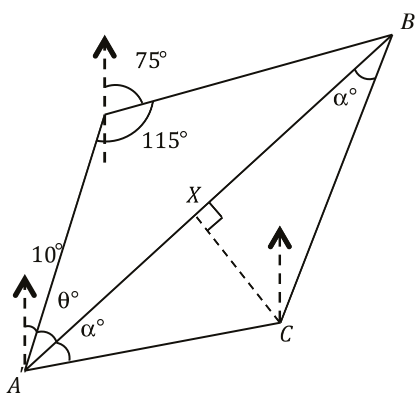
$\cos P = \dfrac{AB^2 - AP^2 - PB^2}{-2AP \times PB}$

and so the magnitude of angle APB is 131.42°

The bearing is therefore given by 131.42 − 100

The bearing is 031°.

d



In this diagram, $AC = CB$ and the bearing of C from A is 80°.

Triangle ACB is isosceles,

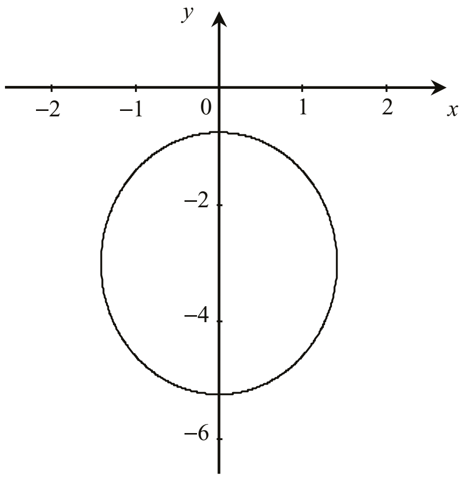
therefore $\cos(\alpha^\circ) = \dfrac{AX}{AC}$

and $AC = \dfrac{AX}{\cos(\alpha^\circ)}$

$AX = \dfrac{1}{2}AB$

From the above, $\alpha = 31.43 \ldots$ and $AX = 5.088 \ldots$
Therefore $AC = 5.963 \ldots$
The total distance travelled = 11.93 km, correct to two decimal places.

3



a **i** The centre of the ellipse is $(0, -3)$ and so the minor axis has endpoints $(\sqrt{2}, -3)$ and $(-\sqrt{2}, -3)$. The domain is $[-\sqrt{2}, \sqrt{2}]$

ii The major axis has endpoints $(0, -3 + \sqrt{5})$ and $(0, -3 - \sqrt{5})$.
The range is $[-3 - \sqrt{5}, -3 + \sqrt{5}]$

iii The centre is $(0, -3)$

b The centre of the ellipse has coordinates
$\left(\frac{-3+1}{2}, \frac{-1+5}{2}\right) = (-1, 2)$
The major axis (parallel to y axis) has length 6 and the minor axis (parallel to x axis) length 4.
Hence the equation of the ellipse is $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$.
So $a = 2$, $b = 3$, $h = 1$, $k = 2$.

c The line $y = x - 2$ intersects the ellipse $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$ at the point $(1, -1)$ and another point.
Substituting, $9(x-1)^2 + 4(x-4)^2 = 36$
Expanding and simplifying gives
$9(x^2 - 2x + 1) \quad +4(x^2 - 8x + 16) = 36$
and $\quad 13x^2 - 50x + 37 = 0 (x - 1)$ is a factor.
Therefore $\quad (x-1)(13x - 37) = 0$
The line intersects the ellipse at $(1, -1)$ and $\left(\frac{37}{13}, \frac{11}{13}\right)$

P has coordinates $\left(\frac{37}{13}, \frac{11}{13}\right)$.

d The line perpendicular to the line with equation $y = x - 2$, and which passes through $\left(\frac{37}{13}, \frac{11}{13}\right)$, has equation

$$y - \frac{11}{13} = -1\left(x - \frac{37}{13}\right)$$

Rearranging gives $y = -x + \frac{48}{13}$

The coordinates of Q are $\left(0, \frac{48}{13}\right)$

e There is a right angle at P and hence AQ is a diameter.

The coordinates of A, P and Q are $(1, -1)$, $\left(\frac{37}{13}, \frac{11}{13}\right)$ and $\left(0, \frac{48}{13}\right)$ respectively.

The centre of AQ is $\left(\frac{1}{2}, \frac{35}{26}\right)$

The diameter $= \sqrt{\left(\frac{61}{13}\right)^2 + 1}$

$$= \frac{\sqrt{3890}}{13}$$

The equation of the circle is $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{35}{26}\right)^2 = \frac{3890}{676}$

4 a $x^2 + y^2 - 2ax - 2ay + a^2 = 0$

Completing the square gives

$x^2 - 2ax + a^2 + y^2 - 2ay + a^2 + a^2 = 2a^2$

$$(x - a)^2 + (y - a)^2 = a^2$$

The centre is at (a, a) and the radius is a.

Therefore the circle touches both axes at $(0, a)$ and $(a, 0)$.

b Suppose $(x - h)^2 + (y - k)^2 = a^2$ touches both axes.

Then, when $y = 0$, there is only one solution to $(x - h)^2 + k^2 = a^2$

This only happens if $k^2 = a^2$, i.e., $k = \pm a$

In the same way, $h = \pm a$. Any combination these is possible as the circle can be in any one of the four quadrants. Therefore the forms could be

$(x - a)^2 + (y - a)^2 = a^2$ or $(x - a)^2 + (y + a)^2 = a^2$

or $(x + a)^2 + (y - a)^2 = a^2$ or $(x + a)^2 + (y + a)^2 = a^2$

c If the circles pass through the point (2, 4) then

$(4 - a)^2 + (2 - a)^2 = a^2$

Expanding gives $a^2 - 12a + 20 = 0$

Therefore $a = 10$ or $a = 2$

So the equations are $x^2 + y^2 - 20x - 20y + 100 = 0$ and $x^2 + y^2 - 4x - 4y + 4 = 0$

d (10, 10) and radius 10, and (2, 2) and radius 2

e For $a = 2$, the gradient is undefined. The point (2, 4) is 'the top of the circle'.

For $a = 10$, the centre is (10, 10). The line joining (10, 10) to (2, 4) has gradient $\frac{3}{4}$.

f For $a = 2$, the tangent is $y = 4$

For $a = 10$, the gradient of the tangent is therefore $-\frac{4}{3}$.

The equation of the tangent is $y = -\frac{4}{3}x + c$

When $x = 2$, $y = 4$ and therefore $4 = -\frac{8}{3} + c$

Hence $c = \frac{20}{3}$, and $y = -\frac{4}{3}x + \frac{20}{3}$

5 a Gradient of a line which passes through $(a\cos\theta,\ a\sin\theta)$ and the origin is $\frac{\sin\theta}{\cos\theta} = \tan\theta$. The equation of the straight line is $y = (\tan\theta)x$.

b The other point is the reflection through the origin and has coordinates $(-a\cos\theta,\ -a\sin\theta)$.

c The tangent at P is perpendicular to the radius and hence has gradient $-\frac{\cos\theta}{\sin\theta}$

Therefore the equation is $y - a\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta)$

d When $y = 0$, $-a\sin^2\theta = -x\cos\theta + a\cos^2\theta$

Therefore $x\cos\theta = a(\sin^2\theta + \cos^2\theta)$

Therefore $x = \frac{a}{\cos\theta}$

The coordinates are $\left(\frac{a}{\cos\theta}, 0\right)$

When $x = 0$, $y - a\sin\theta = -\frac{\cos\theta}{\sin\theta}(-\cos\theta)$

Therefore $y\sin\theta = a(\sin^2\theta + \cos^2\theta)$

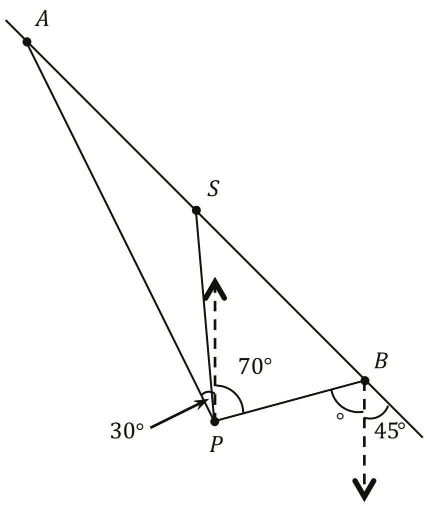
Therefore $y = \frac{a}{\sin\theta}$ and the coordinates are $\left(0, \frac{a}{\sin\theta}\right)$

e The area of the triangle is $\frac{1}{2} \times \frac{a}{\cos\theta} \times \frac{a}{\sin\theta} = \frac{a^2}{\sin 2\theta}$

The triangle has minimum area when $\sin 2\theta = 1$ or when $\theta = \frac{\pi}{4}$

$\left(\right.$This can also be completed by using your CAS calculator to sketch the graph of $y =$
$\left.\frac{1}{2} \times \frac{a}{\cos\theta} \times \frac{a}{\sin\theta}\right)$

6



a From the diagram, $\angle APB = 100°$

$\angle PAB = 15°$

$\angle PBA = 65°$

b In triangle PBA, using the sine rule gives

$$\frac{PB}{\sin(15°)} = \frac{10}{\sin(100°)}$$

$$\therefore PB = \frac{10\sin(15°)}{\sin(100°)}$$

$= 2.63$ km, correct to two decimal places.

Use triangle PSB,

$PS^2 = 25 + PB^2 - 10 \times PB\cos(65°)$

$PS = 4.56$ km, correct to two decimal places.

c From triangle PSB, using the sine rule gives $\frac{PS}{\sin(65°)} = \frac{5}{\sin P}$

$$\therefore \sin P = \frac{5\sin(65°)}{4.56\ldots}$$

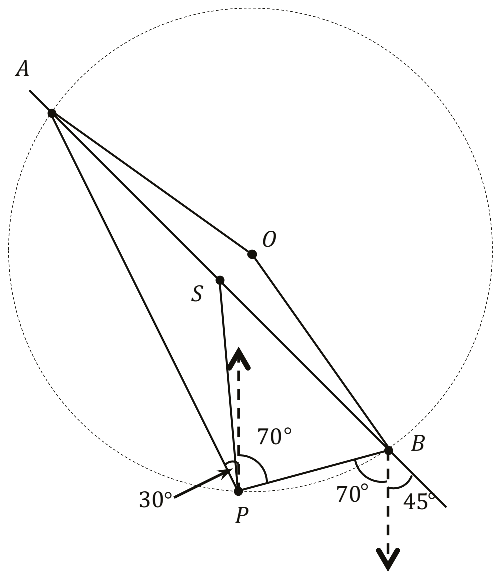
Therefore

$P = 83.5°$, correct to one decimal place.

Therefore, the bearing is given by $360° - (83.5 - 70)°$

The bearing is 346°.

d



Reflex angle $AOB = 200°$ (subtended angle at the centre is twice the angle at the circumference). Therefore, angle $AOB = 160°$.

Triangle AOB is isosceles and so OS is perpendicular to AB.

In triangle OSB, $\sin(80°) = \dfrac{SB}{OB}$

Therefore $OB = \dfrac{5}{\sin(80°)}$ The length of the arc $APB = \dfrac{5}{\sin(80°)} \times \dfrac{160\pi}{180}$

$= \dfrac{40\pi}{9\sin(80°)}$

The length of the track is 14.18 km, correct to two decimal places.

7 $f(x) = |x^2 - ax|$

a $0 = |x^2 - ax|$

$= |x(x - a)|$

$x = 0, a$

$co - ords = (0, 0), (a, 0)$

b $f(0) = |0 - 0|$

$= 0$

$co - ords = (0, 0)$

c $x \in [0, a]$

$$f(x) = ax - x^2$$

$$= -(x^2 - ax + \frac{a^2}{4}) + \frac{a^2}{4}$$

$$= -(x - \frac{a}{2})^2 + \frac{a^2}{4}$$

maximum value is $\frac{a^2}{4}$

d $f(-1) = 4$

$$4 = |(-1)^2 - a(-1)|$$

$$= |1 + a|$$

$$1 + a = \pm 4$$

$$a = -1 \pm 4$$

$$a = -5, 3$$

Chapter 2 – Number and Proof

Solutions to Exercise 2A

1 If n is even then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore

$$n^2 + 2n = (2k)^2 + 2(2k)$$
$$= 4k^2 + 4k$$
$$= 4(k^2 + k)$$

is divisible by 4 .

2 By expanding we obtain,

$$(2m + n)^2 - (2m - n)^2$$
$$= (4m^2 + 4mn + n^2) - (4m^2 - 4mn + n^2)$$
$$= m^2 + 4mn + n^2 - 4m^2 + 4mn - n^2$$
$$= 8\,(mn)$$

is divisible by 8.

3 **a** We have $m = 3p$ and $n = 5q$ for $p, q \in \mathbb{Z}$. Therefore

$$mn = (3p)(5q)$$
$$= 15(pq)$$

is divisible by 15.

b Likewise,

$$mn = (3p)^2(5p)$$
$$= 45(p^2q)$$

is divisible by 45.

4 **a** If n is odd then $n = 2k + 1$ for $k \in \mathbb{Z}$. Therefore

$$n^2 - n = (2k + 1)^2 - (2k + 1)$$
$$= (2k + 1)((2k + 1) - 1)$$
$$= 2(2k + 1)$$

is even. If n is even then $n = 2k$ for $k \in \mathbb{Z}$. Therefore

$$n^2 - n = (2k)^2 - (2k)$$
$$= 2k(2k - 1)$$

is even.

b Since $n^2 - n = n(n - 1)$ is the product of two consecutive numbers, one of these must be even.

5 We have that $a = mp$ and $b = nq$ for $p, q \in \mathbb{Z}$. Multiplying these two equations together gives

$$ab = (mp)(np)$$
$$= (mn)(np)$$

so that mn is a divisor of ab.

6 If $n = 2k + 1$ where $k \in \mathbb{Z}$, then

$$n^2 + 8n + 3$$
$$= (2k + 1)^2 + 8(2k + 1) + 3$$
$$= 4k^2 + 4k + 1 + 16k + 8 + 3$$
$$= 4k^2 + 20k + 4$$
$$= 2(2k^2 + 10k + 2)$$

is even.

7 If $m = a^3$ and $n = b^3$ for $a, b \in \mathbb{Z}$ then

$$mn = (a^3)(b^3)$$
$$= (ab)^3$$

is a perfect cube.

8 **a** $n^4 + 2n^3 - n^2 - 2n =$

$(n-1)n(n+1)(n+2)$.

b This is the product of 4 consecutive numbers. One of these will be divisible by 4, another other be divisible by 2 and another will be divisible by 3. Therefore the product will be divisible by $2 \times 3 \times 4 = 24$.

9 a If $n = 2k + 1$ where $k \in \mathbb{Z}$ then

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4k(k+1) + 1 \end{aligned}$$

Then, since one of k or $k + 1$ must be even, we conclude that $n^2 = 8m + 1$, for some $m \in \mathbb{Z}$.

b As $2^k - 1$ is odd, from the previous question we know that

$$\begin{aligned} n^2 &= 2^k - 1 \\ 8m + 1 &= 2^k - 1 \\ 8m + 2 &= 2^k \\ 4m + 1 &= 2^{k-1} \end{aligned}$$

The left-hand side is an odd number. The right-hand side is even unless $k = 1$. Therefore, $k = 1, m = 0$ and $n = 1$.

10 a If $n = 3k$ then

$$\begin{aligned} n^3 &= (3k)^3 \\ &= 27k^3 \\ &= 9(2k^3) \\ &= 9m \end{aligned}$$

If $n = 3k + 1$ then

$$\begin{aligned} n^3 &= (3k+1)^3 \\ &= 27k^3 + 27k^2 + 9k + 1 \\ &= 9(2k^3 + 9k^2 + k) + 1 \\ &= 9m + 1. \end{aligned}$$

If $n = 3k + 2$ then

$$\begin{aligned} n^3 &= (3k+2)^3 \\ &= 27k^3 + 54k^2 + 18k + 8 \\ &= 9(2k^3 + 6k^2 + 2k) + 8 \\ &= 9m + 8. \end{aligned}$$

b Each of these number is of the form $9m + 2$ for some number m. Therefore, none of these numbers are cubes.

11 If $n = 2k + 1$, where $k \in \mathbb{Z}$ then

$$\begin{aligned} &3n^2 + 7n + 11 \\ &= 3(2k+1)^2 + 7(2k+1) + 11 \\ &= 12k^2 + 14k + 3 + 14k + 7 + 11 \\ &= 12k^2 + 28k + 21 \\ &= 2(6k^2 + 14k + 10) + 1 \end{aligned}$$

is odd. If $n = 2k$, where $k \in \mathbb{Z}$ then

$$\begin{aligned} &3n^2 + 7n + 11 \\ &= 3(2k)^2 + 7(2k) + 11 \\ &= 12k^2 + 14k + 11 \\ &= 2(6k^2 + 7k + 5) + 1 \end{aligned}$$

is also odd.

12 If a and b are not divisible by 3 then each leaves a remainder of 1 or 2 when divided by 3. Without loss of generality, there are just three cases to consider (not four!).

Case 1: $a = 3k + 1, b = 3m + 1$. We find the difference of the two square to be

$$
\begin{aligned}
&a^2 - b^2 \\
&= (3k+1)^2 - (3m+1)^2 \\
&= ((3k+1) - (3m+1))((3k+1) + (3m+1)) \\
&= (3k - 3m)(3k + 3m + 2) \\
&= 3(k - m)(3k + 3m + 2).
\end{aligned}
$$

This is divisible by 3.

Case 2: $a = 3k + 1, b = 3m + 2$. We find the difference of the two square to be

$$
\begin{aligned}
&a^2 - b^2 \\
&= (3k+1)^2 - (3m+2)^2 \\
&= ((3k+1) - (3m+2))((3k+1) + (3m+2)) \\
&= (3k - 3m - 1)(3k + 3m + 3) \\
&= 3(3k - 3m - 1)(k + m + 1).
\end{aligned}
$$

This is divisible by 3.

Case 3: $a = 3k + 2, b = 3m + 2$. We find the difference of the two square to be

$$
\begin{aligned}
&a^2 - b^2 \\
&= (3k+2)^2 - (3m+2)^2 \\
&= ((3k+2) - (3m+2))((3k+2) + (3m+2)) \\
&= (3k - 3m)(3k + 3m + 4) \\
&= 3(k - m)(3k + 3m + 4).
\end{aligned}
$$

This is divisible by 3.

13 a We find n^2 for $n = 5k + r$ where $r = 0, 1, 2, 3, 4$. If $n = 5m$ then

$$
\begin{aligned}
n^2 &= (5m)^2 \\
&= 5(5m^2) \\
&= 5k
\end{aligned}
$$

where $k = 5m^2 \in \mathbb{Z}$. If $n = 5m + 1$ then

$$
\begin{aligned}
n^2 &= (5m+1)^2 \\
&= 25m^2 + 10m + 1 \\
&= 5(5m^2 + 2m) + 1 \\
&= 5k + 1
\end{aligned}
$$

where $k = 5m^2 + 2m \in \mathbb{Z}$. If $n = 5m + 2$ then

$$
\begin{aligned}
n^2 &= (5m+2)^2 \\
&= 25m^2 + 20m + 4 \\
&= 5(5m^2 + 4m) + 4 \\
&= 5k + 1
\end{aligned}
$$

where $k = 5m^2 + 4m \in \mathbb{Z}$. If $n = 5m + 3$ then

$$
\begin{aligned}
n^2 &= (5m+2)^2 \\
&= 25m^2 + 20m + 4 \\
&= 5(5m^2 + 4m) + 4 \\
&= 5k + 4
\end{aligned}
$$

where $k = 5m^2 + 4m \in \mathbb{Z}$. If $n = 5m + 4$ then

$$
\begin{aligned}
n^2 &= (5m+4)^2 \\
&= 25m^2 + 40m + 16 \\
&= 5(5m^2 + 4m + 4) + 1 \\
&= 5k + 1
\end{aligned}
$$

where $k = 5m^2 + 4m + 4 \in \mathbb{Z}$. Therefore every square number is of the form $5k$, $5k + 1$ or $5k + 4$.

b If a square number ends in a 2, then it is of the form $5k + 2$, which is impossible. Likewise, if a square number ends in a 3, then it is of the form $5k + 3$, which is also impossible.

c Note that

$$\begin{aligned} 1! &= 1 = 1^2 \\ 1! + 2! &= 3 \\ 1! + 2! + 3! &= 9 = 3^2 \\ 1! + 2! + 3! + 4! &= 33 \\ 1! + 2! + 3! + 4! + 5! &= 153 \end{aligned}$$

For each $k \geq 5$, $k!$ ends in a zero, therefore $1! + 2! + \cdots + k!$ must end in the digit three. Therefore $1! + 2! + \cdots + n!$ is not a square. It follows that the only squares is the list are $1! = 1^2$ and $1! + 2! + 3! = 3^2$.

14 a If a and b are odd, then ab is odd.

b If a and b are odd, then $a = 2m + 1$ and $b = 2n + 1$, where $m, n \in \mathbb{Z}$. Therefore

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1, \end{aligned}$$

where $k = 2mn + m + n \in \mathbb{Z}$. Therefore ab is odd.

15 a If $m + n$ is odd, then $m^2 + n^2$ is odd.

b If $m + n$ is odd, then one of either m or n must be odd, or else $m + n$ would be even. We can assume without loss of generality that m is odd, and n is even. Therefore $m = 2k + 1$ and $n = 2p$, where $k, p \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m^2 + n^2 &= (2k + 1)^2 + (2p)^2 \\ &= 4k^2 + 4k + 1 + 4p^2 \\ &= 2(2k^2 + 2 + 2p^2) + 1 \\ &= 2r + 1 \end{aligned}$$

where $r = 2k^2 + 2 + 2p^2 \in \mathbb{Z}$. Therefore $m^2 + n^2$ is odd.

16 a If n is even, then $8^n - 1$ is composite.

b If n is even, then $n = 2k$ for $k \in \mathbb{N}$. Therefore

$$\begin{aligned} 8^n - 1 &= 8^{2k} - 1 \\ &= (8^k)^2 - 1 \\ &= (8^k - 1)(8^k + 1) \end{aligned}$$

is composite, since $8^k - 1$ and $8^k + 1$ both exceed 1.

c There is nothing special about the number 8. We generalise as follows. Suppose $n \in \mathbb{N}$ and let a be a positive integer, $a \geq 3$.
Statement. If $a^n - 1$ is prime, then n is odd.
Contrapositive. If n is even, then $a^n - 1$ is composite.
Proof. if n is even, then $n = 2k$ for $k \in \mathbb{N}$. Therefore

$$\begin{aligned} a^n - 1 &= a^{2k} - 1 \\ &= (a^k)^2 - 1 \\ &= (a^k - 1)(a^k + 1) \end{aligned}$$

is composite, since $a^k - 1$ and $a^k + 1$ both exceed 1. Note that the proof fails if we let $a = 2$, since $2^2 - 1 = (2 - 1)(2 + 1) = 3$ is prime.

17 We prove this in the contrapositive.

Suppose n can be expressed as the sum of two consecutive integers. Then

$$n = m + (m + 1) = 2m + 1$$

is odd.

18 We prove this in the contrapositive. Suppose that $2x - 3$ is rational. Then $2x - 3 = \frac{m}{n}$ for integers m and $n \neq 0$. Then

$$\begin{aligned} 2x - 3 &= \frac{m}{n} \\ 2x &= \frac{m}{n} + 3 \\ &= \frac{m + 3n}{n} \\ \Longrightarrow x &= \frac{m + 3n}{2n}, \end{aligned}$$

where $m + 3n$ and $2n \neq 0$ are both integers. Therefore x is rational.

19 **a** Suppose n is the largest natural number. Then $n + 1$ is a larger natural number, which is a contradiction.

b If $a \leq 50$ and $b \leq 50$ then

$$a + b \leq 50 + 50 = 100,$$

which is a contradiction.

c If $a > \sqrt{ab}$ and $b > \sqrt{ab}$, then

$$ab > \sqrt{ab} \cdot \sqrt{ab} = ab,$$

which is a contradiction.

d Suppose by way of contradiction that $\log_2 7$ is rational. Then we can write $2^{\frac{a}{b}} = 7$ for positive integers a and b.

$$(2^{\frac{a}{b}})^b = 7^b \Longrightarrow 2^a = 7^b.$$

As the left-hand side is even and the right-hand side is odd, we have a contradiction.

e Suppose, by way of contradiction that $a + b$ is rational. Then b will *also* be rational since the difference between rational numbers will always be rational:

$$b = \overbrace{(a + b)}^{\text{rational}} - \overbrace{a}^{\text{rational}}$$

This is a contradiction, as b is irrational.

f Let $n \in \mathbb{N}$. Suppose, by way of contradiction that $n(n + 1) = m^2$ for $m \in \mathbb{N}$. Then

$$\begin{aligned} n(n + 1) &= m^2 \\ n^2 + n &= m^2 \\ n &= m^2 - n^2 \\ n &= (m - n)(m + n) \end{aligned}$$

Since $m + n > 0$ and $n > 0$, we must also have $m - n > 0$. Therefore $m - n \geq 1$. Therefore

$$\begin{aligned} n &= \overbrace{(m - n)}^{\geq 1}(m + n) \\ &\geq m + n \\ &> n. \end{aligned}$$

Therefore $n > n$, which is obviously a contradiction.

g Suppose that n is not 3. We will show that one of $n, n + 2$ or $n + 4$ is a multiple of 3. There are three cases to consider: $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$. If $n = 3k$ then n is divisible by 3. However, $n \neq 3$, therefore n is not prime. If $n = 3k + 1$, then $n + 2 = 3k + 3 = 3(k + 1)$ is not prime. If $n = 3k + 2$, then $n + 4 = 3k + 6 = 3(k + 2)$ is not prime. In either case, we obtain a contradiction. This questions show that 3, 5, 7 is the only arithmetic progression of three primes that differ by 2.

20 Suppose that 1 belongs to the range of f Then for some x in the domain of f,

$$\begin{aligned} f(x) &= 1 \\ \frac{x}{x-1} &= 1 \\ x &= x - 1 \\ 0 &= 1, \end{aligned}$$

which is a contradiction.

21 Suppose that 1 belongs to the range of f Then for some x in the domain of f,

$$\begin{aligned} f(x) &= 1 \\ \frac{x^2}{x-1} &= 1 \\ x^2 &= x - 1 \\ x^2 - x + 1 &= 0. \end{aligned}$$

The discriminant of this quadratic equation is

$$\Delta = b^2 - 4ac = (-1)^2 - 4(1)(1) = -3 < 0.$$

Therefore this quadratic has no solution.

22 **a** **Statement.** If $3n$ is odd, then n is odd.
Converse. If n is odd, then $3n$ is odd.
Proof of converse. If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} 3n &= 3(2k + 1) \\ &= 6k + 3 \\ &= 6k + 2 + 1 \\ &= 2(3k + 1) + 1 \\ &= 2m + 1 \end{aligned}$$

where $m = 3k + 1 \in \mathbb{Z}$. Therefore $3n$ is odd.

b **Statement.** If m is even and n is odd, then mn is even.
Converse. If mn is even, then m is even and n is odd.
Counterexample. The converse is not true, since if we let $m = 1$ and $n = 2$, then $mn = 2$ is even. However m is odd and n is even.

c **Statement.** If n is divisible by 6, then n is divisible by 2 and 3.
Converse. If n is divisible by 2 and 3, then n is divisible by 6.
Proof of converse. If n is divisible by 2 and 3, then each of 2 and 3 appear as prime factors of n. Therefore,

$$n = 2 \times 3 \times k = 6k$$

for some integer k. The means that n is divisible by 6.

d **Statement.** If n is divisible by 24, then n is divisible by 4 and 6.
Converse. If n is divisible by 4 and 6, then n is divisible by 24.
Counterexample. The converse is not true, since if $n = 12$, then n is divisible by 4 and 6. However it is not divisible by 24.

23 ($\Rightarrow$) Suppose that n is even. Then $n = 2k$ for some integer k. Therefore $n + 1 = 2k + 1$ is odd.
($\Leftarrow$) Suppose that $n + 1$ is odd. Then $n + 1 = 2k + 1$ for some integer k. Subtracting 1 from each side gives $n = 2k$, which is even.

24 (⇒) Suppose that nm divides am, where $m \neq 0$. Then $am = k(nm)$, for some integer k. Dividing both sides of this equation by $m \neq 0$ gives $a = kn$. Therefore n divides a.
(⇐) Now suppose that n divides a. Then $a = kn$ for some integer k. Multiplying both sides by $m \neq 0$ gives $am = k(nm)$. Therefore nm divides am.

25 **a** Every number n is either of the form $n = 2k$ or $n = 2k + 1$, for some integer k. If $n = 2k$ then$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ is even. Otherwise $n = 2k + 1$, in which case

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

is not divisible by 2. Therefore n is divisible by 2 if and only if n^2 is divisible by 2.

b Suppose, by way of contradiction that $\sqrt{2q} = \frac{m}{n}$ where $m, n \in \mathbb{N}$. We can assume that HCF$(m, n) = 1$, for otherwise we could cancel any factors in common. Therefore,

$$\begin{aligned} \frac{m}{n} &= \sqrt{2q} \\ \frac{m^2}{n^2} &= 2q \\ m^2 &= 2qn^2 \quad (1) \end{aligned}$$

Therefore m^2 is divisible by 2. Therefore m is divisible by 2. If $m = 2k$ for $k \in \mathbb{Z}$, then we substitute this into equation (1) to give

$$\begin{aligned} (2k)^2 &= 2qn^2 \\ 4k^2 &= 2qn^2 \\ qn^2 &= 2k^2. \end{aligned}$$

If n^2 were not divisible by 2, the the left-hand side would be odd. Therefore n^2 is divisible by 2, in which case n is divisible by 2. Therefore HCF(m, n) is at least 2, which is a contradiction.

c If $\sqrt{2} + \sqrt{3}$ were rational then for some $m, n \in \mathbb{N}$,

$$\begin{aligned} \sqrt{2} + \sqrt{3} &= \frac{m}{n} \\ (\sqrt{2} + \sqrt{3})^2 &= \frac{m^2}{n^2} \\ 2 + 2\sqrt{6} + 3 &= \frac{m^2}{n^2} \\ 2\sqrt{6} + 5 &= \frac{m^2}{n^2} \\ 2\sqrt{6} &= \frac{m^2}{n^2} - 5 \\ 2\sqrt{6} &= \frac{m^2 - 5n^2}{n^2} \\ \sqrt{6} &= \frac{m^2 - 5n^2}{2n^2} \end{aligned}$$

Since $m^2 - 5n^2$ and $2n^2$ are integers, this means that $\sqrt{6}$ is rational, which is a contradiction.

26 (⇒) Suppose n that is divisible by 3. Then $n = 3k$ for some integer k. Therefore

$$n^3 = (3k)^3 = 27k^3 = 9(3k^3)$$

is divisible by 9.
(⇐) Suppose n^3 is divisible by 9. If n is not divisible by 3 then $n = 3k + 1$ or

$n = 3k + 2$. In the first case we find that

$$\begin{aligned} n^3 &= (3k+1)^3 \\ &= 27k^3 + 27k^2 + 9k + 1 \\ &= 9(3k^3 + 3k^2 + 1) + 1 \end{aligned}$$

is not divisible by 9, which is a contradiction. In the second case, we find that

$$\begin{aligned} n^3 &= (3k+2)^3 \\ &= 27k^3 + 54k^2 + 36k + 8 \\ &= 9(3k^3 + 6k^2 + 4) + 8 \end{aligned}$$

is not divisible by 9, which is also a contradiction.

27 **a** As $99 = 33 + 33 + 33$, we can write $99 = 32 + 33 + 34$.

b ($\Rightarrow$) Suppose n that is divisible by 3. Then $n = 3k$ for some integer k. Therefore,

$$\begin{aligned} n &= 3k \\ &= k + k + k \\ &= (k-1) + k + (k+1) \end{aligned}$$

which is the sum of three consecutive integers.

($\Leftarrow$) If n is the sum of three consecutive integers, then

$$\begin{aligned} n &= k + (k+1) + (k+2) \\ &= 3k + 3 \\ &= 3(k+1) \end{aligned}$$

is divisible by 3.

28 Let $n-1, n$ and n be the three consecutive integers. Their sum of these integers is

$$(n-1) + n + (n+1) = 3n.$$

The sum of their cubes is

$$\begin{aligned} &(n-1)^3 + n^3 + (n+1)^3 \\ &= (n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1) \\ &= 3n^3 + 6n \\ &= 3n(n^2 + 2) \end{aligned}$$

Comparing these two results, we see that sum of three consecutive positive integers is a divisor of the sum of their cubes.

29 Let the k consecutive integers be $n+1, n+2, \ldots, n+k$, for some integer n. Therefore

$$\begin{aligned} {}^{n+k}C_k &= \frac{(n+k)!}{k!((n+k)-k)} \\ {}^{n+k}C_k &= \frac{(n+k)!}{k!n!} \\ k!\,{}^{n+k}C_k &= \frac{(n+k)!}{n!} \\ k!\,{}^{n+k}C_k &= (n+1)(n+2)\cdots(n+k). \end{aligned}$$

Since the binomial coefficient ${}^{n+k}C_k$ is an integer, we can conclude that $(n+1)(n+2)\cdots(n+k)$ is divisible by $k!$.

30 Let $n - m \geq 2$. Notice that a number is stackable if we can write it in the form

$$\begin{aligned}
&(1 + 2 + \cdots + n) - (1 + 2 + \cdots + m) \\
&= \frac{n(n+1)}{2} - \frac{m(m+1)}{2} \\
&= \frac{n^2 + n - m^2 - m}{2} \\
&= \frac{n^2 + n - m^2 - m}{2} \\
&= \frac{n^2 - m^2 + n - m}{2} \\
&= \frac{(n-m)(n+m) + (n-m)}{2} \\
&= \frac{(n-m)(n+m+1)}{2}
\end{aligned}$$

Therefore, if 2^k is stackable, then

$$\frac{(n-m)((n+m)+1)}{2} = 2^k$$

$$(n-m)((n+m)+1) = 2^{k+1}.$$

As $n - m$ and $n + m$ are either both even or both odd, the left-hand side is the product of one odd number and one even number, with both exceeding 1. On the other hand, the right-hand side has no odd factors except 1. Therefore this is impossible.

31 a Expanding the left-hand side gives

$$\begin{aligned}
&(m+1)^2 - (m+2)^2 \\
&\quad - (m+3)^2 + (m+4)^2 \\
&= (m^2 + 2m + 1) - (m^2 + 4m + 4) \\
&\quad - (m^2 + 6m + 9) + (m^2 + 8m + 16) \\
&= m^2 + 2m + 1 - m^2 - 4m - 4 \\
&\quad - m^2 - 6m - 9 + m^2 + 8m + 16 \\
&= 4.
\end{aligned}$$

b We first note that each of 1, 2 and 3 can be written in the required form:

$$1 = 1^2$$

$$2 = -1^2 - 2^2 - 3^2 + 4^2$$

$$3 = -1^2 + 2^2$$

To obtain larger numbers we first note that

$$\begin{aligned}
4 &= (-1^2 - 2^2 - 3^2 + 4^2) \\
5 &= 1 + 4 \\
&= 1^2 + (-2^2 - 3^2 - 4^2 + 5^2) \\
6 &= 2 + 4 \\
&= (1^2 + 2^2) + (-3^2 - 4^2 - 5^2 + 6^2) \\
7 &= 3 + 4 \\
&= (-1^2 + 2^2) + (-3^2 - 4^2 - 5^2 + 6^2) \\
8 &= 4 + 4 \\
&= (-1^2 - 2^2 - 3^2 + 4^2) + (-5^2 - 6^2 - 7^2 + 8^2) \\
9 &= 1 + 4 + 4 \\
&= 1^2 + (-2^2 - 3^2 - 4^2 + 5^2) \\
&\quad + (-6^2 - 7^2 - 8^2 + 9^2)
\end{aligned}$$

Any other larger number can be obtained by adding more groups of four terms, each of which adds to 4.

32 a If d is a divisor of m then

$$m = dp \qquad (1)$$

for some integer p. If d is a divisor of $m + 1$ then

$$m + 1 = dq \qquad (2)$$

for some integer q. Subtracting (1) from (2) gives

$$1 = dp - dp$$

$$\implies 1 = d(p - q)$$

Therefore d is a divisor of 1. The only possibility is that $d = 1$.

b If $p \leq n$ then p must appear amongst the factors of $n! = 1 \times 2 \times \cdots \times n$. Therefore p is a prime factor of both $n! + 1$ and $n!$. This is impossible, since from part a, the highest common factor of $n!$ and $n! + 1$ is 1.

c For any n, every prime factor of $n! + 1$ must exceed n. Therefore, we can always find a prime factor larger than any given number.

Solutions to Exercise 2B

1 **a** Universal statement

b Existence statement

c Universal statement

d Universal statement

e Existence statement

f Existence statement

g Universal statement

h Universal statement

2 **a** There exists $x \in \mathbb{R}$ for which $x^2 < 0$.

b There exists $n \in \mathbb{N}$ for which $n^2 + n + 11$ is composite.

c For all prime numbers p and q, $p + q \neq 100$.

d There exists $x \in \mathbb{R}$ for which $x > 0$, but $x^3 \leq x$.

e For all $a, b, c \in \mathbb{Z}$, $a^3 + b^3 \neq c^3$.

f There exists $x, y \in \mathbb{R}$ for which $(x + y)^3 \neq x^3 + y^3$.

g There exists $x, y \in \mathbb{R}$ such that $x \geq y$ and $x^2 \leq y^2$.

h For all $x \in \mathbb{R}$, $x^2 + x + 1 \neq 0$.

i There exists $n \in \mathbb{N}$ for which n is divisible by 3, but $n^2 + 2$ is not divisible by 3.

j There exists $m \in \mathbb{Z}$ for which $m > 2$ or $m < -2$, but $m^2 \leq 4$.

k There exists $m, n \in \mathbb{Z}$, for which mn and $m + n$ are odd.

l For all $a \in \mathbb{R}$, $\sqrt{2}a$ is irrational.

m There exists $x \in (-1, 1)$ for which $x^2 \geq 1$.

n For all $x, y \in \mathbb{R}$, either $xy \leq 0$ or $x + y \geq 0$.

3 **a** When $n = 4$ we find that

$$\begin{aligned} n^2 + n + 1 &= 4^2 + 4 + 1 \\ &= 21 \\ &= 3 \times 7 \end{aligned}$$

is not prime.

b If $x = 0$ then, $x^2 = 0$.

c Let $a = \sqrt{2}$ and $a = -\sqrt{2}$, each of which are irrational. Then $a + b = \sqrt{2} - \sqrt{2} = 0$, which is rational.

d Let $a = \sqrt{2}$ and $b = \sqrt{2}$, each of which are irrational. Then $ab = (\sqrt{2})^2 = 2$, which is rational.

e Let $a = 0, b = 1$ and $c = 2$. Then $ab = 0 = ac$, however $b = 1 \neq 2 = c$.

f If $n = 6$, then $n^2 = 36 = 4 \cdot 9$ is divisible by 4, however n is not divisible by 4.

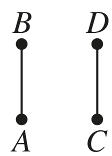
g Let $a = 4, b = 9$ and $c = 6$. Then c is a divisor of $ab = 36$, however c is **not** a divisor of a or b.

h If $n = 8$ and $m = 4$ then $n^2 = 64$ is a divisor of $m^3 = 64$. However n is not a divisor of m.

i The function $f(x) = 2^x$ is increasing and yet does not cross the x-axis.

j If $f(x) = x^3$, then $f'(0) = 0$, however f does not have a turning point at 0

k Each of the vertices in the graph below has degree 1, however the graph is not connected.



4 a If

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \text{ while } \mathbf{BA} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

b If

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

even though $\mathbf{A} \neq \mathbf{O}$.

c Here, the counterexample is harder to find. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$$

then

$$\mathbf{A}^2 = \mathbf{A}$$

even though $\mathbf{A} \neq \mathbf{O}$ and $\mathbf{A} \neq \mathbf{I}$.

5 a We need to show that the negation is the statement is true. That is, we need to show that for all $n \in \mathbb{N}$, $25n^2 - 9$ is composite. To prove this, we note that

$$25n^2 - 9 = (5n - 3)(5n + 3)$$

is composite, since $5n - 3 > 1$ and $5n + 3 > 1$.

b We need to show that the negation is the statement is true. That is, we need to show that for all $n \in \mathbb{N}$, $n^2 + 11n + 30$ is composite. To prove this, we note that

$$n^2 + 11n + 30 = (n + 5)(n + 6)$$

is composite, since $n + 5 > 1$ and $n + 6 > 1$.

c We need to show that the negation is the statement is true. That is, we need to show that for all $x \in \mathbb{R}$, $5 + 2x^2 \neq 1 + x^2$. To prove this, we suppose that $5 + 2x^2 = 1 + x^2$. Then,

$$\begin{aligned} 5 + 2x^2 &= 1 + x^2 \\ x^2 &= -4 \end{aligned}$$

This is impossible, as $x \geq 0$.

Solutions to Exercise 2C

1 There are many ways that we could prove this result. Here is one method:

$$\begin{aligned}\frac{b}{b+1}-\frac{a}{a+1}&=\frac{b(a+1)-a(b+1)}{(a+1)(b+1)}\\&=\frac{ab+b-ab-a}{(a+1)((b+1)}\\&=\frac{b-a}{(a+1)((b+1)}\\&\geq 0,\end{aligned}$$

since $b-a\geq 0$ and $a+1>0$ and $b+1>0$. Another method is to let $f(x)=\dfrac{x}{x+1}$ and show that f is increasing on the interval $x>0$. We can do this by noting that

$$f'(x)=\frac{1}{(x+1)^2}>0$$

for all $x>0$.

2 We will prove the equivalent result,

$$a^3-a^2b+b^3-ab^2>0.$$

We find that

$$\begin{aligned}a^3-a^2b+b^3-ab^2&=a^2(a-b)-b^2(a-b)\\&=(a^2-b^2)(a-b)\\&=(a-b)(a+b)(a-b)\\&=(a-b)^2(a+b)\\&>0,\end{aligned}$$

3 **a** There are a couple of ways of proving this. One method is to note that

$$11\sqrt{10}=\sqrt{(11^2)(10)}=\sqrt{1210},$$
$$10\sqrt{11}=\sqrt{(10)^2(11)}=\sqrt{1100},$$

from which we see that

$$11\sqrt{10}\geq 10\sqrt{11}.$$

b Suppose $a\geq b\geq 0$. We will square both sides and prove the equivalent result,

$$a^2b\geq b^2a.$$

To show this note that

$$a^2b-b^2a=ab(a-b)\geq 0$$

since $a>0, b>0$ and $a-b>0$.

4 We can find that

$$\begin{aligned}&a+\frac{1}{a}-2\\&=\frac{a^2-2a+1}{a^2}\\&=\frac{(a-2)^2}{a^2}\\&>0.\end{aligned}$$

5 **a** Once way of proving this result is to simply let $a=\dfrac{x}{y}$ and then use the previous question. If you didn't spot this, then we can prove it directly by noting that

$$\begin{aligned}&\frac{x}{y}+\frac{y}{x}-2\\&=\frac{x^2}{xy}+\frac{y^2}{xy}-\frac{2xy}{xy}\\&=\frac{x^2-2xy+y^2}{xy}\\&=\frac{(x-y)^2}{xy}\\&\geq 0\end{aligned}$$

since $x-y\geq 0$ and $x, y>0$.

b We find that

$$\left(\frac{1}{x}+\frac{1}{x}\right)(x+y)-4$$
$$=\left(\frac{1}{x}+\frac{1}{y}\right)(x+y)-4$$
$$=\left(\frac{x+y}{xy}\right)(x+y)-\frac{4xy}{xy}$$
$$=\frac{(x+y)^2}{xy}\frac{4xy}{xy}$$
$$=\frac{(x+y)^2-4xy}{xy}$$
$$=\frac{x^2+2xy+y^2-4xy}{xy}$$
$$=\frac{x^2-2xy+y^2}{xy}$$
$$=\frac{(x-y)^2}{xy}$$
$$\geq 0$$

since $(x-y)^2 \geq 0$. We could also expand the brackets and then use part **a**. This gives

$$\left(\frac{1}{x}+\frac{1}{x}\right)(x+y)$$
$$=1+\frac{y}{x}+\frac{x}{y}+1$$
$$=\frac{y}{x}+\frac{x}{y}+2$$
$$\geq 2+2$$
$$=4.$$

c We first expand the brackets and then use part **a**. This gives

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(x+y+z)$$
$$=\left(1+\frac{y}{x}+\frac{z}{x}\right)+\left(\frac{x}{y}+1+\frac{z}{y}\right)+\left(\frac{x}{z}+\frac{y}{z}+1\right)$$
$$=\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+3$$
$$\geq 2+2+2+3$$
$$=9$$

6 a To show this, note that

$$\frac{x^2+y^2}{2}-\left(\frac{x+y}{2}\right)^2$$
$$=\frac{x^2+y^2}{2}-\frac{x^2+2xy+y^2}{4}$$
$$=\frac{2x^2+2y^2}{4}-\frac{x^2+2xy+y^2}{4}$$
$$=\frac{x^2-2xy+y^2}{4}$$
$$=\left(\frac{x-y}{2}\right)^2$$
$$\geq 0.$$

b The total area of the two squares is x^2+y^2. The perimeter of the squares is $4x$ and $4y$ so the total length of string is $4x+4y$, and half that length is $2x+2y$. Each of the new squares therefore has side length

$$\frac{1}{4}(2x+2y)=\frac{x+y}{2}$$

and total area

$$2\left(\frac{x+y}{2}\right)^2.$$

We need to show that

$$2\left(\frac{x+y}{2}\right)^2 \leq x^2+y^2.$$

However, this equivalent to the inequality we already proved in part **a**.

7 a If $ab = 9$ then using the AM-GM inequality (with $x = 2a$ and $y = 2b$) we find

$$\begin{aligned} a + b &= \frac{2a + 2b}{2} \\ &\geq \sqrt{(2a)(2b)} \\ &= \sqrt{4ab} \\ &= \sqrt{36} \\ &= 6, \end{aligned}$$

as required.

b If $a + b = 4$ then using the AM-GM inequality (with $x = a$ and $y = b$) we find

$$\begin{aligned} ab &\leq \left(\frac{a + b}{2}\right)^2 \\ &= \left(\frac{4}{2}\right)^2 \\ &= 4, \end{aligned}$$

as required.

c If $ab = 48$ then using the AM-GM inequality (with $x = 3a$ and $y = 4b$) we find

$$\begin{aligned} 3a + 4b &= \frac{6a + 8b}{2} \\ &\geq \sqrt{(6a)(8b)} \\ &= \sqrt{48ab} \\ &= \sqrt{48^2} \\ &= 48. \end{aligned}$$

d If $3a + 4b = 24$ then using the AM-GM inequality (with $x = 3a$ and $y = 4b$) we find

$$\begin{aligned} ab &= \frac{1}{12}(3a)(4b) \\ &\leq \frac{1}{12}\left(\frac{3a + 4b}{2}\right)^2 \\ &= \frac{1}{12}\left(\frac{24}{2}\right)^2 \\ &= \frac{1}{12}(12)^2 \\ &= 12. \end{aligned}$$

e We use the AM-GM inequality three times to show that

$$\begin{aligned} &\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \\ &\leq \frac{a + b}{2} + \frac{b + c}{2} + \frac{c + a}{2} \\ &= a + b + c \\ &= 1. \end{aligned}$$

8 Use the AM-GM inequality three times to find that

$$\begin{aligned} &(a + b)(b + c)(c + a) \\ &= 8\frac{a + b}{2}\frac{b + c}{2}\frac{c + a}{2} \\ &\geq 8\sqrt{ab}\sqrt{bc}\sqrt{ca} \\ &= 8\sqrt{ab}\sqrt{bc}\sqrt{ca} \\ &= 8\sqrt{a^2b^2c^2} \\ &= 8ab. \end{aligned}$$

9 a Note that $a > 0$ and $1 - a > 0$. Therefore

$$a - a^2 = a(1 - a) > 0$$

b If $0 < \theta < \frac{\pi}{2}$ then $0 < \sin\theta < 1$ and $0 < \cos\theta < 1$. Therefore by part

a we know that $\sin\theta > \sin^2$ and $\cos\theta > \cos^2$. Finally,

$$\cos\theta + \sin\theta > \cos^2 + \sin^2\theta = 1.$$

c We use the AM-GM inequality with $x = \cos\theta$ and $y = \sin\theta$ and also the identity $\sin 2\theta = 2\sin\theta\cos\theta$. We find that

$$\begin{aligned}\cos\theta + \sin\theta &= 2\left(\frac{\cos\theta + \sin\theta}{2}\right)\\ &\geq 2\sqrt{\cos\theta\sin\theta}\\ &= 2\sqrt{\frac{2\cos\theta\sin\theta}{2}}\\ &= 2\sqrt{\frac{\sin(2\theta)}{2}}\\ &\geq 2\sqrt{\frac{1}{2}}\\ &= \sqrt{2}\end{aligned}$$

10 **a**

$$\begin{aligned}(a-b)^2 &\geq 0\\ a^2 - 2ab + b^2 &\geq 0\\ a^2 + b^2 &\geq 2ab\end{aligned}$$

b We have:
$a^2 + b^2 \geq 2ab \ldots (1)$
$a^2 + c^2 \geq 2ac \ldots (2)$
$b^2 + c^2 \geq 2bc \ldots (3)$
Adding (1), (2) and (3).
$2a^2 + 2b^2 + 2c^2 \geq 2ab + 2ac + 2bc$
Hence,
$a^2 + b^2 + c^2 \geq ab + ac + bc$

c Hence prove that $3(a^4 + b^4 + c^4) \geq (a^2 + b^2 + c^2)^2$

$$\begin{aligned}&3(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2\\ &= 2a^4 + 2b^4 + 2c^4 - (2a^2b^2 + 2a^2c^2 + 2c^2b^2)\end{aligned}$$

Solutions to Exercise 2D

1 a We aim to find the partial fraction decomposition of the left-hand side. We have

$$\begin{aligned}\frac{1}{n(n+2)} &= \frac{a}{n}+\frac{b}{n+2}\\ \frac{1}{n(n+2)} &= \frac{a(n+2)+bn}{n(n+2)}\\ 1 &= a(n+2)+bn.\\ 1 &= 2a+(a+b)n\end{aligned}$$

Equating coefficients, we find that $a=\frac{1}{2}$ and $a+b=0$. Therefore $b=-\frac{1}{2}$. We have proved that

$$\frac{1}{n(n+2)}=\frac{\frac{1}{2}}{n}-\frac{\frac{1}{2}}{n+2}$$

b i We find that

$$\begin{aligned}&\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{95\cdot 97}+\frac{1}{97\cdot 99}\\ &=\left(\frac{\frac{1}{2}}{1}-\frac{\frac{1}{2}}{3}\right)+\left(\frac{\frac{1}{2}}{3}-\frac{\frac{1}{2}}{5}\right)+\cdots+\left(\frac{\frac{1}{2}}{97}-\frac{\frac{1}{2}}{99}\right)\\ &=\frac{\frac{1}{2}}{1}-\frac{\frac{1}{2}}{99}\\ &=\frac{49}{99}\end{aligned}$$

ii We find that

$$\begin{aligned}&\frac{1}{2\cdot 4}+\frac{1}{4\cdot 6}+\cdots\frac{1}{98\cdot 100}\\ &=\left(\frac{\frac{1}{2}}{2}-\frac{\frac{1}{2}}{4}\right)+\left(\frac{\frac{1}{2}}{4}-\frac{\frac{1}{2}}{6}\right)+\cdots+\left(\frac{\frac{1}{2}}{98}-\frac{\frac{1}{2}}{100}\right)\\ &=\frac{\frac{1}{2}}{2}-\frac{\frac{1}{2}}{100}\\ &=\frac{49}{200}\end{aligned}$$

2 We find that

$$\begin{aligned}&\log_{10}\left(\frac{1}{2}\right)+\log_{10}\left(\frac{2}{3}\right)+\cdots+\log_{10}\left(\frac{99}{100}\right)\\ &= (\log_{10}1-\log_{10}2)\\ &\quad+(\log_{10}2-\log_{10}3)\\ &\quad\vdots\\ &\quad+(\log_{10}99-\log_{10}100)\\ &= \log_{10}1-\log_{10}100\\ &= 0-2\\ &= -2.\end{aligned}$$

3 a Starting with the right-hand side gives

$$\begin{aligned}(m+1)!-m! &= (m+1)m!-m!\\ &= ((m+1)-1)m!\\ &= m(m!)\end{aligned}$$

b Expanding each term in the sum using the previous result and then cancelling pairs of terms gives

$$\begin{aligned}&1(1!)+2(2!)+\cdots+n(n!)\\ &= (2!-1!)+(3!-2!)+\cdots+((n+1)!-n!)\\ &= -1!+(n+1)!\\ &= (n+1)!-1\end{aligned}$$

4 a Starting with the right-hand side gives

$$\frac{1}{m!} - \frac{1}{(m+1)!}$$
$$= \frac{m+1}{(m+1)!} - \frac{1}{(m+1)!}$$
$$= \frac{m+1-1}{(m+1)!}$$
$$= \frac{m}{(m+1)!}$$

b Expanding each term in the sum using the previous result and then cancelling pairs of terms gives

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!}$$
$$= \left(\frac{1}{1!} - \frac{1}{2!}\right)$$
$$+ \left(\frac{1}{2!} - \frac{1}{3!}\right)$$
$$\vdots$$
$$+ \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$$
$$= \frac{1}{1!} - \frac{1}{(n+1)!}$$
$$= 1 - \frac{1}{(n+1)!}$$

5 a We factorise the right-hand side to give

$$\frac{1}{3}\,(k(k+1)(k+2) - (k-1)k(k+1))$$
$$= \frac{1}{3}k(k+1)[(k+2) - (k-1)]$$
$$= \frac{1}{3}k(k+1)(3)$$
$$= k(k+1)$$

b Expanding each term in the sum using the previous result and then cancelling pairs of terms gives

$$(1 \cdot 2) + (2 \cdot 3) + \cdots + n(n+1)$$
$$= \frac{1}{3}(1 \cdot 2 \cdot 3 - 0 \cdot 1 \cdot 2)$$
$$+ \frac{1}{3}(2 \cdot 3 \cdot 4 - 1 \cdot 2 \cdot 3)$$
$$\vdots$$
$$+ \frac{1}{3}\,(n(n+1)(n+2) - (n-1)n(n+1))$$
$$= \frac{1}{3}n(n+1)(n+2)$$

6 a We aim to find the partial fraction decomposition of the left-hand side. We have

$$\frac{1}{k(k+1)(k+2)} = \frac{a}{k} + \frac{b}{k+1} + \frac{c}{k+2}$$

Therefore

$$1 = a(k+1)(k+2) + bk(k+2) + ck(k+1).$$

This time we find a, b and c by substituting carefully chosen values of k into this equation. If $k = 0$, then

$$1 = a(1)(2) \implies a = \frac{1}{2}.$$

If $k = -1$, then

$$1 = b(-1)(1) \implies b = -1.$$

If $k = -2$, then

$$1 = c(-2)(-1) \implies c = \frac{1}{2}.$$

Therefore,

$$\frac{1}{k(k+1)(k+2)} = \frac{\frac{1}{2}}{k} - \frac{1}{k+1} + \frac{\frac{1}{2}}{k+2}$$

b Expanding each term in the sum using the previous result, then cancelling groups of three terms gives

$$\frac{1}{1\cdot 2\cdot 3}+\cdots+\frac{1}{n(n+1)(n+2)}$$
$$=\frac{\frac{1}{2}}{1}-\frac{1}{2}+\frac{\frac{1}{2}}{3}$$
$$+\frac{\frac{1}{2}}{2}-\frac{1}{3}+\frac{\frac{1}{2}}{4}$$
$$\vdots$$
$$+\frac{\frac{1}{2}}{n}-\frac{1}{n+1}+\frac{\frac{1}{2}}{n+2}$$
$$=\left(\frac{\frac{1}{2}}{1}-\frac{1}{2}+\frac{\frac{1}{2}}{2}\right)$$
$$+\left(\frac{\frac{1}{2}}{n+2}-\frac{1}{n+1}+\frac{\frac{1}{2}}{n+2}\right)$$
$$=\frac{n(n+3)}{4(n+1)(n+2)}$$

7 a We start with the left-hand side to give,

$$\frac{\log_{10}\left(\frac{a}{b}\right)}{\log_{10}(a)\log_{10}(b)}$$
$$=\frac{\log_{10}(a)-\log_{10}(b)}{\log_{10}(a)\log_{10}(b)}$$
$$=\frac{\log_{10}(a)}{\log_{10}(a)\log_{10}(b)}-\frac{\log_{10}(b)}{\log_{10}(a)\log_{10}(b)}$$
$$=\frac{1}{\log_{10}(b)}-\frac{1}{\log_{10}(a)}$$

b Expanding each term in the sum using the previous result, then cancelling pairs of terms gives

$$\frac{\log_{10}\left(\frac{2}{3}\right)}{\log_{10}(2)\log_{10}(3)}+\cdots+\frac{\log_{10}\left(\frac{9}{10}\right)}{\log_{10}(9)\log_{10}(10)}$$
$$=\frac{1}{\log_{10}(3)}-\frac{1}{\log_{10}(2)}$$
$$+\frac{1}{\log_{10}(4)}-\frac{1}{\log_{10}(3)}$$
$$\vdots$$
$$+\frac{1}{\log_{10}(20)}-\frac{1}{\log_{10}(19)}$$
$$=\frac{1}{\log_{10}(20)}-\frac{1}{\log_{10}(2)}$$
$$=\frac{\log_{10}(2)-\log_{10}(20)}{\log_{10}(2)\log_{10}(20)}$$
$$=\frac{\log_{10}\left(\frac{1}{10}\right)}{\log_{10}(2)\log_{10}(20)}$$
$$=-\frac{1}{\log_{10}(2)\log_{10}(20)}.$$

Solutions to Exercise 2E

1 **a** $\boxed{P(n)}$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1$$

and

$$\text{RHS} = \frac{1(1+1)}{2} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1 + 2 + \cdots + k + (k+1)$

$= \dfrac{k(k+1)}{2} + (k+1) \quad$ (by (1))

$= \dfrac{k(k+1)}{2} + \dfrac{2(k+1)}{2}$

$= \dfrac{k(k+1) + 2(k+1)}{2}$

$= \dfrac{(k+1)(k+2)}{2}$

$= \dfrac{(k+1)((k+1)+1)}{2}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1 + 3 + \cdots + (2n-1) = n^2$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1$$

and

$$\text{RHS} = 1^2 = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + 3 + \cdots + (2k-1) = k^2. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1 + 2 + \cdots + (2k-1) + (2k+1)$

$= k^2 + 2k + 1 \quad$ (by (1))

$= (k+1)^2$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^2 - 1$$

and

$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$

$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ (by (1))

$= \dfrac{k(k+1)(2k+1)}{6} + \dfrac{6(k+1)^2}{6}$

$= \dfrac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \dfrac{(k+1)(k(2k+1) + 6(k+1))}{6}$

$= \dfrac{(k+1)(2k^2 + k + 6k + 6)}{6}$

$= \dfrac{(k+1)(2k^2 + 7k + 6)}{6}$

$= \dfrac{(k+1)(k+2)(2k+3)}{6}$

$= \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^3 = 1$$

and

$$\text{RHS} = \frac{1^2(1+1)^2}{4} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}. \quad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3$

$= \dfrac{k^2(k+1)^2}{4} + (k+1)^3$ (by (1))

$= \dfrac{k^2(k+1)^2}{4} + (k+1)^3$

$= \dfrac{k^2(k+1)^2}{4} + \dfrac{4(k+1)^3}{4}$

$= \dfrac{k^2(k+1)^2 + 4(k+1)^3}{4}$

$= \dfrac{(k+1)^2(k^2 + 4(k+1))}{4}$

$= \dfrac{(k+1)^2(k^2 + 4k + 4)}{4}$

$= \dfrac{(k+1)^2(k+2)^2}{4}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

e $\boxed{P(n)}$

$1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$ where $x \neq 1$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 + x$$

and

$$\text{RHS} = \frac{1 - x^2}{1 - x} = \frac{(1-x)(1+x)}{(1-x)} = 1 + x.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x} \quad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1 + x + x^2 + \cdots + x^k + x^{k+1}$

$= \dfrac{1 - x^{k+1}}{1 - x} + x^{k+1}$ (by (1))

$= \dfrac{1 - x^{k+1} + (1 - x)x^{k+1}}{1 - x}$

$= \dfrac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1 - x}$

$= \dfrac{1 - x^{k+2}}{1 - x}$

$= \dfrac{1 - x^{(k+1)+1}}{1 - x}$

= RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f $\boxed{P(n)}$

$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 \cdot 2 = 2$$

and

$$\text{RHS} = \frac{1 \cdot 2 \cdot 3}{3} = 2.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$ (1)

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2)$

$= \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ (by (1))

$= \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$

$= \dfrac{(k+1)(k+2)(k+3)}{3}$

= RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

g $\boxed{P(n)}$

$2 \cdot 2 + \cdots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 2 \cdot 2 = 4$$

and

$$\text{RHS} = 1 \cdot 2^2 = 4.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$2 \cdot 2 + \cdots + (k+1) \cdot 2^k = k \cdot 2^{k+1}$ (1)

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= 2 \cdot 2 + \cdots + (k+1) \cdot 2^k + (k+2) \cdot 2^{k+1}$

$= k \cdot 2^{k+1} + (k+2) \cdot 2^{k+1}$ (by (1))

$= 2^{k+1}(2k+2)$

$= 2(k+1)2^{k+1}$

$= (k+1)2^{k+2}$

= RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$

by the principle of mathematical induction.

h $\boxed{P(n)}$

$$\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = \frac{1}{1\cdot 3} = \frac{1}{3}$$

and

$$\text{RHS} = \frac{1}{2\cdot 1+1} = \frac{1}{3}.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1)\\
&=\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2k-1)(2k+1)}\\
&\quad+\frac{1}{(2k+1)(2k+3)}\\
&=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \text{(by (1))}\\
&=\frac{k(2k+3)+1}{(2k+1)(2k+3)}\\
&=\frac{2k^{2}_{3}k+1}{(2k+1)(2k+3)}\\
&=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}\\
&=\frac{k+1}{2k+3}\\
&=\frac{k+1}{2(k+1)+1}\\
&=\text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 a $\boxed{P(n)}$

$11^n - 1$ is divisible by 10, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $11^1 - 1 = 10$ is divisible by 10. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$11^k - 1 = 10m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}
11^{k+1}-1 &= 11\cdot 11^k - 1\\
&= 11\cdot(1+10m)-1 \quad \text{(by (1))}\\
&= 11+110m-1\\
&= 10+110m\\
&= 10(1+11m)
\end{aligned}$$

We see that this is a multiple of 10. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$7^n - 3^n$ is divisible by 4, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $7^1 - 3^1 = 4$ is divisible by 4. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$7^k - 3^k = 4m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}7^{k+1} - 3^{k+1} &= 7 \cdot 7^k - 3^{k+1}\\ &= 7 \cdot (3^k + 4m) - 3^{k+1} \quad \text{(by (1))}\\ &= 7 \cdot 3^k + 28m - 3 \cdot 3^k\\ &= 4 \cdot 3^k + 28m\\ &= 4(3^k + 7m)\end{aligned}$$

We see that this is a multiple of 4. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$8^n - (-6)^n$ is divisible by 14, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $8^1 - (-6)^1 = 14$ is divisible by 14. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$8^k - (-6)^k = 14m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}&8^{k+1} - (-6)^{k+1}\\ &= 8 \cdot 8^k - (-6)^{k+1}\\ &= 8 \cdot ((-6)^k + 14m) - (-6)^{k+1} \quad \text{(by (1))}\\ &= 8 \cdot (-6)^k + 14(8m) - (-6)(-6)^k\\ &= 8 \cdot (-6)^k + 14(8m) + 6(-6)^k\\ &= 14 \cdot (-6)^k + 14(8m)\\ &= 14((-6)^k + 8m)\end{aligned}$$

We see that this is a multiple of 14. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$2^{3n+1} + 5$ is divisible by 7, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $2^4 + 5 = 21$ is divisible by 7. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$2^{3k+1} + 5 = 7m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}2^{3(k+1)+1} + 5 &= 2^{3k+4} + 5\\ &= 2^3 2^{3k+1} + 5\\ &= 8 \cdot 2^{3k+1} + 5\\ &= 8 \cdot (7m - 5) + 5 \quad \text{(by (1))}\\ &= 7(8m) - 35\\ &= 7(8m - 5)\end{aligned}$$

We see that this is a multiple of 7. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

e $\boxed{P(n)}$

$4^{2n+1} + 5^{2n+1}$ is divisible by 9, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $4^3 + 5^3 = 189$ is divisible by 9. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^{2k+1} + 5^{2k+1} = 9m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}
&4^{2(k+1)+1} + 5^{2(k+1)+1} \\
&= 4^{2k+3} + 5^{2k+3} \\
&= 4^2 2^{2k+1} + 5^{2k+3} \\
&= 16 \cdot 4^{2k+1} + 5^{2k+3} \\
&= 16 \cdot (9m - 5^{2k+1}) + +5^{2k+3} \quad \text{(by (1))} \\
&= 9(16m) - 16 \cdot 5^{2k+1} + 5^2 \cdot 5^{2k+1} \\
&= 9(16m) - 16 \cdot 5^{2k+1} + 25 \cdot 5^{2k+1} \\
&= 9(16m) + 9 \cdot 5^{2k+1} \\
&= 9(16m + 5^{2k+1})
\end{aligned}$$

We see that this is a multiple of 9. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

3 a $\boxed{P(n)}$

$2^{2n} - 1$ is a multiple of 3, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $2^2 - 1 = 3$ is a multiple of 3. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$2^{2k} - 1 = 3m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}
&2^{2(k+1)} - 1 \\
&= 2^{2k+2} - 1 \\
&= 2^2 2^{2k} - 1 \\
&= 4 2^{2k} - 1 \\
&= 4(3m + 1) - 1 \quad \text{(by (1))} \\
&= 12m + 3 \\
&= 3(4m + 1)
\end{aligned}$$

We see that this is a multiple of 3. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising gives

$$\begin{aligned}
2^{2n} - 1 &= (2^2)^2 - 1 \\
&= (2^n - 1)(2^n + 1)
\end{aligned}$$

Note that the number 2^n is not divisible by 3. Therefore either one more or less than this number is a multiple of 3. That is, either $2^n + 1$ or $2^n - 1$ must be divisible by 3.

4 a $\displaystyle\sum_{i=1}^{4} i^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$

b $\displaystyle\sum_{k=1}^{5} 3^k = 3^1 + 3^2 + 3^3 + 3^4 + 3^5 = 363$

c $\displaystyle\sum_{i=0}^{3} (-1)^i\, i = 0 - 1 + 2 - 3 = -2$

d $\displaystyle\frac{1}{5}\sum_{i=1}^{5} i = \frac{1}{5}(1 + 2 + 3 + 4 + 5) = 3$

e $\displaystyle\sum_{i=1}^{4} 2i = 2 + 4 + 6 + 8 = 20$

f $\displaystyle\sum_{k=1}^{4}(k-1)^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14$

g $\displaystyle\sum_{i=1}^{4}(i-2)^2 = (-1)^2 + 0^2 + 1^2 + 2^2 = 6$

h $\displaystyle\sum_{i=1}^{4}(2i-1)^2 = 1^2 + 3^2 + 5^2 + 7^2 = 84$

i $\displaystyle\sum_{i=1}^{3} r^i = r + r^2 + r^3$

j $\displaystyle\sum_{i=1}^{3} i2^i = 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 = 34$

k $\displaystyle\sum_{i=0}^{3} 3^{3-i} = 3^3 + 3^2 + 3^1 + 3^0 = 40$

l $\displaystyle\sum_{i=1}^{2}(x-1)^i = (x-1) + (x-1)^2 = x^2 - x$

m $\displaystyle\sum_{i=3}^{3} i^2 = 3^2 = 9$

n $\displaystyle\sum_{i=-2}^{2} i = (-2) + (-1) + 0 + 1 + 2 = 0$

o $\displaystyle\sum_{i=1}^{n} 1 = \overbrace{1 + \cdots + 1}^{n \text{ times}} = n$

p $\displaystyle\sum_{i=-4}^{-2} 2i = (-8) + (-6) + (-4) = -28$

5 a $\boxed{P(n)}$

$$\sum_{m=1}^{n}(2m-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

$\boxed{P(1)}$

If $n = 1$ then

LHS $= 1^2 = 1$

and

$$\text{RHS} = \frac{1 \cdot 1 \cdot 3}{3} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\sum_{m=1}^{k}(2m-1)^2 = \frac{k(2k-1)(2k+1)}{3} \qquad (1)$$

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= \displaystyle\sum_{m=1}^{k+1}(2m-1)^2$

$= 1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2$

$= \dfrac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \quad$ (by (1))

$= \dfrac{k(2k+3)+1}{3)}$

$= \dfrac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$

$= \dfrac{(2k+1)(k(2k-1) + 3(2k+1))}{3}$

$= \dfrac{(2k+1)(2k^2 + 5k + 3)}{3}$

$= \dfrac{(2k+1)(k+1)(2k+3)}{3}$

$= \dfrac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$\sum_{m=1}^{n}(-1)^{m+1}m^2 = (-1)^{n+1}\frac{n(n+1)}{2}$$

P(1)

If $n = 1$ then

$$\text{LHS} = (-1)^2 \cdot 1^2 = 1$$

and

$$\text{RHS} = (-1)^2 \frac{1 \cdot 2}{2} = 1.$$

Therefore $P(1)$ is true.

P(k)

Assume that $P(k)$ is true so that

$$\sum_{m=1}^{k} (-1)^{m+1} m^2 = (-1)^{k+1} \frac{k(k+1)}{2}$$

P(k + 1)

LHS of $P(k+1)$

$$= \sum_{m=1}^{k+1} (-1)^{m+1} m^2$$

$$= 1^2 + \cdots + (-1)^{k+1} k^2 + (-1)^{k+2} (k+1)^2$$

$$= (-1)^{k+1} \frac{k(k+1)}{2} + (-1)^{k+2} (k+1)^2 \text{ (by (1))}$$

$$= (-1)^{k+2} \left(-\frac{k(k+1)}{2} + (k+1)^2 \right)$$

$$= (-1)^{k+2} \left(\frac{2(k+1)^2 - k(k+1)}{2} \right)$$

$$= (-1)^{k+2} \left(\frac{k^2 + 3k + 2}{2} \right)$$

$$= (-1)^{k+2} \left(\frac{(k+1)(k+2)}{2} \right)$$

$$= (-1)^{k+2} \left(\frac{(k+1)((k+1)+1)}{2} \right)$$

= RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 a $\displaystyle\prod_{i=1}^{4} i = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

b $\displaystyle\prod_{j=1}^{3} 2j = 2 \cdot 4 \cdot 6 = 48$

c $\displaystyle\prod_{k=1}^{3} k^2 = 1^2 \cdot 2^2 \cdot 3^2 = 36$

d $\displaystyle\prod_{i=0}^{2} 10^k = 10^0 \cdot 10^1 \cdot 10^2 = 1000$

e $\displaystyle\prod_{j=1}^{4} \frac{j}{j+1} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$

f $\displaystyle\prod_{k=1}^{5} \sqrt{k} = \sqrt{1}\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5} = \sqrt{120}$

g $\displaystyle\prod_{j=1}^{3} \left(\frac{i+1}{i}\right)^2 = \left(\frac{2}{1}\right)^2 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^2 = 16$

h $\displaystyle\prod_{i=0}^{4} \frac{1}{2^i} = \frac{1}{2^0} \frac{1}{2^1} \frac{1}{2^2} \frac{1}{2^3} \frac{1}{2^4} = \frac{1}{1024}$

7 P(n)

$$\prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}$$

P(2)

If $n = 2$ then

$$\text{LHS} = \prod_{j=2}^{2} \left(1 - \frac{1}{j^2}\right) = 1 - \frac{1}{2^2} = \frac{3}{4}$$

and

$$\text{RHS} = \frac{2+1}{2 \cdot 2} = \frac{3}{4}.$$

Therefore $P(2)$ is true.

P(k)

Assume that $P(k)$ is true so that

$$\prod_{j=2}^{k} \left(1 - \frac{1}{j^2}\right) = \frac{k+1}{2k} \qquad (1)$$

P(k + 1)

LHS of $P(k+1)$

$$= \prod_{j=2}^{k+1}\left(1-\frac{1}{j^2}\right)$$

$$= \left(1-\frac{1}{(k+1)^2}\right)\prod_{j=2}^{k+1}\left(1-\frac{1}{j^2}\right)$$

$$= \left(1-\frac{1}{(k+1)^2}\right)\frac{k+1}{2k} \quad \text{(by (1))}$$

$$= \frac{(k+1)^2-1}{(k+1)^2}\frac{k+1}{2k}$$

$$= \frac{k^2+2k}{2k(k+1)}$$

$$= \frac{k(k+2)}{2k(k+1)}$$

$$= \frac{(k+2)}{2(k+1)}$$

$$= \frac{(k+1)+1}{2(k+1)}$$

= RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

8 a $P(n)$

$n^2 - n$ is even for $n \in \mathbb{N}$

$P(1)$

If $n = 1$ then $1^2 - 1 = 0$ is even. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$k^2 - k = 2m \qquad (1)$$

for some integer m.

$P(k+1)$

Let $n = k + 1$ so that

$$\begin{aligned}(k+1)^2 - (k+1) &= k^2 + 2k + 1 - k - 1\\ &= k^2 + k\\ &= k^2 - k + 2k\\ &= 2m + 2k \quad \text{(by (1))}\\ &= 2(m+k)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^2 - n = n(n-1)$$

Since this is the product of two consecutive numbers, one of these is even. Therefore the product will be even.

9 a $P(n)$

$2^n > n^2$ where $n \geq 5$.

$P(5)$

If $n = 5$ then

$$\text{LHS} = 2^5 = 32 > 25 = 5^2 = \text{RHS}$$

Therefore $P(5)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$2^k > k^2 \quad (1)$$

where $k \geq 5$.

$P(k+1)$

We see that

$$2^{k+1} = 2 \cdot 2^k > 2k^2$$

We now need to prove that $2k^2 > (k+1)^2$ whenever $k \geq 5$. To see this, note that

$$\begin{aligned}
&2k^2 - (k+1)^2 \\
&= 2k^2 - (k^2 + 2k + 1) \\
&= k^2 - 2k - 1 \\
&= (k^2 - 2k + 1) - 1 - 1 \\
&= (k-1)^2 - 2 \\
&\geq (5-1)^2 - 2 \qquad (\text{as } k \geq 5) \\
&> 0
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$n! > 2^n$ where $n \geq 4$.

$\boxed{P(4)}$

If $n = 4$ then

$$\text{LHS} = 4! = 24 > 16 = 2^4 = \text{RHS}$$

Therefore $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k! > 2^k \quad (1)$$

where $k \geq 4$.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}
(k+1)! &= (k+1)k! \\
&> (k+1)2^k \quad (\text{by } (1)) \\
&> 2 \cdot 2^k \quad (\text{as } k \geq 4) \\
&= 2^{k+1}
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$4^n > 2 \times 3^n$ where $n \geq 3$.

$\boxed{P(4)}$

If $n = 4$ then

$$\text{LHS} = 4^3 = 64 > 54 = 2 \times 3^3 = \text{RHS}$$

Therefore $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^k > 2 \times 3^k \quad (1)$$

where $k \geq 3$.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}
4^{k+1} &= 4 \times 4^k \\
&> 4 \times 2 \times 3^k \quad (\text{by } (1)) \\
&= 8 \times 3^k \\
&> 3 \times 3^k \quad (\text{as } 8 > 3) \\
&= 3^{k+1}
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$3^n > 2n + 1$ where $n \geq 2$.

$\boxed{P(42)}$

If $n = 2$ then

$$\text{LHS} = 3^2 = 9 > 5 = 2 \times 2 + 1 = \text{RHS}$$

Therefore $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^k > 2k + 1 \quad (1)$$

where $k \geq 2$.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned} 3^{k+1} &= 3 \times 3^k \\ &> 3 \times (2k+1) \quad \text{(by (1))} \\ &= 6k+3 \\ &= (2(k+1)+1)+4k \\ &> 2(k+1)+1 \quad (\text{as } 4k>0) \end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

10 **a** $\boxed{P(n)}$

$n^3 + 3n^2 + 2n$ is divisible by 6 for $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $1^3 + 3(1)^2 + 2(1) = 6$ is divisible by 3. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k^3 + 3k^2 + 2k = 6m \qquad (1)$$

for some integer m.

$\boxed{P(k+1)}$

$$\begin{aligned} &(k+1)^3 + 3(k+1)^2 + 2(k+1) \\ &= k^3 + 6k^2 + 11k + 6 \\ &= (k^3 + 3k^2 + 2k) + (3k^2 + 9k + 6) \\ &= 6m + 3(k^2 + 3k + 2) \quad \text{(by (1))} \\ &= 6m + 3(k+1)(k+2) \\ &= 6m + 3(2p) \\ &= 6(m+p) \end{aligned}$$

Note that $(k+1)(k+2)$ must be divisible by 2 as it is the product of two consecutive numbers. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^3 + 3n^2 + 2n = n(n+1)(n+2)$$

Since this is the product of three consecutive numbers, one of these is divisible by 3 and at least one of these will be divisible by 2. Therefore the product will be divisible by $2 \times 3 = 6$.

11 Let a, b, x and y be real numbers.

a We expand both the left- and right-hand sides to give

$$\begin{aligned} \text{LHS} &= (a^2 + b^2(x^2 + y^2) \\ &= a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= (ax + by)^2 + (bx - ay)^2 \\ &= a^2x^2 + 2axby + b^2y^2 \\ &\quad + b^2x^2 - 2bxay + a^2y^2 \\ &= a^2x^2 + b^2y^2 + b^2x^2 + a^2y^2 \\ &= \text{LHS}. \end{aligned}$$

b We see that $13 = 2^2 + 3^2$.

c $\boxed{P(n)}$

13^n can be written as the sum of two squares for $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $13^1 = 2^2 + 3^2$ is the sum of two squares. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$13^k = x^2 + y^2 \qquad (1)$$

for integers x, y.

$\boxed{P(k+1)}$

13^{k+1}

$= 13 \cdot 13^k$

$= (2^2 + 3^2)(x^2 + y^2)$ (by (1))

$= (2x + 3y)^2 + (3x - 2y)^2$ (by part **a**)

This is the sum of two squares.
Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

12 $\boxed{P(n)}$

If m is odd, then m^n is odd, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $m^1 = m$ is odd. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that m^k is odd.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned} m^{k+1} &= m^k m \\ &= \text{odd} \times \text{odd} \quad \text{(by (1))} \\ &= \text{odd} \end{aligned}$$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

13 **a** $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55$

b $\boxed{P(n)}$

$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = F_1 = 1$$

while

$$\text{RHS} = F_3 - 1 = 2 - 1 = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$$

where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

LHS of $P(k + 1)$

$= F_1 + F_2 + \cdots + F_k + F_{k+1}$

$= F_{k+2} - 1 + F_{k+1}$ (by (1))

$= F_{k+3} - 1$

$=$ RHS of $P(k + 1)$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c We find that

- $n = 1 : F_1 = 1$
- $n = 2 : F_1 + F_3 = 1 + 2 = 3$
- $n = 3 : F_1 + F_3 + F_5 = 1 + 2 + 5 = 8$
- $n = 4 : F_1 + F_3 + F_5 + F_7 = 1 + 2 + 5 + 13 = 21$

d From the above pattern we conjecture that

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}.$$

e $\boxed{P(n)}$

$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$ where $n \in \mathbb{N}$.

P(1)

If $n = 1$ then

$$\text{LHS} = F_1 = 1$$

while

$$\text{RHS} = F_2 = 1.$$

Therefore $P(1)$ is true.

P(k)

Assume that $P(k)$ is true so that

$$F_1 + F_3 + \cdots + F_{2k-1} = F_{2k}$$

where $k \in \mathbb{N}$.

P(k + 1)

Let $n = k + 1$ so that

$$\begin{aligned} &\text{LHS of } P(k+1) \\ &= F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} \\ &= F_{2k} + F_{2k+1} \quad \text{(by (1))} \\ &= F_{2k+2} \\ &= F_{2(k+1)} \\ &= \text{RHS of } P(k+1) \end{aligned}$$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f P(n)

F_{5n} is divisible by 5, where $n \in \mathbb{N}$.

P(1)

If $n = 1$ then $F_5 = 5$ is divisible by 5. Therefore $P(1)$ is true.

P(k)

Assume that $P(k)$ is true so that $F_{5k} = 5t$ for some integer t.

P(k + 1)

We see that

$$\begin{aligned} F_{5k+5} &= F_{5k+3} + F_{5k+4} \\ &= 2F_{5k+3} + F_{5k+2} \\ &= 2(F_{5k+2} + F_{5k+1}) + F_{5k+1} + F_{5k} \\ &= 3F_{5k+1} + 2F_{5k+1} + F_{5k} + 2F_{5k} \\ &= 5F_{5k+1} + 3(5t) \quad \text{(by (1))} \\ &= 5(F_{5k+1} + 3t) \end{aligned}$$

We see that this is a multiple of 5. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

14 a P(n)

A convex polygonal area with $n \geq 3$ vertices can be triangulated.

P(3)

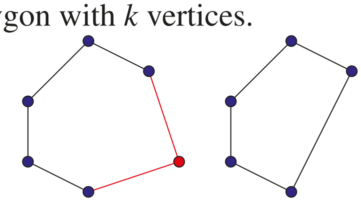
If $n = 3$ then the polygon is a triangle, and so can obviously be triangulated using one triangle (i.e., itself).

P(k)

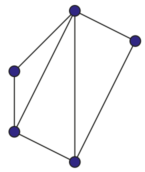
Assume that $P(k)$ is true. Therefore any convex polygonal area with $k \geq 3$ vertices can be triangulated.

P(k + 1)

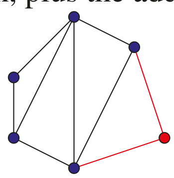
Take any polygon with $k + 1$ vertices. Delete one vertex and its incident edges to create a truncated convex polygon with k vertices.



This can be triangulated by assumption.



Restore the deleted vertex and its two incident edges. The result can be triangulated using the original triangulation, plus the added triangle.



Therefore $P(k+1)$ is true.

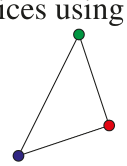
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

Consider the triangulation of a convex polygon with $n \geq 3$ vertices. We can colour its vertices using three colours, so that each vertex has a colour different to each of its neighbours.

$\boxed{P(3)}$

If $n = 3$ then the polygon is a triangle, and so we can obviously colour its vertices using three colours.

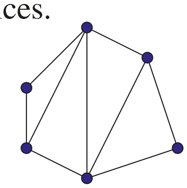


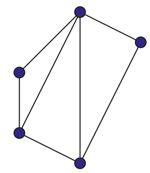
$\boxed{P(k)}$

Assume that $P(k)$ is true. Consider any triangulation of a convex polygon with $k \geq 3$ vertices. Assume that we can colour its vertices using three colours, so that each vertex has a colour different to each of its neighbours.

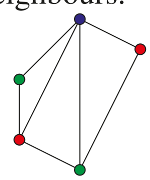
$\boxed{P(k+1)}$

Take any triangulation of a polygon with $k + 1$ vertices. Delete one vertex and its incident edges to create a truncated convex polygon has k vertices.

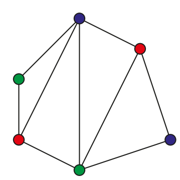




By assumption, we can colour its vertices using three colours, so that each vertex has a colour different to each of its neighbours.



Restore the deleted vertex and its two incident edges. We can colour the restored vertex so that it is different to the two vertices to which it is connected.



Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

15 $\boxed{P(n)}$

$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \geq \sqrt{n}$

$\boxed{P(1)}$

If $n = 1$ then LHS $= \frac{1}{\sqrt{1}} = 1$ and

RHS $= \sqrt{1} = 1$. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true that so that

$$\sum_{j=1}^{k} \frac{1}{\sqrt{j}} \geq \sqrt{k} \quad (1)$$

$\boxed{P(k+1)}$

We now need to prove that

$$\sum_{j=1}^{k+1} \frac{1}{\sqrt{j}} \geq \sqrt{k+1}.$$

To show this we first use (1) to see that

$$\sum_{j=1}^{k+1} \frac{1}{\sqrt{j}} = \sum_{j=1}^{k} \frac{1}{\sqrt{j}} + \frac{1}{\sqrt{k+1}}$$
$$\geq \sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

To complete the proof we need to show that

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}.$$

To show this, note that

$$\begin{aligned}
&\sqrt{k} + \frac{1}{\sqrt{k+1}} \\
&= \frac{1}{\sqrt{k+1}}\left(\sqrt{k}\sqrt{k+1}+1\right) \\
&> \frac{1}{\sqrt{k+1}}\left(\sqrt{k}\sqrt{k}+1\right) \\
&= \frac{1}{\sqrt{k+1}}(k+1) \\
&= \sqrt{k+1}.
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

16 $\boxed{P(n)}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \text{ where } n \in \mathbb{N}.$$

$\boxed{P(2)}$

If $n = 2$ then

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}$$

Therefore $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \quad (1)$$

where $k \geq 2$.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}
&\text{LHS of } P(k+1) \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k+1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \quad \text{(by (1))} \\
&= \begin{bmatrix} F_{k+1}+F_k & F_k+F_{k-1} \\ F_{k+1} & F_k \end{bmatrix} \\
&= \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

17 a $\boxed{P(n)}$

$3^n - (-2)^n$ is divisible by 5, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $3^1 - (-2)^1 = 3 + 2 = 5$ is divisible by 5. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^k - (-2)^k = 5m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned} & 3^{k+1} - (-2)^{k+1} \\ &= 3 \cdot 3^k - (-2)^{k+1} \\ &= 3 \cdot ((-2)^k + 5m) - (-2)^{k+1} \quad \text{(by (1))} \\ &= 3 \cdot (-2)^k + 3(5m) - (-2)(-2)^k \\ &= 3 \cdot (-2)^k + 3(5m) + 2(-2)^k \\ &= 5 \cdot (-2)^k + 3(5m) \\ &= 5((-2)^k + 3m) \end{aligned}$$

We see that this is a multiple of 5. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$4^n - (-3)^n$ is divisible by 7, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $4^1 - (-3)^1 = 4 + 3 = 7$ is divisible by 7. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^k - (-3)^k = 7m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned} & 4^{k+1} - (-3)^{k+1} \\ &= 4 \cdot 4^k - (-3)^{k+1} \\ &= 4 \cdot ((-3)^k + 7m) - (-3)^{k+1} \quad \text{(by (1))} \\ &= 4 \cdot (-3)^k + 4(7m) - (-3)(-3)^k \\ &= 4 \cdot (-3)^k + 4(7m) + 3(-3)^k \\ &= 7 \cdot (-3)^k + 4(7m) \\ &= 7((-3)^k + 4m) \end{aligned}$$

We see that this is a multiple of 7.

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c Let a be an integer. We conjecture that $(a+1)^n - (-a)^n$ is divisible by $2a+1$, where $n \in \mathbb{N}$. We will prove this using induction.

$\boxed{P(n)}$

$(a+1)^n - (-a)^n$ is divisible by $2a+1$, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $(a+1)^1 - (-a)^1 = 2a+1$ is divisible by $2a+1$. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$(a+1)^k - (-a)^k = (2a+1)m \quad (1)$$

for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned} & (a+1)^{k+1} - (-a)^{k+1} \\ &= (a+1)(a+1)^k - (-a)^{k+1} \\ &= (a+1)((-a)^k + (2a+1)m) - (-a)^{k+1} \quad \text{(by (1))} \\ &= (a+1)(-a)^k + (a+1)(2a+1)m - (-a)(-a)^k \\ &= (a+1)(-a)^k + (a+1)(2a+1)m + a(-a)^k \\ &= (2a+1)(-a)^k + (a+1)(2a+1)m \\ &= (2a+1)((-a)^k + (a+1)m) \end{aligned}$$

We see that this is a multiple of $2a+1$. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

18 a $\boxed{P(n)}$

$1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) =$

$\frac{1}{6}n(n+1)(2n+7)$ where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 \cdot 3 = 3$$

and

$$\text{RHS} = \frac{1}{6} \cdot 1 \cdot 2 \cdot 9 = 3.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 \cdot 3 + \cdots + k(k+2) = \frac{1}{6}k(k+1)(2k+7)$$

where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1)\\
&= 1 \cdot 3 + \cdots + k(k+2) + (k+1)(k+3)\\
&= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3) \quad \text{(by (1))}\\
&= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6}\\
&= \frac{(k+1)(k(2k+7) + 6(k+3)}{6}\\
&= \frac{(k+1)(2k^2+13k+18}{6}\\
&= \frac{(k+1)(k+2)(2k+9)}{6}\\
&= \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}\\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n+1) = n(n+1)^2$$

where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 \cdot 4 = 4$$

and

$$\text{RHS} = 1 \cdot 2^2 = 4.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 \cdot 4 + \cdots + k(3k+1) = k(k+1)^2 \quad (1)$$

where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1)\\
&= 1 \cdot 4 + \cdots + k(3k+1) + (k+1)(3k+4)\\
&= k(k+1)^2 + (k+1)(3k+4) \quad \text{(by (1))}\\
&= (k+1)(k(k+1) + 3k + 4)\\
&= (k+1)(k^2 + 4k + 4)\\
&= (k+1)(k+2)^2\\
&= (k+1)((k+1)+1)^2\\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

19 a We find that

b $\boxed{P(n)}$

$$F_n(x) = \frac{x}{2^n - (2^n - 1)x} \text{ where } n \in \mathbb{N}.$$

$\boxed{P(1)}$

If $n = 1$ then

$$\begin{aligned}
\text{LHS} &= F_1(x)\\
&= f(x)\\
&= \frac{x}{2-x}
\end{aligned}$$

and

$$\text{RHS} = \frac{x}{2^1 - (2^1 - 1)x} = \frac{x}{2 - x}.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$F_k(x) = \frac{x}{2^k - (2^k - 1)x}$$

where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$$\begin{aligned}
&= F_{k+1}(x) \\
&= f(F_k(x)) \\
&= f\left(\frac{x}{2^k - (2^k - 1)x}\right) \quad \text{(by (1))} \\
&= \frac{\frac{x}{2^k - (2^k - 1)x}}{2 - \frac{x}{2^k - (2^k - 1)x}} \\
&= \frac{x}{2(2^k - (2^k - 1)x) - x} \\
&= \frac{x}{2^{k+1} - 2^{k+1}x + 2x - x} \\
&= \frac{x}{2^{k+1} - 2^{k+1}x + x} \\
&= \frac{x}{2^{k+1} - (2^{k+1} - 1)x}
\end{aligned}$$

= RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

20 a $R_0 = 1, R_1 = 2, R_2 = 4, R_3 = 7, R_4 = 11$

b If there are n lines already drawn, then next line will intersect each of these lines at n point, creating $n+1$ new regions. Therefore

$$R_{n+1} = R_n + (n+1).$$

Therefore

$$\begin{aligned}
R_0 &= 1 \\
R_1 &= R_0 + 1 = 1 + 1 \\
R_2 &= R_1 + 2 = 1 + 1 + 2 \\
R_3 &= R_2 + 3 = 1 + 1 + 2 + 3.
\end{aligned}$$

More generally,

$$\begin{aligned}
R_n &= 1 + (1 + 2 + 3 + \cdots + n) \\
&= 1 + \frac{n(n+1)}{2}
\end{aligned}$$

This could be formally proved using induction.

21 a Note that

$$\begin{aligned}
{}^pC_i &= \frac{p!}{i!(p-i)!} \\
&= \frac{p(p-1)\cdots(p-i+1)(p-i)!}{i!(p-i)!} \\
&= \frac{p(p-1)\cdots(p-i+1)}{i!}.
\end{aligned}$$

Therefore

$$i!\,{}^pC_i = p(p-1)\cdots(p-i+1).$$

The right-hand sides divisible by p, therefore the left-hand side is divisible by p. Since $i < p$, we know that $i!$ is not divisible by p. Therefore ${}^pC_i$ is divisible by p.

b This will be a proof by induction.

$\boxed{P(n)}$

If p does not divide n then $n^p - n$ is divisible by p

$\boxed{P(1)}$

If $n = 1$ then $1^p - 1 = 0$ is divisible by p (Recall that 0 is divisibly by any integer). Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true. Therefore $k^p - k$ is divisible by p, in which case

$$k^p - k = mp$$

where $k, m \in \mathbb{N}$.

$P(k+1)$

Now suppose that $n = k + 1$. Then

$$\begin{aligned}&(k+1)^p - (k+1)\\&= \left(k^p + {}^pC_1 k^{p-1} + \cdots + {}^pC_{p-1}k + 1\right) - (k+1)\\&= (k^p - k) + \left({}^pC_1 k^{p-1} + \cdots + {}^pC_{p-1}k +\right)\end{aligned}$$

Now that $k^p - k$ is divisible by p as we assumed that $P(k)$ is true. Also, each of the coefficients ${}^pC_i$ are divisible by p, by part **a**. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

22

23 Let $t_n = 2t_{n-1} + 3$ where $t_1 = 3$.

$P(n)$

$t_n = 3 \times 2^n - 3$ where $n \in \mathbb{N}$.

$P(1)$

If $n = 1$ then

$$\text{LHS} = t_1 = 3$$

and

$$\text{RHS} = 3 \times 2^1 - 3 = 3.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$t_k = 3 \times 2^k - 3 \quad (1)$$

where $k \in \mathbb{N}$.

$P(k+1)$

LHS of $P(k+1)$

$= t_{k+1}$

$= 2t_k + 3$ (by definition)

$= 2(3 \times 2^k - 3) + 3$ (by (1))

$= 3 \times 2^{k+1} - 6 + 3$

$= 3 \times 2^{k+1} - 3$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

24 Let $t_n = 2t_{n-1} - n$ and $t_1 = 1$.

$P(n)$

$t_n = n - 2^n + 2$ where $n \in \mathbb{N}$.

$P(1)$

If $n = 1$ then

$$\text{LHS} = t_1 = 1$$

and

$$\text{RHS} = 1 - 2^1 + 2 = 1.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$t_k = k - 2^k + 2 \quad (1)$$

where $k \in \mathbb{N}$.

$P(k+1)$

LHS of $P(k+1)$

$= t_{k+1}$

$= 2t_k - (k+1)$ (by definition)

$= 2(k - 2^k + 2) - (k+1)$ (by (1))

$= 2k - 2^{k+1} + 4 - k - 1$

$= k - 2^{k+1} + 3$

$= (k+1) - 2^{k+1} + 2$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

25 $\boxed{P(n)}$

It is possible to label the teams $T_1, T_2, \ldots, T_n$ so that

$$T_1 > T_2 > \cdots > T_n$$

, where $n \geq 2$.

$\boxed{P(2)}$

If $n = 2$ then there are just two teams. One of these teams beat the other. Label the winning team T_1 and the losing team as T_2.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that it is possible to label the teams $T_1, T_2, \ldots, T_k$ so that

$$T_1 > T_2 > \cdots > T_k,$$

where $k \geq 2$

$\boxed{P(k+1)}$

Now consider a tournament with $k + 1$ teams. Ignore one particular team A. Then each of the remaining k teams played each other exactly once. Each of the remaining teams can be split into two groups: the $a \leq k$ teams that A defeated, and the $b \leq k$ teams that defeated A (where $a + b = k$. By the inductive hypothesis, each of the a teams that A defeated can be labelled so that

$$R_1 > R_2 > \cdots > R_a.$$

Also, each of the b teams that defeated A can be labelled so that

$$S_1 > S_2 > \cdots > S_b.$$

But then we can order each of the $k + 1$ teams as follows:

$$S_1 > S_2 > \cdots > S_b > A > R_1 > R_2 > \cdots > R_a.$$

This gives the required labelling of the teams. Feel free to rewrite this using the letter T instead!

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to Technology-free questions

1 **a** If $(a, a+d, a+2d)$ is a Pythagorean triple, then

$$\begin{aligned} a^2 + (a+d)^2 &= (a+2d)^2 \\ a^2 + a^2 + 2ad + d^2 &= a^2 + 4ad + 4d^2 \\ a^2 - 2ad - 3d^2 &= 0 \\ (a+d)(a-3d) &= 0 \end{aligned}$$

Since $a + d \neq 0$, we conclude that $a = 3d$.

b If (p, q, r) is a Pythagorean triple and p is a prime number then

$$\begin{aligned} p^2 + q^2 &= r^2 \\ p^2 &= r^2 - q^2 \\ p^2 &= (r-q)(r+q) \end{aligned}$$

The only factors of p^2 are $1, p$ and p^2. Also, as $r + q > r - q$, we conclude that $r - q = 1$ and $p^2 = r + q$. Therefore $r = q + 1$ and

$$\begin{aligned} p &= \sqrt{r+q} \\ &= \sqrt{2q+1}. \end{aligned}$$

2 Consider any three-digit number ABC. By this, mean $100A + 10B + C$. Reversing the digits gives $100C + 10B + A$. We can assume the first of these is largest. Therefore

$$\begin{aligned} &(100A + 10B + C) - (100C + 10B + A) \\ &= 99A - 99C \\ &= 99(A - C). \end{aligned}$$

Therefore the difference is divisible by 9. In fact, it's divisible by 99.

3 **a** We find that

$$\begin{aligned} \frac{1}{x(x+3)} &= \frac{a}{x} + \frac{b}{x+3} \\ 1 &= a(x+3) + bx \end{aligned}$$

Let $x = 0$ to find that $a = \frac{1}{3}$. Let $x = -3$ to find that $b = -\frac{1}{3}$. Therefore

$$\frac{1}{x(x+3)} = \frac{1}{3}\left(\frac{1}{x} - \frac{1}{x+3}\right).$$

b We use the above result to expand each term of the series:

$$\begin{aligned} &\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{97 \cdot 100} \\ &= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \cdots + \frac{1}{97} - \frac{1}{100}\right) \\ &= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{100}\right) \\ &= \frac{33}{100}. \end{aligned}$$

4 Let a and b be positive integers.

a First assume $a > b$. Then $a - b > 0$ so that

$$a^2 - b^2 = \overbrace{(a-b)}^{>0}\overbrace{(b+a)}^{>0} > 0.$$

Now we prove the converse. Assume that $a^2 > b^2$. Therefore

$$\begin{aligned} a^2 - b^2 &> 0 \\ (a-b)(a+b) &> 0 \end{aligned}$$

Dividing both sides of the above inequality by $a + b$ (which is positive) gives $a - b > 0$. Therefore $a > b$.

b By the previous result we have that

$$\begin{aligned}
&\sqrt{15} > \sqrt{2} + \sqrt{6} \\
\Leftrightarrow\ &(\sqrt{15})^2 > (\sqrt{2} + \sqrt{6})^2 \\
\Leftrightarrow\ &15 > 2 + 2\sqrt{12} + 6 \\
\Leftrightarrow\ &15 > 8 + 2\sqrt{12} \\
\Leftrightarrow\ &7 > 2\sqrt{12} \\
\Leftrightarrow\ &7^2 > (2\sqrt{12})^2 \\
\Leftrightarrow\ &49 > 48
\end{aligned}$$

As the above six lines are equivalent, and as the final line is true, so too is the first line.

c By the result in part **a** we find that

$$\begin{aligned}
&\sqrt{a} + \sqrt{b} > \sqrt{a+b} \\
\Leftrightarrow\ &(\sqrt{a} + \sqrt{b})^2 > (\sqrt{a+b})^2 \\
\Leftrightarrow\ &a + 2\sqrt{ab} + b > a + b \\
\Leftrightarrow\ &2\sqrt{ab} > 0
\end{aligned}$$

As the above four lines are equivalent, and as the final line is true, so too is the first line.

5 Suppose that $\log_n(n+1) = \frac{a}{b}$ for integers a and b where $b \neq 0$. Then

$$\begin{aligned}
\log_n(n+1) &= \frac{a}{b} \\
n^{\frac{a}{b}} &= n + 1 \\
n^a &= (n+1)^b
\end{aligned}$$

If n is even, then $n + 1$ is odd. Therefore n^a will be even and $(n+1)^b$ will be odd, which is a contradiction. If n is odd, then $n + 1$ is even. Therefore n^a will be odd and $(n+1)^b$ will be even, which is also a contradiction.

6 **a** Suppose $n + 1$ is divisible by 3. Then $n + 1 = 3m$ for some integer m. Therefore,

$$\begin{aligned}
n^3 + 1 &= (n+1)(n^2 - n + 1) \\
&= 3m(n^2 - n + 1)
\end{aligned}$$

is also divisible by 3.

b The contrapositive statement is: If $n^3 + 1$ is not divisible by 3, then $n + 1$ is not divisible by 3.

c The converse statement is: If $n^3 + 1$ is divisible by 3, then $n + 1$ is divisible by 3. This is true, but harder to prove. We will prove the contrapositive statement. Suppose $n + 1$ is not divisible by 3. Then either $n + 1 = 3m + 1$ or $n + 1 = 3m + 2$. In the first case we find that $n = 3m$ so that $n^3 + 1 = 27m^3 + 1 = 3(9m^2) + 1 = 3j + 1$ is not divisible by 3. In the second case we have $n = 3m + 1$ so that

$$\begin{aligned}
n^3 + 1 &= (3m+1)^3 + 1 \\
&= (27m^3 + 9m^2 + 3m + 1) + 1 \\
&= 3(9m^3 + 3m^2 + m) + 2 \\
&= 3j + 2
\end{aligned}$$

is not divisible by 3.

7 Let b be any non-zero rational number.

a Suppose that $\sqrt{2}b$ is rational. Then $\sqrt{2}b = \frac{m}{n}$ for integers m and n where $n \neq 0$. Therefore $\sqrt{2} = \frac{m}{2b}$, in which case $\sqrt{2}$ would be rational. This is a contradiction.

b We note that

$$b = \left(\sqrt{2}b\right)\left(\frac{1}{\sqrt{2}}\right) = \left(\sqrt{2}b\right)\left(\frac{\sqrt{2}}{2}\right)$$

By part **a**, we know that each of the above two numbers on the right is an irrational number. Therefore every non-zero rational number b can be written as the product of two irrational numbers.

8 a If $p = 7$ then $p + 2 = 9$ is not an odd prime.

b Note that $2^3 = 8$ is divisible by 8, although 2 is not divisible by 8.

c Note that $-2 < 1$ and $-3 < 1$, however $(-2)(-3) > (1)(1)$.

d Note that $7, 7 + 4$ and $7 + 6$ are all prime numbers.

e Note that if $n = 6$, then $2^6 + 1 = 65 = 5 \times 13$ is not a prime.

9 a The contrapositive is: If none of a, b or c are even, then $a^2 + b^2 \neq c^2$.

b Assume that none of a, b or c are even. Then each of these is odd. Moreover a^2, b^2 and c^2 will also be odd. Therefore:

$$\begin{aligned} a^2 + b^2 &= (\text{odd})^2 + (\text{odd})^2 \\ &= (\text{odd}) + (\text{odd}) \\ &= (\text{even}) \end{aligned}$$

On the other hand,

$$\begin{aligned} c^2 &= (\text{odd})^2 \\ &= (\text{odd}) \end{aligned}$$

However, as no even number is equal to an odd number, $a^2 + b^2 \neq c^2$.

10 a Every integer is of the form $n = 3m$ or $n = 3m + 1$ or $n = 3m + 2$. Squaring each of these gives

$$n^2 = (3k)^2 = 9k^2 = 3(3k^2) = 3m$$

$$\begin{aligned} n^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2) + 1 \\ &= 3m + 1 \end{aligned}$$

$$\begin{aligned} n^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1 \\ &= 3m + 1. \end{aligned}$$

Therefore the square of any integer n has the form $3k$ or $3k + 1$ where $k \in \mathbb{N}$.

b Note that each of these numbers listed is of the form $3k + 2$. That is,

$$\begin{aligned} 11 &= 3 \times 3 + 2 \\ 101 &= 3 \times 33 + 2 \\ 1001 &= 3 \times 333 + 2 \\ 10001 &= 3 \times 3333 + 2 \\ 100001 &= 3 \times 33333 + 2 \end{aligned}$$

As no square has the form $3k + 2$, the above list contains no squares.

11 a $\boxed{P(n)}$

$1 + 4 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$ for all $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

LHS = 1

and

$$\text{RHS} = \frac{1(3(1) - 1)}{2} = 1.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$1 + 4 + \cdots + (3k - 2) = \frac{k(3k - 1)}{2} \quad (1)$$

where $k \in \mathbb{N}$.

$P(k + 1)$

LHS of $P(k + 1)$

$= 1 + 4 + \cdots + (3k - 2) + (3k + 1)$

$= \dfrac{k(3k - 1)}{2} + (3k + 1) \quad \text{(by (1))}$

$= \dfrac{k(3k - 1)}{2} + \dfrac{2(3k + 1)}{2}$

$= \dfrac{3k^2 + 5k + 2}{2}$

$= \dfrac{(k + 1)(3k + 2)}{2}$

$= \dfrac{(k + 1)(3(k + 1) - 1)}{2}$

= RHS of $P(k + 1)$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $P(n)$

$$3^{-1} + \cdots + 3^{-n} = \frac{3^n - 1}{2(3^n)} \text{ for all } n \in \mathbb{N}.$$

$P(1)$

If $n = 1$ then

$$\text{LHS} = 3^{-1} = \frac{1}{3}$$

and

$$\text{RHS} = \frac{3^1 - 1}{2(3^1)} = \frac{1}{3}.$$

Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$3^{-1} + \cdots + 3^{-k} = \frac{3^k - 1}{2(3^k)} \quad (1)$$

where $k \in \mathbb{N}$.

$P(k + 1)$

LHS of $P(k + 1)$

$= 3^{-1} + \cdots + 3^{-k} + 3^{-k-1}$

$= \dfrac{3^k - 1}{2(3^k)} + 3^{-k-1} \quad \text{(by (1))}$

$= \dfrac{3^k - 1}{2(3^k)} + \dfrac{1}{3^{k+1}}$

$= \dfrac{3^k - 1}{2(3^k)} + \dfrac{1}{3^{k+1}}$

$= \dfrac{(k + 1)(3k + 2)}{2}$

$= \dfrac{(k + 1)(3(k + 1) - 1)}{2}$

= RHS of $P(k + 1)$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

12 $P(n)$

$$\sum_{j=2}^{n} \frac{4}{j^2 - 1} = \frac{(n - 1)(3n + 2)}{n(n + 1)}$$

where $n \geq 2$.

$P(2)$

If $n = 2$ then

$$\text{LHS} = \frac{4}{2^2 - 1} = \frac{4}{3}$$

and

$$\text{RHS} = \frac{(2-1)(3(2)+2)}{2(2+1)} = \frac{4}{3}.$$

Therefore $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\sum_{j=2}^{k} \frac{4}{j^2 - 1} = \frac{(k-1)(3k+2)}{k(k+1)}$$

where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

$$\begin{aligned}
&\text{LHS of } P(k+1) \\
&= \sum_{j=2}^{k+1} \frac{4}{j^2 - 1} \\
&= \sum_{j=2}^{k} \frac{4}{j^2 - 1} + \frac{4}{(k+1)^2 - 1} \\
&= \frac{(k-1)(3k+2)}{k(k+1)} + \frac{4}{k(k+2)} \quad \text{(by (1))} \\
&= \frac{k^2(3k+5)}{k(k+1)(k+2)} \\
&= \frac{k(3k+5)}{(k+1)(k+2)} \\
&= \frac{k(3k+5)}{(k+1)(k+2)} \\
&= \frac{(k+1)(3(k+1)-1)}{2} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

13 a $\boxed{P(n)}$

$n^3 > 2n + 1$ where $n \geq 2$.

$\boxed{P(2)}$

If $n = 2$ then

$$\text{LHS} = 2^3 = 8$$

and

$$\text{RHS} = 2(2) + 1 = 5.$$

As LHS = RHS, we see that $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^3 > 2k + 1$ where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

We first find that

$$\begin{aligned}
&\text{LHS of } P(k+1) \\
&= (k+1)^3 \\
&= k^3 + 3k^2 + 3k + 1 \\
&> 2k + 1 + 3k^2 + 3k + 1 \quad \text{(by (1))} \\
&= 2k + (3k^2 + 3k + 2) \\
&> 2k + 3 \\
&= 2(k+1) + 1 \\
&= (k+1)^2
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$n! > n^2$ where $n \geq 4$.

$\boxed{P(4)}$

If $n = 4$ then

$$\text{LHS} = 4! = 24$$

and

$$\text{RHS} = \frac{(2-1)(3(2)+2)}{2(2+1)} = \frac{4}{3}.$$

Therefore $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that
$k! > k^2$ where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

We first find that

$$\begin{aligned} &\text{LHS of } P(k+1) \\ &= (k+1)! \\ &= (k+1)k! \\ &> (k+1)k^2 \quad \text{(by (1))} \\ &= k^3 + k^2 \\ &> 2k + 1 + k^2 \quad \text{(by part } \mathbf{a}\text{)} \\ &= (k+1)^2 \end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

14 $\boxed{P(n)}$

$3^n \geq n^2 + n$ where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 3^1 = 3 > 1^2 + 1 = \text{RHS}$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that
$3^k > k^2 + k$ where $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

We first find that

$$\begin{aligned} &\text{LHS of } P(k+1) \\ &= 3^{k+1} \\ &= 3 \times 3^k \\ &> 3(k^2 + k) \quad \text{(by (1))} \\ &= 3k^2 + 3k. \end{aligned}$$

Now consider the right-hand side separately. We find that

$$\begin{aligned} &\text{RHS of } P(k+1) \\ &= (k+1)^2 + (k+1) \\ &= k^2 + 3k + 2. \end{aligned}$$

Therefore, we need to show that

$$3k^2 + 3k > k^2 + 3k + 2.$$

To show this, we note that

$$\begin{aligned} &3k^2 + 3k \\ &= k^2 + 3k + 2k^2 \\ &> k^2 + 3k + 2 \quad \text{(as } k \geq 1\text{)} \end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

15 a $\boxed{P(n)}$

$7^{2n-1} + 5$ is divisible by 12.

$\boxed{P(1)}$

If $n = 1$ then

$$7^1 + 5 = 12$$

is divisible by 12. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$7^{2k-1} + 5 = 12m$$

for some integer $m \in \mathbb{Z}$ and $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

We find that

$$
\begin{aligned}
&7^{2k+1} + 5 \\
&= 7^2 \cdot 7^{2k-1} + 5 \\
&= 49 \cdot (12m - 5) + 5 \quad \text{(by (1))} \\
&= 12(49m) - 49 \cdot 5 + 5 \\
&= 12(49m) - 48 \cdot 5 \\
&= 12(49m - 4 \cdot 5)
\end{aligned}
$$

is divisible by 12. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

$\boxed{P(1)}$

If $n = 1$ then

$$1^3 + 2^3 + 3^3 = 36 = 4 \times 9$$

is divisible by 9. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k^3 + (k+1)^3 + (k+2)^3 = 9m$$

for some integer $m \in \mathbb{Z}$ and $k \in \mathbb{N}$.

$\boxed{P(k+1)}$

Let $n = k + 1$ so that

$$
\begin{aligned}
&(k+1)^3 + (k+2)^3 + (k+3)^3 \\
&= k^3 + (k+1)^3 + (k+2)^3 + (k+3)^3 - k^3 \\
&= 9m + k^3 + 9k^2 + 27k + 27 - k^3 \\
&= 9m + 9k^2 + 27k + 27 \\
&= 9(m + k^2 + 3k + 3)
\end{aligned}
$$

is divisible by 9. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to multiple-choice questions

1 E If m is odd and n is even then $2m$ and $3n$ are both even in which case $2m + 3n$ is also even.

2 E To form the converse we switch the hypothesis and the conclusion. This gives: if n^2 is odd, then n is odd.

3 B To form the contrapositive, we switch the hypothesis and the conclusion and negate both. This gives: If a is odd, then $1 + a + a^2$ is even.

4 E The negation becomes the universal statement. We also use one of De Morgen's Laws. We obtain: for all $n \in \mathbb{N}$, we have that n is even or n^2 is odd.

5 A If $ca = cb$ then $ca - cb = 0$ so that $c(a - b) = 0$. Therefore $c = 0$ or $a = c$.

6 C If $f(x) = 2^x$, then $f'(x) > 0$ for all values x. However, the range of f is $(0, \infty)$.

7 D We find that $\sum_{i=3}^{5} i^2 = 3^2 + 4^2 + 5^2 = 50$

8 C We find that $\sum_{i=1}^{4} i = 1 \times 2 \times 3 \times 4 = 24$

Solutions to extended-response questions

1 **a** We expand the right-hand side, and then cancel all bar two terms, to give:

$$\begin{aligned} x^m - 1 &= (x-1)(1 + x + x^2 + \cdots + x^{m-1}) \\ &= (x + x^2 + x^3 + \cdots + x^m) - (1 + x + x^2 + \cdots + x^{m-1}) \\ &= x^m - 1, \end{aligned}$$

as required.

b If n is not prime then $n = mk$ for positive integers $a, b \geq 1$. Therefore, we find that

$$\begin{aligned} 2^n - 1 &= 2^{km} - 1 \\ &= (2^k)^m - 1 \\ &= (2^k - 1)(1 + 2^k + (2^k)^2 + \cdots + (2^k)^{m-1}) \\ &= (2^k - 1)(1 + 2^k + 2^{2k} + \cdots + (2^{(m-1)k}). \end{aligned}$$

As this is the product of two integers, both exceeding 1, the number is not prime.

c $2^{11} - 1 = 2047 = 23 \times 89$

2 **a** If both of a and b are even or if both are odd, then a^2 and b^2 are both even or both odd. In either case $a^2 + b^2 = c^2$ will be even. However, if c^2 is even, then c is even, which is a contradiction. Therefore exactly one of a or b is odd.

b We find that

$$\begin{aligned} \frac{abc}{a+b+c} &= \frac{abc(a+b-c)}{(a+b+c)(a+b-c)} \\ &= \frac{abc(a+b-c)}{(a+b)^2 - c^2)} \\ &= \frac{abc(a+b-c)}{a^2 + 2ab + b^2 - c^2} \\ &= \frac{abc(a+b-c)}{(a^2 + b^2 - c^2) + 2ab} \\ &= \frac{abc(a+b-c)}{2ab} \\ &= \frac{c(a+b-c)}{2}, \end{aligned}$$

as required. There are other approaches to proving the same identity.

c First, if c is even, then the right-hand side is an integer. Then the right-hand side is an integer. Therefore $a + b + c$ divides abc. Otherwise, c is odd. Therefore, from part **a** we know that exactly one of a or b will be odd. Therefore $a + b - c$ is even. Therefore the right-hand side is an integer. Therefore $a + b + c$ divides abc.

3 a We find that

$$f(-x) = (-x)^3 = -(x^3) = -f(x)$$

so that f is odd.

b If f and g are even then $f(-x) = f(x)$ and $g(-x) = g(x)$. Therefore

$$\begin{aligned}(f \cdot g)(-x) &= f(-x)g(-x) \\ &= f(x)g(x) \\ &= (f \cdot g)(x).\end{aligned}$$

Therefore, the product of two even functions is even.

c If f and g are odd then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Therefore

$$\begin{aligned}(f \cdot g)(-x) &= f(-x)g(-x) \\ &= (-f(x))(-g(x)) \\ &= (f \cdot g)(x)\end{aligned}$$

Therefore, the product of two odd functions is even.

d If f and g are odd then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Therefore

$$\begin{aligned}(f + g)(-x) &= f(-x) + g(-x) \\ &= (-f(x)) + (-g(x)) \\ &= -(f(x) + g(x)) \\ &= -(f + g)(x)\end{aligned}$$

Therefore, the sum two odd functions is odd.

e If f and g are even then $f(-x) = f(x)$ and $g(-x) = g(x)$. Therefore

$$\begin{aligned}(f + g)(-x) &= f(-x) + g(-x) \\ &= f(x) + g(x) \\ &= (f + g)(x)\end{aligned}$$

Therefore, the sum two even functions is even.

f Let f be an odd function. Then

$$\begin{aligned}& f(0) = f(-0) \\ \Rightarrow\ & f(0) = -f(0).\end{aligned}$$

Adding $f(0)$ to both sides of this equation gives

$$\begin{aligned}& 2f(0) = 0 \\ \Rightarrow\ & f(0) = 0.\end{aligned}$$

g Suppose that f is both even and odd. Then, for all values of $x \in \mathbb{R}$ we find that

$$\begin{aligned} f(x) &= f(-x) \qquad \text{(as } f \text{ is even)} \\ \Rightarrow f(x) &= -f(x) \qquad \text{(as } f \text{ is odd)} \\ \Rightarrow 2f(x) &= 0 \\ \Rightarrow f(x) &= 0. \end{aligned}$$

That is, f the function equal to 0 for all values of x. Note: this is sometimes called the **zero function**).

4 $\boxed{P(n)}$

If $f(x) = (2x+1)^{-1}$, then $f^{(n)}(x) = (-1)^n \dfrac{2^n n!}{(2x+1)^{n+1}}$.

$\boxed{P(1)}$

If $n = 1$ then

$$\begin{aligned} f^{(1)}(x) &= \frac{d}{dx}\frac{1}{2x+1} \\ &= \frac{d}{dx}(2x+1)^{-1} \\ &= -(2x+1)^{-2} \times 2 \\ &= -\frac{2}{(2x+1)^2} \\ &= (-1)^1 \frac{2^1 1!}{(2x+1)^2} \end{aligned}$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Let $n = k$ and assume that $P(k)$ is true so that

$$f^{(k)}(x) = (-1)^k \frac{2^k k!}{(2x+1)^{k+1}}. \qquad (1)$$

$\boxed{P(k+1)}$

Let $n = k + 1$ so that

$$
\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx}\frac{d^k}{dx^k}f(x) \\
&= \frac{d}{dx}(-1)^k\frac{2^k k!}{(2x+1)^{k+1}} \quad \text{(by (1))} \\
&= (-1)^k 2^k k! \frac{d}{dx}(2x+1)^{-(k+1)} \\
&= (-1)^k 2^k k! \times (-1)(k+1) \times (2x+1)^{-(k+2)} \times 2 \\
&= (-1)^{k+1}\frac{2^{k+1}(k+1)!}{(2x+1)^{(k+1)+1}}
\end{aligned}
$$

Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

5 **a** $P(n)$

$a_{n+1} \geq a_n$, where $n \in \mathbb{N}$.

$P(1)$

If $n = 1$ then

$$
\begin{aligned}
\text{LHS} &= a_{1+1} \\
&= a_2 \\
&= \sqrt{2 + a_1} \\
&= \sqrt{2 + \sqrt{2}} \\
&> \sqrt{2} \\
&= a_1 \\
&= \text{RHS.}
\end{aligned}
$$

Therefore $P(1)$ is true.

$P(k)$

Let $n = k$ and assume that $P(k)$ is true so that

$$a_{k+1} > a_k \qquad (1)$$

$P(k + 1)$

Let $n = k + 1$ so that

$$\begin{aligned}
\text{LHS of } P(k+1) &= a_{(k+1)+1} \\
&= a_{k+2} \\
&= \sqrt{2 + a_{k+1}} \qquad \text{(given recursion)} \\
&> \sqrt{2 + a_k} \quad \text{(by (1))} \\
&= a_{k+1} \qquad \text{(given recursion)} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$a_n < 2$, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$\begin{aligned}
\text{LHS} &= a_1 \\
&= \sqrt{2} \\
&< \sqrt{4} \\
&= 2 \\
&= \text{RHS.}
\end{aligned}$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Let $n = k$ and assume that $P(k)$ is true so that

$$a_k < 2 \qquad (1)$$

$\boxed{P(k+1)}$

Let $n = k + 1$ so that

$$\begin{aligned}
\text{LHS of } P(k+1) &= a_{k+1} \\
&= \sqrt{2 + a_k} \qquad \text{(given recursion)} \\
&< \sqrt{2 + 2} \quad \text{(by (1))} \\
&= \sqrt{4} \\
&= 2 \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 $\boxed{P(n)}$

One can find a set A consisting of n positive integers, such that the sum of the numbers is divisible by every element in A, where $n \geq 3$.

$\boxed{P(3)}$

If we let $n = 3$, then we can take $A = \{1, 2, 3\}$. Notice that the sum $1 + 2 + 3 = 6$ is divisibly by $1, 2$ and 3. Therefore $P(3)$ is true.

$\boxed{P(k)}$

Let $n = k$. If $P(k)$ is true then we have a set $A_k = \{a_1, \ldots, a_k\}$ for which

$$s = a_1 + a_2 + \cdots + a_k$$

is divisible by each of $a_1, a_2, \ldots, a_k$.

$\boxed{P(k+1)}$

Consider the set $A_{k+1} = \{a_1, \ldots, a_k, s\}$. The sum of the entries in A_{n+1} is

$$a_1 + a_2 + \cdots + a_k + s = (a_1 + a_2 + \cdots + a_k) + s = 2s.$$

This is divisible by s and also each of the elements $a_1, a_2, \ldots, a_k$. Thus we have a set with $k + 1$ elements with the sum of the entries divisible by each member. Therefore $P(k + 1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

7 **a** We will show that $x^2 + y^2 = 3$ has no rational points. Suppose to the contrary that there is a rational solution of the equation. Then there exist integers a, b, c, d such that

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3$$

with $b, d \neq 0$. We can also assume that a and b have no factors is common, and that c and d have no factors in common. Therefore

$$a^2 + c^2 = 3b^2d^2$$

This means that a and c must both be divisible by 3. To see this, if an integer m is not divisible by 3 then:

$$m = 3k + 1 \Rightarrow m^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2) + 1$$

$$m = 3k + 2 \Rightarrow m^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = (3k^2 + 4k + 1) + 1$$

In both cases, m^2 has has remainder 1 on division by 3. So if one or both of a and b is not divisible by 3, then $a^2 + c^2$ has remainder 1 or 2 on division by 3. Therefore, a

and b are both divisible by 3. But then $a = 3p$ and $c = 3q$ for integers p and q and

$$\begin{aligned} a^2 + c^2 &= 3b^2d^2 \\ (3p)^2 + (3q)^2 &= 3b^2d^2 \\ 9p^2 + 9q^2 &= 3b^2d^2 \\ 3p^2 + 3q^2 &= b^2d^2 \end{aligned}$$

Therefore the left-hand side is divisible by 3, in which case the right-hand side is divisible by 3. Therefore either b or d is divisible by 3. This is a contradiction as, we assume that a and b have no factors in common, and that c and d have not factors

b We know that the curve $x^2 + y^2 = 3$ has no rational points. The point $(\sqrt{3}, 0)$ is on the circle. Therefore $\sqrt{3}$ is irrational.

c Let $k = 2n + 1$ be odd. Suppose that (a, b) is a rational point on $x^2 + y^2 = 3^k$. Then

$$\begin{aligned} a^2 + b^2 &= 3^k \\ a^2 + b^2 &= 3^{2n+1} \\ a^2 + y^2 &= b^{2n} \cdot 3 \\ \left(\frac{a}{3^n}\right)^2 + \left(\frac{b}{3^n}\right)^2 &= 3 \end{aligned}$$

Note that (a, b) is a rational point on $x^2 + y^2 = 3^k$ if and only if $(\frac{a}{3^n}, \frac{b}{3^n})$ is a rational point on $x^2 + y^2 = 3$. From the previous question, no such points exist.

d The point $(\sqrt{3^k}, 0)$ is on the curve $x^2 + y^2 = 3^k$. When k is odd, this curve contains no rational points. Therefore $\sqrt{3^k}$ is irrational for all odd, positive integers.

Chapter 3 – Circular functions

Solutions to Exercise 3A

1 **a** The graph of $y = \operatorname{cosec}\left(x + \frac{\pi}{4}\right)$ is a translation of the graph of $y = \operatorname{cosec} x$, $\frac{\pi}{4}$ units in the negative direction of the x axis.

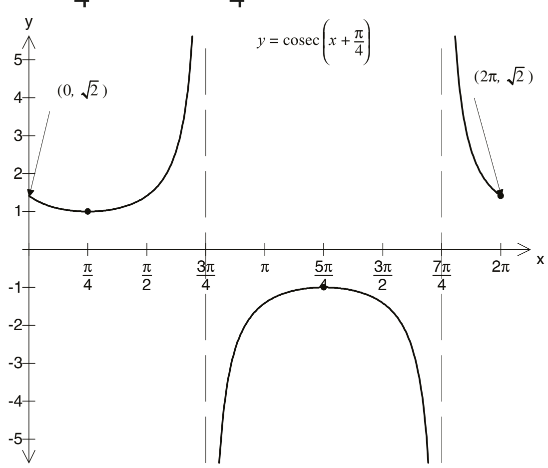
The y axis intercept is:

$\operatorname{cosec}\left(\frac{\pi}{4}\right) = \sqrt{2}$.

Asymptotes will occur when

$\sin\left(x + \frac{\pi}{4}\right) = 0$

Therefore the asymptotes are at $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.



b The graph of $y = \sec\left(x - \frac{\pi}{6}\right)$ is a translation of the graph of $y = \sec x$, $\frac{\pi}{6}$ units in the positive direction of the x axis.

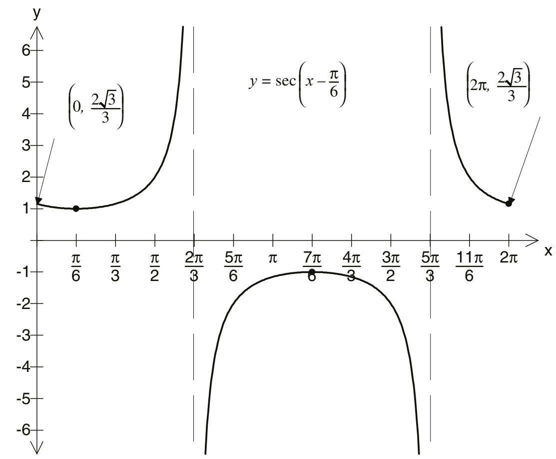
The y axis intercept is

$\sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$.

Asymptotes will occur when

$\cos\left(x - \frac{\pi}{6}\right) = 0$

Therefore the asymptotes are at $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.



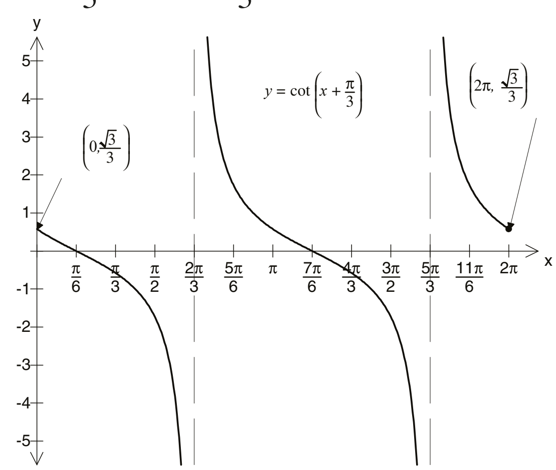
c The graph of $y = \cot\left(x + \frac{\pi}{3}\right)$ is a translation of the graph of $y = \cot x$, $\frac{\pi}{3}$ units in the negative direction of the x axis.

The y axis intercept is $\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$.

Asymptotes will occur when

$\tan\left(x + \frac{\pi}{3}\right) = 0$

Therefore the asymptotes are at $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.



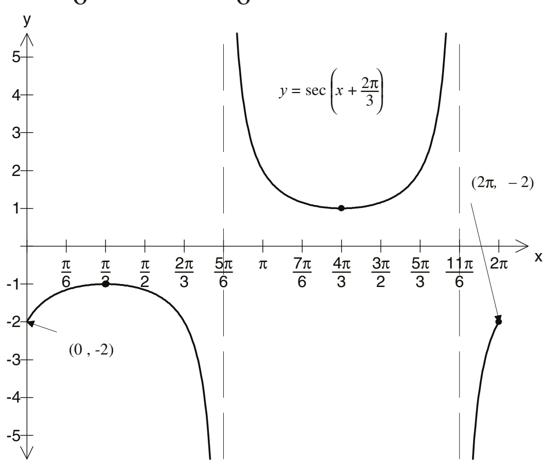
d The graph of $y = \sec\left(x + \dfrac{2\pi}{3}\right)$ is a translation of the graph of $y = \sec x$, $\dfrac{2\pi}{3}$ units in the negative direction of the x axis.

The y axis intercept is $\sec \dfrac{2\pi}{3} = -2$.

Asymptotes will occur when

$\cos\left(x + \dfrac{2\pi}{3}\right) = 0$

Therefore the asymptotes are at $x = \dfrac{5\pi}{6}$ and $x = \dfrac{11\pi}{6}$.



e The graph of $y = \operatorname{cosec}\left(x - \dfrac{\pi}{2}\right)$ is a translation of the graph of $y = \operatorname{cosec} x$, $\dfrac{\pi}{2}$ units in the positive direction of the x axis.

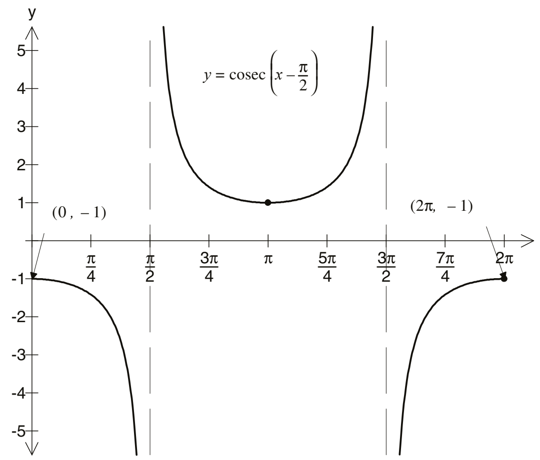
The y axis intercept is

$\operatorname{cosec}\left(-\dfrac{\pi}{2}\right) = -1$.

Asymptotes will occur when

$\sin\left(x - \dfrac{\pi}{2}\right) = 0$

Therefore the asymptotes are at $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$.



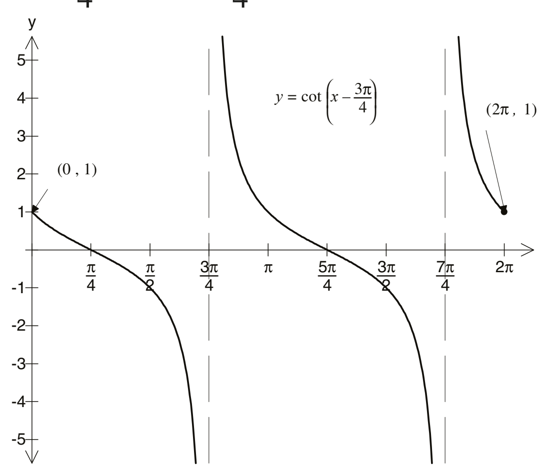
f The graph of $y = \cot\left(x - \dfrac{3\pi}{4}\right)$ is a translation of the graph of $y = \cot x$, $\dfrac{3\pi}{4}$ units in the positive direction of the x axis.

The y axis intercept is $\cot\left(-\dfrac{3\pi}{4}\right) = 1$.

Asymptotes will occur when

$\tan\left(x - \dfrac{3\pi}{4}\right) = 0$

Therefore the asymptotes are at $x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$.



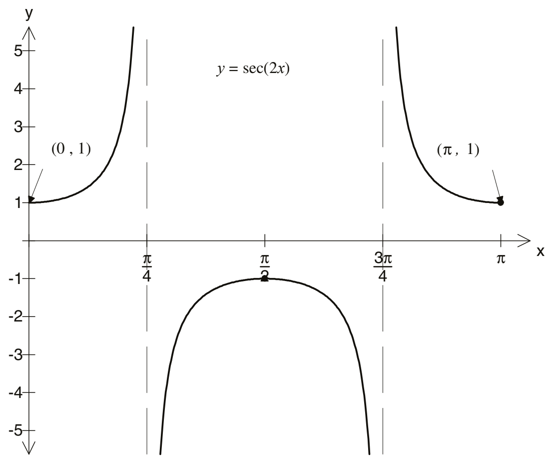
2 a The graph of $y = \sec 2x$ is obtained from the graph of $y = \sec x$ by a dilation of factor $\dfrac{1}{2}$ from the y axis.

The y axis intercept is $\sec(0) = 1$.

Asymptotes will occur when

$\cos(2x) = 0$

Therefore the asymptotes are at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

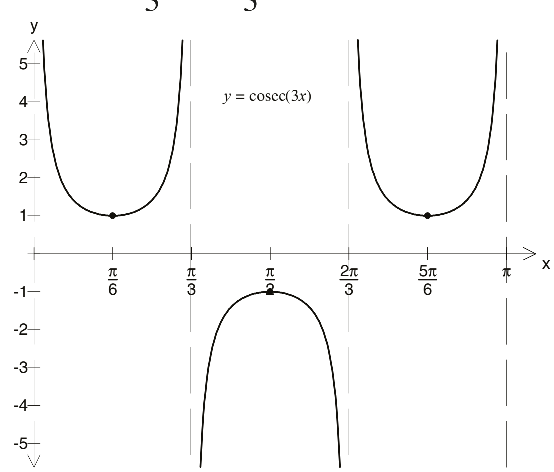


b The graph of $y = \operatorname{cosec}(3x)$ is obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{3}$ from the y axis.

The y axis intercept is cosec(0) which is undefined.

Asymptotes will occur when $\sin(3x) = 0$

Therefore the asymptotes are at $x = 0, x = \frac{\pi}{3}, x = \frac{2\pi}{3}$ and $x = \pi$.

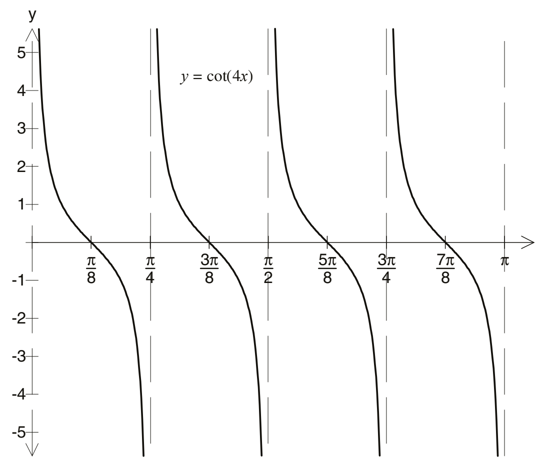


c The graph of $y = \cot(4x)$ is obtained from the graph of $y = \cot x$ by a dilation of factor $\frac{1}{4}$ from the y axis.

The y axis intercept is cot(0) which is undefined.

Asymptotes will occur when $\tan(4x) = 0$

Therefore the asymptotes are at $x = 0, x = \frac{\pi}{4}, x = \frac{\pi}{2}, x = \frac{3\pi}{4}$ and $x = \pi$.



d The graph of $y = \operatorname{cosec}\left(2x + \frac{\pi}{2}\right)$

$$= \operatorname{cosec}\left(2\left(x + \frac{\pi}{4}\right)\right)$$

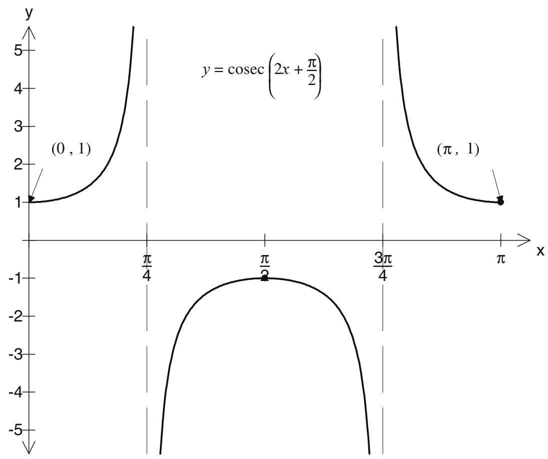
is obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the y axis followed by a translation $\frac{\pi}{4}$ units in the negative direction of the x axis.

The y axis intercept is cosec $\frac{\pi}{2} = 1$.

Asymptotes will occur when $\sin\left(2x + \frac{\pi}{2}\right) = 0$

Therefore the asymptotes are at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

Note: $\operatorname{cosec}\left(2x + \frac{\pi}{2}\right) = \sec 2x$



e The graph of $y = \sec(2x + \pi)$

$$= \sec\left(2\left(x + \frac{\pi}{2}\right)\right)$$

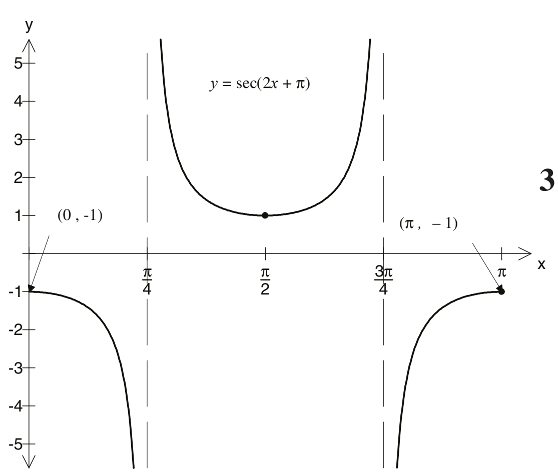
is obtained from the graph of $y = \sec x$ by a dilation of factor $\frac{1}{2}$ from the y axis followed by a translation $\frac{\pi}{2}$ units in the negative direction of the x axis.

The y axis intercept is $\sec(\pi) = -1$.

Asymptotes will occur when $\cos(2x + \pi) = 0$

Therefore the asymptotes are at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

Note: $\sec(2x + \pi) = -\sec 2x$



f The graph of $y = \cot\left(2x - \frac{\pi}{3}\right)$

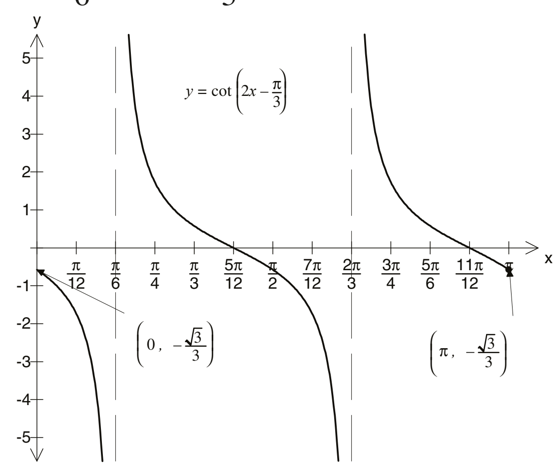
$$= \cot\left(2\left(x - \frac{\pi}{6}\right)\right)$$

is obtained from the graph of $y = \cot x$ by a dilation of factor $\frac{1}{2}$ from the y axis followed by a translation $\frac{\pi}{6}$ units in the positive direction of the x axis.

The y axis intercept is $\cot\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{3}$.

Asymptotes will occur when $\tan\left(2x - \frac{\pi}{3}\right) = 0$

Therefore the asymptotes are at $x = \frac{\pi}{6}$ and $x = \frac{2\pi}{3}$.



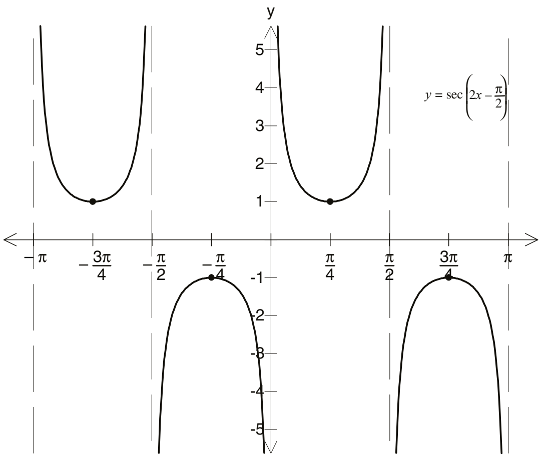
3 a The graph of $y = \sec\left(2x - \frac{\pi}{2}\right)$

$$= \sec\left(2\left(x - \frac{\pi}{4}\right)\right)$$

is obtained from the graph of $y = \sec x$ by a dilation of factor $\frac{1}{2}$ from the y axis followed by a translation $\frac{\pi}{4}$ units in the positive direction of the x axis.

The y axis intercept is $\sec\left(-\frac{\pi}{2}\right)$, which is undefined.
Asymptotes will occur when
$\cos\left(2x - \frac{\pi}{2}\right) = 0$
Therefore the asymptotes are at
$x = -\pi, x = -\frac{\pi}{2}, x = 0, x = \frac{\pi}{2}$ and $x = \pi$.

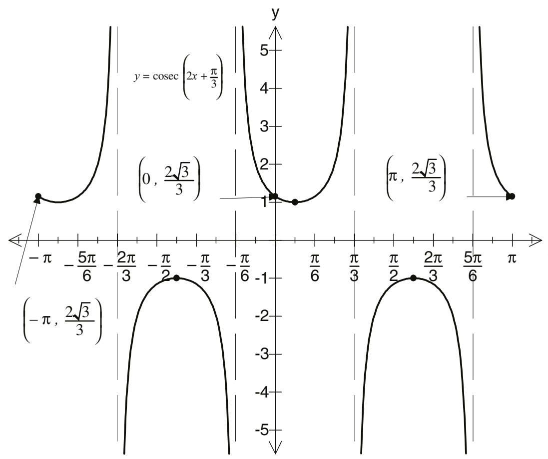
Note: $\sec\left(2x - \frac{\pi}{2}\right) = \text{cosec}(2x)$



b The graph of $y = \text{cosec}\left(2x + \frac{\pi}{3}\right)$

$= \text{cosec}\left(2\left(x + \frac{\pi}{6}\right)\right)$

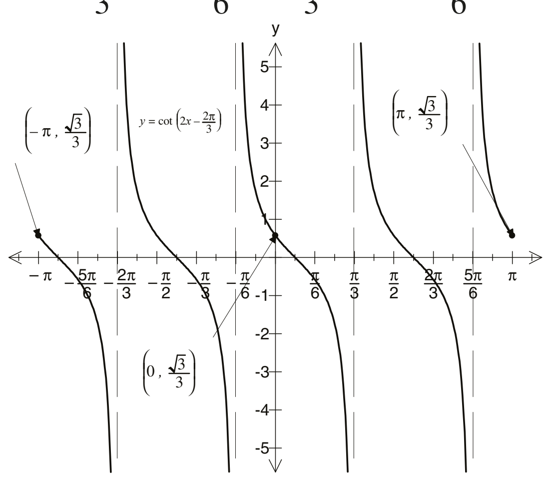
is obtained from the graph of $y = \text{cosec } x$ by a dilation of factor $\frac{1}{2}$ from the y axis followed by a translation $\frac{\pi}{6}$ units in the negative direction of the x axis.
The y axis intercept is
$\text{cosec } \frac{\pi}{3} = \frac{2\sqrt{3}}{3}$.
Asymptotes will occur when
$\sin\left(2x + \frac{\pi}{3}\right) = 0$
Therefore the asymptotes are at
$x = -\frac{2\pi}{3}, x = -\frac{\pi}{6}, x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$.



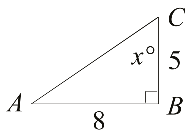
c The graph of $y = \cot\left(2x - \frac{2\pi}{3}\right)$

$= \cot\left(2\left(x - \frac{\pi}{3}\right)\right)$

is obtained from the graph of $y = \cot x$ by a dilation of factor $\frac{1}{2}$ from the y axis followed by a translation $\frac{\pi}{3}$ units in the positive direction of the x axis.
The y axis intercept is
$\cot\left(-\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{3}$.
Asymptotes will occur when
$\tan\left(2x - \frac{2\pi}{3}\right) = 0$
Therefore the asymptotes are at
$x = -\frac{2\pi}{3}, x = -\frac{\pi}{6}, x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$.



4 a



By Pythagoras' theorem,

$AC^2 = AB^2 + BC^2$

$\therefore \quad AC^2 = 25 + 64 = 89$

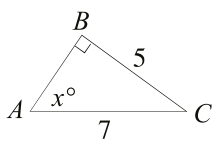
$\therefore \quad AC = \sqrt{89}$

So,

$\cot x° = \frac{5}{8}$, $\sec x° = \frac{\sqrt{89}}{5}$,

$\operatorname{cosec} x° = \frac{\sqrt{89}}{8}$

b



By Pythagoras' theorem,

$AC^2 = AB^2 + BC^2$

$\therefore \quad AB^2 = AC^2 - BC^2$

$\therefore \quad AB^2 = 49 - 25 = 24$

$\therefore \quad AB = \sqrt{24}$

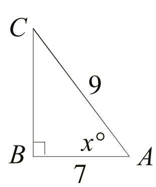
$\therefore \quad AB = 2\sqrt{6}$

So,

$\cot x° = \frac{2\sqrt{6}}{5}$,

$\sec x° = \frac{7\sqrt{6}}{12}$, $\operatorname{cosec} x° = \frac{7}{5}$

c



By Pythagoras' theorem,

$AC^2 = AB^2 + BC^2$

$\therefore \quad BC^2 = AC^2 - AB^2$

$\therefore \quad BC^2 = 81 - 49 = 32$

$\therefore \quad BC = \sqrt{32}$

$\therefore \quad BC = 4\sqrt{2}$

So,

$\cot x° = \frac{7\sqrt{2}}{8}$,

$\sec x° = \frac{9}{7}$, $\operatorname{cosec} x° = \frac{9\sqrt{2}}{8}$

5 a $\sin \frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)$

$= \sin \frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$

b $\cos \frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right)$

$= -\cos \frac{\pi}{4}$

$= -\frac{\sqrt{2}}{2}$

c $\tan\left(-\frac{\pi}{4}\right) = -\tan \frac{\pi}{4} = -1$

d $\operatorname{cosec} \frac{\pi}{6} = \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{0.5} = 2$

e $\sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$

f $\cot\left(\frac{-\pi}{6}\right) = -\frac{1}{\tan \frac{\pi}{6}} = -\frac{1}{\frac{\sqrt{3}}{3}} = -\sqrt{3}$

g $\sin \frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right)$

$= -\sin \frac{\pi}{4}$

$= -\frac{\sqrt{2}}{2}$

h $\tan\frac{5\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right)$

$= -\tan\left(\frac{\pi}{6}\right)$

$= -\frac{\sqrt{3}}{3}$

i $\sec\left(-\frac{\pi}{3}\right) = \frac{1}{\cos\frac{-\pi}{3}}$

$= \frac{1}{\cos\frac{\pi}{3}}$

$= \frac{1}{0.5}$

$= 2$

j $\operatorname{cosec}\frac{3\pi}{4} = \frac{1}{\sin\frac{3\pi}{4}}$

$= \frac{1}{\sin\left(\pi - \frac{\pi}{4}\right)}$

$= \frac{1}{\sin\frac{\pi}{4}}$

$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)}$

$= \sqrt{2}$

k $\cot\frac{9\pi}{4} = \frac{1}{\tan\frac{9\pi}{4}}$

$= \frac{1}{\tan\left(2\pi + \frac{\pi}{4}\right)}$

$= \frac{1}{\tan\frac{\pi}{4}}$

$= \frac{1}{1}$

$= 1$

l $\cos\left(-\frac{7\pi}{3}\right) = \cos\left(-2\pi - \frac{\pi}{3}\right)$

$= \cos\left(-\frac{\pi}{3}\right)$

$= \cos\left(\frac{\pi}{3}\right)$

$= \frac{1}{2}$

6 a $\sec^2 x - \tan^2 x = \frac{1}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}$

$= \frac{1 - \sin^2 x}{\cos^2 x}$

$= \frac{\cos^2 x}{\cos^2 x}$

$= 1$

or

$\sec^2 x - \tan^2 x = 1 + \tan^2 x - \tan^2 x$

$= 1$

b $\cot^2 x - \text{cosec}^2 x = \dfrac{\cos^2 x}{\sin^2 x} - \dfrac{1}{\sin^2 x}$

$= -\dfrac{1-\cos^2 x}{\sin^2 x}$

$= -\dfrac{\sin^2 x}{\sin^2 x}$

$= -1$

or

$\cot^2 x - \text{cosec}^2 x$

$= \cot^2 x - (1 + \cot^2 x)$

$= -1$

c $\dfrac{\tan^2 x + 1}{\tan^2 x} = \dfrac{\sec^2 x}{\tan^2 x}$

$= \dfrac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}$

$= \dfrac{1}{\sin^2 x}$

$= \text{cosec}^2 x$

d $\dfrac{\sin^2 x}{\cos x} + \cos x = \dfrac{\cos^2 x + \sin^2 x}{\cos x}$

$= \dfrac{1}{\cos x}$

$= \sec x$

e $\sin^4 x - \cos^4 x$

$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$

$= \sin^2 x - \cos^2 x$

$= -\cos 2x$

f $\tan^3 x + \tan x = \tan x(1 + \tan^2 x)$

$= \tan x \sec^2 x$

7 $\tan x = -4, x \in \left[-\dfrac{\pi}{2}, 0\right]$

a $\sec^2 x = 1 + \tan^2 x$

$\sec^2 x = 1 + 16 = 17$

$\sec x = \pm\sqrt{17}$

$\sec x = \sqrt{17}$ as $x \in \left[-\dfrac{\pi}{2}, 0\right]$

b $\cos x = \dfrac{1}{\sec x}$

$\cos x = \dfrac{1}{\sqrt{17}} = \dfrac{\sqrt{17}}{17}$

c $\text{cosec}^2 x = 1 + \cot^2 x$

$= 1 + \dfrac{1}{\tan^2 x}$

$= 1 + \dfrac{1}{16}$

$\text{cosec } x = \pm\dfrac{\sqrt{17}}{4}$

$\text{cosec } x = \dfrac{-\sqrt{17}}{4}$ as $x \in \left[-\dfrac{\pi}{2}, 0\right]$

8 $\cot x = 3, x \in \left[\pi, \dfrac{3\pi}{2}\right]$

a $\text{cosec}^2 x = 1 + \cot^2 x$

$\therefore \text{cosec } x = \pm\sqrt{1+3^2} = \pm\sqrt{10}$

$\therefore \text{cosec } x = -\sqrt{10}$ as $x \in \left[\pi, \dfrac{3\pi}{2}\right]$

b $\sin x = \dfrac{1}{\text{cosec } x}$

$= -\dfrac{\sqrt{10}}{10}$

c $\sec^2 x = 1 + \tan^2 x = 1 + \dfrac{1}{\cot^2 x}$

$\therefore \quad \sec x = \pm\sqrt{1 + \left(\dfrac{1}{3}\right)^2} = \pm\dfrac{\sqrt{10}}{3}$

$\therefore \quad \sec x = -\dfrac{\sqrt{10}}{3}$ as $x \in \left[\pi, \dfrac{3\pi}{2}\right]$

9 $\sec x = 10, x \in \left[-\dfrac{\pi}{2}, 0\right]$

a $\tan^2 x = \sec^2 x - 1$

$\therefore \quad \tan x = \pm\sqrt{10^2 - 1} = \pm 3\sqrt{11}$

$\therefore \quad \tan x = -3\sqrt{11}$ as $x \in \left[-\dfrac{\pi}{2}, 0\right]$

b $\sin x = \tan x \div \sec x$

$\therefore \quad \sin x = \dfrac{-3\sqrt{11}}{10}$

10 $\operatorname{cosec} x = -6, x \in \left[\dfrac{3\pi}{2}, 2\pi\right]$

a $\cot^2 x = \operatorname{cosec}^2 x - 1$

$\therefore \quad \cot x = \pm\sqrt{(-6)^2 - 1} = \pm\sqrt{35}$

$\therefore \quad \cot x = -\sqrt{35}$ as $x \in \left[\dfrac{3\pi}{2}, 2\pi\right]$

b $\cos x = \cot x \div \operatorname{cosec} x = \dfrac{\sqrt{35}}{6}$

11 $\sin x^\circ = 0.5, 90 < x < 180$

a $\cos x^\circ = -\sqrt{1 - (0.5)^2} = -\dfrac{\sqrt{3}}{2}$

b $\cot x^\circ = \dfrac{\cos x^\circ}{\sin x^\circ}$

$= -\dfrac{\sqrt{3}}{2} \div \dfrac{1}{2}$

$= -\sqrt{3}$

c $\operatorname{cosec} x^\circ = \dfrac{1}{\sin x^\circ} = \dfrac{1}{0.5} = 2$

12 $\operatorname{cosec} x^\circ = -3, 180 < x < 270$

a $\sin x^\circ = \dfrac{1}{\operatorname{cosec} x^\circ} = -\dfrac{1}{3}$

b $\cos x^\circ = -\sqrt{1 - \left(\dfrac{1}{3}\right)^2}$

$= -\dfrac{\sqrt{8}}{3}$

$= -\dfrac{2\sqrt{2}}{3}$

c $\sec x^\circ = \dfrac{1}{\cos x^\circ} = -\dfrac{3}{\sqrt{8}} = -\dfrac{3\sqrt{2}}{4}$

13 $\cos x^\circ = -0.7, 0 < x < 180$

a $\sin x^\circ = \sqrt{1 - (0.7)^2}$

$= \sqrt{0.51}$

$= \dfrac{\sqrt{51}}{10}$

b $\tan x^\circ = \dfrac{\sin x^\circ}{\cos x^\circ} = \dfrac{\sqrt{0.51}}{-0.7} = -\dfrac{\sqrt{51}}{7}$

c $\cot x^\circ = \dfrac{1}{\tan x^\circ} = -\dfrac{7}{\sqrt{51}} = -\dfrac{7\sqrt{51}}{51}$

14 $\sec x^\circ = 5, 180 < x < 360$

a $\cos x^\circ = \dfrac{1}{\sec x^\circ} = \dfrac{1}{5} = 0.2$

b $\sin x^\circ = -\sqrt{1-\left(\frac{1}{5}\right)^2}$

$= -\frac{\sqrt{24}}{5}$

$= -\frac{2\sqrt{6}}{5}$

c $\cot x^\circ = \frac{1}{5} \div -\frac{2\sqrt{6}}{5} = -\frac{1}{2\sqrt{6}} = -\frac{\sqrt{6}}{12}$

d $\frac{\sec^2\theta - \operatorname{cosec}^2\theta}{\tan^2\theta - \cot^2\theta}$

$= \frac{(1+\tan^2\theta)-(1+\cot^2\theta)}{\tan^2\theta - \cot^2\theta}$

$= \frac{\tan^2\theta - \cot^2\theta}{\tan^2\theta - \cot^2\theta}$

$= 1$

15 a $\sec^2\theta + \operatorname{cosec}^2\theta - \sec^2\theta\operatorname{cosec}^2\theta$

$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} - \frac{1}{\cos^2\theta\sin^2\theta}$

$= \frac{\sin^2\theta + \cos^2\theta - 1}{\cos^2\theta\sin^2\theta}$

$= \frac{1-1}{\cos^2\theta\sin^2\theta}$

$= 0$

b $(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta)$

$= \left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{1}{\sin\theta} - \sin\theta\right)$

$= \left(\frac{1-\cos^2\theta}{\cos\theta}\right)\left(\frac{1-\sin^2\theta}{\sin\theta}\right)$

$= \left(\frac{\sin^2\theta}{\cos\theta}\right)\left(\frac{\cos^2\theta}{\sin\theta}\right)$

$= \sin\theta\cos\theta$

$= \frac{1}{2}\sin 2\theta$

(using a double angle formula)

c $(1-\cos^2\theta)(1+\cot^2\theta)$

$= (\sin^2\theta)(\operatorname{cosec}^2\theta)$

$= (\sin^2\theta)\left(\frac{1}{\sin^2\theta}\right)$

$= 1$

16 If $x = \sec\theta - \tan\theta$

$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$

$= \frac{1-\sin\theta}{\cos\theta}$

then $\frac{1}{x} = \frac{\cos\theta}{1-\sin\theta}$

and

$x + \frac{1}{x} = \frac{1-\sin\theta}{\cos\theta} + \frac{\cos\theta}{1-\sin\theta}$

$= \frac{(1-\sin\theta)^2 + \cos^2\theta}{\cos\theta(1-\sin\theta)}$

$= \frac{1 - 2\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1-\sin\theta)}$

$= \frac{1-2\sin\theta+1}{\cos\theta(1-\sin\theta)}$

$= \frac{2(1-\sin\theta)}{\cos\theta(1-\sin\theta)}$

$= \frac{2}{\cos\theta}$

$= 2\sec\theta$, as required to prove.

$$
\begin{aligned}
x - \frac{1}{x} &= \frac{1 - \sin\theta}{\cos\theta} - \frac{\cos\theta}{1 - \sin\theta} \\
&= \frac{(1 - \sin\theta)^2 - \cos^2\theta}{\cos\theta(1 - \sin\theta)} \\
&= \frac{(\sin^2\theta + \cos^2\theta) - 2\sin\theta + \sin^2\theta - \cos^2\theta}{\cos\theta(1 - \sin\theta)} \\
&= \frac{2\sin^2\theta - 2\sin\theta}{\cos\theta(1 - \sin\theta)} \\
&= \frac{-2\sin\theta(1 - \sin\theta)}{\cos\theta(1 - \sin\theta)} \\
&= \frac{-2\sin\theta}{\cos\theta} \\
&= -2\tan\theta
\end{aligned}
$$

$$\therefore\ x - \frac{1}{x} = -2\tan\theta$$

Solutions to Exercise 3B

1 **a** $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{2}(\sqrt{3}-1)}{4}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

$= \dfrac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$

$= \dfrac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}$

$= \frac{(3+\sqrt{3})^2}{6}$

$= 2 + \sqrt{3}$

c $\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{6\pi}{12} + \frac{\pi}{12}\right)$

$= \cos\left(\frac{\pi}{2} + \frac{\pi}{12}\right)$

$= -\sin\frac{\pi}{12}$

$= -\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ (see **a**)

$= \frac{\sqrt{2}(1-\sqrt{3})}{4}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

d $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$

$= \frac{\sqrt{3}-1}{1+\sqrt{3}}$

$= \frac{(\sqrt{3}-1)(1-\sqrt{3})}{-2}$

$= \frac{2\sqrt{3}-4}{-2}$

$= 2 - \sqrt{3}$

2 **a** $\sin(2x - 5y) = \sin 2x\cos 5y - \cos 2x\sin 5y$

b $\cos(x^2 + y) = \cos(x^2)\cos y - \sin(x^2)\sin y$

c

$\tan(x + (y + z))$

$= \frac{\tan x + \tan(y+z)}{1 - \tan x\tan(y+z)}$

$= \dfrac{\tan x + \frac{\tan y + \tan z}{1 - \tan y\tan z}}{1 - \tan x \times \frac{\tan y + \tan z}{1 - \tan y\tan z}}$

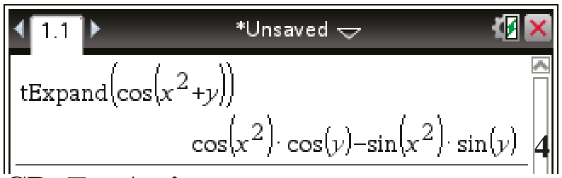
$= \dfrac{\frac{\tan x(1 - \tan y\tan z)}{1 - \tan y\tan z} + \frac{\tan y\tan z}{1 - \tan y\tan z}}{\frac{1 - \tan y\tan z}{1 - \tan y\tan z} - \frac{\tan x(\tan y + \tan z)}{1 - \tan y\tan z}}$

$= \frac{\tan x - \tan x\tan y\tan z + \tan y + \tan z}{1 - \tan y\tan z - \tan x\tan y - \tan x\tan z}$

$= \frac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1 - \tan y\tan z - \tan x\tan y - \tan x\tan z}$

A CAS calculator has the capability to expand and collect some

trigonometric equations.
TI: Press
Menu → 3 : Algebra →
B : Trigonometry → 1 : Expand
For part **b**.
Type tExpand(cos($x^2 + y$))



CP: Tap **Action →**
Transformation → tExpand
and complete the command as per the TI instructions.

3 **a** $\sin x \cos 2y - \cos x \sin 2y = \sin(x - 2y)$

b

$$\cos 3x \cos 2x + \sin 3x \sin 2x = \cos(3x - 2x)$$
$$= \cos x$$

c

$$\frac{\tan A - \tan(A - B)}{1 + \tan A \tan(A - B)} = \tan(A - (A - B))$$
$$= \tan B$$

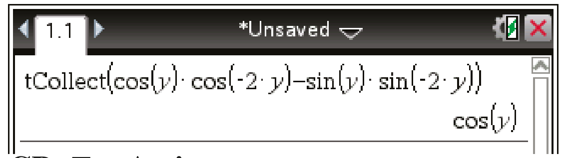
d

$$\sin(A + B)\cos(A - B) + \cos(A + B)\sin(A - B)$$
$$= \sin((A + B) + (A - B))$$
$$= \sin 2A$$

e $\cos y \cos(-2y) - \sin y \sin(-2y)$

$$= \cos(y + (-2y))$$
$$= \cos(-y)$$
$$= \cos y$$

For part **e**.
TI: Press
Menu → 3 : Algebra →
B : Trigonometry → 2 : Collect



CP: Tap **Action →**
Transformation → tCollect
and complete the command as per the TI instructions.

4 **a** $\sin(x+2x) = \sin x \cos 2x + \cos x \sin 2x$

b
$$\begin{aligned}\sin(3x) &= \sin x \cos 2x + \cos x \sin 2x \\ &= \sin x(\cos^2 x - \sin^2 x) \\ &\quad + \cos x \times 2 \sin x \cos x \\ &= \sin x \cos^2 x - \sin^3 x \\ &\quad + 2 \sin x \cos^2 x \\ &= 3 \sin x \cos^2 x - \sin^3 x \\ &= 3 \sin x(1 - \sin^2 x) - \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x\end{aligned}$$

5 **a**
$$\begin{aligned}\cos(x + 2x) &= \cos x \cos 2x \\ &\quad - \sin x \sin 2x\end{aligned}$$

b
$$\begin{aligned}\cos(3x) &= \cos x(\cos^2 x - \sin^2 x) \\ &\quad - 2 \sin^2 x \cos x \\ &= \cos^3 x - 3 \sin^2 x \cos x \\ &= \cos^3 x - 3(1 - \cos^2 x)\cos x \\ &= 4\cos^3 x - 3 \cos x\end{aligned}$$

6 $\sin x = 0.6, x \in \left[\frac{\pi}{2}, \pi\right]$ and

$\tan y = 2.4, y \in \left[0, \frac{\pi}{2}\right]$

a $\cos^2 x = 1 - \sin^2 x$

$\cos x = \pm\sqrt{1 - 0.6^2}$

$= \pm 0.8$

$\therefore \quad \cos x = -0.8$ as $x \in \left[\frac{\pi}{2}, \pi\right]$

b $\sec^2 y = 1 + \tan^2 y$

$\sec y = \pm\sqrt{1 + 2.4^2}$

$= \pm 2.6$

$\therefore \quad \sec y = 2.6$ as $y \in \left[0, \frac{\pi}{2}\right]$

c $\cos y = \frac{1}{\sec y}$

$= \frac{1}{2.6}$

$= \frac{10}{26}$

$= \frac{5}{13}$

d $\sin^2 y = 1 - \cos^2 y$

$\sin y = \pm\sqrt{1 - \left(\frac{5}{13}\right)^2}$

$= \pm\frac{12}{13}$

$\therefore \quad \sin y = \frac{12}{13}$ as $y \in \left[0, \frac{\pi}{2}\right]$

e $\tan^2 x = \sec^2 x - 1$

$= \frac{1}{\cos^2 x} - 1$

$\tan x = \pm\sqrt{\left(\frac{1}{0.8}\right)^2 - 1}$

$= \pm\frac{0.6}{0.8}$

$= \pm\frac{3}{4}$

$= \pm 0.75$

$\therefore \tan x = -0.75$ as $x \in \left[\frac{\pi}{2}, \pi\right]$

f $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$= -\frac{4}{5} \times \frac{5}{13} + \frac{3}{5} \times \frac{12}{13}$

$= -\frac{20}{65} + \frac{36}{65}$

$= \frac{16}{65}$

g $\sin(x - y) = \pm\sqrt{1 - \left(\frac{16}{65}\right)^2}$

$= \pm\frac{63}{65}$

As $\frac{\pi}{2} \le x \le \pi$ and $0 \le y \le \frac{\pi}{2}$,

$0 \le x - y \le \pi$

$\therefore \sin(x - y) = \frac{63}{65}$

or

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$= \frac{3}{5} \times \frac{5}{13} - \left(-\frac{4}{5}\right) \times \frac{12}{13}$

$= \frac{15}{65} + \frac{48}{65}$

$= \frac{63}{65}$

h $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$$= \frac{\frac{-3}{4} + \frac{12}{5}}{1 + \frac{3}{4} \times \frac{12}{5}}$$

$$= \frac{33}{20} \times \frac{5}{14}$$

$$= \frac{33}{56}$$

i $\tan(x+2y) = \tan((x+y)+y)$

$$= \frac{\frac{33}{56} + \frac{12}{5}}{1 - \frac{33}{56} \times \frac{12}{5}}$$

$$= \frac{837}{280} \times -\frac{70}{29}$$

$$= -\frac{837}{116}$$

7 $\cos x = -0.7, x \in \left[\pi, \frac{3\pi}{2}\right]$ and

$\sin y = 0.4, y \in \left[0, \frac{\pi}{2}\right]$

a $\sin x = \pm\sqrt{1 - \cos^2 x}$

$$= \pm\sqrt{1 - (-0.7)^2}$$

$$= \pm\sqrt{0.51}$$

$$= \pm\frac{\sqrt{51}}{10}$$

$\therefore \quad \sin x = -\dfrac{\sqrt{51}}{10} = -0.71$

as $x \in \left[\pi, \frac{3\pi}{2}\right]$

b $\cos y = \pm\sqrt{1 - \sin^2 y}$

$$= \pm\sqrt{1 - (0.4)^2}$$

$$= \pm\sqrt{0.84}$$

$$= \pm\frac{2\sqrt{21}}{10}$$

$\therefore \quad \cos y = \dfrac{\sqrt{21}}{5} = 0.92$ as $x \in \left[0, \frac{\pi}{2}\right]$

c $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

$$= \frac{\frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}}{1 + \frac{\sin x}{\cos x} \times \frac{\sin y}{\cos y}}$$

$$= \frac{\frac{\sqrt{51}}{7} - \frac{2}{\sqrt{21}}}{1 + \frac{\sqrt{51}}{7} \times \frac{2}{21}}$$

$$= \frac{3\sqrt{51} - 2\sqrt{21}}{21}$$

$$\times \frac{49}{49 + 2\sqrt{119}}$$

$$= \frac{21\sqrt{51} - 14\sqrt{21}}{147 + 6\sqrt{119}}$$

$$= 0.40$$

d $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$$= \frac{-7}{10} \times \frac{\sqrt{21}}{5} + \frac{\sqrt{51}}{10} \times \frac{2}{5}$$

$$= \frac{-7\sqrt{21}}{50} + \frac{2\sqrt{51}}{50}$$

$$= -0.36$$

8 **a** $\dfrac{1}{2}\sin x \cos x = \dfrac{1}{4}(2\sin x \ \cos x)$

$$= \frac{1}{4}\sin 2x$$

b $\sin^2 x - \cos^2 x = -(\cos^2 x - \sin^2 x)$

$= -\cos 2x$

c $\dfrac{\tan x}{1-\tan^2 x} = \dfrac{1}{2}\dfrac{2\tan x}{1-\tan^2 x}$

$= \dfrac{1}{2}\tan 2x$

d $\dfrac{\sin^4 x - \cos^4 x}{\cos 2x}$

$= \dfrac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)}{\cos 2x}$

$= \dfrac{(-\cos 2x)(1)}{\cos 2x}$

$= -1$

e $\dfrac{4\sin^3 x - 2\sin x}{\cos x \cos 2x}$

$= \dfrac{2\sin x(2\sin^2 x - 1)}{\cos x \cos 2x}$

$= 2\tan x \cdot \dfrac{-\cos 2x}{\cos 2x}$

$= -2\tan x$

f $\dfrac{4\sin^2 x - 4\sin^4 x}{\sin 2x}$

$= \dfrac{4\sin^2 x(1-\sin^2 x)}{2\sin x \cos x}$

$= \dfrac{2\sin x \cos^2 x}{\cos x}$

$= 2\sin x \cos x$

$= \sin 2x$

9 $\sin x = -0.8,\ x \in \left[\pi, \dfrac{3\pi}{2}\right]$

a

$\sin 2x = 2\sin x \cos x$

$\sin x = -0.8$

$\cos x = \pm\sqrt{1-(0.8)^2}$

$= \pm 0.6$

$\therefore \quad \cos x = -0.6$ as $x \in \left[\pi, \dfrac{3\pi}{2}\right]$

$\therefore \quad \sin 2x = 2 \times (-0.8) \times (-0.6) = 0.96$

b $\cos 2x = 1 - 2\sin^2 x$

$= 1 - 2(-0.8)^2$

$= -0.28$

c $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$

$= \dfrac{0.96}{-0.28}$

$= -\dfrac{24}{7}$

10 $\tan x = 3,\ x \in \left[0, \dfrac{\pi}{2}\right]$

a $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$= \dfrac{6}{1-9}$

$= -\dfrac{6}{8}$

$= -\dfrac{3}{4}$

b $\tan 3x = \tan(2x + x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$= \frac{-\frac{3}{4} + 3}{1 - \left(-\frac{3}{4}\right) \times 3}$$

$$= \frac{9}{13}$$

11 $\sin x = -0.75, x \in \left[\pi, \frac{3\pi}{2}\right]$

a $\cos x = \pm\sqrt{1 - \sin^2 x}$

$= \pm\sqrt{1 - (-0.75)^2}$

$= \pm\frac{\sqrt{7}}{4}$

$= \pm 0.66$ (correct to two decimal places)

$\therefore \cos x = -0.66$ as $x \in \left[\pi, \frac{3\pi}{2}\right]$

b $\cos 2x = 1 - 2\sin^2 x$

$\therefore \cos x = 1 - 2\sin^2 \frac{x}{2}$

$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$

$\sin \frac{x}{2} = \pm\sqrt{\frac{4 + \sqrt{7}}{8}}$

$= \pm 0.91$

as $x \in \left[\pi, \frac{3\pi}{2}\right] \Rightarrow \frac{x}{2} \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$

$\therefore \sin \frac{x}{2} = 0.91$

12 $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Put $x = \frac{\pi}{8}$

$\therefore \tan \frac{\pi}{4} = \frac{2\tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}}$

$1 - \tan^2 \frac{\pi}{8} = 2\tan \frac{\pi}{8}$

Let $\tan \frac{\pi}{8} = x$,

$\therefore 1 - x^2 = 2x$

$x^2 + 2x - 1 = 0$

$x = \frac{-2 \pm \sqrt{4 + 4}}{2}$

$= -1 \pm \sqrt{2}$

as $\frac{\pi}{8} < \frac{\pi}{2}$,

$x = \sqrt{2} - 1$

$\therefore \tan \frac{\pi}{8} = \sqrt{2} - 1$

13 $\cos x = 0.9, x \in \left[0, \frac{\pi}{2}\right]$

Since $\cos 2x = 2\cos^2 x - 1$ then

$\cos x = 2\cos^2 \frac{x}{2} - 1$

$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$

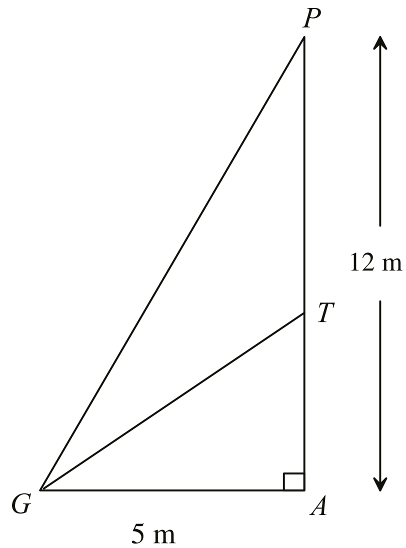
$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + 0.9}{2}}$

$= \pm 0.97$

as $x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \frac{x}{2} \in \left[0, \frac{\pi}{4}\right]$

$\therefore \cos \frac{x}{2} = 0.97$

14



Since $\therefore \angle AGT = \angle TGP = x$

$\therefore \angle AGP = 2x$

a $\tan 2x = \dfrac{\text{opp}}{\text{adj}} = \dfrac{12}{5}$

b $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Let $\tan x = z$

$$\therefore \quad \frac{12}{5} = \frac{2z}{1 - z^2}$$

$$\therefore \quad 6 - 6z^2 = 5z$$

$$\therefore \quad 6z^2 + 5z - 6 = 0$$

$$\therefore \quad z = \frac{-5 \pm \sqrt{25 + 144}}{12}$$

$$= \frac{-5 \pm \sqrt{169}}{12}$$

as $x \in \left(0, \dfrac{\pi}{2}\right)$,

$$z = \frac{-5 + 13}{12} = \frac{8}{12} = \frac{2}{3}$$

$$\therefore \tan x = \frac{2}{3}$$

c $AT = GA \tan x$

$$= 5 \times \frac{2}{3}$$

$$= \frac{10}{3}$$

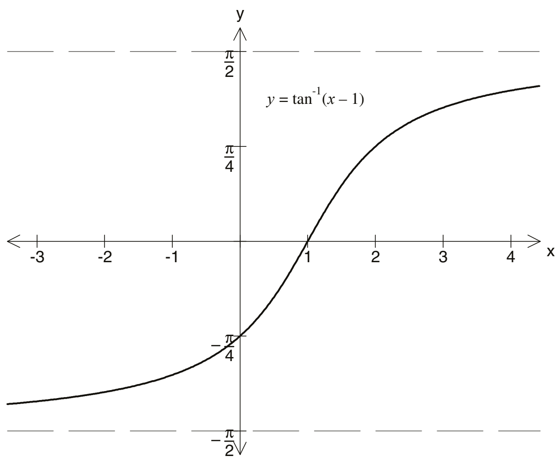
$$= 3\frac{1}{3}$$

Therefore the length of AT is $3\frac{1}{3}$ metres.

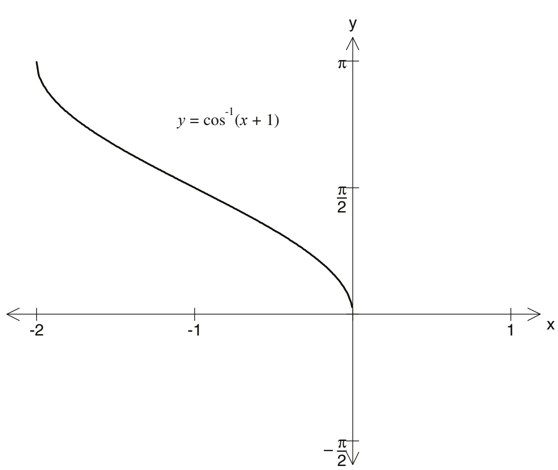
Solutions to Exercise 3C

1 a The graph of $y = \tan^{-1}(x - 1)$ is a translation of the graph of $y = \tan^{-1}(x)$, one unit in the positive direction of the x axis. The x axis intercept is at 1, the y axis intercept is at $\tan^{-1}(-1) = -\frac{\pi}{4}$, the asymptotes remain the same: $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

The range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the domain is R.



b The graph of $y = \cos^{-1}(x + 1)$ is a translation of the graph of $y = \cos^{-1}(x)$ one unit in the negative direction of the x axis. The domain is $[-2, 0]$, the range is $[0, \pi]$



c The graph of $y = 2\sin^{-1}\left(x + \frac{1}{2}\right)$ is a dilation of factor 2 from the x axis of the graph of $y = \sin^{-1}\left(x + \frac{1}{2}\right)$.

That is why the range of the function $y = 2\sin^{-1}\left(x + \frac{1}{2}\right)$ is

$\left[2 \times \left(-\frac{\pi}{2}\right), 2 \times \frac{\pi}{2}\right] = [-\pi, \pi]$.

The graph of $y = \sin^{-1}\left(x + \frac{1}{2}\right)$ is a translation of the graph of $y = \sin^{-1}(x)$, $\frac{1}{2}$ unit in the negative direction of the x axis. Therefore the domain of the function $y = 2\sin^{-1}\left(x + \frac{1}{2}\right)$ is

$\left[-1 - \frac{1}{2}, 1 - \frac{1}{2}\right] = \left[-\frac{3}{2}, \frac{1}{2}\right]$.

When $x = -\frac{3}{2}$, $y = 2\sin^{-1}\left(-\frac{3}{2} + \frac{1}{2}\right)$

$= 2\sin^{-1}(-1)$

$= 2 \times -\frac{\pi}{2}$

$= -\pi$

When $x = \frac{1}{2}$, $y = 2\sin^{-1}\left(\frac{1}{2} + \frac{1}{2}\right)$

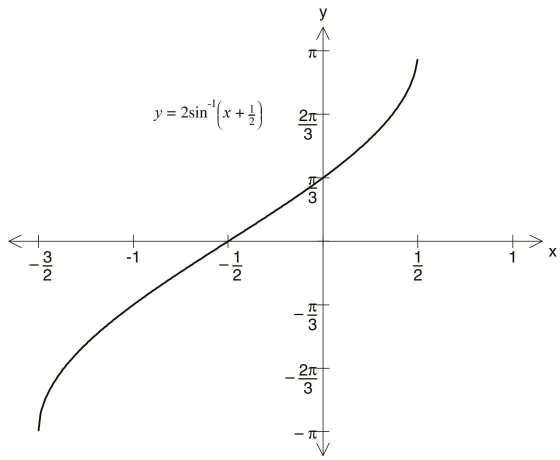
$= 2\sin^{-1}(1)$

$= 2 \times \frac{\pi}{2}$

$= \pi$

x axis intercept is $x = -\frac{1}{2}$

$$y \text{ axis intercept is } y = 2\sin^{-1}\left(\frac{1}{2}\right) = 2 \times \frac{\pi}{6} = \frac{\pi}{3}$$



d The graph of $y = 2\tan^{-1}(x) + \frac{\pi}{2}$ is obtained from the graph of $y = \tan^{-1}(x)$, by a dilation of factor 2 from the x axis followed by a translation of $\frac{\pi}{2}$ units in the positive direction of the y axis. Therefore the domain of the function $y = 2\tan^{-1}(x) + \frac{\pi}{2}$ is R, and the range is $\left(2 \times -\frac{\pi}{2} + \frac{\pi}{2}, 2 \times \frac{\pi}{2} + \frac{\pi}{2}\right) = \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

The asymptotes are at $y = -\frac{\pi}{2}$ and $y = \frac{3\pi}{2}$

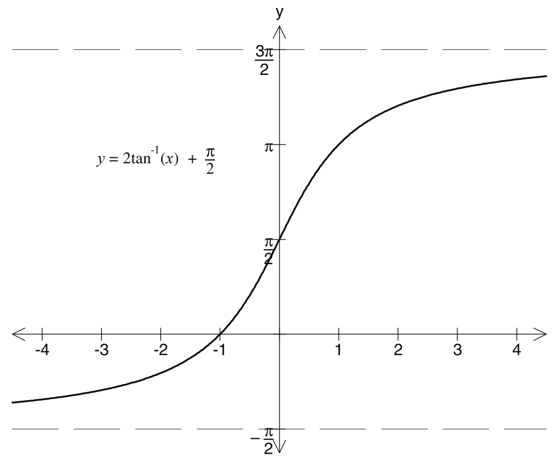
y axis intercept is $2\tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$

x axis intercept can be found from the equation

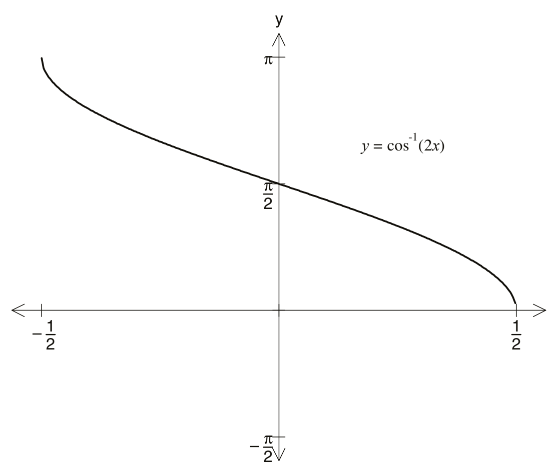
$$2\tan^{-1}(x) = -\frac{\pi}{2}$$

$$\therefore \tan^{-1}(x) = -\frac{\pi}{4}$$

$$\therefore x = \tan\left(-\frac{\pi}{4}\right) = -1$$



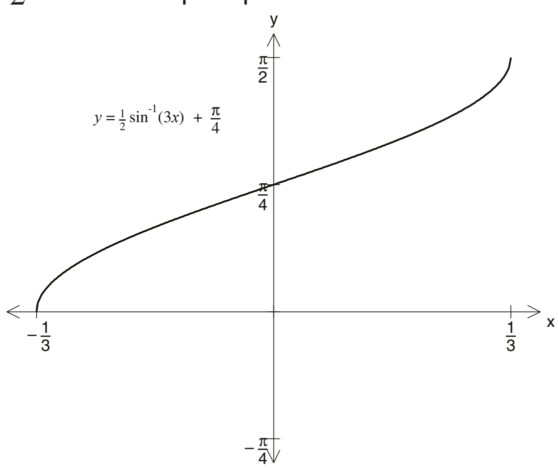
e The graph of $y = \cos^{-1}(2x)$ is obtained from the graph of $y = \cos^{-1}(x)$ by a dilation of factor $\frac{1}{2}$ from the y axis.
The domain of the function $y = \cos^{-1}(2x)$ is $\left[-1 \times \frac{1}{2}, 1 \times \frac{1}{2}\right] = \left[-\frac{1}{2}, \frac{1}{2}\right]$.
The range is $[0, \pi]$.



f The graph of $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$ is a consequence of a dilation of the graph of $y = \sin^{-1}(x)$ of factor $\frac{1}{3}$ from the y axis, then a dilation of $y = \sin^{-1}(3x)$ of factor $\frac{1}{2}$ from the x axis and then a translation $\frac{\pi}{4}$ units in the positive direction of the y axis. Therefore the domain of the function $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$ is $\left[-\frac{1}{3}, \frac{1}{3}\right]$ and the range is

$$\left[-\frac{\pi}{2} \times \frac{1}{2} + \frac{\pi}{4}, \frac{\pi}{2} \times \frac{1}{2} + \frac{\pi}{4}\right] = \left[0, \frac{\pi}{2}\right].$$

The y axis intercept is at $\frac{1}{2}\sin^{-1}(0) + \frac{\pi}{4} = \frac{\pi}{4}$.



2 a Evaluating $\sin^{-1} 1$ is equivalent to solving the equation $\sin x = 1$.

$$\sin\frac{\pi}{2} = 1$$

$$\therefore \sin^{-1} 1 = \frac{\pi}{2}$$

b $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ because $\sin\frac{\pi}{4}$

$$= \frac{\sqrt{2}}{2}$$

$$\therefore \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

c $\sin^{-1} 0.5 = \frac{\pi}{6}$ because $\sin\frac{\pi}{6} = 0.5$

d Evaluating $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is equivalent to solving the equation $\cos x = -\frac{\sqrt{3}}{2}$.

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \cos\left(\pi - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

e $\cos^{-1} 0.5 = \frac{\pi}{3}$ because $\cos\frac{\pi}{3} = 0.5$

f $\tan^{-1} 1 = \frac{\pi}{4}$ because $\tan\frac{\pi}{4} = 1$

g $\tan^{-1}(-\sqrt{3}) = -\tan\sqrt{3} = -\frac{\pi}{3}$ because $\tan\frac{\pi}{3} = \sqrt{3}$

h $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$ because $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

i $\cos^{-1}(-1) = \pi - \cos^{-1} 1 = \pi - 0 = \pi$

3 a $\sin(\cos^{-1} 0.5) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b
$$\sin^{-1}\left(\cos\frac{5\pi}{6}\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{5\pi}{6}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$$

$$= -\frac{\pi}{3}$$

c $\tan\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right) = \tan\left(-\frac{\pi}{4}\right) = -1$

d $\cos(\tan^{-1} 1) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

e $\tan^{-1}\left(\sin\dfrac{5\pi}{2}\right) = \tan^{-1}\left(\sin\left(2\pi + \dfrac{\pi}{2}\right)\right)$

$= \tan^{-1} 1$

$= \dfrac{\pi}{4}$

f $\tan(\cos^{-1} 0.5) = \tan\dfrac{\pi}{3} = \sqrt{3}$

g $\cos^{-1}\left(\cos\dfrac{7\pi}{3}\right) = \cos^{-1}\left(\cos\left(2\pi + \dfrac{\pi}{3}\right)\right)$

$= \cos^{-1}\left(\cos\dfrac{\pi}{3}\right)$

$= \dfrac{\pi}{3}$

h $\sin^{-1}\left(\sin\dfrac{-2\pi}{3}\right) = \sin^{-1}\left(\sin\left(-\pi + \dfrac{\pi}{3}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\dfrac{\pi}{3}\right)\right)$

$= -\dfrac{\pi}{3}$

i $\tan^{-1}\left(\tan\dfrac{11\pi}{4}\right) = \tan^{-1}\left(\tan\left(3\pi - \dfrac{\pi}{4}\right)\right)$

$= \tan^{-1}\left(\tan\left(-\dfrac{\pi}{4}\right)\right)$

$= -\dfrac{\pi}{4}$

j $\cos^{-1}\left(\sin\left(-\dfrac{\pi}{3}\right)\right) = \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

$= \pi - \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

$= \pi - \dfrac{\pi}{6}$

$= \dfrac{5\pi}{6}$

k $\cos^{-1}\left(\tan\left(-\dfrac{\pi}{4}\right)\right) = \cos^{-1}(-1) = \pi$

l $\sin^{-1}\left(\cos\dfrac{-3\pi}{4}\right) = \sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = -\dfrac{\pi}{4}$

4 $f: \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \to R, f(x) = \sin x$

a The range of $f(x) = \sin x$ is $[-1, 1]$
$\therefore$ the domain of f^{-1} is $[-1, 1]$
The range of f^{-1} is $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ as it is a given domain of $f(x)$.
$\therefore f^{-1}(x) = \pi - \sin^{-1}(x)$

b **i** $f\left(\dfrac{\pi}{2}\right) = 1$

ii $f\left(\dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

iii $f\left(\dfrac{7\pi}{6}\right) = -\dfrac{1}{2}$

iv $f^{-1}(-1) = \pi - \sin^{-1}(-1)$

$= \pi - \left(-\dfrac{\pi}{2}\right)$

$= \dfrac{3\pi}{2}$

v $f^{-1}(0) = \pi - \sin^{-1}(0) = \pi$

vi $f^{-1}(0.5) = \pi - \sin^{-1}(0.5)$

$= \pi - \dfrac{\pi}{6}$

$= \dfrac{5\pi}{6}$

5 **a** The domain of $\sin^{-1}(x)$ is $[-1, 1]$

$\therefore \quad -1 \le 2 - x \le 1$

$-3 \le -x \le -1$

$1 \le x \le 3$

$\therefore$ the domain of $\sin^{-1}(2-x)$ is [1, 3]

The range is unchanged at $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b The domain of $\sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad -\frac{\pi}{2} \le x + \frac{\pi}{4} \le \frac{\pi}{2}$

$\therefore \quad -\frac{3\pi}{4} \le x \le \frac{\pi}{4}$

Therefore the domain of $\sin\left(x + \frac{\pi}{4}\right)$ is

$\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

The range is unchanged at [−1, 1].

c As in **a**, the domain of $\sin^{-1}(2x+4)$ can be defined from the inequality

$-1 \le 2x + 4 \le 1$

$-5 \le 2x \le -3$

$-\frac{5}{2} \le x \le -\frac{3}{2}$

The domain of $\sin^{-1}(2x+4)$ is $\left[-\frac{5}{2}, -\frac{3}{2}\right]$, the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

d As in **b**, the domain of $\sin\left(3x - \frac{\pi}{3}\right)$ can be defined from the inequality

$-\frac{\pi}{2} \le 3x - \frac{\pi}{3} \le \frac{\pi}{2}$

$-\frac{\pi}{2} + \frac{\pi}{3} \le 3x \le \frac{\pi}{2} + \frac{\pi}{3}$

$-\frac{\pi}{6} \le 3x \le \frac{5\pi}{6}$

$-\frac{\pi}{18} \le x \le \frac{5\pi}{18}$

So the domain of $\sin\left(3x - \frac{\pi}{3}\right)$ is

$\left[-\frac{\pi}{18}, \frac{5\pi}{18}\right]$, the range is [−1, 1].

e The domain of $\cos x$ is $[0, \pi]$

$\therefore 0 \le x - \frac{\pi}{6} \le \pi$

$\frac{\pi}{6} \le x \le \frac{7\pi}{6}$

$\therefore$ the domain of $\cos\left(x - \frac{\pi}{6}\right)$ is

$\left[\frac{\pi}{6}, \frac{7\pi}{6}\right]$, the range is [−1, 1].

f The domain of $\cos^{-1}(x)$ is [−1, 1]

$\therefore -1 \le x + 1 \le 1$

$-2 \le x \le 0$

$\therefore$ the domain of $\cos^{-1}(x+1)$ is [−2, 0] The range is unchanged at $[0, \pi]$.

g As in **f**, $-1 \le x^2 \le 1$

$\therefore -1 \le x \le 1$

$\therefore$ the domain of $\cos^{-1}(x^2)$ is [−1, 1]

However, when $x \in [-1, 1]$, $x^2 \in [0, 1]$, so the range of $\cos^{-1}(x^2)$ is $\left[0, \frac{\pi}{2}\right]$.

h As in **e**, $0 \le 2x + \frac{2\pi}{3} \le \pi$

$-\frac{2\pi}{3} \le 2x \le \frac{\pi}{3}$

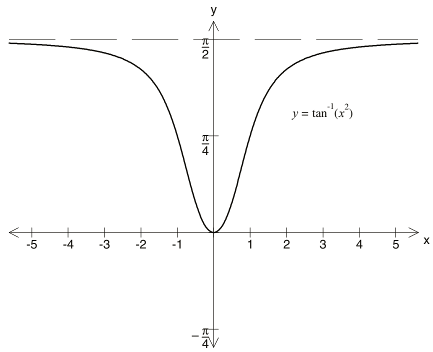
$-\frac{\pi}{3} \le x \le \frac{\pi}{6}$

$\therefore$ the domain of $\cos\left(2x + \frac{2\pi}{3}\right)$ is

$\left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$, the range is [−1, 1].

i The domain of $\tan^{-1}(x)$ is R, so the domain of $\tan^{-1}(x^2)$ is also R.

However when $x \in R$, $x^2 \in R^+ \cup \{0\}$, therefore the range of $\tan^{-1}(x^2)$ is $\left[0, \frac{\pi}{2}\right)$



j The domain of $\tan(x)$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

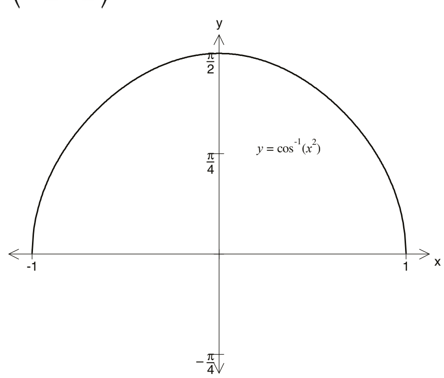
$$\therefore \quad -\frac{\pi}{2} < 2x - \frac{\pi}{2} < \frac{\pi}{2}$$

$$0 < 2x < \pi$$

$$0 < x < \frac{\pi}{2}$$

$\therefore$ the domain of $\tan\left(2x - \dfrac{\pi}{2}\right)$ is $\left(0, \dfrac{\pi}{2}\right)$, the range is R.

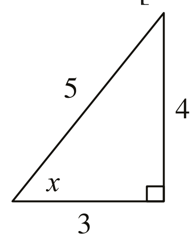
k Both the domain and the range of $\tan^{-1}(2x + 1)$ are the same as those of $\tan^{-1}(x)$: the domain is R, the range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$



l The domain of $\tan x$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, so the domain of $\tan x^2$ is $\left(-\sqrt{\dfrac{\pi}{2}}, \sqrt{\dfrac{\pi}{2}}\right)$.

At the same time $x^2 \in \left[0, \dfrac{\pi}{2}\right)$, therefore the range of $\tan x^2$ is $R^+ \cup \{0\}$.

6 a $\sin^{-1} \dfrac{4}{5} \in \left[0, \dfrac{\pi}{2}\right]$

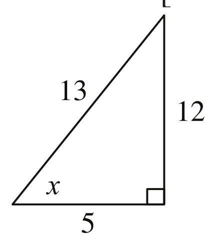


Using a trigonometirc ratio, $\sin x = \dfrac{4}{5}$

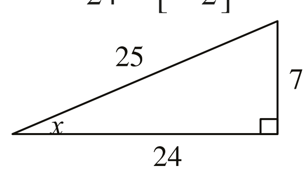
$$\Rightarrow \quad x = \sin^{-1} \frac{4}{5}$$

$$\therefore \quad \cos\left(\sin^{-1} \frac{4}{5}\right) = \cos(x) = \frac{3}{5}$$

b $\cos^{-1} \dfrac{5}{13} \in \left[0, \dfrac{\pi}{2}\right]$



c $\tan^{-1} \dfrac{7}{24} \in \left[0, \dfrac{\pi}{2}\right]$



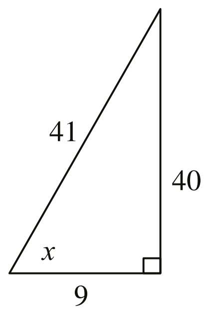
Using a trigonometirc ratio,

$\tan x = \dfrac{7}{24}$

$$\Rightarrow \quad x = \tan^{-1} \frac{7}{24}$$

$$\therefore \quad \cos\left(\tan^{-1} \frac{7}{24}\right) = \cos(x) = \frac{24}{25}$$

d $\sin^{-1} \dfrac{40}{41} \in \left[0, \dfrac{\pi}{2}\right]$



Using a trigonometirc ratio,

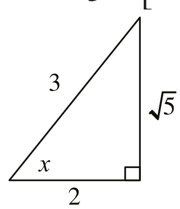
$\sin x = \dfrac{40}{41}$

$\Rightarrow \quad x = \sin^{-1}\dfrac{40}{41}$

$\therefore \quad \tan\left(\sin^{-1}\dfrac{40}{41}\right) = \tan(x) = \dfrac{40}{9}$

e $\tan\left(\cos^{-1}\dfrac{1}{2}\right) = \tan\dfrac{\pi}{3} = \sqrt{3}$

f $\cos^{-1}\dfrac{2}{3} \in \left[0, \dfrac{\pi}{2}\right]$



Using a trigonometirc ratio,

$\cos x = \dfrac{5}{13}$

$\Rightarrow \quad x = \cos^{-1}\dfrac{5}{13}$

$\therefore \quad \tan\left(\cos^{-1}\dfrac{5}{13}\right) = \tan(x) = \dfrac{12}{5}$

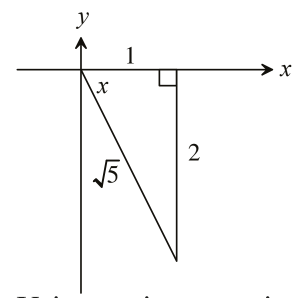
Using a trigonometirc ratio,

$\cos x = \dfrac{2}{3}$

$\Rightarrow \quad x = \cos^{-1}\dfrac{2}{3}$

$\therefore \quad \sin\left(\cos^{-1}\dfrac{2}{3}\right) = \sin(x) = \dfrac{\sqrt{5}}{3}$

g $\tan^{-1}(-2) \in \left[-\dfrac{\pi}{2}, 0\right]$



Using a trigonometirc ratio,

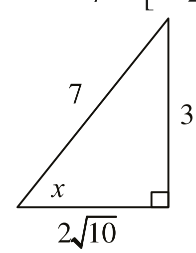
$\tan x = \dfrac{-2}{1}$

$\Rightarrow \quad x = \tan^{-1}\dfrac{-2}{1}$

$\therefore \quad \sin(\tan^{-1}(-2)) = \sin(x) = \dfrac{-2}{\sqrt{5}}$

$= \dfrac{-2\sqrt{5}}{5}$

h $\sin^{-1}\dfrac{3}{7} \in \left[0, \dfrac{\pi}{2}\right]$

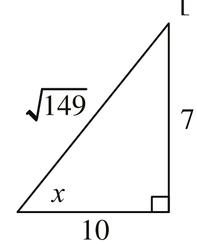


Using a trigonometirc ratio, $\sin x = \dfrac{3}{7}$

$\Rightarrow \quad x = \sin^{-1}\dfrac{3}{7}$

$\therefore \quad \cos\left(\sin^{-1}\dfrac{3}{7}\right) = \cos(x) = \dfrac{2\sqrt{10}}{7}$

i $\tan^{-1}(0.7) \in \left[-\dfrac{\pi}{2}, 0\right]$



Using a trigonometirc ratio,

$\tan x = \dfrac{7}{10}$

$$\Rightarrow \qquad x = \tan^{-1}\frac{7}{10}$$

$$\therefore \quad \sin(\tan^{-1} 0.7) = \sin(x) = \frac{7}{\sqrt{149}}$$

$$= \frac{7\sqrt{149}}{149}$$

7 $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{5}{13}, \alpha \in \left[0, \frac{\pi}{2}\right]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$

a **i** $\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \frac{9}{25}}$

$$= \frac{4}{5}$$

ii $\cos\beta = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$

b **i** To prove the equality we have to prove that $\sin(\alpha - \beta) = \frac{16}{65}$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13}$$

$$= \frac{36 - 20}{65}$$

$$= \frac{16}{65}$$

ii As in **i**,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}$$

$$= \frac{48 - 15}{65}$$

$$= \frac{33}{65}$$

8 **a** The domain of $\cos x$ is $[0, \pi]$ and the range is $[-1, 1]$. As the domain of $\sin^{-1}(x)$ is $[-1, 1]$, the range of the composite function is the same as it is for $\sin^{-1}(x)$.

$\therefore$ the domain of $\sin^{-1}(\cos x)$ is $[0, \pi]$, the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b The domain of $\sin^{-1}(x)$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. As the domain of $\cos(x)$ is $[0, \pi]$, in this composite function it is only $\left[0, \frac{\pi}{2}\right] = [0, \pi] \cap \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So the domain of the composite function is $[0, 1]$ and the range is $[0, 1]$.

c The domain of $\sin 2x$ is $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and the range is $[-1, 1]$. As the domain of $\cos^{-1}$ is $[-1, 1]$, the range of the composite function is the same as it is for $\cos^{-1}(x)$.

$\therefore$ the domain of $\cos^{-1}(\sin 2x)$ is $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, the range is $[0, \pi]$.

d $\sin(-\cos^{-1}(x)) = -\sin(\cos^{-1} x)$

The domain of $\cos^{-1}(x)$ is $[-1, 1]$, the range is $[0, \pi]$. The domain of $\sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore the domain of the composite function is $[0, 1]$ and the range is $[-1, 0]$.

e $\cos(2\sin^{-1}(x))$

The domain of $\sin^{-1}(x)$ is $[-1.1]$, the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So the range of $2\sin^{-1}(x)$ is $[-\pi, \pi]$. However, the domain of $\cos(x)$ is $[0, \pi]$. Therefore the domain of the composite function is [0, 1] and the range is [−1, 1].

f The domain of $\cos x$ is $[0, \pi]$ and the range is $[-1, 1]$. Since the domain of $\tan^{-1}(x)$ is R, the domain of the composite function is $[0, \pi]$ and the range is $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

g The domain of $\tan^{-1}(x)$ is R and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. However, the domain of $\cos x$ is $[0, \pi]$. Therefore the domain of the composite function is $R^+ \cup \{0\}$ and the range is $(0, 1]$.

h Since the range of $\tan^{-1}(x)$ is the same as the domain of $\sin x$ excluding the points $\left\{-\frac{\pi}{2}\right\}$ and $\left\{\frac{\pi}{2}\right\}$, the domain of the composite function is R and the range is $(-1, 1)$.

9 a $\tan\left(\tan^{-1} 3 - \tan^{-1} \frac{1}{2}\right)$

$$= \frac{\tan(\tan^{-1}3) - \tan\left(\tan^{-1}\frac{1}{2}\right)}{1 + \tan(\tan^{-1}3) \times \tan\left(\tan^{-1}\frac{1}{2}\right)}$$

$$= \frac{3 - \frac{1}{2}}{1 + 3 \times \frac{1}{2}}$$

$$= \frac{2.5}{2.5}$$

$$= 1$$

Since $\tan \frac{\pi}{4}$ is 1 and tan is a 1–1 function, the equality is proven.

b As in **a**,

$$\frac{x - \frac{x-1}{x+1}}{1 + \frac{x(x-1)}{x+1}} = \frac{x^2 + x - x + 1}{x + 1 + x^2 - x} = 1$$

10 a $\sin^{-1}(-0.5) = -\frac{\pi}{6}$
However, the domain of $\cos x$ is $[0, \pi]$, so $\cos\left(-\frac{\pi}{6}\right)$ does not exist.

b $\cos^{-1}(-0.2) \in \left(\frac{\pi}{2}, \pi\right) \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
So $\sin(\cos^{-1}(-0.2))$ does not exist.

c $\tan^{-1}(-1) = -\frac{\pi}{4} \notin [0, \pi]$.
So $\cos(\tan^{-1}(-1))$ does not exist.

11 a $[-1, 1]$

b The range of $\cos^{-1}(x)$ is $[0, \pi]$ and $\sin x \geq 0$ for $x \in [0, \pi]$

c $\cos^2(\cos^{-1}(x)) + \sin^2(\cos^{-1} x) = 1$

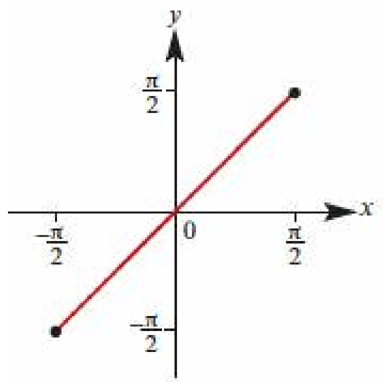
$x^2 + [f(x)]^2 = 1$

$[f(x)]^2 = 1 - x^2$

$f(x) = \sqrt{1 - x^2}$
since $f(x) \geq 0, x \in [-1, 1]$

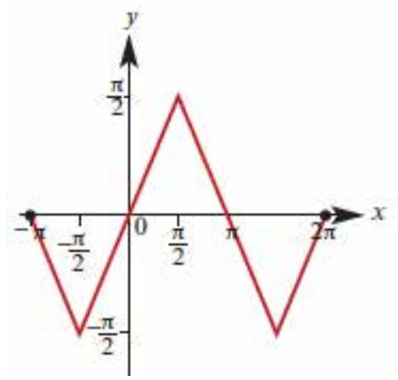
12 a Maximal domain $=\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Range = $[-1, 1]$

b



y
x
$\frac{\pi}{2}$
$-\frac{\pi}{2}$
0
$-\frac{\pi}{2}$
$\frac{\pi}{2}$

c



y
x
$\frac{\pi}{2}$
$-\frac{\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
2π

Solutions to Exercise 3D

1 **a** $\operatorname{cosec} x = -2$

$\therefore \quad \sin x = -\frac{1}{2}$

$\therefore \quad x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

b $\operatorname{cosec}\left(x - \frac{\pi}{4}\right) = -2$

$\therefore \quad \sin\left(x - \frac{\pi}{4}\right) = -\frac{1}{2}$

Now $x \in [0, 2\pi]$

$\therefore \quad x - \frac{\pi}{4} \in \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$

$\therefore \quad x - \frac{\pi}{4} = -\frac{\pi}{6}$ or $\frac{7\pi}{6}$

$\therefore \quad x = \frac{\pi}{12}$ or $\frac{17\pi}{12}$

c $3 \sec x = 2\sqrt{3}$

$\therefore \quad \sec x = \frac{2\sqrt{3}}{3}$

$\therefore \quad \cos x = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$

$\therefore \quad x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$

d $\operatorname{cosec}(2x) + 1 = 2$

$\therefore \quad \operatorname{cosec}(2x) = 1$

$\therefore \quad \sin(2x) = 1$

Now $x \in [0, 2\pi]$

$\therefore \quad 2x \in [0, 4\pi]$

$\therefore \quad 2x = \frac{\pi}{2}$ or $\frac{5\pi}{2}$

$\therefore \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

e $\cot x = -\sqrt{3}$

$\therefore \quad \tan x = -\frac{1}{\sqrt{3}}$

$\therefore \quad x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$

f $\cot\left(2x - \frac{\pi}{3}\right) = -1$

$\therefore \quad \tan\left(2x - \frac{\pi}{3}\right) = -1$

Now $x \in [0, 2\pi]$

$\therefore \quad 2x \in [0, 4\pi]$

$\therefore \quad 2x - \frac{\pi}{3} \in \left[-\frac{\pi}{3}, \frac{11\pi}{3}\right]$

$\therefore \quad 2x - \frac{\pi}{3} = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$ or $\frac{11\pi}{4}$

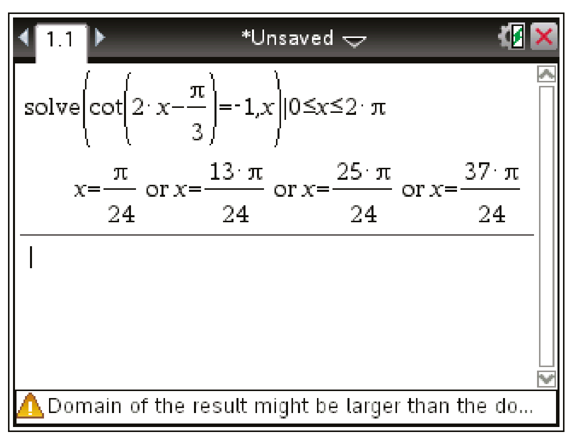
$\therefore \quad 2x = \frac{\pi}{12}$ or $\frac{13\pi}{12}$ or $\frac{25\pi}{12}$ or $\frac{37\pi}{12}$

$\therefore \quad x = \frac{\pi}{24}$ or $\frac{13\pi}{24}$ or $\frac{25\pi}{24}$ or $\frac{37\pi}{24}$

for part **f**.

CAS: type

$\mathbf{solve\left(\cot\left(2x - \frac{\pi}{3}\right) = -1, x\right) \Big| 0 \le x \le 2\pi}$



2 a $\sin x = 0.5, x \in [0, 2\pi]$

$\therefore \quad x = \frac{\pi}{6} \text{ or } \pi - \frac{\pi}{6}$

$\therefore \quad x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$

b $\cos x = \frac{-\sqrt{3}}{2}, x \in [0, 2\pi]$

$\therefore \quad x = \pi - \frac{\pi}{6} \text{ or } \pi + \frac{\pi}{6}$

$\therefore \quad x = \frac{5\pi}{6} \text{ or } \frac{7\pi}{6}$

c $\tan x = \sqrt{3}, x \in [0, 2\pi]$

$\therefore \quad x = \frac{\pi}{3} \text{ or } \pi + \frac{\pi}{3}$

$\therefore \quad x = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$

d $\cot x = -1, x \in [0, 2\pi]$

$\therefore \quad \frac{1}{\tan x} = -1$

$\therefore \quad \tan x = -1$

$\therefore \quad x = \pi - \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$

$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$

e $\sec x = 2, x \in [0, 2\pi]$

$\therefore \quad \frac{1}{\cos x} = 2$

$\therefore \quad \cos x = \frac{1}{2}$

$\therefore \quad x = \frac{\pi}{3} \text{ or } 2\pi - \frac{\pi}{3}$

$\therefore \quad x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$

f $\operatorname{cosec} x = -\sqrt{2}, x \in [0, 2\pi]$

$\therefore \quad \frac{1}{\sin x} = -\sqrt{2}$

$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$

$\therefore \quad x = \pi + \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$

$\therefore \quad x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$

3 a In the interval $[0, 2\pi]$, there are two solutions to $\sin x = \frac{\sqrt{2}}{2}$, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. The period of $\sin x$ is 2π, $\therefore$ the solutions of the equation are $x = (-1)^n \frac{\pi}{4} + \pi n, n \in Z$

Alternatively,

$x = \frac{\pi}{4} + 2n\pi$

or $x = \frac{3\pi}{4} + 2n\pi, n \in Z$

b In the interval $[0, 2\pi]$, $\sec x = 1$ when $x = 0$. The period of $\cos x$ is 2π. Therefore the solutions of the equation are $x = 2\pi n, n \in Z$.

c In the interval $[0, \pi]$, $\cot x = \sqrt{3}$ when $x = \frac{\pi}{6}$. The period of $\cot x$ is π. Therefore the solutions of the equation are $x = \frac{\pi}{6} + \pi n, n \in Z$.

d $x = \frac{(12n-5)\pi}{12}$ or $x = \frac{(4n+1)\pi}{4}$, $n \in \mathbb{Z}$

e $x = \frac{(2n-1)\pi}{3}$ or $x = \frac{2(3n+1)\pi}{9}$, $n \in \mathbb{Z}$

f $x = \frac{2n\pi}{3}$ or $x = \frac{(6n+1)\pi}{9}$, $n \in \mathbb{Z}$

g $x = \dfrac{(3n-2)\pi}{6}, n \in \mathbb{Z}$

h $x = \dfrac{n\pi}{2}, n \in \mathbb{Z}$

i $x = \dfrac{(8n-5)\pi}{8}, n \in \mathbb{Z}$

4 a $\sec x = 2.5$

$\therefore \cos x = 0.4$

$\therefore x = \pm 1.16$

b $\operatorname{cosec} x = -5$

$\sin x = -0.2$

$x = -0.20$

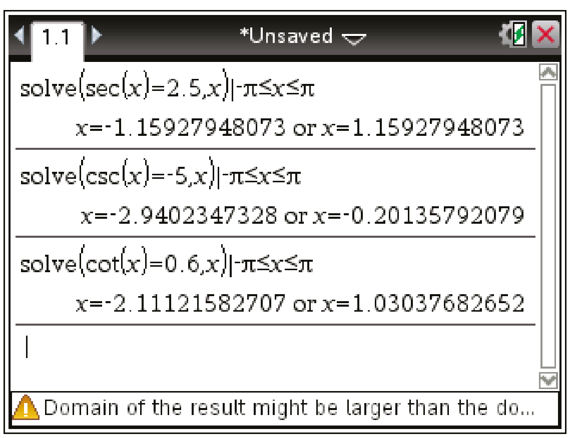
or $x = -3.14 + 0.20 = -2.94$

c $\cot x = 0.6$

$x = \tan^{-1}\left(\dfrac{1}{0.6}\right) = 1.03$

or $x = 1.03 - 3.14 = -2.11$

CAS: Type in as shown below



5 a $\cos^2 x - \cos x \sin x = 0$

$\therefore \cos x(\cos x - \sin x) = 0$

$\therefore \cos x = 0 \text{ or } \cos x - \sin x = 0$

$\therefore \cos x = \sin x$

$\therefore \tan x = 1$

$\therefore x = \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} \text{ or } x = \dfrac{\pi}{4} \text{ or } \dfrac{5\pi}{4}$

$\therefore x = \dfrac{\pi}{2} \text{ or } \dfrac{\pi}{4} \text{ or } \dfrac{5\pi}{4} \text{ or } \dfrac{3\pi}{2}$

b $\sin 2x = \sin x$

$\therefore 2\sin x \cos x = \sin x$

$\therefore 2\sin x \cos x - \sin x = 0$

$\therefore \sin x(2\cos x - 1) = 0$

$\therefore \sin x = 0 \text{ or } 2\cos x - 1 = 0$

$\therefore 2\cos x = 1$

$\therefore \cos x = \dfrac{1}{2}$

$\therefore x = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } x = \dfrac{\pi}{3} \text{ or } \dfrac{5\pi}{3}$

$\therefore x = 0 \text{ or } \dfrac{\pi}{3} \text{ or } \pi \text{ or } \dfrac{5\pi}{3} \text{ or } 2\pi$

c $\sin 2x = \cos x$

$\therefore 2\sin x \cos x = \cos x$

$\therefore 2\sin x \cos x - \cos x = 0$

$\therefore \cos x(2\sin x - 1) = 0$

$\therefore \cos x = 0 \text{ or } 2\sin x - 1 = 0$

$\therefore 2\sin x = 1$

$\therefore \sin x = \dfrac{1}{2}$

$\therefore x = \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} \text{ or } x = \dfrac{\pi}{6} \text{ or } \dfrac{5\pi}{6}$

$\therefore x = \dfrac{\pi}{6} \text{ or } \dfrac{\pi}{2} \text{ or } \dfrac{5\pi}{6} \text{ or } \dfrac{3\pi}{2}$

d $\sin 8x = \cos 4x$

$\therefore 2\sin 4x \cos 4x = \cos 4x$

$\therefore 2\sin 4x \cos 4x - \cos 4x = 0$

$\therefore \cos 4x(2\sin 4x - 1) = 0$

$\therefore \quad \cos 4x = 0 \text{ or } 2\sin 4x - 1 = 0$

$\therefore \quad 2\sin 4x = 1$

$\therefore \quad \sin 4x = \frac{1}{2}$

Now $x \in [0, 2\pi] \quad \therefore 4x \in [0, 8\pi]$

$\therefore \quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2} \text{ or } \frac{15\pi}{2}$

$\text{or } 4x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6} \text{ or } \frac{41\pi}{6}$

$\therefore \quad x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8} \text{ or } \frac{15\pi}{8}$

or $\quad x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24} \text{ or } \frac{41\pi}{24}$

$\therefore \quad x = \frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{5\pi}{8}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24}, \frac{9\pi}{8}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{13\pi}{8}, \frac{41\pi}{24} \text{ or } \frac{15\pi}{8}$

e $\quad \cos 2x = \cos x$

$\therefore \quad 2\cos^2 x - 1 = \cos x$

$\therefore \quad 2\cos^2 x - \cos x - 1 = 0$

Let $a = \cos x$

$\therefore \quad 2a^2 - a - 1 = 0$

$\therefore (2a + 1)(a - 1) = 0$

$\therefore \quad 2a + 1 = 0 \text{ or } a - 1 = 0$

$\therefore \quad 2a = -1 \quad \therefore a = 1$

$\therefore \quad a = -\frac{1}{2}$

$\therefore \quad \cos x = -\frac{1}{2} \text{ or } \cos x = 1$

$\therefore \quad x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } x = 0 \text{ or } 2\pi$

$\therefore \quad x = 0 \text{ or } \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } 2\pi$

f $\quad \cos 2x = \sin x$

$\therefore \quad 1 - 2\sin^2 x = \sin x$

$\therefore \quad 2\sin^2 x + \sin x - 1 = 0$

Let $a = \sin x$

$\therefore \quad 2a^2 + a - 1 = 0$

$\therefore \quad (2a - 1)(a + 1) = 0$

$\therefore \quad 2a - 1 = 0 \text{ or } a + 1 = 0$

$\therefore \quad 2a = 1 \quad \therefore a = -1$

$\therefore \quad a = \frac{1}{2}$

$\therefore \quad \sin x = \frac{1}{2} \text{ or } \sin x = -1$

$\therefore \quad x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$

g $\quad \sec^2 x + \tan x = 1$

$\therefore \quad (1 + \tan^2 x) + \tan x = 1$

$\therefore \quad \tan^2 x + \tan x = 0$

$\therefore \quad \tan x(\tan x + 1) = 0$

$\therefore \quad \tan x = 0 \text{ or } \tan x = -1$

$\therefore \quad x = 0, \pi \text{ or } 2\pi \text{ or } x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$

$\therefore \quad x = 0 \text{ or } \frac{3\pi}{4} \text{ or } \pi \text{ or } \frac{7\pi}{4} \text{ or } 2\pi$

h $\quad \tan x(1 + \cot x) = 0$

$\therefore \quad \tan x\left(1 + \frac{1}{\tan x}\right) = 0$

$\therefore \frac{1}{\tan x} + 1 = 0 \text{ or } \tan x = 0$

Note that if $\tan x = 0$ then $\cot x$ is undefined, thus we only consider the case when

$\tan x = -1$

$\therefore \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

i $\cot x + 3\tan x = 5 \operatorname{cosec} x$

$\therefore \quad \frac{\cos x}{\sin x} + \frac{3\sin x}{\cos x} = \frac{5}{\sin x}$

$\therefore \quad \frac{3\sin x}{\cos x} = \frac{5 - \cos x}{\sin x}$

$\therefore \quad 3\sin^2 x = \cos x(5 - \cos x)$

$= 5\cos x - \cos^2 x$

$\therefore \quad 3(1 - \cos^2 x) = 5\cos x - \cos^2 x$

$\therefore \quad 3 - 3\cos^2 x = 5\cos x - \cos^2 x$

$\therefore 2\cos^2 x + 5\cos x - 3 = 0$

Let $a = \cos x$

$\therefore \quad 2a^2 + 5a - 3 = 0$

$\therefore \quad (2a - 1)(a + 3) = 0$

$\therefore \quad 2a - 1 = 0$ or $a + 3 = 0$

$\therefore \quad 2a = 1$ or $a = -3$

$\therefore \quad a = \frac{1}{2}$

$\therefore \quad \cos x = \frac{1}{2}$ or $\cos x = -3$

$\therefore \quad x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

(as $\cos x \neq -3$)

j $\sin x + \cos x = 1$

Let $x = 2y$ and use double angle formula.

$\sin 2y + \cos 2y = 1$

$2\sin y\cos y + 1 - 2\sin^2 y = 1$

$2\sin y(\cos y - \sin y) = 0$

$2\sin y = 0$ or $\tan y = 1$

$0 \le x \le 2\pi$ so $0 \le y \le \pi$

$\sin y = 0$ gives $y = 0, \pi$

$\tan y = 1$ gives $y = \frac{\pi}{4}$

Hence $x = 2y$, so solutions for x are $0, \frac{\pi}{2}, 2\pi$.

6 **a** $-1 \le \sin\theta \le 1$

$\therefore \quad 1 \le 2 + \sin\theta \le 3$

The maximum and minimum values of $2 + \sin\theta$ are 3 and 1 respectively.

b $1 \le 2 + \sin\theta \le 3$

$\therefore \quad \frac{1}{3} \le \frac{1}{2 + \sin\theta} \le 1$

The maximum and minimum values of $\frac{1}{2 + \sin\theta}$ are 1 and $\frac{1}{3}$ respectively.

c $-1 \le \sin\theta \le 1$

$\therefore \quad 0 \le \sin^2\theta \le 1$

$\therefore \quad 4 \le \sin^2\theta + 4 \le 5$

The maximum and minimum values of $\sin^2\theta + 4$ are 5 and 4 respectively.

d $4 \le \sin^2\theta + 4 \le 5$

$\therefore \quad \frac{1}{5} \le \frac{1}{\sin^2\theta + 4} \le \frac{1}{4}$

The maximum and minimum values of $\frac{1}{\sin^2\theta + 4}$ are $\frac{1}{4}$ and $\frac{1}{5}$ respectively.

e $\cos^2\theta + 2\cos\theta$

$= (\cos^2\theta + 2\cos\theta + 1) - 1$

$= (\cos\theta + 1)^2 - 1$

Now $-1 \le \cos\theta \le 1$

$\therefore \quad 0 \le \cos\theta + 1 \le 2$

$\therefore \quad 0 \le (\cos\theta + 1)^2 \le 4$

$\therefore \quad -1 \le (\cos\theta + 1)^2 - 1 \le 3$

$\therefore \quad -1 \le \cos^2\theta + 2\cos\theta \le 3$

The maximum and minimum values of $\cos^2\theta + 2\cos\theta$ are 3 and -1 respectively.

f $\cos^2\theta + 2\cos\theta + 6$

$= (\cos^2\theta + 2\cos\theta + 1) + 5$

$= (\cos\theta + 1)^2 + 5$

Now $-1 \le \cos\theta \le 1$

$\therefore \quad 0 \le \cos\theta + 1 \le 2$

$\therefore \quad 0 \le (\cos\theta + 1)^2 \le 4$

$\therefore \quad 5 \le (\cos\theta + 1)^2 + 5 \le 9$

$\therefore \quad 5 \le \cos^2\theta + 2\cos\theta + 6 \le 9$

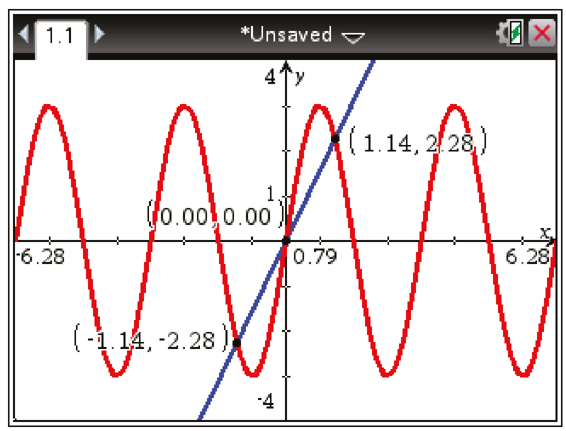
The maximum and minimum values of $\cos^2\theta + 2\cos\theta + 6$ are 9 and 5 respectively.

7 Using a CAS calculator sketch both equations and change the Window settings so that that all points of intersections can be seen.

TI: Change the Document settings to Fix2 and Approximate
Press **Menu** →**6: Analyze Graph** →**4: Intersection**

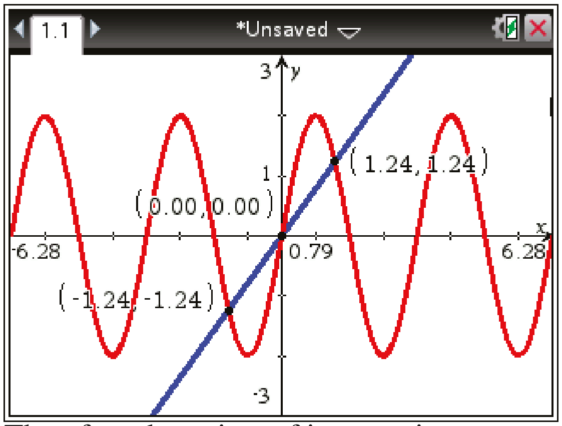
CP: Change the Number Format to Fix2 and set the mode to Decimal.
Tap **Analysis** → **G-Solve** → **Intersect**

a $y = 2x$ and $y = 3\sin(2x)$



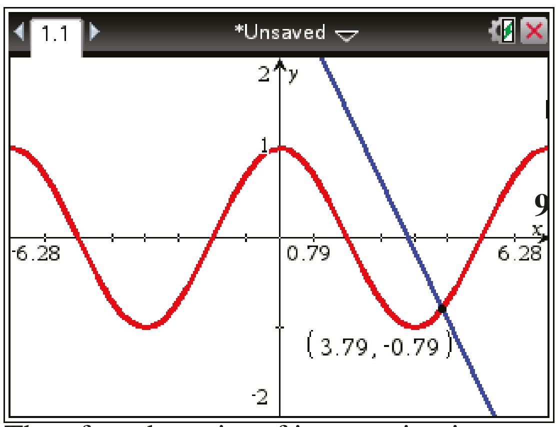
Therefore the points of intersection are
$(-1.14, -2.28)$, $(0, 0)$ and $(1.14, 2.28)$

b $y = x$ and $y = 2\sin(2x)$



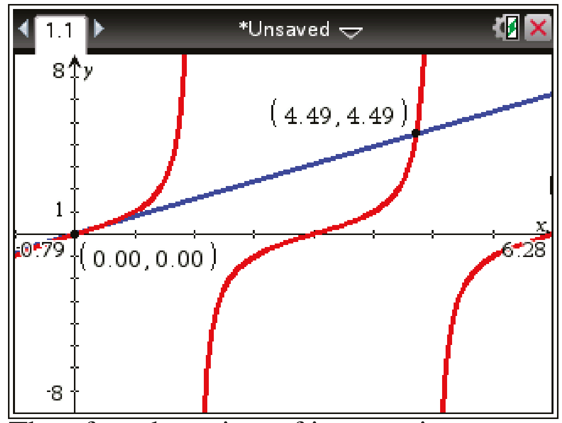
Therefore the points of intersection are
$(-1.24, -1.24)$, $(0, 0)$ and $(1.24, 1.24)$

c $y = 3 - x$ and $y = \cos x$



Therefore the point of intersection is $(3.79, -0.79)$

d $y = x$ and $y = \tan x\ x \in [0, 2\pi]$



Therefore the points of intersection are $(0, 0)$ and $(4.49, 4.49)$

8 $\cos x = a, a \neq -1, x \in [0, 2\pi]$
Since $\cos q = \cos(2\pi - q)$, $2\pi - q$ is the second solution of the equation $\cos x = a$.

9 $\sin\alpha = a, \alpha \in \left[0, \dfrac{\pi}{2}\right]$

a Since $\sin\alpha = -\sin(\pi + \alpha)$ and $\sin\alpha = -\sin(2\pi - \alpha)$, $x = \pi + \alpha$ and $x = 2\pi - \alpha$ are solutions of the equation $\sin x = -a$.

b Since $\sin\alpha = \cos\left(\dfrac{\pi}{2} - \alpha\right)$ and

$$\sin\alpha = \cos\left(2\pi - \left(\frac{\pi}{2} - \alpha\right)\right)$$

$$= \cos\left(\frac{3\pi}{2} + \alpha\right),$$

the solutions are $x = \dfrac{\pi}{2} - \alpha$ and $x = \dfrac{3\pi}{2} + \alpha$.

10 $\sec\beta = b, \beta \in \left[\dfrac{\pi}{2}, \pi\right]$

a Since $\sec(\pi - \beta) = -\sec\beta$ and when $\frac{\pi}{2} \le \beta \le \pi, 0 \le \pi - \beta \le \frac{\pi}{2}$, then $x = \pi - \beta$.

Also $\sec(\beta - \pi) = -\sec\beta$ and when $\frac{\pi}{2} \le \beta \le \pi, -\frac{\pi}{2} \le \beta - \pi \le 0$.

Therefore there are two solutions, $x = \pi - \beta$ and $x = \beta - \pi$.

b Since $\operatorname{cosec}\left(\frac{\pi}{2} - \beta\right) = \sec\beta$ and when $\frac{\pi}{2} \le \beta \le \pi, -\frac{\pi}{2} \le \frac{\pi}{2} - \beta \le 0$, then $x = \frac{\pi}{2} - \beta$. Also

$\operatorname{cosec} x = -\operatorname{cosec}(\pi + x)$

$= \operatorname{cosec}(-\pi - x)$.

Therefore $x = -\pi - \left(\frac{\pi}{2} - \beta\right) = -\frac{3\pi}{2} + \beta$ is a solution if $-\pi < -\frac{3\pi}{2} + \beta < \pi$

As $\frac{\pi}{2} \le \beta \le \pi, \frac{\pi}{2} - \frac{3\pi}{2} \le \beta - \frac{3\pi}{2} \le \pi - \frac{3\pi}{2}$

$\therefore -\pi \le \beta - \frac{3\pi}{2} \le -\frac{\pi}{2}$

Therefore there are two solutions: $x = \frac{\pi}{2} - \beta$ and $x = \beta - \frac{3\pi}{2}$

11 $\tan\gamma = c, \gamma \in \left[\pi, \frac{3\pi}{2}\right]$

a $\tan\gamma = -\tan(2\pi - \gamma)$

As $\pi \le \gamma \le \frac{3\pi}{2}, -\frac{3\pi}{2} \le -\gamma \le -\pi$,

and $\frac{\pi}{2} \le 2\pi - \gamma \le \pi$.

So $x = 2\pi - \gamma$ is the solution.

Also $\tan(x) = \tan(\pi + x)$

$\therefore x = \pi + 2\pi - \gamma = 3\pi - \gamma$ is the second solution.

b $\tan\gamma = \cot\left(\frac{3\pi}{2} - \gamma\right) = \cos\left(\frac{5\pi}{2} - \gamma\right)$

When $\pi \le \gamma \le \frac{3\pi}{2}, 0 \le \frac{3\pi}{2} - \gamma \le \frac{\pi}{2}$ and $\pi \le \frac{5\pi}{2} - \gamma \le \frac{3\pi}{2}$.

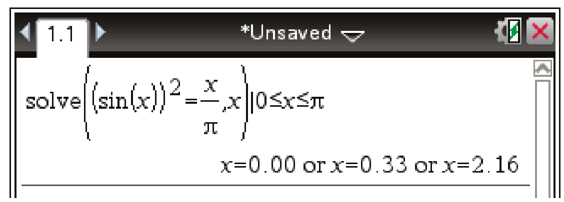
So $x = \frac{3\pi}{2} - \gamma$ and $x = \frac{5\pi}{2} - \gamma$

12 $\sin^2\theta = \frac{\theta}{\pi}, \theta \in [0, \pi]$

CAS: Type

solve $\left(\sin(x)^2 = \frac{x}{\pi}, x\right) \Big| 0 \le x \le \pi$

$\therefore \theta = 0, 0.33$ or 2.16,



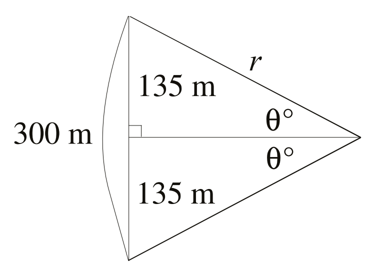
13 $\tan^{-1} x = 4x - 5$

CAS: Type

solve(tan^{-1}(*x*) = 4*x* − 5, *x*)

$\therefore x = 1.50$

14



a circumference $= 2\pi r$

Also, circumference $= \frac{360}{2\theta} \times 300$

$= \frac{54\,000}{\theta}$

$$\therefore 2\pi r = \frac{54\,000}{\theta}$$

$$\therefore r = \frac{54\,000}{2\pi\theta} = \frac{27\,000}{\pi\theta}$$

$$\text{Now } \sin\theta^\circ = \frac{135}{r}$$

$$= 135 \times \frac{1}{r}$$

$$= 135 \times \frac{\pi\theta}{27\,000}$$

$$= \frac{\pi}{200}\theta,$$

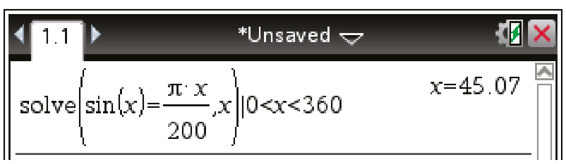
as required to show.

b $\sin\theta^\circ = \frac{\pi}{200}\theta, \theta \in (0, 360)$

CAS: Set to **Degree/Deg** mode and then type

$$\textbf{solve}\left(\sin(x) = \frac{\pi x}{200}, x\right)\Bigg| 0 < x < 360$$

$\therefore \theta = 45.07$

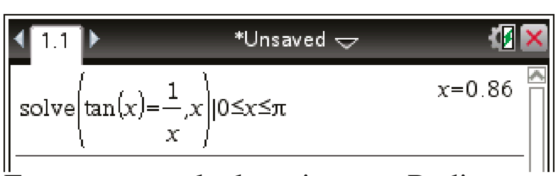


15 $\tan x = \frac{1}{x}, x \in [0, \pi]$

CAS: Type

$$\textbf{solve}\left(\tan(x) = \frac{1}{x}, x\right)\Bigg| 0 \le x \le \pi$$

$\therefore x = 0.86$, correct to two decimal places.



Ensure your calculator is set to Radian mode.

16 $A = \frac{1}{2}r^2(\theta - \sin\theta)$

When $A = 18, r = 6$

$$18 = \frac{1}{2} \times 6^2(\theta - \sin\theta)$$

$$= 18(\theta - \sin\theta)$$

$$\therefore 1 = \theta - \sin\theta$$

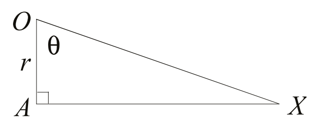
$$\therefore \sin\theta = \theta - 1$$

CAS: Type

solve(sin(x) = x − 1, x)|0 ≤ x ≤ 2π

$\therefore \theta = 1.93$, correct to two decimal places.

17 a



Consider $\triangle AOX$.

$$\text{Area of } \triangle AOX = \frac{1}{2} \text{ base} \times \text{height}$$

$$= \frac{1}{2} \times AX \times r$$

$$\text{and } \tan\theta = \frac{AX}{r}$$

$$\therefore AX = r\tan\theta$$

$$\Rightarrow \Delta AOX = \frac{1}{2}(r\tan\theta) \times r$$

$$= \frac{1}{2}r^2\tan\theta$$

$$\text{Area } AOBX = 2 \times \frac{1}{2}r^2\tan\theta$$

$$= r^2\tan\theta$$

Now, area of remaining region of circle

$$= \left(\frac{2\pi - 2\theta}{2\pi}\right)\pi r^2$$

$$= \frac{2(\pi - \theta)\pi r^2}{2\pi}$$

$$= (\pi - \theta)r^2$$

$$\therefore r^2\tan\theta = (\pi - \theta)r^2$$

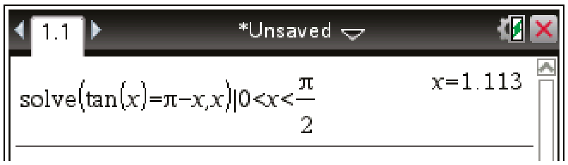
$\therefore \tan\theta = \pi - \theta$, as required to show.

b $\tan\theta = \pi - \theta, \theta \in \left(0, \frac{\pi}{2}\right)$

CAS: Type

solve(tan(x) = π − x, x)|0 < x < $\frac{\pi}{2}$

$\therefore \theta = 1.113$, correct to three decimal places.

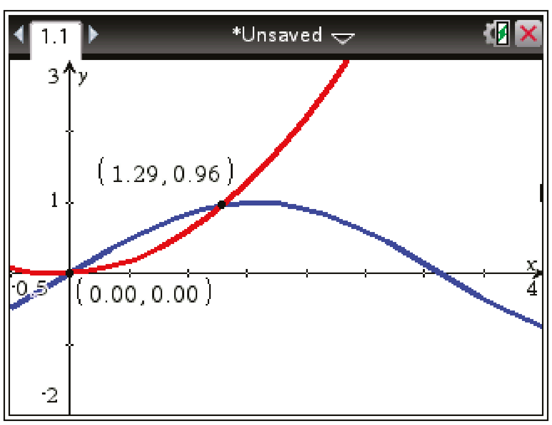


18 $x_A = 0.5\sin t$ and $x_B = 0.25t^2 + 0.05t$

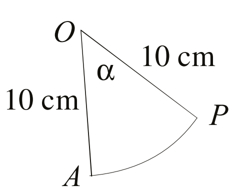
When $x_A = x_B$, $0.5\sin t = 0.25t^2 + 0.05t$

$\therefore \sin t = 0.5t^2 + 0.1t$

Using the CAS calculator procedure outlined in question **7**, the points of intersection are (0, 0) and (1.29, 0.96). The positions of particles A and B are the same at the start ($t = 0$) at the origin ($x_A = x_B = 0$), and after 1.29 seconds when they are 0.48 cm from the origin, correct to two decimal places.



19 **a**

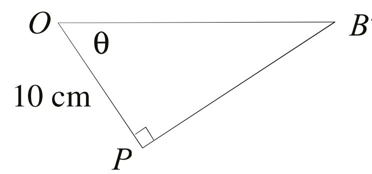


Consider the sector OAP.

$AP = r\alpha$

$= 10\left(\frac{\pi}{2} - \theta\right)$

since $\therefore \angle AOP + \angle POB' = \frac{\pi}{2}$



Consider $\triangle OPB'$.

$\tan\theta = \frac{PB'}{10}$

$= \frac{AB - AP}{10}$

$= \frac{20 - 10\left(\frac{\pi}{2} - \theta\right)}{10}$

$= \frac{10\left(2 - \frac{\pi}{2} + \theta\right)}{10}$

$= 2 - \frac{\pi}{2} + \theta$

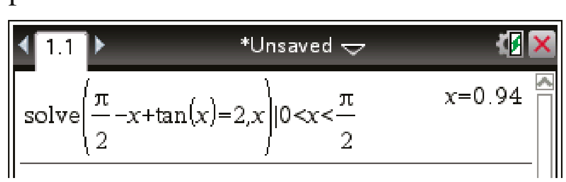
$\therefore \frac{\pi}{2} - \theta + \tan\theta = 2$, as required to show.

b $\frac{\pi}{2} - \theta + \tan\theta = 2, \theta \in \left(0, \frac{\pi}{2}\right)$

CAS: Type

solve$\left(\frac{\pi}{2} - x + \tan(x) = 2, x\right)\Big|$0 < x < $\frac{\pi}{2}$

$\therefore \theta = 0.94$, correct to two decimal places.



Solutions to Exercise 3E

1 **a** $\sin(11\pi t) - \sin(3\pi t)$

b $\frac{1}{2}(\sin 60° + \sin 40°)$

c $\frac{3}{2}\left(\sin(\pi x) + \sin\left(\frac{\pi x}{3}\right)\right)$

d $\sin(A) + \sin(B + C)$

e $\cos(\frac{x}{2}) - cos(\frac{5x}{2})$

f $\cos(\frac{\pi x}{2}) + cos(\pi x)$

2 $\cos(3\theta) - \cos(5\theta)$

3 $\sin x - \sin y$

4

$$\begin{aligned}\text{LHS} &= \cos 75° \cos 15° \\ &= \frac{1}{2}(\cos(75 - 15)° + \cos(75 + 15)°) \\ &= \frac{1}{2}\cos 60° \\ &= \frac{1}{4} \\ &= \text{RHS}\end{aligned}$$

5 **a** $2 \sin 50° \cos 16°$

b $2 \cos 50° \cos 16°$

c $2 \sin 16° \cos 50°$

d $-2 \sin 50° \sin 16°$

6 **a** $2 \sin(5A) \cos(3A)$

b $2 \cos\left(\frac{5x}{2}\right)\cos\left(\frac{3x}{2}\right)$

c $2 \sin(x) \cos(5x)$

d $-2 \sin(4A) \sin(A)$

7

$$\begin{aligned}\text{LHS} &= \sin A + 2 \sin 3A + \sin 5A \\ &= \sin A + \sin 3A + \sin 3A + \sin 5A \\ &= 2 \sin 2A \cos A + \sin 4A \cos A \\ &= 2 \cos A(\sin 2A + \sin 4A) \\ &= 2 \cos A(2 \sin 3A \cos A) \\ &= 4 \cos^2 A \sin 3A \\ &= \text{RHS}\end{aligned}$$

8

LHS

$= \sin(\alpha + \beta)\sin(\alpha - \beta) + \sin(\beta + \gamma)\sin(\beta - \gamma) + \sin(\gamma + \alpha)\sin(\gamma - \alpha)$

$= \frac{1}{2}(\cos 2\alpha - \cos 2\beta + \cos 2\beta - \cos 2\gamma + \cos 2\gamma - \cos 2\alpha)$

$= 0$

$= \text{RHS}$

9

$$\begin{aligned}\text{LHS} &= \cos 70° + \sin 40° \\ &= \cos 70° + \cos 50° \\ &= 2 \cos 10° \cos 60° \\ &= 2 \cos 10° \times \frac{1}{2} \\ &= \cos 10° \\ &= \text{RHS}\end{aligned}$$

10 LHS $= \cos 20^\circ + \cos 100^\circ + \cos 140^\circ$

$= \cos 20^\circ + 2\cos(120^\circ \cos 20^\circ$

$= \cos 20^\circ(1 + 2 \times (-\frac{1}{2})$

$= 0$

$=$ RHS

11 **a**

$$\cos 5x + \cos x = 0$$
$$2\cos 3x \cos 2x = 0$$
$$\cos 3x = 0 \text{ or } \cos 2x = 0$$
$$x = -\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$$

b

$$\cos 5x - \cos x = 0$$
$$-2\sin 3x \sin 2x = 0$$
$$\sin 3x = 0 \text{ or } \sin 2x = 0$$
$$x = -\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$$

c

$$\sin 5x + \sin x = 0$$
$$2\sin 3x \cos 2x = 0$$
$$\sin 3x = 0 \text{ or } \cos 2x = 0$$
$$x = -\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$$

d

$$\sin 5x - \sin x = 0$$
$$2\sin 2x \cos 3x = 0$$
$$\sin 2x = 0 \text{ or } \cos 3x = 0$$
$$x = -\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$$

12

$$\sin 3x + \sin x + \cos 4x = 1$$
$$2\sin 2x \cos x + \cos 4x = 1$$
$$4\sin x \cos^2 x + 1 - 2\sin^2 2x = 1$$
$$4\sin x \cos^2 x - 2\sin^2 2x = 0$$
$$4\sin x \cos^2 x - 8\sin^2 x \cos^2 x = 0$$
$$\sin x \cos^2 x - 2\sin^2 x \cos^2 x = 0$$
$$\sin x \cos^2 x(1 - 2\sin x) = 0$$
$$\sin x = 0 \text{ or } \cos^2 x = 0 \text{ or } (1 - 2\sin x) = 0$$
$$x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$$

13 **a**

$$\cos 2\theta - \sin(\theta) = 0$$
$$1 - 2\sin^2\theta - \sin\theta = 0$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

b

$$\sin 5\theta - \sin(3\theta) + \sin\theta = 0$$
$$2\sin\theta\cos 4\theta + \sin\theta = 0$$
$$\sin\theta(2\cos 4\theta + 1) = 0$$
$$\sin\theta = 0 \text{ or } \cos 4\theta = -\frac{1}{2}$$
$$\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$$

c

$$\sin 7\theta - \sin(\theta) = \sin 3\theta$$
$$2\sin 3\theta\cos 4\theta = \sin 3\theta$$
$$\sin 3\theta(2\cos 4\theta - 1) = 0$$
$$\sin 3\theta = 0 \text{ or } \cos 4\theta = \frac{1}{2}$$
$$\theta = 0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$$

d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

14 a Pairing,

$\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots +$
$\sin 358^\circ + \sin 359^\circ$
$= (\sin 1^\circ + \sin 359^\circ) + (\sin 2^\circ +$
$\sin 358^\circ) + \cdots + \sin(180^\circ)$
$= 2\sin 180^\circ \cos 179^\circ +$
$2\sin 180^\circ \cos 178^\circ + \cdots + \sin(180^\circ)$
$= 0$

b Pairing,

$\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots +$
$\cos 358^\circ + \cos 359^\circ$
$= -2\cos 179^\circ - 2\cos 178^\circ -$
$2\cos 177^\circ + \cdots - 2\cos 1^\circ + \cos(180^\circ)$
Rearrange,
$= -2(\cos 179^\circ + \cos 1^\circ + \cos 178^\circ +$
$\cos 2^\circ + \cdots) + \cos(180^\circ$
$= -2(2\cos 90^\circ \cos 89^\circ +$
$2\cos 90^\circ \cos 88^\circ + \cdots) \cos(180^\circ$
$= -1$

15 a

$$\sin\theta + \sin 2\theta + \cos\theta + \cos 2\theta$$
$$= 2\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + 2\cos\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$
$$= 2\cos\left(\frac{\theta}{2}\right)\left(\sin\left(\frac{3\theta}{2}\right) + \cos\left(\frac{3\theta}{2}\right)\right)$$
$$= 2\cos\left(\frac{\theta}{2}\right)\times \sqrt{2}\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right)$$
$$= 2\sqrt{2}\cos\left(\frac{\theta}{2}\right)\times\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right)$$

b $2\sqrt{2}\cos\left(\frac{\theta}{2}\right)\times\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) = 0$

$$\cos\left(\frac{\theta}{2}\right)\times\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) = 0$$
$$2\sqrt{2}\cos\left(\frac{\theta}{2}\right)\times\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) = 0$$
$$\cos\left(\frac{\theta}{2}\right) = 0 \text{ or } \sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) = 0$$
$$\frac{\theta}{2} = \frac{\pi}{2} \Rightarrow \theta = \pi$$
$$\frac{3\theta}{2} + \frac{\pi}{4} = 0, \pi, 2\pi, 3\pi \ldots$$
$$\frac{3\theta}{2} = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4} \ldots$$
$$\theta = \frac{\pi}{2}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$$

16
$$\text{LHS} = \frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta}$$
$$= \frac{2\sin 3\theta\cos 2\theta + \sin 3\theta}{2\cos 3\theta\cos 2\theta + \cos 3\theta}$$
$$= \frac{\sin 3\theta}{\cos 3\theta}$$

17
$$\frac{b-a}{b+a} = \frac{1 - \frac{a}{b}}{1 + \frac{a}{b}}$$
$$1 - \frac{a}{b} = 1 - \frac{\sin 2A - \cos 2B}{\cos 2A - \sin 2B}$$
$$= \frac{\cos 2A - \sin 2B - \sin 2A + \cos 2B}{\cos 2A - \sin 2B}$$
$$= \frac{\cos 2A + \cos 2B - (\sin 2B + \sin 2A)}{\cos 2A - \sin 2B}$$

$$1 + \frac{a}{b} = 1 + \frac{\sin 2A - \cos 2B}{\cos 2A - \sin 2B}$$
$$= \frac{\cos 2A - \sin 2B + \sin 2A - \cos 2B}{\cos 2A - \sin 2B}$$
$$= \frac{\cos 2A - \cos 2B + (\sin 2A - \sin 2B)}{\cos 2A - \sin 2B}$$

Hence,

$$\frac{b-a}{b+a} = \frac{\cos 2A + \cos 2B - (\sin 2B + \sin 2A)}{\cos 2A - \cos 2B + (\sin 2A - \sin 2B)}$$

$$= \frac{2\cos(A+B)\cos(A-B) - 2\sin(A+B)\cos(A-B)}{-2\sin(A+B)\sin(A-B) + 2\sin(A-B)\cos(A+B)}$$

$$= \frac{2\cos(A-B)(\cos(A+B) - \sin(A+B))}{2\sin(A-B)(\cos(A+B) - \sin(A+B))}$$

$$= \cot(A-B)$$

18 a

$$2\cos B \sin C = \sin A$$

$$2\cos B \sin(180 - (B+A)) = \sin A$$

$$\sin(180 - A) + \sin(180 - (A+2B)) = \sin A$$

$$\sin(180 - (A+2B)) = 0$$

$$A + 2B = 180$$

$$C = 180 - (A+B)$$

$$= 180 - (180 - 2B + B)$$

$$= B$$

Triangle is isosceles

b LHS $= \dfrac{\sin A + \sin B}{\cos A + \cos B}$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$= \frac{\sin\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}$$

$$= \tan\left(\frac{A+B}{2}\right)$$

$$= \tan\left(90 - \frac{C}{2}\right)$$

$$= \cot\left(\frac{C}{2}\right)$$

c

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \sin C$$

$$\frac{\sin\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} = \sin C$$

$$\frac{\sin\left(90 - \frac{C}{2}\right)}{cos\left(90 - \frac{C}{2}\right)} = \sin C$$

$$\frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)} = \sin C$$

$$\cos\left(\frac{C}{2}\right) = \sin\left(\frac{C}{2}\right)\sin C$$

$$\cos\left(\frac{C}{2}\right) = 2\sin^2\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)$$

$$\cos\left(\frac{C}{2}\right)(1 - 2\sin^2\left(\frac{C}{2}\right) = 0$$

$$\sin^2\left(\frac{C}{2}\right) = \frac{1}{2}$$

$$\sin\left(\frac{C}{2}\right) = \frac{1}{\sqrt{2}}$$

$$\frac{C}{2} = 45°$$

$$C = 90°$$

19 a

$$\begin{aligned}
\text{LHS} &= \sin A + \sin B + \sin C \\
&= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\sin\left(\frac{180-C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{C}{2}\right)\right) \\
&= 2\cos\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{180-A-B}{2}\right)\right) \\
&= 2\cos\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{180-A-B}{2}\right)\right) \\
&= 2\cos\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right) \\
&= 4\cos\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)
\end{aligned}$$

b $\sin(2A) + \sin(2B) + \sin(2C)$

$$\begin{aligned}
&= 2\sin(C)\cos(A-B) + 2\sin(C)\cos(C) \\
&= 2\sin(C)\left(\cos(A-B) + \cos(C)\right) \\
&= 2\sin(C)\left(\cos(A-B) + \cos(\pi - (A+B))\right) \\
&= 2\sin(C)\left(\cos(A-B) - \cos(A+B)\right) \\
&= 2\sin(C) \times 2\sin(A)\sin(B) \\
&= 4\sin(A)\sin(B)\sin(C)
\end{aligned}$$

Solutions to Short-answer questions

1 **a** $\sin\theta = \sqrt{1-\cos^2\theta}$

$$= \sqrt{1-\frac{16}{25}} = \frac{3}{5}$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{32}{25} - 1 = \frac{7}{25}$$

b $\sin 2\theta = 2\cos\theta\sin\theta = \frac{24}{25}$

c $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{24}{7}$

d $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{5}{3}$

e $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{4}{3}$

2 **a** $2\sin x\cos x = \sin x$

$\therefore \quad \sin x\,(2\cos x - 1) = 0$

Either $\quad \sin x = 0$

$\therefore \quad x = 0, \pi, 2\pi$

or $\quad 2\cos x = 1$

$\therefore \quad \cos x = \frac{1}{2}$

$\therefore \quad x = \pm\frac{\pi}{3}, \frac{5\pi}{3}$

b $\cos x - 1 = 2\cos^2 x - 1$

$\therefore \quad \cos x\,(2\cos x - 1) = 0$

Either $\quad \cos x = 0$

$\therefore \quad x = \pm\frac{\pi}{2}, \frac{3\pi}{2}$

or $\quad 2\cos x = 1$

$\therefore \quad \cos x = \frac{1}{2}$

$\therefore \quad x = \pm\frac{\pi}{3}, \frac{5\pi}{3}$

c $$2\sin x\cos x = 2\cos x$$

$\therefore$ $$\cos x(\sin x - 1) = 0$$

Either $\cos x = 0$, $\quad\therefore\quad x = \pm\frac{\pi}{2}, \frac{3\pi}{2}$

or $\sin x = 1$, $\quad\therefore\quad x = \frac{\pi}{2}$

d $$\sin^2 x\cos^3 x = \cos x$$

$\cos x(\sin^2 x\cos^2 x - 1) = 0$

Either $\cos x = 0$, $\quad\therefore x = \pm\frac{\pi}{2}, \frac{3\pi}{2}$

or $\sin^2 x\cos^2 x = 1$.

But $$\sin^2 x\cos^2 x = \frac{1}{4}(\sin 2x)^2$$

and $$-1 \le \sin 2x \le 1$$

$\therefore$ $$0 \le (\sin 2x)^2 \le 1$$

$\therefore$ $$0 \le \frac{1}{4}(\sin 2x)^2 \le \frac{1}{4}$$

$\therefore$ $$0 \le \sin^2 x\,\cos^2 x \le \frac{1}{4}$$

$\therefore$ $$\sin^2 x\,\cos^2 x \ne 1$$

e $$\sin^2 x - \frac{1}{2}\sin x - \frac{1}{2} = 0$$

Let $\sin x = t$, then $$t^2 - \frac{1}{2}t - \frac{1}{2} = 0$$

$$2t^2 - t - 1 = 0$$

$$t = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4}$$

$$t = 1 \quad\text{or}\quad t = -\frac{1}{2}$$

$\therefore$ $$\sin x = 1 \quad\text{or}\quad \sin x = -\frac{1}{2}$$

$$x = \frac{\pi}{2} \quad\text{or}\quad x = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

f Let $\cos x = t$

then $2t^2 - 3t + 1 = 0$

$$t = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}$$

$$t = 1 \qquad \text{or } t = \frac{1}{2}$$

$$\therefore \cos x = 1 \qquad \text{or } \cos x = \frac{1}{2}$$

$$x = 0, 2\pi \qquad \text{or } x = \pm\frac{\pi}{3}, \frac{5\pi}{3}$$

3 a The equation $2 - \sin\theta = \cos^2\theta + 7\sin^2\theta$
is rearranged to the form

$$7\sin^2\theta + 1 - \sin^2\theta + \sin\theta - 2 = 0$$

$$\therefore \qquad 6\sin^2\theta + \sin\theta - 1 = 0$$

$$\sin\theta = \frac{-1 \pm \sqrt{1+24}}{12} = \frac{-1 \pm 5}{12}$$

$$\sin\theta = -\frac{1}{2}, \qquad \text{or} \qquad \sin\theta = \frac{1}{3}$$

If $\sin\theta = -\frac{1}{2}$, $\qquad \theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$

or, if $\sin\theta = \frac{1}{3}$, $\qquad \theta = \sin^{-1}\frac{1}{3}$ or $\theta = \pi - \sin^{-1}\frac{1}{3}$

b $\sec 2\theta = 2$

$$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

c $\frac{1}{2}(5\cos\theta - 3\sin\theta) = \sin\theta$

$$5\cos\theta - 3\sin\theta = 2\sin\theta$$

$$5\cos\theta = 5\sin\theta$$

$$\cos\theta = \sin\theta$$

$$1 = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = 1$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

d $\sec\theta = 2\cos\theta$

$$2\cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

4 a $\operatorname{cosec}\left(\frac{-5\pi}{3}\right) = -\frac{1}{\sin\left(\frac{5\pi}{3}\right)}$

$$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

b $\sec\left(\frac{7\pi}{3}\right) = \sec\left(2\pi + \frac{\pi}{3}\right)$

$$= \sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2$$

c $\operatorname{cosec}\left(\frac{5\pi}{6}\right) = \operatorname{cosec}\left(\pi - \frac{\pi}{6}\right)$

$$= \operatorname{cosec}\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{6}\right)} = 2$$

d $\cot\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{5\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1$

e $\cot\left(-\frac{\pi}{6}\right) = -\cot\left(\frac{\pi}{6}\right) = -\frac{1}{\tan\left(\frac{\pi}{6}\right)} = -\sqrt{3}$

5 **a** $\tan(-\alpha) = -\tan\alpha$

$= -p$

b $\tan(\pi - \alpha) = -\tan\alpha$

$= -p$

c $\tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha$

$= \frac{1}{p}$

d $\tan\left(\frac{3\pi}{2} + \alpha\right) = \tan\left(\frac{3\pi}{2} + \alpha - 2\pi\right)$

$= \tan\left(\alpha - \frac{\pi}{2}\right)$

$= -\tan\left(\frac{\pi}{2} - \alpha\right) = -\frac{1}{p}$

e $\tan(2\pi - \alpha) = -\tan\alpha = -p$

6 **a** Domain = $[-1, 1]\backslash\{0\}$. Range = $\mathbb{R}$

b The range of $\cos^{-1} x$ is $[0, \pi]$. Consider the right-angled triangle with hypotenuse of length 1 and adjacent side of length x. Then the opposite side has length $\sqrt{1 - x^2}$ and so

$$f(x) = \frac{\sqrt{1 - x^2}}{x}$$

c see answers

7 **a** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$, because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

b $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}$ under the definition.

c $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$ under the definition.

d $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{2\pi}{3}$ since $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\frac{2\pi}{3} \in [0, \pi]$

e $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

f $\cos(\tan^{-1}(-1)) = \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

8 **a** $-1 \le \sqrt{ax+b} \le 1$

$\Leftrightarrow 0 \le ax + b \le 1$

$\Leftrightarrow -\frac{b}{a} \le x \le \frac{1-b}{a}$

b $-1 \le \frac{2}{ax} \le 1$

$\Leftrightarrow -(ax)^2 \le 2ax \le (ax)^2$

$\Leftrightarrow -ax^2 \le 2x \le ax^2$

Solving the two inequalities simultaneously :

First:

$2x \le ax^2$

$\Leftrightarrow x(2 - ax) \le 0$

$\Leftrightarrow x \le 0$ or $x \ge \frac{a}{2}$

and then

$2x \ge -ax^2$

$\Leftrightarrow x(2 + ax) \ge 0$

$\Leftrightarrow x \ge 0$ or $x \le -\frac{a}{2}$

The intersection of the two solutions gives:

$x \le -\frac{2}{a}$ or $x \ge \frac{2}{a}$

c $-1 \le \frac{ax}{2} - 2 \le 1$

$\Leftrightarrow 1 \le \frac{ax}{2} \le 3$

$\Leftrightarrow \frac{2}{a} \le x \le \frac{6}{a}$

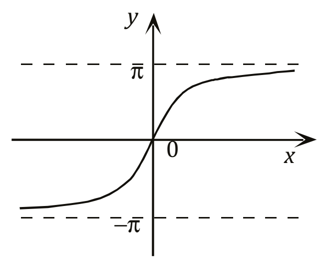
d $-1 \le \sqrt{2 - ax} \le 1$

$\Leftrightarrow 0 \le 2 - ax \le 1$

$\Leftrightarrow -2 \le -ax \le -1$

$\Leftrightarrow \frac{1}{a} \le x \le \frac{2}{a}$

9 a The graph of $y = 2\tan^{-1} x$ is obtained from the graph of $y = \tan^{-1} x$ by a dilation of factor 2 from the x-axis.
The domain is R and the range is $(-\pi, \pi)$.
Asymptotes are $y = \pi$ and $y = -\pi$.
The graph intersects the origin.



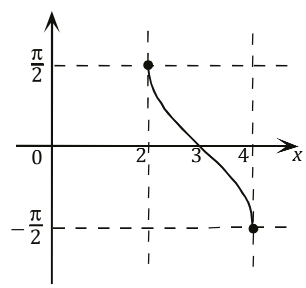
b The graph of $y = \sin^{-1}(3 - x)$ is a translation of the graph of $y = \sin^{-1}(-x)$, three units in the positive direction of the x axis. The graph of $y = \sin^{-1}(-x)$ is a reflection in the y axis of the graph of $y = \sin^{-1}(x)$. The domain can be obtained from the inequality

$$-1 \le 3 - x \le 1$$

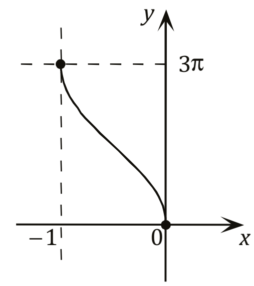
$$\therefore \quad -1 \le x - 3 \le 1$$

$$2 \le x \le 4$$

The domain is [2, 4], the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
The x-axis intercept is at $x = 3$.



c The graph of $y = 3\cos^{-1}(2x+1)$ is obtained from the graph of $y = \cos^{-1}(2x+1)$ by a dilation of factor 3 from the x axis. So the range is $[0, 3\pi]$. The graph touches the origin, so
$y = 3\cos^{-1}(2 \times 0 + 1) = 3\cos^{-1} 1 = 0$
The graph of $y = \cos^{-1}(2x+1)$ is a translation of the graph of $y = \cos^{-1} 2x$ one unit in the negative direction of the x axis, and the graph of $y = \cos^{-1} 2x$ is a dilation of factor $\frac{1}{2}$ from the y axis of the graph of $y = \cos^{-1} x$.

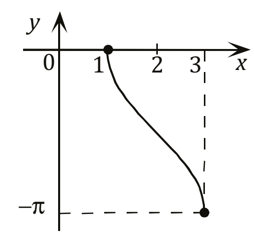


The domain can be obtained from the inequality

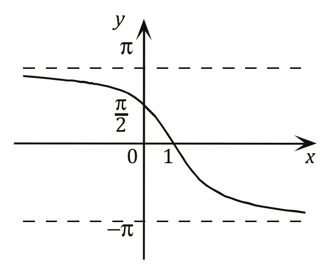
$$-1 \le 2x + 1 \le 1$$
$$\therefore -2 \le 2x \le 0$$
$$-1 \le x \le 0$$

The domain is $[-1, 0]$.

d The graph of $y = -\cos^{-1}(2-x)$ is a reflection in the x axis of the graph of $y = \cos^{-1}(2-x)$, which is the reflection in the y axis and translation two units in the positive direction of the x axis, of the graph of $y = \cos^{-1} x$.
The domain is [1, 3].
The range is $[-\pi, 0]$.



e The graph of $y = 2\tan^{-1}(1-x)$ is obtained from the graph of $y = \tan^{-1}(1-x)$ by a dilation of factor 2 from the x axis.
The range of the given function is $(-\pi, \pi)$.
Asymptotes are $y = \pi$ and $y = -\pi$.
The y-axis intercept is $2\tan^{-1}(1) = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$
The graph of the function $y = \tan^{-1}(1-x)$ is the reflection in the y axis and translation, one unit in the positive direction of the x axis, of the graph of $y = \tan^{-1}(x)$.
The x-axis intercept is at $x = 1$.
The domain is R.



10 $\sin 3x = \sin 5x$

$$\sin 5x - \sin 3x = 0$$

$$2\sin x \cos 4x = 0$$

$$\sin x = 0 \text{ or } \cos 4x = 0$$

$$x = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi$$

11 $\text{LHS} = \dfrac{\sin A + \sin B - \sin(A+B)}{\sin A + \sin B + \sin(A+B)}$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right) + 2\sin\left(\frac{A+B}{2}\right)2\cos\left(\frac{A+B}{2}\right)}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right)}{2\sin\left(\frac{A+B}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right)}$$

$$= \frac{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)}$$

$$= \frac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2} - \left(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2} + \left(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\right)}$$

$$= \frac{2\sin\frac{A}{2}\sin\frac{B}{2}}{2\cos\frac{A}{2}\cos\frac{B}{2}}$$

$$= \tan\frac{A}{2}\tan\frac{B}{2}$$

Solutions to multiple-choice questions

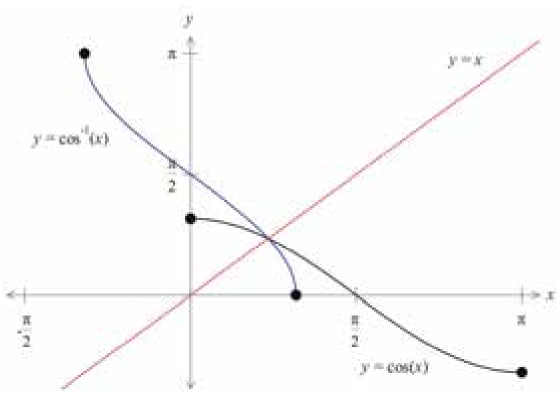
1 C Recall that the graph of $y = \cos^{-1}(x)$ has domain $[-1, 1]$ and range $[0, \pi]$

When $x = 0, y = \cos^{-1}(0) = \frac{\pi}{2}$

When $y = 0, x = \cos(0) = 1$

Therefore response A, B and E are incorrect.

The graph of $y = \cos^{-1}(x)$ is the result of reflecting the graph of $y = \cos(x)$ in the line $y = x$



2 C $\cos x = -\frac{2}{3}, 2\pi < x < 3\pi$

Since cosine is negative we are looking in the second quadrant.

$\therefore \sin x = +\sqrt{1 - \left(-\frac{2}{3}\right)^2}$

$\therefore \sin x = \frac{\sqrt{5}}{3}$

3 E $\cos(x) = -\frac{1}{10}, x \in \left(\frac{\pi}{2}, \pi\right)$

As we are in the second quadrant $\cot(x)$ will be negative.

$\sin(x) = +\sqrt{1 - \left(-\frac{1}{10}\right)^2}$

$\sin(x) = \frac{3\sqrt{11}}{10}$

$\cot(x) = \frac{\cos(x)}{\sin(x)}$

$\therefore \cot(x) = -\frac{1}{10} \div \frac{3\sqrt{11}}{10}$

$\therefore \cot(x) = -\frac{1}{10} \times \frac{10}{3\sqrt{11}}$

$\therefore \cot(x) = -\frac{\sqrt{11}}{33}$

4 D $y = 2 + \sec(3x), x \in \left(-\frac{\pi}{6}, \frac{7\pi}{6}\right)$

The graph of $y = \sec(3x)$ has range $R\backslash(-1, 1)$

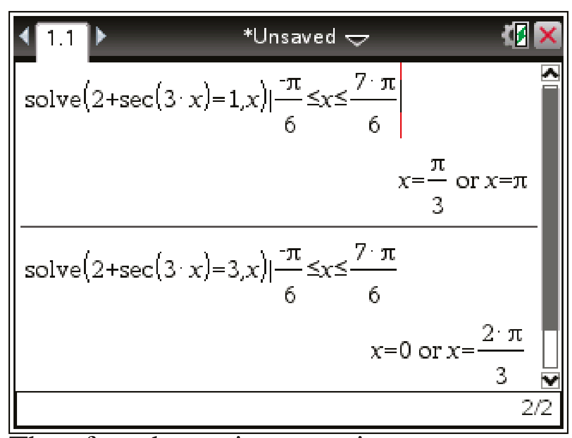
Thus the range of $y = 2 + \sec(3x)$ is $R\backslash(1, 3)$. Which implies that stationary points occur when $y = 1$ and $y = 3$

In this instance the CAS calculator will be used to solve the following equations

for x over $\left(-\frac{\pi}{6}, \frac{7\pi}{6}\right)$

$2 + \sec(3x) = 1$ ①

$2 + \sec(3x) = 3$ ②



Therefore the stationary points are at:

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

5 **A** $\sin x = -\dfrac{1}{3}$

$\therefore \cos x = \pm\sqrt{1 - \left(-\dfrac{1}{3}\right)^2}$

$\therefore \cos x = \pm\dfrac{2\sqrt{2}}{3}$

6 **A** For $y = \cos^{-1}(1 - 5x)$ to be defined

$-1 \le 1 - 5x \le 1$

$\therefore -2 \le -5x \le 0$

$\therefore 0 \le x \le \dfrac{2}{5}$

Therefore the implied domain is

$\left[0, \dfrac{2}{5}\right]$

7 **E** $(1 + \tan x)^2 + (1 - \tan x)^2$

$= 1 + 2\tan x + \tan^2 x + 1$

$\quad - 2\tan x + \tan^2 x$

$= 2 + 2\tan^2 x$

$= 2(1 + \tan^2 x)$

$= 2\sec^2 x$

8 **D** $\cos^2(3x) = \dfrac{1}{4}, 0 \le x \le \pi$

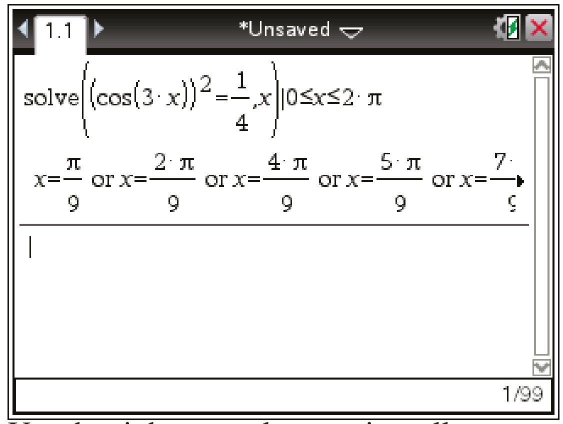
$\Rightarrow\ 0 \le 3x\ \le 3\pi$

$\therefore \cos(3x) = +\dfrac{1}{2}$

$\therefore 3x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{8\pi}{3}$

$\therefore x = \dfrac{\pi}{9}, \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}, \dfrac{8\pi}{9}$

Using **solve** on CAS yields:



Use the right arrow key to view all solutions.

Therefore there are 6 solutions.

9 **E** $\dfrac{\tan(2\theta)}{1 + \sec(2\theta)} = \dfrac{\dfrac{\sin(2\theta)}{\cos(2\theta)}}{1 + \dfrac{1}{\cos(2\theta)}}$

$= \dfrac{\sin(2\theta)}{\cos(2\theta)\left(1 + \dfrac{1}{\cos(2\theta)}\right)}$

$= \dfrac{\sin(2\theta)}{\cos(2\theta) + 1}$

$= \dfrac{2\sin\theta\cos\theta}{(2\cos^2\theta - 1) + 1}$

$= \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$

10 **E** $\dfrac{\pi}{2} < A < \pi, 0 < B < \dfrac{\pi}{2}$,

$\sin A = t$ and $\cos B = t$

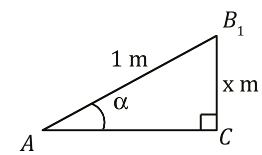
$\therefore \cos A = -\sqrt{1 - t^2}$ and

$\sin B = \sqrt{1 - t^2}$

$$\cos(B + A)$$
$$= \cos B \cos A - \sin B \sin A$$
$$= t \times -\sqrt{1 - t^2} - \sqrt{1 - t^2} \times t$$
$$= -t\sqrt{1 - t^2} - t\sqrt{1 - t^2}$$
$$= -2t\sqrt{1 - t^2}$$

Solutions to extended-response questions

1 a Consider $\triangle AB_1C$ as shown. $AC = \sqrt{1 - x^2}$

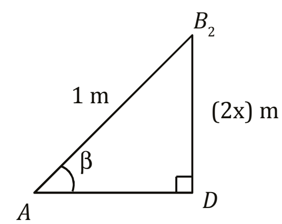


i $\sin\alpha = \dfrac{x}{1} = x$

ii $\cos\alpha = \dfrac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}$

iii $\tan\alpha = \dfrac{x}{\sqrt{1 - x^2}}$

Now consider $\triangle AB_2D$ as shown.
$AD = \sqrt{1 - 4x^2}$



iv $\sin\beta = \dfrac{2x}{1} = 2x$

v $\cos\beta = \dfrac{\sqrt{1 - 4x^2}}{1} = \sqrt{1 - 4x^2}$

vi $\tan\beta = \dfrac{2x}{\sqrt{1 - 4x^2}}$

b **i** $\sin(\beta - \alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha$

$$= 2x\sqrt{1 - x^2} - x\sqrt{1 - 4x^2}$$

ii $\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$

$$= \sqrt{(1 - 4x^2)(1 - x^2)} + 2x^2$$

iii $\tan(\beta - \alpha) = \dfrac{\sin(\beta - \alpha)}{\cos(\beta - \alpha)}$

$$= \frac{2x\sqrt{1 - x^2} - x\sqrt{1 - 4x^2}}{\sqrt{(1 - 4x^2)(1 - x^2)} + 2x^2}$$

iv $\tan(2\alpha) = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$$= \frac{2x}{\sqrt{1-x^2}} \div \left(1 - \left(\frac{x}{\sqrt{1-x^2}}\right)^2\right)$$

$$= \frac{2x}{\sqrt{1-x^2}} \div \left(1 - \frac{x^2}{1-x^2}\right)$$

$$= \frac{2x}{\sqrt{1-x^2}} \div \frac{1-x^2-x^2}{1-x^2}$$

$$= \frac{2x}{\sqrt{1-x^2}} \times \frac{1-x^2}{1-2x^2} = \frac{2x\sqrt{1-x^2}}{1-2x^2}$$

v $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2x\sqrt{1-x^2}$

vi $\cos(2\alpha) = 1 - 2\sin^2\alpha = 1 - 2x^2$

c $\angle B_2AB_1 = \beta - \alpha$

$$= \cos^{-1}(\sqrt{(1-4x^2)(1-x^2)} + 2x^2)$$

$$= \cos^{-1}(\sqrt{(1-4(0.3)^2)(1-0.3^2)} + 2(0.3)^2)$$

$$= \cos^{-1}(\sqrt{0.5824} + 0.18)$$

$$= \cos^{-1}(0.943\,15\ldots) = 0.338\,80$$

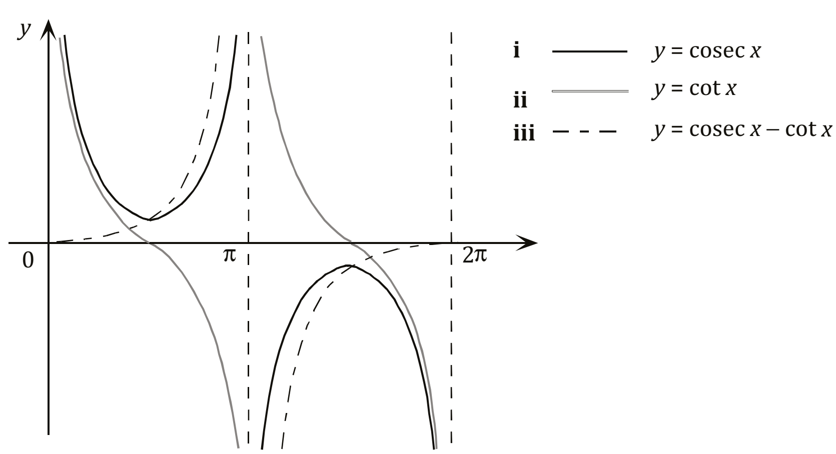
$$2\alpha = \cos^{-1}(1-2x^2)$$

$$= \cos^{-1}(1-2(0.3)^2)$$

$$= \cos^{-1}(0.82) = 0.609\,38\ldots$$

$\angle B_2AB_1 = 0.34$ and $2\alpha = 0.61$, correct to two decimal places.

2 a



$$\operatorname{cosec} x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x}$$

$$= \frac{1 - \cos x}{\sin x} = \frac{1 - \left(1 - 2\sin^2\left(\frac{1}{2}x\right)\right)}{2\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)} = \tan\left(\frac{1}{2}x\right)$$

b **i** Given $0 < x < \pi$

then $0 < \frac{1}{2}x < \frac{\pi}{2}$

$\therefore \quad \tan\left(\frac{1}{2}x\right) > 0$

$\therefore \quad \operatorname{cosec} x - \cot x > 0$

$\therefore \quad \operatorname{cosec} x > \cot x$ for all $x \in (0, \pi)$

ii Given $\pi < x < 2\pi$

then $\frac{\pi}{2} < \frac{1}{2}x < \pi$

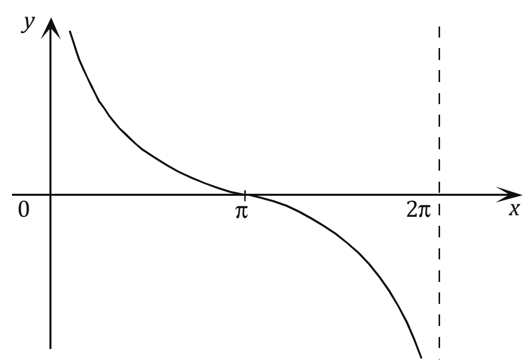
$\therefore \quad \tan\left(\frac{1}{2}x\right) < 0$

$\therefore \quad \operatorname{cosec} x - \cot x < 0$

$\therefore \quad \operatorname{cosec} x < \cot x$ for all $x \in (\pi, 2\pi)$

c $y = \cot\left(\frac{x}{2}\right)$

or $y = \operatorname{cosec} x + \cot x$



d **i** $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}, \sin\theta \neq 0$

$$= \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1+\cos 2\left(\frac{\theta}{2}\right)}{\sin 2\left(\frac{\theta}{2}\right)}$$

$$= \frac{1+\left(2\cos^2\frac{\theta}{2}-1\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2},$$ as required to prove.

ii $\cot\dfrac{\pi}{8} = \operatorname{cosec}\dfrac{\pi}{4} + \cot\dfrac{\pi}{4}$

$= \sqrt{2} + 1$

$\cot\dfrac{\pi}{12} = \operatorname{cosec}\dfrac{\pi}{6} + \cot\dfrac{\pi}{6}$

$= 2 + \sqrt{3}$

iii $1 + \cot^2\dfrac{\pi}{8} = \operatorname{cosec}^2\dfrac{\pi}{8}$

$\therefore \quad 1 + (\sqrt{2}+1)^2 = \dfrac{1}{\sin^2\frac{\pi}{8}}$

$\therefore \quad 1 + 2 + 2\sqrt{2} + 1 = \dfrac{1}{\sin^2\frac{\pi}{8}}$

$\therefore \quad \sin^2\dfrac{\pi}{8} = \dfrac{1}{4+2\sqrt{2}}$

$\therefore \quad \sin\dfrac{\pi}{8} = \sqrt{\dfrac{1}{4+2\sqrt{2}}}$

The negative square root is not appropriate since $\dfrac{\pi}{8}$ is in the first quadrant.

$\therefore \quad \sin\dfrac{\pi}{8} = \dfrac{1}{\sqrt{4+2\sqrt{2}}}$

e $\operatorname{cosec}\theta + \cot\theta = \cot\dfrac{\theta}{2}$ ①

$\therefore \quad \operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ ②

and $\operatorname{cosec} 4\theta + \cot 4\theta = \cot 2\theta$ ③

Adding ①, ② and ③ yields

$$\operatorname{cosec}\theta + \cot\theta + \operatorname{cosec} 2\theta + \cot 2\theta + \operatorname{cosec} 4\theta + \cot 4\theta = \cot\frac{\theta}{2} + \cot\theta + \cot 2\theta$$

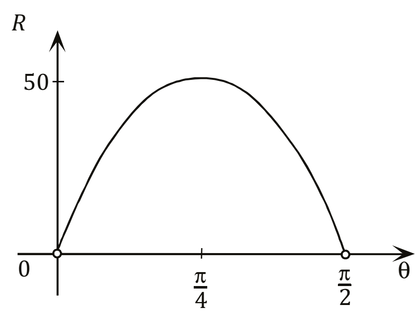
$$\therefore \quad \operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta = \cot\frac{\theta}{2} - \cot 4\theta$$

3 a i $\sin\theta = \dfrac{CD}{10} \qquad \therefore CD = 10\sin\theta$

$\cos\theta = \dfrac{AD}{10} \qquad \therefore AD = 10\cos\theta$

$$\begin{aligned}\text{Area of rectangle} &= AD \times CD \\ &= 100\sin\theta\,\cos\theta \\ &= 50\sin 2\theta\end{aligned}$$

ii



iii From the graph, the maximum value of R is 50 square units.

iv The maximum occurs when $\theta = \dfrac{\pi}{4}$ (when the rectangle is a square).

b i $\cos\theta = \dfrac{AD}{AC}, \qquad \sin\theta = \dfrac{CD}{AC} \qquad$ and $\tan\dfrac{\theta}{2} = \dfrac{CG}{AC}$

$\therefore AD = AC\cos\theta \qquad CD = AC\sin\theta \qquad$ and $CG = AC\tan\dfrac{\theta}{2}$

$\qquad = 10\cos\theta \qquad\qquad = 10\sin\theta \qquad\qquad = 10\tan\dfrac{\theta}{2}$

Volume, $V = AD \times CD \times CG$

$\qquad = 1000\cos\theta\sin\theta\tan\dfrac{\theta}{2}$, as required.

ii $V = 1000\cos\left(2 \times \frac{\theta}{2}\right)\sin\left(2 \times \frac{\theta}{2}\right)\tan\frac{\theta}{2}$

$$= 1000\left(1 - 2\sin^2\frac{\theta}{2}\right) \times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \times \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$\therefore \quad V = 1000\left(1 - 2\sin^2\frac{\theta}{2}\right) \times 2\sin^2\frac{\theta}{2}$

$\therefore \quad V = 2000\sin^2\frac{\theta}{2} - 4000\sin^4\frac{\theta}{2}$

$\therefore \quad a = 2000$ and $b = -4000$

iii Let $p = \sin^2\frac{\theta}{2}$

then $V = 2000p - 4000p^2$

iv For $0 < \theta < \frac{\pi}{2}$

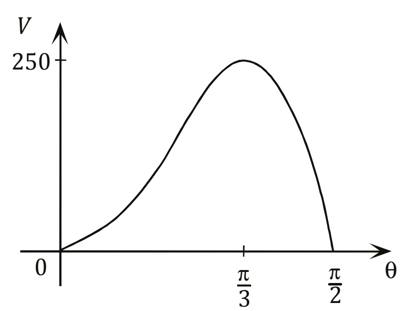
$0 < \frac{\theta}{2} < \frac{\pi}{4}$

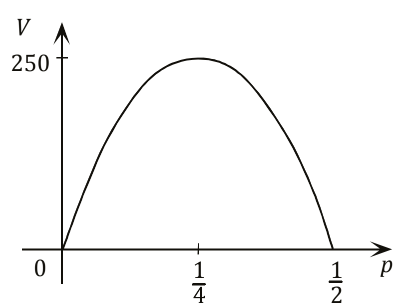
$\therefore \quad 0 < \sin\frac{\theta}{2} < \frac{\sqrt{2}}{2}$

$\therefore \quad 0 < \sin^2\frac{\theta}{2} < \frac{1}{2}$

$\therefore \quad 0 < p < \frac{1}{2}$

v





vi From the graph of V against p, the axis of symmetry gives the maximum volume as 250 cubic units when $p = \frac{1}{4}$.

When $p = \frac{1}{4}$, $\sin^2 \frac{\theta}{2} = \frac{1}{4}$

$\therefore \quad \sin \frac{\theta}{2} = \frac{1}{2} \quad$ as $\sin \frac{\theta}{2} > 0$

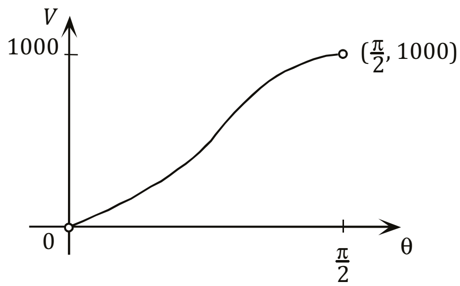
$\therefore \quad \frac{\theta}{2} = \frac{\pi}{6} \quad$ as $0 < \frac{\theta}{2} < \frac{\pi}{4}$

$\therefore \quad \theta = \frac{\pi}{3}$

c **i** If $\angle CAD = \angle GAC = \theta$

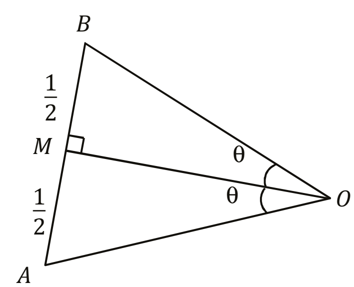
then
$$V = 1000 \cos\theta \sin\theta \tan\theta \text{ (from } \mathbf{b\ i})$$
$$= 1000 \cos\theta \, \sin\theta \times \frac{\sin\theta}{\cos\theta}$$
$$= 1000 \sin^2\theta, 0 < \theta < \frac{\pi}{2}$$

ii



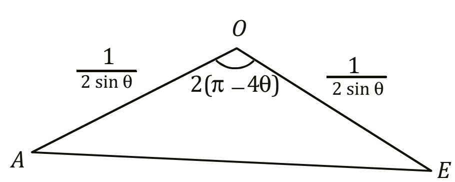
iii V is an increasing function. As the angle θ gets larger, so does the volume of the cuboid. $0 < \theta < \frac{\pi}{2}$ for the cuboid to exist.

4 **a** Consider $\triangle AOB$.
Let M be the midpoint of AB.
Consider $\triangle AMO$.



$$\sin\theta = \frac{\frac{1}{2}}{AO}$$
$$\therefore \quad AO = \frac{1}{2\sin\theta}$$
$$= \text{radius of the circle.}$$

Consider $\triangle AOE$.

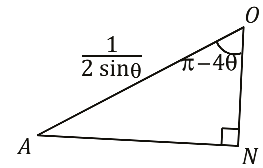


$$EO = AO = \frac{1}{2\sin\theta}$$
$$\angle AOE = 2\pi - 4 \times 2\theta$$
$$= 2\pi - 8\theta$$
$$= 2(\pi - 4\theta)$$

Note: $\pi - 4\theta > 0$ which implies $0 < \theta < \frac{\pi}{4}$.

Let N be the midpoint of AE.

Consider $\triangle ANO$.



$$\sin(\pi - 4\theta) = \frac{AO}{\frac{1}{2\sin\theta}}$$

$$\therefore \quad AN = \frac{\sin(\pi - 4\theta)}{2\sin\theta} = \frac{\sin 4\theta}{2\sin\theta}$$

Now $\quad AE = 2AN$ and $AE = p$

$$\therefore \quad p = 2 \times \frac{\sin 4\theta}{2\sin\theta} = \frac{\sin 4\theta}{\sin\theta}, \text{ as required.}$$

b $p = \frac{\sin 4\theta}{\sin\theta}$

$$= \frac{2\sin 2\theta\cos 2\theta}{\sin\theta}$$

$$= \frac{2(2\sin\theta\cos\theta)(2\cos^2\theta - 1)}{\sin\theta}$$

$$= 4\cos\theta\,(2\cos^2\theta - 1)$$

$$= 8\cos^3\theta - 4\cos\theta$$

c **i** If $x = \cos\theta$, $\quad p = 8x^3 - 4x$

If $p = \sqrt{3}$, $\quad \sqrt{3} = 8x^3 - 4x$

$$\therefore \quad 8x^3 - 4x - \sqrt{3} = 0, \text{ as required.}$$

ii If $x = \frac{\sqrt{3}}{2}$, $8x^3 - 4x - \sqrt{3}$ becomes

$$8\left(\frac{\sqrt{3}}{2}\right)^3 - 4\left(\frac{\sqrt{3}}{2}\right) - \sqrt{3} = 8\left(\frac{3\sqrt{3}}{8}\right) - 2\sqrt{3} - \sqrt{3}$$

$$= 3\sqrt{3} - 3\sqrt{3}$$

$$= 0$$

Therefore $x = \frac{\sqrt{3}}{2}$ is a solution to the equation $8x^3 - 4x - \sqrt{3} = 0$ and $x - \frac{\sqrt{3}}{2}$ is a factor of $8x^3 - 4x - \sqrt{3}$

Dividing to find the quadratic factor yields

$$8x^3 - 4x - \sqrt{3} = \left(x - \frac{\sqrt{3}}{2}\right)(8x^2 + 4\sqrt{3}x + 2)$$

The discriminant $\triangle = (4\sqrt{3})^2 - 4(8)(2)$

$$= 48 - 64, \text{ which is } < 0$$

Therefore the quadratic factor is irreducible and $\frac{\sqrt{3}}{2}$ is the only solution.

iii If $p = \sqrt{3}$, then $x = \frac{\sqrt{3}}{2}$

$\therefore \quad \cos\theta = \frac{\sqrt{3}}{2}$

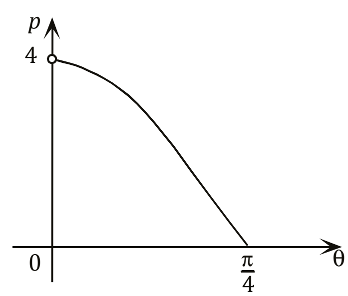
$\therefore \quad \theta = \frac{\pi}{6}$

iv radius $= \frac{1}{2\sin\theta}$

$= \frac{1}{2\sin\frac{\pi}{6}}$

$= 1$

d $p = \frac{\sin 4\theta}{\sin\theta}, \theta \in \left(0, \frac{\pi}{4}\right]$



e If $A = E$, then $4 \times 2\theta = 2\pi$

$\therefore \quad \theta = \frac{\pi}{4}$

f **i** If $AE = 1$, then $p = 8x^3 - 4x$

becomes $1 = 8x^3 - 4x$

$\therefore \quad 8x^3 - 4x - 1 = 0$

Also $5 \times 2\theta = 2\pi$

$\therefore \quad \theta = \frac{\pi}{5}$

$\therefore \quad p = \frac{\sin\left(4 \times \frac{\pi}{5}\right)}{\sin\frac{\pi}{5}} = 1$

ii $8x^3 - 4x - 1 = 0, x = \cos\frac{\pi}{5}$

If $x = \frac{1}{4}(\sqrt{5} + 1)$

then $8x^3 - 4x - 1 = 8\left(\frac{1}{4}(\sqrt{5} + 1)\right)^3 - 4\left(\frac{1}{4}(\sqrt{5} + 1)\right) - 1$

$$= \frac{8(\sqrt{5} + 1)^3}{4^3} - (\sqrt{5} + 1) - 1$$

$$= \frac{8(5\sqrt{5} + 15 + 3\sqrt{5} + 1)}{64} - \sqrt{5} - 1 - 1$$

$$= \frac{16 + 8\sqrt{5}}{8} - \sqrt{5} - 2$$

$$= 2 + \sqrt{5} - \sqrt{5} - 2 = 0$$

$\therefore$ $8x^3 - 4x - 1 = 0$ when $x = \frac{1}{4}(\sqrt{5} + 1)$

and $\frac{1}{4}(\sqrt{5} + 1) = \cos\frac{\pi}{5}$

5 a i $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}, \cos x \neq 0, \sin x \neq 0$

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$

$$= \frac{1}{\frac{1}{2}(2\sin x \cos x)}$$

$$= \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x$$

ii $\tan x = \cot x$ $\therefore$ $\frac{\sin x}{\cos x} = \frac{\cos x}{\sin x}, \cos x \neq 0, \sin x \neq 0$

$\therefore$ $\sin^2 x = \cos^2 x$

$$= 1 - \sin^2 x$$

$\therefore$ $2\sin^2 x = 1$

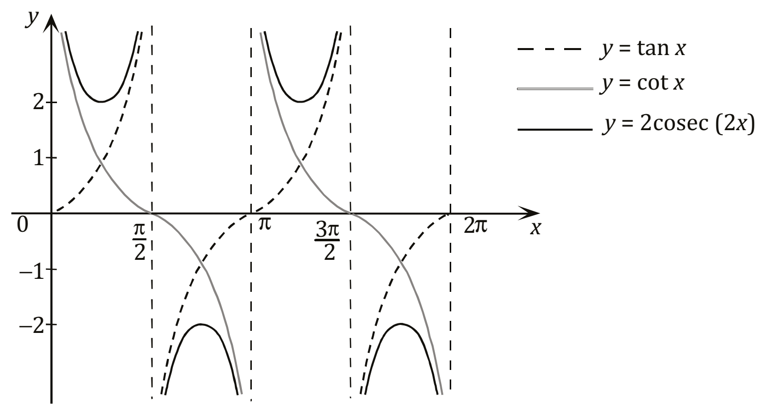
$\therefore$ $\sin^2 x = \frac{1}{2}$

$\therefore$ $\sin x = \pm\frac{1}{\sqrt{2}}$

$\therefore$ $x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}, \ldots$

$$= n\pi \pm \frac{\pi}{4}, n \in Z$$

iii



b **i** $\cot 2x + \tan x = \dfrac{\cos 2x}{\sin 2x} + \dfrac{\sin x}{\cos x}, \sin 2x \neq 0, \cos x \neq 0$

$$= \frac{\cos 2x \cos x + \sin x \sin 2x}{\sin 2x \cos x}$$

$$= \frac{\cos(2x - x)}{\sin 2x \cos x}$$

$$= \frac{1}{\sin 2x} = \text{cosec } 2x, \text{ as required to prove.}$$

ii $\cot 2x = \tan x$

$\therefore \quad \dfrac{\cos 2x}{\sin 2x} = \dfrac{\sin x}{\cos x}, \sin 2x \neq 0, \cos x \neq 0$

$\therefore \quad \cos 2x \cos x = \sin 2x \sin x$

$\therefore \quad (1 - 2\sin^2 x)\cos x = (2\sin x \cos x)\sin x$

$\therefore \quad \cos x - 2\sin^2 x \cos x = 2\sin^2 x \cos x$

$\therefore \quad \cos x = 4\sin^2 x \cos x$

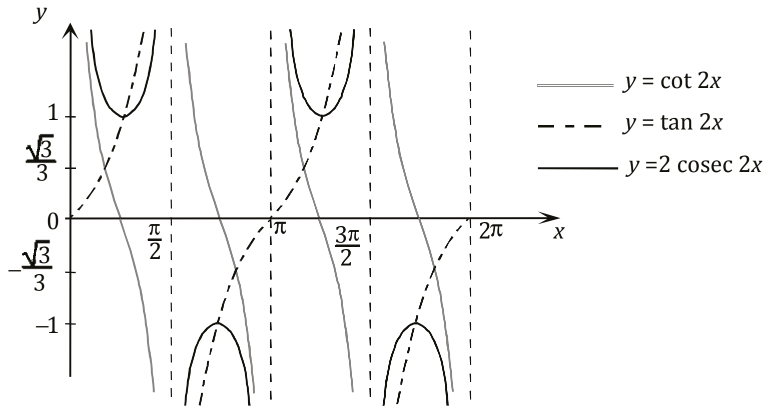
$\therefore \quad 4\sin^2 x = 1$

$\therefore \quad \sin^2 x = \dfrac{1}{4}$

$\therefore \quad \sin x = \pm\dfrac{1}{2}$

$\therefore \quad x = \pm\dfrac{\pi}{6}, \pm\dfrac{5\pi}{6}, \pm\dfrac{7\pi}{6}, \pm\dfrac{11\pi}{6}, \ldots = n\pi \pm \dfrac{\pi}{6}, n \in Z$

iii



c **i** $\cot(mx) + \tan(nx) = \dfrac{\cos(mx)}{\sin(mx)} + \dfrac{\sin(nx)}{\cos(nx)}, \sin(mx) \neq 0, \cos(mx) \neq 0$

$$= \frac{\cos(mx)\cos(nx) + \sin(mx)\sin(nx)}{\sin(mx)\cos(nx)}$$

$$= \frac{\cos(mx - nx)}{\sin(mx)\cos(nx)}$$

$$= \frac{\cos((m-n)x)}{\sin(mx)\cos(nx)},$$ as required to prove.

ii From **c i**, $\cot(6x) + \tan(3x) = \dfrac{\cos((6-3)x)}{\sin(6x)\cos(3x)}$

$$= \frac{\cos(3x)}{\sin(6x)\cos(3x)} = \frac{1}{\sin(6x)}$$

$= \text{cosec}\,(6x)$, as required.

6 **a** **i** $\angle BAE = \angle BEA$ (isosceles △)

$\therefore \quad 2\angle BAE + 36^\circ = 180^\circ$

$\therefore \quad \angle BAE = \left(\dfrac{180 - 36}{2}\right)^\circ = 72^\circ$

$\angle AEC = \angle BEA = \angle BAE = 72^\circ$

$\angle ACE = \angle AEC = 72^\circ$ (isosceles △)

ii $\angle BAC = \angle BAE - \angle CAE$

$= \angle BAE - (180^\circ - 2\angle AEC)$

$= 72^\circ - (180 - 2 \times 72)^\circ = 36^\circ$

b Consider $\triangle ABC$

$\angle ABC = \angle BAC = 36°$

$\therefore \quad \triangle ABC$ is isosceles with $BC = AC = 1$

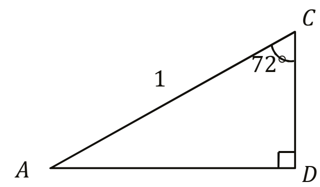
Consider $\triangle ACD$

$\angle CAD = 180° - (90 + 72)° = 18°$

$\sin 18° = \dfrac{CD}{1}$

Now $\quad BD = BC + CD$

$\qquad = 1 + \sin 18°,$ as required.



c Consider $\triangle ADE$

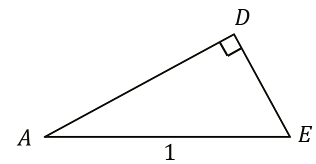
$\angle DAE = \angle CAE$ (isosceles $\triangle ACE$) $= 18°$

$\sin 18° = \dfrac{DE}{1}$

Now $\quad BE = BD + DE$

$\qquad = (1 + \sin 18°) + \sin 18°$

$\qquad = 1 + 2\sin 18°$



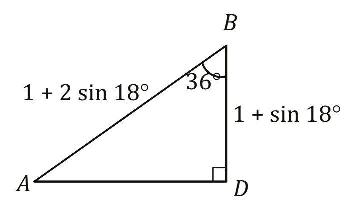
Since $\triangle ABE$ is isosceles,

$AB = 1 + 2\sin 18°$ also.

Now consider $\triangle ABD$

$\cos 36° = \dfrac{1 + \sin 18°}{1 + 2\sin 18°}$

as required to prove.



d

$$\cos 36° = \frac{1 + \sin 18°}{1 + 2\sin 18°}$$

$$\therefore \quad 1 - 2\sin^2 18° = \frac{1 + \sin 18°}{1 + 2\sin 18°}$$

$$\therefore \quad (1 - 2\sin^2 18°)(1 + 2\sin 18°) = 1 + \sin 18°$$

$$\therefore \quad 1 - 2\sin^2 18° + 2\sin 18° - 4\sin^3 18° = 1 + \sin 18°$$

$$\therefore \quad 4\sin^3 18° + 2\sin^2 18° - \sin 18° = 0$$

$$\therefore \quad \sin 18°(4\sin^2 18° + 2\sin 18° - 1) = 0$$

$$\therefore \quad 4\sin^2 18° + 2\sin 18° - 1 = 0, \text{ as required.}$$

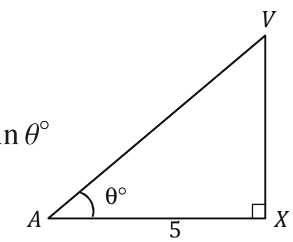
e Let $a = \sin 18° \quad \therefore 4\sin^2 18° + 2\sin 18° - 1 = 0$

$$4a^2 + 2a - 1 = 0$$

Using the general quadratic formula

$$a = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2 \times 4}$$
$$= \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$
$$\therefore \quad \sin 18^\circ = \frac{-1 + \sqrt{5}}{4} \qquad \text{since } \sin 18^\circ > 0$$

7 a i Volume of a pyramid, $V = \frac{1}{3}Ah$, where A is the area of the base and h is the height of the pyramid, VX.



Since $ABCD$ is a rectangle,

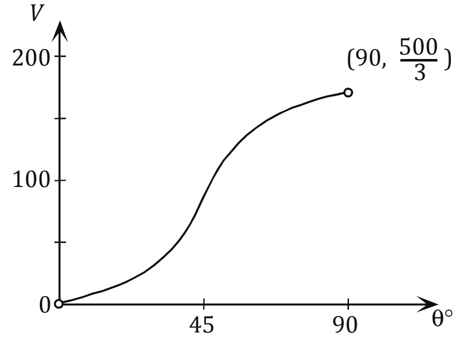
$$A = AD \times CD = AC \cos\theta^\circ \times AC \sin\theta^\circ$$
$$= 10\cos\theta^\circ \times 10\sin\theta^\circ$$
$$= 100\cos\theta^\circ \sin\theta^\circ$$

Consider $\triangle AVX \qquad AX = \frac{1}{2}AC = 5$

$$\tan\theta^\circ = \frac{VX}{5}$$
$$\therefore \quad VX = 5\tan\theta^\circ$$
$$\therefore \quad h = 5\tan\theta^\circ$$

Therefore
$$V = \frac{1}{3} \times 100\cos\theta^\circ \sin\theta^\circ \times 5\tan\theta^\circ$$
$$= \frac{500}{3}\cos\theta^\circ \sin\theta^\circ \times \frac{\sin\theta^\circ}{\cos\theta^\circ}$$
$$= \frac{500}{3}\sin^2\theta^\circ, \text{ as required.}$$

ii



iii V is an increasing function. As the angle θ gets larger, so does the volume of the pyramid. $0 < \theta < 90$ for the pyramid to exist.

b **i** $\tan\dfrac{\theta°}{2} = \dfrac{VX}{5}$

$\therefore \quad VX = 5\tan\dfrac{\theta°}{2} \qquad \therefore h = \dfrac{5\sin\dfrac{\theta°}{2}}{\cos\dfrac{\theta°}{2}}$

From **a i**, $A = 100\cos\theta°\sin\theta°$

$$= 100\left(1 - 2\sin^2\frac{\theta°}{2}\right)\left(2\sin\frac{\theta°}{2}\cos\frac{\theta°}{2}\right)$$

$$= 200\sin\frac{\theta°}{2}\cos\frac{\theta°}{2}\left(1 - 2\sin^2\frac{\theta°}{2}\right)$$

Now $V = \dfrac{1}{3}Ah$

$$= \frac{1}{3} \times 200\sin\frac{\theta°}{2}\cos\frac{\theta°}{2}\left(1 - 2\sin 2\frac{\theta°}{2}\right) \times \frac{5\sin\frac{\theta°}{2}}{\cos\frac{\theta°}{2}}$$

$$= \frac{1000}{3}\sin^2\frac{\theta°}{2}\left(1 - 2\sin^2\frac{\theta°}{2}\right)\text{, as required.}$$

ii The maximal domain is $\theta \in (0, 90)$

iii Let $a = \sin^2\dfrac{\theta°}{2}$

$\therefore \quad V = \dfrac{1000}{3} \times a(1 - 20) = \dfrac{-2000}{3}a^2 + \dfrac{1000}{3}a$

iv V is a concave down parabola with a local maximum tuming point at the axis of symmetry, when

$$a = \frac{-1000}{3} \div \left(2 \times \frac{-2000}{3}\right)$$

$$= \frac{-1000}{3} \div \frac{-4000}{3} = \frac{-1000}{3} \times \frac{-3}{4000} = \frac{1}{4}$$

When $a = \dfrac{1}{4}$, $V = \dfrac{-2000}{3}\left(\dfrac{1}{4}\right)^2 + \dfrac{1000}{3}\left(\dfrac{1}{4}\right)$

$$= \frac{-2000}{48} + \frac{1000}{12}$$

$$= \frac{2000}{48} = \frac{125}{3}$$

When $a = \frac{1}{4}$, $\quad \sin^2 \frac{\theta°}{2} = \frac{1}{4}$

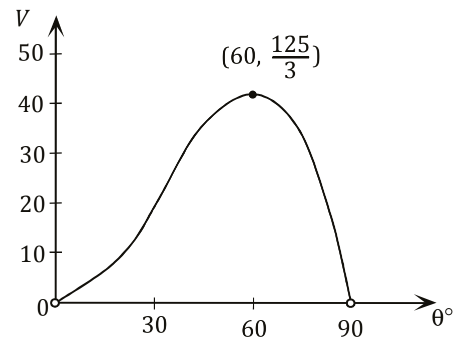
$\therefore \quad \sin \frac{\theta°}{2} = \frac{1}{2}$ since $0 < \frac{\theta}{2} < 45$

$\therefore \quad \frac{\theta}{2} = 30$

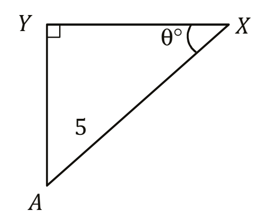
$\therefore \quad \theta = 60$

The maximum value of V is $\frac{125}{3}$ cubic units and the value of θ for which this occurs is 60°.

v $V = \frac{1000}{3} \sin^2 \frac{\theta°}{2}\left(1 - 2\sin^2 \frac{\theta°}{2}\right)$



8 a i $\quad V = \frac{1}{3}Ah$



As in **7 a i**, $\quad A = 100\cos\theta° \sin\theta° \quad$ and $h = vx$

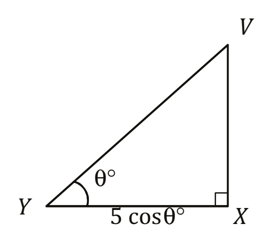
Consider $\triangle AYX$

$\angle AXY = \angle XAD = \theta°$ (alternate angles)

$\cos\theta° = \frac{XY}{5}$

$\therefore \quad XY = 5\cos\theta°$

Now consider $\triangle VYX$



$\tan\theta° = \frac{VX}{5\cos\theta°}$

$\therefore \quad VX = 5\cos\theta° \tan\theta°$

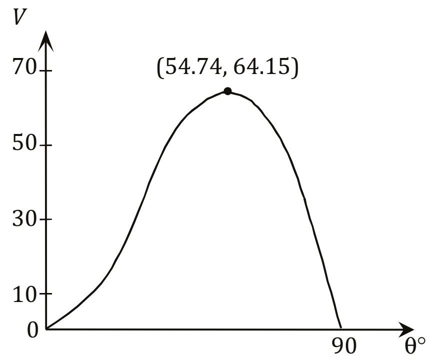
$= 5\cos\theta° \times \frac{\sin\theta°}{\cos\theta°}$

$\therefore \quad h = 5\sin\theta°$

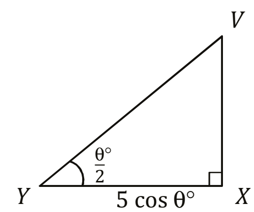
$\therefore \quad V = \frac{1}{3} \times 100\cos\theta° \sin\theta° \times 5\sin\theta°$

$= \frac{500}{3}\cos\theta° \sin^2\theta°$

ii Using a CAS calculator to graph V against θ, $0 < \theta < 90$, the maximum volume is given as 64.15 cubic units when $\theta° = 54.74°$ (correct to two decimal places).



b **i** $V = \frac{1}{3}Ah$



As in **7 a i**, $A = 100\cos\theta°\sin\theta°$ and $h = VX$

As in **8 a i**, $XY = 5\cos\theta°$

Consider $\triangle VYX$

$$\tan\frac{\theta°}{2} = \frac{VX}{5\cos\theta°}$$

$\therefore$ $h = VX$

$\therefore$ $= 5\cos\theta°\tan\frac{\theta°}{2}$

Therefore $V = \frac{1}{3} \times 100\cos\theta°\sin\theta° \times 5\cos\theta°\tan\frac{\theta°}{2}$

$$= \frac{500}{3}\cos^2\theta° \times 2\sin\frac{\theta°}{2}\cos\frac{\theta°}{2} \times \frac{\sin\frac{\theta°}{2}}{\cos\frac{\theta°}{2}}$$

$$= \frac{500}{3}\cos^2\theta° \times 2\sin^2\frac{\theta°}{2}$$

$$= \frac{500}{3}\cos^2\theta°(1 - \cos\theta°), \text{ as required.}$$

ii The implied domain is $0 < \theta < 90$ for the pyramid to exist.

c Let $a = \cos\theta$

Since $0 < \theta < 90$

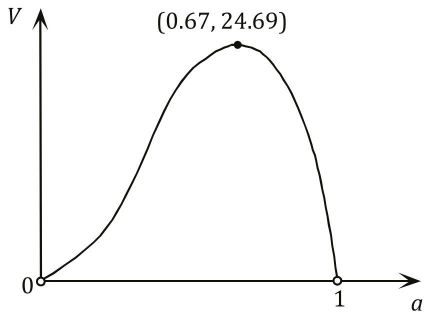
$0 < \cos\theta < 1$

$\therefore$ $0 < a < 1$

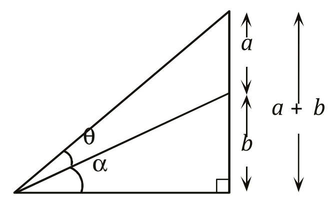
The CAS calculator gives a maximum volume of 24.69 cubic units when $a = 0.666\,66\ldots \approx 0.67$

i.e., $\cos\theta = 0.666\,66\ldots$

$\therefore \quad \theta = 48.19$ (correct to two decimal places)



9 a $\tan(\theta + \alpha) = \dfrac{a + b}{x}$



$$\therefore \quad \theta + \alpha = \tan^{-1}\left(\frac{a+b}{x}\right)$$

$$\tan\alpha = \frac{b}{x}$$

$$\therefore \quad \alpha = \tan^{-1}\left(\frac{b}{x}\right)$$

$$\theta = (\theta + \alpha) - \alpha$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right), \text{ as required.}$$

b $\tan\theta = \tan\left(\tan^{-1}\left(\dfrac{a+b}{x}\right) - \tan^{-1}\left(\dfrac{b}{x}\right)\right)$

$$= \frac{\tan\left(\tan^{-1}\left(\frac{a+b}{x}\right)\right) - \tan\left(\tan^{-1}\left(\frac{b}{x}\right)\right)}{1 + \tan\left(\tan^{-1}\left(\frac{a+b}{x}\right)\right)\tan\left(\tan^{-1}\left(\frac{b}{x}\right)\right)}$$

$$= \frac{\frac{a+b}{x} - \frac{b}{x}}{1 + \frac{a+b}{x} \times \frac{b}{x}}$$

$$= \frac{a + b - b}{x} \div \frac{x^2 + b(a+b)}{x^2}$$

$$= \frac{a}{x} \times \frac{x^2}{x^2 + ab + b^2}$$

$$= \frac{ax}{x^2 + ab + b^2}, \text{ as required.}$$

c **i** If $\theta = \frac{\pi}{4}$, $\tan\theta = \frac{ax}{x^2 + ab + b^2}$

becomes $\tan\frac{\pi}{4} = \frac{ax}{x^2 + ab + b^2}$

$\therefore\quad 1 = \frac{ax}{x^2 + ab + b^2}$

$\therefore\quad x^2 + ab + b^2 = ax$

$\therefore\quad x^2 - ax + ab + b^2 = 0$

Using the general quadratic formula

$$\therefore\quad x = \frac{-(-a) \pm \sqrt{(-a)^2 - 4(1)(ab + b^2)}}{2(1)}$$

$$= \frac{a \pm \sqrt{a^2 - 4(ab + b^2)}}{2}$$

$$= \frac{a \pm \sqrt{a^2 - 4b(a + b)}}{2}$$

ii If $a = 2(1 + \sqrt{2})$ and $b = 1$,

$$\text{then}\quad x = \frac{2(1 + \sqrt{2}) \pm \sqrt{(2(1 + \sqrt{2}))^2 - 4(1)(2(1 + \sqrt{2}) + 1)}}{2}$$

$$= \frac{2(1 + \sqrt{2}) \pm \sqrt{4(1 + \sqrt{2})^2 - 4(2(1 + \sqrt{2}) + 1)}}{2}$$

$$= \frac{2(1 + \sqrt{2}) \pm 2\sqrt{(1 + \sqrt{2})^2 - (2(1 + \sqrt{2}) + 1)}}{2}$$

$$= (1 + \sqrt{2}) \pm \sqrt{1 + 2\sqrt{2} + 2 - 2 - 2\sqrt{2} - 1}$$

$$\therefore\quad x = 1 + \sqrt{2}$$

d If $a = 2(1 + \sqrt{2}), b = 1$ and $x = 1$

then $\theta = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$

$= \tan^{-1}\left(\frac{2(1+\sqrt{2})+1}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right)$

$\therefore \quad \theta = \tan^{-1}(2 + 2\sqrt{2} + 1) - \tan^{-1}(1)$

$= \tan^{-1}(3 + 2\sqrt{2}) - \tan^{-1}(1)$

$= 1.400\,87\ldots - \frac{\pi}{4}$

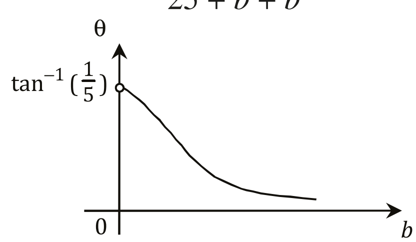
$= 0.615\,47\ldots$

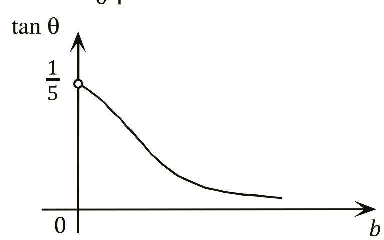
≈ 0.62

e **i** $a = 1, x = 5$

$\therefore \quad \theta = \tan^{-1}\left(\frac{b+1}{5}\right) - \tan^{-1}\left(\frac{b}{5}\right)$

$\therefore \quad \tan\theta = \frac{5}{25 + b + b^2}$

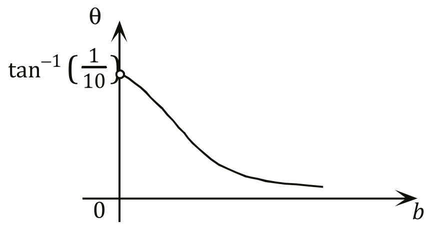


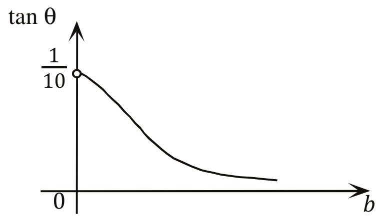


ii $a = 1, x = 10$

$\therefore \quad \theta = \tan^{-1}\left(\frac{b+1}{10}\right) - \tan^{-1}\left(\frac{b}{10}\right)$

$\therefore \quad \tan\theta = \frac{10}{100 + b + b^2}$

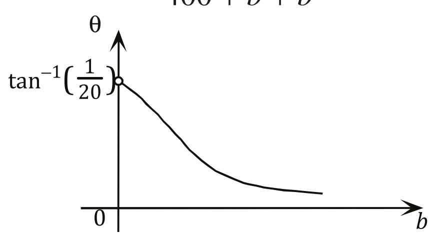


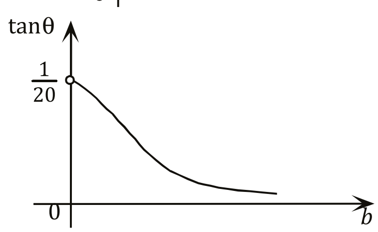


iii $a = 1, x = 20$

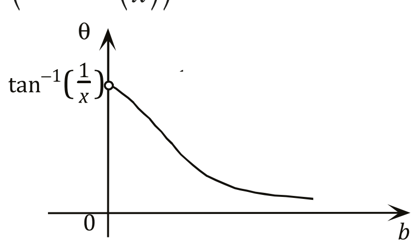
$$\therefore \quad \theta = \tan^{-1}\left(\frac{b+1}{20}\right) - \tan^{-1}\left(\frac{b}{20}\right)$$

$$\therefore \quad \tan\theta = \frac{20}{400 + b + b^2}$$

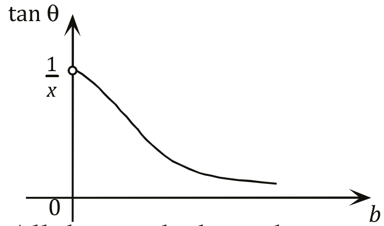




f In general, the graph of $\theta = \tan^{-1}\left(\frac{b+1}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$ has the b axis as a horizontal asymptote. The domain is R^+ and the range is $\left(0, \tan^{-1}\left(\frac{1}{x}\right)\right)$. The graph approaches $\left(0, \tan^{-1}\left(\frac{1}{x}\right)\right)$ as $b \to 0$. The function is decreasing as b increases.



In general, the graph of $\tan\theta = \frac{x}{x^2 + b + b^2}$ has the b axis as a horizontal asymptote, and approaches $\left(0, \frac{1}{x}\right)$ on the vertical axis. The domain is R^+ and the range is $\left(0, \left(\frac{1}{x}\right)\right)$. The function is decreasing as b increases.



All the graphs have the same shape.

10 a $\triangle ACD$ and $\triangle ACB$ have a common angle $\angle CAD$ and each has a right angle, therefore they are similar triangles.

b Consider $\triangle OCD$

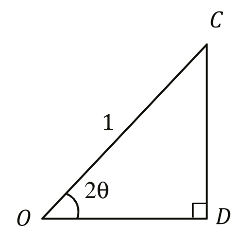
$OC = 1$

since the hemisphere shown has radius 1

$OD = x = \cos 2\theta$

$CD = y = \sin 2\theta$

The coordinates of C are $(\cos 2\theta, \sin 2\theta)$

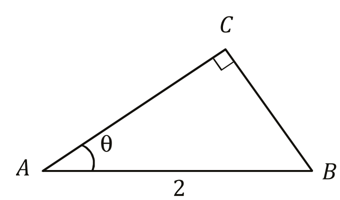


c i Consider $\triangle ABC$

$AB = 2$

as AB is a diameter of the circle $x^2 + y^2 = 1$

$\cos\theta = \frac{CA}{2} \qquad \therefore CA = 2\cos\theta$



ii $\sin\theta = \frac{CB}{2} \qquad \therefore CB = 2\sin\theta$

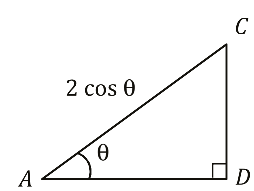
d Consider $\triangle ACD$

$$\sin\theta = \frac{CD}{2\cos\theta}$$

$$\therefore \quad CD = 2\sin\theta\cos\theta$$

From **b**, $\quad CD = \sin 2\theta$

$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta$, as required.



e From $\triangle ACD$, $\quad \cos\theta = \frac{AD}{2\cos\theta}$

$$\therefore \quad AD = 2\cos^2\theta$$

and $\quad OD = AD - AO = 2\cos^2\theta - 1$

since AO is a radius of the circle $x^2 + y^2 = 1$

From **b**, $OD = \cos 2\theta$

$\therefore \quad \cos 2\theta = 2\cos^2\theta - 1$, as required.

11 a To prove:

$$\cos(x)\cos(2x)\cos(4x)\cos(8x) = \frac{\sin(16x)}{16\sin(x)}$$

$$\begin{aligned}
\text{RHS} &= \frac{\sin(16x)}{16\sin(x)} \\
&= \frac{2\sin(8x)\cos(8x)}{16\sin(x)} \\
&= \frac{4\sin(4x)\cos(4x)\cos(8x)}{16\sin(x)} \\
&= \frac{8\sin(2x)\cos(2x)\cos(4x)\cos(8x)}{16\sin(x)} \\
&= \frac{16\sin(x)\cos(x)\cos(2x)\cos(4x)\cos(8x)}{16\sin(x)} \\
&= \cos(x)\cos(2x)\cos(4x)\cos(8x) \\
&= \text{LHS}
\end{aligned}$$

b To Prove:

$$\prod_{i=1}^{n} \cos(2^{i-1}x) = \frac{\sin(2^n x)}{2^n \sin(x)}$$

Step 1 First consider $P(1)$.

$$\cos(2^0 x) = \frac{\sin(2x)}{2\sin(x)}$$

which is true from the known identity

$$\sin 2x = 2\sin x \cos x$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true.

That is,

$$\prod_{i=1}^{k} \cos(2^{i-1}x) = \frac{\sin(2^k x)}{2^k \sin(x)}$$

Step 3 We now have to prove $P(k+1)$ is true.

That is,

$$\prod_{i=1}^{k+1} \cos(2^{i-1}x) = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin(x)}$$

$$\text{LHS} = \prod_{i=1}^{k+1} \cos(2^{i-1}x)$$

$$= \prod_{i=1}^{k} \cos(2^{i-1}x) \times \cos(2^k x)$$

$$= \frac{\sin(2^k x)}{2^k \sin(x)} \times \cos(2^k x)$$

$$= \frac{\sin(2^k x) \times \cos(2^k x)}{2^k \sin(x)}$$

$$= \frac{2 \times \sin(2^k x) \times \cos(2^k x)}{2^{k+1} \sin(x)}$$

$$= \frac{\sin(2^{k+1} x)}{2^{k+1} \sin(x)}$$

$$= \text{RHS}$$

Therefore if $P(k)$ is true, then $P(k+1)$ is true for every natural number k.
It follows from the principle of mathematical induction that $P(n)$ is true for every natural number n.

12 a $t_1 = \sin^2\theta$ and $t_n = 4t_{n-1}(1 - t_{n-1})$

$$t_2 = 4t_1(1 - t_1)$$
$$= 4\sin^2\theta(1 - \sin^2\theta)$$
$$= 4\sin^2\theta\cos^2\theta$$
$$= \sin^2(2\theta)$$
$$t_3 = 4t_2(1 - t_2)$$
$$= 4\sin^2 2\theta(1 - \sin^2 2\theta)$$
$$= 4\sin^2 2\theta\cos^2 2\theta$$
$$= \sin^2(4\theta)$$

b To prove $t_n = \sin^2(2^{n-1}\theta)$

Step 1 First consider $P(1)$.

$t_1 = \sin^2\theta$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true.

That is,

$$t_k = \sin^2(2^{k-1}\theta)$$

Step 3 We now have to prove $P(k+1)$ is true.

That is,

$$t_{k+1} = \sin^2(2^k\theta)$$

$$\begin{aligned} t_{k+1} &= 4t_k(1 - t_k) \\ &= 4\sin^2(2^{k-1}\theta)(1 - \sin^2(2^{k-1}\theta)) \\ &= 4\sin^2(2^{k-1}\theta)\cos^2(2^{k-1}\theta) \\ &= \sin^2(2^k\theta) \end{aligned}$$

Therefore if $P(k)$ is true, then $P(k + 1)$ is true for every natural number k.
It follows from the principle of mathematical induction that $P(n)$ is true for every natural number n.

13 a
$$\begin{aligned} \frac{\sin A + \sin B}{2} &= \sin\frac{A + B}{2}\cos\frac{A - B}{2} \\ &\leq \sin\frac{A + B}{2} \end{aligned}$$

since $\dfrac{A + B}{2} \in [0, \pi]$

b To prove

$$\frac{\sin A + \sin B + \sin C + \sin D}{4} \leq \sin\left(\frac{A + B + C + D}{4}\right)$$

$$\begin{aligned} \frac{\sin A + \sin B + \sin C + \sin D}{4} &= \frac{1}{2}\left(\sin\frac{A + B}{2}\cos\frac{A - B}{2} + \sin\frac{C + D}{2}\cos\frac{C + D}{2}\right) \\ &\leq \frac{1}{2}\left(\sin\frac{A + B}{2} + \sin\frac{C + D}{2}\right) \\ &\leq \sin\left(\frac{A + B + C + D}{4}\right) \end{aligned}$$

14 a $x + \dfrac{1}{x} = 2\cos\theta$

i $x + \dfrac{1}{x} = 2\cos\theta \quad \left(x + \dfrac{1}{x}\right)^2 = 4\cos^2\theta$

$$\begin{aligned} x^2 + 2 + \frac{1}{x^2} &= 4\cos^2\theta \\ x^2 + \frac{1}{x^2} &= 2(2\cos^2\theta - 1) \\ &= 2\cos 2\theta \end{aligned}$$

ii
$$x + \frac{1}{x} = 2\cos\theta$$
$$\left(x + \frac{1}{x}\right)^3 = 8\cos^3\theta$$
$$x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = 8\cos^3\theta$$
$$x^3 + \frac{1}{x^3} = 8\cos^3\theta - 3(x + \frac{1}{x})$$
$$= 8\cos^3\theta - 3(2\cos\theta)$$
$$= 2(4\cos^3\theta - 3\cos\theta)$$
$$= 2\cos 3\theta$$

b **Step 1** First consider $P(1)$.
$x + \frac{1}{x} = 2\cos\theta$ Therefore $P(1)$ is true.
Step 2 We use strong induction. That is Let k be any natural number, and assume $P(1), P(2), \ldots P(k-1), P(k)$ are true.
That is,
$$x^i + \frac{1}{x^i} = 2\cos(i\theta); \quad i \leq k$$
Step 3 We now have to prove $P(k+1)$ is true.
That is,
$$x^{k+1} + \frac{1}{x^{k+1}} = 2\cos((k+1)\theta)$$
First note:
$$(x^k + \frac{1}{x^k})(x + \frac{1}{x}) = x^{k+1} + \frac{1}{x^{k-1}} + x^{k-1} + \frac{1}{x^{k+1}}$$
We can write:
$$2\cos(k\theta)2\cos\theta = (x^{k+1} + \frac{1}{x^{k+1}}) + (\frac{1}{x^{k-1}} + x^{k-1})$$
Use the strong induction to write:
$$2\cos(k\theta)2\cos\theta = (x^{k+1} + \frac{1}{x^{k+1}}) + 2\cos((k-1)\theta)$$
Hence,
$$x^{k+1} + \frac{1}{x^{k+1}} = 2\cos(k\theta)2\cos\theta - 2\cos((k-1)\theta)$$
$$= 2[2\cos(k\theta)2\cos\theta - \cos((k-1)\theta)$$
$$= 2[\cos((k-1)\theta + \cos((k+1)\theta) - 2\cos((k-1)\theta)$$
$$= 2\cos((k+1)\theta)$$
Therefore if $P(i)$ is true for $i \leq k$, then $P(k+1)$ is true for every natural number k.

It follows from the principle of mathematical induction that $P(n)$ is true for every natural number n.

15

16 **a** Using the product to sum identity:

$$2\sin A\cos(kA) = \sin(k+1)A - \sin(k-1)A$$

b $2\sin A\,(\cos A + \cos 3A + \cdots + \cos((2n-1)A))$

$= 2\sin A\cos A + 2\sin A\cos 3A + \cdots + 2\sin A\cos((2n-1)A)$

$= \sin 2A + \sin 0 + \sin 4A - \sin 2A + \sin 6A - \sin 4A + \cdots + \sin(2nA) - \sin((2n-2)A)$

$= \sin(2nA)$

c For each natural number n let $P(n)$ be the proposition.

$$2\sin A(\cos A + \cos 3A + \cdots + \cos((2n-1)A) = \sin(2nA)$$

Step 1 First consider $P(1)$.

$$2\sin A\cos A = \sin 2A$$

Therefore $P(1)$ is true.
Step 2 Let k be any natural number, and assume $P(k)$ is true.
That is,

$$2\sin A(\cos A + \cos 3A + \cdots + \cos((2k-1)A) = \sin(2kA)$$

Step 3 We now have to prove $P(k+1)$ is true.
That is,

$$2\sin A(\cos A + \cos 3A + \cdots + \cos((2k-1)A) + \cos((2k+1)A) = \sin((2n+2)A)$$

LHS $= 2\sin A(\cos A + \cos 3A + \cdots + \cos((2k-1)A) + \cos((2k+1)A)$

$= \sin(2kA) + 2\sin A\cos((2k+1)A)$

$= \sin(2kA) + \sin(2k+2)A - \sin(2kA)$

$= \sin(2k+2)A$

Therefore if $P(k)$ is true, then $P(k+1)$ is true for every natural number k.
It follows from the principle of mathematical induction that $P(n)$ is true for every natural number n.

17 **a** Using the product to sum identity:

$$2\sin A\sin(kA) = \cos(1-k)A - \cos(k+1)A$$

That is

$$2\sin A\sin(kA) = \cos(k-1)A - \cos(k+1)A$$

We first observe that an equivalent identity is

$$\sin A \sin A + \sin A \sin 3A + \sin A \sin 5A + \cdots + \sin A \sin((2n-1)A) = \sin^2(nA)$$

b LHS $= \sin A \sin A + \sin A \sin 3A + \sin A \sin 5A + \cdots + \sin A \sin((2n-1)A)$

$$= \frac{1}{2}(1 - \cos 2A + \cos 2A - \cos 4A + \cos 4A - \cos 6A + \cdots \cos(2n-2)A - \cos(2nA))$$

$$= \frac{1}{2}(1 - \cos(2nA))$$

$$= \frac{1}{2}(1 - (1 - 2\sin^2 nA))$$

$$= \sin^2 nA$$

$$= \text{RHS}$$

c For each natural number n let $P(n)$ be the proposition.

$$\sin A + \sin 3A + \sin 5A + \cdots + \sin((2n-1)A) = \sin^2(nA)\operatorname{cosec} A$$

Step 1 First consider $P(1)$.

$$\sin A = \sin^2 A\operatorname{cosec} A$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true.

That is,

$$\sin A + \sin 3A + \sin 5A + \cdots + \sin((2k-1)A) = \sin^2(kA)\operatorname{cosec} A$$

Step 3 We now have to prove $P(k+1)$ is true.

That is,

$$\sin A + \sin 3A + \sin 5A + \cdots + \sin((2k-1)A) + \sin((2k+1)A) = \sin^2((k+1)A)\operatorname{cosec} A$$

LHS $= \sin A + \sin 3A + \sin 5A + \cdots + \sin((2k-1)A) + \sin((2k+1)A)$

$$= \sin^2(kA)\operatorname{cosec} A + \sin((2k+1)A)$$

$$= (\sin^2(kA) + \sin((2k+1)A)\sin A)\operatorname{cosec} A$$

$$= (\sin^2(kA) + \frac{1}{2}(\cos(2kA) - \cos(2k+2))\operatorname{cosec} A$$

$$= (\sin^2(kA) + \frac{1}{2}(1 - 2\sin^2(kA) - (1 - 2\sin^2((k+1)A)))\operatorname{cosec} A$$

$$= \sin^2((k+1)A))\operatorname{cosec} A = \text{RHS}$$

Therefore if $P(k)$ is true, then $P(k+1)$ is true for every natural number k.

It follows from the principle of mathematical induction that $P(n)$ is true for every natural number n.

18 a Use of the identity to show:

$$2\sin A\cos(2kA) = \sin((2k+1)A) - \sin(2(2k-1)A)$$

b We use this repeatedly.

$$\text{LHS} = \sin A\cos(2A) + \sin A\cos(4A) + \cdots + \sin A\cos(2nA)$$

$$= \frac{1}{2}(\sin(3A) - \sin(A) + \sin(5A) - \sin(3A) + \cdots + \sin((2k+1)A) - \sin((2k-1)A)$$

$$= \frac{1}{2}(-\sin(A) + \sin((2k+1)A))$$

$$= \sin((kA)\cos((k+1)A)$$

$$\therefore \cos(2A) + \cos(4A) + \cdots + \cos(2nA) = \frac{\sin((kA)\cos((k+1)A)}{\sin A}$$

c For each natural number n let $P(n)$ be the proposition.

$$\cos(2A) + \cos(4A) + \cdots + \cos(2nA) = \frac{\sin(nA)\cos((n+1)A)}{\sin A}$$

Step 1 First consider $P(1)$.

$$\cos(2A) = \frac{\sin A\cos(2A)}{\sin A}$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true.

That is,

$$\cos(2A) + \cos(4A) + \cdots + \cos(2kA) = \frac{\sin(kA)\cos((k+1)A)}{\sin A}$$

Step 3 We now have to prove $P(k+1)$ is true.

That is,

$$\cos(2A) + \cos(4A) + \cdots + \cos(2kA) + \cos((2k+2)A) = \frac{\sin((k+1)A)\cos((k+2)A)}{\sin A}$$

$$\text{LHS} = \cos(2A) + \cos(4A) + \cdots + \cos(2kA) + \cos((2k+2)A)$$

$$= \frac{\sin(kA)\cos((k+1)A)}{\sin A} + \cos((2k+2)A$$

$$= \frac{\sin(kA)\cos((k+1)A) + \sin A\cos((2k+2)A}{\sin A}$$

$$= \frac{\frac{1}{2}(\sin((2n+1))A - \sin A + \sin((2n+3)A) - \sin((2n+1)A))}{\sin(A)}$$

$$= \frac{\frac{1}{2}(-\sin A + \sin((2n+3)A))}{\sin(A)}$$

$$= \frac{\sin((n+1))A\cos((n+2)A)}{\sin(A)}$$

$$= \text{RHS}$$

Therefore if $P(k)$ is true, then $P(k+1)$ is true for every natural number k.

It follows from the principle of mathematical induction that $P(n)$ is true for every natural number n.

19 For each natural number n let $P(n)$ be the proposition.

$$\sum_{r=1}^{n} \sin(2rA) = \frac{\cos A - \cos((2n+1)A)}{2\sin(A)}$$

Step 1 First consider $P(1)$.

$$\sin(2A) = \frac{\cos A - \cos(3A)}{2\sin A} = \frac{-2\sin 2A \sin(-A)}{2\sin A} = \frac{2\sin 2A \sin(A)}{2\sin A}$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true.

That is,

$$\sum_{r=1}^{k} \sin(2rA) = \frac{\cos A - \cos((2k+1)A)}{2\sin(A)}$$

Step 3 We now have to prove $P(k+1)$ is true.

That is,

$$\sum_{r=1}^{k+1} \sin(2rA) = \frac{\cos A - \cos((2k+3)A)}{2\sin(A)}$$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k+1} \sin(2rA) \\
&= \sum_{r=1}^{k} \sin(2rA) + \sin((2k+2)A) \\
&= \frac{\cos A - \cos((2k+1)A)}{2\sin(A)} + \sin((2k+2)A) \\
&= \frac{\cos A - \cos((2k+1)A) + 2\sin((2k+2)A\sin(A)}{2\sin(A)} \\
&= \frac{\cos A - \cos((2k+1)A) + \cos((2k+1)A) - \cos((2k+3)A)}{2\sin(A)} \\
&= \frac{\cos A - \cos((2k+3)A)}{2\sin(A)} \\
&= \text{RHS}
\end{aligned}$$

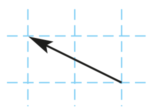
Therefore if $P(k)$ is true, then $P(k+1)$ is true for every natural number k.

It follows from the principle of mathematical induction that $P(n)$ is true for every natural number n.

Chapter 4 – Vectors

Solutions to Exercise 4A

1



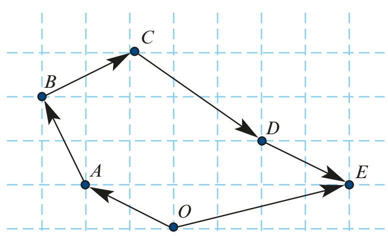
$\overrightarrow{OP} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$|\overrightarrow{OP}| = \sqrt{(-2)^2 + 1} = \sqrt{5}$

2 $\overrightarrow{AB} = \overrightarrow{OC} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\therefore a = 3, b = 2$

3



$(\overrightarrow{OA} + \overrightarrow{AB}) + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

$= (\overrightarrow{OB} + \overrightarrow{BC}) + \overrightarrow{CD} + \overrightarrow{DE}$

$= (\overrightarrow{OC} + \overrightarrow{CD}) + \overrightarrow{DE}$

$= \overrightarrow{OD} + \overrightarrow{DE}$

$= \overrightarrow{OE}$

4 **a** **i** $\overrightarrow{OC} = 2\overrightarrow{OB} = 2\boldsymbol{b}$

ii $\overrightarrow{OE} = 4\overrightarrow{OA} = 4\boldsymbol{a}$

iii $\overrightarrow{OD} = 2\overrightarrow{OA} + \frac{3}{2}\overrightarrow{OB} = 2\boldsymbol{a} + \frac{3}{2}\boldsymbol{b}$

iv $\overrightarrow{DC} = -2\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\boldsymbol{b} - 2\boldsymbol{a}$

v $\overrightarrow{DE} = -\frac{3}{2}\overrightarrow{OB} + 2\overrightarrow{OA} = 2\boldsymbol{a} - \frac{3}{2}\boldsymbol{b}$

b Let $|\boldsymbol{a}| = 1$ and $|\boldsymbol{b}| = 2$

i $|\overrightarrow{OC}| = |2\boldsymbol{b}| = 2|\boldsymbol{b}| = 4$

ii $|\overrightarrow{OE}| = |4\boldsymbol{a}| = 4|\boldsymbol{a}| = 4$

iii $|\overrightarrow{OD}| = \left|2\boldsymbol{a} + \frac{3}{2}\boldsymbol{b}\right|$

$= \sqrt{2^2 + 3^2}$

$= \sqrt{4 + 9}$

$= \sqrt{13}$

5 **a** $|2\boldsymbol{a}| = 2 \times |\boldsymbol{a}|$

$= 2 \times 3$

$= 6$

b $\left|\frac{3}{2}\boldsymbol{a}\right| = \frac{3}{2} \times |\boldsymbol{a}|$

$= \frac{3}{2} \times 3$

$= \frac{9}{2}$

c $\left|-\frac{1}{2}\boldsymbol{a}\right| = \frac{1}{2} \times |\boldsymbol{a}|$

$= \frac{1}{2} \times 3$

$= \frac{3}{2}$

6 **a**

$3(\boldsymbol{a} - \boldsymbol{b} - 2\boldsymbol{c}) + \frac{5}{2}(3\boldsymbol{a} + \boldsymbol{b} - 6\boldsymbol{c})$

$= \left(3 + \frac{15}{2}\right)\boldsymbol{a} + \left(-3 + \frac{5}{2}\right)\boldsymbol{b} + (-6 - 15)\boldsymbol{c}$

$= \frac{21}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} - 21\boldsymbol{c}$

b $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c})+\frac{1}{2}(\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{a})+\frac{1}{2}(\boldsymbol{c}+\boldsymbol{a}-\boldsymbol{b})$

$=\frac{1}{2}\boldsymbol{a}+\frac{1}{2}\boldsymbol{b}+\frac{1}{2}\boldsymbol{c}$

7 a i $\overrightarrow{OA'}=\frac{1}{4}\overrightarrow{OA}=\frac{1}{4}\boldsymbol{a}$

ii $\overrightarrow{OB'}=\frac{1}{4}\overrightarrow{OB}=\frac{1}{4}\boldsymbol{b}$

iii $\overrightarrow{A'B'}=\overrightarrow{OB'}-\overrightarrow{OA'}=\frac{1}{4}(\boldsymbol{b}-\boldsymbol{a})$

iv $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\boldsymbol{b}-\boldsymbol{a}$

b i $\overrightarrow{OA''}=2\overrightarrow{OA'}=2\times\frac{1}{4}\boldsymbol{a}=\frac{1}{2}\boldsymbol{a}$

ii $\overrightarrow{OB''}=2\overrightarrow{OB'}=2\times\frac{1}{4}\boldsymbol{b}=\frac{1}{2}\boldsymbol{b}$

iii $\overrightarrow{A''B''}=\overrightarrow{OB''}-\overrightarrow{OA''}=\frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$

8 a $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\boldsymbol{b}-\boldsymbol{a}$

b $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}=\frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$

c $\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}$

$=\boldsymbol{a}+\frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$

$=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$

9 a $\overrightarrow{XY}=\overrightarrow{XA}+\overrightarrow{AB}+\overrightarrow{BY}$

$=\frac{1}{2}\overrightarrow{DA}+\boldsymbol{a}+\frac{1}{2}\overrightarrow{BC}$

$=\boldsymbol{a}-\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{CB})$ ①

Also $\overrightarrow{XY}=\overrightarrow{XD}+\overrightarrow{DC}+\overrightarrow{CY}$

$=\frac{1}{2}\overrightarrow{AD}+\boldsymbol{b}+\frac{1}{2}\overrightarrow{CB}$

$=\boldsymbol{b}+\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{CB})$ ②

Adding ① and ② yields

$2\overrightarrow{XY}=\boldsymbol{a}-\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{CB})$

$+\boldsymbol{b}+\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{CB})$

$\therefore 2\overrightarrow{XY}=\boldsymbol{a}+\boldsymbol{b}$

$\therefore \overrightarrow{XY}=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$

b $\overrightarrow{XY}=\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$

$=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{DC})$

Since AB and DC are parallel, $\overrightarrow{AB}+\overrightarrow{DC}$ is a vector parallel to $\overrightarrow{AB}$, and $\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{DC})$ is a vector parallel to $\overrightarrow{AB}$. Hence XY is parallel to AB.

10 a $\overrightarrow{OG}=\overrightarrow{OA}+\overrightarrow{AG}$

$=\overrightarrow{OA}+\overrightarrow{BC}$

$=\overrightarrow{OA}+\overrightarrow{OC}-\overrightarrow{OB}$

$=\boldsymbol{a}+\boldsymbol{c}-\boldsymbol{b}$

b $\overrightarrow{CD}=\overrightarrow{BG}$

$=\overrightarrow{OG}-\overrightarrow{OB}$

$=\boldsymbol{a}+\boldsymbol{c}-\boldsymbol{b}-\boldsymbol{b}$

$=\boldsymbol{a}+\boldsymbol{c}-2\boldsymbol{b}$

11 a $\overrightarrow{EF} = \overrightarrow{CO} = -\overrightarrow{OC} = -\boldsymbol{c}$

b $\overrightarrow{AB} = \overrightarrow{OC} = \boldsymbol{c}$

c $\overrightarrow{EM} = \frac{1}{2}\overrightarrow{ED} = \frac{1}{2}\overrightarrow{AO} = -\frac{1}{2}\overrightarrow{OA}$

$= -\frac{1}{2}\boldsymbol{a}$

d $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CD} + \overrightarrow{DM}$

$= \overrightarrow{OC} + \overrightarrow{OG} - \overrightarrow{EM}$

$= \boldsymbol{c} + \boldsymbol{g} - \left(-\frac{1}{2}\boldsymbol{a}\right)$

$= \boldsymbol{c} + \boldsymbol{g} + \frac{1}{2}\boldsymbol{a}$

e $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BE} + \overrightarrow{EM}$

$= \overrightarrow{AB} + \overrightarrow{OG} + \overrightarrow{EM}$

$= \boldsymbol{c} + \boldsymbol{g} - \frac{1}{2}\boldsymbol{a}$

12 a i $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \boldsymbol{b} - \boldsymbol{a}$

ii $\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$

$= \boldsymbol{c} - \boldsymbol{d}$

iii $\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore \boldsymbol{b} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{d}$

b i $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \boldsymbol{c} - \boldsymbol{b}$

ii $\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CB} + \overrightarrow{BN}$

$= \frac{1}{2}\overrightarrow{DC} - \overrightarrow{BC} + \frac{1}{2}\overrightarrow{BO}$

$= \frac{1}{2}\overrightarrow{AB} - \overrightarrow{BC} - \frac{1}{2}\overrightarrow{OB}$

$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) - (\boldsymbol{c} - \boldsymbol{b}) - \frac{1}{2}\boldsymbol{b}$

$= \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a} - \boldsymbol{c} + \boldsymbol{b} - \frac{1}{2}\boldsymbol{b}$

$= -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

13 a $\boldsymbol{a} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix}$

Note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

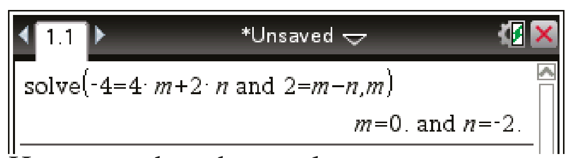
Let $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$

Then $-4 = 4m + 2n$

$2 = m - n$

$6 = 3m + 3n$

Solving the first two equations using a CAS calculator we have $m = 0$ and $n = -2$



However when these values are substituted in the third equation,

$3m + 3n = -6 \neq 6$

There are no solutions which satisfy the three equations.

Therefore the vectors are not linearly dependent.

b $\boldsymbol{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$

Note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

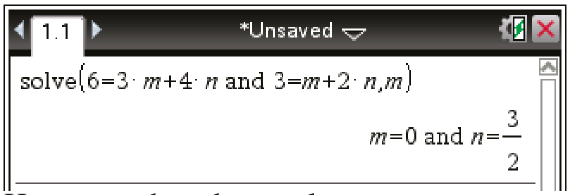
Let $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$

Then $6 = 3m + 4n$

$3 = m + 2n$

$4 = 2m + n$

Solving the first two equations using a CAS calculator we have $m = 0$ and $n = \frac{3}{2}$



However when these values are substituted in the third equation,

$2m + n = \frac{3}{2} \neq 4$

There are no solutions which satisfy the three equations.

Therefore the vectors are not linearly dependent.

c $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$

Note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

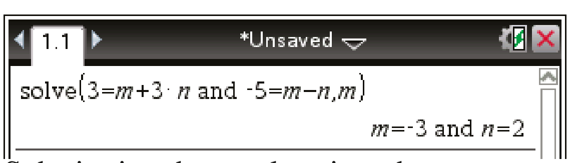
Let $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$

Then $3 = m + 3n$

$-5 = m - n$

$11 = -m + 4n$

Solving the first two equations using a CAS calculator we have $m = -3$ and $n = 2$



Substituting these values into the third equation,

$-m + 4n = 3 + 8 = 11$

As there exist real numbers m and n, both not zero, such that $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$ the set of vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

d

14 a If $k\boldsymbol{a} + l\boldsymbol{b} = 3\boldsymbol{a} + (1 - l)\boldsymbol{b}$, $\boldsymbol{a}, \boldsymbol{b}$ non-zero, non-parallel then $k = 3$ and $l = 1 - l$

$\therefore 2l = 1$

$\therefore l = \frac{1}{2}$

b If $2(l - 1)\boldsymbol{a} + \left(1 - \frac{l}{5}\right)\boldsymbol{b} = -\frac{4}{5}k\boldsymbol{a} + 3\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{b}$ non-zero, non-parallel

then $\quad 2(l - 1) = -\frac{4}{5}k \qquad (1)$

and $\quad 1 - \frac{l}{5} = 3$

$\therefore \frac{l}{5} = -2$

$\therefore l = -10$

Substituting $l = -10$ into (1) yields

$2(-10 - 1) = -\frac{4}{5}k$

$\therefore -22 = -\frac{4}{5}k$

$\therefore k = \frac{55}{2}$

15 a i $\overrightarrow{OS} = k\overrightarrow{OP} = k(2\boldsymbol{a} - \boldsymbol{b}) = 2k\boldsymbol{a} - k\boldsymbol{b}$

ii $\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{RS}$

$= \boldsymbol{a} + 4\boldsymbol{b} + m\overrightarrow{RQ}$

$= \boldsymbol{a} + 4\boldsymbol{b} + m(\overrightarrow{OQ} - \overrightarrow{OR})$

$= \boldsymbol{a} + 4\boldsymbol{b} + m(3\boldsymbol{a} + \boldsymbol{b} - (\boldsymbol{a} + 4\boldsymbol{b}))$

$= \boldsymbol{a} + 4\boldsymbol{b} + m(2\boldsymbol{a} - 3\boldsymbol{b})$

$= (2m + 1)\boldsymbol{a} + (4 - 3m)\boldsymbol{b}$

b Since $\overrightarrow{OS} = 2k\boldsymbol{a} - k\boldsymbol{b}$

and $\overrightarrow{OS} = (2m + 1)\boldsymbol{a} + (4 - 3m)\boldsymbol{b}$

$2k = 2m + 1$ ①

and $-k = 4 - 3m$

$\therefore k = 3m - 4$ ②

Substituting ② into ① yields

$2(3m - 4) = 2m + 1$

$\therefore 6m - 8 = 2m + 1$

$\therefore 4m = 9$

$\therefore m = \frac{9}{4}$

Substituting $m = \frac{9}{4}$ into ② yields

$k = 3 \times \frac{9}{4} - 4$

$= \frac{27}{4} - \frac{16}{4}$

$= \frac{11}{4}$

Hence $k = \frac{11}{4}$ and $m = \frac{9}{4}$

16 a i $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$

$= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

ii $\overrightarrow{OR} = \frac{8}{5}\overrightarrow{OQ}$

$= \frac{8}{5} \times \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{4}{5}(\boldsymbol{a} + \boldsymbol{b})$

iii $\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA}$

$= \frac{4}{5}(\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a}$

$= \frac{1}{5}(4\boldsymbol{b} - \boldsymbol{a})$

iv $\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$

$= 4\overrightarrow{OB} - \overrightarrow{OR}$

$= 4\boldsymbol{b} - \frac{4}{5}(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{4}{5}(4\boldsymbol{b} - \boldsymbol{a})$

b $\overrightarrow{RP} = \frac{4}{5}(4\boldsymbol{b} - \boldsymbol{a})$

$= 4\overrightarrow{AR}$

Hence *RP* is parallel to *AR* and *R* lies on *AP*. *AR* : *RP* = 1 : 4

c $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP}$

$= \lambda\overrightarrow{OQ} - 4\overrightarrow{OB}$

$= \lambda \times \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) - 4\boldsymbol{b}$

$= \frac{\lambda}{2}\boldsymbol{a} + \left(\frac{\lambda}{2} - 4\right)\boldsymbol{b}$

If $\overrightarrow{PS}$ is parallel to $\overrightarrow{AB}$,

then $\overrightarrow{PS} = k\overrightarrow{AB},\ k \in R\backslash\{0\}$

$= k(\overrightarrow{OB} - \overrightarrow{OA})$

$= k(\boldsymbol{b} - \boldsymbol{a})$

$= -k\boldsymbol{a} + k\boldsymbol{b}$

Equating coefficients

$\frac{\lambda}{2} = -k$ ① and $\frac{\lambda}{2} - 4 = k$ ②

From ①, $k = -\frac{\lambda}{2}$

Substituting $k = -\frac{\lambda}{2}$ into ② gives

$-\frac{\lambda}{2} = \frac{\lambda}{2} - 4$

$\therefore \lambda = 4$

17 a $x\boldsymbol{a} = (y - 1)\boldsymbol{b}$

Equating coefficients

$x = 0$ and $y - 1 = 0$

$\therefore x = 0$ and $y = 1$

b $(2 - x)\boldsymbol{a} = 3\boldsymbol{a} + (7 - 3y)\boldsymbol{b}$

Equating coefficients

$2 - x = 3$ and $7 - 3y = 0$

$\therefore x = -1$ and $y = \frac{7}{3}$

c $(5 + 2x)(\boldsymbol{a} + \boldsymbol{b}) = y(3\boldsymbol{a} + 2\boldsymbol{b})$

$\therefore (5 + 2x)\boldsymbol{a} + (5 + 2x)\boldsymbol{b} = 3y\boldsymbol{a} + 2y\boldsymbol{b}$

Equating coefficients

$5 + 2x = 3y$ ① and

$5 + 2x = 2y$ ②

From ①, $2x = 3y - 5$

$\therefore 5 + (3y - 5) = 2y$

$\therefore y = 0$

Substituting $y = 0$ into ① gives

$5 + 2x = 0$

$\therefore x = -\frac{5}{2}$

$\therefore x = -\frac{5}{2}$ and $y = 0$

18 a $\frac{AX}{AB} = k$

b $|\overrightarrow{AX}| < |\overrightarrow{AB}|$

Therefore $k < 1$.

c $\overrightarrow{XB} = (1 - k)\overrightarrow{AX}|$

$\frac{AX}{XB} = \frac{k}{1 - k}$

d $\frac{AX}{XB} = m$

$\frac{k}{1 - k} = m$

$k = m - mk$

$k(m + 1) = m$

$k = \frac{m}{m + 1}$

Solutions to Exercise 4B

1 **a** **i** $\overrightarrow{OA} = 3\boldsymbol{i} + \boldsymbol{j}$

ii $\overrightarrow{OB} = -2\boldsymbol{i} + 3\boldsymbol{j}$

iii $\overrightarrow{OC} = -3\boldsymbol{i} - 2\boldsymbol{j}$

iv $\overrightarrow{OD} = 4\boldsymbol{i} - 3\boldsymbol{j}$

b **i** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-2\boldsymbol{i} + 3\boldsymbol{j}) - (3\boldsymbol{i} + \boldsymbol{j})$

$= -5\boldsymbol{i} + 2\boldsymbol{j}$

ii $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= (4\boldsymbol{i} - 3\boldsymbol{j}) - (-3\boldsymbol{i} - 2\boldsymbol{j})$

$= 7\boldsymbol{i} - \boldsymbol{j}$

iii $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD}$

$= (3\boldsymbol{i} + \boldsymbol{j}) - (4\boldsymbol{i} - 3\boldsymbol{j})$

$= -\boldsymbol{i} + 4\boldsymbol{j}$

c **i** $|\overrightarrow{OA}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

ii $|\overrightarrow{AB}| = \sqrt{(-5)^2 + 2^2} = \sqrt{29}$

iii $|\overrightarrow{DA}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$

2 $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}, \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{c} = 4\boldsymbol{k}$

a $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{i} + 4\boldsymbol{j}$

b $2\boldsymbol{a} + \boldsymbol{c} = 2(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + 4\boldsymbol{k}$

$= 4\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

c $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} - 2\boldsymbol{i}$

$+ 4\boldsymbol{j} + 2\boldsymbol{k} - 4\boldsymbol{k}$

$= 6\boldsymbol{j} - 3\boldsymbol{k}$

d $\boldsymbol{c} - 4\boldsymbol{a} = 4\boldsymbol{k} - 4(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$= -8\boldsymbol{i} - 8\boldsymbol{j} + 8\boldsymbol{k}$

e $|\boldsymbol{b}| = \sqrt{(-1)^2 + 4 + 1} = \sqrt{6}$

f $|\boldsymbol{c}| = \sqrt{16} = 4$

3 $\overrightarrow{OA} = 5\boldsymbol{i}, \overrightarrow{OC} = 2\boldsymbol{j}, \overrightarrow{OG} = 3\boldsymbol{k}$

a **i** $\overrightarrow{BC} = -\overrightarrow{OA} = -5\boldsymbol{i}$

ii $\overrightarrow{CF} = \overrightarrow{OG} = 3\boldsymbol{k}$

iii $\overrightarrow{AB} = \overrightarrow{OC} = 2\boldsymbol{j}$

iv $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = 5\boldsymbol{i} + 3\boldsymbol{k}$

v $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BE}$

$= \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OG}$

$= 5\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

vi $\overrightarrow{GE} = \overrightarrow{GD} + \overrightarrow{DE}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= 5\boldsymbol{i} + 2\boldsymbol{j}$

vii $\overrightarrow{EC} = \overrightarrow{EF} + \overrightarrow{FC}$

$= -\overrightarrow{OA} - \overrightarrow{OG}$

$= -5\boldsymbol{i} - 3\boldsymbol{k}$

viii $\overrightarrow{DB} = \overrightarrow{DE} + \overrightarrow{EB}$

$= \overrightarrow{OC} - \overrightarrow{OG}$

$= 2\boldsymbol{j} - 3\boldsymbol{k}$

ix $\overrightarrow{DC} = \overrightarrow{DG} + \overrightarrow{GF} + \overrightarrow{FC}$

$= -\overrightarrow{OA} + \overrightarrow{OC} - \overrightarrow{OG}$

$= -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

x $\overrightarrow{BG} = \overrightarrow{BC} + \overrightarrow{CO} + \overrightarrow{OG}$

$= -\overrightarrow{OA} - \overrightarrow{OC} + \overrightarrow{OG}$

$= -5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

xi $\overrightarrow{GB} = \overrightarrow{GD} + \overrightarrow{DE} + \overrightarrow{EB}$

$= \overrightarrow{OA} + \overrightarrow{OC} - \overrightarrow{OG}$

$= 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

xii $\overrightarrow{FA} = \overrightarrow{FE} + \overrightarrow{ED} + \overrightarrow{DA}$

$= \overrightarrow{OA} - \overrightarrow{OC} - \overrightarrow{OG}$

$= 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

b **i** $\overrightarrow{OD} = 5\mathbf{i} + 3\mathbf{j}$

$|\overrightarrow{OD}| = \sqrt{(5)^2 + (3)^2} = \sqrt{34}$

ii $\overrightarrow{OE} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$|\overrightarrow{OE}| = \sqrt{(5)^2 + (2)^2 + (3)^2}$

$= \sqrt{38}$

iii $\overrightarrow{GE} = 5\mathbf{i} + 2\mathbf{j}$

$|\overrightarrow{GE}| = \sqrt{(5)^2 + (2)^2} = \sqrt{29}$

c M is the midpoint of CB.

i $\overrightarrow{CB} = \overrightarrow{OA} = 5\mathbf{i}$

$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}(5\mathbf{i}) = \frac{5}{2}\mathbf{i}$

ii $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM}$

$= 2\mathbf{j} + \frac{5}{2}\mathbf{i}$

$= \frac{5}{2}\mathbf{i} + 2\mathbf{j}$

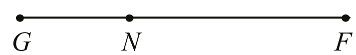
iii $\overrightarrow{DM} = \overrightarrow{DG} + \overrightarrow{GO} + \overrightarrow{OM}$

$= -\overrightarrow{OA} - \overrightarrow{OG} + \overrightarrow{OM}$

$= -5\mathbf{i} - 3\mathbf{k} + \frac{5}{2}\mathbf{i} + 2\mathbf{j}$

$= -\frac{5}{2}\mathbf{i} + 2\mathbf{j} - 3k$

d $\overrightarrow{FN} = 2\overrightarrow{NG}$



i $\overrightarrow{FN} = \frac{2}{3}\overrightarrow{FG}$

and $\overrightarrow{FG} = \overrightarrow{CO} = -2\mathbf{j}$

$\therefore \overrightarrow{FN} = \frac{2}{3}\overrightarrow{FG} = \frac{2}{3} \times -2\mathbf{j} = \frac{-4}{3}\mathbf{j}$

ii $\overrightarrow{GN} = \frac{1}{3}\overrightarrow{GF} = -\frac{1}{3}\overrightarrow{FG} = \frac{2}{3}\mathbf{j}$

iii $\overrightarrow{ON} = \overrightarrow{OG} + \overrightarrow{GN}$

$= 3\mathbf{k} + \frac{2}{3}\mathbf{j} = \frac{2}{3}\mathbf{j} + 3\mathbf{k}$

iv $\overrightarrow{NA} = \overrightarrow{NO} + \overrightarrow{OA}$

$= -\overrightarrow{ON} + \overrightarrow{OA}$

$= -\left(\frac{2}{3}\mathbf{j} + 3\mathbf{k}\right) + 5\mathbf{i}$

$= 5\mathbf{i} - \frac{2}{3}\mathbf{j} - 3\mathbf{k}$

v $\overrightarrow{NM} = \overrightarrow{NF} + \overrightarrow{FC} + \overrightarrow{CM}$

$= -\overrightarrow{FN} - \overrightarrow{OG} + \overrightarrow{CM}$

$= \frac{4}{3}\boldsymbol{j} - 3\boldsymbol{k} + \frac{5}{2}\boldsymbol{i}$

$= \frac{5}{2}\boldsymbol{i} + \frac{4}{3}\boldsymbol{j} - 3\boldsymbol{k}$

e **i** $|\overrightarrow{NM}| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{4}{3}\right)^2 + (-3)^2}$

$= \frac{\sqrt{613}}{6}$

ii $|\overrightarrow{DM}| = \sqrt{\left(\frac{-5}{2}\right)^2 + (2)^2 + (-3)^2}$

$= \frac{\sqrt{77}}{2}$

iii $|\overrightarrow{AN}| = |-\overrightarrow{NA}|$

$= \sqrt{(-5)^2 + \left(\frac{2}{3}\right)^2 + (3)^2}$

$= \frac{\sqrt{310}}{3}$

4 **i** $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j}, \boldsymbol{b} = x\boldsymbol{i} + 3y\boldsymbol{j}, \boldsymbol{a} + \boldsymbol{b} = 7\boldsymbol{i} - 2\boldsymbol{j}$

$\boldsymbol{a} + \boldsymbol{b} = (4 + x)\boldsymbol{i} + (3y - 1)\boldsymbol{j}$

$\therefore (4 + x)\boldsymbol{i} + (3y - 1)\boldsymbol{j} = 7\boldsymbol{i} - 2\boldsymbol{j}$

Equating coefficients

$\therefore 4 + x = 7$ and $3y - 1 = -2$

$\therefore x = 3$ and $y = -\frac{1}{3}$

ii $\boldsymbol{a} = x\boldsymbol{i} + 3\boldsymbol{j}, \boldsymbol{b} = -2\boldsymbol{i} + 5y\boldsymbol{j},$

$\boldsymbol{a} - \boldsymbol{b} = 6\boldsymbol{i} + \boldsymbol{j}$

$\boldsymbol{a} - \boldsymbol{b} = (x + 2)\boldsymbol{i} + (3 - 5y)\boldsymbol{j}$

$\therefore (x + 2)\boldsymbol{i} + (3 - 5y)\boldsymbol{j} = 6\boldsymbol{i} + \boldsymbol{j}$

Equating coefficients

$\therefore x + 2 = 6$ and $3 - 5y = 1$

$\therefore x = 4$ and $y = \frac{2}{5}$

iii $\boldsymbol{a} = 6\boldsymbol{i} + y\boldsymbol{j}, \boldsymbol{b} = x\boldsymbol{i} - 4\boldsymbol{j}, \boldsymbol{a} + 2\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$

$\boldsymbol{a} + 2\boldsymbol{b} = 6\boldsymbol{i} + y\boldsymbol{j} + 2(x\boldsymbol{i} - 4\boldsymbol{j})$

$= (6 + 2x)\boldsymbol{i} + (y - 8)\boldsymbol{j}$

$\therefore (6 + 2x)\boldsymbol{i} + (y - 8)\boldsymbol{j} = 3\boldsymbol{i} - \boldsymbol{j}$

Equating coefficients

$\therefore 6 + 2x = 3$ and $y - 8 = -1$

$\therefore x = -\frac{3}{2}$ and $y = 7$

5 **a** **i** $\overrightarrow{OA} = -2\boldsymbol{i} + 4\boldsymbol{j}$

ii $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \boldsymbol{i} + 6\boldsymbol{j} - (-2\boldsymbol{i} + 4\boldsymbol{j})$

$= 3\boldsymbol{i} + 2\boldsymbol{j}$

iii $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= -\boldsymbol{i} - 6\boldsymbol{j} - (\boldsymbol{i} + 6\boldsymbol{j})$

$= -2\boldsymbol{i} - 12\boldsymbol{j}$

b $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA}$

$= \frac{1}{2}(-2\boldsymbol{i} + 4\boldsymbol{j})$

$= -\boldsymbol{i} + 2\boldsymbol{j}$

c $\overrightarrow{AG} = 3\overrightarrow{BC}$

$= 3(-2\boldsymbol{i} - 12\boldsymbol{j})$

$= -6\boldsymbol{i} - 36\boldsymbol{j}$

6 $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

$= \frac{1}{2}(\boldsymbol{i} - 6\boldsymbol{j} + 7\boldsymbol{k} + 5\boldsymbol{i} - \boldsymbol{j} + 9\boldsymbol{k})$

$= \frac{1}{2}(6\boldsymbol{i} - 7\boldsymbol{j} + 16\boldsymbol{k})$

$= 3\boldsymbol{i} - \frac{7}{2}\boldsymbol{j} + 8\boldsymbol{k}$

$M(3, -\frac{7}{2}, 8)$

7 $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}, \boldsymbol{b} = 5\boldsymbol{i} + \boldsymbol{j} - 6\boldsymbol{k},$
$\boldsymbol{c} = 5\boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{d} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$

a **i** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= \boldsymbol{b} - \boldsymbol{a}$
$= 4\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$

ii $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$= \boldsymbol{c} - \boldsymbol{b}$
$= (0 - 5)\boldsymbol{i} + (5 - 1)\boldsymbol{j}$
$+ (3 + 6)\boldsymbol{k}$
$= -5\boldsymbol{i} + 4\boldsymbol{j} + 9\boldsymbol{k}$

iii $\overrightarrow{CD} = \boldsymbol{d} - \boldsymbol{c}$
$= (2 - 0)\boldsymbol{i} + (4 - 5)\boldsymbol{j}$
$+ (1 - 3)\boldsymbol{k}$
$= 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

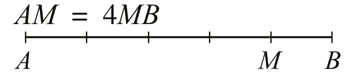
iv $\overrightarrow{DA} = \boldsymbol{a} - \boldsymbol{d}$
$= (1 - 2)\boldsymbol{i} + (3 - 4)\boldsymbol{j}$
$+ (-2 - 1)\boldsymbol{k}$
$= -\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$

b **i** $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$
$= (0 - 1)\boldsymbol{i} + (5 - 3)\boldsymbol{j}$
$+ (3 + 2)\boldsymbol{k}$
$= -\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$
$\therefore |\overrightarrow{AC}| = \sqrt{1 + 4 + 25}$
$= \sqrt{30}$

ii $\overrightarrow{BD} = \boldsymbol{d} - \boldsymbol{b}$
$= (2 - 5)\boldsymbol{i} + (4 - 1)\boldsymbol{j}$
$+ (1 + 6)\boldsymbol{k}$
$= -3\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k}$
$\therefore |\overrightarrow{BD}| = \sqrt{(-3)^2 + 3^2 + 7^2}$
$= \sqrt{67}$

c $2\overrightarrow{CD} = 2(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$
$= 4\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$
$= \overrightarrow{AB}$
$\therefore \overrightarrow{CD} \parallel \overrightarrow{AB}$

8 $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}, \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$
$AM = 4MB$



a **i** $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$
$= (3 - 1)\boldsymbol{i} + (-2 - 1)\boldsymbol{j}$
$+ (-1 - (-5))\boldsymbol{k}$
$= 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$

ii $\overrightarrow{AM} = \frac{4}{5}\overrightarrow{AB}$
$\therefore \overrightarrow{AM} = \frac{4}{5}(2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k})$

iii $\overrightarrow{OM} = \overrightarrow{AM} + \overrightarrow{OA}$
$= \left(\frac{8}{5} + 1\right)\boldsymbol{i} + \left(\frac{-12}{5} + 1\right)\boldsymbol{j}$
$+ \left(\frac{16}{5} - 5\right)\boldsymbol{k}$
$= \frac{13}{5}\boldsymbol{i} - \frac{7}{5}\boldsymbol{j} - \frac{9}{5}\boldsymbol{k}$
$= \frac{1}{5}(13\boldsymbol{i} - 7\boldsymbol{j} - 9\boldsymbol{k})$

b $M = \left(\frac{13}{5}, -\frac{7}{5}, -\frac{9}{5}\right)$

9 **a** Assume $l\boldsymbol{a} + m\boldsymbol{b} = \boldsymbol{c}$

$\therefore 8l + 4m = 2$ ①

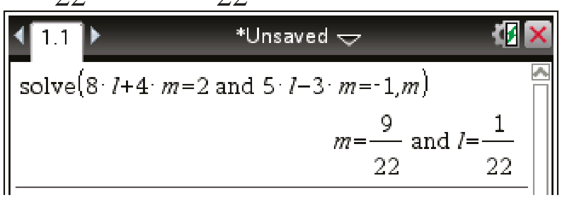
$5l - 3m = -1$ ②

$2l + m = \frac{1}{2}$ ③

① and ③ are identical equations.

Solving ① and ② gives

$l = \frac{1}{22}$ and $m = \frac{9}{22}$.



Since there exists real numbers l and m, not both zero, such that $l\boldsymbol{a} + m\boldsymbol{b} = \boldsymbol{c}$, the set of vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

b Assume $l\boldsymbol{a} + m\boldsymbol{b} = \boldsymbol{c}$

$\therefore 8l + 4m = 2$ ①

$5l - 3m = -1$ ②

$2l + m = 2$ ③

Since ① and ③ are contradictory, $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent.

c Assume $\boldsymbol{c} = \ell\boldsymbol{b} + k\boldsymbol{a}$

$2 = 2\ell + 4k$ ①

$-1 = 5\ell$ ②

$2 = 2\ell + \frac{5}{2}k$ ③

From ②, $\ell = -\frac{1}{5}$

From ① $4k = \frac{11}{5} \Rightarrow k = \frac{11}{20}$

In ③

$2 \neq 2 \times -\frac{1}{5} + \frac{5}{2} \times \frac{11}{20}$

The vectors are linearly independent.

10 Since $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent

$l\boldsymbol{a} + m\boldsymbol{b} = \boldsymbol{c}$

$\therefore 2l + 4m = 2$ ①

$-3l + 3m = -4$ ②

$l - 2m = x$ ③

$3 \times$ ① $6l + 12m = 6$ ④

$2 \times$ ② $-6l + 6m = -8$ ⑤

④ + ⑤ yields $18m = -2$

$\therefore m = \frac{-1}{9}$

Substituting $m = \frac{-1}{9}$ in ① gives

$2l - \frac{4}{9} = 2$

$\therefore l = \frac{11}{9}$

$\therefore x = \frac{11}{9} - \frac{-2}{9} = \frac{13}{9}$

11 **a** **i** $\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j}$

ii $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (\boldsymbol{i} - 3\boldsymbol{j}) - (2\boldsymbol{i} + \boldsymbol{j})$

$= -\boldsymbol{i} - 4\boldsymbol{j}$

iii $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (-5\boldsymbol{i} + 2\boldsymbol{j}) - (\boldsymbol{i} - 3\boldsymbol{j})$

$= -6\boldsymbol{i} + 5\boldsymbol{j}$

iv $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB}$

$= (3\boldsymbol{i} + 5\boldsymbol{j}) - (\boldsymbol{i} - 3\boldsymbol{j})$

$= 2\boldsymbol{i} + 8\boldsymbol{j}$

b $-2\overrightarrow{AB} = -2(-\boldsymbol{i} - 4\boldsymbol{j})$

$= 2\boldsymbol{i} + 8\boldsymbol{j}$

$= \overrightarrow{BD}$

$\therefore \overrightarrow{BD} = -2\overrightarrow{AB}$

$\therefore \overrightarrow{BD}$ is parallel to $\overrightarrow{AB}$

c Points A, B and D are collinear.

12 a i $\overrightarrow{OB} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$

ii $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (-\boldsymbol{j} + 4\boldsymbol{k}) - (\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k})$

$= -\boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k}$

iii $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB}$

$= (4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k})$

$- (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$

$= 2\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$

iv $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= (4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}) - (-\boldsymbol{j} + 4\boldsymbol{k})$

$= 4\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}$

b $2\overrightarrow{OB} = 2(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$

$= 4\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}$

$= \overrightarrow{CD}$

$\therefore \overrightarrow{CD} = 2\overrightarrow{OB}$

$\therefore \overrightarrow{CD}$ is parallel to $\overrightarrow{OB}$

13 a i $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\boldsymbol{i} + 3\boldsymbol{j}) - (\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k})$

$= 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

ii $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (2\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}) - (3\boldsymbol{i} + 3\boldsymbol{j})$

$= -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

iii $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= (6\boldsymbol{j} + \boldsymbol{k}) - (2\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k})$

$= -2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

iv $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD}$

$= (\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}) - (6\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$

b $ABCD$ is a parallelogram.

14 a $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$= \overrightarrow{OC} + \overrightarrow{AB}$ since $\overrightarrow{AB} = \overrightarrow{CD}$

$= \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OA}$

$= (-\boldsymbol{i}) + (4\boldsymbol{j}) - (5\boldsymbol{i} + \boldsymbol{j})$

$= -6\boldsymbol{i} + 3\boldsymbol{j}$

$\therefore D = (-6, 3)$

b $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$

$= \overrightarrow{OA} + (-\overrightarrow{BC})$

$= \overrightarrow{OA} - (\overrightarrow{OC} - \overrightarrow{OB})$

$= \overrightarrow{OA} - \overrightarrow{OC} + \overrightarrow{OB}$

$= (5\boldsymbol{i} + \boldsymbol{j}) - (-\boldsymbol{i}) + (4\boldsymbol{j})$

$= 6\boldsymbol{i} + 5\boldsymbol{j}$

$\therefore E = (6, 5)$

c $\overrightarrow{OG} = \overrightarrow{OC} + \overrightarrow{CG}$

$= \overrightarrow{OC} - \frac{1}{2}(2\overrightarrow{GC})$

$= \overrightarrow{OC} - \frac{1}{2}\overrightarrow{AB}$

$= \overrightarrow{OC} - \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$

$= \overrightarrow{OC} - \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OA}$

$= (-\boldsymbol{i}) - \frac{1}{2}(4\boldsymbol{j}) + \frac{1}{2}(5\boldsymbol{i} + \boldsymbol{j})$

$= \frac{3}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$

$\therefore G = \left(\frac{3}{2}, -\frac{3}{2}\right)$

or Let $\overrightarrow{OG} = x\boldsymbol{i} + y\boldsymbol{j}$
Then $\overrightarrow{AB} = 2\overrightarrow{GC}$
implies $-5\boldsymbol{i} + 3\boldsymbol{j} = 2(\boldsymbol{i} - x\boldsymbol{i} - y\boldsymbol{j})$

$-5\boldsymbol{i} + 3\boldsymbol{j} = -2(x-1)\boldsymbol{i} - 2y\boldsymbol{j}$

Equating coefficients
$\therefore -2(x-1) = -5$ and $-2y = 3$
i.e. $x = \frac{3}{2}$ and $y = -\frac{3}{2}$
$\therefore G = \left(\frac{3}{2}, -\frac{3}{2}\right)$

15 a i $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (\boldsymbol{i} + 7\boldsymbol{j}) - (-5\boldsymbol{i} + 4\boldsymbol{j})$

$= 6\boldsymbol{i} + 3\boldsymbol{j}$

ii $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= (x\boldsymbol{i} + y\boldsymbol{j}) - (2\boldsymbol{i} + \boldsymbol{j})$

$= (x-2)\boldsymbol{i} + (y-1)\boldsymbol{j}$

b $\overrightarrow{BC} = \overrightarrow{AD}$ since $ABCD$ is a parallelogram
$\therefore 6\boldsymbol{i} + 3\boldsymbol{j} = (x-2)\boldsymbol{i} + (y-1)\boldsymbol{j}$
$\therefore x - 2 = 6$ and $y - 1 = 3$
$\therefore x = 8$ and $y = 4$
$\therefore D = (8, 4)$

16 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}) - (\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$

$= \boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

Since M is the midpoint of AB
$\therefore \overrightarrow{OM} = (\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}) + \frac{1}{2}(\boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k})$

$\therefore \overrightarrow{OM} = \left(\frac{3}{2}, \frac{3}{2}, 4\right)$

b $\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$

$= (x_2, y_2, z_2) - (x_1, y_1, z_1)$

$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

$\therefore \overrightarrow{OM} = \overrightarrow{OX} + \overrightarrow{XM}$

$= \overrightarrow{OX} + \frac{1}{2}\overrightarrow{XY}$

Since M is the midpoint of XY
$\therefore \overrightarrow{OM} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

17 $\overrightarrow{AM} = 4\overrightarrow{MB}$
$\therefore \overrightarrow{OM} - \overrightarrow{OA} = 4(\overrightarrow{OB} - \overrightarrow{OM})$

$= 4\overrightarrow{OB} - 4\overrightarrow{OM}$

$\therefore 5\overrightarrow{OM} = \overrightarrow{OA} + 4\overrightarrow{OB}$

$\therefore \overrightarrow{OM} = \frac{1}{5}(\overrightarrow{OA} + 4\overrightarrow{OB})$

$= \frac{1}{5}((5\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) + 4(3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}))$

$= \frac{1}{5}(17\boldsymbol{i} + 8\boldsymbol{j} - 15\boldsymbol{k})$

$= \frac{17}{5}\boldsymbol{j} + \frac{8}{5}\boldsymbol{j} - 3\boldsymbol{k}$

$\therefore M = \left(\frac{17}{5}, \frac{8}{5}, -3\right)$

or Let $\overrightarrow{OM} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$(x-5)\boldsymbol{i} + (y-4)\boldsymbol{j} + (z-1)\boldsymbol{k} =$

$4[(3-x)\boldsymbol{i} + (1-y)\boldsymbol{j} + (-4-z)\boldsymbol{k}]$

Equating coefficients

$x - 5 = 12 - 4x \quad \therefore x = \frac{17}{5}$

$y - 4 = 4 - 4y \quad \therefore y = \frac{8}{5}$

$z - 1 = -16 - 4z \quad \therefore z = -3$

18 $\overrightarrow{AN} = 3\overrightarrow{BN}$

$\therefore \overrightarrow{ON} - \overrightarrow{OA} = 3(\overrightarrow{ON} - \overrightarrow{OB})$

$= 3\overrightarrow{ON} - 3\overrightarrow{OB}$

$\therefore 2\overrightarrow{ON} = 3\overrightarrow{OB} - \overrightarrow{OA}$

$\therefore \overrightarrow{ON} = \frac{1}{2}(3\overrightarrow{OB} - \overrightarrow{OA})$

$= \frac{1}{2}(3(7\boldsymbol{i} + \boldsymbol{j}) - (4\boldsymbol{i} - 3\boldsymbol{j}))$

$= \frac{1}{2}(17\boldsymbol{i} + 6\boldsymbol{j})$

$= \frac{17}{2}\boldsymbol{j} + 3\boldsymbol{j}$

$\therefore N = \left(\frac{17}{2}, 3\right)$

or Let $\overrightarrow{ON} = x\boldsymbol{i} + y\boldsymbol{j}$

Then as $\overrightarrow{AN} = 3\overrightarrow{BN}$

$\therefore \overrightarrow{OM} = (x_1, y_1, z_1) + \frac{1}{2}(x_2 - x_1,$

$y_2 - y_1, z_2 - z_1)$

$(x-4)\boldsymbol{i} + (y+3)\boldsymbol{j} = 3[(x-7)\boldsymbol{i} + (y-1)\boldsymbol{j}]$

Equating coefficients

$\therefore x - 4 = 3x - 21$ and $y + 3 = 3y - 3$

$\therefore x = \frac{17}{2}$ and $y = 3$

19 $x - 6y = 11 \quad \therefore y = \frac{x-11}{6}$

Let $P = (a, b)$

$\therefore b = \frac{a-11}{6}$

$\therefore P = \left(a, \frac{a-11}{6}\right)$

$\overrightarrow{OP}$ is parallel to $3\boldsymbol{i} + \boldsymbol{j}$

$\therefore a\boldsymbol{i} + \left(\frac{a-11}{6}\right)\boldsymbol{j} = k(3\boldsymbol{i} + \boldsymbol{j}),\ k \in R \setminus \{0\}$

$= 3k\boldsymbol{i} + k\boldsymbol{j}$

$\therefore a = 3k$ and $\frac{a-11}{6} = k$

$\therefore a = 3\left(\frac{a-11}{6}\right)$

$= \frac{a-11}{2}$

$\therefore 2a = a - 11$

$\therefore a = -11$ and $b = \frac{-11-11}{6}$

$= \frac{-22}{6}$

$= \frac{-11}{3}$

$\therefore P = \left(-11, \frac{-11}{3}\right)$

20 $\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$

$\therefore \overrightarrow{OB} + \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OC}$

$\therefore \boldsymbol{b} + \boldsymbol{d} = \boldsymbol{a} + \boldsymbol{c}$, as required to show.

21 $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}, \boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}, \boldsymbol{c} = 4\boldsymbol{i} + 5\boldsymbol{j}$

a **i** $\frac{1}{2}\boldsymbol{a} = \frac{1}{2}(2\boldsymbol{i} + 2\boldsymbol{j}) = \boldsymbol{i} + \boldsymbol{j}$

ii $\boldsymbol{b} - \boldsymbol{c} = (3\boldsymbol{i} - \boldsymbol{j}) - (4\boldsymbol{i} + 5\boldsymbol{j})$

$= -\boldsymbol{i} - 6\boldsymbol{j}$

iii

$3\boldsymbol{b} - \boldsymbol{a} - 2\boldsymbol{c}$

$= 3(3\boldsymbol{i} - \boldsymbol{j}) - (2\boldsymbol{i} + 2\boldsymbol{j}) - 2(4\boldsymbol{i} + 5\boldsymbol{j})$

$= 9\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{i} - 2\boldsymbol{j} - 8\boldsymbol{i} - 10\boldsymbol{j}$

$= -\boldsymbol{i} - 15\boldsymbol{j}$

b $k\boldsymbol{a} + l\boldsymbol{b} = \boldsymbol{c}$

$\therefore k(2\boldsymbol{i} + 2\boldsymbol{j}) + l(3\boldsymbol{i} - \boldsymbol{j}) = 4\boldsymbol{i} + 5\boldsymbol{j}$

$\therefore 2k\boldsymbol{i} + 2k\boldsymbol{j} + 3l\boldsymbol{i} - l\boldsymbol{j} = 4\boldsymbol{i} + 5\boldsymbol{j}$

$\therefore (2k + 3l)\boldsymbol{i} + (2k - l)\boldsymbol{j} = 4\boldsymbol{i} + 5\boldsymbol{j}$

Equating coefficients

$\therefore 2k + 3l = 4$ ① and $2k - l = 5$

$\therefore l = 2k - 5$ ②

Substituting ② into ① yields

$2k + 3(2k - 5) = 4$

$\therefore 2k + 6k - 15 = 4$

$\therefore 8k = 19$

$\therefore k = \frac{19}{8}$

Substituting $k = \frac{19}{8}$ in ② yields

$l = 2 \times \frac{19}{8} - 5$

$= \frac{19}{4} - \frac{20}{4}$

$= -\frac{1}{4}$

$\therefore k = \frac{19}{8}$ and $l = -\frac{1}{4}$

22 $\boldsymbol{a} = 5\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}, \boldsymbol{b} = 8\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k},$
$\boldsymbol{c} = \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$

a **i** $2\boldsymbol{a} - \boldsymbol{b}$

$= 2(5\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}) - (8\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= 10\boldsymbol{i} + 2\boldsymbol{j} - 8\boldsymbol{k} - 8\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$= 2\boldsymbol{i} + 4\boldsymbol{j} - 9\boldsymbol{k}$

ii $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$

$= (5\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}) + (8\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$+ (\boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k})$

$= 14\boldsymbol{i} - 8\boldsymbol{j} + 3\boldsymbol{k}$

iii

$0.5\boldsymbol{a} + 0.4\boldsymbol{b}$

$= \frac{1}{2}(5\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}) + \frac{2}{5}(8\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= \frac{5}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} - 2\boldsymbol{k} + \frac{16}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j} + \frac{2}{5}\boldsymbol{k}$

$= \frac{57}{10}\boldsymbol{i} - \frac{3}{10}\boldsymbol{j} - \frac{8}{5}\boldsymbol{k}$

$= 5.7\boldsymbol{i} - 0.3\boldsymbol{j} - 1.6\boldsymbol{k}$

b $k\boldsymbol{a} + l\boldsymbol{b} = \boldsymbol{c}$

$\therefore k(5\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}) + l(8\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$

$\therefore 5k\boldsymbol{i} + k\boldsymbol{j} - 4k\boldsymbol{k} + 8l\boldsymbol{i} - 2l\boldsymbol{j} + l\boldsymbol{k}$

$= \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$

$\therefore (5k + 8l)\boldsymbol{i} + (k - 2l)\boldsymbol{j} + (l - 4k)\boldsymbol{k}$

$= \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$

Equating coefficients

$\therefore 5k + 8l = 1$ ①

$k - 2l = -7$ ②

and $l - 4k = 6$

$\therefore l = 4k + 6$ ③

Substituting ③ in ① yields

$5k + 8(4k + 6) = 1$

$\therefore 5k + 32k + 48 = 1$

$\therefore 37k = -47$

$\therefore k = \frac{-47}{37}$

Substituting $k = \frac{-47}{37}$ in ③ yields

$l = 4 \times \frac{-47}{37} + 6$

$= \frac{-188}{37} + 6$

$= \frac{34}{37}$

Check in ②:

$$\text{LHS} = \frac{-47}{37} - 2 \times \frac{34}{37}$$
$$= \frac{-47 - 68}{37}$$
$$= \frac{-115}{37}$$

RHS = −7

∴ LHS ≠ RHS

Hence there are no values for k and l such that $k\boldsymbol{a} + l\boldsymbol{b} = \boldsymbol{c}$

23 $\boldsymbol{a} = 5\boldsymbol{i} + 2\boldsymbol{j}, \boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j}$,
$\boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{d} = -\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

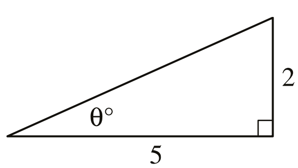
a **i** $|\boldsymbol{a}| = \sqrt{5^2 + 2^2} = \sqrt{29}$

ii $|\boldsymbol{b}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

iii $|\boldsymbol{a} + 2\boldsymbol{b}| = |5\boldsymbol{i} + 2\boldsymbol{j} + 2(2\boldsymbol{i} - 3\boldsymbol{j})|$
$= |9\boldsymbol{i} - 4\boldsymbol{j}|$
$= \sqrt{9^2 + (-4)^2}$
$= \sqrt{97}$

iv $|\boldsymbol{c} - \boldsymbol{d}| = |2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} - (-\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k})|$
$= |3\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}|$
$= \sqrt{3^2 + (-3)^2 + (-1)^2}$
$= \sqrt{19}$

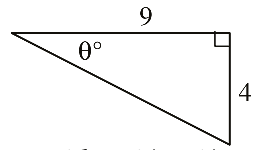
b **i**



$\theta = \tan^{-1}\left(\frac{2}{5}\right)$
≈ 21.80

$\boldsymbol{a}$ makes an angle of 21.80° anticlockwise with the positive direction of the x axis, correct to two decimal places.

ii

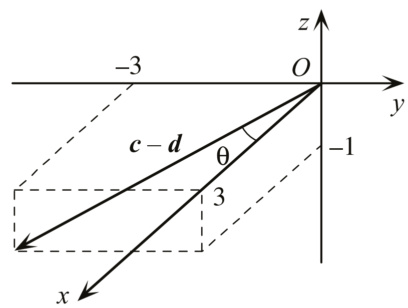


$\boldsymbol{a} + 2\boldsymbol{b} = 9\boldsymbol{i} - 4\boldsymbol{j}$

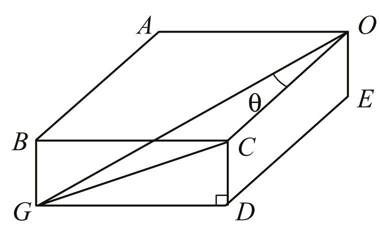
$\theta = \tan^{-1}\left(\frac{9}{4}\right)$
≈ 23.96

$\boldsymbol{a} + 2\boldsymbol{b}$ makes an angle of 23.96° clockwise with the positive direction of the x axis, correct to two decimal places.

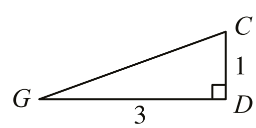
iii $\boldsymbol{c} - \boldsymbol{d} = 3\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$



The above situation can be redrawn as the following triangle in three-dimensions.

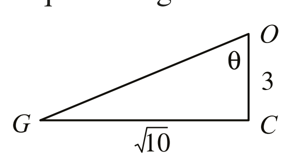


Consider triangle CDG.



$CG = \sqrt{3^2 + 1^2} = \sqrt{10}$

Now consider triangle OCG with the required angle θ.



$$\theta = \tan^{-1}\left(\frac{\sqrt{10}}{3}\right)$$
$$\approx 46.51$$

$\boldsymbol{c} - \boldsymbol{d}$ makes an angle of 46.51° with the positive direction of the x axis.

24 **a** $|\boldsymbol{a}| = 10$, x axis angle $= 110°$,

y axis angle $= 20°$

$\therefore \cos 110° = \frac{a_1}{|\boldsymbol{a}|}$ $\cos 20° = \frac{a_2}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 110° = -3.42$

$a_2 = |\boldsymbol{a}| \cos 20° = 9.40$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} = -3.42\boldsymbol{i} + 9.40\boldsymbol{j}$

b

$|\boldsymbol{a}| = 8.5$, x axis angle $= 250°$,

y axis angle $= 160°$

$\therefore \cos 250° = \frac{a_1}{|\boldsymbol{a}|}$ $\cos 160° = \frac{a_2}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 250° = -2.91$

$a_2 = |\boldsymbol{a}| \cos 160° = -7.99$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} = -2.91\boldsymbol{i} - 7.99\boldsymbol{j}$

c $|\boldsymbol{a}| = 6$, x axis angle $= 40°$,

y axis angle $= 50°$

$\therefore \cos 40° = \frac{a_1}{|\boldsymbol{a}|}$ $\cos 50° = \frac{a_2}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 40° = 4.60$

$a_2 = |\boldsymbol{a}| \cos 50° = 3.86$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} = 4.60\boldsymbol{i} + 3.86\boldsymbol{j}$

d $|\boldsymbol{a}| = 5$, x axis angle $= 300°$,

y axis angle $= 210°$

$\therefore \cos 300° = \frac{a_1}{|\boldsymbol{a}|}$ $\cos 210° = \frac{a_2}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 300° = 2.50$

$a_2 = |\boldsymbol{a}| \cos 210° = -4.33$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} = 2.50\,\boldsymbol{i} - 4.33\boldsymbol{j}$

25 **a** $|\boldsymbol{a}| = 10$,

angle with:

x axis $= 130°$, y axis $= 80°$, z axis $= 41.75°$

$\cos 130° = \frac{a_1}{|\boldsymbol{a}|}$, $\cos 80° = \frac{a_2}{|\boldsymbol{a}|}$,

$\cos 41.75° = \frac{a_3}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 130° = -6.43$

$a_2 = |\boldsymbol{a}| \cos 80° = 1.74$

$a_3 = |\boldsymbol{a}| \cos 41.75° = 7.46$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

$= -6.43\boldsymbol{i} + 1.74\boldsymbol{j} + 7.46\boldsymbol{k}$

b $|\boldsymbol{a}| = 8$,

angle with:

x axis $= 50°$, y axis $= 54.52°$,

z axis $= 120°$

$\cos 50° = \frac{a_1}{|\boldsymbol{a}|}$, $\cos 54.52° = \frac{a_2}{|\boldsymbol{a}|}$,

$\cos 120° = \frac{a_3}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 50° = 5.14$

$a_2 = |\boldsymbol{a}| \cos 54.52° = 4.64$

$a_3 = |\boldsymbol{a}| \cos 120° = -4.00$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

$= 5.14\boldsymbol{i} + 4.64\boldsymbol{j} - 4.00\boldsymbol{k}$

c $|\boldsymbol{a}| = 7$,

angle with:

x axis $= 28.93°$,

y axis $= 110°$, z axis $= 110°$

$\cos 28.93° = \frac{a_1}{|\boldsymbol{a}|}$, $\cos 110° = \frac{a_2}{|\boldsymbol{a}|}$,

$\cos 110° = \frac{a_3}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 28.93° = 6.13$

$a_2 = a_3 = |\boldsymbol{a}| \cos 110° = -2.39$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

$= 6.13\boldsymbol{i} - 2.39\boldsymbol{j} - 2.39\boldsymbol{k}$

d $|\boldsymbol{a}| = 12$,

angle with:

x axis $= 121.43°$,

y axis = 35.5°, z axis = 75.2°

$\cos 121.43° = \dfrac{a_1}{|\boldsymbol{a}|}, \cos 35.5° = \dfrac{a_2}{|\boldsymbol{a}|}$

$\cos 75.2° = \dfrac{a_3}{|\boldsymbol{a}|}$

$a_1 = |\boldsymbol{a}| \cos 121.43° = -6.26$

$a_2 = |\boldsymbol{a}| \cos 35.5° = 9.77$

$a_3 = |\boldsymbol{a}| \cos 75.2° = 3.07$

$\therefore \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

$= -6.26\boldsymbol{i} + 9.77\boldsymbol{j} + 3.07\boldsymbol{k}$

26 a $\cos\alpha = \dfrac{a_1}{|\boldsymbol{a}|}, \cos\beta = \dfrac{a_2}{|\boldsymbol{a}|}, \cos\gamma = \dfrac{a_3}{|\boldsymbol{a}|}$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$= \dfrac{(a_1)^2}{|\boldsymbol{a}|^2} + \dfrac{(a_2)^2}{|\boldsymbol{a}|^2} + \dfrac{(a_3)^2}{|\boldsymbol{a}|^2}$

$= \dfrac{(a_1)^2 + (a_2)^2 + (a_3)^2}{|\boldsymbol{a}|^2}$

$= \dfrac{(a_1)^2 + (a_2)^2 + (a_3)^2}{(a_1) + (a_2)^2 + (a_3)^2}$

$= 1$

b $\cos 60° = \dfrac{1}{2}$

$\dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{4} \neq 1$

c $\dfrac{1}{2}\boldsymbol{i} + \dfrac{1}{2}\boldsymbol{j} + \dfrac{1}{\sqrt{2}}\boldsymbol{k}$

27 $\boldsymbol{a} = -2\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{j} + 3\boldsymbol{k},$
$\boldsymbol{c} = -2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$

a $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$= (0 + 2)\boldsymbol{i} + (2 - 1)\boldsymbol{j} + (3 - 5)\boldsymbol{k}$

$= 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$

$= -2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k} - (-2\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k})$

$= 0\boldsymbol{i} + 3\boldsymbol{j}$

$= 3\boldsymbol{j}$

$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$

$= -2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k} - (2\boldsymbol{j} + 3\boldsymbol{k})$

$= -2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$|\overrightarrow{AB}| = \sqrt{4 + 1 + 4} = 3$

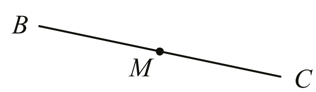
$|\overrightarrow{AC}| = 3$

$|\overrightarrow{BC}| = \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3}$

$\therefore |\overrightarrow{AB}| = |\overrightarrow{AC}| \neq |\overrightarrow{BC}|$

$\therefore \triangle ABC$ is isosceles

b



$\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$

$= 2\boldsymbol{j} + 3\boldsymbol{k} + \dfrac{1}{2}\overrightarrow{BC}$

$= 2\boldsymbol{j} + 3\boldsymbol{k} + (-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$= -\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$

c $\overrightarrow{AM} = \boldsymbol{m} - \boldsymbol{a}$ where $\boldsymbol{m} = \overrightarrow{OM}$

$= (-\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) - (-2\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k})$

$= \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

d $|\overrightarrow{AM}| = \sqrt{1 + 4 + 1}$

$= \sqrt{6}$

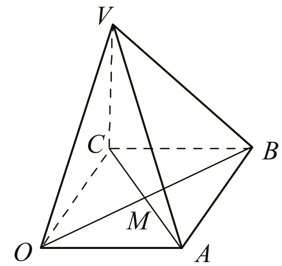
Area $= |\overrightarrow{AM}| \times |\overrightarrow{BM}|$

$= \sqrt{6} \times (\sqrt{(-1)^2 + 1^2 + 1^2})$

$= \sqrt{18}$

$= 3\sqrt{2}$

28 $\overrightarrow{OA} = 5\boldsymbol{i}, \overrightarrow{OC} = 5\boldsymbol{j}, \overrightarrow{MV} = 3\boldsymbol{k}$



a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= 5\boldsymbol{i} + 5\boldsymbol{j}$

b $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$

$= \frac{1}{2}(5\boldsymbol{i} + 5\boldsymbol{j})$

c $\overrightarrow{OV} = \overrightarrow{OM} + \overrightarrow{MV}$

$= \frac{5}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + 3\boldsymbol{k}$

d $\overrightarrow{BV} = \overrightarrow{BM} + \overrightarrow{MV}$

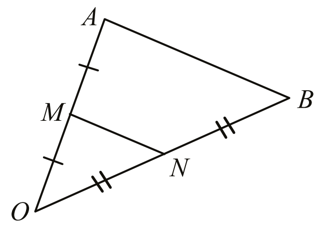
$= -\frac{1}{2}\overrightarrow{OB} + \overrightarrow{MV}$

$= -\frac{5}{2}\boldsymbol{i} - \frac{5}{2}\boldsymbol{j} + 3\boldsymbol{k}$

e $|\overrightarrow{OV}| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + (3)^2}$

$= \frac{\sqrt{86}}{2}$

29



a $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\boldsymbol{a}$

$\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\boldsymbol{b}$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$

$= \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{2}\overrightarrow{AB}$

b $\overrightarrow{MN} \parallel \overrightarrow{AB}$ and $MN = \frac{1}{2}AB$

30 a The unit vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ makes angles of −30° and 120° from the positive directions of the x and y axes respectively.

Now $|\boldsymbol{a}| = 1$

$\therefore \cos(-30°) = a_1$ and $\cos(120°) = a_2$

$\therefore a_1 = \cos 30° \qquad a_2 = -\cos 60°$

$= \frac{\sqrt{3}}{2} \qquad\qquad = -\frac{1}{2}$

$\therefore \boldsymbol{a} = \frac{\sqrt{3}}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$

b Let $\overrightarrow{OA} = 3\boldsymbol{a}$ be the position of the runner with respect to her starting point after she has run three kilometres,

$\therefore \overrightarrow{OA} = 3\left(\frac{\sqrt{3}}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}\right)$

$= \frac{3\sqrt{3}}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$

c Let $\overrightarrow{AB} = 5\boldsymbol{j}$

$\therefore \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \left(\frac{3\sqrt{3}}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j}\right) + 5\boldsymbol{j}$

$= \frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{7}{2}\boldsymbol{j}$

She is now at the position $\frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{7}{2}\boldsymbol{j}$ from her starting point.

d Distance $= |\overrightarrow{OB}| = \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{7}{2}\right)^2}$

$= \sqrt{\frac{27}{4} + \frac{49}{4}}$

$= \sqrt{\frac{76}{4}}$

$= \sqrt{19}$

The runner is $\sqrt{19}$ kilometres from her starting point.

31 a $\overrightarrow{OA} = 50\boldsymbol{k}$

b i $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-80\boldsymbol{i} + 20\boldsymbol{j} + 40\boldsymbol{k}) - (50\boldsymbol{k})$

$= -80\boldsymbol{i} + 20\boldsymbol{j} - 10\boldsymbol{k}$

ii $|\overrightarrow{AB}| = \sqrt{(-80)^2 + 20^2 + (-10)^2}$

$= \sqrt{6400 + 400 + 100}$

$= \sqrt{6900}$

$= 10\sqrt{69}$

The magnitude of $\overrightarrow{AB}$ is $10\sqrt{69}$ metres.

c Let C be the new position of the hang glider.

$\overrightarrow{BC} = 600\boldsymbol{j} + 60\boldsymbol{k}$

$\therefore \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$

$= (-80\boldsymbol{i} + 20\boldsymbol{j} + 40\boldsymbol{k})$

$\quad + (600\boldsymbol{j} + 60\boldsymbol{k})$

$= -80\boldsymbol{i} + 620\boldsymbol{j} + 100\boldsymbol{k},$

the new position vector of the hang glider.

32 $\boldsymbol{r}_1 = 1.5\boldsymbol{i} + 2\boldsymbol{j} + 0.9\boldsymbol{k}$

a $|\boldsymbol{r}_1| = \sqrt{1.5^2 + 2^2 + 0.9^2}$

$= \sqrt{2.25 + 4 + 0.81}$

$= \sqrt{7.06}$

≈ 2.66 km

The distance from the origin is 2.66 kilometres, correct to two decimal places.

b $\boldsymbol{r}_2 = 2\boldsymbol{i} + 3\boldsymbol{j} + 0.8\boldsymbol{k}$

i $\boldsymbol{r}_1 - \boldsymbol{r}_2 = (1.5\boldsymbol{i} + 2\boldsymbol{j} + 0.9\boldsymbol{k})$

$\quad - (2\boldsymbol{i} + 3\boldsymbol{j} + 0.8\boldsymbol{k})$

$= -0.5\boldsymbol{i} - \boldsymbol{j} + 0.1\boldsymbol{k}$

ii $|\boldsymbol{r}_1 - \boldsymbol{r}_2| = \sqrt{(0.5)^2 + (1)^2 + 0.1^2}$

$= \sqrt{0.25 + 1 + 0.01}$

$= \sqrt{1.26}$

≈ 1.12

The distance between the two aircraft is 1.12 kilometres, correct to two decimal places.

c The first aircraft must fly over the point with position vector $\boldsymbol{r}_3 = 0.9\boldsymbol{k}$

$\therefore$ it must fly in the direction

$\boldsymbol{r}_3 - \boldsymbol{r}_1 = 0.9\boldsymbol{k} - (1.5\boldsymbol{i} + 2\boldsymbol{j} + 0.9\boldsymbol{k})$

$= -1.5\boldsymbol{i} - 2\boldsymbol{j}$

A unit vector in this direction is given by

$$\frac{\boldsymbol{r}_3 - \boldsymbol{r}_1}{|\boldsymbol{r}_3 - \boldsymbol{r}_1|} = \frac{-1.5\boldsymbol{i} - 2\boldsymbol{j}}{\sqrt{(-1.5)^2 + (-2)^2}}$$

$$= \frac{-1.5\boldsymbol{i} - 2\boldsymbol{j}}{\sqrt{2.25 + 4}}$$

$$= \frac{1}{\sqrt{6.25}}(-1.5\boldsymbol{i} - 2\boldsymbol{j})$$

$$= \frac{1}{2.5}(-1.5\boldsymbol{i} - 2\boldsymbol{j})$$

$$= -0.6\boldsymbol{i} - 0.8\boldsymbol{j}$$

33 **a** $\overrightarrow{OP} = a_1\boldsymbol{i} + a_2\,\boldsymbol{j}$, where $\dfrac{a_1}{|\overrightarrow{OP}|} = \cos\alpha$

and $\dfrac{a_2}{|OP|} = \cos\beta$

where α and β are the angles $\overrightarrow{OP}$ makes with the easterly and northerly directions

respectively, and $|\overrightarrow{OP}| = 200$.

$\therefore \dfrac{a_1}{200} = \cos 135^\circ$ and $\dfrac{a_2}{200} = \cos 45^\circ$

$\therefore a_1 = 200\cos 135^\circ \quad a_2 = 200\cos 45^\circ$

$= -200\cos 45^\circ \qquad = \dfrac{200\sqrt{2}}{2}$

$= \dfrac{-200\sqrt{2}}{2} \qquad = 100\sqrt{2}$

$= -100\sqrt{2}$

$\therefore \overrightarrow{OP} = -100\sqrt{2}\boldsymbol{i} + 100\sqrt{2}\boldsymbol{j}$

b $\overrightarrow{PQ} = 50\boldsymbol{j}$

c $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$= (-100\sqrt{2}\,\boldsymbol{i} + 100\sqrt{2}\,\boldsymbol{j}) + 50\boldsymbol{j}$

$= -100\sqrt{2}\,\boldsymbol{i} + (50 + 100\sqrt{2})\,\boldsymbol{j}$

d $\overrightarrow{QT} = 30\boldsymbol{k}$

e $\overrightarrow{OT} = \overrightarrow{OQ} + \overrightarrow{QT}$

$= -100\sqrt{2}\,\boldsymbol{i} + (50 + 100\sqrt{2})\boldsymbol{j}$

$+ 30\boldsymbol{k}$

34 a $\overrightarrow{OP} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$, where $\dfrac{a_1}{|\overrightarrow{OP}|} = \cos\alpha$

and $\dfrac{a_2}{|\overrightarrow{OP}|} = \cos\beta$

where α and β are the angles $\overrightarrow{OP}$ makes with the easterly and northerly directions respectively, and $|\overrightarrow{OP}| = 100$.

$\therefore \dfrac{a_1}{100} = \cos 45°$ and $\dfrac{a_2}{200} = \cos 45°$

$\therefore a_1 = 100 \times \dfrac{\sqrt{2}}{2}$ $\quad a_2 = 100 \times \dfrac{\sqrt{2}}{2}$

$= 50\sqrt{2}$ $\quad = 50\sqrt{2}$

$\therefore \overrightarrow{OP} = 50\sqrt{2}\,\boldsymbol{i} + 50\sqrt{2}\,\boldsymbol{j}$,

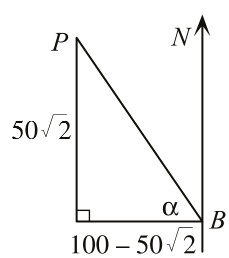
the position vector of point P.

b i $\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$

$= (50\sqrt{2}\,\boldsymbol{i} + 50\sqrt{2}\,\boldsymbol{j}) - (100\boldsymbol{i})$

$= (50\sqrt{2} - 100)\,\boldsymbol{i} + 50\sqrt{2}\,\boldsymbol{j}$

ii



$\alpha = \tan^{-1}\left(\dfrac{50\sqrt{2}}{100 - 50\sqrt{2}}\right)$

$= 67.5$

Bearing $= (270 + 67.5)°$

$= 337.5°$

The bearing of P from B is $337.5°$.

35 $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

There exists real numbers p and q such that:

$\boldsymbol{a} = p\boldsymbol{b} + q\boldsymbol{c}$

Therefore

$\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} = p(\boldsymbol{i} + 2\boldsymbol{j} + m\boldsymbol{k}) + q(3\boldsymbol{i} + n\boldsymbol{j} + \boldsymbol{k})$

$1 = p + 3q \ldots (1)$

$-1 = 2p + nq \ldots (2)$

$2 = mp + q \ldots (3)$

From (2) and (3)

$p = \dfrac{2n + 1}{mn - 2}$ and $q = -\dfrac{m + 4}{mn - 2}$

Substitute in (1)

$m = \dfrac{2n - 9}{n + 3}$

36 a $2\boldsymbol{a} - 3\boldsymbol{b} = 2(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} - 3(\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$

$= -\boldsymbol{i} - 8\boldsymbol{j} + 16\boldsymbol{k}$

b $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

There exist real numbers p and q such that

$m\boldsymbol{i} + 6\boldsymbol{j} - 12\boldsymbol{k} = p\boldsymbol{a} + q\boldsymbol{b}$

$m\boldsymbol{i} + 6\boldsymbol{j} - 12\boldsymbol{k} = p(4\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) + q(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

From **36a**

$2\boldsymbol{a} - 3\boldsymbol{b} = -\boldsymbol{i} - 8\boldsymbol{j} + 16\boldsymbol{k}$

$\therefore -\dfrac{3}{4}(2\boldsymbol{a} - 3\boldsymbol{b}) = \dfrac{3}{4}\boldsymbol{i} + 6\boldsymbol{j} - 12\boldsymbol{k}$

$\therefore m = \dfrac{3}{4}$

37 a $\boldsymbol{c} = m(4\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) + (1 - m)(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

$\therefore \boldsymbol{c} = (3m + 1)\boldsymbol{i} - \boldsymbol{j} + (1 - 3m)\boldsymbol{k}$

b If $\boldsymbol{c} = 7\boldsymbol{i} - \boldsymbol{j} + p\boldsymbol{k}$

$3m + 1 = 7 \ldots (1)$

$1 - 3m = p \ldots (2)$

From (1), $m = 2$

Substitute in (2), $p = -5$

Solutions to Exercise 4C

1 $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k},$
$\boldsymbol{c} = -\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

a $\boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}).(\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k})$

$= 1 \times 1 + -4 \times -4 + 7 \times 7$

$= 1 + 16 + 49$

$= 66$

b $\boldsymbol{b}.\boldsymbol{b} = (2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}).(2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$

$= 2 \times 2 + 3 \times 3 + 3 \times 3$

$= 4 + 9 + 9$

$= 22$

c $\boldsymbol{c}.\boldsymbol{c} = (-\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}).(-\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= -1 \times -1 + -2 \times -2 + 1 \times 1$

$= 1 + 4 + 1$

$= 6$

d $\boldsymbol{a}.\boldsymbol{b} = (\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}).(2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$

$= 1 \times 2 + -4 \times 3 + 7 \times 3$

$= 2 - 12 + 21$

$= 11$

A CAS calculator has the capability to compute the dot product of two vectors.

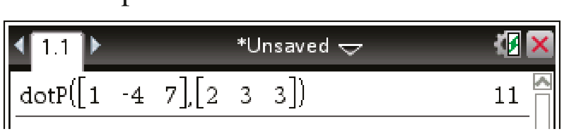
TI: Press→7: Matrix & Vector→ C:Vector→3: Dot Product

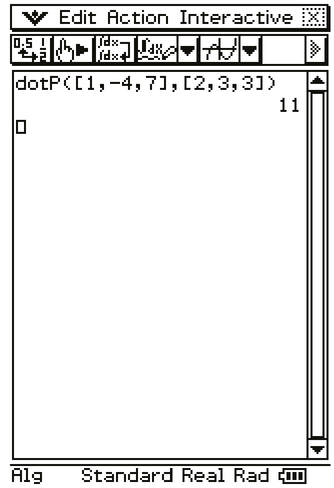
CP: Tap Action→Vector→dotP

The syntax for the Dot Product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is as follows.

dotP([a_1, a_2, a_3], [b_1, b_2, b_3])

Thus for part **d**





e $\boldsymbol{b} + \boldsymbol{c} = (2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) + (-\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$

$\therefore \boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$

$= 1 \times 1 + -4 \times 1 + 7 \times 4$

$= 1 - 4 + 28$

$= 25$

f

$(\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}) + (2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$

$= 3\boldsymbol{i} - \boldsymbol{j} + 10\boldsymbol{k}$

$(\boldsymbol{a} + \boldsymbol{c}) = (\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}) + (-\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= -6\boldsymbol{j} + 8\boldsymbol{k}$

$(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{c})$

$= (3\boldsymbol{i} - \boldsymbol{j} + 10\boldsymbol{k}).(0\boldsymbol{i} - 6\boldsymbol{j} + 8\boldsymbol{k})$

$= 3 \times 0 + -1 \times -6 + 10 \times 8$

$= 6 + 80$

$= 86$

g $(\boldsymbol{a} + 2\boldsymbol{b}) = (\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k})$

$+ 2(2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$

$= 5\boldsymbol{i} + 2\boldsymbol{j} + 13\boldsymbol{k}$

$(3\boldsymbol{c} - \boldsymbol{b}) = 3(-\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$- (2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$

$= -5\boldsymbol{i} - 9\boldsymbol{j}$

$(\boldsymbol{a} + 2\boldsymbol{b}).(3\boldsymbol{c} - \boldsymbol{b})$

$= (5\boldsymbol{i} + 2\boldsymbol{j} + 13\boldsymbol{k}).(-5\boldsymbol{i} - 9\boldsymbol{j} + 0\boldsymbol{k})$

$= 5 \times -5 + 2 \times -9 + 13 \times 0$

$= -25 - 18$

$= -43$

2 $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{k}, \boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$

a $\boldsymbol{a}.\boldsymbol{a} = (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}).(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$= 2 \times 2 + -1 \times -1 + 3 \times 3$

$= 4 + 1 + 9$

$= 14$

b $\boldsymbol{b}.\boldsymbol{b} = (3\boldsymbol{i} + 0\boldsymbol{j} - 2\boldsymbol{k}).(3\boldsymbol{i} + 0\boldsymbol{j} - 2\boldsymbol{k})$

$= 3 \times 3 + 0 + -2 \times -2$

$= 9 + 4$

$= 13$

c $\boldsymbol{a}.\boldsymbol{b} = (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}).(3\boldsymbol{i} + 0\boldsymbol{j} - 2\boldsymbol{k})$

$= 2 \times 3 + 0 + 3 \times -2$

$= 6 - 6$

$= 0$

d $\boldsymbol{a}.\boldsymbol{c} = (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}).(-\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$

$= 2 \times -1 + -1 \times 3 + 3 \times -1$

$= -2 - 3 - 3$

$= -8$

e $\boldsymbol{a} + \boldsymbol{b} = (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$+ (3\boldsymbol{i} + 0\boldsymbol{j} - 2\boldsymbol{k})$

$= 5\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

$\boldsymbol{a}.(\boldsymbol{a} + \boldsymbol{b}) = (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}).(5\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

$= 2 \times 5 + -1 \times -1 + 3 \times 1$

$= 10 + 1 + 3$

$= 14$

3 a $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$

$= 6 \times 7 \times \cos 60^\circ$

$= 21$

b $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$

$= 6 \times 7 \times \cos 120^\circ$

$= -21$

4 a $(\boldsymbol{a} + 2\boldsymbol{b}).(\boldsymbol{a} + 2\boldsymbol{b})$

$= \boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + 2\boldsymbol{b}.\boldsymbol{a} + 4\boldsymbol{b}.\boldsymbol{b}$

$= \boldsymbol{a}.\boldsymbol{a} + 4\boldsymbol{a}.\boldsymbol{b} + 4\boldsymbol{b}.\boldsymbol{b}$

b $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$

$= (\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b}) - (\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b})$

$= \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b}$

$- (\boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} - \boldsymbol{b}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$

$= \boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b} - \boldsymbol{a}.\boldsymbol{a}$

$+ 2\boldsymbol{a}.\boldsymbol{b} - \boldsymbol{b}.\boldsymbol{b}$

$= 4\boldsymbol{a}.\boldsymbol{b}$

c $\boldsymbol{a}.(\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b}.(\boldsymbol{a} + \boldsymbol{b})$

$= (\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b})$

$= \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b}$

$= \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b}$

d $\dfrac{\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b})-\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|}$

$= \dfrac{\boldsymbol{a}.\boldsymbol{a}+\boldsymbol{a}.\boldsymbol{b}-\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|}$

$= \dfrac{\boldsymbol{a}.\boldsymbol{a}}{|\boldsymbol{a}|}$

$= \dfrac{|\boldsymbol{a}|^2}{|\boldsymbol{a}|}$

$= |\boldsymbol{a}|$

5 a $(\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}).(5\boldsymbol{i}+x\boldsymbol{j}+\boldsymbol{k}) = -6$

$\therefore 5+2x-3 = -6$

$\therefore 2x = -8$

$\therefore x = -4$

b $(x\boldsymbol{i}+7\boldsymbol{j}-\boldsymbol{k}).(-4\boldsymbol{i}+x\boldsymbol{j}+5\boldsymbol{k}) = 10$

$\therefore \quad -4x+7x-5 = 10$

$\therefore \quad 3x = 15$

$\therefore \quad x = 5$

c $(x\boldsymbol{i}+0\boldsymbol{j}+5\boldsymbol{k}).(-2\boldsymbol{i}-3\boldsymbol{j}+3\boldsymbol{k}) = x$

$\therefore \quad -2x+15 = x$

$\therefore \quad 3x = 15$

$\therefore \quad x = 5$

d $x(2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}).(\boldsymbol{i}+\boldsymbol{j}+x\boldsymbol{k}) = 6$

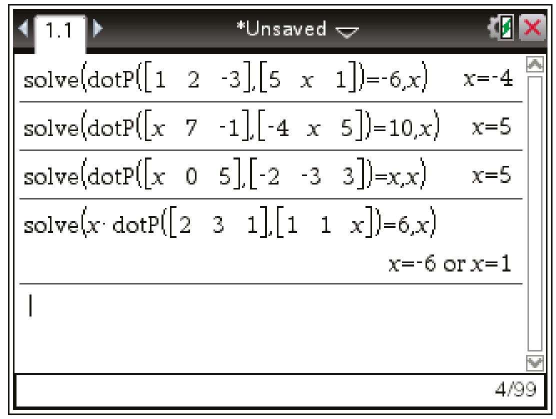
$\therefore \quad x(2+3+x) = 6$

$\therefore \quad x^2+5x-6 = 0$

$\therefore \quad (x+6)(x-1) = 0$

$\therefore \quad x = -6 \text{ or } x = 1$

Using the solve and dot product commands a CAS calculator could be used for question **8**



6 $\boldsymbol{a} = \boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}, \boldsymbol{b} = -\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$

a $\overrightarrow{AB} = \boldsymbol{b}-\boldsymbol{a}$

$= (-\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k})-(\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})$

$= -2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$

b $|\overrightarrow{AB}| = \sqrt{(-2)^2+(-1)^2+(-2)^2}$

$= \sqrt{9}$

$= 3$

c $\cos\theta = \dfrac{\boldsymbol{a}.\overrightarrow{AB}}{|\boldsymbol{a}|\,|\overrightarrow{AB}|}$

$\therefore \cos\theta = \dfrac{(\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}).(-2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})}{\sqrt{6}\times 3}$

$\therefore \cos\theta = \dfrac{-2-2+2}{3\sqrt{6}}$

$\therefore \cos\theta = \dfrac{-2}{3\sqrt{6}}$

$\therefore \cos\theta = -\dfrac{\sqrt{6}}{9}$

$\therefore \theta = \cos^{-1}\left(-\dfrac{\sqrt{6}}{9}\right)$

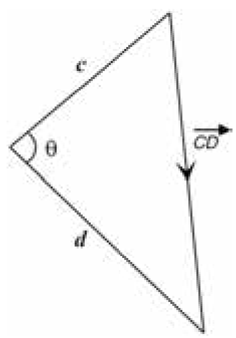
$\therefore \theta = 105.8°$

7 $\cos\theta = \dfrac{\boldsymbol{c}.\boldsymbol{d}}{|\boldsymbol{c}|\,|\boldsymbol{d}|}$

$\therefore \cos\theta = \dfrac{4}{5 \times 7}$

$\therefore \cos\theta = \dfrac{4}{35}$

A visual representation of the problem is:



Using the cosine rule,

$|\overrightarrow{CD}|^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 - 2|\boldsymbol{c}|\,|\boldsymbol{d}|\cos\theta$

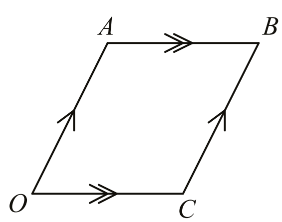
$|\overrightarrow{CD}|^2 = 5^2 + 7^2 - 2(5)(7)\left(\dfrac{4}{35}\right)$

$= 74 - 70\left(\dfrac{4}{35}\right)$

$= 66$

$\therefore |\overrightarrow{CD}| = \sqrt{66}$

8



a **i** $\overrightarrow{AB} = \overrightarrow{OC} = \boldsymbol{c}$

ii $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \boldsymbol{a} + \boldsymbol{c}$

iii $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \boldsymbol{c} - \boldsymbol{a}$

b $\overrightarrow{OB}.\overrightarrow{AC} = (\boldsymbol{a} + \boldsymbol{c}).(\boldsymbol{c} - \boldsymbol{a})$

$= \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{c}.\boldsymbol{c} - \boldsymbol{c}.\boldsymbol{a}$

$= \boldsymbol{c}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{a}$

$= |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2$

As a rhombus has all sides of equal length

$\therefore |\boldsymbol{c}| = |\boldsymbol{a}|$

Hence,

$\overrightarrow{OB}.\overrightarrow{AC} = |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2 = 0$

c $\cos\theta = \dfrac{\overrightarrow{OB}.\overrightarrow{AC}}{|\overrightarrow{OB}|\,|\overrightarrow{AC}|}$

$\therefore \cos\theta = 0$ since $\overrightarrow{OB}.\overrightarrow{AC} = 0$

$\therefore \theta = 90°$

As the angle between the two diagonals is 90°, this implies that the diagonals of a rhombus intersect at right angles.

9 $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}, \boldsymbol{b} = -4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k},$

$\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}, \boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$\boldsymbol{e} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \boldsymbol{f} = -\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$

$\boldsymbol{a}.\boldsymbol{e} = (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}).(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

$= 1 \times 2 + 3 \times -1 + -1 \times -1$

$= 2 - 3 + 1$

$= 0$

$\boldsymbol{b}.\boldsymbol{c} = (-4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(-2\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k})$

$= -4 \times -2 + 1 \times -2 + 2 \times -3$

$= 8 - 2 - 6$

$= 0$

$\boldsymbol{d}.\boldsymbol{f} = (-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).(-\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k})$

$= -1 \times -1 + 1 \times 4 + 1 \times -5$

$= 1 + 4 - 5$

$= 0$

Hence the three pairs of perpendicular vectors are: $\boldsymbol{a}$ and $\boldsymbol{e}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, $\boldsymbol{d}$ and $\boldsymbol{f}$

10 **a** $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = |\boldsymbol{d}| = |\boldsymbol{e}| = |\sqrt{3}$

b $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 1 - 1 = -1$

Therefore $\cos\theta = -\frac{1}{3}$ $\theta = 109.47°$

It is easy to see that this will be the same for all combinations

11 $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{OP} = q\boldsymbol{b}$

a $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= q\boldsymbol{b} - \boldsymbol{a}$

b $\overrightarrow{AP} = q\boldsymbol{b} - \boldsymbol{a}$

$= q(2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}) - (\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k})$

Using $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$

$(2q - 1) \times 2 + (5q - 4) \times 5 + (4 - q) \times -1 = 0$

$\therefore 4q - 2 + 25q - 20 - 4 + q = 0$

$\therefore 30q = 26$

$\therefore q = \frac{13}{15}$

c $\overrightarrow{OP} = \frac{13}{15} - (2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k})$

$= \frac{26}{15}\boldsymbol{i} + \frac{13}{3}\boldsymbol{j} - \frac{13}{15}\boldsymbol{k}$

$\therefore P = \left(\frac{26}{15}, \frac{13}{3}, -\frac{13}{15}\right)$

12 $(x\boldsymbol{i} + 2\boldsymbol{j} + y\boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 0$

$\therefore x + 2 + y = 0$

$\therefore x + y = -2$ ①

$(x\boldsymbol{i} + 2\boldsymbol{j} + y\boldsymbol{k}).(4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\therefore 4x + 2 + 2y = 0$

$\therefore 4x + 2y = -2$ ②

① × 4 − 2 gives

$2y = -6$

$\therefore y = -3$

Substituting $y = -3$ into ① gives

$\therefore x = -2 + 3 = 1$

$\therefore x = 1$ and $y = -3$

$= [(2q - 1)\boldsymbol{i} + (5q - 4)\boldsymbol{j} + (4 - q)\boldsymbol{k}]$

13 Before attempting this question ensure your calculator is set to radian mode.

a $\cos\theta = \frac{(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}).(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})}{\sqrt{6} \times \sqrt{18}}$

$\therefore \cos\theta = \frac{1 - 8 - 1}{6\sqrt{3}}$

$\therefore \cos\theta = -\frac{4}{3\sqrt{3}}$

$\therefore \theta = 2.45^c$

b $\cos\theta = \frac{(-2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}).(-2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})}{\sqrt{14} \times 3}$

$\therefore \cos\theta = \frac{4 - 2 + 3}{3\sqrt{14}}$

$\therefore \cos\theta = \frac{5}{3\sqrt{14}}$

$\therefore \theta = 1.11^c$

c $\cos\theta = \dfrac{(2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}).(4\boldsymbol{i} + 0\boldsymbol{j} - 2\boldsymbol{k})}{\sqrt{14} \times \sqrt{20}}$

$\therefore \cos\theta = \dfrac{8 + 6}{2\sqrt{70}}$

$\therefore \cos\theta = \dfrac{7}{\sqrt{70}}$

$\therefore \theta = 0.580^c$

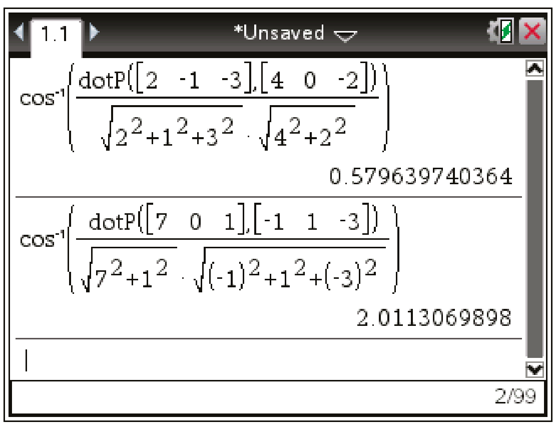
d $\cos\theta = \dfrac{(7\boldsymbol{i} + 0\boldsymbol{j} + \boldsymbol{k}).(-\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k})}{\sqrt{50} \times \sqrt{11}}$

$\therefore \cos\theta = \dfrac{-7 - 3}{5\sqrt{22}}$

$\therefore \cos\theta = -\dfrac{2}{\sqrt{22}}$

$\therefore \theta = 2.01^c$

Using a CAS calculator for part **c** and **d** we have



14 Given: $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{0}$, $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$.

Using the scalar product,

$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$

$\therefore 0 = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$ (since $\boldsymbol{a}.\boldsymbol{b} = 0$)

Now since $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$ this implies that $|\boldsymbol{a}| \neq \boldsymbol{0}$ and $|\boldsymbol{b}| \neq \boldsymbol{0}$.

$\therefore \cos\theta = \dfrac{0}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$

$\therefore \cos\theta = 0$(since $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are both non-zero)

$\therefore \theta = 90°$

Thus, since the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is 90°, they are perpendicular to each other.

15 $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

and M is the midpoint of AB.

a $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$= (2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) - (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - 2\boldsymbol{k}$

$\overrightarrow{OM} = \overrightarrow{OA} + \dfrac{1}{2}\overrightarrow{AB}$

$= (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \dfrac{1}{2}(\boldsymbol{i} - 2\boldsymbol{k})$

$\therefore \overrightarrow{OM} = \dfrac{3}{2}\boldsymbol{i} + \boldsymbol{j}$

b $\cos\theta = \dfrac{\overrightarrow{OA} \cdot \overrightarrow{OM}}{|\overrightarrow{OA}|\,|\overrightarrow{OM}|}$

$\therefore \cos\theta = \dfrac{(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \cdot \left(\dfrac{3}{2}\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}\right)}{\sqrt{3} \times \sqrt{\dfrac{13}{4}}}$

$\therefore \cos\theta = \dfrac{\dfrac{3}{2} + 1}{\dfrac{\sqrt{39}}{2}}$

$\therefore \cos\theta = \dfrac{\dfrac{5}{2}}{\dfrac{\sqrt{39}}{2}}$

$\therefore \cos\theta = \dfrac{5}{\sqrt{39}}$

$\therefore \theta = 36.81°$

c $\cos\theta = \dfrac{\overrightarrow{MB}\cdot\overrightarrow{MO}}{|\overrightarrow{MB}||\overrightarrow{MO}|}$

$\therefore \cos\theta = \dfrac{\left(\frac{1}{2}\boldsymbol{i} + 0\boldsymbol{j} - \boldsymbol{k}\right).\left(-\frac{3}{2}\boldsymbol{i} - \boldsymbol{j} + 0\boldsymbol{k}\right)}{\sqrt{\frac{5}{4}} \times \sqrt{\frac{13}{4}}}$

$\therefore \cos\theta = -\dfrac{3}{4} \times \dfrac{4}{\sqrt{65}}$

$\therefore \cos\theta = -\dfrac{3}{\sqrt{65}}$

$\therefore \theta = 111.85°$

16 a i $\overrightarrow{GB} = \overrightarrow{GF} + \overrightarrow{FB}$

$= \overrightarrow{OA} + \overrightarrow{DO}$

$= \overrightarrow{OA} - \overrightarrow{OD}$

$= 3\boldsymbol{j} - \boldsymbol{i}$

$= -\boldsymbol{i} + 3\boldsymbol{j}$

ii $\overrightarrow{GE} = \overrightarrow{GF} + \overrightarrow{FE}$

$= \overrightarrow{OA} - \overrightarrow{OC}$

$= 3\boldsymbol{j} - 2\boldsymbol{k}$

b $\cos\theta = \dfrac{\overrightarrow{GB}\cdot\overrightarrow{GE}}{|GB||GE|}$

$\therefore \cos\theta = \dfrac{(-\boldsymbol{i} + 3\boldsymbol{j} + 0\boldsymbol{k}).(0\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})}{\sqrt{10} \times \sqrt{13}}$

$\therefore \cos\theta = \dfrac{9}{\sqrt{130}}$

$\therefore \theta = 37.87°$

c $\overrightarrow{CE} = \overrightarrow{CG} + \overrightarrow{GF} + \overrightarrow{FE}$

$= \overrightarrow{OD} + \overrightarrow{OA} - \overrightarrow{OC}$

$= \boldsymbol{j} + 3\boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{GA} = \overrightarrow{GF} + \overrightarrow{FE} + \overrightarrow{EA}$

$= \overrightarrow{OA} - \overrightarrow{OC} - \overrightarrow{OD}$

$= 3\boldsymbol{j} - 2\boldsymbol{k} - \boldsymbol{i}$

$= -\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

$\cos\theta = \dfrac{\overrightarrow{CE}.\overrightarrow{GA}}{|CE||GA|}$

$\therefore \cos\theta = \dfrac{(\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}).(-\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})}{\sqrt{14} \times \sqrt{14}}$

$\therefore \cos\theta = \dfrac{12}{14}$

$\therefore \cos\theta = \dfrac{6}{7}$

$\therefore \theta = 31.00°$

17 a i $\overrightarrow{OM} = \overrightarrow{OA} + \dfrac{1}{2}\overrightarrow{AB}$

$= \overrightarrow{OA} + \dfrac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$= 4\boldsymbol{i} + \dfrac{1}{2}(-4\boldsymbol{i} + 5\boldsymbol{j})$

$\therefore \overrightarrow{OM} = 2\boldsymbol{i} + \dfrac{5}{2}\boldsymbol{j} = \dfrac{1}{2}(4\boldsymbol{i} + 5\boldsymbol{j})$

ii $\overrightarrow{ON} = \overrightarrow{OA} + \dfrac{1}{2}\overrightarrow{AC}$

$= \overrightarrow{OA} + \dfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OA}\right)$

$= 4\boldsymbol{i} + \dfrac{1}{2}(-6\boldsymbol{i} + 7\boldsymbol{k})$

$\therefore \overrightarrow{ON} = \boldsymbol{i} + \dfrac{7}{2}\boldsymbol{k} = \dfrac{1}{2}(2\boldsymbol{i} + 7\boldsymbol{k})$

b

$\cos\theta = \dfrac{\overrightarrow{OM}.\overrightarrow{ON}}{|OM||ON|}$

$\therefore \cos\theta = \dfrac{\left(2\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + 0\boldsymbol{k}\right).\left(\boldsymbol{i} + 0\boldsymbol{j} + \frac{7}{2}\boldsymbol{k}\right)}{\sqrt{\frac{41}{4}} \times \sqrt{\frac{53}{4}}}$

$\therefore \cos\theta = 2 \times \dfrac{4}{\sqrt{2173}} = \dfrac{8}{\sqrt{2173}}$

$\therefore \theta = 80.12°$

c $\cos\theta = \dfrac{\overrightarrow{OM}.\overrightarrow{OC}}{|OM||OC|}$

$\therefore \cos\theta = \dfrac{\left(2\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + 0\boldsymbol{k}\right).(-2\boldsymbol{i} + 0\boldsymbol{j} + 7\boldsymbol{k})}{\sqrt{\frac{41}{4}} \times \sqrt{53}}$

$\therefore \cos\theta = -4 \times \dfrac{2}{\sqrt{2173}} = -\dfrac{8}{\sqrt{2173}}$

$\therefore \theta = 99.88°$

18 $\overrightarrow{CE} = \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AE}$

$= -\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{OD}$

$= -(-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + 3\boldsymbol{j} + (2\boldsymbol{i} - \boldsymbol{j})$

$= 3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OA} + \overrightarrow{AB}$

$= -\overrightarrow{OD} + \overrightarrow{OA} + \overrightarrow{OC}$

$= -(2\boldsymbol{i} - \boldsymbol{j}) + 3\boldsymbol{j} + (-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$= -3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$

Let M be the midpoint of CE.

$\overrightarrow{CE} = 3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

$\Rightarrow \overrightarrow{CM} = \frac{1}{2}(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB}$

$= -\frac{1}{2}(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) + 3\boldsymbol{j}$

$= -\frac{3}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + \boldsymbol{k}$

$= \frac{1}{2}(-3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k})$

$\overrightarrow{DM} = \overrightarrow{DE} + \overrightarrow{EM}$

$= \overrightarrow{OA} - \overrightarrow{CM}$

$= 3\boldsymbol{j} - \frac{1}{2}(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$= -\frac{3}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + \boldsymbol{k}$

$= \frac{1}{2}(-3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k})$

$= \overrightarrow{MB}$

Thus M is the midpoint of DB.
Therefore the diagonals bisect each other.

$\cos\theta = \dfrac{\overrightarrow{CE}.\overrightarrow{DB}}{|CE||DB|}$

$\therefore \cos\theta = \dfrac{(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}).(-3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k})}{\sqrt{14} \times \sqrt{38}}$

$\therefore \cos\theta = \dfrac{-9 + 5 - 4}{2\sqrt{133}}$

$\therefore \cos\theta = -\dfrac{4}{\sqrt{133}}$

$\therefore \theta = 110.29°$

Acute angle = 180 − 110.29 = 69.71°

Solutions to Exercise 4D

1 **a** $\boldsymbol{a} = \boldsymbol{j} + 3\boldsymbol{j} - \boldsymbol{k}$

$|\boldsymbol{a}| = \sqrt{1+9+1} = \sqrt{11}$

$\therefore \hat{\boldsymbol{a}} = \dfrac{1}{\sqrt{11}}(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$

$\therefore \hat{\boldsymbol{a}} = \dfrac{\sqrt{11}}{11}(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$

b $\boldsymbol{b} = \boldsymbol{j} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$|\boldsymbol{b}| = \sqrt{1+4+4} = \sqrt{9} = 3$

$\therefore \hat{\boldsymbol{b}} = \dfrac{1}{3}(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$

c $\boldsymbol{c} = \overrightarrow{AB}$

$= \boldsymbol{b} - \boldsymbol{a}$

$= -\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore |\boldsymbol{c}| = \sqrt{1+9} = \sqrt{10}$

$\therefore \hat{\boldsymbol{c}} = \dfrac{1}{\sqrt{10}}(-\boldsymbol{j} + 3\boldsymbol{k})$

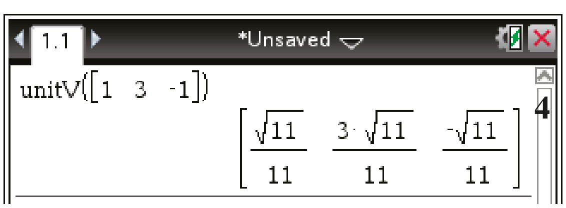
$\therefore \hat{\boldsymbol{c}} = \dfrac{\sqrt{10}}{10}(-\boldsymbol{j} + 3\boldsymbol{k})$

A CAS calculator has the ability to calculate a unit vector as follows:

TI: Press **Menu→**7: **Matrix** & **Vector→C: Vector→**1: **Unit Vector**

CP: Tap **Action → Vector → unitV**

For part **a**. type **unitV([1,3,-1])**



2 $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}, \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

a **i** $|\boldsymbol{a}| = \sqrt{9+16+1} = \sqrt{26}$

$\therefore \hat{\boldsymbol{a}} = \dfrac{1}{\sqrt{26}}(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$

$\therefore \hat{a} = \dfrac{\sqrt{26}}{26}(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$

ii $|\boldsymbol{b}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$

b If a vector is $\sqrt{3} \times \hat{\boldsymbol{a}}$

then $\sqrt{3} \times \hat{\boldsymbol{a}} = \dfrac{\sqrt{78}}{26}(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$

3 **a** $|\hat{\boldsymbol{a}}| = |\hat{\boldsymbol{b}}| = 1.$

Therefore triangle $OA'B'$ is isosceles.

b Let M be the midpoint of $A'B'$

$\overrightarrow{OM'} = \overrightarrow{OA'} + \dfrac{1}{2}\overrightarrow{A'B'}$

$= \hat{\boldsymbol{a}} + \dfrac{1}{2}(\hat{\boldsymbol{b}} - \hat{\boldsymbol{a}})$

$= \dfrac{1}{2}(\hat{\boldsymbol{b}} + \hat{\boldsymbol{a}})$

c $\triangle OA'M'$ is congruent to $\triangle OB'M'$ (SSS).

Hence $\angle A'OM' = \angle B'OM'$

d $\angle A'OB' = \angle AOB$

4 $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}, \boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{k}$

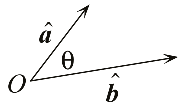
a **i** $|\boldsymbol{a}| = \sqrt{4+4+1} = 3$

$\therefore \hat{\boldsymbol{a}} = \dfrac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$

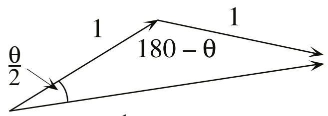
ii $|\boldsymbol{b}| = \sqrt{9+16} = 5$

$\therefore \hat{\boldsymbol{b}} = \dfrac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{k})$

b Consider $\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}$ and let $\theta = \angle AOB$



The resulting vector of adding $\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}$ will bisect $\angle AOB$.



$$\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}} = \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) + \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{k})$$

$$= \frac{5}{15}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) + \frac{3}{15}(3\boldsymbol{i} + 4\boldsymbol{k})$$

$$= \frac{1}{15}(19\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k})$$

$$\therefore |\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}| = \frac{1}{15}\sqrt{19^2 + 10^2 + 7^2}$$

$$= \frac{\sqrt{510}}{15}$$

$\therefore$ the unit vector that bisects $\angle AOB$ is

$$\frac{\frac{1}{15}(19\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k})}{\frac{\sqrt{510}}{15}}$$

$$= \frac{1}{\sqrt{510}}(19\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k})$$

$$= \frac{\sqrt{510}}{510}(19\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k})$$

5 a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b} = \frac{\boldsymbol{a.b}}{\boldsymbol{b.b}}\boldsymbol{b}$

$\boldsymbol{a.b} = 1 - 12 = -11$

$\boldsymbol{b.b} = 1 + 16 + 1 = 18$

$\therefore$ vector $= \frac{-11}{18}(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$

b $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{k}, \boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

$\boldsymbol{a.b} = 1 - 3 = -2$

$\boldsymbol{b.b} = 1 + 16 + 1 = 18$

vector resolute of $\boldsymbol{a}$ in the direction of

$$\boldsymbol{b} = \frac{\boldsymbol{a.b}}{\boldsymbol{b.b}}\boldsymbol{b}$$

$$= \frac{-1}{9}(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$$

c $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{b} = 4\boldsymbol{i} - \boldsymbol{k}$

$\boldsymbol{a.b} = 16 - 3 = 13$

$\boldsymbol{b.b} = 16 + 1 = 17$

vector resolute of $\boldsymbol{a}$ in the direction of

$$\boldsymbol{b} = \frac{\boldsymbol{a.b}}{\boldsymbol{b.b}}\boldsymbol{b}$$

$$= \frac{13}{17}(4\boldsymbol{i} - \boldsymbol{k})$$

6 scalar resolute $= \frac{\boldsymbol{a.b}}{|\boldsymbol{b}|}$

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i}$

$\boldsymbol{a.b} = 2,\ |\boldsymbol{b}| = 1$

$\therefore$ scalar resolute $= 2$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}, \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$

$\boldsymbol{a.c} = 1, |\boldsymbol{c}| = \sqrt{5}$

$\therefore$ scalar resolute $= \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

c $\boldsymbol{b} = 2\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{a} = 2\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$

$\boldsymbol{b.a} = 2\sqrt{3}, |\boldsymbol{a}| = \sqrt{4 + 3} = \sqrt{7}$

$\therefore$ scalar resolute $= \frac{2\sqrt{3}}{\sqrt{7}} = \frac{2\sqrt{21}}{7}$

d $\boldsymbol{b} = \boldsymbol{i} - \sqrt{5}\boldsymbol{j}, \boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$

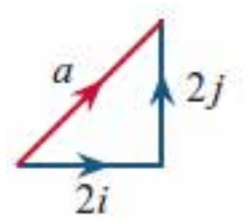
$\boldsymbol{b.c} = -1 - 4\sqrt{5}, |\boldsymbol{c}| = \sqrt{17}$

$\therefore$ scalar resolute $= \frac{-(1 + 4\sqrt{5})}{\sqrt{17}}$

$$= \frac{-\sqrt{17}(1 + 4\sqrt{5})}{17}$$

7 a $2\boldsymbol{i}$

b 2

c 

8 a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = 5\boldsymbol{i} - \boldsymbol{k}$

vector resolute of $\boldsymbol{a}$ in the direction of

$\boldsymbol{b} = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$\boldsymbol{a}.\boldsymbol{b} = 10 - 1 = 9$

$\boldsymbol{b}.\boldsymbol{b} = 25 + 1 = 26$

$\therefore$ vector resolute $= \dfrac{9}{26}(5\boldsymbol{i} - \boldsymbol{k})$

perpendicular component

$= \boldsymbol{a} - \dfrac{9}{26}(5\boldsymbol{i} - \boldsymbol{k})$

$= (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) - \dfrac{9}{26}(5\boldsymbol{i} - \boldsymbol{k})$

$= \dfrac{26}{26}(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) - \dfrac{9}{26}(5\boldsymbol{i} - \boldsymbol{k})$

$= \dfrac{1}{26}(7\boldsymbol{i} + 26\boldsymbol{j} + 35\boldsymbol{k})$

Check:

$(7\boldsymbol{i} + 26\boldsymbol{j} + 35\boldsymbol{k}).(5\boldsymbol{i} - \boldsymbol{k}) = 35 - 35 = 0$

$\therefore$ rectangular components give

$a = \dfrac{9}{26}(5\boldsymbol{i} - \boldsymbol{k}) + \dfrac{1}{26}(7\boldsymbol{i} + 26\boldsymbol{j} + 35\boldsymbol{k})$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{k}$

vector resolute of $\boldsymbol{a}$ in the direction of

$\boldsymbol{b} = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$\boldsymbol{a}.\boldsymbol{b} = 3$

$\boldsymbol{b}.\boldsymbol{b} = 1 + 1 = 2$

$\therefore$ vector resolute $= \dfrac{3}{2}(\boldsymbol{i} + \boldsymbol{k})$

perpendicular component

$= a - \dfrac{3}{2}(\boldsymbol{i} + \boldsymbol{k})$

$= 3\boldsymbol{i} + \boldsymbol{j} - \dfrac{3}{2}\boldsymbol{i} - \dfrac{3}{2}\boldsymbol{k}$

$= \dfrac{3}{2}\boldsymbol{i} + \boldsymbol{j} - \dfrac{3}{2}\boldsymbol{k}$

Check:

$\left(\dfrac{3}{2}\boldsymbol{i} + \boldsymbol{j} - \dfrac{3}{2}\boldsymbol{k}\right).(\boldsymbol{i} + \boldsymbol{k}) = \dfrac{3}{2} - \dfrac{3}{2} = 0$

$\therefore$ rectangular components give

$\boldsymbol{a} = \dfrac{3}{2}(\boldsymbol{i} + \boldsymbol{k}) + \left(\dfrac{3}{2}\boldsymbol{i} + \boldsymbol{j} - \dfrac{3}{2}\boldsymbol{k}\right)$

c $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

vector resolute of $\boldsymbol{a}$ in the direction of

$\boldsymbol{b} = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$\boldsymbol{a}.\boldsymbol{b} = -2 + 2 - 1 = -1$

$\boldsymbol{b}.\boldsymbol{b} = 4 + 4 + 1 = 9$

$\therefore$ vector resolute $= \dfrac{-1}{9}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

perpendicular component

$= a - \left[\dfrac{-1}{9}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})\right]$

$= (-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \dfrac{1}{9}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$= \dfrac{9}{9}(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \dfrac{1}{9}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$= \dfrac{1}{9}(-7\boldsymbol{i} + 11\boldsymbol{j} + 8\boldsymbol{k})$

Check:
$(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}).(-7\boldsymbol{i} + 11\boldsymbol{j} + 8\boldsymbol{k})$

$= (-14 + 22 - 8) = 0$

$\therefore$ rectangular components give

$\boldsymbol{a} = \frac{-1}{9}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$+ \frac{1}{9}(-7\boldsymbol{i} + 11\boldsymbol{j} + 8\boldsymbol{k})$

9 $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}, \boldsymbol{b} = \boldsymbol{j} + \boldsymbol{k}$

a vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b} = \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$\boldsymbol{a}.\boldsymbol{b} = 3 - 1 = 2$

$\boldsymbol{b}.\boldsymbol{b} = 1 + 1 = 2$

$\therefore$ vector resolute $= \boldsymbol{j} + \boldsymbol{k}$

b perpendicular component

$= \boldsymbol{a} - (\boldsymbol{j} + \boldsymbol{k})$

$= (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) - (\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

Magnitude $= \sqrt{1 + 4 + 4} = 3$

$\therefore$ unit vector through A perpendicular to OB is $\frac{1}{3}(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$

10 a $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}, \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

$\boldsymbol{a}.\boldsymbol{b} = (4 \times 1) + (1 \times -1) + (0 \times -1)$

$= 3$

$\boldsymbol{b}.\boldsymbol{b} = (1)^2 + (-1)^2 + (-1)^2 = 3$

vector resolute of $\boldsymbol{a}$ in the direction of

$\boldsymbol{b} = \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$= \frac{3}{3}(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

$= \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

b perpendicular component

$= \boldsymbol{a} - (\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

$= 4\boldsymbol{i} + \boldsymbol{j} - (\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

$= 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

c magnitude of perpendicular component $= \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

11 $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

a i $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$= (2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) - (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

ii $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$

$= (2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) - (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - 5\boldsymbol{j}$

b vector resolute $= \frac{\overrightarrow{AB}.\overrightarrow{AC}}{\overrightarrow{AC}.\overrightarrow{AC}}\overrightarrow{AC}$

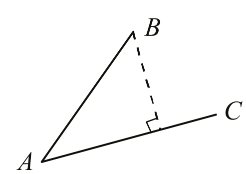
$\overrightarrow{AB}.\overrightarrow{AC} = 1 + 5 = 6$

$\overrightarrow{AC}.\overrightarrow{AC} = 1 + 25 = 26$

$\therefore$ vector resolute $= \frac{6}{26}(\boldsymbol{i} - 5\boldsymbol{j})$

$= \frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$

c



Shortest distance is perpendicular distance. perpendicular component

$= \overrightarrow{AB} - \left(\frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})\right)$

$= (\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) - \frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$

$= \frac{1}{13}(10\boldsymbol{i} + 2\boldsymbol{j} - 26\boldsymbol{k})$

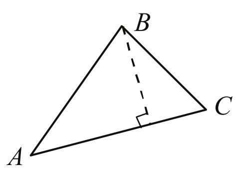
$= \frac{2}{13}(5\boldsymbol{i} + \boldsymbol{j} - 13\boldsymbol{k})$

magnitude $= \frac{2}{13}\sqrt{25 + 1 + 169}$

$= \frac{2}{13}\sqrt{195}$

$\therefore$ shortest distance $= \frac{2\sqrt{195}}{13}$ units

d



$|\overrightarrow{AC}| = \sqrt{1 + 25} = \sqrt{26}$

$\therefore$ area $\triangle ABC = \frac{1}{2}\sqrt{26} \times \frac{2\sqrt{195}}{13}$

$= \frac{\sqrt{5070}}{13}$

$= \sqrt{30}$ square units

12 a $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}, \boldsymbol{b} = 5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$\therefore \boldsymbol{a}.\boldsymbol{b} = 5 - 3 - 2 = 0$

$\therefore \boldsymbol{a} \perp \boldsymbol{b}$

b i $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{k}, \boldsymbol{d} = \frac{\boldsymbol{c}.\boldsymbol{a}}{\boldsymbol{a}.\boldsymbol{a}}\boldsymbol{a}$

$\boldsymbol{c}.\boldsymbol{a} = 2 + 2 = 4$

$\boldsymbol{a}.\boldsymbol{a} = 1 + 9 + 4 = 14$

$\therefore \boldsymbol{d} = \frac{2}{7}(\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$

ii $\boldsymbol{e} = \dfrac{\boldsymbol{c}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$$\boldsymbol{c}.\boldsymbol{b} = (2\boldsymbol{i} - \boldsymbol{k}).(5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$
$$= 10 - 1$$
$$= 9$$
$$\boldsymbol{b}.\boldsymbol{b} = 25 + 1 + 1 = 27$$
$$\therefore \boldsymbol{e} = \frac{1}{3}(5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

c $\boldsymbol{c} = \boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}$

$$\therefore \boldsymbol{f} = \boldsymbol{c} - \boldsymbol{d} - \boldsymbol{e}$$
$$= 2\boldsymbol{i} - \boldsymbol{k} - \frac{2}{7}(\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}) - \frac{1}{3}(5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$
$$= \frac{42}{21}\boldsymbol{i} - \frac{21}{21}\boldsymbol{k} - \frac{6}{21}\boldsymbol{i} + \frac{18}{21}\boldsymbol{j} + \frac{12}{21}\boldsymbol{k} - \frac{35}{21}\boldsymbol{i} - \frac{7}{21}\boldsymbol{j} - \frac{7}{21}\boldsymbol{k}$$
$$= \frac{1}{21}\boldsymbol{i} + \frac{11}{21}\boldsymbol{j} - \frac{16}{21}\boldsymbol{k}$$
$$= \frac{1}{21}(\boldsymbol{i} + 11\boldsymbol{j} - 16\boldsymbol{k})$$

d $\boldsymbol{f}.\boldsymbol{a} = \dfrac{1}{21}(\boldsymbol{i} + 11\boldsymbol{j} - 16\boldsymbol{k}).(\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$

$$= \frac{1}{21}(1 - 33 + 32)$$
$$= 0$$
$$\therefore \boldsymbol{f} \perp \boldsymbol{a}$$
$$\boldsymbol{f}.\boldsymbol{b} = \frac{1}{21}(\boldsymbol{i} + 11\boldsymbol{j} - 16\boldsymbol{k}).(5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$
$$= \frac{1}{21}(5 + 11 - 16)$$
$$= 0$$
$$\therefore \boldsymbol{f} \perp \boldsymbol{b}$$

13 $\boldsymbol{a} \perp \boldsymbol{b}$

a $\boldsymbol{d} = \dfrac{\boldsymbol{c} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a}$

b $\boldsymbol{e} = \dfrac{\boldsymbol{c} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$

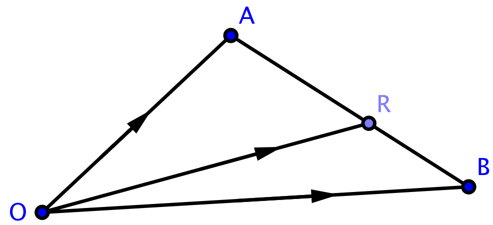
c $\boldsymbol{f} = \boldsymbol{c} - \boldsymbol{d} - \boldsymbol{e} = \boldsymbol{c} - \dfrac{\boldsymbol{c} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} - \dfrac{\boldsymbol{c} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$

d $\boldsymbol{f} \cdot \boldsymbol{a} = (\boldsymbol{c} - \dfrac{\boldsymbol{c} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} - \dfrac{\boldsymbol{c} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}) \cdot \boldsymbol{a}$

$$= \boldsymbol{c} \cdot \boldsymbol{a} - \frac{\boldsymbol{c} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} \cdot \boldsymbol{a}$$
$$= 0$$
$$\boldsymbol{f} \cdot \boldsymbol{b} = (\boldsymbol{c} - \frac{\boldsymbol{c} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} - \frac{\boldsymbol{c} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}) \cdot \boldsymbol{b}$$
$$= \boldsymbol{c} \cdot \boldsymbol{b} - \frac{\boldsymbol{c} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} \cdot \boldsymbol{b}$$
$$= 0$$

Solutions to Exercise 4E

1 a



$AR : RB = 2 : 1$

$\therefore \overrightarrow{OR} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{2}{3}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$

b $AR : RB = 3 : 2$

$\therefore \overrightarrow{OR} = \overrightarrow{OA} + \frac{3}{5}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{3}{5}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{3}{5}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{b}$

2 a $\overrightarrow{AR} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{OR} = \boldsymbol{a} + \frac{1}{2}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

$\therefore \overrightarrow{OR} = \frac{5}{2}\boldsymbol{i} - \boldsymbol{j} + \frac{5}{2}\boldsymbol{k}$

b $\overrightarrow{AR} = \frac{4}{3}\overrightarrow{AB}$

$\overrightarrow{OR} = \boldsymbol{a} + \frac{4}{3}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{4}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= -\frac{1}{3}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}$

$\therefore \overrightarrow{OR} = \frac{5}{3}\boldsymbol{i} - \frac{8}{3}\boldsymbol{j}$

c $\overrightarrow{AR} = -\frac{1}{3}\overrightarrow{AB}$

$\overrightarrow{OR} = \boldsymbol{a} - \frac{1}{3}\overrightarrow{AB}$

$= \boldsymbol{a} - \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{4}{3}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b}$

$\therefore \overrightarrow{OR} = \frac{10}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + 5\boldsymbol{k}$

3 $\overrightarrow{OP} = \boldsymbol{a}, \overrightarrow{OQ} = 3\boldsymbol{a} - 4\boldsymbol{b}, \overrightarrow{OR} = 4\boldsymbol{a} - 6\boldsymbol{b}$

a $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= 2\boldsymbol{a} - 4\boldsymbol{b}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

$= 3\boldsymbol{a} - 6\boldsymbol{b}$

$\therefore \overrightarrow{PQ} = \frac{2}{3}\overrightarrow{PR}$

$\therefore$ P, Q and R are collinear.

b $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

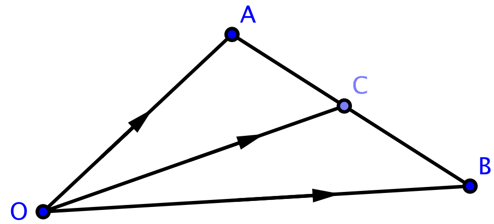
$= \boldsymbol{a} - 2\boldsymbol{b}$

$\overrightarrow{PQ} = 2\boldsymbol{a} - 4\boldsymbol{b}$

$\therefore \overrightarrow{PQ} = 2\overrightarrow{QR}$

$\therefore PQ : QR = 2 : 1$

4



a $\overrightarrow{OC} = \frac{1}{2}((x + a)\boldsymbol{i} + y\boldsymbol{j})$

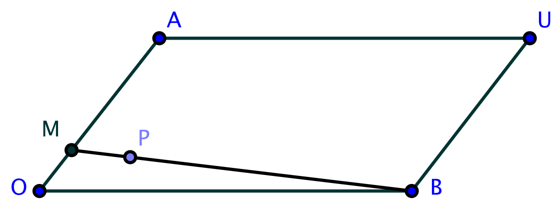
b If $OC \perp AB$ then $\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$

$(\frac{1}{2}((x + a)\boldsymbol{i} + y\boldsymbol{j})) \cdot (x - a)\boldsymbol{i} + y\boldsymbol{j} = 0$

$x^2 - a^2 + y^2 = 0$

$x^2 + y^2 = a^2$

5



a

$$\overrightarrow{OM} = \frac{1}{5}\boldsymbol{a}$$

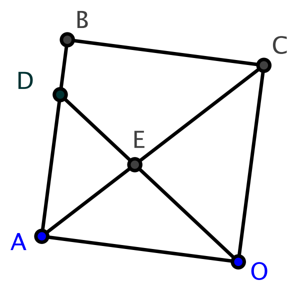
$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OM} + \overrightarrow{MP}\\ &= \frac{1}{5}\boldsymbol{a} + \frac{1}{6}\overrightarrow{MB}\\ &= \frac{1}{5}\boldsymbol{a} + \frac{1}{6}(-\frac{1}{5}\boldsymbol{a} + \boldsymbol{b})\\ &= \frac{1}{6}\boldsymbol{a} + \frac{1}{6}\boldsymbol{b}\end{aligned}$$

Also $\overrightarrow{OU} = \boldsymbol{a} + \boldsymbol{b}$

$\therefore$ P is on diagonal OU.

b $OP : PU = 1 : 5$

6



$\overrightarrow{OA} = -4\boldsymbol{i} + 3\boldsymbol{j}$
$\overrightarrow{OC} = 3\boldsymbol{i} + 4\boldsymbol{j}$

a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC} = -\boldsymbol{i} + 7\boldsymbol{j}$

b $\overrightarrow{BD} = \frac{1}{3}\overrightarrow{BA} = \frac{1}{3}(-3\boldsymbol{i} - 4\boldsymbol{j})$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= (-4\boldsymbol{i} + 3\boldsymbol{j}) + \frac{2}{3}(3\boldsymbol{i} + 4\boldsymbol{j})$

$= -2\boldsymbol{i} + \frac{17}{3}\boldsymbol{j}$

c $\overrightarrow{OE} = (1 - \lambda)\overrightarrow{OA} + \lambda\overrightarrow{OC}$

$= (7\lambda - 4)\boldsymbol{i} + (3 + \lambda)\boldsymbol{j}$

We can also write

$\overrightarrow{OE} = \mu\overrightarrow{OD}$

$\therefore \overrightarrow{OE} = \mu(-2\boldsymbol{i} + \frac{17}{3}\boldsymbol{j})$

$\therefore 7\lambda - 4 = -2\mu \ldots (1)$

$3 + \lambda = \frac{17}{3}\mu \ldots (2)$

Therefore $\lambda = \frac{2}{5}$

7 **a** $\overrightarrow{OA} \cdot \overrightarrow{OB} = |\overrightarrow{OA}||\overrightarrow{OB}| \cos\theta \therefore -5 = 5 \times 3 \cos\theta$

$\cos\theta = -\frac{1}{3}$

$\therefore \theta$ is obtuse.

b **i** $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$

$= 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

ii $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= -2\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}$

Now P, A and B are collinear. Therefore there is a real number λ such that:

$$\overrightarrow{OP} = \lambda\overrightarrow{OA} + (1 - \lambda)\overrightarrow{OB}$$

$$\therefore \overrightarrow{OP} = (1 + 2\lambda)\boldsymbol{i} + (2 - 2\lambda)\boldsymbol{j} + (6\lambda - 2)\boldsymbol{k}$$

$$OP \perp AB \Rightarrow \overrightarrow{OP} \cdot \overrightarrow{AB} = 0$$

$$-2(2 + \lambda) + 2(2 - 2\lambda) - 6(6\lambda - 2) = 0$$

$$\therefore \lambda = \frac{7}{22}$$

$$\therefore \overrightarrow{OP} = \frac{18}{11}\boldsymbol{i} + \frac{15}{11}\boldsymbol{j} - \frac{1}{11}\boldsymbol{k}$$

iii $\overrightarrow{OP} = (1 + 2\lambda)\boldsymbol{i} + (2 - 2\lambda)\boldsymbol{j} + (6\lambda - 2)\boldsymbol{k}$

Because of the bisection of $\angle AOB$

$$\frac{\overrightarrow{OA} \cdot \overrightarrow{OP}}{|\overrightarrow{OA}|} = \frac{\overrightarrow{OB} \cdot \overrightarrow{OP}}{|\overrightarrow{OB}|}$$

$$\therefore \frac{3(1 + 2\lambda) + 4(6\lambda - 2)}{5} = \frac{1 + 2\lambda + 2(2 - 2\lambda) - 2(6\lambda - 2)}{3}$$

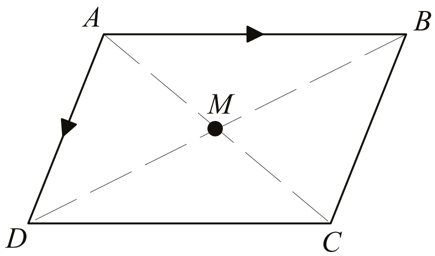
$$\therefore \lambda = \frac{3}{8}$$

$$\therefore \overrightarrow{OP} = \frac{7}{4}\boldsymbol{i} + \frac{5}{4}\boldsymbol{j} + \frac{1}{4}\boldsymbol{k}$$

Solutions to Exercise 4F

Vector proofs in two-dimensional geometry

1 Required to prove that the diagonals of a parallelogram bisect each other.



$ABCD$ is a parallelogram.

Let $\overrightarrow{AD} = \boldsymbol{a}$

Let $\overrightarrow{AB} = \boldsymbol{b}$

Let M be the midpoint of AC.

$$\overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{a}$$

$$\Rightarrow \overrightarrow{AM} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

$$\overrightarrow{BM} = -\overrightarrow{AB} + \overrightarrow{AM}$$

$$= -\boldsymbol{b} + \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

$$= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$$

$$\overrightarrow{MD} = -\overrightarrow{AM} + \overrightarrow{AD}$$

$$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{a}$$

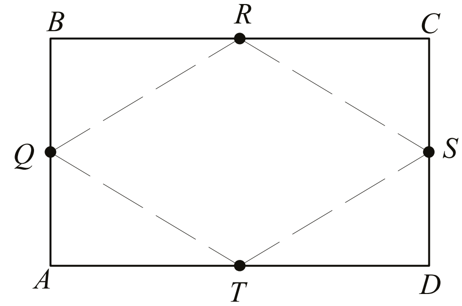
$$= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$$

$$= \overrightarrow{BM}$$

Thus M is the midpoint BD.

Therefore the diagonals of a parallelogram bisect each other.

2 Required to prove that if the midpoints of the sides of a rectangle are joined then a rhombus is formed.



$ABCD$ is a rectangle.

Let Q, R, S and T be the midpoints of AB, BC, CD and DA respectively.

Let $\overrightarrow{AD} = \boldsymbol{a}$

$$\Rightarrow \overrightarrow{AT} = \frac{1}{2}AD = \frac{1}{2}\boldsymbol{a}$$

Let $\overrightarrow{AB} = \boldsymbol{b}$

$\Rightarrow$

$$\overrightarrow{QT} = \overrightarrow{AT} - \overrightarrow{AQ} = \frac{1}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$$

$$\overrightarrow{CR} = -\overrightarrow{AT} = -\frac{1}{2}\boldsymbol{a}$$

$$\overrightarrow{CS} = -\overrightarrow{AQ}- = -\frac{1}{2}\boldsymbol{b}$$

$$\overrightarrow{RS} = \overrightarrow{CS} - \overrightarrow{CR}$$

$$= -\frac{1}{2}\boldsymbol{b} - \left(-\frac{1}{2}\boldsymbol{a}\right)$$

$$= \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$$

$$\therefore \overrightarrow{QT} = \overrightarrow{RS}$$

i.e. QT is parallel to RS and they are equal in length.

$$|\overrightarrow{QT}|^2 = \left(\frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})\right).\left(\frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})\right)$$

$$= \frac{1}{4}(\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b})$$

$$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})$$

$$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$$

since $\boldsymbol{a}.\boldsymbol{b} = 0$ (as they are perpendicular)

$\overrightarrow{TS} = \overrightarrow{AT} + \overrightarrow{AQ}$

$= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

$\left|\overrightarrow{TS}\right|^2 = \left(\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\right).\left(\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\right)$

$= \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})$

$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$

since $\boldsymbol{a}.\boldsymbol{b} = 0$ (as they are perpendicular)

$\therefore |\overrightarrow{QT}|^2 = |\overrightarrow{TS}|^2 \Rightarrow |\overrightarrow{QT}| = |\overrightarrow{TS}|$

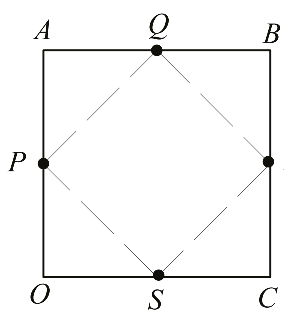
So since

$\overrightarrow{QT} = \overrightarrow{RS}$ *and* $|\overrightarrow{QT}| = |\overrightarrow{TS}|$

i.e. one pair of opposite sides are equal and parallel and adjacent sides are of equal length.

$\therefore$ *QRST* is a rhombus.

3 Required to prove that if the midpoints of the sides of a square are joined then another square are is formed.



Let *P*, *Q*, *R* and *S* be the midpoints of *OA*, *AB*, *BC* and *CO* respectively.

Let

$\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{b}$

$\therefore \overrightarrow{OP} = \frac{1}{2}\boldsymbol{a}$ and $\overrightarrow{OS} = \frac{1}{2}\boldsymbol{b}$

$\therefore \overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{QR} = \overrightarrow{QB} + \overrightarrow{BR}$

$= \overrightarrow{OS} - \overrightarrow{OP}$

$= \frac{1}{2}\boldsymbol{b} - \left(\frac{1}{2}\boldsymbol{a}\right)$

$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$\therefore \overrightarrow{PS} = \overrightarrow{QR}$

i.e. *PS* is parallel to *QR* and they are equal in length.

$|\overrightarrow{PS}|^2 = \left(\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\right).\left(\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\right)$

$= \frac{1}{4}(\boldsymbol{b} - \boldsymbol{a}).(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{4}(\boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})$

$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$

since $\boldsymbol{a}.\boldsymbol{b} = 0$ (as they are perpendicular)

$\overrightarrow{SR} = \overrightarrow{OS} + \overrightarrow{OP}$

$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

$\therefore |\overrightarrow{SR}|^2 = \left(\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\right).\left(\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\right)$

$= \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})$

$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$

since $\boldsymbol{a}.\boldsymbol{b} = 0$ (as they are perpendicular)

$\therefore |\overrightarrow{PS}|^2 = |\overrightarrow{SR}|^2 \Rightarrow |\overrightarrow{PS}| = |\overrightarrow{SR}|$

OABC is a square.

So since

$\overrightarrow{PS} = \overrightarrow{QR}$ and $|\overrightarrow{PS}| = |\overrightarrow{SR}|$, *PSRQ* is a rhombus.

$$\overrightarrow{PS}.\overrightarrow{SR} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}).\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$
$$= \frac{1}{4}(\boldsymbol{b}.\boldsymbol{b} - \boldsymbol{a}.\boldsymbol{a})$$
$$= \frac{1}{4}(|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2)$$

As a rhombus has all sides of equal length.

$\therefore |\boldsymbol{a}| = |\boldsymbol{b}|$

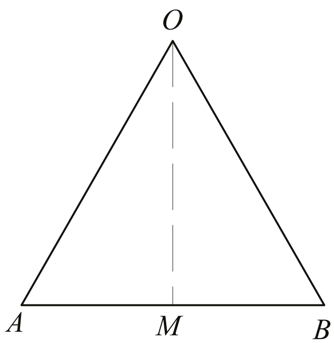
Hence

$\overrightarrow{PS}.\overrightarrow{SR} = 0$

$\therefore \angle PSR = 90°$

Therefore *PSRQ* is a square.

4 Required to prove that the median to the base of an isosceles triangle is perpendicular to the base.



M is the midpoint of *AB*.

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$
$$\overrightarrow{AM}.\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}).\frac{1}{2}(\boldsymbol{b} + \boldsymbol{a})$$
$$= \frac{1}{4}(\boldsymbol{b}.\boldsymbol{b} - \boldsymbol{a}.\boldsymbol{a})$$
$$= \frac{1}{4}(|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2)$$

As an isosceles triangle has two sides of equal length.

$\therefore |\boldsymbol{b}| = |\boldsymbol{a}|$

Hence

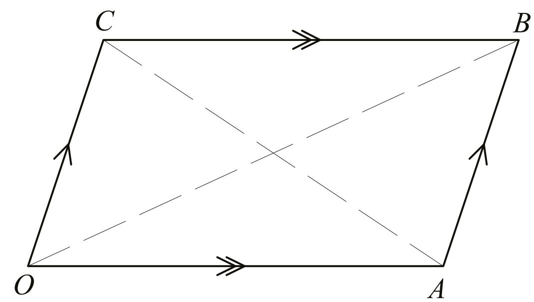
$\overrightarrow{AM}.\overrightarrow{OM} = 0$

$\therefore \angle OMA = 90°$

Thus the median to the base of an isosceles triangle is perpendicular to the base.

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{a})$

5 Required to prove that if the diagonals of a parallelogram are of equal length then the parallelogram is a rectangle.



OABC is a parallelogram.

Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OC} = \boldsymbol{b}$ and $|\overrightarrow{OB}| = |\overrightarrow{CA}|$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC} = \boldsymbol{a} + \boldsymbol{b}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \boldsymbol{a} - \boldsymbol{b}$

$$|\overrightarrow{OB}|^2 = (\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b})$$
$$= \boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$$
$$|\overrightarrow{CA}|^2 = (a - b).(a - b)$$
$$= \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$$

Since

$|\overrightarrow{OB}| = |\overrightarrow{CA}| = |\overrightarrow{OB}|^2 = |CA|^2$

$\therefore \boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b} = \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$

$\therefore 4\boldsymbol{a}.\boldsymbol{b} = 0$

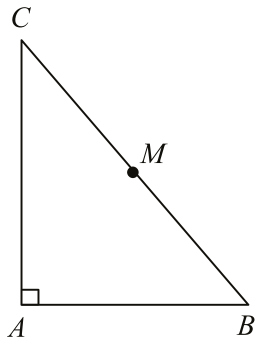
$\therefore \boldsymbol{a}.\boldsymbol{b} = 0$

$\therefore \angle COA = 90°$

Hence the parallelogram *OABC* is a rectangle.

6 Required to prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.

ABC is a triangle.



Let M be the midpoint of BA.

Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$

$$\overrightarrow{CB} = \overrightarrow{AB} - \overrightarrow{AC} = \mathbf{a} - \mathbf{b}$$

$$\therefore \overrightarrow{CM} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$$

$$\overrightarrow{BM} = -\overrightarrow{CM} = -\frac{1}{2}(\mathbf{a} - \mathbf{b})$$

$$\overrightarrow{AM} = \overrightarrow{AC} + \overrightarrow{CM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$|\overrightarrow{CM}|^2 = \frac{1}{2}(\mathbf{a} - \mathbf{b}).\frac{1}{2}(\mathbf{a} - \mathbf{b})$$

$$= \frac{1}{4}(\mathbf{a}.\mathbf{a} - 2\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{b})$$

$$= \frac{1}{4}(\mathbf{a}.\mathbf{a} + \mathbf{b}.\mathbf{b})$$

(since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a}.\mathbf{b} = 0$)

$$|\overrightarrow{BM}|^2 = -\frac{1}{2}(\mathbf{a} - \mathbf{b}). -\frac{1}{2}(\mathbf{a} - \mathbf{b})$$

$$= \frac{1}{4}(\mathbf{a}.\mathbf{a} - 2\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{b})$$

$$= \frac{1}{4}(\mathbf{a}.a + \mathbf{b}.\mathbf{b})$$

(since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a}.\mathbf{b} = 0$)

$$|\overrightarrow{AM}|^2 = \frac{1}{2}(\mathbf{a} + \mathbf{b}).\frac{1}{2}(\mathbf{a} + \mathbf{b})$$

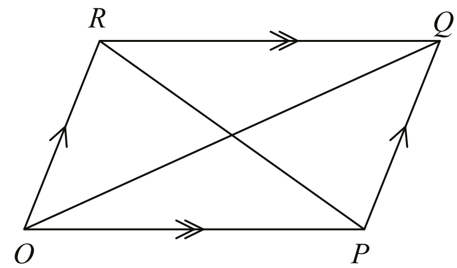
$$= \frac{1}{4}(\mathbf{a}.\mathbf{a} + \mathbf{b}.\mathbf{b})$$

(since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a}.\mathbf{b} = 0$)

$$\therefore |\overrightarrow{CM}|^2 = |\overrightarrow{BM}|^2 = |\overrightarrow{AM}|^2$$

Thus the midpoint of the hypotenuse is equidistant from the three vertices.

7 Required to prove that the sum of the squares of the lengths of the diagonals of any parallelogram is equal to the sum of the squares of the lengths of the sides.



Let $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OR} = \mathbf{b}$

$$\overrightarrow{OQ} = \mathbf{a} + \mathbf{b}$$

$$\overrightarrow{RP} = \mathbf{a} - \mathbf{b}$$

So

$$|\overrightarrow{OQ}|^2 + |\overrightarrow{RP}|^2$$

$$= (\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}).(\mathbf{a} - \mathbf{b})$$

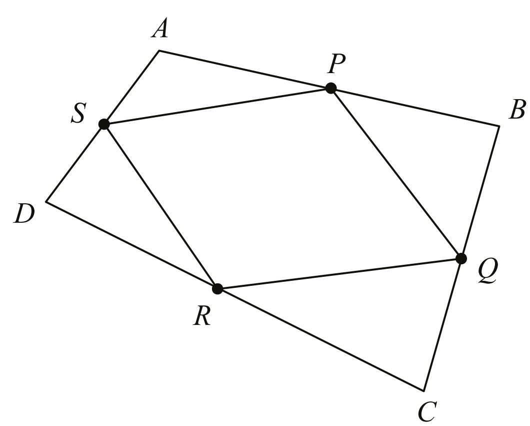
$$= \mathbf{a}.\mathbf{a} + 2\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{b} + \mathbf{a}.\mathbf{a} - 2\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{b}$$

$$= \mathbf{a}.\mathbf{a} + \mathbf{b}.\mathbf{b} + \mathbf{a}.\mathbf{a} + \mathbf{b}.\mathbf{b}$$

$$= |\overrightarrow{OP}|^2 + |\overrightarrow{PQ}|^2 + |\overrightarrow{RQ}|^2 + |\overrightarrow{OR}|^2$$

as required to prove.

8 Required to prove that if the midpoints of the sides of a quadrilateral are joined then a parallelogram if formed.



$ABCD$ is a quadrilateral. P, Q, R and S are the midpoints of the sides AB, BC,

CD and DA respectively.

$\overrightarrow{AS} = \frac{1}{2}\overrightarrow{AD}$

$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{SP} = \overrightarrow{AP} - \overrightarrow{AS}$

$= \frac{1}{2}\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AD}$

$= \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{AD})$

$= \frac{1}{2}\overrightarrow{DB}$

$\therefore \overrightarrow{SP} = \frac{1}{2}\overrightarrow{DB}$

Similarly,

$\overrightarrow{CR} = \frac{1}{2}\overrightarrow{CD}$

$\overrightarrow{CQ} = \frac{1}{2}\overrightarrow{CB}$

$\overrightarrow{RQ} = \overrightarrow{RC} + \overrightarrow{CQ}$

$= \frac{1}{2}\overrightarrow{CB} - \frac{1}{2}\overrightarrow{CD}$

$= \frac{1}{2}(\overrightarrow{CB} - \overrightarrow{CD})2$

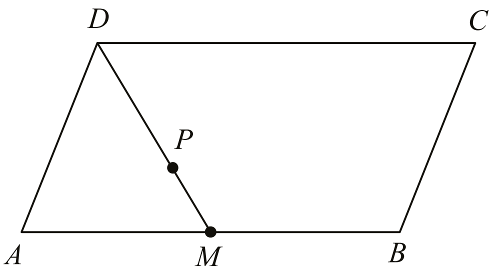
$= \frac{1}{2}\overrightarrow{DB}$

$\therefore \overrightarrow{RQ} = \frac{1}{2}\overrightarrow{DB}$

Thus $\overrightarrow{SP} = \overrightarrow{RQ}$ meaning $SP \parallel RQ$ and $SP = RQ$

Hence $PQRS$ is a parallelogram.

9



Let $\overrightarrow{AD} = \boldsymbol{b}$ and $\overrightarrow{AB} = \boldsymbol{a}$

$\overrightarrow{DM} = \overrightarrow{AM} - \overrightarrow{AD} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b}$

$\overrightarrow{DP} = \frac{2}{3}\overrightarrow{DM} = \frac{2}{3}$

$\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}\right) = \frac{1}{3}(\boldsymbol{a} - 2\boldsymbol{b})$

$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP}$

$= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{1}{3}\overrightarrow{AC}$

$\overrightarrow{PC} = \overrightarrow{DC} - \overrightarrow{DP}$

$= \boldsymbol{a} - \frac{1}{3}(\boldsymbol{a} - 2\boldsymbol{b})$

$= \frac{2}{3}(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{2}{3}\overrightarrow{AC}$

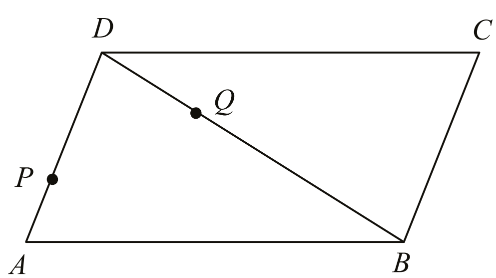
Therefore P is a point of trisection of AC nearer to A.

Since $AP \parallel PC \parallel AC$ this implies that A, P and C are collinear.

Thus A, P and C are collinear and P is a point of trisection of AC.

As required to prove.

10



Let $\overrightarrow{AD} = \boldsymbol{b}$ and $\overrightarrow{AB} = \boldsymbol{a}$

$\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$

$\overrightarrow{DB} = \boldsymbol{a} - \boldsymbol{b}$

$\overrightarrow{DP} = \frac{2}{3}\overrightarrow{DA}$

$= \frac{2}{3}(-\overrightarrow{AD}) = -\frac{2}{3}\boldsymbol{b}$

$\overrightarrow{DQ} = \frac{1}{3}\overrightarrow{DB} = \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$

$\overrightarrow{PQ} = \overrightarrow{DQ} - \overrightarrow{DP}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) - \left(-\frac{2}{3}\boldsymbol{b}\right)$

$= \frac{1}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$

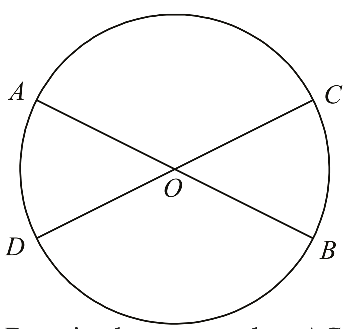
$= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

$= \frac{1}{3}\overrightarrow{AC}$

$\therefore \overrightarrow{PQ} = \frac{1}{3}\overrightarrow{AC}$

Thus PQ is parallel to AC.

11



Required to prove that $ACBD$ is a rectangle.

AB and CD are the diameters of the circle, hence $AB = CD$.

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OD} = \boldsymbol{d}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \boldsymbol{d} - \boldsymbol{a}$

$\overrightarrow{CB} = -\overrightarrow{OA} + \overrightarrow{OD} = \boldsymbol{d} - \boldsymbol{a}$

So since $\overrightarrow{AD} = \overrightarrow{CD} \Rightarrow \overrightarrow{AD} = \overrightarrow{CB}$ and $\overrightarrow{AD}$□$\overrightarrow{CB}$

$\therefore ACBD$ is a parallelogram.

$\overrightarrow{AC} = -\overrightarrow{OA} + \overrightarrow{OC}$

$= \overrightarrow{OA} - \overrightarrow{OD}$

$= -\boldsymbol{d} - \boldsymbol{a}$

$\overrightarrow{AC}.\overrightarrow{AD} = (-\boldsymbol{d} - \boldsymbol{a}).(\boldsymbol{d} - \boldsymbol{a})$

$= -\boldsymbol{d}.\boldsymbol{d} + \boldsymbol{d}.\boldsymbol{a}$

$- \boldsymbol{a}.\boldsymbol{d} + \boldsymbol{a}.\boldsymbol{a}$

$= \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{d}.\boldsymbol{d}$

$= |\boldsymbol{a}|^2 - |\boldsymbol{d}|^2$

Since OA and OD are the radius of the circle

$\therefore |\boldsymbol{a}| = |\boldsymbol{d}|$

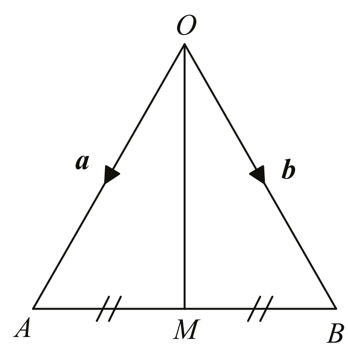
Hence

$\overrightarrow{AC}.\overrightarrow{AD} = 0$

$\therefore \angle CAD = 90°$

Therefore $ACBD$ is a rectangle.

12



a **i** $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$\therefore \overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

ii $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

b $\overrightarrow{AM}.\overrightarrow{AM} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}).\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{4}(\boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{a})$

$$\overrightarrow{OM}.\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$$
$$= \frac{1}{4}(\boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})$$
$$\overrightarrow{AM}.\overrightarrow{AM} + \overrightarrow{OM}.\overrightarrow{OM}$$
$$= 2\left(\frac{1}{4}\boldsymbol{a}.\boldsymbol{a}\right) + 2\left(\frac{1}{4}\boldsymbol{b}.\boldsymbol{b}\right)$$
$$= \frac{1}{2}\boldsymbol{a}.\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}.\boldsymbol{b}$$
$$= \frac{1}{2}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$$
$$\therefore \overrightarrow{AM}.\overrightarrow{AM} + \overrightarrow{OM}.\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$$

c Prove $OA^2 + OB^2 = 2OM^2 + 2AM^2$

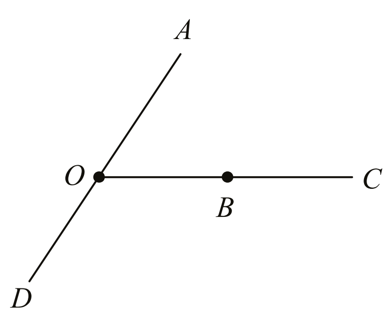
RHS

$$2OM^2 + 2AM^2 = 2(OM^2 + AM^2)$$
$$= 2(OM.OM + AM.AM)$$
$$= 2\left(\frac{1}{2}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})\right)$$

(from part ***b***)

$$= \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b}$$
$$= OA^2 + OB^2$$

13



O is the midpoint of *AD* and *B* is the midpoint of *OC*.

a *A*, *P* and *C* are collinear ifthere exist a $k \in R\backslash\{0\}$ such that $\overrightarrow{AP} = k\overrightarrow{PC}$

$$\overrightarrow{PC} = \overrightarrow{OC} - \overrightarrow{OP}$$
$$= 2\boldsymbol{b} - \frac{1}{3}(\boldsymbol{a} + 4\boldsymbol{b})$$
$$= -\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$
$$= \frac{1}{3}(2\boldsymbol{b} - \boldsymbol{a})$$
$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$$
$$= \frac{1}{3}(\boldsymbol{a} + 4\boldsymbol{b}) - \boldsymbol{a}$$
$$= -\frac{2}{3}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}$$
$$= \frac{2}{3}(2\boldsymbol{b} - \boldsymbol{a})$$
$$= 2\overrightarrow{PC}$$

Thus since $\overrightarrow{AP} = 2\overrightarrow{PC}$, *A*, *P* and *C* are collinear.

b *D*, *B* and *P* are collinear if there exist a $k \in R\backslash\{0\}$ such that $\overrightarrow{DB} = k\overrightarrow{BP}$

$\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$

(Since $\overrightarrow{DO} = \overrightarrow{OA}$)

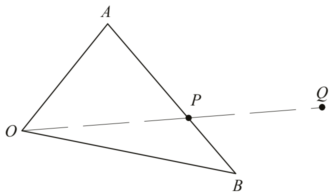
$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$$
$$= \frac{1}{3}(\boldsymbol{a} + 4\boldsymbol{b}) - \boldsymbol{b}$$
$$= \frac{1}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$$
$$= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$$
$$= \overrightarrow{DB}$$

Thus since $\overrightarrow{DB} = 3\overrightarrow{BP}$, *D*, *B* and *P* are collinear.

c Since $\overrightarrow{DB} = \overrightarrow{3BP}$

$\therefore DB{:}BP = 3{:}1$

14



$\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$ and $\overrightarrow{OQ} = 3\overrightarrow{OP}$

a **i** $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \boldsymbol{a} + \frac{2}{3}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{2}{3}\boldsymbol{b} + \frac{1}{3}\boldsymbol{a}$

$\therefore \overrightarrow{OP} = \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

ii $\overrightarrow{OQ} = 3\overrightarrow{OP} = \boldsymbol{a} + 2\boldsymbol{b}$

iii $\overrightarrow{AQ} = \overrightarrow{AP} + \overrightarrow{PQ}$

$= \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a}) + (\overrightarrow{OQ} - \overrightarrow{OP})$

$= \frac{2}{3}(b - \boldsymbol{a})$

$+ \left(\boldsymbol{a} + 2\boldsymbol{b} - \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})\right)$

$= \frac{2}{3}\boldsymbol{b} - \frac{2}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}$

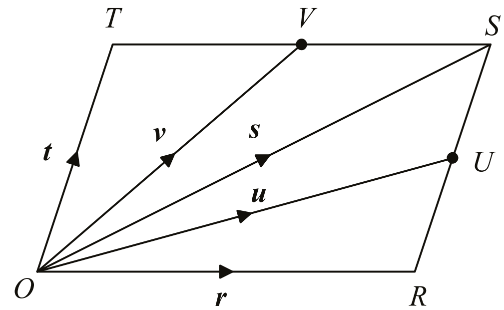
$= 2\boldsymbol{b}$

b $\overrightarrow{AQ} = 2\boldsymbol{b} = 2\overrightarrow{OB}$

$\therefore \overrightarrow{AQ} = 2\overrightarrow{OB}$

Therefore $\overrightarrow{AQ}$ is parallel to $\overrightarrow{OB}$

15



a $\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{RS}$

$= \overrightarrow{OR} + \overrightarrow{OT}$

$= \boldsymbol{r} + \boldsymbol{t}$

$\therefore \boldsymbol{s} = \boldsymbol{r} + \boldsymbol{t}$

b $\overrightarrow{OV} = \overrightarrow{OS} + \overrightarrow{SV}$

$= \boldsymbol{s} - \frac{1}{2}\boldsymbol{r}$

$= \boldsymbol{s} - \frac{1}{2}(\boldsymbol{s} - \boldsymbol{t})$ since $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{t}$

$= \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

$\therefore \boldsymbol{v} = \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

c $\overrightarrow{OU} = \overrightarrow{OR} + \overrightarrow{RS}$

$= \boldsymbol{r} + \frac{1}{2}\boldsymbol{t}$

$= \boldsymbol{r} + \frac{1}{2}(\boldsymbol{s} - \boldsymbol{r})$ since $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{t}$

$= \frac{1}{2}(\boldsymbol{r} + \boldsymbol{s})$

$\therefore \boldsymbol{u} = \frac{1}{2}(\boldsymbol{r} + \boldsymbol{s})$

$$4(\boldsymbol{u}+\boldsymbol{v}) = 4\left(\frac{1}{2}(\boldsymbol{r}+\boldsymbol{s}) + \frac{1}{2}(\boldsymbol{s}+\boldsymbol{t})\right)$$
$$= 2(\boldsymbol{r}+\boldsymbol{s}) + 2(\boldsymbol{s}+\boldsymbol{t})$$
$$= 2\boldsymbol{r} + 2\boldsymbol{s} + 2\boldsymbol{s} + 2\boldsymbol{t}$$
$$= 2\boldsymbol{r} + 3\boldsymbol{s} + \boldsymbol{s} + 2\boldsymbol{t}$$
$$= 2\boldsymbol{r} + 3\boldsymbol{s} + (\boldsymbol{r}+\boldsymbol{t}) + 2\boldsymbol{t}$$
$$= 3\boldsymbol{r} + 3\boldsymbol{s} + 3\boldsymbol{t}$$
$$= 3(\boldsymbol{r}+\boldsymbol{s}+\boldsymbol{t})$$

as required.

16 $\boldsymbol{a} = \boldsymbol{i} + 11\boldsymbol{j}, \boldsymbol{b} = 2\boldsymbol{i} + 8\boldsymbol{j}, \boldsymbol{c} = -\boldsymbol{i} + 7\boldsymbol{j},$

$\boldsymbol{d} = -2\boldsymbol{i} + 8\boldsymbol{j}, \boldsymbol{e} = -4\boldsymbol{i} + 6\boldsymbol{j}$

a $\overrightarrow{AD} = \boldsymbol{d} - \boldsymbol{a} = -3\boldsymbol{i} - 3\boldsymbol{j}$

$\overrightarrow{AE} = \boldsymbol{e} - \boldsymbol{a} = -5\boldsymbol{i} - 5\boldsymbol{j}$

$\therefore \overrightarrow{AE} = \frac{5}{3}\overrightarrow{AD} = \overrightarrow{AE}\square\overrightarrow{AD}$

$\therefore E$ lies on the line DA.

$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -3\boldsymbol{i} - \boldsymbol{j}$

$\overrightarrow{BE} = \boldsymbol{e} - \boldsymbol{b} = -6\boldsymbol{i} - 2\boldsymbol{j}$

$\therefore \overrightarrow{BE} = 2\overrightarrow{BC} \Rightarrow \overrightarrow{BE}\square\overrightarrow{BC}$

$\therefore E$ lies on the line BC.

b $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}$

$\overrightarrow{DC} = \boldsymbol{c} - \boldsymbol{d} = \boldsymbol{i} - \boldsymbol{j}$

c Let the point F have position vector $\boldsymbol{f} = x\boldsymbol{i} + y\boldsymbol{j}$.

As the point F lies on the extended line AB

$\Rightarrow \overrightarrow{AF} = \overrightarrow{AB}$

$\therefore \overrightarrow{AF} = k\overrightarrow{AB}$

$\therefore \overrightarrow{AF} = k(\boldsymbol{i} - 3\boldsymbol{j}) = k\boldsymbol{i} - 3k\boldsymbol{j}$

Also,

$\overrightarrow{AF} = \boldsymbol{f} - \boldsymbol{a} = (x-1)\boldsymbol{i} + (y-11)\boldsymbol{j}$

Hence

$k = x - 1$ and $-3k = y - 11$

$\therefore -3(x-1) = y - 11$

$\therefore 3x + y = 14$ ①

As the point F lies on the extended line DC

$\Rightarrow \overrightarrow{DF} = \alpha\overrightarrow{DC}$

$\therefore \overrightarrow{DF} = \alpha(\boldsymbol{i} - \boldsymbol{j}) = \alpha\boldsymbol{i} - \alpha\boldsymbol{j}$

Also,

$\overrightarrow{DF} = \boldsymbol{f} - \boldsymbol{d} = (x+2)\boldsymbol{i} + (y-8)\boldsymbol{j}$

Hence

$\alpha = x + 2$ and $-\alpha = y - 8$

$\therefore -(x+2) = y - 8$

$\therefore x + y = 6$

$\therefore y = 6 - x$ ②

Substituing ② into ① gives

$2x = 8$

$\therefore x = 4$

Substituting $x = 4$ into ② gives $y = 2$

$\therefore x = 4$ and $y = 2$

Thus $\boldsymbol{f} = 4\boldsymbol{i} + 2\boldsymbol{j}$

d $\overrightarrow{FD} = \boldsymbol{d} - \boldsymbol{f} = -6\boldsymbol{i} + 6\boldsymbol{j}$

$\overrightarrow{EA} = -\overrightarrow{AE} = 5\boldsymbol{i} + 5\boldsymbol{j}$

$\overrightarrow{EB} = -\overrightarrow{BE} = 6\boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{AF} = \boldsymbol{f} - \boldsymbol{a} = 3\boldsymbol{i} - 9\boldsymbol{j}$

$$\overrightarrow{FD}.\overrightarrow{EA} = (-6\boldsymbol{i} + 6\boldsymbol{j}).(5\boldsymbol{i} + 5\boldsymbol{j})$$
$$= -30 + 30$$
$$= 0$$

$\therefore \overrightarrow{FD} \perp \overrightarrow{AF}$

$$\overrightarrow{EB}.\overrightarrow{AF} = (6\boldsymbol{i} + 2\boldsymbol{j})(3\boldsymbol{i} - 9\boldsymbol{j})$$
$$= 18 - 18$$
$$= 0$$

$\therefore \overrightarrow{EB} \perp \overrightarrow{AF}$

as required.

e Since $\angle EDF$ and $\angle EBF$ are at right-angles this implies that EF is the diameter of the circle (angles in a semicircle).

The centre of the circle is the midpoint of EF and has position vector

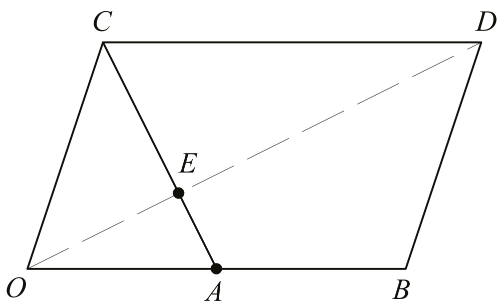
$$\frac{\boldsymbol{e} + \boldsymbol{f}}{2} = \frac{(-4\boldsymbol{i} + 6\boldsymbol{j}) + (4\boldsymbol{i} + 2\boldsymbol{j})}{2}$$

$$= \frac{8\boldsymbol{j}}{2}$$

$$= 4\boldsymbol{j}$$

Hence the position vector of the centre of the circle through E, D, B and F is $4\boldsymbol{j}$.

17



Given:

$\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}, \overrightarrow{OC} = \boldsymbol{c}, \overrightarrow{OD} = \boldsymbol{d},$

$\overrightarrow{OE} = \boldsymbol{e}, e = \frac{1}{3}\boldsymbol{d}, \overrightarrow{AE} = \frac{1}{3}\overrightarrow{AC}$

A is the midpoint of OB.

$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \boldsymbol{d} - \boldsymbol{b}$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AE} + \overrightarrow{EC}$

$$= \boldsymbol{a} + (\overrightarrow{OE} - \overrightarrow{OA}) + (\overrightarrow{ED} + \overrightarrow{DC})$$

$$= \boldsymbol{a} + (\boldsymbol{e} - \boldsymbol{a}) + \left(\frac{2}{3}\boldsymbol{d} - \boldsymbol{b}\right)$$

$$= \boldsymbol{e} + \frac{2}{3}\boldsymbol{d} - \boldsymbol{b}$$

$$= \frac{1}{3}\boldsymbol{d} + \frac{2}{3}\boldsymbol{d} - \boldsymbol{b}$$

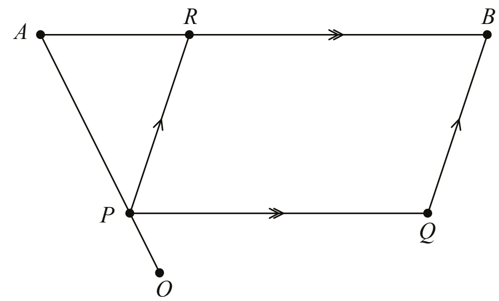
$$= \boldsymbol{d} - \boldsymbol{b}$$

$$= \overrightarrow{BD}$$

$\therefore \overrightarrow{OC} = \overrightarrow{BD} \Rightarrow \overrightarrow{OC} = \overrightarrow{BD}$ and $\overrightarrow{OC} \square \overrightarrow{BD}$

$\therefore$ $OCDB$ is a parallelogram

18



$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$

$$= \frac{1}{4}\overrightarrow{OA} + \overrightarrow{RB}$$

$$= \frac{1}{4}\boldsymbol{a} + \frac{2}{3}\overrightarrow{AB}$$

$$= \frac{1}{4}\boldsymbol{a} + \frac{2}{3}(\overrightarrow{OB} - \overrightarrow{OA})$$

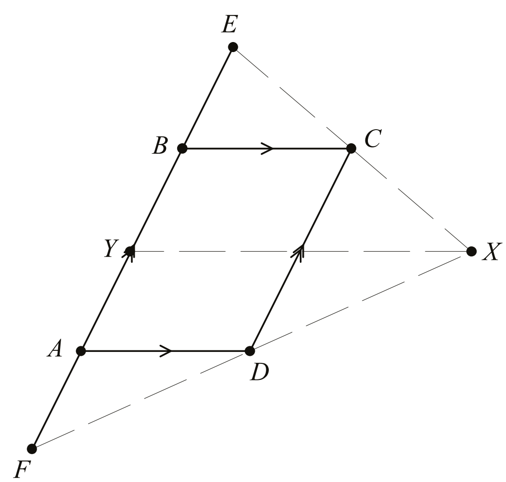
$$= \frac{1}{4}\boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$$

$$= \frac{2}{3}\boldsymbol{b} - \frac{5}{12}\boldsymbol{b}$$

Therefore the position vector of Q is $\frac{2}{3}\boldsymbol{b} - \frac{5}{12}\boldsymbol{a}$

19

20



Given:

$BE = AF = BC$

Let $YX \parallel BC \parallel AD$.

$\triangle EBC$ is similar to $\triangle EYX$ and since

$\overrightarrow{BC} = \overrightarrow{BE} \Rightarrow \overrightarrow{YE} = \overrightarrow{YX}$

Similarly, $YF = YX$.

$\therefore \overrightarrow{YE} = -\overrightarrow{YF}$

Now

$\overrightarrow{XE} = \overrightarrow{YE} - \overrightarrow{YX}$

and

$\overrightarrow{XF} = \overrightarrow{YF} - \overrightarrow{YX}$

$$\begin{aligned}\overrightarrow{XE}.\overrightarrow{XF} &= (\overrightarrow{YE} - \overrightarrow{YX}).(\overrightarrow{YF} - \overrightarrow{YX})\\ &= \overrightarrow{YE}.\overrightarrow{YF} - \overrightarrow{YE}\overrightarrow{YX}\\ &\quad - \overrightarrow{YX}.\overrightarrow{YF} + \overrightarrow{YX}\overrightarrow{YX}\\ &= -\overrightarrow{YE}.\overrightarrow{YE} - \overrightarrow{YE}.\overrightarrow{YX}\\ &\quad + \overrightarrow{YX}.\overrightarrow{YE} + \overrightarrow{YX}.\overrightarrow{YX}\\ &= -\overrightarrow{YE}.\overrightarrow{YE} + \overrightarrow{YX}.\overrightarrow{YX}\\ &= |YX|^2 - |YE|^2\\ &= 0\end{aligned}$$

(Since $\overrightarrow{YE} = \overrightarrow{YX}$)

$\therefore \overrightarrow{EX}$ and $\overrightarrow{FX}$ meet at right angles.

b $|\overrightarrow{AB}| = k|\overrightarrow{BC}| = k|\overrightarrow{BE}| = k|\overrightarrow{FA}|$

$$\begin{aligned}\overrightarrow{FE} &= \overrightarrow{FA} + \overrightarrow{AB} + \overrightarrow{BE}\\ &= \overrightarrow{BE} + k\overrightarrow{BE} + \overrightarrow{BE}\\ &= 2\overrightarrow{BE} + k\overrightarrow{BE}\\ &= (2 + k)\overrightarrow{BE}\end{aligned}$$

$\therefore YE = \frac{1}{2}\overrightarrow{FE} = \frac{1}{2}(2 + k)\overrightarrow{BE}$

As ΔEBC is similar to ΔEYX

$\Rightarrow \frac{YE}{BE} = \frac{EX}{EC}$

Given $\overrightarrow{EX} = \lambda\overrightarrow{EC}$

$\therefore \lambda = \frac{1}{2}(2 + k) = \frac{k + 2}{2}$

$$\begin{aligned}\overrightarrow{FY} &= \overrightarrow{FA} + \overrightarrow{AY}\\ &= \overrightarrow{FA} + \frac{1}{2}\overrightarrow{AB}\\ &= \overrightarrow{FA} + \frac{1}{2}(k\overrightarrow{FA})\\ &= \left(1 + \frac{k}{2}\right)\overrightarrow{FA}\end{aligned}$$

As ΔFAD is similar to ΔFYX

$\Rightarrow \frac{FY}{FA} = \frac{FX}{FD}$

Given $\overrightarrow{FX} = \mu\overrightarrow{ED}$

$\therefore \mu = 1 + \frac{k}{2} = \frac{k + 2}{2}$

c A rhombus has all sides of equal length.

$\therefore |\overrightarrow{AB}| = |\overrightarrow{BC}|$

Given $|\overrightarrow{AB}| = k|\overrightarrow{BC}|,\ k = 1$ if $ABCD$ is a rhombus.

$\therefore \lambda = \frac{3}{2}$ and $\mu = \frac{3}{2}$

d $\overrightarrow{XE} = \overrightarrow{XY} + \overrightarrow{YE}$

$\overrightarrow{FX} = \overrightarrow{FY} + \overrightarrow{YX}$

$$\begin{aligned}|\overrightarrow{XE}|^2 &= (\overrightarrow{XY}.\overrightarrow{YE}).(\overrightarrow{XY}.\overrightarrow{YE})\\ &= |\overrightarrow{XY}|^2 + 2\overrightarrow{XY}.\overrightarrow{YE} + |\overrightarrow{YE}|^2\end{aligned}$$

$$\begin{aligned}|\overrightarrow{FX}|^2 &= (\overrightarrow{FY}.\overrightarrow{YX}).(\overrightarrow{FY}.\overrightarrow{YX})\\ &= |\overrightarrow{FY}|^2 + 2\overrightarrow{FY}.\overrightarrow{YX} + |\overrightarrow{YX}|^2\end{aligned}$$

Since $|\overrightarrow{XY}|^2 = |\overrightarrow{YX}|^2, |\overrightarrow{FY}|^2 = |\overrightarrow{YE}|^2$ and $|XE|^2 = |FX|^2$

$\therefore 2\overrightarrow{XY}.\overrightarrow{YE} = 2\overrightarrow{FY}.\overrightarrow{YX}$

$\therefore \overrightarrow{XY}.\overrightarrow{YE} = \overrightarrow{FY}.\overrightarrow{YX}$

$\therefore -\overrightarrow{YX}.\overrightarrow{YE} = -\overrightarrow{YF}.\overrightarrow{YX}$

$\therefore \overrightarrow{YX}.\overrightarrow{YE} = \overrightarrow{YF}.\overrightarrow{YX}$

$\therefore |\overrightarrow{YX}||\overrightarrow{YE}|\cos(\angle XYE)$

$= |\overrightarrow{YF}||\overrightarrow{YX}|\cos(\angle FYX)$

Since $|\overrightarrow{YX}| = |\overrightarrow{YE}| = |\overrightarrow{YF}|$

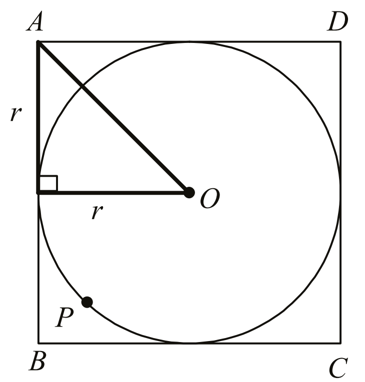
$\therefore \cos(\angle XYE) = \cos(\angle FYX)$

$\therefore \angle XYE = \angle FYX = 90°$

Since $\overrightarrow{BC} \perp \overrightarrow{YX} \Rightarrow \angle YBC = 90°$

$\therefore ABCD$ is a rectangle

21



a $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$\therefore \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$\overrightarrow{AP}.\overrightarrow{AP} = (\overrightarrow{OP} - \overrightarrow{OA}).(\overrightarrow{OP} - \overrightarrow{OA})$

$= \overrightarrow{OP}.\overrightarrow{OP} - 2\overrightarrow{OP}.\overrightarrow{OA} + \overrightarrow{OA}.\overrightarrow{OA}$

$= r^2 - 2\overrightarrow{OP}.\overrightarrow{OA} + |\overrightarrow{OA}|^2$

$= r^2 - 2\overrightarrow{OP}.\overrightarrow{OA} + (r^2 + r^2)$

$= 3r^2 - 2\overrightarrow{OP}.\overrightarrow{OA}$

b $|\overrightarrow{BP}|^2 = 3r^2 - 2|\overrightarrow{OP}||\overrightarrow{OB}|$

$|\overrightarrow{CP}|^2 = 3r^2 - 2|\overrightarrow{OP}||\overrightarrow{OC}|$

$|\overrightarrow{DP}|^2 = 3r^2 - 2|\overrightarrow{OP}||\overrightarrow{OD}|$

$\therefore |\overrightarrow{AP}|^2 + |\overrightarrow{BP}|^2 + |\overrightarrow{CP}|^2 + |\overrightarrow{DP}|^2$

$= 4 \times 3r^2 - 2\overrightarrow{OP}.\overrightarrow{OA} - 2\overrightarrow{OP}.\overrightarrow{OB}$

$- 2\overrightarrow{OP}.\overrightarrow{OC} - 2\overrightarrow{OP}.\overrightarrow{OD}$

$= 12r^2 - 2\overrightarrow{OP}.\overrightarrow{OA}$

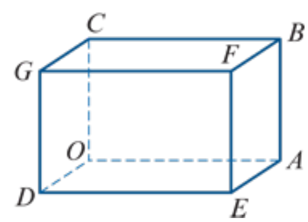
$+ \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$

$= 12r^2 - 2\overrightarrow{OP}.(\mathbf{0})$

$= 12r^2$

Vectorproof in three-dimensional geometry

22



Space diagonls OF and CE are considered. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$, $\overrightarrow{OD} = \mathbf{d}$

$\overrightarrow{OF} = \overrightarrow{OD} + \overrightarrow{DE} + \overrightarrow{EF}$

$= \overrightarrow{OD} + \overrightarrow{OA} + \overrightarrow{OC}$

$= \mathbf{d} + \mathbf{a} + \mathbf{c}$

$\overrightarrow{CE} = \overrightarrow{CO} + \overrightarrow{OD} + \overrightarrow{DE}$

$= \overrightarrow{CO} + \overrightarrow{OD} + \overrightarrow{OA}$

$= -\mathbf{c} + \mathbf{d} + \mathbf{a}$

Let X be the midpoint of OF and Y be the midpoin of CE

Then $\overrightarrow{OX} = \frac{1}{2}\overrightarrow{OF} = \frac{1}{2}(\mathbf{d} + \mathbf{a} + \mathbf{c})$

$$\overrightarrow{OY} = \overrightarrow{OC} + \overrightarrow{CY}$$

$$= \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CY}$$

$$= \mathbf{c} + \frac{1}{2}(-\mathbf{c} + \mathbf{d} + \mathbf{a})$$

$$= \frac{1}{2}(\mathbf{d} + \mathbf{a} + \mathbf{c})$$

$$\therefore X = Y$$

$$|\overrightarrow{OF}|^2 = (\mathbf{d} + \mathbf{a} + \mathbf{c}) \cdot (\mathbf{d} + \mathbf{a} + \mathbf{c})$$

$$= \mathbf{d} \cdot \mathbf{d} + \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}$$

since $\mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{d} = \mathbf{a} \cdot \mathbf{d} = 0$

and

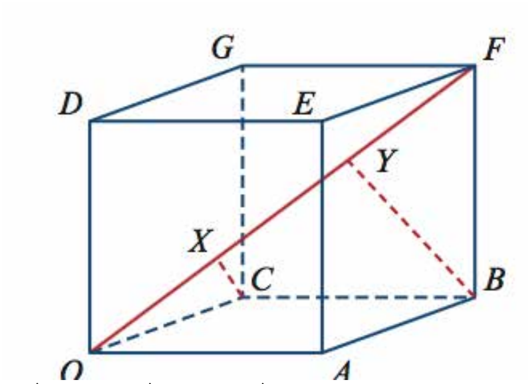
$$|\overrightarrow{CE}|^2 = (\mathbf{d} + \mathbf{a} - \mathbf{c}) \cdot (\mathbf{d} + \mathbf{a} - \mathbf{c})$$

$$= \mathbf{d} \cdot \mathbf{d} + \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}$$

since $\mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{d} = \mathbf{a} \cdot \mathbf{d} = 0$

Therefore $|\overrightarrow{CE}| = |\overrightarrow{OF}|$

23



$\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OC} = \mathbf{c}, \overrightarrow{OD} = \mathbf{d}$
and
$|\mathbf{a}| = a, |\mathbf{c}| = c, |\mathbf{d}| = d$

a $\overrightarrow{OF} = \overrightarrow{OD} + \overrightarrow{DE} + \overrightarrow{EF}$

$= \overrightarrow{OD} + \overrightarrow{OA} + \overrightarrow{OC}$

$= \mathbf{d} + \mathbf{a} + \mathbf{c}$

Let X be the point on OF such that $CX \perp OF$ and $\overrightarrow{OX} = \lambda\overrightarrow{OF}$ for some $\lambda \in \mathbb{R}$

Then $\overrightarrow{CX} = -\mathbf{c} + \overrightarrow{OX} =$ $(\lambda - 1)\mathbf{c} + \lambda\mathbf{a} + \lambda\mathbf{d}$

We use the dot product.

$$\overrightarrow{CX} \cdot \overrightarrow{OF} = 0$$

$$((\lambda - 1)\mathbf{c} + \lambda\mathbf{a} + \lambda\mathbf{d}) \cdot (\mathbf{d} + \mathbf{a} + \mathbf{c}) = 0$$

$$(\lambda - 1)\mathbf{c} \cdot \mathbf{c} + \lambda\mathbf{a} \cdot \mathbf{a} + \lambda\mathbf{d} \cdot \mathbf{d} = 0$$

$$\lambda(a^2 + d^2 + c^2) = c^2$$

$$\lambda = \frac{c^2}{a^2 + d^2 + c^2}$$

Hence $\overrightarrow{OX} = \dfrac{c^2}{a^2 + d^2 + c^2}(\mathbf{d} + \mathbf{a} + \mathbf{c})$

b Let Y be the point on OF such that $BY \perp OF$ and $\overrightarrow{OX} = \mu\overrightarrow{OF}$ for some $\mu \in \mathbb{R}$

Then $\overrightarrow{BY} = \overrightarrow{BA} + \overrightarrow{AO} + \overrightarrow{AY}$

$= -\mathbf{c} - \mathbf{a} + \mu(\mathbf{a} + \mathbf{c} + \mathbf{d})$

$= (\mu - 1)\mathbf{a} + (\mu - 1)\mathbf{c} + \mu\mathbf{d}$

We use the dot product.

$$\overrightarrow{BY} \cdot \overrightarrow{OF} = 0$$

$$((\mu - 1)\mathbf{a} + (\mu - 1)\mathbf{c} + \mu\mathbf{d}) \cdot (\mathbf{d} + \mathbf{a} + \mathbf{c}) = 0$$

$$\mu(a^2 + d^2 + c^2) = a^2 + c^2$$

$$\mu = \frac{c^2 + a^2}{a^2 + d^2 + c^2}$$

Hence $\overrightarrow{OY} = \dfrac{c^2 + a^2}{a^2 + d^2 + c^2}(\mathbf{d} + \mathbf{a} + \mathbf{c})$

c **i** $\overrightarrow{OY} = \dfrac{2}{3}(\mathbf{d} + \mathbf{a} + \mathbf{c})$

and $\overrightarrow{OX} = \dfrac{1}{3}(\mathbf{d} + \mathbf{a} + \mathbf{c})$

ii $\overrightarrow{XC} = \overrightarrow{XO} + \overrightarrow{OC}$

$= -(\dfrac{1}{3}(\mathbf{d} + \mathbf{a} + \mathbf{c}) + \mathbf{c}$

$= -\dfrac{1}{3}\mathbf{d} - \dfrac{1}{3}\mathbf{a} + \dfrac{2}{3}\mathbf{c}$

$|\overrightarrow{XC}|^2 = \dfrac{1}{9} + \dfrac{1}{9} + \dfrac{4}{9}$

$= \dfrac{6}{9}$

$\overrightarrow{XA} = \overrightarrow{XO} + \overrightarrow{OA}$

$= -(\dfrac{1}{3}(\mathbf{d} + \mathbf{a} + \mathbf{c}) + \mathbf{a}$

$= -\dfrac{1}{3}\mathbf{d} - \dfrac{2}{3}\mathbf{a} - \dfrac{1}{3}\mathbf{c}$

$|\overrightarrow{XA}|^2 = \dfrac{1}{9} + \dfrac{4}{9} + \dfrac{1}{9}$

$= \dfrac{6}{9}$

Remembering the magnitudes of **a**, **d** and **c** are each 1:

$\overrightarrow{XC} \cdot \overrightarrow{XA} = -\dfrac{2}{9} + \dfrac{1}{9} - \dfrac{2}{9} = -\dfrac{1}{3}$

$$\overrightarrow{XC} \cdot \overrightarrow{XA} = |\overrightarrow{XC}||\overrightarrow{XA}| \cos \angle CXA$$

$$-\frac{1}{3} = \frac{6}{9} \cos \angle CXA$$

$$\cos \angle CXA = -\frac{1}{2}$$

$$\angle CXA = 120°$$

iii $\overrightarrow{YB} = \overrightarrow{YO} + \overrightarrow{OB}$

$$= -(\frac{2}{3}(\mathbf{d} + \mathbf{a} + \mathbf{c}) + \mathbf{c} + \mathbf{a}$$

$$= \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{c} - \frac{2}{3}\mathbf{d}$$

$$|\overrightarrow{YB}|^2 = \frac{1}{9} + \frac{1}{9} + \frac{4}{9}$$

$$= \frac{6}{9}$$

$$\overrightarrow{YG} = \overrightarrow{YO} + \overrightarrow{OC} + \overrightarrow{OG}$$

$$= -(\frac{2}{3}(\mathbf{d} + \mathbf{a} + \mathbf{c}) + \mathbf{c} + \mathbf{c}$$

$$= -\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{d} + \frac{1}{3}\mathbf{c}$$

$$|\overrightarrow{YG}|^2 = \frac{4}{9} + \frac{1}{9} + \frac{1}{9}$$

$$= \frac{6}{9}$$

Remembering the magnitudes of $\mathbf{a}$, $\mathbf{d}$ and $\mathbf{c}$ are each 1:

$$\overrightarrow{YB} \cdot \overrightarrow{YG} = -\frac{2}{9} + \frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

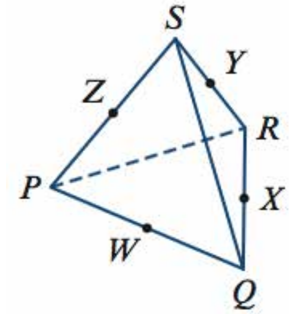
$$\overrightarrow{YB} \cdot \overrightarrow{YG} = |\overrightarrow{YB}||\overrightarrow{YG}| \cos \angle BYG$$

$$-\frac{1}{3} = \frac{6}{9} \cos \angle CXA$$

$$\cos \angle BYG = -\frac{1}{2}$$

$$\angle BYG = 120°$$

24



Let $\overrightarrow{OP} = \mathbf{p}, \overrightarrow{OQ} = \mathbf{q}, \overrightarrow{OR} = \mathbf{r}, \overrightarrow{OS} = \mathbf{s}$
Let W, X, Y and Z be the midpoints of PQ, QR, RS and SP respectively. Then, $\overrightarrow{OW} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$ We prove that $WXYZ$

$$\overrightarrow{OX} = \frac{1}{2}(\mathbf{q} + \mathbf{r})$$

$$\overrightarrow{OZ} = \frac{1}{2}(\mathbf{p} + \mathbf{s})$$

$$\overrightarrow{OY} = \frac{1}{2}(\mathbf{s} + \mathbf{r})$$

is a parallelogram by proving $\overrightarrow{WX} = \overrightarrow{ZY}$ and $\overrightarrow{WZ} = \overrightarrow{XY}$

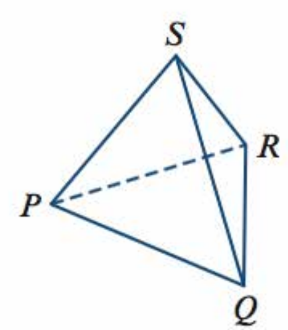
$$\overrightarrow{WX} = \frac{1}{2}(\mathbf{q} + \mathbf{r} - (\mathbf{p} + \mathbf{q})) = \frac{1}{2}(\mathbf{r} - \mathbf{p})$$

$$\overrightarrow{ZY} = \frac{1}{2}(\mathbf{s} + \mathbf{r} - (\mathbf{p} + \mathbf{s})) = \frac{1}{2}(\mathbf{r} - \mathbf{p})$$

$$\overrightarrow{WZ} = \frac{1}{2}(\mathbf{p} + \mathbf{s} - (\mathbf{p} + \mathbf{q})) = \frac{1}{2}(\mathbf{s} - \mathbf{q})$$

$$\overrightarrow{XY} = \frac{1}{2}(\mathbf{s} + \mathbf{r} - (\mathbf{q} + \mathbf{r})) = \frac{1}{2}(\mathbf{s} - \mathbf{q})$$

25



Let $\overrightarrow{OP} = \mathbf{p}, \overrightarrow{OQ} = \mathbf{q}, \overrightarrow{OR} = \mathbf{r}, \overrightarrow{OS} = \mathbf{s}$

Let A be the midpoint of PS and B be the midpoint of QR.

Then,

$\overrightarrow{OA} = \frac{1}{2}(\mathbf{p} + \mathbf{s})$

$\overrightarrow{OB} = \frac{1}{2}(\mathbf{q} + \mathbf{r})$

Let Z be the midpoint of AB

$$\overrightarrow{OZ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$

$$= \frac{1}{2}(\mathbf{p} + \mathbf{s}) + \frac{1}{2}(\frac{1}{2}(\mathbf{q} + \mathbf{r}) - \frac{1}{2}(\mathbf{p} + \mathbf{s}))$$

$$= \frac{1}{4}\mathbf{p} + \frac{1}{4}\mathbf{s} + \frac{1}{4}\mathbf{q} + \frac{1}{4}\mathbf{r}$$

$$= \frac{1}{4}(\mathbf{p} + \mathbf{s} + \mathbf{q} + \mathbf{r})$$

|et C be the midpoint of PR and D be the midpoint of QS.

Then,

$\overrightarrow{OC} = \frac{1}{2}(\mathbf{p} + \mathbf{r})$

$\overrightarrow{OD} = \frac{1}{2}(\mathbf{q} + \mathbf{s})$

Let W be the midpoint of CD

$$\overrightarrow{OW} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CD}$$

$$= \frac{1}{2}(\mathbf{p} + \mathbf{r}) + \frac{1}{2}(\frac{1}{2}(\mathbf{q} + \mathbf{s}) - \frac{1}{2}(\mathbf{p} + \mathbf{r}))$$

$$= \frac{1}{4}\mathbf{p} + \frac{1}{4}\mathbf{s} + \frac{1}{4}\mathbf{q} + \frac{1}{4}\mathbf{r}$$

$$= \frac{1}{4}(\mathbf{p} + \mathbf{s} + \mathbf{q} + \mathbf{r})$$

Therefore $W = Z$

26 The **centroid** is the point of intersection of the three medians of a triangle. A **median** of a triangle is a line from a vertec to the midpoint of the opposite side.

a We refer to Example 31 on page 129.

$\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM}$ where M is the midpoint of AB.

$\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

Therefore,

$\overrightarrow{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$

$\overrightarrow{CG} = \overrightarrow{CO} + \overrightarrow{OG}$

That is, $\overrightarrow{CG} = -\mathbf{c} + \frac{1}{3}(\mathbf{a} + \mathbf{b})$

Consider the dot product. We assume the edge lengths of the tetrahedron are all 1. The angles between edges in triangles are all 60° and $\cos 60° = \frac{1}{2}$.

$$\overrightarrow{CG} \cdot \overrightarrow{OG} = (-\mathbf{c} + \frac{1}{3}(\mathbf{a} + \mathbf{b}) \cdot (\frac{1}{3}(\mathbf{a} + \mathbf{b}))$$

$$= -\frac{1}{3}(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} + \frac{1}{9}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$$

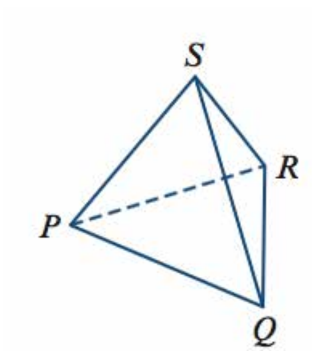
$$= -\frac{1}{3}(\mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}) + \frac{1}{9}(\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b})$$

$$= -\frac{1}{3}(\frac{1}{2} + \frac{1}{2}) + \frac{1}{9}(1 + 1 + 1)$$

$$= -\frac{1}{3} + \frac{1}{3}$$

$$= 0$$

27



$\overrightarrow{PS} = \mathbf{s} - \mathbf{p}$

$\overrightarrow{QR} = \mathbf{r} - \mathbf{q}$

$\overrightarrow{PS} \cdot \overrightarrow{QR} = (\mathbf{s} - \mathbf{p}) \cdot (\mathbf{r} - \mathbf{q})$

$= \mathbf{s} \cdot \mathbf{r} - \mathbf{s} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{r} + \mathbf{p} \cdot \mathbf{q} \ldots (1)$

We now need to use the properties that we can assume all edges have magnitude 1 and angles between edges of a triangle of the the tetrahedron = 60°

$\overrightarrow{SP} \cdot \overrightarrow{SQ} = \frac{1}{2}$

Also

$\overrightarrow{SP} \cdot \overrightarrow{SQ} = (\mathbf{p} - \mathbf{s}) \cdot (\mathbf{q} - \mathbf{s})$

That is,

$\mathbf{p} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{s} - \mathbf{s} \cdot \mathbf{q} + \mathbf{s} \cdot \mathbf{s} = \frac{1}{2} \ldots (2)$

Further,

$\overrightarrow{SP} \cdot \overrightarrow{SR} = \frac{1}{2}$

Also

$\overrightarrow{SP} \cdot \overrightarrow{SR} = (\mathbf{p} - \mathbf{s}) \cdot (\mathbf{r} - \mathbf{s})$

That is,

$\mathbf{p} \cdot \mathbf{r} - \mathbf{p} \cdot \mathbf{s} - \mathbf{s} \cdot \mathbf{r} + \mathbf{s} \cdot \mathbf{s} = \frac{1}{2} \ldots (3)$

Therefore from (2) and (3)

$\mathbf{p} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{s} - \mathbf{s} \cdot \mathbf{q} + \mathbf{s} \cdot \mathbf{s} =$

$\mathbf{p} \cdot \mathbf{r} - \mathbf{p} \cdot \mathbf{s} - \mathbf{s} \cdot \mathbf{r} + \mathbf{s} \cdot \mathbf{s}$

Hence

$\mathbf{p} \cdot \mathbf{q} - \mathbf{s} \cdot \mathbf{q} = \mathbf{p} \cdot \mathbf{r} - \mathbf{s} \cdot \mathbf{r}$

Rearranging,

$\mathbf{s} \cdot \mathbf{r} - \mathbf{s} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{r} + \mathbf{p} \cdot \mathbf{q} = 0$

which gives the result from (1).

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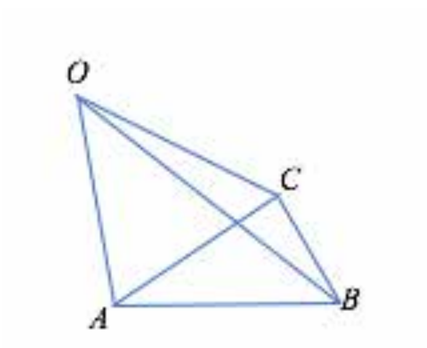


$\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$

$\Rightarrow \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$

$\Rightarrow \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$

$\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$\Rightarrow \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0$

$\Rightarrow \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{a}$

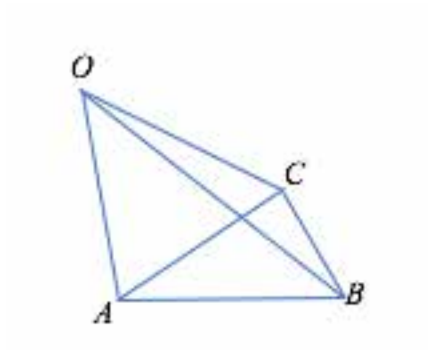
Therefore,

$\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$

$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = 0$

$\Rightarrow \overrightarrow{BA} \cdot \overrightarrow{OC} = 0$

29



$\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$

$\therefore \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0 \ldots (1)$

$\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$\therefore \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0 \ldots (2)$

$\overrightarrow{OC} \cdot \overrightarrow{BA} = 0$

$\therefore \mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0 \ldots (3)$

We prove first that $OA^2 + BC^2 = OB^2 + AC^2$

LHS $= \overrightarrow{OA} \cdot \overrightarrow{OA} + \overrightarrow{BC} \cdot \overrightarrow{BC}$

$= \mathbf{a} \cdot \mathbf{a} + (\mathbf{c} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b})$

$= \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{b}$

From (2), $\mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0 \Rightarrow 2\mathbf{b} \cdot \mathbf{c} = 2\mathbf{a} \cdot \mathbf{c}$

$\therefore \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{b}$

$= \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c} - 2\mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{b}$

$= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{a} - \mathbf{c}) + \mathbf{b} \cdot \mathbf{b}$

$= \overrightarrow{AC} \cdot \overrightarrow{AC} + \overrightarrow{OB} \cdot \overrightarrow{OB}$

We next prove first that

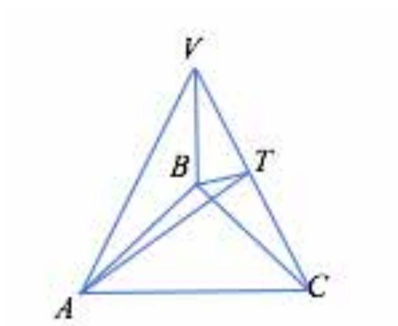
$OB^2 + AC^2 = OC^2 + AB^2$

$$\text{LHS} = \overrightarrow{OB} \cdot \overrightarrow{OB} + \overrightarrow{AC} \cdot \overrightarrow{AC}$$
$$= \mathbf{b} \cdot \mathbf{b} + (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})$$
$$= \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} - 2\mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{a}$$

From (1), $\mathbf{b} \cdot (\mathbf{c} - \mathbf{b}) = 0 \Rightarrow 2\mathbf{a} \cdot \mathbf{c} = 2\mathbf{b} \cdot \mathbf{a}$

$$\therefore \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a}$$
$$= \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a}$$
$$= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) + \mathbf{c} \cdot \mathbf{c}$$
$$= \overrightarrow{AB} \cdot \overrightarrow{AB} + \overrightarrow{OC} \cdot \overrightarrow{OC}$$

30



All edges are taken to be of length 4. All angles between edges in a triangle are 60°

a $\overrightarrow{VT} = \lambda\overrightarrow{VC}$ and $AT \perp VC$

$$\overrightarrow{AT} \cdot \overrightarrow{VC} = 0$$
$$(\overrightarrow{AV} + \overrightarrow{VT}) \cdot \overrightarrow{VC} = 0$$
$$(\overrightarrow{AV} + \lambda\overrightarrow{VC}) \cdot \overrightarrow{VC} = 0$$
$$(\overrightarrow{AV} \cdot \overrightarrow{VC}) + \lambda\overrightarrow{VC} \cdot \overrightarrow{VC} = 0$$
$$-16 \times \frac{1}{2} + 16\lambda = 0$$
$$\lambda = \frac{1}{2}$$

b $\overrightarrow{BT} \cdot \overrightarrow{VC} = (\overrightarrow{BV} + \overrightarrow{VT}) \cdot \overrightarrow{VC}$

$$= \overrightarrow{BV} \cdot \overrightarrow{VC} + \overrightarrow{VC} \cdot \overrightarrow{VT}$$
$$= -\overrightarrow{VB} \cdot \overrightarrow{VC} + \overrightarrow{VC} \cdot \overrightarrow{VT}$$
$$= -8 + 4 \times 2$$
$$= 0$$

c $\overrightarrow{TA} \cdot \overrightarrow{TB} = 4$

$|\overrightarrow{TA}| = |\overrightarrow{TA}| = 2\sqrt{3}$

$\cos(\angle ATB) = \dfrac{4}{12} = \dfrac{1}{3}$

$\angle ATB = \arccos \dfrac{1}{3}$

31 a $$\overrightarrow{OG} = \overrightarrow{OD} + \overrightarrow{DC} + \overrightarrow{CG}$$
$$= \boldsymbol{d} + \boldsymbol{b} + \boldsymbol{e}$$
$$= \boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e}$$
$$\overline{DF} = \overline{DH} + \overline{EF} + \overline{EF}$$
$$= \overline{OE} - \overline{OD} + \overline{OB}$$
$$= \boldsymbol{e} - \boldsymbol{d} + \boldsymbol{b}$$
$$= \boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e}$$
$$\overrightarrow{BH} = \overrightarrow{BO} + \overrightarrow{OD} + \overrightarrow{DH}$$
$$= -\overrightarrow{OB} + \overrightarrow{OD} + \overrightarrow{OE}$$
$$= -\boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e}$$
$$\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DO} + \overrightarrow{OE}$$
$$= -\overrightarrow{OB} - \overrightarrow{OD} + \overrightarrow{OE}$$
$$= -\boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e}$$

b $$|\overrightarrow{OG}|^2 = (\boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e}).(\boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e})$$
$$= \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{d} + \boldsymbol{b}.\boldsymbol{e}$$
$$+ \boldsymbol{d}.\boldsymbol{b} + \boldsymbol{d}.\boldsymbol{d}$$
$$+ \boldsymbol{d}.\boldsymbol{e} + \boldsymbol{e}.\boldsymbol{b} + \boldsymbol{e}.\boldsymbol{d} + \boldsymbol{e}.\boldsymbol{e}$$
$$= |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2$$
$$+ 2(\boldsymbol{b}.\boldsymbol{d} + \boldsymbol{b}.\boldsymbol{e} + \boldsymbol{d}.\boldsymbol{e})$$
$$|\overrightarrow{DF}|^2 = (\boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e}).(\boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e})$$
$$= \boldsymbol{b}.\boldsymbol{b} - \boldsymbol{b}.\boldsymbol{d} + \boldsymbol{b}.\boldsymbol{e}$$
$$- \boldsymbol{d}.\boldsymbol{b} + \boldsymbol{d}.\boldsymbol{d} - \boldsymbol{d}.\boldsymbol{e}$$
$$+ \boldsymbol{e}.\boldsymbol{b} - \boldsymbol{e}.\boldsymbol{d} + \boldsymbol{e}.\boldsymbol{e}$$
$$= |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2$$
$$+ 2(-\boldsymbol{b}.\boldsymbol{d} + \boldsymbol{b}.\boldsymbol{e} - \boldsymbol{d}.\boldsymbol{e})$$
$$|\overrightarrow{BH}|^2 = (-\boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e}).(-\boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e})$$
$$= \boldsymbol{b}.\boldsymbol{b} - \boldsymbol{b}.\boldsymbol{d} - \boldsymbol{b}.\boldsymbol{e}$$
$$- \boldsymbol{d}.\boldsymbol{b} + \boldsymbol{d}.\boldsymbol{d} + \boldsymbol{d}.\boldsymbol{e}$$
$$- \boldsymbol{e}.\boldsymbol{b} + \boldsymbol{e}.\boldsymbol{d} + \boldsymbol{e}.\boldsymbol{e}$$
$$= |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2$$
$$+ 2(-\boldsymbol{b}.\boldsymbol{d} + \boldsymbol{d}.\boldsymbol{e} - \boldsymbol{b}.\boldsymbol{e})$$
$$|\overrightarrow{CE}|^2 = (-\boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e}).(-\boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e})$$
$$= \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{d} - \boldsymbol{b}.\boldsymbol{e}$$
$$+ \boldsymbol{d}.\boldsymbol{b} + \boldsymbol{d}.\boldsymbol{d} - \boldsymbol{d}.\boldsymbol{e}$$
$$- \boldsymbol{e}.\boldsymbol{b} - \boldsymbol{e}.\boldsymbol{d} + \boldsymbol{e}.\boldsymbol{e}$$
$$= |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2$$
$$+ 2(\boldsymbol{b}.\boldsymbol{d} - \boldsymbol{b}.\boldsymbol{e} - \boldsymbol{d}.\boldsymbol{e})$$

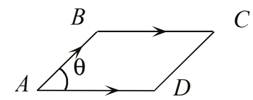
c $$|OG|^2 + |DF|^2 + |BH|^2 + |CE|^2$$
$$= 4|\boldsymbol{b}|^2 + 4|\boldsymbol{d}|^2 + 4|\boldsymbol{e}|^2$$
$$= 4(|\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2)$$
as required.

Solutions to Technology-free questions

1 a $\overrightarrow{AD} = \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (4\boldsymbol{i} - \boldsymbol{k}) - (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$= 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$



b $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) - (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

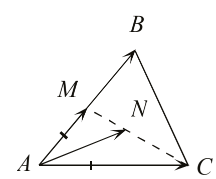
$= \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|}$$

$$= \frac{2 + 1 - 1}{\sqrt{4 + 1 + 1} \times \sqrt{1 + 1 + 1}}$$

$$= \frac{2}{\sqrt{6} \times \sqrt{3}} = \frac{\sqrt{2}}{3}$$

2 a i $\overrightarrow{AM} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \times \overrightarrow{AC}|$



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k})$

$= -3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k})$

$= -\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$

$$\therefore \overrightarrow{AM} = (-3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})\frac{\sqrt{1 + 4 + 4}}{\sqrt{9 + 4 + 36}}$$

$$= \frac{3}{7}(-3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$$

ii $\overrightarrow{AN} = \frac{1}{2}(\overrightarrow{AM} + \overrightarrow{AC})$

$= \frac{1}{2}\left(\frac{3}{7}(-3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}) + (-\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})\right)$

$= \frac{1}{14}(-9\boldsymbol{i} + 6\boldsymbol{j} + 18\boldsymbol{k} - 7\boldsymbol{i} - 14\boldsymbol{j} + 14\boldsymbol{k})$

$= \frac{1}{14}(-16\boldsymbol{i} - 8\boldsymbol{j} + 32\boldsymbol{k})$

$= \frac{1}{7}(-8\boldsymbol{i} - 4\boldsymbol{j} + 16\boldsymbol{k})$

$\overrightarrow{ON} = \overrightarrow{AN} + \overrightarrow{OA}$

$= \frac{1}{7}(-8\boldsymbol{i} - 4\boldsymbol{j} + 16\boldsymbol{k}) + 2\boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k}$

$= \frac{1}{7}(6\boldsymbol{i} - 11\boldsymbol{j} - 12\boldsymbol{k})$

b $\overrightarrow{CM} = \overrightarrow{AM} - \overrightarrow{AC}$

$= \frac{3}{7}(-3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}) - (-\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

$= \frac{1}{7}(-9\boldsymbol{i} + 6\boldsymbol{j} + 18\boldsymbol{k} + 7\boldsymbol{i} + 14\boldsymbol{j} - 14\boldsymbol{k})$

$= \frac{1}{7}(-2\boldsymbol{i} + 20\boldsymbol{j} + 4\boldsymbol{k})$

$\overrightarrow{AN}.\overrightarrow{CM} = \frac{1}{49}(16 - 80 + 64) = 0$

$\therefore \overrightarrow{AN} \perp \overrightarrow{CM}$

3 a $\boldsymbol{a} \perp \boldsymbol{b}$ iff $\boldsymbol{a}.\boldsymbol{b} = 0$

$\therefore \quad 8 - 3 - x = 0 \qquad \therefore \quad x = 5$

b $\boldsymbol{a} \perp \boldsymbol{c} \quad 4y + 3z + 2 = 0$ ①

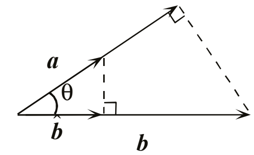
$\boldsymbol{b} \perp \boldsymbol{c} \quad 2y - z - 10 = 0$ ②

① + 3 × ② $10y - 28 = 0 \qquad \therefore y = 2.8$

Substituting $y = 2.8$ into ② gives $5.6 - z - 10 = 0$

$z = -4.4$

4 a $|\boldsymbol{a}|\cos\theta = |\hat{\boldsymbol{b}}| = 1$

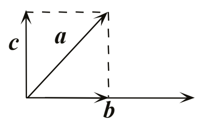


$\therefore \quad \cos\theta = \dfrac{1}{|\boldsymbol{a}|} = \dfrac{1}{\sqrt{1+4+4}} = \dfrac{1}{3}$

b $|\boldsymbol{b}|\cos\theta = 2|\hat{\boldsymbol{a}}| = 2$

$\therefore \quad |\boldsymbol{b}| = \dfrac{2}{\cos\theta} = 2 \div \dfrac{1}{3} = 6$

5 a



$$\boldsymbol{a} = \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b} + \boldsymbol{c}$$

$$\boldsymbol{c} = -\frac{6-6-8}{4+1+4}(2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}) + (3\boldsymbol{i}-6\boldsymbol{j}+4\boldsymbol{k})$$

$$= \frac{8}{9}(2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}) + (3\boldsymbol{i}-6\boldsymbol{j}+4\boldsymbol{k})$$

$$= \frac{1}{9}(16\boldsymbol{i}+8\boldsymbol{j}-16\boldsymbol{k}+27\boldsymbol{i}-54\boldsymbol{j}+36\boldsymbol{k})$$

$$= \frac{1}{9}(43\boldsymbol{i}-46\boldsymbol{j}+20\boldsymbol{k})$$

b $\boldsymbol{d} = \dfrac{\boldsymbol{c}.\boldsymbol{a}}{\boldsymbol{a}.\boldsymbol{a}}\boldsymbol{a}$

$$= \frac{1}{9}\left(\frac{129+276+80}{9+36+16}\right)(3\boldsymbol{i}-6\boldsymbol{j}+4\boldsymbol{k})$$

$$= \frac{485}{549}(3\boldsymbol{i}-6\boldsymbol{j}+4\boldsymbol{k})$$

c $|\boldsymbol{a}| = \sqrt{9+36+16} = \sqrt{61}$

$$|\boldsymbol{d}| = \frac{485}{549} \times \sqrt{61}$$

$$|\boldsymbol{a}||\boldsymbol{d}| = \frac{485 \times 61}{549} = \frac{485}{9}$$

$$|\boldsymbol{c}|^2 = \frac{1}{9^2}(43^2 + 46^2 + 20^2) = \frac{4365}{9^2} = \frac{485}{9}$$

$\therefore |\boldsymbol{a}||\boldsymbol{d}| = |\boldsymbol{c}|^2$

6 a i $\overrightarrow{CA} = \boldsymbol{a} - \boldsymbol{c}$

$$= 2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k} - 2\boldsymbol{i} - (1+3t)\boldsymbol{j} - (-1+2t)\boldsymbol{k}$$

$$= (2-3t)\boldsymbol{j} + (-3-2t)\boldsymbol{k}$$

ii $\overrightarrow{CB} = \boldsymbol{b} - \boldsymbol{c}$

$= 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} - 2\boldsymbol{i} - (1 + 3t)\boldsymbol{j} - (-1 + 2t)\boldsymbol{k}$

$= (-2 - 3t)\boldsymbol{j} + (3 - 2t)\boldsymbol{k}$

b $\angle BCA = 90^\circ \therefore CB \perp CA \qquad \therefore \overrightarrow{CB}.\overrightarrow{CA} = 0$

$\therefore (2 - 3t)(-2 - 3t) + (-3 - 2t)(3 - 2t) = 0$

$-4 + 9t^2 + 4t^2 - 9 = 0$

$13t^2 - 13 = 0$

$t^2 = 1 \qquad \therefore t = \pm 1$

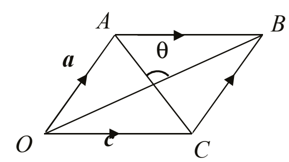
7 a i $\boldsymbol{a} - \boldsymbol{c} = (2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) - (2\boldsymbol{i} - 6\boldsymbol{j} - 3\boldsymbol{k}) = 8\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore |\boldsymbol{a} - \boldsymbol{c}| = \sqrt{64 + 4}$

$= \sqrt{68} = 2\sqrt{17}$

ii $\boldsymbol{a} + \boldsymbol{c} = 4\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$

$|\boldsymbol{a} + \boldsymbol{c}| = 4\sqrt{3}$

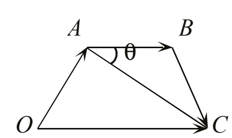


iii $(\boldsymbol{a} - \boldsymbol{c}).(\boldsymbol{a} + \boldsymbol{c}) = -32 - 8 = -40$

b $\cos\theta = \dfrac{(\boldsymbol{a} - \boldsymbol{c}).(\boldsymbol{a} + \boldsymbol{c})}{|\boldsymbol{a} - \boldsymbol{c}||\boldsymbol{a} + \boldsymbol{c}|} = \dfrac{-40}{4\sqrt{3} \times 2\sqrt{17}} = \dfrac{-5}{\sqrt{51}}$

θ is obtuse, $\therefore$ accute angle between diagonals $= \cos^{-1} \dfrac{5}{\sqrt{51}}$

8 a $\overrightarrow{AB} = \dfrac{1}{2}\overrightarrow{OC} = \dfrac{1}{2}(6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}) = 3\boldsymbol{i} - \dfrac{3}{2}\boldsymbol{j} + \boldsymbol{k}$



b $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$\therefore \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{AB}$

$= 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k} - (2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}) - (3\boldsymbol{i} - \dfrac{3}{2}\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - \dfrac{1}{2}\boldsymbol{j} + 4\boldsymbol{k}$

c $\cos\theta = \dfrac{\overrightarrow{AC}.\overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k} - 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

$= 4\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$

$\therefore \cos\theta = \dfrac{12 + 3 + 5}{\sqrt{16 + 4 + 25} \times \sqrt{9 + \frac{9}{4} + 1}}$

$= \dfrac{20}{\sqrt{45} \times \sqrt{\frac{49}{4}}} = \dfrac{40}{21\sqrt{5}} = \dfrac{8\sqrt{5}}{21}$

9 **a** $\overrightarrow{AO} = -\overrightarrow{OA}$

$= -6\boldsymbol{i} - 4\boldsymbol{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 3\boldsymbol{i} + p\boldsymbol{j} - 6\boldsymbol{i} - 4\boldsymbol{j}$

$= -3\boldsymbol{i} + (p - 4)\boldsymbol{j}$

$\overrightarrow{AO}.\overrightarrow{AB} = 18 - 4(p - 4)$

$= 34 - 4p$

b If $\overrightarrow{AO} \perp \overrightarrow{AB}$, then $\therefore \overrightarrow{AO}.\overrightarrow{AB} = 0$

$34 - 4p = 0$

$p = 8.5$

c $\cos\angle OAB = \dfrac{\overrightarrow{AO}.\overrightarrow{AB}}{|\overrightarrow{AO}||\overrightarrow{AB}|}$

$= \dfrac{34 - 24}{\sqrt{36 + 16} \times \sqrt{9 + 4}}$

$= \dfrac{10}{\sqrt{52} \times \sqrt{13}} = \dfrac{5}{13}$

10 To be collinear, A, B and C must lie on the same straight line.

$$\therefore \overrightarrow{AC} = c(\overrightarrow{AB}), c \in R$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= 6p + mq - p - q = 5p + (m-1)p$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= 3p - 2q - p - q$$

$$= 2p - 3q$$

$5 = 2c \qquad \therefore \quad c = 2.5$

$m - 1 = -3c \qquad \therefore \quad m = -3 \times 2.5 + 1 = -6.5$

11 $\boldsymbol{r} + \lambda\boldsymbol{s} + \mu\boldsymbol{t} = 3\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k} + \lambda(\boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}) + \mu(-2\boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k})$

$$= (3 + \lambda - 2\mu)\boldsymbol{i} + (3 - 7\lambda - 5\mu)\boldsymbol{j} + (-6 + 6\lambda + 2\mu)\boldsymbol{k}$$

To be parallel to the x-axis, $\boldsymbol{r} + \lambda\boldsymbol{s} + \mu\boldsymbol{t} = c\boldsymbol{i}, c \in R$

$\therefore \quad 3 + \lambda - 2\mu = c$

$\therefore \quad \lambda - 2u = c - 3 \qquad ①$

$3 - 7\lambda - 5\mu = 0$

$\therefore \quad -7\lambda - 5\mu = -3 \qquad ②$

$-6 + 6\lambda + 2\mu = 0$

$\therefore \quad 6\lambda + 2\mu = 6 \qquad ③$

$6 \times ② + 7 \times ③ \quad -16\mu = 24 \qquad ④$

$$\mu = -\frac{3}{2}$$

$③ \div 2 \qquad 3\lambda + \mu = 3 \qquad ⑤$

Substitute $\mu = -\frac{3}{2}$ in ⑤

$3\lambda - \frac{3}{2} = 3$

$\therefore \lambda = \frac{3}{2}$

12

$$\overrightarrow{AB} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$$

$$\overrightarrow{DC} = 2\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$$

$$\therefore \overrightarrow{DC} = 2\overrightarrow{AB}$$

$\therefore AB \| DC$, $AB{:}CD = 1{:}2$

$ABCD$ is a trapezium.

13

$$\boldsymbol{a} + \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$$

$$(\boldsymbol{a} + \boldsymbol{b}).\boldsymbol{b} = 3 + 2 - 5$$

$$= 0$$

$$\therefore \quad \boldsymbol{a} + \boldsymbol{b} \perp \boldsymbol{b}$$

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{i} + 7\boldsymbol{k}$$

$$\cos\theta = \frac{(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b})}{|\boldsymbol{a} + \boldsymbol{b}||\boldsymbol{a} - \boldsymbol{b}|}$$

$$= \frac{3 + 35}{\sqrt{9 + 4 + 25} \times \sqrt{1 + 49}}$$

$$= \frac{38}{\sqrt{38} \times \sqrt{50}}$$

$$= \frac{19}{\sqrt{19 \times 5}}$$

$$= \frac{\sqrt{19}}{5}$$

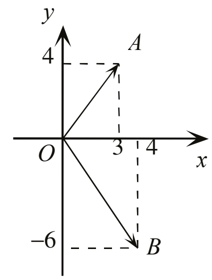
14 a

$$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$$

$$\therefore \quad \overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB} = -i + 10j$$

$$\therefore \quad C(-1, 10)$$

b If $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$



then $1 = 3h + 4k$ ①

$24 = 4h - 6k$ ②

$4 \times ① - 3 \times ②$ $\quad -68 = 34k$

$k = -2$

Substitute in ① $\quad 1 = 3h - 8$

$3h = 9$

$h = 3$

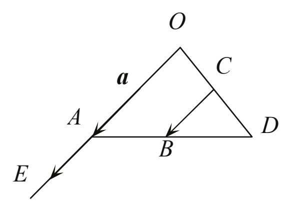
15 **a** $\overrightarrow{OD} = 2c$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= 2\boldsymbol{c} - \boldsymbol{a}$

b $\boldsymbol{b} = \overrightarrow{OB} = \dfrac{\overrightarrow{OA} + \overrightarrow{OD}}{2} = \dfrac{1}{2}\boldsymbol{a} + \boldsymbol{c}$

c $\overrightarrow{OE} = 4\overrightarrow{AE}$



$\therefore \quad \boldsymbol{a} + \overrightarrow{AE} = 4\overrightarrow{AE}$

$\therefore \quad \boldsymbol{a} = 3\overrightarrow{AE}$

$\overrightarrow{AE} = \dfrac{1}{3}\boldsymbol{a}$

$\overrightarrow{CB} = \dfrac{1}{2}\boldsymbol{a}$

$\therefore \quad \dfrac{1}{2}\boldsymbol{a} = \dfrac{k}{3}\boldsymbol{a} \qquad \therefore k = \dfrac{3}{2} = 1.5$

16 $\overrightarrow{QS} = \overrightarrow{OS} - \overrightarrow{OQ} = h\boldsymbol{p} + \dfrac{1}{2}\boldsymbol{q} - \boldsymbol{q} = h\boldsymbol{p} - \dfrac{1}{2}\boldsymbol{q}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \dfrac{1}{3}\boldsymbol{p} + k\boldsymbol{q} - \boldsymbol{q} = \dfrac{1}{3}\boldsymbol{p} + (k-1)\boldsymbol{q}$

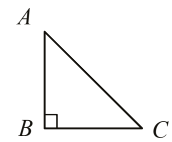
$\overrightarrow{QR} = \dfrac{1}{2}\overrightarrow{QS}$, since R is the midpoint of $\overrightarrow{QS}$

$\therefore \dfrac{1}{3} = \dfrac{1}{2}h \qquad \therefore \; h = \dfrac{2}{3}$

and $\quad k - 1 = -\frac{1}{4}$

$\therefore \quad k = \frac{3}{4}$

17 $\overrightarrow{AC} = 2\boldsymbol{i} + 4\boldsymbol{j}$



$\overrightarrow{AB} = k(\boldsymbol{i} + \boldsymbol{j}),\ k \in R\backslash\{0\}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$= (2\boldsymbol{i} + 4\boldsymbol{j}) - (k\boldsymbol{i} + k\boldsymbol{j})$

$= (2 - k)\boldsymbol{i} + (4 - k)\boldsymbol{j}$

Now $\quad \overrightarrow{BA}.\overrightarrow{BC} = 0$, since $\angle ABC = 90^\circ$

$\therefore \quad (-k\boldsymbol{i} - k\boldsymbol{j}).((2 - k)\boldsymbol{i} + (4 - k)\boldsymbol{j}) = 0$

$\therefore \quad -k(2 - k) - k(4 - k) = 0$

$\therefore \quad -2k + k^2 - 4k + k^2 = 0$

$\therefore \quad 2k^2 - 6k = 0$

$\therefore \quad k^2 - 3k = 0$

$\therefore \quad k(k - 3) = 0$

$\therefore \quad k - 3 = 0$, since $k \neq 0$

$\therefore \quad k = 3$

$\therefore \quad \overrightarrow{AB} = 3(\boldsymbol{i} + \boldsymbol{j})$

18 a $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB}$

$= -\overrightarrow{AD} + \overrightarrow{OC}$

$= -\boldsymbol{a} + \boldsymbol{c}$

b Let $\overrightarrow{OE} = k\,\overrightarrow{OC}$

$= k\boldsymbol{c}, \quad k \in R^+$

and $\overrightarrow{DE} = l\,\overrightarrow{DB}, \quad l \in R^+$

$= l(\boldsymbol{c} - \boldsymbol{a})$

$= l\boldsymbol{c} - l\boldsymbol{a}$

Now $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= 2\overrightarrow{AD} + \overrightarrow{AD}$

$= 3\overrightarrow{AD}$

$= 3\boldsymbol{a}$

$\overrightarrow{OE} = \overrightarrow{OD} + \overrightarrow{DE}$

$\therefore k\boldsymbol{c} = 3\boldsymbol{a} + l\boldsymbol{c} - l\boldsymbol{a}$

$= (3 - l)\boldsymbol{a} + l\boldsymbol{c}$

Since $\boldsymbol{a}$ and $\boldsymbol{c}$ are non-parallel, non-zero vectors

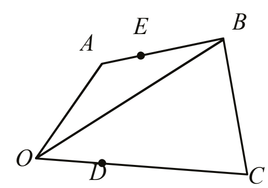
$(3 - l) = 0$, and $l = k$

$\therefore l = 3$, and $k = 3$

$\overrightarrow{OE} = 3\boldsymbol{c}$

$= 3\overrightarrow{OC}$, as required to prove.

19 a i $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OC} = \frac{1}{3}\boldsymbol{c}$



ii $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$

$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{1}{3}(\overrightarrow{OB} - \overrightarrow{OA})$

$= \boldsymbol{a} + \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$

iii $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD}$

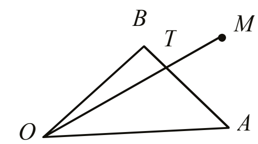
$= \frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{c}$

b $3\overrightarrow{DE} = 3\left(\frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{c}\right) = 2\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

$2\overrightarrow{OA} + \overrightarrow{CB} = 2\overrightarrow{OA} + \overrightarrow{OB} - \overrightarrow{OC} = 2\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

$\therefore 3\overrightarrow{DE} = 2\overrightarrow{OA} + \overrightarrow{CB}$, as required to prove.

20 a $\overrightarrow{OT} = \overrightarrow{OA} + \overrightarrow{AT}$



$$= \overrightarrow{OA} + \frac{3}{4}\overrightarrow{AB}$$

$$= \overrightarrow{OA} + \frac{3}{4}(\overrightarrow{OB} - \overrightarrow{OA})$$

$$= a + \frac{3}{4}(\boldsymbol{b} - \boldsymbol{a})$$

$$= \frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$$

b i $\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB}$

$$= \lambda\overrightarrow{OT} - \overrightarrow{OB}$$

$$= \lambda\left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) - \boldsymbol{b}$$

$$= \frac{\lambda}{4}\boldsymbol{a} + \left(\frac{3\lambda}{4} - 1\right)\boldsymbol{b}$$

ii Let $\overrightarrow{BM} = k\overrightarrow{OA} = k\boldsymbol{a},\ k \in R\backslash\{0\}$

then $ka = \frac{\lambda}{4}\boldsymbol{a} + \left(\frac{3\lambda}{4} - 1\right)\boldsymbol{b}$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel, non-zero vectors

$k = \frac{\lambda}{4}$ and $\frac{3\lambda}{4} - 1 = 0$ $\quad \therefore \lambda = \frac{4}{3}$

21 There exist real numbers p and q such that

$\boldsymbol{a} = p\boldsymbol{b} + q\boldsymbol{c}$

$\therefore \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} = p(\boldsymbol{i} - 2\boldsymbol{j} + m\boldsymbol{k}) + q(-2\boldsymbol{i} + n\boldsymbol{j} + 2\boldsymbol{k})$

$$1 = p - 2q \ldots (1)$$

$$1 = -2p + nq \ldots (2)$$

$$3 = mp + 2q \ldots (3)$$

Eliminate p and q

$$m = \frac{3(n-6)}{(n+2)}$$

22 Vector resolute $= \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b}$

$$= \frac{4}{5}(\boldsymbol{i} + 3\boldsymbol{k})$$

Required vector $\boldsymbol{v} = \boldsymbol{a} - \dfrac{4}{5}(\boldsymbol{i} + 3\boldsymbol{k})$

$$= \frac{6}{5}\boldsymbol{i} + \boldsymbol{j} - \frac{2}{5}\boldsymbol{k}$$

If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{v}$ are linearly dependent then there exists p and q such that: $\boldsymbol{v} = p\boldsymbol{a} + q\boldsymbol{b}$

Hence

$$\frac{6}{5}\boldsymbol{i} + \boldsymbol{j} - \frac{2}{5}\boldsymbol{k} = p(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + q(\boldsymbol{i} + 3\boldsymbol{k})$$

$$= (2p + q)\boldsymbol{i} + p\boldsymbol{j} + (2p + 3q)\boldsymbol{k}$$

Therefore

$$2p + q = \frac{6}{5} \ldots (1)$$

$$p = 1 \ldots (2)$$

$$2p + 3q = -\frac{2}{5} \ldots (3)$$

$p = 1$ and $q = -\dfrac{4}{5}$ satisfy all three equations.

Solutions to multiple-choice questions

1 C $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= (\boldsymbol{a} + 2\boldsymbol{b}) + (\boldsymbol{a} - \boldsymbol{b})$

$= 2\boldsymbol{a} + \boldsymbol{b}$

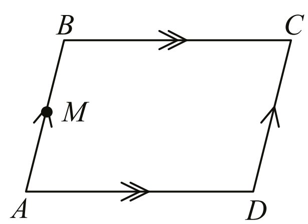
2 D $\overrightarrow{EF} = \overrightarrow{DC} + 3\overrightarrow{AB}$

$= -\overrightarrow{CD} + 3\overrightarrow{AB}$

$= -\boldsymbol{c} + 3\boldsymbol{a}$

$= 3\boldsymbol{a} - \boldsymbol{c}$

3 B



$\overrightarrow{DM} = \overrightarrow{DA} + \overrightarrow{AM}$

$= -\overrightarrow{BC} + \frac{1}{2}\overrightarrow{AB}$

$= \frac{1}{2}\boldsymbol{u} - \boldsymbol{v}$

4 B $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (11, 1) - (3, 6)$

$= (8, -5)$

$= 8\boldsymbol{i} - 5\boldsymbol{j}$

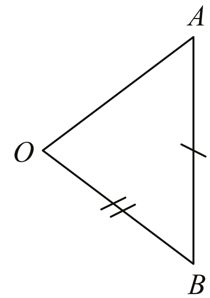
5 C $\cos\theta = \dfrac{(2, 1, -\sqrt{2}).(5, 8, 0)}{\sqrt{7} \times \sqrt{89}}$

$\therefore \cos\theta = \dfrac{10 + 8}{\sqrt{623}}$

$\therefore \cos\theta = \dfrac{18}{\sqrt{623}}$

$\therefore \theta = 43.85°$

6 C As $|\overrightarrow{AB}| \neq |\overrightarrow{OB}|$ the side lengths *AB* and *OB* of triangle *OAB* are different in size.



It is also known that

$\overrightarrow{AO}.\overrightarrow{AB} = \overrightarrow{BO}.\overrightarrow{BA}$

$\therefore \overrightarrow{AO}.\overrightarrow{AB} = \overrightarrow{BO}. -\overrightarrow{AB}$

$\therefore \overrightarrow{AO}.\overrightarrow{AB} = -\overrightarrow{BO}.\overrightarrow{AB}$

$\therefore \overrightarrow{AO} = -\overrightarrow{BO}$

$\therefore \overrightarrow{AO} = \overrightarrow{OB}$

$\Rightarrow |\overrightarrow{AO}| = |\overrightarrow{OB}|$

Thus the side lengths *AO* and *OB* are the same size.

Hence the triangle is isosceles as two sides are identical in length.

7 E $x(\boldsymbol{a} + \boldsymbol{b}) = 2y\boldsymbol{a} + (y + 3)\boldsymbol{b}$

$\therefore x\boldsymbol{a} + x\boldsymbol{b} = 2y\boldsymbol{a} + (y + 3)\boldsymbol{b}$

Equating coefficients

$x = 2y$ ① and $x = y + 3$ ②

Substituting ① into ② gives

$2y = y + 3$

$\therefore y = 3$

Substituting $y = 3$ into ① gives $x = 6$

$\therefore x = 6$ and $y = 3$

8 E $|\overrightarrow{AB}| = \boldsymbol{b} - \boldsymbol{a}$

$= (5\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) - (\boldsymbol{i} + \boldsymbol{j})$

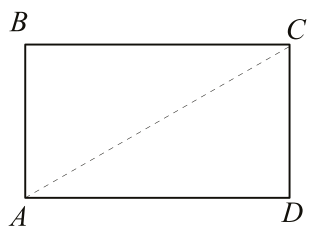
$= 4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

$$\therefore |\overrightarrow{AB}| = \sqrt{4^2 + (-3)^2 + 2^2}$$
$$= \sqrt{16 + 9 + 4}$$
$$= \sqrt{29}$$

9 D $\boldsymbol{x}.\hat{\boldsymbol{y}} = \dfrac{\boldsymbol{x}.\boldsymbol{y}}{|\boldsymbol{y}|}$

$$= \frac{(3, -2, 4).(-5, 1, 1)}{\sqrt{27}}$$
$$= \frac{-15 - 2 + 4}{\sqrt{27}}$$
$$= -\frac{13}{\sqrt{27}}$$
$$= -\frac{13\sqrt{27}}{27}$$

10 B



Given:
$|\overrightarrow{BC}| = 3|\overrightarrow{AB}|, \overrightarrow{AB} = \boldsymbol{a}$

Using Pythagoras' Theorem

$$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$$
$$= |\overrightarrow{AB}|^2 + (3|\overrightarrow{AB}|)^2$$
$$= |\overrightarrow{AB}|^2 + 9|\overrightarrow{AB}|^2$$
$$= 10|\overrightarrow{AB}|^2$$
$$\therefore |\overrightarrow{AC}| = \sqrt{10}|\overrightarrow{AB}|$$
$$\therefore |\overrightarrow{AC}| = \sqrt{10}|\boldsymbol{a}|$$

11 C

Write $\boldsymbol{i} + 2\boldsymbol{j} + a\boldsymbol{k} = \ell(2\boldsymbol{i} - 8\boldsymbol{j} + 10\boldsymbol{k}) + m(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

$$1 = 2\ell + m \ldots (1)$$
$$2 = -8\ell - m \ldots (2)$$
$$a = 10\ell + m \ldots (3)$$

$\therefore \ell = -\dfrac{1}{2}$
$m = 2$
$a = -3$

12 B

13 D

Solutions to extended-response questions

1 a i $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) - (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

ii Length $= |\overrightarrow{AB}|$

$= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

The strand is $\sqrt{3}$ units long.

b i $\overrightarrow{CQ} = \overrightarrow{OQ} - \overrightarrow{OC}$

$= \overrightarrow{OA} + \overrightarrow{AQ} - \overrightarrow{OC}$

$= \overrightarrow{OA} + \lambda\overrightarrow{AB} - \overrightarrow{OC}$

$= (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) - (2.5\boldsymbol{i} + 4\boldsymbol{j} + 1.5\boldsymbol{k})$

$= (\lambda - 0.5)\boldsymbol{i} + (\lambda - 1)\boldsymbol{j} + (\lambda - 0.5)\boldsymbol{k}$

ii $\overrightarrow{CQ}.\overrightarrow{AB} = ((\lambda - 0.5)\boldsymbol{i} + (\lambda - 1)\boldsymbol{j} + (\lambda - 0.5)\boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$= (\lambda - 0.5) \times 1 + (\lambda - 1) \times 1 + (\lambda - 0.5) \times 1$

$= 3\lambda - 2$

But $\overrightarrow{CQ}.\overrightarrow{AB} = 0$

$\therefore \quad 3\lambda - 2 = 0$

$\therefore \quad \lambda = \frac{2}{3}$

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ}$

$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) + \frac{2}{3}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$= \frac{1}{3}(8\boldsymbol{i} + 11\boldsymbol{j} + 5\boldsymbol{k})$

c Let P be the point of contact of AB and MN.

Now $\overrightarrow{AP} = a\overrightarrow{AB},\ a \in R^+$

$\therefore\ \overrightarrow{OP} - \overrightarrow{OA} = a\overrightarrow{AB}$

$\therefore\ \overrightarrow{OP} = a\overrightarrow{AB} + \overrightarrow{OA}$

$= a\,(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$

$= (a + 2)\boldsymbol{i} + (a + 3)\boldsymbol{j} + (a + 1)\boldsymbol{k}$ ①

and $\overrightarrow{MP} = b\overrightarrow{MN},\ b \in R^+$

$\therefore\ \overrightarrow{OP} - \overrightarrow{OM} = b\overrightarrow{MN}$

$\therefore\ \overrightarrow{OP} = \overrightarrow{OM} + b\overrightarrow{MN}$

$= \overrightarrow{OM} + b(\overrightarrow{ON} - \overrightarrow{OM})$

$= (4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + b((6\boldsymbol{i} + 10\boldsymbol{j} + 9\boldsymbol{k}) - (4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}))$

$= (4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + b(2\boldsymbol{i} + 8\boldsymbol{j} + 10\boldsymbol{k})$

$= (4 + 2b)\boldsymbol{i} + (2 + 8b)\boldsymbol{j} + (10b - 1)\boldsymbol{k}$

Equating coefficients:

$a + 2 = 4 + 2b,\ a + 3 = 2 + 8b$ and $a + 1 = 10b - 1$ ②

$\therefore\ a = 2 + 2b$

and $a + 3 = 2 + 8b$

$\therefore\ (2 + 2b) + 3 = 2 + 8b$

$\therefore\ 5 + 2b = 2 + 8b$

$\therefore\ 3 = 6b$

$\therefore\ b = \dfrac{1}{2}$

$\therefore\ a = 2 + \dfrac{2}{2}$

$= 3$ Check in ②

$a + 1 = 3 + 1$

$= 4$

$10b - 1 = \dfrac{10}{2} - 1$

$= 4$

$\therefore$ LHS = RHS

Substituting $a = 3$ in ① yields

$\overrightarrow{OP} = (3 + 2)\boldsymbol{i} + (3 + 3)\boldsymbol{j} + (3 + 1)\boldsymbol{k}$

$= 5\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$, the position vector of the point of contact.

2 a i $|\overrightarrow{OA}| = \sqrt{2^2 + 3^2 + 1^2}$ $\quad |\overrightarrow{OB}| = \sqrt{3^2 + (-2)^2 + 1^2}$

$= \sqrt{4 + 9 + 1} \qquad = \sqrt{9 + 4 + 1}$

$= \sqrt{14} \qquad = \sqrt{14}$

ii $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) - (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - 5\boldsymbol{j}$

b i $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) + \frac{1}{2}(\boldsymbol{i} - 5\boldsymbol{j})$

$= \frac{5}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + \boldsymbol{k}$

$= \frac{1}{2}(5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

ii $\overrightarrow{OX}.\overrightarrow{AB} = \frac{1}{2}(5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(\boldsymbol{i} - 5\boldsymbol{j} + 0\boldsymbol{k})$

$= \frac{1}{2}(5 \times 1 + 1 \times (-5) + 2 \times 0)$

$= \frac{1}{2}(5 - 5)$

$= 0$

Hence $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{AB}$.

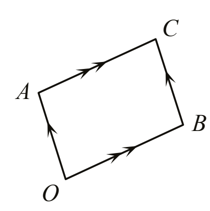
c If *OABC* is a parallelogram

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \overrightarrow{OA} + \overrightarrow{OB}$

$= 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

$= 5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$



i.e. $\overrightarrow{OC} = 5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, the position vector of *C* such that *OACB* is a parallelogram.

d $\overrightarrow{OC}.\overrightarrow{AB} = (5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(\boldsymbol{i} - 5\boldsymbol{j} + 0\boldsymbol{k})$

$= 5 \times 1 + 1 \times (-5) + 2 \times 0$

$= 5 - 5$

$= 0$

$\therefore OC$ is perpendicular to AB.

e **i** Let $\boldsymbol{p} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k},\ a,\ b,\ c \in R$, be the vector with magnitude $\sqrt{195}$ which is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$|\boldsymbol{p}| = \sqrt{a^2 + b^2 + c^2}$ and $|\boldsymbol{p}| = \sqrt{195}$

$\therefore \quad a^2 + b^2 + c^2 = 195 \quad ①$

Now $\quad \overrightarrow{OA}.\boldsymbol{p} = 0$

$\therefore \quad (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}).(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) = 0$

$\therefore \quad 2a + 3b + c = 0 \quad ②$

and $\quad \overrightarrow{OB}.\boldsymbol{p} = 0$

$\therefore \quad (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}).(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) = 0$

$\therefore \quad 3a - 2b + c = 0 \quad ③$

Subtracting ② from ③ yields $\quad a - 5b = 0$

$\therefore \quad a = 5b \quad ④$

From ③ $\quad c = 2b - 3a$

$= 2b - 3(5b)$

$= -13b$

and from ① $\quad c^2 = 195 - a^2 - b^2$

$\therefore \quad (-13b)^2 = 195 - (5b)^2 - b^2$

$\therefore \quad 169b^2 = 195 - 25b^2 - b^2$

$\therefore \quad 195b^2 = 195$

$\therefore \quad b^2 = 1$

$\therefore \quad b = \pm 1$

From ④, when $b = 1, a = 5$ and when $b = -1,\ 0 = -5$

Substituting into ②, when $a = 5$ and $b = 1$,

$2(5) + 3(1) + c = 0$

$\therefore 10 + 3 + c = 0$

$\therefore c = -13$

and when $a = -1$ and $b = -1$,

$$2(-5) + 3(-1) + c = 0$$

$\therefore \quad -10 - 3 + c = 0$

$\therefore \quad c = 13$

$\therefore \quad \boldsymbol{p} = 5\boldsymbol{i} + \boldsymbol{j} - 13\boldsymbol{k}$ or $\boldsymbol{p} = -5\boldsymbol{i} - \boldsymbol{j} + 13\boldsymbol{k}$

ii When $\boldsymbol{p} = 5\boldsymbol{i} + \boldsymbol{j} - 13\boldsymbol{k}$,

$$\begin{aligned}\overrightarrow{AB}.\boldsymbol{p} &= (\boldsymbol{i} - 5\boldsymbol{j} + 0\boldsymbol{k}).(5\boldsymbol{i} + \boldsymbol{j} - 13\boldsymbol{k})\\ &= 1 \times (5) + (-5) \times (1) + 0 \times (-13)\\ &= 5 - 5 = 0\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OC}.\boldsymbol{p} &= (5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(5\boldsymbol{i} + \boldsymbol{j} - 13\boldsymbol{k})\\ &= 5 \times (5) + 1 \times (1) + 2 \times (-13)\\ &= 25 + 1 - 26 = 0\end{aligned}$$

When $\boldsymbol{p} = -5\boldsymbol{i} - \boldsymbol{j} + 13\boldsymbol{k}$,

$$\begin{aligned}\overrightarrow{AB}.\boldsymbol{p} &= (\boldsymbol{i} - 5\boldsymbol{j} + 0\boldsymbol{k}).(-5\boldsymbol{i} - \boldsymbol{j} + 13\boldsymbol{k})\\ &= 1 \times (-5) + (-5) \times (-1) + 0 \times (13)\\ &= -5 + 5 = 0\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OC}.\boldsymbol{p} &= (5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(-5\boldsymbol{i} - \boldsymbol{j} + 13\boldsymbol{k})\\ &= 5 \times (-5) + 1 \times (-1) + 2 \times (13)\\ &= -25 - 1 + 26 = 0\end{aligned}$$

Therefore $\boldsymbol{p}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

iii Since $\boldsymbol{p}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{OC}$, it is perpendicular to the plane containing $OACB$.

3 a

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OC} + \overrightarrow{CY} + \overrightarrow{YX}\\ &= \overrightarrow{OC} + \overrightarrow{OB} + \overrightarrow{OA}\\ &= (\boldsymbol{i} + 4\boldsymbol{j}) + (\boldsymbol{i} + 3\boldsymbol{k}) + 5\boldsymbol{i}\\ &= 7\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OY} &= \overrightarrow{OC} + \overrightarrow{CY}\\ &= \overrightarrow{OC} + \overrightarrow{OB}\\ &= (\boldsymbol{i} + 4\boldsymbol{j}) + (\boldsymbol{i} + 3\boldsymbol{k})\\ &= 2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OZ} &= \overrightarrow{OA} + \overrightarrow{AZ}\\ &= \overrightarrow{OA} + \overrightarrow{OC}\\ &= 5\boldsymbol{i} + (\boldsymbol{i} + 4\boldsymbol{j})\\ &= 6\boldsymbol{i} + 4\boldsymbol{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD}\\ &= \overrightarrow{OA} + \overrightarrow{OB}\\ &= 5\boldsymbol{i} + (\boldsymbol{i} + 3\boldsymbol{k})\\ &= 6\boldsymbol{i} + 3\boldsymbol{k}\end{aligned}$$

Length of $OD = |\overrightarrow{OD}| = \sqrt{6^2 + 3^2}$

$= \sqrt{36 + 9}$

$= \sqrt{45}$

$= 3\sqrt{5}$

Length of $OY = |\overrightarrow{OY}| = \sqrt{2^2 + 4^2 + 3^2}$

$= \sqrt{4 + 16 + 9}$

$= \sqrt{29}$

b $\overrightarrow{ZO}.\overrightarrow{ZY} = |\overrightarrow{ZO}|\,|\overrightarrow{ZY}| \cos \angle OZY$

$\therefore \angle OZY = \cos^{-1}\left(\dfrac{\overrightarrow{ZO}.\overrightarrow{ZY}}{|\overrightarrow{ZO}|\,|\overrightarrow{ZY}|}\right)$

Now $\overrightarrow{ZO} = -6\boldsymbol{i} - 4\boldsymbol{j}$ and $\overrightarrow{ZY} = \overrightarrow{OY} - \overrightarrow{OZ}$

$= (2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}) - (6\boldsymbol{i} + 4\boldsymbol{j})$

$= -4\boldsymbol{i} + 3\boldsymbol{k}$

$\therefore |\overrightarrow{ZO}| = \sqrt{(-6)^2 + (-4)^2}$ and $|\overrightarrow{ZY}| = \sqrt{(-4)^2 + 3^2}$

$= \sqrt{36 + 16}$ $\qquad = \sqrt{16 + 9}$

$= \sqrt{52}$ $\qquad = \sqrt{25}$

$= 2\sqrt{13}$ $\qquad = 5$

and $\overrightarrow{ZO}.\overrightarrow{ZY} = (-6\boldsymbol{i} - 4\boldsymbol{j} + 0\boldsymbol{k}).(-4\boldsymbol{i} + 0\boldsymbol{j} + 3\boldsymbol{k})$

$= (-6) \times (-4) + (-4) \times 0 + 0 \times 3$

$= 24$

$\therefore \angle OZY = \cos^{-1}\left(\dfrac{24}{2\sqrt{13} \times 5}\right)$

$= \cos^{-1}\left(\dfrac{12\sqrt{13}}{65}\right)$

$= (48.268\,53\ldots)^\circ$

Angle OZY is 48.27°, correct to two decimal places.

c **i** $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$

$$= \overrightarrow{OC} + \frac{\lambda}{\lambda + 1}\overrightarrow{CZ}$$

$$= \overrightarrow{OC} + \frac{\lambda}{\lambda + 1}(\overrightarrow{OZ} - \overrightarrow{OC})$$

$$= \boldsymbol{i} + 4\boldsymbol{j} + \frac{\lambda}{\lambda + 1}((6\boldsymbol{i} + 4\boldsymbol{j}) - (\boldsymbol{i} + 4\boldsymbol{j}))$$

$$= \boldsymbol{j} + 4\boldsymbol{j} + \frac{\lambda}{\lambda + 1}(5\boldsymbol{i})$$

$$= \left(\frac{5\lambda}{\lambda + 1} + 1\right)\boldsymbol{i} + 4\boldsymbol{j}$$

ii $\overrightarrow{OP}.\overrightarrow{CZ} = \left(\left(\frac{5\lambda}{\lambda + 1} + 1\right)\boldsymbol{i} + 4\boldsymbol{j}\right).(5\boldsymbol{i} + 0\boldsymbol{j})$

$$= \left(\frac{5\lambda}{\lambda + 1} + 1\right) \times 5 + 4 \times 0$$

$$= 5\left(\frac{5\lambda}{\lambda + 1} + 1\right)$$

If $\overrightarrow{OP} \perp \overrightarrow{CZ}$, then $\overrightarrow{OP}.\overrightarrow{CZ} = 0$

$\therefore \quad 5\left(\frac{5\lambda}{\lambda + 1} + 1\right) = 0$

$\therefore \quad \frac{5\lambda}{\lambda + 1} + 1 = 0$

$\therefore \quad \frac{5\lambda}{\lambda + 1} = -1$

$\therefore \quad 5\lambda = -(\lambda + 1)$

$= -\lambda - 1$

$\therefore \quad 6\lambda = -1$

$\therefore \quad \lambda = \frac{-1}{6}$

Note: P divides CZ extemally.

4 **a** **i** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \boldsymbol{b} - \boldsymbol{a}$$

ii $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$$= \boldsymbol{c} - \boldsymbol{b}$$

iii $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= \boldsymbol{a} - \boldsymbol{c}$

iv $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$

$= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$

$= \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})$

$= \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$

v $\overrightarrow{OQ} = \overrightarrow{OC} + \overrightarrow{CQ}$

$= \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CA}$

$= \boldsymbol{c} + \frac{1}{2}(\boldsymbol{a} - \boldsymbol{c})$

$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$

vi $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

b $$\overrightarrow{OP}.\overrightarrow{BC} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c}).(\boldsymbol{c} - \boldsymbol{b})$$
$$= \frac{1}{2}(\boldsymbol{c}.\boldsymbol{c} - \boldsymbol{b}.\boldsymbol{b})$$
$$= \frac{1}{2}(|\boldsymbol{c}|^2 - |\boldsymbol{b}|^2)$$

Now $\overrightarrow{OR}.\overrightarrow{AB} = 0$ and $\overrightarrow{OQ}.\overrightarrow{AC} = 0$

$\therefore$ $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{b} - \boldsymbol{a}) = 0$ $\quad$ $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}).(\boldsymbol{c} - \boldsymbol{a}) = 0$

$\therefore$ $\frac{1}{2}(\boldsymbol{a}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b}) = 0$ $\quad$ $\frac{1}{2}(\boldsymbol{c}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{a}) = 0$

$\therefore$ $\frac{1}{2}(|\boldsymbol{a}|^2 - |\boldsymbol{b}|^2) = 0$ $\quad$ $\frac{1}{2}(|\boldsymbol{c}|^2 - |\boldsymbol{a}|^2) = 0$

$\therefore$ $|\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 = 0$ $\quad$ $|\boldsymbol{c}|^2 - |\boldsymbol{a}|^2 = 0$

$\therefore$ $|\boldsymbol{a}|^2 = |\boldsymbol{b}|^2$ $\quad$ $|\boldsymbol{c}|^2 = |\boldsymbol{a}|^2$

Therefore $|\boldsymbol{b}|^2 = |\boldsymbol{c}|^2$

and $$\overrightarrow{OP}.\overrightarrow{BC} = \frac{1}{2}(|\boldsymbol{c}|^2 - |\boldsymbol{c}|^2)$$
$$= 0$$

Hence, OP is perpendicular to BC.

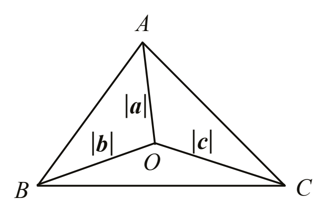
c $OP \perp BC$, therefore the perpendicular bisectors are concurrent.

d From **b**, $|\boldsymbol{a}|^2 = |\boldsymbol{b}|^2$ and $|\boldsymbol{c}|^2 = |\boldsymbol{a}|^2$

$\therefore$ $|\boldsymbol{a}|^2 = |\boldsymbol{b}|^2 = |\boldsymbol{c}|^2$

$\therefore$ $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$

i.e. O is the circumcentre of the triangle.



5 **a** $$\overrightarrow{OL} = \overrightarrow{OB} + \overrightarrow{BL}$$
$$= \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BC}$$
$$= \overrightarrow{OB} + \frac{2}{3}(\overrightarrow{OC} - \overrightarrow{OB})$$
$$= \boldsymbol{b} + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{b})$$
$$= \frac{1}{3}\boldsymbol{b} + \frac{2}{3}\boldsymbol{c}$$

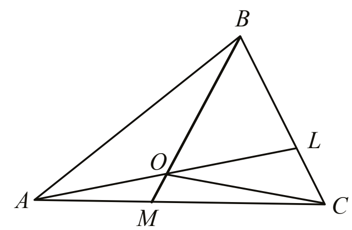
b $\overrightarrow{OL} = -\overrightarrow{OA}$

$\therefore \quad \frac{1}{3}\boldsymbol{b} + \frac{2}{3}\boldsymbol{c} = -\boldsymbol{a}$

$\therefore \quad \boldsymbol{b} + 2\boldsymbol{c} = -3\boldsymbol{a}$

$\therefore \quad 3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c} = 0,$ as required to prove.

c **i** $\overrightarrow{BO} = -\overrightarrow{OB}$



$= -\boldsymbol{b}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \overrightarrow{OA} + \frac{2}{5}\overrightarrow{AC}$

$= \overrightarrow{OA} + \frac{2}{5}(\overrightarrow{OC} - \overrightarrow{OA})$

$= \boldsymbol{a} + \frac{2}{5}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{3}{5}\boldsymbol{a} + \frac{2}{5}\boldsymbol{c}$

$= \frac{1}{5}(3\boldsymbol{a} + 2\boldsymbol{c})$

$= \frac{1}{5}(-\boldsymbol{b})$ since $3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c} = 0$

$= \frac{1}{5}\overrightarrow{BO}$

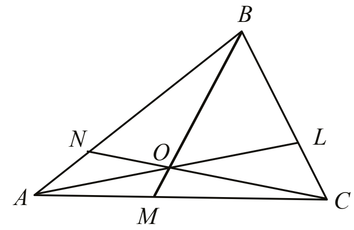
Therefore *OM* is parallel to *BO* and *B*, *O* and *M* are collinear.

ii $BO : OM = BO : \frac{1}{5}BO$

$= 1 : \frac{1}{5}$

$= 5 : 1$

d C, O and N are collinear



$$\therefore \quad \overrightarrow{ON} = p\overrightarrow{CO}, \ p \in R^+$$
$$= -p\overrightarrow{OC}$$
$$= -p\boldsymbol{c}$$
$$= \frac{1}{2}p \times -2\boldsymbol{c}$$
$$= \frac{1}{2}p \times (3\boldsymbol{a} + \boldsymbol{b}), \ \text{since } 3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c} = 0$$
$$= \frac{3p}{2}\boldsymbol{a} + \frac{p}{2}\boldsymbol{b} \qquad ①$$

Also
$$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$$
$$= \overrightarrow{OA} + q\overrightarrow{AB}, \ q \in R^+$$
$$= \overrightarrow{OA} + q(\overrightarrow{OB} - \overrightarrow{OA})$$
$$= \boldsymbol{a} + q(\boldsymbol{b} - \boldsymbol{a})$$
$$= (1 - q)\boldsymbol{a} + q\boldsymbol{b} \qquad ②$$

Equating coefficients yields
$$1 - q = \frac{3p}{2} \quad \text{and} \quad q = \frac{p}{2}$$
$$\therefore \quad 1 - \frac{p}{2} = \frac{3p}{2}$$
$$\therefore \quad 1 = \frac{4p}{2}$$
$$\therefore \quad p = \frac{1}{2} \quad \text{and} \quad q = \frac{1}{4}$$

Therefore $\overrightarrow{AN} = \frac{1}{4}\overrightarrow{AB}$, so $AN : NB = 1 : 3$

6 a i $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$
$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$$
$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$
$$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

ii $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD}$

$$= \lambda\,\overrightarrow{OB} - \overrightarrow{OD}$$

$$= \lambda \boldsymbol{b} - \frac{1}{2}\left(\boldsymbol{a} + \boldsymbol{b}\right)$$

$$= -\frac{1}{2}\boldsymbol{a} + \left(\lambda - \frac{1}{2}\right)\boldsymbol{b}$$

b $\overrightarrow{DE}.\overrightarrow{OB} = \left(-\frac{1}{2}\boldsymbol{a} + \left(\lambda - \frac{1}{2}\right)\boldsymbol{b}\right).(0\boldsymbol{a} + \boldsymbol{b})$

$$= -\frac{1}{2}\boldsymbol{a}.\boldsymbol{b} + \left(\lambda - \frac{1}{2}\right)\boldsymbol{b}.\boldsymbol{b}$$

If $\overrightarrow{DE} \perp \overrightarrow{OB}$,

then $\overrightarrow{DE}.\overrightarrow{OB} = 0$

$\therefore$ $-\frac{1}{2}\boldsymbol{a}.\boldsymbol{b} + \left(\lambda - \frac{1}{2}\right)\boldsymbol{b}.\boldsymbol{b} = 0$

$\therefore$ $\left(\lambda - \frac{1}{2}\right)\boldsymbol{b}.\boldsymbol{b} = \frac{1}{2}\boldsymbol{a}.\boldsymbol{b}$

$\therefore$ $\lambda \boldsymbol{b}.\boldsymbol{b} - \frac{1}{2}\boldsymbol{b}.\boldsymbol{b} = \frac{1}{2}\boldsymbol{a}.\boldsymbol{b}$

$\therefore$ $\lambda \boldsymbol{b}.\boldsymbol{b} = \frac{1}{2}(\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})$

$\therefore$ $\lambda = \dfrac{\frac{1}{2}(\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})}{\boldsymbol{b}.\boldsymbol{b}}$ as required.

c **i** If $\overrightarrow{DE} \perp \overrightarrow{OB}$,

then $$\lambda = \frac{\frac{1}{2}(\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b})}{\boldsymbol{b}.\boldsymbol{b}}$$

Let $a = |\boldsymbol{a}|$ and $b = |\boldsymbol{b}|$

$$\therefore \quad \lambda = \frac{\frac{1}{2}(ab\cos\theta + b^2)}{b^2}$$

$$= \frac{\frac{1}{2}(b^2\cos\theta + b^2)}{b^2} \text{ since } a = b$$

$$= \frac{\frac{1}{2}b^2(\cos\theta + 1)}{b^2}$$

$$= \frac{1}{2}\cos\theta + \frac{1}{2}$$

Now $$\lambda = \frac{5}{6}$$

$$\therefore \quad \frac{1}{2}\cos\theta + \frac{1}{2} = \frac{5}{6}$$

$$\therefore \quad \frac{1}{2}\cos\theta = \frac{2}{6}$$

$$= \frac{1}{3}$$

$$\therefore \quad \cos\theta = \frac{2}{3}, \text{ as required.}$$

ii $\overrightarrow{OF} = \overrightarrow{OE} + \overrightarrow{EF}$ and $\overrightarrow{AE} = \overrightarrow{OE} - \overrightarrow{OA}$

$$
\begin{aligned}
\overrightarrow{OF} &= \overrightarrow{OE} + \overrightarrow{EF} \\
&= \frac{5}{6}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{ED} \\
&= \frac{5}{6}\overrightarrow{OB} - \frac{1}{2}\overrightarrow{DE} \\
&= \frac{5}{6}\boldsymbol{b} - \frac{1}{2}\left(-\frac{1}{2}\boldsymbol{a} + \left(\frac{5}{6} - \frac{1}{2}\right)\boldsymbol{b}\right) \\
&= \frac{5}{6}\boldsymbol{b} + \frac{1}{4}\boldsymbol{a} - \frac{1}{6}\boldsymbol{b} \\
&= \frac{1}{4}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}
\end{aligned}
\qquad
\begin{aligned}
\overrightarrow{AE} &= \overrightarrow{OE} - \overrightarrow{OA} \\
&= \frac{5}{6}\overrightarrow{OB} - \overrightarrow{OA} \\
&= \frac{5}{6}\boldsymbol{b} - \boldsymbol{a}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{OF}.\overrightarrow{AE} &= \left(\frac{1}{4}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}\right).\left(\frac{5}{6}\boldsymbol{b} - \boldsymbol{a}\right) \\
&= \frac{1}{4}\boldsymbol{a}.\frac{5}{6}\boldsymbol{b} + \frac{2}{3}\boldsymbol{b}.\frac{5}{6}\boldsymbol{b} - \frac{1}{4}\boldsymbol{a}.\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}.\boldsymbol{a} \\
&= \frac{-11}{24}\boldsymbol{a}.\boldsymbol{b} - \frac{1}{4}\boldsymbol{a}.\boldsymbol{a} + \frac{5}{9}\boldsymbol{b}.\boldsymbol{b} \\
&= \frac{-11}{24}\boldsymbol{a}.\boldsymbol{b} + \frac{11}{36}\boldsymbol{b}.\boldsymbol{b}, \text{ as } |\boldsymbol{b}| = |\boldsymbol{a}|
\end{aligned}
$$

As $\cos\theta = \frac{2}{3}$, $\boldsymbol{a}.\boldsymbol{b} = \frac{2}{3}\boldsymbol{b}.\boldsymbol{b}$

$\therefore\ \overrightarrow{OF}.\overrightarrow{AE} = 0$

Since $\overrightarrow{OF}.\overrightarrow{AE} = 0$, $OF \perp AE$, as required.

7 a i $\overrightarrow{OA}.\overrightarrow{OB} = (3\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}).(2\boldsymbol{i} + a\boldsymbol{j} + 2\boldsymbol{k})$

$$
\begin{aligned}
&= 3 \times 2 + (-12) \times a + 3 \times 2 \\
&= 6 - 12a + 6 = 12(1 - a)
\end{aligned}
$$

ii $\overrightarrow{OA} \perp \overrightarrow{OB}$ $\therefore$ $\overrightarrow{OA}.\overrightarrow{OB} = 0$

$\therefore$ $12(1 - a) = 0$

$\therefore$ $1 - a = 0$

$\therefore$ $a = 1$

b i $\overrightarrow{OA} \perp \overrightarrow{OC}$ $\therefore$ $\overrightarrow{OA}.\overrightarrow{OC} = 0$

$\therefore$ $(3\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}).(x\boldsymbol{i} + y\boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\therefore$ $3x - 12y + 6 = 0$

$\therefore$ $x - 4y + 2 = 0$ ①

ii $\overrightarrow{OB} \perp \overrightarrow{OC}$ $\quad\therefore\quad \overrightarrow{OB}.\overrightarrow{OC} = 0$

$\therefore \quad (2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(x\boldsymbol{i} + y\boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\therefore \quad 2x + y + 4 = 0$

$\therefore \quad y = -2x - 4 \qquad ②$

Substituting ② in ① yields

$x - 4(-2x - 4) + 2 = 0$

$\therefore \quad x + 8x + 16 + 2 = 0$

$\therefore \quad 9x + 18 = 0$

$\therefore \quad x = \dfrac{-18}{9} = -2$

$\therefore \quad y = -2(-2) - 4 = 0$

c **i** $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$

$= \overrightarrow{OB} + \overrightarrow{OC}$

$= (2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + (-2\boldsymbol{i} + 2\boldsymbol{k}) = \boldsymbol{j} + 4\boldsymbol{k}$

ii $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= (3\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}) + (-2\boldsymbol{i} + 2\boldsymbol{k})$

$= \boldsymbol{i} - 12\boldsymbol{j} + 5\boldsymbol{k}$

iii $\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{AZ} + \overrightarrow{ZY}$

$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$

$= (3\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}) + (2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + (-2\boldsymbol{i} + 2\boldsymbol{k})$

$= 3\boldsymbol{i} - 11\boldsymbol{j} + 7\boldsymbol{k}$

d The heights above ground are given by the $\boldsymbol{k}$ components.
Hence X is 5 units above the ground and Y is 7 units above the ground.

8 a i $\overrightarrow{BD} = \frac{3}{4}\overrightarrow{BC} = \frac{3}{4}\boldsymbol{c}$

ii $\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AE}$

$= \overrightarrow{BA}\ \frac{3}{5}\overrightarrow{AC}$

$= \overrightarrow{BA} + \frac{3}{5}(\overrightarrow{BC} - \overrightarrow{BA})$

$= \boldsymbol{a} + \frac{3}{5}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$

iii $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$= -\overrightarrow{BA} + \frac{3}{4}\overrightarrow{BC} = -\boldsymbol{a} + \frac{3}{4}\boldsymbol{c}$

b $\overrightarrow{BP} = \mu\,\overrightarrow{BE}$ and $\overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP}$

$= \mu\left(\frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}\right)$ $\qquad = \overrightarrow{BA} + \lambda\,\overrightarrow{AD}$

$= \frac{2\mu}{5}\boldsymbol{a} + \frac{3\mu}{5}\boldsymbol{c}$ $\qquad = \boldsymbol{a} + \lambda\left(-\boldsymbol{a} + \frac{3}{4}\boldsymbol{c}\right)$

$= (1 - \lambda)\boldsymbol{a} + \frac{3\lambda}{4}\boldsymbol{c}$

Equating coefficients:

$\frac{2\mu}{5} = 1 - \lambda$ and $\frac{3\mu}{5} = \frac{3\mu}{4}$ $\therefore$ $\lambda = \frac{4\mu}{5}$

$= 1 - \frac{4\mu}{5}$

$\therefore \frac{6\mu}{5} = 1$

$\therefore 6\mu = 5$ $\qquad\therefore\qquad$ $\mu = \frac{5}{6}$

So $\lambda = \frac{4 \times \frac{5}{6}}{5} = \frac{2}{3}$

9 a $\boldsymbol{a} = p\boldsymbol{i} + q\boldsymbol{j}$

$\boldsymbol{b} = q\boldsymbol{i} - p\boldsymbol{j}$

$\boldsymbol{c} = -q\boldsymbol{i} + p\boldsymbol{j}$

b **i** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\quad$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= -\boldsymbol{i} - (x\boldsymbol{i} + y\boldsymbol{j})$ $\quad$ $= \boldsymbol{i} - (x\boldsymbol{i} + y\boldsymbol{j})$

$= -(x + 1)\boldsymbol{i} - y\boldsymbol{j}$ $\quad$ $= (1 - x)\boldsymbol{i} - y\boldsymbol{j}$

ii $\overrightarrow{AE} = y\boldsymbol{i} + (1 - x)\boldsymbol{j}$ $\quad$ $\overrightarrow{AF} = -y\boldsymbol{i} + (x + 1)\boldsymbol{j}$

($\overrightarrow{AC}$ is rotated 90° anticlockwise about A.) $\quad$ ($\overrightarrow{AB}$ is rotated 90° clockwise about A.)

c **i**

$$\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{AF} - \overrightarrow{AE}\\ &= (-y\boldsymbol{i} + (x + 1)\boldsymbol{j}) - (y\boldsymbol{i} + (1 - x)\boldsymbol{j})\\ &= -2y\boldsymbol{i} + 2x\boldsymbol{j}\\ &= 2(-y\boldsymbol{i} + x\boldsymbol{j})\end{aligned}$$

$$\begin{aligned}\overrightarrow{OA}.\overrightarrow{EF} &= (x\boldsymbol{i} + y\boldsymbol{j}).2(-y\boldsymbol{i} + x\boldsymbol{j})\\ &= 2(-xy + xy)\\ &= 0\end{aligned}$$

Since $\overrightarrow{OA}.\overrightarrow{EF} = 0$, $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{EF}$.

ii $|\overrightarrow{EF}| = \sqrt{(-2y)^2 + (2x)^2}$ and $|\overrightarrow{OA}| = \sqrt{x^2 + y^2}$

$= \sqrt{4y^2 + 4x^2}$

$= \sqrt{4(x^2 + y^2)}$

$= 2\sqrt{x^2 + y^2}$

$= 2|\overrightarrow{OA}|$, as required to prove.

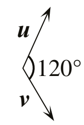
10 **a** **i** $\overrightarrow{BC} = m\boldsymbol{v}$

$\overrightarrow{BE} = n\boldsymbol{v}$

$\overrightarrow{CA} = m\boldsymbol{w}$

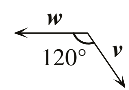
$\overrightarrow{CF} = n\boldsymbol{w}$

ii $|\overrightarrow{AE}| = |\overrightarrow{AB} + \overrightarrow{BE}|$



$$= |m\boldsymbol{u} + n\boldsymbol{v}|$$
$$= \sqrt{(m\boldsymbol{u} + n\boldsymbol{v})^2}$$
$$= \sqrt{m^2\boldsymbol{u}.\boldsymbol{u} + 2mn\boldsymbol{u}.\boldsymbol{v} + n^2\boldsymbol{v}.\boldsymbol{v}}$$
$$= \sqrt{m^2|\boldsymbol{u}|^2 + 2mn|\boldsymbol{u}||\boldsymbol{v}|\cos 120^\circ + n^2|\boldsymbol{v}|^2}$$
$$= \sqrt{m^2 + 2mn \times \frac{-1}{2} + n^2}, \text{ since } |\boldsymbol{u}| = |\boldsymbol{v}| = 1$$
$$= \sqrt{m^2 - mn + n^2}$$

$|\overrightarrow{FB}| = |\overrightarrow{FC} + \overrightarrow{CB}|$



$$= |-\overrightarrow{CF} - \overrightarrow{BC}|$$
$$= |-n\boldsymbol{w} - m\boldsymbol{v}|$$
$$= \sqrt{(-n\boldsymbol{w} - m\boldsymbol{v})^2}$$
$$= \sqrt{n^2|\boldsymbol{w}|^2 + 2mn|\boldsymbol{w}||\boldsymbol{v}|\cos 120^\circ + m^2|\boldsymbol{v}|^2}$$
$$= \sqrt{n^2 + 2mn \times \frac{-1}{2} + m^2}, \text{ since } |\boldsymbol{v}| = |\boldsymbol{w}| = 1$$
$$= \sqrt{m^2 - mn + n^2}$$

b $\overrightarrow{AE}.\overrightarrow{FB} = (m\boldsymbol{u} + n\boldsymbol{v}).(-n\boldsymbol{w} - m\boldsymbol{v})$

$$= -mn\boldsymbol{u}.\boldsymbol{w} - n^2\boldsymbol{v}.\boldsymbol{w} - m^2\boldsymbol{u}.\boldsymbol{v} - mn\boldsymbol{v}.\boldsymbol{v}$$
$$= -mn|\boldsymbol{u}||\boldsymbol{w}|\cos 120^\circ - n^2|\boldsymbol{v}||\boldsymbol{w}|\cos 120^\circ - m^2|\boldsymbol{u}||\boldsymbol{v}|\cos 120^\circ - mn|\boldsymbol{v}|^2$$
$$= -mn \times \frac{-1}{2} - n^2 \times \frac{-1}{2} - m^2 \times \frac{-1}{2} - mn, \text{ since } |\boldsymbol{u}| = |\boldsymbol{v}| = |\boldsymbol{w}| = 1$$
$$= \frac{1}{2}mn + \frac{1}{2}n^2 + \frac{1}{2}m^2 - mn$$
$$= \frac{1}{2}(m^2 - mn + n^2), \text{ as required.}$$

c $$\overrightarrow{AE}.\overrightarrow{FB} = |\overrightarrow{AE}||\overrightarrow{FB}|\cos G$$
$$= \sqrt{m^2 - mn + n^2}\sqrt{m^2 - mn + n^2}\cos G$$
$$= (m^2 - mn + n^2)\cos G$$

But $$\overrightarrow{AE}.\overrightarrow{FB} = \frac{1}{2}(m^2 - mn + n^2)$$

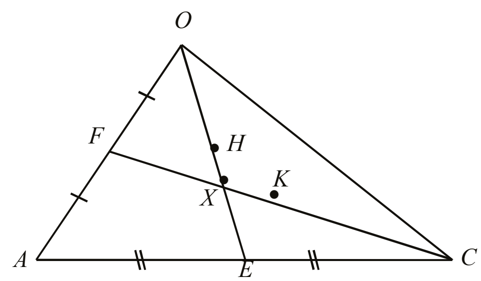
$\therefore$ $$(m^2 - mn + n^2)\cos G = \frac{1}{2}(m^2 - mn + n^2)$$

$\therefore$ $$\cos G = \frac{1}{2}$$

$\therefore$ $$G = 60°$$

By symmetry, H and K are also angles of 60°, hence $\triangle GHK$ is equilateral.

11 a In the diagram $\overrightarrow{OC} = \boldsymbol{c}$ and $\overrightarrow{OA} = \boldsymbol{a}$



$$\overrightarrow{CF} = \overrightarrow{CO} + \overrightarrow{OF}$$
$$= -\boldsymbol{c} + \frac{1}{2}\boldsymbol{a}$$
$$\overrightarrow{OE} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$$

b **i** $\overrightarrow{OE}$ is perpendicular to $\overrightarrow{AC}$

which implies $$\overrightarrow{OE}.\overrightarrow{AC} = 0$$

which can be written as $$\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})(\boldsymbol{c} - \boldsymbol{a}) = 0$$

Hence $$\boldsymbol{c}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{a} = 0$$

which implies $$|\boldsymbol{c}|^2 = |\boldsymbol{a}|^2$$

and $$OA = OC$$

The triangle OAC is isosceles.

ii Let $\angle AOC = \theta$. If $\overrightarrow{CF}$ is perpendicular to $\overrightarrow{OA}$

then $\cos\theta = \dfrac{\frac{1}{2}|\boldsymbol{a}|}{|\boldsymbol{c}|} = \dfrac{1}{2}$, as $oA = OC$

Therefore $\theta = 60°$

Hence all angles are 60° and triangle AOC is equilateral.

c i

$$\overrightarrow{OH} = \frac{1}{4}(\boldsymbol{a} + \boldsymbol{c})$$

$$\overrightarrow{CK} = \frac{1}{2}\overrightarrow{CF}$$

$$= \frac{1}{2}\left(-\boldsymbol{c} + \frac{1}{2}\boldsymbol{a}\right)$$

Now $$\overrightarrow{HK} = \overrightarrow{HO} + \overrightarrow{OC} + \overrightarrow{CK}$$

$$= -\frac{1}{4}(\boldsymbol{a} + \boldsymbol{c}) + \boldsymbol{c} + \frac{1}{2}\left(-\boldsymbol{c} + \frac{1}{2}\boldsymbol{a}\right)$$

$$= \frac{1}{4}\boldsymbol{c}$$

Since $\overrightarrow{HK} = \lambda \boldsymbol{c}, \lambda = \frac{1}{4}$

Also $$\overrightarrow{FE} = \overrightarrow{FA} + \overrightarrow{AE}$$

$$= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$$

$$= \frac{1}{2}\boldsymbol{c}$$

Since $\overrightarrow{FE} = \mu \boldsymbol{c},\ \mu = \frac{1}{2}$

ii HK is parallel to FE

$\angle XEF = \angle XHK$ (alternate angles)

$\angle XFE = \angle XKH$ (alternate angles)

Therefore triangle HXK is similar to triangle EXF.

iii As $|\overrightarrow{HK}| : |\overrightarrow{FE}| = 1 : 2$ (from **c**)

$|\overrightarrow{HX}| : |\overrightarrow{XE}| = 1 : 2$ (similar triangles)

Therefore $$|\overrightarrow{XE}| = \frac{2}{3}|\overrightarrow{HE}|$$

$$= \frac{1}{3}|\overrightarrow{OE}|$$

Therefore $$|\overrightarrow{OX}| = \frac{2}{3}|\overrightarrow{OE}|$$

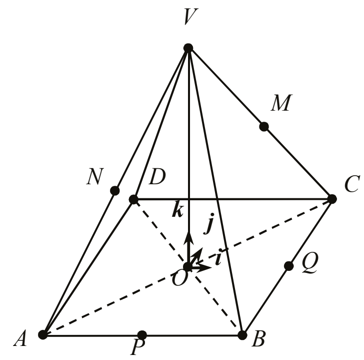
Hence $OX : XE = 2 : 1$

12 a $\overrightarrow{OA} = -2\boldsymbol{i} - 2\boldsymbol{j}$

$\overrightarrow{OB} = 2\boldsymbol{i} - 2\boldsymbol{j}$

$\overrightarrow{OC} = 2\boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OD} = -2\boldsymbol{i} + 2\boldsymbol{j}$



b $\overrightarrow{PM} = \overrightarrow{PB} + \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CV}$

$= 2\boldsymbol{i} + 4\boldsymbol{j} + \frac{1}{2}(-2\boldsymbol{i} - 2\boldsymbol{j} + 2h\boldsymbol{k})$

$= \boldsymbol{i} + 3\boldsymbol{j} + h\boldsymbol{k}$

$\overrightarrow{QN} = \overrightarrow{QB} + \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AV}$

$= -2\boldsymbol{j} - 4\boldsymbol{i} + \frac{1}{2}(2\boldsymbol{i} + 2\boldsymbol{j} + 2h\boldsymbol{k})$

$= -3\boldsymbol{i} - \boldsymbol{j} + h\boldsymbol{k}$

c Write $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AP} + \lambda\overrightarrow{PM}$

$= -2\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{i} + \lambda(\boldsymbol{i} + 3\boldsymbol{j} + h\boldsymbol{k})$

$= \lambda\boldsymbol{i} + (3\lambda - 2)\boldsymbol{j} + \lambda h\boldsymbol{k}$ ①

Also $\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BQ} + \mu\overrightarrow{QN}$

$= 2\boldsymbol{i} + \mu(-3\boldsymbol{i} - \boldsymbol{j} + h\boldsymbol{k})$

$= (2 - 3\mu)\boldsymbol{i} - \mu\boldsymbol{j} + \mu h\boldsymbol{k}$ ②

From ① and ②

$\lambda = 2 - 3\mu$, $3\lambda - 2 = -\mu$ and $\lambda h = \mu h$

$\lambda = \mu$

Therefore

and $4\lambda = 2$

which implies $\lambda = \frac{1}{2}$

Therefore $\mu = \frac{1}{2}$

Therefore $\overrightarrow{OX} = \frac{1}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{1}{2}h\boldsymbol{k}$

d **i** If $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{VB}$

$$\left(\frac{1}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{1}{2}h\boldsymbol{k}\right).(2\boldsymbol{i} - 2\boldsymbol{j} - 2h\boldsymbol{k}) = 0$$

Therefore $1 + 1 - h^2 = 0$

Therefore $h = \sqrt{2}$ as $h > 0$

ii Let θ be the angle between PM and QM.

Consider $\overrightarrow{PM}.\overrightarrow{QN} = |\overrightarrow{PM}||\overrightarrow{QN}| \cos\theta$

Therefore $-3 - 3 + 2 = \sqrt{12}\sqrt{12}\cos\theta$

Therefore $\cos\theta = \frac{-4}{12}$

$= \frac{-1}{3}$

θ is obtuse. The acute angle between PM and QN is $\cos^{-1}\left(\frac{1}{3}\right) = 71°$, to the nearest degree.

e **i** PM and QN are diagonals of quadrilateral $NMPQ$ and are of equal length and bisect each other at $X(\mathbf{c})$. Therefore $NMPQ$ is a rectangle.

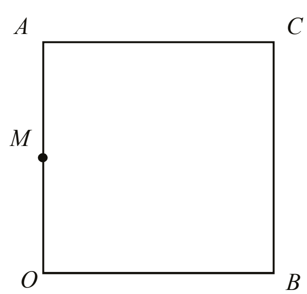
ii $NMPQ$ is a square if the diagonals bisect each other at right angles.

i.e. $\overrightarrow{PM}.\overrightarrow{QN} = 0$

This implies $-3 - 3 + h^2 = a$

and therefore $h = \sqrt{6}$ as $h > 0$

13



a **i** $\overrightarrow{OM} = \frac{1}{2}a\boldsymbol{j}$

ii $\overrightarrow{MC} = \overrightarrow{MA} + \overrightarrow{AC}$

$= a\boldsymbol{i} + \frac{1}{2}a\boldsymbol{j}$

b $\overrightarrow{MP} = \lambda\left(a\boldsymbol{i} + \frac{1}{2}a\boldsymbol{j}\right)$

$$\overrightarrow{BP} = \frac{1}{2}a\boldsymbol{j} - a\boldsymbol{i} + \lambda\left(a\boldsymbol{i} + \frac{1}{2}a\boldsymbol{j}\right)$$

$= (\lambda - 1)a\boldsymbol{i} + \frac{1}{2}(1 + \lambda)a\boldsymbol{j}$ Also $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$

$$= \lambda a\boldsymbol{i} + \frac{1}{2}(1 + \lambda)a\boldsymbol{j}$$

c **i** $\overrightarrow{BP}.\overrightarrow{MC} = 0$ implies $\frac{1}{4}a^2(1 + \lambda) + a^2(\lambda - 1) = 0$ $(a \neq 0)$

Therefore $(1 + \lambda) + 4(\lambda - 1) = 0$

which implies $5\lambda = 3$

Therefore $\lambda = \frac{3}{5}$

$$|\overrightarrow{BP}| = \sqrt{\frac{4}{25}a^2 + \frac{16}{25}a^2} = \frac{2\sqrt{5}}{5}a$$

$$|\overrightarrow{OP}| = \sqrt{\frac{9}{25}a^2 + \frac{16}{25}a^2} = a$$

$$|\overrightarrow{OB}| = a$$

ii

$$\overrightarrow{BP}.\overrightarrow{BO} = |\overrightarrow{BP}||\overrightarrow{BO}|\cos\theta$$

$$-\frac{2}{5}a \times -a = \frac{2\sqrt{5}}{5}a^2\cos\theta$$

Therefore $\cos\theta = \frac{\sqrt{5}}{5}$

d

$$|\overrightarrow{OP}| = \frac{a}{2}\sqrt{4\lambda^2 + 1 + 2\lambda + \lambda^2}$$

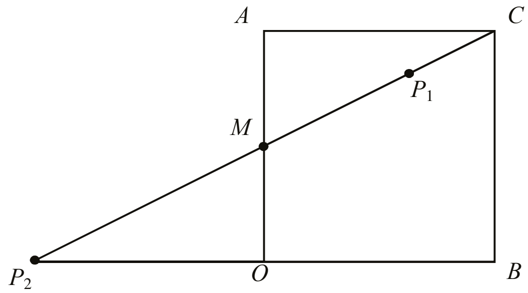
$$= \frac{a}{2}\sqrt{5\lambda^2 + 2\lambda + 1}$$

$|\overrightarrow{OP}| = |\overrightarrow{OB}|$ implies $a = \frac{a}{2}\sqrt{5\lambda^2 + 2\lambda + 1}$

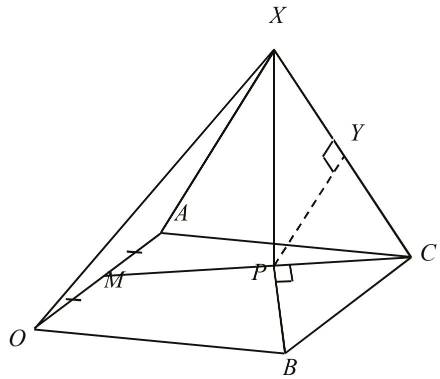
Squaring both sides gives $4 = 5\lambda^2 + 2\lambda + 1$

which implies $\lambda = \frac{3}{5}$ or $\lambda = -1$

P_1 corresponds to $\lambda = \frac{3}{5}$ and P_2 corresponds to $\lambda = -1$



e



$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$$

$$= \lambda a\boldsymbol{i} + \frac{1}{2}(1 + \lambda)a\boldsymbol{j}$$

From **c**, $\lambda = \frac{3}{5}$, therefore $\overrightarrow{OP} = \frac{3}{5}a\boldsymbol{i} + \frac{4}{5}a\boldsymbol{j}$

Now $\overrightarrow{CX} = \overrightarrow{CP} + \overrightarrow{PX}$

$$= -\frac{2}{5}\overrightarrow{MC} + a\boldsymbol{k}$$

$$= -\frac{2}{5}a\boldsymbol{i} - \frac{1}{5}a\boldsymbol{j} + a\boldsymbol{k}$$

$$\overrightarrow{PY} = \overrightarrow{PC} + \overrightarrow{CY}$$

$$= \frac{2}{5}\left(a\boldsymbol{i} + \frac{1}{2}a\boldsymbol{j}\right) + \mu\left(-\frac{2}{5}a\boldsymbol{i} - \frac{1}{5}a\boldsymbol{j} + a\boldsymbol{k}\right)$$

$$= \frac{2}{5}a(1 - \mu)\boldsymbol{j} + \frac{1}{5}a(1 - \mu)\boldsymbol{j} + \mu a\boldsymbol{k}$$

$\overrightarrow{CX}.\overrightarrow{PY} = 0$

$$\therefore \quad -\frac{4}{25}a^2(1-\mu) - \frac{1}{25}a^2(1-\mu) + \mu a^2 = 0$$

$$\therefore \quad -\frac{4}{25}a^2 + \frac{4}{25}a^2\mu - \frac{1}{25}a^2 + \frac{1}{25}a^2\mu + \mu a^2 = 0$$

$$\therefore \quad \frac{a^2}{25}(-4 + 4\mu - 1 + \mu + 25\mu) = 0$$

$$\therefore \quad -5 + 30\mu = 0$$

$$\therefore \quad 30\mu = 5$$

$$\therefore \quad \mu = \frac{1}{6}$$

Therefore $$\overrightarrow{OY} = a\boldsymbol{i} + a\boldsymbol{j} + \frac{1}{6}\left(-\frac{2}{5}a\boldsymbol{i} - \frac{1}{5}a\boldsymbol{j} + a\boldsymbol{k}\right)$$

$$= \frac{14}{15}a\boldsymbol{i} + \frac{29}{30}a\boldsymbol{j} + \frac{1}{6}a\boldsymbol{k}$$

Chapter 5 – Vector equations of lines and planes

Solutions to Exercise 5A

1 a If so, $4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} =$
$(1 - 3t)\boldsymbol{i} + (3 + t)\boldsymbol{j} + (-1 - 2t)\boldsymbol{k}$

$$\therefore \begin{cases} 1 - 3t = 4 \\ 2 = 3 + t \\ 1 = -1 - 2t \end{cases} \quad \therefore \begin{cases} t = -1 \\ t = -1 \\ t = -1 \end{cases}$$

$\therefore$ Point is on line. $\therefore$ Yes

b If so, $\begin{cases} 3 = 6 + t \\ -3 = 3 + 2t \\ -4 = -1 + t \end{cases} \quad \therefore \begin{cases} t = -3 \\ t = -3 \\ t = -3 \end{cases}$

$\therefore$ Point is on line. $\therefore$ Yes

c If so $\begin{cases} 3 = -1 - t \\ -1 = 2 + t \\ -1 = -3 - 2t \end{cases} \quad \therefore \begin{cases} t = -4 \\ t = -3 \end{cases}$

$\therefore$ Point is not on line. $\therefore$ No

2 a $\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j}$, $\overrightarrow{OB} = \boldsymbol{i} + 3\boldsymbol{j}$, $\therefore \overrightarrow{AB} = 2\boldsymbol{j}$.

$\therefore \boldsymbol{r} = \boldsymbol{i} + \boldsymbol{j} + t(2\boldsymbol{j})$

$\therefore$ or $\boldsymbol{r} = \boldsymbol{i} + (2t + 1)\boldsymbol{j}$

b $\overrightarrow{OA} = \boldsymbol{i} - 3\boldsymbol{k}$, $\overrightarrow{OB} = 2\boldsymbol{j} + \boldsymbol{j} - \boldsymbol{k}$

$\therefore \overrightarrow{AB} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \boldsymbol{r} = \boldsymbol{i} - 3\boldsymbol{k} + t(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

or $\boldsymbol{r} = (1 + t)\boldsymbol{i} + t\boldsymbol{j} + (2t - 3)\boldsymbol{k}$

c

$\overrightarrow{AB} = -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$\therefore \boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} + t(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

or $\boldsymbol{r} = (2 - t)\boldsymbol{i} + (2t - 1)\boldsymbol{j} + (2 - t)\boldsymbol{k}$

d $\overrightarrow{AB} = -4\boldsymbol{i} + 3\boldsymbol{j}$

$\therefore \boldsymbol{r} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} + t(-4\boldsymbol{i} + 3\boldsymbol{j})$

or $\boldsymbol{r} = (2 - 4t)\boldsymbol{i} + (3t - 2)\boldsymbol{j} + \boldsymbol{k}$

3 a $\overrightarrow{OA} = 3\boldsymbol{i} + \boldsymbol{j}$, $\overrightarrow{OB} = -2\boldsymbol{i} + 2\boldsymbol{j}$

$\therefore \overrightarrow{AB} = -5\boldsymbol{i} + \boldsymbol{j}$

$\therefore \boldsymbol{r} = 3\boldsymbol{i} + \boldsymbol{j} + t(-5\boldsymbol{i} + \boldsymbol{j})$

or $\boldsymbol{r} = (3 - 5t)\boldsymbol{i} + (1 + t)\boldsymbol{j}$

b $\overrightarrow{OA} = -\boldsymbol{i} + 5\boldsymbol{j}$, $\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j}$

$\therefore \overrightarrow{AB} = 3\boldsymbol{i} - 6\boldsymbol{j}$

$\therefore \boldsymbol{r} = -\boldsymbol{i} + 5\boldsymbol{j} + t(3\boldsymbol{i} - 6\boldsymbol{j})$

or $\boldsymbol{r} = (3t - 1)\boldsymbol{i} + (5 - 6t)\boldsymbol{j}$

c $\overrightarrow{OA} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{k}$

$\therefore \overrightarrow{AB} = \boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$

$\therefore \boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k})$

or $\boldsymbol{r} = (1 + t)\boldsymbol{i} + (2 - 2t)\boldsymbol{j} + (3 - 4t)\boldsymbol{k}$

d $\overrightarrow{OA} = \boldsymbol{i} - 4\boldsymbol{j}$, $\overrightarrow{OB} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$

$\therefore \overrightarrow{AB} = \boldsymbol{i} + 7\boldsymbol{j} + \boldsymbol{k}$

$\therefore \boldsymbol{r} = \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 7\boldsymbol{j} + \boldsymbol{k})$

or $\boldsymbol{r} = (t + 1)\boldsymbol{i} + (7t - 4)\boldsymbol{j} + t\boldsymbol{k}$

4 a $\boldsymbol{r} = (3 - 5t)\boldsymbol{i} + (1 + t)\boldsymbol{j}$

i $x = 3 - 5t, y = 1 + t$

ii $t = \dfrac{3 - x}{5}$

Therefore,

$y = 1 + \frac{3-x}{5} = \frac{1}{5}(8-x)$

b $\boldsymbol{r} = (3t-1)\boldsymbol{i} + (5-6t)\boldsymbol{j}$

i $x = 3t-1, y = 5-6t$

ii $t = \frac{x+1}{3}$

Therefore,

$y = 5 - 6 \times \frac{x+1}{3} = 3 - 2x$

c $\boldsymbol{r} = (1+t)\boldsymbol{i} + (2-2t)\boldsymbol{j} + (3-4t)\boldsymbol{k}$

i $x = 1+t, y = 2-2t, z = 3-4t$

ii $t = x-1$

$t = \frac{2-y}{2}$

$t = \frac{3-z}{4}$

Therefore,

$x - 1 = \frac{2-y}{2} = \frac{3-z}{4}$

d $\boldsymbol{r} = (t+1)\boldsymbol{i} + (7t-4)\boldsymbol{j} + t\boldsymbol{k}$

i $x = 1+t, y = 7t-4, z = t$

ii $t = x-1$

$t = \frac{y+4}{7}$

$t = z$

Therefore,

$x - 1 = \frac{y+4}{7} = z$

5 a Parallel $\therefore$ same direction vector
$= -3\boldsymbol{i} + \boldsymbol{j}$
$\therefore \boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(-3\boldsymbol{i} + \boldsymbol{j})$

b Let vector $\boldsymbol{d} = d_1\boldsymbol{i} + d_2\boldsymbol{j}$
(parallel to $x-y$ plane $\therefore k = 0$)
Now direction vector of $\boldsymbol{r} = -3\boldsymbol{i} + \boldsymbol{j}$

$\therefore\ \boldsymbol{d} \cdot \boldsymbol{r} = 0$

$\therefore\ -3d_1 + d_1 = 0$

Let $d_1 = 1 \quad \therefore\ d_2 = 3$

$\therefore\ \boldsymbol{d} = \boldsymbol{i} + 3\boldsymbol{j}$

$\therefore\ \boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(\boldsymbol{i} + 3\boldsymbol{j})$.

c $\boldsymbol{r} = 6\boldsymbol{i} + t(-6\boldsymbol{i} + 6\boldsymbol{j})$

6 a $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(-3\boldsymbol{i} + \boldsymbol{j})$

b $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(\boldsymbol{i} + 3\boldsymbol{j})$

7 a Direction vector $= 2\boldsymbol{j} - \boldsymbol{k}$

$\therefore\ \boldsymbol{r} = t(2\boldsymbol{j} - \boldsymbol{k})$

b direction vector in $2\boldsymbol{j} - \boldsymbol{k}$
Let $\boldsymbol{d}$ = required vector =
$d_1\boldsymbol{i} + d_2\boldsymbol{j} + d_3\boldsymbol{k}$
In $y-z$ plane $\therefore\ d_1 = 0$
$\boldsymbol{d} = d_2\boldsymbol{j} + d_3\boldsymbol{k}$
Now perpendicular to
$\boldsymbol{r} \therefore\ 2d_2 - d_3 = 0$
Let $d_2 = 1, d_3 = 2$
$\boldsymbol{d} = \boldsymbol{j} + 2\boldsymbol{k}$ $\therefore$ Passes through
$0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$
$\therefore\ \boldsymbol{r} = t(\boldsymbol{j} + 2\boldsymbol{k})$

8 a $\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j},\ \overrightarrow{OB} = -\boldsymbol{i} + 3\boldsymbol{j}$
$\therefore\ \overrightarrow{AB} = -3\boldsymbol{i} + 2\boldsymbol{j}$

$\therefore\ \boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(-3\boldsymbol{i} + 2\boldsymbol{j})$

b **i** $(5, 0)$ $5\boldsymbol{i} + 0\boldsymbol{j} = (2 - 3t)\boldsymbol{i} + (1 + 2t)j$

$\therefore \begin{cases} 2 - 3t = 5 \\ 1 + 2t = 0 \end{cases} \quad \therefore \begin{cases} t = -1 \\ t = -\frac{1}{2} \end{cases}$

No!!

ii $(0, 7)$

$\begin{cases} 2 - 3t = 0 \\ 1 + 2t = 7 \end{cases} \quad \therefore \begin{cases} t = \frac{2}{3} \\ t = 3 \end{cases}$

No!!

iii $(8, -3)$

$\therefore \begin{cases} 2 - 3t = 8 \\ 1 + 2t = -3 \end{cases} \quad \therefore$

$\begin{cases} t = -2 \\ t = -2 \end{cases}$

Yes!!

9 **a** Parallel to ℓ $\therefore$ parallel to $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

$\therefore\ \boldsymbol{r} = \boldsymbol{j} + \boldsymbol{k} + t(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

b For ℓ

$\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k} + t(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$= (3t + 1)\boldsymbol{i} + (t - 2)\boldsymbol{j} + (-1 - t)\boldsymbol{k}$

For the second line

$\boldsymbol{r} = \boldsymbol{j} + \boldsymbol{k} + t(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$= 3t\,\boldsymbol{i} + (t + 1)\boldsymbol{j} + (1 - t)\boldsymbol{k}$

Not the same line

c l $\boldsymbol{r} = (3t + 1)\boldsymbol{i} + (t - 2)\boldsymbol{j} + (-1 - t)\boldsymbol{k}$

$= 2\boldsymbol{i} + m\boldsymbol{j} + nk$

$\therefore\ 3t + 1 = 2 \ \therefore\ t = \frac{1}{3}.$

$\therefore\ m = -\frac{5}{3}, n = -1 - \frac{1}{3} = -\frac{4}{3}$

10 **a** let $\boldsymbol{a}, = a_1\boldsymbol{i} + \mathrm{a}_2 j$

$\boldsymbol{a} \cdot \boldsymbol{v} = 0$

$\therefore 3a_1 - 4a_2 = 0 (a_2 = \frac{3}{4}a_1)$

Now $|\boldsymbol{a}| = 5$

$\therefore\ \sqrt{a_1^2 + a^2} = 5$

$a_1^2 + a_2^2 = 25$

But $a_2 = \frac{3}{4}a_1 \quad \therefore\ a_1^2 + \frac{9}{16}a_1^2 = 25$

$\therefore\ \frac{25}{16}a_1^2 = 25$

$\therefore\ a_1 = \pm 4.$

So possible vector is $4\boldsymbol{i} + 3\boldsymbol{j}$

(In general $a_2 = \frac{3}{4}a_1$)

$\therefore$ All vector possible are $a_1\boldsymbol{i} + \frac{3}{4}a_1\boldsymbol{j}$

b $\overrightarrow{OA} = 2\boldsymbol{i} - 3\boldsymbol{j}, \overrightarrow{OB} = -\boldsymbol{i} + \boldsymbol{j}$

$\therefore\ \overrightarrow{AB} = -3\boldsymbol{i} + 4\boldsymbol{j} \quad \therefore\ \overrightarrow{BA} = 3\boldsymbol{i} - 4\boldsymbol{j}$

Vector parallel is

$\boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{j} + t(-3\boldsymbol{i} - 4\boldsymbol{j})$

$\therefore$ Vector equal of perpendicular line in $k = -\boldsymbol{i} + \boldsymbol{j} + t(4\boldsymbol{i} + 3\boldsymbol{j})$

c $r = (4t - 1)\boldsymbol{i} + (3t + 1)j$

$\therefore\ 4t - 1 = 0 \quad \therefore\ t = \frac{1}{4}$

$\therefore\ r = 0\boldsymbol{i} + \frac{7}{4}j$

$\therefore\ y$ intercept $= (0, \frac{7}{4})$

and $3t + 1 = 0 \ \therefore\ t = -\frac{1}{3}$

$\therefore\ r = -\frac{7}{3}\boldsymbol{i} + 0\boldsymbol{j}$

$\therefore\ x$ intercept $= (-\frac{7}{3}, 0)$

11 **a** $r = 2\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k} + t(-3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$= (-3t + 2)\boldsymbol{i} + (t + 5)\boldsymbol{j} + (4 - 2t)\boldsymbol{k}$

Parametric equations:

$x = 2 - 3t, y = t + 5,\ z = 4 - 2t$

Cartesian equation

$t = y - 5,\ t = \dfrac{2 - x}{3},$

$t = \dfrac{4 - z}{2}.$

$\therefore\ \dfrac{2 - x}{3} = y - 5 = \dfrac{4 - z}{2}.$

b $\boldsymbol{r} = 2\boldsymbol{j} - \boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$

$= (2t)\boldsymbol{i} + (t + 2)\boldsymbol{j} + (4t - 1)\boldsymbol{k}$

Parametric equations

$x = 2t, y = t + 2, z = 4t - 1$

Cartesian equations

$t = \dfrac{x}{2} = y - 2 = \dfrac{z + 1}{4}$

12 a $O = (0, 0, 0)$

$\boldsymbol{r} = (4 - 3t)\boldsymbol{i} + (1 + 2t)\boldsymbol{j} + (5t - 3)\boldsymbol{k}$

$\boldsymbol{d} = -3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$

Require

$(-3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}) \cdot ((4 - 3t)\boldsymbol{i} + (1 + 2t)\boldsymbol{i} + (5t - 3)\boldsymbol{k}) = 0$

$\therefore\ -3(4 - 3t) + 2(1 + 2t) + 5(5t - 3) = 0$

$-12 + 9t + 2 + 4t + 25t - 15 = 0$

$\therefore\ 38t - 25 = 0$

$\therefore\ t = \dfrac{25}{38}$

$\therefore$ Required vector is $\dfrac{77}{38}\boldsymbol{i} + \dfrac{44}{19}\boldsymbol{j} + \dfrac{11}{38}\boldsymbol{k}$

$\therefore$ distance $= \sqrt{\left(\dfrac{77}{38}\right)^2 + \left(\dfrac{44}{19}\right)^2 - \left(\dfrac{11}{38}\right)^2}$

$= \dfrac{11\sqrt{114}}{38}$

b $\boldsymbol{r} = (4 - 3t)\boldsymbol{i} + (2t + 1)\boldsymbol{j} + (5t - 3)\boldsymbol{k}$

$= \overrightarrow{OP}$

Vector parallel to line is $-3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$

$\overrightarrow{OA} = \boldsymbol{i} + 10\boldsymbol{j} - 2\boldsymbol{k}$

If P is on line l

Then

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= (3 - 3t)\boldsymbol{i} + (2t - 9)\boldsymbol{j} + (5t - 1)\boldsymbol{k}$

$\therefore$ Direction of line is $-3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$

For perpendicular

$(-3t)(-3) + 2(2t - 9) + 5(5t - 1) = 0$

$\therefore\ -9 + 9t + 4t - 18 + 25t - 5 = 0$

$\therefore\ 38t - 32 = 0$

$\therefore\ t = \dfrac{16}{19}$

$\therefore\ \overrightarrow{AP} = \left(3 - \dfrac{48}{19}\right)\boldsymbol{i} + \left(\dfrac{32}{19} - 9\right)\boldsymbol{j} + \left(\dfrac{80}{19} - 1\right)\boldsymbol{k}$

$= \dfrac{9}{19}\boldsymbol{i} - \dfrac{139}{19}\boldsymbol{j} + \dfrac{61}{19}k$

$|\overrightarrow{AP}| = \sqrt{\left(\dfrac{9}{19}\right)^2 + \left(\dfrac{139}{19}\right)^2 + \left(\dfrac{61}{19}\right)^2} = \dfrac{\sqrt{23123}}{19}$

13

c $r = (1+t)\boldsymbol{i} + (2t-4)\boldsymbol{j} + (1-t)\boldsymbol{k}$

For B $\quad 2t - 4 = 0$

$\therefore \; t = 2$

For C $\quad 1 + t = -2$

$\therefore \; t = -3$

$\therefore \; \overrightarrow{BC} = (1+t)\boldsymbol{i} + (2t-4)\boldsymbol{j} + (1-t)\boldsymbol{k}$

For $t \in [-3, 2]$

14 $\overrightarrow{OA} = 3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}, \overrightarrow{OB} = 5\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}$

$\therefore \; \overrightarrow{AB} = 2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$

$\therefore \; \boldsymbol{r} = 3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} + t(2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k})$

$= (3+2t)\boldsymbol{i} + (4-3t)\boldsymbol{j} + (1+5t)\boldsymbol{k}$

Crosses x, y plane $\boldsymbol{k}$ component = 0

$\therefore \; 1 + 5t = 0 \; \therefore \; t = -\frac{1}{5}$

$\therefore \; \boldsymbol{r} = \left(3 - \frac{2}{5}\right)\boldsymbol{i} + \left(4 + \frac{3}{5}\right)\boldsymbol{j}$

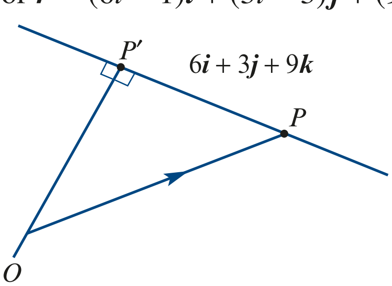
$= \frac{13}{5}\boldsymbol{i} + \frac{23}{5}\boldsymbol{j}$

$\therefore$ point is $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

15 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 6\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k}$

$\therefore \; \boldsymbol{r} = -\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} + \mathrm{t}(6\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k})$

or $\boldsymbol{r} = (6t-1)\boldsymbol{i} + (3t-3)\boldsymbol{j} + (9t-3)\boldsymbol{k}$



Require $\overrightarrow{OP'} \perp \overrightarrow{PP'}$

$\therefore \; \overrightarrow{OP'} \cdot \overrightarrow{PP'} = 0$

$\therefore \; 6(6t-1) + 3(3t-3) + 9(9t-3) = 0$

$\therefore \; 36t - 6 + 9t - 9 + 81t - 27 = 0$

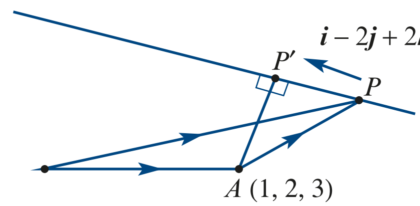
$\therefore \; t = \frac{1}{3}$

$\therefore \; \overrightarrow{OP'} = \left(6 \times \frac{1}{3} - 1\right)\boldsymbol{i} + \left(3 \times \frac{1}{3} - 3\right)\boldsymbol{j} + \left(9 \times \frac{1}{3} - 3\right)\boldsymbol{k}$

$\therefore \; \overrightarrow{OP'} = \boldsymbol{i} - 2\boldsymbol{j}$

$\therefore \; \left|\overrightarrow{OP'}\right| = \sqrt{1+4} = \sqrt{5}$

16



$\overrightarrow{OP} = (3+t)\boldsymbol{i} + (4-2t)\boldsymbol{j} + (2t-2)\boldsymbol{k}$

$\overrightarrow{OA} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore \; \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (2+t)\boldsymbol{i} + (2-2t)\boldsymbol{j} + (2t-5)\boldsymbol{k}$

Vector parallel to line = $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$

For P' = point on line nearest to A

Require $((2+t)\boldsymbol{i} + (2-2t)\boldsymbol{j} + (2t-5)\boldsymbol{k}) \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\therefore \; 2 + t - 2(2-2t) + 2(2t-5) = 0$

$\therefore \; 2 + t - 4 + 4t + 4t - 10 = 0$

$\therefore \; 9t - 12 = 0$

$\therefore \; t = \frac{4}{3}$

$\therefore \; \overrightarrow{AP'} = \left(2 + \frac{4}{3}\right)\boldsymbol{i} + \left(2 - \frac{8}{3}\right)\boldsymbol{j} + \left(\frac{8}{3} - 5\right)\boldsymbol{k}$

$\therefore \; \overrightarrow{AP'} = \frac{10}{3}\boldsymbol{i} - \frac{2}{3}j - \frac{7}{3}\boldsymbol{k}$

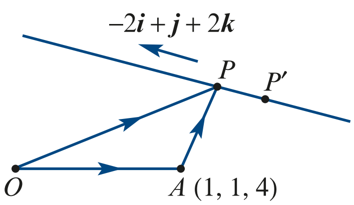
$$\therefore \text{ Distance} = \left|\overrightarrow{AP'}\right| = \sqrt{\left(\frac{10}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{7}{3}\right)^2}$$

$$= \sqrt{\frac{100}{9} + \frac{4}{9} + \frac{49}{9}}$$

$$= \frac{\sqrt{153}}{3}$$

$$= \sqrt{17}$$

17*



$\overrightarrow{OP} = (1 - 2t)\boldsymbol{i} + (t - 2)\boldsymbol{j} + (1 + 2t)\boldsymbol{k}$

$\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$\therefore\ \overrightarrow{AP} = -2t\boldsymbol{i} + (t - 3)\boldsymbol{j} + (2t - 3)k$

$\overrightarrow{OP}$ parallel $-2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

Require $\overrightarrow{AP}$ where $\overrightarrow{OP} \perp \overrightarrow{AP'}$

$\therefore\ -2(2t) - 2t + (t - 3) + 2(2t - 3) = 0$

$\therefore\ -4t + t - 3 + 4t - 6 = 0$

$\therefore\ 9t - 9 = 0$

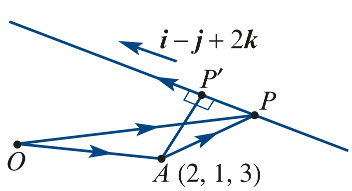
$\therefore\ t = 1$

$\therefore\ \overrightarrow{AP'} = -2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$

$\therefore$ Distance $= \left|\overrightarrow{AP'}\right| = \sqrt{4 + 4 + 1}$

$= 3$

18*



$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$\overrightarrow{OP} = (t + 1)\boldsymbol{i} + (2 - t)\boldsymbol{j} + 2tk$

$\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

$\therefore\ \overrightarrow{AP} = (t - 1)\boldsymbol{i} + (1 - t)\boldsymbol{j} + (2t - 3)\boldsymbol{k}$

To find P' require $\overrightarrow{AP} \cdot (\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\therefore\ (t - 1) - (1 - t) + 2(2t - 3) = 0$

$\therefore\ t - 1 - 1 + t + 4t - 6 = 0$

$\therefore\ 6t - 8 = 0 \quad \therefore\ t = \frac{4}{3}$

$\therefore\ \overrightarrow{OP'} = \left(\frac{4}{3} + 1\right)\boldsymbol{i} + \left(2 - \frac{4}{3}\right)\boldsymbol{j} + \frac{8}{3}k$

$\therefore\ \overrightarrow{OP'} = \frac{7}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{8}{3}k$

$\therefore$ Coordinates of nearest point $= \left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$

17 Parallel to x axis

$\therefore$ direction vector passes through

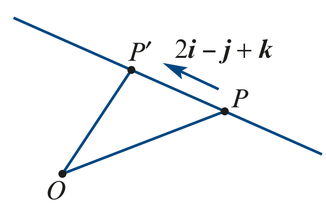
$P(-2, 2, 1)$ and $Q(0, 2, 1)$

$\therefore$ Direction vector is $\overrightarrow{PQ} = 2\boldsymbol{i}$ or $\boldsymbol{i}$

$\therefore \boldsymbol{r} = -2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t\boldsymbol{i}$.

$\therefore \boldsymbol{r} = (t - 2)\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

18



$\overrightarrow{OP} = 3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

$= (3 + 2t)\boldsymbol{i} + (1 - t)\boldsymbol{j} + (5 + t)\boldsymbol{k}$

For $\overrightarrow{OP'}$ $\quad \overrightarrow{OP} \cdot (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 0$

$\therefore\ 2(3 + 2t) - (-1 - t) + 5 + t = 0$

$\therefore\ 6 + 4t + t - 1 + t + 5 = 0$

$\therefore\ 6t + 10 = 0$

$\therefore\ t = -\frac{5}{3}$

$$\therefore\ \overrightarrow{OP'} = \left(3 - \frac{10}{3}\right)\boldsymbol{i} + \left(1 + \frac{5}{3}\right)\boldsymbol{j} + \left(-\frac{5}{3} + 5\right)\boldsymbol{k}$$

$$\therefore\ \overrightarrow{OP'} = -\frac{1}{3}\boldsymbol{i} + \frac{8}{3}\boldsymbol{j} + \frac{10}{3}k$$

$$\therefore \text{ Distance } = \left|\overrightarrow{OP'}\right| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{8}{3}\right)^2 + \left(\frac{10}{3}\right)^2}$$

$$= \sqrt{\frac{1}{9} + \frac{64}{9} + \frac{100}{9}}$$

$$= \sqrt{\frac{165}{9}}$$

$$= \frac{\sqrt{165}}{3}$$

19a Now $\boldsymbol{r} = (1 - 2t)\boldsymbol{i} + (t - 2)\boldsymbol{j} + (2t + 1)\boldsymbol{k}$

$\therefore$ At $t = 1$ $\quad \boldsymbol{r} = -\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

$\therefore$ End point $= (-1, -1, 3)$

At $t = 3$ $\quad \boldsymbol{r} = -5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}$

$\therefore$ Other end point is $(-5, 1, 7)$

19b Now $\boldsymbol{r} = (3 + t)\boldsymbol{i} + (4 - 2t)\boldsymbol{j} + (2t - 2)\boldsymbol{k}$

Now at $t = -1$ $\quad r = 2\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k}$

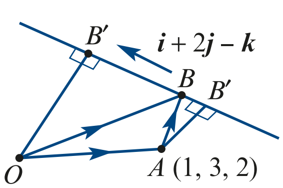
$\therefore$ point $= (2, 6, -4)$

At $t = 2$, $\quad r = 5\boldsymbol{i} + 2\boldsymbol{k}$

$\therefore$ point $= (5, 0, 2)$

20

21



a $\overrightarrow{OB} = (1 + t)\boldsymbol{i} + (2t - 4)\boldsymbol{j} + (1 - t)\boldsymbol{k}$

b $\left|\overrightarrow{OB}\right| = \sqrt{(1 + t)^2 + (2t - 4)^2 + (1 - t)^2}$

$$= \sqrt{6t^2 - 16t + 18}$$

c $\left|\overrightarrow{OB}\right| = \sqrt{6\left(t - \frac{4}{3}\right)^2 + \frac{22}{3}}$

$\therefore$ Min value of $\left|\overrightarrow{OB}\right| = \sqrt{\frac{22}{3}}$

$$= \frac{\sqrt{66}}{3}$$

d Now $\overrightarrow{OB} = (1 + t)\boldsymbol{i} + (2t - 4)\boldsymbol{j} + (1 - t)\boldsymbol{k}$

and $\overrightarrow{OA} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} =$

$t\boldsymbol{i} + (2t - 7)\boldsymbol{j} + (-1 - t)\boldsymbol{k}$

B' is point on line nearest A

$\therefore$ Require $\overrightarrow{AB} \cdot (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) = 0$

$$\therefore t + 2(2t - 7) + 1 + t = 0$$

$$\therefore 6t - 13 = 0$$

$$\therefore t = \frac{13}{6}$$

$$\therefore\ \overrightarrow{AB'} = \frac{13}{6}\boldsymbol{i} + \left(\frac{13}{3} - 7\right)\boldsymbol{j} + \left(-1 - \frac{13}{6}\right)\boldsymbol{k}$$

$$= \frac{13}{6}\boldsymbol{i} - \frac{8}{3}j - \frac{19}{6}k$$

$\therefore$ Shortest distance $= \left|\overrightarrow{AB'}\right|$

$$= \sqrt{\left(\frac{13}{6}\right)^2 + \left(\frac{8}{3}\right)^2 + \left(\frac{19}{6}\right)^2}$$

$$= \frac{\sqrt{786}}{6}$$

Solutions to Exercise 5B

1 At point of intersection

$3\boldsymbol{i} + 5\boldsymbol{j} + \lambda(2\boldsymbol{i} - \boldsymbol{j}) = -2\boldsymbol{j} + \mu(4\boldsymbol{i} + 2\boldsymbol{j})$

$\therefore\ (2\lambda + 3)\boldsymbol{i} + (5 - \lambda)j = 4\mu\boldsymbol{i} + (2\mu - 2)i$

$\therefore \begin{cases} 2\lambda + 3 = 4\mu \\ 5 - \lambda = 2\mu - 2 \end{cases}$

$\therefore\ \lambda = \dfrac{11}{4}$

$\therefore$ Position vector of intersection is

$3\boldsymbol{i} + 5\boldsymbol{j} + \dfrac{11}{4}(2\boldsymbol{i} - \boldsymbol{j}) = \dfrac{17}{2}\boldsymbol{i} + \dfrac{9}{4}j$

2 At Point of intersection

$\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + \lambda(-2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) =$

$-3\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$\therefore\ (-2\lambda + 1)\boldsymbol{i} + (3 - \lambda)\boldsymbol{j} + (1 + 2\lambda)\boldsymbol{k} =$

$(\mu - 3)\boldsymbol{i} + (4 - \mu)\boldsymbol{j} + (7 - 2\mu)\boldsymbol{k}$

For intersection $\begin{cases} -2\lambda + 1 = \mu - 3 & (1) \\ 3 - \lambda = 4 - \mu & (2) \\ 1 + 2\lambda = 7 - 2\mu & (3) \end{cases}$

(1) + (2) Gives $-3\lambda + 4 = 1 \quad \therefore \lambda = 1$

And $\mu = 2$

Check in (3) $1 + 2\lambda = 3, 7 - 2\mu = 3$

$\therefore$ point of intersection at $(-1, 2, 3)$

3 $\boldsymbol{r}_1 = (3 + 2\lambda)\boldsymbol{i} + (2 - 3\lambda)\boldsymbol{j} + (1 + \lambda)\boldsymbol{k}$

$\boldsymbol{r}_2 = (1 + \mu)\boldsymbol{i} + (-3 - 2\mu)\boldsymbol{j} + (2 + 3\mu)\boldsymbol{k}$

"Not parallel"

For intersection

$\begin{cases} 3 + 2\lambda = 1 + \mu & (1) \\ 2 - 3\lambda = -3 - 2\mu & (2) \\ 1 + \lambda = 2 + 3\mu & (3) \end{cases}$

From (1) & (2) $\quad$ (1) × 2 + (2)

$\therefore\ 6 + 4\lambda + 2 - 3\lambda = 2 + 2\mu - 3 - 2\mu$

$\therefore\ \lambda + 8 = -1$

$\therefore\ \lambda = -9$

$\therefore$ From (1): $3 - 18 = 1 + \mu$

$\therefore\ \mu = -16.$

Check in (3) $\quad 1 + \lambda = -8$

$2 + 3\mu = -46$

$\therefore$ No point of intersection

$\therefore$ Skew lines

4 **a** $\boldsymbol{r}_1 = (1 + t)\boldsymbol{i} + (2 + t)\boldsymbol{j}$

$\boldsymbol{r}_2 = (-1 + s)\boldsymbol{i} + (6 + 2s)\boldsymbol{j}$

At point of intersection

$\begin{cases} 1 + t = -1 + s & (1) \\ 2 + t = 6 + 2s & (2) \end{cases}$

$\therefore\ s = -6, t = -8$

$\therefore$ point of intersection $= -7\boldsymbol{i} - 6\boldsymbol{j}$

$\therefore$ (i) No, (ii) No, (iii) No, (iv) point of intersection $= (-7, -6)$

b Lines are perpendicular, since Dot product of direction vectors is zero.

Now $\boldsymbol{r}_1 = (t - 1)\boldsymbol{i} + (1 + 2t)\boldsymbol{j}$

and $\boldsymbol{r}_2 = (3 - 2s)\boldsymbol{i} + (s - 1)\boldsymbol{j}$

$\therefore$ At point of intersection $\begin{cases} t - 1 = 3 - 2s & (1) \\ 1 + 2t = s - 1 & (2) \end{cases}$

$\therefore\ s = 2,\ t = 0$

$\therefore$ Point of intersection is $(-1, 1)$

c $\boldsymbol{r}_1 = (5 - 2t)\boldsymbol{i} + (9 - 3t)\boldsymbol{j}$

$\boldsymbol{r}_2 = (1 + 4s)\boldsymbol{i} + (3 + 6s)\boldsymbol{j}$

These lines are parallel (direction vectors are scalar multiples) and coincident as both yield $2s + t = 2$.

d $\boldsymbol{r}_1 = (1 + 2t)\boldsymbol{i} + (-4 - t)\boldsymbol{j}$

$\boldsymbol{r}_2 = (7 - 2s)\boldsymbol{i} + (8 + s)\boldsymbol{j}$

These are parallel (direction vectors are scalar multiples)

$\begin{cases} 1 + 2t = 7 - 2s & (1) \\ -4 - t = 8 + s & (2) \end{cases}$

$\therefore \begin{cases} 2t + 2s = 6 & (1) \\ t + s = -12 & (2) \end{cases}$

Hence contradiction so not coincident

∴ No points of intersection.

e $\begin{cases} \boldsymbol{r}_1 = 5\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) \\ \boldsymbol{r}_2 = 4\boldsymbol{j} + \boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) \end{cases}$

$\therefore \begin{cases} \boldsymbol{r}_1 = (5 + t)\boldsymbol{i} + (5 + 2t)\boldsymbol{j} + (-4 - t)\boldsymbol{k} \\ \boldsymbol{r}_2 = s\boldsymbol{i} + (4 - s)\boldsymbol{j} + (1 - s)\boldsymbol{k} \end{cases}$

At Point of intersection

$\begin{cases} 5 + t = s & (1) \\ 5 + 2t = 4 - s & (2) \\ -4 - t = 1 - s & (3) \end{cases}$

From (1) + (2)$t = 2, s = 3$

Check in (3) Left side = −2,

Right side = −2

Lines are perpendicular as dot product of direction vectors is zero.

Point of intersection is $3\boldsymbol{i} + \boldsymbol{i} - 2\boldsymbol{k}$

= point (3, 1, −2)

f $\boldsymbol{r}_1 = (7 + 3t)\boldsymbol{i} + (4 + t)\boldsymbol{j} + (5 - t)\boldsymbol{k}$

$\boldsymbol{r}_2 = s\boldsymbol{i} + (1 + 4s)\boldsymbol{j} + (2s - 3)\boldsymbol{k}.$

Not parallel , not ⊥.

At point of intersection

$\begin{cases} 7 + 3t = s & (1) \\ 4 + t = 1 + 4s & (2) \\ 5 - t = 2s - 3 & (3) \end{cases}$

(2) + (3) $9 = 6s - 2$

$\therefore\ s = \frac{11}{6}$

$\therefore\ 4 + t = 1 + \frac{22}{3}$

$\therefore\ t = -3 + \frac{22}{3}$

$= \frac{13}{3}$

Substitute in (1)

$\text{LHS} = 7 + 3 \times \frac{13}{3} = 20$

$\text{RHS} = s = \frac{11}{6}$

Therefore, skew lines (no point of intersection).

g $\boldsymbol{r}_1 = (6 + t)\boldsymbol{i} + (-6 - 2\text{t})\boldsymbol{j} + (5 + 2t)\boldsymbol{k}$

$\boldsymbol{r}_2 = (1 + s)\boldsymbol{i} + (2 - s)\boldsymbol{j} + (-5 + 2s)\boldsymbol{k}$

Not parallel, Not ⊥.

$\begin{cases} 6 + t = 1 + s & (1) \\ -6 - 2t = 2 - s & (2) \\ 5 + 2t = -5 + 2s & (3) \end{cases}$

(1) + (2) $\therefore -t = 3 \quad \therefore t = -3$

Substitute in (1) $3 = 1 + s$

$\therefore\ s = 2$

Check in (3)

Left side = 5 − 6 = −1

Right side = −5 + 4 = −1

∴ Point of intersection = $3\boldsymbol{i} - \boldsymbol{k}$

∴ Point of intersection = (3, 0, −1)

h Parallel lines

$\boldsymbol{r}_1 = (4 + 2t)\boldsymbol{i} + (-5 - 4t)\boldsymbol{j} + (1 - 2t)\boldsymbol{k}$

$\boldsymbol{r}_2 = (-1 - s)\boldsymbol{i} + (5 + 2s)\boldsymbol{j} + (6 + s)\boldsymbol{k}$

$\therefore \begin{cases} 4 + 2t = -1 - s & (1) \\ -5 - 4t = 5 + 2s & (2) \\ 1 - 2t = 6 + s & (3) \end{cases}$

(1) + (3) $\therefore\ 5 = 5(2) - 2 \times (3) - 7 = -7$

∴ Lines are coincident

i $\boldsymbol{r}_1 = (-3 + 3t)\boldsymbol{i} + (-1 + 2t)\boldsymbol{j} - 2t\boldsymbol{k}$ Not parallel

$\boldsymbol{r}_2 = (4 + s)\boldsymbol{i} + \boldsymbol{j} + (-6 - s)k$ Not ⊥.

$\therefore \begin{cases} -3 + 3t = 4 + s & (1) \\ -1 + 2t = 1 & (2) \\ -2t = -6 - s & (3) \end{cases}$

From (2) $t = 1$ substitute in (1)

$\therefore\ s = -4$

Check in (3) Left side = −2

Right side = −2

∴ Point of intersection is

$\therefore\ 0\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

∴ Point of intersection at (0, 1, −2)

j Parallel lines

$\boldsymbol{r}_1 = (7 + 2t)\boldsymbol{i} + (-6 - 2t)\boldsymbol{j} + (-6 + t)\boldsymbol{k}$

$\boldsymbol{r}_2 = (-3 + 2s)\boldsymbol{i} + (4 - 2s)\boldsymbol{j} + (-5 + s)\boldsymbol{k}$

$$\therefore \begin{cases} 7 + 2t = -3 + 2s & (1) \\ -6 - 2t = 4 - 2s & (2) \\ -6 + t = -5 + s & (3) \end{cases}$$

∴ Coincident Lines

5 **a** Three lines intersect at $(1, 2, -1)$

b, c, d Three lines are not concurrent.

6 **a**

$$\boldsymbol{d_1} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$

$$\boldsymbol{d_2} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$$

$$\therefore\ \boldsymbol{d_1} \cdot \boldsymbol{d_2} = 3 + 4 + 12 = 19$$

$$|\boldsymbol{d_1}| = \sqrt{9} = 3$$

$$|\boldsymbol{d_2}| = \sqrt{9 + 4 + 36} = 7$$

$$\therefore\ 19 = 21 \cos\theta$$

$$\therefore\ \cos\theta = \frac{19}{21}$$

Angle = 25.21°

b

$$\boldsymbol{d_1} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$$

$$\boldsymbol{d_2} = 2\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$$

$$\therefore\ \boldsymbol{d_1} \cdot \boldsymbol{d_2} = 2 + 8 + 8 = 18$$

$$|\boldsymbol{d_1}| = \sqrt{1 + 4 + 4} = 3$$

$$|\boldsymbol{d_2}| = \sqrt{4 + 16 + 16} = 6$$

$$\therefore\ 18 = 18 \cos\theta$$

$$\therefore\ \cos\theta = 1$$

$$\therefore\ \theta = 0$$

Angle is 0° as parallel lines

7 $\boldsymbol{d_1} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\quad \boldsymbol{d_2} = \boldsymbol{i} + \boldsymbol{k}$

$\boldsymbol{d_1} \cdot \boldsymbol{d_2} = 2 + 1 = 3$

$$|\boldsymbol{d_1}| = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$|\boldsymbol{d_2}| = \sqrt{2}$$

a $\boldsymbol{d_1} \cdot \boldsymbol{d_2} = |\boldsymbol{d_1}|\,|\boldsymbol{d_2}|.\cos\theta$

$$\therefore\ 3 = \sqrt{2} \cdot \sqrt{6} \cos\theta$$

$$\therefore\ \cos\theta = \frac{3}{2\sqrt{3}}$$

$$\therefore\ \cos\theta = \frac{\sqrt{3}}{2}$$

$$\therefore\ \theta = 30°$$

b $\boldsymbol{r}_1 = (1 + 2t)\boldsymbol{i} + (6 - t)\boldsymbol{j} + (3 + t)\boldsymbol{k}$

$\boldsymbol{r}_2 = (3 + s)\boldsymbol{i} + 3\boldsymbol{j} + (8 + s)\boldsymbol{k}$

$$\therefore\ \text{If point of intersection} \begin{cases} 1 + 2t = 3 + s & (1) \\ 6 - t = 3 & (2) \\ 3 + t = 8 + s & (3) \end{cases}$$

From (2) $t = 3$ in (1) $s = 3$

Using (3) as a check.

Left side = 6, Right side = 11

∴ No point of intersection.

∴ Skew lines

8 $\boldsymbol{r}_1 = 3\boldsymbol{i} + (1 + 2t)\boldsymbol{j} + t\boldsymbol{k}$

$\boldsymbol{r}_2 = s\boldsymbol{i} + s\boldsymbol{j} + (4 - s)\boldsymbol{k}$

a $$\text{At point of intersection} \begin{cases} 3 = s & (1) \\ 1 + 2t = s & (2) \\ t = 4 - s & (3) \end{cases}$$

From (1) & (3) $s = 3,\ t = 1$

Check in (2) Left side = 3

Right side = 3

∴ Point of intersection

$3\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} = (3, 3, 1)$

b $\boldsymbol{d_1} = 2\boldsymbol{j} + \boldsymbol{k}$, $\quad \boldsymbol{d_2} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

$d_1 \cdot d_2 = 2 - 1 = 1$

$|\boldsymbol{d_1}| = \sqrt{5}$, $\quad |\boldsymbol{d_2}| = \sqrt{3}$

$\therefore\ 1 = \sqrt{5} \cdot \sqrt{3} \cdot \cos\theta$

$$\therefore\ \cos\theta = \frac{1}{\sqrt{15}}$$

9 a Check l_1 and l_2

$\boldsymbol{r}_1 = (1 + t_1)\boldsymbol{i} + (3t_1)\boldsymbol{j} + (-2 + t_1)\boldsymbol{k}$

$\boldsymbol{r}_2 = (2 - t_2)\boldsymbol{i} + (-1 + 2t_2)\boldsymbol{j} + (1 + t_2)\boldsymbol{k}$

$$\therefore \begin{cases} 1 + t_1 = 2 - t_2 & (1) \\ 3t_1 = -1 + 2t_2 & (2) \\ -2 + t_1 = 1 + t_2 & (3) \end{cases}$$

No solutions $\therefore l_1$ & l_2 so not intersect

Check l_2 & l_3

$\boldsymbol{r}_2 = (2 - t_2)\boldsymbol{i} + (-1 + 2t_2)\boldsymbol{j} + (1 + t_2)\boldsymbol{k}$

$\boldsymbol{r}_3 = (3 + t_3)\boldsymbol{i} + (-1 - 4t_3)\boldsymbol{j} - \boldsymbol{k}$

$$\therefore \begin{cases} 2 - t_2 = 3 + t_3 & (1) \\ -1 + 2t_2 = -1 + 4t_3 & (2) \\ 1 + t_2 = -1. & (3) \end{cases}$$

$\therefore$ From (3) $t_2 = -2$

$\therefore$ From (1) $4 = 3 + t_3 \quad \therefore t_3 = 1$

Check in (2)

Left side $= -1 - 4 = -5$

Right side $= -1 - 4 = -5$

Intersect at $4\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k} = (4, -5, -1)$

$$\text{Check } l_1 \ \& \ l_3 \begin{cases} 1 + t_1 = 3 + t_3 & (1) \\ 3t_1 = -1 - 4t_3 & (2) \\ -2 + t_1 = -1 & (3) \end{cases}$$

From (3) $t_1 = 1$,

From (1) $\quad 2 = 3 + t_3$

$\therefore \ t_3 = -1$

Check in (2) Left side $= 3$

Right side $= -1 + 4 = 3$

$\therefore$ Point of intersection

$2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} \ \therefore \ (2, 3, -1)$

Solutions to Exercise 5C

1 a $a \times b = \begin{bmatrix} i & j & k \\ 1 & -4 & 1 \\ 4 & 3 & 0 \end{bmatrix}$

$\therefore\ a \times b = \begin{vmatrix} -4 & 1 \\ 3 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} j$

$+ \begin{vmatrix} 1 & -4 \\ 4 & 3 \end{vmatrix} k$

$= -3i + 4j + 19k$

b $a \times b = i - 7j - 4k$

c $a \times b = i - j$

d $a \times b = 2i + 4k$ or $i + 2k$

2 a $a \times b = -9i - 26j - 12k$

b $a \times b = -4j - 2k$ or $2j + k$

c $a \times b = -2j - k$ or $2j + k$

d $a \times b = 2i - 4k$ or $i - 2k$

3 a $(a + b) \times b = a \times b + b \times b$

$= a \times b$

b $(a \times b) \times (a + b)$

$= (a + b) \times a + (a + b) \times b$

$= a \times a + b \times a + a \times b + b \times b$

$= -a \times b + a \times b$

$= \mathbf{0}$

c $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b})$

$= (\boldsymbol{a} - \boldsymbol{b}) \times \boldsymbol{a} + (\boldsymbol{a} - \boldsymbol{b}) \times \boldsymbol{b}$

$= \boldsymbol{a} \times \boldsymbol{a} - \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{b}$

$= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}$

$= 2(\boldsymbol{a} \times \boldsymbol{b})$

d $(\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})) \cdot \boldsymbol{b}$

$= (\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$

$= (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{b} + (\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$

$= (\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot ((\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a})$

$= \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{a})$

$= \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{a}) + \boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{a})$

$= 0 + 0$

$= 0$

f $((\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{a}) + (\boldsymbol{b}.(\boldsymbol{a} \times \boldsymbol{b}))$

$= 0 + 0$

$= 0$

4 $\boldsymbol{a} \times \boldsymbol{b} = 4\boldsymbol{i} - 5\boldsymbol{j} - 7\boldsymbol{k}$

Now $|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{16 + 25 + 49}$

$= \sqrt{90}$

$= 3\sqrt{10}$

$\therefore$ Unit vector parallel $a \times b = \dfrac{1}{3\sqrt{10}}(4\boldsymbol{i} - 5\boldsymbol{j} - 7\boldsymbol{k})$

$\therefore \dfrac{5}{3\sqrt{10}}(4\boldsymbol{i} - 5\boldsymbol{j} - 7\boldsymbol{k})$ has magnitude $= 5$

$= \dfrac{5\sqrt{10}}{30}(4\boldsymbol{i} - 5\boldsymbol{j} - 7\boldsymbol{k})$

$= \dfrac{\sqrt{10}}{6}(4\boldsymbol{i} - 5\boldsymbol{j} - 7\boldsymbol{k})$

5 $\boldsymbol{a} \times \boldsymbol{b}$ = area of parallelogram
bounded by

$\boldsymbol{a}$ and $\boldsymbol{b} = \frac{1}{2}$ area of $\triangle$ bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$

Also $\boldsymbol{c} \times \boldsymbol{a}$ = area of parallelogram bounded by

$\boldsymbol{c} \times \boldsymbol{a} = \frac{1}{2}$ area of $\triangle$ bounded by $\boldsymbol{c}$ &$\boldsymbol{a}$

And $\boldsymbol{b} \times \boldsymbol{c}$ = area of parallelogram bounded by $\boldsymbol{b}$&$\boldsymbol{c}$

$= \frac{1}{2}$ area of $\triangle$ bounded by $\boldsymbol{b}$ &$\boldsymbol{c}$

$\therefore$ area of $\triangle$ ABC $= \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}|$

6 $\boldsymbol{c} = \boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{j} + 5\boldsymbol{k}$

Now $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{i}$

So $|\boldsymbol{a} \times \boldsymbol{b}| = |-\boldsymbol{i}| = 1$

= area parallelogram $OABC$

7 $\boldsymbol{a} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}, \boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$

Now $\boldsymbol{a} \times \boldsymbol{b} = 15\boldsymbol{i} - 7\boldsymbol{j} - 10\boldsymbol{k}$

$\therefore\ |\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{15^2 - 7^2 + 10^2}$

$= \sqrt{374}$

= area of parallelogram bounded by $\boldsymbol{a}$&$\boldsymbol{b}$

$\therefore$ area $\triangle = \frac{\sqrt{374}}{2}$

8 Let $\overrightarrow{OP} = \boldsymbol{u}$

$\left|\overrightarrow{PL}\right| = |\boldsymbol{u}| \cdot \sin\theta$

$= \frac{|\boldsymbol{u}| \cdot \sin\theta|\boldsymbol{v}|}{|\boldsymbol{v}|}$

Now $\frac{|\boldsymbol{u} \times \boldsymbol{v}|}{|\boldsymbol{v}|}$

$= \frac{|\boldsymbol{u}| \cdot |\boldsymbol{v}| \cdot \sin\theta}{|\boldsymbol{v}|}$

$\therefore$ Equal

9 a $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

$\boldsymbol{c} = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$

$\boldsymbol{a} \cdot \boldsymbol{c} = (a_2b_3 - a_3b_2)a_1 - (a_1b_3 - a_3b_1)a_2 + (a_1b_2 - a_2b_1)a_3$

$= (a_2b_3a_1 - a_3b_2a_1) - (a_1a_2b_3 - a_3a_2b_1) + (a_1b_2a_3 - a_2b_1a_3)$

$= 0$

$\boldsymbol{b} \cdot \boldsymbol{c} = (a_2b_3 - a_3b_2)b_1 - (a_1b_3 - a_3b_1)b_2 + (a_1b_2 - a_2b_1)b_3$

$= (a_2b_3b_1 - a_3b_2b_1) - (a_1b_2b_3 - a_3b_2b_1) + (a_1b_2b_3 - a_2b_1b_3)$

$= 0$

b Swapping $\boldsymbol{a}$ and $\boldsymbol{b}$

swap $\boldsymbol{c} = (b_2a_3 - b_3a_2)\boldsymbol{i} - (b_1a_3 - b_3a_1)\boldsymbol{j} + (b_1a_2 - b_2a_1)\boldsymbol{k}$

Clearly this is $-\boldsymbol{c}$

c $|\boldsymbol{c}|^2 = \boldsymbol{c} \cdot \boldsymbol{c}$

$|\boldsymbol{a}|^2|\boldsymbol{b}|^2 \cos^2 \theta = (a_1b_1 + a_2b_2 + a_3b_3)^2$

$|\boldsymbol{a}|^2|\boldsymbol{b}|^2(1 - \ sin^2\theta = (a_1b_1 + a_2b_2 + a_3b_3)^2$

$|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (a_1b_1 + a_2b_2 + a_3b_3)^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 \sin^2 \theta$

It remains to show $\boldsymbol{c} \cdot \boldsymbol{c} = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (a_1b_1 + a_2b_2 + a_3b_3)^2$

That is:

$\boldsymbol{c} \cdot \boldsymbol{c} = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

Solutions to Exercise 5D

1 a $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = k$

But $(\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}) \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = k$

$\therefore\ 1 - 2 + 4 = k$

$\therefore\ k = 3$

$\therefore\ \boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 3$ is equation of plane

& Cartesian equation is $x + y + z = 3$

b $\boldsymbol{r} \cdot \boldsymbol{n} = r \cdot (\boldsymbol{i} - 2\boldsymbol{k}) = k$

But $(3\boldsymbol{i} + \boldsymbol{j}) \cdot (\boldsymbol{i} - 2\boldsymbol{k}) = k$

$\therefore\ k = 3$

$\therefore\ \boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{k}) = 3$ is equation of plane

& Cartesian equation is $x - 2z = 3$

c $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r} \cdot (2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = k$

But $(2\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}) \cdot (2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = 0$

$\boldsymbol{r} \cdot (2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = 0$ is equation of plan & Cartesian equation is

$2x + 3y - z = 0$

d $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r} \cdot (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = k$

But $(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) \cdot (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) =$

$1 - 6 - 3 = -8$

$\therefore$ Equation of plan is

$r \cdot (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = -8$

And Cartesian equation is

$x + 3y - z = -8$

2 $\overrightarrow{AB} = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}, \quad \overrightarrow{AC} = \boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = 12\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$

$\therefore$ Unit vector normal to plane is

$\dfrac{1}{\sqrt{170}}(12\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k})$

So vector equation is $r \cdot (12\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}) = \boldsymbol{k}$

Where k $= (2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}).(12\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k})$

$= 24 + 5 - 1$

$= 28$

$\therefore\ r \cdot (12\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}) = 28$ is equation of plane

3 Now $\boldsymbol{r}_1 = (1 + 2\lambda)\boldsymbol{i} + (-10 - \lambda)\boldsymbol{j} + (4 + \lambda)\boldsymbol{k}$

And $\boldsymbol{r}_2 = (-3 + \mu)\boldsymbol{i} + (-2 - 2\mu)\boldsymbol{j} + \mu\boldsymbol{k}$

$$\therefore \begin{cases} 1 + 2\lambda = -3 + \mu & (1) \\ -10 - \lambda = -2 - 2\mu & (2) \\ 4 + \lambda = \mu & (3) \end{cases}$$

$\therefore\ \lambda = 0, \mu = 4$

$\therefore$ Point of intersection is $(1, -10, 4)$

Now $\boldsymbol{n} = (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) \times (\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

$= \boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$

$\therefore$ Vector equation is $\boldsymbol{r} \cdot (\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}) = \boldsymbol{k}$

Now $(\boldsymbol{i} - 10\boldsymbol{j} + 4\boldsymbol{k}) \cdot (\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}) = -1$

$\therefore$ Equation is $r \cdot (\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}) = -1$

4 a $\boldsymbol{r} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$\therefore\ \boldsymbol{d} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

And $\overrightarrow{OB} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ gives a point in plane

Also $\overrightarrow{OA} = -3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ gives another point

$\therefore$ A line in the plane

$= \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 4\boldsymbol{i} - 5\boldsymbol{j}$

$\therefore$ Normal to plane, $\boldsymbol{r} = \boldsymbol{d} \times \overrightarrow{AB}$

$= (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) \times (4\boldsymbol{i} - 5\boldsymbol{j})$

$= 5\boldsymbol{i} + 4\boldsymbol{j} + 13\boldsymbol{k}$

$\therefore$ Line through A that is normal to plane is

$\boldsymbol{r} = -3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + t(5\boldsymbol{i} + 4\boldsymbol{j} + 13\boldsymbol{k})$

5

$\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}, \overrightarrow{OB} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k},$

$\overrightarrow{OC} = 3\boldsymbol{j} - \boldsymbol{k}$

$\therefore\ \overrightarrow{AB} = 4\boldsymbol{j} - 5\boldsymbol{k}$ is in plane

and $\overrightarrow{AC} = -\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$ also

$\therefore \boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$= -6\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}$

$\therefore$ Unit vector normal to plane is

$\frac{1}{\sqrt{77}}(6\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k})$

Vector equation of plane is

$\boldsymbol{r} \cdot (-6\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}) = k$

Where

$k = (\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) \cdot (-6\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}) = 11$

$\therefore$ Equation is $\boldsymbol{r} \cdot (-6\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}) = 11$

6 $\boldsymbol{r} \cdot (2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}) = 7$,

$\therefore \boldsymbol{n} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$

a $\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = 2 + 2 + 3 = 7$

b $\boldsymbol{r} = 3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = 6 + 4 - 3 = 7$

c $\boldsymbol{r} = -\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = -2 - 3 + 12 = 7$

d $\boldsymbol{r} = 2\boldsymbol{j} - 3\boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = -2 + 9 = 7$

7 $\boldsymbol{r} \cdot (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 10$

$\therefore \boldsymbol{n} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

a $\boldsymbol{r} = 2\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = 6 + 2 + 2 = 10$

b $\boldsymbol{r} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = 3 + 5 + 2 = 10$

c $\boldsymbol{r} = 3\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = 9 + 4 - 3 = 10$

d $\boldsymbol{r} = 2\boldsymbol{i} - 4\boldsymbol{k}$

$\therefore\ \boldsymbol{r} \cdot \boldsymbol{n} = 6 + 4 = 10$

8 a $(\boldsymbol{i} + x\boldsymbol{j} + 2\boldsymbol{k}) \cdot (-\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) = 5$

$\therefore\ 5 = -1 + x + 6$

$\therefore\ x = 0$

b $(2\boldsymbol{i} - \boldsymbol{j}).(3\boldsymbol{i} + 2\boldsymbol{k}) = x$

$\therefore\ x = 6$

c $(\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}) \cdot (2\boldsymbol{i} + xk) = 8$

$\therefore\ 2 + 2x = 8$

$\therefore\ x = 3$

d $(x\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) \cdot (\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) = 5$

$\therefore\ x + 3 - 2 = 5$

$\therefore\ x = 4$

9 $\overrightarrow{OA} = 3\boldsymbol{j} + 4\boldsymbol{k}, \overrightarrow{OB} = \boldsymbol{i} + 2\boldsymbol{j}$,

$\overrightarrow{OC} = -\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$

$\therefore\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k}$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\boldsymbol{i} + 3\boldsymbol{j}$.

$\therefore\ \overrightarrow{AB} \times \overrightarrow{AC} = 12\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k} = \boldsymbol{n}$

$\therefore$ Equation is $r \cdot (12\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) = k$

Where $k = (3\boldsymbol{j} + 4\boldsymbol{k}) \cdot (12\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k})$

$= 12 + 8 = 20$

$\therefore\ \boldsymbol{r} \cdot (12\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) = 20$

$\therefore$ Cartesian equation is

$12x + 4y + 2z = 20$ or $6x + 2y + z = 10$

10 $\boldsymbol{r} = (4 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j} + 8t\boldsymbol{k}$

$= 4\boldsymbol{i} + \boldsymbol{j} + t(\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k})$

$\therefore\ \boldsymbol{d} = \boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$

But plane perpendicular to this line

$\therefore\ \boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$

$\therefore$ Plane equation is $\boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}) = k$

Now plane passes through $(3, 2, 1)$

$\therefore\ \boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\therefore\ k = (3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k})$

$\therefore\ k = 3 - 4 + 8 = 7$

$\therefore\ \boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}) = 7$

$\therefore$ Cartesian equation is $x - 2y + 8z = 7$

11 So plane is $\boldsymbol{r} \cdot (5p\boldsymbol{i} - 3p\boldsymbol{j} + 2p\boldsymbol{k}) = 6$
Now $(4, -1, 2)$ is in plane
$\therefore (4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) \cdot (5p\boldsymbol{i} - 3p\boldsymbol{j} + 2pk) = 6$
$\therefore 20p + 3p + 4p = 6$
$\therefore 27p = 6$
$\therefore p = \frac{6}{27} = \frac{2}{9}$
$\therefore$ Plane is $\boldsymbol{r} \cdot (\frac{10}{9}\boldsymbol{i} - \frac{6}{9}\boldsymbol{j} + \frac{4}{9}\boldsymbol{k}) = 6$
$\therefore$ Cartesian equation is
$\frac{10}{9}x - \frac{6}{9}y + \frac{4}{9}z = 6$
or $10x - 6y + 4z = 54$
or $5x - 3y + 2z = 27$

12 $\boldsymbol{r}_1 = 4\boldsymbol{i} + \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$

$\boldsymbol{r}_2 = 4\boldsymbol{i} + \boldsymbol{k} + s(-3\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k})$
Point of intersection $= (4, 0, 1)$
$\therefore\ \boldsymbol{d}_1 = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$

$\boldsymbol{d}_2 = -3\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$
Now $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = 13\boldsymbol{i} + 7\boldsymbol{j} + 9\boldsymbol{k}$
$\therefore$ Equation is $\boldsymbol{r} \cdot (13\boldsymbol{i} + 7\boldsymbol{j} + 9\boldsymbol{k}) = k$
Where $k = (4\boldsymbol{i} + \boldsymbol{k}) \cdot (13\boldsymbol{i} + 7\boldsymbol{j} + 9\boldsymbol{k}) = 61$
$\therefore\ \boldsymbol{r} \cdot (13\boldsymbol{i} + 7\boldsymbol{j} + 9\boldsymbol{k}) = 61$
$\therefore$ Cartesian equation is
$13x + 7y + 9z = 61$

13 $\overrightarrow{OA} = \boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}, \quad \overrightarrow{OB} = -\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$
$\therefore\ \overrightarrow{AB} = -2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$
Given plane is $\boldsymbol{r} \cdot (3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) = 4$
$\therefore$ Perpendicular plane contains
$3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$
$\therefore\ \boldsymbol{n} = (3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) \times (-2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$= -3\boldsymbol{i} + 8\boldsymbol{j} + 7\boldsymbol{k}$
$\therefore$ Plane equation is $-3x + 8y + 7z = k$
But$(1, 2, 4)$is in the plane.
$\therefore\ k = -3 + 16 + 28 = 41$
$\therefore\ -3x + 8y + 7z = 41$

14 $\boldsymbol{r}_1(t) = 3\boldsymbol{i} + \boldsymbol{j} + t(4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$
$\boldsymbol{r}_1(t) = 5\boldsymbol{i} + \boldsymbol{k} + s(4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$ We know that the vector $4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ is in the plane.
Also the vector $5\boldsymbol{i} + \boldsymbol{k} - (3\boldsymbol{i} + \boldsymbol{j})$ is in the plane.
We use the cross product to find a normal vector to the plane.
This is $\boldsymbol{n} = -\boldsymbol{i} - 2\boldsymbol{j}$
Then the equation is given by :
$\boldsymbol{r} \cdot (-\boldsymbol{i} - 2\boldsymbol{j}) = (3\boldsymbol{i} + \boldsymbol{j}) \cdot (-\boldsymbol{i} - 2\boldsymbol{j})$
$-x - 2y = -5$
which we neaten to
$x + 2y = 5$

Solutions to Exercise 5E

1 a $\boldsymbol{n} = 7\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$

Plane $7x + 4y + 4z = 9$

$\therefore\ \hat{\boldsymbol{n}} = \dfrac{1}{9}(7\boldsymbol{i} + 4y + 4\boldsymbol{k})$

Now $P = (1, 3, 2)\ \therefore\ \overrightarrow{OP} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

Let $Q(x, y, z)$ be in plane,

$\overrightarrow{OQ} = x\boldsymbol{i} + y\boldsymbol{j} + 7\boldsymbol{k}$

$\therefore\ 7x + 4y + 4z = 9$

Now $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (x - 1)\boldsymbol{i} + (y - 3) + (z - 2)\boldsymbol{k}$

Require projection of $\overrightarrow{PQ}$ onto $\hat{\boldsymbol{n}}$

$= \overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}$

$= \dfrac{7}{9}(x - 1) + \dfrac{4}{9}(y - 3) + \dfrac{4}{9}(z - 2)$

$= \dfrac{1}{9}(7x - 7 + 4y + 2 + 4z - 8)$

$= \dfrac{1}{9}(7x + 4y + 4z - 27)$

$= \dfrac{1}{9}(9 - 27)$

$= \dfrac{1}{9}(-18)$

$= -2$

$\therefore$ Distance $= 2$

b Plane is $6x + 6y + 3z = 8$

$\therefore\ \boldsymbol{n} = 6\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k}, |\hat{r}| = 9$

$\therefore\ \hat{\boldsymbol{n}} = \dfrac{1}{9}(6\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k})$

Now $P = (1, 3, 2)\quad \therefore\ \overrightarrow{OP} =$
$\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

Let $Q(x, y, z)$ be point in plane

$\therefore\ \overrightarrow{OQ} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

and $6x + 6y + 3z = 8$

Now $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (x - 1)\boldsymbol{i} + (y - 3)\boldsymbol{j} + (z - 2)\boldsymbol{k}$

Require projection of $\overrightarrow{PQ}$ onto $\hat{\boldsymbol{n}}$

$= \dfrac{1}{9}(6(x - 1) + 6(y - 3) + 3(z - 2))$

$= \dfrac{1}{9}(6x - 6 + (6y - 18) + 3z - 6)$

$= \dfrac{1}{9}(6x + 6y + 3z - 30)$

$= \dfrac{1}{9}(8 - 30)$

$= \dfrac{-22}{9}$

$\therefore$ Distance $= \dfrac{22}{9}$

2 $\boldsymbol{n}$ for both planes is $\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

$\therefore\ |\boldsymbol{n}| = 3$

$\therefore$ Distance of π_1 from origin is $\dfrac{4}{|\boldsymbol{n}|} = \dfrac{4}{3}$

Distance of π_2 from origin is $\dfrac{12}{3} = 4$

$\therefore$ Distance apart $= 4 - \dfrac{4}{3} = \dfrac{8}{3}$

3 a Now line is $(3 + t)\boldsymbol{i} + (-1 + 2t)\boldsymbol{j} +$
$(-1 - 2t)\boldsymbol{k}$

If on plane (intersection)

$((3 + t)\boldsymbol{i} + (-1 + 2t)\boldsymbol{j} + (-1 - 2t)\boldsymbol{k}) \cdot$
$(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) = 4$

$\therefore\ 3 + t - 1 + 2t + 2(-1 - 2t) = 4$

$\therefore\ -t = 4$

$\therefore\ t = -4$

$\therefore$ Point of intersection $= -\boldsymbol{i} - 9\boldsymbol{j} + 7\boldsymbol{k}$

$\therefore$ Point of intersection $(-1, -9, 7)$

b Now for line $d = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

and $\boldsymbol{n} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

Now $\boldsymbol{d} \cdot \boldsymbol{n} = 1 + 2 - 4 = -1$

$= |\boldsymbol{d}| \cdot |\boldsymbol{n}| \cdot \cos\theta$

$\therefore\ -1 = 3 \cdot \sqrt{6}\cos\theta$

$\therefore\ \theta = \cos^{-1}\left(-\dfrac{1}{3\sqrt{6}}\right)$

$\approx 97.82°$

$\therefore$ Acute angle is 7.82°

4 a For π, normal $\boldsymbol{n}_1 = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$
For π_2 normal $\boldsymbol{n}_2 = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$
Find angle between normals

$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2| \cdot \cos\theta$

$\therefore\ 2 - 1 - 2 = \sqrt{6} \cdot \sqrt{6}\cos\theta$

$\therefore\ \cos\theta = -\dfrac{1}{6}$

$\therefore\ \theta \approx 99° \cdot 5941°$

Angle between planes = 80.41°

b Cartesian equation of π_1 is
$2x + y - z = 8$ (1)
Cartesian equation of π_2 is
$x - y + 2z = 6$ (2)
(1) + (2) gives $3x + z = 14$
$\therefore$ If $x = \lambda,\ z = 14 - 3\lambda$
and $\lambda - y + 2(14 - 3\lambda) = 6$ from (2).
$\therefore\ y = 22 - 5\lambda$
$\therefore$ Line of intersection has parametric equations
$x = \lambda,\ y = 22 - 5\lambda,\ z = 14 - 3\lambda$
$\therefore$ Vector equation is
$\boldsymbol{r} = \lambda\boldsymbol{i} + (22 - 5\lambda)\boldsymbol{j} + (14 - 3\lambda)\boldsymbol{k}$
$\therefore\ \boldsymbol{r} = 22\boldsymbol{j} + 14\boldsymbol{k} + \lambda(\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k})$

5 a $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{k}, \overrightarrow{OB} = \boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, \overrightarrow{OC} = -\boldsymbol{j} + 2\boldsymbol{k}$
and $\overrightarrow{OD} = 3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$
$\therefore \overrightarrow{AB} = -\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and
$\overrightarrow{AC} = -2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$
Now $\overrightarrow{AB} \times \overrightarrow{AC} = -7\boldsymbol{i} - \boldsymbol{j} - 5\boldsymbol{k}$
$\therefore\ \boldsymbol{n}_1 = 7\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$

b Now $\overrightarrow{BC} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}, \overrightarrow{BD} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
Now $\overrightarrow{BC} \times \overrightarrow{BD} = \boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k} = \boldsymbol{n_2}$

c Angle between normals
$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2| \cdot \cos\theta$
$\therefore\ \theta = \cos^{-1}\left(\dfrac{-\sqrt{105}}{35}\right) = 107.024°$
$\therefore$ Angle between planes = 72.98°.

6 a Point of intersection of line and plane:
$((1 + t)\boldsymbol{i} + (-3 + t)\boldsymbol{j} + (2 - 3t)\boldsymbol{k}) \cdot (2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) = 7$
$\therefore\ 2(1 + t) - (t + 3) + (3t - 2) = 7$
$\therefore\ 4t + 3 = 7 \quad \therefore\ t = 1$
$\therefore$ Point of intersection = (2, −2, −1)
$\therefore$ angle between line and plane = angle between $\boldsymbol{d}$) for line = $\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$
and $\boldsymbol{n} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$
Now $(\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) \cdot (2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) = 4$
$\therefore\ 4 = \sqrt{6} \cdot \sqrt{11} \cdot \cos\theta$
$\therefore\ \theta = \cos^{-1}\left(\dfrac{4}{\sqrt{66}}\right)$

$= 60.5038$

$\therefore$ angle $\approx 29.50°$

b Point of intersection of line and plane
Line: $(3 - t)\boldsymbol{i} + (-1 + t)\boldsymbol{j} + (-2 + t)\boldsymbol{k}$
$\boldsymbol{r} \cdot \boldsymbol{n} = 7$
$\therefore\ \boldsymbol{r} \cdot (\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}) = 7$
$\therefore\ (3 - t) - 4(-1 + t) + (-2 + t) = 7$
$\therefore\ -4t = 2$
$\therefore\ t = -\dfrac{1}{2}$
$\therefore$ Point of intersection = $\left(\dfrac{7}{2}, -\dfrac{3}{2}, -\dfrac{5}{2}\right)$
Now $\boldsymbol{d}$ for line = $-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
and $\boldsymbol{n}$ for plane = $\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

$\therefore\ \boldsymbol{d}\cdot\boldsymbol{n} = -1 - 4 + 1 = -4$

$\therefore\ -4 = \sqrt{3}\cdot\sqrt{8}\cdot\cos\theta$

$\therefore\ \theta = \cos^{-1}\left(-\dfrac{4}{\sqrt{3}\sqrt{18}}\right)$

$\therefore\ \theta = 122.98°$

$\therefore$ angle is 32.98°

c Point of intersection of line & plane

For line $\boldsymbol{r} = (-1 + 3t)\boldsymbol{i} + (2 - t)\boldsymbol{j} + (-4 + t)\boldsymbol{k}$

Now $\boldsymbol{r}\cdot\boldsymbol{n} = 4$

$\therefore ((-1 + 3t)\boldsymbol{i} + (2 - t)\boldsymbol{j} + (-4 + t)\boldsymbol{k})\cdot(-2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 4$

$\therefore -2(-1 + 3t) + (2 - t) - (-4 + t) = 4$

$\therefore 2 - 6t + 2 - t + 4 - t = 4$

$\therefore -8t = -4$

$\therefore t = \dfrac{1}{2}$

$\therefore$ Point of intersection $= \left(\dfrac{1}{2}, \dfrac{3}{2}, -\dfrac{7}{2}\right)$

Now $\boldsymbol{d}$ for line $= 3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}\cdot 8\boldsymbol{n} = -2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

$\therefore\ \boldsymbol{d}\cdot\boldsymbol{n} = -6 - 1 - 1 = -8$

$\therefore\ -8 = \sqrt{11}\sqrt{6}\cos\theta$

$\therefore\ \theta = \cos^{-1}\left(-\dfrac{8}{\sqrt{66}}\right)$

$\therefore\ \theta = 169.98°$

angle required is 79.98°

d For line $\boldsymbol{r} = (-1 + 2t)\boldsymbol{i} + (-5 - 3t)\boldsymbol{j} + (3 + 2t)\boldsymbol{k}$

Now $\boldsymbol{r}\cdot(3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) = -10$ for plane.

$\therefore 3(-1 + 2t) + 2(-5 - 3t) - (3 + 2t) = -10$

$\therefore -3 + \cancel{6t} - 10 - \cancel{6t} - 3 - 2t = -10$

$\therefore -6 - 2t = 0$

$\therefore t = -3$

$\therefore$ Point of intersection $= (-7, 4, -3)$

Now $\boldsymbol{d}$ for line $= 2\boldsymbol{i} - 3\boldsymbol{i} + 2\boldsymbol{k}$

and $\boldsymbol{n}$ for plane $= 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$\therefore \boldsymbol{d}\cdot\boldsymbol{n} = -2,\ |\boldsymbol{d}| = \sqrt{17}, |\boldsymbol{n}| = \sqrt{14}$

$\therefore -2 = \sqrt{17}\cdot\sqrt{14}\cdot\cos\theta$

$\therefore \theta = \cos^{-1}\left(-\dfrac{2}{\sqrt{17}\sqrt{14}}\right)$

$\therefore \theta = 97.45°$

$\therefore \angle$ is 7.45°

7 **a** Vector equation is

$\boldsymbol{r}\cdot\boldsymbol{n} = k$

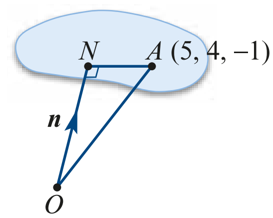
$\therefore (5\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})\cdot(\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}) = k$

$\therefore k = -9.$

$\therefore$ Equation of plane is

$\boldsymbol{r}\cdot(\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}) = -9$

b $\overrightarrow{OA} = 5\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}\quad \therefore |\overrightarrow{OA}| = \sqrt{42}$



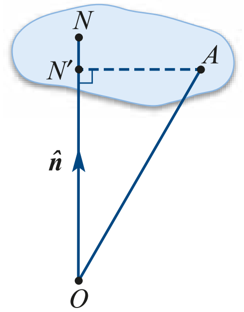
Now $|\overrightarrow{ON}|$ = projection of $\overrightarrow{OA}$ onto $\boldsymbol{n}$

$= \dfrac{\overrightarrow{OA}\cdot\overrightarrow{ON}}{|\overrightarrow{ON}|}$

$= \dfrac{(5\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})\cdot(\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k})}{\sqrt{41}}$

$\therefore$ Distance $= \dfrac{9}{\sqrt{41}}$

8 **a**



$\boldsymbol{n} = 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

A point on the plane, A, is (x, y, z).

Where $2x - y - 2z = 7$.

Let $x = 0,\ y = 0 \quad \therefore \quad z = -\dfrac{7}{2}$

$\therefore$ Point on plane is $\left(0, 0, -\dfrac{7}{2}\right)$

$\therefore \quad \overrightarrow{OA} = -\dfrac{7}{2}\boldsymbol{j}$

$\therefore$ Projection of $\overrightarrow{OA}$ onto $\boldsymbol{n}$

$$= \frac{\overrightarrow{OA} \cdot \boldsymbol{n}}{|\boldsymbol{n}|} = \frac{7}{3}$$

b $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{n} = 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

Vector projection of

$\boldsymbol{a}$ parallel $\boldsymbol{n} = \overrightarrow{ON'}$

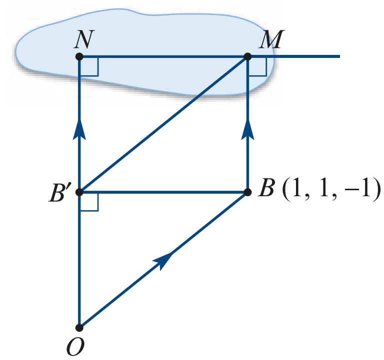
$$= \left(\frac{\boldsymbol{a} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\right) \cdot \boldsymbol{n}$$

$$= \left(\frac{2 - 1 + 2}{4 + 1 + 4}\right)(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$$

$$= \frac{1}{3}(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$$

c $|\overrightarrow{ON'}| = \sqrt{\dfrac{4}{9} + \dfrac{1}{9} + \dfrac{4}{9}} = 1$

d Let $B = (1, 1, -1) \quad \therefore \overrightarrow{OB} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$



Projection of $\overrightarrow{OB}$ parallel

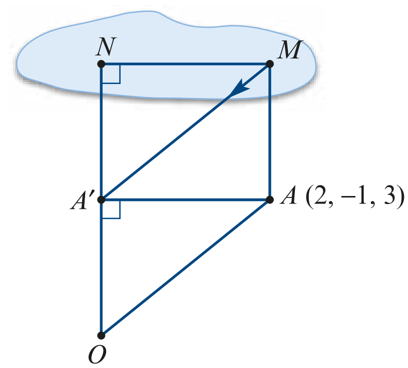
$\boldsymbol{n} = \dfrac{\overrightarrow{OB} \cdot \boldsymbol{n}}{|\boldsymbol{n}|} = \dfrac{1}{3}$

Now $OB'MB$ is a parallelogram

$\therefore |\overrightarrow{BM}| = \dfrac{1}{3}$

9 Let $A = (2, -1, 3) \quad \therefore \overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

$\boldsymbol{n} = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$



So $OA'MA$ is a parallelogram.

Now $|\overrightarrow{OA'}|$ = projection of $\overrightarrow{OA}$ parallel $\boldsymbol{n}$

$$= \frac{\overrightarrow{OA} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$$

$$= \frac{(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) \cdot (-\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})}{3}$$

$$= \frac{-2 - 2 + 6}{3}$$

$$= \frac{2}{3}$$

$\therefore |\overrightarrow{AM}| = \dfrac{2}{3}$

10 a Now $\boldsymbol{r} = (1 + 2t)\boldsymbol{i} - \boldsymbol{j} + (1 - t)\boldsymbol{k}$

Now $\boldsymbol{r} \cdot (3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = 11$

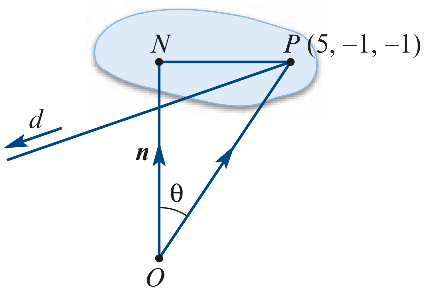
$\therefore 3(1 + 2t) - 2 + 2(1 - t) = 11$

$\therefore t = 2$

$\therefore$ Point of intersection $= 5\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

$\therefore \ P = (5, -1, -1)$

b



Now for line $\boldsymbol{d} = 2\boldsymbol{i} - \boldsymbol{k}$

$\therefore$ Angel between $\boldsymbol{d}$ and $\boldsymbol{n}$, given by

$\boldsymbol{d} \cdot \boldsymbol{n} = |\boldsymbol{d}| \cdot |\boldsymbol{n}| \cdot \cos\theta$

$4 = \sqrt{5} \cdot \sqrt{17} \cdot \cos\theta$

$\therefore \theta = \cos^{-1}\left(\dfrac{4}{\sqrt{5} \cdot \sqrt{17}}\right) = 64.28°$

$\therefore$ angle between line & plane $= 25.7°$

11 a Now $\overrightarrow{OA} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}, \overrightarrow{OB} =$
$3\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\overrightarrow{OC} = \boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$

$\therefore \ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\boldsymbol{i} + 2\boldsymbol{k}$

$\therefore \ \overrightarrow{AB} \times \overrightarrow{AC} = -4\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$

$\therefore \ \boldsymbol{n} = 4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$

Now $\boldsymbol{r} \cdot \boldsymbol{n} = k$

$\therefore (3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) \cdot (4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k})$

$= 12 - 4 + 8$

$= 16$

Therefore equation of plane is

$\therefore \ \boldsymbol{r} \cdot (4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}) = 16$

or $4x + 4y + 4z = 16$

or $x + y + z = 4$

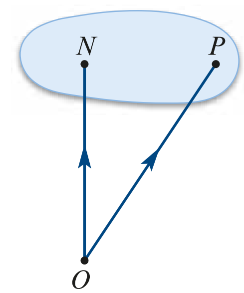
b area Δ ABC $= \dfrac{1}{2} \times |\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \dfrac{1}{2} \times \sqrt{16 + 16 + 16}$

$= \dfrac{1}{2}\sqrt{48} = \dfrac{1}{2} \times 4\sqrt{3}$

$\therefore$ Area $\triangle ABC = 2\sqrt{3}$

c



$\overrightarrow{ON} =$ vector resolute of
$\overrightarrow{OA}$ parallel $\boldsymbol{n}$

$= \left(\dfrac{\overrightarrow{OA} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\right)\boldsymbol{n}$

$= \dfrac{(3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) \cdot (4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k})}{48} \cdot \boldsymbol{n}$

$= \left(\dfrac{12 - 4 + 8}{48}\right) \cdot (4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k})$

$= \dfrac{16}{48}(4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k})$

$\therefore \ \overrightarrow{ON} = \dfrac{4}{3}\boldsymbol{i} + \dfrac{4}{3}\boldsymbol{j} + \dfrac{4}{3}k$

12 a $\boldsymbol{n}_1$ For $\pi_1 = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$

$\boldsymbol{n}_2$ For $\pi_2 = 8\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

Find angle between normals

Now $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2| \cdot \cos\theta$

$\therefore \ -2 = 7 \cdot \cos\theta$

$\therefore \ \theta = \cos^{-1}(-\dfrac{2}{63})$

$\therefore \ \theta = 91.82°$

$\therefore$ angle between planes $= 180 - 91.82°$

$= 88.18°$

b Cartesian equation of π_1 is

$3x + 6y - 2z = 3 \qquad (1)$

Cartesian equation of π_2 is

$8x - 4y + z = 1 \quad (2)$

$(1) + 2 \times (2)$ gives $\quad 19x - 2y = 5$

If $x = \lambda, \quad 19x - 2y = 5$

$\therefore \ -2y = 5 - 19\lambda$

$\therefore \ y = \dfrac{19\lambda - 5}{2}$

$8\lambda - 4(\dfrac{19\lambda - 5}{2}) + 2 = 1$

$\therefore \ 16\lambda - 4(19\lambda - 5) + 2z = 2$

$\therefore \ z = 2 - 16\lambda + 4(19\lambda - 5)$

$= 2 - 16\lambda + 76\lambda - 20$

$\therefore \ 2z = 60\lambda - 18$

$\therefore \ z = 30\lambda - 9$

$\therefore$ A vector equation is

$$\boldsymbol{r} = \lambda \boldsymbol{i} + \left(\frac{19\lambda - 5}{2}\right)\boldsymbol{j} + (30\lambda - 9)\boldsymbol{k}$$

or $\boldsymbol{r} = \dfrac{5}{2}\boldsymbol{i} - 9\boldsymbol{k} + \lambda\left(\boldsymbol{i} + \dfrac{19}{2}\boldsymbol{j} + 30\boldsymbol{k}\right)$

13 a $\overrightarrow{OA} = 2\boldsymbol{j} - \boldsymbol{k}, \overrightarrow{OB} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \overrightarrow{OC} = -\boldsymbol{i} + 2\boldsymbol{k}$

$\overrightarrow{OD} = 2\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$

Now $\overrightarrow{AB} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

and $\overrightarrow{AC} = -\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k}$

$\therefore \ \boldsymbol{n}_1 = \boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k}$

b Now $\overrightarrow{BC} = -2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

and $\overrightarrow{BD} = \boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Now $\overrightarrow{BC} \times \overrightarrow{BD} = 2\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k}$

$\therefore \ \boldsymbol{n}_2 = 2\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k}$

c Now $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2| \cdot \cos\theta$

$\therefore \ 2 - 15 - 21 = \sqrt{35}.\sqrt{62}\cos\theta$

$\therefore \ \theta = \cos^{-1}\left(\dfrac{-34}{\sqrt{85} \cdot \sqrt{62}}\right)$

$\therefore \ \theta = 136.88°$

$\therefore$ Angle between planes $= 43.12°$

14 a Now $\boldsymbol{d}_1 = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}, \boldsymbol{d}_2 = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \ \boldsymbol{d}_1 \times \boldsymbol{d}_2 = -6\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k} =$ line perpendicular to both lines

b Now $\boldsymbol{r}_1 = (2 + t_1)\boldsymbol{i} + (1 - t_1)\boldsymbol{j} + (-2 + 2t_1)\boldsymbol{k}$

and $\boldsymbol{r}_2 = (2 - t_2)\boldsymbol{i} + (1 + 2t_2)\boldsymbol{j} + (-2 + 2t_2)\boldsymbol{k}$

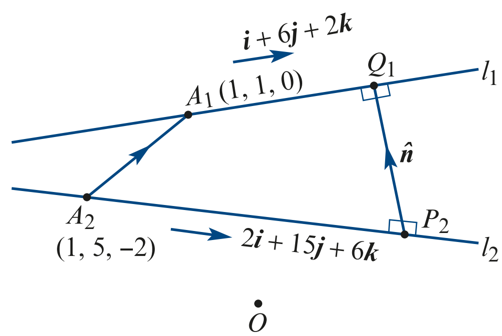
At point of intersection, $t_1 = 0, \ t_2 = 0$

$\therefore$ Point of intersection $= (2, 1, -2)$

$\therefore$ Vector equation required is

$\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t(-6\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$

15 a



Now $\overrightarrow{OA_1} = \boldsymbol{i} + \boldsymbol{j}, \ \overrightarrow{OA_2} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

$\therefore \quad \overrightarrow{A_2A_1} = \overrightarrow{OA_1} - \overrightarrow{OA_2}$

$= -4\boldsymbol{j} + 2\boldsymbol{k}$

Now $\boldsymbol{d}_1 = \boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}, \ \boldsymbol{d}_2 = 2\boldsymbol{i} + 15\boldsymbol{j} + 6\boldsymbol{k}$

Now $\overrightarrow{P_2Q_1} = \boldsymbol{d_1} \times \boldsymbol{d_2} = 6\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} = \boldsymbol{n}$

$\therefore \hat{\boldsymbol{n}} = \dfrac{1}{7}(6\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$

So distance between lines

$= |\overrightarrow{A_2A_1} \cdot \hat{\boldsymbol{n}}|$ (scalar resolute of $\overrightarrow{A_1A_2}$ parallel $\hat{\boldsymbol{n}}$)

$= \left|(-4\boldsymbol{j} + 2\boldsymbol{k}) \cdot \dfrac{1}{7}(6\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})\right|$

$= \left|\dfrac{1}{7}(8 + 6)\right|$

$= 2$

b $\overrightarrow{OA_1} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}_1,\ \overrightarrow{OA_2} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

$\therefore\ \overrightarrow{A_1A_2} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$

Now $\boldsymbol{d}_1 = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k},\ \boldsymbol{d}_2 = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

$\therefore\quad \overrightarrow{P_2Q_1} = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = -3\boldsymbol{i} + 3\boldsymbol{k} = \boldsymbol{n}$

$$\therefore\quad \hat{\boldsymbol{n}} = \frac{1}{3\sqrt{2}}(-3\boldsymbol{i} + 3\boldsymbol{k})$$

So distance between lines

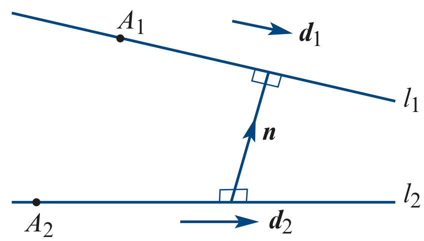
$$= \left|\overrightarrow{A_2A_1} \cdot \hat{\boldsymbol{n}}\right|$$

$$= \left|(\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}) \cdot \frac{1}{3\sqrt{2}}(-3\boldsymbol{i} + 3\boldsymbol{k})\right|$$

$$= \left|\frac{1}{3\sqrt{2}}(-9)\right| = \frac{3}{\sqrt{2}}$$

c $\boldsymbol{r}_1 = (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) + t(-\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$\boldsymbol{r}_2 = (\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) + s(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$



Now $\overrightarrow{OA_1} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k},\ \overrightarrow{OA_2} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

$$\therefore\ \overrightarrow{A_1A_2} = (\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) - (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$$

$$= \boldsymbol{j} - 4\boldsymbol{k}$$

Also $\boldsymbol{d}_1 = -\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k},\ \boldsymbol{d}_2 = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$$\therefore\quad \boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = 6\boldsymbol{i} - 3\boldsymbol{k}$$

Now scalar resolute of $\overrightarrow{A_1A_2}$ parallel $\boldsymbol{n}$

$$= \frac{\overrightarrow{A_1A_2} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$$

$$= \frac{(\boldsymbol{j} - 4\boldsymbol{k}) \cdot (6\boldsymbol{i} - 3\boldsymbol{k})}{\sqrt{45}}$$

$$= \frac{12}{\sqrt{45}}$$

$$= \frac{12}{3\sqrt{5}}$$

$$= \frac{4}{\sqrt{5}}$$

Solutions to short-answer questions

1 $\boldsymbol{r}_1 = (1 + 3\lambda)\boldsymbol{i} + (1 - \lambda)\boldsymbol{j} - \boldsymbol{k}$

$\boldsymbol{r}_2 = (4 + 2\mu)\boldsymbol{i} + (-1 + 3\mu)\boldsymbol{k}$

$\therefore$ At point of intersection

$$\begin{cases} 1 + 3\lambda = 4 + 2\mu \\ 1 - \lambda = 0 \\ -1 + 3\mu = -1 \end{cases}$$

$$\therefore \begin{cases} \lambda = 1 \\ \mu = 0 \end{cases}$$

$\therefore$ Point of intersection has posn vector $4\boldsymbol{i} - \boldsymbol{k}$

2 $\boldsymbol{r}_1 = (1 + 2\lambda)\boldsymbol{i} - \boldsymbol{j} + \lambda\boldsymbol{k}$

$\boldsymbol{r}_2 = (2 + \mu)\boldsymbol{i} + (-1 + \mu)\boldsymbol{j} - \mu\boldsymbol{k}$

$$\text{For intersection} \begin{cases} 1 + 2\lambda = 2 + \mu & (1) \\ -1 + \mu = -1 & (2) \\ \lambda = -\mu & (3) \end{cases}$$

From (2) $\mu = 0$

substitute in (1) $\quad \therefore\ 1 + 2\lambda = 2$

$\therefore\ 2\lambda = 1$

$\therefore\ \lambda = \dfrac{1}{2}$

This contradicts (3)

$\therefore$ No point of intersection

3 Vector equation is $\boldsymbol{r} \cdot (4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}) = k$

But $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) \cdot (4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}) = k$

$\therefore \quad k = 4 + 10 + 18 = 32$

$\therefore$ Cartesian equation is

$4x + 5y + 6z = 32$

4 Equation is $\boldsymbol{r} = -2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i})$

A direction vector is $\boldsymbol{i}$

or $\boldsymbol{r} = (t - 2)\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

5 $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + t(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\therefore\ \overrightarrow{OP} = (1 + t)\boldsymbol{i} + (2 - t)\boldsymbol{j} + 2t\boldsymbol{k}$ (since P is on the line)

$\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

$\therefore\ \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$\therefore\ \overrightarrow{AP} = (-1 + t)\boldsymbol{i} + (1 - t)\boldsymbol{j} + (2t - 3)\boldsymbol{k}$

(For shortest distance vectors are perpendicular)

$\therefore \overrightarrow{AP} \cdot (\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\therefore (-1 + t) - (1 - t) + 2(2t - 3) = 0$

$\therefore -2 + 2t + 4t - 6 = 0$

$\therefore 6t - 8 = 0$

$\therefore t = \frac{4}{3}$

$\therefore$ nearest point is $\frac{7}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{8}{3}\boldsymbol{k}$

$= (\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$

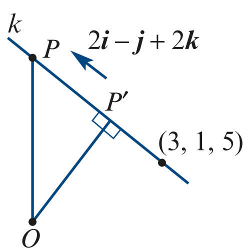
6 Now $\overrightarrow{OP} = (3 + 2\lambda)\boldsymbol{i} + (1 - \lambda)\boldsymbol{j} + (5 + \lambda)\boldsymbol{k}$

Now at P', nearest point on line to O

$\overrightarrow{OP} \cdot \boldsymbol{d} = 0$

$\therefore\ 2(3 + 2\lambda) - (1 - \lambda)(5 + \lambda) = 0$

$\therefore\ \lambda = -\frac{5}{3}$



$\therefore OP' = -\frac{1}{3}\boldsymbol{i} + \frac{8}{3}\boldsymbol{j} + \frac{10}{3}\boldsymbol{k}$

$\therefore$ distance $= \sqrt{\frac{1}{9} + \frac{64}{9} + \frac{100}{9}}$

$= \frac{\sqrt{165}}{3}$

7 $\boldsymbol{r} = (1 + 2t)\boldsymbol{i} + t\boldsymbol{j} + (1 - 3t)\boldsymbol{k}$

$\therefore$ At point of intersection $((1 + 2t)\boldsymbol{i} + t\boldsymbol{j} + (1 - 3t)\boldsymbol{k}) \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) = 13$

$\therefore (1 + 2t) - 2t + 3(1 - 3t) = 13$

$\therefore t = -1$

$\therefore$ Point of intersection $= (1 - 2)\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k} = (-1, -1, 4)$

8 $\boldsymbol{n}$ is perpendicular $(8\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) \times (7\boldsymbol{i} - 2\boldsymbol{j}) = \boldsymbol{n}$

$\therefore \quad \boldsymbol{n} = 2\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}$

Using the cross product =

$$\begin{bmatrix} i & j & k \\ 8 & -3 & 1 \\ 7 & -2 & 0 \end{bmatrix} = \begin{vmatrix} -3 & 1 \\ -2 & 0 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} 8 & 1 \\ 7 & 0 \end{vmatrix}\boldsymbol{j}$$

$$+ \begin{vmatrix} 8 & -3 \\ 7 & -2 \end{vmatrix}\boldsymbol{k}$$

$$= 2\boldsymbol{i} - (-7\boldsymbol{j}) + (-16 + 21)\boldsymbol{k}$$

$$= 2\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}$$

9 $\boldsymbol{d} = 6\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k}$

$\therefore \boldsymbol{r} = -\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} + \lambda(6\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k})$

$\therefore \boldsymbol{r} = (-1 + 6\lambda)\boldsymbol{i} + (-3 + 3\lambda)j$

$\quad + (-3 + 9\lambda)\boldsymbol{k}$

For nearest point on the line to 0

$(6\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k}) \cdot \boldsymbol{r} = 0$

$\therefore 6(-1 + 6\lambda) + 3(-3 + 3\lambda) +$
$9(-3 + 9\lambda) = 0$

$\therefore \lambda = \frac{1}{3}$

$\therefore \overrightarrow{OP} = (-1 + 6 \times \frac{1}{3})\boldsymbol{i} + (-3 + 3 \times \frac{1}{3})\boldsymbol{j}$

$\quad + (-3 + 9 \times \frac{1}{3})\boldsymbol{k}$

$\therefore \overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} \quad \therefore P = (1, -2, 0)$

10 $\boldsymbol{r}_1 = (3 + 2\lambda)\boldsymbol{i} + (4 - \lambda)\boldsymbol{j} + (1 + \lambda)\boldsymbol{k}$

$\boldsymbol{r}_2 = (1 + \mu)\boldsymbol{i} + 5\boldsymbol{j} + (7 + \mu)\boldsymbol{k}$

If point of intersection

$$\begin{cases} 3 + 2\lambda = 1 + \mu & (1) \\ 4 - \lambda = 5 & (2) \\ 1 + \lambda = 7 + \mu & (2) \end{cases}$$

From (2) $\lambda = -1$

In (1) $1 = 1 + \mu \quad \therefore \ \mu = 0$

In (2) $0 = 7 + \mu \quad \therefore \ \mu = -7$

$\therefore$ contradiction $\therefore$ skew lines

Now $\boldsymbol{d}_1 = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ & $\boldsymbol{d}_2 = \boldsymbol{i} + \boldsymbol{k}$

$\therefore \ \boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = |\boldsymbol{d}_1| . |\boldsymbol{d}_2| . \cos\theta$

$\therefore \ 3 = \sqrt{6}. \sqrt{2} \cos\theta$

$$\therefore \ \cos\theta = \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

11 $\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j}$, $\overrightarrow{OQ} = 3\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$, $\overrightarrow{OR} = -\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \ \overrightarrow{PQ} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$

and $\overrightarrow{PR} = -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \ \overrightarrow{PQ} \times \overrightarrow{PR} = 2\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k} = \boldsymbol{n}$

$\therefore$ Plane equation $\boldsymbol{r} \cdot (2\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}) = k$

Now $(\boldsymbol{i} - 2\boldsymbol{j}) \cdot (2\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}) = 18$

$\therefore \ \boldsymbol{r} \cdot (2\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}) = 18$

$\therefore$ Equation is $2x - 8y + 5z = 18$

12 $\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}, \overrightarrow{OQ} = -2\boldsymbol{i} + 5\boldsymbol{j}, \overrightarrow{OR} = -4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \ \overrightarrow{PQ} = -3\boldsymbol{i} + 7\boldsymbol{j} - \boldsymbol{k}, \ \overrightarrow{PR} = -5\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$

Now $\overrightarrow{PQ} \times \overrightarrow{PR} = \boldsymbol{n}$

$$= 12\boldsymbol{i} + 8\boldsymbol{j} + 20\boldsymbol{k}$$

$\therefore$ Equation of plane is $\boldsymbol{r} \cdot (12\boldsymbol{i} + 8\boldsymbol{j} + 20\boldsymbol{k}) = k$

But $k = (-2\boldsymbol{i} + 5\boldsymbol{j}) \cdot (12\boldsymbol{i} + 8\boldsymbol{j} + 40\boldsymbol{k})$

$\therefore \ k = -24 + 40 = 16$

$\therefore \ \boldsymbol{r} \cdot (12\boldsymbol{i} + 8\boldsymbol{j} + 20\boldsymbol{k}) = 16$

$\therefore$ Cartesian equation in $12x + 8y + 20z = 16$

or $3x + 2y + 5 = 4$

13 $\overrightarrow{OA} = -\boldsymbol{i} + 2\boldsymbol{j}, \ \overrightarrow{OB} = 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \ \overrightarrow{OC} = \boldsymbol{i} + 3\boldsymbol{k}$

$\therefore \ \overrightarrow{AB} = 4\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \overrightarrow{AC} = 2\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \boldsymbol{n}$

$\therefore \ \boldsymbol{n} = -\boldsymbol{i} - 10\boldsymbol{j} - 6\boldsymbol{k}$

$\therefore$ Vector equation is $\boldsymbol{r} \cdot (-\boldsymbol{i} - 10\boldsymbol{j} - 6\boldsymbol{k}) = k$

use $\boldsymbol{r} = \boldsymbol{i} + 3\boldsymbol{k}$

$\therefore\ k = (\boldsymbol{i} + 3\boldsymbol{k})(-\boldsymbol{i} - 10\boldsymbol{j} - 6\boldsymbol{k})$

$\therefore\ k = -1 - 18 = -19$

Vector equation is $\boldsymbol{r} \cdot (-\boldsymbol{i} - 10\boldsymbol{j} - 6\boldsymbol{k}) = -19$

or $\boldsymbol{r} \cdot (\boldsymbol{i} + 10\boldsymbol{j} + 6\boldsymbol{k}) = 19$

and cartesian equation is $-x - 10y - 6z = -19$

or $x + 10y + 6z = 19$

14 a $\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \overrightarrow{OB} = \boldsymbol{j} + 2\boldsymbol{k}$

$\therefore\ \boldsymbol{n} = \overrightarrow{OA} \times \overrightarrow{OB} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

$\therefore$ Equation is $\boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = k$

$\therefore\ k = 0$

$\therefore x(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 0$

$\therefore$ Cartesian equation is $x - 2y + z = 0$

b Area $\triangle OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

$= \frac{1}{2} \times \sqrt{6}$

$= \frac{\sqrt{6}}{2}$

c Now $\overrightarrow{OC} = -2\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k},\ C = (-2, 2, 6)$

Now $x - 2y + z = -2 - 4 + 6 = 0$

$\therefore$ C is on plane

Now $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (-2\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$

$- (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$= -3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$

and $\overrightarrow{OB} = \boldsymbol{j} + 2\boldsymbol{k}$

Now $\overrightarrow{AC}$ passes through $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{d} = -3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$

$\therefore\ \boldsymbol{r}_1 = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + \lambda(-3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k})$

and vector equation of $\overrightarrow{OB} = \boldsymbol{r}_2 = \mu(\boldsymbol{j} + 2\boldsymbol{k})$

$$\therefore \begin{cases} \boldsymbol{r}_1 = (1 - 3\lambda)\boldsymbol{i} + (1 + \lambda)\boldsymbol{j} + (1 + 5\lambda)\boldsymbol{k} \\ \boldsymbol{r}_2 = \mu\boldsymbol{j} + 2\mu\boldsymbol{k} \end{cases}$$

$\therefore$ At point of intersection

$$\begin{cases} 1 - 3\lambda = 0 & (1) \\ 1 + \lambda = \mu & (2) \\ 1 + 5\lambda = 2\mu & (3) \end{cases}$$

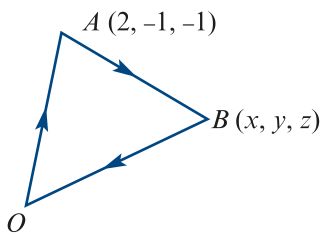
From (1) $\lambda = \frac{1}{3}$

Sub in (2) $\mu = \frac{4}{3}$

Check in (3) Left side $= 1 + \frac{5}{3} = \frac{8}{3}$, RS $= \frac{8}{3}$

$\therefore$ Point of intersection $= (0, \frac{4}{3}, \frac{8}{3})$

15



In $x + y + z = 0$ plane

$\therefore\ B = (x, y, -x - y)$

Now $|\overrightarrow{OA}| = \sqrt{6}$

So $|\overrightarrow{OB}| = \sqrt{6}$

$\therefore\ x^2 + y^2 + (-x - y)^2 = 6$

$\therefore\ x^2 + y^2 + x^2 + z \times y + y^2 = 6$

$\therefore\ x^2 + y^2 + xy = 3 \qquad (1)$

Now $\overrightarrow{AB} = (x - 2)\boldsymbol{i} + (y + 1)\boldsymbol{j} + (z + 1)\boldsymbol{k}$

Now $|\overrightarrow{AB}| = \sqrt{6}$(since equilateral $\triangle$)

$\therefore\ (x - 2)^2 + (y + 1)^2 + (z + 1)^2 = 6$

But $z = -x - y$

$\therefore\ x^2 - 4x + 4 + y^2 + 2y + 1 + x^2 + y^2 + 2xy - 2x - 2y + 1 = 6$

$\therefore\ x^2 + y^2 - 3x + xy = 0 \quad (2)$

Now (2) – (1) gives $-3x = -3$

$\therefore\ x = 1$

Sub in (1) $\quad \therefore\ 1 + y^2 + y = 3$

$\therefore\ y^2 + y - 2 = 0$

$\therefore\ (y + 2)(y - 1) = 0$

$\therefore\ y = -2$ or $y = 1$

If $x = 1,\ y = -2,\ z = 1$

If $x = 1,\ y = 1,\ z = -2$

$\therefore\ B = (1, -2, 1)$ or $(1, 1, -2)$

16 Show $(1, 0, 0), (2, 1, 0), (3, 2.1)$ and $(4, 3, 2)$ are coplanar

Equation of plane is $ax + by + cz = d$

Let $a = 1\ \therefore\ x + by + cz = d$

Using $(1, 0, 0)\ \therefore\ d = 1$

Using $(2, 1, 0)\ \therefore\ 2 + b = d$

$$\therefore\ 2 + b = 1$$

$$\therefore\ b = -1$$

Using $(3, 2, 1)\ \therefore\ 3 + 2b + c = d$

$$\text{But } b = -1, d = 1$$

$$\therefore\ 3 - 2 + c = 1$$

$$\therefore\ c = 0$$

$\therefore$ Equation of plane through these points is

$x - y = 1$

Check for $(4, 3, 2)$ here also $x - y = 1$

$\therefore$ All 4 points are coplanar

17 Assume D is not collinear with A, B and C. Then the points A, B and D define a plane. C must also be on this plane because it it is collinear with A and B.

18 Now $\quad \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{b} \times (-\boldsymbol{a} - \boldsymbol{b})$ since $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$

$$= -\boldsymbol{b} \times \boldsymbol{a} - \boldsymbol{b} \times \boldsymbol{b}$$

$$= -\boldsymbol{b} \times \boldsymbol{a} \quad \text{since } \boldsymbol{b} \times \boldsymbol{b} = 0$$

$$= \boldsymbol{a} \times \boldsymbol{b}$$

Now $\quad \boldsymbol{c} \times \boldsymbol{a} = \boldsymbol{c} \times (-\boldsymbol{b} - \boldsymbol{c})$

$$= -\boldsymbol{c} \times \boldsymbol{b} - \boldsymbol{c} \cdot \boldsymbol{c}$$

$$= -\boldsymbol{c} \times \boldsymbol{b} \quad \text{since } \boldsymbol{c} \cdot \boldsymbol{c} = 0$$

$$= \boldsymbol{b} \times \boldsymbol{c}$$

$\therefore\ \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$

Solutions to multiple-choice questions

1 C Sides are $\boldsymbol{b}-\boldsymbol{a}, \boldsymbol{c}-\boldsymbol{a}$ and $\boldsymbol{c}-\boldsymbol{b}$

Now area $= \frac{1}{2}|(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})|$

or $\frac{1}{2}|(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{b})|$

2 D $\boldsymbol{r}\cdot(\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}) = 5$

$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{6}}(\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})$

Distance $= \frac{5}{\sqrt{6}}$

3 D $X = (1,0,0)$, $Y = (0,2,0)$ and $Z = (0,0,3)$ are intercepts

For X : $x+2y+3z = 1$,

For Y : $x+2y+3z = 4$

For Z : $x+2y+3z = 9$

so not (A), (B), or (C)

For X : $6x+3y+2z = 6$, Y,

$6x+3y+2z = 6$

For Z : $6x+3y+2z = 6$

(D) is correct answer

4 D Now $\boldsymbol{r} = (5+2\lambda)\boldsymbol{i}+(-3-2\lambda)\boldsymbol{j}+(1-\lambda)\boldsymbol{k}$

At point of intersection

$2(5+2\lambda)+(-3-2\lambda)-3(1-\lambda) = -6$

Since $\boldsymbol{r}\cdot(2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}) = -6$

$\therefore 10+4\lambda-3-2\lambda-3+3\lambda = -6$

$\therefore 5\lambda = -10 \ \therefore\ \lambda = -2$

$\therefore$ Point of intersection $= (1,1,3)$

5 D $P = (1,5)$, Let $Q = (x,x)$

Gradient of $\overrightarrow{PQ} = -1$

Equation of $\overrightarrow{PQ}$ is $y = -x+6$

$\therefore$ At Q : $x = -x+6$

$\therefore\ x = 3 \quad \therefore\ Q = (3,3)$

$d = \sqrt{4+4}$

$= \sqrt{8}$

$= 2\sqrt{2}$

6 C For a line passing through $(2,0,1)$ and $(3,3,3)$, $d = \boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$

$\therefore \boldsymbol{r} = 2\boldsymbol{i}+\boldsymbol{k}+\lambda(\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k})$

$= (2+\lambda)\boldsymbol{i}+3\lambda\boldsymbol{j}+(1+2\lambda)\boldsymbol{k}$

or $r = 3\boldsymbol{i}+3\boldsymbol{j}+3\boldsymbol{k}+\lambda(\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k})$

$= (3+\lambda)\boldsymbol{i}+(3+3\lambda)+(3+2\lambda)\boldsymbol{k}$

$\therefore$ (A) is valid

(B) is also valid (C) is not valid

$\therefore$ (C) is the correct answer

7 A $\begin{cases}\text{For (A)}(0,0,1),(1,1,1)(2,0,0)\\ x-y+2z- = 2\end{cases}$

$\therefore$ Correct answer is (A)

8 D

At intersection $\begin{cases}2x+y+z = 6\\ x+z = 0 \ \therefore\ z = -x\end{cases}$

$\therefore\ 2x+y-x = 6$

$\therefore\ x+y = 6$

Let $x = \lambda$

$\therefore$ points of intersection $= (\lambda, 6-\lambda, -\lambda)$

Equation of line $= 6\boldsymbol{j}+\lambda(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$

$\therefore$ Correct answer is (D)

9 E Plane is $\boldsymbol{r}\cdot(\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}) = 5$

where $\boldsymbol{n} = \boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$

$\therefore\ \hat{\boldsymbol{n}} = \frac{1}{3}(\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k})$

$\therefore$ distance $= \frac{5}{3}\ \therefore$ (E)

10 C $\overrightarrow{OA} = \boldsymbol{k},\ \overrightarrow{OB} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k},\ \overrightarrow{OC} = 2\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore\ \overrightarrow{AB} = \boldsymbol{i} + \boldsymbol{j}, \overrightarrow{AC} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

$\therefore$ Area $\Delta = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$\therefore$ area $= \frac{1}{2} \times \sqrt{3}$

$= \frac{\sqrt{3}}{2}$ $\quad \therefore$ (C) is correct

Solutions to extended-response questions

1 $\boldsymbol{r}_1 = (1 + t_1)\boldsymbol{i} + (1 - t_1)\boldsymbol{j} + (-2 + 2t_1)\boldsymbol{k}$

$\boldsymbol{r}_2 = (2 - t_2)\boldsymbol{i} + (1 + 2t_2)\boldsymbol{j} + (4 + 2t_2)\boldsymbol{k}$

a At point of intersection $\begin{cases} 1 + t_1 = 2 - t_2 & (1) \\ 1 - t_1 = 1 + 2t_2 & (2) \\ -2 + 2t_1 = 4 + 2t_2 & (3) \end{cases}$

(1) + (2) gives $2 = 3 + t_2$

$\therefore\ t_2 = -1$

Substitute in (1) $\therefore\ 1 + t_1 = 3$

$\therefore\ t_1 = 2$

Check in (3) Left side $= -2 + 4 = 2$

Right side $= 4 + 2 \times -1 = 2$ $\quad\therefore$ Point of intersection exists

$\therefore$ Point of intersection $= (3, -1, 2) = P$

b $\boldsymbol{r}_3 = t_3(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

From **a** $\overrightarrow{OP} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

and for a point A on the line $\overrightarrow{OA} = t_3\boldsymbol{i} - t_3\boldsymbol{j} + 2t_3\boldsymbol{k}$

$\therefore\ \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (3 - t_3)\boldsymbol{i} + (-1 + t_3)\boldsymbol{j} + (2 - 2t_3)\boldsymbol{k}$

If AP is perpendicular to the line $\overrightarrow{AP} \cdot (\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\overrightarrow{AP} \cdot (\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$

$(3 - t_3) - (t_3 - 1) + 2(2 - 2t_3) = 0$

$8 - 6t_3 = 0$

$t_3 = \dfrac{4}{3}$

$\therefore\ \overrightarrow{AP} = \left(3 - \dfrac{4}{3}\right)\boldsymbol{i} + \left(-1 + \dfrac{4}{3}\right)\boldsymbol{j} + \left(2 - \dfrac{8}{3}\right)\boldsymbol{k}$

$\therefore\ \overrightarrow{AP} = \dfrac{5}{3}\boldsymbol{i} + \dfrac{1}{3}\boldsymbol{j} - \dfrac{2}{3}\boldsymbol{k}$

So distance $= \sqrt{\dfrac{25}{9} + \dfrac{1}{9} + \dfrac{4}{9}} = \sqrt{\dfrac{30}{9}}$

$= \dfrac{\sqrt{30}}{3}$

2 **a** $\overrightarrow{OC} = 3\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k}$

Now Π_1 is $\boldsymbol{r} \cdot (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 6$

Now $\overrightarrow{OC} \cdot (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$$= (3\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k}) \cdot (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$$

$$= 9 + 4 - 7$$

$$= 6$$

$\therefore$ C is on plane, Π_1

b $\overrightarrow{OB} = -\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

Now M is on Π_1 nearest to B

$\therefore\ \overrightarrow{MB} = p(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

Now $\overrightarrow{OM} = \overrightarrow{OB} - \overrightarrow{MB}$

$$\therefore\ \overrightarrow{OM} = -\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} - 3p\boldsymbol{i} - p\boldsymbol{j} + p\boldsymbol{k}$$

$$\therefore\ \overrightarrow{OM} = (-1 - 3p)\boldsymbol{i} + (1 - p)\boldsymbol{j} + (3 + p)\boldsymbol{k}$$

But $\boldsymbol{r} \cdot (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 6$

$\therefore\ 3(-1 - 3p) + (1 - p) - (3 + p) = 6$

$\therefore\ p = -1$

$\therefore\ \overrightarrow{MB} = -3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

and $\overrightarrow{OM} = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

Now $\overrightarrow{MA} = \overrightarrow{OA} - \overrightarrow{OM}$

$$= (5\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) - (2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$$

$$= 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$$

$$= -\overrightarrow{MB}$$

$\therefore B$ is a reflection of A in plane Π_1

c Now $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$$= 2\boldsymbol{i} - \boldsymbol{j} - 6\boldsymbol{k}$$

$$\overrightarrow{CM} = \overrightarrow{OM} - \overrightarrow{OC}$$

$$= -\boldsymbol{i} - 2\boldsymbol{j} - 5\boldsymbol{k}$$

$\therefore$ Scalar resolute of $\overrightarrow{CA}$ parallel $\overrightarrow{CM}$ (length of projection required)

$$= \frac{\overrightarrow{CA} \cdot \overrightarrow{CM}}{\left|\overrightarrow{CM}\right|}$$

$$= \frac{30}{\sqrt{30}}$$

$$= \sqrt{30}$$

d Π_2 is $12x - 4y + 3z = k$

$\therefore\ \boldsymbol{r} \cdot (12\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}) = k$

Now Angle between normals $\boldsymbol{n}_1 = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{n}_2 = 12\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$

Now $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1|\ |\boldsymbol{n}_2| \cdot \cos\theta$

Now $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 36 - 4 - 3 = 29$

$|\boldsymbol{n}_1| = \sqrt{11}, \quad \boldsymbol{m}_2 = \sqrt{144 + 16 + 9}$

$= \sqrt{169}$

$= 13$

$\therefore\ 29 = 13\sqrt{11}\cos\theta$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{29}{13\sqrt{11}}\right)$

$\therefore\ \theta = 47.73°$

$\therefore$ Angle between Π_1 and Π_2

$\therefore$ Angle required is $47.73°$

e Now $\boldsymbol{n} = 12\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore\ \hat{n} = \dfrac{1}{13}(12\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k})$

Now $\overrightarrow{OC} = 3\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k}$, $\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ (P is the closest to C in the plane)

Now $\overrightarrow{PC} = (3 - x)\boldsymbol{i} + (4 - y)\boldsymbol{j} + (7 - z)\boldsymbol{k}$

Now $d = \left|\overrightarrow{PC} \cdot \hat{\boldsymbol{n}}\right| \quad \therefore\ 3 = \left|\overrightarrow{PC} \cdot \dfrac{1}{13}(12\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k})\right|$

$\therefore\ 39 = |12(3 - x) - 4(4 - y) + 3(7 - z)|$

$\therefore\ 39 = |36 - 12x - 16 + 4y + 21 - 3z|$

But $12x - 4y + 3z = k$

$\therefore\ 39 = |41 - \boldsymbol{k}|$

$\therefore\ 39 = 41 - k \quad \text{or} \quad 39 = 41 - \boldsymbol{k}$

$\therefore\ k = 2 \quad \text{or} \quad k = 80$

3 a $l_1 \quad \boldsymbol{r}_1 = (3 + 5s)\boldsymbol{i} + (2 + 4t)\boldsymbol{j} + (1 + 3t)\boldsymbol{k}$

$l_2 \quad \boldsymbol{r}_2 = (16 + 3s)\boldsymbol{i} + (-10 + 2s)\boldsymbol{j} + (2 - s)\boldsymbol{k}$

If Point of intersection $\begin{cases} 3 + 5t = 16 + 3s & (1) \\ 2 + 4t = -10 + 2s & (2) \\ 1 + 3t = 2 - s & (3) \end{cases}$

From (1) & (2) $t = -31$, $s = -56$

Check in (3) Left side $= 1 - 93$, Right side $= 58$

$\therefore$ No point of intersection $\quad\therefore$ skew lines

b For l_1, $\boldsymbol{d}_1 = 5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore\ \boldsymbol{d}_1 \cdot (5\boldsymbol{i} - 7\boldsymbol{j} + \boldsymbol{k}) = 25 - 28 + 3$

$= 0$

$\therefore\ l_1$ and $\boldsymbol{n}$ are perpendicular

For $l_2, \boldsymbol{d}_2 = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$\therefore\ \boldsymbol{d}_2 \cdot (5\boldsymbol{i} - 7\boldsymbol{j} + \boldsymbol{k}) = 15 - 14 - 1$

$= 0$

$\therefore\ l_2$ and $\boldsymbol{n}$ are perpendicular.

c $l_3 \quad \boldsymbol{r}_3 = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + \lambda(5\boldsymbol{i} - 7\boldsymbol{j} + \boldsymbol{k})$

d Now $\boldsymbol{r}_3 = (3 + 5\lambda)\boldsymbol{i} + (2 - 7\lambda)\boldsymbol{j} + (1 + \lambda)\boldsymbol{k}$

and $\boldsymbol{r}_2 = (16 + 3s)\boldsymbol{i} + (-10 + 2s)\boldsymbol{j} + (2 - s)\boldsymbol{k}$

At point of intersection $\begin{cases} 3 + 5\lambda = 16 + 3s & (1) \\ 2 - 7\lambda = -10 + 2s & (2) \\ 1 + \lambda = 2 - s & (3) \end{cases}$

$\therefore \lambda = 2,\ s = -1$

$\therefore$ point of intersection $= (13, -12, 3) = B$

$\left|\overrightarrow{AB}\right| = \sqrt{10^2 + 14^2 + 2^2}$

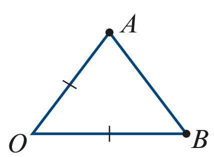
$A = (3, 2, 1)$

$= 10\sqrt{3}$

4 a $\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$

$\overrightarrow{OB} = 6\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \overrightarrow{AB} = 2\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$



$\therefore\ \left|\overrightarrow{OA}\right| = \sqrt{16 + 9 + 16} = \sqrt{41}$

$\left|\overrightarrow{OB}\right| = \sqrt{36 + 1 + 4} = \sqrt{41}$

$\therefore\ \left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right|\ \therefore\ \Delta\, OAB$ is isosceles

b Now for plane , $\boldsymbol{n} = \overrightarrow{OA} \times \overrightarrow{OB}$

$\therefore\ \boldsymbol{n} = 2\boldsymbol{i} + 16\boldsymbol{j} - 14\boldsymbol{k}$

$\therefore$ plane OAB has equation

$\boldsymbol{r} \cdot (2\boldsymbol{i} + 16\boldsymbol{j} - 14\boldsymbol{k}) = k$
where $(4\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) \cdot (2\boldsymbol{i} + 16\boldsymbol{j} - 14\boldsymbol{k})$

$= 8 + 48 - 56$

$= 0$

$\therefore\ \boldsymbol{r} \cdot (2\boldsymbol{i} + 16\boldsymbol{j} - 14\boldsymbol{k}) = 0$ is plane OAB

Check D

$(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \cdot (2\boldsymbol{i} + 16\boldsymbol{j} - 14\boldsymbol{k})$

$= -2 + 16 - 14$

$= 0$

$\therefore\ D$ is also in plane OAB.

c $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -\boldsymbol{i} - 8\boldsymbol{j} + 7\boldsymbol{k}$
Now $\overrightarrow{CD} = -2\boldsymbol{n}$ $\quad\therefore\ \overrightarrow{CD}$ is perpendicular to OAB
as parallel to normal vector to plane OAB

d $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -4\boldsymbol{i} + 6\boldsymbol{j} - 10\boldsymbol{k}$
Now $\overrightarrow{AC} \cdot n = (-4\boldsymbol{i} + 6\boldsymbol{j} - 10\boldsymbol{k}) \cdot (2\boldsymbol{i} + 16\boldsymbol{j} - 14\boldsymbol{k})$

$= -8 + 96 + 140$

$= 228$

$\therefore\ 228 = \sqrt{16 + 36 + 100} \cdot \sqrt{4 + 256 + 196} \cdot \cos\theta$

$\therefore\ \cos\theta = \dfrac{228}{\sqrt{152} \cdot \sqrt{456}} = \dfrac{\sqrt{3}}{2} \quad \therefore\ \theta = 60°$

5 a Π_1 cartesian equation $y + z = 0$
or $\boldsymbol{r} \cdot (\boldsymbol{j} + \boldsymbol{k}) = 0$
Equation of $\boldsymbol{l}$ $\quad \boldsymbol{r} = 5\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$= (5 + 2t)\boldsymbol{i} + (2 - t)\boldsymbol{j} + (2 + 3t)\boldsymbol{k}$

$\therefore$ At point of intersection $(5 + 2t) \cdot 0 + (2 - t) \cdot 1 + (2 + 3t) \cdot 1 = 0$

$\therefore\ 2 - t + 2 + 3t = 0$

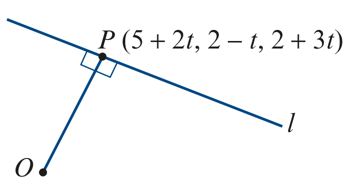
$\therefore\ 4 + 2t = 0$

$\therefore\ t = -2$

$\therefore$ point of intersection is $(1, 4, -4)$

$\therefore$ position vector of point of intersection is $\boldsymbol{i}+4\boldsymbol{j}-4\boldsymbol{k}$

b



For perpendicular $\overrightarrow{OP} \cdot \boldsymbol{d} = 0$

$\therefore\ (5 + 2t)\dot{2} - (2 - t) + 3(2 + 3t) = 0$

$\therefore\ 10 + 4t - 2 + t + 6 + 9t = 0$

$\therefore\ 14t + 14 = 0$

$\therefore\ t = -1$

$\therefore\ P = (3, 3, -1)$

$\therefore\ \left|\overrightarrow{OP}\right| = \sqrt{9 + 9 + 1} = \sqrt{19}$

c π_2 includes $(0, 0, 0)$ and $\boldsymbol{r} = 5\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$
Now $\boldsymbol{n} = (3\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) \times (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) = 8\boldsymbol{i} - 11\boldsymbol{j} - 9\boldsymbol{k}$
$\therefore\ \boldsymbol{r} \cdot (8\boldsymbol{i} - 11\boldsymbol{j} - 9\boldsymbol{k}) = k$
But plane must include $(5, 2, 2)$ $\quad \therefore\ k = 40 - 22 - 18 = 0$
$\therefore$ Equation is $8x - 11y - 9z = 0$

d π_1 equation is $\boldsymbol{r} \cdot (\boldsymbol{j} + \boldsymbol{k}) = 0$
π_2 equation is $\boldsymbol{r} \cdot (8\boldsymbol{i} - 11\boldsymbol{j} - 9\boldsymbol{k}) = 0$
$\therefore\ \boldsymbol{n}_1 = \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{n}_2 = 8\boldsymbol{i} - 11\boldsymbol{j} - 9\boldsymbol{k}$
$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = -11 - 9 = -20, |\boldsymbol{n}_1| = \sqrt{2}$
and
$|\boldsymbol{n}_2| = \sqrt{266}$ $\therefore$ Find angle between normals
$-20 = \sqrt{2} \cdot \sqrt{266} \cos\theta$

$\therefore\ \theta = \cos^{-1}\left(-\frac{20}{\sqrt{532}}\right) = 150.125°$
$\therefore$ angle is 29.88°

6 a If $a = b = 0$ then $a^2 + b^2 = 0$
Conversely if $a^2 + b^2 = 0$. We know $a^2 > 0$ and $b^2 > 0$ for all non-zero values of a and b. Hence $a^2 + b^2 = 0 \Rightarrow a = b = 0$

b If (x, y, z) is on the z-axis if and only if $x = y = 0$. Hence (x, y, z) is on the z-axis if and only if $x^2 + y^2 = 0$.

c From the equations
$2y - 8 = x - 3 \Leftrightarrow x - 2y + 5 = 0$ and $z + 1 = x - 3 \Leftrightarrow x - z - 4 = 0$
Hence from **a** $(x - 2y + 5)^2 + (x - z - 4)^2 = 0$
$\Leftrightarrow x - 2y + 5 = 0$ and $x - z - 4 = 0$
$\Leftrightarrow 2y - 8 = x - 3 = z + 1$

d $y - \frac{x}{2} - 3 = 0$ and $4z + 5 - \frac{x}{2} = 0$
Hence, $\left(y - \frac{x}{2} - 3\right)^2 + \left(4z + 5 - \frac{x}{2}\right)^2 = 0$ is a quadratic equation for the line.

e $x = 2 + t, y = 3 - t, z = 5t$

Therefore,

$t = x - 2 = 3 - y = \dfrac{z}{5}$

Hence $(x + y - 5)^2 + (x - \dfrac{z}{5} - 2)^2 = 0$ is cartesian quadratic equation for the line.

7 **a** Let P be a point on the line $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{d}$

$|\overrightarrow{OP}|^2 = (\boldsymbol{a} + t\boldsymbol{d}) \cdot (\boldsymbol{a} + t\boldsymbol{d})$

$= \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{d}t + t^2\boldsymbol{d} \cdot \boldsymbol{d}$

$= |\boldsymbol{d}|^2t^2 + 2\boldsymbol{a} \cdot \boldsymbol{d}t + |\boldsymbol{a}|^2$

b Quadratic in t and the minimum occurs when $t = -\dfrac{\boldsymbol{a} \cdot \boldsymbol{d}}{|\boldsymbol{d}|^2}$

c $\boldsymbol{r}\left(-\dfrac{\boldsymbol{a} \cdot \boldsymbol{d}}{|\boldsymbol{d}|^2}\right)$ is the position vector correspomding to this value of t.

d First assume $\overrightarrow{OP}$ is perpendicular to $\boldsymbol{d}$.

This implies $\overrightarrow{OP} \cdot \boldsymbol{d} = 0$

That is, $(\boldsymbol{a} + t\boldsymbol{d}) \cdot \boldsymbol{d} = 0$

$\Leftrightarrow \boldsymbol{a} \cdot \boldsymbol{d} + t\boldsymbol{d} \cdot \boldsymbol{d} = 0$

$\Leftrightarrow t = -\dfrac{\boldsymbol{a} \cdot \boldsymbol{d}}{\boldsymbol{d} \cdot \boldsymbol{d}} = -\dfrac{\boldsymbol{a} \cdot \boldsymbol{d}}{|\boldsymbol{d}|^2}$

The converse is clear by working back through the steps.

8 **a** Let X be a point on the line AB.

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\boldsymbol{a} + \boldsymbol{b}$

$\overrightarrow{AX} = \lambda\overrightarrow{AB}$ for some $\mu \in \mathbb{R}\backslash\{0\}$

$\therefore \overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$= \boldsymbol{a} + \mu(-\boldsymbol{a} + \boldsymbol{b})$

$= (1 - \mu)\boldsymbol{a} + \mu\boldsymbol{b}$

Let $\lambda = 1 - \mu$

$\therefore \overrightarrow{OX} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}$

b Let Y be a point on the line CX.
$\overrightarrow{OY} = \overrightarrow{OX} + k\overrightarrow{XC}$, for some $k \in \mathbb{R}\backslash\{0\}$

$= (\lambda \boldsymbol{a} + \mu \boldsymbol{b}) + k(\overrightarrow{XO} + \overrightarrow{OC})$

$= (\lambda \boldsymbol{a} + \mu \boldsymbol{b}) + k(-\lambda \boldsymbol{a} - \mu \boldsymbol{b}) + k\boldsymbol{c}$

$= (\lambda - k\lambda)\boldsymbol{a} + (\mu - k\mu)\boldsymbol{b} + k\boldsymbol{c}$

$(\lambda - k\lambda) + (\mu - k\mu) + k = (\lambda + \mu) - k(\lambda + \mu) + k$

$= 1 - k + k$

$= 1$

Let $\alpha = \lambda - k\lambda, \beta = \mu - k\mu$ and $\gamma = k$.
We have the result.

c Let P be a point in the plane containing A, B and C. We see that it can be described by a vector $\alpha \boldsymbol{a} + \beta \boldsymbol{b} + \gamma \boldsymbol{c}$ as before.

9 **a** Assume $\boldsymbol{v} \neq \boldsymbol{0}$. First, $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel. Therefore $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{0}$. The vector $\boldsymbol{v}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. Therefore $\boldsymbol{v} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \neq 0$ a contradiction

b Let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
RHS $= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2 \cos^2\theta$

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1 - \cos^2\theta)$

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 \sin^2\theta$

$= |\boldsymbol{a} \times \boldsymbol{b}|^2$

$=$ LHS

c Assume $r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$
Taking scalar producrs with $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$
$r\boldsymbol{a} \cdot \boldsymbol{a} + s\boldsymbol{b} \cdot \boldsymbol{a} = 0 \ldots (1)$
$r\boldsymbol{a} \cdot \boldsymbol{b} + s\boldsymbol{b} \cdot \boldsymbol{b} = 0 \ldots (2)$
$t|(\boldsymbol{a} \times \boldsymbol{b})|^2 = 0 \ldots (3)$
From (3) $t = 0$
Multiply (1) by$\boldsymbol{a} \cdot \boldsymbol{b}$ and (2) by $\boldsymbol{a} \cdot \boldsymbol{a}$ and subtract.
$s[(\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b})] = 0$
Therefore $s|(\boldsymbol{a} \times \boldsymbol{b})|^2 = 0$ (From **b**)
Therefore $s = 0$
Similarly $r = 0$
The vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ are linearly independent

d **i** Let $\boldsymbol{v} = r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b})$

$\boldsymbol{a} \cdot \boldsymbol{v} = r|\boldsymbol{a}|^2 + s(\boldsymbol{a} \cdot \boldsymbol{b}) \ldots (1)$

$\boldsymbol{b} \cdot \boldsymbol{v} = r(\boldsymbol{b} \cdot \boldsymbol{a}) + s|\boldsymbol{b}|^2 \ldots (2)$

$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{v} = t|\boldsymbol{a} \times \boldsymbol{b}|^2 \ldots (3)$

From (3) $t = \dfrac{(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{v}}{|\boldsymbol{a} \times \boldsymbol{b}|^2}$

Multiply (1) by $\boldsymbol{a} \cdot \boldsymbol{b}$ and (2) by $|\boldsymbol{a}|^2$

$(\boldsymbol{a} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b}) = r|\boldsymbol{a}|^2 \boldsymbol{a} \cdot \boldsymbol{b} + s(\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{b}) \ldots (4)$

$(\boldsymbol{b} \cdot \boldsymbol{v})|\boldsymbol{a}|^2 = r(\boldsymbol{b} \cdot \boldsymbol{a})|\boldsymbol{a}|^2 + s|\boldsymbol{b}|^2|\boldsymbol{a}|^2 \ldots (5)$

Subtract (4) from (5)

$(\boldsymbol{b} \cdot \boldsymbol{v})|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b}) = s(|\boldsymbol{b}|^2|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2)$

Therefore $s = \dfrac{(\boldsymbol{b} \cdot \boldsymbol{v})|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{b}|^2|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2} = \dfrac{(\boldsymbol{b} \cdot \boldsymbol{v})|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2}$

Similarly,

$r = \dfrac{(\boldsymbol{a} \cdot \boldsymbol{v})|\boldsymbol{b}|^2 - (\boldsymbol{b} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{b}|^2|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2} = \dfrac{(\boldsymbol{a} \cdot \boldsymbol{v})|\boldsymbol{b}|^2 - (\boldsymbol{b} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2}$

ii First consider,

$(r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b}) - \boldsymbol{v}) \cdot (\boldsymbol{a} \times \boldsymbol{b})$

$= t|\boldsymbol{a} \times \boldsymbol{b}|^2 - \boldsymbol{v} \cdot (\boldsymbol{a} \times \boldsymbol{b})$

$= \dfrac{(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{v}}{|\boldsymbol{a} \times \boldsymbol{b}|^2}|\boldsymbol{a} \times \boldsymbol{b}|^2 - \boldsymbol{v} \cdot (\boldsymbol{a} \times \boldsymbol{b})$

$= 0$

Next,

$(r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b}) - \boldsymbol{v}) \cdot \boldsymbol{a}$

$= r|\boldsymbol{a}|^2 + s\boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{v} \cdot \boldsymbol{a}$

$= \dfrac{(\boldsymbol{a} \cdot \boldsymbol{v})|\boldsymbol{b}|^2 - (\boldsymbol{b} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2}|\boldsymbol{a}|^2 + \dfrac{(\boldsymbol{b} \cdot \boldsymbol{v})|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2}\boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{v} \cdot \boldsymbol{a}$

$= \dfrac{(\boldsymbol{a} \cdot \boldsymbol{v})|\boldsymbol{b}|^2|\boldsymbol{a}|^2 - (\boldsymbol{b} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})|\boldsymbol{a}|^2 + (\boldsymbol{b} \cdot \boldsymbol{v})|\boldsymbol{a}|^2(\boldsymbol{a} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2} - \boldsymbol{v} \cdot \boldsymbol{a}$

$= \dfrac{(\boldsymbol{a} \cdot \boldsymbol{v})|\boldsymbol{b}|^2|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{v})(\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2} - \boldsymbol{v} \cdot \boldsymbol{a}$

$= \dfrac{(\boldsymbol{a} \cdot \boldsymbol{v})(|\boldsymbol{b}|^2|\boldsymbol{a}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{b}))}{|\boldsymbol{a} \times \boldsymbol{b}|^2} - \boldsymbol{v} \cdot \boldsymbol{a}$

$= \dfrac{(\boldsymbol{a} \cdot \boldsymbol{v})|\boldsymbol{a} \times \boldsymbol{b}|^2}{|\boldsymbol{a} \times \boldsymbol{b}|^2} - \boldsymbol{v} \cdot \boldsymbol{a}$

$= 0$

Similarly,

$(r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b}) - \boldsymbol{v}) \cdot \boldsymbol{b} = 0$

iii From part**a** $\boldsymbol{v} = r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b})$

e $\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ ar linearly independent. Hence r, s and t are unique.

10 a ℓ_1 and ℓ_2 are skew lines.

b If $\boldsymbol{r}_2(\mu) - \boldsymbol{r}_1(\lambda)$ is perpendicular to the vectors $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$ then $\boldsymbol{r}_2(\mu) - \boldsymbol{r}_1(\lambda)$ is parallel to $\boldsymbol{d_1} \times \boldsymbol{d_2}$. The converse is clear.
Hence

$$\boldsymbol{r}_2(\mu) - \boldsymbol{r}_1(\lambda) = k(\boldsymbol{d_1} \times \boldsymbol{d_2}) \text{ for some } k \in \mathbb{R}$$

if and only if the line segment is perpendicular to both lines.

c

$$\boldsymbol{r}_2(\mu) - \boldsymbol{r}_1(\lambda) = k(\boldsymbol{d_1} \times \boldsymbol{d_2})$$
$$\Leftrightarrow \boldsymbol{a_2} + \mu\boldsymbol{d_2} - (\boldsymbol{a_1} + \lambda\boldsymbol{d_1}) = k(\boldsymbol{d_1} \times \boldsymbol{d_2})$$
$$\Leftrightarrow \boldsymbol{a_2} - \boldsymbol{a_1} = \lambda\boldsymbol{d_1} - \mu\boldsymbol{d_2} + k(\boldsymbol{d_1} \times \boldsymbol{d_2})$$

d From **Question 9** λ, μ and k are unique.

e Let P be the point defined by $r_1(\lambda)$ and Q be the point defined by $r_2(\mu)$

11 a $\boldsymbol{r}_1 = \boldsymbol{p} + \lambda\boldsymbol{d}_1$ and $\boldsymbol{r}_2 = \boldsymbol{q} + \mu\boldsymbol{d}_2$
$\boldsymbol{q} - \boldsymbol{p}$ is the vector $\overrightarrow{PQ}$ and PQ is perpendicular to ℓ_1 and ℓ_2.
Therefore $(\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{d}_1 = 0$ and $(\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{d}_2 = 0$

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= (\boldsymbol{q} - \boldsymbol{p}) + \mu\boldsymbol{d}_2 - \lambda\boldsymbol{d}_1$
$|\overrightarrow{AB}|^2 = \overrightarrow{AB} \cdot \overrightarrow{AB} = ((\boldsymbol{q} - \boldsymbol{p}) + \mu\boldsymbol{d}_2 - \lambda\boldsymbol{d}_1) \cdot ((\boldsymbol{q} - \boldsymbol{p}) + \mu\boldsymbol{d}_2 - \lambda\boldsymbol{d}_1)$
$= ((\boldsymbol{q} - \boldsymbol{p}) \cdot ((\boldsymbol{q} - \boldsymbol{p}) + (\mu\boldsymbol{d}_2 - \lambda\boldsymbol{d}_1) \cdot (\mu\boldsymbol{d}_2 - \lambda\boldsymbol{d}_1)$
$= |\boldsymbol{q} - \boldsymbol{p}|^2 + |\mu\boldsymbol{d}_2 - \lambda\boldsymbol{d}_1|^2$

c Hence $|\overrightarrow{AB}|^2 \geq |\overrightarrow{PQ}|^2$.
If $|\overrightarrow{AB}|^2 = |\overrightarrow{PQ}|^2, |\mu\boldsymbol{d}_2 - \lambda\boldsymbol{d}_1|^2 = 0$ which implies $\mu\boldsymbol{d}_2 = \lambda\boldsymbol{d}_1$.
But the lines are skew. Hence $\lambda = \mu = 0$ and $\overrightarrow{OA} = \overrightarrow{OP}$ and $\overrightarrow{OB} = \overrightarrow{OQ}$.
Conversely if $A = P$ and $B = Q$ then clearly $|\overrightarrow{AB}| = |\overrightarrow{AB}|$

12 a $\overrightarrow{OA} = \boldsymbol{i}$
$\overrightarrow{OB} = \boldsymbol{j}$
$\overrightarrow{OC} = \boldsymbol{k}$
$\overrightarrow{OD} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
First consider, $\overrightarrow{AB} = \boldsymbol{j} - \boldsymbol{i}$
$|\overrightarrow{AB}| = \sqrt{1 + 1} = \sqrt{2}$
$\overrightarrow{AC} = \boldsymbol{k} - \boldsymbol{i}$
$|\overrightarrow{AC}| = \sqrt{1 + 1} = \sqrt{2}$
$\overrightarrow{AD} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} - \boldsymbol{i} = \boldsymbol{j} + \boldsymbol{k}$
$|\overrightarrow{AD}| = \sqrt{1 + 1} = \sqrt{2}$
It is easy to see that the other edges will all have length $\sqrt{2}$

b Consider $\overrightarrow{AB} \times \overrightarrow{AC} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{3}$

Therefore area of triangle is $\dfrac{\sqrt{3}}{2}$

c We work with the Cartesian form for plane ABC

$ax + by + cz = k$

$a = k$

$b = k$

$c = k$

The equation is $x + y + z = 1$

For plane BCD $b = k$

$c = k$

$a + b + c = k \Rightarrow a = -k$

The equation is $-x + y + z = 1$ or equivalently $x - y - z = -1$

Consider the vectors normal to each plane. These are $\dfrac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

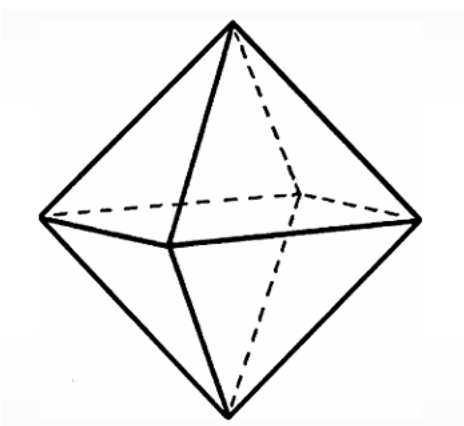
and $\dfrac{1}{\sqrt{3}}(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

Use the scalar product:

$\dfrac{1}{3}(-1 + 1 + 1) = \cos(\theta)$

$\cos\theta = \dfrac{1}{3}$

$\theta = \cos^{-1}\dfrac{1}{3}$

13 

Take the origin as the centre of the square,

a $\overrightarrow{OA} = \boldsymbol{i}$

$\overrightarrow{OA'} = -\boldsymbol{i}$

$\overrightarrow{OB} = \boldsymbol{j}$

$\overrightarrow{OB} = -\boldsymbol{j}$

$\overrightarrow{OC} = \boldsymbol{k}$

$\overrightarrow{OC'} = \boldsymbol{k}$

First consider,

From the 'top vertex' C

$\overrightarrow{CA} = -\boldsymbol{k} + \boldsymbol{i}$

$\overrightarrow{CA'} = -\boldsymbol{k} - \boldsymbol{i}$

$\overrightarrow{CB} = -\boldsymbol{k} + \boldsymbol{j}$

$\overrightarrow{CB'} = -\boldsymbol{k} - \boldsymbol{j}$

From the 'top vertex' C'

$\overrightarrow{C'A} = \boldsymbol{k} + \boldsymbol{i}$

$\overrightarrow{C'A'} = \boldsymbol{k} - \boldsymbol{i}$

$\overrightarrow{C'B} = \boldsymbol{k} + \boldsymbol{j}$

$\overrightarrow{C'B'} = \boldsymbol{k} - \boldsymbol{j}$

$|\overrightarrow{CA}| = \sqrt{1+1} = \sqrt{2}$

The square is $ABA'B'$ $\overrightarrow{AB} = -\boldsymbol{i} + \boldsymbol{j}$

$|\overrightarrow{AB}| = \sqrt{2}$

t It is easy to see that the other edges will all have length $\sqrt{2}$

b For example, for triangular face CAB:

$\overrightarrow{CA} \times \overrightarrow{CB} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$|\overrightarrow{CA} \times \overrightarrow{CB}| = \sqrt{3}$

Therefore area of triangle $ABC = \dfrac{\sqrt{3}}{2}$

Clearly all of the triangles will have this area.

c Consider $\overrightarrow{AB} \times \overrightarrow{AC} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{3}$

Therefore, area of triangle is $\dfrac{\sqrt{3}}{2}$.

d Consider opposite facesCBA' and $C'B'A$. We find vectors normal to these planes For face CBA',

$\overrightarrow{CB} \times \overrightarrow{CA'} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

For face $C'B'A$,

$\overrightarrow{C'B'} \times \overrightarrow{C'A} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

These opposite planes are parallel.

Same for the other pairs of opposite planes

e Using $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

For plane ABC, normal is given by,

$\boldsymbol{n_1} = \overrightarrow{CB} \times \overrightarrow{CA} = -\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

$\boldsymbol{r} \cdot \boldsymbol{n_1} = -x - y - z$

$\boldsymbol{a} \cdot \boldsymbol{n_1} = -1$

Therefore equation of plane is $x + y + z = 1$
For plane $A'BC$, normal is given by,
$\boldsymbol{n_2} = \overrightarrow{CB} \times \overrightarrow{CA'} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$
$\boldsymbol{r} \cdot \boldsymbol{n_2} = -x - y + z$
$\boldsymbol{a'} \cdot \boldsymbol{n_2} = -1$
Therefore equation of plane is $x - y - z = 1$
Angle between normals $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$
$\boldsymbol{n_1} \cdot \boldsymbol{n_2} = 1$
$|\boldsymbol{n_1}| = |\boldsymbol{n_2}| = \sqrt{3}$
Therefore as $\boldsymbol{n_1} \cdot \boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}| \cos\theta$
$\cos\theta = \frac{1}{3}$.
The obtuse angle between them is $\cos^{-1}\left(-\frac{1}{3}\right)$

f Calculations are the same.

Chapter 6 – Complex numbers

Solutions to Exercise 6A

1 a $\text{Re}(z) = 6$

b $\text{Im}(z) = -7$

c $\text{Re}(z) - \text{Im}(z) = 13$

2 a $\sqrt{-25} = \sqrt{25} \times \sqrt{-1} = 5i$

b $\sqrt{-27} = \sqrt{27} \times \sqrt{-1} = 3\sqrt{3}i$

c $2i - 7i = -5i$

d $5\sqrt{-16} - 7i = 20i - 7i = 13i$

e $\sqrt{-8} + \sqrt{-18} = 2\sqrt{2}i + 3\sqrt{2}i$

$= 5\sqrt{2}i$

f $i\sqrt{-12} = i\left(2\sqrt{3}i\right) = 2\sqrt{3}i^2$

$= 2\sqrt{3} \times -1 = -2\sqrt{3}$

g $i(2 + i) = 2i + i^2 = -1 + 2i$

h $\text{Im}(2\sqrt{-4}) = 4$

i $\text{Re}(5\sqrt{-49}) = \text{Re}(5 \times 7i)$

$= \text{Re}(35i) = 0$

3 a $x + iy = 5 + 0i$

$\therefore\ x = 5,\ y = 0$

b $x + iy = 0 + 2i$

$\therefore\ x = 0,\ y = 2$

c $x = iy$

$\therefore\ x - iy = 0 + 0i$

$\therefore\ x = 0,\ y = 0$

d $x + iy = (2 + 3i) + 7(1 - i)$

$\therefore\ x + iy = 9 - 4i$

$\therefore\ x = 9,\ y = -4$

e $2x + 3 + 8i = -1 + (2 - 3y)i$

$\therefore\ 2x + 3 + 8i = -1 + 2i - 3yi$

$\therefore\ 2x + 3yi = -4 - 6i$

$\therefore\ 2x = -4, 3y = -6$

$\therefore\ x = -2,\ y = -2$

f $x + iy = (2y + 1) + (x - 7)i$

$= 2y + 1 + xi - 7i$

$\therefore\ (x - 2y) + (y - x)i = 1 - 7i$

Equating corresponding components gives:

$x - 2y = 1$ ①

$y - x = -7$ ②

① + ② gives

$-y = -6$

$\therefore\ y = 6$

Substituting $y = 6$ into ①

$x - 2(6) = 1$

$\therefore\ x - 12 = 1$

$\therefore\ x = 13$

4 a $z_1 + z_2 = (2 - i) + (3 + 2i) = 5 + i$

b $z_1 + z_2 + z_3 = (2 - i) + (3 + 2i)$

$+ (-1 + 3i)$

$= 4 + 4i$

c $2z_1 - z_3 = 2(2 - i) - (-1 + 3i)$
$= 4 - 2i + 1 - 3i$
$= 5 - 5i$

d $3 - z_3 = 3 - (-1 + 3i) = 4 - 3i$

e $4i - z_2 + z_1 = 4i - (3 + 2i)$
$+ (2 - i)$
$= -1 + i$

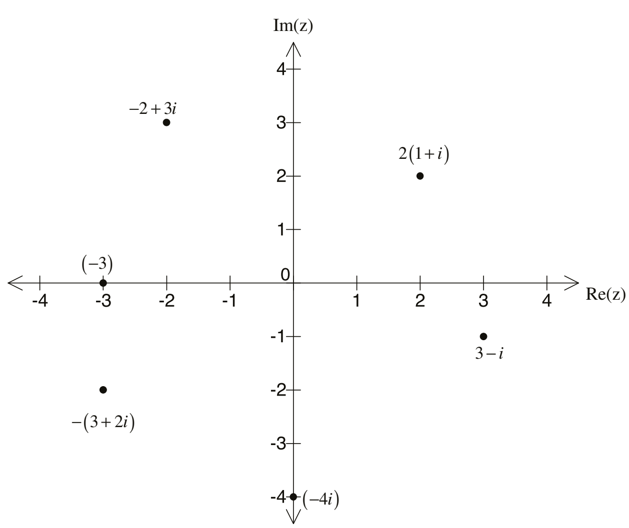
f $\text{Re}(z_1) = \text{Re}(2 - i) = 2$

g $\text{Im}(z_2) = \text{Im}(3 + 2i) = 2$

h $\text{Im}(z_3 - z_2) = \text{Im}((-1 + 3i)$
$- (3 + 2i))$
$= \text{Im}(-4 + i)$
$= 1$

i $\text{Re}(z_2) - i\text{Im}(z_2) = 3 - i(2) = 3 - 2i$

5

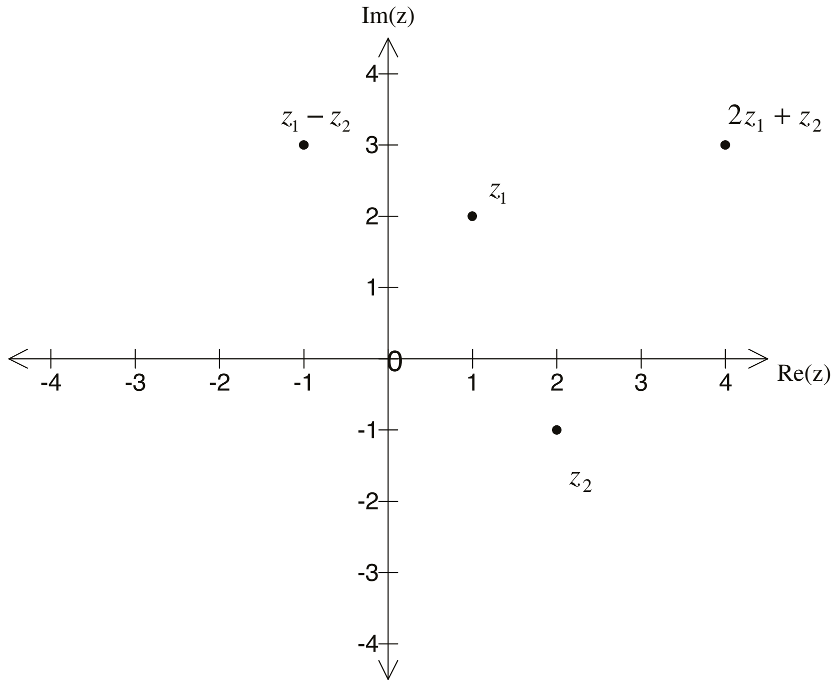


6 **a** **i** $z_1 = 1 + 2i$

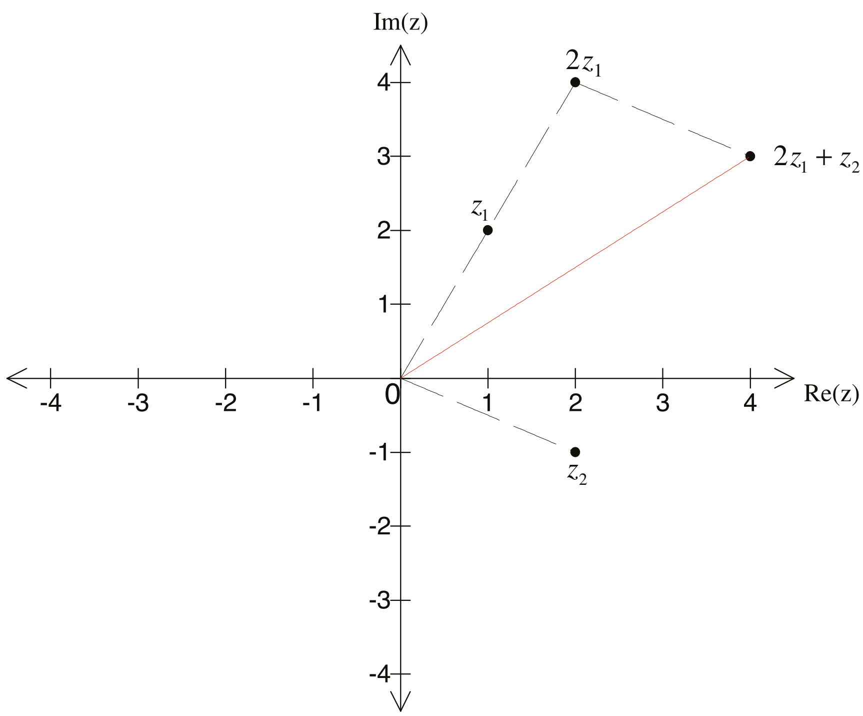
ii $z_2 = 2 - i$

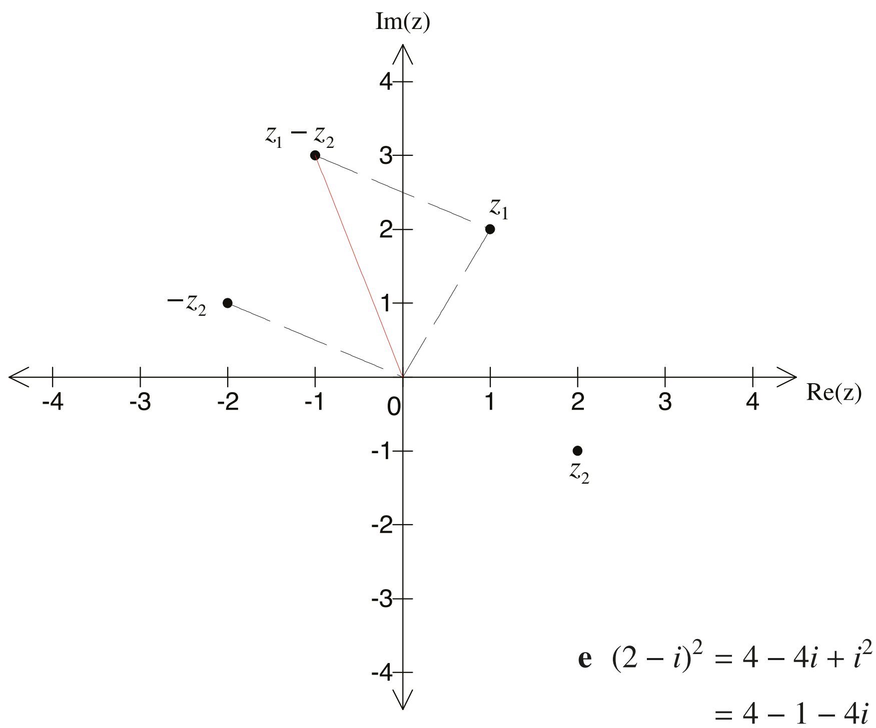
iii $2z_1 + z_2 = 2 + 4i + (2 - i)$
$= 4 + 3i$

iv $z_1 - z_2 = (1 + 2i) - (2 - i)$
$= -1 + 3i$


Im(z)
4
3
2
1
0
-1
-2
-3
-4
Re(z)
$z_1 - z_2$
$2z_1 + z_2$
z_1
z_2

b


Im(z)
4
3
2
1
0
-1
-2
-3
-4
Re(z)
$2z_1$
$2z_1 + z_2$
z_1
z_2



7 **a** $(5 - i)(2 + i) = 10 + 5i - 2i - i^2$

$= 10 - (-1) + 3i$

$= 11 + 3i$

b $(4 + 7i)(3 + 5i) = 12 + 20i$

$+ 21i + 35i^2$

$= -23 + 41i$

$= 47 - i$

c $(2 + 3i)(2 - 3i) = 4 - 6i + 6i - 9i^2$

$= 4 - (-9) + 0i$

$= 4 + 9$

$= 13$

d $(1 + 3i)^2 = 1 + 6i + 9i^2$

$= 1 - 9 + 6i$

$= -8 + 6i$

e $(2 - i)^2 = 4 - 4i + i^2$

$= 4 - 1 - 4i$

$= 3 - 4i$

f Recall $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

So,

$(1 + i)^3 = 1^3 + 3 \times 1^2 \times i + 3 \times i^2 + i^3$

$= 1 + 3i + 3i^2 + i^3$

$= 1 + 3i - 3 - i$

$= -2 + 2i$

g $i^4 = i^2 \times i^2 = -1 \times -1 = 1$

h $i^{11}(6 + 5i) = i^{10} \times i(6 + 5i)$

$= -i(6 + 5i)$

$= -6i - 5i^2$

$= 5 - 6i$

i $i^{70} = (i^2)^{35}$

$= (-1)^{35}$

$= -1$

8 **a** $2x + (y + 4)i = (3 + 2i)(2 - i)$

$\therefore \quad 2x + yi + 4i = 8 + i$

$\therefore \quad 2x + yi = 8 - 3i$

$\therefore \quad 2x = 8, \; y = -3$

$\therefore \quad x = 4, \; y = -3$

b

$(x + yi)(3 + 2i) = -16 + 11i$

$\therefore \quad 3x + 2xi + 3yi - 2y = -16 + 11i$

$\therefore \quad (3x - 2y) + (2x + 3y)i = -16 + 11i$

Equating corresponding components gives:

$3x - 2y = -16$ [1]

$2x + 3y = 11$ [2]

$2 \times [1] - 3 \times [2]$:

$\therefore \quad -13y = -65$

$\therefore \quad y = 5$

Substituting $y = 5$ into [2] yields

$x = -2$

$\therefore \quad x = -2, \; y = 5$

c $(x + 2i)^2 = 5 - 12i$

$\therefore \quad x^2 + 4xi - 4 = 5 - 12i$

$\therefore \quad (x^2 - 4) + 4xi = 5 - 12i$

Equating components gives:

$x^2 - 4 = 5$ [1]

$4x = -12$ [2]

From [2], $x = -3$

From [1], $x = \pm 3$

$\therefore \quad x = -3$

d $(x + iy)^2 = -18i$

$\therefore \quad x^2 + 2xyi - y^2 = -18i$

$(x^2 - y^2) = 2xyi = -18i$

Equating components gives:

$x^2 = y^2$ [1]

$2xy = -18$ [2]

From [2]:

$y = -\dfrac{9}{x}$ [3]

Substitute [3] into [1]:

$x^2 = \dfrac{81}{x^2}$

$\therefore \quad x^4 = 81$

$\therefore \quad x = \pm 3$

When $x = 3, \; y = -3$

and when $x = -3, \; y = 3$

$\therefore \quad x = 3, \; y = -3$ or $x = -3, \; y = 3$

e $i(2x - 3yi) = 6(1 + i)$

$\therefore \quad 3y + 2xi = 6 + 6i$

Equating components gives:

$3y = 6$ and $2x = 6$

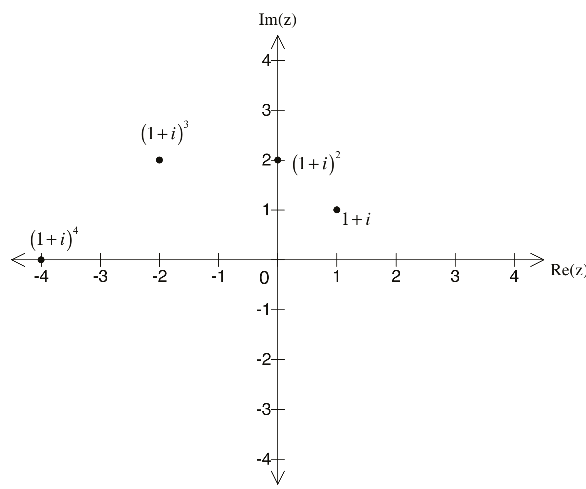
$\therefore \quad x = 3, \; y = 2$

9 **a** **i** $1 + i$

ii $(1 + i)^2 = 2i$

iii $(1 + i)^3 = -2 + 2i$

iv $(1 + i)^4 = -4$



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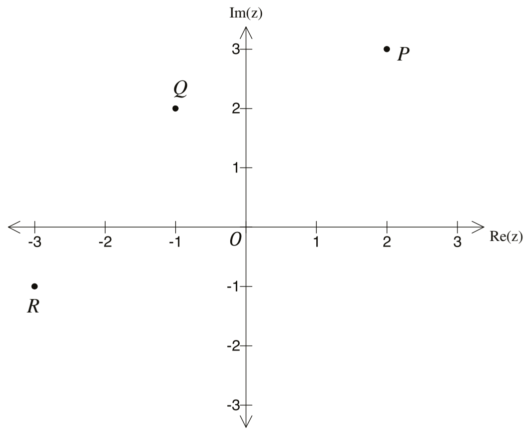


$P = z_1 = 2 + 3i$

$Q = z_2 = -1 + 2i$

$R = z_2 - z_1$

$= (-1 + 2i) - (2 + 3i)$

$= -3 - i$

a $\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ}$

$= -\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

$= \overrightarrow{OR}$

b $\overrightarrow{QP} = -\overrightarrow{PQ}$

$= -\begin{bmatrix} -3 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

and $|QP| = QP = \sqrt{3^2 + 1^2} = \sqrt{10}$

11 Use,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{100} = \frac{1(i^{101} - 1)}{i - 1}$$

$$= \frac{i - 1}{i - 1}$$

$$= 1$$

Solutions to Exercise 6B

1 a $\overline{\sqrt{3}} = \overline{\sqrt{3} + 0i} = \sqrt{3}$

b $\overline{8i} = -8i$

c $\overline{4 - 3i} = 4 + 3i$

d $\overline{-(1 + 2i)} = \overline{-1 - 2i} = -1 + 2i$

e $\overline{4 + 2i} = 4 - 2i$

f $\overline{-3 - 2i} = -3 + 2i$

2 a $\dfrac{2 + 3i}{3 - 2i} = \dfrac{(2 + 3i)(3 + 2i)}{9 + 4}$
$= \dfrac{6 + 4i + 9i - 6}{13}$
$= \dfrac{13i}{13}$
$= i$

b $\dfrac{i}{-1 + 3i} = \dfrac{i(-1 - 3i)}{1 + 9}$
$= \dfrac{3 - i}{10}$
$= \dfrac{3}{10} - \dfrac{1}{10}i$

c $\dfrac{-4 - 3i}{i} = \dfrac{-i(-4 - 3i)}{1} = -3 + 4i$

d $\dfrac{3 + 7i}{1 + 2i} = \dfrac{(3 + 7i)(1 - 2i)}{1 + 4}$
$= \dfrac{3 - 6i + 7i + 14}{5}$
$= \dfrac{17 + i}{5}$
$= \dfrac{17}{5} + \dfrac{1}{5}i$

e $\dfrac{\sqrt{3} + i}{-1 - i} = \dfrac{(\sqrt{3} + i)(-1 + i)}{2}$
$= \dfrac{-\sqrt{3} + \sqrt{3}i - i - 1}{2}$
$= \dfrac{-1 - \sqrt{3}}{2} + \dfrac{\sqrt{3} - 1}{2}i$

f $\dfrac{17}{4 - i} = \dfrac{17(4 + i)}{16 + 1}$
$= \dfrac{68 + 17i}{17}$
$= 4 + i$

3 a $\overline{z + w} = \overline{a + bi + c + di}$
$= \overline{(a + c) + (b + d)i}$
$= (a + c) - (b + d)i$
$= (a - bi) + (c - di)$
$= \overline{z} + \overline{w}$

b $\overline{zw} = \overline{(a + bi)(c + di)}$
$= ac + bdi^2 + (ad + bc)i$
$= ac + bdi^2 - (ad + bc)i$
$= (a - bi)(c - di)$
$= \overline{z}\,\overline{w}$

c $\overline{\left(\dfrac{z}{w}\right)} = \overline{\dfrac{(a + bi)(c - di)}{c^2 + d^2}}$
$= \dfrac{\overline{(a + bi)(c - di)}}{c^2 + d^2}$ (see **b**)
$= \dfrac{(a - bi)(c + di)}{c^2 + d^2}$
$= \dfrac{a - bi}{c - di}$
$= \dfrac{\overline{z}}{\overline{w}}$

d $|zw|^2 = |(a + bi)(c + di)|^2$

$= |ac - bd + (ad + bc)i|^2$

$= (ac - bd)^2 + (ad + bc)^2$

$= (a^2c^2 - 2acbd + b^2d^2$

$+ a^2b^2 + 2adbc + b^2c^2)^2$

$= (a^2c^2 + b^2d^2 + a^2b^2 + b^2c^2)^2$

$= ((a^2 + b^2)(c^2 + d^2))^2$

$= |z|^2|w|^2$

e $\left|\frac{z}{w}\right|^2 = \left|\frac{(a + bi)(c - di)}{c^2 + d^2}\right|^2$

$= \left(\frac{|(a + bi)(c - di)|}{c^2 + d^2}\right)^2$

$= \left(\frac{|(a + bi)||(c - di)|}{c^2 + d^2}\right)^2$

$= \left(\frac{|a + bi|}{|c - di|}\right)^2$

$= \frac{|z|^2}{|w|^2}$

4 a $z(z + 1) = (2 - i)(3 - i)$

$= 6 - 2i - 3i + i^2$

$= 5 - 5i$

b $\overline{z + 4} = \overline{6 - i} = 6 + i$

c $\overline{z - 2i} = \overline{2 - 3i} = 2 + 3i$

d $\frac{z - 1}{z + 1} = \frac{1 - i}{3 - i}$

$= \frac{(1 - i)(3 + i)}{9 + 1}$

$= \frac{3 + i - 3i - i^2}{10}$

$= \frac{4 - 2i}{10}$

$= \frac{2 - i}{5}$

e $(z - i)^2 = (2 - 2i)^2$

$= 4 - 8i + 4i^2$

$= -8i$

f $(z + 1 + 2i)^2 = (3 + i)^2$

$= 9 + 6i + i^2$

$= 8 + 6i$

5 a $z\overline{z} = (a + ib)(a - ib)$

$= a^2 - abi + abi - b^2i^2$

$= a^2 + b^2$

c $z + \overline{z} = (a + bi) + (a - bi) = 2a$

d $z - \overline{z} = (a + bi) - (a - bi) = 2bi$

e $\frac{z}{\overline{z}} = \frac{a + bi}{a - bi}$

$= \frac{(a + bi)(a + bi)}{a^2 + b^2}$

$= \frac{a^2 + 2abi - b^2}{a^2 + b^2}$

$= \frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$

f $\frac{\overline{z}}{z} = \frac{a - bi}{a + bi}$

$= \frac{(a - bi)(a - bi)}{a^2 + b^2}$

$= \frac{a^2 - 2abi - b^2}{a^2 + b^2}$

$= \frac{a^2 - b^2}{a^2 + b^2} - \frac{2ab}{a^2 + b^2}i$

6 $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_2}$

$= |z_1|^2 + 2\text{Re}(z_1\overline{z_2}) + |z_2|^2$

$\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$= (|z_1| + |z_2|)^2$

This implies $|z_1 + z_2| \leq |z_1| + |z_2|$

7 $P(n)$ is the statement $(\overline{z})^n = \overline{z^n}$
$P(1)$ is true.
Assume $P(k)$ is true.
$(\overline{z})^k = \overline{z^k}$
To prove $P(k + 1)$ is true.
$(\overline{z})^{k+1} = (\overline{z})^k \times (\overline{z})$

$= \overline{z^k} \times \overline{z}$

$= \overline{z^k \times z}$ (see **3b** in this exercise)

$= \overline{z^{k+1}}$

Therefore $P(n)$ is true for all $n \in \mathbb{N}$

8 **a** $z = a + bi$ and $w = c + di$
$|z| = a^2 + b^2$ and $|w| = c^2 + d^2$
$zw = (a + bi)(c + di) = ac + (bc + ad)i - bd = ac - bd + (ad + bc)i$
$|zw| = (ac - bd)^2 + (ad + bc)^2$
Since $|zw| = |z||w|$
$(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2)$

b $65 = 13 \times 5 = (9 + 4) \times (4 + 1)$ Using the result of **a**
$a = 3, b = 2, c = 2, d = 1$
$(3^2 + 2^2) \times (2^2 + 1^2)$
$= (6 - 2)^2 + (3 + 4)^2$
$= 7^2 + 4^2$

c $n^4 + 5n^2 + 4 = (n^2 + 1)(n^2 + 4)$
In this case $a = c = n, b = 1$ and $d = 2$
Therefore,
$n^4 + 5n^2 + 4 = (n^2 - 2)^2 + (2n + n)^2$
$= (n^2 - 2)^2 + (3n)^2$

Solutions to Exercise 6C

1 a $|-3| = 3$ and $\text{Arg}(-3) = \pi$

b $|5i| = 5$ and $\text{Arg}(5i) = \dfrac{\pi}{2}$

c $|i - 1| = \sqrt{1+1} = \sqrt{2}$ and
$\text{Arg}(i - 1) = \dfrac{3\pi}{4}$

d $|\sqrt{3} + i| = \sqrt{3+1} = 2$ and
$\text{Arg}(\sqrt{3} + i) = \dfrac{\pi}{6}$

e $|2 - 2\sqrt{3}i| = \sqrt{4 + 12} = 4$ and
$\theta = \tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$
Since complex number is situated in the fourth quadrant
$\therefore$ $\text{Arg}(2 - 2\sqrt{3}i) = -\dfrac{\pi}{3}$

f $z = (2 - 2\sqrt{3}i)^2$

$= 4 - 8\sqrt{3}i - 12$

$= -8 - 8\sqrt{3}i$

$|z| = \sqrt{64 + 192} = \sqrt{256} = 16$
Because z is situated
in the third quadrant:
$\theta = \pi + \tan^{-1}(\sqrt{3}) = \pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3}$
$\therefore$ $\text{Arg}\, z = -\dfrac{2\pi}{3}$ $\quad\because -\pi < \text{Arg}(z) \le \pi$

2 a Let $z = 5 + 12i$ and $\theta = \text{Arg}\, z$
Then $\cos\theta = \dfrac{5}{\sqrt{25 + 144}}$

$\sin\theta = \dfrac{12}{\sqrt{25 + 144}}$

$\theta = \cos^{-1}\dfrac{5}{13} \approx 1.18$

b Let $z = -8 + 15i$ and $\theta = \text{Arg}\, z$
Then $\cos\theta = \dfrac{-8}{\sqrt{64 + 225}}$,

$\sin\theta = \dfrac{15}{\sqrt{64 + 225}}$

$\theta \in \left(\dfrac{\pi}{2}, \pi\right)$

so $\theta = \cos^{-1}\dfrac{-8}{17} \approx 2.06$

c Let $z = -4 - 3i$ and $\theta = \text{Arg}\, z$
Then $\cos\theta = \dfrac{-4}{\sqrt{16 + 9}}$,

$\sin\theta = \dfrac{-3}{\sqrt{16 + 9}}$

$\theta \in \left(-\pi, -\dfrac{\pi}{2}\right)$

so $\theta = -\cos^{-1}\dfrac{-4}{5} \approx -2.50$

d Let $z = 1 - \sqrt{2}i$ and $\theta = \text{Arg}\, z$
Then $\cos\theta = \dfrac{1}{\sqrt{3}}$

$\sin\theta = -\dfrac{\sqrt{2}}{\sqrt{3}}$

$\theta \in \left(-\dfrac{\pi}{2}, 0\right)$

so $\theta = -\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \approx -0.96$

e Let $z = \sqrt{2} + \sqrt{3}i$ and $\theta = \text{Arg}\, z$
Then $\cos\theta = \dfrac{\sqrt{2}}{\sqrt{5}}$,

$\sin\theta = \dfrac{\sqrt{3}}{\sqrt{5}}$

so $\theta = \cos^{-1}\left(\sqrt{\dfrac{2}{5}}\right) \approx 0.89$

f Let $z = -(3 + 7i)$ and $\theta = \text{Arg } z$

Then $\cos\theta = \dfrac{-3}{\sqrt{9+49}}$,

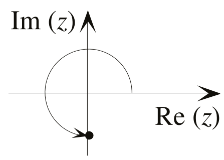
$\sin\theta = \dfrac{-7}{\sqrt{9+49}}$

so $\theta = -\cos^{-1}\dfrac{-3}{\sqrt{58}} \approx -1.98$

3 a $\cos\theta = \dfrac{1}{2},\ \sin\theta = -\dfrac{\sqrt{3}}{2},$

$\therefore\quad \theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$

b



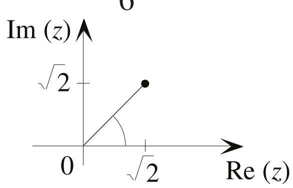
$\therefore\quad \theta = \dfrac{3\pi}{2}$

c $\cos\theta = \dfrac{-3}{\sqrt{9+3}} = \dfrac{-3}{\sqrt{12}}$

$= \dfrac{-3}{2\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$

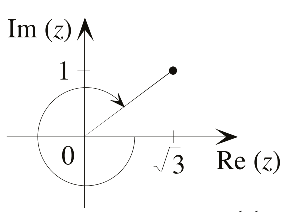
$\sin\theta = \dfrac{\sqrt{3}}{\sqrt{12}} = \dfrac{1}{2}$

$\therefore\quad \theta = \dfrac{5\pi}{6}$



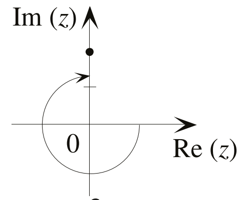
$\therefore\quad \theta = \dfrac{\pi}{4}$

e



$\therefore\quad \theta = -2\pi + \dfrac{\pi}{6} = -\dfrac{11\pi}{6}$

f



$\therefore\quad \theta = -\dfrac{3\pi}{2}$

4 a $\dfrac{5\pi}{4} - 2\pi = -\dfrac{3\pi}{4} \in (-\pi,\ \pi]$

b $\dfrac{17\pi}{6} - 2\pi = \dfrac{5\pi}{6} \in (-\pi,\ \pi]$

c $-\dfrac{15\pi}{8} + 2\pi = \dfrac{\pi}{8} \in (-\pi,\ \pi]$

d $-\dfrac{5\pi}{2} + 2\pi = -\dfrac{\pi}{2} \in (-\pi,\ \pi]$

5 a $|z| = |-1-i| = \sqrt{2}$

$\sin\theta = -\dfrac{1}{\sqrt{2}}$

$\cos\theta = -\dfrac{1}{\sqrt{2}}$

$z = \sqrt{2}\text{cis}\left(-\dfrac{3\pi}{4}\right)$

b $|z| = \left|\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right| = \sqrt{\dfrac{1}{4} + \dfrac{3}{4}} = 1$

$\sin\theta = -\dfrac{\sqrt{3}}{2}$

$\cos\theta = \dfrac{1}{2}$

$\theta = \text{Arg}\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right) = -\dfrac{\pi}{3}$

$\therefore\quad z = \text{cis}\left(-\dfrac{\pi}{3}\right)$

c $|z| = |\sqrt{3} - \sqrt{3}i| = \sqrt{3+3} = \sqrt{6}$

$\sin\theta = -\frac{\sqrt{3}}{\sqrt{6}} = -\frac{1}{\sqrt{2}}$

$\cos\theta = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$

$\theta = \text{Arg}(\sqrt{3} - \sqrt{3}i) = -\frac{\pi}{4}$

$\therefore \quad z = \sqrt{6}\text{cis}\left(-\frac{\pi}{4}\right)$

d $|z| = \left|\frac{1}{\sqrt{3}} + \frac{1}{3}i\right| = \sqrt{\frac{1}{3} + \frac{1}{9}} = \frac{2}{3}$

$\sin\theta = \frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\theta = \text{Arg}\left(\frac{1}{\sqrt{3}} + \frac{1}{3}i\right) = \frac{\pi}{6}$

$\therefore \quad z = \frac{2}{3}\text{cis}\left(\frac{\pi}{6}\right)$

e $|z| = |\sqrt{6} - \sqrt{2}i| = \sqrt{8}$

$\cos\theta = -\frac{1}{\sqrt{2}}$

$\theta = \text{Arg}(-1 - i) = -\frac{3\pi}{4}$

$\therefore \quad z = \sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$

f $|z| = |-2\sqrt{3} + 2i| = \sqrt{12 + 4} = 4$

$\sin\theta = \frac{1}{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = \frac{5\pi}{6}$

$\therefore \quad z = 4\text{cis}\left(\frac{5\pi}{6}\right)$

6 a $2\text{cis}\,\frac{3\pi}{4} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= -\sqrt{2} + \sqrt{2}i$

b $5\text{cis}\left(\frac{-\pi}{3}\right) = 5\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

c $2\sqrt{2}\text{cis}\,\frac{\pi}{4} = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\sin\theta = -\frac{\sqrt{2}}{\sqrt{8}} = -\frac{\sqrt{2}}{\sqrt{8}}$

$\cos\theta = \frac{\sqrt{6}}{\sqrt{8}} = \frac{\sqrt{3}}{2}$

$\theta = -\frac{\pi}{6}$

$\therefore \quad z = \sqrt{8}\text{cis}\left(-\frac{\pi}{6}\right)$

$\therefore \quad z = 2\sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$

d $3\text{cis}\left(-\frac{5\pi}{6}\right) = 3\left(\cos\left(-\frac{5\pi}{6}\right)\right.$

$\left.+ i\sin\left(-\frac{5\pi}{6}\right)\right)$

$= 3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

e $6\text{cis}\,\frac{\pi}{2} = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 6(0 + i)$

$= 6i$

f $4\text{cis}\,\pi = 4(\cos\pi + i\sin\pi)$

$= 4(-1 + 0i)$

$= -4$

7 a $|z| = |\cos\theta + i\sin\theta|$

$= \sqrt{\cos^2\theta + \sin^2\theta}$

$= 1$

$= 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$= 2 + 2i$

b $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \bar{z}$ since $|z| = 1$

If $z = \text{cis}\,\theta$, $\bar{z} = \text{cis}(-\theta)$

$\therefore \quad \frac{1}{z} = \bar{z} = \text{cis}(-\theta)$

8 a Let $z = 2\text{cis}\frac{3\pi}{4}$,

then $\bar{z} = 2\text{cis}\left(-\frac{3\pi}{4}\right)$

b Let $z = 7\text{cis}\left(-\frac{2\pi}{3}\right)$,

then $\bar{z} = 7\text{cis}\,\frac{2\pi}{3}$

c Let $z = -3\text{cis}\,\frac{2\pi}{3}$,

then $\bar{z} = -3\text{cis}\left(-\frac{2\pi}{3}\right) = 3\text{cis}\,\frac{\pi}{3}$

d Let $z = 5\text{cis}\left(-\frac{\pi}{4}\right)$,

then $\bar{z} = 5\text{cis}\,\frac{\pi}{4}$

Solutions to Exercise 6D

1

$$4\text{cis}\left(\frac{\pi}{6}\right) + 6\text{cis}\left(\frac{2\pi}{3}\right) = 4\cos\left(\frac{\pi}{6}\right) + 4\sin\left(\frac{\pi}{6}\right)i$$
$$+6\cos\left(\frac{2\pi}{3}\right) + 6\sin\left(\frac{2\pi}{3}\right)i$$
$$= (2\sqrt{3} + 2i) + (-3 + 3\sqrt{3}i)$$
$$= (2\sqrt{3} - 3) + (2 + 3\sqrt{3}i)$$

2 4

a $4\text{cis}\,\frac{2\pi}{3} \times 3\text{cis}\,\frac{3\pi}{4}$

$$= 12\text{cis}\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)$$
$$= 12\text{cis}\left(\frac{17\pi}{12} - 2\pi\right)$$

Note : $\frac{17\pi}{12} \notin (-\pi, \pi]$

$$= 12\text{cis}\left(-\frac{7\pi}{12}\right)$$

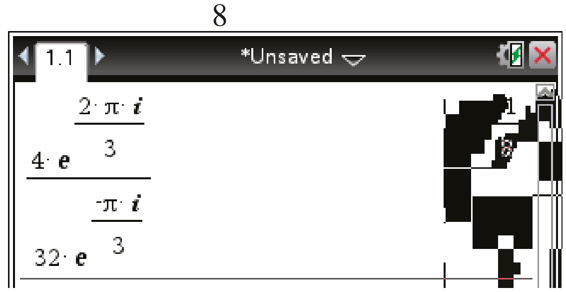
b $$\frac{\sqrt{2}\text{cis}\,\frac{\pi}{2}}{\sqrt{8}\text{cis}\,\frac{5\pi}{6}} = \frac{1}{2}\text{cis}\left(\frac{\pi}{2} - \frac{5\pi}{6}\right)$$
$$= \frac{1}{2}\text{cis}\left(-\frac{\pi}{3}\right)$$

c $\frac{1}{2}\text{cis}\left(-\frac{2\pi}{5}\right) \times \frac{7}{3}\text{cis}\left(\frac{\pi}{3}\right)$

$$= \frac{7}{6}\text{cis}\left(-\frac{2\pi}{5} + \frac{\pi}{3}\right)$$
$$= \frac{7}{6}\text{cis}\left(-\frac{\pi}{15}\right)$$

d $$\frac{4\text{cis}\left(-\frac{\pi}{4}\right)}{\frac{1}{2}\text{cis}\left(\frac{7\pi}{10}\right)} = 8\text{cis}\left(-\frac{\pi}{4} - \frac{7\pi}{10}\right)$$
$$= 8\text{cis}\left(-\frac{19\pi}{20}\right)$$

e $$\frac{4\text{cis}\left(\frac{2\pi}{3}\right)}{32\text{cis}\left(-\frac{\pi}{3}\right)} = \frac{1}{8}\text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$$
$$= \frac{1}{8}\text{cis}(\pi)$$
$$= \frac{1}{8}(\cos\pi + i\sin\pi)$$
$$= -\frac{1}{8}$$



3 **a** $2\text{cis}\left(\frac{5\pi}{6}\right) \times \left(\sqrt{2}\text{cis}\left(\frac{7\pi}{8}\right)\right)^4$

$$= 2\text{cis}\left(\frac{5\pi}{6}\right) \times 4\text{cis}\left(\frac{7\pi}{2}\right)$$
$$= 8\text{cis}\left(\frac{5\pi}{6} + \frac{7\pi}{2}\right)$$
$$= 8\text{cis}\left(\frac{13\pi}{3} - 4\pi\right)$$
$$= 8\text{cis}\left(\frac{\pi}{3}\right)$$

b $\dfrac{1}{\left(\frac{3}{2}\text{cis}\left(\frac{5\pi}{8}\right)\right)^3}$

$= \left(\frac{3}{2}\text{cis}\left(\frac{5\pi}{8}\right)\right)^{-3}$

$= \frac{8}{27}\text{cis}\left(-\frac{15\pi}{8} + 2\pi\right)$

$= \frac{8}{27}\text{cis}\left(\frac{\pi}{8}\right)$

c $\left(\text{cis}\left(\frac{\pi}{6}\right)\right)^8 \times \left(\sqrt{3}\text{cis}\left(\frac{\pi}{4}\right)\right)^6$

$= \text{cis}\left(\frac{4\pi}{3}\right) \times 27\text{cis}\left(\frac{3\pi}{2}\right)$

$= 27\text{cis}\left(\frac{4\pi}{3} + \frac{3\pi}{2}\right)$

$= 27\text{cis}\left(\frac{17\pi}{6} - 2\pi\right)$

$= 27\text{cis}\left(\frac{5\pi}{6}\right)$

d $\left(\frac{1}{2}\text{cis}\left(\frac{\pi}{2}\right)\right)^{-5}$

$= 32\text{cis}\left(-\frac{5\pi}{2} + 2\pi\right)$

$= 32\text{cis}\left(-\frac{\pi}{2}\right)$

$= 32\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$= -32i$

e $\left(2\text{cis}\left(\frac{3\pi}{2}\right) \times 3\text{cis}\left(\frac{\pi}{2}\right)\right)^3$

$= \left(6\text{cis}\left(\frac{3\pi}{2} + \frac{\pi}{6}\right)\right)^3$

$= \left(6\text{cis}\left(\frac{5\pi}{3} - 2\pi\right)\right)^3$

$= \left(6\text{cis}\left(-\frac{\pi}{3}\right)\right)^3$

$= 216\text{cis}(-\pi)$

$= 216\text{cis}(\pi)$

$= 216(\cos(\pi) + i\sin(\pi))$

$= -216$

f $\left(\frac{1}{2}\text{cis}\left(\frac{\pi}{8}\right)\right)^{-6} \times \left(4\text{cis}\left(\frac{\pi}{3}\right)\right)^2$

$= 64\text{cis}\left(-\frac{6\pi}{8}\right) \times 16\text{cis}\left(\frac{2\pi}{3}\right)$

$= 1024\text{cis}\left(\frac{2\pi}{3} - \frac{3\pi}{4}\right)$

$= 1024\text{cis}\left(-\frac{\pi}{12}\right)$

g $\dfrac{\left(6\text{cis}\left(\frac{2\pi}{5}\right)\right)^3}{\left(\frac{1}{2}\text{cis}\left(\frac{\pi}{4}\right)\right)^{-5}}$

$= \left(6\text{cis}\left(\frac{2\pi}{5}\right)\right)^3 \times \left(\frac{1}{2}\text{cis}\left(-\frac{\pi}{4}\right)\right)^5$

$= 216\text{cis}\left(\frac{6\pi}{5}\right) \times \frac{1}{32}\text{cis}\left(-\frac{5\pi}{4}\right)$

$= \frac{27}{4}\text{cis}\left(\frac{6\pi}{5} - \frac{5\pi}{4}\right)$

$= \frac{27}{4}\text{cis}\left(-\frac{\pi}{20}\right)$

4 a $\text{Arg}(z_1) + \text{Arg}(z_2) = \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$

$z_1 z_2 = \text{cis}\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \text{cis}\left(\frac{7\pi}{12}\right)$

$\text{Arg}\,(z_1 z_2) = \frac{7\pi}{12}$

$\therefore \quad \text{Arg}\,(z_1 z_2) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2)$

b $\text{Arg}\,(z_1) + \text{Arg}\,(z_2) = -\frac{2\pi}{3} - \frac{3\pi}{4}$

$= -\frac{17\pi}{12}$

$z_1 z_2 = \text{cis}\left(-\frac{2\pi}{3} - \frac{3\pi}{4}\right)$

$= \text{cis}\left(-\frac{17\pi}{12} + 2\pi\right)$

$= \text{cis}\left(\frac{7\pi}{2}\right)$

$\text{Arg}\,(z_1 z_2) = \frac{7\pi}{12}$

$\therefore \quad \text{Arg}\,(z_1 z_2) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2) + 2\pi$

c $\text{Arg}\,(z_1) + \text{Arg}\,(z_2) = \frac{2\pi}{3} + \frac{\pi}{2} = \frac{7\pi}{6}$

$z_1 z_2 = \text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{2}\right)$

$= \text{cis}\left(\frac{7\pi}{6} - 2\pi\right)$

$= \text{cis}\left(-\frac{5\pi}{6}\right)$

$\text{Arg}\,(z_1 z_2) = -\frac{5\pi}{6}$

$\therefore \quad \text{Arg}\,(z_1 z_2) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2) - 2\pi$

5 Let $z_1 = r_1 \text{cis}\,\theta_1$ and $z_2 = r_2 \text{cis}\,\theta_2$

Then, $z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$

and $\text{Arg}\,(z_2 z_2) = \theta_1 + \theta_2$

Since $-\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \theta_2 < \frac{\pi}{2}$

$\Rightarrow -\pi < \theta_1 + \theta_2 < \pi$

$\therefore \quad \text{Arg}\,(z_1 z_2) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2) = \theta_1 + \theta_2$

Also,

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$

$\Rightarrow \text{Arg}\left(\frac{z_1}{z_2}\right) = (\theta_1 - \theta_2)$

$\Rightarrow -\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}$ and $-\frac{\pi}{2} < -\theta_2 < \frac{\pi}{2}$

$\therefore \quad -\pi < \theta_1 - \theta_2 < \pi$

$\therefore \quad \text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}\,(z_1) - \text{Arg}\,(z_2) = \theta_1 - \theta_2$

6 a $\text{Arg}\,(z) = \text{Arg}\,(1 + i)$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \quad \theta = \frac{\pi}{4}$

$\therefore \quad \text{Arg}\,(1 + i) = \frac{\pi}{4}$

b $\text{Arg}\,(-z) = \text{Arg}\,(-1 - i)$

$\cos\theta = -\frac{1}{\sqrt{2}}$

$\sin\theta = -\frac{1}{\sqrt{2}}$

$\therefore \quad \theta = -\frac{3\pi}{4}$

$\therefore \quad \text{Arg}\,(-1 - i) = -\frac{3\pi}{4}$

c $\text{Arg}\left(\frac{1}{z}\right) = \text{Arg}\left(\frac{1}{1 + i}\right)$

$= \text{Arg}\left(\frac{1}{2} - \frac{1}{2}i\right)$

$\cos\theta = \left(\dfrac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right) = \frac{\sqrt{2}}{2}$

$\sin\theta = \left(\dfrac{-\frac{1}{\sqrt{2}}}{\frac{1}{2}}\right) = -\frac{\sqrt{2}}{2}$

$\therefore \quad \theta = -\frac{\pi}{4}$

$$\therefore \quad \text{Arg}\left(\frac{1}{1+i}\right) = -\frac{\pi}{4}$$

7 The point $(2, 3)$ is rotated by $\frac{\pi}{6}$ clockwise. By multiplying two complex numbers, find the image of this rotation.
Solution. We note that multiplication by the complex number $\text{cis}\,\theta = \cos\theta + i\sin\theta$ will rotate any given complex number by angle θ anti-clockwise. Therefore we will write $z = 2 + 3i$, and then rotate this point by angle $\theta = -\frac{\pi}{6}$. Therefore,

$$\begin{aligned}(2+3i)\text{cis}\left(-\frac{\pi}{6}\right) &= (2+3i)(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right))\\ &= (2+3i)\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\\ &= \frac{2\sqrt{3}+3}{2} + i\frac{3\sqrt{3}-2}{2}\end{aligned}$$

Therefore, the point is rotated to $\left(\frac{2\sqrt{3}+3}{2}, \frac{3\sqrt{3}-2}{2}\right)$.

8 **a** $\sin\theta + i\cos\theta$

$$\begin{aligned}&= \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\\ &= \text{cis}\left(\frac{\pi}{2} - \theta\right)\end{aligned}$$

b **i** $$\begin{aligned}(\sin\theta + i\cos\theta)^7 &= \left(\text{cis}\left(\frac{\pi}{2} - \theta\right)\right)^7\\ &= \text{cis}\left(\frac{7\pi}{2} - 7\theta\right)\\ &= \text{cis}\left(\frac{3\pi}{2} - 7\theta\right)\end{aligned}$$

ii $(\sin\theta + i\cos\theta)(\cos\theta + i\sin\theta)$

$$\begin{aligned}&= \text{cis}\left(\frac{\pi}{2} - \theta\right) \times \text{cis}\,\theta\\ &= \text{cis}\left(\frac{\pi}{2}\right)\\ &= i\end{aligned}$$

iii $(\sin\theta + i\cos\theta)^{-4}$

$$\begin{aligned}&= \left(\text{cis}\left(\frac{\pi}{2} - \theta\right)\right)^{-4}\\ &= \text{cis}\left(-\frac{4\pi}{2} + 4\theta\right)\\ &= \text{cis}(-2\pi + 4\theta)\\ &= \text{cis}(4\theta)\end{aligned}$$

iv $(\sin\theta + i\cos\theta)(\sin\phi + i\cos\phi)$

$$\begin{aligned}&= \text{cis}\left(\frac{\pi}{2} - \theta\right) \times \text{cis}\left(\frac{\pi}{2} - \phi\right)\\ &= \text{cis}\,(\pi - \theta - \phi)\end{aligned}$$

9 **a** $$\begin{aligned}\cos\theta - i\sin\theta &= \cos(-\theta) + i\sin(-\theta)\\ &= \text{cis}(-\theta)\end{aligned}$$

b **i** $$\begin{aligned}(\cos\theta - i\sin\theta)^5 &= (\text{cis}(-\theta))^5\\ &= \text{cis}(-5\theta)\end{aligned}$$

ii $$\begin{aligned}(\cos\theta - \sin\theta)^{-3} &= (\text{cis}(-\theta))^{-3}\\ &= \text{cis}(3\theta)\end{aligned}$$

iii $$\begin{aligned}&= \text{cis}(-\theta) \times \text{cis}(\theta)\\ &= \text{cis}0\\ &= 1\end{aligned}$$

iv $(\cos\theta - i\sin\theta)(\sin\theta + i\cos\theta)$

$= \text{cis}(-\theta) \times \text{cis}\left(\frac{\pi}{2} - \theta\right)$

(from question 5)

$= \text{cis}\left(\frac{\pi}{2} - 2\theta\right)$

10 a $\sin\theta - i\cos\theta = \cos\left(\theta - \frac{\pi}{2}\right)$

$+ i\sin\left(\theta - \frac{\pi}{2}\right)$

$= \text{cis}\left(\theta - \frac{\pi}{2}\right)$

b i $(\sin\theta - i\cos\theta)^6 = \left(\text{cis}\left(\theta - \frac{\pi}{2}\right)\right)^6$

$= \text{cis}\left(6\theta - \frac{6\pi}{2}\right)$

ii $(\sin\theta - i\cos\theta)^{-2} = \left(\text{cis}\left(\theta - \frac{\pi}{2}\right)\right)^{-2}$

$= \text{cis}\ (-2\theta + \pi)$

$= \text{cis}\ (\pi - 2\theta)$

iii $(\sin\theta - i\cos\theta)^2(\cos\theta - i\sin\theta)$

$= \text{cis}\ (2\theta - \pi) \times \text{cis}\ (-\theta)$

(from question 6)

$= \text{cis}\ (\theta - \pi)$

iv $\dfrac{\sin\theta - i\cos\theta}{\cos\theta + i\sin\theta} = \dfrac{\text{cis}\left(\theta - \frac{\pi}{2}\right)}{\text{cis}\ (\theta)}$

$= \text{cis}\left(-\frac{\pi}{2}\right)$

$= -i$

11 a i $|1 + i\tan\theta| = \sqrt{1 + \tan^2\theta} = \sec\theta$

$1 + i\tan\theta = \sec\theta(\cos\theta + i\sin\theta)$

$= \sec\theta\text{cis}\ \theta$

ii $|1 + i\cot\theta| = \sqrt{1 + \cot^2\theta}$

$= \text{cosec}\ \theta$

$1 + i\cot\theta = \text{cosec}\ \theta$

$\times (\sin\theta + i\cos\theta)$

$= \text{cosec}\ \theta\ \text{cis}\left(\frac{\pi}{2} - \theta\right)$

iii $\dfrac{1}{\sin\theta} + \dfrac{i}{\cos\theta} = \dfrac{\cos\theta + i\sin\theta}{\sin\theta\cos\theta}$

$= \dfrac{\text{cis}\ \theta}{\sin\theta\cos\theta}$

$= \text{cosec}\theta\sec\theta\ \text{cis}\ \theta$

b i $(1 + i\tan\theta)^2 = \sec^2\theta\ \text{cis}\ 2\theta$

ii $(1 + i\cot\theta)^{-3}$

$= \sin^3\theta\ \text{cis}\left(3\theta - \frac{3\pi}{2}\right)$

iii $\dfrac{1}{\sin\theta} - \dfrac{i}{\cos\theta}$

$= \text{cosec}\ \theta\sec\theta\text{cis}(-\theta)$

12 a $|1 + i\sqrt{3}| = \sqrt{1 + 3} = 2$

$\sin\theta = \frac{\sqrt{3}}{2},\ \cos\theta = \frac{1}{2}$

$\therefore$ $\text{Arg}\ (1 + i\sqrt{3}) = \theta = \frac{\pi}{3}$

$\therefore$ $(1 + i\sqrt{3})^6 = 2^6\text{cis}\left(\frac{6\pi}{3}\right)$

$= 64\text{cis}(2\pi)$

$= 64\text{cis}\ 0$

b $|1 - i| = \sqrt{2}$

$\sin\theta = -\frac{1}{\sqrt{2}},\ \cos\theta = \frac{1}{\sqrt{2}}$

$\therefore \quad \text{Arg}(1-i) = \theta = -\frac{\pi}{4}$

$\therefore \quad (1-i)^{-5} = (\sqrt{2})^{-5}\text{cis}\left(\frac{5\pi}{4} - 2\pi\right)$

$= \frac{\sqrt{2}}{8}\text{cis}\left(-\frac{3\pi}{4}\right)$

c $|\sqrt{3} - i| = \sqrt{3+1} = 2$

$\sin\theta = -\frac{1}{2}, \ \cos\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad \text{Arg}(\sqrt{3} - i) = \theta = -\frac{\pi}{6}$

$\therefore \quad \sqrt{3} - i = 2\text{cis}\left(-\frac{\pi}{6}\right)$

and $i = \text{cis}\left(\frac{\pi}{2}\right)$

$\therefore \quad i(\sqrt{3} - i)^7 = 2^7\text{cis}\left(-\frac{7\pi}{6} + \frac{\pi}{2}\right)$

$= 128\text{cis}\left(-\frac{4\pi}{6}\right)$

$= 128\text{cis}\left(-\frac{2\pi}{3}\right)$

d $(-3 + i\sqrt{3})^{-3} = (\sqrt{3}(-\sqrt{3} + i))^{-3}$

$= (2\sqrt{3}\text{cis}\left(\frac{5\pi}{6}\right))^{-3}$

$= \frac{\sqrt{3}}{72}\text{cis}\left(-\frac{5\pi}{2}\right))$

(add II to argument in part c)

e $1 + i\sqrt{3} = 2\text{cis}\left(\frac{\pi}{3}\right)$

$i = \text{cis}\left(\frac{\pi}{2}\right)$

(see part **a** and **b**)

$\therefore \quad \frac{(1 + i\sqrt{3})^3}{i(1-i)^5} = \frac{8\text{cis}(\pi)}{(\sqrt{2})^5\text{cis}\left(-\frac{5\pi}{4} + \frac{\pi}{2}\right)}$

$= \sqrt{2}\text{cis}\left(\pi + \frac{5\pi}{4} - \frac{\pi}{2}\right)$

$= \sqrt{2}\text{cis}\left(\frac{7\pi}{4} - 2\pi\right)$

$= \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

f $-1 + i\sqrt{3} = 2\text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore \quad (-1 + i\sqrt{3})^4 = 16\text{cis}\left(\frac{8\pi}{3} - 2\pi\right)$

$= 16\text{cis}\left(\frac{2\pi}{3}\right)$

$-\sqrt{2} - i\sqrt{2} = 2\text{cis}\left(-\frac{3\pi}{4}\right)$

$\therefore \quad (-\sqrt{2} - i\sqrt{2})^3 = 8\text{cis}\left(-\frac{9\pi}{4} + 2\pi\right)$

$= 8\text{cis}\left(-\frac{\pi}{4}\right)$

$\sqrt{3} - 3i = 2\sqrt{3}\text{cis}\left(-\frac{\pi}{3}\right)$

$\therefore \quad \frac{(-1 + i\sqrt{3})^4(-\sqrt{2} - i\sqrt{2})^3}{\sqrt{3} - 3i}$

$= \frac{128\text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)}{2\sqrt{3}\text{cis}\left(-\frac{\pi}{3}\right)}$

$= \frac{64\sqrt{3}}{3}\text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{4} + \frac{\pi}{3}\right)$

$= \frac{64\sqrt{3}}{3}\text{cis}\left(\frac{3\pi}{4}\right)$

g $-1 + i = \sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

$$\therefore \quad (-1+i)^5\left(\frac{1}{2}\text{cis}\left(\frac{\pi}{4}\right)\right)$$

$$= 4\sqrt{2}\times\frac{1}{8}\text{cis}\left(\frac{15\pi}{4}+\frac{3\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}\text{cis}\left(\frac{9\pi}{2}-4\pi\right)$$

$$= \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2}\right)$$

h $1-i\sqrt{3} = 2\text{cis}\left(-\frac{\pi}{3}\right)$

$$\therefore \quad \frac{\left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^3}{(1-i\sqrt{3})^2} = \frac{1}{4}\text{cis}\left(\frac{6\pi}{5}+\frac{2\pi}{3}\right)$$

$$= \frac{1}{4}\text{cis}\left(\frac{28\pi}{15}-2\pi\right)$$

$$= \frac{1}{4}\text{cis}\left(-\frac{2\pi}{15}\right)$$

i $1-i = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

$$\therefore \quad (1-i)\text{cis}\left(\frac{2\pi}{3}\right) = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right)$$

$$= \sqrt{2}\text{cis}\left(\frac{5\pi}{12}\right)$$

$$\therefore \quad \left((1-i)\text{cis}\left(\frac{2\pi}{3}\right)\right)^7$$

$$= (\sqrt{2})^7\text{cis}\left(\frac{35\pi}{12}\right)$$

$$= 8\sqrt{2}\text{cis}\left(\frac{35\pi}{12}-2\pi\right)$$

$$= 8\sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right)$$

13 We first find that

$$1+i = \sqrt{2}\,\text{cis}\,(\tfrac{\pi}{4})$$

$$\implies \qquad (1+i)^{20} = (\sqrt{2})^{10}\,\text{cis}\,5\pi$$

$$1-i = \sqrt{2}\,\text{cis}\,(-\tfrac{\pi}{4})$$

$$\implies \qquad (1-i)^{20} = (\sqrt{2})^{21}\,\text{cis}\,(-\tfrac{21\pi}{4})$$

Therefore

$$\frac{(1+i)^{20}}{(1-i)^{21}} = \frac{(\sqrt{2})^{10}\,\text{cis}\,5\pi}{(\sqrt{2})^{21}\,\text{cis}\,(-\frac{21\pi}{4})}$$

$$= \frac{1}{\sqrt{2}}\,\text{cis}\,(\tfrac{41\pi}{4})$$

$$= \frac{1}{\sqrt{2}}\,\text{cis}\,(\tfrac{\pi}{4})$$

$$= \frac{1}{\sqrt{2}}(\cos(\tfrac{\pi}{4}) + i\sin(\tfrac{\pi}{4}))$$

$$= \frac{1}{2}+\frac{i}{2}.$$

14 **a** $1+i = \sqrt{2}\text{cis}\frac{\pi}{4}$

$$(1+i)^{11} = 2^{\frac{11}{2}}\text{cis}\frac{11\pi}{4}$$

$$= 2^{\frac{11}{2}}\text{cis}\frac{3\pi}{4}$$

$$= 2^{\frac{11}{2}}\times\frac{1}{2^{\frac{1}{2}}}(-1+i)$$

$$= 2^5(-1+i)$$

$$= -32+32i$$

b $P(1)$ LHS = 1: RHS = $\frac{1}{1} = 1$

$P(1)$ is true.

Assume $P(k)$ is true:

$$1+z+z^2+\cdots+z^{k-1} = \frac{1-z^k}{1-z}$$

For $P(k+1)$

$$\begin{aligned}\text{LHS} &= 1 + z + z^2 + \cdots + z^{k-1} + z^k \\ &= \frac{1 - z^k}{1 - z} + z^k \\ &= \frac{1 - z^k + z^k - z^{k+1}}{1 - z} \\ = &= \frac{1 - z^{k+1}}{1 - z}\end{aligned}$$

c Using the formula

$$\begin{aligned}&= \frac{1 - (1 + i)^{11}}{1 - 1 - i} \\ &= \frac{1 - (-32 + 32i}{-i} \\ &= \frac{33 - 32i)}{-i} \\ &= 33i + 32 \\ &= 32 + 33i\end{aligned}$$

Solutions to Exercise 6E

1 **a** $z^2 + 16 = z^2 - (4i)^2 = (z - 4i)(z + 4i)$

b $z^2 + 5 = (z - i\sqrt{5})(z + i\sqrt{5})$

c $z^2 + 2z + 5 = (z^2 + 2z + 1) + 4$

$= (z + 1)^2 - (2i)^2$

$= (z + 1 - 2i)(z + 1 + 2i)$

d $z^2 - 3z + 4 = \left(z^2 - 3z + \frac{9}{4}\right) + 4 - \frac{9}{4}$

$= \left(z - \frac{3}{2} + \frac{i\sqrt{7}}{2}\right)$

$\times \left(z - \frac{3}{2} - \frac{i\sqrt{7}}{2}\right)$

e $2z^2 - 8z + 9 = 2\left[(z^2 - 4z + 4) + \frac{1}{2}\right]$

$= 2\left(z - 2 - \frac{i\sqrt{2}}{2}\right)$

$\times \left(z - 2 + \frac{i\sqrt{2}}{2}\right)$

f $3\left[(z^2 + 2z + 1) + \frac{1}{3}\right]$

$= 3\left(z + 1 + \frac{i\sqrt{3}}{3}\right)\left(z + 1 - \frac{i\sqrt{3}}{3}\right)$

g $3\left[\left(z^2 + \frac{2}{3}z + \frac{1}{9}\right) + \frac{5}{9}\right]$

$= 3\left(z + \frac{1}{3} + \frac{i\sqrt{5}}{3}\right)\left(z + \frac{1}{3} - \frac{i\sqrt{5}}{3}\right)$

h $2\left[\left(z^2 - \frac{1}{2}z + \frac{1}{16}\right) + \frac{3}{2} - \frac{1}{16}\right]$

$= 2\left(z - \frac{1}{4} - \frac{i\sqrt{23}}{4}\right)\left(z - \frac{1}{4} + \frac{i\sqrt{23}}{4}\right)$

2 **a** $x^2 + 25 = 0$

$x^2 = -25$

$x = \pm 5i$

b $x^2 + 8 = 0$

$x^2 = -8$

$x = \pm 2i\sqrt{2}$

c $x^2 - 4x + 5 = 0$

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 5}}{2 \times 1}$

$= \dfrac{4 \pm \sqrt{16 - 20}}{2}$

$= \dfrac{4 \pm 2i}{2}$

$x = 2 \pm i$

d $3x^2 + 7x + 5 = 0$

$x = \dfrac{-7 \pm \sqrt{49 - 3 \times 4 \times 5}}{6}$

$= \dfrac{-7 \pm \sqrt{-11}}{6}$

$x = \dfrac{-7 \pm i\sqrt{11}}{6}$

e $x^2 = 2x - 3$

$\therefore \quad x^2 - 2x + 3 = 0$

$x = \dfrac{2 \pm \sqrt{4 - 12}}{2}$

$= \dfrac{2 \pm \sqrt{-8}}{2}$

$x = 1 \pm i\sqrt{2}$

f $5x^2 + 1 = 3x$

$\therefore \quad 5x^2 - 3x + 1 = 0$

$x = \dfrac{3 \pm \sqrt{9 - 4 \times 5}}{10}$

$= \dfrac{3 \pm \sqrt{-11}}{10}$

$x = \dfrac{3 + i\sqrt{11}}{10}$

g $z^2 + (1 + 2i)z + (-1 + i) = 0$

$$z = \frac{-1 - 2i \pm \sqrt{(1 + 2i)^2 - 4(-1 + i)}}{2}$$

$$z = \frac{-1 - 2i \pm \sqrt{-3 + 4i - 4(-1 + i)}}{2}$$

$$z = \frac{-1 - 2i \pm \sqrt{1}}{2}$$

$z = -i$ or $z = -1 - i$

h $z^2 + z + (1 - i) = 0$

$$z = \frac{-1 - 2i \pm \sqrt{1 - 4(1 - i)}}{2}$$

$$z = \frac{-1 - 2i \pm \sqrt{-3 + 4i}}{2}$$

$$z = \frac{-1 - 2i \pm (1 + 2i)}{2}$$

$z = i$ or $z = -1 - i$

3 Let $\alpha = a + id$ where $a, d \in \mathbb{R}$

$\overline{\alpha} = a - id$

$(z - \alpha)(z - \overline{\alpha}) = z^2 - (\alpha + \overline{\alpha})z + \alpha\overline{\alpha}$

$= z^2 - 2az + a^2 + d^2$

Therefore, $b = 2a \in \mathbb{R}$ and

$c = a^2 + d^2 \in \mathbb{R}$

Alternatively substitute α and $\overline{\alpha}$ into $z^2 + bz + c = 0$

$\alpha^2 + b\alpha + c = 0 \ldots (1)$

$\overline{\alpha}^2 + b\overline{\alpha} + c = 0 \ldots (2)$

Solutions to Exercise 6F

1 **a** Let $P(z) = z^3 - 4z^2 - 4z - 5$

Possible factors are $\pm 1, \ \pm 5$

$P(1) = 1 - 4 - 4 - 5 \neq 0$

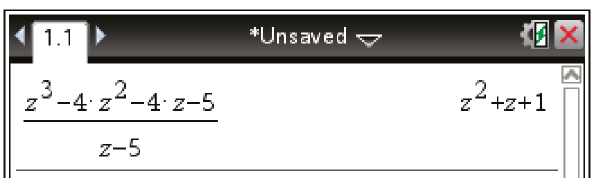
$P(-1) = -1 - 4 + 4 - 5 \neq 0$

$P(5) = 125 - 100 - 20 - 5 = 0$

$\therefore$ $(z - 5)$ is a factor.

By long division

$$\begin{array}{r|l} & z^2 + z + 1 \\ \hline z - 5 & z^3 - 4z^2 - 4z - 5 \\ & \underline{z^3 - 5z^2} \\ & \quad z^2 - 4z \\ & \quad \underline{z^2 - 5z} \\ & \qquad z - 5 \\ & \qquad \underline{z - 5} \\ & \qquad\quad 0 \end{array}$$



$P(z) = (z - 5)(z^2 + z + 1)$

$$z^2 + z + 1 = \left(z^2 + z + \frac{1}{4}\right) + \frac{3}{4}$$

$$= \left(z + \frac{1}{2}\right)^2 - \left(\frac{i\sqrt{3}}{2}\right)^2$$

$$= \left(z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$\times \left(z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$\therefore \quad P(z) = (z - 5)\left(z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$\times \left(z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

b Let $P(z) = z^3 - z^2 - z + 10$

Possible factors are $\pm 1, \ \pm 2, \ \pm 5$

(Only ± 2 needs to be tried because of 3 of 4 odd coefficients.)

Use a CAS calculator to help find the ‘first’ factor.

$P(2) = 8 - 4 - 2 + 10 \neq 0$

$P(-2) = -8 - 4 + 2 + 10 = 0$

$\therefore$ $(z + 2)$ is a factor.

By long division

$$\begin{array}{r|l} & z^2 + 3z + 5 \\ \hline z + 2 & z^3 - z^2 - z + 10 \\ & \underline{z^3 + 2z^2} \\ & \quad -3z^2 - z \\ & \quad \underline{-3z^2 - 6z} \\ & \qquad 5z + 10 \\ & \qquad \underline{5z + 10} \\ & \qquad\quad 0 \end{array}$$

$P(z) = (z + 2)(z^2 - 3z + 5)$

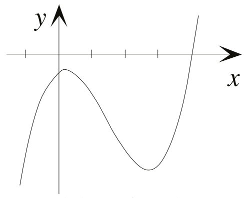
$$z^2 - 3z + 5 = \left(z^2 - 3z + \frac{9}{4}\right) + 5$$

$$= \left(z - \frac{3}{2}\right)^2 - \left(\frac{i\sqrt{11}}{2}\right)^2$$

$$\therefore \quad P(z) = (z + 2)\left(z - \frac{3}{2} - \frac{i\sqrt{11}}{2}\right)$$

$$\times \left(z - \frac{3}{2} + \frac{i\sqrt{11}}{2}\right)$$

c Using a CAS calculator, the graph of the function of $y = 3x^3 - 13x^2 + 5x - 4$ is as shown. Therefore, the most probable factor of the polynomial $P(z) = 3z^3 - 13z^2 + 5z - 4$ is $(z - 4)$.



A table of values also helps.

$$P(4) = 3 \times 64 - 13 \times 16 + 20 - 4$$
$$= 192 - 208 + 20 - 4$$
$$= 0$$

$\therefore$ $z - 4$ is a factor.

By long division

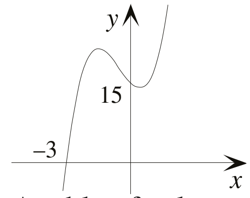
$$\begin{array}{r|l} & 3z^2 - z + 1 \\ \hline z - 4 & 3z^3 - 13z^2 + 5z - 4 \\ & \underline{3z^3 - 12z^2} \\ & \quad -z^2 + 5z \\ & \quad \underline{-z^2 + 4z} \\ & \qquad z - 4 \\ & \qquad \underline{z - 4} \\ & \qquad\ 0 \end{array}$$

$$P(z) = (z - 4)(3z^2 - z + 1)$$

$$3z^2 - z + 1 = 3\left[\left(z^2 - \frac{1}{3}z + \frac{1}{36}\right) + \frac{1}{3} - \frac{1}{36}\right]$$
$$= 3\left[\left(z - \frac{1}{6}\right)^2 - \left(\frac{i\sqrt{11}}{6}\right)^2\right]$$

$$\therefore \quad P(z) = 3(z - 4)\left(z - \frac{1}{6} - \frac{i\sqrt{11}}{6}\right) \times \left(z - \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)$$

d Using a CAS calculator, the graph of the function of $y = 2x^3 + 3x^2 - 4x + 15$ is as shown. Therefore, the most probable factor of the polynomial $P(z) = 2z^3 + 3z^2 - 4z + 15$ is $(z + 3)$.



A table of values also helps.

$$P(-3) = -2 \times 27 + 3 \times 9 + 4 \times 3 + 15$$
$$= -54 + 27 + 12 + 15$$
$$= 0$$

$\therefore$ $z + 3$ is a factor.

By long division

$$\begin{array}{r|l} & 2z^2 - 3z + 5 \\ \hline z + 3 & 2z^3 + 3z^2 - 4z + 15 \\ & \underline{2z^3 + 6z^2} \\ & \quad -3z^2 - 4z \\ & \quad \underline{-3z^2 - 9z} \\ & \qquad 5z + 15 \\ & \qquad \underline{5z + 15} \\ & \qquad\ 0 \end{array}$$

$$P(z) = (z + 3)(2z^2 - 3z + 5)$$

$$2z^2 - 3z + 5 = 2\left[\left(z^2 - \frac{3}{2}z + \frac{9}{16}\right) + \frac{5}{2} - \frac{9}{16}\right]$$
$$= 2\left[\left(z - \frac{3}{4}\right)^2 - \left(\frac{i\sqrt{31}}{4}\right)^2\right]$$

$$\therefore \quad P(z) = 2(z + 3)\left(z - \frac{3}{4} - \frac{i\sqrt{31}}{4}\right) \times \left(z - \frac{3}{4} + \frac{i\sqrt{31}}{4}\right)$$

e $z^3 - (2 - i)z^2 + z - 2 + i$

$= z^2(z - 2 + i) + z - 2 + i$

(factorise by grouping)

$= (z - 2 + i)(z^2 + 1)$

$= (z - 2 + i)(z^2 - i^2)$

$= (z - 2 + i)(z + i)(z - i)$

2 a $P(1+i) = (1+i)^3 + 4(1+i)^2$

$- 10(i+1) + 12$

$= 1 + 3i^2 + 3i + i^3$

$+ 4(1 + 2i + i^2)$

$- 10i - 10 + 12$

$= 7 + 7i^2 + i + i^3$

$= 7 - 7 + i - i = 0$

$\therefore$ $z - (1+i) = z - 1 - i$ is a factor.

b $z - (1-i) = z - 1 + i$ is another factor because of the rule of conjugate pairs.

c $(z - 1 - i)(z - 1 + i) = (z-1)^2 + 1 = z^2 - 2z + 2$

By long division

$$\begin{array}{r|l} & z + 6 \\ z^2 - 2z + 2 & z^3 + 4z^2 - 10z + 12 \\ & z^3 - 2z^2 + 2z \\ & \quad 6z^2 - 12z + 12 \\ & \quad 6z^2 - 12z + 12 \\ & \qquad\qquad 0 \end{array}$$

$P(z) = (z+6)(z - 1 - i)(z - 1 + i)$

3 a $P(-2+i) = 2(i-2)^3 + 9(i-2)^2$

$+ 14(i-2) + 5$

$= 2(i^3 - 6i^2 + 12i - 8)$

$+ 9(i^2 - 4i + 4)$

$+ 14i - 28 + 5$

$= 2i^3 - 3i^2 + (24-36+14)i$

$+ (-16 + 36 - 28 + 5)$

$= 2i^3 - 3i^2 + 2i - 3$

$= -2i + 3 + 2i - 3 = 0$

b Another factor of $P(z)$ can be obtained by conjugate pairs rule

$z - \overline{(-2+i)} = z - (-2 - i) = z + 2 + i$

c $(z + 2 - i)(z + 2 + i) = (z+2)^2 + 1 = z^2 + 4z + 5$

$\therefore$ $P(z) = (z^2 + 4z + 5)(2z + 1)$ by division.

$\therefore$ $P(z) = (2z + 1)(z + 2 - i)(z + 2 + i)$

4 a $P(1-3i) = (1-3i)^4 + 8(1-3i)^2$

$+ 16(1-3i) + 20$

$= (1-3i)^2[(1-3i)^2 + 8]$

$+ 16 - 48i + 20$

$= (1 - 6i - 9)(1 - 6i - 9 + 8)$

$+ 36 - 48i$

$= (-8 - 6i)(-6i) + 36 - 48i$

$= 48i - 36 + 36 - 48i = 0$

$\therefore$ $z - (1-3i) = z - 1 + 3i$ is a factor.

b $z - (1+3i) = z - 1 - 3i$ is another factor because of the rule of conjugate pairs.

c $(z-1+3i)(z-1-3i) = (z-1)^2+9$

$$= z^2-2z+10$$

$$\begin{array}{r|l}
 & z+2z+2 \\ \hline
z^2-2z+10 & z^4+8z^2-16z+20 \\
 & \underline{z^4-2z^3+10z^2} \\
 & \quad 2z^3-2z^2+16z \\
 & \quad \underline{2z^3-4z^2+20z} \\
 & \qquad 2z^2-4z+20 \\
 & \qquad \underline{2z^2-4z+20} \\
 & \qquad\qquad 0
\end{array}$$

$\therefore \quad P(z) = (z^2+2z+2)$

$\times(z^2-2z+10)$

$z^2+2z+2 = (z+1)^2+1$

$= (z+1+i)(z+1-i)$

$\therefore \quad P(z) = (z+1+i)(z+1-i)$

$\times(z-1+3i)(z-1-3i)$

5 a $z^4-81 = (z^2-9)(z^2+9)$

$= (z-3)(z+3)(z-3i)(z+3i)$

b $z^6-64 = (z^3-8)(z^3+8)$

$= (z-2)(z^2+2z+4)(z+2)$

$\times(z^2-2z+4)$

$= (z+2)(z-2)[(z^2+2z+1)$

$+3][(z^2-2z+1)+3]$

$= (z+2)(z-2)(z+1+i\sqrt{3})$

$\times(z+1-i\sqrt{3})(z-1$

$+i\sqrt{3})(z-1-i\sqrt{3})$

6 a $P(z) = z^3+(1-i)z^2+(1-i)z-i$

$$\begin{array}{r|l}
 & z^2+z+1 \\ \hline
z-i & z^3+(1-i)z^2+(1-i)z-i \\
 & \underline{z^3-iz^2} \\
 & \quad z^2+(1-i)z \\
 & \quad \underline{z^2-iz} \\
 & \qquad z-i \\
 & \qquad \underline{z-i} \\
 & \qquad\quad 0
\end{array}$$

$\therefore \quad P(z) = (z-i)(z^2+z+1)$

$$= (z-i)\left(\left(z+\frac{1}{2}\right)^2+\frac{3}{4}\right)$$

$$= (z-i)\left(\left(z+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}i\right)^2\right)$$

$$= (z-i)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$\times\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

b

$$\begin{array}{r|l}
 & z^2+2z-1 \\ \hline
z+i & z^3-(2-i)z^2-(1+2i)z-i \\
 & \underline{z^3+iz^2} \\
 & \quad -2z^2-(1+2i)z \\
 & \quad \underline{-2z^2-2zi} \\
 & \qquad -z-i \\
 & \qquad \underline{-z-i} \\
 & \qquad\quad 0
\end{array}$$

$\therefore \quad P(z) = (z+i)(z^2-2z-1)$

$= (z+i)((z-1)^2-(\sqrt{2})^2)$

$= (z+i)(z-1+\sqrt{2})$

$\times(z-1-\sqrt{2})$

c $P(z) = z^3-(2+2i)z^2-(3-4i)z+6i$

$$z^2 + 2z - 3$$
$$z - 2i \,\overline{\big)\, z^3 - (2 + 2i)z^2 - (3 - 4i)z + 6i}$$
$$\underline{z^3 - 2z^2 i}$$
$$-2z^2 - (3 - 4i)z$$
$$\underline{-2z^2 + 4zi}$$
$$-3z + 6i$$
$$\underline{-3z + 6i}$$
$$0$$

$$\begin{aligned} \therefore\quad P(z) &= (z - 2i)(z^2 - 2z - 3) \\ &= (z - 2i)((z - 1)^2 - 4) \\ &= (z - 2i)((z - 1)^2 - 2^2) \\ &= (z - 2i)(z - 1 + 2)(z - 1 - 2) \\ &= (z - 2i)(z + 1)(z - 3) \end{aligned}$$

d $P(z) = 2z^3 + (1 - 2i)z^2 - (5 + i)z + 5i$

$$2z^2 + z - 5$$
$$z - i \,\overline{\big)\, 2z^3 - (1 - 2i)z^2 - (5 - i)z + 5i}$$
$$\underline{2z^3 - 2z^2 i}$$
$$z^2 - (5 + i)$$
$$\underline{z^2 - zi}$$
$$-5z + 5i$$
$$\underline{-5z + 5i}$$
$$0$$

$$\begin{aligned} \therefore\quad P(z) &= (z - i)(2z^2 + z - 5) \\ &= 2(z - i)\left(z^2 + \frac{1}{2}z - \frac{5}{2}\right) \\ &= 2(z - i)\left(\left(z + \frac{1}{4}\right)^2 - \frac{41}{16}\right) \\ &= 2(z - i)\left(\left(z + \frac{1}{4}\right)^2 - \left(\frac{\sqrt{41}}{4}\right)^2\right) \\ &= 2(z - i)\left(z + \frac{1}{4} + \frac{\sqrt{41}}{4}\right) \\ &\quad \times \left(z + \frac{1}{4} - \frac{\sqrt{41}}{4}\right) \end{aligned}$$

7 a Let $P(z) = z^3 + 3z^2 + pz + 12$

$P(-2) = 0$ because $z + 2$ is a factor.

$P(-2) = -8 + 12 - 2p + 12 = -2p + 16$

$\therefore\quad -2p + 16 = 0$

$p = 8$

b Let $P(z) = z^3 + pz^2 + z - 4$

$P(i) = 0$ because $z - i$ is a factor of $P(z)$.

$P(i) = (i)^3 + pi^2 + i - 4 = -p - 4$

$\therefore\quad -p - 4 = 0$

$p = -4$

c Let $P(z) = 2z^3 + z^2 - 2z + p$

$P(i - 1) = 0$ because $z - (i - 1) = z + 1 - i$ is a factor.

$$\begin{aligned} P(i - 1) &= 2(i - 1)^3 + (i - 1)^2 \\ &\quad - 2(i - 1) + p \\ &= (i - 1)^2(2i - 2 + 1) \\ &\quad - 2i + 2 + p \\ &= (-2i)(2i - 1) \\ &\quad - 2i + 2 + p \\ &= -4i^2 + 2i - 2i + 2 + p \\ &= 6 + p \end{aligned}$$

$\therefore\quad 6 + p = 0$

$p = -6$

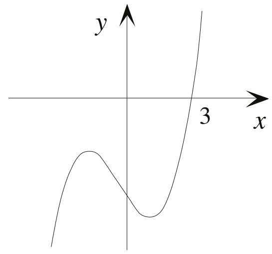
8 a Let $P(x) = x^3 + x^2 - 6x - 18$

Possible factors are

$\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm 9,\ \pm 18$

Using a CAS calculator, the graph of the function of $y = x^3 + x^2 - 6x - 18$ is as shown.

Therefore, the most probable real solution of the equation $x^3 + x^2 - 6x - 18 = 0$ is $x = 3$



$P(3) = 3^3 + 3^2 - 6 \times 3 - 18$

$\quad = 27 + 9 - 18 - 18 = 0$

$\therefore$ $x = 3$ is a solution of the equation

$\therefore$ $x - 3$ is a factor of the polynomial.

To find two other solutions, find the quadratic polynomial

$P_1(x) = P(x) \div (x - 3)$

$$\begin{array}{r l} & x^2 + 4x + 6 \\ x - 3 & \overline{)\,x^3 + x^2 - 6x - 18} \\ & \underline{x^3 - 3x^2} \\ & \quad 4x^2 - 6x \\ & \quad \underline{4x^2 - 12x} \\ & \qquad 6x - 18 \\ & \qquad \underline{6x - 18} \\ & \qquad\quad 0 \end{array}$$

$P_1(x) = x^2 + 4x + 6$

$\therefore$ $x^2 + 4x + 6 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 6}}{2}$$

$$= \frac{-4 \pm \sqrt{-8}}{2}$$

$$= -2 \pm i\sqrt{2}$$

The solutions of the given equation are $x = 3$, $x = -2 \pm i\sqrt{2}$

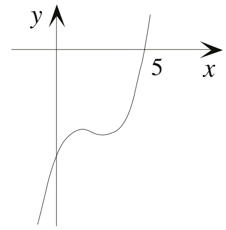
b Let $P(x) = x^3 - 6x^2 + 11x - 30$

Using a CAS calculator, the graph of the function of $y = x^3 - 6x^2 + 11x - 30$ is as shown. From the graph, the real solution appears to be $x = 5$.

$P(5) = 125 - 6 \times 25 + 55 - 30$

$\quad = 180 - 150 - 30$

$\quad = 0$



$\therefore$ $x = 5$ is a solution of the equation $x^3 - 6x^2 + 11x - 30$

$\therefore$ $x - 5$ is a factor of $P(x)$

To find two other solutions, find the quadratic polynomial

$P_1(x) = P(x) \div (x - 5)$

$$\begin{array}{r l} & x^2 - x + 6 \\ x - 5 & \overline{)\,x^3 - 6x^2 + 11x - 30} \\ & \underline{x^3 - 5x^2} \\ & \quad -x^2 + 11x \\ & \quad \underline{-x^2 + 5x} \\ & \qquad 6x - 30 \\ & \qquad \underline{6x - 30} \\ & \qquad\quad 0 \end{array}$$

$\therefore$ $P_1(x) = x^2 - x + 6$

$x^2 - x + 6 = 0$

$$x = \frac{1 \pm \sqrt{1 - 24}}{2}$$

$$= \frac{1 \pm i\sqrt{23}}{2}$$

$x = 5, \ \dfrac{1 \pm i\sqrt{23}}{2}$ are the solutions of the equation $x^3 - 6x^2 + 11x - 30 = 0$

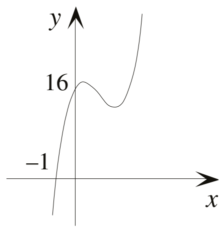
c The equation

$2x^3 + 3x^2 = 11x^2 - 6x - 16$

is rearranged to the form

$2x^3 - 8x^2 + 6x + 16 = 0$

Using a CAS calculator, the graph of the function of $y = 2x^3 - 8x^2 + 6x + 16$ is as shown. Therefore, $x = -1$ appears to be a real solution of the equation.



In fact, $2(-1)^3 - 8(-1)^2 + 6(-1) + 16 = -2 - 8 - 6 + 16 = 0$

$\therefore$ $x = -1$ is a solution, so $x + 1$ is a factor of the polynomial.

To find two other solutions, use long division as before.

$$\begin{array}{r} 2x^2 - 10x + 16 \\ x+1 \overline{\smash{)}\,2x^3 - 8x^2 + 6x + 16} \\ \underline{2x^3 + 2x^2 \qquad\qquad\quad} \\ -10x^2 + 6x \qquad \\ \underline{-10x^2 - 10x \qquad} \\ 16x + 16 \\ \underline{16x + 16} \\ 0 \end{array}$$

$\therefore$ $2x^2 - 10x + 16 = 0$

$x^2 - 5x + 8 = 0$

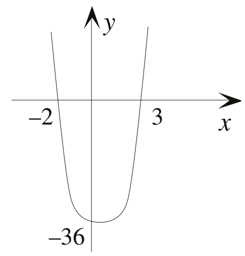
$x = \frac{5 \pm \sqrt{25 - 32}}{2}$

$= \frac{5 \pm i\sqrt{7}}{2}$

The solutions are $x = -1,\ x = \frac{5 \pm i\sqrt{7}}{2}$

d The equation $x^4 + x^2 = 2x^3 + 36$ is rearranged to the form $x^4 - 2x^3 + x^2 - 36 = 0$

According to the graph (using a CAS calculator), the equation of $y = x^4 - 2x^3 + x^2 - 36$ has two real solutions, $x = -2$ and $x = 3$



In fact, $(-2)^4 - 2(-2)^3 + (-2)^2 - 36 = 16 + 16 + 4 - 36 = 0$

$\therefore$ $x = -2$ is a solution and $x + 2$ is a factor of the polynomial $P(x) = x^4 - 2x^3 + x^2 - 36$

Also $3^4 - 2 \times 3^3 + 3^2 - 36 = 81 - 54 + 9 - 36 = 0$

$\therefore$ $x = 3$ is a solution and $x - 3$ is a factor of $P(x)$

$\therefore$ $(x + 2)(x - 3) = x^2 - x - 6$ is a quadratic factor of $P(x)$

Another quadratic factor of $P(x)$ can be obtained by long division.

$$\begin{array}{r} x^2 - x + 6 \qquad\quad \\ x^2 - x - 6 \overline{\smash{)}\,x^4 - 2x^3 + x^2 - 36} \\ \underline{x^4 - x^3 - 6x^2 \qquad\qquad} \\ -x^3 + 7x^2 \qquad\quad \\ \underline{-x^3 + x^2 + 6x \qquad} \\ 6x^2 - 6x - 36 \\ \underline{6x^2 - 6x - 36} \\ 0 \end{array}$$

$\therefore$ $x^2 - x + 6 = 0$

$x = \frac{1 \pm \sqrt{1 - 24}}{2}$

$= \frac{1 \pm i\sqrt{23}}{2}$

The solutions of the given equation are $x = -2,\ x = 3,\ x = \frac{1 \pm i\sqrt{23}}{2}$

9 a If one of the solutions is $2i$, the other is $-2i$ according to the conjugate factor theorem. Therefore, the polynomial $P(z) = z^2 + az + b$ can be factorised as

$P(z) = (z - 2i)(z + 2i) = z^2 + 4$

$\therefore \quad z^2 + az + b = z^2 + 4$

$\therefore \quad a = 0, b = 4$

b If one of the solutions is $3 + 2i$, the other is $3 - 2i$ according to the conjugate factor theorem.
Therefore, the polynomial $P(z) = z^2 + az + b$ can be factorised as

$P(z) = (z - 3 - 2i)(z - 3 + 2i)$

$= (z - 3)^2 + 4$

$= z^2 - 6z + 13$

$\therefore \quad z^2 + az + b = z^2 - 6z + 13$

$\therefore \quad a = -6, b = 13$

c If one of the solutions is $-1 + 3i$, the other is $-1 - 3i$

So $z^2 + az + b = (z + 1 - 3i)$

$\times (z + 1 + 3i)$

$= (z + 1)^2 + 9$

$= z^2 + 2z + 10$

$\therefore \quad a = 2, \; b = 10$

10 a If $1 + 3i$ is a solution of the equation $3z^3 - 7z^2 + 32z - 10 = 0$ then $1 - 3i$ is also a solution (the conjugate factor theorem).
Then $(z - 1 - 3i)(z - 1 + 3i) = (z - 1)^2 + 9 = z^2 - 2z + 10$ is a quadratic factor of the polynomial $P(z) = 3z^3 - 7z^2 + 32z - 10$
The linear factor of $P(z)$ will be in this case $3\left(z - \frac{1}{3}\right) = 3z - 1$ and the third solution is $z = \frac{1}{3}$

b If $-2 - i$ is a solution of the equation $z^4 - 5z^2 + 4z + 30 = 0$ then $-2 + i$ is also a solution (the conjugate factor theorem).
Then $(z + 2 + i)(z + 2 - i) = (z + 2)^2 + 1 = z^2 + 4z + 5$ is a quadratic factor of the polynomial $P(z) = z^4 - 5z^2 + 4z + 30$
Another quadratic factor is obtained by long division.

$$
\begin{array}{r}
z^2 - 4z + 6 \\
z^2 + 4z + 5 \overline{\big) z^4 - 5z^2 + 4z + 30} \\
\underline{z^4 + 4z^3 + 5z^2} \\
-4z^3 - 10z^2 + 4z \\
\underline{-4z^3 - 16z^2 - 20z} \\
6z^2 + 24z + 3 \\
\underline{6z^2 + 24z + 3} \\
0
\end{array}
$$

$\therefore \quad z^2 - 4z + 6 = 0$

$$z = \frac{4 \pm \sqrt{16 - 24}}{2}$$

$= 2 \pm i\sqrt{2}$

Therefore the solutions are $z = -2 \pm i$ and $z = 2 \pm i\sqrt{2}$

11 $P(0) = 10 \;\therefore\; d = 10$

$P(1) = 0$

$P(2 + i) = 0$

$\therefore$ $(z - (2 - i))$ is also a factor

$\therefore$ $(z - (2 - i))(z - (2 + i))$ is a factor

i.e. $z^2 - 4z + 5$ is a factor.

Also $z - 1$ is a factor.

$\therefore \quad P(z) = k(z - 1)(z^2 - 4z + 5)$

$= k(z^3 - 5z^2 + 9z - 5)$

But $P(0) = 10$

$\therefore \quad k = -2$

$\therefore \quad P(x) = -2x^3 + 10x^2 - 18x + 10$

Since $P(2 + i) = 0$ and the coefficients are real, then $P(2 - i) = 0$

Hence the solutions to the equation

$P(x) = 0$ are $x = 1$ or $x = 2 \pm i$

12 $P(1 + i) = 0$

$\therefore \quad (1 + i)^3 + a(1 + i)^2$

$+ b(1 + i) + 10 - 6i = 0$

$\therefore \quad 1 + 3i + 3i^2 + i^3 + a(1 + 2i + i^2)$

$+ b + bi + 10 - 6i = 0$

$\therefore \quad 1 + 3i - 3 - i + a(1 + 2i - 1)$

$+ b + bi + 10 - 6i = 0$

$\therefore \quad 8 - 4i + 2ai + b + bi = 0$

$\therefore \quad (b + 8) + (2a + b - 4)i = 0$

$\therefore \quad b + 8 = 0$

$\therefore \quad b = -8$

$\therefore \quad 2a + b - 4 = 0$

$\therefore \quad 2a - 8 - 4 = 0$

$\therefore \quad a = 6$

$\therefore \quad P(z) = z^3 + 6z^2 - 8z + 10 - 6i$

13 a $2 + i$ is another zero of $P(z)$ since the coefficients of $P(z)$ are real and $2 - i$ is a zero of $P(z)$.

Hence a quadratic factor of $P(z)$ is given by

$(z - (2 + i))(z - (2 - i))$

$= z^2 - (2 + i)z - (2 - i)z$

$+ (2 + i)(2 - i)$

$= z^2 - 2z - iz - 2z + iz + 4 - i^2$

$= z^2 - 4z + 5$

Hence $2z^3 + az^2 + bz + 5 =$ $(z^2 - 4x + 5)Q(z)$ where $Q(z)$ is a linear factor.

Hence $Q(z) = 2z + 1$ and

$2z^3 + az^2 + bz + 5 = 2z^3 - 7z^2 + 6z + 5$

$\therefore \quad a = -7$ and $b = 6$

Alternatively, consider:

$$\begin{array}{r l}
 & 2z + (a + 8) \\
z^2 - 4z + 5 \,\big|\!\! & \overline{2z^3 + az^2 + bz + 5} \\
 & \underline{2z^3 - 8z^2 + 10z} \\
 & (a + 8)z^2 + (b - 10)z + 5 \\
 & \underline{(a + 8)z^2 - 4(a - 8)z + 5(a + 8)} \\
 & (b - 10 + 4(a + 8))z + 5 - 5(a + 8)
\end{array}$$

Now $(b - 10 + 4(a + 8))z + 5 - 5(a + 8) = 0$

$\therefore \quad b - 10 + 4(a + 8) = 0$ and $5 - 5(a + 8) = 0$

$\therefore \quad b - 10 + 4a + 32 = 0 \qquad 5 = 5(a + 8)$

$\therefore \quad b + 22 + 4a = 0 \qquad 1 = a + 8$

$\therefore \quad a = -7$

$\therefore \quad b + 22 + 4(-7) = 0$

$\therefore \quad b + 22 - 28 = 0$

$\therefore \quad b - 6 = 0$

$\therefore \quad b = 6$

$\therefore \quad P(z) = 2z^3 - 7z^2 + 6z + 5$

b $P(z) = 0$

$\therefore \quad 2z^3 - 7z^2 + 6z + 5 = 0$

$\therefore \quad (z - (2 + i))(z - (2 - i))(2z + 1) = 0$

since $2z + (a + 8)$

$= 2z + (-7 + 8)$

$= 2z + 1$

$\therefore \quad z = 2 \pm i$ or $z = -\dfrac{1}{2}$

14 a $P(1 + i) = a(1 + i)^4 + a(1 + i)^2$

$- 2(1 + i) + d$

$= a(1 + 4i + 6i^2 + 4i^3 + i^4)$

$+ a(1 + 2i + i^2)$

$- 2 - 2i + d$

$= a(1 + 4i - 6 - 4i + 1)$

$+ a(1 + 2i - 1)$

$- 2 - 2i + d$

$= -4a + 2ai$

$- 2 - 2i + d$

$= (-4a + d - 2) + 2(a - 1)i$

b Given $P(1+i)=0$,
$-4a+d-2=0$ and $2(a-1)=0$

$$\therefore \quad d = 4a+2 \quad a = 1$$
$$= 4(1)+2$$
$$= 6$$

$P(z) = z^4 + z^2 - 2z + 6$

c $P(1-i)=0$ as $P(1+i)=0$ and the coefficients of $P(z)$ are real.
Hence a factor of $P(z)$ is given by
$(z-(1-i))(z-(1+i))$

$$= z^2 - (1-i)z - (1+i)z + (1-i)(1+i)$$
$$= z^2 - z + iz - z - iz + 1 - i^2$$
$$= z^2 - 2z + 2$$

$$\begin{array}{r} z^2 + 2z + 3 \\ z^2 - 2z + 2 \overline{\big)\, z^4 + z^2 - 2z + 6} \\ \underline{z^4 - 2z^3 + 2z^2} \\ 2z^3 - z^2 - 2z \\ \underline{2z^3 - 4z^2 + 4z} \\ 3z^2 - 6z + 6 \\ \underline{3z^2 - 6z + 6} \\ 0 \end{array}$$

$\therefore \quad P(z) = (z^2 - 2z + 2)(z^2 + 2z + 3)$
Hence for $P(z)=0$, $z = 1 \pm i$

$$\text{or } z = \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2}$$
$$= \frac{-2 \pm \sqrt{4-12}}{2}$$
$$= \frac{-2 \pm \sqrt{8i^2}}{2}$$
$$= \frac{-2 \pm i2\sqrt{2}}{2}$$
$$= -1 \pm i\sqrt{2}$$

15 Since $P(1+i)=0$ and $P(4+3i)=0$, where $P(z) = z^2 + pz + q$
then $P(z) = (z-(1+i))(z-(4+3i))$

$$= z^2 - (1+i)z - (4+3i)z + (1+i)(4+3i)$$
$$= z^2 - (1+i+4+3i)z + 4 + 4i + 3i + 3i^2$$
$$= z^2 - (5+4i)z + 4 + 7i - 3$$
$$= z^2 - (5+4i)z + (1+7i)$$

$\therefore \quad p = -(5+4i)$ and $q = 1+7i$

16 Let $P(z) = z^3 - 4z^2 + 6z - 4$
Since the coefficients are all real, and $P(1-i)=0$, then $P(1+i)=0$

$\therefore \quad (z-(1-i))(z-(1+i))$

$$= z^2 - (1-i)z - (1+i)z + (1-i)(1+i)$$
$$= z^2 - (1-i+1+i)z + 1 - i^2$$
$$= z^2 - 2z + 2$$

$$\begin{array}{r} z - 2 \\ z^2 - 2z + 2 \overline{\big)\, z^3 - 4z^2 + 6z - 4} \\ \underline{z^3 - 2z^2 + 2z} \\ -2z^2 + 4z - 4 \\ \underline{-2z^2 + 4z - 4} \\ 0 \end{array}$$

$\therefore \quad P(z) = (z-(1-i))(z-(1+i))(z-2)$
When $P(z)=0$, $z = 1 \pm i$ or $z = 2$

17 **a** For $z^2 - (6+2i)z + (8+6i) = 0$, use the general quadratic formula where $a=1, b=-(6+2i), c=8+6i$

$$\therefore \quad z = \frac{6+2i \pm \sqrt{(6+2i)^2 - 4(8+6i)}}{2}$$
$$= \frac{6+2i \pm \sqrt{36+24i-4-32-24i}}{2}$$
$$= \frac{6+2i \pm \sqrt{0}}{2}$$
$$= 3+i$$

Alternatively, note $(3 + i)^2$

$$= 9 + 6i - 1$$
$$= 8 + 6i$$

$\therefore \quad z^2 - (6 + 2i)z + (8 + 6i) = 0$ implies

$$(z - (3 + i))^2 = 0$$
$$\therefore \quad z = 3 + i$$

b $z^3 - 2iz^2 - 6z + 12i = 0$

Factorise by grouping

$z^2(z - 2i) - 6(z - 2i) = 0$

$\therefore \quad (z^2 - 6)(z - 2i) = 0$

$\therefore \quad z = -\sqrt{6}$ or $\sqrt{6}$ or $2i$

c Let $P(z) = z^3 - z^2 + 6z - 6$

$P(1) = 0 \therefore \quad z - 1$ is a factor of $P(z)$

Also observe

$P(z) = z^2(z - 1) + 6(z - 1)$

$= (z - 1)(z^2 + 6)$

$= (z - 1)(z + i\sqrt{6})(z - i\sqrt{6})$

If $P(z) = 0$, then $z = 1$ or $-i\sqrt{6}$ or $i\sqrt{6}$

d Let $P(z) = z^3 - z^2 + 2z - 8$

$P(2) = 0 \therefore \quad z - 2$ is a factor of $P(z)$

Also observe

$P(z)$

$= z^3 - 8 - (z^2 - 2z)$

$= (z - 2)(z^2 + 2z + 4) - (z - 2)z$

$= (z - 2)(z^2 + z + 4)$

$= (z - 2)\left(z^2 + z + \frac{1}{4} - \frac{15}{4}i^2\right)$

$= (z - 2)\left(\left(z + \frac{1}{2}\right)^2 - \left(i\frac{\sqrt{15}}{2}\right)^2\right)$

$= (z - 2)\left(z + \frac{1}{2} + i\frac{\sqrt{15}}{2}\right)$

$\times\left(z + \frac{1}{2} - i\frac{\sqrt{15}}{2}\right)$

If $P(z) = 0$, then $z = 2$ or $-\frac{1}{2} - i\frac{\sqrt{15}}{2}$ or $-\frac{1}{2} + i\frac{\sqrt{15}}{2}$

e $6z^2 - 3\sqrt{2}z + 6 = 0$

$\therefore \quad 6\left(z^2 - \frac{\sqrt{2}}{2}z + 1\right) = 0$

$\therefore \quad 6\left(z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{8} - \frac{7}{8}i^2\right) = 0$

$\therefore \quad 6\left(\left(z - \frac{\sqrt{2}}{4}\right)^2 - \left(\frac{i\sqrt{7}}{2\sqrt{2}}\right)^2\right) = 0$

$\therefore \quad 6\left(\left(z - \frac{\sqrt{2}}{4}\right)^2 - \left(\frac{i\sqrt{14}}{4}\right)^2\right) = 0$

$\therefore \quad 6\left(z - \frac{\sqrt{2}}{4} + \frac{i\sqrt{14}}{4}\right)\left(z - \frac{\sqrt{2}}{4} - \frac{i\sqrt{14}}{4}\right) = 0$

$\therefore \quad z = \frac{\sqrt{2}}{4} - \frac{i\sqrt{14}}{4}$ or $\frac{\sqrt{2}}{4} + \frac{i\sqrt{14}}{4}$

f $z^3 + 2z^2 + 9z = 0$

$\therefore \quad z(z^2 + 2z + 9) = 0$

$\therefore \quad z(z^2 + 2z + 1 - 8i^2) = 0$

$\therefore \quad z((z + 1)^2 - (2\sqrt{2}i)^2) = 0$

$\therefore \quad z(z + 1 + 2\sqrt{2}i)(z + 1 - 2\sqrt{2}i) = 0$

$\therefore \quad z = 0$ or $-1 - 2\sqrt{2}i$ or $-1 + 2\sqrt{2}i$

Solutions to Exercise 6G

1 a $z^2 + 1 = 0$

$\therefore\ z^2 = -1$

$\therefore\ (r\text{cis }\theta)^2 = \text{cis }\pi$ where $z = r\text{ cis }\theta$

$\therefore\ r^2\text{cis }2\theta = \text{cis }\pi$

$\therefore\ r^2 = 1$ and $2\theta = \pi + 2\pi k,\ k \in Z$

$\therefore\ r = 1$ and $\theta = \dfrac{\pi}{2} + \pi k,\ k \in Z$

$\therefore\ z = \text{cis}\left(\dfrac{\pi}{2} + \pi k\right),\ k \in Z$

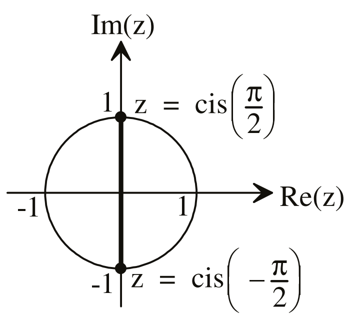
When $k = 0,\ z = \text{cis }\dfrac{\pi}{2}$

When $k = 1,\ z = \text{cis }\dfrac{3\pi}{2} = \text{cis}\left(-\dfrac{\pi}{2}\right)$

When $k = 2,\ z = \text{cis }\dfrac{5\pi}{2} = \text{cis }\dfrac{\pi}{2}$

Note: There are, at most, two solutions of a quadratic equation.

Also $\text{cis}\left(-\dfrac{\pi}{2}\right)$ and $\text{cis}\left(\dfrac{\pi}{2}\right)$ are conjugate $z = \pm i$



b $z^3 = 27i$

$\therefore\ (r\text{cis }\theta)^3 = 27\text{cis}$ where $z = r\text{ cis }\theta$

$\therefore\ r^3\text{cis }3\theta = 27\text{cis }\dfrac{\pi}{2}$

$\therefore\ r^3 = 27$ and $3\theta = \dfrac{\pi}{2} + 2\pi k,\ k \in Z$

$\therefore\ r = 3$ and $\theta = \dfrac{\pi}{6} + \dfrac{2\pi k}{3},\ k \in Z$

When $k = 0,\ z = 3\text{cis }\dfrac{\pi}{6}$

When $k = 1,\ z = 3\text{cis }\dfrac{5\pi}{6}$

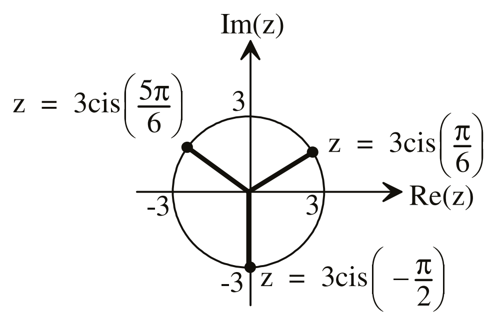
When $k = 2,\ z = 3\text{cis }\dfrac{9\pi}{6}$

$= 3\text{cis }\dfrac{3\pi}{2}$

$= 3\text{cis}\left(-\dfrac{\pi}{2}\right)$

When $k = 3,\ z = 3\text{cis }\dfrac{13\pi}{6} = 3\text{cis }\dfrac{\pi}{6}$

$\therefore\ z = 3\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)$ or $z = 3\left(\dfrac{-\sqrt{3}}{2} + \dfrac{i}{2}\right)$ or $z = -3i$



c $z^2 = 1 + \sqrt{3}i$

$\therefore\ (r\text{cis }\theta)^2 = 2\text{cis }\dfrac{\pi}{3}$ where $z = r\text{cis }\theta$

$\therefore\ r^2\text{cis }2\theta = 2\text{cis }\dfrac{\pi}{3}$

$\therefore\ r^2 = 2$ and $2\theta = \dfrac{\pi}{3} + 2\pi k,\ k \in Z$

$\therefore\ r = \sqrt{2}$ and $\theta = \dfrac{\pi}{6} + \pi k,\ k \in Z$

$\therefore\ z = \sqrt{2}\text{cis}\left(\dfrac{\pi}{6} + \pi k\right),\ k \in Z$

When $k = 0,\ z = \sqrt{2}\text{cis }\dfrac{\pi}{6}$

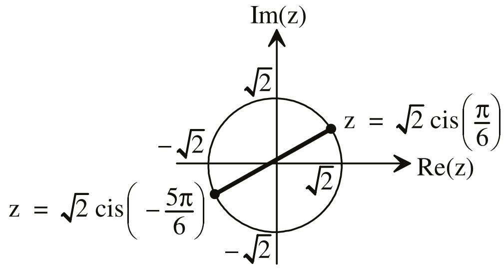
When $k = 1,\ z = \sqrt{2}\text{cis }\dfrac{7\pi}{6}$

$= \sqrt{2}\text{cis}\left(-\dfrac{5\pi}{6}\right)$

When $k = 2,\ z = \sqrt{2}\text{cis }\dfrac{13\pi}{6} =$ $\sqrt{2}\text{cis }\dfrac{\pi}{6}$

$\therefore \quad z = \sqrt{2}\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right)$ or

$z = \sqrt{2}\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right)$



d $z^2 = 1 - \sqrt{3}i$

$\therefore \quad (r\text{cis }\theta)^2 = 2\text{cis}\left(-\frac{\pi}{3}\right)$ where $z = r\text{cis }\theta$

$\therefore \quad r^2\text{cis }2\theta = 2\text{cis}\left(-\frac{\pi}{3}\right)$

$\therefore \quad r^2 = 2$ and $2\theta = -\frac{\pi}{3} + 2\pi k,\ k \in Z$

$\therefore \quad r = \sqrt{2}$ and $\theta = -\frac{\pi}{6} + \pi k,\ k \in Z$

$\therefore \quad z = \sqrt{2}\text{cis}\left(-\frac{\pi}{6} + \pi k\right),\ k \in Z$

When $k = 0,\ z = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$

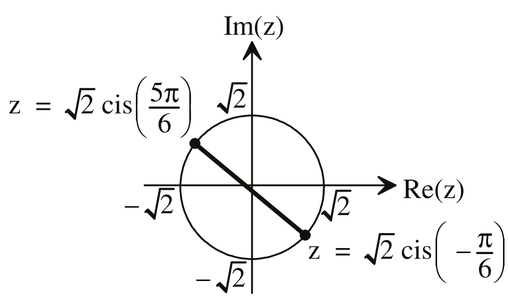
When $k = 1,\ z = \sqrt{2}\text{cis }\frac{5\pi}{6}$

When $k = 2,\ z = \sqrt{2}\text{cis }\frac{11\pi}{6}$

$= \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$

Solutions are $z = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$ or

$z = \sqrt{2}\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right)$



e $z^3 = i$

$\therefore \quad (r\text{cis }\theta)^3 = \text{cis }\frac{\pi}{2}$ where $z = r\text{cis }\theta$

$\therefore \quad r^3\text{cis }3\theta = \text{cis }\frac{\pi}{2}$

$\therefore \quad r^3 = 1$ and $3\theta = \frac{\pi}{2} + 2\pi k,\ k \in Z$

$\therefore \quad r = 1$ and $\theta = \frac{\pi}{6} + \frac{2\pi k}{3},\ k \in Z$

$\therefore \quad z = \text{cis}\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = \text{cis }\frac{\pi}{6}$

When $k = 1,\ z = \text{cis }\frac{5\pi}{6}$

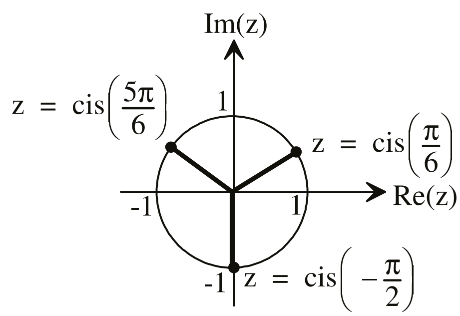
When $k = 2,\ z = \text{cis }\frac{9\pi}{6} = \text{cis }\frac{3\pi}{2}$

$= \text{cis}\left(-\frac{\pi}{2}\right)$

When $k = 3,\ z = \text{cis }\frac{13\pi}{6} = \text{cis }\frac{\pi}{6}$

Solutions are $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ or

$z = \frac{-\sqrt{3}}{2} + \frac{i}{2}$ or $z = -i$



f $z^3 + i = 0$

$\therefore \quad z^3 = -i$

$\therefore \quad (r\text{cis }\theta)^3 = \text{cis}\left(-\frac{\pi}{2}\right)$ where $z = r\text{cis }\theta$

$\therefore \quad r^3\text{cis }3\theta = \text{cis}\left(-\frac{\pi}{2}\right)$

$\therefore \quad r^3 = 1$ and $3\theta = -\frac{\pi}{2} + 2\pi k,\ k \in Z$

$\therefore \quad r = 1$ and $\theta = -\frac{\pi}{6} + \frac{2\pi k}{3},\ k \in Z$

$\therefore \quad z = \text{cis}\left(-\frac{\pi}{6} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = \text{cis}\left(-\frac{\pi}{6}\right)$

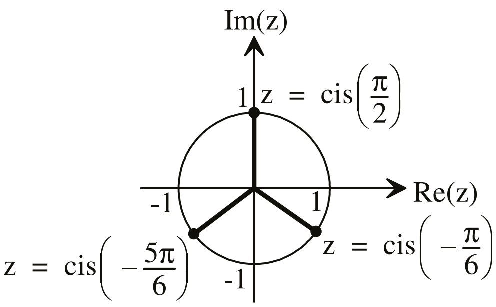
When $k = 1,\ z = \text{cis}\ \frac{3\pi}{6} = \text{cis}\ \frac{\pi}{2}$

When $k = 2,\ z = \text{cis}\ \frac{7\pi}{6} = \text{cis}\left(-\frac{5\pi}{6}\right)$

When $k = 3,\ z = \text{cis}\ \frac{11\pi}{6} = \text{cis}\left(-\frac{\pi}{6}\right)$

Solutions are $z = \frac{\sqrt{3}}{2} - \frac{i}{2}$ or $z = i$

or $z = \frac{-\sqrt{3}}{2} - \frac{i}{2}$



2 a Let $z = r\text{cis}\ \theta$

Also $4\sqrt{2} - 4\sqrt{2}i = 8\text{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \quad (r\text{cis}\ \theta)^3 = 8\text{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \quad r^3\text{cis}\ 3\theta = 8\text{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \quad r^3 = 8 \text{ and } 3\theta = -\frac{\pi}{4} + 2\pi k,\ k \in Z$

$\therefore \quad r = 2 \text{ and } \theta = -\frac{\pi}{12} + \frac{2\pi k}{3},\ k \in Z$

$\therefore \quad z = 2\text{cis}\left(-\frac{\pi}{12} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = 2\text{cis}\left(-\frac{\pi}{12}\right)$

When $k = 1,\ z = 2\text{cis}\ \frac{7\pi}{12}$

When $k = 2,\ z = 2\text{cis}\ \frac{15\pi}{12} =$

$2\text{cis}\ \frac{5\pi}{4} = 2\text{cis}\left(-\frac{3\pi}{4}\right)$

When $k = 3,\ z = 2\text{cis}\ \frac{23\pi}{12} =$

$2\text{cis}\left(-\frac{\pi}{12}\right)$

Hence the cube roots of $4\sqrt{2} - 4\sqrt{2}i$ are $2\ \text{cis}\left(-\frac{\pi}{12}\right), 2\ \text{cis}\ \frac{7\pi}{12}$ and

$2\ \text{cis}\left(-\frac{3\pi}{4}\right)$

b Let $z = r\text{cis}\ \theta$

Also $-4\sqrt{2} + 4\sqrt{2}i = 8\text{cis}\ \frac{3\pi}{4}$

$\therefore \quad (r\text{cis}\ \theta)^3 = 8\text{cis}\ \frac{3\pi}{4}$

$\therefore \quad r^3\text{cis}\ 3\theta = 8\text{cis}\ \frac{3\pi}{4}$

$\therefore \quad r^3 = 8 \text{ and } 3\theta = \frac{3\pi}{4} + 2\pi k,\ k \in Z$

$\therefore \quad r = 2 \text{ and } \theta = \frac{\pi}{4} + \frac{2\pi k}{3},\ k \in Z$

$\therefore \quad z = 2\text{cis}\left(\frac{\pi}{4} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = 2\text{cis}\ \frac{\pi}{4}$

When $k = 1,\ z = 2\text{cis}\ \frac{11\pi}{12}$

When $k = 2,\ z = 2\text{cis}\ \frac{19\pi}{12} =$

$2\text{cis}\left(-\frac{5\pi}{12}\right)$

When $k = 3,\ z = 2\text{cis}\ \frac{9\pi}{4} = 2\text{cis}\ \frac{\pi}{4}$

Hence the cube roots of $-4\sqrt{2} + 4\sqrt{2}i$ are $2\text{cis}\ \frac{\pi}{4}, 2\text{cis}\ \frac{11\pi}{12}$ and $2\text{cis}\left(-\frac{5\pi}{12}\right)$

c Let $z = r\ \text{cis}\ \theta$

Also $-4\sqrt{3} - 4i = 8\text{cis}\left(-\frac{5\pi}{6}\right)$

$\therefore \quad (r\text{cis}\ \theta)^3 = 8\text{cis}\left(-\frac{5\pi}{6}\right)$

$\therefore \quad r^3 \text{cis } 3\theta = 8\text{cis}\left(-\frac{5\pi}{6}\right)$

$\therefore \quad r^3 = 8 \text{ and } 3\theta = -\frac{5\pi}{6} + 2\pi k,\ k \in Z$

$\therefore \quad r = 2 \text{ and } \theta = -\frac{5\pi}{18} + \frac{2\pi k}{3},\ k \in Z$

$\therefore \quad z = 2\text{cis}\left(-\frac{5\pi}{18} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = 2\text{cis}\left(-\frac{5\pi}{18}\right)$

When $k = 1,\ z = 2\text{cis } \frac{7\pi}{18}$

When $k = 2,\ z = 2\text{cis } \frac{19\pi}{18} = 2\text{cis}\left(-\frac{17\pi}{18}\right)$

When $k = 3,\ z = 2\text{cis } \frac{31\pi}{18} = 2\text{cis}\left(-\frac{5\pi}{18}\right)$

Hence the cube roots of $-4\sqrt{3} - 4i$ are $2\text{cis}\left(-\frac{5\pi}{18}\right), 2\text{cis } \frac{7\pi}{18}$ and $2\text{cis}\left(-\frac{17\pi}{18}\right)$

d Let $z = r \text{ cis } \theta$

Also $4\sqrt{3} - 4i = 8\text{cis}\left(-\frac{\pi}{6}\right)$

$\therefore \quad (r\text{cis } \theta)^3 = 8\text{cis}\left(-\frac{\pi}{6}\right)$

$\therefore \quad r^3 \text{cis } 3\theta = 8\text{cis}\left(-\frac{\pi}{6}\right)$

$\therefore \quad r^3 = 8 \text{ and } 3\theta = -\frac{\pi}{6} + 2\pi k,\ k \in Z$

$\therefore \quad r = 2 \text{ and } \theta = -\frac{\pi}{18} + \frac{2\pi k}{3},\ k \in Z$

$\therefore \quad z = 2\text{cis}\left(-\frac{\pi}{18} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = 2\text{cis}\left(-\frac{\pi}{18}\right)$

When $k = 1,\ z = 2\text{cis } \frac{11\pi}{18}$

When $k = 2,\ z = 2\text{cis } \frac{23\pi}{18} = 2\text{cis}\left(-\frac{13\pi}{18}\right)$

When $k = 3,\ z = 2\text{cis } \frac{35\pi}{18} = 2\text{cis}\left(-\frac{\pi}{18}\right)$

Hence the cube roots of $4\sqrt{3} - 4i$ are $2\text{cis}\left(-\frac{\pi}{18}\right), 2\text{cis } \frac{11\pi}{18}$ and $2\text{cis}\left(-\frac{13\pi}{18}\right)$

e Let $z = r\text{cis } \theta$

Also $-125i = 125\text{cis}\left(-\frac{\pi}{2}\right)$

$\therefore \quad (r\text{cis } \theta)^3 = 125\text{cis}\left(-\frac{\pi}{2}\right)$

$\therefore \quad r^3 \text{cis } 3\theta = 125\text{cis}(-)$

$\therefore \quad r^3 = 125 \text{ and } 3\theta = -\frac{\pi}{2} + 2\pi k,\ k \in Z$

$\therefore \quad r = 5 \text{ and } \theta = -\frac{\pi}{6} + \frac{2\pi k}{3},\ k \in Z$

$\therefore \quad z = 5\text{cis}\left(-\frac{\pi}{6} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = 5\text{cis}\left(-\frac{\pi}{6}\right)$

When $k = 1,\ z = 5\text{cis } \frac{3\pi}{6} = 5\text{cis } \frac{\pi}{2}$

When $k = 2,\ z = 5\text{cis } \frac{7\pi}{6} = 5\text{cis}\left(-\frac{5\pi}{6}\right)$

When $k = 3,\ z = 5\text{cis } \frac{11\pi}{6} = 5\text{cis}\left(-\frac{\pi}{6}\right)$

Hence the cube roots of $-125i$ are $5\text{cis}\left(-\frac{\pi}{6}\right), 5\text{cis}\left(\frac{\pi}{2}\right)$ and $5\text{cis}\left(-\frac{5\pi}{6}\right)$

f Let $z = r\text{cis } \theta$

Also $-1 + i = 2^{\frac{1}{2}}\text{cis } \frac{3\pi}{4}$

$\therefore\ (r\text{cis }\theta)^3 = 2^{\frac{1}{2}}\text{cis }\frac{3\pi}{4}$

$\therefore\ r^3\text{cis }3\theta = 2^{\frac{1}{2}}\text{cis }\frac{3\pi}{4}$

$\therefore\ r^3 = 2^{\frac{1}{2}}$ and $3\theta = \frac{3\pi}{4} + 2\pi k,\ k \in Z$

$\therefore\ r = 2^{\frac{1}{6}}$ and $\theta = \frac{\pi}{4} + \frac{2\pi k}{3},\ k \in Z$

$\therefore\ z = 2^{\frac{1}{6}}\text{cis}\left(\frac{\pi}{4} + \frac{2\pi k}{3}\right),\ k \in Z$

When $k = 0,\ z = 2^{\frac{1}{6}}\text{cis }\frac{\pi}{4}$

When $k = 1,\ z = 2^{\frac{1}{6}}\text{cis }\frac{11\pi}{12}$

When $k = 2,\ z = 2^{\frac{1}{6}}\text{cis }\frac{19\pi}{12} = 2^{\frac{1}{6}}\text{cis}\left(-\frac{5\pi}{12}\right)$

When $k = 3,\ z = 2^{\frac{1}{6}}\text{cis }\frac{9\pi}{4} = 2^{\frac{1}{6}}\text{cis }\frac{\pi}{4}$

Hence the cube roots of $-l + i$ are $2^{\frac{1}{6}}\text{cis }\frac{\pi}{4}, 2^{\frac{1}{6}}\text{cis }\frac{11\pi}{12}$ and $2^{\frac{1}{6}}\text{cis}\left(-\frac{5\pi}{12}\right)$

3 a $z^2 = (a + ib)^2$

$= a^2 + 2abi + b^2i^2$

$= (a^2 - b^2) + 2abi$

Now $z^2 = 3 + 4i$

$\therefore\ a^2 - b^2 = 3$ and $2ab = 4$

$\therefore\ b = \frac{2}{a}$

$\therefore\ a^2 - \left(\frac{2}{a}\right)^2 = 3$

$\therefore\ a^2 - \frac{4}{a^2} = 3$

$\therefore\ a^4 - 4 = 3a^2$

$\therefore\ a^4 - 3a^2 - 4 = 0$

b Let $x = a^2$

$\therefore\ x^2 - 3x + 4 = 0$

$\therefore\ (x - 4)(x + 1) = 0$

$\therefore\ x = -1$ or 4

$\therefore\ a^2 = -1$ or 4

$\therefore\ a^2 = 4$ as $a \in R$

$\therefore\ a = \pm 2$

When $a = 2,\ b = \frac{2}{2} = 1$

$\therefore\ z = 2 + i$

When $a = -2,\ b = \frac{2}{-2} = -1$

$\therefore\ z = -2 + (-1)i$

$= -(2 + i)$

$\therefore\ z = \pm(2 + i)$,

the square roots of $3 + 4i$

4 a Let $z = a + bi$ and $z^2 = -15 - 8i$

$\therefore\ z^2 = (a^2 - b^2) + 2abi$

$\therefore\ a^2 - b^2 = -15$ and $2ab = -8$

$\therefore\ b = \frac{-4}{a}$

$\therefore\ a^2 - \left(\frac{-4}{a}\right)^2 = -15$

$\therefore\ a^2 - \frac{16}{a^2} = -15$

$\therefore\ a^4 - 16 = -15a^2$

$\therefore\ a^4 + 15a^2 - 16 = 0$

Let $x = a^2\ \therefore\ x^2 + 15x - 16 = 0$

$\therefore\ (x + 16)(x - 1) = 0$

$\therefore\ x = 1$ or -16

$\therefore\ a^2 = 1$ or -16

$\therefore\ a = \pm 1$ as $a \in R$

When $a = 1,\ b = \frac{-4}{1} = -4$

$\therefore\ z = 1 - 4i$

When $a = -1,\ b = \frac{-4}{-1} = 4$

$\therefore\ z = -1 + 4i$

Hence $z = \pm(1 - 4i)$

b Let $z = a + bi$ and $z^2 = 24 + 7i$

$\therefore\ z^2 = (a^2 - b^2) + 2abi$

$\therefore\ a^2 - b^2 = 24$ and $2ab = 7$

$\therefore\ b = \frac{7}{2a}$

$\therefore \quad a^2 - \left(\frac{7}{2a}\right)^2 = 24$

$\therefore \quad a^2 - \frac{49}{4a^2} = 24$

$\therefore \quad 4a^4 - 49 = 96a^2$

$\therefore \quad 4a^4 - 96a^2 - 49 = 0$

Let $x = a^2 \ \therefore \quad 4x^2 - 96x - 49 = 0$

$\therefore \quad 4\left(x^2 - 24x - \frac{49}{4}\right) = 0$

$\therefore \quad x^2 - 24x + 12^2 - \frac{49}{4} - 12^2 = 0$

$\therefore \quad (x-12)^2 - \frac{625}{4} = 0$

$\therefore \quad (x-12)^2 = \frac{625}{4}$

$\therefore \quad x - 12 = \pm\frac{25}{2}$

$\therefore \quad x = 12 \pm \frac{25}{2}$

$\therefore \quad x = \frac{49}{2}$ or $-\frac{1}{2}$

$\therefore \quad a^2 = \frac{49}{2}$ or $-\frac{1}{2}$

$\therefore \quad a^2 = \frac{49}{2}$ as $a \in R$

$\therefore \quad a = \pm\frac{7}{\sqrt{2}} = \pm\frac{7\sqrt{2}}{2}$

When $a = \pm\frac{7\sqrt{2}}{2}$, $b = \frac{7}{2\left(\pm\frac{7\sqrt{2}}{2}\right)} =$

$\pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

Hence $z = \pm\left(\frac{7\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)i =$

$\pm\frac{\sqrt{2}}{2}(7 + i)$

c Let $z = a + bi$ and $z^2 = -3 + 4i$

$\therefore \quad z^2 = (a^2 - b^2) + 2abi$

$\therefore \quad a^2 - b^2 = -3$ and $2ab = 4$

$\therefore \quad b = \frac{2}{a}$

$\therefore \quad a^2 - \left(\frac{2}{a}\right)^2 = -3$

$\therefore \quad a^2 - \frac{4}{a^2} = -3$

$\therefore \quad a^4 - 4 = -3a^2$

$\therefore \quad a^4 + 3a^2 - 4 = 0$

Let $x = a^2$, $\therefore \quad x^2 + 3x - 4 = 0$

$\therefore \quad (x+4)(x-1) = 0$

$\therefore \quad x = -4$ or 1

$\therefore \quad a^2 = -4$ or 1

$\therefore \quad a^2 = 1$ as $a \in R$

$\therefore \quad a = \pm 1$

When $a = \pm 1$, $b = \pm\frac{2}{1} = \pm 2$

Hence $z = \pm(1 + 2i)$

d Let $z = a + bi$ and $z^2 = -7 + 24i$

$\therefore \quad z^2 = (a^2 - b^2) + 2abi$

$\therefore \quad a^2 - b^2 = -7$ and $2ab = 24$

$\therefore \quad b = \frac{12}{a}$

$\therefore \quad a^2 - \left(\frac{12}{a}\right)^2 = -7$

$\therefore \quad a^2 - \frac{144}{a^2} = -7$

$\therefore \quad a^4 - 144 = -7a^2$

$\therefore \quad a^4 + 7a^2 - 144 = 0$

$\therefore \quad (a^2 + 16)(a^2 - 9) = 0$

$\therefore \quad a^2 = -16$ or 9

$\therefore \quad a^2 = 9$ as $a \in R$

$\therefore \quad a = \pm 3$

When $a = \pm 3$, $b = \pm\frac{12}{3} = \pm 4$

Hence $z = \pm(3 + 4i)$

5 Let $x = z^2 \ \therefore \quad x^2 - 2x + 4 = 0$

$\therefore \quad x^2 - 2x + 1 + 4 - 1 = 0$

$\therefore \quad (x-1)^2 + 3 = 0$

$\therefore \quad (x-1)^2 = -3$

$\therefore \quad x - 1 = \pm\sqrt{-3}$

$= \pm\sqrt{3}i$

$\therefore \quad x = 1 \pm \sqrt{3}i$

$\therefore \quad z^2 = 1 \pm \sqrt{3}i$

Let $z = r\text{cis}\,\theta$

When $z^2 = 1 + \sqrt{3}i$

and $1 + \sqrt{3}i = 2\text{cis}\,\frac{\pi}{3}$

$(r\text{cis}\,\theta)^2 = 2\text{cis}\,\frac{\pi}{3}$

$\therefore \quad r^2\text{cis}\,2\theta = 2\text{cis}\,\frac{\pi}{3}$

$\therefore \quad r^2 = 2 \text{ and } 2\theta = \frac{\pi}{3} + 2\pi k,\ k \in Z$

$\therefore \quad r = \sqrt{2} \text{ and } \theta = \frac{\pi}{6} + \pi k,\ k \in Z$

$\therefore \quad z = \sqrt{2}\text{cis}\left(\frac{\pi}{6} + \pi k\right),\ k \in Z$

When $k = 0,\ z = \sqrt{2}\text{cis}\,\frac{\pi}{6}$

When $k = 1,\ z = \sqrt{2}\text{cis}\,\frac{7\pi}{6} = \sqrt{2}\text{cis}\left(-\frac{5\pi}{6}\right)$

When $k = 2,\ z = \sqrt{2}\text{cis}\,\frac{13\pi}{6} = \sqrt{2}\text{cis}\,\frac{\pi}{6}$

When $z^2 = 1 - \sqrt{3}i$

and $1 - \sqrt{3}i = 2\text{cis}\left(-\frac{\pi}{3}\right)$

$(r\text{cis}\,\theta)^2 = 2\text{cis}\left(-\frac{\pi}{3}\right)$

$\therefore \quad r^2\text{cis}\,2\theta = 2\text{cis}\left(-\frac{\pi}{3}\right)$

$\therefore \quad r^2 = 2 \text{ and } 2\theta = -\frac{\pi}{3} + 2\pi k,\ k \in Z$

$\therefore \quad r = \sqrt{2} \text{ and } \theta = -\frac{\pi}{6} + \pi k,\ k \in Z$

$\therefore \quad z = \sqrt{2}\text{cis}\left(-\frac{\pi}{6} + \pi k\right),\ k \in Z$

When $k = 0,\ z = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$

When $k = 1,\ z = \sqrt{2}\text{cis}\,\frac{5\pi}{6}$

When $k = 2,\ z = \sqrt{2}\text{cis}\,\frac{11\pi}{6} = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$

Hence the solutions of $z^4 - 2z^2 + 4 = 0$ are $\sqrt{2}\text{cis}\,\frac{\pi}{6},\ \sqrt{2}\text{cis}\left(-\frac{5\pi}{6}\right),\ \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$ and $\sqrt{2}\text{cis}\,\frac{5\pi}{6}$

6 a Fifth roots of unity are

$1, \text{cis}\left(\frac{2\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right), \text{cis}\left(\frac{4\pi}{5}\right), \text{cis}\left(-\frac{4\pi}{5}\right)$

Take the product of the factors with conjugate roots to form the quadratic:

$(z - \text{cis}\left(\frac{2\pi}{5}\right))(z - \text{cis}\left(-\frac{2\pi}{5}\right))$

$= z^2 - (\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right))z + 1$

$= z^2 - (\cos\left(\frac{2\pi}{5}\right) + \cos\left(-\frac{2\pi}{5}\right))z + 1$

$= z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1$

$= z^2 - \frac{1}{2}(\sqrt{5} - 1)z + 1$

$= z^2 + \frac{1}{2}(1 - \sqrt{5})z + 1$

$\therefore a = \frac{1}{2}(1 - \sqrt{5})$

$(z - \text{cis}\left(\frac{4\pi}{5}\right))(z - \text{cis}\left(-\frac{4\pi}{5}\right))$

$= z^2 - (\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right))z + 1$

$= z^2 - (\cos\left(\frac{4\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right))z + 1$

$= z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1$

Use $\cos 2x = 2\cos^x - 1$ to find

$\cos\left(\frac{4\pi}{5}\right) = -\frac{1}{4}(\sqrt{5} + 1)$

So

$z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1$

$= z^2 + \frac{1}{2}(1 + \sqrt{5})z + 1$

$\therefore b = \frac{1}{2}(1 + \sqrt{5})$

b Sixth roots of unity are
$1, -1, \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(-\frac{\pi}{3}\right),$
$\text{cis}\left(\frac{2\pi}{3}\right), \text{cis}\left(-\frac{2\pi}{3}\right)$
Take the product of the factors with conjugate roots to form the quadratic:
$(z - \text{cis}\left(\frac{\pi}{3}\right))(z - \text{cis}\left(-\frac{\pi}{3}\right))$
$= z^2 - (\text{cis}\left(\frac{\pi}{3}\right) + \text{cis}\left(-\frac{\pi}{3}\right))z + 1$
$= z^2 - (\cos\left(\frac{\pi}{3}\right) + \cos\left(-\frac{\pi}{3}\right))z + 1$
$= z^2 - 2\cos\left(\frac{\pi}{3}\right)z + 1$
$= z^2 - z + 1$
$\therefore a = -1$
$(z - \text{cis}\left(\frac{2\pi}{3}\right))(z - \text{cis}\left(-\frac{2\pi}{3}\right))$
$= z^2 - (\text{cis}\left(\frac{2\pi}{3}\right) + \text{cis}\left(-\frac{2\pi}{3}\right))z + 1$
$= z^2 - (\cos\left(\frac{2\pi}{3}\right) + \cos\left(-\frac{2\pi}{3}\right))z + 1$
$= z^2 - 2\cos\left(\frac{2\pi}{3}\right)z + 1$
$= z^2 + z + 1$
$\therefore b = 1$

7 a When n is odd
$1 \times \omega \times \omega^2 \times \omega^3 \times \cdots \omega^{n-1}$
$= \omega^{1+2+\cdots+n-1}$
$= \omega^{\frac{n(n-1)}{2}}$
$= (\omega^n)^{\frac{(n-1)}{2}}$
$= (\text{cis}(2\pi))^{\frac{n-1}{2}}$
$= (\text{cis}(\pi))^{n-1}$
$= (-1)^{n-1}$
$= 1$ when n is odd..

b $= (-1)^{n-1}$
$= -1$ when n is even

8 $z^2 = i$
$\therefore \quad (r\text{cis}\,\theta)^2 = \text{cis}\left(\frac{\pi}{2}\right)$ where
$\therefore \quad r^2\text{cis}\,2\theta = \text{cis}\left(\frac{\pi}{2}\right)$
$\therefore \quad r^2 = 1$ and $2\theta = \frac{\pi}{2} + 2k\pi,\ k \in Z$
$\therefore \quad r = 1$ and $\theta = \frac{\pi}{4} + k\pi,\ k \in Z$
$\therefore \quad z = \text{cis}\left(\frac{\pi}{4} + k\pi\right),\ k \in Z$
When $k = 0,\ z = \text{cis}\left(\frac{\pi}{4}\right)$
When $k = 1,\ z = \text{cis}\left(\frac{5\pi}{4}\right) = \text{cis}\left(-\frac{3\pi}{4}\right)$
Hence solutions of $z^2 - i = 0$ are:
$\text{cis}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$
$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and
$\text{cis}\left(-\frac{3\pi}{4}\right) = \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$
$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$
Hence $z^2 - i = \left(z - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$
$\times\left(z + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

9 $z^8 = -1$
$\therefore \quad (r\text{cis}\,\theta)^8 = \text{cis}\,(-\pi)$ where $z = r\text{cis}\,\theta$
$\therefore \quad r^8\text{cis}\,8\theta = \text{cis}\,(-\pi)$
$\therefore \quad r^8 = 1$ and $8\theta = -\pi + 2k\pi,\ k \in Z$
$\therefore \quad r = 1$ and $\theta = -\frac{\pi}{8} + \frac{k\pi}{4},\ k \in Z$
$\therefore \quad z = \text{cis}\left(-\frac{\pi}{8} + \frac{k\pi}{4}\right),\ k \in Z$
When $k = 0,\ z = \text{cis}\left(-\frac{\pi}{8}\right)$
When $k = 1,\ z = \text{cis}\left(\frac{\pi}{8}\right)$

When $k = 2,\ z = \text{cis}\left(\frac{3\pi}{8}\right)$

When $k = 3,\ z = \text{cis}\left(\frac{5\pi}{8}\right)$

When $k = 4,\ z = \text{cis}\left(\frac{7\pi}{8}\right)$

When $k = 5,\ z = \text{cis}\left(\frac{9\pi}{8}\right) = \text{cis}\left(-\frac{7\pi}{8}\right)$

When $k = 6,\ z = \text{cis}\left(\frac{11\pi}{8}\right) = \text{cis}\left(-\frac{5\pi}{8}\right)$

When $k = 7,\ z = \text{cis}\left(\frac{13\pi}{8}\right) = \text{cis}\left(-\frac{3\pi}{8}\right)$

When $k = 8,\ z = \text{cis}\left(\frac{15\pi}{8}\right) = \text{cis}\left(-\frac{\pi}{8}\right)$

Hence solutions of $z^8 + 1 = 0$ are:

$\text{cis}\left(\frac{\pi}{8}\right)$, $\text{cis}\left(\frac{3\pi}{8}\right)$, $\text{cis}\left(\frac{5\pi}{8}\right)$, $\text{cis}\left(\frac{7\pi}{8}\right)$, $\text{cis}\left(\frac{9\pi}{8}\right)$, $\text{cis}\left(\frac{11\pi}{8}\right)$, $\text{cis}\left(\frac{13\pi}{8}\right)$ and $\text{cis}\left(\frac{15\pi}{8}\right)$

Factors are:

$z - \text{cis}\left(\frac{\pi}{8}\right)$, $z - \text{cis}\left(\frac{3\pi}{8}\right)$, $z - \text{cis}\left(\frac{5\pi}{8}\right)$, $z - \text{cis}\left(\frac{7\pi}{8}\right)$, $z - \text{cis}\left(\frac{9\pi}{8}\right)$, $z - \text{cis}\left(\frac{11\pi}{8}\right)$, $z - \text{cis}\left(\frac{13\pi}{8}\right)$ and $z - \text{cis}\left(\frac{15\pi}{8}\right)$

10 **a** Let $z = a + bi,\ a,\ b \in R$ and $z^2 = 1 + i$

$\therefore\ z^2 = (a^2 - b^2) + 2abi$

$\therefore\ a^2 - b^2 = 1$ and $2ab = 1$

$\therefore\ b = \frac{1}{2a}$

$\therefore\ a^2 - \left(\frac{1}{2a}\right)^2 = 1$

$\therefore\ a^2 - \frac{1}{4a^2} = 1$

$\therefore\ 4a^4 - 1 = 4a^2$

$\therefore\ 4a^4 - 4a^2 - 1 = 0$

Let $x = a^2$ $\therefore\ 4x^2 - 4x - 1 = 0$

$\therefore\ 4\left(x^2 - x - \frac{1}{4}\right) = 0$

$\therefore\ x^2 - x + \left(\frac{1}{2}\right)^2 - \frac{1}{4} - \left(\frac{1}{2}\right)^2 = 0$

$\therefore\ \left(x - \frac{1}{2}\right)^2 - \frac{1}{2} = 0$

$\therefore\ x - \frac{1}{2} = \pm\sqrt{\frac{1}{2}}$

$\therefore\ x = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$

$= \frac{1}{2} \pm \frac{\sqrt{2}}{2}$

$= \frac{1 \pm \sqrt{2}}{2}$

$\therefore\ a^2 = \frac{1 \pm \sqrt{2}}{2}$

$\therefore\ a = \pm\sqrt{\frac{1 \pm \sqrt{2}}{2}}$

but $a \in R$ $\therefore\ a = \pm\sqrt{\frac{1 \pm \sqrt{2}}{2}}$

When a

$$= \pm\sqrt{\frac{1 \pm \sqrt{2}}{2}},$$

$$b = \frac{\pm 1}{2\sqrt{\frac{1 \pm \sqrt{2}}{2}}}$$

$$= \frac{\pm 1}{\sqrt{\frac{4(1 \pm \sqrt{2})}{2}}}$$

$$= \frac{\pm 1}{\sqrt{2(1 \pm \sqrt{2})}}$$

$$= \pm\sqrt{\frac{1}{2(1 + \sqrt{2})} \times \frac{1 - \sqrt{2}}{1 - \sqrt{2}}}$$

$$= \pm\sqrt{\frac{1 - \sqrt{2}}{2(1 - 2)}}$$

$$= \pm\sqrt{\frac{1 - \sqrt{2}}{-2}}$$

$$= \pm\sqrt{\frac{\sqrt{2} - 1}{2}}$$

$\therefore$ $z = \pm\left(\sqrt{\frac{1 + \sqrt{2}}{2}} + \sqrt{\frac{\sqrt{2} - 1}{2}}i\right)$, the square roots of $1 + i$

b Let $z = r\text{cis}\,\theta$ and $z^2 = 1 + i$

Also $1 + i = \sqrt{2}\text{cis}\,\frac{\pi}{4}$

$\therefore$ $(r\text{cis}\,\theta)^2 = \sqrt{2}\text{cis}\,\frac{\pi}{4}$

$\therefore$ $r^2\text{cis}\,2\theta = 2\frac{1}{2}\text{cis}\,\frac{\pi}{4}$

$\therefore$ $r^2 = 2^{\frac{1}{2}}$ and $2\theta = \frac{\pi}{4} + 2\pi k,\ k \in Z$

$\therefore$ $r = 2^{\frac{1}{4}}$ and $\theta = \frac{\pi}{8} + \pi k,\ k \in Z$

$\therefore$ $z = 2^{\frac{1}{4}}\text{cis}\left(\frac{\pi}{8} + \pi k\right),\ k \in Z$

When $k = 0,\ z = 2^{\frac{1}{4}}\text{cis}\,\frac{\pi}{8}$

When $k = 1,\ z = 2^{\frac{1}{4}}\text{cis}\,\frac{9\pi}{8} = 2^{\frac{1}{4}}\text{cis}\left(-\frac{7\pi}{8}\right)$

When $k = 2,\ z = 2^{\frac{1}{4}}\text{cis}\,\frac{17\pi}{8} = 2^{\frac{1}{4}}\text{cis}\,\frac{\pi}{8}$

Hence the square roots of $1 + i$ are $2^{\frac{1}{4}}\text{cis}\,\frac{\pi}{8}$ and $2^{\frac{1}{4}}\text{cis}\left(-\frac{7\pi}{8}\right)$

c $2^{\frac{1}{4}}\text{cis}\,\frac{\pi}{8}$

$$= 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

$$= 2^{\frac{1}{4}}\cos\frac{\pi}{8} + 2\frac{1}{4}\sin\frac{\pi}{8}i$$

$$= \sqrt{\frac{1 \pm \sqrt{2}}{2}} + \sqrt{\frac{\sqrt{2} - 1}{2}}i$$

(from part **a**)

$\therefore$ $2^{\frac{1}{4}}\cos\frac{\pi}{8} = \left(\frac{1 \pm \sqrt{2}}{2}\right)^{\frac{1}{2}}$ and

$2^{\frac{1}{4}}\sin\frac{\pi}{8} = \left(\frac{\sqrt{2} - 1}{2}\right)^{\frac{1}{2}}$

$\therefore$ $\cos\frac{\pi}{8} = \frac{\left(\frac{1 \pm \sqrt{2}}{2}\right)^{\frac{1}{2}}}{2^{\frac{1}{4}}}$

$$= \frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2}$$

$$\text{and } \sin\frac{\pi}{8} = \frac{\left(\frac{\sqrt{2}-1}{2}\right)^{\frac{1}{2}}}{2^{\frac{1}{4}}}$$

$$= \left(\frac{\frac{1 \pm \sqrt{2}}{2}}{\frac{2}{\sqrt{2}}}\right)^{\frac{1}{2}} = \left(\frac{\frac{\sqrt{2}-1}{2}}{\sqrt{2}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1 \pm \sqrt{2}}{2\sqrt{2}}\right)^{\frac{1}{2}} = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{1 \pm \sqrt{2}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)^{\frac{1}{2}}$$

$$= \frac{(2 \pm \sqrt{2})^{\frac{1}{2}}}{2} = \frac{(2-\sqrt{2})^{\frac{1}{2}}}{2}$$

d

$$z^4 = i$$

$$r^4 \text{cis } 4\theta = \text{cis}\left(\frac{\pi}{2}\right)$$

$$r = 1 \text{ and } \theta = \frac{\pi}{8} \text{ or } \frac{5\pi}{8} \text{ or } -\frac{3\pi}{8} \text{ or } -\frac{3\pi}{8}$$

$$\therefore z = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)$$

$$= \frac{(2+\sqrt{2})^{\frac{1}{2}}}{2} + \frac{(2+\sqrt{2})^{\frac{1}{2}}}{2} i$$

$$\text{or } = -\frac{(2-\sqrt{2})^{\frac{1}{2}}}{2} + \frac{(2+\sqrt{2})^{\frac{1}{2}}}{2} i$$

$$\text{or } = -\frac{(2+\sqrt{2})^{\frac{1}{2}}}{2} - \frac{(2-\sqrt{2})^{\frac{1}{2}}}{2} i$$

$$\text{or } = \frac{(2-\sqrt{2})^{\frac{1}{2}}}{2} - \frac{(2+\sqrt{2})^{\frac{1}{2}}}{2} i$$

Solutions to Exercise 6H

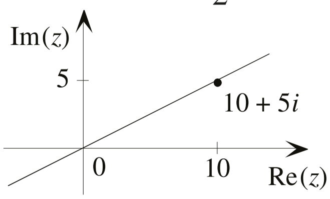
1 **a** Let $z = x + yi$

then $2\text{Im}(z) = 2y$

and $\text{Re}(z) = x$

$\therefore \quad 2y = x$

$\therefore \quad y = \dfrac{x}{2}$



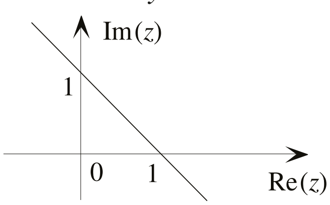
b Let $z = x + yi$

then $\text{Im}(z) = y$

and $\text{Re}(z) = x$

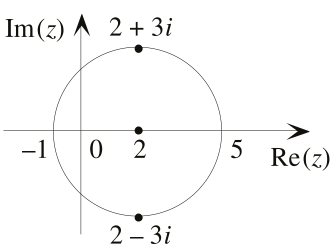
$\therefore \quad y + x = 1$

$\therefore \quad y = 1 - x$



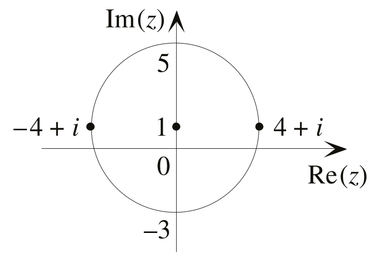
c $|z - 2| = 3$

a circle with centre (2, 0) and radius 3



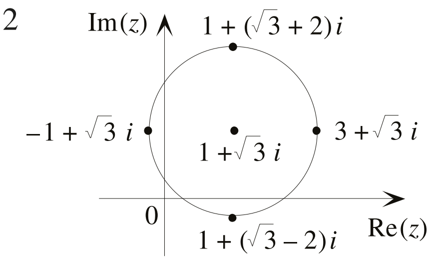
d $|z - i| = 4$

a circle with centre (0, 1) and radius 4



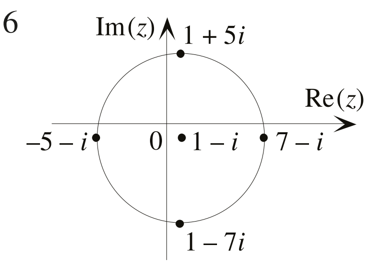
e $|z - (1 + \sqrt{3}i)| = 2$

a circle with centre $(1, \sqrt{3})$ and radius 2



f $|z - (1 - i)| = 6$

a circle with centre (1, −1) and radius 6



2 $z = i\bar{z}$

Let $z = x + yi$

$\therefore \quad \bar{z} = x - yi$

$\therefore \quad x + yi = i(x - yi)$

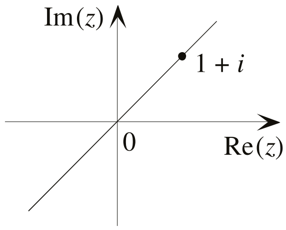
$= xi - yi^2$

$= xi + y$

$\therefore \quad (x - y) + (y - x)i = 0$

$\therefore \quad x - y = 0$

$\therefore \quad x = y$



3

$$|z - 1| = |z + 1|$$

Let $z = x + yi$

$$\therefore \quad |x + yi - 1| = |x + yi + 1|$$

$$\therefore \quad |(x - 1) + yi| = |(x + 1) + yi|$$

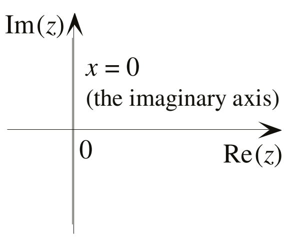
$$\therefore \quad \sqrt{(x-1)^2 + y^2} = \sqrt{(x+1)^2 + y^2}$$

$$\therefore \quad (x - 1)^2 + y^2 = (x + 1)^2 + y^2$$

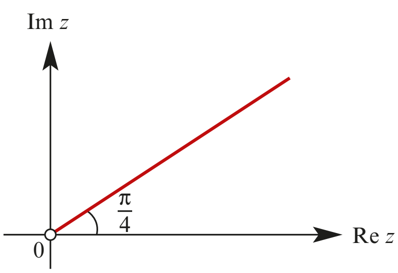
$$\therefore \quad x^2 - 2x + 1 = x^2 + 2x + 1$$

$$\therefore \quad 0 = 4x$$

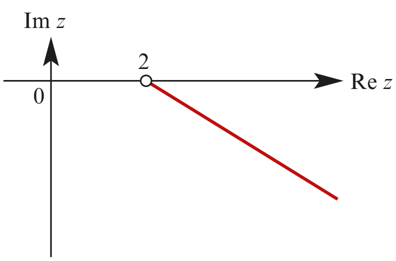
$$\therefore \quad x = 0$$



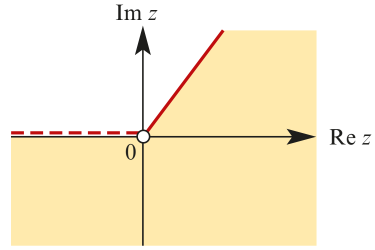
4 a



b



c



5 We are required to prove that $3|z - 1|^2 = |z + 1|^2$ and $|z - 2|^2 = 3$ are equivalent statements.
i.e. $3|z - 1|^2 = |z + 1|^2 \Leftrightarrow |z - 2|^2 = 3$

Let $z = x + yi$

$$\begin{aligned}
\text{then } 3|z - 1|^2 &= 3|x + yi - 1|^2 \\
&= 3|(x - 1) + yi|^2 \\
&= 3\left(\sqrt{(x-1)^2 + y^2}\right)^2 \\
&= 3((x - 1)^2 + y^2) \\
&= 3(x^2 - 2x + 1 + y^2) \\
&= 3x^2 - 6x + 3 + 3y^2
\end{aligned}$$

$$\begin{aligned}
\text{and } |z + 1|^2 &= |x + yi + 1|^2 \\
&= |(x + 1) + yi|^2 \\
&= \left(\sqrt{(x+1)^2 + y^2}\right)^2 \\
&= (x + 1)^2 + y^2 \\
&= x^2 + 2x + 1 + y^2
\end{aligned}$$

If $3|z - 1|^2 = |z + 1|^2$
then $3x^2 - 6x + 3 + 3y^2 = x^2 + 2x + 1 + y^2$

$$\therefore \quad 2x^2 - 8x + 2 + 2y^2 = 0$$

$$\therefore \quad 2(x^2 - 4x + 1 + y^2) = 0$$

$$\therefore \quad x^2 - 4x + 1 + y^2 = 0$$

$$\therefore \quad x^2 - 4x + 4 + y^2 = 3$$

$$\therefore \quad (x - 2)^2 + y^2 = 3$$

Thus $3|z - 1|^2 = |z + 1|^2$ represents a circle with centre $(2, 0)$ and radius $\sqrt{3}$ and

$$|z - 2|^2 = 3$$

$$\therefore \quad |x + yi - 2|^2 = 3$$

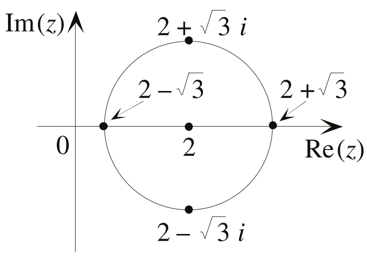
$$\therefore \quad |(x - 2) + yi|^2 = 3$$

$$\therefore \quad \left(\sqrt{(x - 2)^2 + y^2}\right)^2 = 3$$

$$\therefore \quad (x - 2)^2 + y^2 = 3$$

Thus $|z - 2|^2 = 3$ represents a circle with centre $(2, 0)$ and radius $\sqrt{3}$

Hence $3|z - 1|^2 = |z + 1|^2$ and $|z - 2|^2 = 3$ are equivalent statements



6 a

$$|z + 2i| = 2|z - i|$$

$$\text{Let } z = x + yi$$

$$\therefore \quad |x + yi + 2i| = 2|x + yi - i|$$

$$\therefore \quad |x + (y + 2)i| = 2|x + (y - 1)i|$$

$$\therefore \quad \sqrt{x^2 + (y + 2)^2} = 2\sqrt{x^2 + (y - 1)^2}$$

$$\therefore \quad x^2 + (y + 2)^2 = 4(x^2 + (y - 1)^2)$$

$$\therefore \quad x^2 + y^2 + 4y + 4 = 4(x^2 + y^2 - 2y + 1)$$

$$= 4x^2 + 4y^2 - 8y + 4$$

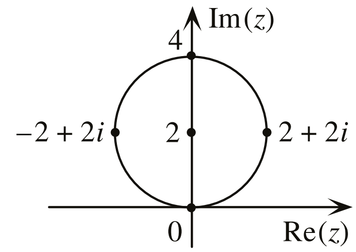
$$\therefore \quad 0 = 3x^2 + 3y^2 - 12y$$

$$= 3(x^2 + y^2 - 4y)$$

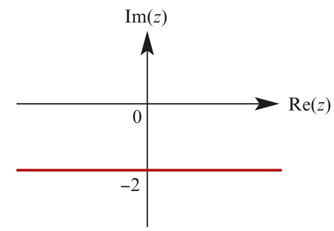
$$= x^2 + y^2 - 4y + 4 - 4$$

$$\therefore \quad 4 = x^2 + (y - 2)^2$$

a circle with centre $(0, 2)$ and radius 2



b



c

$$z + \overline{z} = 5$$

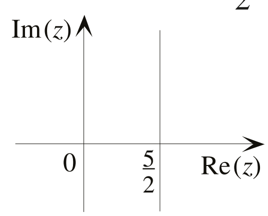
$$\text{Let } z = x + yi$$

$$\text{then } \overline{z} = x - yi$$

$$\therefore \quad x + yi + x - yi = 5$$

$$\therefore \quad 2x = 5$$

$$\therefore \quad x = \frac{5}{2}$$



d

$$z\overline{z} = 5$$

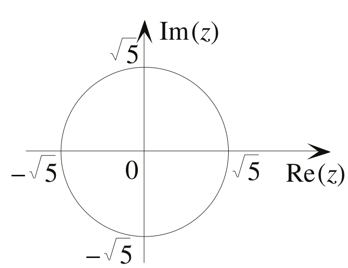
$$\text{Let } z = x + yi$$

$$\text{then } \overline{z} = x - yi$$

$$\therefore \quad (x + yi)(x - yi) = 5$$

$$\therefore \quad x^2 + y^2 = 5$$

a circle with centre $(0, 0)$ and radius $\sqrt{5}$



e $\quad \text{Re}(z^2) = \text{Im}(z)$

Let $z = x + yi$

$\therefore \quad z^2 = (x + yi)^2$

$= x^2 + 2xyi + y^2i^2$

$= (x^2 - y^2) + 2xyi$

$\therefore \quad \text{Re}(z^2) = x^2 - y^2$

$\therefore \quad x^2 - y^2 = y$

$\therefore \quad x^2 = y^2 + y$

$= y^2 + y + \frac{1}{4} - \frac{1}{4}$

$= \left(y + \frac{1}{2}\right)^2 - \frac{1}{4}$

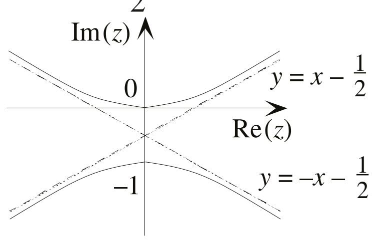
$\therefore \quad \left(y + \frac{1}{2}\right)^2 - x^2 = \frac{1}{4}$

$\left(y + \frac{1}{2}\right)^2 = x^2 + \frac{1}{4}$

$\left(y + \frac{1}{2}\right)^2 = x^2\left(1 + \frac{1}{4x^2}\right)$

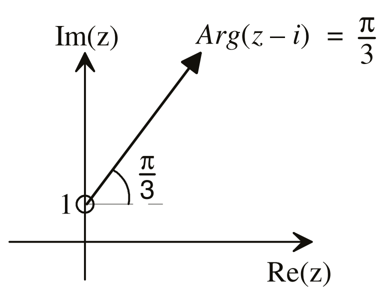
The equations of the asymptotes are

$y = \pm x - \frac{1}{2}$



f $\text{Arg}(z - i) = \frac{\pi}{3}$

The equation is $y = \sqrt{3}x + 1$ for $x > 0$



7 a $\quad \left|\frac{z - 2}{z}\right| = 1$

$\therefore \quad |z - 2| = |z|$

Let $z = x + iy$

$\therefore \quad |x + iy - 2| = |x + iy|$

$\therefore \quad |(x - 2) + iy| = |x + iy|$

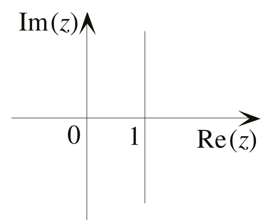
$\therefore \quad \sqrt{(x - 2)^2 + y^2} = \sqrt{x^2 + y^2}$

$\therefore \quad (x - 2)^2 + y^2 = x^2 + y^2$

$\therefore \quad x^2 - 4x + 4 = x^2$

$\therefore \quad 4 = 4x$

$\therefore \quad x = 1$



b $\quad \left|\frac{z - 1 - i}{z}\right| = 1$

$\therefore \quad |z - 1 - i| = |z|$

$\therefore \quad |x + iy - 1 - i| = |x + iy|$

$\therefore \quad |(x - 1) + (y - 1)i| = |x + iy|$

$\therefore \quad \sqrt{(x - 1)^2 + (y - 1)^2} = \sqrt{x^2 + y^2}$

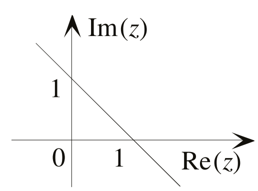
$\therefore \quad (x - 1)^2 + (y - 1)^2 = x^2 + y^2$

$\therefore \quad x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 + y^2$

$\therefore \quad -2x - 2y + 2 = 0$

$\therefore \quad x + y = 1$

$\therefore \quad y = -x + 1$



8 Let $z_1 = \dfrac{z+1}{z-1}$ where $z = x + yi$

$$\therefore \quad z_1 = \frac{x+yi+1}{x+yi-1}$$

$$= \frac{(x+1)+yi}{(x-1)+yi}$$

$$= \frac{(x+1)+yi}{(x-1)+yi} \times \frac{(x-1)-yi}{(x-1)-yi}$$

$$= \frac{(x+1)(x-1)+(x-1)yi-(x+1)yi+y^2}{(x-1)^2+y^2}$$

$$= \frac{x^2-1+xyi-yi-xyi-yi+y^2}{(x-1)^2+y^2}$$

$$= \frac{(x^2+y^2-1)-2yi}{(x-1)^2+y^2}$$

Now Re $(z_1) = 0$

$$\therefore \quad \frac{x^2+y^2-1}{(x-1)^2+y^2} = 0$$

$$\therefore \quad x^2+y^2-1 = 0$$

$$\therefore \quad x^2+y^2 = 1$$

a circle with centre $(0, 0)$ and radius 1

9 Let $z = x + yi$

then $2|z-2| = 2|x+yi-2|$

$$= 2|\,(x-2)+yi|$$

$$= 2\sqrt{(x-2)^2+y^2}$$

and $|z-6i| = |x+yi-6i|$

$$= |x+(y-6)i|$$

$$= \sqrt{x^2+(y-6)^2}$$

If $2|z-2| = |z-6i|$

then $2\sqrt{(x-2)^2+y^2} = \sqrt{x^2+(y-6)^2}$

$$\therefore \quad 4((x-2)^2+y^2) = x^2+(y-6)^2$$

$$\therefore \quad 4(x^2-4x+4+y^2)$$

$$= x^2+y^2-12y+36$$

$$\therefore \quad 4x^2-16x+16+4y^2$$

$$= x^2+y^2-12y+36$$

$$\therefore \quad 3x^2-16x+3y^2+12y = 20$$

$$\therefore \quad 3\left(x^2-\frac{16}{3}x+\frac{64}{9}-\frac{64}{9}\right)$$

$$+\,3(y^2+4y+4-4) = 20$$

$$\therefore \quad 3\left(x-\frac{8}{3}\right)^2-\frac{64}{3}+3(y+2)^2-12 = 20$$

$$\therefore \quad 3\left(x-\frac{8}{3}\right)^2+3(y+2)^2 = \frac{160}{3}$$

$$\therefore \quad \left(x-\frac{8}{3}\right)^2+(y+2)^2 = \frac{160}{9}$$

a circle with centre $\left(\dfrac{8}{3}, -2\right)$ and radius $\dfrac{4\sqrt{10}}{3}$

10 Let $z = x + yi$

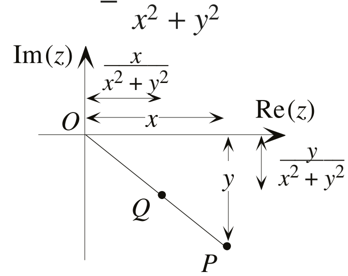
$$\therefore \quad \bar{z} = x - yi$$

and $\dfrac{1}{z} = \dfrac{1}{x+yi}$

$$= \frac{1}{x+yi} \times \frac{x-yi}{x-yi}$$

$$= \frac{x-yi}{x^2+y^2}$$

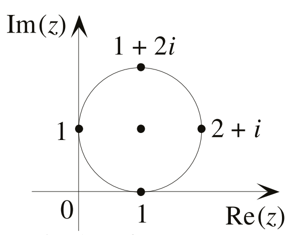
$$= \frac{\bar{z}}{x^2+y^2}$$



Hence O, P and Q are collinear.

$$OP = \sqrt{x^2 + y^2}$$
$$= |z|$$
$$OQ = \sqrt{\left(\frac{x}{x^2 + y^2}\right)^2 + \left(\frac{y}{x^2 + y^2}\right)^2}$$
$$= \frac{1}{x^2 + y^2}\sqrt{x^2 + y^2}$$
$$= \frac{|z|}{x^2 + y^2}$$
$$= \frac{|z|}{|z|^2}$$
$$= \frac{1}{|z|}$$
$$\therefore \quad OP : OQ = |z| : \frac{1}{|z|} = |z|^2 : 1$$

11 a $|z - (1 + i)| = 1$

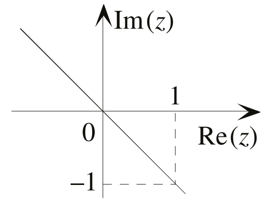


a circle of centre (1, 1) and radius 1

$(x - 1)^2 + (y - 1)^2 = 1$

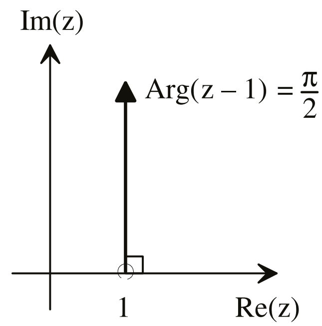
b
$$|z - 2| = |z + 2i|$$
Let $z = x + yi$

then $|x + yi - 2| = |x + yi + 2i|$
$$\therefore \quad |(x - 2) + yi| = |x + (y + 2)i|$$
$$\therefore \quad \sqrt{(x - 2)^2 + y^2} = \sqrt{x^2 + (y + 2)^2}$$
$$\therefore \quad (x - 2)^2 + y^2 = x^2 + (y + 2)^2$$
$$\therefore \quad x^2 - 4x + 4 + y^2 = x^2 + y^2 + 4y + 4$$
$$\therefore \quad -4x = 4y$$
$$\therefore \quad y = -x$$



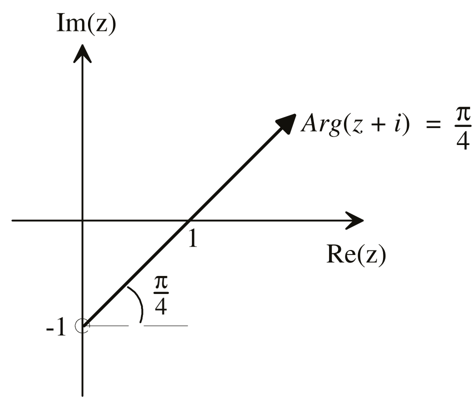
c $\text{Arg}(z - 1) = \dfrac{\pi}{2}$

$\therefore \quad x = 1, \ y > 0$

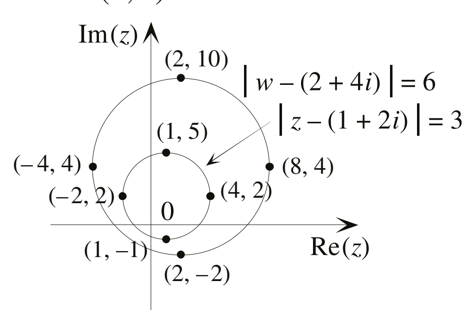


d $\text{Arg}(z + i) = \dfrac{\pi}{4}$

$\therefore \quad y = x - 1, \ x > 0$



12 If $w = 2z$ then w describes a circle with centre $(2, 4)$ and radius 6.



13 a

$$z^2 + 2z + 4 = 0$$

$$\therefore \quad z^2 + 2z + 1 + 3 = 0$$

$$\therefore \quad (z+1)^2 - 3i^2 = 0$$

$$\therefore \quad (z + 1 + \sqrt{3}i)(z + 1 - \sqrt{3}i) = 0$$

$$\therefore \quad z = -1 \pm \sqrt{3}i$$

b i $|z| = \sqrt{(-1)^2 + (\pm\sqrt{3})^2}$

$= \sqrt{1+3}$

$= 2$

ii $z - 1 = -1 \pm \sqrt{3}i - 1$

$= -2 \pm \sqrt{3}$

$|z - 1| = \sqrt{(-2)^2 + (\pm\sqrt{3})^2}$

$= \sqrt{4+3}$

$= \sqrt{7}$

iii If $z = -1 \pm \sqrt{3}i$

then $\bar{z} = -1 \mp \sqrt{3}i$

and $z + \bar{z} = -2$

c $|z| = 2$

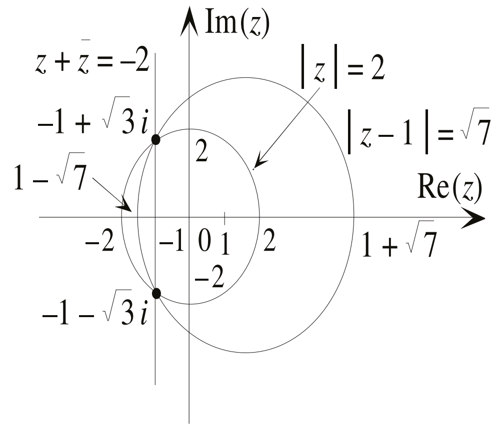
a circle with centre (0, 0) and radius 2

$|z - 1| = \sqrt{7}$

a circle with centre (1, 0) and radius $\sqrt{7}$

$z + z = -2$

a straight line with equation $x = -1$



Solutions to short-answer questions

1 a $3 + 2i + 5 - 7i = (3 + 5) + (2i - 7i)$
$= 8 - 5i$

b $i^3 = i^2 \times i$
$= -1 \times i = -i$

c $(3 - 2i)(5 + 7i)$
$= 3(5 + 7i) - 2i(5 + 7i)$
$= 15 + 21i - 10i - 14i^2$
$= 29 + 11i$

d $(3 - 2i)(3 + 2i) = 3^2 - (2i)^2$
$= 9 - 4i^2 = 13$

e $$\frac{2}{3-2i} = \frac{2}{3-2i} \times \frac{3+2i}{3+2i} = \frac{2(3+2i)}{(3-2i)(3+2i)} = \frac{6+4i}{13} \text{ or } \frac{6}{13} + \frac{4}{13}i$$

f $$\frac{5-i}{2+i} = \frac{5-i}{2+i} \times \frac{2-i}{2-i} = \frac{(5-i)(2-i)}{(2+i)(2-i)} = \frac{5(2-i) - i(2-i)}{2^2 - i^2} = \frac{10 - 5i - 2i + i^2}{5} = \frac{9-7i}{5} \text{ or } \frac{9}{5} - \frac{7}{5}i$$

g $$\frac{3i}{2+i} = \frac{3i}{2+i} \times \frac{2-i}{2-i} = \frac{3i(2-i)}{(2+i)(2-i)} = \frac{6i - 3i^2}{5} = \frac{3+6i}{5} \text{ or } \frac{3}{5} + \frac{6}{5}i$$

h $(1 - 3i)^2 = (1 - 3i)(1 - 3i)$
$= 1(1 - 3i) - 3i(1 - 3i)$
$= 1 - 3i - 3i + 9i^2$
$= -8 - 6i$

i $$\frac{(5+2i)^2}{3-i}$$
$$= \frac{5(5+2i) + 2i(5+2i)}{3-i} \times \frac{3+i}{3+i}$$
$$= \frac{25 + 10i + 10i + 4i^2}{3-i} \times \frac{3+i}{3+i}$$
$$= \frac{(21+20i)(3+i)}{(3-i)(3+i)}$$
$$= \frac{21(3+i) + 20i(3+i)}{3^2 - i^2}$$
$$= \frac{63 + 21i + 60i + 20i^2}{10}$$
$$= \frac{43+81i}{10} \text{ or } \frac{43}{10} + \frac{81}{10}i$$

2 a $(z - 2)^2 + 9 = 0$
$\therefore\ (z - 2)^2 - 9i^2 = 0$
$\therefore\ (z - 2)^2 - (3i)^2 = 0$
$\therefore\ ((z - 2) + 3i)((z - 2) - 3i) = 0$
$\therefore\ (z - 2 + 3i)(z - 2 - 3i) = 0$
$\therefore\ z - 2 + 3i = 0 \text{ or } z - 2 - 3i = 0$
$\therefore\ z = 2 - 3i \text{ or } z = 2 + 3i$

b $\dfrac{z - 2i}{z + (3 - 2i)} = 2$

$\therefore \quad z - 2i = 2(z + (3 - 2i))$

$\therefore \quad = 2z + 6 - 4i$

$\therefore \quad -2i - 6 + 4i = 2z - z$

$\therefore \quad z = -6 + 2i$

c $z^2 + 6z + 12 = 0$

$\therefore \quad z^2 + 6z + 9 + 3 = 0$

$\therefore \quad (z + 3)^2 - 3i^2 = 0$

$\therefore \quad (z + 3)^2 - (\sqrt{3}i)^2 = 0$

$\therefore \quad ((z + 3) + \sqrt{3}i)((z + 3) - \sqrt{3}i) = 0$

$\therefore \quad (z + 3 + \sqrt{3}i)(z + 3 - \sqrt{3}i) = 0$

$\therefore \quad z + 3 + \sqrt{3}i = 0$ or $z + 3 - \sqrt{3}i = 0$

$\therefore \quad z = -3 - \sqrt{3}i$ or $z = -3 + \sqrt{3}i$

d $z^4 + 81 = 0$

$\therefore \quad z^4 = -81$

Let $z = r\text{cis}\,\theta$,

$\therefore \quad (r\text{cis}\,\theta)^4 = 81\text{cis}\,\pi$

$\therefore \quad r^4\text{cis}\,4\theta = 3^4\text{cis}\,\pi$

$\therefore \quad r^4 = 3^4$ and $4\theta = \pi + 2k\pi,\ k \in Z$

$= \pi(2k + 1)$

$\therefore \quad r = 3$ and $\theta = \dfrac{\pi(2k + 1)}{4}$

When $k = 0$, $\quad \theta = \dfrac{\pi}{4}$

When $k = 1$, $\quad \theta = \dfrac{3\pi}{4}$

When $k = 2$, $\quad \theta = \dfrac{5\pi}{4}$ or $-\dfrac{3\pi}{4}$

When $k = 3$, $\quad \theta = \dfrac{7\pi}{4}$ or $-\dfrac{\pi}{4}$

$\therefore \quad z = 3\text{cis}\,\dfrac{\pi}{4}$,

$\text{cis}\dfrac{3\pi}{4}, 3\text{cis}\left(-\dfrac{3\pi}{4}\right)$ or $3\text{cis}\left(-\dfrac{\pi}{4}\right)$

or $z = 3\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)$,

$3\left(-\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right),\ 3\left(-\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right)$

or $3\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right)$

$= \dfrac{3\sqrt{2}}{2}(1 + i),\ \dfrac{3\sqrt{2}}{2}(-1 + i),$

$-\dfrac{3\sqrt{2}}{2}(1 + i)$ or $\dfrac{3\sqrt{2}}{2}(1 - i)$

e $z^3 - 27 = 0$

$\therefore \quad z^3 - 3^3 = 0$

$\therefore \quad (z - 3)(z^2 + 3z + 9) = 0$

$\therefore \quad (z - 3)\left(z^2 + 3z + \dfrac{9}{4} + \dfrac{27}{4}\right) = 0$

$\therefore \quad (z - 3)\left(\left(z + \dfrac{3}{2}\right)^2 - \dfrac{27}{4}i^2\right) = 0$

$\therefore \quad (z - 3)\left(\left(z + \dfrac{3}{2}\right)^2 - \left(\dfrac{3\sqrt{3}}{2}i\right)^2\right) = 0$

$\therefore \quad (z - 3)\left(z + \dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i\right)$

$\left(z + \dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i\right) = 0$

$\therefore \quad z - 3 = 0$ or $z + \dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i = 0$

or $z + \dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i = 0$

$\therefore \quad z = 3$ or $z = -\dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i$

or $z = -\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i$

f $8z^3 + 27 = 0$

$\therefore\ (2z)^3 + 3^3 = 0$

$\therefore\ (2z + 3)(4z^2 - 6z + 9) = 0$

$\therefore\ 4(2z + 3)\left(z^2 - \frac{3}{2}z + \frac{9}{16} + \frac{27}{16}\right) = 0$

$\therefore\ 4(2z + 3)\left(\left(z - \frac{3}{4}\right)^2 - \frac{27}{16}i^2\right) = 0$

$\therefore\ 4(2z + 3)\left(\left(z - \frac{3}{4}\right)^2 - \left(\frac{3\sqrt{3}}{4}i\right)^2\right) = 0$

$\therefore\ 4(2z + 3)\left(z - \frac{3}{4} + \frac{3\sqrt{3}}{4}i\right)$

$\left(z - \frac{3}{4} - \frac{3\sqrt{3}}{4}i\right) = 0$

$\therefore\ 2z + 3 = 0$ or $z - \frac{3}{4} + \frac{3\sqrt{3}}{4}i = 0$

or $z - \frac{3}{4} - \frac{3\sqrt{3}}{4}i = 0$

$\therefore\ z = -\frac{3}{2}$ or $z = \frac{3}{4} - \frac{3\sqrt{3}}{4}i$

or $z = \frac{3}{4} + \frac{3\sqrt{3}}{4}i$

$= \frac{3}{4}(1 - \sqrt{3}i) \quad = \frac{3}{4}(1 + \sqrt{3}i)$

3 a Let $P(z) = z^3 - 2z^2 - 3z + 10$.
If $2 - i$ is a root of the equation $P(z) = 0$, then $P(2 - i) = 0$

$P(2 - i)$

$= (2 - i)^3 - 2(2 - i)^2 - 3(2 - i) + 10$

$= 2^3 - 12i + 6i^2 - i^3 - 2(4 - 4i + i^2)$

$\quad - 6 + 3i + 10$

$= 8 - 12i - 6 + i - 8 + 8i$

$\quad + 2 + 4 + 3i$

$= 0$

Since $P(2 - i) = 0, 2 - i$ is a root of the equation $z^3 - 2z^2 - 3z + 10 = 0$.
By the conjugate factor theorem, $2 + i$ is also a root of $P(z)$.
Therefore two linear factors of $P(z)$ are $z - (2 - i)$ and $z - (2 + i)$.
Multiply these two factors to get the quadratic factor:

$(z - (2 - i))(z - (2 + i))$

$= z^2 - (2 + i)z - (2 - i)z$

$\quad + (2 - i)(2 + i)$

$= z^2 - (2 + i + 2 - i)z + 4 - i^2$

$= z^2 - 4z + 5$

By division, $P(z) = (z^2 - 4z + 5)(z + 2)$
Therefore $P(z) = 0$ implies $z = -2, 2 - i$ or $2 + i$.

b Let $P(x) = x^3 - 5x^2 + 7x + 13$.
If $3 - 2i$ is a root of the equation $P(x) = 0$, then $P(3 - 2i) = 0$.

$P(3 - 2i)$

$= (3 - 2i)^3 - 5(3 - 2i)^2$

$\quad + 7(3 - 2i) + 13$

$= 3^3 - 54i + 36i^2 - (2i)^3$

$\quad - 5(9 - 12i + 4i^2) + 21 - 14i + 13$

$= 27 - 54i - 36 + 8i - 45 + 60i$

$\quad + 20 + 34 - 14i$

$= 0$

Since $P(3 - 2i) = 0, 3 - 2i$ is a root of the equation $x^3 - 5x^2 + 7x + 13 = 0$.
By the conjugate factor theorem, $3 + 2i$ is also a root of $P(x)$.
Therefore two linear factors of $P(x)$ are $x - (3 - 2i)$ and $x - (3 + 2i)$.
Multiply these two factors to get the quadratic factor:

$(x-(3-2i))(x-(3+2i))$

$= x^2-(3+2i)x-(3-2i)x$

$\quad +(3-2i)(3+2i)$

$= x^2-(3+2i+3-2i)x+9-4i^2$

$= x^2-6x+13$

By division, $P(x) =$ $(x^2-6x+13)(x+1)$

Therefore $P(x)=0$ implies $z=-1, 3-2i$ or $3+2i$.

c Let $P(z)=z^3-4z^2+6z-4$.

If $1+i$ is a root of the equation $P(z)=0$, then $P(1+i)=0$.

$P(1+i)$

$= (1+i)^3-4(1+i)^2+6(1+i)-4$

$= 1^3+3i+3i^2+i^3-4(1+2i+i^2)$

$\quad +6+6i-4$

$= 1+3i-3-i-4-8i+4+2+6i$

$= 0$

Since $P(1+i)=0$, $1+i$ is a root of the equation $z^3-4z^2+6z-4=0$. By the conjugate factor theorem, $1-i$ is also a root of $P(z)$.

Therefore two linear factors of $P(z)$ are $z-(1+i)$ and $z-(1-i)$.

Multiply these two factors to get the quadratic factor:

$(z-(1+i))(z-(1-i))$

$= z^2-(1-i)z-(1+i)z$

$\quad +(1+i)(1-i)$

$= z^2-(1-i+1+i)z+1-i^2$

$= z^2-2z+2$

By division, $P(z)=(z^2-2z+2)(z-2)$

Therefore $P(z)=0$ implies $z=2, 1+i$ or $1-i$.

4 **a** $2x^2+3x+2$

$$= 2\left(x^2+\frac{3}{2}x+\frac{9}{16}+1-\frac{9}{16}\right)$$

$$= 2\left(\left(x+\frac{3}{4}\right)^2+\frac{7}{16}\right)$$

$$= 2\left(x+\frac{3}{4}+\frac{i\sqrt{7}}{4}\right)\left(x+\frac{3}{4}-\frac{i\sqrt{7}}{4}\right)$$

b Let $P(x)=x^3-x^2+x-1$

$P(1)=1-1+1-1=0$

$\therefore$ $x-1$ is a factor

$P(x)=(x-1)(x^2+1)$

$=(x-1)(x+i)(x-i)$

c Let $x^3+2x^2-4x-8=P(x)$

Possible solutions of the equation $P(x)=0$ are $\pm 1, \pm 2, \pm 4, \pm 8$

A check shows that $P(-2)=0$, $\therefore$ $x+2$ is a factor of $P(x)$.

By division, $P(x)=(x+2)(x^2-4)$

$\therefore$ $P(x)=(x+2)(x+2)(x-2)$ or $(x+2)^2(x-2)$

5 $(a+ib)^2=3-4i$

$\therefore$ $a^2+2iab-b^2=3-4i$

$\therefore$ $(a^2-b^2)+2abi=3-4i$

By equating real and imaginary parts, $a^2-b^2=3$ and $2ab=-4$

$$\therefore\quad b=-\frac{2}{a}$$

$$\therefore\quad a^2-\frac{4}{a^2}=3$$

$\therefore$ $a^4-4=3a^2$

$\therefore$ $a^4-3a^2-4=0$

$\therefore$ $(a^2-4)(a^2+1)=0$

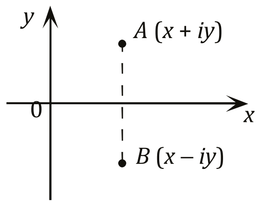
$\therefore$ $a^2=4$ since $a \in R$

$\therefore$ $a=\pm 2$

When $a = 2,\ b = -1$ and when $a = -2$, $b = 1$

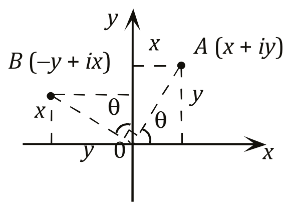
6 a $\bar{z} = x - iy$

$\therefore$ (iv) take the conjugate



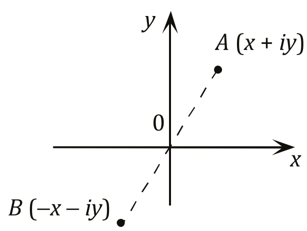
b $i(x + iy) = ix - y$

$\therefore$ (ii) multiply by i



c $-1(x + iy) = -x - iy$

$\therefore$ (i) multiply by $-l$



d Anticlockwise rotation about O through 270° is a sequence of anticlockwise rotation about O by 90° and then 180° (see **b** and **c**). Therefore it is equivalent to sequence of multiplication by i and $-l$

$\therefore$ (iii) multiply by $-i$

7
$$(a + ib)^2 = -24 - 10i$$
$$\therefore \quad a^2 + 2iab - b^2 = -24 - 10i$$
$$\therefore \quad (a^2 - b^2) + 2abi = -24 - 10i$$

By equating real and imaginary parts,
$a^2 - b^2 = -24$ and $2ab = -10$

$$\therefore \quad b = -\frac{5}{a}$$
$$\therefore \quad a^2 - \frac{25}{a^2} = -24$$
$$a^4 - 25 = -24a^2$$
$$a^4 + 24a^2 - 25 = 0$$
$$\therefore \quad (a^2 - 1)(a^2 + 25) = 0$$
$$\therefore \quad a^2 = 1 \text{ since } a \in R$$
$$a = \pm 1$$

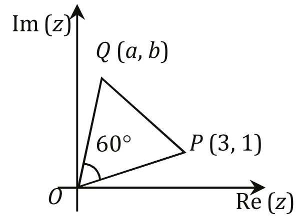
When $a = 1,\ b = -5$ and when $a = -1$, $b = 5$

8 If $z = -1 - 2i$ is a solution of the equation $f(z) = 0$, then $-1 + 2i$ is also a solution (conjugate factor theorem).
Therefore $f(z) = z^2 + az + b$

$$= (z + 1 + 2i)(z + 1 - 2i)$$
$$= (z + 1)^2 + 4$$
$$= z^2 + 2z + 5$$
$$\therefore \quad a = 2,\ b = 5$$

9
$$\frac{1}{1 + i\sqrt{3}} \times \frac{1 - i\sqrt{3}}{1 - i\sqrt{3}} = \frac{1 - i\sqrt{3}}{1 + 3}$$
$$= \frac{1}{4} - \frac{\sqrt{3}}{4}i$$
$$\therefore \quad r = \sqrt{\frac{1}{16} + \frac{3}{16}} = \frac{1}{2},$$
$$\cos\theta = \frac{1}{2},$$
$$\sin\theta = -\frac{\sqrt{3}}{2}$$
$$\therefore \quad \theta = -\frac{\pi}{3}$$
$$\therefore \quad \frac{1}{1 + i\sqrt{3}} = \frac{1}{2}\text{cis}\left(-\frac{\pi}{3}\right)$$

10 $|\overrightarrow{OP}| = |\overrightarrow{OQ}| = |\overrightarrow{PQ}|$



$$\text{and } |\overrightarrow{OP}|^2 = 1^2 + 3^2$$
$$= 10$$
$$|\overrightarrow{QO}|^2 = a^2 + b^2$$
$$\therefore \quad |\overrightarrow{PQ}|^2 = (a-3)^2 + (b-1)^2$$
$$a^2 + b^2 = 10,$$
because $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$

$$(a-3)^2 + (b-1)^2 = 10,$$
because $|\overrightarrow{OP}| = |\overrightarrow{PQ}|$

$$a^2 + b^2 = 10 \qquad ①$$
$$a^2 - 60 + 9 + b^2 - 2b + 1 = 10$$
$$\therefore \quad a^2 - 6a + b^2 - 2b = 0 \qquad ②$$
① − ② gives

$$6a + 2b = 10$$
$$\therefore \quad b = -30 + 5$$
Substitute in ①

$$a^2 + (-3a+5)^2 = 10$$
$$10a^2 - 30a + 15 = 0$$
$$a^2 - 3a + 1.5 = 0$$
$$a = \frac{3 \pm \sqrt{9 - 4 \times 1.5}}{2}$$
$$= \frac{3}{2} \pm \frac{\sqrt{3}}{2}$$
When $a = \frac{3}{2} + \frac{\sqrt{3}}{2}$, $b = -\frac{9}{2} + 5 - \frac{3\sqrt{3}}{2}$

$$= \frac{1}{2} - \frac{3\sqrt{3}}{2}$$
but $b > 0$ so this is not a solution.

When $a = \frac{3}{2} - \frac{\sqrt{3}}{2}$, $b = -\frac{9}{2} + 5 + \frac{3\sqrt{3}}{2}$

$$= \frac{1}{2} + \frac{3\sqrt{3}}{2}$$

11 a $2 \times (1+i) = 2(1+i)$

$$= 2 + 2i$$

b $\frac{1}{1-i} = \frac{1}{1-1} \times \frac{1+i}{1+i}$

$$= \frac{1+i}{1+1} = \frac{1}{2} + \frac{1}{2}i$$

c $1 - i = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$,

$$\therefore \quad (1-i)^7 = (\sqrt{2})^7 \text{cis}\left(-\frac{7\pi}{4}\right)$$
$$= 8\sqrt{2}\text{cis}\,\frac{\pi}{4}$$
$$\therefore \quad |z^7| = 8\sqrt{2}$$

d $\text{Arg}\,(z^7) = \frac{\pi}{4}$

12

$$\frac{1}{a+3i} + \frac{1}{a-3i} = \frac{4}{13}$$
$$\frac{a-3i}{(a+3i)(a-3i)} + \frac{a+3i}{(a-3i)(a+3i)} = \frac{4}{13}$$
$$\frac{a}{a^2+9} = \frac{2}{13}$$
$$2a^2 + 18 = 13a$$
$$2a^2 - 13a + 18 = 0$$
$$(a-2)(2a-9) = 0$$
$$a = 2, \frac{9}{2}.$$

13 a i $|1+i| = \sqrt{1+1} = \sqrt{2}$

ii $|1 - i\sqrt{3}| = \sqrt{1+3} = 2$

iii $\cos\theta_1 = \dfrac{\sqrt{2}}{2},\ \sin\theta_1 = \dfrac{\sqrt{2}}{2}$

$\therefore \quad \theta_1 = \dfrac{\pi}{4}$

iv $\cos\theta_2 = \dfrac{1}{2},\ \sin\theta_2 = -\dfrac{\sqrt{3}}{2}$

$\therefore \quad \theta_2 = -\dfrac{\pi}{3}$

b $\left|\dfrac{w}{z}\right| = \dfrac{\sqrt{2}}{2}$, $\text{Arg}\,(wz) = \dfrac{\pi}{4} - \dfrac{\pi}{3} = \dfrac{-\pi}{12}$

14 $\sqrt{3} + i = 2\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = 2\text{cis}\,\dfrac{\pi}{6}$

$(\sqrt{3} + i)^7 = 2^7\text{cis}\,\dfrac{7\pi}{6}$

$= 128\text{cis}\left(-\dfrac{5\pi}{6}\right)$

$= 128\left(-\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)$

$= -64\sqrt{3} - 64i$

15 $r^4(i)^4 - 2r^3(i)^3$

$+ 11r^2(i)^2 - 18ri + 18 = 0$

$\therefore \quad (r^4 - 11r^2 + 18) + (2r^3 - 18r)i = 0 + 0i$

By equating real and imaginary parts,

$r^4 - 11r^2 + 18 = 0$ and $2r^3 - 18r = 0$

$\therefore \quad r = \pm 3$ and

$\therefore \quad z = \pm 3i$ are solutions.

$(z - 3i)(z + 3i) = z^2 + 9$

By division $(z^2 + 9)(z^2 - 2z + 2)$ are factors,

$\therefore \quad (z - 3i)(z + 3i)(z - 1 + i)(z - 1 - i)$ are factors, and the solutions of the equation $z^4 - 2z^3 + 11z^2 - 18z + 18 = 0$ are

$z = \pm 3i,\ 1 \pm i$

16 $1 - i = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right)$

$= \sqrt{2}\text{cis}\left(-\dfrac{\pi}{4}\right)$

$(1 - i)^9 = (\sqrt{2})^9\text{cis}\left(-\dfrac{9\pi}{4}\right)$

$= 16\sqrt{2}\text{cis}\left(-\dfrac{\pi}{4}\right)$

$= 16\sqrt{2}\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right)$

$= 16 - 16i$

17 $k^3(i)^3 + (2 + i)k^2(i)^2 + (2 + 2i)ki + 4 = 0$

$\therefore \quad (-k^3 - k^2 + 2k)$

$+ (-2k^2 - 2k + 4)i = 0 + 0i$

By equating real and imaginary parts,

$-k^3 - k^2 + 2k = 0$ and $-2k^2 - 2k + 4 = 0$

$\therefore \quad k^2 + k - 2 = 0$

$k = -2,\ k = 1$

Therefore $z = -2i$ and $z = i$ are the roots of the given equation, and $(z + 2i)(z - i) = z^2 + iz + 2$ is a quadratic factor of the polynomial.

By division, $P(z) = (z + 2i)(z - i)(z + 2)$

Therefore the three roots are $-2i,\ i, -2$.

18 **a** $P(z) = z^3 - 2z + 4$

Now $P(-2) = 0$, $\therefore \quad z + 2$ is a factor of $P(z)$.

By division,

$P(z) = (z + 2)(z^2 - 2z + 2)$

$\therefore \quad P(z) = (z + 2)(z - 1 + i)(z - 1 - i)$

b $P(3) = 27 - 6 + 4$

$= 25$

$\therefore$ 25 is a remainder when $P(z)$ is divided by $z - 3$

19

$$a = x + 2i, \; b = -1 + iy$$
$$a + b = x - 1 + (2 + y)i$$
$$ab = (x + 2i)(-1 + iy)$$
$$= -x - 2y - 2i + ixy$$
$$x - 1 + (2 + y)i = x + 2y + 2i - ixy$$

since $a + b = -ab$

$$x - 1 = x + 2y$$

therefore $y = -\frac{1}{2}$

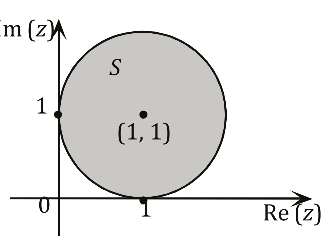
$$2 + y = 2 - xy$$

therefore $x = -1$

$$\therefore \quad a = -1 + 2i$$
$$b = -1 - \frac{1}{2}i$$

20 a $|z - (1 + i)| \leq 1$ can be represented by a disc with centre (1, 1) and radius 1, i.e., $(x - 1)^2 + (y - 1)^2 \leq 1$

b



21

$$|z + i| = |z - i|$$

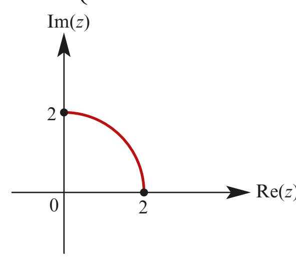
Let $z = x + iy$

then $|x + iy + i| = |x + iy - i|$

$$\therefore \quad |x + (y + 1)i| = |x + (y - 1)i|$$
$$\therefore \quad \sqrt{x^2 + (y + 1)^2} = \sqrt{x^2 + (y - 1)^2}$$
$$\therefore \quad x^2 + (y + 1)^2 = x^2 + (y - 1)^2$$
$$\therefore \quad x^2 + y^2 + 2y + 1 = x^2 + y^2 - 2y + 1$$
$$\therefore \quad 2y = -2y$$
$$\therefore \quad 4y = 0$$
$$\therefore \quad y = 0$$

So the set describes the Re(z) axis.

22 a $S = \left\{z : z = 2\text{cis}\,\theta, \; 0 \leq \theta \leq \frac{\pi}{2}\right\}$

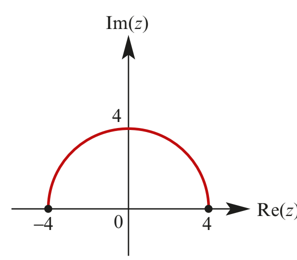


b

$$w = z^2$$
$$= (2\text{cis}\,\theta)^2$$
$$= 2^2\text{cis}\,2\theta$$
$$= 4\text{cis}\,2\theta$$

Now $0 \leq \theta \leq \frac{\pi}{2}$

$$\therefore \quad 0 \leq 2\theta \leq \pi$$



c $v = \frac{2}{Z}$

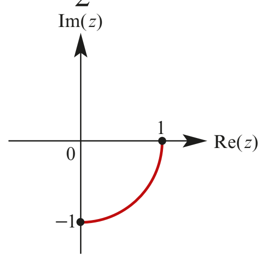
$= \frac{2}{2\text{cis}\theta}$

$= \frac{1}{\text{cis}\theta}$

$= \text{cis}(-\theta)$

Now $0 \leq \theta \leq \frac{\pi}{2}$

$\therefore \quad -\frac{\pi}{2} \leq -\theta \leq 0$



23 (0, −2), (1, 0) and (2, −1) are all points on the circle with cartesian equation $(x-h)^2 + (y-k)^2 = r^2$

i.e. $(0-h)^2 + (-2-k)^2 = r^2$

$\therefore \quad h^2 + 4 + 4k + k^2 = r^2$ ①

$(1-h)^2 + (0-k)^2 = r^2$

$\therefore \quad 1 - 2h + h^2 + k^2 = r^2$ ②

and $(2-h)^2 + (-1-k)^2 = r^2$

$\therefore \quad 4 - 4h + h^2 + 1 + 2k + k^2 = r^2$ ③

Subtracting ② from ① yields

$3 + 4k + 2h = 0$ ④

Subtracting ③ from ① yields

$2k + 4h - 1 = 0$

$\therefore \quad 4k + 8h - 2 = 0$ ⑤

Subtracting ④ from ⑤

yields $6h - 5 = 0$

$\therefore \quad 6h = 5$

$\therefore \quad h = \frac{5}{6}$

Substituting $h = \frac{5}{6}$ into ④ yields

$3 + 4k + 2 \times \frac{5}{6} = 0$

$\therefore \quad 3 + 4k + \frac{5}{3} = 0$

$\therefore \quad 4k + \frac{14}{3} = 0$

$\therefore \quad 4k = -\frac{14}{3}$

$\therefore \quad k = -\frac{7}{6}$

The centre of the circle is $\left(\frac{5}{6}, -\frac{7}{6}\right)$.

24 a $i(a-b) = i((5+2i) - (8+6i))$

$= i(-3-4i)$

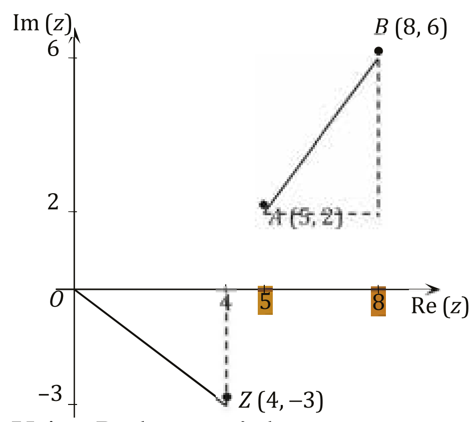
$= -3i - 4i^2$

$= -3i + 4$

$= 4 - 3i$

Let Z be the point $4 - 3i$.

The triangles in the diagram are congruent since they are both right-angled and have two pairs of side lengths the same, $\therefore \quad OZ = AB$.



Using Pythagoras' theorem,

$$|\overrightarrow{OZ}| = |\overrightarrow{AB}|$$
$$= \sqrt{3^2 + 4^2}$$
$$= 5$$

Now gradient of $\overrightarrow{AB} = \frac{4}{3} = m_1$

and gradient of $\overrightarrow{OZ} = \frac{-3}{4} = m_2$

Since $m_1 m_2 = -1$, $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{OZ}$, the vector that represents the point $i(a - b)$ and is the same length as $\overrightarrow{AB}$.

b Since OZ is perpendicular to AB, let $d = a \pm z$

and $c = b \pm z$ where $z = 4 - 3i$

$\therefore \quad d = (5 + 2i) \pm (4 - 3i)$

and $c = (8 + 6i) \pm (4 - 3i)$

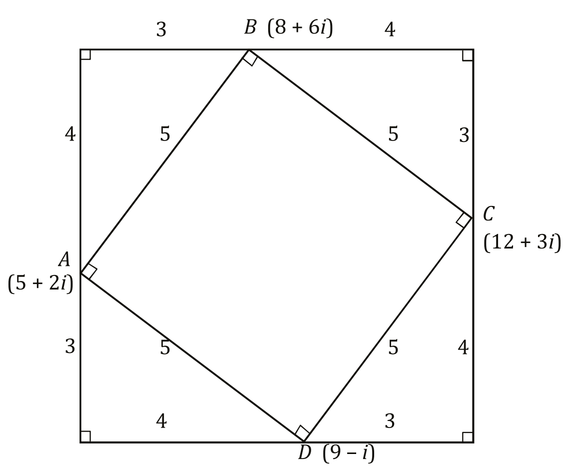
$= 9 - i$ or $1 + 5i$

$= 12 + 3i$ or $4 + 9i$

Now AD and BC are both perpendicular to AB, and using Pythagoras' theorem $\quad AD = \sqrt{3^2 + 4^2} = 5$

and $BC = \sqrt{3^2 + 4^2} = 5$

Hence CD is parallel to AB and $ABCD$ is a square.



Similarly, for $c = 4 + 9i$ and $d = 1 + 5i$, it can be shown that CD is parallel to AB and that $ABCD$ is a square.

25 a $\quad z^3 = -8$

Let $z = r\text{cis}\,\theta$

$\therefore \quad z^3 = r^3\text{cis}\,3\theta$

Now $-8 = 8\text{cis}\,\pi$

$\therefore \quad r^3\text{cis}\,3\theta = 8\text{cis}\,\pi$

$\therefore \quad r^3 = 8$ and $3\theta = \pi + 2\pi k$, $k \in Z$

$\therefore \quad r = 2$ and $\theta = \frac{\pi}{3} + \frac{2\pi k}{3}$

$= \frac{\pi}{3}(1 + 2k)$

Therefore the solutions are in the form $z = 2\text{ cis }\frac{\pi}{3}(1 + 2k),\ k \in Z$

When $k = 0, z = 2\text{cis}\,\frac{\pi}{3}$

$k = 1,\ z = 2\text{cis}\,\pi$

$k = 2,\ z = 2\text{cis}\,\frac{5\pi}{3}$

$= 2\text{cis}\left(-\frac{\pi}{3}\right)$

$k = 3,\ z = 2\text{cis}\,\frac{7\pi}{3}$

$= 2\text{cis}\,\frac{\pi}{3}$ as before.

The three solutions are $2\text{cis}\,\frac{\pi}{3}$, $2\text{cis}\,\pi$ and $2\text{cis}\left(-\frac{\pi}{3}\right)$

b $z^2 = 2 + 2\sqrt{3}i$

and $2 + 2\sqrt{3}i = 4\text{cis}\,\frac{\pi}{3}$

$\therefore \quad z^2 = 4\text{cis}\frac{\pi}{3}$

Now if $z = r\text{cis}\,\theta$,

$z^2 = r^2\text{cis}\,2\theta$

and $r^2\text{cis}\,2\theta = 4\text{cis}\left(\frac{\pi}{3} + 2\pi k\right),\ k \in Z$

$\therefore \quad r^2 = 4$ and $2\theta = \frac{\pi}{3} + 2\pi k$

$\therefore \quad r = 2$ and $\theta = \frac{\pi}{6} + \pi k$

$= \frac{\pi}{6}(1 + 6k)$

Therefore the solutions are in the form $z = 2\text{ cis}\,\frac{\pi}{6}(1 + 6k),\ k \in Z$

When $k = 0,\ z = 2\text{cis}\,\frac{\pi}{6}$

$k = 1,\ z = 2\text{cis}\frac{7\pi}{6}$

$= 2\text{cis}\left(-\frac{5\pi}{6}\right)$

The two solutions are $2\text{cis}\,\frac{\pi}{6}$ and $2\text{cis}\left(-\frac{5\pi}{6}\right)$

26 a $x^6 - 1 = (x^3 + 1)(x^3 - 1)$

$= (x + 1)(x^2 - x + 1)$

$\times (x - 1)(x^2 + x + 1)$

The discriminant of $x^2 - x + 1$ is $(-1)^2 - 4(1)(1) = -3 < 0$

Similarly, the discriminant of $x^2 + x + 1$ is $(1) - 4(1)(1) = -3 < 0$

So the factors $(x^2 - x + 1)$ and $(x^2 + x + 1)$ are irreducible for R.

b $(x^2 - x + 1)$ and $(x^2 + x + 1)$ can be further factorised for C.

$$x^2 - x + 1 = x^2 - x + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2$$

$$= \left(x - \frac{1}{2}\right)^2 + 1 - \frac{1}{4}$$

$$= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \left(x - \frac{1}{2}\right)^2 - \frac{3}{4}i^2$$

$$= \left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2$$

$$= \left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\times \left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$x^2 + x + 1 = x^2 + x + \left(\frac{1}{2}\right)^2$$

$$+ 1 - \left(\frac{1}{2}\right)^2$$

$$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2$$

$$= \left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\times \left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$\therefore \quad x^6 - 1 = (x + 1)(x - 1)\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\times \left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$\times \left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\times \left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

c Let $x^6 = 1$ where x represents the sixth roots of unity,
then $x^6 - 1 = 0$

$$\therefore \quad (x+1)(x-1)\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$\times\left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$\times\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$\times\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$$

$$\therefore \quad x = -1 \text{ or } 1 \text{ or } \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \text{ or } -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

27 Let $z = x + iy$

a Then
$$\left|\frac{\bar{z}}{z}\right| = \left|\frac{x - iy}{x + iy}\right|$$
$$= \frac{|(x - iy)^2|}{x^2 + y^2}$$
$$= \frac{|(x - iy)|^2}{x^2 + y^2}$$
$$= \frac{x^2 + y^2}{x^2 + y^2}$$
$$= 1$$

or $|z| = \sqrt{x^2 + y^2}$ and $|\bar{z}| = \sqrt{x^2 + y^2}$,
therefore $\left|\frac{\bar{z}}{z}\right| = \frac{|\bar{z}|}{|z|} = 1$

or let $z = r\text{cis}\,\theta$

then $\bar{z} = r\text{cis}(-\theta)$

$$\frac{\bar{z}}{z} = \text{cis}\,2\theta \text{ and } \left|\frac{\bar{z}}{z}\right| = 1$$

b
$$\frac{i(\text{Re}\,(z) - z)}{\text{Im}\,(z)} = \frac{i(x - (x + iy))}{y}$$
$$= \frac{i \times (-iy)}{y}$$
$$= 1$$

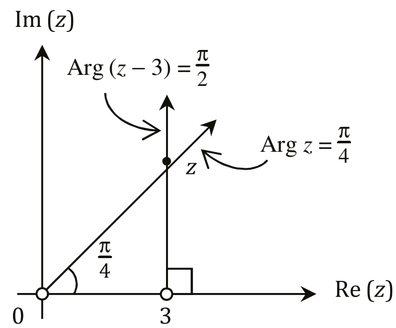
c Let $z = r\text{cis}\,\theta$

then $\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$

$$\therefore \quad \text{Arg}\,z + \text{Arg}\frac{1}{z} = \theta - \theta$$
$$= 0$$

28
$$\left|\frac{1}{z} + \frac{1}{w}\right| = \left|\frac{w + z}{zw}\right| = \frac{|w + z|}{|zw|} = \frac{|w + z|}{|z||w|} = \frac{3}{2 \cdot 2} = \frac{3}{4}$$

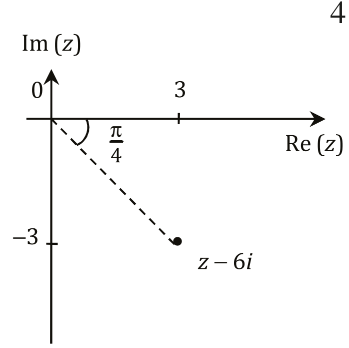
29



From the diagram, $z = 3 + 3i$

therefore $z - 6i = 3 + 3i - 6i$
$= 3 - 3i$

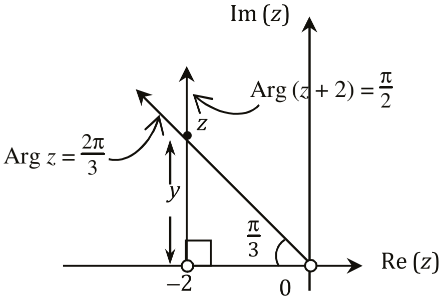
$$\text{Arg}\,(z - 6i) = \text{Arg}\,(3 - 3i)$$
$$= \tan^{-1}\left(\frac{-3}{3}\right)$$
$$= \tan^{-1}(-1)$$
$$= -\frac{\pi}{4}$$



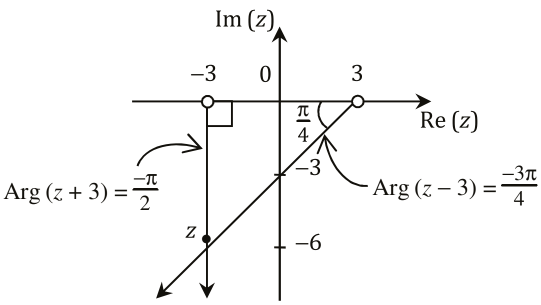
30 a From the diagram, $y = 2\tan\frac{\pi}{3}$

$= 2\sqrt{3}$

$\therefore \quad z = -2 + 2\sqrt{3}i$



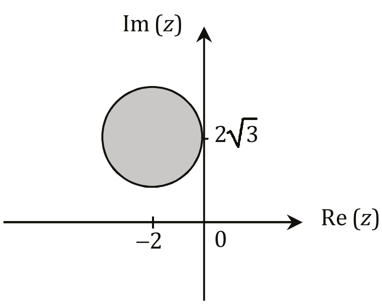
b From the diagram, $z = -3 - 6i$



31 a $|z + 2 - 2\sqrt{3}i| \le 2$

$|z - (-2 + 2\sqrt{3}i)| \le 2$

The distance between z and the point with coordinates $(-2, 2\sqrt{3})$ is less than or equal to 2. This may be represented by a disc with centre $(-2, 2\sqrt{3})$ and radius 2.



b i $a^2 = 2^2 + (2\sqrt{3})^2$

$= 4 + 12$

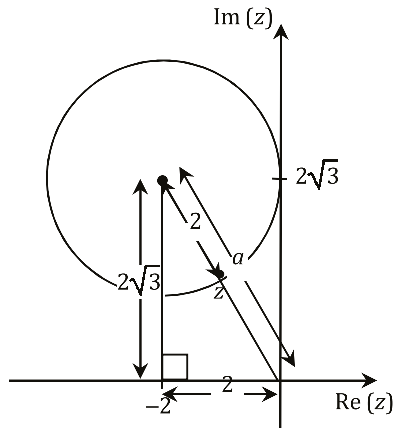
$= 16$

Therefore $a = 4$

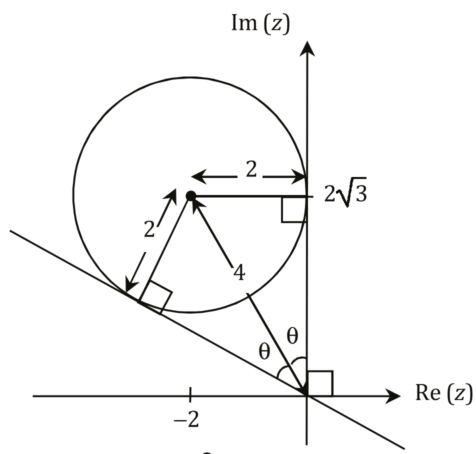
Therefore $a - r = 4 - 2$

$= 2$

The least possible value of $|z|$ is 2



ii



$\sin\theta = \frac{2}{4}$

$= \frac{1}{2}$

Therefore $\theta = \frac{\pi}{6}$

The maximum value of Arg (z) is

$\frac{\pi}{2} + \frac{\pi}{6} + \frac{\pi}{6} = \frac{5\pi}{6}$

Solutions to multiple-choice questions

1 E $\therefore \quad z_1 z_2 = 10\text{cis}\left(\frac{\pi}{3} + \frac{3\pi}{4}\right)$

$$= 10\text{cis}\left(\frac{13\pi}{12}\right)$$

$$= 10\text{cis}\left(-\frac{11\pi}{12}\right)$$

$$-\pi < \text{Arg}(z) \le \pi$$

2 C

$z = -3 + 4i$

Using the TI-nspire CAS calculator to convert $-3 + 4i$ into polar form

$\therefore \quad z = 5e^{2.21i} = 5\text{cis}(2.21)$

3 D $(x + iy)^2 = -32i$

$\therefore \quad (x^2 - y^2) + 2xyi = -32i$

$\therefore \quad x^2 - y^2 = 0 \qquad (1)$ and

$2xy = -32 \qquad (2)$

From (2): $y = -\frac{16}{x}$

$\therefore \quad x^2 - \left(-\frac{16}{x}\right)^2 = 0$

$\therefore \quad x^2 = \frac{256}{x^2}$

$\therefore \quad x^4 = 256$

$\therefore \quad x = \pm 4$

When $x = 4,\ y = -4$

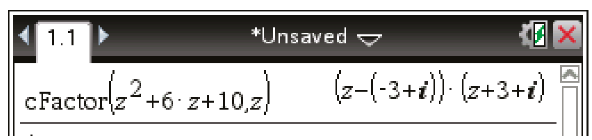
When $x = -4,\ y = 4$

4 E $u = 1 - i$

$$\frac{1}{3-u} = \frac{1}{3-(1-i)} = \frac{1}{2+i} = \frac{2-i}{4+1} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i$$

Alternatively, using a CAS calculator to determine $\frac{1}{3-u}$ we have:

5 D The linear factors of $z^2 + 6z + 10$ can be obtained by using the cFactor command on the TI-nspire CAS calculator.



6 B $z^3 = -8i$

$\therefore \quad (r\text{cis}\,\theta)^3 = 8\text{cis}\left(-\frac{\pi}{2}\right)$ where $z = r\text{cis}\,\theta$

$\therefore \quad r^3 = 8$ and $3\theta = -\frac{\pi}{2} + 2k\pi,\ k \in Z$

$\therefore \quad r = 2$ and $\theta = -\frac{\pi}{6} + \frac{2k\pi}{3},\ k \in Z$

$\therefore \quad z = 2\text{cis}\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right),\ k \in Z$

When $k = 0$,

$$z = 2\text{cis}\left(-\frac{\pi}{6}\right)$$
$$= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$
$$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$
$$= \sqrt{3} - i$$

When $k = 1$,

$$z = 2\text{cis}\left(-\frac{\pi}{6} + \frac{2\pi}{3}\right)$$
$$= 2\text{cis}\left(\frac{\pi}{2}\right)$$
$$= 2i$$

When $k = 2$,

$$z = 2\text{cis}\left(-\frac{\pi}{6} + \frac{4\pi}{3}\right)$$
$$= 2\text{cis}\left(\frac{7\pi}{6}\right)$$
$$= 2\text{cis}\left(-\frac{5\pi}{6}\right)$$
$$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$
$$= -\sqrt{3} - i$$

Hence the cube roots of $z^3 + 8i = 0$ are $\sqrt{3} - i, -\sqrt{3} - i$ and $2i$ (Alternatively, use the cSolve command To solve the equation with a TI-Nspire CAS calculator)

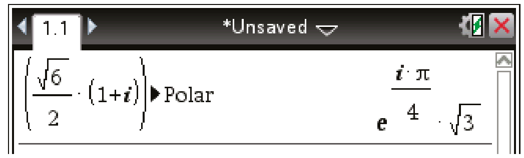
7 B

Using a CAS calculator to convert $\frac{\sqrt{6}}{2}(1 + i)$ into polar form we have

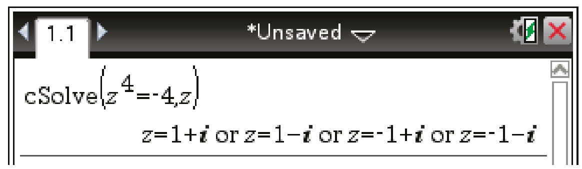
$$\therefore \quad \frac{\sqrt{6}}{2}(1 + i) = \sqrt{3}e^{\frac{\pi}{4}i} = \sqrt{3}\text{cis}\left(\frac{\pi}{4}\right)$$

Note that $\sqrt{3}\ \text{cis}\left(\frac{\pi}{4}\right)$ is not listed as a response. Look for an alternate form for the argument.

$$\therefore \quad \sqrt{3}\text{cis}\left(\frac{\pi}{4}\right) = \sqrt{3}\text{cis}\left(-\frac{7\pi}{4}\right)$$



8 C If $z = 1 + i$ is a solution to the equation $z^4 = a$ then $a = (1 + i)^4 = -4$. Hence solutions to the equation $z^4 = -4$ using the TI-nspire CAS calculator are



9 B

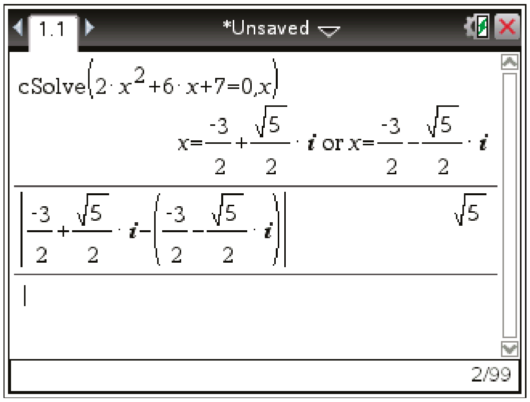
$$z^2 = -2 - 2j\sqrt{3}$$
$$\therefore \quad (r\text{cis}\,\theta)^2 = -2 - 2i\sqrt{3}$$
$$\therefore \quad r^2\text{cis}\,2\theta = 4\text{cis}\left(-\frac{2\pi}{3}\right)$$
$$\therefore \quad r^2 = 4 \text{ and } 2\theta = -\frac{2\pi}{3} + 2k\pi,\ k \in Z$$
$$\therefore \quad r = 2 \text{ and } \theta = -\frac{\pi}{3} + k\pi,\ k \in Z$$
$$\therefore \quad z = 2\text{cis}\left(-\frac{\pi}{3} + k\pi\right),\ k \in Z$$

When $k = 0,\ z = 2\text{cis}\left(-\frac{\pi}{3}\right)$

When $k = 1,\ z = 2\text{cis}\left(\frac{2\pi}{3}\right)$

Hence the square roots of $-2 - 2i\sqrt{3}$ are $2\text{cis}\left(-\frac{\pi}{3}\right)$ and $2\text{cis}\left(\frac{2\pi}{3}\right)$

10 A Using a CAS calculator to calculate $|\alpha - \beta|$ where α and β are the roots of the equation $2x^2 + 6x + 7 = 0$ we have

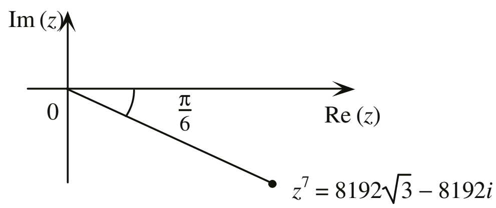


$\therefore \quad |\alpha - \beta| = \sqrt{5}$

Solutions to extended-response questions

1 **a**
$$z^7 = \left(4\text{cis}\,\frac{5\pi}{6}\right)^7$$
$$= 4^7\text{cis}\left(7 \times \frac{5\pi}{6}\right)$$
$$= 16\,384\text{cis}\,\frac{35\pi}{6}$$
$$= 16\,384\text{cis}\left(\frac{-\pi}{6}\right)$$
$$\therefore\quad |z^7| = 16\,384$$
$$\text{and Arg}(z^7) = \frac{-\pi}{6}$$

b



c
$$\frac{z}{w} = \frac{4\text{cis}\,\frac{5\pi}{6}}{\sqrt{2}\text{cis}\,\frac{\pi}{4}}$$
$$= \frac{4}{\sqrt{2}}\text{cis}\left(\frac{5\pi}{6} - \frac{\pi}{4}\right) = 2\sqrt{2}\text{cis}\frac{7\pi}{12}$$

d
$$z = 4\text{cis}\,\frac{5\pi}{6}$$
$$= 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$= 4\left(\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right) = -2\sqrt{3} + 2i$$
$$w = \sqrt{2}\text{cis}\,\frac{\pi}{4}$$
$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 1 + i$$

$$\frac{z}{w} = \frac{-2\sqrt{3} + 2i}{1 + i}$$

$$= \frac{-2\sqrt{3} + 2i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{-2\sqrt{3} + 2i + 2\sqrt{3} + 2}{2}$$

$$= \frac{2((1 - \sqrt{3}) + i(1 + \sqrt{3}))}{2}$$

$$= (1 - \sqrt{3}) + (1 + \sqrt{3})i$$

e $$\frac{z}{w} = 2\sqrt{2}\text{cis}\,\frac{7\pi}{12} = 2\sqrt{2}\cos\frac{7\pi}{12} + 2\sqrt{2}\sin\frac{7\pi}{12}i$$

$$\therefore \quad 2\sqrt{2}\cos\frac{7\pi}{12} = 1 - \sqrt{3} \text{ and } 2\sqrt{2}\sin\frac{7\pi}{12} = 1 + \sqrt{3}$$

$$\therefore \quad \cos\frac{7\pi}{12} = \frac{1 - \sqrt{3}}{2\sqrt{2}} \quad \therefore \quad \sin\frac{7\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

$$\text{Now } \tan\frac{7\pi}{12} = \frac{\sin\frac{7\pi}{12}}{\cos\frac{7\pi}{12}}$$

$$= \frac{1 + \sqrt{3}}{2\sqrt{2}} \div \frac{1 - \sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \times \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{1 + 2\sqrt{3} + 3}{-2}$$

$$= \frac{2(2 + \sqrt{3})}{-2} = -2 - \sqrt{3}$$

f $\tan\frac{7\pi}{6} = \tan\left(2 \times \frac{7\pi}{12}\right)$

$$= \frac{2\tan\frac{7\pi}{12}}{1 - \tan^2\frac{7\pi}{12}}$$

$$= \frac{-4 - 2\sqrt{3}}{1 - (-2 - \sqrt{3})^2}$$

$$= \frac{-4 - 2\sqrt{3}}{1 - (7 + 4\sqrt{3})}$$

$$= \frac{4 + 2\sqrt{3}}{6 + 4\sqrt{3}}$$

$$= \frac{2 + \sqrt{3}}{3 + 2\sqrt{3}} \times \frac{3 - 2\sqrt{3}}{3 - 2\sqrt{3}} = \frac{\sqrt{3}}{3}$$

2 a $P(2 + i) = (2 + i)^3 - 7(2 + i)^2 + 17(2 + i) - 15$

$= 8 + 12i + 6i^2 + i^3 - 7(4 + 4i + i^2) + 34 + 17i - 15$

$= 8 + 12i - 6 - i - 7(4 + 4i - 1) + 17i + 19$

$= 11i + 2 - 7(3 + 4i) + 17i + 19$

$= 28i + 21 - 21 - 28i = 0$

b Since the coefficients of $P(z)$ are real and $2 + i$ is a solution then so must $2 - i$ be a solution.

$(z - (2 + i))(z - (2 - i)) = z^2 - (2 + i)z - (2 - i)z + (2 + i)(2 - i)$

$= z^2 - 2z - iz - 2z + iz + 4 - i^2$

$= z^2 - 4z + 5$

By division, $P(z) = (z^2 - 4z + 5)(z - 3)$

$\therefore \quad P(z) = (z - (2 + i))(z - (2 - i))(z - 3)$

and $z = 3, 2 \pm i$ are solutions of $P(z) = 0$. The other two roots are $2 - i$ and 3.

c Multiply $1 - 2i$ by i to produce a complex number that is a rotation of B anticlockwise by $\frac{\pi}{2}$ about the origin

$\therefore \quad (1 - 2i)i = i - 2i^2 = 2 + i$

This corresponds to A, and hence $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$.

d A polynomial $P(z)$ with real coefficients and with 3, $1 - 2i$, and $2 + i$ as roots must have $1 + 2i$ and $2 - i$ as other roots (by the conjugate factor theorem).

$\therefore \quad P(z) = (z-3)((z-(1-2i))(z-(1+2i))(z-(2+i))(z-(2-i))$

$= (z-3)(z^2-(1-2i)z-(1+2i)z+(1-2i)(1+2i))$

$\times (z^2-(2+i)z-(2-i)z+(2+i)(2-i))$

$= (z-3)(z^2-(1-2i+1+2i)z+1-4i^2)\times(z^2-(2+i+2-i)z+4-i^2)$

$= (z-3)(z^2-2z+5)(z^2-4z+5)$

$= (z-3)(z^4-4z^3+5z^2-2z^3+8z^2-10z+5z^2-20z+25)$

$= (z-3)(z^4-6z^3+18z^2-30z+25)$

$= (z^5-6z^4+18z^3-30z^2+25z-3z^4+18z^3-54z^2+90z-75$

$= z^5-9z^4+36z^3-84z^2+115z-75$

3 a $\quad z^2-2\sqrt{3}z+4=0$

$\therefore \quad z^2-2\sqrt{3}z+(\sqrt{3})^2+4-(\sqrt{3})^2=0$

$\therefore \quad (z-\sqrt{3})^2+4-3=0$

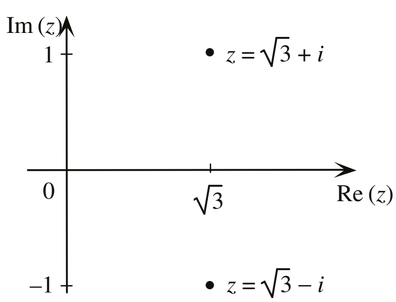
$\therefore \quad (z-\sqrt{3})^2+1=0$

$\therefore \quad (z-\sqrt{3})^2-i^2=0$

$\therefore \quad (z-\sqrt{3}+i)(z-\sqrt{3}-i)=0$

$\therefore \quad z=\sqrt{3}\pm i$

b i

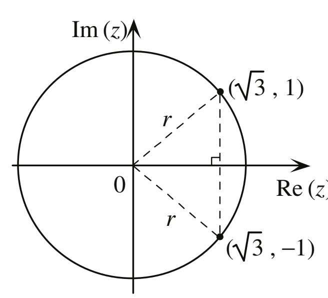


ii $\quad r = \sqrt{(\sqrt{3})^2+1^2}$

$= \sqrt{3+1}$

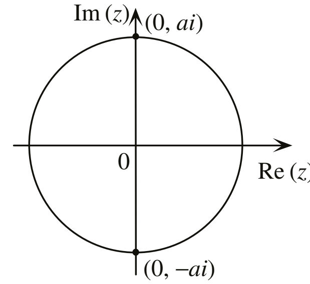
$= 2$

$\therefore \quad x^2+y^2=4$



iii $a = r$

$\therefore \quad a = 2$



iv $Q(z)P(z) = z^6 + 64$

$$\therefore \quad P(z) = \frac{z^6 + 64}{Q(z)}$$

$$= \frac{z^6 + 64}{(z^2 + 4)(z^2 - 2\sqrt{3}z + 4)}$$

$$\begin{array}{r|l}
 & z^4 - 4z^2 + 16 \\
z^2 + 4 & z^6 \qquad\qquad + 64 \\
 & \underline{z^6 + 4z^4} \\
 & -4z^4 \qquad + 64 \\
 & \underline{-4z^4 - 16z^2} \\
 & 16z^2 + 64 \\
 & \underline{16z^2 + 64} \\
 & 0
\end{array}$$

$$\therefore \quad P(z) = \frac{(z^2 + 4)(z^4 - 4z^2 + 16)}{(z^2 + 4)(z^2 - 2\sqrt{3}z + 4)}$$

$$= \frac{z^4 - 4z^2 + 16}{z^2 - 2\sqrt{3}z + 4}$$

$$\begin{array}{r|l}
 & z^2 + 2\sqrt{3}z \\
z^2 - 2\sqrt{3}z + 4 & z^4 \qquad - 4z^2 \qquad + 16 \\
 & \underline{z^4 - 2\sqrt{3}z^3 + 4z^2} \\
 & 2\sqrt{3}z^3 - 8z^2 \qquad + 16 \\
 & \underline{2\sqrt{3}z^3 - 12z^2 + 8\sqrt{3}z} \\
 & 4z^2 - 8\sqrt{3} + 16 \\
 & \underline{4z^2 - 8\sqrt{3} + 16} \\
 & 0
\end{array}$$

$$\therefore \quad P(z) = z^2 + 2\sqrt{3}z + 4$$

So $z^6 + 64 = (z^2 + 4)(z^2 - 2\sqrt{3}z + 4)(z^2 + 2\sqrt{3}z + 4)$

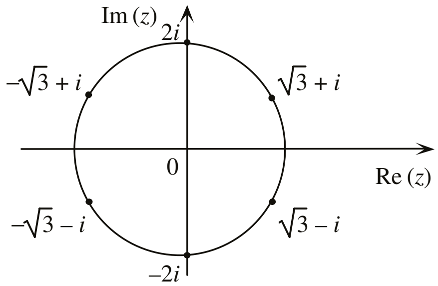
When $z^6 + 64 = 0$

$$(z^2 + 4)(z^2 - 2\sqrt{3}z + 4)(z^2 + 2\sqrt{3}z + 4) = 0$$

$$\therefore \quad (z + 2i)(z - 2i)(z - \sqrt{3} + i)(z - \sqrt{3} - i)(z + \sqrt{3} + i)(z + \sqrt{3} - i) = 0$$

$\therefore \quad z = \pm 2i,\ \sqrt{3} \pm i, -\sqrt{3} \pm i$

On an Argand diagram, these solutions are equally spaced around the circumference of the circle $x^2 + y^2 = 4$, and represent the sixth roots of -64. Three of these solutions are the conjugates of the other three solutions.



4 a Let $z = x + yi$

Also $z = -4\sqrt{3} - 4i \therefore \quad x = -4\sqrt{3},\ y = -4$

When $z = r \text{ cis } \theta, r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$

$$\begin{aligned} \therefore \quad r &= \sqrt{(-4\sqrt{3})^2 + (-4)^2} \\ &= \sqrt{48 + 16} \\ &= \sqrt{64} = 8 \end{aligned}$$

and $\tan\theta = \dfrac{-4}{-4\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\therefore \quad \theta = \dfrac{-5\pi}{6}$ since $-4\sqrt{3} - 4i$ is in the third quadrant

$$\therefore \quad z = 8\text{cis}\left(\frac{-5\pi}{6}\right)$$

b Now let $z^3 = -4\sqrt{3} - 4i = 8\text{cis}\left(\dfrac{-5\pi}{6}\right)$

If $z = r \text{ cis } \theta$, then $(r \text{ cis } \theta)^3 = 8\text{cis}\left(\dfrac{-5\pi}{6}\right)$

$$\therefore \quad r^3 \text{ cis } 3\theta = 8\text{cis}\left(\frac{-5\pi}{6}\right)$$

$\therefore \quad r^3 = 8$ and $3\theta = \dfrac{-5\pi}{6} + 2\pi k,\ k \in Z$

$\therefore \quad r = 2$ and $\theta = \dfrac{-5\pi}{18} + \dfrac{2\pi k}{3},\ k \in Z$

$$\therefore \quad z = 2\text{cis}\left(\frac{-5\pi}{18} + \frac{2\pi k}{3}\right),\ k \in Z$$

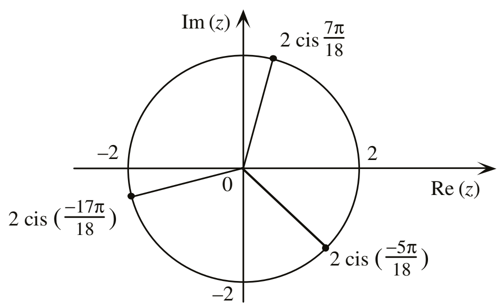
When $k = 0,\ z = 2\text{cis}\left(\dfrac{-5\pi}{18}\right)$

When $k = 1,\ z = 2\text{cis}\ \dfrac{7\pi}{18}$

When $k = 2,\ z = 2\text{cis}\left(\dfrac{-17\pi}{18}\right)$

Hence the cube roots of $-4\sqrt{3} - 4i$ are $2\text{cis}\left(\dfrac{-5\pi}{18}\right)$, $2\text{cis}\ \dfrac{7\pi}{18}$, $2\text{cis}\left(\dfrac{-17\pi}{18}\right)$

c



d **i** $(z - w)^3 = z^3 - 3z^2w + 3zw^2 - w^3 = z^3 - 3wz^2 + 3w^2z - w^3$

Let $(z - w)^3 = z^3 - 3\sqrt{3}iz^2 - 9z + 3\sqrt{3}i$

Equating coefficients $3w = 3\sqrt{3}i$

$$\therefore \quad w = \sqrt{3}i$$

$$3w^2 = -9$$

$$\therefore \quad w^2 = -3$$

$$\therefore \quad w = \sqrt{3}i$$

$$\text{and } -w^3 = 3\sqrt{3}i = -3\sqrt{3}i^3$$

$$\therefore \quad w^3 = 3\sqrt{3}i^3$$

$$\therefore \quad w = \sqrt{3}i$$

So $(z - \sqrt{3}i)^3 = -4\sqrt{3} - 4i$

ii $z^3 - 3\sqrt{3}iz^2 - 9z + (3\sqrt{3} + 4)i + 4\sqrt{3} = 0$

$\therefore \quad z^3 - 3\sqrt{3}iz^2 - 9z + 3\sqrt{3}i = -4\sqrt{3} - 4i$

$\therefore \quad (z - \sqrt{3}i)^3 = -4\sqrt{3} - 4i$

$\therefore \quad z - \sqrt{3}i = 2\text{cis}\left(\frac{-5\pi}{18}\right)$

$\therefore \quad z = 2\cos\left(\frac{-5\pi}{18}\right) + \left(2\sin\left(\frac{-5\pi}{18}\right) + \sqrt{3}\right)i$

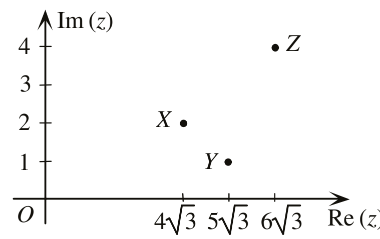
or $z - \sqrt{3}i = 2\text{cis}\frac{7\pi}{18}$

$\therefore \quad z = 2\cos\frac{7\pi}{18} + \left(2\sin\frac{7\pi}{18} + \sqrt{3}\right)i$

or $z - \sqrt{3}i = 2\text{cis}\left(\frac{-17\pi}{18}\right)$

$\therefore \quad z = 2\cos\left(\frac{-17\pi}{18}\right) + \left(2\sin\left(\frac{-17\pi}{18}\right) + \sqrt{3}\right)i$

5 a Let $\boldsymbol{i}$ be the unit vector in the positive direction of the Re(z) axis and let $\boldsymbol{j}$ be the unit vector in the positive direction of the Im(z) axis.



$\therefore \quad \overrightarrow{OX} = 4\sqrt{3}\boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{OY} = 5\sqrt{3}\boldsymbol{i} + \boldsymbol{j}$

$\overrightarrow{OZ} = 6\sqrt{3}\boldsymbol{i} + 4\boldsymbol{j}$

$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$

$= (5\sqrt{3}\boldsymbol{i} + \boldsymbol{j}) - (4\sqrt{3}\boldsymbol{i} + 2\boldsymbol{j}) = \sqrt{3}\boldsymbol{i} - \boldsymbol{j}$

$\overrightarrow{XZ} = \overrightarrow{OZ} - \overrightarrow{OX}$

$= (6\sqrt{3}\boldsymbol{i} + 4\boldsymbol{j}) - (4\sqrt{3}\boldsymbol{i} + 2\boldsymbol{j})$

$= 2\sqrt{3}\boldsymbol{i} + 2\boldsymbol{j}$

b $z_1 = \sqrt{3} - i$

$z_2 = 2\sqrt{3} + 2i$

$$z_3 = \frac{z_2}{z_1}$$

$$= \frac{2\sqrt{3} + 2i}{\sqrt{3} - i} \times \frac{\sqrt{3} + i}{\sqrt{3} + i}$$

$$= \frac{6 + 2\sqrt{3}i + 2\sqrt{3}i - 2}{4}$$

$$= \frac{4 + 4\sqrt{3}i}{4} = 1 + \sqrt{3}i$$

c Let $z_3 = r \text{ cis } \theta$

then $|z_3| = \sqrt{1^2 + (\sqrt{3})^2}$

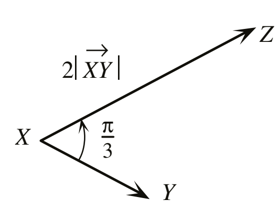
$= 2$

$$\tan\theta = \frac{\sqrt{3}}{1}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

$$\therefore \quad z_3 = 2\text{cis } \frac{\pi}{3}$$

The geometric interpretation is an enlargement of $\overrightarrow{XY}$ by a factor of 2 and a rotation of $\overrightarrow{XY}$, $\frac{\pi}{3}$ units anticlockwise about X, to produce the vector $\overrightarrow{XZ}$.

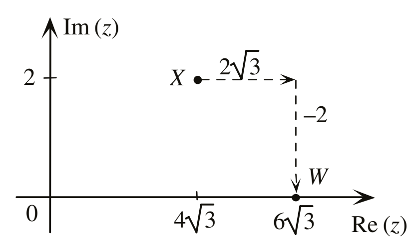


Hence $\angle ZXY = \frac{\pi}{3}$ and $XZ = 2XY$, so XYZ is half an equilateral triangle.

Now $\overrightarrow{XW} = 2\overrightarrow{XY}$

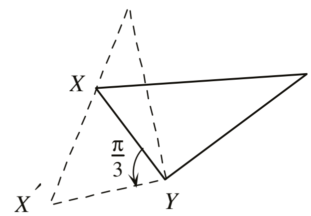
so $2z_1 = 2(\sqrt{3} - i)$

$= 2\sqrt{3} - 2i$



The complex number to which W corresponds is $6\sqrt{3}$.

d Let X' be the new position of X.



The vector $\overrightarrow{YX} = -\overrightarrow{XY}$ can be represented by the complex number $-z_1 = -(\sqrt{3} - i)$

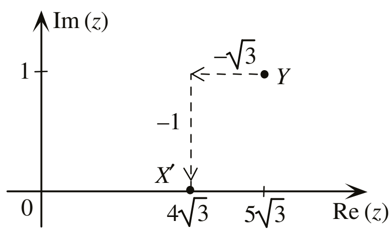
$$= -\sqrt{3} + i$$

$$= 2\text{cis}\left(\frac{-\pi}{6}\right)$$

$\overrightarrow{YX'}$ is produced by rotating YX, $\frac{\pi}{3}$ anticlockwise about Y, and can be represented by the complex number

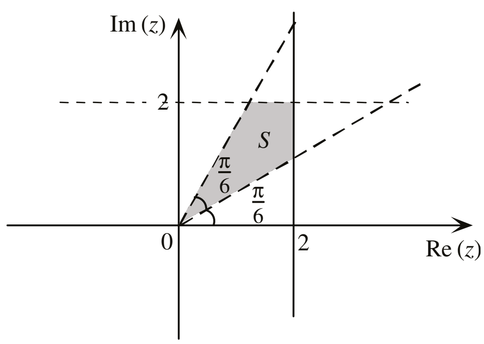
$$\left(\text{cis}\frac{\pi}{3}\right)\left(2\text{cis}\left(\frac{-\pi}{6}\right)\right) = 2\text{cis}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 2\text{cis}\ \frac{\pi}{6}$$

$$= -\sqrt{3} - i \text{ as } X' \text{ is below and to the left of } Y.$$



The new position of X can be represented by the complex number $4\sqrt{3}$.

6 a First sketch $S = \{z : \text{Re}(z) \le 2\} \cap \{z : \text{Im}(z) < 2\} \cap \left\{z : \frac{\pi}{6} < \text{Arg}(z) < \frac{\pi}{3}\right\}$



Now $|z + i| = |z - 1|$

Let $z = x + yi$

$\therefore \quad |x + yi + i| = |x + yi - 1|$

$\therefore \quad |x + (y + 1)i| = |(x - 1) + yi|$

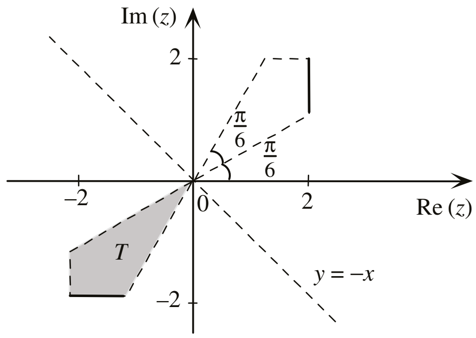
$$\therefore \quad \sqrt{x^2 + (y+1)^2} = \sqrt{(x-1)^2 + y^2}$$

$$\therefore \quad x + (y+1)^2 = (x-1)^2 + y^2$$

$$\therefore \quad x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$$

$$\therefore \quad 2y = -2x$$

$$\therefore \quad y = -x$$



b $T = \{z : \text{Re}(z) > -2\} \cap \{z : \text{Im}(z) \geq -2\} \cap \left\{Z : \dfrac{-5\pi}{6} < \text{Arg}(z) < \dfrac{-2\pi}{3}\right\}$

7 $x^2 + 4x - 1 + k(x^2 + 2x + 1) = 0$

$\therefore \quad x^2 + 4x - 1 + kx^2 + 2kx + k = 0$

$\therefore \quad (k+1)x^2 + 2(k+2)x + (k-1) = 0 \qquad ①$

The discriminant is given by

$$\begin{aligned}\triangle &= (2(k+2))^2 - 4(k+1)(k-1)\\ &= 4(k^2 + 4k + 4) - 4(k^2 - 1)\\ &= 4(k^2 + 4k + 4 - k^2 + 1)\\ &= 4(4k + 5)\end{aligned}$$

a For real and distinct roots $\triangle > 0$

$\therefore \quad 4(4k+5) > 0$

$\therefore \quad 4k + 5 > 0$

$\therefore \quad 4k > -5$

$\therefore \quad k > -\dfrac{5}{4}$

b For real and equal roots $\triangle = 0$

$\therefore \quad k = -\dfrac{5}{4}$

c For complex roots $\triangle < 0$

$\therefore \quad k < -\frac{5}{4}$

Using the general quadratic formula in ①

$$x = \frac{-2(k+2) \pm \sqrt{(2(k+2))^2 - 4(k+1)(k-1)}}{2(k+1)}$$

$$= \frac{-2(k+2) \pm \sqrt{4k^2 + 16k + 16 - 4k^2 + 4}}{2(k+1)}$$

$$= \frac{-2(k+2) \pm 2\sqrt{4k+5}}{2(k+1)}$$

$$= \frac{-(k+2) \pm \sqrt{-(4k+5)i^2}}{k+1}$$

$$= \frac{-(k+2)}{k+1} \pm \frac{\sqrt{-(4k+5)}}{k+1}i, \text{ with } k < -\frac{5}{4} \text{ for complex solutions}$$

$$\therefore \quad \text{Re}(x) = \frac{-(k+2)}{k+1}, \ k < -\frac{5}{4}$$

$$\text{Im}(x) = \frac{\pm\sqrt{-(4k+5)}}{k+1}, \ k < -\frac{5}{4}$$

If $\text{Re}(x) > 0$

then $\frac{-(k+2)}{k+1} > 0$

$\therefore \quad k + 2 > 0, \text{ as } k + 1 < 0$

$\therefore \quad k > -2$

$\therefore \quad \text{Re}(x) > 0 \text{ for } -2 < k < -\frac{5}{4}$

Hence for complex roots with positive real part, $-2 < k < -\frac{5}{4}$.

8 a If $z = \cos\theta + i\sin\theta$

$$\frac{1+z}{1-z} = \frac{1 + \cos\theta + i\sin\theta}{1 - (\cos\theta + i\sin\theta)}$$

$$= \frac{1 + \left(2\cos^2\frac{\theta}{2} - 1\right) + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 - \left(1 - 2\sin^2\frac{\theta}{2}\right) - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\therefore \quad \frac{1+z}{1-z} = \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)} \times \frac{\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}$$

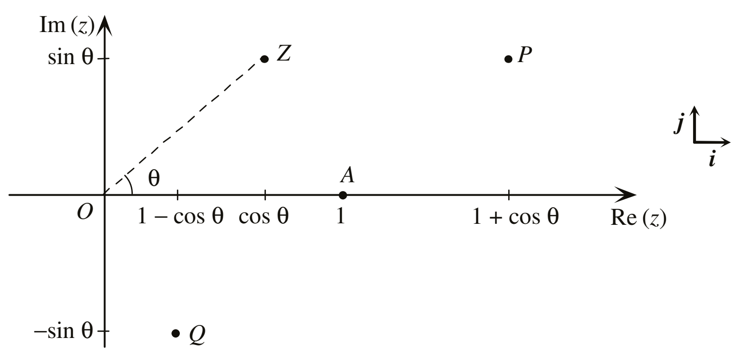
$$= \frac{\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2} + i\sin^2\frac{\theta}{2} + i\cos^2\frac{\theta}{2} + i^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} - i^2\cos^2\frac{\theta}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2} + i\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2} \times i}{\sin\frac{\theta}{2}}$$

$$= i\cot\frac{\theta}{2}$$

b



c $\overrightarrow{OP} \cdot \overrightarrow{OQ} = ((1+\cos\theta)\boldsymbol{i} + \sin\theta\boldsymbol{j})((1-\cos\theta)\boldsymbol{i} - \sin\theta\boldsymbol{j})$

$= 1 - \cos^2\theta - \sin^2\theta$

$= 1 - (\cos^2\theta + \sin^2\theta)$

$= 1 - 1$

$= 0$

Hence OP is perpendicular to OQ.

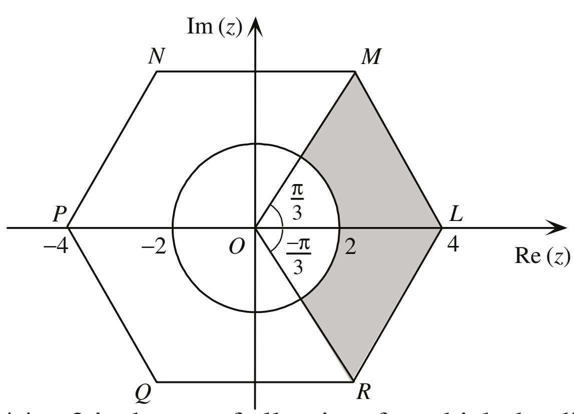
$\therefore \quad \angle POQ = \frac{\pi}{2}$, as required.

$$\text{Now } |OP| = \sqrt{(1+\cos\theta)^2 + \sin^2\theta}$$
$$= \sqrt{2 + 2\cos\theta}$$
$$= \sqrt{2(1+\cos\theta)}$$

$$\text{Also } |OQ| = \sqrt{(1-\cos\theta)^2 + \sin^2\theta}$$
$$= \sqrt{2 - 2\cos\theta}$$
$$= \sqrt{2(1-\cos\theta)}$$

$$\frac{|OP|}{|OQ|} = \frac{\sqrt{2(1+\cos\theta)}}{\sqrt{2(1-\cos\theta)}}$$
$$= \frac{\sqrt{1+\cos\theta}}{\sqrt{1-\cos\theta}}$$
$$= \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}}$$
$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$
$$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$
$$= \frac{1+\cos\theta}{\sin\theta}$$
$$= \operatorname{cosec}\theta + \cot\theta$$
$$= \cot\left(\frac{\theta}{2}\right)$$

9 a



$|z| \geq 2$ is the set of all points for which the distance from $(0, 0)$ is greater than or equal to 2.

b $\triangle OLM$ and $\triangle OLR$ are equilateral, therefore LM, LO and LR are radial lengths of the circle with centre (4, 0) and radius 4.
Hence $|z - 4| = 4$ is the required equation.

c N is the point corresponding to $4\text{cis}\frac{2\pi}{3}$.

Q is the point corresponding to $4\text{cis}\left(\frac{-2\pi}{3}\right)$.

d Let N' and Q' be the new positions of N and Q respectively.
ON' and OQ' are a rotation of ON and OQ respectively, $\frac{\pi}{4}$ clockwise about O, and can be represented by the complex number

$$\therefore \quad N' = 4\text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$$

$$= 4\text{cis}\,\frac{5\pi}{12}$$

$$\text{and } Q' = 4\text{cis}\left(-\frac{2\pi}{3} - \frac{\pi}{4}\right)$$

$$= 4\text{cis}\left(\frac{-11\pi}{12}\right)$$

Hence the new positions of N and Q correspond with the complex numbers $4\text{cis}\,\frac{5\pi}{12}$ and $4\text{cis}\left(\frac{-11\pi}{12}\right)$ respectively.

10 a

$$z = a + bi$$

$$|z| = \sqrt{a^2 + b^2} \text{ and } |z| = 1$$

$$\therefore \quad \sqrt{a^2 + b^2} = 1$$

$$\therefore \quad a^2 + b^2 = 1$$

$$\frac{1}{z} = \frac{1}{a + bi}$$

$$= \frac{1}{a + bi} \times \frac{a - bi}{a - bi}$$

$$= \frac{a - bi}{a^2 + b^2}$$

$$= a - bi, \text{ since } a^2 + b^2 = 1$$

$$= \bar{z}, \text{ as required.}$$

b $|z_1| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= \sqrt{1}$

$= 1$

$\therefore \quad \frac{1}{z_1} = \overline{z_1}$ from **a** above.

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$|z_2| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \sqrt{\frac{3}{4} + \frac{1}{4}}$

$= \sqrt{1}$

$= 1$

$\therefore \quad \frac{1}{z_2} = \overline{z_2}$ from **a** above.

$= \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Now $z_3 = \frac{1}{z_1} + \frac{1}{z_2}$

$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i$

Let $z_3 = r\cos\theta$

where $r = \sqrt{x^2 + y^2}$

$$= \sqrt{\left(\frac{\sqrt{3}+1}{2}\right)^2 + \left(\frac{\sqrt{3}-1}{2}\right)^2}$$

$$= \sqrt{\frac{3 + 2\sqrt{3} + 1}{4} + \frac{3 - 2\sqrt{3} + 1}{4}}$$

$$= \sqrt{\frac{8}{4}}$$

$$= \sqrt{2}$$

and $\tan\theta = \frac{y}{x} = \frac{\sqrt{3}-1}{2} \div \frac{\sqrt{3}+1}{2}$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= 2 - \sqrt{3}$$

$\therefore \quad \theta = \tan^{-1}(2 - \sqrt{3})$

$\therefore \quad z_3 = \sqrt{2}\operatorname{cis}(\tan^{-1}(2 - \sqrt{3}))$

Note: $\tan^{-1}(2 - \sqrt{3}) = \frac{\pi}{12}$

c $z_4 = \frac{1}{z_3}$

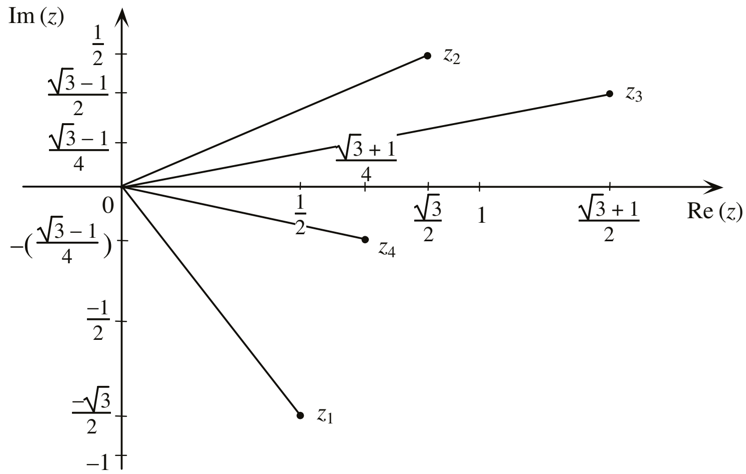
$$= \frac{1}{\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i}$$

$$= \frac{2}{(\sqrt{3}+1) + (\sqrt{3}-1)i} \times \frac{(\sqrt{3}+1) - (\sqrt{3}-1)i}{(\sqrt{3}+1) - (\sqrt{3}-1)i}$$

$$= \frac{2((\sqrt{3}+1) - (\sqrt{3}-1)i)}{(\sqrt{3}+1)^2 - (\sqrt{3}-1)^2}$$

$$= \frac{2((\sqrt{3}+1) - (\sqrt{3}-1)i)}{4 + 2\sqrt{3} + 4 - 2\sqrt{3}}$$

$$= \frac{\sqrt{3}+1}{4} - \frac{\sqrt{3}-1}{4}i$$



11 a i $P(z) = (z-k)^2(z-a)$

$= (z^2 - 2kz + k^2)(z-a)$

$= z^3 - 2kz^2 + k^2z - az^2 + 2akz - ak^2$

$= z^3 - (a+2k)z^2 + (2ak + k^2)z - ak^2$

Also $P(z) = z^3 + 3pz + q$

Equating coefficients $a + 2k = 0$

$\therefore \quad a = -2k$

$2ak + k^2 = 3p$

$\therefore \quad 2(-2k)k + k^2 = 3p$

$\therefore \quad -4k^2 + k^2 = 3p$

$\therefore \quad -3k^2 = 3p$

$\therefore \quad p = -k^2$, as required.

ii and $-ak^2 = q$

$\therefore \quad -(-2k)k^2 = q$

$\therefore \quad q = 2k^3$

iii $4p^3 + q^2 = 4(-k^2)^3 + (2k^3)^2$

$= -4k^6 + 4k^6 = 0$, as required.

b From **a** $3p = -6i$ and $q = 4 - 4i$

$\therefore \quad p = -2i$

Also $p = -b^2$ ①

and $q = 2b^3$ ②

Dividing ② by ① $\dfrac{2b^3}{-b^2} = \dfrac{q}{p}, \; b \neq 0$

$$\therefore \quad -2b = \frac{4 - 4i}{-2i}$$

$$= \frac{-2 + 2i}{i} \times \frac{i}{i}$$

$$= \frac{-2 - 2i}{-1}$$

$$= 2 + 2i$$

$$\therefore \quad b = -1 - i$$

From **a ii** $-cb^2 = q$

$$\therefore \quad -c(-1 - i)^2 = 4 - 4i$$

$$-2ic = 4 - 4i$$

$$\therefore \quad c = \frac{-2 + 2i}{i} \times \frac{i}{i}$$

$$= \frac{-2 - 2i}{-1}$$

$$= 2 + 2i$$

12 a i $|(1 + i)z| = |1 + i|\,|z|$

$$= \sqrt{2} \times 6$$

$$= 6\sqrt{2}$$

ii $|(1 + i)z - z| = |z + iz - z|$

$$= |iz|$$

$$= |i|\,|z|$$

$$= 1 \times 6$$

$$= 6$$

iii Since $|z| = 6$, the distance from the origin O to a point A is 6 units.
Since $|(1 + i)z| = 6\sqrt{2}$, the distance OB is $6\sqrt{2}$ units.
Since $|(1 + i)z - z| = 6$, the distance AB is 6 units.

Now $|OB|^2 = |OA|^2 + |AB|^2$.
Hence, OAB is an isosceles right triangle.

b **i** $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$

$\therefore \quad (\alpha z_2)^2 - 2(\alpha z_2)z_2 + 2z_2^2 = 0, \text{ since } z_1 = \alpha z_2$

$\therefore \quad \alpha^2 z_2^2 - 2\alpha z_2^2 + 2z_2^2 = 0$

$\therefore \quad z_2^2(\alpha^2 - 2\alpha + 2) = 0$

$\therefore \quad z_2^2(\alpha^2 - 2\alpha + 1 + 1) = 0$

$\therefore \quad z_2^2((\alpha - 1)^2 - i^2) = 0$

$\therefore \quad z_2^2(\alpha - 1 + i)(\alpha - 1 - i) = 0$

$\therefore \quad z_2 = 0 \text{ or } \alpha = 1 \pm i, \text{ but } z_2 \neq 0$

$\therefore \quad \alpha = 1 \pm i$

ii Let A and B be the points represented by z_2 and z_1 respectively.

$OA = |z_2|$

If $\alpha = 1 + i$, then $OB = |z_1|$

$= |(1 + i)z_2|$

$= \sqrt{2}|z_2|$

and $AB = |z_1 - z_2|$

$= |(1 + i)z_2 - z_2|$

$= |z_2|$

$\therefore \quad |OB|^2 = |OA|^2 + |AB|^2$

Hence, OAB has two sides the same length and is a right isosceles triangle.

If $\alpha = 1 - i$, then $OB = |z_1|$

$= |(1 - i)z_2|$

$= |(1 - i| \, |z_2|$

$= \sqrt{1^2 + (-1)^2}|z_2|$

$= \sqrt{2}|z_2|$

and $AB = |z_1 - z_2|$

$$= |(1 - i)z_2 - z_2|$$

$$= |z_2 - iz_2 - z_2|$$

$$= |-iz_2|$$

$$= |-i|\,|z_2|$$

$$= |z_2|$$

$\therefore \quad |OB|^2 = |OA|^2 + |AB|^2$

Again, OAB has two sides the same length and is a right isosceles triangle.

13 a i $|z| = \sqrt{(-12)^2 + 5^2}$

$$= \sqrt{144 + 25}$$

$$= \sqrt{169}$$

$$= 13$$

ii Let $\quad \text{Arg } z = \phi$

$$\tan\phi = \frac{y}{x}$$

$$= \frac{5}{-12}$$

$$\therefore \quad \phi = \tan^{-1}\left(\frac{5}{-12}\right)$$

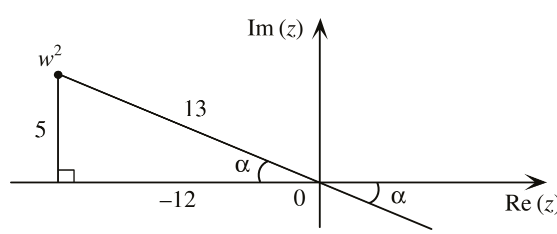
$$\therefore \quad \text{Arg }(z) = \tan^{-1}\left(\frac{5}{-12}\right)$$

$\approx 2.75^c$, since z is in the second quadrant.

$= 157.38°$ correct to decimal places.

b i $\cos\alpha = \dfrac{-12}{13}$

$\sin\alpha = \dfrac{5}{13}$



ii $r^2(\cos 2\theta + i\sin 2\theta) = |w^2|(\cos\alpha + i\sin\alpha)$

$$= 13\left(\frac{-12}{13} + i\frac{5}{13}\right)$$

$$\therefore \quad r^2 = 13,\ \cos 2\theta = \frac{-12}{13} \text{ and } \sin 2\theta = \frac{5}{13}$$

$$\therefore \quad r = \sqrt{13}$$

iii $\cos 2\theta = \dfrac{-12}{13}$ from **b ii** above.

$$2\cos^2\theta - 1 = \frac{-12}{13}$$

$$\cos^2\theta = \frac{1}{26}$$

$$\cos\theta = \pm\sqrt{\frac{1}{26}}$$

$$= \pm\frac{1}{\sqrt{26}} = \pm\frac{\sqrt{26}}{26}$$

Now $\sin 2\theta = \dfrac{5}{13}$ from **b ii** above.

$$\therefore \quad 2\sin\theta\cos\theta = \frac{5}{13}$$

$$\therefore \quad \sin\theta = \frac{5}{26\cos\theta}$$

$$= \pm\frac{5}{\sqrt{26}} = \pm\frac{5\sqrt{26}}{26}$$

iv From **b**, $w^2 = z$

$$= |z|\text{cis}\,\phi$$

$$= r^2\text{cis}\,2\theta$$

$$\therefore \quad w = r\text{cis}\,\theta$$

$$= \sqrt{13}(\cos\theta + i\sin\theta)$$

$$= \pm\sqrt{13}\left(\frac{1}{\sqrt{26}} + i\frac{5}{\sqrt{26}}\right)$$

$$= \pm\left(\frac{1}{\sqrt{2}} + i\frac{5}{\sqrt{2}}\right) = \pm\frac{\sqrt{2}}{2}(1 + 5i)$$

c Let $w = a + bi,\ a,\ b \in R$ and $w^2 = -12 + 5i$

$$\therefore \quad (a + bi)^2 = -12 + 5i$$

$$\therefore \quad a^2 + 2abi + b^2i^2 = -12 + 5i$$

$$\therefore \quad (a^2 - b^2) + 2abi = -12 + 5i$$

Equating coefficients, $a^2 - b^2 = -12$ and $2ab = 5$

$$\therefore \quad a = \frac{5}{2b} \qquad ①$$

$$\therefore \quad \left(\frac{5}{2b}\right)^2 - b^2 = -12$$

$$\therefore \quad \frac{25}{4b^2} - b^2 = -12$$

$$\therefore \quad 25 - 4b^4 = -48b^2$$

$$\therefore \quad 4b^4 - 48b^2 - 25 = 0$$

$$\therefore \quad (2b^2 + 1)(2b^2 - 25) = 0$$

$$\therefore \quad b^2 = \frac{-1}{2} \text{ or } \frac{25}{2} \text{ but } b^2 > 0 \text{ since } b \in R$$

$$\therefore \quad b^2 = \frac{25}{2}$$

$$\therefore \quad b = \pm\frac{5}{\sqrt{2}} = \pm\frac{5\sqrt{2}}{2}$$

From ① $a = \dfrac{5}{2b} = \pm\dfrac{5}{2 \times \dfrac{5\sqrt{2}}{2}}$

$$= \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\therefore \quad w = \pm\frac{\sqrt{2}}{2}(1 + 5i)$$

d Let $v^2 = 12 + 5i$ where $v = c + di,\ c,\ d \in R$

$$\therefore \quad (c + di)^2 = 12 + 5i$$

$$\therefore \quad (c^2 - d^2) + 2cdi = 12 + 5i$$

Equating coefficients

$c^2 - d^2 = 12$ and $2cd = 5$

$$\therefore \quad c = \frac{5}{2d} \qquad ①$$

$$\therefore \quad \left(\frac{5}{2d}\right)^2 - d^2 = 12$$

$$\therefore \quad \frac{25}{4d^2} - d^2 = 12$$

$$\therefore \quad 25 - 4d^4 = 48d^2$$

$$\therefore \quad 4d^4 + 48d^2 - 25 = 0$$

$$\therefore \quad (2d^2 + 25)(2d^2 - 1) = 0$$

$$\therefore \quad d^2 = \frac{-25}{2} \text{ or } \frac{1}{2} \text{ but } d^2 > 0 \text{ since } d \in R$$

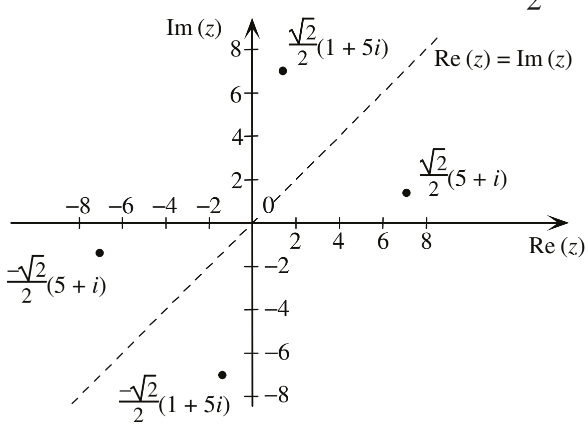
$$\therefore \quad d^2 = \frac{1}{2}$$

$$\therefore \quad d = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\text{From ① } c = \pm\frac{5}{2 \times \frac{\sqrt{2}}{2}} = \pm\frac{5\sqrt{2}}{2}$$

$$\therefore \quad v = \pm\frac{\sqrt{2}}{2}(5 + i)$$

Hence the square roots of $12 + 5i$ are $\pm\frac{\sqrt{2}}{2}(5 + i)$



Geometrically the square roots of $12 + 5i$ are the reflection of the square roots of $-12 + 5i$, in the line $\text{Re}(z) = \text{Im}(z)$.

14 Let $z = x + yi,\ a,\ b \in R \ \therefore \quad \bar{z} = x - yi$

a

$$2z\bar{z} + 3z + 3\bar{z} - 10 = 0$$

$$\therefore \quad 2(x + yi)(x - yi) + 3(x + yi) + 3(x - yi) - 10 = 0$$

$$\therefore \quad 2(x^2 + y^2) + 3x + 3yi + 3x - 3yi - 10 = 0$$

$$\therefore \quad 2x^2 + 6x + 2y^2 - 10 = 0$$

$$\therefore \quad x^2 + 3x + y^2 - 5 = 0$$

$$\therefore \quad x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + y^2 - 5 = 0$$

$$\therefore \quad \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + y^2 - 5 = 0$$

$$\therefore \quad \left(x + \frac{3}{2}\right)^2 + y^2 = \frac{29}{4}$$

a circle with centre $\left(-\frac{3}{2},\ 0\right)$ and radius $\frac{\sqrt{29}}{2}$.

b

$$2z\bar{z} + (3 + i)z + (3 - i)\bar{z} - 10 = 0$$

$$\therefore \quad 2(x + yi)(x - yi) + (3 + i)(x + yi) + (3 - i)(x - yi) - 10 = 0$$

$$\therefore \quad 2(x^2 + y^2) + 3x + xi + 3yi - y + 3x - xi - 3yi - y - 10 = 0$$

$$\therefore \quad 2x^2 + 2y^2 + 6x - 2y - 10 = 0$$

$$\therefore \quad x^2 + y^2 + 3x - y - 5 = 0$$

$$\therefore \quad x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + y^2 - y + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 5 = 0$$

$$\therefore \quad \left(x + \frac{3}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{9}{4} - \frac{1}{4} - 5 = 0$$

$$\therefore \quad \left(x + \frac{3}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{15}{2}$$

a circle with centre $\left(-\frac{3}{2},\ \frac{1}{2}\right)$ and radius $\sqrt{\frac{15}{2}} = \frac{\sqrt{30}}{2}$.

c

$$\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$$

$$\therefore \quad \alpha(x + yi)(x - yi) + \beta(x + yi) + \beta(x - yi) + \gamma = 0$$

$$\therefore \quad \alpha(x^2 + y^2) + \beta x + \beta yi + \beta x - \beta yi + \gamma = 0$$

$$\therefore \quad \alpha(x^2 + y^2) + 2\beta x + \gamma = 0$$

$$\therefore \quad x^2 + y^2 + \frac{2\beta x}{\alpha}x + \frac{\gamma}{\alpha} = 0$$

$$\therefore \quad x^2 + \frac{2\beta}{\alpha}x + \left(\frac{\beta}{\alpha}\right)^2 - \left(\frac{\beta}{\alpha}\right)^2 + y^2 + \frac{\gamma}{\alpha} = 0$$

$$\therefore \quad \left(x + \frac{\beta}{\gamma}\right)^2 + y^2 - \frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha} = 0$$

$$\therefore \quad \left(x + \frac{\beta}{\alpha}\right)^2 + y^2 = \frac{\beta^2 - \alpha\gamma}{\alpha^2}$$

a circle with centre $\left(-\frac{\beta}{\alpha}, 0\right)$ and radius $\frac{\sqrt{\beta^2 - \alpha\gamma}}{\alpha}$.

d Let $\beta = a + bi \;\therefore\; \bar{z} = a - bi$

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$$

$$\therefore \quad \alpha(x + yi)(x - yi) + (a + bi)(x + yi) + (a - bi)(x - yi) + \gamma = 0$$

$$\therefore \quad \alpha(x^2 + y^2) + ax + bxi + ayi - by + ax - bxi - ayi - by + \gamma = 0$$

$$\therefore \quad \alpha(x^2 + y^2) + 2ax - 2by + \gamma = 0$$

$$\therefore \quad x^2 + y^2 + \frac{2a}{\alpha}x - \frac{2b}{\alpha}y + \frac{\gamma}{\alpha} = 0$$

$$\therefore \quad x^2 + \frac{2a}{\alpha}x + \left(\frac{a}{\alpha}\right)^2 - \left(\frac{a}{\alpha}\right)^2 + y^2 - \frac{2b}{\alpha}y + \left(\frac{b}{\alpha}\right)^2 - \left(\frac{b}{\alpha}\right)^2 + \frac{\gamma}{\alpha} = 0$$

$$\therefore \quad \left(x + \frac{a}{\alpha}\right)^2 + \left(y - \frac{b}{\alpha}\right)^2 - \frac{a^2}{\alpha^2} - \frac{b^2}{\alpha^2} + \frac{\gamma}{\alpha} = 0$$

$$\therefore \quad \left(x + \frac{a}{\alpha}\right)^2 + \left(y - \frac{b}{\alpha}\right)^2 = \frac{a^2 + b^2 - \alpha\gamma}{\alpha^2}$$

a circle with centre $\left(-\frac{a}{\alpha}, \frac{b}{\alpha}\right)$ and radius $\frac{\sqrt{a^2 + b^2 - \alpha\gamma}}{\alpha}$.

15 a $(\cos\theta + i\sin\theta)^5$

$$= \binom{5}{0}(\cos\theta)^5(i\sin\theta)^0 + \binom{5}{1}(\cos\theta)^4(i\sin\theta)^1 + \binom{5}{2}(\cos\theta)^3(i\sin\theta)^2$$

$$+ \binom{5}{3}(\cos\theta)^2(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)^1(i\sin\theta)^4 + \binom{5}{5}(\cos\theta)^0(i\sin\theta)^5$$

$$= \cos^5\theta + 5\cos^4\theta\ \sin\theta\, i + 10\cos^3\theta\ \sin^2\theta\, i^2 + 10\cos^2\theta\ \sin^3\theta\, i^3$$

$$+ 5\cos\theta\sin^4\theta i^4 + \sin^5\theta i^5$$

$$= \cos^5\theta + 5\cos^4\theta\ \sin\theta\, i - 10\cos^3\theta\sin^2\theta - 10\cos^2\theta\sin^3\theta\, i$$

$$+ 5\cos\theta\sin^4\theta + \sin^5\theta i$$

$$= (\cos^5\theta - 10\cos^3\theta\ \sin^2\theta + 5\cos\theta\ \sin^4\theta)$$

$$+ (5\cos^4\theta\ \sin\theta - 10\cos^2\theta\ \sin^3\theta + \sin^5\theta)i \qquad ①$$

b $(\cos\theta + i\sin\theta)^5 = (\text{cis}\ \theta)^5$

$$= \text{cis}\ 5\theta = \cos 5\theta + i\sin 5\theta$$

From ① in **a**

i $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$$= \cos^5\theta - 10\cos^3\theta\,(1 - \cos^2\theta) + 5\cos\theta\,(\sin^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\ \cos\theta\,(1 - \cos^2\theta)^2$$

$$= 11\cos^5\theta - 10\cos^3\theta + 5\ \cos\theta\,(1 - 2\cos^2\theta + \cos^4\theta)$$

$$= 11\cos^5\theta - 10\cos^3\theta + 5\ \cos\theta - 10\ \cos^3\theta + 5\cos^5\theta$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,\ \text{as required.}$$

ii $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$$= \sin\theta\,(5\cos^4\theta - 10\cos^2\theta\ \sin^2\theta + \sin^4\theta)$$

$$\therefore\quad \frac{\sin 5\theta}{\sin\theta} = 5\cos^4\theta - 10\cos^2\theta(1 - \cos^2\theta) + (\sin^2\theta)^2,\ \text{if}\ \sin\theta \neq 0$$

$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + (1 - \cos^2\theta)^2$$

$$= 15\cos^4\theta - 10\cos^2\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 16\cos^4\theta - 12\cos^2\theta + 1,\ \text{as required.}$$

16 a $(1+i)z+(1-i)\bar{z}=-2$

$\therefore \quad (1+i)(x+iy)+(1-i)(x-iy)=-2$ since $z=x+iy$ and $\bar{z}=x-iy$

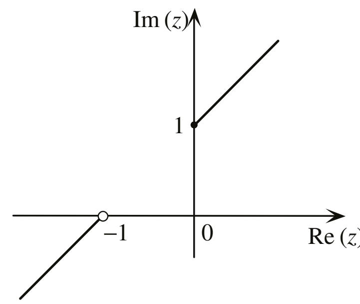
$\therefore \quad x+ix+iy-y+x-ix-iy-y=-2$

$\therefore \quad 2x-2y=-2$

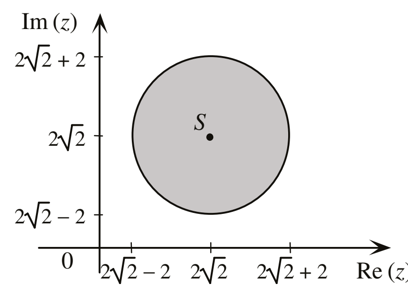
$\therefore \quad x-y=-1$

$\therefore \quad y=x+1$

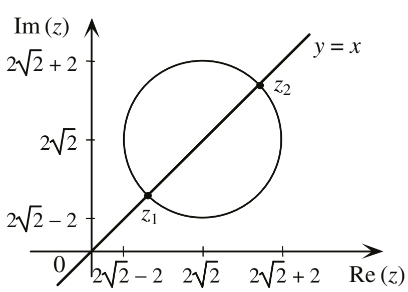
$\left\{z:(1+i)z+(1-i)\bar{z}=-2,\ \text{Arg}\, z\le\frac{\pi}{2}\right\}$



b **i** $|z-(2\sqrt{2}+i2\sqrt{2})|\le 2$ is the set of all points for which the distance from $(2\sqrt{2},2\sqrt{2})$ is less than or equal to 2. It is represented by $(x-2\sqrt{2})^2+(y-2\sqrt{2})^2\le 4$, a disc with centre $(2\sqrt{2},2\sqrt{2})$ and radius 2.
$S=\{z:|z-(2\sqrt{2}+i2\sqrt{2})|\le 2\}$



ii The minimum and maximum values of $|z|$ occur along the line $y=x$, at z_1 and z_2 respectively on the diagram below.



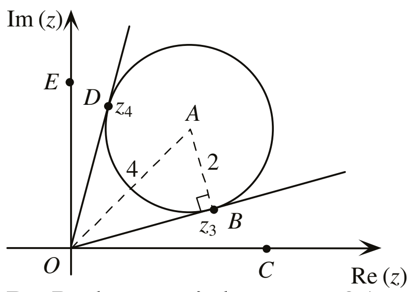
Along the line $y = x,\ z = r\text{cis}\,\frac{\pi}{4}$

At the centre of the circle $z = 2\sqrt{2} + i2\sqrt{2}$

$$\text{and } r = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2}$$
$$= \sqrt{8+8} = 4$$
$$\therefore \quad |z_1| = 4 - 2 = 2 \text{ and } |z_2| = 4 + 2 = 6$$

The minimum and maximum values of $|z|$ are 2 and 6.

iii The minimum and maximum values of Arg(z) occur at the points of intersection of the tangents to the circle $(x - 2\sqrt{2})^2 + (y - 2\sqrt{2})^2 = 4$ that also go through the origin as shown in the following diagram by z_3 and z_4.



By Pythagoras' theorem, $OA = 4$ and $AB = 2$

$$\therefore \quad \angle AOB = \sin^{-1}\frac{1}{2}$$
$$= \frac{\pi}{6}$$
$$\text{Now } \angle AOC = \frac{\pi}{4}$$
$$\therefore \quad \angle BOC = \frac{\pi}{12}$$

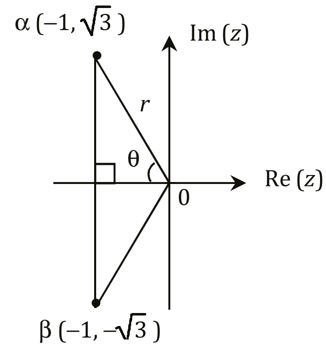
By symmetry, $\angle DOE = \frac{\pi}{12}$

$$\therefore \quad \angle DOC = \frac{\pi}{2} - \frac{\pi}{12}$$
$$= \frac{5\pi}{12}$$

The minimum and maximum values of Arg(z) are $\frac{\pi}{12}$ and $\frac{5\pi}{12}$.

17 a
$$z^2 + 2z + 4 = 0$$
$$z^2 + 2z + 1 + 3 = 0$$
$$(z+1)^2 - (\sqrt{3}i)^2 = 0$$
$$(z + 1 - \sqrt{3}i)(z + 1 + \sqrt{3}i) = 0$$
$$z = -1 \pm \sqrt{3}i$$

$$\theta = \tan^{-1} \sqrt{3}$$
$$= \frac{\pi}{3}$$
$$r = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= 2$$



Therefore $\alpha = 2\text{cis}\left(\frac{2\pi}{3}\right)$ or $\alpha = 2\text{cis}\left(\frac{-2\pi}{3}\right)$

$$\beta = 2\text{cis}\left(\frac{-2\pi}{3}\right) \text{ or } \beta = 2\text{cis}\left(\frac{2\pi}{3}\right)$$

$$\beta^3 = \left(2\text{cis}\left(\frac{-2\pi}{3}\right)\right)^3$$
$$= 2^3\text{cis}\left(3 \times \frac{-2\pi}{3}\right)$$
$$= 8\text{cis}(-2\pi)$$
$$= 8\text{cis } 0$$
$$= 8$$

b $\alpha^3 = \left(2\text{cis}\left(\frac{2\pi}{3}\right)\right)^3$
$$= 2^3\text{cis}\left(3 \times \frac{2\pi}{3}\right)$$
$$= 8\text{cis } 2\pi$$
$$= 8\text{cis } 0$$
$$= 8$$

Therefore $\alpha^3 = \beta^3$

c $\alpha + \beta = (-1 + \sqrt{3}i) + (-1 - \sqrt{3}i)$
$$= -2$$
$$\alpha - \beta = (-1 + \sqrt{3}i) - (-1 - \sqrt{3}i)$$
$$= 2\sqrt{3}i$$

$(z-(\alpha+\beta))(z-(\alpha-\beta)) = 0$

$(z-(-2))(z-(2\sqrt{3}i)) = 0$

$(z+2)(z-2\sqrt{3}i) = 0$

$z^2+2z-2\sqrt{3}iz-4\sqrt{3}i = 0$

$z^2+(2-2\sqrt{3}i)z-4\sqrt{3}i = 0$

Alternatively, if $\alpha = \text{cis}\left(\frac{-2\pi}{3}\right)$ and $\beta = \text{cis}\left(\frac{2\pi}{3}\right)$,

then $\alpha+\beta = -2$ and $\alpha-\beta = -2\sqrt{3}i$.

In this case, the quadratic equation is $z^2+(2+2\sqrt{3}i)z+4\sqrt{3}i$

d $\alpha\overline{\beta}+\beta\overline{\alpha} = (-1+\sqrt{3}i)(-1+\sqrt{3}i)+(-1-\sqrt{3}i)(-1-\sqrt{3}i)$

$= 1-2\sqrt{3}i+3i^2+1+2\sqrt{3}i+3i^2$

$= 2-6$

$= -4$

18 a i $z = w+\frac{1}{w}$

$= 2\text{cis}\,\theta+\frac{1}{2\text{cis}\,\theta}$

$= 2\text{cis}\,\theta+\frac{1}{2}\text{cis}(-\theta)$

ii $z = 2(\cos\theta+i\sin\theta)+\frac{1}{2}(\cos(-\theta)+i\sin(-\theta))$

$= 2\cos\theta+2i\sin\theta+\frac{1}{2}\cos\theta-\frac{1}{2}i\sin\theta$

$= \frac{5}{2}\cos\theta+\frac{3}{2}i\sin\theta = x+iy$

where $x = \frac{5}{2}\cos\theta$ and $y = \frac{3}{2}\sin\theta$

$x^2 = \frac{25}{4}\cos^2\theta$ $\qquad y^2 = \frac{9}{4}\sin^2\theta$

$\frac{x^2}{25} = \frac{1}{4}\cos^2\theta$ $\qquad \frac{y^2}{9} = \frac{1}{4}\sin^2\theta$

$\frac{x^2}{25}+\frac{y^2}{9} = \frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$

$= \frac{1}{4}(\cos^2\theta+\sin^2\theta) = \frac{1}{4}$

Therefore z lies on the ellipse with equation $\frac{x^2}{25}+\frac{y^2}{9} = \frac{1}{4}$.

iii $|z-2|^2 = \left|\frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta - 2\right|^2$

$= \left|\left(\frac{5}{2}\cos\theta - 2\right) + \frac{3}{2}i\sin\theta\right|^2$

$= \left(\frac{5}{2}\cos\theta - 2\right)^2 + \left(\frac{3}{2}\sin\theta\right)^2$

$= \frac{25}{4}\cos^2\theta - 10\cos\theta + 4 + \frac{9}{4}\sin^2\theta$

$= \frac{9}{4}(\cos^2\theta + \sin^2\theta) + 4\cos^2\theta - 10\cos\theta + 4$

$= \frac{9}{4} + 4 - 10\cos\theta + 4\cos^2\theta$

$= \frac{25}{4} - 10\cos\theta + 4\cos^2\theta$

$= \left(\frac{5}{2} - 2\cos\theta\right)^2$, as required.

iv $|z-2| = \left|\frac{5}{2} - 2\cos\theta\right|$

Since $-1 \le \cos\theta \le 1$

$-2 \le 2\cos\theta \le 2$

$-2 \le -2\cos\theta \le 2$

$\frac{1}{2} \le \frac{5}{2} - 2\cos\theta \le \frac{9}{2}$

Therefore $\frac{5}{2} - 2\cos\theta > 0$ for all θ

Therefore $\left|\frac{5}{2} - 2\cos\theta\right| = \frac{5}{2} - 2\cos\theta$

and hence $|z-2| = \frac{5}{2} - 2\cos\theta$ ①

$$\text{Now } |z+2|^2 = \left|\frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta + 2\right|^2$$

$$= \left|\left(\frac{5}{2}\cos\theta + 2\right) + \frac{3}{2}i\sin\theta\right|^2$$

$$= \left(\frac{5}{2}\cos\theta + 2\right)^2 + \left(\frac{3}{2}\sin\theta\right)^2$$

$$= \frac{25}{4}\cos^2\theta + 10\cos\theta + 4 + \frac{9}{4}\sin^2\theta$$

$$= \frac{9}{4}(\cos^2\theta + \sin^2\theta) + 4\cos^2\theta + 10\cos\theta + 4$$

$$= \frac{9}{4} + 4 + 10\cos\theta + 4\cos^2\theta$$

$$= \frac{25}{4} + 10\cos\theta + 4\cos^2\theta$$

$$= \left(\frac{5}{2} + 2\cos\theta\right)^2$$

$$|z+2| = \left|\frac{5}{2} + 2\cos\theta\right|$$

Since $-1 \le \cos\theta \le 1$

$$-2 \le 2\cos\theta \le 2$$

$$\frac{1}{2} \le \frac{5}{2} + 2\cos\theta \le \frac{9}{2}$$

Therefore $\frac{5}{2} + 2\cos\theta > 0$ for all θ

Therefore $\left|\frac{5}{2} + 2\cos\theta\right| = \frac{5}{2} + 2\cos\theta$

and hence $|z+2| = \frac{5}{2} + 2\cos\theta$ ②

From ① and ②

$$|z-2| + |z+2| = \frac{5}{2} - 2\cos\theta + \frac{5}{2} + 2\cos\theta$$

$$= 5, \text{ as required.}$$

b **i** $z = w - \frac{1}{w}$

$$= 2i\operatorname{cis}\theta - \frac{1}{2i\operatorname{cis}\theta} = 2i\operatorname{cis}\theta + \frac{1}{2}i\operatorname{cis}(-\theta)$$

ii $z = 2i(\cos\theta + i\sin\theta) + \frac{1}{2}i(\cos(-\theta) + i\sin(-\theta))$

$= 2i\cos\theta - 2\sin\theta + \frac{1}{2}i\cos\theta + \frac{1}{2}\sin\theta$

$= -\frac{3}{2}\sin\theta + \frac{5}{2}i\cos\theta$

$= x + iy$

where $x = -\frac{3}{2}\sin\theta$ and $y = \frac{5}{2}\cos\theta$

$x^2 = \frac{9}{4}\sin^2\theta$ $\quad y^2 = \frac{25}{4}\cos^2\theta$

$\frac{x^2}{9} = \frac{1}{4}\sin^2\theta$ $\quad \frac{y^2}{25} = \frac{1}{4}\cos^2\theta$

$\frac{y^2}{25} + \frac{x^2}{9} = \frac{1}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta$

$= \frac{1}{4}(\cos^2\theta + \sin^2\theta) = \frac{1}{4}$

Therefore z lies on the ellipse with equation $\frac{y^2}{25} + \frac{x^2}{9} = \frac{1}{4}$

iii $|z - 2i|^2 = \left|-\frac{3}{2}\sin\theta + \frac{5}{2}i\cos\theta - 2i\right|^2$

$= \left|-\frac{3}{2}\sin\theta + i\left(\frac{5}{2}\cos\theta - 2\right)\right|^2$

$= \left(-\frac{3}{2}\sin\theta\right)^2 + \left(\frac{5}{2}\cos\theta - 2\right)^2$

$= \frac{9}{4}\sin^2\theta + \frac{25}{4}\cos^2\theta - 10\cos\theta + 4$

$= \frac{9}{4}(\sin^2\theta + \cos^2\theta) + 4\cos^2\theta - 10\cos\theta + 4$

$= \frac{9}{4} + 4 - 10\cos\theta + 4\cos^2\theta$

$= \frac{25}{4} - 10\cos\theta + 4\cos^2\theta$

$= \left(\frac{5}{2} - 2\cos\theta\right)^2$, as required.

$$|z-2i| = \left|\frac{5}{2} - 2\cos\theta\right|$$

Since $$-1 \le \cos\theta \le 1$$

$$-2 \le 2\cos\theta \le 2$$

$$-2 \le -2\cos\theta \le 2$$

$$\frac{1}{2} \le \frac{5}{2} - 2\cos\theta \le \frac{9}{2}$$

Therefore $\frac{5}{2} - 2\cos\theta > 0$ for all θ

Therefore $\left|\frac{5}{2} - 2\cos\theta\right| = \frac{5}{2} - 2\cos\theta$

and hence $|z-2i| = \frac{5}{2} - 2\cos\theta$ ①

Now $$|z+2i|^2 = \left|-\frac{3}{2}\sin\theta + \frac{5}{2}i\cos\theta + 2i\right|^2$$

$$= \left|-\frac{3}{2}\sin\theta + i\left(\frac{5}{2}\cos\theta + 2\right)\right|^2$$

$$= \left(-\frac{3}{2}\sin\theta\right)^2 + \left(\frac{5}{2}\cos\theta + 2\right)^2$$

$$= \frac{9}{4}\sin^2\theta + \frac{25}{4}\cos^2\theta + 10\cos\theta + 4$$

$$= \frac{9}{4}(\sin^2\theta + \cos^2\theta) + 4\cos^2\theta + 10\cos\theta + 4$$

$$= \frac{9}{4} + 4 + 10\cos\theta + 4\cos^2\theta$$

$$= \frac{25}{4} + 10\cos\theta + 4\cos^2\theta$$

$$= \left(\frac{5}{2} + 2\cos\theta\right)^2$$

$$|z+2i| = \left|\frac{5}{2} + 2\cos\theta\right|$$

Since $$-1 \le \cos\theta \le 1$$

$$-2 \le 2\cos\theta \le 2$$

$$\frac{1}{2} \le \frac{5}{2} + 2\cos\theta \le \frac{9}{2}$$

Therefore $\frac{5}{2} + 2\cos\theta > 0$ for all θ

Therefore $\left|\frac{5}{2} + 2\cos\theta\right| = \frac{5}{2} + 2\cos\theta$

and hence $|z + 2i| = \frac{5}{2} + 2\cos\theta$ ②

From ① and ②

$$|z - 2i| + |z + 2i| = \frac{5}{2} - 2\cos\theta + \frac{5}{2} + 2\cos\theta$$

$$= 5, \text{ as required.}$$

Chapter 7 – Revision of Chapters 1 to 6

Solutions to Short-answer questions

Logic and proof

1 a Contrapositive. If n is even, then $n^2 - 6n + 5$ is odd.

b Proof. Suppose $n = 2k$ for integer k. Then

$$n^2 - 6n + 5 = (2k)^2 - 6(2k) + 5 \quad \text{is}$$
$$= 4k^2 - 12k + 5$$
$$= 2(2k^2 - 6k + 2) + 1$$

odd, as required.

c Converse. If n is odd, then $n^2 - 6n + 5$ is even.

d Proof. Suppose $n = 2k + 1$ for integer k. Then

$$n^2 - 6n + 5 = (2k + 1)^2 - 6(2k + 1) + 5$$
$$= 4k^2 + 4k + 1 - 12k - 6 + 1$$
$$= 4k^2 - 8k$$
$$= 2(2k^2 - 4k)$$

is even, as required.

2 a Let (m, n) be a solution to the equation in positive integers. Then $m^2 - n^2 = 1$ There-

$$(m - n)(n + n) = 1$$

fore $m - n = 1$ and $m + n = 1$ or $m - n = -1$ and $m + n = -1$. Adding the first pair of equations gives

$$2m = 2 \Rightarrow m = 1.$$

Therefor,e $n = 0$, which is a contradiction as 0 is not positive. Adding the second pair of equations gives

$$2m = -2 \Rightarrow m = -1,$$

which is also a contradiction as -1 is not positive.

b To find a counterexample to the claim we simple solve the equation. This gives $2m^2 - mn = 1$

$$m(2m - n) = 1$$

Therefore $m = 1$ and $2m - n = 1$ or $m = -1$ and $2m - n = -1$. Clearly, the second pair cannot yield a solution in the positive integers. From the first pair we find that $m = 1$ and $n = 1$. Therefore the solution $(1, 1)$ gives the required counterexample.

3 a $\boxed{P(n)}$

$(1 + i)^{4n} = (-4)^n$ where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$(1 + i)^4 = 1 + 4i + 6i^2 + 4i^3 + i^4$$
$$= 1 + 4i - 6 - 4i + 1$$
$$= -4$$
$$= (-4)^1$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$(1 + i)^{4k} = (-4)^k \quad (1)$

$\boxed{P(k+1)}$

We see that

LHS of $P(k+1)$

$= (1+i)^{4(n+1)}$

$= (1+i)^{4n}(1+i)^4$

$= (-4)^k(-4)$ (by (1) and P(1))

$= (-4)^{k+1}$.

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b We first write $(1+i)$ in polar form. This gives $1 + i = \sqrt{2}\text{cis}(\frac{\pi}{4})$.

Therefore,

$(1+i)^{4n} = (\sqrt{2}\text{cis}(\frac{\pi}{4}))^{4n}$

$= (2^{\frac{1}{2}}\text{cis}(\frac{\pi}{4}))^{4n}$

$= 2^{2n}\text{cis}(\frac{4n\pi}{4})$

$= (2^2)^n\text{cis}(n\pi)$

$= 4^n(\cos(n\pi) + i\sin(n\pi))$

$= 4^n(-1)^n$

$= (-4)^n$,

4 We will use the fact that for non-negative real numbers a and b, $a \geq b$ if and only if $a^2 \geq b^2$. The equivalence signs in the following proof are essential as they show that each step in the proof is reversible. We find that

$$\sqrt{2} + \sqrt{6} > \sqrt{14}$$

$$\Leftrightarrow (\sqrt{2} + \sqrt{6})^2 > (\sqrt{14})$$

$$\Leftrightarrow 2 + 2\sqrt{12} + 6 > 14$$

$$\Leftrightarrow 8 + 2\sqrt{12} > 14$$

$$\Leftrightarrow \sqrt{12} > 3$$

$$\Leftrightarrow 12 > 9$$

As the final line is true, and each of the steps is reversible, the first line is also true.

5 $\boxed{P(n)}$

$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^1 = \begin{bmatrix} 3^1 & 3^1 - 2^1 \\ 0 & 2^1 \end{bmatrix}$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix} \quad (1)$$

where $k \geq 1$.

$\boxed{P(k+1)}$

We see that

LHS of $P(k+1)$

$$= \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{k+1}$$

$$= \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{k}$$

$$= \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix} \quad \text{(by (1))}$$

$$= \begin{bmatrix} 3^{k+1} & 3(3^k - 2^k) - 2^k \\ 0 & 2^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 3(2^k) - 2^k \\ 0 & 2^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 2(2^k) \\ 0 & 2^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 2^{k+1} \\ 0 & 2^{k+1} \end{bmatrix}$$

= RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 **a** Suppose that $\log_6 3 = \frac{m}{n}$, where m, n are integers. Then

$$\log_6 3 = \frac{m}{n}$$

$$6^{\frac{m}{n}} = 3$$

$$(6^{\frac{m}{n}})^n = 3^n$$

$$6^m = 3^n.$$

The left-hand side is even, the right-hand side is odd, which is impossible. Now suppose $\log_6 2 = \frac{m}{n}$, where m, n are integers. Then

$$\log_6 2 = \frac{m}{n}$$

$$6^{\frac{m}{n}} = 2$$

$$(6^{\frac{m}{n}t})^n = 2^n$$

$$6^m = 2^n.$$

left-hand side is divisible by 3, the right-hand side is not, which is a contradiction.

b Taking the sum of these two numbers gives

$\log_6 3 + \log_6 2 = \log_6 6 = 1$,

which is rational.

c Assume a and b are irrational. Suppose, by way of contradiction, that each of $a+b$ and $a-b$ are rational. Then we can write a as the sum of two rational numbers:

$a = \frac{a+b}{2} + \frac{a-b}{2}$. As the sum of two rational numbers is rational, this gives a contradiction.

7 **a** $P(n)$

$7^n + 2$ is divisible by 3, where $n \in \mathbb{N}$.

$P(1)$

If $n = 1$ then $7^1 + 2 = 9$ is divisible by 3. Therefore $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true so that

$$7^k + 2 = 3m \quad (1)$$

for some integer m.

$P(k+1)$

We see that

$$\begin{aligned} &7^{k+1} + 2 \\ &= 7 \cdot 7^k + 2 \\ &= 7 \cdot (3m - 2) + 2 \quad \text{(by (1))} \\ &= 21m - 14 + 2 \\ &= 21m - 12 \\ &= 3(7m - 4) \end{aligned}$$

is a multiple of 3. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$5^{2n} + 3n - 1$ is divisible by 9, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then
$5^2 + 3(1) - 1 = 27 = 3 \times 9$ is divisible by 5. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that
$5^{2k} + 3k - 1 = 9m \quad (1)$
for some integer m.

$\boxed{P(k+1)}$

We see that

$$\begin{aligned}
&5^{2(k+1)} + 3(k+1) - 1\\
&= 5^{2k+2} + 3k + 2\\
&= 5^2 5^{2k} + 3k + 2\\
&= 25(9m - 3k + 1) + 3k + 2 \quad \text{(by (1))}\\
&= 9(25m) - 75k + 25 + 3k + 2\\
&= 9(25m) - 72k + 27\\
&= 9(25m - 8k + 3)
\end{aligned}$$

is divisible by 9. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

8 **a** If $p = 3$ and $q = 5$, then
$pq + 1 = 3 \times 5 + 1 = 16 = 4 \times 4$
is not a prime.

b The number $6^2 = 36$ is divisible by 9, however 6 is not divisible by 9.

9 $\boxed{P(n)}$

$1 \times 4 + 2 \times 5 + \cdots + n(n+3) = \frac{1}{3}n(n+1)(n+5)$

$\boxed{P(1)}$

If $n = 1$ thenLHS $= 1 \cdot 4 = 4$
and RHS $= \frac{1}{3} \times 1 \times 2 \times 6 = 4$.
Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that
$1 \times 4 + 2 \times 5 + \cdots + k(k+3) = \frac{1}{3}k(k+1)(k+5) \qquad (1)$

$\boxed{P(k+1)}$

LHS of $P(k+1)$ t

$$\begin{aligned}
&= 1 \times 4 + 2 \times 5 + \cdots + k(k+3) + (k+1)(k+4)\\
&= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4) \quad \text{(by (1))}\\
&= \frac{k(k+1)(k+5) + 3(k+1)(k+4)}{3}\\
&= \frac{(k+1)(k(k+5) + 3(k+4))}{3}\\
&= \frac{(k+1)(k^2 + 8k + 12)}{3}\\
&= \frac{(k+1)(k+2)(k+6)}{3}\\
&= \frac{(k+1)((k+1)+1)((k+1)+5)}{3}
\end{aligned}$$

$=$ RHS of $P(k+1)$
Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

10 $\boxed{P(n)}$

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

$\boxed{P(1)}$

If $n = 1$ then

LHS $= \frac{1}{2!} = \frac{1}{2}$

and RHS $= 1 - \frac{1}{(1+1)!} = \frac{1}{2}$.

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} =$

$1 - \frac{1}{(k+1)!}$ (1)

$\boxed{P(k+1)}$

LHS of $P(k+1)$

$= \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$ (by (1))

$= 1 - \frac{k+2}{(k+1)!(k+2)} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{(k+2)-(k+1)}{(k+2)!}$

$= 1 - \frac{1}{(k+2)!}$

$= 1 - \frac{1}{((k+1)+1)!}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

11 $\boxed{P(n)}$

$\sum_{r=1}^{n} r(2r+1) = \frac{1}{6}n(n+1)(4n+5)$

$\boxed{P(1)}$

If $n = 1$ then

LHS $= 1 \times 3 = 3$

and RHS $= \frac{1}{6} \times 1 \times 2 \times 9 = 3$.

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true

so that $\sum_{r=1}^{k} r(2r+1) =$

$\frac{1}{6}k(k+1)(4k+5)$ (1) $\boxed{P(k+1)}$

LHS of $P(k+1)$

$= \sum_{r=1}^{k+1} r(2r+1)$

$= \sum_{r=1}^{k} r(2r+1) + (k+1)(2k+3)$

$= \frac{1}{6}k(k+1)(4k+5) + (k+1)(2k+3)$ (by (1))

$= \frac{k(k+1)(4k+5)}{6} + \frac{6(k+1)(2k+3)}{6}$

$= \frac{k(k+1)(4k+5) + 6(k+1)(2k+3)}{6}$

$= \frac{(k+1)(4k^2+17k+18}{6}$

$= \frac{(k+1)(k+2)(4k+9)}{6}$

$= \frac{(k+1)((k+1)+1)(4(k+1)+5)}{6}$

$=$ RHS of $P(k+1)$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

12 $\boxed{P(n)}$

$5^{2n-1} + 2^{2n-1} - 7$ is divisible by 6, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$5^1 + 2^1 - 7 = 0 = 0 \times 6$is divisible by 6.

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$5^{2k-1} + 2^{2k-1} - 7 = 6m$ (1) for some integer m.

$\boxed{P(k+1)}$

This one is a little harder than your average induction proof. Here, we find that

$$\begin{aligned} & 5^{2k+1} + 2^{2k+1} - 7 \\ &= 25(5^{2k-1}) + 2^{2k+1} - 7 \\ &= 25(6m + 7 - 2^{2k-1}) + 2^{2k+1} - 7 \quad \text{(by (1))} \\ &= 6(25m) + 25(7) - 25(2^{2k-1}) + 4(2^{2k-1}) - 7 \\ &= 6(25m) + 24(7) - 24(2^{2k-1}) \\ &= 6(25m + 28 - 4(2^{2k-1}) \end{aligned}$$

is divisible by 6. Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

13 **a** Every number is n is either even or odd. If n is even then $n = 2k$ for integer k. Therefore $n^2 = (2k)^2 = 4k$ leaves a remainder of 0 when divided by 2. If n is odd, then $n = 2k + 1$ for integer k. Therefore

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$

leaves a remainder of 1 when divided by 4.

b Suppose by way of contradiction that $\sqrt{4n+3}$ were rational. Then $\sqrt{4n+3} = \frac{m}{n}$ for integers m and n. We can assume that m and n have no factors in common. Therefore

$$\sqrt{4n+3} = \frac{m}{n}$$

$$4n + 3 = \frac{m^2}{n^2}$$ Note that n^2 and

$n^2(4n+3) = m^2$.

m^2 both leave a remainder of 0 or 1 when divided by 4. If both leave a remainder of 0, then both are divisible by 4. Therefore n and m are both even, which is impossible. Therefore, there are three cases to consider:

Case 1. If $n^2 = 4k+1$ and $m^2 = 4j+1$ then

$(4k+1)(4n+3) = (4j+1)$.

The left-hand side gives a remainder of 3 when divided by 4, the right-hand side has a remainder of 1. This a contradiction.

Case 2. If $n^2 = 4k + 1$ and $m^2 = 4j$ then

$(4k+1)(4n+3) = 4j$.

The left-hand side gives a remainder of 3 when divided by 4, the right-hand side has a remainder of 0. This a contradiction.

Case 3. If $n^2 = 4k$ and $m^2 = 4j + 1$ then

$(4k)(4n+3) = 4j + 1$.

The left-hand side gives a remainder of 0 when divided by 4, the right-hand side has a remainder of 1. This a contradiction.

Circular functions

14 **a**

$$\begin{aligned} \cos^2\left(\frac{\pi}{12}\right) + \sin^2\left(\frac{\pi}{12}\right) &= 1 \\ \cos^2\left(\frac{\pi}{12}\right) &= 1 - \left(\frac{-1+\sqrt{3}}{2\sqrt{2}}\right)^2 \\ &= 1 - \frac{1 - 2\sqrt{3} + 3}{8} \\ &= \frac{2+\sqrt{3}}{4} \end{aligned}$$

b $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$

i $\sec\left(\frac{\pi}{5}\right) = \frac{4}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$

$= \sqrt{5} - 1$

ii $\tan^2\left(\frac{\pi}{5}\right) = \sec^2\left(\frac{\pi}{5}\right) - 1$

$= 5 - 2\sqrt{5}$

15 $f(x) = 3\arcsin(2x+1) + 4$

For maximal domain

$-1 \le 2x + 1 \le 1$

$-2 \le 2x \le 0$

$-1 \le x \le 0$

Maximal domain = $[-1, 0]$

Range:

$f(-1) = 3\arcsin(-1) + 4 = 4 - \frac{3\pi}{2}$

$f(0) = 3\arcsin(1) + 4 = 4 + \frac{3\pi}{2}$

Range = $\left[\frac{8-3\pi}{2}, \frac{8+3\pi}{2}\right]$

16 $\sec^2\left(\frac{\pi x}{3}\right) = 2 \qquad 0 < x < 6$

$\sec^2\left(\frac{\pi x}{3}\right) = \pm\sqrt{2}$

$\cos\left(\frac{\pi x}{3}\right) = \pm\frac{1}{\sqrt{2}}$

First consider

$\cos\left(\frac{\pi x}{3}\right) = \frac{1}{\sqrt{2}}$

$\frac{\pi x}{3} = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

$x = \frac{3}{4}$ or $\frac{21}{4}$

Next consider

$\cos\left(\frac{\pi x}{3}\right) = \frac{1}{\sqrt{2}}$

$\frac{\pi x}{3} = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

$x = \frac{9}{4}$ or $\frac{15}{4}$

Points of intesection are:

$\left(\frac{3}{4}, 2\right), \left(\frac{9}{4}, 2\right), \left(\frac{15}{4}, 2\right), \left(\frac{21}{4}, 2\right)$

17

$4\cos x = 2\cot x$

$4\cos x - 2\frac{\cos x}{\sin x} = 0$

$4\cos x \sin x - 2\cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = (2n+1)\frac{\pi}{2}, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, n \in \mathbb{Z}.$

18 a $\sin(4x) = \cos(2x)$

$$2\sin 2x\cos 2x - \cos 2x = 0$$

$$\cos 2x(2\sin 2x - 1) = 0$$

$$\cos 2x = 0 \text{ or } \sin 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } 2x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$x = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \text{ or } x = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

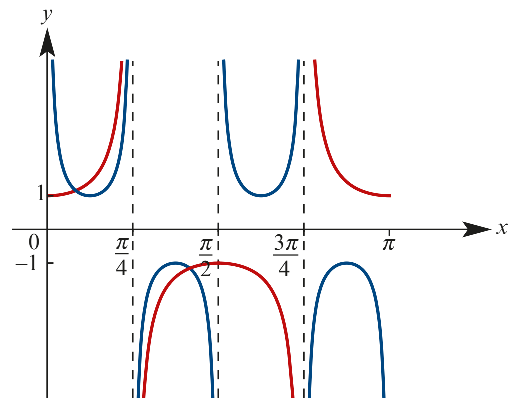
b **i** $\operatorname{cosec}(4x) = \sec(2x) \Rightarrow \sin(4x) = \cos(2x)$

The equation components are not defined at $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

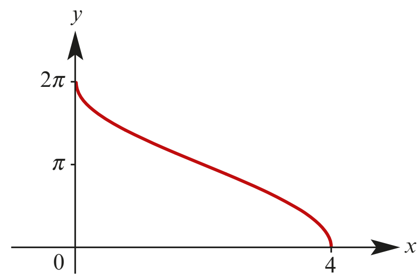
Therefore x values of points of intersection are $x = \frac{\pi}{12}$ or $\frac{5\pi}{12}$

Points of intersection are $\left(\frac{\pi}{12}, \frac{2\sqrt{3}}{3}\right), \left(\frac{5\pi}{12}, -\frac{2\sqrt{3}}{2}\right)$

ii

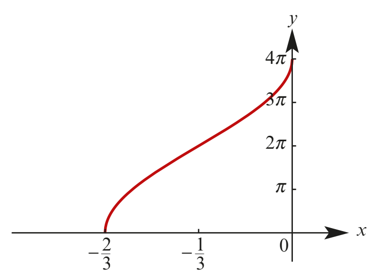


c



19 a $\left[-\frac{1-d}{c}, \frac{1-d}{c}\right], \left[a - \frac{\pi b}{2}, a + \frac{\pi b}{2}\right]$

b



Vectors

20 $\boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{k}$ and $\boldsymbol{c} = m\boldsymbol{i} + n\boldsymbol{j}$

Therefore by inspection if $\frac{m}{n} = -1$ they are linearly dependent.

Suppose that there exists real numbers p and q such that

$\boldsymbol{c} = p\boldsymbol{a} + q\boldsymbol{b}$

$m\boldsymbol{i} + n\boldsymbol{j} = p(-2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) + q(\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k})$

$$m = -2p + q \ldots (1)$$

$$n = 3p - 3q \ldots (2)$$

$$0 = -p + 2q \ldots (3)$$

From (3), $p = 2q$

Substitute in (1) and (2)

$$m = -4q + q = -3q$$

$$n = 6q - 3q = 3q$$

$$\therefore \frac{m}{n} = -1$$

The three vectors are linearly dependent if $\frac{m}{n} = -1$.

21 One diagonal of the parallelogram is either AB or AC or BC. If the diagonal. is AB, then D is on the opposite side of the diagonal to point C. Therefore

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$$

$$= \overrightarrow{OB} + \overrightarrow{CA}$$

$$= (-3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}) + (-2\boldsymbol{i} - 4\boldsymbol{j} + 4\boldsymbol{k})$$

$$= -5\boldsymbol{i} - 2\boldsymbol{j} + 9\boldsymbol{k},$$

in which case the coordinates are $D(-5, -2, 9)$. If the diagonal is AC, then D is on the opposite side of the diagonal to point B. Therefore

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$

$$= \overrightarrow{OA} + \overrightarrow{BC}$$

$$= (2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + (7\boldsymbol{i} + 3\boldsymbol{j} - 7\boldsymbol{k})$$

$$= 9\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}),$$

in which case the coordinates are $D(9, 4, -5)$, If the diagonal is BC, then D is on the opposite side of the diagonal to point A. Therefore

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$$

$$= \overrightarrow{OB} + \overrightarrow{AC}$$

$$= (-3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}) + (2\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k})$$

$$= -\boldsymbol{i} + 6\boldsymbol{j} + \boldsymbol{k},$$

22 Let $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

Vector resolute $= \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$

$$= \frac{2}{3}(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$$

$$= \frac{4}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$$

Vector resolute perpendicular to $\boldsymbol{b}$

$$= 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} - (\frac{4}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k})$$

$$= \frac{5}{3}\boldsymbol{i} + \frac{4}{3}\boldsymbol{j} - \frac{7}{3}\boldsymbol{k}$$

23 **a** $\overrightarrow{OA} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\overrightarrow{OB} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\overrightarrow{AB} = \overrightarrow{OB} + \overrightarrow{AO} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} = -\boldsymbol{i}$

b Let $\angle AOB = \theta$

The scalar product gives that

$$\cos\theta = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}$$

$$\therefore \cos\theta = \frac{7}{3\sqrt{6}} = \frac{7\sqrt{6}}{18}$$

c Area $\triangle AOB =$

$|\overrightarrow{OA}||\overrightarrow{OB}|\sin\left(\arccos\left(\frac{7\sqrt{6}}{18}\right)\right)$

$\sin\left(\arccos\left(\frac{7\sqrt{6}}{18}\right)\right) = \sqrt{1-\left(\frac{7\sqrt{6}}{18}\right)^2}$

Area $\triangle AOB = \frac{\sqrt{5}}{2}$

24 a $|\boldsymbol{a}| = \sqrt{4+9+m^2}$

$\sqrt{4+9+m^2} = \sqrt{38}$

$13 + m^2 = 38$

$m^2 = 25$

$m = \pm 5$

b $\boldsymbol{a}$ perpendicular to $\boldsymbol{b} \Rightarrow \boldsymbol{a}\cdot\boldsymbol{b} = 0$

$\Rightarrow -2 + \frac{9}{2} + 2m = 0$

$m = -\frac{5}{4}$

c $-2\boldsymbol{b} + 3\boldsymbol{c} = (-2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) + (6\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}) = 4\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k}$

d From (c), $-2\boldsymbol{b} + 3\boldsymbol{c} = 4\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k} = -2\boldsymbol{a}$ if $m = \frac{7}{2}$

25 a Let $\boldsymbol{a} = \overrightarrow{OA} = \boldsymbol{i} + \sqrt{3}\boldsymbol{j}$
and $\boldsymbol{b} = \overrightarrow{OB} = 3\boldsymbol{i} - 4\boldsymbol{k}$
If P is a point on AB the there exists $\lambda \in \mathbb{R}$ such that
$\overrightarrow{AP} = \lambda\overrightarrow{AB}$
$\therefore \overrightarrow{OP} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

$= \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$

$= (1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}$

$= (1-\lambda)(\boldsymbol{i} + \sqrt{3}\boldsymbol{j}) + \lambda(3\boldsymbol{i} - 4\boldsymbol{k})$

$= (1+2\lambda)\boldsymbol{i} + \sqrt{3}(1-\lambda)\boldsymbol{j} - 4\lambda\boldsymbol{k}$

b Let $\overrightarrow{OA'} = \hat{\boldsymbol{a}}$ be the unit vector in the dirction of $\boldsymbol{a}$

Let $\overrightarrow{OB'} = \hat{\boldsymbol{b}}$ be the unit vector in the dirction of $\boldsymbol{b}$
The $\triangle A'OB'$ is isosceles.
Let M be the midpoint of $A'B'$ then OM bisects angle $\angle AOB$.

$\overrightarrow{OM} = \frac{1}{2}(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}})$

$= \frac{1}{2}(\frac{1}{2}(\boldsymbol{i} + \sqrt{3}\boldsymbol{j}) + \frac{1}{5}(3\boldsymbol{i} - 4\boldsymbol{k}))$

$= \frac{11}{20}\boldsymbol{i} + \frac{\sqrt{3}}{4}\boldsymbol{j} - \frac{2}{5}\boldsymbol{k}$

Let OM extended meet AB at P
Then for some real number μ, $\overrightarrow{OP} = \mu\left(\frac{11}{20}\boldsymbol{i} + \frac{\sqrt{3}}{4}\boldsymbol{j} - \frac{2}{5}\boldsymbol{k}\right)$
From (a)
$\overrightarrow{OP} = (1+2\lambda)\boldsymbol{i} + \sqrt{3}(1-\lambda)\boldsymbol{j}) - 4\lambda\boldsymbol{k}$
Hence $\lambda = \frac{2}{7}$

26 a The line has gradient $-\frac{3}{2}$. Therefore vector perpendicular $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j}$.
Therefore $\hat{\boldsymbol{a}} = \frac{\sqrt{13}}{13}(3\boldsymbol{i} + 2\boldsymbol{j})$

b Let $P(x, y)$ be the point. Then
$\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j}$ and $\overrightarrow{OA} = 2\boldsymbol{i} - 5\boldsymbol{j}$
Hence $\overrightarrow{AP} = (x-2)\boldsymbol{i} + (y+5)\boldsymbol{j}$. It is parallel to $\hat{\boldsymbol{a}}$
$\therefore (x-2)\boldsymbol{i} + (y+5)\boldsymbol{j} = k(3\boldsymbol{i} + 2\boldsymbol{j})$
$k \in \mathbb{R}\backslash\{0\}$.
$\therefore \frac{x-2}{y+5} = \frac{3}{2}$
$2x - 3y = 19$ and we know
$2y - 3x = 6$.
Solving the simultaneous equations:
$x = \frac{30}{13}$ and $y = \frac{20}{13}$
$\overrightarrow{AP} = \frac{10}{13}(3\boldsymbol{i} + 2\boldsymbol{j})$

c $|\overrightarrow{AP}| = \frac{10}{13} \times \sqrt{13} = \frac{10\sqrt{13}}{13}$

27 a $-4i + 2j - 3k = m(2i - 2j + 5k) + n(-i + 2j - 6k)$

$$-4 = 2m - n \ldots (1) \quad 1$$

$$2 = -2m + 2n \ldots (2)$$

$$-3 = 5m - 6n \ldots (3)$$

Add (1) and (2)

$$n = -2$$

From (1)$m = -3$

b

$$\overrightarrow{OP} = \lambda c$$

$$\therefore \overrightarrow{OP} = -4\lambda i + 2\lambda j - 3\lambda k$$

P is a point on AB
There is a real number μ such that
$\overrightarrow{OP} = (1 - \mu)a + \mu b$
$\therefore \overrightarrow{OP} = (2 - 3\mu)i + (4\mu - 2)j + (5 - 11\mu)k$
Hence

$$2 - 3\mu = -4\lambda \ldots (1)$$

$$4\mu - 2 = 2\lambda \ldots (2)$$

$$5 - 11\mu = -3\lambda \ldots (3)$$

$$\lambda = -\frac{1}{5}$$

28 a $a = 2mi + 3mj + 6mk$
$a| = 1 \Rightarrow 4m^2 + 9m^2 + 36m^2 = 1$
$\Rightarrow 49m^2 = 1$
$\Rightarrow m = \pm\frac{1}{7}.$
Therefore $m = \frac{1}{7}$

b i

$$a \cdot b = \frac{1}{7}(2i + 3j + 6k) \cdot (2i + j + 2k)$$

$$= \frac{1}{7}(4 + 3 + 12)$$

$$= \frac{19}{7}$$

ii

$$a \times b = \frac{1}{7}(2i + 3j + 6k) \times (2i + j + 2k)$$

$$= \frac{1}{7}(0i + 8j - 4k)$$

$$= \frac{4}{7}(2j - k)$$

29 $\overrightarrow{OA} \times \overrightarrow{OB} = -3i$
Area of parallelogram $= |-3i| = 3$

30 a $\overrightarrow{QP} = 2j + 7k$
$\overrightarrow{QR} = 4i + 3j + k$
$\overrightarrow{QP} \times \overrightarrow{QR}$
$= (2 \times 1)i - (2 \times 1)j + (2 \times 3 - 4 \times 2)k$
$= 2i - 2j - 2k$
$|\overrightarrow{QP} \times \overrightarrow{QR}| = \sqrt{4 + 4 + 4} = 2\sqrt{3}$
Therefore area of triangle $= \sqrt{3}$

b $\overrightarrow{QP} = 2i$
$\overrightarrow{QR} = i + j + k$
$\overrightarrow{QP} \times \overrightarrow{QR}$
$= -2j + 2k$
$|\overrightarrow{QP} \times \overrightarrow{QR}| = \sqrt{4 + 4} = 2\sqrt{2}$
Therefore area of triangle $= \sqrt{2}$

31 We first find

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - a & -1 & -1 \\ -a & 0 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + a\mathbf{k}.$$

Therefore, as the area is 1, we find that

$$\text{area}(\triangle CAB) = 1$$

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = 1$$

$$\frac{1}{2}\sqrt{1^2 + 1^2 + a^2} = 1$$

$$\sqrt{2 + a^2} = 2$$

$$2 + a^2 = 4$$

$$a^2 = 2$$

$$a = \pm\sqrt{2}.$$

Vector Equations of lines and planes

32 a $\boldsymbol{d} = 3\boldsymbol{i} + 4\boldsymbol{k}$

$\therefore \boldsymbol{r} = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k} + t(3\boldsymbol{i} + 4\boldsymbol{k})$

or $\boldsymbol{r} = t(3\boldsymbol{i} + 4\boldsymbol{k}), \quad t \in \mathbb{R}$

b $\boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

$\therefore \boldsymbol{r} = 2\boldsymbol{j} + \boldsymbol{k} + t(-\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}), \quad t \in \mathbb{R}$

c $\boldsymbol{d} = -3\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}$

$\therefore \boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k} + t(-3\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}), \; t \in \mathbb{R}$

33 a $\boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 0 \quad (\boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

b $\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \overrightarrow{OB} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

$\therefore \boldsymbol{n} = -2\boldsymbol{i} + 2\boldsymbol{k} = \overrightarrow{OA} \times \overrightarrow{OB}$

and $(2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}) \cdot (-2\boldsymbol{i} + 2\boldsymbol{k}) = 6$

$\therefore \boldsymbol{r} \cdot (-2\boldsymbol{i} + 2\boldsymbol{k}) = 6$

c $\overrightarrow{OA} = -3\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}, \quad \overrightarrow{OB} = 4\boldsymbol{j} - 4\boldsymbol{k}$

$\therefore \boldsymbol{n} = 16\boldsymbol{i} - 12\boldsymbol{j} - 12\boldsymbol{k} = \overrightarrow{OA} \times \overrightarrow{OB}$

or choose $\boldsymbol{n} = 4\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k}$

and $(3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}) \cdot \boldsymbol{n} = 12 - 6 - 12$

$= -6$

$\therefore \boldsymbol{r} \cdot (4\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k}) = -6$

34 a $\Pi_1 \quad 2x + 2y + z = 6$

or $\boldsymbol{r_1} \cdot (2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = 6$

$\Pi_2 \quad 2x + 2y + z = 10$

or $\boldsymbol{r_2} \cdot (2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = 10$

$\boldsymbol{n} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ for both planes.

Now if P is any point on Π_1

$\therefore \overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

(where $2x + 2y + 2 = 6$)

Now scalar resolute $\overrightarrow{OP}$ parallel

$$\boldsymbol{n} = \frac{\overrightarrow{OP} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$$

$$= \frac{(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot (2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})}{3}$$

$$= \frac{2x + 2y + z}{3} = \frac{6}{3} = 2$$

= distance of P from O

Similarly if Q is any point on Π_2

$\overrightarrow{OQ} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

(where $2x + 2y + z = 10$)

$\therefore$ Scalar resolute of $\overrightarrow{OQ}$ parallel to $\boldsymbol{n}$

$$= \frac{\overrightarrow{OQ} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$$

$$= \frac{(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})}{3}$$

$$= \frac{2x + 2y + z}{3}$$

$$= \frac{10}{3}$$

= distance of Q from O

$$= \frac{10}{3}$$

$$\therefore \text{ Distance between planes} = \frac{10}{3} - 2$$

(Since both on same side of O) $= \dfrac{4}{3}$

b The plane has equation $x + 2y + 3z = 6$. By observation $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ is the position vector of the point $Q(1, 1, 1)$ on the plane. Unit vector perpendicular to the plane is $\hat{\boldsymbol{n}} = \dfrac{1}{\sqrt{14}}(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$

$\overrightarrow{PQ} = -\boldsymbol{i} - \boldsymbol{k} + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} = \boldsymbol{j}$

$$\boldsymbol{j} \cdot \hat{\boldsymbol{n}} = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$$

c The lines intersect. Shortest distance is 0.

d $\dfrac{x + 1}{2} = y = z - 1 = t$

Vector equation is:
$\boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
$\overrightarrow{OP} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$
Let Q be the point on the line so that PQ is perpendicular to the line.
$\overrightarrow{PQ} = -2\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
$= (2t-2)\boldsymbol{i} + (t-2)\boldsymbol{j} + (t-2)\boldsymbol{k}$
PQ is perpendicular to $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
$((2t-2)\boldsymbol{i} + (t-2)\boldsymbol{j} + (t-2)\boldsymbol{k}) \cdot (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 0$
$\Leftrightarrow 6t - 8 = 0$
$\Leftrightarrow t = \frac{4}{3}$
$\overrightarrow{PQ} = \frac{2}{3})\boldsymbol{i} - \frac{2}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k}$
$|\overrightarrow{PQ}| = \frac{2\sqrt{3}}{3}$

35 $l_1: \quad \boldsymbol{r}_1 = 2\boldsymbol{j} + 3\boldsymbol{k} + t(2\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k})$
$l_2: \quad \boldsymbol{r}_2 = 3\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k} + s(\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$
$\therefore l_1$ is not parallel to l_2
At point of intersection
$$\begin{cases} 2t = 3 + s & (1) \\ 2 - 2t = -1 + s & (2) \\ 3 - 4t = 4s - 3 & (3) \end{cases}$$
$(1) - (2) \quad \therefore 4t - 2 = 4$
$\therefore t = \frac{3}{2}$
Substitute in (1) $\therefore s = 0$
Check in (3)
$\therefore$ Point of intersection is $(3, -1, -3)$

36 **a** Equation of line:
$\boldsymbol{r}(t) = \boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$
Therefore $x = \lambda, y = -\lambda + 1, z = \lambda$
Therefore for the intersection:
$\lambda - \lambda + 1 + \lambda = 1$
Therefore $\lambda = 0$
Intersects at $(0, 1, 0)$

b Equation of line: $\boldsymbol{r}(t) = \boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$
Therefore $x = \lambda, y = -\lambda + 1, z = \lambda$
Therefore for the intersection:
$\lambda - \lambda + 1 + \lambda = 3$
Therefore $\lambda = 2$
Intersects at $(2, -1, 2)$

c Equation of line: $\boldsymbol{r}(t) = \boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$
Therefore $x = \lambda, y = -\lambda + 1, z = \lambda$
Therefore for the intersection:
$\lambda + \lambda - 1 + \lambda = 1$
Therefore $\lambda = \frac{2}{3}$
Intersects at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

37 For plane $\boldsymbol{n} = 3\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}$ with $|\boldsymbol{n}| = 7$
Let P be a point on the plane
$= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$
Now $A = (4, 0, 1) \quad \therefore \overrightarrow{OA} = 4\boldsymbol{i} + \boldsymbol{k}$
$\therefore \overrightarrow{PA} = (4 - x)\boldsymbol{i} - y\boldsymbol{j} + (1 - z)\boldsymbol{k}$
Now scalar resolute of $\overrightarrow{PA}$ parallel
$\boldsymbol{n} = \frac{\overrightarrow{PA} \cdot \boldsymbol{n}}{|\boldsymbol{n}|} = 3$
$\therefore$ Distance required is 3.

38 **a** $$\begin{cases} 2x - y + 5z = 7 & (1) \\ 5x - 3y - z = 4 & (2) \\ 3x + 4y - 6z = a & (3) \end{cases}$$
$$= \begin{cases} 2x - y + 5z = 7 & (1) \\ 11y - 27z = -27 & (2)' = 5\left(\frac{2}{5}(2) - (1)\right) \\ 11y - 27z = 2a - 21 & (3)' = 3\left(\frac{2}{3}(3) - (1)\right) \end{cases}$$
For intersection to be a line
$-27 = 2a - 21$
$\therefore 2a = -6$
$\therefore a = -3$

b $\therefore$ Line is $x = \dfrac{-14\lambda + 25}{11}$,

$y = \dfrac{27\lambda - 27}{11}$, $z = \lambda$

or $\boldsymbol{r} = \dfrac{-14\lambda + 25}{11}\boldsymbol{i} + \dfrac{27\lambda - 27}{11}\boldsymbol{j} + \lambda\boldsymbol{k}$

or $\boldsymbol{r} = \dfrac{25}{11}\boldsymbol{i} - \dfrac{27}{11}\boldsymbol{j} + \lambda\left(\dfrac{-14}{11}\boldsymbol{i} + \dfrac{27}{11}\boldsymbol{j} + \boldsymbol{k}\right)$

or $\boldsymbol{r} = \dfrac{1}{11}((25\boldsymbol{i} - 27\boldsymbol{j}) + \lambda(-14\boldsymbol{i} + 27\boldsymbol{j} + 11\boldsymbol{k}))$

39 a $(2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})) \cdot (-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = 3$

$|2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})| = \sqrt{14}$

$|-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})| = \sqrt{6}$

Therefore $\cos\theta = \dfrac{3}{\sqrt{84}} = \dfrac{\sqrt{21}}{14}$

b Cartesian equations are:

$2x + y + 3z = 1 \ldots (1)$

$-2x + 4y + 2z = 4 \ldots (2)$

Multiply (2) by 2 and add to (1)

$5y + 5z = 5$

$y + z = 1$

Therefore $y = 1 - z$

Substituting in (1) for z gives $x = -z$

In parametric form we have the line

$x = -t, y = 1 - t, z = t$

We have the vector equation,

$\boldsymbol{r} = \boldsymbol{j} + t(-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) \quad t \in \mathbb{R}$

40 a All four points satisfy the equation

$x - 2y + 2z = 0$

b $\overrightarrow{AB} = -4\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

$|\overrightarrow{AB}| = \sqrt{16 + 4 + 16} = 6$

$\overrightarrow{BC} = -4\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k}$

$|\overrightarrow{BC}| = \sqrt{16 + 16 + 4} = 6$

$\overrightarrow{CD} = 4\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$

$|\overrightarrow{CD}| = \sqrt{16 + 4 + 16} = 6$

$\overrightarrow{DA} = 4\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

$|\overrightarrow{DA}| = \sqrt{16 + 16 + 4} = 6$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 16 - 8 - 8 = 0$

$\overrightarrow{BC} \cdot \overrightarrow{CD} = -16 + 8 + 8 = 0$

$ABCD$ is a square.

c Vector normal to the plane is

$\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$

Equation of line $\boldsymbol{r}(t) =$

$8\boldsymbol{j} + 5\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) \quad t \in \mathbb{R}$

41 a For π_1, $\boldsymbol{n}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

For π_2, $\boldsymbol{n}_2 = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

Angle between Π_1 & Π_2 is angle between $\boldsymbol{n}_1$ & $\boldsymbol{n}_2$

Now $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2| \cos\theta$

$\therefore \cos\theta = \dfrac{1}{\sqrt{51}}$

b For Π_1, $\boldsymbol{n}_1 = 4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

For Π_2, $\boldsymbol{n}_2 = 2\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$

Now $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 2$

and $|\boldsymbol{n}_1| = \sqrt{29}$, $|\boldsymbol{n}_2| = \sqrt{29}$

$\therefore 2 = \sqrt{29} \cdot \sqrt{29} \cdot \cos\theta$

$\therefore \cos\theta = \dfrac{2}{\sqrt{29}}$

42 a $\overrightarrow{AB} = -2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$

Therefore vector equation of line:

$\boldsymbol{r}(\lambda) = 3\boldsymbol{i} + 5\boldsymbol{j} + 9\boldsymbol{k} + \lambda(-2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$

b $x = 3 - 2\lambda, y = 5 + 4\lambda, z = 9 + \lambda$

Therefore,

$\dfrac{x - 3}{-2} = \dfrac{y - 5}{4} = z - 9$

c $x = 3 - 2\lambda, y = 5 + 4\lambda, z = 9 + \lambda$

Complex numbers

43 $z^4 - z^2 - 12 = 0$

$(z^2 - 4)(z^2 + 3) = 0$

$z^2 = 4$ or $z^2 = -3$

$z = \pm 2, \pm\sqrt{3}i$

44 $z = \dfrac{\sqrt{3} - i}{1 - i}$

$z = \dfrac{2\text{cis}\left(-\frac{\pi}{6}\right)}{\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)} = \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$

$\therefore \text{Arg } z = \dfrac{\pi}{12}$

45 $P(z) = z^3 - 6z^3 - 2z^2 + 17z - 10$

$P(1) = 0 \Rightarrow z - 1$ is a factor.

$P(2) = 0 \Rightarrow z - 2$ is a factor.

$\therefore P(z) = (z - 1)(z - 2)Q(z) =$
$(z^2 - 3z + 2)Q(z)$ where $Q(z)$ is a cubic factor.

By inspection or long division or by using the method of equating coefficients

$Q(z) = z^3 + 3z^2 + z - 5$

Also $Q(z) = (z - 1)(z^2 + 4z + 5) =$
$(z - 1)((z + 2)^2 + 1)$

Hence

$P(z) = 0$

$\Leftrightarrow (z - 1)^2(z - 2)((z + 2)^2 + 1) = 0$

$\Leftrightarrow z = 1$ or $z = 2$ or $z = -2 \pm i$

46 **a** $z^3 - 2z^2 + 2z - 1 = (z - 1)(z^2 - z + 1)$

$z^3 - 2z^2 + 2z - 1 = 0$

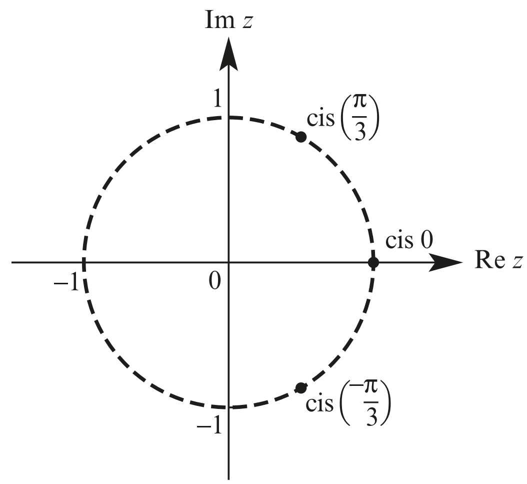
$(z - 1)(z^2 - z + 1) = 0$

$z = 1$ or $z^2 - z + 1 = 0$

$z = 1$ or $z = \dfrac{1 \pm \sqrt{3}i}{2}$

b $z = \text{cis}0,\ \text{cis}\left(\frac{\pi}{3}\right),\ \text{cis}\left(-\frac{\pi}{3}\right)$

c



47 $\dfrac{1}{\sqrt{2}}(1 + i)(-1 + 3i) = \dfrac{1}{\sqrt{2}}(-4 + 2i)$

$= -2\sqrt{2} + \sqrt{2}i$

The point has coordinates:

$(-2\sqrt{2}, \sqrt{2})$

48 $\dfrac{\cos(2\theta) + i\sin(2\theta)}{\cos(3\theta) + i\sin(3\theta)} = \dfrac{\text{cis}(2\theta)}{\text{cis}(3\theta)}$

$= \text{cis}(2\theta - 3\theta)$

$= \text{cis}(-\theta)$

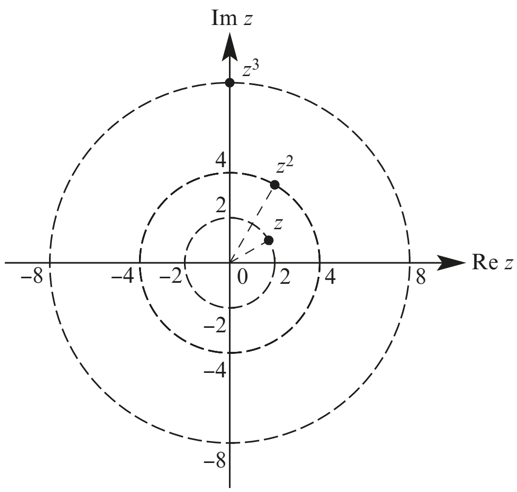
$= \cos(-\theta) + i\sin(-\theta)$

$= \cos(\theta) - i\sin(\theta).$

49 $z = \sqrt{3} + i = 2\text{cis}\left(\frac{\pi}{6}\right)$

$z^2 = 4\text{cis}\left(\frac{\pi}{3}\right)$

$z^3 = 8\text{cis}\left(\frac{\pi}{2}\right)$



50 a

$f(1+i)$

$= (1+i)^3 - (5+i)(1+i)^2 + (17+4i)(1+i) - 13 - 13i$

Now $(1+i)^2 = 2i$

$(1+i)^3 = -2 + 2i$

$f(1+i) = -2 + 2i - 2i(5+i) + (17+4i)(1+i) - 13 - 13i$

$= 0$

$\therefore z - 1 - i$ is a factor.

b $f(z) = (z - 1 - i)(z + az + b)$ for some a, b

We can see that

$b = \dfrac{-13-13i}{-1-i} = 13 \times \dfrac{1+i}{1+i} = 13$

Similarly $a = -4$

Finally,

$$
\begin{aligned}
f(z) &= (z-1-i)(z^2 - 4z + 13) \\
&= (z-1-i)(z^2 - 4z + 4 + 9) \\
&= (z-1-i)((z-2)^2 - (3i)^2) \\
&= (z-1-i)(z-2-3i)(z-2+3i)
\end{aligned}
$$

51 a $f(z) = z^2 + aiz + b, a \neq 0$

$z^2 + aiz + b = 0$

$$
\begin{aligned}
z &= \frac{-ai \pm \sqrt{a^2i^2 - 4b}}{2} \\
&= \frac{-ai \pm \sqrt{-a^2 - 4b}}{2} \\
&= \frac{-ai \pm i\sqrt{a^2 + 4b}}{2}
\end{aligned}
$$

Imaginary solutions $b \geq -\dfrac{a^2}{4}$

b **i** $z^2 + 2iz + 1 = 0$

$a = 2, b = 1$

$\therefore z = \dfrac{-2i \pm i\sqrt{2^2+4}}{2} = (-1 \pm \sqrt{2})i$

ii $z^2 - 2iz - 1 = 0$

$a = -2, b = -1$

$\therefore z = \dfrac{2i \pm i\sqrt{(-2)^2 - 4}}{2} = i$

iii $z^2 + 2iz - 2 = 0$

$a = 2, b = -2$

$\therefore z = \dfrac{-2i \pm i\sqrt{-4}i}{2} = \pm 1 - i$

52 a

$$
\begin{aligned}
f(z) &= (z+1)(z+(1-i))(z+(1+i)) \\
&= (z+1)(z^2 + z(1 - i + 1 + i) + 2) \\
&= (z+1)(z^2 + 2z + 2) \\
&= z^3 + 2z^2 + 2z + z^2 + 2z + 2 \\
&= z^3 + 3z^2 + 4z + 2
\end{aligned}
$$

Therefore $a = 3, b = 4$ and $c = 2$.

b The solutions have the same modulus and there Arguments differ by $\dfrac{2\pi}{3}$

The two given solutions are $2\text{cis}\left(\dfrac{\pi}{6}\right)$ and $2\text{cis}\left(-\dfrac{\pi}{2}\right)$. The third solution is $2\text{cis}\left(\dfrac{5\pi}{6}\right) = -\sqrt{3} + i$

53 $z^5 = 1 + i$

$(r\text{cis}\,\theta)^5 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$

$r^5\text{cis}\,(5\theta) = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$

$r = 2^{\frac{1}{10}}$

$5\theta = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$

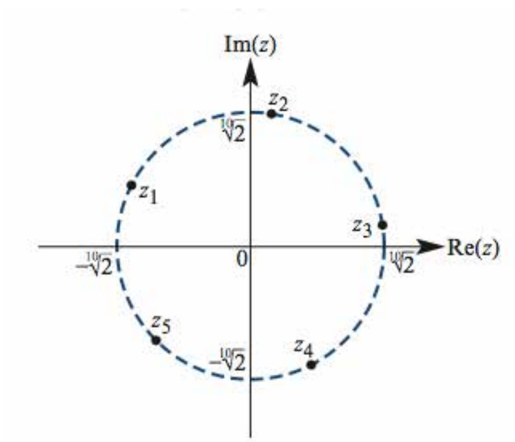
$\theta = \frac{\pi}{20} + \frac{2k\pi}{5}$

Therefore solutions are:

$z_1 = 2^{\frac{1}{10}}\text{cis}\,\frac{17\pi}{20}, \quad z_2 = 2^{\frac{1}{10}}\text{cis}\,\frac{9\pi}{20},$

$z_3 = 2^{\frac{1}{10}}\text{cis}\,\frac{\pi}{20}, \quad z_4 = 2^{\frac{1}{10}}\text{cis}\,\frac{-7\pi}{20}$

$, \quad z_5 = 2^{\frac{1}{10}}\text{cis}\,\frac{-15\pi}{20}$



54 a $z_1 = \sqrt{3} - i = 2(\frac{\sqrt{3}}{2} - \frac{1}{2}i) = 2\text{cis}\left(-\frac{\pi}{6}\right)$

$z_1 = -1 - i = \sqrt{2}(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) = \sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$

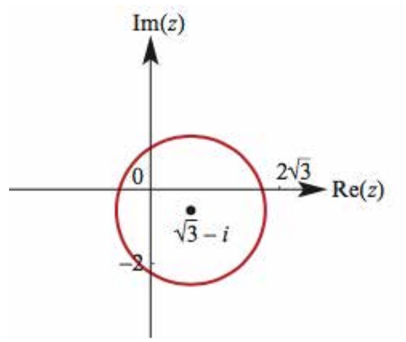
b $\frac{z_1}{z_2} = \frac{2\text{cis}\left(-\frac{\pi}{6}\right)}{\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)} = \sqrt{2}\text{cis}\,\frac{7\pi}{12}$

c $\sqrt{2}\text{cis}\,\frac{-7\pi}{12}$

d i $|z - (\sqrt{3} - i)| = 2$ is a circle of radius 2 and centre at $\sqrt{3} - i$. Cartesian equation is

$$(x - \sqrt{3})^2 + (y + 1)^2 = 16$$

The axis intercepts are at
$(\sqrt{3} + \sqrt{15}, 0)$,
$(\sqrt{3} - \sqrt{15}, 0)$,
$(0, -1 + \sqrt{13}), (0, -1 + \sqrt{13})$



ii $|z - (\sqrt{3} - i)| = 2$ is a circle. The question should be $\text{Arg}(z - z_2) = \frac{\pi}{4}$. Graph is as in the answers to the chapter

55 a i $2\text{cis}\left(\frac{\pi}{3}\right)$

ii $2^n\text{cis}\left(\frac{n\pi}{3}\right)$

iii Multiples of 3

iv None

b $z_1^2 = (1 + \sqrt{3}i)(1 + \sqrt{3}i)$
$= 1 + 2\sqrt{3}i - 3$
$= -2 + 2\sqrt{3}i$
$z_1^3 = (1 + \sqrt{3}i)(-2 + 2\sqrt{3}i)$
$= -2 - 2\sqrt{3}i + 2\sqrt{3}i - 6$
$= -8$

c Let$P(z) = 2z^3 + az^2 + bz + 20$

$P(1 + \sqrt{3}i)$

$= 2 \times (-8) + a(-2 + 2\sqrt{3}i) + b(1 + \sqrt{3}i) + 20$

$= -16 - 2a + b + 20 + (-2a\sqrt{3} + b\sqrt{3})i$

$P(1 + \sqrt{3}i) = 0$

$\Rightarrow 4 - 2a + b = 0 \ldots (1)$

and $2a\sqrt{3} + b\sqrt{3}) = 0 \ldots (2)$

From (2), $b = -2a$

Therefore substituting in (1)

$4 - 4a = 0 \Rightarrow a = 1$ and $b = -2$

d Coefficient of $P(z)$ are also real.

Therefore $1 - \sqrt{3}i$ is also a solution and

$$(z - (1 + \sqrt{3}i))(z + (1 + \sqrt{3}i)) = z^2 - 2z + 4$$

$P(z) = (z^2 - 2z + 4)(2z + 5)$

$\therefore$ solutions are $z = 1 + \sqrt{3}i.1 - \sqrt{3}i. - \dfrac{5}{2}$

56 a To go from A to B one can think of it as '2 to the left' and '4 down'.

AD is perpendiclar and of the same length.

So to get to D' 4 to the right and 2 down' from $A(-1 + 4i)$

D is the point $3 + 2i$

Similarly for C from B.

B is the point $1 - 2i$.

b We know tha the centre is at the midpoint of both AC and BD

Midpoint of AC =

$\frac{1}{2}(-1 + 4i + 1 - 2i) = i$

Solutions to multiple-choice questions

1 C When $t = 2$, $x = 2^2 + 2 = 6$ and
$y = 6 - 2^3 = -2$.
The point is $(6, -2)$

2 B
$x = 2\cos t$ and
$y = 2\cos 2t = 2(2\cos^2 t - 1)$
$\cos t = \frac{x}{2}$.
Hence $y = 2(2\left(\frac{x}{2}\right)^2 - 1)$
$= 4 \times \frac{x^2}{4} - 2$
$= x^2 - 2$
The cartesian equation is$y = x^2 - 2$

3 B
When $t = -\frac{\pi}{3}$, $x = 2\sec\left(-\frac{\pi}{3}\right) = 4$
$t = -\frac{\pi}{3}$, $y = 3\tan\left(-\frac{\pi}{3}\right) = -3\sqrt{3}$
The point is $(4, -3\sqrt{3})$

4 D $x = 2 \times 3^t + 1 \Rightarrow 3^t = \dfrac{x-1}{2}$
$\therefore y = 2 \times 3^{-2t} = \dfrac{2}{3^{2t}}$
$\therefore y = \dfrac{2}{\left(\frac{x-1}{2}\right)^2}$ $\therefore y = \dfrac{8}{(x-1)^2}$

5 C $x = t - 3 \Rightarrow t = x + 3$
$\therefore y = t^2 + 5$
$= (x + 3)^2 + 5$
$= x^2 + 6x + 14$

6 A Both $x = \sqrt{t}$ and $y = 4t + 2$ are increasing functions for $t \in [0, 4]$
When $t = 0$, $x = 0$ and $y = 2$
When $t = 4$, $x = 2$ and $y = 18$
Therefore, Domain $= [0, 2]$ and Range $= [2.18]$

7 A

8 E We remember that for statements A and B,
$A \Rightarrow B \Leftrightarrow \neg A \vee B$
and $\neg(\neg A \vee B) = A \wedge \neg B$

9 B $\mathbb{R}\backslash((-\infty, -2) \cup (2, \infty)) = [-2, 2]$

10 E

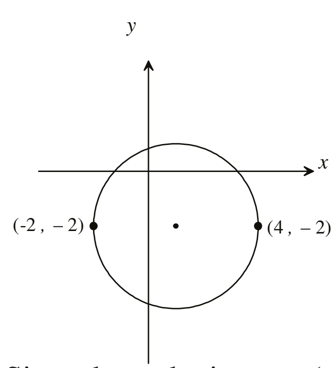
11 D $(1)^2 < (-2)^2$

12 E Take $n = 1$

13 E $41^2 - 41 + 41$ is clearly divisible by 41.

14 C $\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{16} = 1$
The equations of asymptotes of a hyperbola are given by the equation
$y = \pm\frac{b}{a}(x - h) + k$.
Here, $a = 3, b = 4, h = -1, k = 2$
Therefore the asymptotes are:
$y = \pm\frac{4}{3}(x + 1) + 2$
$y = \frac{4}{3}(x + 1) + 2$ and $y = -\frac{4}{3}(x + 1) + 2$
$\therefore$ $y = \frac{4}{3}x + \frac{10}{3}$ and $y = -\frac{4}{3}x + \frac{2}{3}$

15 D



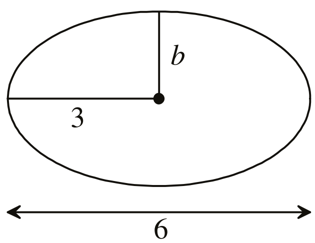
Since the endpoints are $(-2, -2)$ and $(4, -2)$, the diameter is 6 implying that the radius of the circle is 3.
Also, the centre of the circle is the

midpoint of the two endpoints.

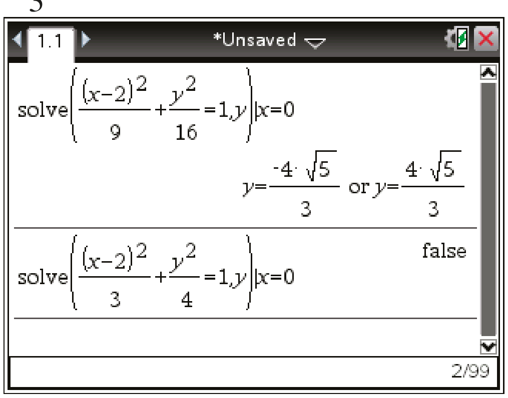
$$M = \left(\frac{-2+4}{2}, \frac{-2-2}{2}\right) = (1, -2)$$

Therefore the equation of the circle must have radius 3 and centre $(1, -2)$.

16 B Response A and C are both incorrect as the given graph does not have its centre at $(-2, 0)$.
Response E is also incorrect as represents the graph of a hyperbola.
Since the centre is on the x-axis the length of the major axes is 6 units.
Half of this is 3 which implies that $a = 3$.



For response D, $a = \sqrt{3}$. Thus response D is incorrect.
For response B, $a = 3$.
Also, by using a CAS calculator it is clear that response B has a y-intercept of $\frac{4\sqrt{5}}{3}$ and response B does not.



17 D $\frac{x^2}{9} + \frac{y^2}{25} = 1$

When $y = 0$:

$x^2 = 9$

$\therefore \quad x = \pm 3$

Therefore the x-axis intercepts are $(-3, 0)$ and $(3, 0)$.

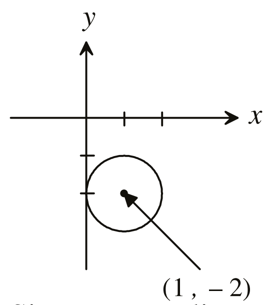
18 D $x^2 + y^2 - 6x + 8y = 0$

$\therefore \quad (x^2 - 6x + 9) + (y^2 + 8y + 16) = 25$

$\therefore \quad (x - 3)^2 + (y + 4)^2 = 25$

Centre is $(3, -4)$

19 E $(x - 1)^2 + (y + 2)^2 = 1$



Since tangent lines are of the form $x = k$, from the sketch above this implies that $x = 0$ or $x = 2$.

20 B $x^2 - 2x = y^2$

$\therefore \quad (x^2 - 2x + 1) - y^2 = 1$

$\therefore \quad (x - 1)^2 - y^2 = 1$

i.e. a hyperbola with centre $(1, 0)$

21 C $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = -3\mathbf{j} + 4\mathbf{k}$

$$\begin{aligned}\mathbf{a} - 2\mathbf{b} - \mathbf{c} &= (2, 3, -4) \\ &\quad - (-2, 4, -4) \\ &\quad - (0, -3, 4) \\ &= (4, 2, -4) \\ &= 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\end{aligned}$$

22 D Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$\therefore \quad \hat{\mathbf{u}} = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

Opposite direction:

$\Rightarrow \frac{1}{3}(-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

Magnitude of 6:

$\Rightarrow \frac{6}{3}(-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

$\therefore \quad -2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

Therefore $-2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$ is a vector of magnitude 6 and with direction opposite to $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

23 B $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}, \mathbf{b} = -2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

$|\mathbf{b}| = 7$

$\mathbf{a} \cdot \mathbf{b} = -4 - 9 + 6 = -7$

$$\therefore \quad (\mathbf{a} \cdot \hat{\mathbf{b}}) \cdot \hat{\mathbf{b}} = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}|^2}\mathbf{b}$$

$$= \frac{-7}{49}\mathbf{b}$$

$$= -\frac{1}{7}\mathbf{b}$$

$$= -\frac{1}{7}(-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$$

$$= \frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$$

24 A $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

By inspection, response A is in the form $k\mathbf{a}$ and is therefore a vector that is parallel to $\mathbf{a}$. Hence response A is a vector which is not perpendicular to $\mathbf{a}$.

25 E $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$$\therefore \quad |\mathbf{a}| = \sqrt{1^2 + (-3)^2 + 5^2} = \sqrt{35}$$

26 D $\mathbf{u} = 2\mathbf{i} - \sqrt{2}\mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} + \sqrt{2}\mathbf{j} - \mathbf{k}$

$|\mathbf{u}| = \sqrt{7}$ and $|\mathbf{v}| = 2$

$\mathbf{u} \cdot \mathbf{v} = 2 - 2 - 1 = -1$

$$\therefore \quad \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

$$\therefore \quad \cos\theta = -\frac{1}{2\sqrt{7}}$$

$$\therefore \quad \theta = \cos^{-1}\left(-\frac{1}{2\sqrt{7}}\right)$$

$$\therefore \quad \theta = 100.89^\circ$$

27 C $\mathbf{u} = 2\mathbf{i} - a\mathbf{j} - \mathbf{k}, \mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - b\mathbf{k}$

$\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other when $\mathbf{u} \cdot \mathbf{v} = 0$

$\therefore \quad 6 - 2a + b = 0$

$\therefore \quad b = 2a - 6$

So, for $\mathbf{u}$ and $\mathbf{v}$ to be perpendicular to each other the equation $b = 2a - 6$ must be satisfied.

Testing all responses with this equation reveals that C is the correct response.

28 E $\mathbf{u} = \mathbf{i} + a\mathbf{j} - 4\mathbf{k}, \mathbf{v} = b\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\mathbf{u}$ and $\mathbf{v}$ are parallel to each other when $\mathbf{u} = k\mathbf{v}$, where $k \in R\backslash\{0\}$

$\therefore \quad (1, a, -4) = k(b, -2, 3)$

$\Rightarrow 1 = kb \quad (1)$

$a = -2k \quad (2)$

$-4 = 3k \quad (3)$

From (3), $k = -\frac{4}{3}$

$\therefore \quad a = \frac{8}{3}$ and $b = -\frac{3}{4}$

29 A $\mathbf{a} = \mathbf{i} - 5\mathbf{j} + \mathbf{k}, \mathbf{b} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

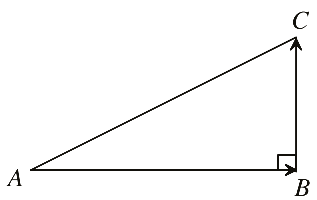
$|\mathbf{b}| = 3$

$\mathbf{a} \cdot \mathbf{b} = 2 + 5 + 2 = 9$

Perpendicular component is

$$\mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$
$$= \mathbf{a} - \left(\frac{9}{9}\right)\mathbf{b}$$
$$= \mathbf{a} - \mathbf{b}$$
$$= (1, -5, 1) - (2, -1, 2)$$
$$= (-1, -4, -1)$$
$$= -\mathbf{i} - 4\mathbf{j} - \mathbf{k}$$

30 C $\overrightarrow{AB}\cdot\overrightarrow{BC} = 0$



For this situation to be true the vector resolute of $\overrightarrow{AC}$ in the direction of $\overrightarrow{AB}$ must be $\overrightarrow{AB}$.

31 C $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{v} = 4\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{i} - \mathbf{j} - \mathbf{k})\cdot(4\mathbf{i} + 12\mathbf{j} - 3\mathbf{k})$$
$$= 1 \times 4 + -1 \times 12 + -1 \times -3$$
$$= 4 - 12 + 3$$
$$= -5$$

32 B $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

The scalar resolute is $\mathbf{a}\cdot\hat{\mathbf{b}}$

$$\therefore\ \mathbf{a}\cdot\hat{\mathbf{b}} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot\frac{1}{7}(6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$$
$$= \frac{1}{7}(18 - 6 - 2)$$
$$= \frac{10}{7}$$

33 C $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

$$\mathbf{a} - \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

Let $\mathbf{u} = \mathbf{a} - \mathbf{b}$

Then,

$$\hat{\mathbf{u}} = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

Thus a unit vector in the direction of $\mathbf{a} - \mathbf{b}$ is $\frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

34 C $\overrightarrow{OP} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \overrightarrow{OQ} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{OR} = 2\boldsymbol{i} + p\boldsymbol{j} + q\boldsymbol{k}$

Since P, Q and R are collinear then $PQ = nQR$, where $n \in R\backslash\{0\}$.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$$
$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \boldsymbol{i} + (p + 2)\boldsymbol{j} + (q - 1)\boldsymbol{k}$$

Applying $PQ = nQR$, we have

$-2 = n$ ①

$-3 = n(p + 2)$ ②

$2 = n(q - 1)$ ③

Substituting ① into ② gives

$-3 = -2(p + 2)$

$\therefore\ \frac{3}{2} = p + 2$

$\therefore\ p = -\frac{1}{2}$

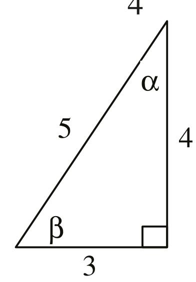
Substituting ① into ③ gives

$2 = -2(q - 1)$

$\therefore\ -1 = q - 1$

$\therefore\ q = 0$

35 E $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{4}{3}$



From the above triangle:

$\sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}, \sin\beta = \frac{4}{5}$ and $\cos\beta = \frac{3}{5}$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \frac{3}{5}\times\frac{3}{5}+\frac{4}{5}\times\frac{4}{5}\\ &= \frac{9}{25}+\frac{16}{25}\\ &= \frac{25}{25}\\ &= 1\end{aligned}$$

36 C $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{b} = 2\mathbf{i} - \mathbf{j},\ \mathbf{x} = \mathbf{i} + 5\mathbf{j}$

and $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$

$\therefore\ \mathbf{i} + 5\mathbf{j} = (3s + 2t)\mathbf{i} + (4s - t)\mathbf{j}$

Equating components gives:

$3s + 2t = 1$ ①

$4s - t = 5$ ②

$2\times$② + ①:

$11s = 11$

$\therefore\ s = 1$

Substituting $s = 1$ into ① gives

$2t = -2$

$\therefore\ t = -1$

37 C $\overrightarrow{OP} = p, \overrightarrow{OQ} = q$ and O, P and Q are not collinear.

Let R be the vector for each response.

We are required to find the response that is **not** collinear with P and Q.

i.e. there is **no** such k such that

$PQ = kQR$

For response A:

$$\begin{aligned}\mathbf{q}-\mathbf{p} &= k\left(\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{q}-\mathbf{q}\right)\\ &= k\left(\frac{1}{2}\mathbf{p}-\frac{1}{2}\mathbf{q}\right)\\ &= \frac{k}{2}(\mathbf{p}-\mathbf{q})\end{aligned}$$

$\therefore\ k = -2$

For response B:

$$\begin{aligned}\mathbf{q}-\mathbf{p} &= k(3\mathbf{p}-2\mathbf{q}-\mathbf{q})\\ &= k(3\mathbf{p}-3\mathbf{q})\\ &= 3k(\mathbf{p}-\mathbf{q})\end{aligned}$$

$\therefore\ k = -\frac{1}{3}$

For response C:

$$\begin{aligned}\mathbf{q}-\mathbf{p} &= k(\mathbf{p}-\mathbf{q}-\mathbf{q})\\ &= k(\mathbf{p}-2\mathbf{q})\end{aligned}$$

No k exists

For response D:

$$\begin{aligned}\mathbf{q}-\mathbf{p} &= k\left(\frac{1}{3}\mathbf{p}+\frac{2}{3}\mathbf{q}-\mathbf{q}\right)\\ &= k\left(\frac{1}{3}\mathbf{p}-\frac{1}{3}\mathbf{q}\right)\\ &= \frac{k}{3}(\mathbf{p}-\mathbf{q})\end{aligned}$$

$\therefore\ k = -3$

For response E:

$$\begin{aligned}\mathbf{q}-\mathbf{p} &= k(2\mathbf{p}-\mathbf{q}-\mathbf{q})\\ &= k(2\mathbf{p}-2\mathbf{q})\\ &= 2k(\mathbf{p}-\mathbf{q})\end{aligned}$$

$\therefore\ k = -\frac{1}{2}$

Response C contains the vector that is not collinear with P and Q.

38 B $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$

$\therefore$ (B) is the answer as the line must include $\boldsymbol{a}$

39 B $\boldsymbol{r}_1 = (9 + 3\lambda)\boldsymbol{i} + (-2 - \lambda)\boldsymbol{i}$,

$\boldsymbol{s} = (3 + 3\mu)\boldsymbol{i} + (-2 + \mu)\boldsymbol{j}$

At point of intersection

$$\begin{cases}9 + 3\lambda = 3 + 3\mu\\ -2 - \lambda = -2 + \mu\end{cases}$$

$\therefore \mu = -\lambda$ and $\lambda = -1$

$\therefore$ At point of intersection $\boldsymbol{r}_1 = 6\boldsymbol{i} - \boldsymbol{j}$

$\therefore$ point of intersection = $(6, -1)$

40 D

Plane has equation $x - y + z = 2$

For $(1, -1, 1)$ $\quad 1 + 1 + 1 = 3$

$\therefore$ Not on plane

For $(-1, 1, 0)$ $\quad -1 - 1 + 0 = -2$

$\therefore$ Not on plane

For $(0, 1, 1)$ $\quad 1 - 1 + 1 = 0$

$\therefore$ Not on plane

For $(2, 0, 0)$ $\quad 2 - 0 + 0 = 2$

$\therefore$ On plane

41 E A False $\boldsymbol{d} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

$\therefore$ Not parallel to $-2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$

B False $(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) \cdot (2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) \neq 0$

C $\boldsymbol{n} = (-1 + 2t)\boldsymbol{i} + (-3 + t)\boldsymbol{j} + (-3 + 3t)\boldsymbol{k}$

$x = -2 \quad \therefore -1 + 2t = -2$

$\therefore 2t = -1 \quad \therefore t = \frac{1}{2}$

But $\quad -3 + \frac{1}{2} \neq -3 \quad \therefore$ False

D $\quad -1 + 2t = 0 = x \quad \therefore t = \frac{1}{2}$

$y = -3 + t \neq 0 \quad \therefore$ False

42 C $\therefore |\boldsymbol{u}|\,|\boldsymbol{v}| \cdot \cos\theta = |\boldsymbol{u}| \cdot |\boldsymbol{v}| \cdot \sin\theta$

$$\therefore \cos\theta = \sin\theta$$

$$\therefore \theta = \frac{\pi}{4}$$

43 E $\cos^2\theta + 3\sin^2\theta$

$$= \cos^2\theta + 3(1 - \cos^2\theta)$$

$$= 3 - 2\cos^2\theta$$

$$= 3 - 2\left(\frac{1}{2}(\cos 2\theta + 1)\right)$$

$$= 3 - \cos 2\theta - 1$$

$$= 2 - \cos 2\theta$$

You may also sketch the graph of $\cos^2\theta + 3\sin^2\theta$ against the graph of all of the responses to confirm this answer.

44 C

We know that $\boldsymbol{d_1}$ is parallel to $\boldsymbol{d_2}$ and $\boldsymbol{a}$ cannot be the zero vector as the line does not pass through the origin.

45 E $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

The restricted domain of $\sin x$ and $\cos x$ are $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively.

$$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{6} - \left(-\frac{\pi}{3}\right)$$

$$= \frac{5\pi}{6} + \frac{\pi}{3}$$

$$= \frac{7\pi}{6}$$

46 B $\overrightarrow{PQ} = 2\overrightarrow{QR}, \overrightarrow{OQ} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{OR} = \boldsymbol{i} + 3\boldsymbol{j}$

Let P be the position vector $x\mathbf{i} + y\mathbf{j}$

$\overrightarrow{QR} = -2\boldsymbol{i} + 5\boldsymbol{j}$ and

$\overrightarrow{PQ} = (3 - x)\boldsymbol{i} - (2 + y)\boldsymbol{j}$

Since $PQ = 2QR$ and PQR is a straight line

$3 - x = -4 \quad$ and $\quad 2 + y = -10$

$\therefore \quad x = 7 \quad$ and $\quad y = -12$

$\therefore \quad \overrightarrow{OP} = 7\boldsymbol{i} - 12\boldsymbol{j}$

47 E $\overrightarrow{OP} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}, \overrightarrow{PQ} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = 4\boldsymbol{i}$$

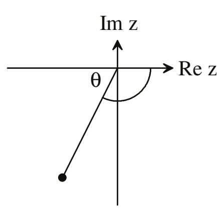
$$\therefore \quad \overrightarrow{OQ} = \sqrt{4^2} = 4$$

48 B $z_1 = 2 - i, z_2 = 3 + 4i$

$$\frac{z_2}{z_1} = \frac{3 + 4i}{2 - i}$$
$$= \frac{(3 + 4i)(2 + i)}{2^2 + 1^2}$$
$$= \frac{2 + 11i}{5}$$
$$= \frac{2}{5} + \frac{11}{5}i$$
$$\therefore \quad \left|\frac{z_2}{z_1}\right|^2 = \left[\sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{11}{5}\right)^2}\right]^2$$
$$= \frac{4}{25} + \frac{121}{25}$$
$$= \frac{125}{25}$$
$$= 5$$

Or using CAS

49 A $z = -1 - i\sqrt{3}$



$$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$
$$\therefore \quad \text{Arg}(z) = -\frac{\pi}{2} - \frac{\pi}{6} = -\frac{2\pi}{3}$$

Or using CAS

50 C $(p\mathbf{i} + 2\mathbf{j} - 3p\mathbf{k}) \cdot (\rho\mathbf{i} + \mathbf{k}) = p^2 - 3p$

For the two vectors to be perpendicular to each other the dot product of the two vectors must equal zero.

$$\therefore \quad p^2 - 3p = 0$$
$$\therefore \quad p(p - 3) = 0$$
$$\therefore \quad p = 0 \text{ or } p = 3$$

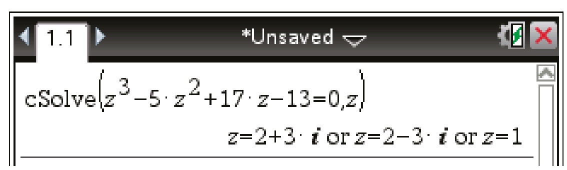
51 B $2 + 3i$ is a root and since all the coefficients of the polynomial are real this implies that $2 - 3i$ is also a root.

$$\therefore \quad (z - 2 - 3i)(z - 2 + 3i)$$
$$= z^2 - 2z + 3zi - 2z$$
$$+ 4 - 6i - 3zi + 6i + 9$$
$$= z^2 - 4z + 13$$

$$\begin{array}{r l}
 & z - 1 \\
z^2 - 4z + 13 \,\big) & z^3 - 5z^2 + 17z - 13 \\
 & z^3 - 4z^2 + 13z \\
 & \quad -z^2 + 4z - 13 \\
 & \quad -z^2 + 4z - 13 \\
 & \qquad\qquad 0
\end{array}$$

Therefore the other two roots are $2 - 3i$ and 1

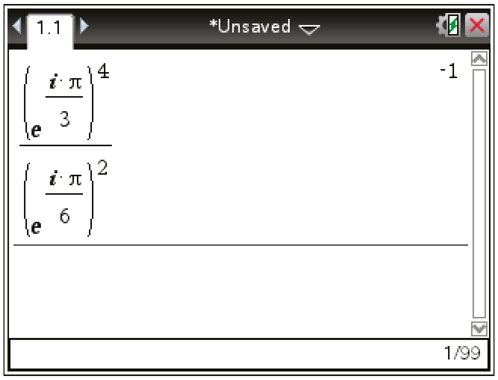
Or using the TI-nspire CAS calculator



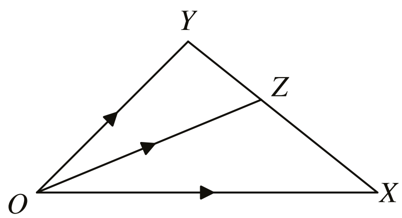
52 A $\dfrac{(\cos 60° + i\sin 60°)^4}{(\cos 30° + i\sin 30°)^2}$

$$= \frac{(\text{cis } 60°)^4}{(\text{cis } 30°)^2}$$
$$= \frac{\text{cis}(4 \times 60°)}{\text{cis}(2 \times 30°)}$$
$$= \frac{\text{cis } 240°}{\text{cis } 60°}$$
$$= \text{cis}(240° - 60°)$$
$$= \text{cis } 180°$$
$$= -1$$

Or using CAS



53 D $3\overrightarrow{OX} + 4\overrightarrow{OY} = 7\overrightarrow{OZ}$



$\overrightarrow{XZ} = \overrightarrow{OZ} - \overrightarrow{OX}$

$= \frac{1}{7}(3\overrightarrow{OX} + 4\overrightarrow{OY}) - \overrightarrow{OX}$

$= \frac{1}{7}(4\overrightarrow{OY} - 4\overrightarrow{OX})$

$= \frac{1}{7}(\overrightarrow{OY} - \overrightarrow{OX})$

$\overrightarrow{ZY} = \overrightarrow{OY} - \overrightarrow{OZ}$

$= \overrightarrow{OY} - \frac{1}{7} - (3\overrightarrow{OX} + 4\overrightarrow{OY})$

$= \frac{3}{7}\overrightarrow{OY} - \frac{3}{7}\overrightarrow{OX}$

$= \frac{3}{7}(\overrightarrow{OY} - \overrightarrow{OX})$

Hence $\overrightarrow{XZ} = \frac{4}{3}\overrightarrow{ZY}$, so that

$|\overrightarrow{XZ}| = \frac{4}{3}|\overrightarrow{ZY}|$

i.e. $\dfrac{\overrightarrow{XZ}}{\overrightarrow{ZY}} = \dfrac{4}{3}$

54 C $\tan\frac{\pi}{4} = 1$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \quad \cos\left(\tan^{-1}(1) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$

$= \cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$

$= \cos\left(\frac{\pi}{2}\right)$

$= 0$

55 A Let $x + iy = \dfrac{1}{3 + 4i}$

$\dfrac{1}{3 + 4i} = \dfrac{3 - 4i}{3^2 + 4^2}$

$\therefore \quad \dfrac{1}{3 + 4i} = \dfrac{3}{25} - \dfrac{4}{25}i$

$\therefore \quad x = \dfrac{3}{25}$ and $y = -\dfrac{4}{45}$

56 B $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \mathbf{b} = \mathbf{i} + p\mathbf{j} + \mathbf{k}$

If **a** and **b** are perpendicular to each other then $\mathbf{a} \cdot \mathbf{b} = 0$

$\mathbf{a} \cdot \mathbf{b} = 2 + 3p + 4 = 3p + 6$

$\Rightarrow 3p + 6 = 0$

$\therefore \quad p = -2$

57 E $z = \dfrac{1}{1 - i}$

$= \dfrac{\operatorname{cis} 0}{\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}$

$= \dfrac{1}{\sqrt{2}}\operatorname{cis}\left(0 - \left(-\frac{\pi}{4}\right)\right)$

$= \dfrac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{\pi}{4}\right)$

$\therefore \quad r = \dfrac{1}{\sqrt{2}}$ and $\theta = \dfrac{\pi}{4}$

58 D The restricted domain and range of $\sin x$ are $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[-1, 1]$ respectively.

Thus the restricted domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore $\sin^{-1}(2x - 1)$ is defined when

$$-1 \le 2x - 1 \le 1$$
$$\therefore \quad 0 \le 2x \le 2$$
$$\therefore \quad 0 \le x \le 1$$

59 E $u = 3\operatorname{cis}\left(\frac{\pi}{4}\right), v = 2\operatorname{cis}\left(\frac{\pi}{2}\right)$

$$\therefore \quad uv = (3 \times 2)\operatorname{cis}\left(\frac{\pi}{4} + \frac{\pi}{2}\right)$$
$$= 6\operatorname{cis}\left(\frac{3\pi}{4}\right)$$

60 A The restricted domain of $\cos x$ is $[0, \pi]$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$
$$\therefore \quad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$
$$\therefore \quad \sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left(\frac{2\pi}{3}\right)$$
$$= \sin\left(\pi - \frac{\pi}{3}\right)$$
$$= \sin\left(\frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2}$$

61 A Start by considering the right angle triangle and arcsin(x) as an angle, and with hypotenuse of length 1 and opposite side x. Then the adjacent sside has length $\sqrt{1 - x^2}$

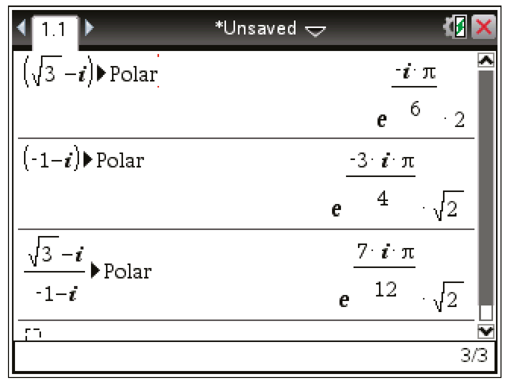
62 C $|12 - 5i| = \sqrt{12^2 + (-5)^2}$
$= \sqrt{169} = 13$

63 C $\sqrt{3} - j = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$

$$-1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

$$\therefore \quad \frac{\sqrt{3} - i}{-1 - i} = \frac{2\operatorname{cis}\left(-\frac{\pi}{6}\right)}{\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)}$$
$$\therefore \quad \frac{\sqrt{3} - i}{-1 - i} = \frac{2}{\sqrt{2}}\operatorname{cis}\left(-\frac{\pi}{6} + \frac{3\pi}{4}\right)$$
$$= \sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$$

Or using CAS



64 A $|z + 3| = |z + 4i + 3 - 4i|$

$$\le |z + 4i| + |3 - 4i|$$
$$= 3 + 5$$
$$= 8$$

Maximum value is 8

$$|z + 3| = |z + 4i + 3 - 4i|$$
$$\ge ||z + 4i| - |3 - 4i||$$
$$= |3 - 5|$$
$$= 2$$

Minimum value is 2.

65 C Conjugate root theorem makes C impossible.

66 B $1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

$$\sqrt{3} + j = 2\operatorname{cis}\left(\frac{\pi}{6}\right)$$

$\therefore \quad \dfrac{1-i}{\sqrt{2}} \times \dfrac{\sqrt{3}+i}{2}$

$$= \frac{\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}{\sqrt{2}} \times \frac{2\operatorname{cis}\left(\frac{\pi}{6}\right)}{2}$$

$$= \operatorname{cis}\left(-\frac{\pi}{4}\right) \times \operatorname{cis}\left(\frac{\pi}{6}\right)$$

$$= \operatorname{cis}\left(-\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= \operatorname{cis}\left(-\frac{\pi}{12}\right)$$

67 C $\tan\theta = \dfrac{1}{3}$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2 \times \frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}$$

$$= \frac{2}{3} \times \frac{9}{8}$$

$$= \frac{3}{4}$$

68 E

$\cos^4\theta - \sin^4\theta$

$$= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos 2\theta$$

Response A is identical to response C.

Recall that $\cos(2k\theta) = 2\cos^2(k\theta) - 1$

Putting $k = \dfrac{1}{2}$ gives:

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$\therefore \quad 1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$$

Response D is identical to B.

Therefore response E is not identical to any of the others.

69 C $1 + \cos 2\theta + i\sin 2\theta, 0 < \theta < \dfrac{\pi}{2}$

$1 + \cos 2\theta + i\sin 2\theta$

$$= 1 + 2\cos^2\theta - 1 + i\,2\sin\theta\cos\theta$$

$$= 2\cos^2\theta + i\,2\sin\theta\cos\theta$$

$\therefore \quad |1 + \cos 2\theta + i\sin 2\theta|$

$$= \sqrt{(2\cos^2\theta)^2 + (2\sin\theta\cos\theta)^2}$$

$$= \sqrt{4\cos^4\theta + 4\sin^2\theta\cos^2\theta}$$

$$= \sqrt{4\cos^2\theta(\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{4\cos^2\theta}$$

$$= 2\cos\theta, \text{ since } \cos\theta > 0$$

70 D Let $z = 1 + \cos\theta + i\sin\theta$

$$|z| = \sqrt{(1+\cos\theta)^2 + \sin^2\theta}$$

$$= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \sqrt{2 + 2\cos\theta}$$

$$= \sqrt{2(1+\cos\theta)}$$

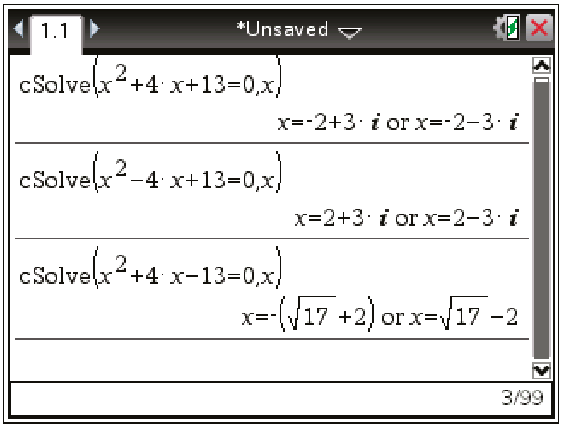
From question **59**:

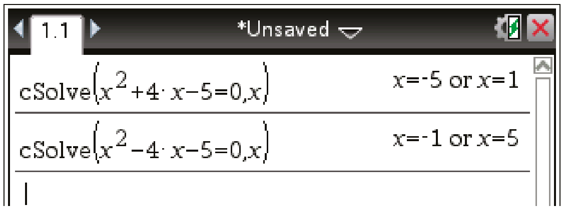
$$2\cos^2\left(\frac{\theta}{2}\right) = 1 + \cos\theta$$

$$\therefore \quad |z| = \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}$$

$$\therefore \quad |z| = 2\cos\left(\frac{\theta}{2}\right)$$

71 B Using a CAS calculator for each response we have:





Therefore response B is the quadratic with roots $2 + 3i$ and $2 - 3i$.

72 A $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1} x$

$$\therefore \quad x = \tan\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$$

$$= \frac{\tan\left(\tan^{-1}\frac{1}{2}\right) + \tan\left(\tan^{-1}\frac{1}{3}\right)}{1 - \tan\left(\tan^{-1}\frac{1}{2}\right)\tan\left(\tan^{-1}\frac{1}{3}\right)}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}}$$

$$\therefore \quad x = \frac{5}{6} \times \frac{6}{5} = 1$$

73 D $\operatorname{cosec}^2\theta - \cot^2\theta$

$= 1 + \cot^2\theta - \cot^2\theta = 1$

Response B is identical to response C.

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{\cos\theta\sin\theta}$$

$$= \frac{2}{2\cos x\sin x}$$

$$= \frac{2}{2\sin\theta}$$

$$= 2\operatorname{cosec} 2\theta$$

Response A is identical to response E.

Therefore response D is not identical to any of the others.

74 C $|z - 2| - |z + 2| = 0 \therefore x = 0$

75 B A relation in the form

$|z - (a + bi)| = r$

defines a circle with centre (a, b) and radius r.

$\therefore$ $|z - (2 - i)| = 6$ is a circle with centre $(2, -1)$ and radius 6.

76 D $z = x + iy$

The given graph appears to be the equation $y = -x$ (in Cartesian form).

So, using complex relations,

$\text{Im}(z) = -\text{Re}(z)$

$\therefore$ $\text{Im}(z) + \text{Re}(z) = 0$

77 C $|z - 2| - |z - 2i| = 0$

$\therefore$ $|z - 2| = |z - 2i|$

Let $z = x + iy$

$\therefore$ $|(x - 2) + iy| = |x + (y - 2)i|$

Applying the modulus

$\therefore$ $\sqrt{(x-2)^2 + y^2} = \sqrt{x^2 + (y-2)^2}$

Squaring both sides gives:

$(x - 2)^2 + y^2 = x^2 + (y - 2)^2$

$\therefore$ $x^2 - 4x + 4 + y^2 = x^2 + y^2 - 4y + 4$

$\therefore$ $y = x$

i.e. a straight line

78 E Response A

$|z| = 2$ is a circle with centre $(0, 0)$ and radius 2.

Response B

$|z - i| = 2$ is a circle with centre $(0, 1)$ and radius 2.

Response C

Let $z = x + iy$

$iz = ix - y$

$\therefore\ 2\text{Re}(iz) = -2y$

$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$

$\therefore\ z\bar{z} + 2\text{Re}(iz) = 0$ becomes

$x^2 + y^2 - 2y = 0$

$\therefore\ x^2 + (y^2 - 2y + 1) = 1$

$\therefore\ x^2 + (y - 1)^2 = 1$

Circle with centre $(0, 1)$ and radius 1.

Response D

$|z - 1| = 2$ is a circle with centre $(1, 0)$ and radius 2.

Response E

$|z| = 2i$

Let $z = x + iy$

$\therefore\ \sqrt{x^2 + y^2} = 2i$

$\therefore\ x^2 + y^2 = 4i^2$

$\therefore\ x^2 + y^2 = -4$

Not a circle

79 D Let $z = x + iy$

Response A

$\text{Im}(z) = 0 \Rightarrow y = 0$ i.e. a line

Response B

$\text{Im}(z) + \text{Re}(z) = 1$

$\Rightarrow y + x = 1$ i.e. a line

Response C

$z + \bar{z} = 4$

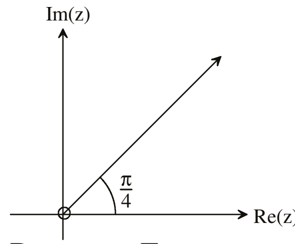
$\Rightarrow (x + iy) + (x - iy) = 4$

$\therefore\ 2x = 4$ i.e. a line

Response D

$\text{Arg}(z) = \frac{\pi}{4}$

Is a ray (not a line) starting from the point $(0, 0)$ following the direction $\frac{\pi}{4}$.



Response E

$\text{Re}(z) = \text{Im}(z)$

$\Rightarrow x = y$ i.e. a line

80 C $\overrightarrow{PQ} = 5\boldsymbol{i}, \overrightarrow{PR} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}, \overrightarrow{RM} = \lambda\boldsymbol{i}$

For the angle RQM to be a right angle $\overrightarrow{RQ} \cdot \overrightarrow{QM} = 0$

$$\begin{aligned}\overrightarrow{RQ} &= \overrightarrow{RP} + \overrightarrow{PQ} \\ &= -\mathbf{i} - \mathbf{j} - 2\mathbf{k} + 5\mathbf{i} \\ &= 4\mathbf{i} - \mathbf{j} - 2\mathbf{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{QM} &= \overrightarrow{QP} + \overrightarrow{PR} + \overrightarrow{RM} \\ &= -5\mathbf{i} + \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda\mathbf{i} \\ &= (\lambda - 4)\mathbf{i} + \mathbf{j} + 2\mathbf{k}\end{aligned}$$

$\overrightarrow{RQ} \cdot \overrightarrow{QM} = 4(\lambda - 4) - 1 - 4$

$\therefore\ 4(\lambda - 4) - 5 = 0$

$\therefore\ \lambda - 4 = \frac{5}{4}$

$\therefore\ \lambda = \frac{21}{4}$

81 D $\overrightarrow{OA} = 6\boldsymbol{i} - \boldsymbol{j} + 8\boldsymbol{k}, \overrightarrow{OB} = -3 + 4\boldsymbol{j} - 2\boldsymbol{k}$

and $AP{:}PB = 1{:}2$

$\Rightarrow \overrightarrow{BP} = \frac{2}{3}\overrightarrow{BA}$

$\overrightarrow{BA} = 9\boldsymbol{i} - 5\boldsymbol{j} + 10\boldsymbol{k}$

$\therefore \quad \overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$

$$= -3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \frac{2}{3}(9\mathbf{i} - 5\mathbf{j} + 10\mathbf{k})$$

$$= 3\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{14}{3}\mathbf{k}$$

82 B $P = 2 + i = z$ and $Q = 1 + 2i$

Response A

$\overline{z} = 2 - i \neq Q$

Response B

$\overline{iz} = i(2 - i) = 1 + 2i = Q$

Response C

$-\overline{z} = -(2 - i) = -2 + i \neq Q$

Response D

$-\overline{iz} = -i(2 - i) = -1 - 2i \neq Q$

Response E

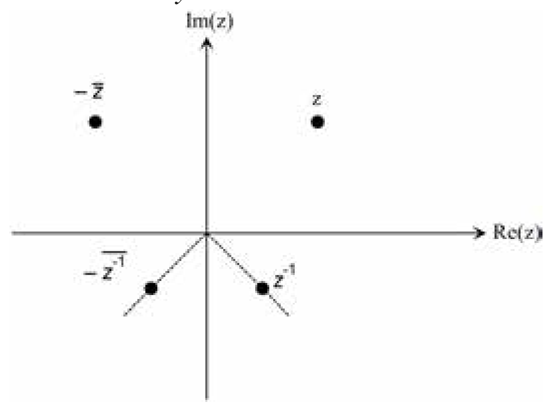
$z\overline{z} = (2 + i)(2 - i) = 5 \neq Q$

83 E Let $z = x + iy$

$-\overline{z} = -x + iy$

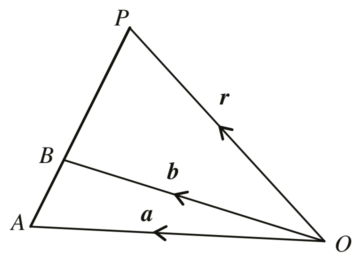
$z^{-1} = \dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2}$

$-\overline{z^{-1}} = \dfrac{-x - yi}{x^2 + y^2}$



z^{-1} and $-\overline{z^{-1}}$ run along the dotted lines but never move past z and $-\overline{z}$ respectively. Hence the points make a trapezium.

Solutions to extended-response questions

1 

The point C has been labelled C.

a **i** $\overrightarrow{AP} = \frac{3}{2}\overrightarrow{AB}$

$= \frac{3}{2}(\overrightarrow{OB} - \overrightarrow{OA})$

$= \frac{3}{2}(\boldsymbol{b} - \boldsymbol{a})$

ii $\overrightarrow{OP} = r$ and $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\therefore \quad r = \boldsymbol{a} + \frac{3}{2}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{2}(3\boldsymbol{b} - \boldsymbol{a})$

b **i** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (2\boldsymbol{i} + 2\boldsymbol{j}) - \boldsymbol{i}$

$= \boldsymbol{i} + 2\boldsymbol{j}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (4\boldsymbol{i} + \boldsymbol{j}) - (2\boldsymbol{i} + 2\boldsymbol{j})$

$= 2\boldsymbol{i} - \boldsymbol{j}$

ii $|\overrightarrow{AB}| = \sqrt{1^2 + 2^2}$

$= \sqrt{5}$

$|\overrightarrow{BC}| = \sqrt{2^2 + (-1)^2}$

$= \sqrt{5}$

Hence $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same magnitude.

iii $\overrightarrow{AB} \cdot \overrightarrow{BC} = (\boldsymbol{i} + 2\boldsymbol{j}) \cdot (2\boldsymbol{i} - \boldsymbol{j})$

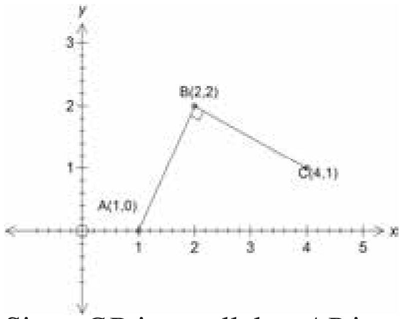
$= 1 \times 2 + 2 \times (-1)$

$= 0$

Since the scalar product $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, AB is perpendicular to BC.

iv Method 1:

Plot the points A, B, C using their Cartesian coordinates.



Since CD is parallel to AB its gradient is 2.

For C, move 2 down and 1 back.

This gives $D(3, -1)$ and checking shows the gradient of AD is $-\frac{1}{2}$, The same as BC.

So $\overrightarrow{OD} = 3\boldsymbol{o} - \boldsymbol{j}$

Method 2:

As in given solution, $\overrightarrow{CD} = (x-4)\boldsymbol{i} + (y-1)\boldsymbol{j}$

And the dot product gives $y = 2x - 7$ ①

Similarly, $\overrightarrow{AD} = (x-1)\boldsymbol{i} + y\boldsymbol{j}$ and

The dot product with $\overrightarrow{CD}$ gives $(x-1)(x-4) + y(y-1) = 0$ ②

Substituting ① into ② and simplifying gives

$x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$x = 3$ or $x = 4$

But $x = 4$ corresponds to point D, which would make $\overrightarrow{CD} = 0$.

So $x = 3$ and then $y = -1$.

Hence $\overrightarrow{CD} = 3\boldsymbol{i} - \boldsymbol{j}$.

c $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) - (8\boldsymbol{i})$

$= (x-8)\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$

$= (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) - (10\boldsymbol{j})$

$= x\boldsymbol{i} + (y-10)\boldsymbol{j} + z\boldsymbol{k}$

P is equidistant from O, A and B,

$\therefore \quad OP = AP = BP$

$$OP = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2} \qquad ①$$

$$AP = |\overrightarrow{AP}| = \sqrt{(x-8)^2 + y^2 + z^2} \qquad ②$$

$$BP = |\overrightarrow{BP}| = \sqrt{x^2 + (y-10)^2 + z^2} \qquad ③$$

Equating ① and ② yields

$$\sqrt{x^2 + y + z^2} = \sqrt{(x-8)^2 + y^2 + z^2}$$

$\therefore \quad x^2 + y^2 + z^2 = (x-8)^2 + y^2 + z^2$

$\therefore \quad x^2 = x^2 - 16x + 64$

$\therefore \quad 16x = 64$

$\therefore \quad x = 4$

Equating ① and ③ yields

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + (y-10)^2 + z^2}$$

$\therefore \quad x^2 + y^2 + z^2 = x^2 + (y-10)^2 + z^2$

$\therefore \quad y^2 = y^2 - 20y + 100$

$\therefore \quad 20y = 100$

$\therefore \quad y = 5$

Since ΔOAB is in the ***i*–*j*** plane, and P is at a distance of 2 above the triangle, the ***k*** component of $\overrightarrow{OP}$ is 2, i.e., $z = 2$.

2 a $2z^2 - 4z + 6 = 0$

$z^2 - 2z + 3 = 0$

$z^2 - 2z + 1 = -2$

$(z-1)^2 = -2$

$z = 1 \pm i\sqrt{2}$

b The circle has radius $\sqrt{2}$.
The Cartesian equation is
$(x-1)^2 + y^2 = 2$

c $2z^2 - 4z + d = 0$

$$z^2 - 2z + \frac{d}{2} = 0$$

$$z^2 - 2z + 1 = 1 - \frac{d}{2}$$

$$(z-1)^2 = 1 - \frac{d}{2}$$

First assume $d > 2$

$$z = 1 \pm i\sqrt{\frac{d}{2} - 1}$$

Therefore consider $\sqrt{\frac{d}{2} - 1} \le \sqrt{2}$

$\frac{d}{2} \le 3$

$d \le 6$

Now consider $d \le 2$

$z = 1 \pm \sqrt{1 - \frac{d}{2}}$

$\sqrt{1 - \frac{d}{2}} \le \sqrt{2}$

$1 - \frac{d}{2} \le 2$

$d \ge -2$

Therefore $-2 \le d \le 6$

d $az^2 - bz + c = 0$

$$z^2 - \frac{b}{a}z + \frac{c}{a} = 0$$

$$z^2 - \frac{b}{a}z + \frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2}{4a^2}$$

$$\left(z - \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$z = \frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

e $b^2 < 4ac$

f $\left(x - \frac{b}{2a}\right)^2 + y^2 = \frac{4ac - b^2}{4a^2}$

3 a i $|z| \le 2$ is represented by a disc with centre $(0, 0)$ and radius 2, i.e. $x^2 + y^2 \leq 2$

Let $z = x + iy, x, y \in R$

$\text{Re}(z) = x, \text{Im}(z) = y$

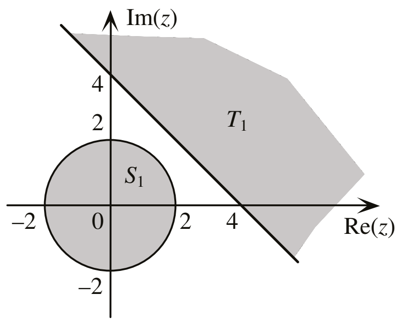
$\therefore \quad \text{Im}(z) + \text{Re}(z) = y + x$

$\therefore \quad \text{Im}(z) + \text{Re}(z) \geq 4$

becomes $y + x \geq 4$

$\therefore \quad y \geq 4 - x$

$\therefore \quad S_1 = \{(x, y) : x^2 + y^2 \leq 2\}$ and $T_1 = \{(x, y) : y \geq 4 - x\}$



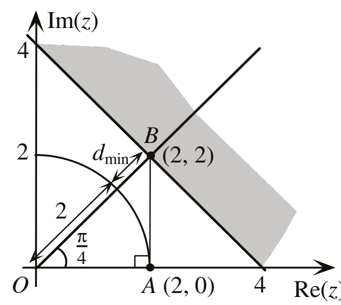
ii d is the distance between $z_1 \in S_1$ and $z_2 \in T_1$
The minimum distance is represented on the above diagram by the smallest gap between the shaded areas S_1 and T_1.
In ΔOAB, using Pythagoras' theorem,

$OB = \sqrt{2^2 + 2^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

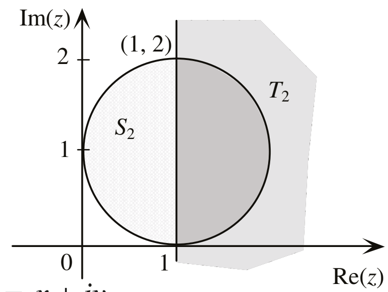
and the minimum value of d is
$d_{\min} = 2\sqrt{2} - 2$



b **i** $|z-1-i| \le 1$ is represented by a disc with centre (1, 1) and radius 1,

i.e. $(x-1)^2 + (y-1)^2 \le 1$

$\therefore$ $S_2 = \{(x, y) : (x-1)^2 + (y-1)^2 \le 1\}$



Now $|z-2-i| \le |z-i|$

$\therefore$ $|x+iy-2-i| \le |x+iy-i|$ where $z = x+iy$

$\therefore$ $|(x-2)+i(y-1)| \le |x+i(y-1)|$

$\therefore$ $\sqrt{(x-2)^2+(y-1)^2} \le \sqrt{x^2+(y-1)^2}$

$\therefore$ $(x-2)^2+(y-1)^2 \le x^2+(y-1)^2$

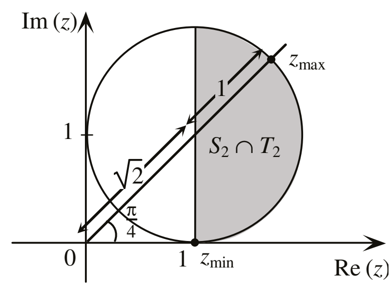
$\therefore$ $x^2-4x+4 \le x^2$

$\therefore$ $4 \le 4x$

$\therefore$ $x \ge 1$

$\therefore$ $T_2 = \{(x, y) : x \ge 1\}$

ii In the diagram above, the maximum $|z_{max}|$ and minimum $|z_{min}|$ values of $|z|$ are represented, respectively, by the greatest and least straight line distances from the point (0, 0) to the intersecting shaded area.



From the diagram,

$z_{min} = 1 + 0i = 1$

$\therefore$ $|z_{min}| = \sqrt{1^2+0^2} = 1$

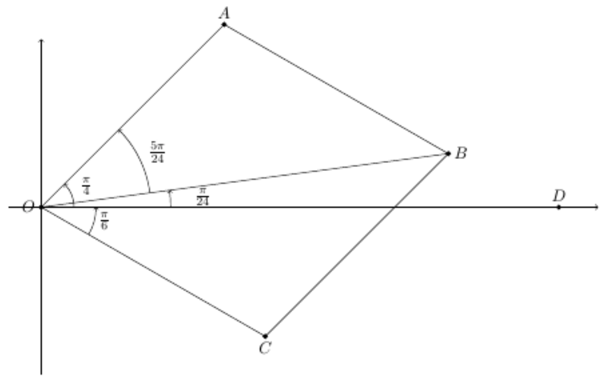
The point z_{max} lies on the circle with equation $(x-1)^2+(y-1)^2 = 1$ and the line with equation $y = x$.

By inspection, $|z_{max}| = 1+\sqrt{2}$

$\therefore$ $z_{max} = (1+\sqrt{2})\text{cis}\dfrac{\pi}{4}$

Hence the maximum and minimum values of $|z|$ are $1+\sqrt{2}$ and 1 respectively.

4



Now note that

$$\angle COA = \frac{\pi}{6} + \frac{\pi}{4} = \frac{10\pi}{24} = \frac{5\pi}{12}.$$

Therefore, as diagonal OB bisects $\angle COA$, we know that

$$\angle BOA = \frac{1}{2}\angle COA = \frac{5\pi}{24}.$$

Therefore,

$$\angle DOB = \frac{\pi}{4} - \frac{5\pi}{24} = \frac{\pi}{24}.$$

However we also know that

$$\angle DOB = \text{Arg}\,(z + w) = \arctan\left(\frac{\sqrt{2} - 1}{\sqrt{2} + \sqrt{3}}\right) = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2.$$

Therefore

$$\tan(\tfrac{\pi}{24}) = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2,$$

as required.

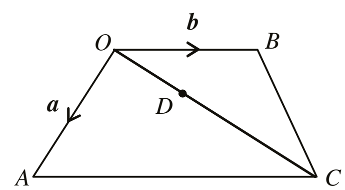
5 a i $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (\overrightarrow{OA} + \overrightarrow{AC}) - \overrightarrow{OB}$

$= (\overrightarrow{OA} + 2\overrightarrow{OB}) - \overrightarrow{OB}$

$= \overrightarrow{OA} + \overrightarrow{OB}$

$= \boldsymbol{a} + \boldsymbol{b}$



ii $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB}$

$$= \frac{1}{3}\overrightarrow{OC} - \overrightarrow{OB}$$

$$= \frac{1}{3}(\overrightarrow{OA} + 2\overrightarrow{OB}) - \overrightarrow{OB}$$

$$= \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b}) - \boldsymbol{b}$$

$$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$$

iii $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD}$

$$= \boldsymbol{a} - \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$$

$$= \frac{2}{3}(\boldsymbol{a} - \boldsymbol{b})$$

b $\overrightarrow{DA} = \frac{2}{3}(\boldsymbol{a} - \boldsymbol{b})$

$$= 2 \times \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$$

$$= 2\overrightarrow{BD}$$

Hence, A, B and D are collinear.

6 a i $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$

$$\therefore \quad \cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$= \frac{(i - 2j + 2k) \cdot (0i + 12j - 5k)}{\sqrt{1^2 + (-2)^2 + 2^2} \times \sqrt{0^2 + 12^2 + (-5)^2}}$$

$$= \frac{1 \times 0 + (-2) \times 12 + 2 \times (-5)}{\sqrt{9} \times \sqrt{169}}$$

$$= \frac{-34}{3 \times 13}$$

$$= \frac{-34}{39}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-34}{39}\right)$$

$$= 150.66788\ldots^\circ$$

The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ has magnitude 151°, to the nearest degree.

ii The vector resolute of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$ is given by

$$\boldsymbol{b} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} = (12\boldsymbol{j} - 5\boldsymbol{k}) - \frac{-34}{(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) \cdot (i - 2\boldsymbol{j} + 2\boldsymbol{k})}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$$

$$= 12\boldsymbol{j} - 5\boldsymbol{k} + \frac{34}{1^2 + (-2)^2 + 2^2}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$$

$$= 12\boldsymbol{j} - 5\boldsymbol{k} + \frac{34}{9}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$$

$$= \frac{34}{9}\boldsymbol{j} + \frac{40}{9}\boldsymbol{j} + \frac{23}{9}\boldsymbol{k}$$

iii

$$x\boldsymbol{a} + y\boldsymbol{b} = x(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) + y(12\boldsymbol{j} - 5\boldsymbol{k})$$

$$= x\boldsymbol{i} + (12y - 2x)\boldsymbol{j} + (2x - 5y)\boldsymbol{k}$$

If $x\boldsymbol{a} + y\boldsymbol{b} = 3\boldsymbol{i} - 30\boldsymbol{j} + z\boldsymbol{k}$

then $x\boldsymbol{i} + (12y - 2x)\boldsymbol{j} + (2x - 5y)\boldsymbol{k} = 3\boldsymbol{i} - 30\boldsymbol{j} + z\boldsymbol{k}$

Equating coefficients

$$x = 3$$

and $12y - 2x = -30$

$\therefore$ $12y - 2(3) = -30$

$\therefore$ $12y = -24$

$\therefore$ $y = -2$

and $2x - 5y = z$

$\therefore$ $2(3) - 5(-2) = z$

$\therefore$ $z = 16$

b **i** $\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA}$

$$= \frac{3}{2}\overrightarrow{OP} - \overrightarrow{OA}$$

$$= \frac{3}{2}(\overrightarrow{OA} + \overrightarrow{AP}) - \overrightarrow{OA}$$

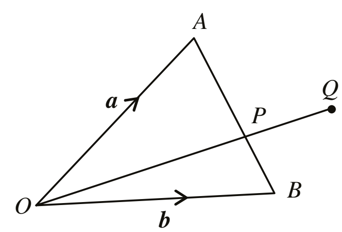
$$= \frac{3}{2}\left(\overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}\right) - \overrightarrow{OA}$$

$$= \frac{3}{2}\left(\overrightarrow{OA} + \frac{2}{3}(\overrightarrow{OB} - \overrightarrow{OA})\right) - \overrightarrow{OA}$$

$$= \frac{3}{2}\left(\boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})\right) - \boldsymbol{a}$$

$$= \left(\frac{3}{2}\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{a}\right) - \boldsymbol{a}$$

$$= \boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$$



ii $\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB}$

$$= \frac{1}{2}\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{b}$$

$$= \frac{1}{2}\overrightarrow{OA}$$

i.e. $\overrightarrow{OA} = 2\overrightarrow{BQ}$

Hence, $\overrightarrow{OA}$ is parallel to $\overrightarrow{BQ}$, as required.

7 a $2a + b - c = 0$ ①

$a - 4b - 2c = 0$ ②

Multiply ② by 2 $2a - 8b - 4c = 0$ ③

Subtract ③ from ① $9b + 3c = 0$

$\therefore$ $c = -3b$

Substitute in ① $2a + b + 3b = 0$

$\therefore$ $a = -2b$

$\therefore$ $a : b : c = 2 : -1 : 3$

b $(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot (2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = 0$

and $(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot (\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) = 0$

$\Rightarrow$ $2x + y - 3z = 0$ ①

and $x - y - z = 0$ ②

Add ① and ② $3x - 4z = 0$

$\therefore$ $x = \frac{4}{3}z$

Substitute in ② $\frac{4}{3}z - y - z = 0$

$\therefore$ $\frac{z}{3} = y$

$\therefore$ $x = 4y$

$\therefore$ $x : y : z = 4 : 1 : 3$

c $(4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) \cdot (2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = 8 + 1 - 9$

$= 0$

and $(4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) \cdot (\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) = 4 - 1 - 3$

$= 0$

i.e. $4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ is perpendicular to both vectors.

d $(4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}) \cdot (4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) = 16 + 5 - 21$

$= 0$

$\therefore$ $4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}$ is perpendicular to $\boldsymbol{v}$.

e $4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k} = s(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + t(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

implies $4 = 2s + t$ ①

$5 = s - t$ ②

$-7 = -3s - t$ ③

Add ① and ② $9 = 3s$

$\therefore$ $s = 3$ and $t = -2$, and these satisfy ③.

f $\boldsymbol{r} = t(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + s(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

$$\begin{aligned}\boldsymbol{r} \cdot \boldsymbol{v} &= (t(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + s(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})) \cdot (4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) \\ &= 4(2t + s) + (t - s) + 3(-3t - s) \\ &= 8t + 4s + t - s - 9t - 3s = 0\end{aligned}$$

$\therefore$ $\boldsymbol{r} \perp \boldsymbol{v}$

8 a $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$

$$\therefore \quad \cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

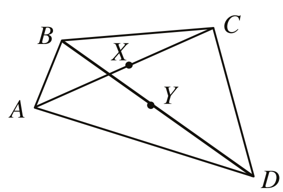
b $\sin^2\theta + \cos^2\theta = 1$

$$\begin{aligned}\therefore \quad \sin^2\theta &= 1 - \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right)^2 \\ &= \frac{(\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2}{|\boldsymbol{a}|^2|\boldsymbol{b}|^2}\end{aligned}$$

$$\therefore \quad \sin\theta = \frac{\sqrt{(\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

c
$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}| \sin\theta \\ &= \frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}|\frac{\sqrt{(\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2}}{|\boldsymbol{a}||\boldsymbol{b}|} \\ &= \frac{1}{2}\sqrt{(\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2}\end{aligned}$$

9 a



X and Y are the midpoints of AC and BD respectively. We are required to show $\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BX}$

Now $\quad \overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{AX}$

and $\quad \overrightarrow{BX} = \overrightarrow{BC} + \overrightarrow{CX}$

$\therefore \quad 2\overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{AX} + \overrightarrow{BC} + \overrightarrow{CX}$

but $\quad \overrightarrow{AX} = -\overrightarrow{CX}$ as X is the midpoint

$\therefore \quad 2\overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{BC} + \mathbf{0}$

i.e. $\quad 2\overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{BC}$

b Now $\quad \overrightarrow{YX} = \overrightarrow{YB} + \overrightarrow{BC} + \overrightarrow{CX}$ and $\overrightarrow{YX} = \overrightarrow{YD} + \overrightarrow{DA} + \overrightarrow{AX}$

Also $\quad \overrightarrow{YX} = \overrightarrow{YB} + \overrightarrow{BA} + \overrightarrow{AX}$ and $\overrightarrow{YX} = \overrightarrow{YD} + \overrightarrow{DC} + \overrightarrow{CX}$

Adding gives $\quad 4\overrightarrow{YX} = \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{BA} + \overrightarrow{DC}$
(as $\overrightarrow{YB} + \overrightarrow{YD} = \mathbf{0}$ and $\overrightarrow{CX} + \overrightarrow{AX} = \mathbf{0}$)

10 a $\boldsymbol{a} \times \boldsymbol{b} = 3\boldsymbol{a} \times \boldsymbol{c}$

$\therefore \boldsymbol{a} \times \boldsymbol{b} - 3\boldsymbol{a} \times \boldsymbol{c} = 0$

$\therefore \boldsymbol{a} \times (\boldsymbol{b} - 3\boldsymbol{c}) = 0$

$\therefore \boldsymbol{a}$ parallel $\boldsymbol{b} - 3\boldsymbol{c}$

$\therefore \boldsymbol{b} - 3\boldsymbol{c} = k\boldsymbol{a}$

b i $\quad \boldsymbol{b} \cdot \boldsymbol{c} = |\boldsymbol{b}| \cdot |\boldsymbol{c}| \cdot \cos\theta$

$$\therefore \boldsymbol{b} \cdot \boldsymbol{c} = 3 \cdot 1 \cdot \frac{1}{3} = 1$$

ii $\quad |\boldsymbol{b} - 3\boldsymbol{c}|^2 = (\boldsymbol{b} - 3\boldsymbol{c}) \cdot (\boldsymbol{b} - 3\boldsymbol{c})$

$$= \boldsymbol{b} \cdot \boldsymbol{b} - 6\boldsymbol{b} \cdot \boldsymbol{c} + 9\boldsymbol{c} \cdot \boldsymbol{c}$$

$$= 9 - 6 \times 3 \times \frac{1}{3} + 9$$

$$= 12$$

$$\therefore |\boldsymbol{b} - 3\boldsymbol{c}| = 2\sqrt{3}$$

iii Now $\quad |\boldsymbol{b} - 3\boldsymbol{c}| = b\boldsymbol{a} \qquad$ (From (a) above)

and $\quad |\boldsymbol{a}| = 1$

$\therefore k = \pm 2\sqrt{3} \quad$ (Depending on direction of $\boldsymbol{b} - 3\boldsymbol{c}$)

c $\boldsymbol{a}$ is parallel to $\boldsymbol{b} - 3\boldsymbol{c}$ (in the same or opposite direction)

$$
\begin{aligned}
(\boldsymbol{b} - 3\boldsymbol{c}) \cdot \boldsymbol{c} &= |\boldsymbol{b} - 3\boldsymbol{c}| \times |\boldsymbol{c}| \times \cos\theta \\
\boldsymbol{b} \cdot \boldsymbol{c} - 3\boldsymbol{c} \cdot \boldsymbol{c} &= |\boldsymbol{b} - 3\boldsymbol{c}| \times |\boldsymbol{c}| \times \cos\theta \\
-2 &= |\boldsymbol{b} - 3\boldsymbol{c}| \times |\boldsymbol{c}| \times \cos\theta \\
-2 &= 2\sqrt{3}\cos\theta \\
\cos\theta &= -\frac{1}{\sqrt{3}}
\end{aligned}
$$

$$
\text{or } \cos\theta = \frac{1}{\sqrt{3}}
$$

11 a $\boldsymbol{r} \cdot \boldsymbol{n} = k$

Now at point of intersection $(\boldsymbol{a} + t\boldsymbol{b})\boldsymbol{n} = k$

$$\therefore \boldsymbol{a} \cdot \boldsymbol{n} + t\boldsymbol{b} \cdot \boldsymbol{n} = k$$

$$\therefore t = \frac{k - \boldsymbol{a} \cdot \boldsymbol{n}}{\boldsymbol{b} \cdot \boldsymbol{n}}$$

Let Q be point of intersection;

$$\therefore \overrightarrow{OQ} = \boldsymbol{r} = \boldsymbol{a} + \left(\frac{k - \boldsymbol{a} \cdot \boldsymbol{n}}{\boldsymbol{b} \cdot \boldsymbol{n}}\right) \cdot \boldsymbol{b}$$

$$\therefore \overrightarrow{OQ} = \frac{(\boldsymbol{b} \cdot \boldsymbol{n})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{n})\,\boldsymbol{b} + k\boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{n}}$$

b i The position vector is $\boldsymbol{r} = \boldsymbol{p} + t\boldsymbol{n}$ where $t = \dfrac{k - \boldsymbol{p} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}$

$$\boldsymbol{r} = \boldsymbol{p} + \frac{k - \boldsymbol{p} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\boldsymbol{n}$$

ii Let R be the point of intersection with the plane.

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$$

$$= -\boldsymbol{p} + \boldsymbol{p} + \frac{k - \boldsymbol{p} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\boldsymbol{n}$$

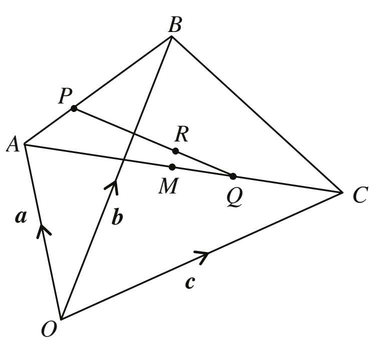
$$= \frac{k - \boldsymbol{p} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\boldsymbol{n}$$

$$\text{Distance}^2 = \overrightarrow{PR} \cdot \overrightarrow{PR}$$

$$= \left(\frac{k - \boldsymbol{p} \cdot \boldsymbol{n}}{\boldsymbol{n} \cdot \boldsymbol{n}}\right)^2 \times \boldsymbol{n} \cdot \boldsymbol{n}$$

That is Distance$= \left|\dfrac{k - \boldsymbol{p} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}\right|$

12



$AP : PB = 1 : 2$

$AQ : QC = 2 : 1$

$PR : RQ = 2 : 1$

a $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AP} + \overrightarrow{PR}$

$$= \boldsymbol{a} + \frac{1}{3}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{PQ}$$

$$= \boldsymbol{a} + \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a}) + \frac{2}{3}(\overrightarrow{PA} + \overrightarrow{AQ})$$

$\therefore \quad \overrightarrow{OR} = \boldsymbol{a} + \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a}) + \frac{2}{3}\left(\frac{1}{3}\overrightarrow{BA} + \frac{2}{3}\overrightarrow{AC}\right)$

$= \boldsymbol{a} + \frac{1}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{a} + \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{4}{9}\boldsymbol{a} + \frac{1}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$

b $\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM}$

$= \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$

$= \boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c} - \boldsymbol{b}$

$\overrightarrow{BR} = \overrightarrow{BP} + \overrightarrow{PR}$

$= \frac{2}{3}\overrightarrow{BA} + \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{2}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{4}{9}\boldsymbol{a} - \frac{8}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$

$= \frac{8}{9}\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c} - \boldsymbol{b}\right)$

$\therefore \quad \overrightarrow{BR} = \frac{8}{9}\overrightarrow{BM}$, and R lies on BM.

c $BM : RM = 8 : 1$

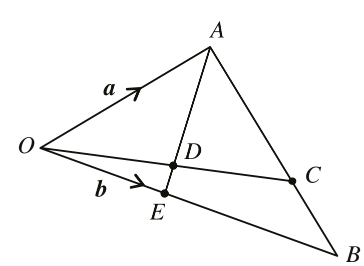
13 $AC : CB = 2 : 1$

a **i** $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \boldsymbol{a} + \frac{2}{3}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$



ii $\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OC}$

$= \frac{1}{6}(\boldsymbol{a} + 2\boldsymbol{b})$

$\therefore \quad \overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$

$= -\boldsymbol{a} + \frac{1}{6}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$

$= \frac{1}{6}(2\boldsymbol{b} - 5\boldsymbol{a})$

b **i** $\overrightarrow{OE} = \lambda\boldsymbol{b}$

also $\overrightarrow{OE} = \overrightarrow{OA} + k\overrightarrow{AD}$

$= \boldsymbol{a} + \frac{k}{6}(2\boldsymbol{b} - 5\boldsymbol{a})$

$\therefore \quad \lambda\boldsymbol{b} = \boldsymbol{a} + \frac{k}{3}\boldsymbol{b} - \frac{5k}{6}\boldsymbol{a}$

$= \left(1 - \frac{5k}{6}\right)\boldsymbol{a} + \frac{k}{3}\boldsymbol{b}$

$\therefore \quad \lambda = \frac{k}{3}$

and $0 = 1 - \frac{5k}{6}$

i.e., $k = \frac{6}{5}$

and $\lambda = \frac{2}{5}$

$\therefore \quad \overrightarrow{OE} = \frac{2}{5}\overrightarrow{OB}$

$\therefore \quad OE : EB = 2 : 3$

ii $\overrightarrow{AE} = k\overrightarrow{AD}$

$= \frac{6}{5}\overrightarrow{AD}$

and $\overrightarrow{AD} = \frac{5}{6}\overrightarrow{AE}$

$\therefore \quad AE : ED = 6 : 1$

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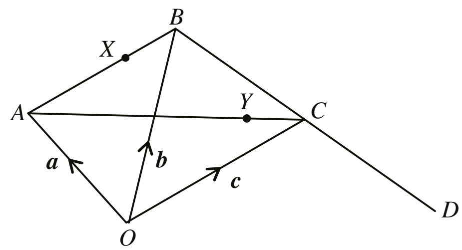


a **i** $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$= \overrightarrow{OC} + \overrightarrow{BC}$ as $\overrightarrow{CD} = \overrightarrow{BC}$

$= \boldsymbol{c} + (\boldsymbol{c} - \boldsymbol{b})$

$= 2\boldsymbol{c} - \boldsymbol{b}$

ii $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AE}$

$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

iii $\overrightarrow{OY} = \overrightarrow{OA} + \frac{4}{5}\overrightarrow{AC}$

$= \boldsymbol{a} + \frac{4}{5}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{1}{5}(\boldsymbol{a} + 4\boldsymbol{c})$

b $\overrightarrow{X}Y = \overrightarrow{XO} + \overrightarrow{OY}$

$= \frac{-1}{3}(\boldsymbol{a} + 2\boldsymbol{b}) + \frac{1}{5}(\boldsymbol{a} + 4\boldsymbol{c})$

$= \frac{-5}{15}(\boldsymbol{a} + 2\boldsymbol{b}) + \frac{3}{15}(\boldsymbol{a} + 4\boldsymbol{c})$

$= \frac{-2}{15}(\boldsymbol{a} + 5\boldsymbol{b} - 6\boldsymbol{c})$

$\overrightarrow{XD} = \frac{-1}{3}(\boldsymbol{a} + 2\boldsymbol{b}) + 2\boldsymbol{c} - \boldsymbol{b}$

$= \frac{-1}{3}(\boldsymbol{a} + 5\boldsymbol{b} - 6\boldsymbol{c})$

$\therefore$ $\overrightarrow{XD} = \frac{5}{2}\overrightarrow{XY}$

$\therefore$ X, D and Y are collinear.

15 **a** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-\boldsymbol{i} - \boldsymbol{j}) - (\boldsymbol{j} + 2\boldsymbol{k})$

$= -\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (4\boldsymbol{i} + \boldsymbol{k}) - (\boldsymbol{j} + 2\boldsymbol{k})$

$= 4\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (4\boldsymbol{i} + \boldsymbol{k}) - (-\boldsymbol{i} - \boldsymbol{j})$

$= 5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = -5 - 2 - 2$

$\neq 0$

$\overrightarrow{AC} \cdot \overrightarrow{AB} = -4 + 2 + 2$

$= 0$

$\therefore \quad AC \perp AB$

b $|\overrightarrow{AB}| = \sqrt{9}$

$= 3$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= 3\boldsymbol{i}$

$\therefore \quad |\overrightarrow{AD}| = \sqrt{9}$

$= 3$

$\therefore \quad \triangle ABD$ is isosceles.

c $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB}$

$= 4\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

E is the midpoint of AC

$\therefore \quad \overrightarrow{BE} = \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$

$= \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{j} + 2\boldsymbol{k} + \frac{1}{2}(-\boldsymbol{j} - 2\boldsymbol{k} + 4\boldsymbol{i} + \boldsymbol{k})$

$= 3\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} + \frac{3}{2}\boldsymbol{k}$

$= \frac{3}{4}(4\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$

$= \frac{3}{4}\overrightarrow{BD}$

$\therefore$ E lies on BD.

The ratio $BE : ED = 3 : 1$

16 a $\alpha = 1 - \sqrt{3}i$

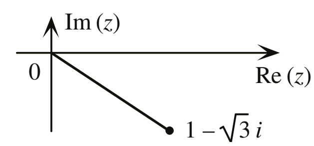
$\overline{\alpha} = 1 + \sqrt{3}i$

$(z - \alpha)(z - \overline{\alpha}) = z^2 - (\alpha + \overline{\alpha})z + \alpha\overline{\alpha}$

$= z^2 - 2z + (1 - \sqrt{3}i)(1 + \sqrt{3}i)$

$= z^2 - 2z + 4$

b i $\alpha = 2\operatorname{cis}\left(\frac{-\pi}{3}\right)$



ii $\alpha^2 = 4\operatorname{cis}\left(\frac{-2\pi}{3}\right)$

$= 4\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

$= -2 - 2\sqrt{3}i$

$\alpha^3 = 8\operatorname{cis}(-\pi) = -8$

iii $\alpha^3 - \alpha^2 + 2\alpha + 4 = -8 + 2 + 2\sqrt{3}i + 2(1 - \sqrt{3}i) + 4 = 0$

$\therefore$ α is a root.

Since α is a root, $\overline{\alpha}$ is also a root (conjugate root theorem)

$\therefore$ $z^2 - 2z + 4$ is a factor.

By division, $z^3 - z^2 + 2z + 4 = (z^2 - 2z + 4)(z + 1)$

$\therefore$ $z^3 - z^2 + 2z + 4 = (z + 1)(z - (1 - \sqrt{3}i))(z - (1 + \sqrt{3}i))$

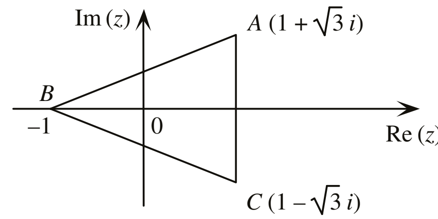
and the three roots are -1, $1 - \sqrt{3}i$ and $1 + \sqrt{3}i$

c **i** $AB = \sqrt{4+3}$

$= \sqrt{7}$

$BC = \sqrt{4+3}$

$= \sqrt{7}$



ii $\triangle ABC$ is isosceles.

17 **a** $\overrightarrow{AB} = \mathbf{i} + \mathbf{j}$ and $\overrightarrow{AC} = 2\mathbf{i} - \mathbf{k}$

b $\overrightarrow{AB} \times \overrightarrow{AC} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Use $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix}$

c Cartesian equation is $-x + y - 2z = k$

Use the point $(0, -1, 2)$

$-1 - 4 = k$

$k = -5$

Equation of plane $= -x + y - 2z = -5$ or $x - y + 2z = 5$

d $\mathbf{r_A} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \quad \lambda \in \mathbb{R}$

Equation:

$\mathbf{r_A} = (1 + \lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (1 + 2\lambda)\mathbf{k})$

e Use the parametric equations

$x = 1 + \lambda$

$y = 2 - \lambda$

$z = 1 + 2\lambda$

Substitute in the equation of the plane.

$1 + \lambda - (2 - \lambda) + 2(1 + 2\lambda) = 5$

$1 + 6\lambda = 5$

$\lambda = \frac{2}{3}$

Therefore position vector of point of intersection is

$\frac{5}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$

f We can use the reult of **d**

Minimum distance $= \sqrt{(1 - \frac{5}{3})^2 + (2 - \frac{4}{3})^2 + (1 - \frac{7}{3})^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{16}{9}} = \frac{2\sqrt{6}}{3}$

18 a $z = 1 + i\sqrt{2}$

$$\frac{1}{z} = \frac{1}{1 + i\sqrt{2}} \times \frac{1 - i\sqrt{2}}{1 - i\sqrt{2}}$$

$$= \frac{1 - i\sqrt{2}}{3}$$

$$z + \frac{1}{z} = 1 + i\sqrt{2} + \frac{1 - i\sqrt{2}}{3}$$

$$= \frac{3 + 3\sqrt{2}i + 1 - i\sqrt{2}}{3}$$

$$= \frac{4}{3} + \frac{2\sqrt{2}i}{3}$$

i.e. $p = \frac{1}{3}(4 + 2\sqrt{2}i)$

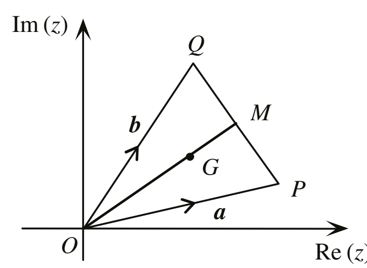
$$z - \frac{1}{z} = 1 + i\sqrt{2} - \frac{1 - i\sqrt{2}}{3}$$

$$= \frac{3 + 3\sqrt{2}i - 1 + i\sqrt{2}}{3}$$

$$= \frac{1}{3}(2 + 4\sqrt{2}i)$$

i.e. $q = \frac{1}{3}(2 + 4\sqrt{2}i)$

b $\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM}$



i $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$= -\boldsymbol{a} + \boldsymbol{b}$

ii $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$

$= \boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

iii $\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

iv $\overrightarrow{GP} = \overrightarrow{GO} + \overrightarrow{OP}$

$$= \frac{-1}{3}(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{a}$$

$$= \frac{2}{3}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b} = \frac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$$

v $\overrightarrow{GQ} = \overrightarrow{GO} + \overrightarrow{OQ}$

$$= \frac{-1}{3}(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{b}$$

$$= \frac{2}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{a} = \frac{1}{3}(2\boldsymbol{b} - \boldsymbol{a})$$

c From **a**, $PQ = \sqrt{\left(\frac{4}{3} - \frac{2}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3} - \frac{4\sqrt{2}}{3}\right)^2}$

$$= \sqrt{\frac{4}{9} + \frac{8}{9}}$$

$$= \frac{2\sqrt{3}}{3}$$

$\therefore\ PM = MQ = \frac{\sqrt{3}}{3}$

M represents the point $\left(\frac{\frac{4}{3} + \frac{2}{3}}{2}, \frac{\frac{2\sqrt{2}}{3} + \frac{4\sqrt{2}}{3}}{2}\right) = (1, \sqrt{2})$

$\therefore\ OM = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$

$\therefore\ OG = \frac{2\sqrt{3}}{3}$ and $GM = \frac{\sqrt{3}}{3}$

Since $PM = MQ = GM$, G, P and Q represent points on the circumference of a circle with centre at M and radius $\frac{\sqrt{3}}{3}$. PQ is a diameter of the circle, and the angle in a semicircle is a right angle. Hence $\angle PGQ$ is a right angle.

19 **a** $z^2 + 4 = (z - 2i)(z + 2i)$

b $z^4 + 4 = (z^2 - 2i)(z^2 + 2i)$

c **i** $(1 + i)^2 = 1 + 2i - 1 = 2i$

ii $(1 - i)^2 = 1 - 2i - 1 = -2i$

d $z^2 - 2i = (z - (1 + i))(z + (1 + i))$ as $2i = (1 + i)^2$

$z^2 + 2i = (z - (1 - i))(z + (1 - i))$ as $-2i = (1 - i)^2$

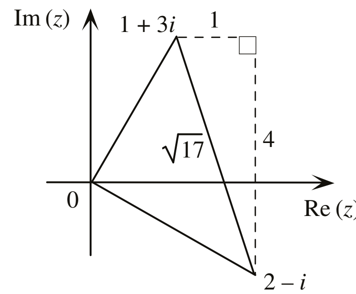
$z^4 + 4 = (z - (1 + i))(z - (1 - i))(z + (1 + i))(z + (1 - i))$

e $z^4 + 4 = (z - (1 + i))(z - (1 - i))(z + (1 + i))(z + (1 - i))$

$= (z^2 - z(1 + i + 1 - i) + 2)(z^2 + z(1 + i + 1 - i) + 2)$

$= (z^2 - 2z + 2)(z^2 + 2z + 2)$

20 a $z_1 = 1 + 3i$ and $z_2 = 2 - i$



$|z_1 - z_2| = |1 + 3i - 2 + i|$

$= |-1 + 4i|$

$= \sqrt{1 + 16} = \sqrt{17}$

b $|z - (2 - i)| = \sqrt{5}$

$\Rightarrow |x + iy - 2 + i| = \sqrt{5}$

$\Rightarrow |x - 2 + (y + 1)i| = \sqrt{5}$

$\Rightarrow \sqrt{(x - 2)^2 + (y + 1)^2} = \sqrt{5}$

$\Rightarrow (x - 2)^2 + (y + 1)^2 = 5$

Circle centre $(2, -1)$ and radius $\sqrt{5}$. The set of all points a distance of $\sqrt{5}$ from the point $2 - i$.

c $|z - (1 + 3i)| = |z - (2 - i)|$

$\Rightarrow |x + iy - 1 - 3i| = |x + iy - 2 + i|$

$\Rightarrow \sqrt{(x - 1)^2 + (y - 3)^2} = \sqrt{(x - 2)^2 + (y + 1)^2}$

$\Rightarrow x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 - 4x + 4 + y^2 + 2y + 1$

$\Rightarrow 2x - 8y = -5$

This equation represents the set of all points equidistant from $1 + 3i$ and $2 - i$, i.e., the perpendicular bisector of the line connecting $1 + 3i$ and $2 - i$.

21 a Let $z = 2 + i$

$\therefore \quad z^3 = (2 + i)(2 + i)(2 + i)$

$= (4 + 4i - 1)(2 + i)$

$= (3 + 4i)(2 + i)$

$= 2 + 11i$

b **i** $z = 2 + i$

$= r \operatorname{cis}(\alpha)$

$\therefore \quad 2 + 11i = r^3 \operatorname{cis} 3\alpha$

$= (\sqrt{5})^3 \operatorname{cis} 3\alpha$

$= 5\sqrt{5}(\cos 3\alpha + i \sin 3\alpha)$

Equating real and imaginary parts

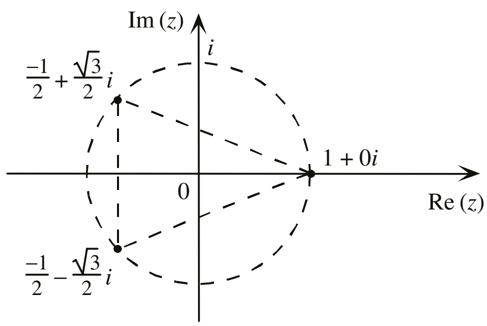
$\therefore \quad \cos 3\alpha = \dfrac{2}{5\sqrt{5}}$

$= \dfrac{2\sqrt{5}}{25}$

ii $\sin 3\alpha = \dfrac{11}{5\sqrt{5}}$

$= \dfrac{11\sqrt{5}}{25}$

22 **a** **i**



ii $(w^2)^2 = w^3 \times w$

$= w$ since $w^3 = 1$

b $z^3 - 1 = (z - 1)(z^2 + z + 1)$

As $w^3 = 1$, $0 = w^3 - 1$

$= (w - 1)(w^2 + w + 1)$

and as $w \neq 1$, $w^2 + w + 1 = 0$

c **i** $(1 + w)(1 + w^2) = 1 + w^2 + w + w^3$

$= 0 + w^3$ from **b** above

$= 1$ since $w^3 = 1$

ii $(1 + w^2)^3 = (1 + 2w^2 + w^4)(1 + w^2)$

$= 1 + 2w^2 + w^4 + w^2 + 2w^4 + w^6$

$= 1 + 3w^2 + 3w^4 + w^6$

$= 1 + 3w^2 + 3w + 1$ since $w^3 = 1$

$= 2 + 3(w^2 + w)$

$= 2 + 3(-1)$ since $w^2 + w + 1 = 0$

$= -1$

d **i** $2 + w = \frac{3}{2} + \frac{\sqrt{3}}{2}i$

$2 + w^2 = \frac{3}{2} - \frac{\sqrt{3}}{2}i$

so the required equation is $\left(z - \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(\frac{3}{2} - \frac{\sqrt{3}}{2}i\right)\right) = 0$

$\therefore \quad z^2 - z\left(\frac{3}{2} + \frac{\sqrt{3}}{2}i + \frac{3}{2} - \frac{\sqrt{3}}{2}i\right) + \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{3}{2} - \frac{\sqrt{3}}{2}i\right) = 0$

$z^2 - 3z + 3 = 0$

ii $3w - w^2 = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \frac{1}{2} + \frac{\sqrt{3}}{2}i = -1 + 2\sqrt{3}i$

$3w^2 - w = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i + \frac{1}{2} - \frac{\sqrt{3}}{2}i = -1 - 2\sqrt{3}i$

Required equation is $(z - (3w - w^2))(z - (3w^2 - w)) = 0$

$\therefore \quad (z - (-1 + 2\sqrt{3}i))(z - (-1 - 2\sqrt{3}i)) = 0$

$\therefore \quad z^2 - z(-1 + 2\sqrt{3} + -1 - 2\sqrt{3}i) + (-1 + 2\sqrt{3}i)(-1 - 2\sqrt{3}i) = 0$

$\therefore \quad z^2 + 2z + 13 = 0$

e $1 + w^n + w^{2n} = 1 + \left(\text{cis}\,\frac{2\pi}{3}\right)^n + \left(\text{cis}\,\frac{4\pi}{3}\right)^n$

$= 1 + \text{cis}\,\frac{2\pi n}{3} + \text{cis}\,\frac{4\pi n}{3}$

If $n = 3k$, i.e., a multiple of 3,

$1 + w^n + w^{2n} = 1 + \text{cis}(2k\pi) + \text{cis}(4k\pi)$

$= 1 + 1 + 1 = 3$

$$\text{If } n = 3k+1,\ 1 + w^n + w^{2n} = 1 + \text{cis}\left(\frac{2\pi}{3}(3k+1)\right) + \text{cis}\left(\frac{4\pi}{3}(3k+1)\right)$$

$$= 1 + \text{cis}\left(2\pi k + \frac{2\pi}{3}\right) + \text{cis}\left(4\pi k + \frac{4\pi}{3}\right)$$

$$= 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$$

$$\text{If } n = 3k+2,\ 1 + w^n + w^{2n} = 1 + \text{cis}\left(\frac{2\pi}{3}(3k+2)\right) + \text{cis}\left(\frac{4\pi}{3}(3k+2)\right)$$

$$= 1 + \text{cis}\left(2\pi k + \frac{4\pi}{3}\right) + \text{cis}\left(4\pi k + \frac{8\pi}{3}\right) = 0$$

23 a $z^5 - 1 = (z-1)P(z)$

$\therefore \quad P(z) = \dfrac{z^5 - 1}{z - 1}$

By division,

```
         z^4 + z^3 + z^2 + z + 1
z - 1 ) z^5 + 0z^4 + 0z^3 + 0z^2 + 0z - 1
        z^5 - z^4
        ---------------------------------
              z^4 + 0z^3
              z^4 - z^3
        ---------------------------------
                    z^3 + 0z^2
                    z^3 - z^2
        ---------------------------------
                          z^2 + 0z
                          z^2 - z
        ---------------------------------
                                z - 1
                                z - 1
        ---------------------------------
                                    0
```

$\therefore \quad P(z) = z^4 + z^3 + z^2 + z + 1$

b If $z = \text{cis}\left(\dfrac{2\pi}{5}\right)$

$z^5 = \text{cis } 2\pi$

$= 1$

$\therefore \quad z^5 - 1 = 0$ has $z = \text{cis}\left(\dfrac{2\pi}{5}\right)$ as a solution.

c By the conjugate root theorem, $\text{cis}\left(-\dfrac{2\pi}{5}\right)$ is also a solution.

d We note $\left(\text{cis}\left(\dfrac{4\pi}{5}\right)\right)^5 = \text{cis } 4\pi$

$= 1$

Therefore the other two solutions are $\operatorname{cis}\left(\frac{4\pi}{5}\right)$ and $\operatorname{cis}\left(-\frac{4\pi}{5}\right)$.

The solutions are $\operatorname{cis}\left(\frac{4\pi}{5}\right)$, $\operatorname{cis}\left(-\frac{4\pi}{5}\right)$, $\operatorname{cis}\left(\frac{2\pi}{5}\right)$, $\operatorname{cis}\left(-\frac{2\pi}{5}\right)$ and 1.

e $$\therefore P(z) = \left(z - \operatorname{cis}\left(\frac{4\pi}{5}\right)\right) \times \left(z - \operatorname{cis}\left(-\frac{4\pi}{5}\right)\right) \times \left(z - \operatorname{cis}\left(\frac{2\pi}{5}\right)\right) \times \left(z - \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)$$

$$= \left(z^2 - z\left(\operatorname{cis}\left(\frac{4\pi}{5}\right) + \operatorname{cis}\left(-\frac{4\pi}{5}\right)\right) + \operatorname{cis}\left(\frac{4\pi}{5}\right) \times \operatorname{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$\times \left(z^2 - z\left(\operatorname{cis}\left(\frac{2\pi}{5}\right) + \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right) + \operatorname{cis}\left(\frac{2\pi}{5}\right) \times \operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)$$

$$= \left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)$$

24 a $w = \dfrac{az + b}{z + c}$

When $z = -3i, w = 3i$,

$\therefore \quad 3i = \dfrac{a(-3i) + b}{(-3i) + c}$ ①

and when $z = 1 + 4i, w = 1 - 4i$,

$\therefore \quad 1 - 4i = \dfrac{a(1 + 4i) + b}{(1 + 4i) + c}$ ②

① becomes

$3i(-3i + c) = a(-3i) + b$

i.e. $9 + 3ci = b - 3ai$

Equating real and imaginary parts,

$b = 9$ and $a = -c$

② becomes

$(1 - 4i)((1 + 4i) + c) = a(1 + 4i) + b$

$17 + c(1 - 4i) = a(1 + 4i) + b$

$\therefore \quad (17 + c) - 4ci = (a + b) + 4ai$

Equating real and imaginary parts,

$$17 + c = a + b \text{ and } -c = a$$

As $b = 9$ and $a = -c$, $17 + c = a + b$

becomes $17 - a = a + 9$

$\therefore \quad 2a = 8$

which implies $a = 4, b = 9, c = -4$

b $w = \dfrac{4z + 9}{z - 4}$

If $w = \bar{z}$ where $z = x + iy$,

then $\dfrac{4(x+iy)+9}{(x+iy)-4} = x - iy$

$$4(x+iy)+9 = (x-iy)((x+iy)-4)$$

$$4x + 9 + 4iy = x^2 + y^2 - 4(x - iy)$$

$$\therefore \quad x^2 - 8x + y^2 = 9$$

$$x^2 - 8x + 16 + y^2 = 25$$

$$(x-4)^2 + y^2 = 25$$

Circle centre (4, 0), radius 5

25 **a** $(1 + i\tan\theta)^5 = \left(1 + i\dfrac{\sin\theta}{\cos\theta}\right)^5$

$$= \left(\frac{1}{\cos\theta}(\cos\theta + i\sin\theta)\right)^5$$

$$= \frac{1}{\cos^5\theta} \times \text{cis}(5\theta) \quad \text{De Moivre's theorem}$$

$$= \frac{\text{cis}\,5\theta}{\cos^5\theta}$$

b $(1 + i\tan\theta)^5 = \dfrac{1}{\cos^5\theta}(\cos 5\theta + i\sin 5\theta)$

Note: $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$\therefore \quad (1 + i\tan\theta)^5 = 1 + 5i\tan\theta - \tan^2\theta - 10i\tan^3\theta + 5\tan^4\theta + i\tan^5\theta$

Equating real and imaginary parts gives

$$\frac{\cos 5\theta}{\cos^5\theta} = 1 - 10\tan^2\theta + 5\tan^4\theta$$

i.e. $\cos 5\theta = \cos^5\theta(1 - 10\tan^2\theta + 5\tan^4\theta)$

and $\dfrac{\sin 5\theta}{\cos^5\theta} = (5\tan\theta - 10\tan^3\theta + \tan^5\theta)$

$$\sin 5\theta = \cos^5\theta\,(5\tan\theta - 10\tan^3\theta + \tan^5\theta)$$

c $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$$= \frac{\cos^5\theta(5\tan\theta - 10\tan^3\theta + \tan^5\theta)}{\cos^5\theta(1 - 10\tan^2\theta + 5\tan^4\theta)}$$

Let $t = \tan\theta$,

$$\therefore \quad \tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

d Let $\theta = \dfrac{\pi}{5}$, $5\theta = \pi$ and $\tan 5\theta = 0$

$$\therefore \quad 0 = 5t - 10t^3 + t^5$$

$$\Rightarrow 0 = 5 - 10t^2 + t^4 \text{ as } t \neq 0$$

$$\Rightarrow t^4 - 10t^2 + 5 = 0$$

$$\therefore \quad t^4 - 10t^2 + 25 - 20 = 0$$

$$\therefore \quad (t^2 - 5)^2 = 20$$

$$\therefore \quad t^2 - 5 = \pm 2\sqrt{5}$$

$$\therefore \quad t^2 = 5 \pm 2\sqrt{5}$$

$$\therefore \quad t = \pm\sqrt{5 \pm 2\sqrt{5}}$$

$$\text{i.e. } \tan\frac{\pi}{5} = \pm\sqrt{5 \pm 2\sqrt{5}}$$

but $0 < \frac{\pi}{5} < \frac{\pi}{4}$,

$\therefore \quad 0 < \tan\frac{\pi}{5} < 1$

$\tan\frac{\pi}{5} = (5 - 2\sqrt{5})^{\frac{1}{2}}$

26 a
$$z + \frac{1}{z} = 2\cos\theta$$

$$\therefore \quad z^2 + 1 = 2z\cos\theta$$

$$\therefore \quad z^2 - 2z\cos\theta + 1 = 0$$

$$\therefore \quad z = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{\cos^2\theta - 1}}{2}$$

$$= \cos\theta \pm i\sqrt{\sin^2\theta}$$

$$= \cos\theta \pm i|\sin\theta|$$

$$= \text{cis }\theta \text{ or cis}(-\theta)$$

Also note: $\text{cis }\theta + \text{cis}(-\theta) = 2\cos\theta$

Hence the roots are cis θ and cis $(-\theta)$.

b P is the point representing $\alpha^n + \beta^n$

Q is the point representing $\alpha^n - \beta^n$

$$PQ = |\alpha^n + \beta^n - (\alpha^n - \beta^n)| = 2|\beta^n|$$

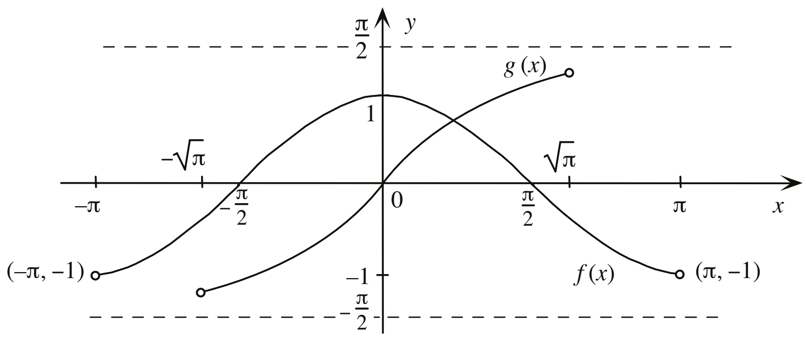
Let $\beta = \text{cis }\ \theta$

Then $PQ = 2|(\text{cis }\theta)^n| = 2|\text{cis }n\theta|$

$$= 2\sqrt{\cos^2(n\theta) + \sin^2(n\theta)} = 2$$

(The same result is valid if $\beta = \text{cis}(\theta)$.)

27 a i,ii



b **i** $\tan^{-1}\left(\frac{\pi}{4}\right) = 0.67$

ii $\cos(1) = 0.54$

c $\tan^{-1}(0) = 0$ and $\cos(0) = 1$

$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.7071 > \tan^{-1}\left(\frac{\pi}{4}\right)$$

$$\tan^{-1}(1) = \frac{\pi}{4} \approx 0.7853 > \cos(1)$$

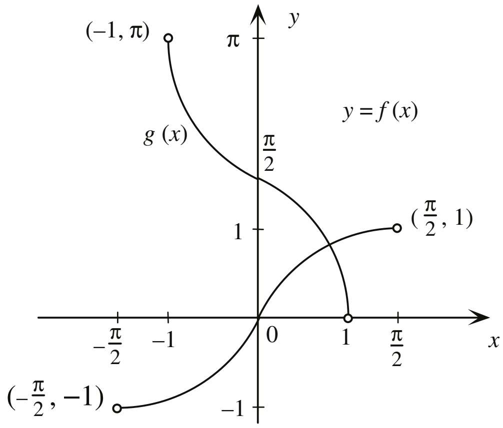
$\therefore$ solution in the interval $\left[\frac{\pi}{4}, 1\right]$.

d By using a CAS calculator, $f(x) = g(x)$ for $x = 0.82$

e $\tan^{-1}(x) > \tan^{-1}(a)$ for $x > a$ and $\tan^{-1}\left(\frac{\pi}{2}\right) = 1.004$

No other solution for $x > a$ where $f(a) = g(a)$. $\tan^{-1}\left(-\frac{\pi}{2}\right) = -1.004$, and thus it becomes clear there is only one point of intersection.

28 **a i,ii**



b **i** $\sin(0.5) = 0.48$

ii $\cos^{-1}\left(\frac{\pi}{4}\right) = 0.67$

c $\cos^{-1}(0.5) = \frac{\pi}{3}$

$\approx 1.041\,97$

i.e., $\cos^{-1}(0.5) > \sin(0.5)$

$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

≈ 0.7071

$\therefore \quad \sin\left(\frac{\pi}{4}\right) > \cos^{-1}\left(\frac{\pi}{4}\right)$

d The point of intersection is (0.768, 0.695), correct to three decimal places.

29 **a** $f(x) = a\sec\left(\frac{\pi}{15}x\right) + d$

period $= 2\pi \div \frac{\pi}{15} = 30$

when $x = 0$, $f(x) = -5$,

$\therefore \quad -5 = a\ \sec\left(\frac{\pi}{15} \times 0\right) + d$

$-5 = a + d$ ①

when $y = a$, $x = \pm 5$,

$\therefore \quad 0 = a\sec\left(\pm\frac{\pi}{3}\right) + d$

$0 = 2a + d$ ②

subtract ① from ②

$$5 = a$$

$$\therefore \quad d = -10$$

$$\therefore \quad f(x) = 5\sec\left(\frac{\pi}{15}x\right) - 10$$

b **i** When width is 7 m, $x = 3.5$,

$$\therefore \quad f(3.5) = 5\sec\left(\frac{\pi}{15} \times 3.5\right) - 10$$

$$= -3.2718$$

$\therefore$ depth is $-3.2718 - (-5) = 1.728$ metres, or 1.73 m, correct to two decimal places.

ii When depth is 2.5 metres, $y = -2.5$,

$$\therefore \quad -2.5 = 5\sec\left(\frac{\pi}{15}x\right) - 10$$

$$\frac{7.5}{5} = \sec\left(\frac{\pi}{15}x\right)$$

$$1.5 = \sec\left(\frac{\pi}{15}x\right)$$

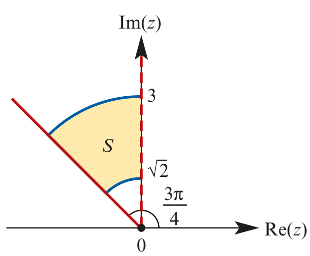
$$\frac{2}{3} = \cos\left(\frac{\pi}{15}x\right)$$

$$x = \pm\frac{\pi}{15}\cos^{-1}\left(\frac{2}{3}\right)$$

$$= \pm 4.0158$$

$\therefore$ width is 8.0316 metres, or 8.03 m (correct to two decimal places).

30 **a**



b From the diagram in **a**, $x < 0$, $y > 0$.

As $\sqrt{2} \le |z| \le 3,\ \ \sqrt{2} \le |x + iy| \le 3$

$$\therefore \quad \sqrt{2} \le \sqrt{x^2 + y^2} \le 3$$

$$\therefore \quad 2 \le x^2 + y^2 \le 9$$

Also, as $\dfrac{\pi}{2} < \text{Arg}\, z \le \dfrac{3\pi}{4},\ -\infty < \tan(\text{Arg}\, z) \le -1$

$$\therefore \quad \frac{y}{x} \le -1$$

$$\therefore \quad y \ge -x\ (x < 0)$$

Different values of x and y can be tested systematically to find z.

$x(< 0)$	$y(\ge -x)$	$x^2 + y^2$	$2 \le x^2 + y^2 \le 9$	z
-1	1	2	yes	$-1 + i$
-1	2	5	yes	$-1 + 2i$
-1	3	10	no	
-2	2	8	yes	$-2 + 2i$
-2	3	13	no	
-3	3	18	no	

The solutions are $\{z : z = -1 + i, -1 + 2i \text{ or } -2 + 2i\}$

c

$$z\bar{z} + 2\,\text{Re}(iz) \le 0$$

$$\therefore \quad (x + iy)(x - iy) + 2\,\text{Re}(i(x + iy)) \le 0$$

$$\therefore \quad x^2 - i^2y^2 + 2\,\text{Re}\,(ix + i^2y) \le 0$$

$$\therefore \quad x^2 + y^2 + 2\,\text{Re}(-y + ix) \le 0$$

$$\therefore \quad x^2 + y^2 + 2(-y) \le 0$$

$$\therefore \quad x^2 + y^2 - 2y \le 0$$

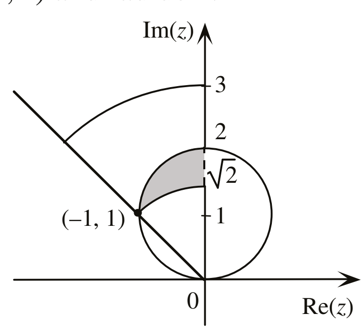
$$\therefore \quad x^2 + y^2 - 2y + 1 - 1 \le 0$$

$\therefore \quad x^2 + (y - 1)^2 \le 1$, a disc with centre (0, 1) and radius 1.

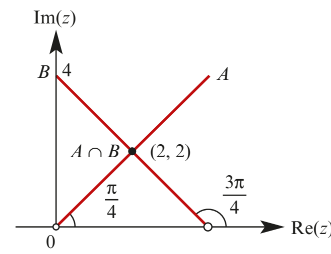
For the circle $x^2 + y^2 = 2$, when $x = -1,\ y = 1 (y > 0)$

For the circle $x^2 + (y - 1)^2 = 1$, when $x = -1, y = 1$

$S \cap T$ is represented by the shaded region.



31 a $A \cap B = \{2 + 2i\}$



b

$$\text{Let } z = x + iy$$

$$\text{Then } \left|\frac{z - \bar{z}}{z + \bar{z}}\right| \leq 1$$

$$\text{becomes } \left|\frac{(x + iy) - (x - iy)}{(x + iy) + (x - iy)}\right| \leq 1$$

$$\therefore \quad \left|\frac{2yi}{2x}\right| \leq 1$$

$$\therefore \quad \sqrt{\frac{y^2}{x^2}} \leq 1$$

$$\therefore \quad y^2 \leq x^2$$

$$\therefore \quad C = \{(x, y) : y^2 \leq x^2\}$$

$$\text{Now } z^2 + \bar{z}^2 \leq 2$$

$$\text{becomes } (x + iy)^2 + (x - iy)^2 \leq 2$$

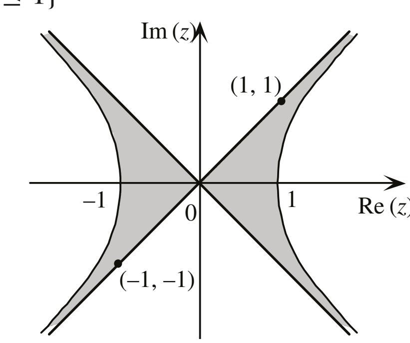
$$\therefore \quad x^2 + 2xyi + i^2y^2 + x^2 - 2xyi + i^2y^2 \leq 2$$

$$\therefore \quad 2(x^2 - y^2) \leq 2$$

$$\therefore \quad x^2 - y^2 \leq 1$$

$$\therefore \quad D = \{(x, y) : x^2 - y^2 \leq 1\}$$

$C \cap D$ is represented by the shaded region.



32 a $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$

$= \boldsymbol{i} + \sqrt{\lambda}\boldsymbol{k}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= (\boldsymbol{i} + \sqrt{\lambda}\boldsymbol{k}) - (-\boldsymbol{i} + 3\boldsymbol{j})$

$= 2\boldsymbol{i} - 3\boldsymbol{j} + \sqrt{\lambda}\boldsymbol{k}$

b $\overrightarrow{BC} \cdot \overrightarrow{BO} = |\overrightarrow{BC}||\overrightarrow{BO}| \cos \angle CBO$

$\therefore \quad \cos \angle CBO = \dfrac{\overrightarrow{BC} \cdot \overrightarrow{BO}}{|\overrightarrow{BC}||\overrightarrow{BO}|}$

Now $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$ and $\overrightarrow{BO} = -\overrightarrow{OB}$

$= (-\boldsymbol{i} + 3\boldsymbol{j}) - \boldsymbol{i}$ $\quad = -\boldsymbol{i}$

$= -2\boldsymbol{i} + 3\boldsymbol{j}$

$\therefore \quad |\overrightarrow{BC}| = \sqrt{(-2)^2 + 3^2}$ and $|\overrightarrow{BO}| = \sqrt{(-1)^2}$

$= \sqrt{13}$ $\quad = 1$

Hence $\cos \angle CBO = \dfrac{(-2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (-\boldsymbol{i} + 0\boldsymbol{j})}{\sqrt{13} \times 1}$

$= \dfrac{1}{\sqrt{13}}((-2) \times (-1) + 3 \times 0) = \dfrac{2}{\sqrt{13}}$

$\therefore \quad \angle CBO = \cos^{-1}\left(\dfrac{2}{\sqrt{13}}\right) = 56.309\,93\ldots^\circ$

The magnitude of angle CBO is 56°, to the nearest degree.

c $\overrightarrow{AO} \cdot \overrightarrow{AC} = |\overrightarrow{AO}||\overrightarrow{AC}| \cos \angle OAC$

Now $\overrightarrow{AO} = -\overrightarrow{OA}$ and $\overrightarrow{AC} = -\overrightarrow{CA}$

$= -(\boldsymbol{i} + \sqrt{\lambda}\boldsymbol{k})$ $\quad = -(2\boldsymbol{i} - 3\boldsymbol{j} + \sqrt{\lambda}\boldsymbol{k})$

$= -\boldsymbol{i} - \sqrt{\lambda}\boldsymbol{k})$ $\quad = -2\boldsymbol{i} + 3\boldsymbol{j} - \sqrt{\lambda}\boldsymbol{k}$

$\therefore \quad |\overrightarrow{AO}| = \sqrt{(-1)^2 + (\sqrt{-\lambda})^2)}$ and $|\overrightarrow{AC}| = \sqrt{(-2)^2 + 3^2 + (-\sqrt{\lambda})^2}$

$= \sqrt{1 + \lambda}$ $\quad = \sqrt{13 + \lambda}$

and as $\angle OAC = 30^\circ$, $\cos \angle OAC = \dfrac{\sqrt{3}}{2}$.

Hence $\overrightarrow{AO} \cdot \overrightarrow{AC} = |\overrightarrow{AO}||\overrightarrow{AC}| \cos \angle OAC$ becomes

$$(-\boldsymbol{i} + 0\boldsymbol{j} - \sqrt{\lambda}\boldsymbol{k}) \cdot (-2\boldsymbol{i} + 3\boldsymbol{j} - \sqrt{\lambda}\boldsymbol{k}) = \sqrt{1+\lambda}\,\sqrt{13+\lambda} \times \frac{\sqrt{3}}{2}$$

$$\therefore \quad (-1) \times (-2) + 0 \times 3 + (-\sqrt{\lambda})(-\sqrt{\lambda}) = \frac{\sqrt{3(1+\lambda)(13+\lambda)}}{2}$$

$$\therefore \quad 2 + \lambda = \frac{\sqrt{3(13 + 14\lambda + \lambda^2)}}{2}$$

$$\therefore \quad 4(2+\lambda)^2 = 3(13 + 14\lambda + \lambda^2)$$

$$\therefore \quad 16 + 16\lambda + 4\lambda^2 = 39 + 42\lambda + 3\lambda^2$$

$$\therefore \quad \lambda^2 - 26\lambda - 23 = 0$$

Using the general quadratic formula

$$\lambda = \frac{-(-26) \pm \sqrt{(-26)^2 - 4(1)(-23)}}{2 \times 1}$$

$$= \frac{26 \pm \sqrt{676 + 92}}{2} = \frac{26 \pm \sqrt{768}}{2}$$

$$= \frac{26 \pm 16\sqrt{3}}{2} = 13 \pm 8\sqrt{3}$$

$$\therefore \quad \lambda = 13 + 8\sqrt{3}, \text{ as } \lambda > 0$$

33 a $\qquad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a}$

and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \boldsymbol{d} - \boldsymbol{c}$

Now $\overrightarrow{AB} \perp \overrightarrow{CD} \;\therefore\; \overrightarrow{AB} \cdot \overrightarrow{CD} = 0$

$$\therefore \quad (\boldsymbol{b} - \boldsymbol{a})(\boldsymbol{d} - \boldsymbol{c}) = 0$$

$$\therefore \quad \boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{a} \cdot \boldsymbol{d} - \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{c} = 0 \qquad ①$$

Also $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \boldsymbol{d} - \boldsymbol{a}$

and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \boldsymbol{c} - \boldsymbol{b}$

$\overrightarrow{AD} \perp \overrightarrow{BC} \;\therefore\; \overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

$$\therefore \quad (\boldsymbol{d} - \boldsymbol{a})(\boldsymbol{c} - \boldsymbol{b}) = 0$$

$$\therefore \quad \boldsymbol{c} \cdot \boldsymbol{d} - \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{d} + \boldsymbol{a} \cdot \boldsymbol{b} = 0 \qquad ②$$

For AC to be perpendicular to BD, $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= \boldsymbol{c} - \boldsymbol{a}$$
and $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB}$
$$= \boldsymbol{d} - \boldsymbol{b}$$
$$\therefore \quad \overrightarrow{AC} \cdot \overrightarrow{BD} = (\boldsymbol{c} - \boldsymbol{a})(\boldsymbol{d} - \boldsymbol{b})$$
$$= \boldsymbol{c} \cdot \boldsymbol{d} - \boldsymbol{a} \cdot \boldsymbol{d} - \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{b}$$

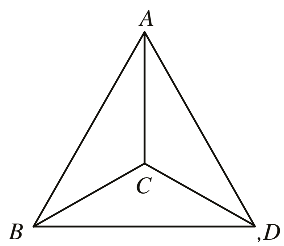
From ① $\boldsymbol{a} \cdot \boldsymbol{d} = \boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{c}$

From ② $\boldsymbol{c} \cdot \boldsymbol{d} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{a} \cdot \boldsymbol{b}$

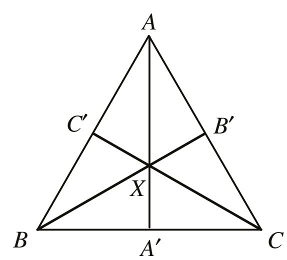
$$\therefore \quad \overrightarrow{AC} \cdot \overrightarrow{BD} = (\boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{a} \cdot \boldsymbol{b}) - (\boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{b}$$
$$= 0$$

Hence $\overrightarrow{AC} \perp \overrightarrow{BD}$, as required.

b



i Consider face *ABC*.
For an equilateral triangle, the perpendicular bisectors of the triangle coincide with the medians. It has been proved earlier that



$$\overrightarrow{BX} = \frac{2}{3}\overrightarrow{BB'}$$
$$\overrightarrow{BB'} = \overrightarrow{BO} + \overrightarrow{OB'}$$
$$= \overrightarrow{BO} + \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$$

$$\therefore \quad \overrightarrow{BB'} = -\boldsymbol{b} + \boldsymbol{a} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$$

$$= -\boldsymbol{b} + \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$$

$$\therefore \quad \overrightarrow{BX} = \frac{2}{3}\left(-\boldsymbol{b} + \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}\right)$$

$$= -\frac{2}{3}\boldsymbol{b} + \frac{1}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{c}$$

and $\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}$

$$= \boldsymbol{b} + \left(-\frac{2}{3}\boldsymbol{b} + \frac{1}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{c}\right)$$

$$= \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$$

Similarly $\overrightarrow{OY} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})$, $\overrightarrow{OZ} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{d})$

and $\overrightarrow{OW} = \frac{1}{3}(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$

ii $\overrightarrow{DX} = \overrightarrow{DO} + \overrightarrow{OX}$ $\qquad$ $\overrightarrow{BY} = \overrightarrow{BO} + \overrightarrow{OY}$

$= -\boldsymbol{d} + \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ $\qquad$ $= -\boldsymbol{b} + \frac{1}{3}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})$

$\overrightarrow{CZ} = \overrightarrow{CO} + \overrightarrow{OZ}$ $\qquad$ $\overrightarrow{AW} = \overrightarrow{AO} + \overrightarrow{OW}$

$= -\boldsymbol{c} + \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{d})$ $\qquad$ $= -\boldsymbol{a} + \frac{1}{3}(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$

iii $\overrightarrow{DP} = \frac{3}{4}\overrightarrow{DX}$

$$= \frac{3}{4}\left(-\boldsymbol{d} + \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\right)$$

$$= -\frac{3}{4}\boldsymbol{d} + \frac{1}{4}\left(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}\right)$$

$$\therefore \quad \overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP}$$

$$= \boldsymbol{d} + -\frac{3}{4}\boldsymbol{d} + \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$$

$$= \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$$

iv $\overrightarrow{OQ} = \overrightarrow{OR} = \overrightarrow{OS}$

$$= \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$$

v Hence $P = Q = R = S$.

This point is the centre of the sphere passing through each of the vertices.

34 **a** Let $P(n)$ be the proposition that

$1 + \text{cis}\theta + \text{cis}2\theta + \cdots + \text{cis}n\theta$

$= \dfrac{1 - \text{cis}((n+1)\theta)}{1 - \text{cis}\theta}$, where $n \in \mathbb{N}$.

For the base case, we let $n = 1$ then LHS of $P(k+1) = 1 + \text{cis}\theta$

and

$$\text{RHS of } P(k+1) = \frac{1 - \text{cis}2\theta}{1 - \text{cis}\theta}$$

$$= \frac{1 - \text{cis}^2\theta}{1 - \text{cis}\theta}$$

$$= \frac{(1 - \text{cis}\theta)(1 + \text{cis}\theta)}{1 - \text{cis}\theta}$$

$$= 1 + \text{cis}\theta$$

Therefore $P(1)$ is true.

For the inductive hypothesis, we assume that $P(k)$ is true so that

$1 + \text{cis}(\theta) + \text{cis}(2\theta) + \cdots + \text{cis}(k\theta)$

$= \dfrac{1 - \text{cis}((k+1)\theta)}{1 - \text{cis}(\theta)}$

for integer k.

For the inductive step, we let $n = k + 1$. Then

LHS of $P(k+1)$

$$= 1 + \text{cis}(\theta) + \text{cis}(2\theta) + \cdots + \text{cis}(k\theta) + \text{cis}((k+1)\theta)$$

$$= \frac{1 - \text{cis}((k+1)\theta)}{1 - \text{cis}(\theta)} + \text{cis}((k+1)\theta) \quad \text{(by (1))}$$

$$= \frac{1 - \text{cis}((k+1)\theta)}{1 - \text{cis}(\theta)} + \frac{\text{cis}((k+1)\theta)(1 - \text{cis}(\theta))}{1 - \text{cis}(\theta)}$$

$$= \frac{1 - \text{cis}((k+1)\theta) + \text{cis}((k+1)\theta)(1 - \text{cis}(\theta))}{1 - \text{cis}(\theta)}$$

$$= \frac{1 - \text{cis}((k+1)\theta) + \text{cis}((k+1)\theta) - \text{cis}((k+1)\theta)\text{cis}(\theta)}{1 - \text{cis}(\theta)}$$

$$= \frac{1 - \text{cis}((k+1)\theta)\text{cis}(\theta)}{1 - \text{cis}(\theta)}$$

$$= \frac{1 - \text{cis}((k+1)\theta + \theta)}{1 - \text{cis}(\theta)}$$

$$= \frac{1 - \text{cis}((k+2))\theta}{1 - \text{cis}(\theta)}$$

$= \text{RHS of } P(k+1)$

Therefore $P(k+1)$ is true.

In conclusion, $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b We find that

$$\begin{aligned}
\frac{\text{cis}\theta + 1}{\text{cis}\theta - 1} &= \frac{(1 + \cos\theta) + i\sin\theta}{(1 + \cos\theta) + i\sin\theta} \\
&= \frac{(1 + \cos\theta) + i\sin\theta}{(1 + \cos\theta) + i\sin\theta} \cdot \frac{(1 + \cos\theta) - i\sin\theta}{(1 + \cos\theta) - i\sin\theta} \\
&= \frac{\cos^2\theta - 1 + \sin^2\theta + i(\sin\theta(\cos\theta - 1) - \sin\theta(1 + \cos\theta))}{(\cos\theta - 1)^2 + \sin^2\theta} \\
&= \frac{-2i\sin\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} \\
&= \frac{-2i\sin\theta}{2 - 2\cos\theta} \\
&= \frac{-i\sin\theta}{1 - \cos\theta}\frac{1 + \cos\theta}{1 + \cos\theta} \\
&= \frac{-i\sin\theta(1 + \cos\theta)}{1 - \cos^2\theta} \\
&= \frac{-i\sin\theta(1 + \cos\theta)}{\sin^2\theta} \\
&= \frac{-i(1 + \cos\theta)}{\sin\theta} \\
&= \frac{-2i\cos^2\frac{\theta}{2}}{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})} \\
&= \frac{-i\cos\frac{\theta}{2}}{\sin(\frac{\theta}{2})} \\
&= -i\cot\frac{\theta}{2},
\end{aligned}$$

as required.

35 **a** Let x and y be positive numbers. We prove the contrapositive. That is, if $x > y$, then $x^2 > y^2$. To show this, we have $x^2 - y^2 = (x - y)(x + y) > 0$ since $x - y > 0$ and $x + y > 0$.

b $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

Hence $\boldsymbol{a} \cdot \boldsymbol{b} \le |\boldsymbol{a}||\boldsymbol{b}|$

c

$$\begin{aligned}
|\boldsymbol{a} + \boldsymbol{b}|^2 &= (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) \\
&= |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + |\boldsymbol{b}|^2 \\
&\le |\boldsymbol{a}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}| + |\boldsymbol{b}|^2 \qquad \text{by } \mathbf{b} \\
&= (|\boldsymbol{a}| + |\boldsymbol{b}|)^2
\end{aligned}$$

Therefore $|\boldsymbol{a} + \boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$ by **a**

d Using the triangle inequality on $\triangle AXB$, we know that

$BX \leq XA + AB \qquad (1)$

Let p be the perimeter of $\triangle ABC$. Using the triangle inequality on $\triangle CXB$, we know that $BX \leq XC + CB \qquad (2)$

Adding equations (1) and (2) together gives

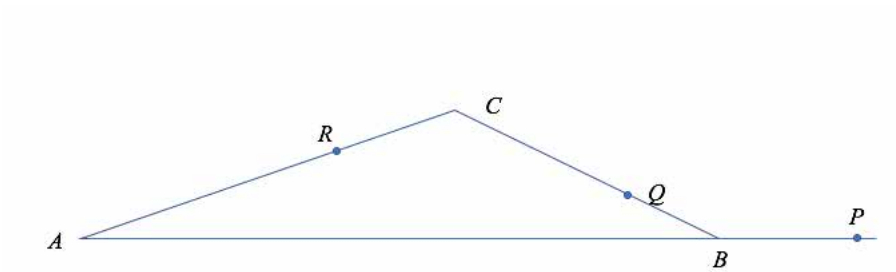
$$2\,BX \leq XA + AB + XC + CB \quad \text{as required.}$$

$$= (XA + XC) + AB + CB$$

$$= p$$

$$\Rightarrow BX = \frac{p}{2},$$

36



a $\overrightarrow{CR} = r\overrightarrow{CA} = r\boldsymbol{a}$

$\overrightarrow{CQ} = \overrightarrow{CB} - \overrightarrow{QB}$

$= \boldsymbol{b} + \overrightarrow{BQ}$

$= \boldsymbol{b} + q\overrightarrow{BC}$

$= \boldsymbol{b} - q\boldsymbol{b}$

$= (1 - q)\boldsymbol{b}$

b Line QR

$\boldsymbol{r}(\lambda) = (1 - q)\boldsymbol{b} + \lambda((q - 1)\boldsymbol{b} + r\boldsymbol{a})$

$= r\lambda\boldsymbol{a} + (1 - q)(1 - \lambda)\boldsymbol{b}$

c $\overrightarrow{AP} = p\overrightarrow{AB}$

$= p(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{CP} = \overrightarrow{CA} + p(\boldsymbol{b} - \boldsymbol{a})$

$= \boldsymbol{a} + p(\boldsymbol{b} - \boldsymbol{a})$

$= (1 - p)\boldsymbol{a} + p\boldsymbol{b}$

d P, Q and R are collinear if an only if P lies on the line RQ.

That is if and only if there exists $\lambda \in \mathbb{R}$ such that:

$(1 - p)\boldsymbol{a} + p\boldsymbol{b} = r\lambda\boldsymbol{a} + (1 - q)(1 - \lambda)\boldsymbol{b}$

If and only if there exists $\lambda \in \mathbb{R}$ such that

$1 - p = r\lambda$ and $(1 - \lambda)(1 - q) = p$

e Eliminate λ to find $(1 - p)(1 - q) = r(1 - p - q) \ldots (1)$

f We use equation (1) several times.

$$
\begin{aligned}
\text{LHS} &= \frac{p}{1-p} \times \frac{q}{1-q} \times \frac{r}{1-r} \\
&= \frac{pq}{(1-p)(1-q)} \times \frac{r}{1-r} \\
&= \frac{pq}{r(1-p-q)} \times \frac{}{1-r} \\
&= \frac{pq}{(1-p-q)(1-r)} \\
&= \frac{pq}{1-p-q-r(1-p-q)} \\
&= \frac{pq}{1-p-q-(1-p)(1-q)} \\
&= \frac{pq}{-pq} \\
&= -1 \\
&= \text{RHS}
\end{aligned}
$$

37 **a** You may like to undertake a geometric proof first but here we fully utilise the cube roots of unity.

The cube roots of unity are $1, \omega_1, \omega_2$

$z^3 - 1 = (z - 1)(z^2 + z + 1) = (z - 1)(z - \omega_1)(z - \omega_2)$

$\therefore |(z^2 + z + 1)| = |(z - \omega_1)(z - \omega_2)|$

Let $z = 1$

$3 = |1 - \omega_1| \times |1 - \omega_2|$

$3 = P_0P_1 \times P_0P_2$

b As before

$z^4 - 1 = (z - 1)(z^3 + z^2 + z + 1) = (z - 1)(z - \omega_1)(z - \omega_2)(z - \omega_3)$

$\therefore |(z^4 + z^3 + z^2 + z + 1)| = |(z - \omega_1)(z - \omega_2)(z - \omega_3)|$

Let $z = 1$

$4 = |1 - \omega_1| \times |1 - \omega_2| \times |1 - \omega_3|$

$4 = P_0P_1 \times P_0P_2 \times P_0P_3$

c $z^n - 1 = (z-1)(z^{n-1} + \cdots + z^3 + z^2 + z + 1) = (z-1)(z-\omega_1)(z-\omega_2)(z-\omega_3)\cdots(z-\omega_{n-1})$

$\therefore |z^{n-1} + \cdots + z^3 + z^2 + z + 1)| = |(z-\omega_1)(z-\omega_2)(z-\omega_3)\cdots(z-\omega_{n-1}))|$

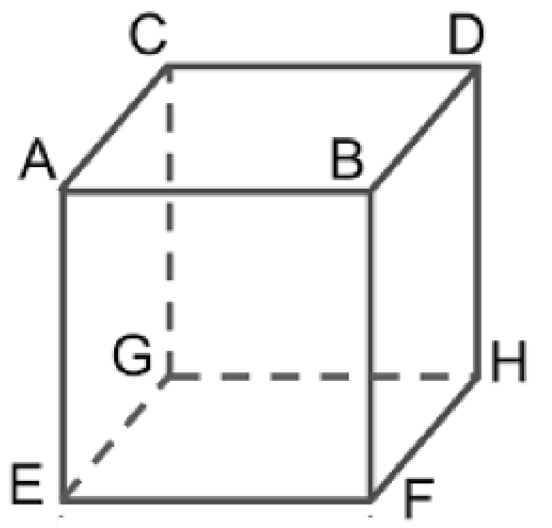
Let $z = 1$

$n = |1-\omega_1| \times |1-\omega_2| \times |1-\omega_3| \times \cdots \times |1-\omega_{n-1}|$

$n = P_0P_1 \times P_0P_2 \times P_0P_3 \cdots \times P_0P_{n-1}$

d The product of the length of the diagonals from a vertex P_0 of a regular n-sided polygon is n.

38 The origin is at the intersection of the space diagonals. Each edge has length 2.



a $|\boldsymbol{a}| = |\boldsymbol{b}| = \cdots = |\boldsymbol{g}| = |\boldsymbol{h}| = \sqrt{1+1+1} = \sqrt{3}$

b
- $-\boldsymbol{a} = \boldsymbol{h}$
- $-\boldsymbol{b} = \boldsymbol{g}$
- $-\boldsymbol{c} = \boldsymbol{f}$
- $-\boldsymbol{d} = \boldsymbol{e}$

c Examples

$\boldsymbol{a} \cdot \boldsymbol{a} = 3$ Occurs when $\boldsymbol{v} = \boldsymbol{w}$

$\boldsymbol{h} \cdot \boldsymbol{b} = 1$ Angle BOH is acute

$\boldsymbol{a} \cdot \boldsymbol{d} = -1$ Angle AOD is obtuse

$\boldsymbol{a} \cdot \boldsymbol{h} = -3$ Occurs when $\boldsymbol{v} = -\boldsymbol{w}$

d When $\boldsymbol{v} \neq \pm\boldsymbol{w}$,

$$\boldsymbol{v} \cdot \boldsymbol{w} = \pm 1$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = \pm 1$$

$$3 \cos \theta = \pm 1$$

$$\theta = \cos^{-1} \pm \frac{1}{\sqrt{3}}$$

Solutions for algorithms and pseudocode

Python programs for each of the questions are given at the end of this section. You are advised to look at the Pseudocode appendix to this book and the appropriate programming appendix.

1 a i $count \leftarrow 0$

```
for a from 1 to 100
    for b from a to 100
        for c from b to 100
            if a^2 + b^2 = c^2 and a < b then
                print (a, b, c)
            end if
            count ← count + 1
        end for
    end for
end for
```

The value of count is 171 700 down from 1 000 000

ii We define a highest common factor function.

```
define HCF(x, y)
    if x > y then
        smaller ← y
    else
        smaller ← x
    end if
    i ← 1
    while i ≤ (smaller + 1)
        if ((x is divisible by i ) and (y is divisible by i)) then
            hcf ← i
        end if
        i ← i + 1
    end while
    return hcf
```

New algorithm for Pythagorean triples

```
count ← 0
for a from 1 to 100
    for b from a to 100
        for c from b to 100
            if a^2 + b^2 = c^2 and a < b and HCF(a, b) = 1 then
                print (a, b, c)
            end if
```

```
            count ← count + 1
        end for
    end for
end for
```
[(3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (9, 40, 41), (11, 60, 61), (12, 35, 37), (13, 84, 85), (16, 63, 65), (20, 21, 29), (28, 45, 53), (33, 56, 65), (36, 77, 85), (39, 80, 89), (48, 55, 73), (65, 72, 97)]

b **i**
```
count ← 0
for a from 1 to 100
    for b from a to 100
        for c from b to 100
            if a^2 + b^2 = c^2 and a < b and HCF(a, b) = 1 then
                print (a * b * c/(a + b + c))
            end if
            count ← count + 1
        end for
    end for
end for
```
[5.0, 26.0, 75.0, 51.0, 164.0, 305.0, 185.0, 510.0, 455.0, 174.0, 530.0, 780.0, 1190.0, 1335.0, 1095.0, 1940.0]

ii
```
count ← 0
for a from 1 to 100
    for b from a to 100
        for c from b to 100
            if a^2 + b^2 = c^2 and a < b and HCF(a, b) = 1 then
                print (a * b * c/60)
            end if
            count ← count + 1
        end for
    end for
end for
```
[1.0, 13.0, 70.0, 34.0, 246.0, 671.0, 259.0, 1547.0, 1092.0, 203.0, 1113.0, 2002.0, 3927.0, 4628.0, 3212.0, 7566.0]

c $a = m^2 - n^2, b = 2mn, c = m^2 + n^2$

$a + b + c = m^2 - n^2 + 2mn + m^2 + n^2$

$= 2m^2 + 2mn$

$= 2m(m + n)$

$abc = (m^2 - n^2) \times 2mn \times (m^2 + n^2)$

$= 2mn(m+n)(m-n)(m^2+n^2)$

Therefore,

$$\frac{abc}{a+b+c} = \frac{2mn(m+n)(m-n)(m^2+n^2)}{2m(m+n)}$$

$= n(m-n)(m^2+n^2)$

$$\frac{abc}{60} = \frac{2mn(m+n)(m-n)(m^2+n^2)}{60}$$

We show that $2mn(m+n)(m-n)(m^2+n^2)$ is divisible by $3, 4$ and 5. The proof of this is through modular arithmetic but we don't employ the machinery of this here.

Divisible by 4

If m or n is even we are done.

If m and n ar odd $m+n$ is even.

Therefore divisible by 4.

Divisible by 3

If m or n is divisible by 3 we are done.

If not:

$m = 3k+1$ or $m = 3k+2$

$n = 3\ell+1$ or $n = 3\ell+2$

Considering $m^2 - n^2$:

All combinations are divisible by 3:

For example if $m = 3k+2$ and $n = 3\ell+1$

$m^2 - n^2 = 9k^2 + 12k + 4 - (9\ell^2 + 6\ell + 1) = 9(k^2 - \ell^2) + 6(2k - \ell) + 3$

Divisible by 5

If m or n is divisible by 5 we are done.

If not:

$m = 5k \pm 1$ or $m = 5k \pm 2$

$n = 5\ell \pm 1$ or $n = 5\ell \pm 2$

Considering $m^4 - n^4$:

All combinations are divisible by 5:

For example if $m = 5k+2$ and $n = 5\ell+1$

$m^4 - n^4 = 625(k^4 - \ell^4) + \cdots + 15$

d **i**

```
count ← 0
b ← 0
for a from 1 to 100
    for c from a to 100
        b ← c − 1
        if a^2 + b^2 = c^2 and a < b then
            count ← count + 1
            print (count, a, b, c)
        end if
```

```
            count ← count + 1
        end for
end for
```
[(3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61), (13, 84, 85), (15, 112, 113), (17, 144, 145), (19, 180, 181), (21, 220, 221), (23, 264, 265), (25, 312, 313), (27, 364, 365), (29, 420, 421), (31, 480, 481), (33, 544, 545), (35, 612, 613), (37, 684, 685), (39, 760, 761), (41, 840, 841), (43, 924, 925)]

There 21 of these with $c \leq 1000$

ii We see that for odd numbers, 2n+1, greater than or equal to 3 there is a Pythagorean triple.

$$\begin{aligned}(2n+1)^2 + (c-1)^2 &= c^2 \\ 4n^2 + 4n + 1 + c^2 - 2c + 1 &= c^2 \\ 2c - 1 &= 4n^2 + 4n + 1 \\ c &= 2n^2 + 2n + 1 \\ \therefore c - 1 &= 2n^2 + 2n\end{aligned}$$

Verify that $(2n+1)^2 + (2n^2 + 2n)^2 = (2n^2 + 2n + 1)^2$

There are infinitely many Pythagorean triples of this form.

iii

$$\begin{aligned}a^2 + (c-1)^2 &= c^2 \\ a^2 + c^2 - 2c + 1 &= c^2 \\ a^2 &= 2c - 1 \\ &= c + c - 1 \\ &= b + c\end{aligned}$$

2 a i

```
for a from 1 to 100
    for b from 1 to 100
        for c from 1 to 100
            for d from 1 to 100
            if a^3 + b^3 + c^3 = d^3 then
                print (a, b, c, d)
            end if
            end for
        end for
    end for
end for
```

The first few:

[(1, 6, 8, 9), (2, 12, 16, 18), (2, 17, 40, 41), (3, 4, 5, 6), (3, 10, 18, 19), . . .

ii
```
for a from 1 to 100
    for b from 1 to 100
        for c from 1 to 100
            for d from 1 to 100
            if a^3 + b^3 + c^3 = d^3 then
```

$$e \leftarrow \frac{abcd}{a+b+c+d}$$

```
                print (e, a, b, c, d)
            end if
            end for
        end for
    end for
end for
```

It does not always hold. For example it is not true for (3, 10, 18, 19).

but $3^3 + 10^3 + 18^3 = 19^3$ and $\dfrac{3 \times 10 \times 18 \times 19}{3 + 10 + 18 + 19} = 205.2$.

b
```
L ← []
for n from 3 to 8
    for x from 1 to 100
        for y from 1 to 100
            for z from 1 to 100
            if x^n + y^n = z^n then
                append (x, y, z, n) to L
            end if
            end for
        end for
    end for
end for
if L = [] then
    print("no solutions")
```

c **i**
```
L ← []
for a from 1 to 150
    for b from 1 to 1000
        for q from 1 to 100
            if a^2 + b^2 = q(1 + ab) and a < b then
                append (a, b, q) to L
            end if
```

```
        end for
    end for
end for
```
print(L)
The result is:
$[(2, 8, 4), (3, 27, 9), (4, 64, 16), (5, 125, 25), (6, 216, 36), (7, 343, 49), (8, 30, 4), \ldots$

ii We see that q is always a square.
One family of solutions is (n, n^3, n^2) since $\dfrac{a^2 + b^2}{ab + 1} = \dfrac{n^2 + n^6}{1 + n^4} = n^2$ but there are other solutions. For example $(8, 30, 4)$.

Chapter 8 – Differentiation and rational functions

Solutions to Exercise 8A

1 a Let $f(x) = x^5 \sin x$

Then by the product rule

$$\therefore \quad f'(x) = 5x^4 \sin x + x^5 \cos x$$
$$= x^4(5 \sin x + x \cos x)$$

b Let $f(x) = \sqrt{x} \cos x$

Then by the product rule

$$\therefore \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \cos x + \sqrt{x}(-\sin x),$$
$$x \neq 0$$
$$= \frac{\cos x}{2\sqrt{x}} - \sqrt{x} \sin x$$
$$= \sqrt{x}\left(\frac{\cos x}{2x} - \sin x\right)$$

c Let $f(x) = e^x \cos x$

Then by the product rule

$$\therefore \quad f'(x) = e^x \cos x + e^x(-\sin x)$$
$$= e^x(\cos x - \sin x)$$

d Let $f(x) = x^3 e^x$

Then by the product rule

$$\therefore \quad f'(x) = 3x^2 e^x + x^3 e^x$$
$$= x^2 e^x (3 + x)$$

e Let $f(x) = \sin x \cos x$

Then by the product rule

$$\therefore \quad f'(x) = \cos x \cos x + \sin x(-\sin x)$$
$$= \cos^2 x - \sin^2 x$$
$$= \cos 2x$$

2 a Let $f(x) = e^x \tan x$

Then by the product rule

$$\therefore \quad f'(x) = e^x \tan x + e^x \sec^2 x$$
$$= e^x(\tan x + \sec^2 x)$$

b Let $f(x) = x^4 \tan x$

Then by the product rule

$$\therefore \quad f'(x) = 4x^3 \tan x + x^4 \sec^2 x$$
$$= x^3(4 \tan x + x \sec^2 x)$$

c Let $f(x) = \tan x \log_e x$

Then by the product rule

$$\therefore \quad f'(x) = \sec^2 x \log_e x + \tan x \frac{1}{x}, x \neq 0$$
$$= \sec^2 x \log_e x + \frac{\tan x}{x}$$

d Let $f(x) = \sin x \tan x$

Then by the product rule

$$\therefore \quad f'(x) = \cos x \tan x + \sin x \sec^2 x$$
$$= \cos x \frac{\sin x}{\cos x} + \sin x \sec^2 x$$
$$= \sin x\,(1 + \sec^2 x)$$

e Let $f(x) = \sqrt{x} \tan x$

Then by the product rule

$$\therefore \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \tan x + \sqrt{x} \sec^2 x,$$
$$x \neq 0$$
$$= \frac{\tan x}{2\sqrt{x}} + \sqrt{x} \sec^2 x$$
$$= \sqrt{x}\left(\frac{\tan x}{2x} + \sec^2 x\right)$$

3 a Let $f(x) = \dfrac{x}{\log_e x}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{\log_e x \times 1 - x \times \frac{1}{x}}{(\log_e x)^2}, x \neq 0$$

$$= \frac{\log_e x - 1}{(\log_e x)^2}$$

b Let $f(x) = \dfrac{\sqrt{x}}{\tan x}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{\tan x \times \frac{1}{2}x^{-\frac{1}{2}} - \sqrt{x}\sec^2 x}{\tan^2 x}$$

$$= \frac{\tan x}{2\sqrt{x}\tan^2 x} - \frac{\sqrt{x}\sec^2(x)}{\tan^2(x)}$$

$$= \frac{1}{2\sqrt{x}\tan x} - \frac{\sqrt{x}\cos^2 x}{\cos^2 x \sin^2 x}$$

$$= \frac{\sqrt{x}}{2x\tan x} - \frac{\sqrt{x}}{\sin^2 x}$$

$$= \sqrt{x}\left(\frac{\cot x}{2x} - \operatorname{cosec}^2 x\right)$$

c Let $f(x) = \dfrac{e^x}{\tan x}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{\tan x\, e^x - e^x \sec^2 x}{\tan^2 x}$$

$$= \frac{e^x}{\tan x} - \frac{e^x \cos^2 x}{\cos^2 x \sin^2 x}$$

$$= e^x(\cot x - \operatorname{cosec}^2 x)$$

d Let $f(x) = \dfrac{\tan x}{\log_e x}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{\log_e x \sec^2 x - \tan x \times \frac{1}{x}}{(\log_e x)^2},$$

$$x \neq 0$$

$$= \frac{\sec^2 x}{\log_e x} - \frac{\tan x}{x(\log_e x)^2}$$

$$\text{or } f'(x) = \frac{x\log_e x \sec^2 x - \tan x}{x(\log_e x)^2}$$

e Let $f(x) = \dfrac{\sin x}{x^2}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{x^2\cos x - \sin x \times 2x}{x^4}$$

$$= \frac{\cos x}{x^2} - \frac{2\sin x}{x^3}$$

$$\text{or } = \frac{x\cos x - 2\sin x}{x^3}$$

f Let $f(x) = \dfrac{\tan x}{\cos x}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{\cos x \sec^2 x - \tan x(-\sin x)}{\cos^2 x}$$

$$= \frac{\sec^2 x}{\cos x} + \frac{\tan^2 x}{\cos x}$$

$$= \sec x(\sec^2 x + \tan^2 x)$$

g Let $f(x) = \dfrac{\cos x}{e^x}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{e^x(-\sin x) - \cos x\, e^x}{(e^x)^2}$$

$$= \frac{-(\sin x + \cos x)}{e^x}$$

h Let $f(x) = \dfrac{\cos x}{\sin x}$

Then by the quotient rule

$$\therefore \quad f'(x) = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$$

4 a Let $f(x) = \tan(x^2 + 1)$

Then by the chain rule with

$g(x) = x^2 + 1$

$$\therefore \quad f'(x) = \sec^2(x^2 + 1) \times 2x$$

$$= 2x\sec^2(x^2 + 1)$$

b Let $f(x) = \sin^2 x$

Then by the chain rule with

$g(x) = \sin x$

$\therefore \quad f'(x) = 2\sin x\cos x$

$= \sin 2x$

c Let $f(x) = e^{\tan x}$

Then by the chain rule with

$g(x) = \tan x$

$\therefore \quad f'(x) = e^{\tan x}\sec^2 x$

d Let $f(x) = \tan^5 x$

Then by the chain rule with

$g(x) = \tan x$

$\therefore \quad f'(x) = 5\tan^4 x\sec^2 x$

e Let $f(x) = \sin(\sqrt{x})$

Then by the chain rule with

$g(x) = \sqrt{x}$

$\therefore \quad f'(x) = \cos(\sqrt{x})\frac{1}{2}x^{-\frac{1}{2}}, x \neq 0$

$= \dfrac{\sqrt{x}\cos(\sqrt{x})}{2x}$

f Let $f(x) = \sqrt{\tan x}$

Then by the chain rule with

$g(x) = \tan x$

$\therefore \quad f'(x) = \frac{1}{2}(\tan x)^{-\frac{1}{2}}\sec^2 x$

$= \dfrac{\sec^2 x}{2\sqrt{\tan x}}$

$= \frac{1}{2}\sec^2 x\sqrt{\cot x}$

g Let $f(x) = \cos\left(\frac{1}{x}\right)$

Then by the chain rule with

$g(x) = \dfrac{1}{x}$

$\therefore \quad f'(x) = -\sin\left(\frac{1}{x}\right) \times -1x^{-2}$

$= \dfrac{\sin\left(\frac{1}{x}\right)}{x^2}$

h Let $f(x) = \sec^2 x = (\cos x)^{-2}$

Then by the chain rule with

$g(x) = \cos x$

$\therefore \quad f'(x) = -2(\cos x)^{-3}(-\sin x)$

$= 2\tan x\sec^2 x$

i Let $f(x) = \tan\left(\frac{x}{4}\right)$

Then by the chain rule with

$g(x) = \dfrac{x}{4}$

$\therefore \quad f'(x) = \sec^2\left(\frac{x}{4}\right) \times \frac{1}{4}$

$= \frac{1}{4}\sec^2\left(\frac{x}{4}\right)$

j Let $f(x) = \cot x = \tan\left(\frac{\pi}{2} - x\right)$

Then by the chain rule with

$g(x) = \dfrac{\pi}{2} - x$

$\therefore \quad f'(x) = \sec^2\left(\frac{\pi}{2} - x\right) \times -1$

$= -\sec^2\left(\frac{\pi}{2} - x\right)$

$= -\text{cosec}^2 x$

5 a Let $f(x) = \tan(kx), k \in R$

Then by the chain rule with

$g(x) = kx$

$\therefore \quad f'(x) = \sec^2(kx) \times k$

$= k\sec^2(kx)$

b Let $f(x) = e^{\tan(2x)}$
Then by the chain rule with
$g(x) = \tan(2x)$

$$\therefore \quad f'(x) = e^{\tan(2x)} g'(x)$$
$$= e^{\tan(2x)} \times \sec^2(2x) \times 2$$
(using the chain rule to find $g'(x)$)
$$= 2\sec^2(2x)\, e^{\tan(2x)}$$

c Let $f(x) = \tan^2(3x)$
Then by the chain rule with
$g(x) = \tan(3x)$

$$\therefore \quad f'(x) = 2\tan(3x) \times g'(x)$$
$$= 2\tan(3x)\sec^2(3x) \times 3$$
(using the chain rule to find $g'(x)$)
$$= 6\tan(3x)\sec^2(3x)$$

d Let $f(x) = \log_e x e^{\sin x}$ where
$g(x) = \log_e x$ and $h(x) = e^{\sin x}$
Then by the product rule

$$\therefore \quad f'(x) = \frac{1}{x} e^{\sin x} + \log_e x \times h'(x),$$
$$x \neq 0$$
$$= \frac{1}{x} e^{\sin x} + \log_e x e^{\sin x} \cos x$$
(using the chain rule to find $h'(x)$)
$$= e^{\sin x}\left(\frac{1}{x} + \log_e x \cos x\right)$$

e Let $f(x) = \sin^3(x^2)$
Then by the chain rule with
$g(x) = \sin(x^2)$

$$\therefore \quad f'(x) = 3\sin^2(x^2) \times g'(x)$$
$$= 3\sin^2(x^2)\cos(x^2) \times 2x$$
(using the chain rule to find $g'(x)$)
$$= 6x\ \sin^2(x^2)\cos(x^2)$$

f Let $f(x) = \dfrac{e^{3x+1}}{\cos x}$ where $g(x) = e^{3x+1}$
and $h(x) = \cos x$
Then by the quotient rule

$$\therefore \quad f'(x) = \frac{\cos x g'(x) - e^{3x+1}(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos x\, e^{3x+1} \times 3 + \sin x\, e^{3x+1}}{\cos^2 x}$$
(using the chain rule to find $g'(x)$)
$$= e^{3x+1} \sec^2 x\,(3\cos x + \sin x)$$

g Let $f(x) = e^{3x}\tan(2x)$ where
$g(x) = e^{3x}$ and $h(x) = \tan(2x)$
Then by the product rule

$$\therefore \quad f'(x) = g'(x)\tan(2x) + e^{3x} h'(x)$$
$$= e^{3x} \times 3\tan(2x)$$
$$+ e^{3x}\sec^2(2x) \times 2$$
(using the chain rule to find $g'(x)$ and $h'(x)$)
$$= e^{3x}(3\tan(2x) + 2\sec^2(2x))$$

h Let $f(x) = \sqrt{x}\tan\sqrt{x}$ where
$g(x) = \sqrt{x}$ and $h(x) = \tan\sqrt{x}$
Then by the product rule

$$\therefore \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}\tan\sqrt{x} + \sqrt{x}h'(x),$$
$$x \neq 0$$
$$= \frac{\tan\sqrt{x}}{2\sqrt{x}} + \sqrt{x}\sec^2(\sqrt{x})\frac{1}{2}x^{-\frac{1}{2}}$$
(using the chain rule to find $h'(x)$)
$$= \frac{\sqrt{x}\tan\sqrt{x}}{2x} + \frac{\sqrt{x}\sec^2\sqrt{x}}{2\sqrt{x}}$$
$$= \frac{\sqrt{x}\tan\sqrt{x}}{2x} + \frac{\sec^2\sqrt{x}}{2}$$

i Let $f(x) = \dfrac{\tan^2 x}{(x+1)^3}$ where
$g(x) = \tan^2 x$ and $h(x) = (x+1)^3$
Then by the quotient rule

$$\therefore \quad f'(x) = \frac{(x+1)^3 g'(x) - \tan^2 x h'(x)}{(x+1)^6}$$
$$= \frac{(x+1)^3 2\tan x\sec^2 x - \tan^2 x \times 3(x+1)^2 \times 1}{(x+1)^6}$$
(using the chain rule to find $g'(x)$ and $h'(x)$)
$$= \frac{2(x+1)\tan x\sec^2 x - 3\tan^2 x}{(x+1)^4}$$

j Let $f(x) = \sec^2(5x^2)$
where $g(x) = \sec(5x^2) = (\cos(5x^2))^{-1}$
Then by the chain rule

$\therefore \quad f'(x) = 2\sec(5x^2)g'(x)$

$= 2\sec(5x^2) \times -(\cos(5x^2))^{-2}$

$\times -\sin(5x^2) \times 10x$

(using the chain rule to find $g'(x)$)

$$= \frac{20x\sec(5x^2)\sin(5x^2)}{\cos^2(5x^2)}$$

$= 20x\ \sec^3(5x^2)\sin(5x^2)$

6 a $y = (x-1)^5$
Then by the chain rule with
$g(x) = x - 1$

$$\therefore \quad \frac{dy}{dx} = 5(x-1)^4 \times g'(x)$$

$= 5(x-1)^4 \times 1$

$= 5(x-1)^4$

b $y = \log_e(4x)$
Then by the chain rule with
$g(x) = 4x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4x} \times g'(x)$$

$$= \frac{1}{4x} \times 4$$

$$= \frac{1}{x}$$

c $y = e^x\tan(3x)$
Then by the product rule with
$g(x) = e^x$ and $h(x) = \tan(3x)$

$$\therefore \quad \frac{dy}{dx} = e^x\tan(3x) + e^x \times h'(x)$$

$= e^x\tan(3x) + e^x \times \sec^2(3x) \times 3$

(using the chain rule to find $h'(x)$)

$= e^x(\tan(3x) + 3\sec^2(3x))$

d $y = e^{\cos x}$
Then by the chain rule with
$g(x) = \cos x$

$$\therefore \quad \frac{dy}{dx} = e^{\cos x} \times g'(x)$$

$= e^{\cos x} \times -\sin x$

$= -\sin x\, e^{\cos x}$

e $y = \cos^3(4x)$
Then by using the chain rule with
$g(x) = \cos 4x$

$$\therefore \quad \frac{dy}{dx} = 3\cos^2(4x) \times g'(x)$$

$= 3\cos^2(4x) \times -\sin(4x) \times 4$

(using the chain rule to find $g'(x)$)

$= -12\cos^2(4x)\sin(4x)$

f $y = (\sin x + 1)^4$
Then by the chain rule with
$g(x) = \sin x + 1$

$$\therefore \quad \frac{dy}{dx} = 4 \times (\sin x + 1)^3 \times g'(x)$$

$= 4(\sin x + 1)^3 \times \cos x$

$= 4\cos x(\sin x + 1)^3$

g $y = \sin(2x)\cos x$
where $g(x) = \sin(2x)$ and $h(x) = \cos x$
Then by the product rule

$$\therefore \quad \frac{dy}{dx} = \cos x \times g'(x) + \sin(2x) \times h'(x)$$

$= \cos x \times 2\cos(2x)$

$+ \sin(2x) \times -\sin x$

$= -\sin x\sin(2x) + 2\cos(2x)\cos x$

h $$y = \frac{x^2+1}{x}$$

$$= x + \frac{1}{x}$$

$= x + x^{-1}$

$$\therefore \quad \frac{dy}{dx} = 1 - x^{-2}$$

$$= 1 - \frac{1}{x^2}$$

i $y = \dfrac{x^3}{\sin x}$
where $g(x) = x^3$ and $h(x) = \sin x$
Then by the quotient rule

$$\therefore \quad \frac{dy}{dx} = \frac{\sin x \times 3x^2 - x^3 \times \cos x}{\sin^2 x}$$

$$= \frac{x^2(3\sin x - x\cos x)}{\sin^2 x}$$

or

$$\frac{dy}{dx} = x^2 \operatorname{cosec} x(3 - x\cot x)$$

j $y = \dfrac{1}{x\log_e x}$
where $g(x) = 1$ and $h(x) = x\log_e x$
Then by the quotient rule

$$\therefore \quad \frac{dy}{dx} = \frac{x\log_e x \times 0 - 1 \times h'(x)}{(x\log_e x)^2}$$

$$= \frac{-h'(x)}{(x\log_e x)^2}$$

$$= \frac{-\left(1 \times \log_e x + x \times \frac{1}{x}\right)}{(x\log_e x)^2}$$

(using the product rule to find $h'(x)$)

$$= \frac{-(1 + \log_e x)}{(x\log_e x)^2}$$

7 **a** $\dfrac{d}{dx}(x^3) = 3x^2$

b $\dfrac{d}{dy}(2y^2 + 10y) = 4y + 10$

c $\dfrac{d}{dz}(\cos^2(z)) = 2\cos(z) \times -\sin(z)$
(using the chain rule)

$$= -2\sin(z)\cos(z)$$

$$= -\sin(2z)$$

d $\dfrac{d}{dx}(e^{\sin^2 x}) = e^{\sin^2 x} \cdot \dfrac{du}{dx}$
(using the chain rule where
$u = \sin^2 x$)
$= e^{\sin^2 x} 2\sin x\cos x$
$\left(\text{using the chain rule to find } \dfrac{du}{dx}\right)$
$= \sin(2x)e^{\sin^2 x}$

e $\dfrac{d}{dz}(1 - \tan^2 z) = -2\tan z\sec^2 z$
(using the chain rule)

f $\dfrac{d}{dy}(\operatorname{cosec}^2 y) = \dfrac{d}{dy}(\sin y)^{-2}$

$$= -2(\sin y)^{-3} \times \cos y$$

(using the chain rule)

$$= \frac{-2\cos y}{\sin^3 y}$$

$$= -2\cos y \operatorname{cosec}^3 y$$

8 Recall that

$$\frac{d}{dx}[\log_e f(x)] = \frac{f'(x)}{f(x)} \quad \text{for} \quad f(x) > 0$$

It is possible to show that

$$\boxed{\frac{d}{dx}[\log_e |f(x)|] = \frac{f'(x)}{f(x)} \quad \text{for} \quad f(x) \neq 0}$$

This result will be used throughout this question.

a Let $y = \log_e |2x + 1|$

Put $f(x) = 2x + 1 \Rightarrow f'(x) = 2$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{2x + 1} \quad \text{for} \quad x \neq -\frac{1}{2}$$

b Let $y = \log_e |-2x + 1|$

Put $f(x) = -2x + 1$

$$\Rightarrow f'(x) = -2$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2}{-2x + 1}$$

$$= \frac{2}{2x - 1} \quad \text{for} \quad x \neq \frac{1}{2}$$

c Let $y = \log_e|\sin x|$

Put $f(x) = \sin x \Rightarrow f'(x) = \cos x$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{\sin x}$$

$$= \cot x \quad \text{for} \quad \sin x \neq 0$$

d

Let $y = \log_e|\sec x + \tan x|$

Put $f(x) = \sec x + \tan x$

$$= (\cos x)^{-1} + \tan x$$

$$\therefore \quad f'(x) = -(\cos x)^{-2} - \sin x + \sec^2 x$$

$$= \frac{\sin x}{\cos^2 x} + \sec^2 x$$

$$= \sec x \tan x + \sec^2 x$$

$$= \sec x(\tan x + \sec x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$= \sec x \quad \text{for} \quad \sec x \neq -\tan x$$

e Let $y = \log_e|\operatorname{cosec} x + \tan x|$

Put $f(x) = \operatorname{cosec} x + \tan x$

$$= \frac{1}{\sin x} + \tan x$$

$$= (\sin x)^{-1} + \tan x$$

$$\therefore \quad f'(x) = -(\sin x)^{-2}\cos x + \sec^2 x$$

$$= -\frac{\cos x}{\sin^2 x} + \sec^2 x$$

$$= -\frac{\cos x}{\sin^2 x} + \frac{1}{\cos^2 x}$$

$$= \frac{\sin^2 x - \cos^3 x}{\sin^2 x \cos^2 x}$$

Re-writing $f(x)$ in terms of sine and cosine:

$f(x) = \operatorname{cosec} x + \tan x$

$$= \frac{1}{\sin x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos x + \sin^2 x}{\sin x \cos x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{\sin^2 x - \cos^3 x}{\sin^2 x \cos^2 x}}{\dfrac{\cos x + \sin^2 x}{\sin x \cos x}}$$

$$= \frac{\sin^2 x - \cos^3 x}{\sin^2 x \cos^2 x} \times \frac{\sin x \cos x}{\cos x + \sin^2 x}$$

$$= \frac{\sin^2 x - \cos^3 x}{\sin x \cos x(\cos x + \sin^2 x)}$$

f Let $y = \log_e\left|\tan \frac{1}{2}x\right|$

Put $f(x) = \tan \frac{1}{2}x$

$$\therefore \quad f'(x) = \frac{1}{2}\sec^2 \frac{1}{2}x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{1}{2}\sec^2\left(\frac{1}{2}x\right)}{\tan\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2\cos^2\left(\frac{1}{2}x\right)} \div \frac{\sin\left(\frac{1}{2}x\right)}{\cos\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2\cos^2\left(\frac{1}{2}x\right)} \times \frac{\cos\left(\frac{1}{2}x\right)}{\sin\left(\frac{1}{2}x\right)}$$

$$= \frac{1}{2\cos\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}x\right)}$$

Using the fact that $\sin(2kx) = 2\cos(kx)\sin(kx)$ and putting $k = \frac{1}{2}$, we have:

$$\sin(x) = 2\cos\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}x\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sin x} = \operatorname{cosec} x$$

g Let $y = \log_e |\text{cosec } x - \cot x|$

Put $f(x) = \text{cosec } x - \cot x$

It was established in question **3** part **h** that $\frac{d}{dx}\cot x = -\text{cosec}^2 x$

and that

$\frac{d}{dx}\text{cosec } x = -\frac{\cos x}{\sin^2 x} =$
$-\cot x \text{ cosec } x$ from question **8** part **e**.

$$\therefore \quad f'(x) = -\cot x \text{ cosec } x + \text{cosec}^2 x$$

$$= \text{cosec } x(\text{cosec } x - \cot x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\text{cosec } x(\text{cosec } x - \cot x)}{\text{cosec } x - \cot x}$$

$$= \text{cosec } x$$

h Let $y = \log_e \left| x + \sqrt{x^2 - 4} \right|$

Put $f(x) = x + (x^2 - 4)^{\frac{1}{2}}$

$$\therefore \quad f'(x) = 1 + \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}} \times 2x$$

$$= 1 + \frac{x}{\sqrt{x^2 - 4}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2 - 4}}}{x + \sqrt{x^2 - 4}}$$

$$= \frac{\frac{\sqrt{x^2 - 4}}{\sqrt{x^2 - 4}} + \frac{x}{\sqrt{x^2 - 4}}}{x + \sqrt{x^2 - 4}}$$

$$= \frac{\frac{x + \sqrt{x^2 - 4}}{\sqrt{x^2 - 4}}}{x + \sqrt{x^2 - 4}}$$

$$= \frac{x + \sqrt{x^2 - 4}}{(\sqrt{x^2 - 4})(x + \sqrt{x^2 - 4})}$$

$$= \frac{1}{\sqrt{x^2 - 4}} \quad \text{where } x \neq \pm 2$$

i Let $y = \log_e |x + \sqrt{x^2 + 4}|$

Put $f(x) = x + (x^2 + 4)^{\frac{1}{2}}$

$$\therefore \quad f'(x) = 1 + \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}} \times 2x$$

$$= 1 + \frac{x}{\sqrt{x^2 + 4}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2 + 4}}}{x + \sqrt{x^2 + 4}}$$

$$= \frac{\frac{\sqrt{x^2 + 4}}{\sqrt{x^2 + 4}} + \frac{x}{\sqrt{x^2 + 4}}}{x + \sqrt{x^2 + 4}}$$

$$= \frac{\frac{x + \sqrt{x^2 + 4}}{\sqrt{x^2 + 4}}}{x + \sqrt{x^2 + 4}}$$

$$= \frac{x + \sqrt{x^2 + 4}}{\left(\sqrt{x^2 + 4}\right)\left(x + \sqrt{x^2 + 4}\right)}$$

$$= \frac{1}{\sqrt{x^2 + 4}}$$

9 The gradient of the graph of $f(x) = \tan\left(\frac{x}{2}\right)$ is given by $f'(x)$

By the chain rule $f'(x) = \sec^2\left(\frac{x}{2}\right) \times \frac{1}{2}$

$$= \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

a When $x = 0$,

$$f'(0) = \frac{1}{2}\sec^2 0$$

$$= \frac{1}{2} \times 1$$

$$= \frac{1}{2}$$

The gradient at the point where $x = 0$ is $\frac{1}{2}$

b When $x = \frac{\pi}{3}$,

$$f'\left(\frac{\pi}{3}\right) = \frac{1}{2}\sec^2\left(\frac{\pi}{6}\right)$$
$$= \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2$$
$$= \frac{2}{3}$$

The gradient at the point where $x = \frac{\pi}{3}$ is $\frac{2}{3}$

c When $x = \frac{\pi}{2}$,

$$f'\left(\frac{\pi}{2}\right) = \frac{1}{2}\sec^2\left(\frac{\pi}{4}\right)$$
$$= \frac{1}{2}\left(\sqrt{2}\right)^2$$
$$= 1$$

The gradient at the point where $x = \frac{\pi}{2}$ is 1.

10 $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R, f(x) = \tan x$

a $f'(x) = \sec^2 x$

When $f'(x) = 4$,

$$\sec^2 x = 4$$
$$\therefore \quad \cos^2 x = \frac{1}{4}$$
$$\therefore \quad \cos x = \pm\frac{1}{2}$$
$$\left(\text{but } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \therefore \quad \cos x > 0\right)$$
$$\therefore \quad \cos x = \frac{1}{2}$$
$$\therefore \quad x = -\frac{\pi}{3} \quad \text{or} \quad \frac{\pi}{3}$$

$$f\left(-\frac{\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3} \quad \text{and} \quad f\left(\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

The coordinates of the points on the graph where the gradient is 4 are $\left(-\frac{\pi}{3}, -\sqrt{3}\right)$ and $\left(\frac{\pi}{3}, \sqrt{3}\right)$

b At the point $\left(-\frac{\pi}{3}, -\sqrt{3}\right)$ where the gradient is 4, the equation of the tangent is given by

$$y - \left(-\sqrt{3}\right) = 4\left(x - \left(-\frac{\pi}{3}\right)\right)$$
$$\therefore \quad y = 4x + \frac{4\pi}{3} - \sqrt{3}$$

At the point $\left(\frac{\pi}{3}, \sqrt{3}\right)$ where the gradient is 4, the equation of the tangent is given by

$$y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$$
$$\therefore \quad y = 4x - \frac{4\pi}{3} + \sqrt{3}$$

11 $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R, f(x) = \tan x - 8\sin x$

a i $f'(x) = \sec^2 x - 8\cos x$

The stationary points occur where $f'(x) = 0$

i.e. $\sec^2 x - 8\cos x = 0$

$$\therefore \quad \sec^2 x = 8\cos x$$
$$\therefore \quad \frac{1}{8} = \cos^3 x$$
$$\therefore \quad \cos x = \frac{1}{2}$$
$$\therefore \quad x = -\frac{\pi}{3} \quad \text{or} \quad \frac{\pi}{3}, \quad \text{since}$$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f\left(-\frac{\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right) - 8\sin\left(-\frac{\pi}{3}\right)$$
$$= -\sqrt{3} - 8\left(-\frac{\sqrt{3}}{2}\right)$$
$$= -\sqrt{3} + 4\sqrt{3}$$
$$= 3\sqrt{3}$$

and

$$f\left(\frac{\pi}{3}\right) = \tan\frac{\pi}{3} - 8\sin\frac{\pi}{3}$$
$$= \sqrt{3} - 8\left(\frac{\sqrt{3}}{2}\right)$$
$$= \sqrt{3} - 4\sqrt{3}$$
$$= -3\sqrt{3}$$

Stationary points are found at $\left(-\frac{\pi}{3}, 3\sqrt{3}\right)$ and $\left(\frac{\pi}{3}, -3\sqrt{3}\right)$

ii Consider the gradient on either side of the points where $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$

$$f'\left(-\frac{5\pi}{6}\right) = \sec^2\left(-\frac{5\pi}{6}\right) - 8\cos\left(-\frac{5\pi}{6}\right)$$
$$= -\sec^2\left(\frac{\pi}{6}\right) + 8\cos\left(\frac{\pi}{6}\right)$$
$$= -\left(\frac{2}{\sqrt{3}}\right)^2 + 8\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{4}{3} + 4\sqrt{3} > 0$$

$$f'(0) = \sec^2(0) - 8\cos(0)$$
$$= 1 - 8$$
$$= -7$$
$$< 0$$

$$f'\left(\frac{5\pi}{6}\right) = \sec^2\left(\frac{5\pi}{6}\right) - 8\cos\left(\frac{5\pi}{6}\right)$$
$$= -\sec^2\left(\frac{\pi}{6}\right) + 8\cos\left(\frac{\pi}{6}\right)$$
$$= -\frac{4}{3} + 4\sqrt{3}$$
$$> 0$$

x	$-\frac{5\pi}{6}$	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$	$\frac{5\pi}{6}$
$f'(x)$	> 0	0	< 0	0	> 0
Slope	/	—	\	—	/

Hence $\left(-\frac{\pi}{3}, 3\sqrt{3}\right)$ is a local maximum and $\left(\frac{\pi}{3}, -3\sqrt{3}\right)$ is a local minimum turning point.

b $f(x) = \tan x - 8\sin x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As $x \to -\frac{\pi}{2}, f(x) \to -\infty$

As $x \to \frac{\pi}{2}, f(x) \to \infty$

There are vertical asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

$$f(0) = \tan(0) - 8\sin(0)$$
$$= 0 - 8 \times 0$$
$$= 0$$

The y-axis intercept is 0

Let $f(x) = 0$

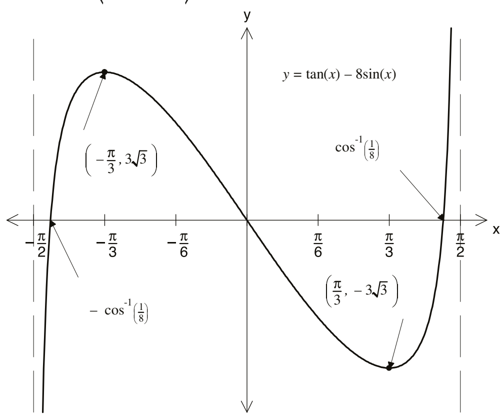
$\therefore \quad \tan x - 8\sin x = 0$

$\therefore \quad \tan x = 8\sin x$

$\therefore \quad \frac{\sin x}{\cos x} = 8\sin x$

$\therefore \quad \cos x = \frac{1}{8}$

$\therefore \quad x = \cos^{-1}\left(\frac{1}{8}\right)$ or $-\cos^{-1}\left(\frac{1}{8}\right)$, as $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



12 $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R, f(x) = e^x \sin x$

a By the product rule

$$f'(x) = e^x \cos x + e^x \sin x$$
$$= e^x(\cos x + \sin x)$$
$$f'\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$$
$$= e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$
$$= \sqrt{2}e^{\frac{\pi}{4}}$$

The gradient when $x = \frac{\pi}{4}$ is $\sqrt{2}e^{\frac{\pi}{4}}$

b When $f'(x) = 0$,

$$e^x(\cos x + \sin x) = 0$$
$$\therefore \quad \cos x + \sin x = 0$$

since $e^x > 0$ for all x

$$\therefore \quad \sin x = -\cos x$$
$$\therefore \quad \tan x = -1, \cos x \neq 0$$
$$\therefore \quad x = -\frac{\pi}{4}, \quad \text{since } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$f\left(-\frac{\pi}{4}\right) = e^{\left(-\frac{\pi}{4}\right)}\sin\left(-\frac{\pi}{4}\right)$$
$$= \frac{-\sqrt{2}}{2}e^{\left(-\frac{\pi}{4}\right)}$$

The coordinates of the point where the gradient is zero are

$$\left(-\frac{\pi}{4}, -\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}\right)$$

13 $f: \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to R, f(x) = \tan(2x)$

By the chain rule

$f'(x) = 2\sec^2(2x)$

The tangent to the graph of $f(x)$ that makes an angle of 70° with the positive direction of the x-axis intersects with $f(x)$ at $x = a$ where

$$f'(a) = \tan 70° = \tan\frac{7\pi}{18}$$
$$\therefore \quad 2\sec^2(2a) = \tan\frac{7\pi}{18}$$
$$\therefore \quad \sec^2(2a) = \frac{1}{2}\tan\frac{7\pi}{18}$$
$$\therefore \quad \sec(2a) = \pm\sqrt{\frac{\tan\frac{7\pi}{18}}{2}}$$
$$\therefore \quad \cos(2a) = \sqrt{\frac{2}{\tan\frac{7\pi}{18}}}$$
$$\left(\cos 2x > 0 \quad \text{for} \quad x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right)$$
$$\therefore \quad 2a = \cos^{-1}\left(\sqrt{\frac{2}{\tan\frac{7\pi}{18}}}\right)$$
$$\text{or } -\cos^{-1}\left(\sqrt{\frac{2}{\tan\frac{7\pi}{18}}}\right)$$
$$\therefore \quad a = \pm\frac{1}{2}\cos^{-1}\left(\sqrt{\frac{2}{\tan\frac{7\pi}{18}}}\right)$$
$$\therefore \quad a = \pm\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{2\tan\frac{7\pi}{18}}}{\tan\frac{7\pi}{18}}\right)$$

(by rationalising the denominator)

14 a $f(x) = \sec \dfrac{x}{4}$

$$= \left(\cos\left(\frac{x}{4}\right)\right)^{-1}, \cos\frac{x}{4} \neq 0$$

By the chain rule with $g(x) = \cos\left(\dfrac{x}{4}\right)$

$$f'(x) = -1\left(\cos\left(\frac{x}{4}\right)\right)^{-2} \times g'(x)$$

$$= -\sec^2\left(\frac{x}{4}\right) \times -\sin\left(\frac{x}{4}\right) \times \frac{1}{4}$$

(using the chain rule to find $g'(x)$)

$$= \frac{1}{4}\sin\left(\frac{x}{4}\right)\sec^2\left(\frac{x}{4}\right)$$

b $f'(\pi) = \dfrac{1}{4}\sin\left(\dfrac{\pi}{4}\right)\sec^2\left(\dfrac{\pi}{4}\right)$

$$= \frac{1}{4} \times \frac{1}{\sqrt{2}} \times \left(\sqrt{2}\right)^2$$

$$= \frac{\sqrt{2}}{4}$$

c $f(\pi) = \sec\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

The equation of the tangent to $y = f(x)$ at $\left(\pi, \sqrt{2}\right)$ with gradient $\dfrac{\sqrt{2}}{4}$ is given by

$$y - \sqrt{2} = \frac{\sqrt{2}}{4}(x - \pi)$$

$$\therefore \quad y = \frac{\sqrt{2}}{4}x - \frac{\sqrt{2}\pi}{4} + \sqrt{2}$$

$$= \frac{\sqrt{2}}{4}(x - \pi + 4)$$

Solutions to Exercise 8B

1 **a** $x = 2y + 6$

$$\frac{dx}{dy} = 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}$$

b $x = y^2$

$$\frac{dx}{dy} = 2y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2y}$$

c $x = (2y - 1)^2$

$$\frac{dx}{dy} = 4(2y - 1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4(2y - 1)}$$

d $x = e^y$

$$\frac{dx}{dy} = e^y$$

$$\therefore \quad \frac{dy}{dx} = e^{-y}$$

e $x = \sin 5y$

$$\frac{dx}{dy} = 5\cos 5y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{5\cos 5y}$$

f $x = \log_e y$

$$\frac{dx}{dy} = \frac{1}{y}$$

$$\therefore \quad \frac{dy}{dx} = y$$

g $x = \tan y$

$$\frac{dx}{dy} = \sec^2 y$$

$$\therefore \quad \frac{dy}{dx} = \cos^2 y$$

h $x = y^3 + y - 2$

$$\frac{dx}{dy} = 3y^2 + 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3y^2 + 1}$$

i $x = \frac{y - 1}{y} = 1 - \frac{1}{y}$

$$\frac{dx}{dy} = \frac{1}{y^2}$$

$$\therefore \quad \frac{dy}{dx} = y^2$$

j $x = ye^y$

$$\frac{dx}{dy} = ye^y + e^y = e^y(y + 1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{e^y(y + 1)}$$

2 **a** $x = y^3$

$$\frac{dx}{dy} = 3y^2$$

At $y = \frac{1}{8}, \frac{dx}{dy} = \frac{3}{64}$

$$\therefore \quad \frac{dy}{dx} = \frac{64}{3}$$

b $x = y^3$

$\dfrac{dx}{dy} = 3y^2$

At $x = \dfrac{1}{8}, y = \dfrac{1}{2}$ and $\dfrac{dx}{dy} = \dfrac{3}{4}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{4}{3}$

c $x = e^{4y}$

$\dfrac{dx}{dy} = 4e^{4y}$

At $y = 0, \dfrac{dx}{dy} = 4$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{4}$

d $x = e^{4y},$

$\dfrac{dx}{dy} = 4e^{4y}$

$= 4x$

As $x = \dfrac{1}{4}, \dfrac{dx}{dy} = 1$

$\therefore \quad \dfrac{dy}{dx} = 1$

e $x = (1 - 2y)^2$

$\dfrac{dx}{dy} = -4(1 - 2y)$

At $y = 1, \dfrac{dx}{dy} = 4$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{4}$

f $x = (1 - 2y)^2$

At $x = 4,$

$(1 - 2y)^2 = 4$

$1 - 2y = \pm 2$

$\therefore \quad \dfrac{dx}{dy} = \pm 8$ since

$\dfrac{dx}{dy} = -4(1 - 2y)$

$\therefore \quad \dfrac{dy}{dx} = \pm\dfrac{1}{8}$

g $x = \cos 2y$

$\dfrac{dx}{dy} = -2\sin 2y$

At $y = \dfrac{\pi}{6}, \dfrac{dx}{dy} = -2 \times \dfrac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$

h $x = \cos 2y$

At $x = 0,$

$\cos 2y = 0$

$2y = \dfrac{\pi}{2} + \pi k, \quad k \in Z$

$\therefore \quad \dfrac{dx}{dy} = \pm 2$ since $\dfrac{dx}{dy} = -2\sin 2y$

$\therefore \quad \dfrac{dy}{dx} = \pm\dfrac{1}{2}$

3 a $x = (2y - 1)^3$

$\dfrac{dx}{dy} = 6(2y - 1)^2$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{6(2y - 1)^2}$

b $x = e^{2y+1}$

$\dfrac{dx}{dy} = 2e^{2y+1}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{2e^{2y+1}}$

c $x = \log_e(2y - 1)$

$$\frac{dx}{dy} = \frac{2}{2y-1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(2y-1)$$

d $x = \log_e(2y) - 1$

$$\frac{dx}{dy} = \frac{1}{y}$$

$$\therefore \quad \frac{dy}{dx} = y$$

4 a $x = (2y-1)^3$

$$\therefore \quad x^{\frac{1}{3}} = 2y - 1$$

$$\therefore \quad 2y = x^{\frac{1}{3}} + 1$$

$$\therefore \quad y = \frac{1}{2}x^{\frac{1}{3}} + \frac{1}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{6}x^{-\frac{2}{3}} \quad \text{or} \quad \frac{1}{6\sqrt[3]{x^2}}$$

b $x = e^{2y+1}, x > 0$

$$\therefore \quad \log_e x = 2y + 1$$

$$\therefore \quad 2y = \log_e x - 1$$

$$\therefore \quad y = \frac{1}{2}\log_e x - \frac{1}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2} \times \frac{1}{x}$$

$$= \frac{1}{2x}$$

c $x = \log_e(2y - 1)$

$$\therefore \quad e^x = 2y - 1$$

$$\therefore \quad 2y = e^x + 1$$

$$\therefore \quad y = \frac{1}{2}e^x + \frac{1}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}e^x$$

d $x = \log_e 2y - 1$

$$\therefore \quad x + 1 = \log_e 2y$$

$$\therefore \quad e^{x+1} = 2y$$

$$\therefore \quad y = \frac{1}{2}e^{x+1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}e^{x+1}$$

5 $x = 2 - 3y^2$

When $x = -1$,

$$-1 = 2 - 3y^2$$

$$\therefore \quad 3y^2 = 3$$

$$\therefore \quad y^2 = 1$$

$$\therefore \quad y = \pm 1$$

Now $\frac{dx}{dy} = -6y$

$$\frac{dy}{dx} = \frac{-1}{6y}$$

When $y = -1 \quad \frac{dy}{dx} = \frac{1}{6}$

When $y = 1, \quad \frac{dy}{dx} = -\frac{1}{6}$

The equation of the tangent at $(-1, -1)$ with gradient $\frac{1}{6}$ is given by

$$y - (-1) = \frac{1}{6}(x - (-1))$$

$$\therefore \quad y = \frac{1}{6}x + \frac{1}{6} - 1$$

$$= \frac{1}{6}x - \frac{5}{6}$$

The equation of the tangent at $(-1, 1)$ with gradient $-\frac{1}{6}$ is given by

$$y - 1 = -\frac{1}{6}(x - (-1))$$

$$\therefore \quad y = -\frac{1}{6}x - \frac{1}{6} + 1$$

$$= -\frac{1}{6}x + \frac{5}{6}$$

6 a $x = y^2 - 4y$ and $x = y + 6$ ①

At the points of intersection

$$y^2 - 4y = y + 6$$

$$\therefore \quad y^2 - 5y - 6 = 0$$

$$\therefore \quad (y - 6)(y + 1) = 0$$

$$\therefore \quad y = -1 \quad \text{or} \quad 6$$

Substituting into ①

When $y = -1$,

$$x = -1 + 6 = 5$$

When $y = 6$,

$$x = 6 + 6 = 12$$

The points of intersection are $(5, -1)$ and $(12, 6)$

b $x = y^2 - 4y$

$$\frac{dx}{dy} = 2y - 4$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2y - 4}$$

The gradient of the tangent is that of the line $y = x - 6$, i.e. the gradient is 1

$\therefore \quad \frac{dy}{dx} = 1$ where the tangent meets the curve $x = y^2 - 4y$

$$\therefore \quad \frac{1}{2y - 4} = 1$$

$$\therefore \quad 2y - 4 = 1$$

$$\therefore \quad y = \frac{5}{2}$$

When $y = \frac{5}{2}$,

$$x = \left(\frac{5}{2}\right)^2 - 4 \times \frac{5}{2}$$

$$= \frac{25}{4} - \frac{20}{2} = -\frac{15}{4}$$

The coordinates of the point are $\left(-\frac{15}{4}, \frac{5}{2}\right)$

c The gradient of the tangent is $\frac{-1}{1} = -1$

$$\therefore \quad \frac{dy}{dx} = -1$$

$$\therefore \quad \frac{1}{2y - 4} = -1$$

$$\therefore \quad 2y - 4 = -1$$

$$\therefore \quad 2y = 3$$

$$\therefore \quad y = \frac{3}{2}$$

When $y = \frac{3}{2}$,

$$x = \left(\frac{3}{2}\right)^2 - 4 \times \frac{3}{2}$$

$$= \frac{9}{4} - \frac{12}{2} = -\frac{15}{4}$$

The coordinates of the point are $\left(-\frac{15}{4}, \frac{3}{2}\right)$

7 a

$x = y^2 - y$ and $\frac{1}{2}x = y - 1$

$$\therefore \quad x = 2y - 2 \qquad ①$$

At the points of intersection

$$y^2 - y = 2y - 2$$

$$\therefore \quad y^2 - 3y + 2 = 0$$

$$\therefore \quad (y - 2)(y - 1) = 0$$

$$\therefore \quad y = 1 \quad \text{or} \quad 2$$

Substituting into ①

When $y = 1$,

$$x = 2 \times 1 - 2 = 0$$

When $y = 2$,

$$x = 2 \times 2 - 2 = 2$$

The points of intersection are $(0, 1)$ and $(2, 2)$.

Hence it is shown that the graphs intersect where $x = 2$ at the point $(2, 2)$

b Let θ be the angle between the line $y = \frac{1}{2}x + 1$ and the positive direction of the x-axis.

$\therefore \quad \tan\theta = \frac{1}{2}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{2}\right)$

Consider $x = y^2 - y$

$\frac{dx}{dy} = 2y - 1$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2y - 1}$

At the point $(2, 2)$,

$\frac{dy}{dx} = \frac{1}{2 \times 2 - 1} = \frac{1}{3}$

i.e. the tangent to the curve $x = y^2 - y$ at $(2, 2)$ has gradient $\frac{1}{3}$, and

$\tan\alpha = \frac{1}{3}$ where α is the angle between the tangent and the positive direction of the x-axis.

$\therefore \quad \alpha = \tan^{-1}\left(\frac{1}{3}\right)$

$\therefore \quad \theta - \alpha = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right)$

$= (8.13010\ldots)^\circ$

Therefore the angle between the line $y = \frac{1}{2}x + 1$ and the tangent is 8.13°, correct to two decimal places.

Solutions to Exercise 8C

1 a Let $f(x) = \sin^{-1}\left(\frac{x}{2}\right)$

then $f'(x) = \frac{1}{\sqrt{2^2 - x^2}}, x \in (-2, 2)$

$= \frac{1}{\sqrt{4 - x^2}}$

b Let $f(x) = \cos^{-1}\left(\frac{x}{4}\right)$

then $f'(x) = \frac{-1}{\sqrt{4^2 - x^2}}, x \in (-4, 4)$

$= \frac{-1}{\sqrt{16 - x^2}}$

c Let $f(x) = \tan^{-1}\left(\frac{x}{3}\right)$

then $f'(x) = \frac{3}{3^2 + x^2}$

$= \frac{3}{9 + x^2}$

d Let $f(x) = \sin^{-1}(3x)$

then by the chain rule

$f'(x) = \frac{1}{\sqrt{1 - (3x)^2}} \times 3, 3x \in (-1, 1)$

$= \frac{3}{\sqrt{1 - 9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

e Let $f(x) = \cos^{-1}(2x)$

then by the chain rule

$f'(x) = \frac{-1}{\sqrt{1 - (2x)^2}} \times 2, 2x \in (-1, 1)$

$= \frac{-2}{\sqrt{1 - 4x^2}}, x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

f Let $f(x) = \tan^{-1}(5x)$

then by the chain rule

$f'(x) = \frac{1}{1 + (5x)^2} \times 5$

$= \frac{5}{1 + 25x^2}$

g Let $f(x) = \sin^{-1}\left(\frac{3x}{4}\right)$

then by the chain rule

$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{3}{4}\right)^2}} \times \frac{3}{4},$

$\frac{3x}{4} \in (-1, 1)$

$= \frac{3}{4\sqrt{1 - \frac{9x^2}{16}}}, x \in \left(-\frac{4}{3}, \frac{4}{3}\right)$

$= \frac{3}{\sqrt{16 - 9x^2}}$

h Let $f(x) = \cos^{-1}\left(\frac{3x}{2}\right)$

then by the chain rule

$f'(x) = \frac{-1}{\sqrt{1 - \left(\frac{3x}{2}\right)^2}} \times \frac{3}{2},$

$\frac{3x}{2} \in (-1, 1)$

$= \frac{-3}{2\sqrt{1 - \frac{9x^2}{4}}}, x \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

$= \frac{-3}{\sqrt{4 - 9x^2}}$

i Let $f(x) = \tan^{-1}\left(\frac{2x}{5}\right)$

then by the chain rule

$$f'(x) = \frac{1}{1 + \left(\frac{2x}{5}\right)^2} \times \frac{2}{5}$$

$$= \frac{2}{5\left(1 + \frac{4x^2}{25}\right)}$$

$$= \frac{10}{25 + 4x^2}$$

j Let $f(x) = \sin^{-1}(0.2x) = \sin^{-1}\left(\frac{x}{5}\right)$

$$f'(x) = \frac{1}{\sqrt{5^2 - x^2}}, x \in (-5, 5)$$

$$= \frac{1}{\sqrt{25 - x^2}}$$

k Let $f(x) = 3\cos^{-1}(x^2 - 1)$

then by the chain rule

$$f'(x) = -\frac{1}{\sqrt{1 - (x^2 - 1)^2}} \times 6x,$$

$$= \frac{-6x}{|x|\sqrt{2 - x^2}},$$

2 a Let $f(x) = \sin^{-1}(x + 1)$

then by the chain rule

$$f'(x) = \frac{1}{\sqrt{1 - (x + 1)^2}} \times 1,$$

$$x + 1 \in (-1, 1)$$

$$= \frac{1}{\sqrt{1 - (x^2 + 2x + 1)}}, \quad x \in (-2, 0)$$

$$= \frac{1}{\sqrt{-x^2 - 2x}}$$

$$= \frac{1}{\sqrt{-x(x + 2)}}$$

b Let $f(x) = \cos^{-1}(2x + 1)$

then by the chain rule

$$f'(x) = \frac{-1}{\sqrt{1 - (2x + 1)^2}} \times 2,$$

$$2x + 1 \in (-1, 1)$$

$$= \frac{-2}{\sqrt{1 - (4x^2 + 4x + 1)}},$$

$$x \in (-1, 0)$$

$$= \frac{-2}{\sqrt{-4x^2 - 4x}}$$

$$= \frac{-2}{2\sqrt{-x^2 - x}}$$

$$= \frac{-1}{\sqrt{-x(x + 1)}}$$

c Let $f(x) = \tan^{-1}(x + 2)$

then by the chain rule

$$f'(x) = \frac{1}{1 + (x + 2)^2} \times 1$$

$$= \frac{1}{1 + x^2 + 4x + 4}$$

$$= \frac{1}{x^2 + 4x + 5}$$

d Let $f(x) = \sin^{-1}(4 - x)$

then by the chain rule

$$f'(x) = \frac{1}{\sqrt{1 - (4 - x)^2}} \times -1,$$

$$4 - x \in (-1, 1)$$

$$= \frac{-1}{\sqrt{-x^2 + 8x - 15}}, x \in (3, 5)$$

e Let $f(x) = \cos^{-1}(1 - 3x)$

then by the chain rule

$$f'(x) = \frac{-1}{\sqrt{1 - (1 - 3x)^2}} \times -3,$$

$$1 - 3x \in (-1, 1)$$

$$= \frac{3}{\sqrt{-9x^2 + 6x}}, x \in \left(0, \frac{2}{3}\right)$$

f Let $f(x) = 3\tan^{-1}(1 - 2x)$

then by the chain rule

$$f'(x) = 3 \times \frac{1}{1 + (1 - 2x)^2} \times -2$$

$$= \frac{-6}{1 + (1 - 4x + 4x^2)}$$

$$= \frac{-6}{2 - 4x + 4x^2}$$

$$= \frac{-3}{2x^2 - 2x + 1}$$

g Let $f(x) = 2\sin^{-1}\left(\frac{3x+1}{2}\right)$

then by the chain rule

$$f'(x) = 2 \times \frac{1}{\sqrt{1 - \left(\frac{3x+1}{2}\right)^2}} \times \frac{3}{2},$$

$$\frac{3x+1}{2} \in (-1, 1)$$

$$= \frac{3}{\sqrt{1 - \frac{1}{4}(9x^2 + 6x + 1)}},$$

$$x \in \left(-1, \frac{1}{3}\right)$$

$$= \frac{6}{\sqrt{4 - 9x^2 - 6x - 1}}$$

$$= \frac{6}{\sqrt{-9x^2 - 6x + 3}}$$

$$= \frac{6}{\sqrt{-3(3x^2 + 2x - 1)}}$$

h Let $f(x) = -4\cos^{-1}\left(\frac{5x-3}{2}\right)$

then by the chain rule

$$f'(x) = -4 \times \frac{-1}{\sqrt{1 - \left(\frac{5x-3}{2}\right)^2}} \times \frac{5}{2},$$

$$\frac{5x-3}{2} \in (-1, 1)$$

$$= \frac{10}{\sqrt{1 - \frac{1}{4}(25x^2 - 30x + 9)}},$$

$$x \in \left(\frac{1}{5}, 1\right)$$

$$= \frac{20}{\sqrt{4 - 25x^2 + 30x - 9}}$$

$$= \frac{20}{\sqrt{-25x^2 + 30x - 5}}$$

$$= \frac{20}{\sqrt{-5(5x^2 - 6x + 1)}}$$

i Let $f(x) = 5\tan^{-1}\left(\frac{1-x}{2}\right)$

then by the chain rule

$$f'(x) = 5 \times \frac{1}{1 + \left(\frac{1-x}{2}\right)^2} \times -\frac{1}{2}$$

$$= \frac{-5}{2\left(1 + \frac{1}{4}(1 - 2x + x^2)\right)}$$

$$= \frac{-10}{x^2 - 2x + 5}$$

j Let $f(x) = -\sin^{-1}(x^2)$

then by the chain rule

$$f'(x) = -\frac{1}{\sqrt{1 - (x^2)^2}} \times 2x,$$

$$x^2 \in (0, 1)$$

$$= \frac{-2x}{\sqrt{1 - x^4}}, x \in (-1, 1)$$

3 a $y = \cos^{-1}\left(\frac{3}{x}\right), x > 3$

By the chain rule

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{3}{x}\right)^2}} \times -3x^{-2}$$

$$= \frac{3}{x^2\sqrt{1-\frac{9}{x^2}}}$$

$$= \frac{3}{x\sqrt{x^2-9}}$$

b $y = \sin^{-1}\left(\frac{5}{x}\right), \quad x > 5$

By the chain rule

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{5}{x}\right)^2}} \times -5x^{-2}$$

$$= \frac{-5}{x^2\sqrt{1-\frac{25}{x^2}}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-5}{x\sqrt{x^2-25}}$$

c $y = \cos^{-1}\left(\frac{3}{2x}\right), x > \frac{3}{2}$

By the chain rule

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{3}{2x}\right)^2}} \times -\frac{3}{2}x^{-2}$$

$$= \frac{3}{2x^2\sqrt{1-\frac{9}{4x^2}}}$$

$$= \frac{3}{x\sqrt{4x^2-9}}$$

4 **a** Let $f(x) = \sin^{-1}(ax), a > 0$

then by the chain rule

$$f'(x) = \frac{1}{\sqrt{1-(ax)^2}} \times a, ax \in (-1, 1)$$

$$= \frac{a}{\sqrt{1-a^2x^2}}, x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$$

b Let $f(x) = \cos^{-1}(ax), a > 0$

then by the chain rule

$$f'(x) = \frac{-1}{\sqrt{1-(ax)^2}} \times a, ax \in (-1, 1)$$

$$= \frac{-a}{\sqrt{1-a^2x^2}}, x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$$

c Let $f(x) = \tan^{-1}(ax), a > 0$

then by the chain rule

$$f'(x) = \frac{1}{1+(ax)^2} \times a$$
$$= \frac{a}{1+a^2x^2}$$

5 $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

a **i** $\sin^{-1}\left(\frac{x}{2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad \frac{x}{2} \in [-1, 1]$

$\therefore \quad x \in [-2, 2]$

The maximal domain of f is $[-2, 2]$

ii $\sin^{-1}\left(\frac{x}{2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad 3\sin^{-1}\left(\frac{x}{2}\right) \in \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

The range of f is $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

b $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

By the chain rule

$$f'(x) = 3 \times \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2},$$

$\frac{x}{2} \in (-1, 1)$

$$= \frac{3}{2\sqrt{1-\frac{x^2}{4}}}, x \in (-2, 2)$$

$$= \frac{3}{\sqrt{4-x^2}}$$

The domain for which the derivative exists is $(-2, 2)$.

c $f'(x) = \frac{3}{\sqrt{4-x^2}}, x \in (-2, 2)$

$= 3(4-x^2)^{-\frac{1}{2}}$

As $x \to \pm 2$, $f'(x) \to \infty$

There are vertical asymptotes at $x = -2$ and $x = 2$

The 'gradient function of $f'(x)$'

$$= -\frac{3}{2}(4-x^2)^{-\frac{3}{2}} \times -2x$$

$$= \frac{3x}{\sqrt{(4-x^2)^3}}$$

When $\frac{3x}{\sqrt{(4-x^2)^3}} = 0$,

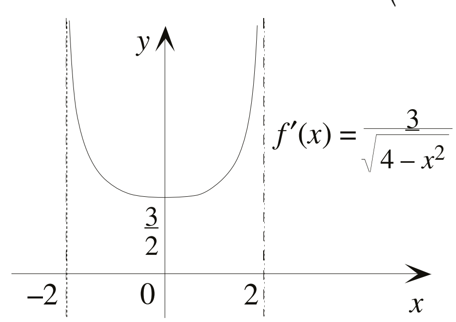
$3x = 0$

$\therefore \quad x = 0$

$$f'(0) = \frac{3}{\sqrt{4-0}}$$

$$= \frac{3}{2}$$

There is a stationary point at $\left(0, \frac{3}{2}\right)$



6 $f(x) = 4\cos^{-1}(3x)$

a **i** $\cos^{-1}(3x) \in [0, \pi]$

$\therefore \quad 3x \in [-1, 1]$

$\therefore \quad x \in \left[-\frac{1}{3}, \frac{1}{3}\right]$

The maximal domain of f is

$\left[-\frac{1}{3}, \frac{1}{3}\right]$

ii $\cos^{-1}(3x) \in [0, \pi]$

$\therefore \quad 4\cos^{-1}(3x) \in [0, 4\pi]$

The range of f is $[0, 4\pi]$

b $f(x) = 4\cos^{-1}(3x)$

By the chain rule

$$f'(x) = 4 \times \frac{-1}{\sqrt{1-(3x)^2}} \times 3,$$

$$3x \in (-1, 1)$$

$$= \frac{-12}{\sqrt{1-9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$$

The domain for which the derivative exists is $\left(-\frac{1}{3}, \frac{1}{3}\right)$

c $f'(x) = \dfrac{-12}{\sqrt{1-9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

$$= -12(1-9x^2)^{-\frac{1}{2}}$$

As $x \to \pm\frac{1}{3}, f(x) \to -\infty$

There are vertical asymptotes at $x = -\frac{1}{3}$ and $x = \frac{1}{3}$

The 'gradient function of $f'(x)$'

$$= -12 \times -\frac{1}{2}(1-9x^2)^{-\frac{3}{2}} \times -18x$$

$$= \frac{-108x}{\sqrt{(1-9x)^3}}$$

When $\dfrac{-108x}{\sqrt{(1-9x)^3}} = 0,$

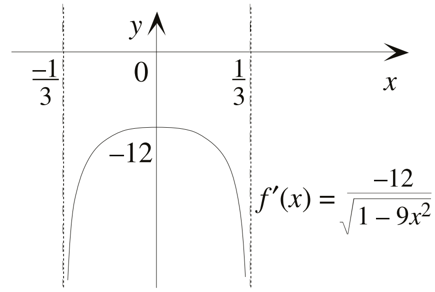
$$-108x = 0$$

$$\therefore \quad x = 0$$

$$f'(0) = \frac{-12}{\sqrt{1-0}}$$

$$= -12$$

There is a stationary point at $(0, -12)$



7 $f(x) = 2\tan^{-1}\left(\dfrac{x+1}{2}\right)$

a **i** $\tan^{-1}\left(\dfrac{x+1}{2}\right) \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$$\frac{x+1}{2} \in (-\infty, \infty)$$

$$\therefore \quad x \in (-\infty, \infty)$$

The maximal domain of f is R.

ii $\tan^{-1}\left(\dfrac{x+1}{2}\right) \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\therefore \quad 2\tan^{-1}\left(\dfrac{x+1}{2}\right) \in (-\pi, \pi)$

The range of f is $(-\pi, \pi)$

b $f(x) = 2\tan^{-1}\left(\dfrac{x+1}{2}\right)$

By the chain rule

$$f'(x) = 2 \times \frac{1}{1+\left(\frac{x+1}{2}\right)^2} \times \frac{1}{2}$$

$$= \frac{1}{1+\frac{1}{4}(x^2+2x+1)}$$

$$= \frac{4}{4+x^2+2x+1}$$

$$= \frac{4}{4+(x+1)^2}$$

$$= \frac{4}{x^2+2x+5}$$

c $f'(x) = \dfrac{4}{x^2 + 2x + 5}$

$= 4(x^2 + 2x + 5)^{-1}$

The 'gradient function of $f'(x)$'

$= -4(x^2 + 2x + 5)^{-2} \times (2x + 2)$

$= \dfrac{-8(x + 1)}{(x^2 + 2x + 5)^2}$

When $\dfrac{-8(x + 1)}{(x^2 + 2x + 5)^2} = 0$,

$-8(x + 1) = 0$

$\therefore \quad x + 1 = 0$

$\therefore \quad x = -1$

$f'(-1) = \dfrac{4}{(-1)^2 + 2(-1) + 5}$

$= \dfrac{4}{1 - 2 + 5}$

$= 1$

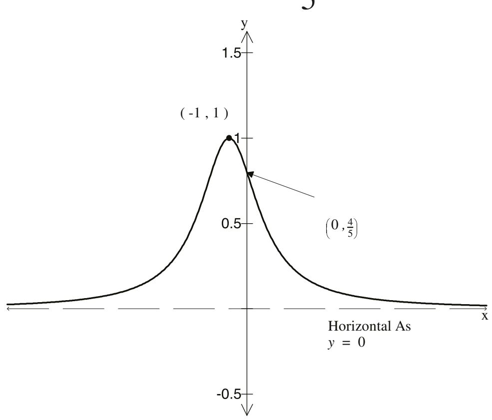
There is a stationary point at $(-1, 1)$

$f'(x) \neq 0$ for all $x \in R$ so there is a horizontal asymptote at $y = 0$, (the x-axis.)

$f'(0) = \dfrac{4}{0^2 + 2(0) + 5}$

$= \dfrac{4}{5}$

The y-axis intercept is $\dfrac{4}{5}$



8 **a** Let $f(x) = (\sin^{-1} x)^2$

then by the chain rule

$f'(x) = 2\sin^{-1} x \times \dfrac{1}{\sqrt{1 - x^2}}$,

$x \in (-1, 1)$

$= \dfrac{2\sin^{-1} x}{\sqrt{1 - x^2}}$

b Let $f(x) = \sin^{-1} x + \cos^{-1} x$

then $f'(x) = \dfrac{1}{\sqrt{1 - x^2}} + \dfrac{-1}{\sqrt{1 - x^2}}$,

$x \in (-1, 1)$

$= 0$

c Let $f(x) = \sin(\cos^{-1} x)$

then by the chain rule

$f'(x) = \cos(\cos^{-1} x) \times \dfrac{-1}{\sqrt{1 - x^2}}$,

$x \in (-1, 1)$

$= \dfrac{-x}{\sqrt{1 - x^2}}$

d Let $f(x) = \cos(\sin^{-1} x)$

then by the chain rule

$f'(x) = -\sin(\sin^{-1} x) \times \dfrac{1}{\sqrt{1 - x^2}}$,

$x \in (-1, 1)$

$= \dfrac{-x}{\sqrt{1 - x^2}}$

Note: This answer is the same as **8 c**.

e Let $f(x) = e^{\sin^{-1} x}$

then by the chain rule

$f'(x) = e^{\sin^{-1} x} \times \dfrac{1}{\sqrt{1 - x^2}}, x \in (-1, 1)$

$= \dfrac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$

f Let $f(x) = \tan^{-1}(e^x)$

then by the chain rule

$$f'(x) = \frac{1}{1 + (e^x)^2} \times e^x$$
$$= \frac{e^x}{1 + e^{2x}}$$

9 **a** $f(x) = \sin^{-1}\left(\frac{x}{3}\right)$

By the chain rule

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \times \frac{1}{3}, \frac{x}{3} \in (-1, 1)$$
$$= \frac{1}{3\sqrt{1 - \frac{x^2}{9}}}, x \in (-3, 3)$$
$$= \frac{1}{\sqrt{9 - x^2}}$$

Note: This answer could be obtained directly from the rule.

$$f'(1) = \frac{1}{\sqrt{9 - 1^2}}$$
$$= \frac{1}{\sqrt{8}}$$
$$= \frac{\sqrt{2}}{4}$$
$$= 0.35355\ldots$$

The gradient of $f(x)$ at $x = 1$ is 0.35, correct to two decimal places.

b $f(x) = 2\cos^{-1}(3x)$

By the chain rule

$$f'(x) = 2 \times \frac{-1}{\sqrt{1 - (3x)^2}} \times 3,$$
$$3x \in (-1, 1)$$
$$= \frac{-6}{\sqrt{1 - 9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$$

$$f'(0.1) = \frac{-6}{\sqrt{1 - 9(0.1)^2}}$$
$$= \frac{-6}{\sqrt{1 - 0.09}}$$
$$= \frac{-6}{\sqrt{0.91}}$$
$$= \frac{-60}{\sqrt{91}}$$
$$= -6.28970\ldots$$

The gradient of $f(x)$ at $x = 0.1$ is -6.29, correct to two decimal places.

c $f(x) = 3\tan^{-1}(2x + 1)$

By the chain rule

$$f'(x) = 3 \times \frac{1}{1 + (2x + 1)^2} \times 2$$
$$= \frac{6}{1 + 4x^2 + 4x + 1}$$
$$= \frac{6}{4x^2 + 4x + 2}$$
$$= \frac{3}{2x^2 + 2x + 1}$$
$$f'(1) = \frac{3}{2(1)^2 + 2(1) + 1}$$
$$= \frac{3}{5}$$

The gradient of $f(x)$ at $x = 1$ is $\frac{3}{5}$

10 **a** $f(x) = 2\sin^{-1} x$

$$\therefore \quad f'(x) = \frac{2}{\sqrt{1 - x^2}}, x \in (-1, 1)$$

Now $f'(a) = 4$

$$\therefore \quad \frac{2}{\sqrt{1 - a^2}} = 4$$
$$\therefore \quad \sqrt{1 - a^2} = \frac{1}{2}$$
$$\therefore \quad 1 - a^2 = \frac{1}{4}$$

$\therefore \quad a^2 = \frac{3}{4}$

$\therefore \quad a = \pm\frac{\sqrt{3}}{2}$

b $f(x) = 3\cos^{-1}\frac{x}{2}$

$\therefore \quad f'(x) = \frac{-3}{\sqrt{2^2 - x^2}}, x \in (-2, 2)$

$= \frac{-3}{\sqrt{4 - x^2}}$

Now $f'(a) = -10$

$\therefore \quad \frac{-3}{\sqrt{4 - a^2}} = -10$

$\therefore \quad \sqrt{4 - a^2} = \frac{3}{10}$

$\therefore \quad 4 - a^2 = \frac{9}{100}$

$\therefore \quad a^2 = \frac{391}{100}$

$\therefore \quad a = \pm\frac{\sqrt{391}}{10}$

c $f(x) = \tan^{-1}(3x)$

$\therefore \quad f'(x) = \frac{1}{1 + (3x)^2} \times 3$

$= \frac{3}{1 + 9x^2}$

Now $f'(a) = 0.5$

$\therefore \quad \frac{3}{1 + 9a^2} = 0.5$

$\therefore \quad 1 + 9a^2 = \frac{3}{0.5}$

$\therefore \quad 9a^2 = 5$

$\therefore \quad a^2 = \frac{5}{9}$

$\therefore \quad a = \pm\frac{\sqrt{5}}{3}$

d $f(x) = \sin^{-1}\left(\frac{x + 1}{2}\right)$

$\therefore \quad f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x + 1}{2}\right)^2}} \times \frac{1}{2},$

$\frac{x + 1}{2} \in (-1, 1)$

$= \frac{1}{2\sqrt{1 - \frac{1}{4}(x^2 + 2x + 1)}},$

$x \in (-3, 1)$

$= \frac{1}{\sqrt{4 - x^2 - 2x - 1}}$

$= \frac{1}{\sqrt{-x^2 - 2x + 3}}$

Now $f'(a) = 20$

$\therefore \quad \frac{1}{\sqrt{-a^2 - 2a + 3}} = 20$

$\therefore \quad \sqrt{-a^2 - 2a + 3} = \frac{1}{20}$

$\therefore \quad -a^2 - 2a + 3 = \frac{1}{400}$

$\therefore \quad -a^2 - 2a + \frac{1199}{400} = 0$

$\therefore \quad a^2 + 2a - \frac{1199}{400} = 0$

$\therefore \quad a^2 + 2a + 1 - 1 - \frac{1199}{400} = 0$

$\therefore \quad (a + 1)^2 - \frac{1599}{400} = 0$

$\therefore \quad (a + 1)^2 = \frac{1599}{400}$

$\therefore \quad a + 1 = \pm\frac{\sqrt{1599}}{20}$

$\therefore \quad a = -1 \pm \frac{\sqrt{1599}}{20}$

e $f(x) = 2\cos^{-1}\left(\dfrac{2x}{3}\right)$

$$\therefore \quad f'(x) = 2 \times \frac{-1}{\sqrt{1 - \left(\frac{2x}{3}\right)^2}} \times \frac{2}{3},$$

$$\frac{2x}{3} \in (-1, 1)$$

$$= \frac{-4}{3\sqrt{1 - \frac{4x^2}{9}}}, x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$$

$$= \frac{-4}{\sqrt{9 - 4x^2}}$$

Now $f'(a) = -8$

$$\therefore \quad \frac{-4}{\sqrt{9 - 4a^2}} = -8$$

$$\therefore \quad \sqrt{9 - 4a^2} = \frac{1}{2}$$

$$\therefore \quad 9 - 4a^2 = \frac{1}{4}$$

$$\therefore \quad 4a^2 = \frac{35}{4}$$

$$\therefore \quad a^2 = \frac{35}{16}$$

$$\therefore \quad a = \pm\frac{\sqrt{35}}{4}$$

f $f(x) = 4\tan^{-1}(2x - 1)$

$$\therefore \quad f'(x) = 4 \times \frac{1}{1 + (2x - 1)^2} \times 2$$

$$= \frac{8}{1 + 4x^2 - 4x + 1}$$

$$= \frac{8}{4x^2 - 4x + 2}$$

$$= \frac{4}{2x^2 - 2x + 1}$$

Now $f'(a) = \dfrac{4}{2a^2 - 2a + 1} = 1$

$$\therefore \quad 2a^2 - 2a + 1 = 4$$

$$\therefore \quad 2a^2 - 2a - 3 = 0$$

$$\therefore \quad a^2 - a - \frac{3}{2} = 0$$

$$\therefore \quad a^2 - a + \frac{1}{4} - \frac{1}{4} - \frac{3}{2} = 0$$

$$\therefore \quad \left(a - \frac{1}{2}\right)^2 - \frac{7}{4} = 0$$

$$\therefore \quad \left(a - \frac{1}{2}\right)^2 = \frac{7}{4}$$

$$\therefore \quad a - \frac{1}{2} = \pm\frac{\sqrt{7}}{2}$$

$$\therefore \quad a = \frac{1}{2} \pm \frac{\sqrt{7}}{2}$$

$$\therefore \quad a = \frac{1}{2}(1 \pm \sqrt{7})$$

11 The gradient of the tangent is given by $\dfrac{dy}{dx}$

a $y = \sin^{-1}(2x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - (2x)^2}} \times 2,$$

$$2x \in (-1, 1)$$

$$= \frac{2}{\sqrt{1 - 4x^2}}, x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

When $x = \dfrac{1}{4}$,

$$y = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

and $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1 - 4\left(\frac{1}{4}\right)^2}}$

$$= \frac{2}{\sqrt{1 - \frac{1}{4}}}$$

$$= \frac{2}{\sqrt{\frac{3}{4}}}$$

$$= \frac{4\sqrt{3}}{3}$$

Hence the equation of the tangent is given by

$$y - \frac{\pi}{6} = \frac{4\sqrt{3}}{3}\left(x - \frac{1}{4}\right)$$

$$= \frac{4\sqrt{3}}{3}x - \frac{\sqrt{3}}{3}$$

$$\therefore \quad y = \frac{4\sqrt{3}}{3}x - \frac{\sqrt{3}}{3} + \frac{\pi}{6}$$

b $y = \tan^{-1}(2x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1 + (2x)^2} \times 2$$

$$= \frac{2}{1 + 4x^2}$$

When $x = \frac{1}{2}$,

$$y = \tan^{-1}(1) = \frac{\pi}{4}$$

and $\frac{dy}{dx} = \frac{2}{1 + 4\left(\frac{1}{2}\right)^2}$

$$= \frac{2}{1 + 1}$$

$$= 1$$

Hence the equation of the tangent is given by

$$y - \frac{\pi}{4} = 1\left(x - \frac{1}{2}\right)$$

$$= x - \frac{1}{2}$$

$$\therefore \quad y = x - \frac{1}{2} + \frac{\pi}{4}$$

c $y = \cos^{-1}(3x)$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1 - (3x)^2}} \times 3,$$

$3x \in (-1, 1)$

$$= \frac{-3}{\sqrt{1 - 9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$$

When $x = \frac{1}{6}$,

$$y = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

and $\frac{dy}{dx} = \frac{-3}{\sqrt{1 - 9\left(\frac{1}{6}\right)^2}}$

$$= \frac{-3}{\sqrt{1 - \frac{1}{4}}}$$

$$= \frac{-3}{\sqrt{\frac{3}{4}}}$$

$$= -2\sqrt{3}$$

Hence the equation of the tangent is given by

$$y - \frac{\pi}{3} = -2\sqrt{3}\left(x - \frac{1}{6}\right)$$

$$= -2\sqrt{3}x + \frac{\sqrt{3}}{3}$$

$$\therefore \quad y = -2\sqrt{3}x + \frac{\sqrt{3}}{3} + \frac{\pi}{3}$$

$$\therefore \quad y = -2\sqrt{3}x + \frac{\sqrt{3} + \pi}{3}$$

d $y = \cos^{-1}(3x)$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1 - (3x)^2}} \times 3,$$

$3x \in (-1, 1)$

$$= \frac{-3}{\sqrt{1 - 9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$$

When $x = \frac{1}{2\sqrt{3}}$,

$$y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

and $$\frac{dy}{dx} = \frac{-3}{\sqrt{1 - 9\left(\frac{1}{2\sqrt{3}}\right)^2}}$$

$$= \frac{-3}{\sqrt{1 - \frac{9}{12}}}$$

$$= -6$$

Hence the equation of the tangent is given by

$$y - \frac{\pi}{6} = -6\left(x - \frac{1}{2\sqrt{3}}\right)$$

$$= -6x + \sqrt{3}$$

$$\therefore \quad y = -6x + \sqrt{3} + \frac{\pi}{6}$$

12 $f(x) = \cos^{-1}\left(\frac{6}{x}\right)$

a $\cos^{-1}\left(\frac{6}{x}\right) \in [0, \pi]$

$$\therefore \quad \frac{6}{x} \in [-1, 1] \setminus \{0\}$$

$$\therefore \quad x \leq -6 \quad \text{or} \quad x \geq 6$$

The maximal domain is

$\{x : x \leq -6\} \cup \{x : x \geq 6\}$

b $$f'(x) = \frac{-1}{\sqrt{1 - \left(\frac{6}{x}\right)^2}} \times -6x^{-2},$$

$$\frac{6}{x} \in [-1, 1], \frac{6}{x} \neq 0$$

$$= \frac{6}{x^2\sqrt{1 - \frac{36}{x^2}}}$$

$$= \frac{6}{x\sqrt{x^2 - 36}},$$

$$x < -6 \text{ or } x > 6$$

for $x > 6$, $x^2 > 36$

$$\therefore \quad \sqrt{x^2 - 36} > 0$$

and $$\frac{6}{x\sqrt{x^2 - 36}} > 0$$

$$\therefore \quad f'(x) > 0$$

c

$$f(x) = \cos^{-1}\left(\frac{6}{x}\right), x \le -6 \quad \text{or} \quad x \ge 6$$

$$f(-6) = \cos^{-1}(-1)$$

$$= \pi$$

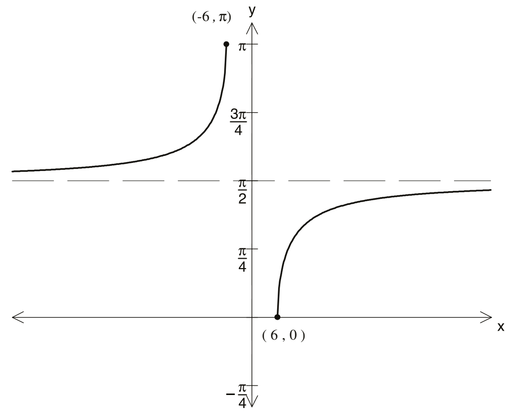
$$f(6) = \cos^{-1}(1)$$

$$= 0$$

As $x \to +\infty, f(x) \to \frac{\pi}{2}$ from below.

As $x \to -\infty, f(x) \to \frac{\pi}{2}$ from above.

Hence $y = \frac{\pi}{2}$ is a horizontal asymptote.



Solutions to Exercise 8D

1 a Let $f(x) = 2x + 5$

Then $f'(x) = 2$

and $f''(x) = 0$

b Let $f(x) = x^8$

Then $f'(x) = 8x^7$

and $f''(x) = 56x^6$

c Let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

Then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \frac{-1}{4\sqrt{x^3}}$

d Let $f(x) = (2x + 1)^4$

Then $f'(x) = 4(2x + 1)^3 \times 2$

$= 8(2x + 1)^3$

and $f''(x) = 24(2x + 1)^2 \times 2$

$= 48(2x + 1)^2$

e Let $f(x) = \sin x$

Then $f'(x) = \cos x$

and $f''(x) = -\sin x$

f Let $f(x) = \cos x$

Then $f'(x) = -\sin x$

and $f''(x) = -\cos x$

g Let $f(x) = e^x$

Then $f'(x) = e^x$

and $f''(x) = e^x$

h Let $f(x) = \log_e x$

Then $f'(x) = \frac{1}{x} = x^{-1}$

and $f''(x) = -x^{-2} = \frac{-1}{x^2}$

i Let $f(x) = \frac{1}{x + 1} = (x + 1)^{-1}$

Then $f'(x) = -1(x + 1)^{-2} \times 1$

$= -(x + 1)^{-2}$

and $f''(x) = 2(x + 1)^{-3} \times 1$

$= \frac{2}{(x + 1)^3}$

j Let $f(x) = \tan x$

Then $f'(x) = \sec^2 x = (\cos x)^{-2}$

and $f''(x) = -2(\cos x)^{-3} \times -\sin x$

$= 2\sin x \sec^3 x$

2 a Let $y = \sqrt{x^5} = x^{\frac{5}{2}}$

Then $\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$

and $\frac{d^2y}{dx^2} = \frac{15}{4}x^{\frac{1}{2}} = \frac{15\sqrt{x}}{4}$

b Let $y = (x^2 + 3)^4$

Then $\frac{dy}{dx} = 4(x^2 + 3)^3 \times 2x$

$= 8x(x^2 + 3)^3$

and
$\frac{d^2y}{dx^2} = 8x \times 3(x^2 + 3)^2 \times 2x$

$+ 8(x^2 + 3)^3$

$= 48x^2(x^2 + 3)^2 + 8(x^2 + 3)^3$

$= 8(x^2 + 3)^2(7x^2 + 3)$

c Let $y = \sin\frac{x}{2}$

Then $\frac{dy}{dx} = \frac{1}{2}\cos\frac{x}{2}$

and $\frac{d^2y}{dx^2} = -\frac{1}{4}\sin\frac{x}{2}$

d Let $y = 3\cos(4x + 1)$

Then $\frac{dy}{dx} = -3\sin(4x + 1) \times 4$

$= -12\sin(4x + 1)$

and $\frac{d^2y}{dx^2} = -12\cos(4x + 1) \times 4$

$= -48\cos(4x + 1)$

e Let $y = \frac{1}{2}e^{2x+1}$

Then $\frac{dy}{dx} = \frac{1}{2}e^{2x+1} \times 2 = e^{2x+1}$

and $\frac{d^2y}{dx^2} = e^{2x+1} \times 2 = 2e^{2x+1}$

f Let $y = \log_e(2x + 1)$

Then $\frac{dy}{dx} = \frac{1}{2x+1} \times 2 = 2(2x + 1)^{-1}$

and $\frac{d^2y}{dx^2} = -2(2x + 1)^{-2} \times 2$

$= \frac{-4}{(2x + 1)^2}$

g Let $y = 3\tan(x - 4)$

Then $\frac{dy}{dx} = 3\sec^2(x - 4)$

$= 3(\cos(x - 4))^{-2}$

and $\frac{d^2y}{dx^2} = -6(\cos(x - 4))^{-3}$

$\times (-\sin(x - 4))$

$= 6\sin(x - 4)\sec^3(x - 4)$

h Let $y = 4\sin^{-1}(x)$

Then $\frac{dy}{dx} = \frac{4}{\sqrt{1 - x^2}}, x \in (-1, 1)$

$= 4(1 - x^2)^{-\frac{1}{2}}$

and $\frac{d^2y}{dx^2} = -2(1 - x^2)^{-\frac{3}{2}} \times (-2x)$

$= \frac{4x}{\sqrt{(1 - x^2)^3}}$

i Let $y = \tan^{-1}(x)$

Then $\frac{dy}{dx} = \frac{1}{1 + x^2} = (1 + x^2)^{-1}$

and

$\frac{d^2y}{dx^2} = -(1 + x^2)^{-2} \times 2x = \frac{-2x}{(1 + x^2)^2}$

j Let $y = 2(1 - 3x)^5$

Then $\frac{dy}{dx} = 10(1 - 3x)^4 \times (-3)$

$= -30(1 - 3x)^4$

and $\frac{d^2y}{dx^2} = -120(1 - 3x)^3 \times (-3)$

$= 360(1 - 3x)^3$

3 a $f(x) = 6e^{3-2x}$

$f'(x) = 6e^{3-2x} \times (-2) = -12e^{3-2x}$

$f''(x) = -12e^{3-2x} \times (-2) = 24e^{3-2x}$

b $f(x) = -8e^{-0.5x^2}$

$f'(x) = -8e^{-0.5x^2} \times (-x) = 8x\, e^{-0.5x^2}$

$f''(x) = 8x\, e^{-0.5x^2} \times (-x) + 8e^{-0.5x^2}$

$= -8x^2e^{-0.5x^2} + 8e^{-0.5x^2}$

$= 8e^{-0.5x^2}(1 - x^2)$

c $f(x) = e^{\log_e x} = x$

$f'(x) = 1$

$f''(x) = 0$

d $f(x) = \log_e(\sin x)$

$f'(x) = \frac{1}{\sin x} \times \cos x = (\tan x)^{-1}$

$f''(x) = -(\tan x)^{-2} \times \sec^2 x$

$= \frac{-\cos^2 x}{\sin^2 x \cos^2 x}$

$= -\operatorname{cosec}^2 x$

e $f(x) = 3\sin^{-1}\left(\frac{x}{4}\right)$

$f'(x) = \frac{3}{\sqrt{16 - x^2}}, x \in (-4, 4)$

$= 3(16 - x^2)^{-\frac{1}{2}}$

$f''(x) = -\frac{3}{2}(16 - x^2)^{-\frac{3}{2}} \times -2x$

$= \frac{3x}{\sqrt{(16 - x^2)^3}}$

f $f(x) = \cos^{-1}(3x)$

$f'(x) = \frac{-1}{\sqrt{1 - (3x)^2}} \times 3,$

$3x \in (-1, 1)$

$= \frac{-3}{\sqrt{1 - 9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

$= -3(1 - 9x^2)^{-\frac{1}{2}}$

$f''(x) = \frac{3}{2}(1 - 9x^2)^{-\frac{3}{2}} \times (-18x)$

$= \frac{-27x}{\sqrt{(1 - 9x^2)^3}}$

g $f(x) = 2\tan^{-1}\left(\frac{2x}{3}\right)$

$f'(x) = \frac{2}{1 + \left(\frac{2x}{3}\right)^2} \times \frac{2}{3}$

$= \frac{4}{3\left(1 + \frac{4x^2}{9}\right)}$

$= \frac{12}{9 + 4x^2}$

$= 12(9 + 4x^2)^{-1}$

$f''(x) = -12(9 + 4x^2)^{-2}(8x)$

$= \frac{-96x}{(9 + 4x^2)^2}$

h $f(x) = \frac{1}{\sqrt{1 - x}} = (1 - x)^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}(1 - x)^{-\frac{3}{2}} \times (-1)$

$= \frac{1}{2}(1 - x)^{-\frac{3}{2}}$

$f''(x) = -\frac{3}{4}(1 - x)^{-\frac{5}{2}} \times (-1)$

$= \frac{3}{4\sqrt{(1 - x)^5}}$

i $f(x) = 5\sin(3 - x)$

$f'(x) = 5\cos(3 - x) \times (-1)$

$= -5\cos(3 - x)$

$f''(x) = -5 \times (-\sin(3 - x)) \times (-1)$

$= -5\sin(3 - x)$

j $f(x) = \tan(1 - 3x)$

$$f'(x) = \sec^2(1 - 3x) \times (-3)$$
$$= -3\sec^2(1 - 3x)$$
$$= -3(\cos(1 - 3x))^{-2}$$
$$f''(x) = 6(\cos(1 - 3x))^{-3} \times (-\sin(1 - 3x)) \times (-3)$$
$$= 18\sin(1 - 3x)\sec^3(1 - 3x)$$

k $f(x) = \sec\left(\frac{x}{3}\right) = \left(\cos\left(\frac{x}{3}\right)\right)^{-1}$

$$f'(x) = -\left(\cos\left(\frac{x}{3}\right)\right)^{-2} \times \left(-\sin\left(\frac{x}{3}\right)\right) \times \frac{1}{3}$$
$$= \frac{1}{3}\sin\left(\frac{x}{3}\right)\sec^2\left(\frac{x}{3}\right)$$
$$f''(x) = \frac{1}{3}\sin\left(\frac{x}{3}\right) \times -2\left(\cos\frac{x}{3}\right)^{-3} \times \left(-\sin\frac{x}{3}\right) \times \frac{1}{3} + \frac{1}{3}\cos\left(\frac{x}{3}\right) \times \frac{1}{3} \times \sec^2\left(\frac{x}{3}\right)$$
$$= \frac{2}{9}\sec^3\left(\frac{x}{3}\right)\sin^2\left(\frac{x}{3}\right) + \frac{1}{9}\sec\left(\frac{x}{3}\right)$$
$$= \frac{1}{9}\sec\left(\frac{x}{3}\right)\left(2\tan^2\left(\frac{x}{3}\right) + 1\right)$$

l $f(x) = \operatorname{cosec}\left(\frac{x}{4}\right) = \left(\sin\left(\frac{x}{4}\right)\right)^{-1}$

$$f'(x) = -\left(\sin\left(\frac{x}{4}\right)\right)^{-2} \times \cos\left(\frac{x}{4}\right) \times \frac{1}{4}$$
$$= -\frac{1}{4}\cos\left(\frac{x}{4}\right)\left(\sin\left(\frac{x}{4}\right)\right)^{-2}$$
$$f''(x) = -\frac{1}{4}\cos\left(\frac{x}{4}\right) \times -2\left(\sin\left(\frac{x}{4}\right)\right)^{-3} \times \frac{1}{4}\cos\left(\frac{x}{4}\right) + \frac{1}{4}\left(\sin\left(\frac{x}{4}\right)\right)^{-2} \times \frac{1}{4}\sin\left(\frac{x}{4}\right)$$
$$= \frac{1}{8}\cos^2\left(\frac{x}{4}\right)\left(\sin\left(\frac{x}{4}\right)\right)^{-3} + \frac{1}{16}\sin\left(\frac{x}{4}\right)^{-1}$$
$$= \frac{2\cos^2\left(\frac{x}{4}\right) + \sin^2\left(\frac{x}{4}\right)}{16\sin^3\left(\frac{x}{4}\right)}$$
$$= \frac{1 + \cos^2\left(\frac{x}{4}\right)}{16\sin^3\left(\frac{x}{4}\right)}$$

4 a Let $f(x) = e^{\sin x}$

Then $f'(x) = \cos x\, e^{\sin x}$
and
$$f''(x) = -\sin x \times e^{\sin x} + \cos x \times e^{\sin x} \times \cos x$$
$$= e^{\sin x}(\cos^2 x - \sin x)$$
$$f''(0) = e^0(1 - 0) = 1$$

b Let $f(x) = e^{-\frac{1}{2}x^2}$

Then $f'(x) = e^{-\frac{1}{2}x^2} \times (-x)$

$= -x\, e^{-\frac{1}{2}x^2}$

and

$f''(x) = -1 \times e^{-\frac{1}{2}x^2} - x \times e^{-\frac{1}{2}x^2} \times -x$

$= x^2 e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2}$

$= e^{-\frac{1}{2}x^2}(x^2 - 1)$

$f''(0) = e^0(0 - 1) = -1$

c Let $f(x) = \sqrt{1 - x^2} = (1 - x^2)^{\frac{1}{2}}$

Then $f'(x) = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \times (-2x)$

$= -x(1 - x^2)^{-\frac{1}{2}}$

and

$f''(x) = -(1 - x^2)^{-\frac{1}{2}}$

$+ \frac{1}{2}x(1 - x^2)^{-\frac{3}{2}} \times -2x$

$= -\frac{1}{\sqrt{1 - x^2}} - \frac{x^2}{\sqrt{(1 - x^2)^3}}$

$f''(0) = -1 - 0 = -1$

d Let $f(x) = \tan^{-1}\left(\frac{1}{x - 1}\right)$

$= \tan^{-1}((x - 1)^{-1})$

Then $f'(x) = \frac{-(x - 1)^{-2}}{1 + ((x - 1)^{-1})^2}$

$= \frac{-1}{(x - 1)^2(1 + (x - 1)^{-2})}$

$= \frac{-1}{(x - 1)^2 + 1}$

$= \frac{-1}{x^2 - 2x + 2}$

$= -(x^2 - 2x + 2)^{-1}$

and

$f''(x) = (x^2 - 2x + 2)^{-2} \times (2x - 2)$

$= \frac{2x - 2}{(x^2 - 2x + 2)^2}$

$f''(0) = \frac{-2}{4} = -\frac{1}{2}$

5 $y = e^{\sin^{-1} x}$

Let $u = \sin^{-1} x$ .Then $y = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \frac{1}{\sqrt{1 - x^2}} \quad = \frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$

$\frac{d^2y}{dx^2} = \frac{(\sqrt{1 - x^2} + x)e^{\sin^{-1} x}}{(1 - x^2)^{\frac{3}{2}}}$ (Calc used)

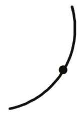
LHS $= (1 - x^2)\frac{(\sqrt{1 - x^2} + x)e^{\sin^{-1} x}}{(1 - x^2)^{\frac{3}{2}}} - x\frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}} - e^{\sin^{-1} x}$

$= \frac{(\sqrt{1 - x^2} + x)e^{\sin^{-1} x}}{(1 - x^2)^{\frac{1}{2}}} - x\frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}} - e^{\sin^{-1} x}$

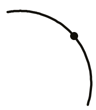
$= 0$

Solutions to Exercise 8E

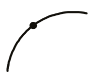
1 **a** Since $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$ at $x = a$
The small portion of graph surrounding $x = a$ is a rising curve (concave upwards)



b Since $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$ at $x = a$
The small portion of graph surrounding $x = a$ is a falling curve (concave downwards)



c Since $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$ at $x = a$
The small portion of graph surrounding $x = a$ is a rising curve (concave downwards)



d Since $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ at $x = a$
The small portion of graph surrounding $x = a$ is a falling curve (concave upwards)

2 **a** $f'(x) = 3x^2 - 1$
$f''(x) = 6x$
$f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$
Point of inflection $(0, 0)$;
Concave up on $(0, \infty)$

b $f'(x) = 3x^2 - 2x$
$f''(x) = 6x - 2$
$f''(x) < 0$ for $x < \frac{1}{3}$ and $f''(x) > 0$ for $x > \frac{1}{3}$
Point of inflection $\left(\frac{1}{3}, -\frac{2}{27}\right)$;
Concave up on $\left(\frac{1}{3}, \infty\right)$

c $f'(x) = 2x - 3x^2$
$f''(x) = 2 - 6x$
$f''(x) < 0$ for $x > \frac{1}{3}$ and $f''(x) > 0$ for $x < \frac{1}{3}$
Point of inflection $\left(\frac{1}{3}, \frac{2}{27}\right)$;
Concave up on $\left(-\infty, \frac{1}{3}\right)$

d $f'(x) = 4x^3 - 3x^2$
$f''(x) = 12x^2 - 6x = 6x(2x - 1)$
Points of inflection $(0, 0)$, $\left(\frac{1}{2}, -\frac{1}{16}\right)$;
Concave up on $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$

3 $y = \frac{1}{1 + x + x^2} = (1 + x + x^2)^{-1}$

a $\dfrac{dy}{dx} = -(1 + x + x^2)^{-2}$

$\times (1 + 2x)$

$= (-1-2x)(1 + x + x^2)^{-2}$

$\dfrac{d^2y}{dx^2} = -2(1 + x + x^2)^{-2}$

$+ (-1 - 2x)$

$\times (1 + x + x^2)^{-3}$

$\times -2 \times (1 + 2x)$

$= \dfrac{2(1 + 2x)^2}{(1 + x + x^2)^3}$

$- \dfrac{2}{(1 + x + x^2)^2}$

For $\dfrac{d^2y}{dx^2} = 0$

$\therefore \quad \dfrac{2(1 + 2x)^2}{(1 + x + x^2)^3} = \dfrac{2}{(1 + x + x^2)^2}$

$\therefore \quad 2(1 + 2x)^2 = 2(1 + x + x^2)$

$\therefore \quad 8x^2 + 8x + 2 = 2x^2 + 2x + 2$

$\therefore \quad 6x^2 + 6x = 0$

$\therefore \quad 6x(x + 1) = 0$

$\therefore \quad x = -1$ or $x = 0$

$y(-1) = 1$ and $y(0) = 1$

$\left.\dfrac{dy^2}{dx^2}\right|_{x=-2} = \dfrac{4}{9}$ and

$\left.\dfrac{d^2y}{dx^2}\right|_{x=-\frac{1}{2}} = -\dfrac{32}{9}$

Hence there is a point of inflection at $(-1, 1)$.

$\left.\dfrac{d^2y}{dx^2}\right|_{x=1} = \dfrac{4}{9}$

Hence there is a point of inflection at $(0, 1)$.

b $\dfrac{dy}{dx} = \dfrac{-(1 + 2x)}{(1 + x + x^2)^2}$

When $x = -1$,

$\dfrac{dy}{dx} = 1$

The equation of the tangent at the point $(-1, 1)$ is

$y - 1 = 1(x - (-1))$

$\therefore \quad y = x + 2$

When $x = 0$,

$\dfrac{dy}{dx} = -1$

The equation of the tangent at the point $(0, 1)$ is

$y - 1 = -1(x - 0)$

$\therefore \quad y = 1 - x$

The two tangents will intersect when

$x + 2 = 1 - x$

$\therefore \quad x = -\dfrac{1}{2}$

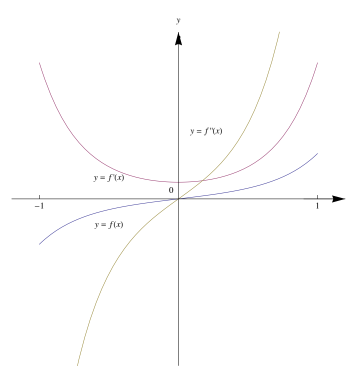
When $x = -\dfrac{1}{2}, y = \dfrac{3}{2}$

Therefore the tangents at the point of inflection intersect at the point $\left(-\dfrac{1}{2}, \dfrac{3}{2}\right)$

4 **a** **i** $(2x^2 + 1)e^{x^2}$ (By calc in this case)

ii $2x(2x^2 + 3)e^{x^2}$ (By calc in this case)

b



c $2x^2 + 1 \geq 1$ for all $x \in \mathbb{R}$ and $e^{x^2} > 0$ for all $x \in \mathbb{R}$

d $f''(0) = 0$ and $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$

e **i** $f''(x) > 0 \Rightarrow x > 0$. Concave up for $x \in (0, \infty)$

ii $f''(x) < 0 \Rightarrow x < 0$. Concave up for $x \in (-\infty, 0)$

5 $f : [0, 20] \to R, f(x) = \dfrac{x^2}{10}(20 - x)$

a $f(x) = \dfrac{x^2}{10}(20 - x)$

$= 2x^2 - \dfrac{1}{10}x^3$

$f'(x) = 4x - \dfrac{3}{10}x^2$

$f''(x) = 4 - \dfrac{3}{5}x$

When $f'(x) = 0$,

$4x - \dfrac{3}{10}x^2 = 0$

$\therefore \quad x\left(4 - \dfrac{3}{10}x\right) = 0$

$\therefore \quad x = 0 \text{ or } x = \dfrac{40}{3}$

$f\left(\dfrac{40}{3}\right) = 2\left(\dfrac{40}{3}\right)^2 - \dfrac{1}{10}\left(\dfrac{40}{3}\right)^3$

$= \dfrac{3200}{9} - \dfrac{6400}{27}$

$= \dfrac{3200}{27}$

$f''(0) = 4 > 0$ and $f''(\dfrac{40}{3}) = -4 < 0$

Therefore, local min $(0, 0)$; local max $\left(\dfrac{40}{3}, \dfrac{3200}{27}\right)$

b $f''(x) = 0$,

$4 - \dfrac{3}{5}x = 0$

$\therefore \quad 4 = \dfrac{3}{5}x$

$\therefore \quad x = \dfrac{20}{3}$

$f\left(\dfrac{20}{3}\right) = 2\left(\dfrac{20}{3}\right)^2 - \dfrac{1}{10}\left(\dfrac{20}{3}\right)^3$

$= \dfrac{800}{9} - \dfrac{800}{27}$

$= \dfrac{1600}{27}$

$f''(x) > 0 \Leftrightarrow 4 - \dfrac{3}{5}x > 0 \Leftrightarrow x < \dfrac{20}{3}$

$f''(x) < 0 \Leftrightarrow 4 - \dfrac{3}{5}x < 0 \Leftrightarrow x > \dfrac{20}{3}$

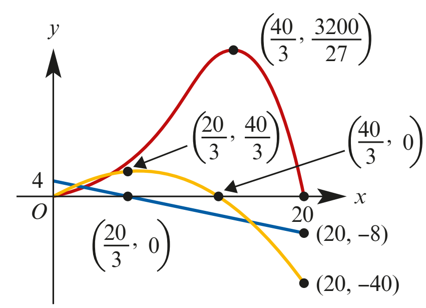
Point of inflection $\left(\dfrac{20}{3}, \dfrac{1600}{27}\right)$

$f'\left(\dfrac{20}{3}\right) = 4\left(\dfrac{20}{3}\right) - \dfrac{3}{10}\left(\dfrac{20}{3}\right)^2$

$= \dfrac{80}{3} - \dfrac{40}{3}$

$= \dfrac{40}{3}$

c



6 $f(x) = 2x^3 + 6x^2 - 12$

a **i** $f'(x) = 6x^2 + 12x$

ii $f''(x) = 12x + 12$

b $f'(x) = 0 \quad f(0) = -12$ and

$6x^2 + 12x = 0$

$6x(x + 2) = 0$

$x = 0$ or $x = -2$
$f(-2) = -4$
Second derivative test.
$f''(x) = 12x + 12$
$f''(0) = 12 > 0$ and
$f''(-2) = -12 < 0.$
Local min $(0, -12)$; local max $(-2, -4)$

c For inflection point.
$f''(x) = 0$ and concavity changes.
$12x + 12 = 0$

$\therefore \quad x = -1$

$f(-1) = 2(-1)^3 + 6(-1)^2 - 12$

$= -2 + 6 - 12$

$= -8$
$f''(x) > 0 \Leftrightarrow 12x + 12 > 0 \Leftrightarrow x > -1$
$f''(x) < 0 \Leftrightarrow 12x + 12 < 0 \Leftrightarrow x < -1$
Point of inflection is $(-1, -8)$

7 Part a

a $f : [0, 2\pi] \to R, f(x) = \sin x.$

i $f'(x) = \cos x$

ii $f''(x) = -\sin x$

b Stationary points where $\cos x = 0$
Stationary points $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$
$f''(\left(\frac{\pi}{2}\right) = -1$ Therefore local max.
$f''(\left(\frac{3\pi}{2}\right) = 1$ Therefore local min.

c $f(x)$ has point of inflection where $f''(x) = 0$ and concavity changes.

$\therefore \quad -\sin x = 0$

$\therefore \quad \sin x = 0$

$\therefore \quad x = \pi$

$f(\pi) = \sin \pi$

$= 0$
Concavity changes at $x = \pi$
Point of inflection at $(\pi, 0)$.

7 part b

a $f : R \to R, f(x) = x\,e^x$

i $f'(x) = x\,e^x + e^x = e^x(x + 1)$

ii $f''(x) = e^x \times 1 + (x + 1)e^x$

$= e^x(x + 2)$

b $f'(x) = 0$ implies $x = -1$
Therefore stationary point at $(-1, -e^{-1})$
$f''(-1) > 0.$ Therefore local minimum.

c $f(x)$ has a point of inflection where $f''(x) = 0$ and concavity changes.

$\therefore \quad e^x(x + 2) = 0$

$\therefore \quad x + 2 = 0$ ($e^x > 0$ for all x)

$\therefore \quad x = -2$

$f(-2) = -2e^{-2}$
Concavity changes at $x = -2.$
There is an inflection point $(-2, -2e^{-2})$ for the graph of $y = f(x)$.

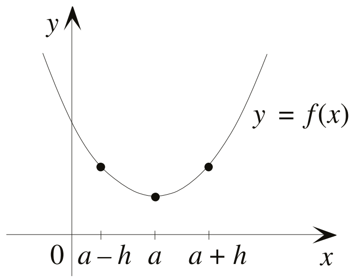
8 $f(x)$ has a local minimum at $x = a$ and no other stationary point 'close' to a

a $f'(x)$ is the gradient function of $f(x)$.

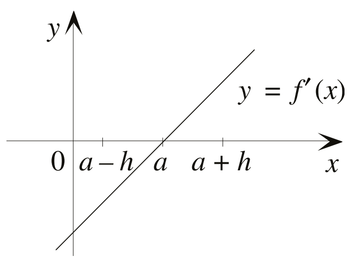
i $f'(a - h) < 0$ since a point on the left of a minimum turning point has a negative gradient.

ii $f'(a) = 0$ since the turning point has zero gradient.

iii $f'(a+h) > 0$ since a point on the right of a minimum turning point has a positive gradient.



b The gradient of the graph of $y = f'(x)$ for $x \in [a-h, a+h]$ is always non-negative.



c $f''(a) \geq 0$

d **i** $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$

$f''(0) = 2 > 0$

ii $f(x) = -\cos x$

$f'(x) = \sin x$

$f''(x) = \cos x$

$f''(0) = \cos 0 = 1 > 0$

iii $f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

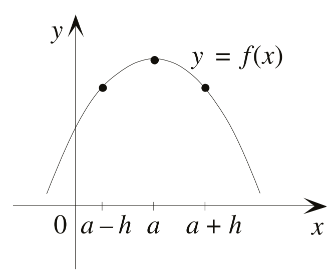
$f''(0) = 12(0)^2 = 0$

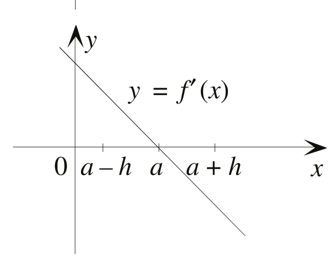
e If the graph of $y = f(x)$ has a stationary point at $x = a$, then $f'(a) = 0$.
If $f''(a) < 0$, then $f'(a-h) > 0$ and $f'(a+h) < 0$, and $(a-h, f(a-h))$ is a point of positive gradient and $(a+h, f(a+h))$ is a point of negative gradient.
Therefore $(a, f(a))$ would be a local maximum turning point.
Hence $f''(a)$ can never be less than zero if the graph of $y = f(x)$ has a local minimum at $x = a$.

9 Consider the interval $x \in [a-h, a+h]$ where the graph of $y = f(x)$ has a local maximum at $x = a$.

x	$a-h$	a	$a+h$
$f'(x)$	> 0	0	< 0
$f''(x)$	≤ 0	≤ 0	≤ 0

$f''(a) \leq 0$ if the graph of $y = f(x)$ has a local maximum at $x = a$.





10 $f : [0, 10] \to R, f(x) = x(10 - x)e^x$

a
$$\begin{aligned} f(x) &= x(10 - x)e^x \\ &= (10x - x^2)e^x \\ f'(x) &= (10x - x^2)e^x + (10 - 2x)e^x \\ &= e^x(10 + 8x - x^2) \\ f''(x) &= e^x(8 - 2x) + e^x(10 + 8x - x^2) \\ &= e^x(18 + 6x - x^2) \end{aligned}$$

b
$$\begin{aligned} f(0) &= 0 \\ f''(0) &= 18 \\ f(10) &= 0 \\ f''(10) &= e^{10}(18 + 6(10) - 10^2) \\ &= -22e^{10} \\ &\approx -484\,582 \end{aligned}$$

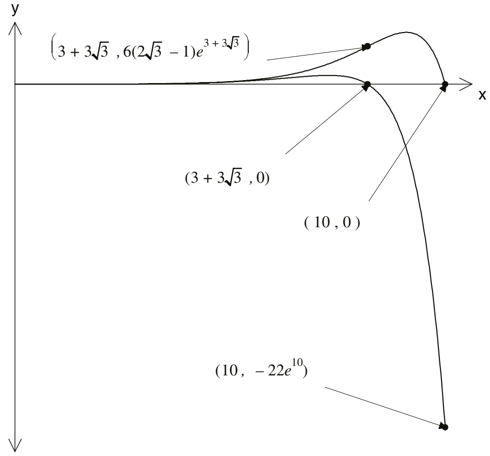
When $f''(x) = 0$

$$e^x(18 + 6x - x^2) = 0$$

$$\therefore \quad 18 + 6x - x^2 = 0 \qquad \because e^x \neq 0$$

$$\begin{aligned} \therefore \quad x &= \frac{-6 \pm \sqrt{36 + 4 \times 18}}{-2} \\ &= \frac{-6 \pm \sqrt{108}}{-2} \\ &= 3 \pm 3\sqrt{3} \\ &= 3 + 3\sqrt{3} \text{ since } x > 0 \end{aligned}$$

$$\begin{aligned} f(3 + 3\sqrt{3}) &= 6(2\sqrt{3} - 1)e^{3+3\sqrt{3}} \\ &\approx 53\,623 \end{aligned}$$



c Gradient is a maximum when $f''(x) = 0$, i.e. $x = 3 + 3\sqrt{3}$

$$\begin{aligned} & f(3 + 3\sqrt{3}) \\ &= (3 + 3\sqrt{3})(10 - (3 + 3\sqrt{3}))e^{3+3\sqrt{3}} \\ &= (3 + 3\sqrt{3})(7 - 3\sqrt{3})e^{3+3\sqrt{3}} \\ &= (21 + 21\sqrt{3} - 9\sqrt{3} - 27)e^{3+3\sqrt{3}} \\ &= (12\sqrt{3} - 6)e^{3+3\sqrt{3}} \\ &= 6(2\sqrt{3} - 1)e^{3+3\sqrt{3}} \\ &\approx 53\,623 \end{aligned}$$

The point of maximum gradient, i.e. $(3 + 3\sqrt{3}, 6(2\sqrt{3} - 1)e^{3+3\sqrt{3}})$ is marked on the graph in **b**.

11
$$y = x - \sin x, x \in [0, 4\pi]$$

$$\frac{dy}{dx} = 1 - \cos x$$

$$\frac{d^2y}{dx^2} = \sin x$$

For $\frac{d^2y}{dx^2} = 0$

$$\sin x = 0, x \in [0, 4\pi]$$

$$\therefore \quad x = 0, \pi, 2\pi, 3\pi, 4\pi$$

$$y(0) = 0, y(\pi) = \pi, y(2\pi) = 2\pi,$$
$$y(3\pi) = 3\pi, y(4\pi) = 4\pi$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{\pi}{2}} = -1 \text{ and}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{2}} = 1$$

Hence the point $(0, 0)$ is a point of inflection.

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{3\pi}{2}} = -1$$

Hence the point (π, π) is a point of inflection.

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{5\pi}{2}} = 1$$

Hence the point $(2\pi, 2\pi)$ is a point of inflection.

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{7\pi}{2}} = -1$$

Hence the point $(3\pi, 3\pi)$ is a point of inflection.

The point $(4\pi, 4\pi)$ is a point of inflection.

12 a

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} = -\sin x$$

For $\frac{d^2y}{dx^2} = 0$

$$-\sin x = 0$$

$$\therefore \quad x = k\pi, k \in \mathbb{Z}$$

Therefore the points of inflection will occur when $x = k\pi, k \in \mathbb{Z}$

b

$$y = \tan x$$

$$\frac{dy}{dx} = \sec^2 x = (\cos x)^{-2}$$

$$\frac{d^2y}{dx^2} = -2(\cos x)^{-3} \times -\sin x$$

$$= 2\sin x \sec^3 x$$

For $\frac{d^2y}{dx^2} = 0$

$$\frac{2\sin x}{\cos^3 x} = 0, \cos x \neq 0$$

$$\therefore \quad 2\sin x = 0$$

$$\therefore \quad x = k\pi, k \in \mathbb{Z}$$

Therefore the points of inflection will occur when $x = k\pi, k \in \mathbb{Z}$.

c

$$y = \sin^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \times -2x$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{x}{\sqrt{(1-x^2)^3}}$$

For $\frac{d^2y}{dx^2} = 0$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{9\pi}{2}} = 1$$

Hence the point $(4\pi, 4\pi)$ is a point of inflection.

$$\frac{x}{\sqrt{(1-x^2)^3}} = 0$$

$$\therefore \quad x = 0$$

Therefore the points of inflection will occur when $x = 0$.

d

$$y = \sin(2x)$$

$$\frac{dy}{dx} = 2\cos(2x)$$

$$\frac{d^2y}{dx^2} = -4\sin(2x)$$

For $\frac{d^2y}{dx^2} = 0$

$$-4\sin(2x) = 0$$

$$\therefore \quad 2x = k\pi, k \in \mathbb{Z}$$

$$\therefore \quad x = \frac{1}{2}k\pi, k \in \mathbb{Z}$$

Therefore the points of inflection will occur when $x = \frac{1}{2}k\pi, k \in \mathbb{Z}$

13 $y = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$

$\frac{d^2y}{dx^2} = 2a$ i.e. constant

For $\frac{d^2y}{dx^2} = 0$

$\therefore$ $2a = 0$ but $a \neq 0$

Since the variable x does not appear in the second derivative there are no points of inflection.

14 $y = 2x^3 - 9x^2 + 12x + 8$

$\frac{dy}{dx} = 6x^2 - 18x + 12$

$\frac{d^2y}{dx^2} = 12x - 18$

a For $\frac{dy}{dx} < 0$,

$6x^2 - 18x + 12 = 0$

$\therefore$ $x^2 - 3x + 2 = 0$

$\therefore$ $(x - 1)(x - 2) = 0$

$\therefore$ $x = 1$ or $x = 2$

$\therefore$ $\frac{dy}{dx} < 0$ when $1 < x < 2$

For $\frac{d^2y}{dx^2} > 0$,

$12x - 18 > 0$

$\therefore$ $x > \frac{3}{2}$

$\therefore$ for $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$

$\frac{3}{2} < x < 2$

b For $\frac{d^2y}{dx^2} < 0$,

$12x - 18 < 0$

$\therefore$ $x < \frac{3}{2}$

$\therefore$ for $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$

$1 < x < \frac{3}{2}$

15 a $y = x^3 - 6x$

$\frac{dy}{dx} = 3x^2 - 6$

$\frac{d^2y}{dx^2} = 6x$

For $\frac{d^2y}{dx^2} = 0, x = 0$

$y(0) = 0$

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -6$ and

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 6$

Therefore the point $(0, 0)$ is a point of inflection.

The gradient when $x = 0$ is -6.

b $y = x^4 - 6x^2 + 4$

$\frac{dy}{dx} = 4x^3 - 12x$

$\frac{d^2y}{dx^2} = 12x^2 - 12$

For $\frac{d^2y}{dx^2} = 0, x = \pm 1$

$y(-1) = -1$ and $y(1) = -1$

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 36$, $\left.\frac{d^2y}{dx^2}\right|_{x=0} = -12$

and $\left.\frac{d^2y}{dx^2}\right|_{x=2} = 36$

Therefore the points $(-1, -1)$ and $(1, -1)$ are the points of inflection.

The gradient when $x = -1$ is 8.
The gradient when $x = 1$ is -8

c

$$y = 3 - 10x^3 + 10x^4 - 3x^5$$
$$\frac{dy}{dx} = -30x^2 + 40x^3 - 15x^4$$
$$\frac{d^2y}{dx^2} = -60x + 120x^2 - 60x^3$$
$$\text{For } \frac{d^2y}{dx^2} = 0, -x(x^2 - 2x + 1) = 0$$
$$\therefore \quad -x(x-1)^2 = 0$$
$$\therefore \quad x = 0 \text{ or } x = 1$$
$$y(0) = 3 \text{ and } y(1) = 0$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 240 \quad \text{and}$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = -\frac{15}{2}$$

Therefore the point $(0, 3)$ is a point of inflection.

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = -120$$

Therefore the point $(1, 0)$ is **not** a point of inflection since the value of the second derivative on either side of $x = 1$ are the same sign.
The gradient when $x = 0$ is 0.

d

$$y = (x^2 - 1)(x^2 + 1) = x^4 - 1$$
$$\frac{dy}{dx} = 4x^3$$
$$\frac{d^2y}{dx^2} = 12x^2$$
$$\text{For } \frac{d^2y}{dx^2} = 0, x = 0$$
$$y(0) = -1$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 12 \text{ and } \left.\frac{d^2y}{dx^2}\right|_{x=1} = 12$$

Therefore the point $(0, -1)$ is **not** a point of inflection since the value of the second derivative on either side of $x = 0$ are the same sign.
Hence there are no points of inflection.

e

$$y = \frac{x+1}{x-1} = 1 + \frac{2}{x-1}$$
$$\frac{dy}{dx} = -2(x-1)^{-2}$$
$$\frac{d^2y}{dx^2} = 4(x-1)^{-3}$$
$$= \frac{4}{(x-1)^3}, x \neq 1$$
$$\text{For } \frac{d^2y}{dx} = 0, \frac{4}{(x-1)^3} = 0$$

$\therefore$ 4 = 0 which is a false statement.
Hence there are no points of inflection.

f

$$y = x\sqrt{x+1}$$
$$\frac{dy}{dx} = \sqrt{x+1} + x(x+1)^{-\frac{1}{2}} \times \frac{1}{2}$$
$$= \sqrt{x+1} + \frac{x}{2}(x+1)^{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{x+1}} + \frac{1}{2}(x+1)^{-\frac{1}{2}}$$
$$+ \frac{x}{2}(x+1)^{-\frac{3}{2}} \times -\frac{1}{2}$$
$$= \frac{1}{\sqrt{x+1}} - \frac{x}{4\sqrt{(x+1)^3}},$$
$$x > -1$$
$$\text{For } \frac{d^2y}{dx^2} = 0,$$
$$\frac{1}{\sqrt{x}+1} = \frac{x}{4(\sqrt{(x+1)^3})}$$
$$\therefore \quad 4(x+1) = x$$
$$\therefore \quad 3x = -4$$
$$\therefore \quad x = -\frac{4}{3}$$

But $x > -1$ for $\frac{d^2y}{dx^2}$ to exist.
Therefore a point of inflection does not exist when $x = -\frac{4}{3}$.
Hence there are no points of inflection.

g $y = \frac{2x}{x^2 + 1} = 2x(x^2 + 1)^{-1}$

$$\frac{dy}{dx} = 2(x^2+1)^{-1} + 2x(x^2+1)^{-2} \times -1 \times 2x$$
$$= 2(x^2 + 1)^{-1} - 4x^2(x^2 + 1)^{-2}$$
$$\frac{d^2y}{dx^2} = -4x(x^2 + 1)^{-2} - 8x(x^2 + 1)^{-2} - 4x^2(x^2 + 1)^{-3} \times -2 \times 2x$$
$$= \frac{16x^3}{(x^2 + 1)^3} - \frac{12x}{(x^2 + 1)^2}$$

For $\frac{d^2y}{dx^2} = 0, 16x^3 = 12x(x^2 + 1)$

$\therefore$ $16x^3 = 12x^3 + 12x$

$\therefore$ $4x^3 - 12x = 0$

$\therefore$ $4x(x^2 - 3) = 0$

$\therefore$ $x = 0$ or $x = \pm\sqrt{3}$

$y(0) = 0, y(-\sqrt{3}) = -\frac{\sqrt{3}}{2}$ and

$y(\sqrt{3}) = \frac{\sqrt{3}}{2}$

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = -\frac{8}{125}$ and $\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 1$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = -1$ and $\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{8}{125}$

Therefore the points of inflection are;
$\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$, $(0, 0)$ and $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$

The gradient when $x = -\sqrt{3}$ is $-\frac{1}{4}$.
The gradient when $x = 0$ is 2.
The gradient when $x = \sqrt{3}$ is $-\frac{1}{4}$

h $y = \sin^{-1} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2} = -\frac{1}{2}(1 - x^2)^{-\frac{3}{2}} \times -2x$$
$$\therefore \quad \frac{d^2y}{dx^2} = \frac{x}{\sqrt{(1 - x^2)^3}}$$

For $\frac{d^2y}{dx^2} = 0$

$$\frac{x}{\sqrt{(1 - x^2)^3}} = 0$$

$\therefore$ $x = 0$

$y(0) = \sin^{-1} 0 = k\pi, k \in Z$

However, since the range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow y(0) = 0$

$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{1}{2}} = -\frac{4\sqrt{3}}{9}$ and

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = \frac{4\sqrt{3}}{9}$

Therefore the point $(0, 0)$ is a point of inflection.
The gradient when $x = 0$ is 1.

i $y = \dfrac{x-2}{(x+2)^2}$

$\dfrac{dy}{dx} = (x+2)^{-2}$

$+ (x-2)(x+2)^{-3} \times -2$

$= (x+2)^{-2}$

$+ (4-2x)(x+2)^{-3}$

$\dfrac{dy^2}{dx^2} = -2(x+2)^{-3} - 2(x+2)^{-3}$

$+ (4-2x)(x+2)^{-4} \times -3$

$= -\dfrac{4}{(x+2)^3} + \dfrac{6x-12}{(x+2)^4}$

$= \dfrac{2x-20}{(x+2)^4}$

For $\dfrac{d^2y}{dx^2} = 0, 2x - 20 = 0$

$\therefore \quad x = 10$

$y(10) = \dfrac{1}{18}$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=9} = -\dfrac{2}{14\,641}$ and

$\left.\dfrac{d^2y}{dx^2}\right|_{x=11} = \dfrac{2}{28\,561}$

Therefore the point $\left(10, \dfrac{1}{18}\right)$ is a point of inflection.

The gradient when $x = 10$ is $-\dfrac{1}{432}$

16 $y = e^{-x}\sin x$

a

$\dfrac{dy}{dx} = e^{-x}\cos x - e^{-x}\sin x$

$= e^{-x}(\cos x - \sin x)$

$\dfrac{dy}{dx} = 0$ for stationary points.

$\therefore \quad e^{-x}(\cos x - \sin x) = 0$

$\therefore \quad \cos x - \sin x = 0 \qquad \because e^{-x} \neq 0$

$\therefore \quad \tan x = 1$

$\therefore \quad x = \dfrac{\pi}{4} + k\pi, k \in \mathbb{Z}$

Therefore stationary points will occur when
$x = \dfrac{\pi}{4} + k\pi, k \in \mathbb{Z}$.

b $\dfrac{d^2y}{dx^2} = -e^{-x}(\cos x - \sin x)$

$+ e^{-x}(-\sin x - \cos x)$

$= -2\cos x\, e^{-x}$

For $\dfrac{d^2y}{dx^2} = 0, -2\cos x = 0$

(since $e^{-x} \neq 0$)

$\therefore \quad \cos x = 0$

Therefore points of inflection will occur when
$x = \dfrac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

17 $f(x) = x^3 + bx^2 + cx$ and $b^2 > 3c$

a

$$\frac{dy}{dx} = 3x^2 + 2bx + c$$

$$\frac{dy}{dx} = 0 \text{ for stationary points.}$$

$\therefore \quad 3x^2 + 2bx + c = 0$

$$\begin{aligned}\Delta &= (2b)^2 - 4 \times 3 \times c\\ &= 4b^2 - 12c\\ &= 4(b^2 - 3c)\\ &> 0 \text{ (since } b^2 > 3c)\end{aligned}$$

Since the discriminant is greater than zero, there are two real solutions. Thus there are two stationary points.

b

$$\frac{d^2y}{dx^2} = 6x + 2b$$

For $\frac{d^2y}{dx^2} = 0, 6x + 2b = 0$

$\therefore \quad x = -\frac{b}{3}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{b}{3}-1} = -6 \text{ and}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{b}{3}+1} = 6$$

Therefore the point of inflection occurs when $x = -\frac{b}{3}$. Thus there is one point of inflection.

$\therefore \quad x = \frac{\pi}{2} + k\pi, k \in Z$

c From part **b**, when $x = -\frac{b}{3}, f(x) = \frac{2b^3 - 9bc}{27}$

Thus the point of inflection occurs at $\left(-\frac{b}{3}, \frac{2b^3 - 9bc}{27}\right)$

Stationary point occur when

$3x^2 + 2bx + c = 0$

$$\begin{aligned}\therefore \quad x &= \frac{-2b \pm \sqrt{4(b^2 - 3c)}}{6}\\ &= \frac{-2b \pm 2\sqrt{b^2 - 3c}}{6}\\ &= \frac{-b \pm \sqrt{b^2 - 3c}}{3}\end{aligned}$$

When $x = \frac{-b - \sqrt{b^2 - 3c}}{3}$,

$$f(x) = \frac{2\sqrt{(b^2 - 3c)^3} + 2b^3 - 9bc}{27}$$

When $x = \frac{-b + \sqrt{b^2 - 3c}}{3}$,

$$f(x) = \frac{-2\sqrt{(b^2 - 3c)^3} + 2b^3 - 9bc}{27}$$

Thus, the stationary points are:

$$\left(\frac{-b - \sqrt{b^2 - 3c}}{3}, \frac{2\sqrt{(b^2 - 3c)^3} + 2b^3 - 9bc}{27}\right)$$

and $$\left(\frac{-b + \sqrt{b^2 - 3c}}{3}, \frac{-2\sqrt{(b^2 - 3c)^3} + 2b^3 - 9bc}{27}\right)$$

The midpoint of the two stationary points can be calculated by evaluating $\frac{1}{2}(x_1 + x_2, y_1 + y_2)$

$\therefore \quad \frac{1}{2}\left(\frac{-2b}{3}, \frac{2(2b^3 - 9bc)}{27}\right)$

$\therefore \quad \left(-\frac{b}{3}, \frac{2b^3 - 9bc}{27}\right)$ is the midpoint of the two stationary points.

Therefore the point of inflection is the midpoint of the interval joining the two stationary points.

18 $f(x) = 2x^2 \log_e(x)$

a $f'(x) = 4x\log_e(x) + 2x^2 \times \dfrac{1}{x}$

$= 4x\log_e(x) + 2x$

$= 2x(1 + 2\log_e(x))$

b $f''(x) = 2(1 + 2\log_e x) + 2x \times \dfrac{2}{x}$

$= 2(1 + 2\log_e x) + 4$

$= 2(3 + 2\log_e x)$

c When $f'(x) = 0$,

$2x(1 + 2\log_e(x) = 0$

$x = 0$ or $\log_e(x) = -\dfrac{1}{2}$

$x = 0$ or $x = e^{-\frac{1}{2}}$

$f(x)$ is not defined when $x = 0$

Stationary point at$(e^{-\frac{1}{2}}, -e^{-1})$

When $f''(x) = 0$,

$3 + 2\log_e x = 0$

$\therefore \quad \log_e x = -\dfrac{3}{2}$

$\therefore \quad x = e^{-\frac{3}{2}}$

When $x = e^{-\frac{3}{2}}$

$f(x) = 2 \times e^{-3} \times -\dfrac{3}{2} = -3e^{-3}$

$f''\left(\dfrac{1}{10}\right) = 6 - 4\log_e(10) < 0$

$f''\left(\dfrac{1}{2}\right) = 6 - 4\log_e(2) > 0$

Therefore the point of inflection occurs at $\left(e^{-\frac{3}{2}}, -3e^{-3}\right)$

19 Let $f(x) = xe^{\frac{x}{3}}$

$f'(x) = e^{\frac{x}{3}} + \dfrac{1}{3}xe^{\frac{x}{3}}$

$f''(x) = \dfrac{1}{3}e^{\frac{x}{3}} + \dfrac{1}{3}(e^{\frac{x}{3}} + \dfrac{1}{3}xe^{\frac{x}{3}})$

$= \dfrac{2}{3}e^{\frac{x}{3}} + \dfrac{1}{9}xe^{\frac{x}{3}}$

$= \dfrac{1}{9}e^{\frac{x}{3}}(6 + x)$

$f'(x) = 0 \Rightarrow e^{\frac{x}{3}}(1 + \dfrac{x}{3}) = 0$

$\Rightarrow x = -3$

and $f(-3) = -3e^{-1}$

$f''(-3) = \dfrac{1}{3}e^{-1} > 0$

Therefore local minimum at $\left(-3, -3e^{-1}\right)$

Point of inflection at $(-6, -6e^{-2})$

20 $f'(x) = -10x\sin(5x) + 2\cos 5x +$
$10x\sin 5x + 5(5x^2 - 6)\cos 5x$

$= (2 + 25x^2 - 30)\cos 5x$

$= (25x^2 - 28)\cos 5x$

$f''(x) = 50x\cos 5x - 5(25x^2 - 28)\sin 5x$

Point of inflection at $(a, f(a))$ implies

$f''(a) = 0$

$f''(a) = 0$

$\Rightarrow 10a\cos 5x - (25a^2 - 28)\sin 5a = 0$

$\Rightarrow \tan 5a = \dfrac{10a}{25a^2 - 28}$

Solutions to Exercise 8F

1 $V = \frac{4}{3}\pi r^3$

a $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

When $\frac{dV}{dt} = 0.1 \text{ m}^3/\text{min}, r = 2.5 \text{ m}$

$$\begin{aligned}\therefore \quad \frac{dr}{dt} &= \frac{0.1}{4\pi \times (2.5)^2}\\ &= \frac{1}{250\pi}\\ &\approx 0.00127 \text{ m/min}\end{aligned}$$

b $A = 4\pi r^2;\ \frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$

$$\begin{aligned}\therefore \quad \frac{dA}{dt} &= 8\pi r \frac{dr}{dt}\\ &= \frac{8\pi \times 2.5}{250\pi}\\ &= 0.08 \text{ m}^2/\text{min}\end{aligned}$$

2 $V = 4x^{\frac{3}{2}}$

$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$

$\frac{dV}{dt} = 6\sqrt{x}\frac{dx}{dt}$

When $x = 9$ cm, $\frac{dV}{dt} = 10 \text{ cm}^3/\text{s}$

So
$\frac{dx}{dt} = \frac{10}{6\sqrt{9}} = \frac{5}{9} \approx 0.56 \text{ cm/s}$

3 $y = 2x^2 + 5x + 2$

$\frac{dy}{dx} = 4x + 5$

When $x = 2$, $\frac{dx}{dt} = 3$ and $\frac{dy}{dx} = 13$

$$\begin{aligned}\therefore \quad \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt}\\ &= 13 \times 3\\ &= 39 \text{ units/s}\end{aligned}$$

4 $V = \frac{1}{3}\pi x^2(18 - x)$

$\frac{dV}{dt} = \left(\frac{2}{3}\pi\kappa \times (18 - x) - \frac{1}{3}\pi x^2\right)\frac{dx}{dt}$

When $x = 2$ cm, $\frac{dV}{dt} = 3 \text{ cm}^3/\text{s}$

$$\begin{aligned}\therefore \quad 3 &= \left(\frac{2}{3}\pi \times 2 \times 16 - \frac{4}{3}\pi\right)\frac{dx}{dt}\\ &= (20\pi)\frac{dx}{dt}\end{aligned}$$

$\therefore \quad \frac{dx}{dt} = \frac{3}{20\pi} \approx 0.048 \text{ cm/s}$

5 $p = \frac{1500}{v}$

$\frac{dp}{dv} = \frac{-1500}{v^2}$

When $p = 60, v = \frac{1500}{60} = 25$

$$\begin{aligned}\therefore \quad \frac{dv}{dt} &= \frac{dv}{dp}\frac{dp}{dt}\\ &= \frac{-625}{1500} \times 2\\ &= \frac{-5}{6} \text{ units/min}\end{aligned}$$

6 $A = \pi r^2$

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr}\frac{dr}{dt}\\ &= 2\pi r\frac{dr}{dt}\end{aligned}$$

$$\therefore \quad \frac{dA}{dt} = 2\pi \times 4 \times 0.01$$
$$= 0.08\pi \text{ cm}^2/\text{h}$$
$$\approx 0.25 \text{ cm}^2/\text{h}$$

7 $A = \pi r^2$ and $C = 2\pi r$

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$
$$\therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$
$$\text{and } \frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}$$
$$\therefore \quad \frac{dC}{dt} = 2\pi\frac{dr}{dt}$$
$$\therefore \quad \frac{dC}{dt} = \frac{dA}{dt} \div r = \frac{4}{8} = \frac{1}{2} \text{ cm/s}$$

8 $x = \dfrac{1}{1+t^2}$ and $y = \dfrac{t}{1+t^2}$

a
$$\frac{dx}{dt} = -(1+t^2)^{-2} \times 2t$$
$$= \frac{-2t}{(1+t^2)^2}$$
$$\frac{dy}{dt} = (1+t^2)^{-1} - t(1+t^2)^{-2} \times 2t$$
$$= \frac{1}{1+t^2} - \frac{2t^2}{(1+t^2)^2}$$
$$= \frac{(1+t^2) - 2t^2}{1+(1+t^2)^2}$$
$$= \frac{1-t^2}{(1+t^2)^2}$$

b
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{1-t^2}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-2t}$$
$$= \frac{1-t^2}{-2t}$$
$$= \frac{t^2-1}{2t}$$

9 $x = 2t + \sin 2t$ and $y = \cos 2t$

$$\frac{dx}{dt} = 2 + 2\cos 2t$$
$$\frac{dy}{dt} = -2\sin 2t$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{-2\sin 2t}{2+2\cos 2t}$$
$$= \frac{-4\sin t\cos t}{2(1+\cos 2t)}$$
$$= \frac{-2\sin t\cos t}{2\cos^2 t}$$
$$= -\tan t$$

using $\cos 2t = 2\cos^2 t - 1$

10 $x = t - \cos t$ and $y = \sin t$

When $t = \dfrac{\pi}{6}$,

$$x = \frac{\pi}{6} - \frac{\sqrt{3}}{2} \text{ and } y = \frac{1}{2}$$
$$\Rightarrow \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$
$$\frac{dx}{dt} = 1 + \sin t \text{ and } \frac{dy}{dt} = \cos t$$
$$\therefore \quad \frac{dy}{dx} = \frac{\cos t}{1+\sin t}$$

When $x = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = \dfrac{\sqrt{3}}{2} \div \dfrac{3}{2} = \dfrac{\sqrt{3}}{3}$

The equation of the tangent is given by

$$y - \frac{1}{2} = \frac{\sqrt{3}}{3}\left(x - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)\right)$$
$$\therefore \quad y = \frac{\sqrt{3}}{3}x - \frac{\pi\sqrt{3}}{18} + \frac{3}{6} + \frac{1}{2}$$
$$\therefore \quad y = \frac{\sqrt{3}}{3}x + 1 - \frac{\pi\sqrt{3}}{18}$$

11 $y = x^2$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

a $\frac{dy}{dt} = 2 \times 3 \times 2 = 12$ cm/s

b When $y = 16, x = \pm 4$

$\therefore \quad \frac{dy}{dt} = 2 \times \pm 4 \times 2 = \pm 16$ cm/s

12 $y = \frac{2x - 6}{x}$

$$\frac{dy}{dt} = \frac{6}{x^2}\frac{dx}{dt}\left(y = 2 - \frac{6}{x}\right)$$

Given $x = f(t)$ and $y = g(t)$, then

$\therefore \quad g'(t) = \frac{6}{x^2}f'(t)$

$\therefore \quad f'(t) = \frac{x^2 g'(t)}{6}$

When $y = 1, 1 = 2 - \frac{6}{x}$

$$-1 = -\frac{6}{x}$$

$$x = 6$$

$\therefore \quad f'(t) = 6g'(t) = 6 \times 0.4 = 2.4$

13 $y = 10\cos^{-1}\left(\frac{x - 5}{5}\right)$

a $\frac{dx}{dt} = 3$

$$\frac{dy}{dx} = 10 \times \frac{-1}{\sqrt{1 - \left(\frac{x-5}{5}\right)^2}} \times \frac{1}{5}$$

$$= \frac{-10}{5\sqrt{1 - \frac{1}{25}(x^2 - 10x + 25)}}$$

$$= \frac{-10}{\sqrt{25 - x^2 + 10x - 25}}$$

$$= \frac{-10}{\sqrt{x(10 - x)}}$$

$\therefore \quad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

$$= \frac{-30}{\sqrt{x(10 - x)}}$$

When $x = 6$,

$$\frac{dy}{dt} = \frac{-30}{\sqrt{6(10 - 6)}}$$

$$= \frac{-30}{\sqrt{24}} = \frac{-5\sqrt{6}}{2}$$

The velocity parallel to the y-axis, when $x = 6$, is $\frac{-5\sqrt{6}}{2}$ cm/s.

b When $y = \frac{10\pi}{3}$,

$$10\cos^{-1}\left(\frac{x - 5}{5}\right) = \frac{10\pi}{3}$$

$\therefore \quad \cos^{-1}\left(\frac{x - 5}{5}\right) = \frac{\pi}{3}$

$\therefore \quad \frac{x - 5}{5} = \cos\frac{\pi}{3}$

$$= \frac{1}{2}$$

$\therefore \quad x - 5 = \frac{5}{2}$

$\therefore \quad x = \frac{15}{2}$

When $x = \frac{15}{2}$, $\frac{dy}{dt} = \frac{-30}{\sqrt{\frac{15}{2}\left(10 - \frac{15}{2}\right)}}$

$$= \frac{-30}{\sqrt{\frac{75}{4}}}$$

$$= \frac{-12\sqrt{3}}{3}$$

$$= -4\sqrt{3}$$

The velocity parallel to the y-axis, when $y = \frac{10\pi}{3}$, is $-4\sqrt{3}$ cm/s.

14 $\frac{dr}{dt} = 2$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 8\pi r^2$

When $V = 36\pi$,

$\frac{4}{3}\pi r^3 = 36\pi$

$\therefore \quad r^3 = 27$

$\therefore \quad r = 3$

When $r = 3$, $\frac{dV}{dt} = 8\pi(3)^2 = 72\pi$

The rate at which the volume is increasing, at the instant when the volume is 36π cm^3, is 72π cm^3/s.

15 $V = \frac{1}{2}(h^2 + 4h)$

$\frac{dV}{dt} = 12$

a When $V = 16$,

$\frac{1}{2}(h^2 + 4h) = 16$

$\therefore \quad h^2 + 4h = 32$

$\therefore \quad h^2 + 4h - 32 = 0$

$\therefore \quad (h + 8)(h - 4) = 0$

$\therefore \quad h + 8 = 0$ or $h - 4 = 0$

$\therefore \quad h = -8$ or 4 but $h > 0$

$\therefore \quad h = 4$

b $V = \frac{1}{2}h^2 + 2h$

$\frac{dV}{dh} = h + 2$

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$

$= \frac{1}{h+2} \times 12$

$= \frac{12}{h+2}$

When $V = 16$, $h = 4$ and

$\frac{dh}{dt} = \frac{12}{4+2} = 2$

The rate at which h is increasing, when $V = 16$, is 2 cm/s.

16 Let A = the area of the inkblot (cm^2)

r = the radius of the inkblot (cm)

and t = time (seconds)

$\frac{dA}{dt} = 3.5$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = \frac{dr}{dA}\frac{dA}{dt}$

$= \frac{1}{2\pi r} \times 3.5$

$= \frac{7}{4\pi r}$

When $r = 3$, $\frac{dr}{dt} = \frac{7}{4\pi \times 3} = \frac{7}{12\pi}$

The rate of increase of the radius, when the radius is 3 cm, is $\frac{7}{12\pi}$ cm/s.

17 $V = Ah$

$\frac{dV}{dh} = A$

$\therefore \quad \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$

$= A\frac{dh}{dt}$

18 Let V = volume of water (m^3)

$$\frac{dV}{dt} = -\sqrt{h}$$

$$V = \pi r^2 h = \pi(2)^2 h = 4\pi h$$

a $\frac{dV}{dh} = 4\pi$

$$\therefore \quad \frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$

$$= \frac{1}{4\pi} \times -\sqrt{h}$$

$$= \frac{-\sqrt{h}}{4\pi}$$

b **i** When $V = 10\pi, 4\pi h = 10\pi$

$$\therefore \quad h = \frac{5}{2}$$

When $h = \frac{5}{2}$,

$$\frac{dV}{dt} = -\sqrt{\frac{5}{2}} = \frac{-\sqrt{10}}{2} \text{ m}^3/\text{h}$$

ii When $V = 10\pi, h = \frac{5}{2}$

and $\frac{dh}{dt} = \frac{-\sqrt{\frac{5}{2}}}{4\pi} = \frac{-\sqrt{10}}{8\pi}$ m/h

19 $x = 2\cos t$ and $y = \sin t$

a At $\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$

$2\cos t = \sqrt{2}$ and $\sin t = \frac{\sqrt{2}}{2}$

$\therefore \quad \cos t = \frac{\sqrt{2}}{2}$ and $\sin t = \frac{\sqrt{2}}{2}$

$$\therefore \quad t = \frac{\pi}{4}$$

$\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = \cos t$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos t}{-2\sin t}$$

When $t = \frac{\pi}{4}, \frac{dy}{dx} = \frac{\sqrt{2}}{-2\sqrt{2}} = -\frac{1}{2}$

The equation of the tangent is given by

$$y - \frac{\sqrt{2}}{2} = -\frac{1}{2}(x - \sqrt{2})$$

$$\therefore \quad y = -\frac{1}{2}x + \sqrt{2}$$

b At the point $(2\cos t, \sin t)$ the equation of the tangent is given by

$$y - \sin t = -\frac{\cos t}{2\sin t}(x - 2\cos t)$$

$$\therefore \quad y = -\frac{\cos t}{2\sin t}x + \frac{\cos^2 t}{\sin t} + \sin t$$

$$\therefore \quad y = -\frac{\cos t}{2\sin t}x + \frac{\cos^2 t + \sin^2 t}{\sin t}$$

$$\therefore \quad y = -\frac{\cos t}{2\sin t}x + \frac{1}{\sin t}$$

or

$$y = -\frac{x}{2}\cot t + \operatorname{cosec} t$$

20 $x = 2\sec\theta$ and $y = \tan\theta$

$$\frac{dx}{dt} = -2(\cos\theta)^{-2} \times -\sin\theta$$

$$= 2\sin\theta\sec^2\theta$$

$$\frac{dy}{dt} = \sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sec^2\theta}{2\sin\theta\sec^2\theta} = \frac{1}{2\sin\theta}$$

a When $\theta = \frac{\pi}{4}, \frac{dy}{dx} = \frac{\sqrt{2}}{2}$,

$x = 2\sqrt{2}$ and $y = 1$

The equation of the tangent is given by

$$y - 1 = \frac{\sqrt{2}}{2}(x - 2\sqrt{2})$$

$$\therefore \quad y = \frac{\sqrt{2}}{2}x - 1$$

b The gradient of the normal is equal to $-\frac{2}{\sqrt{2}}$ or more simply $-\sqrt{2}$.
Thus the equation of the normal is given by
$y - 1 = -\sqrt{2}(x - 2\sqrt{2})$
$\therefore \quad y = -\sqrt{2}x + 5$

c At the point $(2\sec\theta, \tan\theta)$ the equation of the tangent is given by
$$y - \tan\theta = \frac{1}{2\sin\theta}(x - 2\sec\theta)$$
$$\therefore \quad y = \frac{1}{2\sin\theta}x - \frac{1}{\sin\theta\cos\theta} + \tan\theta$$
$$\therefore \quad y = \frac{1}{2\sin\theta}x + \frac{\sin^2\theta - 1}{\sin\theta\cos\theta}$$
$$\therefore \quad y = \frac{1}{2\sin\theta}x - \frac{1 - \sin^2\theta}{\sin\theta\cos\theta}$$
$$\therefore \quad y = \frac{1}{2\sin\theta}x - \frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$\therefore \quad y = \frac{1}{2\sin\theta}x - \frac{\cos\theta}{\sin\theta}$$
or
$$y = \frac{x}{2}\operatorname{cosec}\theta - \cot\theta$$

21 a $x = 2\sec t, y = 4\tan t + 2$
$$\frac{dx}{dt} = 2\tan t\sec t$$
$$\frac{dy}{dt} = 4\sec^2 t$$
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{4\sec^2 t}{2\tan t\sec t}$$
$$= \frac{2}{\sin t} = 2\operatorname{cosec} t$$

b Gradient of the tangent when $t = \frac{\pi}{4}$ is
$$\frac{2}{\sin\frac{\pi}{4}} = 2\sqrt{2}$$
The point on the curve is
$$\left(2\sec\frac{\pi}{4}, 4\tan\frac{\pi}{4} + 2\right) = (2\sqrt{2} - 3, 6)$$
Equation of tangent
$y - 6 = 2\sqrt{2}(x - 2\sqrt{2} + 3)$
$\therefore y = 2\sqrt{2}x - 2 + 6\sqrt{2}$

22 a $x = \sec t, y = \tan t$
$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= \frac{\sec^2 t}{\sec t\tan t}$$
$$= \operatorname{cosec} t$$
Therefore, gradient of normal $= -\sin t$
Equation of normal
$y - \tan t = -\sin t(x - \sec t)$
Therefore
$y = -\sin(t)x + 2\tan t$

b When $x = 0, y = 2\tan t$
When $y = 0, x = 2\sec t$
Therefore $\triangle OAB$ is right-angled at O and has vertices:
$O(0, 0), A(2\sec t, 0), B(0, 2\tan t)$
$$\text{The area} = \frac{1}{2}|4\sec t\tan t|$$
$$= |2\sec t\tan t|$$
$$= 2\frac{|\sin t|}{\cos^2 t}$$

c

$$2\frac{|\sin t|}{\cos^2 t} = 4\sqrt{3}$$

$$\text{Assume } 0 \le t < \frac{\pi}{2}$$

$$\sin t = 2\sqrt{3}(1 - \sin^2)$$

$$2\sqrt{3}\sin^2 t + \sin t - 2\sqrt{3} = 0$$

$$\therefore \sin t = \frac{-1 \pm 7}{4\sqrt{3}}$$

$$\therefore \sin t = \frac{\sqrt{3}}{2}$$

$$t = \frac{\pi}{3}$$

23 $x = e^{2t} + 1, y = 2e^t + 1$

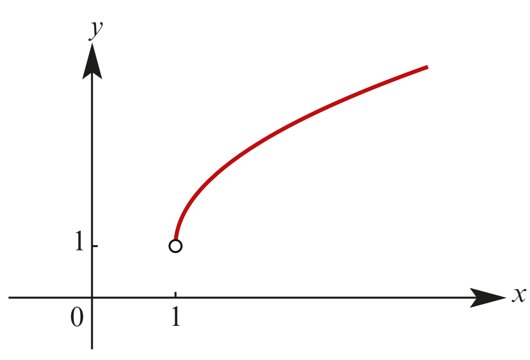
a $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

$= \dfrac{2e^t}{2e^{2t}}$

$= e^{-t}$

b Domain $= (1, \infty)$ since $e^{2t} + 1 > 1$ for all x.

c



d When $t = \log_e\left(\dfrac{1}{2}\right)$

Gradient$= e^{\log_e 2} = 2$

$x = e^{2\log_e \frac{1}{2}} + 1 = e^{\log_e \frac{1}{4}} + 1 = \dfrac{5}{4}$

$y = 2e^{\log_e \frac{1}{2}} + 1 = 2$

Equation of tangent

$y = 2x - \dfrac{1}{2}$

24 $x = t^2 + 1, y = t(t-3)^2 = t^3 - 6t^2 + 9t$

a $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

$= \dfrac{3t^2 - 12 + 9}{2t}, t \neq 0$

$= \dfrac{3(t-3)(t-1)}{2t}, t \neq 0$

b $\dfrac{dy}{dx} = 0$

$\Rightarrow 3(t-3)(t-1) = 0$

$t = 3$ or $t = 3$

Stationary points are $(2, 4), (10, 0)$

c $\dfrac{dy}{dx} = \dfrac{3(t-3)(t-1)}{2t}$

$= \dfrac{3}{2}(t - 4 + 3t^{-1})$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\dfrac{3}{2}(t - 4 + 3t^{-1}))$

$= \dfrac{d}{dt}(\dfrac{3}{2}(t - 4 + 3t^{-1})) \div \dfrac{dx}{dt}$

$= \dfrac{3(t^2 - 3)}{4t^3}$

$\dfrac{d^2y}{dx^2} = 0$

$\Rightarrow t = \pm\sqrt{3}$

d Points of inflection are $(4, 12\sqrt{3} - 18)$ and $(4, -12\sqrt{3} - 18)$

Solutions to Exercise 8G

1 a Here use the reciprocals of ordinates approach, using the graph $y = x^2 - 2x$

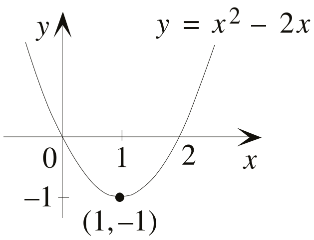
As $y = 0 + \dfrac{1}{x^2 - 2x}$, $y = 0$ is a horizontal asymptote.

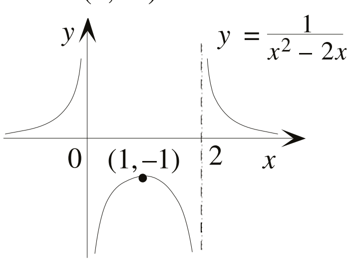
When $x^2 - 2x = 0$

$x(x - 2) = 0$

$\therefore$ $x = 0$ and $x = 2$ are the vertical asymptotes.

From the parabola $y = x^2 - 2x$ using the reciprocal of ordinates, obtain the graph of the given function.

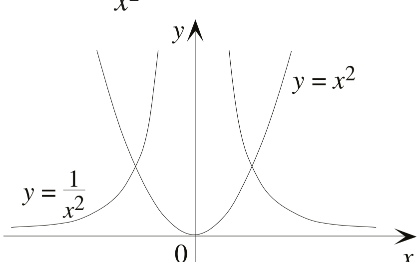




b $y = \dfrac{x^4 + 1}{x^2} = x^2 + \dfrac{1}{x^2}$

So asymptotes are $x = 0$ and $y = x^2$

Add ordinates of the graphs of $y = x^2$ and $y = \dfrac{1}{x^2}$



There are no y or x-axis intercepts. Only turning points need to be found.

$$\frac{dy}{dx} = \frac{4x^3 \times x^2 - 2x(x^4 + 1)}{x^4}$$

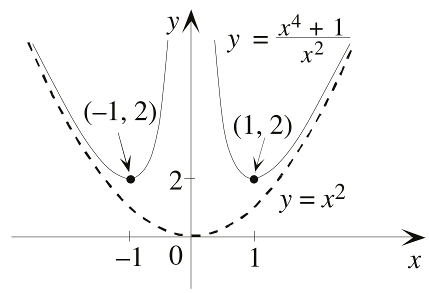
$$= \frac{4x^5 - 2x^5 - 2x}{x^4}$$

$$= \frac{2x(x^4 - 1)}{x^4}$$

$$= \frac{2(x^4 - 1)}{x^3}$$

$\dfrac{dy}{dx} = 0$ when $x = \pm 1, y = 2$

Turning points are $(-1, 2), (1, 2)$



c The graph of the function $y = \dfrac{1}{(x - 1)^2 + 1}$ does not have vertical asymptote.

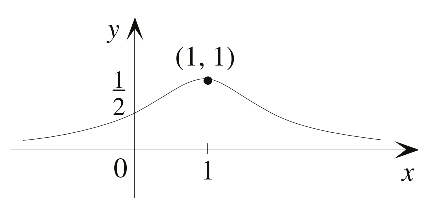
Because $(x - 1)^2 + 1 > 0$, $y = 0$ is a horizontal asymptote.

y-axis intercept is at

$y = \dfrac{1}{(-1)^2 + 1} = \dfrac{1}{2}$

Turning point $\dfrac{dy}{dx} = \dfrac{-2(x - 1)}{((x - 1)^2 + 1)^2}$

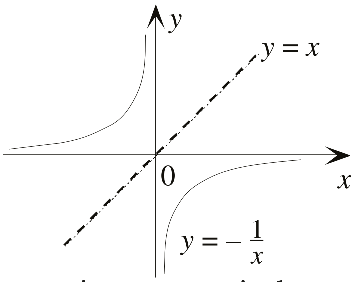
$\dfrac{dy}{dx} = 0$ when $x = 1, y = 1$. It is obviously a maximum.



d $\dfrac{x^2 - 1}{x} = x - \dfrac{1}{x}$

Add ordinates of the graphs of $y = x$

and $y = -\dfrac{1}{x}$

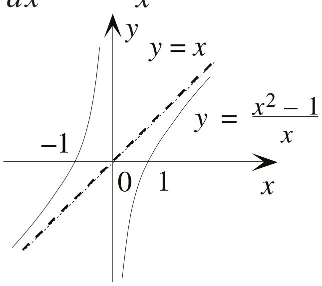


$y = x$ is a non-vertical asymptote,
$x = 0$ is a vertical asymptote.
x-axis intercepts are at

$\dfrac{x^2 - 1}{x} = 0, x = \pm 1$

There are obviously no turning points. This can be proved algebraically.

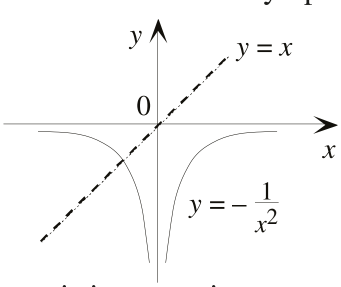
$\dfrac{dy}{dx} = 1 + \dfrac{1}{x^2} > 0, x \in R\backslash\{0\}$



e $y = \dfrac{x^3 - 1}{x^2} = x - \dfrac{1}{x^2}$

As in **d**, add ordinates of the graphs of $y = x$ and $y = -\dfrac{1}{x^2}$

$y = x$ is a non-vertical asymptote and $x = 0$ is a vertical asymptote.

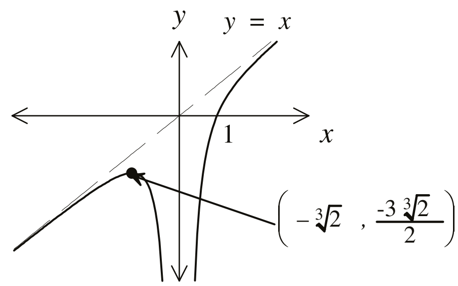


x-axis intercept is at

$\dfrac{x^3 - 1}{x^2} = 0, x = 1$, turning point is at $\dfrac{dy}{dx} = 0$.

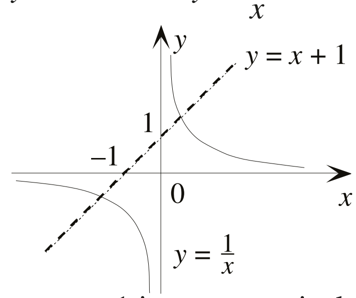
$\dfrac{dy}{dx} = 1 + \dfrac{2}{x^3} = \dfrac{x^3 + 2}{x^3} = 0$

$x = -\sqrt[3]{2}, y = \dfrac{-3}{\sqrt[3]{4}} = \dfrac{-3\sqrt[3]{2}}{2}$



f $\dfrac{x^2 + x + 1}{x} = x + 1 + \dfrac{1}{x}$

Again add ordinates of the graphs of $y = x + 1$ and $y = \dfrac{1}{x}$

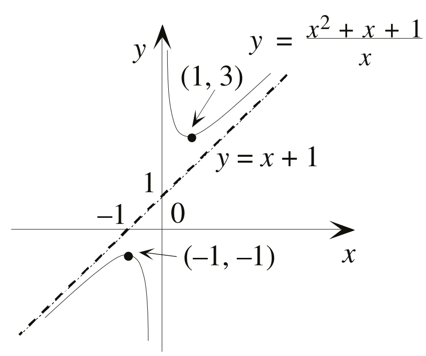


$y = x + 1$ is a non-vertical asymptote and $x = 0$ is a vertical asymptote.
There are no x or y-axis intercepts on the graph because $x^2 + x + 1 > 0, x \in R$.
Turning point

$\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2}$,

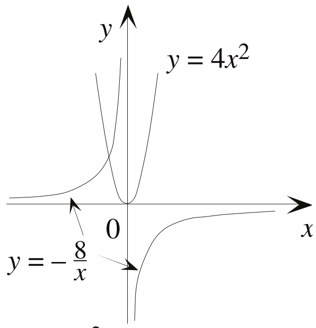
$\dfrac{dy}{dx} = 0$ when $\dfrac{x^2 - 1}{x^2} = 0, x = \pm 1$

$y = 3$ when $x = 1$, $y = -1$ when $x = -1$



g $\dfrac{4x^3 - 8}{x} = 4x^2 - \dfrac{8}{x}$

Add ordinates of the graphs of $y = 4x^2$ and $y = -\dfrac{8}{x}$

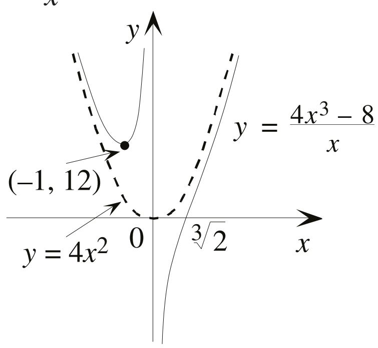


$y = 4x^2$ is a non-vertical asymptote and $x = 0$ is a vertical asymptote.

x-axis intercept is at $\dfrac{4x^3 - 8}{x} = 0, x = \sqrt[3]{2}$

Turning point is at $\dfrac{dy}{dx} = 0, \dfrac{dy}{dx} = 8x + \dfrac{8}{x^2};$

$\dfrac{8x^3 + 8}{x^2} = 0, x = -1, y = 12$



h For the graph of $y = \dfrac{1}{x^2 + 1}$ use the reciprocals of ordinates approach, using the 'simpler' graph $y = x^2 + 1$.

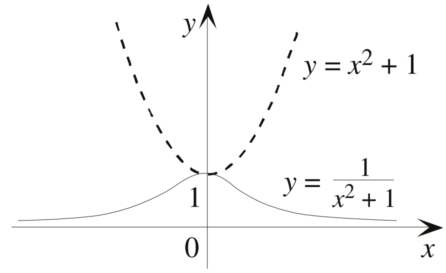
As $y = 0 + \dfrac{1}{x^2 + 1}, y = 0$ is a horizontal asymptote.

As $x^2 + 1 > 0, x \in R$, there are no vertical asymptote.

Turning point:

$\dfrac{dy}{dx} = \dfrac{-2x}{(x^2 + 1)^2}$

$\dfrac{dy}{dx} = 0$ when $x = 0, y = 1$



i Again use the reciprocals of ordinates approach.

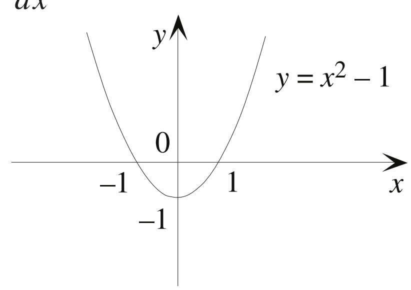
As $y = 0 + \dfrac{1}{x^2 - 1}, y = 0$ is a horizontal asymptote.

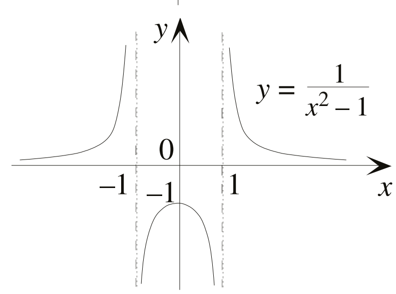
When $x^2 - 1 = 0, x = \pm 1$, the vertical asymptote.

Turning point:

$\dfrac{dy}{dx} = \dfrac{-2x}{(x^2 - 1)^2}$

$\dfrac{dy}{dx} = 0$ at $x = 0, y = -1$

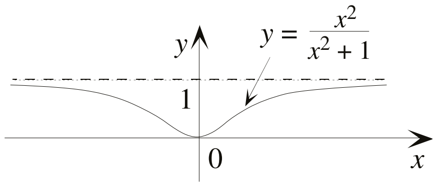




j $\dfrac{x^2}{x^2 + 1} = \dfrac{x^2 + 1 - 1}{x^2 + 1} = 1 - \dfrac{1}{x^2 + 1}$

In **h** we have already the graph of $y = \dfrac{1}{x^2 + 1}$. The given function can be graph as a reflection of the graph of $y = \dfrac{1}{x^2 + 1}$ in the x-axis (see **h**) and translation of the graph of $y = -\dfrac{1}{x^2 + 1}$ one unit up.

In this case, the asymptote is $y = 1$ and the turning point moves to the origin $(0, 1) \to (0, -1) \to (0, 0)$

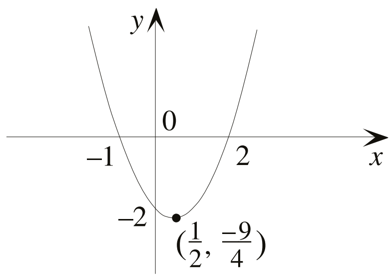


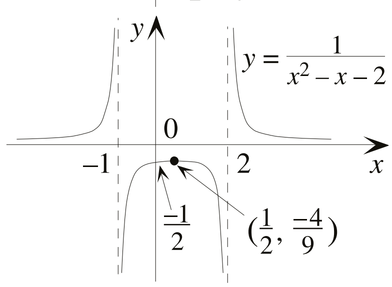
k $y = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x + 1)(x - 2)}$

Vertical asymptotes have equations $x = -1$ and $x = 2$.

The non-vertical asymptote is $y = 0$ as $y \to 0$ as $x \to \pm\infty$

The graph is produced by first sketching the graph of $y = x^2 - x - 2$



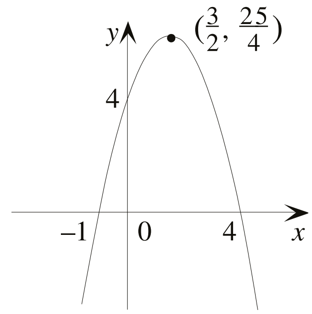


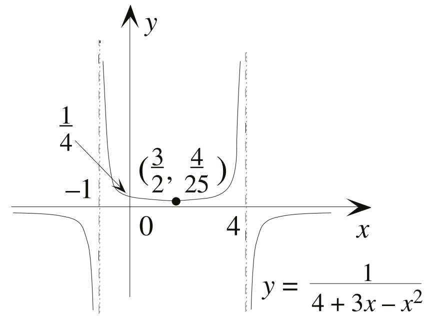
l $y = \dfrac{1}{4 + 3x - x^2} = \dfrac{1}{(4 - x)(x + 1)}$

Vertical asymptotes have equations $x = -1$ and $x = 4$.

The non-vertical asymptote is $y = 0$ as $y \to 0$ as $x \to \pm\infty$

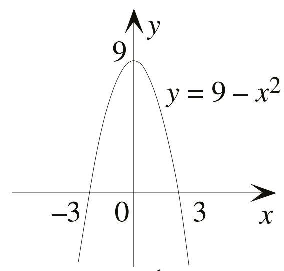
The graph is produced by first sketching the graph of $y = -x^2 + 3x + 4$





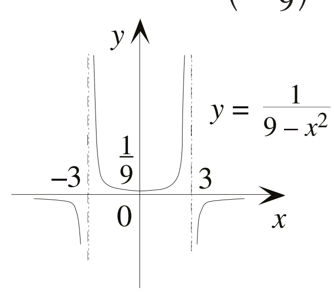
2 For **2 a – d** use the reciprocals of ordinates approach.

a The 'simpler' function is $y = 9 - x^2$.



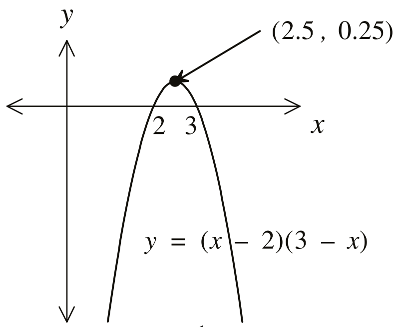
For $y = \dfrac{1}{9 - x^2}$, vertical asymptotes are at $x = \pm 3$, horizontal asymptote at $y = 0$.

Turning point at $\left(0, \dfrac{1}{9}\right)$

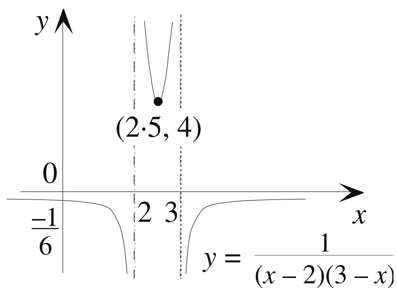


b The simpler function is

$y = (x - 2)(3 - x)$.



For $y = \dfrac{1}{(x-2)(3-x)}$, vertical asymptotes are at $x = 2$ and $x = 3$. Horizontal asymptote at $y = 0$. y-axis intercept at $y = -\dfrac{1}{6}$ since the simpler graph has y-axis intercept at $y = -6$. Turning point is at $x = 2.5, y = 4$ since the turning point of the simpler graph is at $\left(2.5, \dfrac{1}{4}\right)$

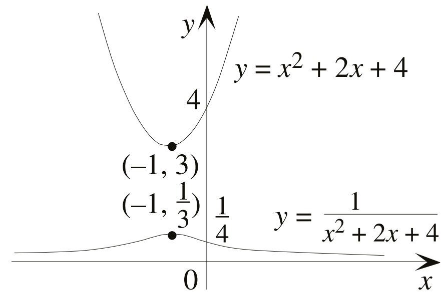


c The simpler function is $y = x^2 + 2x + 4 = (x + 1)^2 + 3$

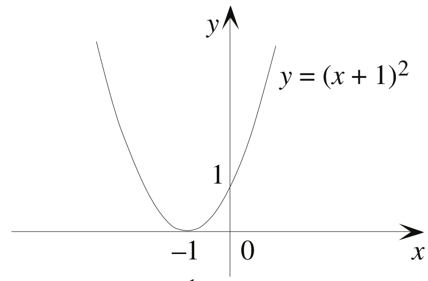
For $y = \dfrac{1}{(x+1)^2 + 3}$, there are no vertical asymptotes since $(x + 1)^2 + 3 > 0$

The horizontal asymptote is at $y = 0$

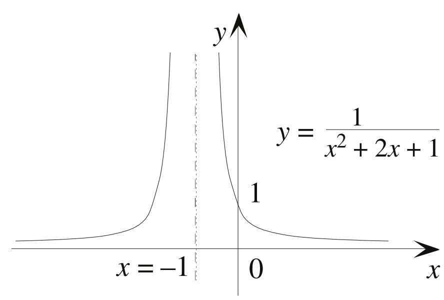
Turning point is at $\left(-1, \dfrac{1}{3}\right)$ y-axis intercept is at $y = \dfrac{1}{4}$



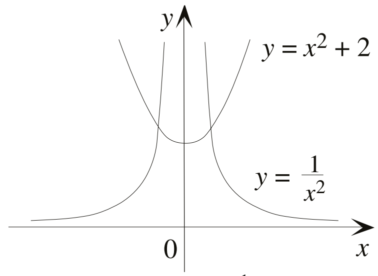
d The simpler function is $y = x^2 + 2x + 1$.



For $y = \dfrac{1}{x^2 + 2x + 1}$, the vertical asymptote is $x = -1$, the horizontal asymptote is $y = 0$. There are no turning points since $\dfrac{dy}{dx} > 0$ when $x < -1$, and $\dfrac{dy}{dx} < 0$ when $x > -1$. $\dfrac{dy}{dx} \neq 0$ for any $x \in R\backslash\{-1\}$. y-axis intercept is at $y = 1$.



e Add the ordinates of the graph of $y = x^2 + 2$ and $y = \dfrac{1}{x^2}$

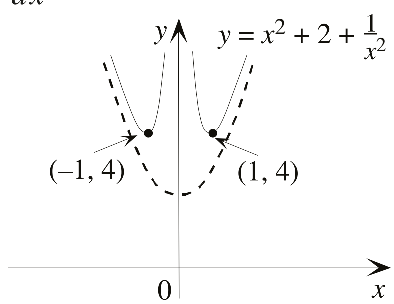


For $y = x^2 + 1 + \frac{1}{x^2}$, asymptotes are $y = x^2 + 2$ and $x = 0$

Turning point

$\frac{dy}{dx} = 2x - \frac{2}{x^3} = \frac{2(x^4 - 1)}{x^3}$

$\frac{dy}{dx} = 0$ when $x = \pm 1, y = 4$



3 $y = 4x + \frac{1}{x}$

a $\frac{dy}{dx} = 4 - \frac{1}{x^2}$

$\frac{dy}{dx} = 0$ when $x = \pm\frac{1}{2}, y = \pm 4$

x	−1	−0.5	−0.25	0.5	1
$\frac{dy}{dx}$	3	0	−12	0	3
Slope	/	–	\	–	/

$\therefore$ $\left(-\frac{1}{2}, -4\right)$ is a maximum and $\left(\frac{1}{2}, 4\right)$ is a minimum.

b When $x = 2$, $\frac{dy}{dx} = 4 - \frac{1}{4} = \frac{15}{4}$

and $y = 8\frac{1}{2} = \frac{17}{2}$

The equation of the tangent line is given by $y - \frac{17}{2} = \frac{15}{4}(x - 2)$

$\therefore \quad 4y - 34 = 15x - 30$

$\therefore \quad 4y - 15x = 4$

or

$y = \frac{15}{4}x + 1$

4 $y = \frac{x^2 - 1}{x} = x - \frac{1}{x}$

$\therefore \quad \frac{dy}{dx} = 1 + \frac{1}{x^2}$

$1 + \frac{1}{x^2} = 5$ when $x = \pm\frac{1}{2}$

5 The curve crosses the x-axis at

$\frac{2x - 4}{x^2} = 0,$

$\therefore \quad x = 2$

$$\frac{dy}{dx} = \frac{2x^2 - 2x(2x - 4)}{x^4} = \frac{2x - 2(2x - 4)}{x^3} = \frac{-2x + 8}{x^3}$$

When $x = 2$,

$$\frac{dy}{dx} = \frac{4}{8} = \frac{1}{2}$$

6 $y = x - 5 + \frac{4}{x}$

a x-axis intercepts occur when

$x - 5 + \frac{4}{x} = 0$

$x^2 - 5x + 4 = 0, x \neq 0$

$(x - 1)(x - 4) = 0$

$x = 1$ and $x = 4$

There are no y-axis intercepts since the domain of the function is $R\backslash\{0\}$.

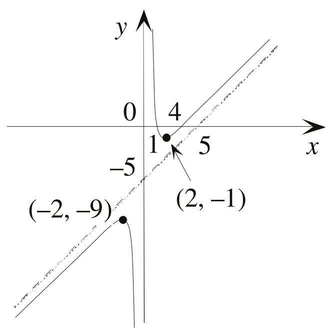
The x intercepts are $(1, 0)$ and $(4, 0)$.

b The equation of the non-vertical asymptote is $y = x - 5$.
The equation of the vertical asymptote is $x = 0$ since the graph of the function $y = x - 5 + \frac{4}{x}$ can be obtained as addition of the ordinates of the graphs of $y = x - 5$ and $y = \frac{4}{x}$
Asymptotes are $y = x - 5$ and $x = 0$

c $\frac{dy}{dx} = 1 - \frac{4}{x^2}$
$\frac{dy}{dx} = 0$ when $x = \pm 2$

x	-3	-2	-1	2	3
$\frac{dy}{dx}$	> 0	0	< 0	0	> 0
Slope	/	–	\	–	/

$\therefore$ $(-2, -9)$ is a maximum and $(2, -1)$ is a minimum



7 Let $y = x + \frac{4}{x^2}, x > 0$

$\therefore \quad \frac{dy}{dx} = 1 - \frac{8}{x^3}$

$= \frac{x^3 - 8}{x^3}$

$\frac{dy}{dx} = 0$ when $x = 2, y = 3$

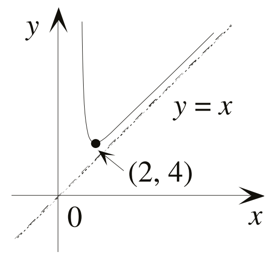
x	1	2	3
$\frac{dy}{dx}$	< 0	0	> 0
Slope	\	–	/

Thus the point $(2, 3)$ is a minimum.
Therefore the least value of $x + \frac{4}{x^2}$ is 3.

8 $y = x + \frac{4}{x}, x > 0$
The non-vertical asymptote is $y = x$
The vertical asymptote is $x = 0$
$\frac{dy}{dx} = 1 - \frac{4}{x^2}$
$\frac{dy}{dx} = 0$ when $x = 2, y = 4$

x	1	2	3
$\frac{dy}{dx}$	< 0	0	> 0
Slope	\	–	/

Thus the point $(2, 4)$ is a minimum.
Therefore the least value of y is 4.



9 **a** $y = \frac{(x-3)^2}{x}$

$\frac{dy}{dx} = \frac{2x(x-3) - (x-3)^2}{x^2}$

$= \frac{(x-3)(3+x)}{x^2}$

$\therefore \quad \frac{dy}{dx} = 0$ when $x = 3$ and $x = -3$
$y = 0$ and $y = -12$

x	-4	-3	1	3	4
$\frac{dy}{dx}$	> 0	0	< 0	0	> 0
Slope	/	–	\	–	/

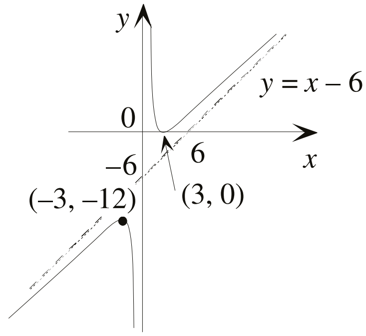
Turning points:
$(3, 0)$ minimum and $(-3, -12)$ maximum

b $\frac{(x-3)^2}{x} = \frac{x^2 - 6x + 9}{x}$

$= x - 6 + \frac{9}{x}$

Add the ordinates of the graphs of $y = x - 6$ and $y = \frac{9}{x}$

Non-vertical asymptote is $y = x - 6$,
vertical asymptote is $x = 0$.
x-axis intercept is at $x = 0$



10 **a** $y = 8x + \dfrac{1}{2x^2}$

$\dfrac{dy}{dx} = 8 - \dfrac{1}{x^3}$

$\dfrac{dy}{dx} = 0$ when $x = \dfrac{1}{2}, y = 6$

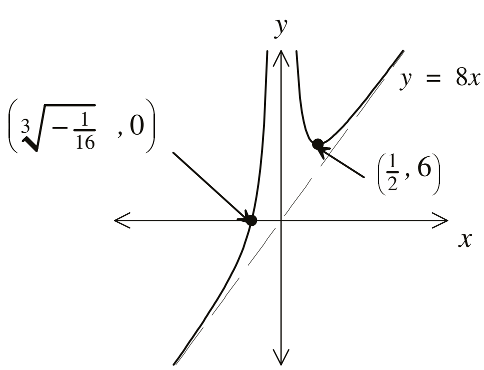
x	0.25	0.5	1
$\dfrac{dy}{dx}$	< 0	0	> 0
Slope	\	–	/

The point $\left(\dfrac{1}{2}, 6\right)$ is a minimum.

b $y = 0$ when $16x^3 + 1 = 0, x = \sqrt[3]{-\dfrac{1}{16}}$

Non-vertical asymptote is $y = 8x$

Vertical asymptote is $x = 0$



11 $y = \dfrac{x^3 + 3x^2 - 4}{x^2}$

$y = x + 3 - \dfrac{4}{x^2}$

Asymptotes: $y = x + 3, x = 0$;

No y-intercept

When $y = 0, x^3 + 3x^2 - 4 = 0$

x-intercepts $(-4, 0), (1, 0)$;

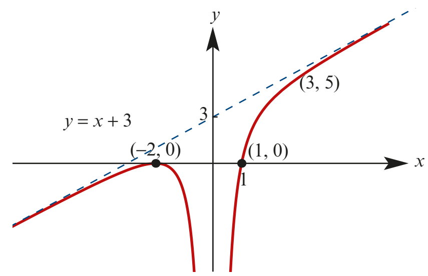
$\dfrac{dy}{dx} = 1 + \dfrac{8}{x^3}$

$\dfrac{dy}{dx} = 0$

$\Rightarrow 1 + \dfrac{8}{x^3} = 0$

$\Rightarrow x = -2$

Stationary points: local max $(-2, 0)$



12 $y = \dfrac{4x^2 + 8}{2x + 1} = \dfrac{9}{2x + 1} + 2x - 1$

a $\mathbb{R} \setminus \{-\frac{1}{2}\}$

b $\dfrac{dy}{dx} = \dfrac{8(x^2 + x - 2)}{(2x + 1)^2}$

c $\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{8(x^2 + x - 2)}{(2x + 1)^2} = 0$

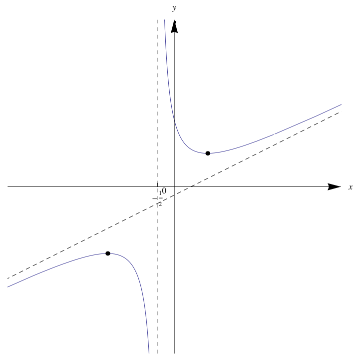
$\Rightarrow (x + 2)(x - 1) = 0$

$\Rightarrow x = -2$ or $x = 1$

Local min $(1, 4)$; local max $(-2, -8)$

d $x = 0, y = 2x - 1$

e $\mathbb{R} \setminus (-8, 4)$



13 $f(x) = \dfrac{x^2 + 4}{x^2 - 5x + 4}$

$= 1 + \dfrac{5x}{x^2 - 5x + 4}$

$= 1 + \dfrac{5x}{(x - 4)(x - 1)}$

a $x = 4, x = 1, y = 1$

b $f'(x) = \dfrac{-5(x^2 - 4)}{(x^2 - 5x + 4)^2}$

$f'(x) = 0$

$\Rightarrow \dfrac{-5(x^2 - 4}{(x^2 - 5x + 4)^2} = 0$

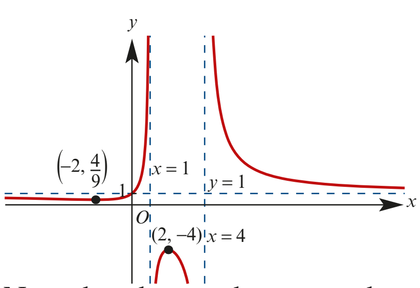
$\Rightarrow x = -2 \text{ or } x = 2$

$f''(x) = \dfrac{10(x^3 - 12x + 20}{(x^2 - 5x + 4)^3}$

$f''(2) < 0, f''(-2) > 0$

Local max $(2, -4)$; local min $\left(-2, \dfrac{4}{9}\right)$

c



Note that the graph crosses the horizontal asymptote at $(0, 1)$

14 $y = \dfrac{2x^2 + 2x + 3}{2x^2 - 2x + 5} = \dfrac{22x - 1}{2x^2 - 2x + 5} + 1$

a $y = 1$

b $\dfrac{dy}{dx} = \dfrac{-8(x^2 - x - 2)}{(2x^2 - 2x + 5)^2}$

$\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{-8(x^2 - x - 2)}{(2x^2 - 2x + 5)^2} = 0$

$\Rightarrow (x - 2)(x + 1) = 0$

$\Rightarrow x = 2 \text{ or } x = -1$

$\dfrac{d^2y}{dx^2} = \dfrac{8(2x - 1)(2x^2 - 2x - 13)}{(2x^2 - 2x + 5)^2}$

$\dfrac{d^2y}{dx^2} < 0$ when $x = 2$

$\dfrac{d^2y}{dx^2} > 0$ when $x = -1$

Local min $\left(-1, \dfrac{1}{3}\right)$; local max $\left(2, \dfrac{5}{3}\right)$

c

$\dfrac{d^2y}{dx^2} = 0$

$\Rightarrow 8(2x - 1)(2x^2 - 2x - 13) = 0$

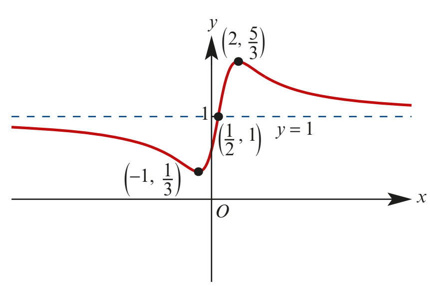
$\Rightarrow x = \dfrac{1}{2} \text{ or } x = \dfrac{1 \pm 3\sqrt{3}}{2}$

Points of inflection $\left(\dfrac{1}{2}, 1\right)$,

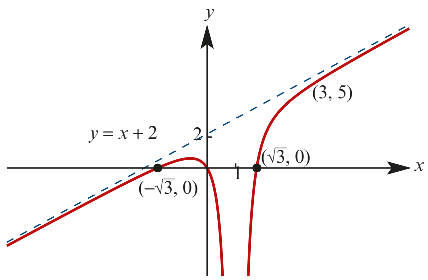
$\left(\dfrac{1 - 3\sqrt{3}}{2}, \dfrac{3 - 3\sqrt{3}}{3}\right)$,

$\left(\dfrac{1 + 3\sqrt{3}}{2}, \dfrac{3 + 3\sqrt{3}}{3}\right)$

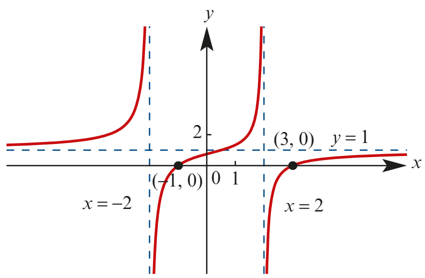
d



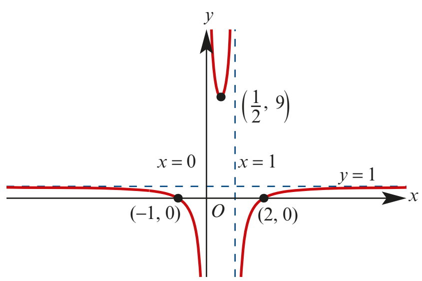
15 **a**



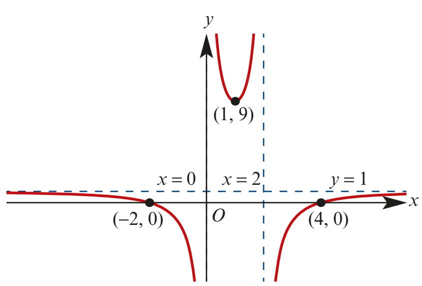
b



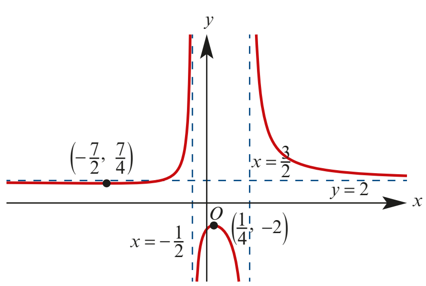
c



d



e



16 $f(x) = \dfrac{x}{\sqrt{x-2}}$

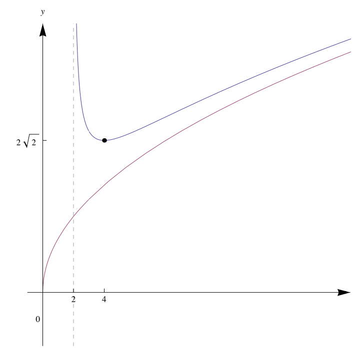
a Maximal domain:$x > 2$

b $f'(x) = \dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$

c $(4, 2\sqrt{2})$, local minimum

d Vertical asymptote $x = 2$

e $f(x) \to \sqrt{x}$ as $x \to \infty$



17 $f(x) = \dfrac{x^2 + x + 7}{\sqrt{2x+1}}$

a Maximal domain: $x > -\dfrac{1}{2}$

b $f(0) = 7$

c $f'(x) = \dfrac{3x^2 + 3x - 6}{(2x+1)^{\frac{3}{2}}}$

d $(1, 3\sqrt{3})$, local minimum ($f''(1) > 0$)

e Vertical asymptote $x = -\dfrac{1}{2}$

Solutions to Exercise 8H

1 **a** Let $f(x) = x^{10}$

$$f'(x) = 10x^9$$
$$f''(x) = 90x^8$$

b Let $f(x) = (2x + 5)^8$

$$f'(x) = 8(2x + 5)^7 \times 2 = 16(2x + 5)^7$$
$$f''(x) = 112(2x + 5)^6 \times 2 = 224(2x + 5)^6$$

c Let $f(x) = \sin(2x)$

$$f'(x) = \cos(2x) \times 2 = 2\cos(2x)$$
$$f''(x) = 2 \times -\sin(2x) \times 2 = -4\sin(2x)$$

d Let $f(x) = \cos\left(\frac{x}{3}\right)$

$$f'(x) = -\sin\left(\frac{x}{3}\right) \times \frac{1}{3} = -\frac{1}{3}\sin\left(\frac{x}{3}\right)$$
$$f''(x) = -\frac{1}{3}\cos\left(\frac{x}{3}\right) \times \frac{1}{3} = -\frac{1}{9}\cos\left(\frac{x}{3}\right)$$

e Let $f(x) = \tan\left(\frac{3x}{2}\right), \cos\left(\frac{3x}{2}\right) \neq 0$

$$f'(x) = \sec^2\left(\frac{3x}{2}\right) \times \frac{3}{2} = \frac{3}{2}\left(\cos\left(\frac{3x}{2}\right)\right)^{-2}$$
$$f''(x) = \frac{3}{2} \times -2\left(\cos\left(\frac{3x}{2}\right)\right)^{-3} \times \left(-\sin\left(\frac{3x}{2}\right)\right) \times \frac{3}{2} = \frac{9}{2}\sin\left(\frac{3x}{2}\right)\sec^3\left(\frac{3x}{2}\right)$$

f Let $f(x) = e^{-4x}$

$$f'(x) = e^{-4x} \times (-4) = -4e^{-4x}$$
$$f''(x) = -4e^{-4x} \times (-4) = 16e^{-4x}$$

g Let $f(x) = \log_e(6x), x > 0$

$$f'(x) = \frac{1}{6x} \times 6 = \frac{1}{x} = x^{-1}$$
$$f''(x) = -x^{-2} = \frac{-1}{x^2}$$

h

Let $f(x) = \sin^{-1}\left(\frac{x}{4}\right)$

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} \times \frac{1}{4}, \frac{x}{4} \in (-1, 1)$$
$$= \frac{1}{4\sqrt{\left(1 - \frac{x^2}{16}\right)}}, x \in (-4, 4)$$
$$= \frac{1}{\sqrt{16 - x^2}}$$
$$= (16 - x^2)^{-\frac{1}{2}}$$
$$f''(x) = -\frac{1}{2}(16 - x^2)^{-\frac{3}{2}} \times (-2x) = \frac{x}{\sqrt{(16 - x^2)^3}}$$

i Let $f(x) = \cos^{-1}(2x)$

$$f'(x) = \frac{-1}{\sqrt{1 - (2x)^2}} \times 2, 2x \in (-1, 1)$$
$$= \frac{-2}{\sqrt{1 - 4x^2}}, x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$
$$= -2(1 - 4x^2)^{-\frac{1}{2}}$$

$$f''(x) = -2 \times -\frac{1}{2}(1-4x^2)^{-\frac{3}{2}} \times (-8x)$$
$$= \frac{-8x}{\sqrt{(1-4x^2)^3}}$$

j Let $f(x) = \tan^{-1}\left(\frac{x}{2}\right)$

$$f'(x) = \frac{2}{4+x^2}$$
$$= 2(4+x^2)^{-1}$$
$$f''(x) = -2(4+x^2)^{-2} \times 2x$$
$$= \frac{-4x}{(4+x^2)^2}$$

k

Let $f(x) = (x+2)\arctan(x-4)$

Using the product rule

$$f'(x) = \arctan(x-4) + \frac{x+2}{1+(x-4)^2}$$
$$f''(x) = \frac{1}{1+(x-4)^2} + \frac{x^2-8x+17-(2x-8)(x+2)}{(1+(x-4)^2)^2}$$
$$= \frac{1}{x^2-8x+17} + \frac{x^2-8x+17-(2x^2-4x-16)}{(x^2-8x+17)^2}$$
$$= \frac{x^2-8x+17+(-x^2-4x+33}{(x^2-8x+17)^2}$$
$$= \frac{-12x+50}{(x^2-8x+17)^2}$$

2 **a** Let $y = (1-4x^2)^3$

$$\therefore \quad \frac{dy}{dx} = 3(1-4x^2)^2 \times (-8x)$$
$$= -24x(1-4x^2)^2$$

b Let $y = \frac{1}{\sqrt{2-x}}, x < 2$

$$= (2-x)^{-\frac{1}{2}}$$
$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}(2-x)^{-\frac{3}{2}} \times (-1)$$
$$= \frac{1}{2\sqrt{(2-x)^3}}$$

c Let $y = \sin(\cos x)$

$$\therefore \quad \frac{dy}{dx} = \cos(\cos x) \times (-\sin x)$$
$$= -\sin x \cos(\cos x)$$

d Let $y = \cos(\log_e x), x > 0$

$$\therefore \quad \frac{dy}{dx} = -\sin(\log_e x) \times \frac{1}{x}$$
$$= \frac{-\sin(\log_e x)}{x}$$

e Let $y = \tan\frac{1}{x}, x \neq 0, \cos\left(\frac{1}{x}\right) \neq 0$

$$\therefore \quad \frac{dy}{dx} = \sec^2\left(\frac{1}{x}\right) \times (-x^{-2})$$
$$= \frac{-\sec^2\left(\frac{1}{x}\right)}{x^2}$$

f Let $y = e^{\cos x}$

$$\therefore \quad \frac{dy}{dx} = e^{\cos x} \times (-\sin x)$$
$$= -\sin x\, e^{\cos x}$$

g Let $y = \log_e(4-3x), x < \frac{4}{3}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4-3x} \times (-3)$$
$$= \frac{-3}{4-3x} = \frac{3}{3x-4}$$

h Let $y = \sin^{-1}(1 - x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (1 - x)^2}} \times (-1), 1 - x \in (-1, 1)$$

$$= \frac{-1}{\sqrt{1 - (1 - 2x + x^2)}},$$

$x \in (0, 2)$

$$= \frac{-1}{\sqrt{x(2 - x)}}$$

i Let $y = \cos^{-1}(2x + 1)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - (2x + 1)^2}} \times 2, 2x + 1 \in (-1, 1)$$

$$= \frac{-2}{\sqrt{1 - (4x^2 + 4x + 1)}},$$

$x \in (-1, 0)$

$$= \frac{-2}{\sqrt{-4x(x + 1)}}$$

j Let $y = \tan^{-1}(x + 1)$

$$\frac{dy}{dx} = \frac{1}{1 + (x + 1)^2} \times 1$$

$$= \frac{1}{1 + x^2 + 2x + 1}$$

$$= \frac{1}{x^2 + 2x + 2}$$

k Let $y = \cos^{-1}\left(\frac{9}{x}\right)$

$$\frac{dy}{dx} = -\frac{9}{x^2} \times -\frac{1}{\sqrt{1 - \left(\frac{9}{x}\right)^2}}$$

$$= \frac{9}{x^2} \times \frac{|x|}{\sqrt{x^2 - 81}}$$

$$= \frac{9}{|x|^2} \times \frac{|x|}{\sqrt{x^2 - 81}}$$

$$= \frac{9}{|x|\sqrt{x^2 - 81}}$$

3 **a** $y = \frac{\log_e x}{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{1}{x}\cdot x - \log_e x}{x^2} = \frac{1 - \log_e x}{x^2}$$

b $y = \frac{x^2 + 2}{x^2 + 1}$

$$\therefore \quad \frac{dy}{dx} = \frac{2x(x^2 + 1) - 2x(x^2 + 2)}{(x^2 + 1)^2}$$

$$= \frac{-2x}{(x^2 + 1)^2}$$

c $y = 1 - \tan^{-1}(1 - x)$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{1 + (1 - x)^2} \times -1$$

$$= \frac{1}{x^2 - 2x + 2}$$

d $y = \log_e\left(\dfrac{e^x}{e^x + 1}\right)$

$= \log_e e^x - \log_e(e^x + 1)$

$= x - \log_e(e^x + 1)$

$\therefore \quad \dfrac{dy}{dx} = 1 - \dfrac{e^x}{e^x + 1}$

$= \dfrac{e^x + 1 - e^x}{e^x + 1}$

$= \dfrac{1}{e^x + 1}$

e $x = \sqrt{\sin y + \cos y}$

$= (\sin y + \cos y)^{\frac{1}{2}}$

$\dfrac{dx}{dy} = \dfrac{\cos y - \sin y}{2\sqrt{\sin y + \cos y}}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2\sqrt{\sin y + \cos y}}{\cos y - \sin y}$

f $y = \log_e(x + \sqrt{1 + x^2})$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1 + \dfrac{2x}{2\sqrt{1 + x^2}}}{x + \sqrt{1 + x^2}}$

$= \dfrac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}(x + \sqrt{1 + x^2})}$

$= \dfrac{1}{\sqrt{1 + x^2}}$

g $y = \sin^{-1} e^x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{e^x}{\sqrt{1 - e^{2x}}}$

h $y = \dfrac{\sin x}{e^x + 1}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(e^x + 1)\cos x - \sin x(e^x)}{(e^x + 1)^2}$

$= \dfrac{e^x(\cos x - \sin x) + \cos x}{(e^x + 1)^2}$

4 $y = ax + \dfrac{b}{x}$

a **i** $\dfrac{dy}{dx} = a - \dfrac{b}{x^2}$

ii $\dfrac{d^2y}{dx^2} = \dfrac{2b}{x^3}$

b $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = x^2\left(\dfrac{2b}{x^3}\right) + x\left(a - \dfrac{b}{x^2}\right)$

$= \dfrac{2b}{x} + ax - \dfrac{b}{x}$

$= ax + \dfrac{b}{x}$

$= y$ (as required)

5 $y = \sin(2x) + 3\cos(2x)$

a **i** $\dfrac{dy}{dx} = \cos(2x) \times 2$

$+ 3(-\sin(2x)) \times 2$

$= 2\cos(2x) - 6\sin(2x)$

ii $\dfrac{d^2y}{dx^2} = 2(-\sin(2x)) \times 2$

$- 6\cos(2x) \times 2$

$= -4\sin(2x) - 12\cos(2x)$

b $\dfrac{d^2y}{dx^2} + 4y = -4\sin(2x) - 12\cos(2x)$

$+ 4(\sin(2x) + 3\cos(2x))$

$= -4\sin(2x) - 12\cos(2x)$

$+ 4\sin(2x) + 12\cos(2x)$

$= 0$, as required to show.

Solutions to Exercise 8I

1 a $x^2 - 2y = 3$

$$2x - 2\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = x$$

Alternatively,

$$2y = x^2 - 3$$

$$y = \frac{1}{2}(x^2 - 3)$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2x = x$$

b $x^2 y = 1$

$$2xy + x^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2} = -\frac{2y}{x}$$

c $x^3 + y^3 = 1$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-x^2}{y^2}$$

Alternatively,

$$y = \sqrt[3]{1 - x^3}$$

$$\frac{dy}{dx} = \frac{-3x^2}{3\sqrt[3]{(1 - x^3)^2}} = \frac{-x^2}{\sqrt[3]{(1 - x^3)^2}}$$

$$= \frac{-x^2}{y^2}$$

d $y^3 = x^2$

$$3y^2\frac{dy}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \frac{2x}{3y^2} \quad ①$$

Alternatively,

$$y = x^{\frac{2}{3}}$$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} \quad ②$$

Substituting $y = x^{\frac{2}{3}}$ into ① yields ②

e $x - \sqrt{y} = 2$

$$1 - \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 2\sqrt{y}$$

Alternatively,

$$y = (x - 2)^2$$

$$\frac{dy}{dx} = 2(x - 2)$$

$$= 2\sqrt{y}$$

since $\sqrt{y} = x - 2$,

f $xy - 2x + 3y = 0$

$$y + x\frac{dy}{dx} - 2 + 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3 + x) = 2 - y$$

$$\frac{dy}{dx} = \frac{2 - y}{3 + x}$$

Alternatively

$$y = \frac{2x}{x + 3}$$

$$\frac{dy}{dx} = \frac{2(x + 3) - 2x}{(x + 3)^2}$$

$$= \frac{6}{(x + 3)^2}$$

$$\frac{2 - y}{3 + x} = \frac{2 - \frac{2x}{3 + x}}{3 + x}$$

$$= \frac{2(x + 3) - 2x}{(x + 3)^2}$$

$$= \frac{6}{(x + 3)^2}$$

g $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$= \frac{2a}{y}$$

Alternatively,

$$y = \pm\sqrt{4ax}$$

$$\frac{dy}{dx} = \pm\frac{4a}{2\sqrt{4ax}}$$

$$= \frac{2a}{y}$$

h $4x + y^2 - 2y - 2 = 0$

$$4 + 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2}{1-y}$$

Alternatively,

$$x = -\frac{1}{4}(y^2 - 2y - 2)$$

$$\frac{dx}{dy} = -\frac{1}{4}(2y - 2)$$

$$\frac{dy}{dx} = -\frac{2}{y-1} = \frac{2}{1-y}$$

2 a $(x+2)^2 - y^2 = 4$

$$2(x+2) - 2y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{x+2}{y}$$

b $\frac{1}{x} = \frac{1}{y} = 1$

$$-\frac{1}{x^2} - \frac{1}{y^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y^2}{x^2}$$

c $y = (x+y)^2$

$$\frac{dy}{dx} = 2(x+y)\left(1 + \frac{dy}{dx}\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x+2y}{1-2x-2y}$$

$$= \frac{2(x+y)}{1-2(x+y)}$$

d $x^2 - xy + y^2 = 1$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{y-2x}{2y-x}$$

e $y = x^2e^y$

$$\frac{dy}{dx} = 2x\,e^y + x^2e^y\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x\,e^y}{1 - x^2e^y}$$

f $\sin y = \cos^2 x$

$$\cos y\frac{dy}{dx} = -2\cos x \sin x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\sin 2x}{\cos y}$$

g $\sin(x-y) = \sin x - \sin y$

$$\cos(x-y)\left(1 - \frac{dy}{dx}\right) = \cos x - \cos y\frac{dy}{dx}$$

$$(-\cos(x-y) + \cos y)\frac{dy}{dx} = \cos x - \cos(x-y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x - \cos(x-y)}{\cos y - \cos(x-y)}$$

h $y^5 - x\sin y + 3y^2 = 1$

$$5y^4\frac{dy}{dx} - \sin y$$

$$-x\cos y\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(5y^4 - x\cos y + 6y) = \sin y$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin y}{5y^4 - x\cos y + 6y}$$

3 **a** $y^2 = 8x$ at $(2, -4)$

$$2y\frac{dy}{dx} = 8$$

At $(2, -4)$, $\frac{dy}{dx} = -1$

The equation of the tangent is

$y + 4 = -(x - 2)$

$\therefore \quad y = -x - 2$

$\therefore \quad x + y = -2$

b $x^2 - 9y^2 = 9$ at $\left(5, \frac{4}{3}\right)$

$$2x - 18y\frac{dy}{dx} = 0$$

At $\left(5, \frac{4}{3}\right)$, $\frac{dy}{dx} = \frac{x}{9y} = \frac{5}{12}$

The equation of the tangent is

$$y - \frac{4}{3} = \frac{5}{12}(x - 5)$$

$\therefore \quad 12y - 16 = 5x - 25$

$\therefore \quad 12y - 5x = -9$

$\therefore \quad 5x - 12y = 9$

c $xy - y^2 = 1$ at $\left(\frac{17}{4}, 4\right)$

$$y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

Substituting $x = \frac{17}{4}$ and $y = 4$ yields

$$4 + \frac{17}{4}\frac{dy}{dx} - 8\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{16}{15}$$

The equation of the tangent is

$$y - 4 = \frac{16}{15}\left(x - \frac{17}{4}\right)$$

$\therefore \quad 15y - 60 = 16x - 68$

$\therefore \quad 16x - 15y = 8$

d $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at $(0, -3)$

$$\frac{x}{8} + \frac{2y}{9}\frac{dy}{dx} = 0$$

Substituting $x = 0$ and $y = -3$ yields

$$\frac{dy}{dx} = 0$$

The equation of the tangent is $y = -3$

4 $\log_e y = \log_e x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}$$

5 $x^3 + y^3 = 9$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2}{y^2}$$

At $(1, 2)$,

$$\frac{dy}{dx} = -\frac{1}{4}$$

6 $x^3 + y^3 + 3xy - 1 = 0$

$$3x^2 + 3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx}(3y^2 + 3x) = -3x^2 - 3y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(3x^2 + 3y)}{3y^2 + 3x}$$

At $(2, -1)$,

$$\frac{dy}{dx} = \frac{-9}{9} = -1$$

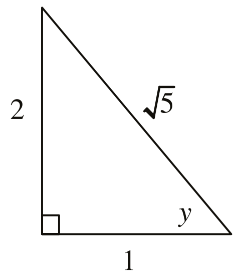
7 $\tan x + \tan y = 3$

$$\sec^2 x + \sec^2 y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\sec^2 x}{\sec^2 y} = -\frac{\cos^2 y}{\cos^2 x}$$

When $x = \frac{\pi}{4}$, $1 + \tan y = 3$

$\therefore \quad \tan y = 2$



$\Rightarrow \cos y = \frac{1}{\sqrt{5}}$

$\therefore \quad$ when $x = \frac{\pi}{4}$, $\cos y = \frac{1}{\sqrt{5}}$

$$\therefore \quad \frac{dy}{dx} = -\frac{\left(\frac{1}{\sqrt{5}}\right)^2}{\left(\cos\frac{\pi}{4}\right)^2}$$

$$= -\frac{1}{5} \times \frac{2}{1}$$

$$= -\frac{2}{5}$$

8 $\quad y^2 + xy - 2x^2 = 4$

$$2y\frac{dy}{dx} + y + x\frac{dy}{dx} - 4x = 0$$

$$\therefore \quad \frac{dy}{dx}(2y + x) = -y + 4x$$

$$\therefore \quad \frac{dy}{dx} = \frac{4x - y}{2y + x}$$

At $(1, -3)$,

$$\frac{dy}{dx} = \frac{7}{-5} = -\frac{7}{5}$$

9 $x^3 + y^3 = 28$

a $3x^2 + 3y^2\frac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2}{y^2}$$

b $\frac{dy}{dx} = -\frac{x^2}{y^2}$

$$= -\left(\frac{x}{y}\right)^2$$

$$< 0$$

Therefore $\frac{dy}{dx}$ cannot be positive.

c When $x = 1$,

$$y^3 = 27$$

$$\therefore \quad y = 3$$

At $(1, 3)$,

$$\frac{dy}{dx} = -\frac{1}{9}$$

10 $\quad 2x^2 + 8xy + 5y^2 = -3 \quad ①$

$$4x + 8y + 8x\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx}(8x + 10y) = -4x - 8y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(2x + 4y)}{4x + 5y}$$

Tangents are in the form $y = k$,

i.e. $\frac{dy}{dx} = 0$

$\therefore \quad 2x + 4y = 0$

$\therefore \quad x = -2y$

Substituiting into ①

$$8y^2 - 16y^2 + 5y^2 = -3$$

$$\therefore \quad -3y^2 = -3$$

$$\therefore \quad y^2 = 1$$

$$\therefore \quad y = \pm 1$$

The equation of the two tangents that are parallel to the x-axis are $y = -1$ and $y = 1$.

11 $x^3 + xy + 2y^3 = k, k \in R$ ①

a $3x^2 + y + x\dfrac{dy}{dx} + 6y^2\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx}(6y^2 + x) = -3x^2 - y$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-(3x^2 + y)}{6y + x}$

b Tangent is parallel to the y-axis

i.e. $\dfrac{dy}{dx}$ = undefined

$\Rightarrow 6y^2 + x = 0$

$\therefore \quad x = -6y^2$

Substituiting into ①

$-216y^6 - 6y^3 + 2y^3 = k$

$\therefore \quad -216y^6 - 4y^3 = k$

$\therefore \quad -216y^6 - 4y^3 - k = 0$

$\therefore \quad 216y^6 + 4y^3 + k = 0$

as required to show.

c Let $z = y^3$ in part b giving

$216z^2 + 4z + k = 0$

For y to exist, z must exist so this Quadratic must have solutions.

i.e.

$b^2 - 4ac \geq 0$

$16 - 4 \times 216k \geq 0$

$54k \leq 1$

$k \leq \dfrac{1}{54}$

d Let $x = -6$ be the tangent.

$\therefore \quad 6y^2 - 6 = 0$ for $\dfrac{dy}{dx}$ = undefined

$\therefore \quad 6y^2 = 6$

$\therefore \quad y = \pm 1$

When $x = -6$ and $y = -1$,

$\therefore \quad k = (-6)^3 + (-6) \times (-1) + 2(-1)^3$

$= -212$

When $x = -6$ and $y = 1$,

$\therefore \quad k = (-6)^3 + (-6) \times (1) + 2(1)^3$

$= -220$

12 $x^2 - 2xy + 2y^2 = 4$ ①

a $2x - 2y - 2x\dfrac{dy}{dx} + 4y\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx}(4y - 2x) = 2y - 2x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{y - x}{2y - x}$

b Tangents are parallel to the x-axis when

$\dfrac{dy}{dx} = 0$

$\therefore \quad x = y$

Substituting into ①

$y^2 - 2y^2 + 2y^2 = 4$

$\therefore \quad y^2 = 4$

$\therefore \quad y = \pm 2$

When $y = -2, x = -2$

When $y = 2, x = 2$

$\therefore \quad (-2, -2)$ and $(2, 2)$

13 $y^2 + x^3 = 1$ ①

a $2y\dfrac{dy}{dx} + 3x^2 = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{3x^2}{2y}$

b When $\frac{dy}{dx} = 0, x = 0$

Substituting $x = 0$ into ①

$\therefore \quad y^2 = 1$

$\therefore \quad y = \pm 1$

$\therefore \quad (0, -1)$ and $(0, 1)$

c $\frac{dx}{dy} = -\frac{2y}{3x^2}$

For $\frac{dx}{dy} = 0, y = 0$

When $y = 0, x^3 = 1$

$\therefore \quad x = 1$

$\therefore \quad (1, 0)$

d for $x \to -\infty, y^2 \to \infty$

$\therefore \quad y \to \pm\infty$

Also as $x \to -\infty, \frac{dy}{dx} \to \frac{-\infty}{2y}$

if y is positive then $\frac{dy}{dx} \to -\infty$

if $y = 0$ then $\frac{dy}{dx} =$ undefined

if y is negative then $\frac{dy}{dx} \to \infty$

e $y^2 + x^3 = 1$

$y^2 = 1 - x^3$

$\therefore \quad y = \pm\sqrt{1 - x^3}$

f If $y = \pm\sqrt{1 - x^3}$

Then $\frac{dy}{dx} = \pm\frac{1}{2}(1 - x^3)^{-\frac{1}{2}} \times -3x^2$

$= \pm\frac{3x^2}{2}(1 - x^3)^{-\frac{1}{2}}$

$= \pm\frac{3x^2}{2\sqrt{1 - x^3}}$

and from part **b**. stationary points occur at $(0, -1)$ and $(0, 1)$.

X	$-\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{dy}{dx} = +\ldots$	< 0	0	< 0
Slope	\	–	\

Thus the point $(0, 1)$ is a point of inflection.

X	$-\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{dy}{dx} = -\ldots$	> 0	0	> 0
Slope	/	–	/

Thus the point $(0, -1)$ is a point of inflection.

g Using a CAS calculator to sketch a graph of the equation $y^2 + x^3 = 1$

Input the following function into $f1(x) : f1(x) = \sqrt{1 - x^3}$

Input the following function into $f2(x) : f2(x) = -\sqrt{1 - x^3}$

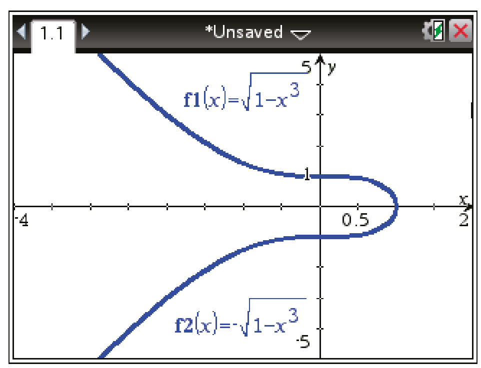
Change the window settings to:

$X_{min} = -4$

$X_{max} = 2$

$Y_{min} = -5$

$Y_{max} = 5$



14 a $2(x - 2) + 2(y - 2)\frac{dy}{dx} = 0$

Therefore $\frac{dy}{dx} = \frac{2 - x}{y - 2}$

b When $x = 1$,

$1 + (y - 2)^2 = 9$

$(y-2)^2 = 8$

$y = 2 \pm 2\sqrt{2}$

In the first quadrant the point is $(1, 2 + 2\sqrt{2})$

Hence gradient $= \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

15 $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(6-y)^2$

$= -2(6-y)\dfrac{dy}{dx}$

$= -2(6-y)^3$

16 $8x + (3y + 3x\dfrac{dy}{dx}) + 2y\dfrac{dy}{dx} = 0$

$8x + 3y + (3x+2y)\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{8x+3y}{3x+2y}$

When $x = 1$ and $y = 2$,

$\dfrac{dy}{dx} = -2$

For the second derivative:

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(-\dfrac{8x+3y}{3x+2y}\right)$

$= \dfrac{7x\dfrac{dy}{dx} - 7y}{(3x+2y)^2}$

When $x = 1$ and $y = 2$,

$\dfrac{d^2y}{dx^2} = \dfrac{7 \times (-2) - 14}{49}$

$= -\dfrac{28}{49}$

$= -\dfrac{4}{7}$

17 $2x^2\cos y + 2xy = \dfrac{\pi^2}{3}$

$4x\cos y - 2x^2\sin y\dfrac{dy}{dx} + 2x\dfrac{dy}{dx} + 2y = 0$

$4x\cos y + 2y = (2x^2\sin y - 2x)\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{4x\cos y + 2y}{2x^2\sin y - 2x}$

$= \dfrac{2x\cos y + y}{x^2\sin y - x}$

When $x = \dfrac{\pi}{3}$ and $y = \dfrac{\pi}{3}$

$\dfrac{dy}{dx} = \dfrac{12}{\pi\sqrt{3} - 6}$

18 $x^2 + y^2 = 169$

$2x + 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x}{y}$

When $x = 5$ and $y = 12$,

$\dfrac{dy}{dx} = -\dfrac{5}{12}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(-\dfrac{x}{y}\right)$

$= -\dfrac{y - x\dfrac{dy}{dx}}{y^2}$

When $x = 5$ and $y = 12$,

$\dfrac{d^2y}{dx^2} = -\dfrac{12 + \dfrac{25}{12}}{144}$

$= -\dfrac{169}{1728}$

19

$2(x^2+y^2)^2 = 25(x^2-y^2)$

$2(x^4 + 2x^2y^2 + y^4) = 25x^2 - 25y^2$

$2x^4 + 4x^2y^2 + 2y^4 - 25x^2 + 25y^2 = 0$

Implicitly differentiating

$8x^3 + 4(2xy^2 + 2x^2y\dfrac{dy}{dx}) + 8y^3\dfrac{dy}{dx} - 50x + 50y\dfrac{dy}{dx} = 0$

$(8x^2y + 8y^3 + 50y)\dfrac{dy}{dx} = 50x - 8x^3 - 8xy^2$

$\dfrac{dy}{dx} = \dfrac{50x - 8x^3 - 8xy^2}{8x^2y + 8y^3 + 50y}$

$= \dfrac{x(25 - 4x^2 - 4y^2)}{y(4x^2 + 4y^3 + 25}$

When $x = 3$ and $y = 1$, $\dfrac{dy}{dx} = \dfrac{-9}{13}$

So quation of tangent:

$$y - 1 = \frac{-9}{13}(x - 3)$$
$$13y - 13 = -9x + 27$$
$$13y + 9x = 40$$

20 $4x^2 + y^2 = 1$

Implicitly differentiating

$$8x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{4x}{y}$$

$$\frac{d}{dx}\left(\frac{-4x}{y}\right) = -\frac{4y - 4x\frac{dy}{dx}}{y^2}$$
$$= -\frac{4y + 4 \times \frac{4x}{y}}{y^2}$$
$$= -\frac{4y + \frac{16x^2}{y}}{y^2}$$
$$= -\frac{4y^2 + 16x^2}{y^3}$$

Solutions to Technology-free questions

1 a $\frac{dy}{dx} = \tan x + x\sec^2 x$

b $y = x \Rightarrow \frac{dy}{dx} = 1$

c $y = \sqrt{1 - x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}}$

d $\frac{dy}{dx} = \frac{2}{\sqrt{1 - (2x - 1)^2}}$

$= \frac{2}{\sqrt{1 - 4x^2 + 4x - 1}}$

$= \frac{1}{\sqrt{x - x^2}}$

2 a $f'(x) = \sec^2 x$

$f''(x) = 2\sec x\left(\frac{\sin x}{\cos^2 x}\right)$

$= 2\sec^2 x\ \tan x$

b $f'(x) = \frac{\sec^2 x}{\tan x}$

$= \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}$

$= \frac{2}{2\sin x\cos x}$

$= \frac{2}{\sin 2x}$

$f''(x) = \frac{-4\cos 2x}{\sin^2 2x}$

$= -4\cot 2x \operatorname{cosec} 2x$

Alternative solution:

$f(x) = \log_e \sin x - \log_e \cos x$

$f'(x) = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

$= \cot x + \tan x$

$f''(x) = -\operatorname{cosec}^2 x + \sec^2 x$

c $f'(x) = \sin^{-1} x + \frac{x}{\sqrt{1 - x^2}}$

(product rule)

$f''(x) = \frac{1}{\sqrt{1 - x^2}} + \frac{\sqrt{1 - x^2} + \frac{2x^2}{2\sqrt{1 - x^2}}}{1 - x}$

$= \frac{1 - x^2 + 1 - x^2 + x^2}{(1 - x^2)\sqrt{1 - x^2}}$

$= \frac{2 - x^2}{(1 - x^2)^{\frac{3}{2}}}$

d $f'(x) = \cos e^x \times e^x$ (chain rule)

$f''(x) = -e^{2x}\sin e^x + e^x\cos e^x$

$= e^x(\cos e^x - e^x \sin e^x)$

3 a $y = x^3 - 8x^2$

$\frac{dy}{dx} = 3x^2 - 16x$

$\frac{d^2y}{dx^2} = 6x - 16$

When $\frac{d^2y}{dx^2} = 0, 6x - 16 = 0$

$\therefore \quad x = \frac{8}{3}$

There is a change of sign for $\frac{d^2y}{dx^2}$.

The point of inflection is $\left(\frac{8}{3}, \frac{-1024}{27}\right)$.

b $y = \sin^{-1}(x-2)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x-2)^2}}$$

$$\frac{d^2y}{dx^2} = \frac{2-x}{(4x-3-x^2)^{\frac{3}{2}}}$$

When $\frac{d^2y}{dx^2} = 0$, $\frac{2-x}{(4x-3-x^2)^{\frac{3}{2}}} = 0$

$\therefore \quad x = 2$

There is a change of sign for $\frac{d^2y}{dx^2}$.
The point of inflection is $(2, 0)$.

c $y = \log_e(x) + \frac{1}{x}$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2} + \frac{2}{x^3}$$

When $\frac{d^2y}{dx^2} = 0$, $\frac{-1}{x^2} + \frac{2}{x^3} = 0$

$\therefore \quad -x + 2 = 0$

$\therefore \quad x = 2$

There is a change of sign for $\frac{d^2y}{dx^2}$. The point of inflection is $\left(2, \log_e 2 + \frac{1}{2}\right)$.

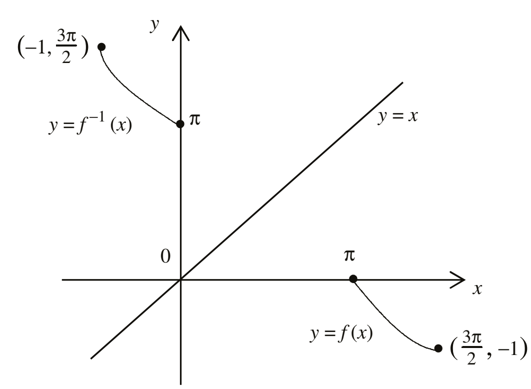
4 If the graph of $y = f(x)$ is concave up then $f''(x) > 0$
Let $g(x) = e^{f(x)}$. Then,
$g'(x) = f'(x)e^{f(x)}$
$g''(x) = f''(x)e^{f(x)} + (f'(x))^2 e^{f(x)}$
We have,
$f''(x)e^{f(x)} > 0$ and $(f'(x))^2 e^{f(x)} \geq 0$
Hence, $g''(x) > 0$

5 Let $g(x) = (x-a)^3 f(x)$
$g'(x) = 3(x-a)^2 f(x) + (x-a)^3 f'(x)$
$g''(x) = 6(x-a)f(x) + 3(x-a)^2 f'(x) + 3(x-a)^2 f'(x) + (x-a)^3 f''(x)$
$g''(a) = 0$ since $(x-a)$ is a factor of each term.

6 Given $f(x) > 0, x \in \mathbb{R}$ and $f''(x) < 0, x \in \mathbb{R}$
Let $g(x) = \log_e(f(x))$. Then,
$g'(x) = \frac{f'(x)}{f(x)}$
$g''(x) = \frac{f''(x)f(x) - [f'(x)]^2}{(f(x))^2}$
Since $f(x) > 0, x \in \mathbb{R}$ and
$f''(x) < 0, x \in \mathbb{R}$
$f''(x)f(x) - [f'(x)]^2 < 0$ and $(f(x)) > 0$.
Hence, $g''(x) < 0, x \in \mathbb{R}$

7 The domain of arccos is $[-1, 1]$ and the range is $[0, \pi]$ and $\arccos(1) = 0$. The derivative of is $\arccos(x) - \frac{1}{\sqrt{1-x^2}}$.
Therefore domain is $(-1, 1)$ and range is $(-\infty, 1]$.
The derivative of $f(x) = \frac{1}{\arccos(x)}$ is
$f'(x) = -\frac{1}{\sqrt{1-x^2} \times -(\arccos(x))^2}$
$= \frac{1}{\sqrt{1-x^2}(\arccos(x))^2}$
Domain $= (-1, 1)$

8 a



b For f, $\dfrac{dy}{dx} = \cos x$

$= -\sqrt{1-\sin^2 x}$

since $\cos x \le 0$

when $\pi \le x \le \dfrac{3\pi}{2}$

For f^{-1}, $x = \sin y$

$\therefore \quad \dfrac{dx}{dy} = \cos y$

$\therefore \quad \dfrac{dx}{dy} = \dfrac{1}{\cos y}$

$= \dfrac{1}{-\sqrt{1-\sin^2 y}}$

since $\dfrac{dy}{dx} < 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

c $-\dfrac{1}{\sqrt{1-x^2}} = -2$

$\sqrt{1-x^2} = \dfrac{1}{2}$

$1 - x^2 = \dfrac{1}{4}$

$x^2 = \dfrac{3}{4}$

$x = \dfrac{-\sqrt{3}}{2}$ since $x < 0$

$y = \pi + \dfrac{\pi}{3}$

$= \dfrac{4\pi}{3}$

9 $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{1-t^2}}$ and

$\dfrac{dy}{dt} = \dfrac{-2t}{1-t^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{-2t}{1-t^2} \times \sqrt{1-t^2}$

$= \dfrac{-2t}{\sqrt{1-t^2}}$

a When $t = \dfrac{1}{2}$, $x = \dfrac{\pi}{6}$ and $y = \log_e\left(\dfrac{3}{4}\right)$

$\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{\left(\dfrac{3}{4}\right)}}$

$= -\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$

b $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{-2t}{\sqrt{1-t^2}}\right)$

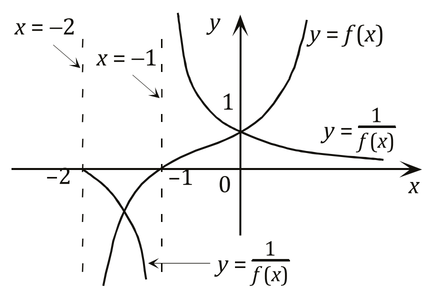
$= \dfrac{d}{dt}\left(\dfrac{-2t}{\sqrt{1-t^2}}\right) \times \dfrac{dt}{dx}$

$= -\dfrac{2}{(1-t^2)^{\frac{3}{2}}} \times \sqrt{1-t^2}$

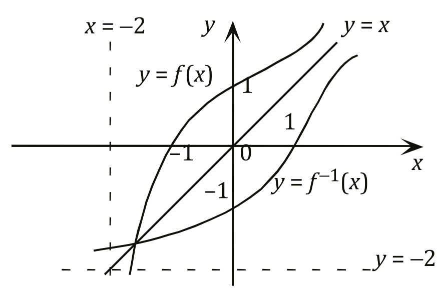
$= -\dfrac{2}{(1-t^2)}$

When $t = \dfrac{1}{2}$, $\dfrac{d^2y}{dx^2} = -\dfrac{8}{3}$

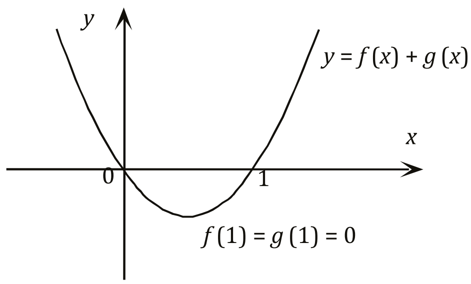
10 a

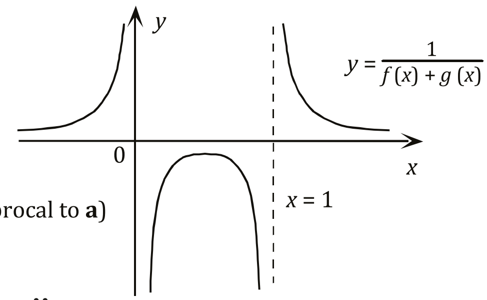


b



11 a i

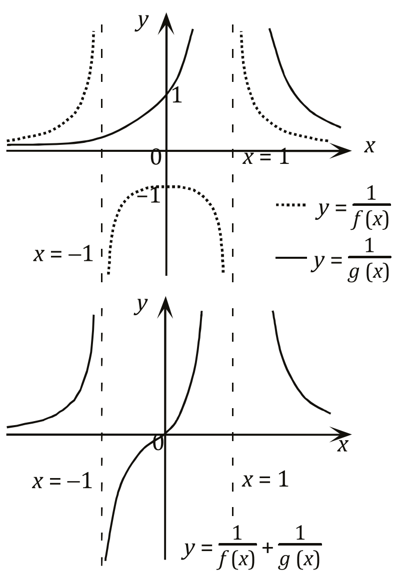




(reciprocal to **a**)

ii

iii



b $f(x)$ is a translation of $y = x^2$ one unit in the negative direction of the y-axis.
$g(x)$ is a translation of $y = x^2$ one unit in the positive direction of the x-axis.

$\therefore\ f(x) = x^2 - 1, g(x) = (x - 1)^2$

c **i** $f(x) + g(x) = 2x^2 - 2x$

ii $\dfrac{1}{f(x) + g(x)} = \dfrac{1}{2x^2 - 2x}$

iii $\dfrac{1}{f(x)} + \dfrac{1}{g(x)} = \dfrac{f(x) + g(x)}{f(x)g(x)}$

$= \dfrac{2x(x - 1)}{(x^2 - 1)(x - 1)^2}$

$= \dfrac{2x}{(x^2 - 1)(x - 1)}$

$= \dfrac{2x}{(x - 1)^2(x + 1)}$

12 **a** $x^2 + 2xy + y^2 = 1$

Differentiate both sides with respect to x.

$2x + 2y + 2x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$

$\therefore\ \dfrac{dy}{dx} = \dfrac{-2x - 2y}{2x + 2y}$

$= -1$

This problem can be also be done by observing that

$x^2 + 2xy + y^2 = 1$

which implies

$(x + y)^2 = 1$

b $x^2 + 2x + y^2 + 6y = 10$
Differentiate both sides with respect to x.

$$2x + 2 + 2y\frac{dy}{dx} + 6\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(2x+2)}{2y+6} = -\left(\frac{x+1}{y+3}\right)$$

c $\frac{2}{x} + \frac{1}{y} = 4$
Differentiate both sides of the equation with respect to x.

$$\frac{-2}{x^2} + \frac{-1}{x^2} \times \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2y^2}{x^2}$$

d $(x+1)^2 + (y-3)^2 = 1$
Differentiate both sides with respect to x.

$$2(x+1) + 2(y-3)\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(x+1)}{y-3}$$

e $\cos(x) + \sin(y) = 1$
Differentiate both sides with respect to x.

$$-\sin(x) + \cos(y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\sin(x)}{\cos(y)}$$

f $x \log_e y = 10$
Differentiate both sides with respect to x.

$$\log_e y + \frac{x}{y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y\log_e(y)}{x}$$

13 $y = x^3$

$$\therefore \quad \frac{dy}{dx} = 3x^2$$

Now $\frac{dx}{dt} = 3$

and $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = 3x^2 \times 3 = 9x^2$

a When $x = 6$, $\frac{dy}{dt} = 324$ cm/s

b When $y = 8$, $x = 2$ and $\frac{dy}{dt} = 36$ cm/s

14 a Let $f(x) = \frac{x+1}{x^2-9}$

$$f'(x) = \frac{x^2 - 9 - 2x(x+1)}{(x^2-9)^2} = \frac{-x^2-2x-9}{(x^2-9)^2} = -\frac{x^2+2x+9}{(x^2-9)^2} = -\frac{(x+1)^2+8}{(x^2-9)^2} \leq 0$$

b $f(0) = -\frac{1}{9}$
$f(x) = 0 \Rightarrow x = -1$
Coordinates of axis intercepts:
$(-1, 0)$ and $\left(0, -\frac{1}{9}\right)$

c $x = \pm 3$ $y = 0$

15 Let $y = x^3 + ax^2 + bx - 5 \ldots (1)$

$$\frac{dy}{dx} = 3x^2 + 2ax + b$$

$$\frac{d^2y}{dx^2} = 6x + 2a \ldots (2)$$

Point of inflection at $(1, 5)$.
Hence from (2), $6 + 2a = 0 \Rightarrow a = -3$
From (1)

$5 = 1 - 3 + b - 5$
$b = 12$

16 $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{3t}{2}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3t}{2}\right)$

$= \frac{d}{dt}\left(\frac{3t}{2}\right) \times \frac{dt}{dx}$

$= \frac{3}{2} \times \frac{1}{2t}$

$= \frac{3}{4t}$

17 Given $\frac{dy}{dx} = e^{2x} \arctan y$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(e^{2x} \arctan y)$

$= 2e^{2x} \arctan y + e^{2x} \times \frac{1}{1 + y^2} \frac{dy}{dx}$

$= 2e^{2x} \arctan y + e^{2x} \times \frac{1}{1 + y^2} \times$
$e^{2x} \arctan y$

$= e^{2x} \arctan y \left(2 + \frac{e^{2x}}{1 + y^2}\right)$

Solutions to multiple-choice questions

1 E $x^2 + y^2 = 1$

Using implicit differentiation

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}$$

At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\frac{dy}{dx} = -1$

The equation of the tangent is given by

$$y - \frac{1}{\sqrt{2}} = -\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\therefore \quad y = -x + \frac{2}{\sqrt{2}}$$

$$\therefore \quad y = -x + \sqrt{2}$$

2 E

$$f(x) = 2x^2 + 3x - 20$$

$$= (x+4)(2x-5)$$

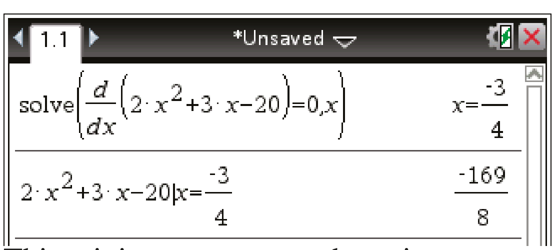
Therefore $f(x)$ has x-axis intercepts at $x = -4$

$x = \frac{5}{2}$.

Thus $\frac{1}{f(x)}$ will have asymptotes at $x = -4$ and $x = \frac{5}{2}$ since $\frac{1}{0}$ is undefined

Since the coefficient of the x^2 term is positive,

$f(X)$ has a local minimum.



This minimum occurs at the point

Thus $\frac{1}{f(x)}$ has a local maximum at the point $\left(-\frac{3}{4}, -\frac{8}{169}\right)$.

3 B

$$y = \sin x, \quad x \in [0, 2\pi]$$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} = -\sin x$$

For $\frac{d^2y}{dx^2} = 0, \sin x = 0$

$$\therefore \quad x = 0, \pi, 2\pi \quad x \in [0, 2\pi]$$

$$f(0) = \sin 0$$
$$= 0$$

$$f(\pi) = \sin \pi$$
$$= 0$$

$$f(2\pi) = \sin 2\pi$$
$$= 0$$

Consider $f''(x)$ for values of x on either side of $0, \pi$ and 2π to determine the nature of the stationary points. Noting that due to the restricted domain we cannot determine the value of $f''(x)$ for x values lying outside the interval $[0, 2\pi]$.

$$f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2}$$
$$= -1$$
$$< 0$$

$$f''\left(\frac{3\pi}{2}\right) = -\sin\frac{3\pi}{2}$$
$$= 1$$
$$> 0$$

X	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f''(X)$	0	< 0	0	> 0	0
Slope	–	\	–	/	–

There is a point of minimum gradient at $(\pi, 0)$ i.e. there is a point of inflection at the point $(\pi, 0)$.
$\left(-\frac{3}{4}, -\frac{169}{8}\right)$.

4 E

$$g(x) = e^{-x}f(x)$$
$$g'(x) = -e^{-x}f(x) + e^{-x}f'(x)$$
$$= e^{-x}(f'(x) - f(x))$$
$$g''(x) = -e^{-x}(f'(x) - f(x)) + e^{-x}(f''(x) - f'(x))$$
$$g''(x) = e^{-x}(f''(x) - 2f'(x) + f(x))$$
$$\therefore \quad g''(x)e^{x} = f''(x) - 2f'(x) + f(x)$$

When $x = a$,

$$g''(a)e^{a} = f''(a) - 2f'(a) + f(a)$$

Since there is a point of inflection at $(a, g(a))$, this implies that $g''(a) = 0$.

$$\therefore \quad f''(a) - 2f'(a) + f(a) = 0$$
$$\therefore \quad f''(a) = 2f'(a) - f(a)$$

5 B

$x = t^2$ ① and $y = t^3$ ②

From ② : $y^{\frac{2}{3}} = t^2$

$$\therefore \quad x = y^{\frac{2}{3}}$$
$$\frac{dx}{dy} = \frac{2}{3}y^{-\frac{1}{3}}$$

Also from ② : $y^{-\frac{1}{3}} = t^{-1} = \frac{1}{t}$

$$\therefore \quad \frac{dx}{dy} = \frac{2}{3t}$$

Alternatively,

$$\frac{dx}{dy} = \frac{dx}{dt} \times \frac{dt}{dy}$$
$$= 2t \times \frac{1}{\frac{dy}{dt}}$$
$$= 2t \times \frac{1}{3t^2}$$
$$= \frac{2}{3t}$$

6 D

$$y = \cos^{-1}\left(\frac{4}{x}\right), x > 4$$

Let $g(x) = 4x^{-1}$ then $g'(x) = -\frac{4}{x^2}$

Using the chain rule,

$$\frac{dy}{dx} = -\frac{g'(x)}{\sqrt{1 - [g(x)]^2}}$$
$$= \frac{4}{x^2\sqrt{1 - \frac{16}{x^2}}}$$
$$= \frac{4}{x\sqrt{x^2 - 16}}, x > 4$$

7 C

$$y = x^2 + \frac{54}{x}$$
$$\frac{dy}{dx} = 2x - \frac{54}{x^2}$$

$\frac{dy}{dx} = 0$ for stationary points.

$$\therefore \quad 2x = \frac{54}{x^2}$$
$$\therefore \quad 2x^3 = 54$$
$$\therefore \quad x^3 = 27$$
$$\therefore \quad x = 3$$

When $x = 3, y = 27$
Therefore the coordinates of the turning point are (3, 27).

8 B

$y = \sin^{-1}\left(\dfrac{x}{2}\right), x \in [0, 1]$ and $y \in \left[0, \dfrac{\pi}{2}\right]$

$$\frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}$$

$$= (4 - x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(4 - x^2)^{-\frac{3}{2}} \times -2x$$

$$= \frac{x}{\sqrt{(4 - x^2)^3}}$$

$$= \frac{x}{(4 - x^2)^{\frac{3}{2}}}$$

9 D

$$y = \tan^{-1}\left(\frac{1}{3x}\right)$$

Let $g(x) = \dfrac{1}{3}x^{-1}$ then $g'(x) = -\dfrac{1}{3x^2}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{g'(x)}{1 + [g(x)]^2}$$

$$= \frac{-1}{3x^2\left(1 + \left(\frac{1}{3x}\right)^2\right)}$$

$$= \frac{-1}{3x^2\left(1 + \frac{1}{9x^2}\right)}$$

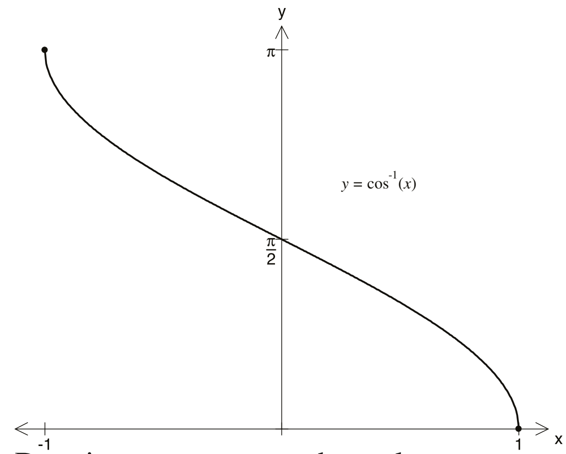
$$= \frac{-3}{9x^2\left(1 + \frac{1}{9x^2}\right)}$$

$$= \frac{-3}{9x^2 + 1}$$

10 C

$y = \cos^{-1} x$

A sketch of $y = f(X)$ is



Drawing tangents anywhere along this curve (except at $x = \pm 1$) reveals that the gradient is always negative. Therefore response A is true.

By observation there is a point of inflection at $\left(0, \dfrac{\pi}{2}\right)$. Therefore response B is true.

The gradient of the graph at $x = \pm 1$ is undefined. Therefore response D is true.

When $x = \dfrac{1}{2}, y = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$.

Therefore response E is true.

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}(1 - x^2)^{-\frac{3}{2}} \times -2x$$

$$= \frac{x}{\sqrt{(1 - x^2)^3}}$$

X	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f''(X)$	> 0	0	< 0
Slope	/	–	\

When $x = 0, \dfrac{dy}{dx} = -1$

Therefore the gradient of the graph has a **maximum** value of -1.

Response C is false.

11 C Differentiating with respect to x.

$$\frac{d^2y}{dx^2} = \frac{d(e^x \cos^2 y)}{dx}$$

$$= e^x \cos^2(y) + e^x(-2 \sin y \cos y)\frac{dy}{dx}$$

$$= e^x \cos^2(y) - e^x \sin 2y\frac{dy}{dx}$$

$$= e^x \cos^2(y) - e^x \sin 2y \times e^x \cos^2(y)$$

When $x = 0, y = \frac{\pi}{6}$

$$\frac{d^2y}{dx^2} = \frac{3}{4} - \frac{\sqrt{3}}{2} \times \frac{3}{4}$$

$$= \frac{3}{4} - \frac{3\sqrt{3}}{8}$$

$$= \frac{3}{8}(2 - 3\sqrt{3})$$

12 B Differentiating both sides with respect to x.

$$\cos x = e^y \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{\cos x}{e^y}$$

But $e^y = \sin x$

$$\therefore \frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

13 B Applying chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dw} \times \frac{dw}{dx}$$

$$= \sec^2 t \times (1 + \frac{1}{w^2}) \times \frac{1}{x}$$

When $x = e, w = 1, t = 0, y = 0$

$$\therefore \frac{dy}{dx} = \frac{2}{e}$$

14 C With calculator.

Solutions to extended-response questions

1 a i $f'(x) = 6x^2 - 10x - 4$

$$f'(x) = 0$$
$$\Rightarrow 3x^2 - 5x - 2 = 0$$
$$\Rightarrow (3x + 1)(x - 2) = 0$$
$$\Rightarrow x = -\frac{1}{3} \text{ or } x = 2$$
$$f\left(-\frac{1}{3}\right) = \frac{19}{27}$$
$$f(2) = -12$$

Two turning points $\left(-\frac{1}{3}, \frac{19}{27}\right)$ and $(2, -12)$

ii $f''(x) = 12x - 10.$

Point of inflection $\left(\frac{5}{6}, -\frac{305}{54}\right)$

iii Midpoint of segment joining turning points:

$\left(\frac{1}{2}(-\frac{1}{3} + 2), \frac{1}{2}(\frac{19}{27} - 12)\right) = \left(\frac{5}{6}, -\frac{305}{54}\right)$

b i $f'(x) = 3ax^2 + 2bx + c$

Discriminant $> 0 \Rightarrow 4b^2 - 4ac > 0 \Rightarrow b^2 - ac > 0$

ii Turning points when $f'(x) = 0$

$x = \dfrac{-2b \pm \sqrt{4b^2 - 4ac}}{6a} = \dfrac{-b \pm \sqrt{b^2 - ac}}{3a}$

$f''(x) = 6ax + 2b$

Point of inflection when: $x = -\dfrac{b}{3a}$

$\dfrac{1}{2}\left(\dfrac{-b + \sqrt{b^2 - ac}}{3a} + \dfrac{-b - \sqrt{b^2 - ac}}{3a}\right) = -\dfrac{b}{3a}$

2 a Let $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f''(x) = 0 \Rightarrow x = -\dfrac{b}{3a}$

$f\left(-\dfrac{b}{3a}\right) = \dfrac{27a^2d - 9abc + 2b^3}{27a^2}$

$p = -\dfrac{b}{3a}$ and $q = \dfrac{27a^2d - 9abc + 2b^3}{27a^2}$

b $g(x) = f(x+p) - q = \dfrac{3a^2x^3 + (3ac - b^2)x}{3a}$

$g(-x) = f(-x+p) - q = \dfrac{-3a^2x^3 - (3ac - b^2)x}{3a} = -g(x)$

The function is odd.

3 a $f(x) = xe^x$

$f'(x) = e^x + xe^x$

$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$

$f^{(3)}(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$

b $f^{(n)}(x) = ne^x + xe^x = e^x(x+n)$

c

d $f(x) = x^2e^x$

$f'(x) = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$f''(x) = x^2e^x + 2xe^x + 2e^x + 2xe^x = e^x(x^2 + 4x + 2)$

$f^{(3)}(x) = e^x(x^2+4x+2) + e^x(2x+4) = e^x(x^2+6x+6)$ $f^{(n)}(x) == e^x(x^2+2nx+n^2-n)$

4

5 a $x = \sin t$ and $y = \sin\left(t + \dfrac{\pi}{3}\right)$

Domain = (0, 1) and range = $\left(\dfrac{1}{2}, 1\right]$

$t = \sin^{-1} x$ and $\sin^2 t + \cos^2 t = 1 \Rightarrow \cos t = \sqrt{1 - x^2}$

$y = \sin t \cos\dfrac{\pi}{3} + \cos t \sin\dfrac{\pi}{3}$

$= \dfrac{x}{2} + \dfrac{\sqrt{3}}{2}\sqrt{1 - x^2}$

b $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$= \cos\left(t + \dfrac{\pi}{3}\right) \times \dfrac{1}{\cos t}$ When $t = \dfrac{\pi}{6}$

$\dfrac{dy}{dx} = 0$

Equation of tangent $y = 1$

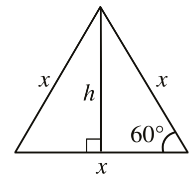
c Let $y = \dfrac{x}{2} + \dfrac{\sqrt{3}}{2}\sqrt{1 - x^2}$

$\dfrac{dy}{dx} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} \times -\dfrac{x}{\sqrt{1 - x^2}}$

$= \dfrac{1}{2} - \dfrac{\sqrt{3}x}{2\sqrt{1 - x^2}}$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{3}}{2(1 - x^2)^{\frac{3}{2}}} < 0$$

6 **a** Let S be the surface area of the given triangle.



$$S = \frac{xh}{2} \text{ and } h = x \sin 60°$$

$$= \frac{x\sqrt{3}}{2}$$

$$\therefore \quad S = \frac{x^2\sqrt{3}}{4}$$

and $$A = 3xy + 2 \times \frac{x^2\sqrt{3}}{4}$$

$$= 3xy + \frac{x^2\sqrt{3}}{2}$$

b The volume of the prism is $V = \dfrac{x^2\sqrt{3}}{4} \times y$

$$\therefore \quad \frac{x^2\sqrt{3}}{4}y = 2000$$

$$\therefore \quad y = \frac{8000}{x^2\sqrt{3}}$$

$$= \frac{8000\sqrt{3}}{3x^2}$$

c $$A = 3x\left(\frac{8000}{x^2\sqrt{3}}\right) + \frac{x^2\sqrt{3}}{2}$$

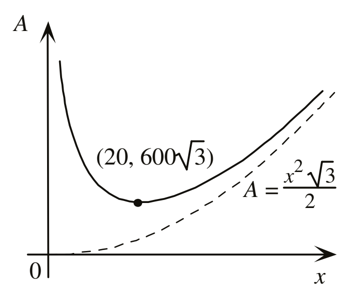
$$= \frac{8000\sqrt{3}}{x} + \frac{x^2\sqrt{3}}{2}$$

d $$\frac{dA}{dx} = -\frac{8000\sqrt{3}}{x^2} + x\sqrt{3}$$

$$\frac{dA}{dx} = 0 \text{ when } x = 20$$

$$A = 400\sqrt{3} + 200\sqrt{3}$$

$$= 600\sqrt{3}$$



e The minimum surface area is $600\sqrt{3}$ cm^2.

7 a i Volume of prism = Area of triangular base $\times$ height of prism

$$\therefore \quad 3000 = \frac{1}{2}(12x)(5x) \times y$$
$$= 30x^2 y$$
$$\therefore \quad y = \frac{3000}{30x^2}$$
$$= \frac{100}{x^2}$$

ii $S = 2 \times 30x^2 + 5x \times y + 12x \times y + 13x \times y$

$$= 60x^2 + 30xy$$
$$= 60x^2 + 30x \times \frac{100}{x^2}$$
$$= 60x^2 + \frac{3000}{x}$$

iii $S = 60x^2 + \dfrac{3000}{x}, x > 0$

As $x \to \infty, \dfrac{3000}{x} \to 0, \therefore \quad S \to 60x^2$

$S = 60x^2$ is a curved asymptote.

As $x \to 0^+, S \to +\infty \therefore \quad x = 0$ is a vertical asymptote.

$$\frac{dS}{dx} = 120x - \frac{3000}{x^2}$$

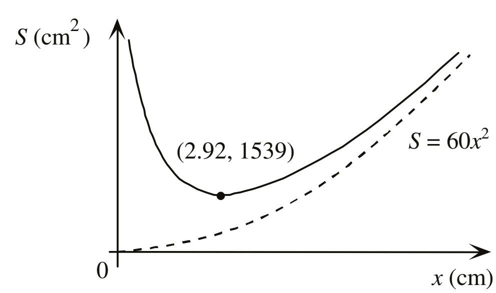
When $\dfrac{dS}{dx} = 0, 120x - \dfrac{3000}{x^2} = 0$

$$\therefore \quad x^3 = \frac{3000}{120}$$
$$= 25$$
$$\therefore \quad x = \sqrt[3]{25} = 2.924\,01\ldots$$

When $x = \sqrt[3]{25}$,

$$S = 60\sqrt[3]{25^2} + \frac{3000}{\sqrt[3]{25}}$$
$$= 1538.978\ldots$$

There is a minimum turning point at (2.92, 1539).



b $\frac{dx}{dt} = 0.5$, find $\frac{dS}{dt}$

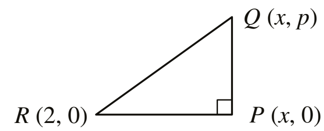
$$\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt}$$
$$= \left(120x - \frac{3000}{x^2}\right) \times 0.5$$
$$= 60x - \frac{1500}{x^2}$$

When $x = 9$, $\frac{dS}{dt} = 60 \times 9 - \frac{1500}{9^2}$
$$= 540 - \frac{1500}{81}$$
$$= \frac{14\,080}{27} \text{ or } 521\frac{13}{27}$$

When $x = 9$, the rate at which S is increasing is $521\frac{13}{27}$ cm^2/s.

c Using a CAS calculator, graph $S = 60x^2 + \dfrac{3000}{x}$ and $S = 2000$.
The points of intersection are given as (1.629 898, 2000) and (4.783 346 1, 2000).
Thus the surface area is 2000 cm^2 when x is either 1.63 cm or 4.78 cm, correct to two decimal places.
(Alternatively, use the 'solve' command.)

8 **a** $x^2 - y^2 = 4$



$\therefore \quad x^2 = y^2 + 4$

$\therefore \quad x = \pm\sqrt{y^2 + 4}$

At the point Q, $x = \sqrt{p^2 + 4}$, therefore the x coordinate of point p is also $\sqrt{p^2 + 4}$, and $PQ = p$.

$RP = x - 2$

$\quad = \sqrt{p^2 + 4} - 2$

$$\therefore \quad A = \frac{1}{2}(RP)(PQ)$$

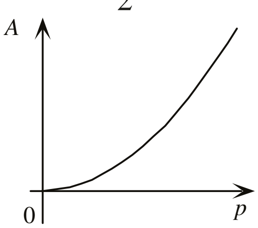
$$= \frac{1}{2}(\sqrt{p^2 + 4} - 2)(p)$$

$$= \frac{p\sqrt{p^2 + 4}}{2} - p$$

b **i** $\dfrac{dA}{dp} = \dfrac{1}{2}p \times \dfrac{1}{2}(p^2 + 4)^{\frac{-1}{2}} \times 2p + \dfrac{1}{2}(p^2 + 4)^{\frac{1}{2}} - 1$ by product and chain rules

$$= \frac{p^2}{2\sqrt{p^2 + 4}} + \frac{\sqrt{p^2 + 4}}{2} - 1$$

ii $A = \dfrac{p\sqrt{p^2 + 4}}{2} - p, \ p > 0$



iii Using a CAS calculator and plotting $A = \frac{p\sqrt{p^2+4}}{2} - p$, and $A = 50$, the point of intersection is found to be (10.95, 50). (Alternatively, use the 'solve' command.) The value of p for which $A = 50$ is 10.95, correct to two decimal places.

iv $$\frac{dA}{dp} = \frac{p^2}{2\sqrt{p^2+4}} + \frac{\sqrt{p^2+4}}{2} - 1$$

Now $p^2 \geq 0$ and $\sqrt{p^2+4} \geq 2$

$$\therefore \quad \frac{p^2}{2\sqrt{p^2+4}} \geq 0 \text{ and } \frac{\sqrt{p^2+4}}{2} \geq 1$$

$$\therefore \quad \frac{dA}{dp} \geq 0 + 1 - 1 = 1$$

Thus $\frac{dA}{dp} \geq 0$ for all p.

c $\frac{dp}{dt} = 0.2$, find $\frac{dA}{dt}$

$$\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$$

$$= \left(\frac{p^2}{2\sqrt{p^2+4}} + \frac{\sqrt{p^2+4}}{2} - 1\right) \times 0.2$$

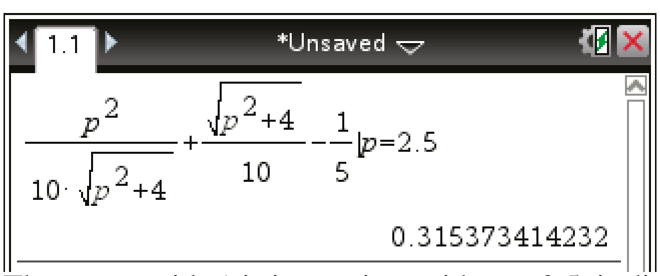
$$= \frac{p^2}{10\sqrt{p^2+4}} + \frac{\sqrt{p^2+4}}{10} - \frac{1}{5}$$

i When $p = 2.5$, $\frac{dA}{dt} = \frac{2.5^2}{10\sqrt{2.5^2+4}} + \frac{\sqrt{2.5^2+4}}{10} - \frac{1}{5}$

$$= \frac{6.25}{10\sqrt{10.25}} + \frac{\sqrt{10.25}}{10} - \frac{1}{5}$$

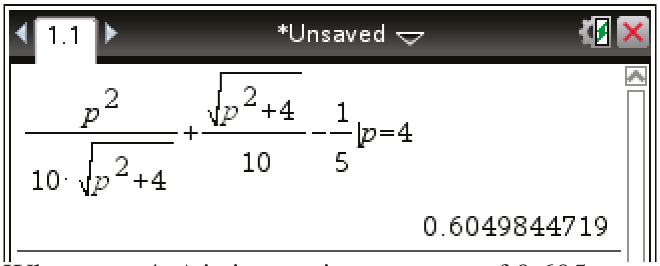
$$= 0.315\,37\ldots$$

Using a CAS calculator complete as follows,



The rate at with A is increasing, with $p = 2.5$, is displayed on the screen and is 0.315 square units per second, correct to three decimal places.

ii On a CAS calculator complete as follows,



When $p = 4$, A is increasing at a rate of 0.605 square units per second, correct to three decimal places.

iii Repeating the procedure in **ii** above, with $p = 50$, A is increasing at a rate of 9.800 square units per second.

iv Repeating the procedure in **ii** above, with $p = 80$, A is increasing at a rate of 15.800 square units per second.

9 **a** $g(x) = 4 - \dfrac{8}{2 + x^2}$

$g(0) = 4 - \dfrac{8}{2}$

$\quad = 0$

So the y-axis intercept is at $(0, 0)$.

When $g(0) = 0, 4 - \dfrac{8}{2 + x^2} = 0$

$\therefore \quad 2 + x^2 = 2$

$\therefore \quad x = 0$

So the x-axis intercept is at $(0, 0)$.

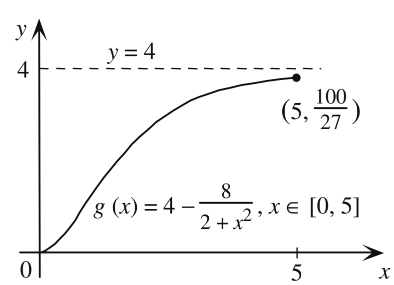
As $x \to \pm\infty, -\dfrac{8}{2 + x^2} \to 0$, so $y = 4$ is a horizontal asymptote.

The above information together with a CAS calculator yields the graph shown.

$g(5) = 4 - \dfrac{8}{2 + 5^2}$

$\quad = 4 - \dfrac{8}{27}$

$\quad = \dfrac{100}{27}$



b **i** $g(x) = 4 - 8(2 + x^2)^{-1}$

$g'(x) = 8(2 + x^2)^{-2} \times 2x$ by the chain rule

$\quad = \dfrac{16x}{(2 + x^2)^2}$ or $16x(2 + x^2)^{-2}$

ii $g''(x) = 16x \times -2(2 + x^2)^{-3} \times 2x + 16(2 + x^2)^{-2}$ by the product rule

$$= \frac{-64x^2}{(2 + x^2)^3} + \frac{16}{(2 + x^2)^2}$$

$$= \frac{16}{(2 + x^2)^2}\left(1 - \frac{4x^2}{2 + x^2}\right)$$

c $g'(x)$ is a maximum when $g''(x) = 0$

i.e. $\frac{16}{(2 + x^2)^2}\left(1 - \frac{4x^2}{2 + x^2}\right) = 0$

$$\therefore \quad 1 - \frac{4x^2}{2 + x^2} = 0$$

$$\therefore \quad 4x^2 = 2 + x^2$$

$$\therefore \quad 3x^2 = 2$$

$$\therefore \quad x = \pm\sqrt{\frac{2}{3}}$$

$$= \pm\frac{\sqrt{6}}{3}$$

$$\therefore \quad x = \frac{\sqrt{6}}{3} \text{ since } 0 \le x \le 5$$

$$g''(1) = \frac{16}{9}\left(1 - \frac{4}{3}\right) < 0$$

$$g''\left(\frac{1}{2}\right) = \frac{1280}{729} > 0$$

Therefore the gradient of the graph of $y = g(x)$ is a maximum when $x = \frac{\sqrt{6}}{3}$.

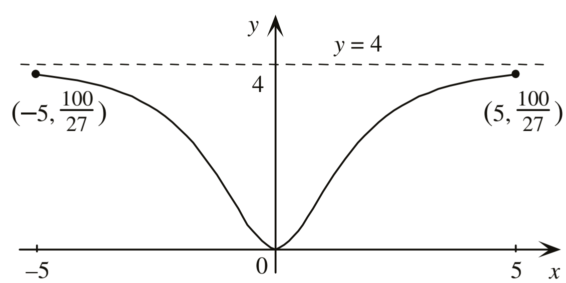
d $g(-5) = 4 - \frac{8}{2 + 5^2}$

$$= \frac{100}{27}$$

In general, $g(-x) = g(x)$

So the graph is symmetrical

About the y-axis.



10 a $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

b $f''(x) = 6ax + 2b$

c For the cubic function $f(x)$ to have no turning points, $f'(x) = 0$ must have at most one solution,

i.e. $(2b)^2 - 4(3a)c \leq 0$ (discriminant of $f'(x)$)

$$\therefore \quad 4b^2 \leq 12ac$$

$$\therefore \quad b^2 \leq 3ac$$

The family has no turning points when $b^2 \leq 3ac$.

The family has no stationary points when $b^2 < 3ac$.

d **i** The gradient of the family, i.e., $f'(x)$, has a local maximum or minimum at x_1 when

$f''(x_1) = 0 \therefore \quad 6ax_1 + 2b = 0$

$$\therefore \quad x_1 = \frac{-b}{3a}$$

ii When $a > 0$, the gradient is a local minimum.

When $a < 0$, the gradient is a local maximum.

e If $a = 1, x_1 = \dfrac{-b}{3}$

i.e. the x coordinate of the stationary point of $y = f'(x)$ is $\dfrac{-b}{3}$.

f **i** To find the x-axis intercept for $y = x^3 + bx^2 + cx$, let $y = 0$

$\therefore \quad x^3 + bx^2 + cx = 0$

$\therefore \quad x(x^2 + bx + c) = 0$

$$\therefore \quad x = 0 \text{ or } x^2 + bx + c = 0$$

If there is only one x-axis intercept (at $x = 0$) there must be no solutions to the equation $x^2 + bx + c = 0$, i.e. $b^2 - 4c < 0$

$\therefore \quad b^2 < 4c$

ii If there are two turning points then there are two solutions to the equation $\dfrac{dy}{dx} = 0$, i.e.

$$\frac{dy}{dx} = 3x^2 + 2bx + c$$

When $\dfrac{dy}{dx} = 0, 3x^2 + 2bx + c = 0$

$$\therefore \quad x = \frac{-2b \pm \sqrt{4b^2 - 12c}}{6}$$

$$= \frac{-b \pm \sqrt{b^2 - 3c}}{3}$$

For two solutions $b^2 - 3c > 0$

$\therefore \quad b^2 > 3c$

Now $b^2 < 4c$ since there is only one x-axis intercept

$\therefore\quad 3c < b^2 < 4c$

11 a i $y = \dfrac{1 - x^2}{1 + x^2}$

$= \dfrac{u}{v}$ where $u = 1 - x^2$ and $v = 1 + x^2$

$\therefore\quad \dfrac{du}{dx} = -2x \; \dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1 + x^2)(-2x) - (1 - x^2)(2x)}{(1 + x^2)^2}$$

$$= \frac{-2x - 2x^3 - 2x + 2x^3}{(1 + x^2)^2}$$

$= \dfrac{-4x}{(1 + x^2)^2}$ as required.

ii $\dfrac{dy}{dx} = \dfrac{-4x}{(1 + x^2)^2}$

$= \dfrac{u}{v}$

where $u = -4x$ and $v = (1 + x^2)^2$

$\therefore\quad \dfrac{du}{dx} = -4 \; \dfrac{dv}{dx} = 2(1 + x^2)(2x)$

$= 4x(1 + x^2)$

$$\frac{d^2y}{dx^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1 + x^2)^2(-4) - (-4x)(4x)(1 + x^2)}{(1 + x^2)^4}$$

$$= \frac{-4(1 + x^2)^2 + 16x^2(1 + x^2)}{(1 + x^2)^4}$$

$$= \frac{-4(1 + x^2) + 16x^2}{(1 + x^2)^3}$$

$$= \frac{12x^2 - 4}{(1 + x^2)^3}$$

$$= \frac{4(3x^2 - 1)}{(1 + x^2)^3}$$

b $$y = \frac{1 - x^2}{1 + x^2}$$
$$= -1 + \frac{2}{1 + x^2}$$

When $y = 0, 1 - x^2 = 0$

$\therefore \quad x = \pm 1$

The x-axis intercepts are at $x = \pm 1$

When $x = 0, y = 1$

The y-axis intercept is at $y = 1$.

As $x \to \pm\infty, y \to -1$ from above.

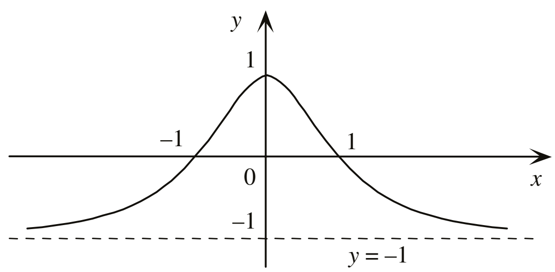
Thus $y = -1$ is a horizontal asymptote.

$$\frac{dy}{dx} = \frac{-4x}{(1 + x^2)^2}$$

When $\frac{dy}{dx} = 0, \frac{-4x}{(1 + x^2)^2} = 0$

$\therefore \quad -4x = 0$

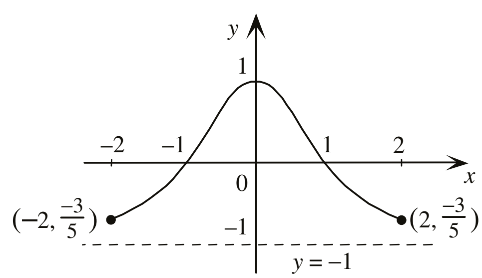
$\therefore \quad x = 0$



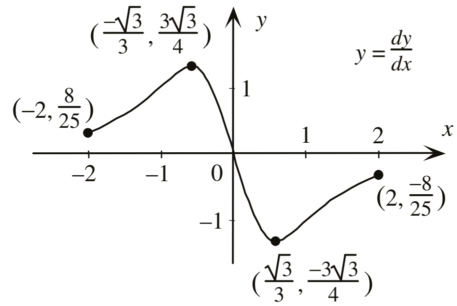
c $$y = \frac{1 - x^2}{1 + x^2}$$

When $x = \pm 2, y = \frac{1 - 4}{1 + 4} = \frac{-3}{5}$

$$\frac{dy}{dx} = \frac{-4x}{(1 + x^2)^2}$$



When $x = \frac{1}{\sqrt{3}}, \frac{dy}{dx} = \frac{-\frac{4}{\sqrt{3}}}{\left(1 + \frac{1}{3}\right)^2} = \frac{-3\sqrt{3}}{4}$

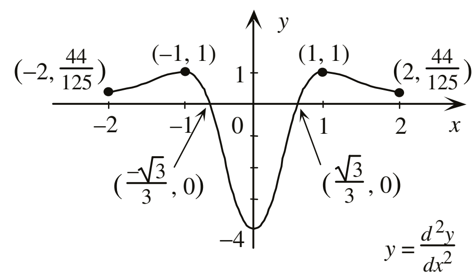


When $x = \frac{-1}{\sqrt{3}}, \frac{dy}{dy} = \frac{\frac{4}{\sqrt{3}}}{\left(1 + \frac{1}{3}\right)^2} = \frac{3\sqrt{3}}{4}$

When $x = 2, \frac{dy}{dx} = \frac{-8}{(1+4)^2} = \frac{-8}{25}$

When $x = -2, \frac{dy}{dx} = \frac{8}{(1+4)^2} = \frac{8}{25}$

$$\frac{d^2y}{dx^2} = \frac{4(3x^2 - 1)}{(1 + x^2)^3}$$



When $\frac{d^2y}{dx^2} = 0$,

$3x^2 - 1 = 0$

$\therefore \quad x^2 = \frac{1}{3}$

$\therefore \quad x = \pm\frac{\sqrt{3}}{3}$When $x \quad = \pm 2, \frac{d^2y}{dx^2} = \frac{4(11)}{(1+4)^3} = \frac{44}{125}$

d **i** $y = \frac{1 - x^2}{1 + x^2}$

The x-axis intercepts are at $x = -1$ and $x = 1$

$\frac{dy}{dx} = \frac{-4x}{(1 + x^2)^2}$

At $x = -1, \frac{dy}{dx} = \frac{4}{4} = 1$, and $y = 0$

Therefore the tangent to the curve at $x = -1$ has equation

$y - 0 = 1(x - (-1))$

$\therefore \quad y = x + 1 \qquad ①$

At $x = 1, \frac{dy}{dx} = \frac{-4}{4} = -1$, and $y = 0$

Therefore the tangent to the curve at $x = 1$ has equation

$y - 0 = -1(x - 1)$

$\therefore \quad y = -x + 1$

ii The point of intersection of the tangents is where

$x + 1 = -x + 1$

$\therefore \quad 2x = 0$

$\therefore \quad x = 0$

Substituting $x = 0$ into ① in **d i** gives $y = 1$

Therefore the point of intersection of the tangents is at $C(0, 1)$, the y-axis intercept of the graph of $y = \dfrac{1 - x^2}{1 + x^2}$.

12 a i $V = -3000\pi(\log_e(1 - h) + h)$

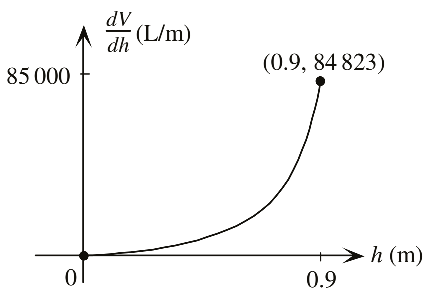
$$\frac{dV}{dh} = -3000\pi\left(\frac{1}{1-h} \times -1 + 1\right), h < 1$$

$$= -3000\pi\left(\frac{-1}{1-h} + \frac{1-h}{1-h}\right)$$

$$= -3000\pi\left(\frac{-h}{1-h}\right)$$

$$= \frac{3000\pi h}{1-h}$$

ii From a CAS calculator, the graph of $\dfrac{dV}{dh}$ against h is as shown.



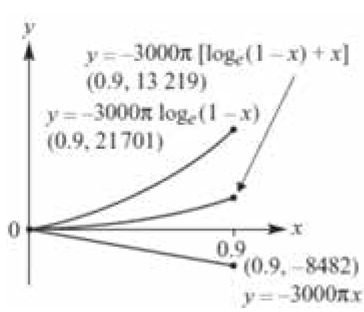
b i When $h = 0.9$, $V = -3000\pi(\log_e(1 - 0.9) + 0.9)$

$= -3000\pi(\log_e(0.1) + 0.9)$

$= 13\,219.053\,07\ldots$

The maximum volume of the pool is 13 219 litres, to the nearest litre.

ii



c Let t = time (in minutes)

$$\frac{dV}{dt} = 15$$

Find $\frac{dh}{dt}$ $\quad \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$$= \frac{1-h}{3000\pi h} \times 15$$

$$= \frac{1-h}{200\pi h}$$

When $h = 0.2$, $\frac{dh}{dt} = \frac{1-0.2}{200\pi \times 0.2}$

$$= 0.006\,366\,1\ldots$$

The rate at which the depth is increasing when $h = 0.2$ is 0.0064 m/min, correct to two significant figures.

13 a i $f(x) = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right), \quad x \neq 0$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \frac{-1}{x^2}$$

$$= \frac{1}{1+x^2} + \frac{-1}{x^2+1}$$

$$= 0$$

ii $f(x) = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$

If $x > 0$, $f(x) = \tan^{-1}(x) + \frac{\pi}{2} - \tan^{-1}(x)$

$$= \frac{\pi}{2}$$

iii If $x < 0$, $f(x) = \tan^{-1}(x) + \frac{-\pi}{2} - \tan^{-1}(x)$

$$= \frac{-\pi}{2}$$

b **i** $y = \cot x, x \in (0, \pi)$

$= \dfrac{\cos x}{\sin x}$

$= \dfrac{u}{v}$ where $u = \cos x$ and $v = \sin x$

$\dfrac{du}{dx} = -\sin x \quad \dfrac{dv}{dx} = \cos x$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\sin x \times -\sin x - \cos x \times \cos x}{\sin^2 x}$

$= \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= -\text{cosec}^2 x, x \in (0, \pi)$

ii $\dfrac{dy}{dx} = -\text{cosec}^2 x$

$= -\cot^2 x - 1$

$= -y^2 - 1$

c If $y = \cot^{-1} x, y \in (0, \pi)$

then $x = \cot y$

$\therefore \quad \dfrac{dx}{dy} = -\text{cosec}^2 y$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-1}{\text{cosec}^2 y}$

$= \dfrac{-1}{\cot^2 y + 1}$

$= \dfrac{-1}{x^2 + 1}$

d Let $y = \cot(x) + \tan(x), x \in \left(0, \dfrac{\pi}{2}\right)$

From **b i** $\dfrac{d}{dx}\cot(x) = -\text{cosec}^2 x$

$\therefore \quad \dfrac{dy}{dx} = -\text{cosec}^2 x + \sec^2 x$

14 For $f(x) = \dfrac{8}{x^2} - 32 + 16\log_e(2x)$,

a $f'(x) = \dfrac{16}{x} - \dfrac{16}{x^3}$

b $f''(x) = \dfrac{48}{x^4} - \dfrac{16}{x^2}$

c $\quad f'(x) = 0$ implies $\dfrac{16}{x} - \dfrac{16}{x^3} = 0$

$\therefore\ 16x^2 - 16 = 0$

$\therefore\ x = \pm 1$

$\therefore\ x = 1, \quad$ since $x > 0$

When $x = 1$, $f(1) = 16 \log_e 2 - 24$

The coordinates of the one stationary point are $(1, 16 \log_e 2 - 24)$.

d For the inflection point consider $\dfrac{48}{x^4} - \dfrac{16}{x^2} = 0.$

$\therefore\ 48 - 16x^2 = 0$

$\therefore\ x = \pm \sqrt{3}$

$\therefore\ x = \sqrt{3}, \quad$ since $x > 0$

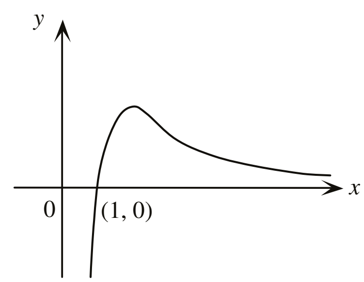
Now $f''(1) > 0$ and $f''(1) < 0$, so there is a sign change.

So a point of inflection exists at $x = \sqrt{3}$.

e $f'(x) = \dfrac{16}{x} - \dfrac{16}{x^3}$

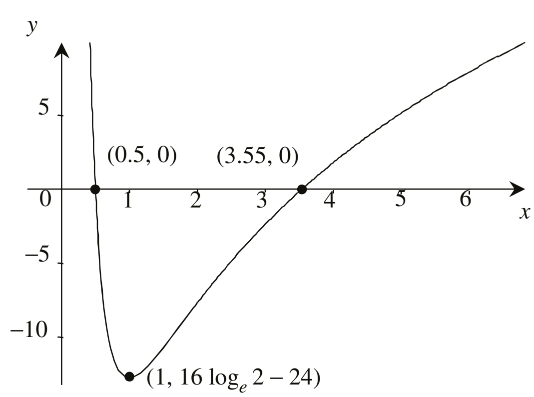
This is the graph of $y = f'(x)$.

So $f'(x) > 0 \quad$ for $\quad x > 1$.



f Using the 'solve' command of a CAS calculator, there is a second x-intercept at $x = 3.55$, correct to 2 decimal places.

g



15 a For $x = 3\cos\theta$ and $y = 2\sin\theta$

$$\frac{dx}{d\theta} = -3\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 2\cos\theta$$

Therefore
$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= 2\cos\theta \times \frac{1}{-3\sin\theta}$$
$$= -\frac{2\cos\theta}{3\sin\theta}$$

Therefore the tangent has equation

$$y - 2\sin\theta = -\frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$

Multiplying both sides of the equation by $3\sin\theta$

$$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$

Then rearranging gives

$$3y\sin\theta + 2x\cos\theta = 6\sin^2\theta + 6\cos^2\theta$$

$\therefore$ $3y\sin\theta + 2x\cos\theta = 6$, as required.

b i When $x = 3$, $3y\sin\theta + 6\cos\theta = 6$

This implies
$$y = \frac{6 - 6\cos\theta}{3\sin\theta}$$
$$= \frac{2 - 2\cos\theta}{\sin\theta}$$

The point T has coordinates $\left(3, \dfrac{2 - 2\cos\theta}{\sin\theta}\right)$.

ii OT has gradient $\dfrac{2 - 2\cos\theta}{3\sin\theta} = \dfrac{2(1 - \cos\theta)}{3\sin\theta}$

AP has gradient $\dfrac{2\sin\theta}{3(\cos\theta + 1)}$

We are required to prove that $\dfrac{2(1 - \cos\theta)}{3\sin\theta} = \dfrac{2\sin\theta}{3(\cos\theta + 1)}$

$$\text{LHS} = \frac{2(1 - \cos\theta)}{3\sin\theta}$$

$$= \frac{2\left(1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\right)}{6\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2}{3}\tan\frac{\theta}{2}$$

$$\text{RHS} = \frac{2\sin\theta}{3(\cos\theta + 1)}$$

$$= \frac{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{3\left(2\cos^2\frac{\theta}{2} - 1 + 1\right)}$$

$$= \frac{2}{3}\tan\frac{\theta}{2}$$

So OT is parallel to AP as both have the same gradient of $\frac{2}{3}\tan\frac{\theta}{2}$.

c **i** The tangent has equation $3y\sin\theta - 2x\cos\theta = 6$.

It intersects the x-axis then $y = 0$.

then $y = 0, x = \dfrac{3}{\cos\theta}$ (Q)

It intersects the y-axis then $x = 0$.

When $x = 0, y = \dfrac{2}{\sin\theta}$ (R)

Coordinates of Q are $\left(\dfrac{3}{\cos\theta}, 0\right)$ and the coordinates of R are $\left(0, \dfrac{2}{\sin\theta}\right)$.

The midpoint M has coordinates $\left(\dfrac{3}{2\cos\theta}, \dfrac{1}{\sin\theta}\right)$.

ii To find the locus, let $x = \dfrac{3}{2\cos\theta}$ and $y = \dfrac{1}{\sin\theta}$.

Rearrange to give $\cos\theta = \dfrac{3}{2x}$ and $\sin\theta = \dfrac{1}{y}$.

Squaring and adding gives

$$\frac{9}{4x^2} + \frac{1}{y^2} = 1 \quad \text{since } \sin^2\theta + \cos^2\theta = 1$$

d **i** $W(-3\sin\theta, 2\cos\theta)$ and $P(3\cos\theta, 2\sin\theta)$ are points on the ellipse.

For $x = -3\sin\theta$ and $y = 2\cos\theta$,

$$\frac{dx}{d\theta} = -3\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = -2\sin\theta$$

Therefore $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$

$$= -2\sin\theta \times \frac{1}{-3\cos\theta} = \frac{2\sin\theta}{3\cos\theta}$$

Therefore the tangent has equation

$y - 2\cos\theta = \dfrac{2\sin\theta}{3\cos\theta}(x + 3\sin\theta)$ ①

ii Multiplying both sides of equation ① in **d i** by $3\cos\theta$,

$3y\cos\theta - 6\cos^2\theta = 2x\sin\theta + 6\sin^2\theta$

Then rearranging gives

$$\begin{aligned}3y\cos\theta + 2x\sin\theta &= 6\sin^2\theta + 6\cos^2\theta \\ &= 6\end{aligned}$$

Thus $3y\cos\theta - 2x\sin\theta = 6$ ②

The equation of the tangent at P is

$3y\sin\theta + 2x\ \cos\theta = 6$ ③ from **a** above.

Multiplying ② by $\cos\theta$ and ③ by $\sin\theta$, and then adding,

$3y\cos^2\theta + 3y\sin^2\theta = 6(\cos\theta + \sin\theta)$

Therefore $y = 2(\cos\theta + \sin\theta)$

Multiply ② by $\sin\theta$ and ③ by $\cos\theta$, and subtract

$-2x\sin^2\theta - 2x\sin^2\theta = 6\sin\theta - 6\cos\theta$

This implies $x = 3(\cos\theta - \sin\theta)$

The point Z has coordinates $(3(\cos\theta - \sin\theta), 2(\cos\theta + \sin\theta))$.

iii $x = 3(\cos\theta - \sin\theta)$ and $y = 2(\cos\theta + \sin\theta)$

$\dfrac{x}{3} = \cos\theta - \sin\theta$ and $\dfrac{y}{2} = \cos\theta + \sin\theta$

Therefore $\dfrac{x}{3} + \dfrac{y}{2} = 2\cos\theta$ and $\dfrac{x}{3} - \dfrac{y}{2} = -2\sin\theta$ or $\dfrac{y}{2} - \dfrac{x}{3} = 2\sin\theta$

Squaring and adding these new equations gives

$\left(\dfrac{x}{3} + \dfrac{y}{2}\right)^2 + \left(\dfrac{y}{2} - \dfrac{x}{3}\right)^2 = 4$ or $(2x + 3y)^2 + (3y - 2x)^2 = 144$

16 a For $x = a\cos\theta$ and $y = b\sin\theta$

$\dfrac{dx}{d\theta} = -a\sin\theta$ and $\dfrac{dy}{d\theta} = b\cos\theta$

$$\begin{aligned}\text{Therefore } \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= b\cos\theta \times \frac{1}{-a\sin\theta} \\ &= -\frac{b\cos\theta}{a\sin\theta}\end{aligned}$$

Therefore the tangent has equation

$y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\ \cos\theta)$

Multiplying both sides of the equation by $a\sin\theta$,

$ay\sin\theta - ba\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

Then rearranging gives

$$ay\sin\theta + bx\cos\theta = ab\sin^2\theta + ab\cos^2\theta$$
$$= ab$$

$\therefore \quad ay\sin\theta + bx\cos\theta = ab$

It intersects the x-axis when $y = 0$.

When $y = 0$, $\quad x = \dfrac{a}{\cos\theta} \quad (M)$

It intersects the y-axis when $x = 0$.

When $x = 0$, $\quad y = \dfrac{b}{\sin\theta} \quad (N)$

Therefore, the area of the triangle $= \dfrac{1}{2} \times \left|\dfrac{a}{\cos\theta}\right| \times \left|\dfrac{b}{\sin\theta}\right|$

$$= \left|\frac{ab}{\sin 2\theta}\right|$$

b Minimum area is ab when $\sin 2\theta = \pm 1, 2\theta = \dfrac{\pi}{2} + k\pi$, where k is an integer.

Therefore $\theta = (2k+1)\dfrac{\pi}{4}$, where k is an integer.

17 a $\quad x = a\sec\theta \quad$ and $\quad y = b\tan\theta$

That is, $x = \dfrac{a}{\cos\theta} \quad$ and $\quad y = \dfrac{b\sin\theta}{\cos\theta}$

$$\frac{dx}{d\theta} = \frac{a\sin\theta}{\cos^2\theta} \quad \text{and} \quad \frac{dy}{d\theta} = b\sec^2\theta$$

Therefore $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$

$$= b\sec^2\theta \times \frac{\cos^2\theta}{a\sin\theta}$$
$$= \frac{b}{a\sin\theta}$$

Therefore the equation of the tangent at $P(a\sec\theta, b\tan\theta)$ is

$$y - \frac{b\sin\theta}{\cos\theta} = \frac{b}{a\sin\theta}\left(x - \frac{a}{\cos\theta}\right)$$

Multiplying both sides by $a\sin\theta$ gives

$$ya\sin\theta - \frac{ab\sin^2\theta}{\cos\theta} = b\left(x - \frac{a}{\cos\theta}\right)$$

$$\therefore \quad ya\sin\theta - \frac{ab\sin^2\theta}{\cos\theta} = bx - \frac{ab}{\cos\theta}$$

Dividing both sides by $\cos\theta$ and rearranging gives

$$\frac{ya\sin\theta}{\cos\theta} - \frac{bx}{\cos\theta} = \frac{ab\sin^2\theta}{\cos\theta} - \frac{ba}{\cos^2\theta}$$

Dividing both sides by ab,

$$\frac{y\tan\theta}{b} - \frac{x\sec\theta}{a} = \tan^2\theta - \sec^2\theta$$

and therefore

$$\therefore \quad \frac{y\tan\theta}{b} - \frac{x\sec\theta}{a} = -1$$

$$\therefore \quad \frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1, \text{ as required.}$$

b First consider the intersection with the asymptote with equation $y = \dfrac{bx}{a}$.

Substitute for y in $\dfrac{y\tan\theta}{b} - \dfrac{x\sec\theta}{a} = -1$

to give $\dfrac{x\tan\theta}{a} - \dfrac{x\sec\theta}{a} = -1$

which implies $x(\sec\theta - \tan\theta) = a$

$$\therefore \quad x = \frac{a}{\sec\theta - \tan\theta} = \frac{a\cos\theta}{1 - \sin\theta}$$

and substituting in $y = \dfrac{bx}{a}$ gives $y = \dfrac{ba\cos\theta}{a(1 - \sin\theta)}$

$$= \frac{b}{\sec\theta - \tan\theta}$$

and hence the coordinates of Q are $\left(\dfrac{a}{\sec\theta - \tan\theta}, \dfrac{b}{\sec\theta - \tan\theta}\right)$.

Now consider the intersection with the asymptote with equation $y = \dfrac{-bx}{a}$.

Substitute for y in $\dfrac{y\tan\theta}{b} - \dfrac{x\sec\theta}{a} = -1$

to give $\dfrac{-x\tan\theta}{a} - \dfrac{x\sec\theta}{a} = -1$

which implies $x(\sec\theta + \tan\theta) = a$

$$\therefore \quad x = \frac{a}{\sec\theta + \tan\theta} = \frac{a\cos\theta}{1 + \sin\theta}$$

and substituting in $y = \dfrac{-bx}{a}$ gives $y = \dfrac{-b}{\sec\theta + \tan\theta}$

$$= \frac{-b\cos\theta}{1 + \sin\theta}$$

and hence the coordinates of R are $\left(\dfrac{a}{\sec\theta + \tan\theta}, \dfrac{-b}{\sec\theta + \tan\theta}\right)$

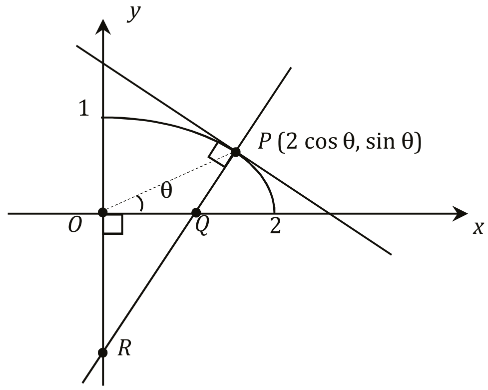
c The midpoint of QR is the point with coordinates

$$\left(\frac{1}{2}\left(\frac{a}{\sec\theta - \tan\theta} + \frac{a}{\sec\theta + \tan\theta}\right), \frac{1}{2}\left(\frac{b}{\sec\theta - \tan\theta} - \frac{b}{\sec\theta + \tan\theta}\right)\right)$$

Obtaining common denominator yields

$$\left(\frac{a\sec\theta}{\sec^2\theta-\tan^2\theta}, \frac{b\tan\theta}{\sec^2\theta-\tan^2\theta}\right) = (a\ \sec\theta, b\ \tan\theta)$$

18 a



For $x = 2\cos\theta$ and $y = \sin\theta$

$$\frac{dx}{d\theta} = -2\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \cos\theta$$

Therefore $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$$= \cos\theta \times \frac{1}{-2\sin\theta}$$

$$= -\frac{\cos\theta}{2\sin\theta}$$

Gradient of the normal is $2\tan\theta$.

Therefore the normal has equation

$$y - \sin\theta = \frac{2\sin\theta}{\cos\theta}(x - 2\cos\theta)$$

Multiplying both sides of the equation by $\cos\theta$ gives

$$y\cos\theta - \sin\theta\ \cos\theta = 2x\sin\theta - 4\cos\theta\ \sin\theta$$

Then rearranging gives

$$-2x\sin\theta + y\cos\theta = -3\cos\theta\ \sin\theta$$

Making y the subject

$$y = 2x\tan\theta - 3\sin\theta$$

When $x = 0, y = -3\sin\theta$ $\quad (R)$

and, when $y = 0, x = \frac{3\cos\theta}{2}$ (Q)

Therefore: area of triangle $OQR = \frac{9}{4}\sin\theta\cos\theta$

$$= \frac{9}{8}\sin 2\theta$$

b Therefore the maximum area is $\frac{9}{8}$ which occurs where $2\theta = \frac{\pi}{2}$,

$\therefore \quad \theta = \frac{\pi}{4}$.

c The coordinates of R are $(0, -3\sin\theta)$ and the coordinates of Q are $\left(\frac{3\cos\theta}{2}, 0\right)$.

Therefore the midpoint M of QR has coordinates $\left(\frac{3\cos\theta}{4}, \frac{-3\sin\theta}{2}\right)$.

d The parametric equations of the midpoint are $x = \frac{3\cos\theta}{4}$ and $y = \frac{-3\sin\theta}{2}$.

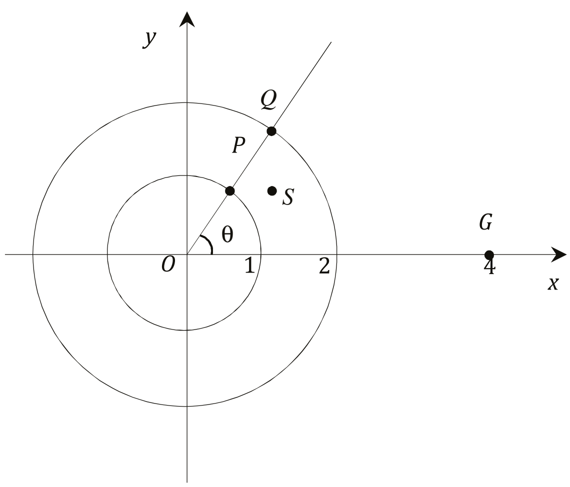
Therefore $\cos\theta = \frac{4x}{3}$ and $\sin\theta = \frac{-2y}{3}$.

Squaring and adding

$$\frac{16x^2}{9} + \frac{4y^2}{9} = \cos^2\theta + \sin^2\theta$$

That is, $16x^2 + 4y^2 = 9$

19



a The coordinates of S are $(2\ \cos\theta, \sin\theta)$.

The parametric equations of C are $x = 2\cos\theta$ and $y = \sin\theta$.

$\therefore$ $\frac{x^2}{4} + y^2 = 1$ is the equation of the path C.

b Using implicit differentiation gives $\frac{x}{2} + 2y\frac{dy}{dx} = 0$

Therefore $\frac{dy}{dx} = \frac{-x}{4y}$

At the point (u, v) the equation of the tangent is

$$y - v = \frac{-u}{4v}(x - u)$$

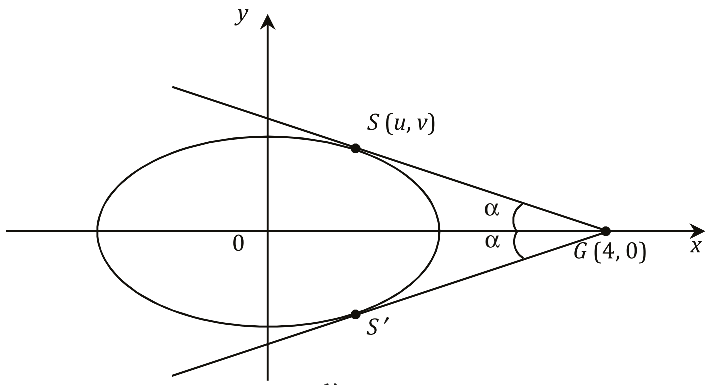
Therefore $y = \frac{-u}{4v}x + \frac{u^2}{4v} + v$

$$\text{But } \frac{u^2}{4} + v^2 = 1$$

Divide both sides of this by v to see $\frac{u^2}{4v} + v = \frac{1}{v}$

$$\text{The equation of the tangent is } y = \frac{-u}{4v}x + \frac{1}{v}$$

c This diagram shows extreme positions of the direction in which the gun is pointing.



The gradient of $GS = \dfrac{v}{u-4}$

Therefore $\tan\alpha = \dfrac{v}{4-u}$

For the tangent to pass through $G(4, 0)$, $0 = \dfrac{-u}{4v} \times 4 + \dfrac{1}{v}$ from **b** above

This gives $u = 1$

and substituting in the equation of the ellipse gives

$$v = \frac{\sqrt{3}}{2}$$

$$\text{Therefore } \tan\alpha = \frac{\frac{\sqrt{3}}{2}}{4-1}$$

$$= \frac{1}{6}\sqrt{3}$$

Chapter 9 – Techniques of integration

Solutions to Exercise 9A

1 a $\int \sin\left(2x + \frac{\pi}{4}\right)dx$

$= -\frac{1}{2}\cos\left(2x + \frac{\pi}{4}\right) + c$

$= -\frac{1}{2}\cos\left(2x + \frac{\pi}{4}\right)$, (where $c = 0$)

b $\int \cos(\pi x)dx = \frac{1}{\pi}\sin \pi x + c$

$= \frac{1}{\pi}\sin \pi x,$

(where $c = 0$)

c $\int \sin\left(\frac{2\pi}{3}x\right)dx = -\frac{3}{2\pi}\cos\left(\frac{2\pi}{3}x\right) + c$

$= -\frac{3}{2\pi}\cos\left(\frac{2\pi x}{3}\right),$

(where $c = 0$)

d $\int e^{3x+1}dx = \frac{1}{3}e^{3x+1} + c$

$= \frac{1}{3}e^{3x+1}$, (where $c = 0$)

e $\int e^{5(x+4)}dx = \frac{1}{5}e^{5(x+4)} + c$

$= \frac{1}{5}e^{5(x+4)},$

(where $c = 0$)

f $\int \frac{3dx}{2x^2} = \frac{3}{2}\int \frac{dx}{x^2} = \frac{3}{2}\int x^{-2}dx$

$= \frac{3}{2}\frac{x^{-1}}{-1} + c$

$= -\frac{3}{2x} + c$

$= -\frac{3}{2x},$

(where $c = 0$)

g

$\int (6x^3 - 2x^2 + 4x + 1)dx$

$= 6\int x^3dx - 2\int x^2dx + 4\int x\,dx$

$+ \int dx$

$= \frac{6x^4}{4} - \frac{2x^3}{3} + \frac{4x^2}{2} + x + c$

$= \frac{3x^4}{2} - \frac{2x^3}{3} + 2x^2 + x + c;$

$= \frac{3x^4}{2} - \frac{2x^3}{3} + 2x^2 + x,$

(where $c = 0$)

2 a

$\int_{-1}^{1}(e^x - e^{-x})dx = [e^x + e^{-x}]_{-1}^{1}$

$= (e + e^{-1}) - (e^{-1} + e)$

$= 0$

b $\int_0^2 (3x^2 + 2x + 4)dx$

$= [x^3 + x^2 + 4x]_0^2$

$= 8 + 4 + 8$

$= 20$

c $\int_0^{\frac{\pi}{2}} \sin 2x\, dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= -\frac{1}{2}(\cos\pi - \cos 0)$

$= -\frac{1}{2} \times -2$

$= 1$

d $\int_2^3 \frac{3}{x^3}dx = 3\int_2^3 x^{-3}dx$

$= -\frac{3}{2}[x^{-2}]_2^3$

$= -\frac{3}{2}\left(\frac{1}{9} - \frac{1}{4}\right)$

$= -\frac{3}{2} \times -\frac{5}{36}$

$= \frac{5}{24}$

e $\int_0^{\frac{\pi}{4}} (\cos x + 2x)dx$

$= [\sin x + x^2]_0^{\frac{\pi}{4}}$

$= \frac{\sqrt{2}}{2} + \frac{\pi^2}{16} \approx 1.324$

f $\int_0^1 (e^{3x} + x)dx = \left[\frac{1}{3}e^{3x} + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{3}e^3 + \frac{1}{2} - \frac{1}{3}$

$= \frac{e^3}{3} + \frac{1}{6}$

g $\int_0^{\frac{\pi}{2}} \cos 4x\, dx = \left[\frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}} = 0$

h $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\frac{1}{2}x\, dx = -2\left[\cos\frac{1}{2}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0,$

because $\cos\frac{1}{2}x$ is an even function.

i $\int_0^{\frac{\pi}{4}} \sec^2 x\, dx = [\tan x]_0^{\frac{\pi}{4}} = 1$

3 a $\int \left(\frac{1}{2x-5}\right) dx = \frac{1}{2}\log_e|2x-5| + c$

b $\int_0^1 \left(\frac{1}{2x-5}\right) dx = \frac{1}{2}\log_e\frac{3}{5}$

c $\int_{-2}^{-1} \left(\frac{1}{2x-5}\right) dx = \frac{1}{2}\log_e\frac{7}{9}$

4 a $\int_0^1 \frac{1}{3x+2}dx = \left[\frac{1}{3}\log_e|3x+2|\right]_0^1$

$= \frac{1}{3}(\log_e 5 - \log_e 2)$

$= \frac{1}{3}\log_e\frac{5}{2} \approx 0.305$

b $\int_{-3}^{-1} \frac{1}{3x-2}dx$

$= \left[\frac{1}{3}\log_e|3x-2|\right]_{-3}^{-1}$

$= \frac{1}{3}(\log_e|-5| - \log_e|-11|)$

$= \frac{1}{3}\log_e\frac{5}{11}$

c $\int_{-1}^0 \frac{1}{4-3x}dx = \left[-\frac{1}{3}\log_e|4-3x|\right]_{-1}^0$

$= -\frac{1}{3}(\log_e 4 - \log_e 7)$

$= -\frac{1}{3}\log_e\frac{4}{7}$

$= \frac{1}{3}\log_e\frac{7}{4}$

5 a $\int (3x+2)^5 dx = \frac{1}{3}\frac{(3x+2)^6}{6} + c$

$= \frac{(3x+2)^6}{18} + c$

$= \frac{1}{18}(3x+2)^6,$

(where $c = 0$)

b $\int \frac{dx}{3x-2} = \frac{1}{3}\log_e |3x-2| + c$

$= \frac{1}{3}\log_e |3x-2|,$

(where $c = 0$)

c $\int \sqrt{3x+2}\,dx$

$= \int (3x+2)^{\frac{1}{2}} dx$

$= \frac{1}{3}\frac{(3x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{2\sqrt{(3x+2)^3}}{9} + c$

$= \frac{2}{9}(3x+2)^{\frac{3}{2}},$ (where $c = 0$)

d $\int \frac{dx}{(3x+2)^2} = \int (3x+2)^{-2} dx$

$= \frac{1}{3}\frac{(3x+2)^{-1}}{-1} + c$

$= -\frac{1}{3(3x+2)} + c$

$= -\frac{1}{3(3x+2)},$

(where $c = 0$)

e $\int \frac{3x+1}{x+1} dx = \int \left(3 - \frac{2}{x+1}\right) dx$

$= 3x - 2\log_e |x+1| + c$

$= 3x - 2\log_e |x+1|,$

(where $c = 0$)

f $\int \cos\frac{3x}{2} dx = \frac{2}{3}\sin\frac{3x}{2} + c$

$= \frac{2}{3}\sin\frac{3x}{2},$

(where $c = 0$)

g $\int (5x-1)^{\frac{1}{3}} dx = \frac{1}{5}\frac{(5x-1)^{\frac{4}{3}}}{\frac{4}{3}} + c$

$= \frac{3\sqrt[3]{(5x-1)^4}}{20} + c$

$= \frac{3}{20}(5x-1)^{\frac{4}{3}},$

(where $c = 0$)

h $\int \frac{2x+1}{x+3} dx = \int \left(2 - \frac{5}{x+3}\right) dx$

$= 2\int dx - 5\int \frac{dx}{x+3}$

$= 2x - 5\log_e |x+3| + c;$

$= 2x - 5\log_e |x+3|,$

(where $c = 0$)

6 a $f(x) = 2x$

$$F(x) = \int f(x)dx$$
$$= \int 2x\, dx$$
$$= x^2 + c$$

Now $F(-1) = 4$

$\therefore\ (-1)^2 + c = 4$

$\therefore\ c = 3$

$\therefore\ F(x) = x^2 + 3$

b $f(x) = 4x^2$

$$F(x) = \int 4x^2 dx$$
$$= \frac{4}{3}x^3 + c$$

Now $F(0) = 0$

$\therefore\ \frac{4}{3}(0)^2 + c = 0$

$\therefore\ c = 0$

$\therefore\ F(x) = \frac{4}{3}x^3$

c $f(x) = -2(x-2)^2$

$= -2(x^2 - 4x + 4)$

$= -2x^2 + 8x - 8$

$$F(x) = \int -2x^2 + 8x - 8dx$$
$$= -\frac{2}{3}x^3 + 4x^2 - 8x + c$$

Now $F(2) = 4$

$\therefore\ -\frac{2}{3}(2)^3 + 4(2)^2 - 8(2) + c = 4$

$\therefore\ -\frac{16}{3} + 16 - 16 + c = 4$

$\therefore\ c = \frac{28}{3}$

$\therefore\ F(x) = -\frac{2}{3}x^3 + 4x^2 - 8x + \frac{28}{3}$

d $f(x) = ae^{bx}$

$f(0) = -1$

$\therefore\ ae^{b\times 0} = -1$

$\therefore\ a = -1$

$\therefore\ f(x) = -e^{bx}$

$f(-\log_e 2) = -2$

$\therefore\ -e^{b(-\log_e 2)} = -2$

$\therefore\ -(e^{\log_e 2})^{-b} = -2$

$\therefore\ -2^{-b} = -2$

$\therefore\ b = -1$

$\therefore\ f(x) = -e^{-x}$

$$F(x) = \int -e^{-x}dx$$
$$= e^{-x} + c$$

The graph of $F(x)$ is a translation 3 units in the positive direction of the y axis of the graph of

$y = e^{-x}$.

$\therefore\ c = 3$

$\therefore\ F(x) = e^{-x} + 3$

e $f(x) = 2\sin x,\ x \in (0, 2\pi)$

$$F(x) = \int 2\sin x\, dx$$
$$= -2\cos x + c$$

Now $F(\pi) = 4$

$\therefore\ -2\cos\pi + c = 4$

$\therefore\ 2 + c = 4$

$\therefore\ c = 2$

$\therefore\ F(x) = -2\cos x + 2$

f $$f(x) = \frac{a}{b + x^2}$$

Now $f(1) = 0.4$

$$\therefore \frac{a}{b + 1^2} = 0.4$$

$$\therefore a = 0.4(b + 1) \qquad ①$$

Also $f(0) = 0.5$

$$\therefore \frac{a}{b + 0^2} = 0.5$$

$$\therefore a = \frac{b}{2} \qquad ②$$

Substituting ② in ① yields

$$\frac{b}{2} = 0.4(b + 1)$$

$$= \frac{2b}{5} + \frac{2}{5}$$

$$\therefore 5b = 4b + 4$$

$$\therefore b = 4$$

Substituting $b = 4$ in ② yields

$$a = \frac{4}{2} = 2$$

$$\therefore f(x) = \frac{2}{4 + x^2}$$

$$\therefore F(x) = \int \frac{2}{4 + x^2} dx$$

$$= \tan^{-1}\left(\frac{x}{2}\right) + c$$

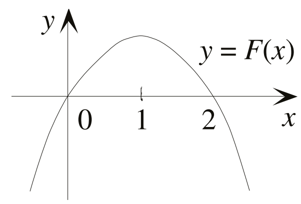
Now $F(0) = \frac{\pi}{2}$

$$\therefore \tan^{-1}(0) + c = \frac{\pi}{2}$$

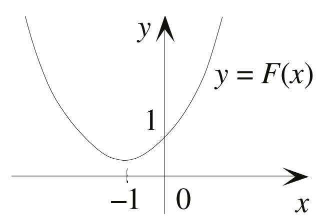
$$\therefore c = \frac{\pi}{2}$$

$$\therefore F(x) = \tan^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{2}$$

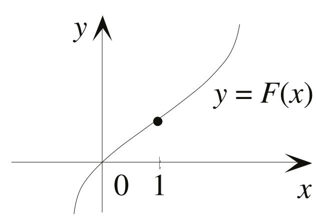
7 a $y = f(x)$ is the gradient graph of $F(x)$.
Therefore the gradient of $y = F(x)$ is negative for $x > 1$, zero for $x = 1$ and positive for $x < 1$. Since $y = f(x)$ is linear, the graph of $F(x)$ is a parabola. $F(0) = 0$ is given.
A possible graph is shown.



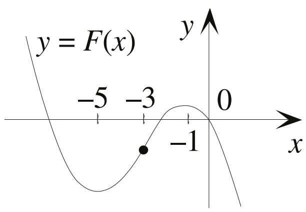
b $y = f(x)$ is the gradient graph of $F(x)$.
Therefore the gradient of $y = F(x)$ is positive for $x > -1$, zero for $x = -1$ and negative for $x < -1$. Since $y = f(x)$ is linear, the graph of $F(x)$ is a parabola. $F(0) = 1$ is given.
A possible graph is shown.



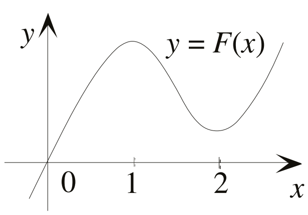
c $y = f(x)$ is the gradient graph of $F(x)$.
Therefore the gradient of $y = F(x)$ is positive for all $x \in R$.. The gradient is at a minimum of 2 when $x = 1$, and $F(0) = 0$ is given.
A possible graph is shown.



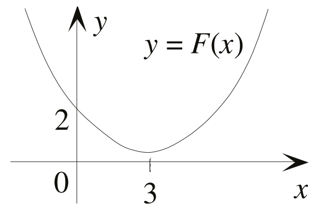
d $y = f(x)$ is the gradient graph of $F(x)$.
Therefore the gradient of $y = F(x)$ is positive for $-5 < x < -1$, zero for $x = -5$ and $x = -1$, and negative for $x < -5$ and $x > -1$. The gradient is at a maximum of 4 when $x = -3$ and $F(0) = 0$ is given.
A possible graph is shown.



e $y = f(x)$ is the gradient graph of $F(x)$.
Therefore the gradient of $y = F(x)$ is positive for $x < 1$ and $x > 2$, negative for $1 < x < 2$ and zero for $x = 1$ and $x = 2$. $F(0) = 0$ is given.
A possible graph is shown.



f $y = f(x)$ is the gradient graph of $F(x)$.
Therefore the gradient of $y = F(x)$ is positive for $x > 3$, negative for $x < 3$ and zero for $x = 3$. $F(0) = 2$ is given.
A possible graph is shown.



Solutions to Exercise 9B

1 **a** $\displaystyle\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}}$

$\displaystyle = \sin^{-1}\frac{x}{3} + c$

b $\displaystyle\int \frac{dx}{5+x^2} = \int \frac{dx}{(\sqrt{5})^2+x^2}$

$\displaystyle = \frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + c$

$\displaystyle = \frac{\sqrt{5}}{5}\tan^{-1}\frac{\sqrt{5}x}{5} + c$

c $\displaystyle\int \frac{dt}{1+t^2} = \tan^{-1} t + c$

d $\displaystyle\int \frac{5}{\sqrt{5-x^2}}dx = 5\sin^{-1}\frac{x}{\sqrt{5}} + c$

$\displaystyle = 5\sin^{-1}\frac{\sqrt{5}x}{5} + c$

e $\displaystyle\int \frac{3}{16+x^2}dx = \frac{3}{4}\tan^{-1}\frac{x}{4} + c$

f $\displaystyle\int \frac{dx}{\sqrt{16-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{4-x^2}}$

$\displaystyle = \frac{1}{2}\sin^{-1}\frac{x}{2} + c$

g $\displaystyle\int \frac{10dt}{\sqrt{10-t^2}} = 10\sin^{-1}\frac{t}{\sqrt{10}} + c$

$\displaystyle = 10\sin^{-1}\frac{\sqrt{10}t}{10} + c$

h $\displaystyle\int \frac{dt}{9+16t^2} = \frac{1}{16}\int \frac{dt}{\frac{9}{16}+t^2}$

$\displaystyle = \frac{1}{16}\times\frac{4}{3}\tan^{-1}\frac{t}{\frac{3}{4}} + c$

$\displaystyle = \frac{1}{12}\tan^{-1}\frac{4t}{3} + c$

i $\displaystyle\int \frac{dx}{\sqrt{5-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{5}{2}-x^2}}$

$\displaystyle = \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{\frac{5}{2}}} + c$

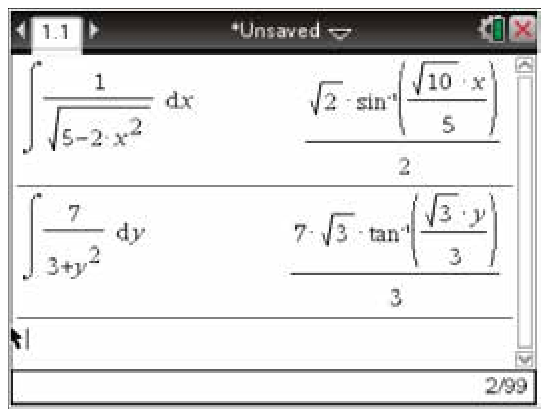
$\displaystyle = \frac{\sqrt{2}}{2}\sin^{-1}\frac{x\sqrt{10}}{5} + c$

j $\displaystyle\int \frac{7}{3+y^2}dy = \frac{7}{\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + c$

$\displaystyle = \frac{7\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}y}{3} + c$

CAS:

For part **i** and **j**



2 **a** $\displaystyle\int_0^1 \frac{2}{1+x^2}dx = 2[\tan^{-1} x]_0^1$

$\displaystyle = 2\times\frac{\pi}{4}$

$\displaystyle = \frac{\pi}{2}$

b $\displaystyle\int_0^{\frac{1}{2}} \frac{3}{\sqrt{1-x^2}}dx = 3[\sin^{-1} x]_0^{\frac{1}{2}}$

$\displaystyle = 3\times\frac{\pi}{6}$

$\displaystyle = \frac{\pi}{2}$

c $\displaystyle\int_0^1 \frac{5}{\sqrt{4-x^2}}dx = 5\left[\sin^{-1}\frac{x}{2}\right]_0^1$

$$= 5 \times \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

d $\displaystyle\int_0^5 \frac{6dx}{25+x^2} = \frac{6}{5}\left[\tan^{-1}\frac{x}{5}\right]_0^5$

$$= \frac{6}{5} \times \frac{\pi}{4}$$

$$= \frac{3\pi}{10}$$

e $\displaystyle\int_0^{\frac{3}{2}} \frac{3dx}{9+4x^2} = \frac{3}{4}\int_0^{\frac{3}{2}} \frac{dx}{\left(\frac{3}{2}\right)^2 + x^2}$

$$= \frac{3}{4} \times \frac{2}{3}\left[\tan^{-1}\frac{x}{\frac{3}{2}}\right]_0^{\frac{3}{2}}$$

$$= \frac{1}{2} \times \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

f $\displaystyle\int_0^2 \frac{dx}{8+2x^2} = \frac{1}{2}\int_0^2 \frac{dx}{4+x^2}$

$$= \frac{1}{2} \times \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{1}{4} \times \frac{\pi}{4}$$

$$= \frac{\pi}{16}$$

g $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{\frac{3}{2}}$

$$= \frac{\pi}{6}$$

h $\displaystyle\int_0^{\frac{3\sqrt{2}}{4}} \frac{dx}{\sqrt{9-4x^2}}$

$$= \frac{1}{2}\int_0^{\frac{3\sqrt{2}}{4}} \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{x}{\frac{3}{2}}\right]_0^{\frac{3\sqrt{2}}{4}}$$

$$= \frac{1}{2}\sin^{-1}\frac{\sqrt{2}}{2}$$

$$= \frac{\pi}{8}$$

i $\displaystyle\int_0^{\frac{1}{3}} \frac{3dy}{\sqrt{1-9y^2}} = \int_0^{\frac{1}{3}} \frac{dy}{\sqrt{\left(\frac{1}{3}\right)^2 - y^2}}$

$$= \left[\sin^{-1}\frac{y}{\frac{1}{3}}\right]_0^{\frac{1}{3}}$$

$$= \frac{\pi}{2}$$

j $\displaystyle\int_0^2 \frac{dx}{1+3x^2} = \frac{1}{3}\int_0^2 \frac{dx}{\left(\frac{1}{\sqrt{3}}\right)^2 + x^2}$

$$= \frac{1}{3} \times \sqrt{3}\left[\tan^{-1}\frac{x}{\frac{1}{\sqrt{3}}}\right]_0^2$$

$$= \frac{\sqrt{3}}{3}\tan^{-1} 2\sqrt{3}$$

$$\approx 0.745$$

Solutions to Exercise 9C

1 **a**

Let $u = x^2 + 1$

Then $\dfrac{du}{dx} = 2x$ and

$$f(u) = u^3 = (x^2+1)^3$$

$$\therefore \int 2x(x^2+1)^3 dx = \int \frac{du}{dx} u^3 dx = \int u^3 du = \frac{u^4}{4} + c = \frac{(x^2+1)^4}{4} + c$$

b Let $u = x^2 + 1$

Then $\dfrac{du}{dx} = 2x$ and

$$f(u) = \frac{1}{u^2} = \frac{1}{(x^2+1)^2}$$

$$\therefore \int \frac{x\,dx}{(x^2+1)^2} = \int \frac{du}{2u^2} = \frac{1}{2}\int u^{-2} du = -\frac{1}{2}u^{-1} + c = -\frac{1}{2(x^2+1)} + c$$

c Let $\sin x = u$

Then $\dfrac{du}{dx} = \cos x$

$$\therefore \int \cos x \sin^3 x\, dx = \int u^3 du = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c$$

d Let $\sin x = u$

Then $\dfrac{du}{dx} = \cos x$ and

$$f(u) = \frac{1}{u^2} = \frac{1}{\sin^2 x}$$

$$\therefore \int \frac{\cos x dx}{\sin^2 x} = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\frac{1}{\sin x} + c$$

e $$\int (2x+1)^5 dx = \frac{1}{2} \times \frac{1}{6}(2x+1)^6 + c$$

(linear substitution)

$$= \frac{(2x+1)^6}{12} + c$$

f Let $9 + x^2 = u$

Then $\dfrac{du}{dx} = 2x$, and

$$f(u) = \sqrt{u} = \sqrt{9+x^2}$$

$$\therefore \int 5x\sqrt{9+x^2}dx = \frac{5}{2}\int \sqrt{u}du = \frac{5}{2} \times \frac{2}{3}u^{\frac{3}{2}} + c = \frac{5}{3}\sqrt{(9+x^2)^3} + c$$

g Let $x^2 - 3 = u$

Then $\dfrac{du}{dx} = 2x$, $f(u) = u^5 = (x^2-3)^5$

$$\therefore \int x(x^2-3)^5 dx = \frac{1}{2}\int u^5 du = \frac{1}{2} \times \frac{1}{6}u^6 + c = \frac{1}{12}(x^2-3)^6 + c$$

h Let $x^2 + 2x = u$

Then $\frac{du}{dx} = 2x + 2 = 2(x + 1)$

$\therefore\ 2(x + 1) = \frac{du}{dx},\ f(u) = \frac{1}{u^3} = \frac{1}{(x^2 + 2x)^3}$

$$\therefore \int \frac{x + 1}{(x^2 + 2x)^3} dx = \frac{1}{2} \int \frac{du}{u^3}$$
$$= \frac{1}{2} \int u^{-3} du$$
$$= \frac{1}{2} \times \frac{-1}{2} u^{-2} + c$$
$$= -\frac{1}{4(x^2 + 2x)^2} + c$$

i Let $3x + 1 = u$

Then $\frac{du}{dx} = 3$

$f(u) = \frac{1}{u^3} = \frac{1}{(3x + 1)^3}$

$$\therefore \int \frac{2}{(3x + 1)^3} dx = \frac{2}{3} \int \frac{du}{u^3}$$
$$= \frac{2}{3} \times -\frac{1}{2} u^{-2} + c$$
$$= \frac{-1}{3(3x + 1)^2} + c$$

j Let $1 + x = u$

Then $\frac{du}{dx} = 1$

$\therefore\ f(u) = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{x + 1}}$

$$\therefore \int \frac{dx}{\sqrt{1 + x}} = \int \frac{du}{\sqrt{u}}$$
$$= 2\sqrt{u} + c$$
$$= 2\sqrt{1 + x} + c$$

k Let $x^3 - 3x^2 + 1 = u$

Then $\frac{du}{dx} = 3x^2 - 6x = 3(x^2 - 2x)$

$\therefore\ f(u) = u^4 = (x^3 - 3x^2 + 1)^4$

$$\therefore \int (x^2 - 2x)(x^3 - 3x^2 + 1)^4 dx$$
$$= \frac{1}{3} \int u^4 du$$
$$= \frac{1}{15} u^5 + c$$
$$= \frac{(x^3 - 3x^2 + 1)^5}{15} + c$$

l Let $x^2 + 1 = u$

Then $\frac{du}{dx} = 2x$

$f(u) = \frac{1}{u} = \frac{1}{x^2 + 1}$

$$\therefore \int \frac{3x}{x^2 + 1} dx = \frac{3}{2} \int \frac{1}{u} du$$
$$= \frac{3}{2} \log_e |u| + c$$
$$= \frac{3}{2} \log_e (x^2 + 1) + c$$

(since $x^2 + 1 > 0$)

m Let $2 - x^2 = u$

Then $\frac{du}{dx} = -2x$

$f(u) = \frac{1}{u} = \frac{1}{2 - x^2}$

$$\therefore \int \frac{3x}{2 - x^2} dx = -\frac{3}{2} \int \frac{1}{u} du$$
$$= -\frac{3}{2} \log_e |2 - x^2| + c$$

n Let $u = \log_e x$. Then $\frac{du}{dx} = \frac{1}{x}$

$$\therefore \int \frac{\log_e x}{x}\,dx = \int u\,du$$
$$= \frac{u^2}{2} + c$$
$$= \frac{(\log_e x)^2}{2} + c$$

o Let $u = -4x^2$. Then $\frac{du}{dx} = -8x$

$$\therefore \int e^{-4x^2}\,dx = -\frac{1}{8}\int e^u\,du$$
$$= -\frac{1}{8}e^u + c$$
$$= -\frac{1}{8}e^{-4x^2} + c$$

2 a $$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{(x+1)^2 + 1}$$
$$= \tan^{-1}(x+1) + c$$

b $$\int \frac{dx}{x^2 - x + 1}$$
$$= \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + 1 - \frac{1}{4}}$$
$$= \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$
$$= \frac{2\sqrt{3}}{3} \tan^{-1} \frac{2x - 1}{\sqrt{3}} + c$$
$$= \frac{2\sqrt{3}}{3} \tan^{-1} \frac{\sqrt{3}(2x - 1)}{3} + c$$

c $21 - 4x - x^2 = -(x^2 + 4x + 4) + 25$

$= 25 - (x+2)^2$

Let $x + 2 = u$

Then $\frac{du}{dx} = 1$

$$\therefore \int \frac{dx}{\sqrt{21 - 4x - x^2}} = \int \frac{du}{\sqrt{25 - u^2}}$$
$$= \sin^{-1} \frac{u}{5} + c$$
$$= \sin^{-1} \frac{x+2}{5} + c$$

d $10x - x^2 - 24 = -(x^2 - 10x + 25) + 1$

$$= 1 - (x-5)^2$$

Let $x - 5 = u$

Then $\frac{du}{dx} = 1$

and $f(u) = \frac{1}{\sqrt{1-u^2}}$

$$= \frac{1}{\sqrt{1-(x-5)^2}}$$

$$= \frac{1}{\sqrt{10x - x^2 - 24}}$$

$$\therefore \int \frac{dx}{\sqrt{10x - x^2 - 24}}$$

$$= \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + c$$

$$= \sin^{-1}(x-5) + c$$

e $40 - x^2 - 6x = -(x^2 + 6x + 9) + 49$

$$= 7^2 - (x+3)^2$$

Let $x + 3 = u$

Then $\frac{du}{dx} = 1$

and $f(u) = \frac{1}{\sqrt{7^2 - u^2}}$

$$= \frac{1}{\sqrt{7^2 - (x+3)^2}}$$

$$= \frac{1}{\sqrt{40 - x^2 - 6x}}$$

$$\therefore \int \frac{dx}{\sqrt{40 - x^2 - 6x}} = \int \frac{du}{\sqrt{7^2 - u^2}}$$

$$= \sin^{-1}\frac{u}{7} + c$$

$$= \sin^{-1}\frac{x+3}{7} + c$$

f $\int \frac{dx}{3x^2 + 6x + 7}$

$$= \frac{1}{3}\int \frac{dx}{x^2 + 2x + \frac{7}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{(x+1)^2 + \frac{7}{3} - 1}$$

$$= \frac{1}{3}\int \frac{dx}{(x+1)^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{3} \times \frac{\sqrt{3}}{2}\tan^{-1}\frac{x+1}{\frac{2}{\sqrt{3}}} + c$$

$$= \frac{\sqrt{3}}{6}\tan^{-1}\frac{\sqrt{3}(x+1)}{2} + c$$

3 a Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$

$$\int \sin x \cos x \, dx = \int u \, du$$

$$= \frac{1}{2}u^2 + c_1$$

$$= \frac{1}{2}\sin^2 x + c_1$$

b $u = \cos x$. Then $\frac{du}{dx} = -\sin x$

$$\int \sin x \cos x \, dx = -\int u \, du$$

$$= -\frac{1}{2}u^2 + c_2$$

$$= -\frac{1}{2}\cos^2 x + c_2$$

c $-\frac{1}{2}\cos^2 x + c_2 = \frac{1}{2}\sin^2 x + c_1$

$\therefore \cos^2 x + \sin^2 x = C$

d If $x = 0$ we find $C = 1$ and

$\cos^2 x + \sin^2 x = 1$

4 a Let $2x + 3 = u$

Then $\frac{du}{dx} = 2,\ x = \frac{u-3}{2}$

$$\therefore \int x\sqrt{2x+3}\,dx$$
$$= \int \frac{u-3}{2}\sqrt{u}\frac{du}{2}$$
$$= \frac{1}{4}\int\left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)du$$
$$= \frac{1}{4}\times\frac{2}{5}u^{\frac{5}{2}} - \frac{3}{4}\times\frac{2}{3}u^{\frac{3}{2}} + c$$
$$= \frac{1}{10}(2x+3)^{\frac{5}{2}} - \frac{1}{2}(2x+3)^{\frac{3}{2}} + c$$

b Let $1 - x = u$

Then $\frac{du}{dx} = -1,\ x = 1 - u$

$$\therefore \int x\sqrt{1-x}\,dx$$
$$= \int (1-u)\sqrt{u}(-du)$$
$$= \int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$
$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$$
$$= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$$

c Let $3x - 7 = u$

Then $\frac{du}{dx} = 3,\ x = \frac{u+7}{3}$

$$\therefore \int \frac{6x}{(3x-7)^{\frac{1}{2}}}dx$$
$$= \int \frac{6(u+7)}{3}u^{-\frac{1}{2}}\frac{du}{3}$$
$$= \frac{2}{3}\int\left(u^{\frac{1}{2}} + 7u^{-\frac{1}{2}}\right)du$$
$$= \left(\frac{2}{3} \div \frac{3}{2}\right)u^{\frac{3}{2}} + \left(\frac{2\times 7}{3} \div \frac{1}{2}\right)u^{\frac{1}{2}} + c$$
$$= \frac{4}{9}(3x-7)^{\frac{3}{2}} + \frac{28}{3}(3x-7)^{\frac{1}{2}} + c$$

d Let $3x - 1 = u$

Then $\frac{du}{dx} = 3,\ x = \frac{u+1}{3}$

$$\therefore \int (2x+1)\sqrt{3x-1}\,dx$$
$$= \int\left(\frac{2(u+1)}{3} + 1\right)u^{\frac{1}{2}}\frac{du}{3}$$
$$= \frac{1}{9}\int (2u+5)u^{\frac{1}{2}}du$$
$$= \frac{2}{9}\int u^{\frac{3}{2}}du + \frac{5}{9}\int u^{\frac{1}{2}}du$$
$$= \frac{2}{9}\times\frac{2}{5}u^{\frac{5}{2}} + \frac{5}{9}\times\frac{2}{3}u^{\frac{3}{2}} + c$$
$$= \frac{4}{45}(3x-1)^{\frac{5}{2}} + \frac{10}{27}(3x-1)^{\frac{3}{2}} + c$$

e $\int \frac{2x-1}{(x-1)^2}dx$

$= \int \frac{2x-2+1}{(x-1)^2}dx$

$= 2\int \frac{x-1}{(x-1)^2}dx + \int \frac{1}{(x-1)^2}dx$

$= 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$

$= 2\log_e|x-1| - \frac{1}{x-1} + c$

f Let $3x+1 = u$

Then $\frac{du}{dx} = 3,\ x = \frac{u-1}{3},$

$x+3 = \frac{u+8}{3}$

$\int (x+3)\sqrt{3x+1}dx$

$= \int \frac{u+8}{3}u^{\frac{1}{2}}\frac{du}{3}$

$= \frac{1}{9}\int u^{\frac{3}{2}}du + \frac{8}{9}\int u^{\frac{1}{2}}du$

$= \frac{1}{9} \times \frac{2}{5}u^{\frac{5}{2}} + \frac{8}{9} \times \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{45}(3x+1)^{\frac{5}{2}} + \frac{16}{27}(3x+1)^{\frac{3}{2}} + c$

g $\int (x+2)(x+3)^{\frac{1}{3}}dx$

$= \int (x+3-1)(x+3)^{\frac{1}{3}}dx$

$= \int (x+3)^{\frac{4}{3}}dx - \int (x+3)^{\frac{1}{3}}dx$

$= \frac{3}{7}(x+3)^{\frac{7}{3}} - \frac{3}{4}(x+3)^{\frac{4}{3}} + c$

h Let $2x+1 = u$

Then $\frac{du}{dx} = 2,\ x = \frac{u-1}{2},$

$5x - 1 = \frac{5(u-1)}{2} - 1$

$= \frac{5u-5-2}{2}$

$= \frac{5u-7}{2}$

$f(u) = \frac{1}{u^2} = \frac{1}{(2x+1)^2}$

$\therefore \int \frac{5x-1}{(2x+1)^2}dx$

$= \int \frac{5u-7}{2} \times \frac{1}{u^2} \times \frac{du}{2}$

$= \frac{5}{4}\int \frac{du}{u} - \frac{7}{4}\int \frac{du}{u^2}$

$= \frac{5}{4}\log_e|u| + \frac{7}{4}\frac{1}{u} + c$

$= \frac{5}{4}\log_e|2x+1| + \frac{7}{4(2x+1)} + c$

i Let $x - 1 = u$

Then $\frac{du}{dx} = 1,\ x = u+1$

$\therefore\ x^2 = (u+1)^2$

$\therefore \int x^2\sqrt{x-1}dx$

$= \int (u+1)^2\sqrt{u}du$

$= \int u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + \sqrt{u}du$

$= \frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{105}u^{\frac{3}{2}}(15u^2 + 42u + 35) + c$

$= \frac{2}{105}(x-1)^{\frac{3}{2}}(15(x-1)^2$

$+ 42(x-1) + 35) + c$

$= \frac{2}{105}(x-1)^{\frac{3}{2}}(15x^2 + 12x + 8) + c$

j Let $x - 1 = u$

Then $\frac{du}{dx} = 1, \ x = u + 1$

$\therefore \ x^2 = (u + 1)^2$

$$\therefore \int \frac{x^2}{\sqrt{x-1}}dx$$

$$= \int \frac{(u+1)^2}{\sqrt{u}}du$$

$$= \int (u+1)^2 u^{-\frac{1}{2}} du$$

$$= \int u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + u^{-\frac{1}{2}} du$$

$$= \frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + c$$

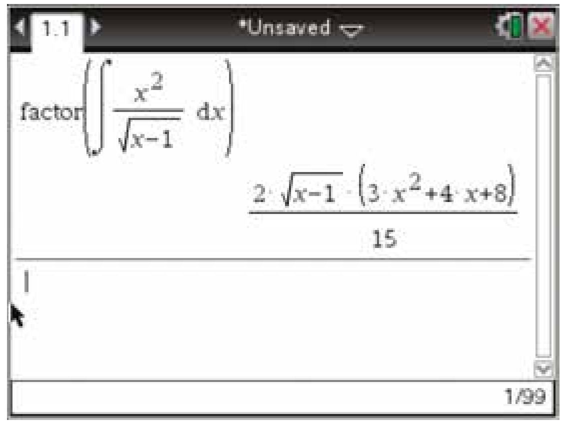
$$= \frac{2}{15}u^{\frac{1}{2}}(3u^2 + 10u + 15) + c$$

$$= \frac{2}{15}(x-1)^{\frac{1}{2}}(3(x-1)^2$$

$$+ 10(x-1) + 15) + c$$

$$= \frac{2\sqrt{x-1}}{15}(3x^2 + 4x + 8) + c$$

CAS:



Solutions to Exercise 9D

1 **a** Let $u = x^2 + 16$

Then $\frac{du}{dx} = 2x$

When $x = 0,\ u = 16$

and when $x = 3,\ u = 25$

$$\therefore \int_0^3 x\sqrt{x^2+16}\,dx$$
$$= \int_{16}^{25} \sqrt{u}\frac{du}{2}$$
$$= \frac{1}{2} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_{16}^{25}$$
$$= \frac{1}{3}(125 - 64)$$
$$= \frac{61}{3}$$

b Let $u = \sin x$

Then $\frac{du}{dx} = \cos x$

When $x = 0,\ u = 0$

and when $x = \frac{\pi}{4},\ u = \frac{\sqrt{2}}{2}$

$$\therefore \int_0^{\frac{\pi}{4}} \cos x \sin^3 x\,dx$$
$$= \int_0^{\frac{\sqrt{2}}{2}} u^3 du$$
$$= \left[\frac{u^4}{4}\right]_0^{\frac{\sqrt{2}}{2}}$$
$$= \frac{1}{16}$$

c Let $\cos x = u$

Then $\frac{du}{dx} = -\sin x$

When $x = 0,\ u = 1$

and when $x = \frac{\pi}{2},\ u = 0$

$$\therefore \int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$$
$$= -\int_1^0 u^2 du$$
$$= \int_0^1 u^2 du$$
$$= \left[\frac{u^3}{3}\right]_0^1$$
$$= \frac{1}{3}$$

d Let $x - 3 = u$

Then $\frac{du}{dx} = 1,\ x = u + 3$

When $x = 3,\ u = 0$

and when $x = 4,\ u = 1$

$$\therefore \int_3^4 x(x-3)^{17}dx$$
$$= \int_0^1 (u+3)u^{17}du$$
$$= \int_0^1 u^{18}du + 3\int_0^1 u^{17}du$$
$$= \left[\frac{u^{19}}{19}\right]_0^1 + 3\left[\frac{u^{18}}{18}\right]_0^1$$
$$= \frac{1}{19} + \frac{3}{18}$$
$$= \frac{25}{114}$$

e Let $1 - x = u$

Then $\frac{du}{dx} = -1, \; x = 1 - u$

When $x = 0, \; u = 1$

and when $x = 1, \; u = 0$

$$\therefore \int_0^1 x\sqrt{1-x}\,dx$$

$$= -\int_1^0 (1-u)\sqrt{u}\,du$$

$$= \int_0^1 (1-u)\sqrt{u}\,du$$

$$= \int_0^1 u^{\frac{1}{2}}\,du - \int_0^1 u^{\frac{3}{2}}\,du$$

$$= \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1 - \frac{2}{5}\left[u^{\frac{5}{2}}\right]_0^1$$

$$= \frac{2}{3} - \frac{2}{5}$$

$$= \frac{4}{15}$$

f Let $\log_e x = u$

Then $\frac{du}{dx} = \frac{1}{x}$

When $x = e, \; u = 1$

and when $x = e^2, \; u = 2$

$$\therefore \int_e^{e^2} \frac{dx}{x\log_e x}$$

$$= \int_1^2 \frac{du}{u}$$

$$= \left[\log_e |u|\right]_1^2$$

$$= \log_e 2$$

g Let $3x + 4 = u$

Then $\frac{du}{dx} = 3$

When $x = 0, \; u = 4$

and when $x = 4, \; u = 16$

$$\therefore \int_0^4 \frac{dx}{\sqrt{3x+4}} = \frac{1}{3}\int_4^{16} \frac{du}{\sqrt{u}}$$

$$= \frac{2}{3}\left[\sqrt{u}\right]_4^{16}$$

$$= \frac{2}{3}(4-2)$$

$$= \frac{4}{3}$$

h Let $u = e^x + 1$

Then $\frac{du}{dx} = e^x$

When $x = -1, \; u = \frac{1}{e} + 1$

and when $x = 1, \; u = e + 1$

$$\therefore \int_{-1}^1 \frac{e^x\,dx}{e^x+1} = \int_{\frac{1}{e}+1}^{e+1} \frac{du}{u}$$

$$= \left[\log_e |u|\right]_{\frac{1}{e}+1}^{e+1}$$

$$= \log_e(e+1) - \log_e\left(\frac{1}{e}+1\right)$$

$$= \log_e \frac{e+1}{\frac{1}{e}+1}$$

$$= \log_e \frac{e(e+1)}{1+e}$$

$$= \log_e e$$

$$= 1$$

i Let $u = \cos x$

Then $\dfrac{du}{dx} = -\sin x$

When $x = 0,\ u = 1$

and when $x = \dfrac{\pi}{4},\ u = \dfrac{\sqrt{2}}{2}$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin x\,dx}{\cos^3 x} = -\int_1^{\frac{\sqrt{2}}{2}} \frac{du}{u^3}$$

$$= -\int_1^{\frac{\sqrt{2}}{2}} u^{-3}du$$

$$= \left[\frac{u^{-2}}{2}\right]_1^{\frac{\sqrt{2}}{2}}$$

$$= \frac{2}{1} - \frac{1}{2}$$

$$= \frac{1}{2}$$

j Let $x^2 + 3x + 4 = u$

Then $\dfrac{du}{dx} = 2x + 3$

When $x = 0,\ u = 4$

and when $x = 1,\ u = 1 + 3 + 4 = 8$

$$\therefore \int_0^1 \frac{2x+3}{x^2+3x+4}dx = \int_4^8 \frac{du}{u}$$

$$= \left[\log_e |u|\right]_4^8$$

$$= \log_e 8 - \log_e 4$$

$$= \log_e 2$$

k Let $u = \sin x$

Then $\dfrac{du}{dx} = \cos x$

When $x = \dfrac{\pi}{4}, u = \dfrac{1}{\sqrt{2}}$

and when $x = \dfrac{\pi}{3}, u = \dfrac{\sqrt{3}}{2}$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x\,dx}{\sin x} = \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{du}{u}$$

$$= \left[\log_e |u|\right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}}$$

$$= \log_e \frac{\sqrt{3}}{2} - \log_e \frac{1}{\sqrt{2}}$$

$$= \log_e \frac{\sqrt{3}}{\sqrt{2}}$$

$$= \log_e \frac{\sqrt{6}}{2}$$

l Let $1 - x^2 = u$

Then $\dfrac{du}{dx} = -2x$

When $x = -4,\ u = -15$

and when $x = -3,\ u = -8$

$$\therefore \int_{-4}^{-3} \frac{2x}{1-x^2}dx = -\int_{-15}^{-8} \frac{du}{u}$$

$$= \int_{-8}^{-15} \frac{du}{u}$$

$$= \left[\log_e |u|\right]_{-8}^{-15}$$

$$= \log_e 15 - \log_e 8$$

$$= \log_e \frac{15}{8}$$

m Let $1 - e^x = u$

Then $\dfrac{du}{dx} = -e^x$

When $x = -2,\ u = 1 - \dfrac{1}{e^2}$

and when $x = -1,\ u = 1 - \dfrac{1}{e}$

$$\therefore \int_{-2}^{-1} \frac{e^x}{1 - e^x} dx$$

$$= -\int_{1-\frac{1}{e^2}}^{1-\frac{1}{e}} \frac{du}{u}$$

$$= \int_{1-\frac{1}{e}}^{1-\frac{1}{e^2}} \frac{du}{u}$$

$$= \left[\log_e |u|\right]_{1-\frac{1}{e}}^{1-\frac{1}{e^2}}$$

$$= \log_e\left(1 - \frac{1}{e^2}\right) - \log_e\left(1 - \frac{1}{e}\right)$$

$$= \log_e\left(\frac{e^2 - 1}{e^2}\right) - \log_e\left(\frac{e - 1}{e}\right)$$

$$= \log_e\left(\frac{(e + 1)(e - 1)}{e^2}\right) - \log_e\left(\frac{e - 1}{e}\right)$$

$$= \log_e\left(\frac{(e + 1)(e - 1)}{e^2} \times \frac{e}{e - 1}\right)$$

$$= \log_e\left(\frac{e + 1}{e}\right)$$

$$= \log_e(e + 1) - \log_e e$$

$$= \log_e(e + 1) - 1$$

2 $\displaystyle\int_0^{\frac{\pi}{3}} \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$

Let$u = cosx, \dfrac{du}{dx} = -sinx$

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\, dx = \int_1^{\frac{1}{2}} -\frac{1}{u}\, du$$

$$= -[\log_e u]_1^{\frac{1}{2}}$$

$$= -\log_e \frac{1}{2}$$

$$= \log_e 2$$

Solutions to Exercise 9E

Some useful formulae:

For even powers of sine and cosine use:

- $\sin kx \cos kx = \frac{1}{2}\sin 2kx$
- $\sin^2 kx = \frac{1}{2}(1 - \cos 2kx)$
- $\cos^2 kx = \frac{1}{2}(1 + \cos 2kx)$

For odd powers of sine and cosine use:

- $\sin^2 kx + \cos^2 kx = 1$

Secant, tangent, cosecant and cotangent

- $1 + \tan^2 kx = \sec^2 kx$
- $1 + \cot^2 kx = \operatorname{cosec}^2 kx$

Integrals

- $\int \sin kx dx = -\frac{1}{k}\cos kx + c$
- $\int \cos kx dx = \frac{1}{k}\sin kx + c$
- $\int \sec^2 kx dx = \frac{1}{k}\tan kx + c$
- $\int \operatorname{cosec}^2 kx dx = -\frac{1}{k}\cot kx + c$

1 **a** $\int \sin^2 x\,dx$

$$= \int \frac{1}{2}(1 - \cos 2x)dx$$
$$= \frac{1}{2}\int 1 - \cos 2x\,dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

b

$\int \sin^4 x\,dx$

$$= \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 dx$$
$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\,dx$$
$$= \frac{1}{4}\int 1 - 2\cos 2x$$
$$+ \frac{1}{2}(1 + \cos 4x)dx$$
$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right] + c$$
$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

c $\int 2\tan^2 x\,dx$

$$= \int 2\left(\sec^2 x - 1\right)dx$$
$$= 2\tan x - 2x + c$$

d $\int 2\sin 3x\cos 3x\,dx$

$= 2\int \sin 3x\cos 3x\,dx$

$= 2\int \frac{1}{2}(\sin 6x)dx$

$= \int \sin 6x dx$

$= -\frac{1}{6}\cos 6x + c$

e $\int \sin^2 2x\,dx$

$= \frac{1}{2}\int 1 - \cos 4x\,dx$

$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right] + c$

$= \frac{1}{2}x - \frac{1}{8}\sin 4x + c$

f $\int \tan^2 2x\,dx$

$= \int \sec^2 2x - 1dx$

$= \frac{1}{2}\tan 2x - x + c$

g $\int \sin^2 x\cos^2 x\,dx$

$= \int (\sin x\cos x)^2 dx$

$= \int \left(\frac{1}{2}\sin 2x\right)^2 dx$

$= \frac{1}{4}\int \sin^2 2x\,dx$

$= \frac{1}{8}\int 1 - \cos 4x\,dx$

$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + c$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + c$

h $\int \cos^2 x - \sin^2 x\,dx$

$= \int \cos 2x\,dx$

$= \frac{1}{2}\sin 2x + c$

i $\int \cot^2 x\,dx = \int \left(\text{cosec}^2 x - 1\right)dx$

$= -\cot x - x + c$

Alternatively,

$\int \cot^2 x\,dx = \int \left(\text{cosec}^2 x - 1\right)dx$

$= \int \left(\frac{1}{\sin^2 x} - 1\right)dx$

$= \int \left(\frac{\sec^2 x}{\tan^2 x} - 1\right)dx$

$= \int \frac{\sec^2 x}{\tan^2 x}dx - x + c$

Let $u = \tan x$

Then $\frac{du}{dx} = \sec^2 x$

$\therefore \int \cot^2 x\,dx = \int \frac{1}{u^2}du - x + c$

$= -\frac{1}{u} - x + c$

$= -\frac{1}{\tan x} - x + c$

$= -\cot x - x + c$

j $\int \cos^3 2x\,dx$

$= \int \cos 2x \cdot \cos^2 2x\,dx$

$= \int \cos 2x\left(1 - \sin^2 2x\right)dx$

Let $u = \sin 2x$

$\frac{du}{dx} = 2\cos 2x$

$= \int \left(\frac{1}{2}\frac{du}{dx}\right)\left(1 - u^2\right)dx$

$= \frac{1}{2}\int 1 - u^2 du$

$= \frac{1}{2}\left[u - \frac{1}{3}u^3\right] + c$

$= \frac{1}{2}u - \frac{1}{6}u^3 + c$

$\therefore \int \cos^3 2x\, dx$

$= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + c$

2 **a** $\int \sec^2 x\, dx = \tan x + c$

An antiderivative of $\sec^2 x$ is $\tan x$ $(c = 0)$

b $\int \sec^2(2x)dx = \int \sec^2 u dx$

where $u = 2x$ and $\frac{du}{dx} = 2$

$= \frac{1}{2}\int \sec^2 u du$

$= \frac{1}{2}\tan u + c$

$= \frac{1}{2}\tan(2x) + c$

An antiderivative of $\sec^2(2x)$ is $\frac{1}{2}\tan(2x)$ $(c = 0)$

c $\int \sec^2\left(\frac{1}{2}x\right)dx = \int \sec^2 u dx$

where $u = \frac{1}{2}x$ and $\frac{du}{dx} = \frac{1}{2}$

$= 2\int \sec^2 u du$

$= 2\tan u + c$

$= 2\tan\left(\frac{1}{2}x\right) + c$

An antiderivative of $\sec^2\left(\frac{1}{2}x\right)$ is $2\tan\left(\frac{1}{2}x\right)$ $(c = 0)$

d $\int \sec^2(kx)dx = \int \sec^2 u dx$

where $u = kx$ and $\frac{du}{dx} = k$

$= \frac{1}{k}\int \sec^2 u du$

$= \frac{1}{k}\tan u + c$

$= \frac{1}{k}\tan(kx) + c$

An antiderivative of $\sec^2(kx)$ is $\frac{1}{k}\tan(kx)$ $(c = 0)$

e $\int \tan^2(3x)dx = \int \sec^2(3x) - 1dx$

$= \int \sec^2(3x)dx - \int 1dx$

$= \frac{1}{3}\tan(3x) - x + c$

An antiderivative of $\tan^2(3x)$ is $\frac{1}{3}\tan(3x) - x(c = 0)$

f $\int 1 - \tan^2 x\, dx$

$= \int 1 - (\sec^2 x - 1)dx$

$= \int 2 - \sec^2 x\, dx$

$= \int 2dx - \int \sec^2 x\, dx$

$= 2x - \tan x + c$

An antiderivative of $1 - \tan^2 x$ is $2x - \tan x$ $(c = 0)$

g $\int \tan^2 x - \sec^2 x\, dx = \int (\sec^2 x - 1) - \sec^2 x\, dx$

$= \int -1 dx$

$= -x + c$

An antiderivative of $\tan^2 x - \sec^2 x$ is $-x$ $(c = 0)$

h $\int \operatorname{cosec}^2\left(x - \frac{\pi}{2}\right) dx$

$= \int \frac{1}{\sin^2\left(x - \frac{\pi}{2}\right)} dx$

$= \int \frac{1}{\left(\sin(x - \frac{\pi}{2})\right)^2} dx$

$= \int \frac{1}{(-\cos x)^2} dx$

$= \int \frac{1}{\cos^2 x} dx$

$= \int \sec^2 x\, dx$

$= \tan x + c$

An antiderivative of $\operatorname{cosec}^2\left(x - \frac{\pi}{2}\right)$ is $\tan x$ $(c = 0)$

3 a $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} dx$

$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4}$

b $\int_0^{\frac{\pi}{4}} \tan^3 x\, dx$

$= \int_0^{\frac{\pi}{4}} \tan x \tan^2 x\, dx$

$= \int_0^{\frac{\pi}{4}} \tan x(\sec^2 x - 1) dx$

$= -\int_0^{\frac{\pi}{4}} \tan x\, dx + \int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$

$= -\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx + \int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$

For the first integral, let $z = \cos x$

Then $\frac{dz}{dx} = -\sin x$

When $x = 0$, $z = 1$

and when $x = \frac{\pi}{4}$, $z = \frac{1}{\sqrt{2}}$

For the second integral, let $u = \tan x$

Then $\frac{du}{dx} = \sec^2 x$

When $x = 0$, $u = 0$

and when $x = \frac{\pi}{4}$, $u = 1$

$\therefore \int_0^{\frac{\pi}{4}} \tan^3 x\, dx = \int_1^{\frac{1}{\sqrt{2}}} \frac{dz}{z} + \int_0^1 u du$

$= \left[\log_e |z|\right]_1^{\frac{1}{\sqrt{2}}} + \left[\frac{u^2}{2}\right]_0^1$

$= -\log_e \sqrt{2} + \frac{1}{2}$

$= \frac{1 - \log_e 2}{2}$

$= \frac{1}{2} - \frac{1}{2}\log_e 2$

c Let $u = \sin x$

Then $\frac{du}{dx} = \cos x$

When $x = 0,\ u = 0$

and when $x = \frac{\pi}{2},\ u = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x \cos x\, dx = \int_0^1 u^2 du$$

$$= \left[\frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

d $\cos^4 x = \left(\frac{\cos 2x + 1}{2}\right)^2$

$$= \frac{1}{4}\left(\cos^2 2x + 2\cos 2x + 1\right)$$

$$= \frac{1}{4}\left(\frac{\cos 4x + 1}{2} + 2\cos 2x + 1\right)$$

$$= \frac{1}{8}\left(3 + 4\cos 2x + \cos 4x\right)$$

$$= \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}$$

$$\therefore \int_0^{\frac{\pi}{4}} \cos^4 x\, dx$$

$$= \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{3\pi}{32} + \frac{1}{4}$$

$$\approx 0.545$$

e $\int_0^{\pi} \sin^3 x\, dx$

$$= \int_0^{\pi} \sin^2 x \sin x\, dx$$

$$= \int_0^{\pi} \left(1 - \cos^2 x\right) \sin x\, dx$$

Let $u = \cos x$

Then $\frac{du}{dx} = -\sin x$

When $x = 0,\ u = 1$

and when $x = \pi,\ u = -1$

$$\int_0^{\pi} \sin^3 x\, dx = -\int_1^{-1} (1 - u^2) du$$

$$= \int_{-1}^{1} (1 - u^2) du$$

$$= \left[u - \frac{u^3}{3}\right]_{-1}^{1}$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$$

$$= \frac{2}{3} - \left(-\frac{2}{3}\right)$$

$$= \frac{4}{3}$$

f $\int_0^{\frac{\pi}{2}} \sin^2 2x\, dx = \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4x}{2} dx$

$$= \left[\frac{x}{2} - \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

g $\int_0^{\frac{\pi}{3}} \sin^2 x \cos^2 x\, dx$

$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} \sin^2 2x\, dx$$

$$= \frac{1}{4}\left[\frac{x}{2} - \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{64}$$

$$\approx 0.158$$

h $\int_0^1 (\sin^2 x + \cos^2 x) dx = \int_0^1 1 dx$

$$= [x]_0^1$$

$$= 1$$

4 a

$$\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$$
$$= \int \left(1 - \sin^2 x\right) \cos x\,dx$$

Let $u = \sin x$

Then $\frac{du}{dx} = \cos x$

$$= \int (1 - u^2)du$$
$$= u - \frac{u^3}{3} + c$$
$$= \sin x - \frac{\sin^3 x}{3} + c$$

b $\int \sin^3 \frac{x}{4}dx$

$$= \int \sin^2 \frac{x}{4} \sin \frac{x}{4}dx$$
$$= \int \left(1 - \cos^2 \frac{x}{4}\right) \sin \frac{x}{4}dx$$

Let $\cos \frac{x}{4} = u$

Then $\frac{du}{dx} = -\frac{1}{4} \sin \frac{x}{4}$

$$= -4 \int \left(1 - u^2\right)du$$
$$= -4 \left(u - \frac{u^3}{3}\right) + c$$
$$= -4 \cos \frac{x}{4} + \frac{4}{3} \cos^3 \frac{x}{4} + c$$

c $\int \cos^2(4\pi x)dx$

$$= \int \frac{1 + \cos(8\pi x)}{2}dx$$
$$= \frac{x}{2} + \frac{\sin 8\pi x}{16\pi} + c$$

d $\int 7 \cos^7 t dt$

$$= 7 \int \cos^6 t \cos t dt$$
$$= 7 \int \left(1 - \sin^2 t\right)^3 \cos t dt$$

Let $\sin t = u$

Then $\frac{du}{dt} = \cos t$

$$= 7 \int \left(1 - u^2\right)^3 du$$
$$= 7 \int \left(1 - 3u^2 + 3u^4 - u^6\right)du$$
$$= 7u - 7u^3 + \frac{21}{5}u^5 - u^7 + c$$
$$= 7 \sin t - 7 \sin^3 t + \frac{21}{5} \sin^5 t$$
$$- \sin^7 t + c$$
$$= 7 \sin t\Big(\left(1 - \sin^2 t\right)$$
$$+ \frac{3}{5} \sin^4 t - \frac{\sin^6 t}{7}\Big) + c$$
$$= 7 \sin t\left(\cos^2 t + \frac{3}{5} \sin^4 t - \frac{\sin^6 t}{7}\right) + c$$

e

$$\int \cos^3 5x\,dx = \int\left(1 - \sin^2 5x\right) \cos 5x\,dx$$

Let $\sin 5x = u$

Then $\frac{du}{dx} = 5 \cos 5x$

$$= \frac{1}{5} \int \left(1 - u^2\right)du$$
$$= \frac{u}{5} - \frac{u^3}{15} + c$$
$$= \frac{1}{5} \sin 5x - \frac{\sin^3 5x}{15} + c$$

f

$$\int 8 \sin^4 x\,dx$$
$$= \int 8\left(\frac{1 - \cos 2x}{2}\right)^2 dx$$
$$= \int 2\left(1 - 2 \cos 2x + \cos^2 2x\right)dx$$
$$= 2x - 2 \sin 2x + 2 \int \frac{1 + \cos 4x}{2}dx$$
$$= 2x - 2 \sin 2x + x + \frac{\sin 4x}{4} + c$$
$$= 3x - 2 \sin 2x + \frac{\sin 4x}{4} + c$$

g $\int \sin^2 x \cos^4 x \, dx$

$= \int \left(\sin^2 x \cos^2 x\right) \times \cos^2 x \, dx$

$= \int \left(\frac{1}{4} \sin^2 2x\right) \cos^2 x \, dx$

$= \frac{1}{4} \int \sin^2 2x \frac{1 + \cos 2x}{2} dx$

$= \frac{1}{8} \int \sin^2 2x \, dx$

$+ \frac{1}{8} \int \sin^2 2x \cos 2x \, dx$

For the first integral, use the formula

$\sin^2 2x = \frac{1 - \cos 4x}{2}$

For the second integral, use substitution $\sin 2x = u$

Then $2 \cos 2x = \frac{du}{dx}$,

$\cos 2x \, dx = \frac{1}{2} du$

$\therefore \int \sin^2 x \cos^4 x \, dx$

$= \frac{1}{16} \int \left(1 - \cos 4x\right) dx$

$+ \frac{1}{16} \int u^2 du$

$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{u^3}{48} + c$

$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + c$

h

$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx$

$= \int (1 - \sin^2 x)^2 \cos x \, dx$

Let $\sin x = u$

Then $\frac{du}{dx} = \cos x$

$= \int \left(1 - u^2\right)^2 du$

$= \int \left(1 - 2u^2 + u^4\right) du$

$= u - \frac{2u^3}{3} + \frac{u^5}{5} + c$

$= \sin x - \frac{2 \sin^3 x}{3} + \frac{\sin^5 x}{5} + c$

5 **a**

$\int \sin(4x) \sin(2x) \, dx = \frac{1}{2} \int \cos(2x) - \cos(6x) \, dx$

$= \frac{1}{2}(\frac{1}{2} \sin(2x) - \frac{1}{6} \sin(6x)) + c$

$= \frac{1}{4} \sin(2x) - \frac{1}{12} \sin(6x) + c$

b

$\int \cos(4x) \cos(2x) \, dx = \frac{1}{2} \int \cos(2x) + \cos(6x) \, dx$

$= \frac{1}{2}(\frac{1}{2} \sin(2x) + \frac{1}{6} \sin(6x)) + c$

$= \frac{1}{4} \sin(2x) + \frac{1}{12} \sin(6x) + c$

c

$\int \sin(4x) \cos(2x) \, dx = \frac{1}{2} \int \sin(2x) + \sin(6x) \, dx$

$= -\frac{1}{2}(\frac{1}{2} \cos(2x) + \frac{1}{6} \cos(6x)) + c$

$= -\frac{1}{4} \cos(2x) - \frac{1}{12} \cos(6x) + c$

d $\int \sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)\ dx$

$= \frac{1}{2}\int \sin(2x) + \sin(x)\ dx$

$= -\frac{1}{2}(\frac{1}{2}\cos(2x) + \cos(x)) + c$

$= -\frac{1}{4}\cos(2x) - \frac{1}{2}\cos(x) + c$

e $\int \cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)\ dx$

$= \frac{1}{2}\int \cos(2x) + \cos(x)\ dx$

$= \frac{1}{2}(\frac{1}{2}\sin(2x) + \sin(x)) + c$

$= \frac{1}{4}\sin(2x) + \frac{1}{2}\sin(x) + c$

f $\int \sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)\ dx$

$= \frac{1}{2}\int \cos(x) - \cos(2x)\ dx$

$= \frac{1}{2}(\sin x - \frac{1}{2}\sin(2x)) + c$

$= -\frac{1}{4}\sin(2x) + \frac{1}{2}\sin(x) + c$

6 **a** $\int_0^{\pi} \cos(2x)\cos\left(\frac{x}{2}\right)\ dx$

$= \frac{1}{2}\int_0^{\pi} \cos\left(\frac{3x}{2}\right) + \cos\left(\frac{5x}{2}\right)\ dx$

$= \frac{1}{2}\left[\frac{2}{3}\sin\left(\frac{3x}{2}\right) + \frac{2}{5}\sin\left(\frac{5x}{2}\right)\right]_0^{\pi}$

$= -\frac{1}{3} + \frac{1}{5}$

$= -\frac{2}{15}$

b $\int_0^{\frac{\pi}{2}} \sin(2x)\cos(6x)\ dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} -\sin 4x + \sin 8x\ dx$

$= \frac{1}{2}\left[\frac{1}{4}\cos 4x - \frac{1}{8}\cos 8x\right]_0^{\frac{\pi}{2}}$

$= 0$

c $\int_0^{\frac{\pi}{2}} \sin(8x)\cos(10x)\ dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} -\sin 2x + \sin 18x\ dx$

$= \frac{1}{2}\left[\frac{1}{2}\cos 2x - \frac{1}{18}\cos 18x\right]_0^{\frac{\pi}{2}}$

$= -\frac{4}{9}$

Solutions to Exercise 9F

1 $$\int \frac{1}{x^2+9}\,dx$$

Let $x = 3\tan u, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$$\frac{dx}{du} = 3\sec^2 u$$

$$\begin{aligned}\int \frac{1}{x^2+9}\,dx &= \int \frac{1}{9\tan^2 u+9} \times 3\sec^2 u\,du \\ &= \int \frac{1}{9(\tan^2 u+1)} \times 3\sec^2 u\,du \\ &= \int \frac{1}{9(\sec^2 u)} \times 3\sec^2 u\,du \\ &= \int \frac{1}{3}\,du \\ &= \frac{u}{3} + c\end{aligned}$$

$$\therefore \int \frac{1}{x^2+9}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$$

2 $$\int \frac{-1}{\sqrt{4-x^2}}\,dx$$

Let $x = 2\cos u, \quad 0 \le u \le \pi$

$$\frac{dx}{du} = -2\sin u$$

$$\begin{aligned}\int \frac{-1}{\sqrt{4-x^2}}\,dx &= \int \frac{-1}{\sqrt{4-4\cos u^2}} \times (-2\sin u)\,du \\ &= \int \frac{-1}{\sqrt{4(1-\cos u^2)}} \times (-2\sin u)\,du \\ &= \int 1\,du \\ &= u + c\end{aligned}$$

$$\therefore \int \frac{-1}{\sqrt{4-x^2}}\,dx = \arccos\left(\frac{x}{2}\right) + c$$

3 $$\int \frac{1}{x+\sqrt{x}}\,dx$$

Let $x = u^2, \quad u > 0$

$$\frac{dx}{du} = 2u$$

$$\begin{aligned}\int \frac{}{x+\sqrt{x}}\,dx &= \int \frac{1}{u^2+u} \times (2u)\,du \\ &= \int \frac{2u}{u^2+u}\,du \\ &= \int \frac{2}{u+1}\,du \\ &= 2\log_e(u+1) + c \\ &= 2\log_e(\sqrt{x}+1) + c\end{aligned}$$

$$\therefore \int \frac{1}{x+\sqrt{x}}\,dx = 2\log_e(\sqrt{x}+1) + c$$

4 $$\int \frac{1}{3\sqrt{x}+4x}\,dx$$

Let $x = u^2, \quad u > 0$

$$\frac{dx}{du} = 2u$$

$$\begin{aligned}\int \frac{1}{3\sqrt{x}+4x}\,dx &= \int \frac{1}{3u+4u^2} \times (2u)\,du \\ &= \int \frac{2u}{3u+4u^2}\,du \\ &= \int \frac{2}{3+4u}\,du \\ &= \frac{1}{2}\log_e(3+4u) + c\end{aligned}$$

$$\begin{aligned}\therefore \int \frac{1}{3\sqrt{x}+4x}\,dx &= \frac{1}{2}\log_e(3+4u) + c \\ &= \frac{1}{2}\log_e(3+4\sqrt{x}) + c\end{aligned}$$

5

$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$

Let $x = 3\sin u, \qquad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$$\frac{dx}{du} = 3\cos u$$

$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \quad \int \frac{1}{\sqrt{9-9\sin^2 u}} \times (3\cos u)\,du$$

$$= \int \frac{1}{\sqrt{9(1-\sin^2 u)}} \times (3\cos u)\,du$$

$$= \int 1\,du$$

$$= u + c$$

$$\therefore \int \frac{1}{\sqrt{9-x^2}}\,dx = \arcsin\left(\frac{x}{3}\right) + c$$

6

$$\int \sqrt{9-x^2}\,dx$$

Let $x = 3\sin u, \qquad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$$\frac{dx}{du} = 3\cos u$$

$$\int \sqrt{9-x^2}\,dx$$

$$= \quad \int \sqrt{9-9\sin^2 u} \times (3\cos u)\,du$$

$$= \int 9\cos^2 u)\,du$$

$$= \frac{9}{2}\int \cos 2u + 1\,du$$

$$= \frac{9}{2}(\frac{1}{2}\sin(2u) + u) + c$$

$$= \frac{9}{2}(\sin(u)\cos(u) + u) + c$$

$$= \frac{9}{2}\sin(\sin^{-1}\frac{x}{3})\cos(\sin^{-1}\frac{x}{3}) + \sin^{-1}\frac{x}{3} + c$$

$$= \frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$$

$$\int \sqrt{9-x^2}\,dx = \frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$$

7 $\displaystyle\int \frac{1}{x(x+\sqrt[3]{x})}\,dx$

Let $x = u^3$, $\quad u > 0$

$$\frac{dx}{du} = 3u^2$$

$$\int \frac{1}{x(x+\sqrt[3]{x})}\,dx$$

$$= \int \frac{1}{u^3(1+u)} \times (3u^2)\,du$$

$$= \int \frac{3}{u(1+u)}\,du$$

$$= \int \frac{3}{u} - \frac{3}{u+1}\,du$$

$$= 3\log_e(u) - 3\log_e(1+u) + c$$

$$= 3\log_e\left(\frac{u}{u+1}\right) + c$$

$$\therefore \int \frac{1}{x(x+\sqrt[3]{x})}\,dx = 3\log_e\left(\frac{x}{(1+\sqrt[3]{x}))^3}\right) + c$$

8 $\displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$

Let $x = \sin u$, $\quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$$\frac{dx}{du} = \cos u$$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$$

$$= \int \frac{1}{(1-\sin^2 u)^{\frac{3}{2}}} \times (\cos u)\,du$$

$$= \int \frac{1}{\cos^2 u}\,du$$

$$= \int \sec^2 u\,du$$

$$= \tan u + c$$

$$= \tan(\arcsin x) + c$$

$$= \frac{x}{\sqrt{1-x^2}}$$

Solutions to Exercise 9G

1 a

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$
$$= \frac{Ax+Bx+2A-B}{(x-1)(x+2)}$$

$$A + B = 5 \quad ①$$
$$2A - B = 1 \quad ②$$

① + ②:

$$3A = 6$$
$$A = 2$$
$$2 + B = 5$$
$$B = 3$$

$$\therefore \quad \frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$$

b $\dfrac{-1}{(x+1)(2x+1)}$

$$= \frac{A}{x+1} + \frac{B}{2x+1}$$
$$= \frac{A(2x+1)+B(x+1)}{(x+1)(2x+1)}$$
$$= \frac{2Ax+Bx+A+B}{(x+1)(2x+1)}$$

$$2A + B = 0 \quad ①$$
$$A + B = -1 \quad ②$$

① − ②:

$$A = 1$$
$$1 + B = -1$$
$$B = -2$$

$$\therefore \quad \frac{-1}{(x+1)(2x+1)} = \frac{1}{x+1} - \frac{2}{2x+1}$$

c

$$\frac{3x-2}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$
$$= \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$
$$= \frac{Ax+Bx-2A+2B}{(x+2)(x-2)}$$

$$A + B = 3$$
$$2A + 2B = 6 \quad ①$$
$$-2A + 2B = -2 \quad ②$$

① + ②:

$$4B = 4$$
$$B = 1$$
$$A + 1 = 3$$
$$A = 2$$

$$\therefore \quad \frac{3x-2}{(x+2)(x-2)} = \frac{2}{x+2} + \frac{1}{x-2}$$

d

$$\frac{4x+7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$
$$= \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}$$
$$= \frac{Ax+Bx-2A+3B}{(x+3)(x-2)}$$

$$A + B = 4$$
$$2A + 2B = 8 \quad ①$$
$$-2A + 3B = 7 \quad ②$$

① + ②:

$$5B = 15$$
$$B = 3$$
$$A + 3 = 4$$
$$A = 1$$

$$\therefore \quad \frac{4x+7}{(x+3)(x-2)} = \frac{1}{x+3} + \frac{3}{x-2}$$

e

$$\frac{7-x}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$$
$$= \frac{Ax+Bx+A-4B}{(x-4)(x+1)}$$

$A + B = -1$ ①

$A - 4B = 7$ ②

① − ②:

$5B = -8$

$B = -\frac{8}{5}$

$A - \frac{8}{5} = -1$

$A = \frac{3}{5}$

$$\therefore \frac{7-x}{(x-4)(x+1)} = \frac{3}{5(x-4)} - \frac{8}{5(x+1)}$$

2 a $$\frac{2x+3}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$
$$= \frac{A(x-3)+B}{(x-3)^2}$$
$$= \frac{Ax-3A+B}{(x-3)^2}$$

$A = 2$

$-3A + B = 3$

$-6 + B = 3$

$B = 9$

$$\therefore \frac{2x+3}{(x-3)^2} = \frac{2}{x-3} + \frac{9}{(x-3)^2}$$

b $$\frac{9}{(1+2x)(1-x)^2}$$
$$= \frac{A}{1+2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$
$$= \frac{A(1-x)^2 + B(1+2x)(1-x) + C(1+2x)}{(1+2x)(1-x)^2}$$
$$= \frac{A - 2Ax + Ax^2 + B + Bx - 2Bx^2 + C + 2Cx}{(1+2x)(1-x)^2}$$

$A - 2B = 0$ ①

$-2A + B + 2C = 0$ ②

$A + B + C = 9$ ③

$2A + 2B + 2C = 18$ ④

④ − ②:

$4A + B = 18$

① × ④: $4A - 8B = 0$

$9B = 18$

$B = 2$

$4A + 2 = 18$

$A = 4$

$4 + 2 + C = 9$

$C = 3$

$$\therefore \frac{9}{(1+2x)(1-x)^2} = \frac{4}{1+2x} + \frac{2}{1-x} + \frac{3}{(1-x)^2}$$

c $$\frac{2x-2}{(x+1)(x-2)^2}$$
$$= \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$
$$= \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}$$
$$= \frac{Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C}{(x+1)(x-2)^2}$$

$$A + B = 0 \quad ①$$
$$-4A - B + C = 2 \quad ②$$
$$4A - 2B + C = -2 \quad ③$$
$$③ - ②: 8A - B = -4 \quad ④$$
$$④ + ①: 9A = -4$$
$$A = -\frac{4}{9}$$
$$A + B = 0$$
$$B = \frac{4}{9}$$
$$4A - 2B + C = -2$$
$$-\frac{16}{9} - \frac{8}{9} + C = -2$$
$$C = -2 + \frac{24}{9} = \frac{2}{3}$$
$$\therefore \quad \frac{2x-2}{(x+1)(x-2)^2} = -\frac{4}{9(x+1)} + \frac{4}{9(x-2)} + \frac{2}{3(x-2)^2}$$

3 a
$$\frac{3x+1}{(x+1)(x^2+x+1)}$$
$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$
$$= \frac{A(x^2+x+1) + (Bx+C)(x+1)}{(x+1)(x^2+x+1)}$$
$$= \frac{Ax^2 + Ax + A + Bx^2 + Bx + Cx + C}{(x+1)(x^2+x+1)}$$

$$A + B = 0 \quad ①$$
$$A + B + C = 3 \quad ②$$
$$A + C = 1 \quad ③$$
$$② - ①: C = 3$$
$$A + 3 = 1$$
$$A = -2$$
$$A + B + C = 3$$
$$-2 + B + 3 = 3$$
$$B = 2$$
$$\therefore \quad \frac{3x+1}{(x+1)(x^2+x+1)} = -\frac{2}{x+1} + \frac{2x+3}{x^2+x+1}$$

b
$$\frac{3x^2+2x+5}{(x^2+2)(x+1)}$$
$$= \frac{Ax+B}{x^2+2} + \frac{C}{x+1}$$
$$= \frac{(Ax+B)(x+1) + C(x^2+2)}{(x^2+2)(x+1)}$$
$$= \frac{Ax^2 + Ax + Bx + B + Cx^2 + 2C}{(x^2+2)(x+1)}$$
$$A + C = 3 \quad ①$$
$$A + B = 2 \quad ②$$
$$B + 2C = 5 \quad ③$$
$$① - ②:$$
$$C - B = 1 \quad ④$$
$$③ + ④:$$
$$3C = 6$$
$$C = 2$$
$$A + 2 = 3$$

$$A = 1$$

$$1 + B = 2$$

$$B = 1$$

$$\therefore \quad \frac{3x^2 + 2x + 5}{(x^2 + 2)(x + 1)} = \frac{x + 1}{x^2 + 2} + \frac{2}{x + 1}$$

c Factorise the denominator:

$$2x^3 + 6x^2 + 2x + 6$$

$$= 2x^2(x + 3) + 2(x + 3)$$

$$= 2(x^2 + 1)(x + 3)$$

The 2 factor can be put with either fraction.

$$\frac{x^2 + 2x - 13}{2(x^2 + 1)(x + 3)}$$

$$= \frac{Ax + B}{x^2 + 1} + \frac{C}{2(x + 3)}$$

$$= \frac{2(Ax + B)(x + 3) + C(x^2 + 1)}{2(x^2 + 1)(x + 3)}$$

$$= \frac{2Ax^2 + 6Ax + 2Bx + 6B + Cx^2 + C}{2(x^2 + 1)(x + 3)}$$

$2A + C = 1$ ①

$6A + 2B = 2$

$9A + 3B = 3$ ②

$6B + C = -13$ ③

① − ③:

$2A - 6B = 14$

$A - 3B = 7$ ④

② + ④:

$$10A = 10$$

$$A = 1$$

$$2 + C = 1$$

$$C = -1$$

$$3A + B = 1$$

$$A + B = 1$$

$$B = -2$$

$$\therefore \quad \frac{x^2 + 2x - 13}{2(x^2 + 1)(x + 3)} = \frac{x - 2}{x^2 + 1} - \frac{1}{2(x + 3)}$$

4 $(x - 1)(x - 2) = x^2 - 3x + 2$

First divide:

$$3x^2 - 4x - 2 = 3(x^2 - 3x + 2) + 5x - 8$$

$$\frac{3x^2 - 4x - 2}{(x - 1)(x - 2)} = \frac{5x - 8}{(x - 1)(x - 2)}$$

$$\frac{5x - 8}{(x - 1)(x - 2)} = \frac{A}{x - 1} + \frac{B}{x - 2}$$

$$= \frac{A(x - 2) + B(x - 1)}{(x - 1)(x - 2)}$$

$$= \frac{Ax + Bx - 2A - B}{(x - 1)(x - 2)}$$

$A + B = 5$ ①

$-2A - B = -8$ ②

① + ②:

$$-A = -3$$

$$A = 3$$

$$3 + B = 5$$

$$B = 2$$

$$\therefore \quad \frac{5x - 8}{(x - 1)(x - 2)} = \frac{3}{x - 1} + \frac{2}{x - 2}$$

Use the previous working:

$$\frac{3x^2 - 4x - 2}{(x - 1)(x - 2)} = 3 + \frac{3}{x - 1} + \frac{2}{x - 2}$$

5 $\dfrac{9}{(x-10)(x-1)} = \dfrac{1}{x-10} - \dfrac{1}{x-1}$

$\displaystyle\int \frac{9}{(x-10)(x-1)}\,dx$

$= \displaystyle\int \frac{1}{x-10} - \frac{1}{x-1}\,dx$

$= \log_e|x-10| - \log_e|x-1| + c$

$= \log_e \dfrac{|x-10|}{|x-1|} + c$

6 $\dfrac{x^4+1}{(x+2)^2} = x^2 - 4x + 12 - \dfrac{32x+47}{(x+2)^2}$

Consider$-\dfrac{32x+47}{(x+2)^2} = \dfrac{a}{x+2} + \dfrac{b}{(x+2)^2}$

Therefore $-32x + 47 = a((x+2) = b$

$\therefore b = 17$ and $a = -32$

$\displaystyle\int \frac{x^4+1}{(x+2)^2}\,dx$

$= \displaystyle\int x^2 - 4x + 12 - \frac{32}{x+2} + \frac{17}{(x+2)^2}\,dx$

$= \dfrac{1}{3}x^3 - 2x^2 + 12x - \dfrac{17}{x+2} - 32\log_e|x+2| + c$

7 $\dfrac{7x+1}{(x+2)^2)} = \dfrac{7}{x+2} - \dfrac{13}{(x+2)^2}$

$\displaystyle\int \frac{7x+1}{(x+2)^2)}\,dx = \int \frac{7}{x+2} - \frac{13}{(x+2)^2}\,dx$

$= 7\log_e|x+2| + \dfrac{13}{x+2} + c$

8 $\dfrac{5}{(x^2+2)(x-4))}$

$= \dfrac{-5x-20}{18(x^2+2)} + \dfrac{5}{18(x-4)}$

$\displaystyle\int \frac{5}{(x^2+2)(x-4))}\,dx$

$= \displaystyle\int \frac{-5(x+4)}{18(x^2+2)} + \frac{5}{18(x-4)}\,dx$

$= \dfrac{5}{18}\displaystyle\int \frac{1}{x-4} - \frac{x+4}{x^2+2}\,dx$

$= \dfrac{5}{18}\displaystyle\int \frac{1}{x-4} - \frac{x}{x^2+2} - \frac{4}{x^2+2}\,dx$

$= \dfrac{5}{18}\Bigg(\log_e|x-4| - \dfrac{1}{2}\log_e(x^2+2)$

$- 2\sqrt{2}\arctan\left(\dfrac{\sqrt{2}x}{2}\right)\Bigg)$

9 a To decompose $\dfrac{7}{(x-2)(x+5)}$ into partial fractions, find A and B such that

$\dfrac{7}{(x-2)(x+5)} = \dfrac{A}{x-2} + \dfrac{B}{x+5}$

$\therefore\ 7 = A(x+5) + B(x-2)$

When $x = 2, A = 1$

When $x = -5, B = -1$

$\therefore\ \dfrac{7}{(x-2)(x+5)} = \dfrac{1}{x-2} - \dfrac{1}{x+5}$

$\displaystyle\int \frac{7}{(x-2)(x+5)}dx$

$= \displaystyle\int \frac{dx}{x-2} - \int \frac{dx}{x+5}$

$= \log_e|x-2| - \log_e|x+5| + c$

$= \log_e\left|\dfrac{x-2}{x+5}\right| + c$

b To decompose $\dfrac{x+3}{x^2-3x+2}$ into partial fractions, factorise $x^2 - 3x + 2$

$x^2 - 3x + 2 = (x-1)(x-2)$

Find A and B such that

$$\frac{x+3}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$\therefore\ x + 3 = A(x-2) + B(x-1)$

When $x = 2, B = 5$

When $x = 1, A = -4$

$$\therefore\ \frac{x+3}{(x-1)(x-2)} = \frac{-4}{(x-1)} + \frac{5}{(x-2)}$$

$$\int \frac{x+3}{x^2-3x+2}dx$$

$$= -4\int \frac{dx}{x-1} + 5\int \frac{dx}{x-2}$$

$$= -4\log_e|x-1| + 5\log_e|x-2| + c$$

$$= \log_e\left|\frac{(x-2)^5}{(x-1)^4}\right| + c$$

c $$\frac{2x+1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

$\therefore\ 2x + 1 = A(x-1) + B(x+1)$

When $x = 1, B = \dfrac{3}{2}$

When $x = -1, A = \dfrac{1}{2}$

$$\int \frac{2x+1}{(x+1)(x-1)}dx$$

$$= \frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{dx}{x-1}$$

$$= \frac{1}{2}\log_e|x+1| + \frac{3}{2}\log_e|x-1| + c$$

$$= \frac{1}{2}\log_e|(x+1)(x-1)^3| + c$$

d $$\frac{2x^2}{x^2-1} = \frac{2(x^2-1)+2}{x^2-1}$$

$$= 2 + \frac{2}{(x-1)(x+1)}$$

Find A and B such that

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$\therefore\ 2 = A(x+1) + B(x-1)$

When $x = 1, A = 1$

When $x = -1, B = -1$

$$\therefore\ \frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

CAS:

$$\int \frac{2x^2}{x^2-1}dx$$

$$= 2\int dx + \int \frac{dx}{x-1} - \int \frac{dx}{x+1}$$

$$= 2x + \log_e|x-1| - \log_e|x+1| + c$$

$$= 2x + \log_e\left|\frac{x-1}{x+1}\right| + c$$

e Since $x^2 + 4x + 4 = (x+2)^2$

$$\frac{2x+1}{x^2+4x+4} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$\therefore\ 2x + 1 = A(x+2) + B$

When $x = -2, B = -3$

When $x = 1, 3 = 3A - 3$

$A = 2$

$$\therefore\ \frac{2x+1}{x^2+4x+4} = \frac{2}{x+2} - \frac{3}{(x+2)^2}$$

$$\int \frac{2x+1}{x^2+4x+4}dx$$

$$= \int \frac{2}{x+2} - \frac{3}{(x+2)^2}dx$$

$$= 2\log_e|x+2| + \frac{3}{x+2} + c$$

f $$\frac{4x-2}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}$$

$\therefore\ 4x - 2 = A(x+4) + B(x-2)$

When $x = 2, A = 1$

When $x = -4, B = 3$

$$\int \frac{4x-2}{(x-2)(x+4)}dx$$
$$= \int \frac{dx}{x-2} + 3\int \frac{dx}{x+4}$$
$$= \log_e|x-2| + 3\log_e|x+4| + c$$
$$= \log_e|(x-2)(x+4)^3| + c$$

10 **a** Since $x^2 - 5x + 6 = (x-2)(x-3)$

$$\frac{2x-3}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$
$\therefore\ 2x - 3 = A(x-3) + B(x-2)$

When $x = 2,\ A = -1$

When $x = 3,\ B = 3$

$$\therefore\ \frac{2x-3}{x^2-5x+6} = -\frac{1}{x-2} + \frac{3}{x-3}$$
$$\int \frac{2x-3}{x^2-5x+6}dx$$
$$= -\int \frac{dx}{x-2} + 3\int \frac{dx}{x-3}$$
$$= -\log_e|x-2| + 3\log_e|x-3| + c$$
$$= \log_e\left|\frac{(x-3)^3}{x-2}\right| + c$$

b $$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$\therefore\ 5x + 1 = A(x+2) + B(x-1)$

When $x = 1, A = 2$

When $x = -2, B = 3$

$$\int \frac{5x+1}{(x-1)(x+2)}dx$$
$$= \int \frac{2}{x-1}dx + \int \frac{3}{x+2}dx$$
$$= 2\log_e|x-1| + 3\log_e|x+2| + c$$
$$= \log_e|(x-1)^2(x+2)^3| + c$$

c Dividing through

$$\begin{array}{r} x-2 \\ x^2-4\,\overline{)\,x^3-2x^2-3x+9} \\ \underline{x^3 \qquad\quad -4x} \\ -2x^2+x+9 \\ \underline{-2x^2 \qquad +8} \\ x+1 \end{array}$$

$$\therefore\ \frac{x^3-2x^2-3x+9}{x^2-4} = x - 2 + \frac{x+1}{x^2-4}$$
$$\frac{x+1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$
$\therefore\ x + 1 = A(x+2) + B(x-2)$

When $x = 2, A = \frac{3}{4}$

When $x = -2, B = \frac{1}{4}$

$$\therefore\ \frac{x^3-2x^2-3x+9}{x^2-4}$$
$$= x - 2 + \frac{3}{4(x-2)} + \frac{1}{4(x+2)}$$
$$\int \frac{x^3-2x^2-3x+9}{x^2-4}dx$$
$$= \int (x-2)dx + \frac{3}{4}\int \frac{dx}{x-2} + \frac{1}{4}\int \frac{dx}{x+2}$$
$$= \frac{x^2}{2} - 2x + \frac{3}{4}\log_e|x-2| + \frac{1}{4}\log_e|x+2| + c$$
$$= \frac{x^2}{2} - 2x + \frac{1}{4}\log e|(x+2) \times (x-2)^3| + c$$
$$= \frac{x^2}{2} - 2x + \log_e\left|(x+2)^{\frac{1}{4}} \times (x-2)^{\frac{3}{4}}\right| + c$$

d Since $x^2 + 5x + 4 = (x+1)(x+4)$

$$\frac{4x+10}{x^2+5x+4} \equiv \frac{A}{x+1} + \frac{B}{x+4}$$

$\therefore\ 4x + 10 = A(x + 4) + B(x + 1)$
When $x = -1, A = 2$
When $x = -4, B = 2$

$$\int \frac{4x + 10}{x^2 + 5x + 4} dx$$
$$= 2\int \frac{dx}{x + 1} + 2\int \frac{dx}{x + 4}$$
$$= 2\log_e|x + 1| + 2\log_e|x + 4| + c$$
$$= \log_e((x + 1)^2(x + 4)^2) + c$$

Alternate solution:
Let $x^2 + 5x + 4 = u$
Then $\dfrac{du}{dx} = 2x + 5$

$$2\int \frac{du}{u} = 2\log_e|u| + c$$
$$= 2\log_e|x^2 + 5x + 4| + c$$
$$= 2\log_e|(x + 1)(x + 4)| + c$$
$$= \log_e((x + 1)^2(x + 4)^2) + c$$

e Dividing through

$$\begin{array}{r|l} & x^2 - x - 1 \\ x + 2 & \overline{x^3 + x^2 - 3x + 3} \\ & \underline{x^3 + 2x^2} \\ & \quad - x^2 - 3x \\ & \quad \underline{-x^2 - 2x} \\ & \qquad - x + 3 \\ & \qquad \underline{-x - 2} \\ & \qquad\quad 5 \end{array}$$

$\therefore\ \dfrac{x^3 + x^2 - 3x + 3}{x + 2} =$
$x^2 - x - 1 + \dfrac{5}{x + 2}$

$$\int \frac{x^3 + x^2 - 3x + 3}{x + 2} dx$$
$$= \int (x^2 - x - 1)dx + 5\int \frac{dx}{x + 2}$$
$$= \frac{x^3}{3} - \frac{x^2}{2} - x + 5\log_e|x + 2| + c$$

f Dividing through

$$\begin{array}{r|l} & x + 1 \\ x^2 - x & \overline{x^3 + 3} \\ & \underline{x^3 - x^2} \\ & \quad x^2 + 3 \\ & \quad \underline{x^2 - x} \\ & \qquad x + 3 \end{array}$$

$\therefore\ fracx^3 + 3x^2 - x = x + 1 + \dfrac{x + 3}{x^2 - x}$

$$\frac{x^3 + 3}{x^2 - x} = \frac{A}{x - 1} + \frac{B}{x}$$

$x + 3 = Ax + B(x - 1)$
When $x = 0, B = -3$
When $x = 1, A = 4$

$$\int \frac{x^3 + 3}{x^2 - x} dx$$
$$= \int (x + 1)dx + 4\int \frac{dx}{x - 1}$$
$$- 3\int \frac{dx}{x}$$
$$= \frac{x^2}{2} + x + 4\log_e|x - 1|$$
$$- 3\log_e|x| + c$$
$$= \frac{x^2}{2} + x + \log_e\left|\frac{(x - 1)^4}{x^3}\right| + c$$

11 a $\dfrac{3x}{(x + 1)(x^2 + 2)} = -\dfrac{1}{x + 1} + \dfrac{x + 2}{x^2 + 2}$

$$\int \frac{3x}{(x + 1)(x^2 + 2)} dx$$
$$= \int -\frac{1}{x + 1} + \frac{x + 2}{x^2 + 2} dx$$
$$= \int -\frac{1}{x + 1} + \frac{x}{x^2 + 2} + \frac{2}{x^2 + 2} dx$$
$$= \frac{1}{2}\log_e(x^2 + 2) + \sqrt{2}\arctan\left(\frac{\sqrt{2}x}{2}\right) - \log_e|x + 1| + c$$
$$= \log_e\left(\frac{\sqrt{x^2 + 2}}{|x + 1|}\right) + \sqrt{2}\arctan\left(\frac{\sqrt{2}x}{2}\right) + c$$

b $\dfrac{2}{(x+1)^2(x^2+1)} =$

$\dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} - \dfrac{x}{x^2+1}$

$\displaystyle\int \frac{2}{(x+1)^2(x^2+1)}\,dx$

$\displaystyle = \int \frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{x}{x^2+1}\,dx$

$= \log_e(|x+1) - -\log_e \sqrt{x^2+1} - \dfrac{1}{x+1} + c$

$= \log_e\left(\dfrac{|x+1|}{\sqrt{x^2+1}}\right) - \dfrac{1}{x+1} + c$

c $\dfrac{5x^3}{(x-1)(x^2+4)} = 5 + \dfrac{1}{x-1} + \dfrac{4x-16}{x^2+4}$

$\displaystyle\int \frac{5x^3}{(x-1)(x^2+4)}\,dx$

$\displaystyle = \int 5 + \frac{1}{x-1} + \frac{4x-16}{x^2+4}\,dx$

$\displaystyle = \int 5 + \frac{1}{x-1} + \frac{4x}{x^2+4} - \frac{16}{x^2+4}\,dx$

$= 5x + \log_e|x-1| + 2\log_e(x^2+4)$

$- 8\tan^{-1}\dfrac{x}{2} + c$

$= 5x + \log_e(|x-1|(x^2+4)^2)) - 8\tan^{-1}\dfrac{x}{2} + c$

d $\displaystyle\int \frac{16(4x+1)}{(x-2)^2(x^2+4)}\,dx$

$\displaystyle = \int -\frac{1}{x-2} + \frac{18}{(x-2)^2} + \frac{x-16}{x^2+4}\,dx$

$\displaystyle = \int -\frac{1}{x-2} + \frac{18}{(x-2)^2}$

$+ \dfrac{x}{x^2+4} - \dfrac{16}{x^2+4}\,dx$

$= -\log_e|x-2| - \dfrac{18}{x-2}$

$+ \dfrac{1}{2}\log_e(x^2+4) - 8\tan^{-1}\left(\dfrac{x}{2}\right) + c$

e

$\displaystyle\int \frac{24(x+2)}{(x+2)^2(x^2+2)}\,dx$

$\displaystyle = \int \frac{4}{x+2} + \frac{8-4x}{(x^2+2)}\,dx$

$\displaystyle = \int \frac{4}{x+2} + \frac{8}{(x^2+2)} - \frac{4x}{(x^2+2)}\,dx$

$= 4\log_e|x+2| - \log_e(x^2+2)^2 + 4\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{2}x}{2}\right) + c$

f $\dfrac{1}{2}\left(\log_e\left|\dfrac{x-1}{x+1}\right|\right) + \dfrac{3x^2+9x+10}{3(x+1)^3}$

12 a $\displaystyle\int_1^2 \frac{1}{x(x+1)}dx$

$\dfrac{1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$

$1 = A(x+1) + Bx$

When $x = 0, A = 1$

When $x = -1, B = -1$

$\displaystyle\therefore \int_1^2 \frac{1}{x(x+1)}dx$

$\displaystyle = \int_1^2 \frac{dx}{x} - \int_1^2 \frac{dx}{x+1}$

$= [\log_e|x| - \log_e|x+1|]_1^2$

$= \left[\log_e\left|\dfrac{x}{x+1}\right|\right]_1^2$

$= \log_e\dfrac{2}{3} - \log_e\dfrac{1}{2}$

$= \log_e\dfrac{4}{3}$

b $\int_0^1 \frac{1}{(x+1)(x+2)}dx$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$\therefore\ 1 = A(x+2) + B(x+1)$

When $x = -1, A = 1$

When $x = -2, B = -1$

$$\therefore \int_0^1 \frac{dx}{(x+1)(x+2)}$$

$$= \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{x+2}$$

$$= [\log_e|x+1| - \log_e|x+2|]_0^1$$

$$= \left[\log_e\left|\frac{x+1}{x+2}\right|\right]_0^1$$

$$= \log_e \frac{2}{3} - \log_e \frac{1}{2}$$

$$= \log_e \frac{4}{3}$$

c $\int_2^3 \frac{x-2}{(x-1)(x+2)}dx$

$$\frac{x-2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$\therefore\ x - 2 = A(x+2) + B(x-1)$

When $x = -2, -3B = -4$

$$\therefore\ B = \frac{4}{3}$$

When $x = 1, 3A = -1$

$$\therefore\ A = -\frac{1}{3}$$

$$\therefore \int_2^3 \frac{x-2}{(x-1)(x+2)}dx$$

$$= \int_2^3 -\frac{1}{3(x-1)} + \frac{4}{3(x+2)}dx$$

$$= \left[-\frac{1}{3}\log_e|x-1| + \frac{4}{3}\log_e|x+2|\right]_2^3$$

$$= \left(-\frac{1}{3}\log_e 2 + \frac{4}{3}\log_e 5\right)$$

$$-\left(-\frac{1}{3}\log_e 1 + \frac{4}{3}\log_e 4\right)$$

$$= \log_e 5^{\frac{4}{3}} - \log_e 2^{\frac{1}{3}} - \log_e 4^{\frac{4}{3}}$$

$$= \frac{1}{3}\log_e 5^4 - \frac{1}{3}\log_e 2 - \frac{1}{3}\log_e 4$$

$$= \frac{1}{3}\log_e\left(\frac{5^4}{2 \times 4^4}\right)$$

$$= \frac{1}{3}\log_e \frac{625}{512}$$

d $\frac{x^2}{x^2+3x+2} = \frac{(x^2+3x+2) - 3x - 2}{x^2+3x+2}$

$$= 1 - \frac{3x+2}{(x+1)(x+2)}$$

$$\frac{3x+2}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$3x + 2 = A(x+2) + B(x+1)$

When $x = -1, A = -1$

When $x = -2, B = 4$

$$\therefore \int_0^1 \frac{x^2}{x^2+3x+2}dx$$

$$= \int_0^1 dx + \int_0^1 \frac{dx}{x+1} - 4\int_0^1 \frac{dx}{x+2}$$

$$= [x]_0^1 + [\log_e|x+1|]_0^1$$

$$- 4[\log_e|x+2|]_0^1$$

$$= 1 + \log_e 2 - 4\log_e 3 + 4\log_e 2$$

$$= 1 + \log_e \frac{2^5}{3^4}$$

$$= 1 + \log_e \frac{32}{81}$$

e $\dfrac{x+7}{(x+3)(x-1)} = \dfrac{A}{x+3} + \dfrac{B}{x-1}$

$x+7 = A(x-1) + B(x+3)$

When $x = 1, B = 2$

When $x = -3, A = -1$

$$\therefore \int_2^3 \frac{x+7}{(x+3)(x-1)}dx$$
$$= -\int_2^3 \frac{dx}{x+3} + 2\int_2^3 \frac{dx}{x-1}$$
$$= [-\log_e|x+3| + 2\log_e|x-1|]_2^3$$
$$= \left[\log_e\left|\frac{(x-1)^2}{x+3}\right|\right]_2^3$$
$$= \log_e\frac{4}{6} - \log_e\frac{1}{5}$$
$$= \log_e\frac{10}{3}$$

f $\dfrac{2x+6}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$\therefore\ 2x+6 = A(x-1) + B$

When $x = 1, B = 8$

When $x = 0, 6 = -A + 8$

$\therefore\ A = 2$

$\dfrac{2x+6}{(x-1)^2} = \dfrac{2}{x-1} + \dfrac{8}{(x-1)^2}$

$$\therefore \int_2^3 \frac{2x+6}{(x-1)^2}dx$$
$$= 2\int_2^3 \frac{1}{x-1}dx + 8\int_2^3 \frac{dx}{(x-1)^2}$$
$$= \left[2\log_e|x-1| - \frac{8}{x-1}\right]_2^3$$
$$= \log_e 4 - 4 + 8$$
$$= \log_e 4 + 4$$

g $\dfrac{x+2}{x(x+4)} = \dfrac{A}{x} + \dfrac{B}{x+4}$

$x+2 = A(x+4) + Bx$

When $x = 0, A = \dfrac{1}{2}$

When $x = -4, B = \dfrac{1}{2}$

$$\therefore \int_2^3 \frac{x+2}{x(x+4)}dx$$
$$= \frac{1}{2}\int_2^3 \frac{dx}{x} + \frac{1}{2}\int_2^3 \frac{dx}{x+4}$$
$$= \left[\frac{1}{2}(\log_e|x| + \log_e|x+4|)\right]_2^3$$
$$= [\log_e \sqrt{x(x+4)}]_2^3$$
$$= \log_e\sqrt{\frac{21}{12}}$$
$$= \log_e\left(\frac{\sqrt{7}}{2}\right) \approx 0.28$$

h $$\int_0^1 \frac{1-4x}{3+x-2x^2}dx$$
$$= \int_0^1 \frac{(4x-1)dx}{2x^2-x-3}$$
$$= \int_0^1 \frac{(4x-1)dx}{2\left(x^2-\frac{1}{2}x+\frac{1}{16}\right)-\frac{25}{8}}$$
$$= \int_0^1 \frac{4\left(x-\frac{1}{4}\right)dx}{2\left(\left(x-\frac{1}{4}\right)^2-\frac{25}{16}\right)}$$
$$= 2\int_0^1 \frac{\left(x-\frac{1}{4}\right)dx}{\left(x-\frac{1}{4}-\frac{5}{4}\right)\left(x-\frac{1}{4}+\frac{5}{4}\right)}$$
$$= 2\int_0^1 \frac{\left(x-\frac{1}{4}\right)dx}{\left(x-\frac{3}{2}\right)(x+1)}$$
$$= \int_0^1 \frac{4x-1}{(2x-3)(x+1)}dx$$

$\dfrac{4x-1}{(2x-3)(x+1)} = \dfrac{A}{(2x-3)} + \dfrac{B}{x+1}$

$4x-1 = A(x+1) + B(2x-3)$

When $x = -1, B = 1$

When $x = \frac{3}{2}, A = 2$

$$\therefore \int_0^1 \frac{1-4x}{3+x-2x^2}dx$$

$$= 2\int_0^1 \frac{dx}{2x-3} + \int_0^1 \frac{dx}{x+1}$$

$$= [\log_e|2x-3| + \log_e|x+1|]_0^1$$

$$= \log_e|-1| + \log_e 2 - \log_e|-3|$$

$$- \log_e 1$$

$$= \log_e \frac{2}{3}$$

i $\frac{1}{x(x-4)} \equiv \frac{A}{x} + \frac{B}{x-4}$

$\therefore\ 1 = A(x-4) + Bx$

When $x = 0, A = -\frac{1}{4}$

When $x = 4, B = \frac{1}{4}$

$$\frac{1}{x(x-4)} = -\frac{1}{4x} + \frac{1}{4(x-4)}$$

$$\therefore \int_1^2 \frac{1}{x(x-4)}dx$$

$$= -\frac{1}{4}\int_1^2 \frac{1}{x}dx + \frac{1}{4}\int_1^2 \frac{1}{x-4}dx$$

$$= \frac{1}{4}[\log_e|x-4| - \log_e|x|]_1^2$$

$$= \frac{1}{4}[\log_e|-2| - \log_e 2 - \log_e|-3|$$

$$+ \log_e 1]$$

$$= -\frac{1}{4}\log_e 3$$

$$= \frac{1}{4}\log_e \frac{1}{3}$$

j $\frac{1-4x}{(x+6)(x+1)} \equiv \frac{A}{x+6} + \frac{B}{x+1}$

$\therefore\ 1-4x = A(x+1) + B(x+6)$

When $x = -1, B = 1$

When $x = -6, A = -5$

$$\frac{1-4x}{(x+6)(x+1)} = -\frac{5}{x+6} + \frac{1}{x+1}$$

$$\therefore \int_{-3}^{-2} \frac{1-4x}{(x+6)(x+1)}dx$$

$$= -5\int_{-3}^{-2} \frac{1}{x+6}dx + \int_{-3}^{-2} \frac{1}{x+1}dx$$

$$= \Big[-5\log_e|x+6| + \log_e|x+1|\Big]_{-3}^{-2}$$

$$= -5\log_e 4 + \log_e|-1| + 5\log_e 3$$

$$- \log_e|-2|$$

$$= 5\log_e \frac{3}{4} - \log_e 2$$

k $\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{)x+2)(x^2+3)^2}$

$$= \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

$$\int_0^1 \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}\ dx$$

$$= \Big[\log_e[(x^2+3)|x+2|] - \frac{2}{x^2+3}\Big]_0^1$$

$$= (\log_e 12 - \frac{2}{4} - (\log_e 6) - \frac{2}{3})$$

$$= \log_e \frac{12}{6} - \frac{1}{2} + \frac{2}{3}$$

$$= \log_e 2 + \frac{1}{6}$$

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$$\int_1^0 \frac{10}{(x+1)(x^2+1)}\,dx$$

$$= \int_1^0 -\frac{5}{x+1}dx + \int_0^1 \frac{5x+5}{x^{+}1}\,dx$$

$$= \left[-5\log_e|x+1| + \frac{5}{2}\log_e(x^2+1) + 5\tan^{-1}x\right]_0^1$$

$$= -5\log_e 2 + \frac{5}{2}\log_e 2 + 5\tan^{-1}1 - (-5\log_e 1$$

$$+ \frac{5}{2}\log_e 1 + 5\tan^{-1}0$$

$$= -\frac{5}{2}\log_e 2 + \frac{5\pi}{4} - 0$$

$$= \frac{5\pi}{4} - \frac{5}{2}\log_e 2$$

b $$\int_0^{\sqrt{3}} \frac{x^3-8}{(x-1)(x^2+1)}\,dx$$

$$= \int_0^{\sqrt{3}} \frac{(x-2)(x^2+2x+4)}{(x-2)(x^2+1)}\,dx$$

$$= \int_0^{\sqrt{3}} \frac{x^2+2x+4}{x^2+1}\,dx$$

$$= \int_0^{\sqrt{3}} 1 + \frac{2x+3}{x^2+1}\,dx$$

$$= \left[x + \log_e(x^2+1) + 3\tan^{-1}x\right]_0^{\sqrt{3}}$$

$$= \sqrt{3} + \log_e 4 + \pi$$

c $$\int_0^1 \frac{x^2-1}{x^2+1}dx$$

$$= \int_0^1 1 - \frac{2}{x^2+1}\,dx$$

$$= \left[x - 2\tan^{-1}x\right]_0^1$$

$$= 1 - \frac{\pi}{2}$$

$$= \frac{2-\pi}{2}$$

d

$$\int_{-\frac{1}{2}}^1 \frac{6}{(x^2+x+1)(x-1)}dx$$

$$= \int_{-\frac{1}{2}}^1 \left(\frac{-2x-4}{x^2+x+1} + \frac{2}{x-1}\right)dx$$

$$= \int_{-\frac{1}{2}}^1 \frac{-4}{(x+\frac{1}{2})^2+\frac{3}{4}} - \frac{-2x}{x^2+x+1} + \frac{2}{x-1}\,dx$$

$$= \int_{-\frac{1}{2}}^1 \frac{-4}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$-\left(\frac{2x+1}{x^2+x+1} - \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\right) + \frac{2}{x-1}\,dx$$

$$= \int_{-\frac{1}{2}}^1 \frac{-3}{(x+\frac{1}{2})^2+\frac{3}{4}} - \frac{2x+1}{x^2+x+1} + \frac{2}{x-1}\,dx$$

$$= \Big[-2\sqrt{3}\tan^{-1}\left(\tfrac{\sqrt{3}}{3}(2x+1)\right) - \log_e(x^2+x+1)$$

$$+ 2\log_e|x-1|\Big]_{-\frac{1}{2}}^0$$

$$= -\frac{\sqrt{3}\pi}{3} - \log_e 3$$

14 $$f(x) = \frac{x^2+6x+5}{(x-2)(x^2+x+1)}$$

a Let $f(x) = \dfrac{a}{x-2} + \dfrac{bx+c}{x^2+x+1}$

$\therefore x^2+6x+5$

$= a(x^2+x+1) + (bx+c)(x-2)$

Let $x = 2, \therefore 21 = 7a \Rightarrow a = 3$

Let $x = 0, \therefore 5 = 3 - 2c \Rightarrow c = -1$

Let $x = 1$,

$\therefore 12 = 9 + (b-1)(-3) \Rightarrow b = 0$

Therefore $f(x) = \dfrac{3}{x-2} - \dfrac{2x+1}{x^2+x+1}$

b $\int f(x)\,dx = \int \dfrac{3}{x-2} - \dfrac{2x+1}{x^2+x+1}\,dx$

$= 3\log_e|x-2| - \log_e(x^2+x+1) + c$

c $\int_{-2}^{-1} f(x)\,dx = \Big[3\log_e|x-2| -$

$$\log_e(x^2 + x + 1)\Big]_{-2}^{-1}$$
$$= 2\log_e\left(\frac{9}{8}\right)$$

15 a
$$\text{RHS} = \frac{\cos x}{(1 - \sin x)(1 + \sin x)}$$
$$= \frac{\cos x}{1 - \sin^2 x}$$
$$= \frac{\cos x}{\cos^2 x}$$
$$= \frac{1}{\cos x}$$
$$= \sec x$$
$$= \text{LHS}$$

b
$$\int \sec x\, dx = \int \frac{\cos x}{(1 - \sin x)(1 + \sin x)}\, dx$$

Let $u = \sin x$, then $\dfrac{du}{dx} = \cos x$

$$\int \frac{\cos x}{(1 - \sin x)(1 + \sin x)}\, dx = \int \frac{1}{(1 - u)(1 + u)}\, du$$
$$= \int \frac{1}{(1 - u)(1 + u)}\, du = \frac{1}{2}\int \frac{1}{1 - u} + \frac{1}{1 + u}\, du$$
$$= \frac{1}{2}(-\log_e|1 - u| + \log_e|1 + u|) + c$$
$$= \frac{1}{2}\log_e\left(\frac{|1 + u|}{|1 - u|}\right) + c$$
$$= \frac{1}{2}\log_e\left(\frac{|1 + \sin x|}{|1 - \sin x|}\right) + c$$

Divide numerator and denominator by $\cos x$

$$= \frac{1}{2}\log_e\left(\frac{|\sec x + \tan x|}{|\sec x - \tan x|}\right) + c$$

Multiply numerator and denominator by $|\sec x + \tan x|$

$$= \frac{1}{2}\log_e\left(\frac{|\sec x + \tan x|^2}{|\sec^2 x - \tan^2 x|}\right) + c$$
$$= \log_e|\sec x + \tan x| + c$$

16 a i
$$\frac{2t}{1 + t^2} = \frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$
$$= \frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \div \sec^2\left(\frac{x}{2}\right)$$
$$= \frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \times \cos^2\left(\frac{x}{2}\right)$$
$$= 2\sin \div \sec^2\left(\frac{x}{2}\right)\cos \div \sec^2\left(\frac{x}{2}\right)$$
$$= \sin x$$

ii
$$\frac{1 - t^2}{1 + t^2} = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$
$$= \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} \div \sec^2\left(\frac{x}{2}\right)$$
$$= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$
$$= \cos x$$

b
$$\int \frac{1}{\cos x}\, dx = \int \frac{1 + t^2}{1 - t^2}\, dx$$
$$\frac{dx}{dt} = \frac{2}{1 + t^2}$$
$$= 2\int \frac{1 + t^2}{1 - t^2} \times \frac{1}{1 + t^2}\, dt$$
$$= 2\int \frac{1}{1 - t^2}\, dt$$
$$= \int \frac{1}{1 + t} + \frac{1}{1 - t}\, dt$$
$$= \log_e\left|\frac{1 + t}{1 - t}\right| + c$$

c

$$\int \frac{\sin x}{1 + \sin x}\, dx = \int (\frac{2t}{1 + t^2}) \div (1 + \frac{2t}{1 + t^2})\, dx$$

$$= \int (\frac{2t}{1 + t^2}) \div (\frac{(1 + t)^2}{1 + t^2})\, dx$$

$$= \int (\frac{2t}{(1 + t)^2})\, dx$$

$$\frac{dx}{dt} = \frac{2}{1 + t^2}$$

$$= 2\int \frac{2t}{(1 + t)^2(1 + t^2)}\, dt$$

$$= 2\int -\frac{1}{(1 + t)^2} + \frac{1}{(1 + t^2)}\, dt$$

$$= 2\tan^{-1} t + \frac{2}{1 + t} + c$$

Solutions to Exercise 9H

1 **a** Let $u = x$ and $\frac{dv}{dx} = e^{-x}$.

Then $\frac{du}{dx} = 1$ and $v = -e^{-x}$

So
$$\begin{aligned}\int xe^{-x}\,dx &= \int u\frac{dv}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx \\ &= -xe^{-x} - \int -e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + c \\ &= e^{-x}(-x-1) + c\end{aligned}$$

b Let $u = \ln(x)$ and $\frac{dv}{dx} = 1$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

So
$$\begin{aligned}\int \ln(x)\,dx &= \int u\frac{dv}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx \\ &= x\ln(x) - \int 1\,dx \\ &= x\ln(x) - x + c\end{aligned}$$

c Let $u = x$ and $\frac{dv}{dx} = \sin x$.

Then $\frac{du}{dx} = 1$ and $v = -\cos x$

So
$$\begin{aligned}\int x\sin x\,dx &= \int u\frac{dv}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx \\ &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x + \sin x + c\end{aligned}$$

d Let $u = \arccos(x)$ and $\frac{dv}{dx} = 1$.

Then $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$ and $v = x$

So
$$\begin{aligned}\int \arccos(x)\,dx &= \int u\frac{dv}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx \\ &= x\arccos(x) - \int \frac{-x}{\sqrt{1-x^2}}\,dx \\ &= x\arccos(x) + \int \frac{x}{\sqrt{1-x^2}}\,dx \\ &= x\arccos(x) - \sqrt{1-x^2}\end{aligned}$$

e Let $u = x$ and $\frac{dv}{dx} = \cos 3x$.

Then $\frac{du}{dx} = 1$ and $v = \frac{1}{3}\sin 3x$

So
$$\begin{aligned}\int x\cos 3x\,dx &= \int u\frac{dv}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx \\ &= \frac{x}{3}\sin 3x - \frac{1}{3}\int \sin 3x\,dx \\ &= \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + c\end{aligned}$$

f Let $u = x$ and $\frac{dv}{dx} = \sec^2 x$.

Then $\frac{du}{dx} = 1$ and $v = \tan x$

So
$$\begin{aligned}\int x\sec^2 x\,dx &= \int u\frac{dv}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx \\ &= x\tan x - \int \tan x\,dx \\ &= x\tan x - \ln|\cos x| + c\end{aligned}$$

g

$$\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx$$
$$= \int x\sec^2 x\,dx - \int x\,dx$$
$$= -\frac{x^2}{2} + x\tan x - \ln|\cos x| + c$$

h Let $u = \arcsin(2x)$ and $\frac{dv}{dx} = 1$.

Then $\frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}$ and $v = x$

So $\int \arcsin(2x)\,dx$

$$= \int u\frac{dv}{dx}\,dx$$
$$= x\arcsin(2x) - \int v\frac{du}{dx}\,dx$$
$$= x\arcsin(2x) - \int \frac{x}{\sqrt{1-4x^2}}\,dx$$
$$= x\arcsin(2x) + \frac{1}{2}\sqrt{1-4x^2} + c$$

i Let $u = \arctan(x)$ and $\frac{dv}{dx} = 1$.

Then $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = x$

So $\int \arctan(x)\,dx$

$$= \int u\frac{dv}{dx}\,dx$$
$$= x\arctan(x) - \int v\frac{du}{dx}\,dx$$
$$= x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$
$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c$$

j

$$\int (x+1)e^{-x}\,dx = \int xe^{-x} + e^{-x})\,dx$$
$$= \int xe^{-x}\,dx + \int e^{-x})\,dx$$
$$= e^{-x}(-x-1) - e^{-x} + c$$
$$= (-x-2)e^{-x}$$

k Let $u = \arctan(x)$ and $\frac{dv}{dx} = x$.

Then $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = \frac{x^2}{2}$

So

$$\int \frac{x^2}{2}\arctan(x)\,dx$$
$$= \int u\frac{dv}{dx}\,dx$$
$$= \frac{x^2}{2}\arctan(x) - \int v\frac{du}{dx}\,dx$$
$$= \frac{x^2}{2}\arctan(x) - \int \frac{x^2}{2(1+x^2)}\,dx$$
$$= \frac{x^2}{2}\arctan(x) - \frac{1}{2}\int 1 - \frac{1}{(1+x^2)}\,dx$$
$$= \frac{x^2}{2}\arctan(x) - \frac{1}{2}(x - \arctan x) + c$$
$$= \frac{1}{2}(x^2\arctan(x) - x + \arctan x) + c$$

l Let $u = \ln x$ and $\frac{dv}{dx} = x$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^2}{2}$

So $\int x\ln(x)\,dx = \int u\frac{dv}{dx}\,dx$

$$= uv - \int v\frac{du}{dx}\,dx$$
$$= \frac{x^2}{2}\ln(x) - \int \frac{x}{2}\,dx$$
$$= \frac{x^2}{4}(2\ln(x) - 1) + c$$

m Let $u = \ln x$ and $\frac{dv}{dx} = x^2$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^3}{3}$

So

$$\int x\ln(x)\,dx = \int u\frac{dv}{dx}\,dx$$
$$= uv - \int v\frac{du}{dx}\,dx$$
$$= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx$$
$$= \frac{x^3}{3}\ln(x) - \frac{x^2}{9} + c$$
$$= \frac{x^3}{9}(3\ln(x) - 1) + c$$

n Let $u = \ln x$ and $\frac{dv}{dx} = x^{-\frac{1}{2}}$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = 2x^{\frac{1}{2}}$

So

$$\int x^{-\frac{1}{2}}\ln(x)\,dx = \int u\frac{dv}{dx}\,dx$$
$$= uv - \int v\frac{du}{dx}\,dx$$
$$= 2x^{\frac{1}{2}}\ln(x) - \int 2x^{-\frac{1}{2}}\,dx$$
$$= 2x^{\frac{1}{2}}(\ln(x) - 2) + c$$

o

$$\int (x+3)e^x\,dx = \int xe^x + 3e^x)\,dx$$
$$= \int xe^x\,dx + \int 3e^x)\,dx$$
$$= xe^x - e^x + 3e^x + c$$
$$= (x+2)e^x + c$$

p Let $u = \ln x$ and $\frac{dv}{dx} = x^5$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^6}{6}$

So

$$\int x^5\ln(x)\,dx = \int u\frac{dv}{dx}\,dx$$
$$= uv - \int v\frac{du}{dx}\,dx$$
$$= \frac{x^6}{6}\ln(x) - \int \frac{x^5}{6}\,dx$$
$$= \frac{x^6}{6}\ln(x) - \frac{x^6}{36}\,dx$$
$$= \frac{x^6}{36}(6\ln(x) - 1) + c$$

q Let $u = x$ and $\frac{dv}{dx} = e^{2x+1}$.

Then $\frac{du}{dx} = 1$ and $v = \frac{1}{2}e^{2x+1}$

So

$$\int xe^{2x+1}\,dx = \int u\frac{dv}{dx}\,dx$$
$$= uv - \int v\frac{du}{dx}\,dx$$
$$= \frac{x}{2}e^{2x+1} - \int \frac{1}{2}e^{2x+1}\,dx$$
$$= \frac{x}{2}e^{2x+1} - \frac{1}{4}e^{2x+1}\,dx$$
$$= \frac{1}{4}(2x-1)e^{2x+1} + c$$

r Let $u = \ln 2x$ and $\frac{dv}{dx} = x$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^2}{2}$

So

$$\int x\ln(2x)\,dx = \int u\frac{dv}{dx}\,dx$$
$$= uv - \int v\frac{du}{dx}\,dx$$
$$= \frac{x^2}{2}\ln(2x) - \int \frac{x}{2}\,dx$$
$$= \frac{x^2}{2}\ln(2x) - \frac{x^2}{4}\,dx$$
$$= \frac{x^2}{4}(2\ln(2x) - 1) + c$$

2 a $\int x^2 e^{-x}\,dx$

$= -x^2e^{-x} + \int 2xe^{-x}\,dx$

$= -x^2e^{-x} + 2e^{-x}(-x-1) + c$

$= (-x^2 - 2x - 2)e^{-x}$

b $\int x^2 \sin x\,dx$

$= -x^2\cos x + \int 2x\cos x\,dx$

$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$

$= -x^2\cos x + 2x\sin x + 2\cos x$

3 a Let $u = e^x$ and $\dfrac{dv}{dx} = \sin x$.

Then $\dfrac{du}{dx} = e^x$and $v = -\cos x$

So

$\int e^x \sin x\,dx$

$= \int u\dfrac{dv}{dx}\,dx$

$= uv - \int v\dfrac{du}{dx}\,dx$

$= -e^x\cos x + \int e^x\cos x\,dx$

$= -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$

$\therefore \int e^x\sin x\,dx = \dfrac{1}{2}e^x(\sin x - \cos x)$

b Let $u = e^{2x}$ and $\dfrac{dv}{dx} = \cos 3x$.

Then $\dfrac{du}{dx} = 2e^{2x}$and $v = \dfrac{1}{3}\sin 3x$

So

$\int e^{2x}\cos 3x\,dx$

$= \int u\dfrac{dv}{dx}\,dx$

$= uv - \int v\dfrac{du}{dx}\,dx$

$= \dfrac{1}{3}e^{2x}\sin 3x - \int \dfrac{2}{3}e^{2x}\sin 3x\,dx$

$= \dfrac{1}{3}e^{2x}\sin 3x - \dfrac{2}{3}(-\dfrac{1}{3}e^{2x}\cos 3x + \dfrac{2}{3}\int e^{2x}\cos 3x\,dx)$

$= \dfrac{1}{3}e^{2x}\sin 3x + \dfrac{2}{9}e^{2x}\cos 3x - \dfrac{4}{9}\int e^{2x}\cos 3x\,dx$

$\therefore \int e^{2x}\cos 3x\,dx =$

$\dfrac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x)$

c $-\dfrac{1}{10}e^{3x}(\cos x - 3\sin x)$

d $-\dfrac{2}{5}e^x\left(\cos\left(\dfrac{x}{2}\right) - 2\sin\left(\dfrac{x}{2}\right)\right)$

4 It will help to denote $I_n = \int x^n e^{2x}\,dx$.

We then integrate by parts to find that

$I_n = \int x^n e^{2x}\,dx$ We now

$= \int x^n \dfrac{d}{dx}\dfrac{1}{2}e^{2x}\,dx$

$= \dfrac{1}{2}x^n e^{2x} - \dfrac{1}{2}\int e^{2x}\dfrac{d}{dx}x^n\,dx$

$= \dfrac{1}{2}x^n e^{2x} - \dfrac{1}{2}\int e^{2x}nx^{n-1}\,dx$

$= \dfrac{1}{2}x^n e^{2x} - \dfrac{n}{2}\int x^{n-1}e^{2x}\,dx$

$= \dfrac{1}{2}x^n e^{2x} - \dfrac{n}{2}I_{n-1}\,dx.$

use this formula repeatedly to find I_3,

$$
\begin{aligned}
I_3 &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}I_2 \\
&= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \frac{2}{2}I_1\right) \\
&= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}I_1 \\
&= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\left(\frac{1}{2}xe^{2x} - \frac{1}{2}I_0\right) \\
&= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}I_0 \\
&= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\int e^{2x}\,dx \\
&= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + c.
\end{aligned}
$$

5 Let $I_n = \int 1 \cdot (\log_e(x))^n\,dx$. We then integrate by parts to find that

$$
\begin{aligned}
I_n &= \int 1 \cdot (\log_e(x))^n\,dx \\
&= \int \frac{d}{dx}(x) \cdot (\log_e(x))^n\,dx \\
&= x(\log_e(x))^n - \int x \cdot \frac{d}{dx}(\log_e(x))^n\,dx \\
&= x(\log_e(x))^n - \int x \cdot n(\log_e(x))^{n-1}\frac{1}{x}\,dx \\
&= x(\log_e(x))^n - n\int (\log_e(x))^{n-1}\,dx \\
&= x(\log_e(x))^n - nI_{n-1}.
\end{aligned}
$$

We now use this formula repeatedly to find I_3,

$$
\begin{aligned}
I_3 &= x(\log_e(x))^3 - 3I_2. \\
&= x(\log_e(x))^3 - 3(x(\log_e(x))^2 - 2I_1 \\
&= x(\log_e(x))^3 - 3x(\log_e(x))^2 + 6I_1 \\
&= x(\log_e(x))^3 - 3x(\log_e(x))^2 + 6(x\log_e(x) - I_0) \\
&= x(\log_e(x))^3 - 3x(\log_e(x))^2 + 6x(\log_e(x)) - 6I_0 \\
&= x(\log_e(x))^3 - 3x(\log_e(x))^2 + 6x(\log_e(x)) - 6\int 1\,dx \\
&= x(\log_e(x))^3 - 3x(\log_e(x))^2 + 6x(\log_e(x)) - 6x + c.
\end{aligned}
$$

6 a Let $I_n = \int \sin^n x\,dx$. We then find that

$$\begin{aligned}
I_n &= \int \sin^n x\,dx \\
&= \int \sin^{n-1} x \sin x\,dx \\
&= \int \sin^{n-1} x \frac{d}{dx}(-\cos x)\,dx \\
&= -\sin^{n-1} x \cos x + \int \frac{d}{dx}(\sin^{n-1} x)\cos x\,dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\ \cos^2 x\,dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x - \sin^n x\,dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx \\
&= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n
\end{aligned}$$

Therefore we can now solve for I_n:

$$\begin{aligned}
I_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n \\
I_n + (n-1)I_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2} \\
nI_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2} \\
I_n &= -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}I_{n-2}.
\end{aligned}$$

b

$$\begin{aligned}
I_5 &= -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}I_3 \\
&= -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}\left(-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}I_1\right) \\
&= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x + \frac{8}{15}I_1 \\
&= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x + \frac{8}{15}\int \sin x\,dx \\
&= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x.
\end{aligned}$$

7 a Let $I_n = \int \cos^n x\,dx$. We then find that

$$\begin{aligned}
I_n &= \int \cos^n x\,dx \\
&= \int \cos^{n-1} x \cos x\,dx \\
&= \int \cos^{n-1} x \frac{d}{dx}(\sin x)\,dx \\
&= \cos^{n-1} x \sin x - \int \frac{d}{dx}(\cos^{n-1} x) \sin x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, \sin^2 x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x - \cos^n x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n
\end{aligned}$$

Therefore we can now solve for I_n:

$$\begin{aligned}
I_n &= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n \\
I_n + (n-1)I_n &= \cos^{n-1} x \sin x + (n-1)I_{n-2} \\
nI_n &= \cos^{n-1} x \sin x + (n-1)I_{n-2} \\
I_n &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}I_{n-2}.
\end{aligned}$$

We can then find that

$$\begin{aligned}
I_5 &= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}I_3 \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}I_1\right) \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}I_1 \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\int \cos x\,dx \\
&= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x
\end{aligned}$$

8 **a** We first find that
$$I_0 = \int_0^{\pi} \sin(x)\,dx = [-\cos(x)]_0^{\pi} = -\cos\pi + \cos 0 = 2.$$

We first find that
$$\begin{aligned} I_1 &= \int_0^{\pi} \sin^3(x)\,dx \\ &= \int_0^{\pi} \sin^2(x)\sin(x)\,dx \\ &= \int_0^{\pi} (1-\cos^2(x))\sin(x)\,dx \end{aligned}$$

Let $u = \cos(x)$ so that $\frac{du}{dx} = -\sin(x)$. If $x = 0$ then $u = 1$ and if $x = \pi$ then $u = -1$. Therefore
$$\begin{aligned} I_1 &= -\int_1^{-1} (1-u^2)\,du \\ &= \int_{-1}^{1} (u^2-1)\,du \\ &= [\frac{u^3}{3} - u]_{-1}^{1} \\ &= (\tfrac{(-1)^3}{3}+1) - (\tfrac{1^3}{3}+1) \\ &= \frac{4}{3} \end{aligned}$$

b We now find that
$$\begin{aligned} I_n &= \int_0^{\pi} \sin^{2n+1}(x)\,dx \\ &= \int_0^{\pi} \sin^{2n}(x)\sin(x)\,dx \\ &= \int_0^{\pi} \sin^{2n}(x)\frac{d}{dx}(-\cos(x))\,dx \\ &= [-\sin^{2n}(x)\cos(x)]_0^{\pi} + \int_0^{\pi} \cos(x)\frac{d}{dx}(\sin^{2n}(x))\,dx \\ &= 2n\int_0^{\pi} (\sin^{2n-1}(x))\cos^2(x)\,dx \\ &= 2n\int_0^{\pi} (\sin^{2n-1}(x))(1-\sin^2(x))\,dx \\ &= 2n\int_0^{\pi} (\sin^{2n-1}(x) - \sin^{2n+1}(x))\,dx \\ &= 2n\int_0^{\pi} \sin^{2n-1}(x)\,dx - 2n\int_0^{\pi} \sin^{2n+1}(x)\,dx \\ &= 2nI_{n-1} - 2nI_n \end{aligned}$$

Therefore
$$\begin{aligned} I_n &= 2nI_{n-1} - 2nI_n \\ I_n + 2nI_n &= 2nI_{n-1} \\ I_n &= \frac{2n}{1+2n}I_{n-1}, \end{aligned}$$
as required.

c We can now find that
$$\begin{aligned} I_2 &= \frac{4}{5}I_1 = \frac{4}{5}\frac{4}{3} = \frac{16}{15}, \\ I_3 &= \frac{6}{7}I_2 = \frac{6}{7}\frac{16}{15} = \frac{32}{35}. \end{aligned}$$

d More generally, we find that
$$\begin{aligned} I_n &= \frac{2n}{2n+1}I_{n-1} \\ &= \frac{2n}{2n+1}I_{n-1} \\ &= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{4}{3}\cdot\frac{2}{1} \end{aligned}$$
as required.

9 **a** Let $u = x$ and $\frac{dv}{dx} = e^{2x}$.

Then $\frac{du}{dx} = 1$ and $v = \frac{1}{2}e^{2x}$

$$\int_0^2 xe^{2x}\,dx = \left[\frac{x}{2}e^{2x}\right]_0^2 - \int_0^2 \frac{1}{2}e^{2x}\,dx$$
$$= e^4 - \left(\frac{1}{4}e^4 - \frac{1}{4}\right)$$
$$= \frac{1}{4}(1 + 3e^4)$$

b Let $u = x$ and $\frac{dv}{dx} = \sin 4x$.

Then $\frac{du}{dx} = 1$ and $v = -\frac{1}{4}\cos 4x$

$$\int_0^{2\pi} x\sin 4x\,dx$$
$$= \left[-\frac{x}{4}\cos 4x\right]_0^{2\pi} + \int_0^{2\pi} \frac{1}{4}\cos 4x\,dx$$
$$= -\frac{\pi}{2}$$

c Let $u = x$ and $\frac{dv}{dx} = \cos 4x$.

Then $\frac{du}{dx} = 1$ and $v = \frac{1}{4}\sin 4x$

$$\int_0^{\frac{\pi}{4}} x\cos 4x\,dx$$
$$= \left[-\frac{x}{4}\sin 4x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{4}\sin 4x\,dx$$
$$= -\frac{1}{8}$$

d Let $u = 2x$ and $\frac{dv}{dx} = e^{3x}$.

Then $\frac{du}{dx} = 2$ and $v = \frac{1}{3}e^{3x}$

$$\int_0^1 2xe^{3x}\,dx = \left[\frac{2}{3}xe^{3x}\right]_0^1 - \int_0^1 \frac{2}{3}e^{3x}\,dx$$
$$= \frac{2}{3}e^3 - \left[\frac{2}{9}e^{3x}\right]_0^1$$
$$= \frac{2}{9}(2e^3 + 1)$$

e

$$\int_0^{\pi} (4x - 3)\sin\left(\frac{x}{4}\right)\,dx$$
$$= \int_0^{\pi} 4x\sin\left(\frac{x}{4}\right)\,dx - \int_0^{\pi} 3\sin\left(\frac{x}{4}\right)\,dx$$
$$= \int_0^{\pi} 4x\sin\left(\frac{x}{4}\right)\,dx + 6\sqrt{2} - 12$$

Consider

$$\int_0^{\pi} 4x\sin\left(\frac{x}{4}\right)\,dx$$

Let $u = 4x$ and $\frac{dv}{dx} = \sin\left(\frac{x}{4}\right)$

Then $\frac{du}{dx} = 4$ and $v = -4\cos\left(\frac{x}{4}\right)$

$$\int_0^{\pi} 4x\sin\left(\frac{x}{4}\right)\,dx$$
$$= \int_0^{\pi} u\frac{dv}{dx}\,dx$$
$$= [uv]_0^{\pi} - \int_0^{\pi} v\frac{du}{dx}\,dx$$
$$= \left[-16x\cos\left(\frac{x}{4}\right)\right]_0^{\pi} + \int_0^{\pi} 16\cos\left(\frac{x}{4}\right)\,dx$$
$$= -16\pi \times \frac{1}{\sqrt{2}} + 32\sqrt{2}$$
$$= -8\sqrt{2}\pi + 32\sqrt{2}$$

Therefore,

$$\int_0^{\pi}(4x - 3)\sin\left(\frac{x}{4}\right)\,dx$$
$$= -8\sqrt{2}\pi + 32\sqrt{2} + 6\sqrt{2} - 12$$
$$= -12 + 38\sqrt{2} - 8\sqrt{2}\pi$$

f Let $u = x^2$ and $\frac{dv}{dx} = e^{3x-1}$

$\frac{du}{dx} = 2x$ and $v = \frac{1}{3}e^{3x-1}$

$$\int_0^1 x^2e^{3x-1}\,dx$$
$$= \left[\frac{1}{3}x^2e^{3x-1}\right]_0^1 - \frac{2}{3}\int_0^1 xe^{3x-1}\,dx$$
$$= \frac{1}{3}e^2 - \frac{2}{3}\int_0^1 xe^{3x-1}\,dx$$

Use integration by parts again.

$$\int_0^1 x^2 e^{3x-1}\, dx$$

$$= \frac{1}{3}e^2 - \frac{2}{27}\left[(3x-1)e^{3x-1}\right]_0^1$$

$$= \frac{1}{3}e^2 - \frac{2}{27}(2e^2 - (-1e^{-1}))$$

$$= \frac{1}{3}e^2 - \frac{4e^2}{27} - \frac{2}{27e}$$

$$= \frac{1}{27}(9e^2 - 4e^2 - 2)$$

$$= \frac{1}{27}(5e^2 - 2)$$

g Let $u = \log_e 3x$ and $\dfrac{dv}{dx} = 1$

$$\frac{du}{dx} = \frac{1}{x} \text{ and } v = x$$

$$\int_1^2 \log_e(3x)\, dx$$

$$= [x \log_e 3x]_1^2 - \int_1^2 x \times \frac{1}{x}\, dx$$

$$= 2\log_e 6 - \log_e 3 - 1$$

$$= \log_e(12) - 1$$

h $\dfrac{1}{4}(5e^4 - 1)$

i $9\ln(3) - \dfrac{26}{9}$

Solutions to Exercise 9I

1 $\int_0^1 \frac{1}{(x+1)(x+2)}dx = \log_e p$

$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$1 = A(x+2) + B(x+1)$

When $x = -2,\ B = -1$

When $x = -1,\ A = 1$

$\therefore \int_0^1 \frac{1}{(x+1)(x+2)}dx$

$= \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{x+2}$

$= [\log_e|x+1| - \log_e|x+2|]_0^1$

$= \left[\log_e\left|\frac{x+1}{x+2}\right|\right]_0^1$

$= \log_e\frac{2}{3} - \log_e\frac{1}{2}$

$= \log_e\frac{4}{3}$

$\therefore\ p = \frac{4}{3}$

2 a $\int_0^{\frac{\pi}{6}} \sin^2 x\cos x dx$

Let $\sin x = t$

Then $\frac{dt}{dx} = \cos x$

When $x = 0,\ t = 0$

and when $x = \frac{\pi}{6}, t = \frac{1}{2}$

$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\cos x dx$

$= \int_0^{\frac{1}{2}} t^2 dt$

$= \left[\frac{t^3}{3}\right]_0^{\frac{1}{2}}$

$= \frac{1}{24}$

b $\int_0^1 \frac{e^{2x}}{1+e^x}dx$

Let $e^x = t$, then $\frac{dt}{dx} = e^x$

$\frac{e^{2x}}{1+e^x} = \frac{e^x \times e^x}{1+e^x}$

$= \frac{te^x}{t+1}$

$= \frac{(t+1-1)e^x}{t+1}$

$= \left(1 - \frac{1}{t+1}\right)e^x$

When $x = 0,\ t = 1$

and when $x = 1,\ t = e$

$\therefore \int_0^1 \frac{e^{2x}dx}{e^x+1}$

$= \int_1^e dt - \int_1^e \frac{d}{t+1}$

$= [t - \log_e|t+1|]_1^e$

$= e - \log_e(e+1) - 1 + \log_e 2$

$= e - 1 + \log_e\frac{2}{e+1}$

$= e - 1 - \log_e\left(\frac{e+1}{2}\right)$

c $\int_0^{\frac{\pi}{3}} \sin^3 x \cos x dx$

Let $\sin x = t$

Then $\frac{dt}{dx} = \cos x$

When $x = 0,\ t = 0$

and when $x = \frac{\pi}{3},\ t = \frac{\sqrt{3}}{2}$

$\therefore \int_0^{\frac{\pi}{3}} \sin^3 x \cos x dx$

$= \int_0^{\frac{\sqrt{3}}{2}} t^3 dt$

$= \left[\frac{t^4}{4}\right]_0^{\frac{\sqrt{3}}{2}}$

$= \frac{9}{4 \times 16}$

$= \frac{9}{64}$

d $\frac{x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$x = A(x+1) + B(x-2)$

When $x = 2,\ A = \frac{2}{3}$

When $x = -1,\ B = \frac{1}{3}$

$\therefore \int_3^4 \frac{x}{(x-2)(x+1)} dx$

$= \frac{2}{3}\int_3^4 \frac{dx}{(x-2)} + \frac{1}{3}\int_3^4 \frac{dx}{(x+1)}$

$= \left[\frac{2}{3}\log_e |x-2| + \frac{1}{3}\log_e |x+1|\right]_3^4$

$= \left[\frac{1}{3}\log_e |(x-2)^2(x+1)|\right]_3^4$

$= \frac{1}{3}(\log_e 20 - \log_e 4)$

$= \frac{1}{3}\log_e 5 \approx 0.536$

3 $\int_0^{\frac{\pi}{6}} \frac{(\cos x)dx}{1 + \sin x} = \log_e c$

Let $1 + \sin x = u$

Then $\frac{du}{dx} = \cos x$

When $x = 0,\ u = 1$

and when $x = \frac{\pi}{6},\ u = \frac{3}{2}$

$\therefore \int_0^{\frac{\pi}{6}} \frac{\cos x dx}{1 + \sin x}$

$= \int_1^{\frac{3}{2}} \frac{du}{u}$

$= [\log_e |u|]_1^{\frac{3}{2}}$

$= \log_e \frac{3}{2}$

$\therefore c = \frac{3}{2}$

4 $\int \sin 3x \cos^5 3x\, dx$

Let $\cos 3x = u$

Then $\frac{du}{dx} = -3\sin 3x$

$\therefore \int \cos^5 3x \sin 3x\, dx = -\frac{1}{3}\int u^5 du$

$= -\frac{u^6}{18} + c$

$= -\frac{1}{18}\cos^6 3x + c$

5 $\int_4^6 \frac{2}{x^2-4}dx = \log_e p$

$\frac{2}{x^2-4} = \frac{2}{(x-2)(x+2)}$

$= \frac{A}{x-2} + \frac{B}{x+2}$

$2 = A(x+2) + B(x-2)$

When $x = 2,\ A = \frac{1}{2}$

When $x = -2,\ B = -\frac{1}{2}$

$\therefore \int_4^6 \frac{2}{x^2-4}dx$

$= \frac{1}{2}\int_4^6 \frac{dx}{x-2} - \frac{1}{2}\int_4^6 \frac{dx}{x+2}$

$= \frac{1}{2}[\log_e|x-2| - \log_e|x+2|]_4^6$

$= \frac{1}{2}\left[\log_e\left|\frac{x-2}{x+2}\right|\right]_4^6$

$= \frac{1}{2}\left(\log_e\frac{4}{8} - \log_e\frac{2}{6}\right)$

$= \frac{1}{2}\log_e\frac{3}{2}$

$= \log_e\sqrt{\frac{3}{2}}$

$\therefore\ p = \sqrt{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{1}{2}} = \frac{\sqrt{6}}{2}$

6 $\int_5^6 \frac{3}{x^2-5x+4}dx = \log_e p$

$\frac{3}{x^2-5x+4} = \frac{3}{(x-1)(x-4)}$

$= \frac{A}{x-1} + \frac{B}{x-4}$

$3 = A(x-4) + B(x-1)$

When $x = 4,\ B = 1$

When $x = 1,\ A = -1$

$\therefore \int_5^6 \frac{3}{x^2-5x+4}dx$

$= -\int_5^6 \frac{dx}{x-1} + \int_5^6 \frac{dx}{x-4}$

$= [-\log_e|x-1| + \log_e|x-4|]_5^6$

$= \left[\log_e\left|\frac{x-4}{x-1}\right|\right]_5^6$

$= \log_e\frac{2}{5} - \log_e\frac{1}{4}$

$= \log_e\frac{8}{5}$

$\therefore\ p = \frac{8}{5}$

7 a $\int \frac{\cos x}{\sin^3 x}dx$

Let $\sin x = u$

Then $\frac{du}{dx} = \cos x$

$\therefore \int \frac{\cos x dx}{\sin^3 x} = \int \frac{du}{u^3}$

$= \frac{u^{-2}}{-2} + c$

$= -\frac{1}{2\sin^2 x} + c$

$= -\frac{1}{2}\operatorname{cosec}^2 x + c$

b $\int x(4x^2+1)^{\frac{3}{2}}dx$

Let $4x^2 + 1 = u$

Then $\frac{du}{dx} = 8x$

$\therefore \int x(4x^2+1)^{\frac{3}{2}}dx$

$= \frac{1}{8}\int u^{\frac{3}{2}}du$

$= \frac{2}{5\times 8}u^{\frac{5}{2}} + c$

$= \frac{1}{20}(4x^2+1)^{\frac{5}{2}} + c$

c $\int \sin^2 x \cos^3 x \, dx$

Let $\sin x = u$

Then $\frac{du}{dx} = \cos x$,

$\sin^2 x \cos^3 x = \sin^2 x \cos^2 x \cos x$

$= u^2(1 - u^2) \cos x$

Therefore,

$$\int \sin^2 x \cos^3 x \, dx = \int (u^2 - u^4) du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + c$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

d $\int \frac{e^x}{e^{2x} - 2e^x + 1} dx$

Let $e^x = u$

Then $\frac{du}{dx} = e^x$

$$\therefore \int \frac{e^x \, dx}{e^{2x} - 2e^x + 1} = \int \frac{du}{u^2 - 2u + 1}$$
$$= \int \frac{du}{(u-1)^2}$$
$$= -\frac{1}{u-1} + c$$
$$= \frac{1}{1 - e^x} + c$$

8 $\int_0^3 \frac{x}{\sqrt{25 - x^2}} dx$

Let $25 - x^2 = u$

Then $\frac{du}{dx} = -2x$

When $x = 0,\ u = 25$

and when $x = 3,\ u = 16$

$$\int_0^3 \frac{x \, dx}{\sqrt{25 - x^2}} = -\frac{1}{2} \int_{25}^{16} \frac{du}{\sqrt{u}}$$
$$= [-\sqrt{u}]_{25}^{16}$$
$$= -4 + 5$$
$$= 1$$

9 **a** $\int \frac{dx}{(x+1)^2 + 4} = \int \frac{dx}{(x+1)^2 + 2^2}$

$$= \frac{1}{2} \tan^{-1} \frac{x+1}{2} + c$$

b $\int \frac{1}{\sqrt{1 - 9x^2}} dx = \frac{1}{3} \int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}$

$$= \frac{1}{3} \sin^{-1}\left(\frac{x}{\frac{1}{3}}\right) + c$$
$$= \frac{1}{3} \sin^{-1} 3x + c$$

c $\int \frac{1}{\sqrt{1 - 4x^2}} dx = \frac{1}{2} \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}}$

$$= \frac{1}{2} \sin^{-1}\left(\frac{x}{\frac{1}{2}}\right) + c$$
$$= \frac{1}{2} \sin^{-1} 2x + c$$

d $\int \frac{dx}{(2x+1)^2 + 9}$

$$= \frac{1}{4} \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$
$$= \frac{1}{4} \times \frac{2}{3} \tan^{-1} \frac{2}{3}\left(x + \frac{1}{2}\right) + c$$
$$= \frac{1}{6} \tan^{-1} \frac{2x+1}{3} + c$$

10 $f : (1,\ \infty) \to R, f(x) = \sin^{-1}\left(\frac{1}{\sqrt{x}}\right)$

a By the chain rule

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x}}\right)^2}} \times \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$$

$$= -\frac{1}{2\sqrt{x^3}\sqrt{1-\frac{1}{x}}}$$

$$= -\frac{1}{2x\sqrt{x-1}}$$

b $$\int_2^4 \frac{dx}{x\sqrt{x-1}} = -2\int_2^4 \frac{dx}{-2x\sqrt{x-1}}$$

$$= \left[-2\sin^{-1}\left(\frac{1}{\sqrt{x}}\right)\right]_2^4$$

$$= -2 \times \frac{\pi}{6} + 2 \times \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{3}$$

$$= \frac{\pi}{6}$$

11 For each of the following let $f(x) = u$.
Then $f'(x)\,dx = du$
Therefore

a $$\int f'(x)(f(x))^2 dx = \int u^2 du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{1}{3}[f(x)]^3 + c$$

b $$\int \frac{f'(x)dx}{(f(x))^2} = \int \frac{du}{u^2}$$

$$= -\frac{1}{u} + c$$

$$= -\frac{1}{f(x)} + c$$

c $$\int \frac{f'(x)dx}{f(x)} = \log_e(f(x)) + c$$

(for $f(x) > 0$)

Also note that

$$\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)|$$

(for $f(x) \neq 0$)

d $$\int \sin(f(x))f'(x)dx$$

$$= \int \sin u\,du$$

$$= -\cos u + c$$

$$= -\cos[f(x)] + c$$

12 $y = x\sqrt{4-x}$

$$\frac{dy}{dx} = \sqrt{4-x} - \frac{x}{2\sqrt{4-x}} \text{ (product rule)}$$

$$= \frac{2(4-x)-x}{2\sqrt{4-x}}$$

$$= \frac{8-3x}{2\sqrt{4-x}}$$

$$\therefore \int_0^2 \frac{8-3x}{\sqrt{4-x}}dx = 2\left[x\sqrt{4-x}\right]_0^2$$

$$= 4\sqrt{2}$$

13

$$\begin{array}{r|l} & \quad 2x-3 \\ x^2-4x+4 & 2x^3 - 11x^2 + 20x - 13 \\ & \underline{2x^3 - 8x^2 + 8x} \\ & \qquad -3x^2 + 12x - 13 \\ & \qquad \underline{-3x^2 + 12x - 12} \\ & \qquad\qquad\qquad -1 \end{array}$$

$\therefore\ a = 2,\ b = -3,\ c = -1$

$$\therefore \int \frac{2x^3 - 11x^2 + 20x - 13}{(x-2)^2}dx$$

$$= \int (2x-3)dx - \int \frac{dx}{(x-2)^2}$$

$$= x^2 - 3x + \frac{1}{x-2} + c$$

14 a $\int_0^{\frac{\pi}{4}} \sin^2 2x\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos 4x}{2}dx$

$= \left[\frac{1}{2}x - \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{8}$

b Let $x^2 - 14x + 1 = u$

Then $\frac{du}{dx} = 2x - 14$

When $x = -1,\ u = 16$

and when $x = 0,\ u = 1$

$\therefore \int_{-1}^{0}(14-2x)\sqrt{x^2-14x+1}dx$

$= -\int_{16}^{1}\sqrt{u}du$

$= -\frac{2}{3}\left[u^{\frac{3}{2}}\right]_{16}^{1}$

$= -\frac{2}{3}(1-64)$

$= 42$

c Let $\cos x = u$

Then $\frac{du}{dx} = -\sin x$

When $x = -\frac{1}{3}\pi,\ u = \frac{1}{2}$

and when $x = \frac{1}{3}\pi,\ u = \frac{1}{2}$

$\therefore\ 9\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}\frac{\sin x}{\sqrt{\cos x}}dx$

$= 9\int_{\frac{1}{2}}^{\frac{1}{2}}\frac{-du}{\sqrt{u}}$

$= 0$

(since $\int_a^a f(x)dx = 0$)

d Let $\log_e x = u$

Then $\frac{du}{dx} = \frac{1}{x}$

When $x = e,\ u = 1$

and when $x = e^2,\ u = 2$

$\therefore\ \int_e^{e^2}\frac{dx}{x\log_e x} = \int_1^2 \frac{du}{u}$

$= [\log_e|u|]_1^2$

$= \log_e 2 \approx 0.693$

e $\int_0^{\frac{\pi}{4}}\tan^2 x\,dx = \int_0^{\frac{\pi}{4}}(\sec^2 x - 1)dx$

$= [\tan x - x]_0^{\frac{\pi}{4}}$

$= 1 - \frac{\pi}{4} \approx 0.215$

f Let $2 + \cos x = u$

Then $\frac{du}{dx} = -\sin x$

When $x = 0,\ u = 3$

and when $x = \frac{\pi}{2}, u = 2$

$\therefore\ \int_0^{\frac{\pi}{2}}\frac{\sin x}{2+\cos x}dx = -\int_3^2\frac{du}{u}$

$= -[\log_e|u|]_3^2$

$= -\log_e 2 + \log_e 3$

$= \log_e\frac{3}{2} \approx 0.405$

15 a $\int \sin x\cos x\,dx$

Let $u = \sin x$

Then $\frac{du}{dx} = \cos x$

$\therefore\ \int \sin x\cos x\,dx = \int u\cos x\,dx$

$= \int udu$

$= \frac{1}{2}u^2 + c$

$= \frac{1}{2}\sin^2 x + c$

b $\int \sin x \cos x \, dx$

$= \frac{1}{2}\int 2\sin x \cos x \, dx$

$= \frac{1}{2}\int \sin 2x \, dx$

$= -\frac{1}{4}\cos 2x + c$

16 a $y = \log_e(x + \sqrt{x^2+1})$

$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}}$

$\times\left(1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x\right)$

$= \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$

$= \frac{1}{x + \sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right)$

$= \frac{1}{\sqrt{x^2+1}}$

$\therefore \int \frac{1}{\sqrt{x^2+1}}dx$

$= \int \frac{dy}{dx}dx$

$= y + c$

$= \log_e|x + \sqrt{x^2+1}| + c$

b $y = \log_e(x + \sqrt{x^2-1})$

$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2-1}}$

$\times\left(1 + \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \times 2x\right)$

$= \frac{1}{x + \sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$

$= \frac{1}{x + \sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1} + x}{\sqrt{x^2-1}}\right)$

$= \frac{1}{\sqrt{x^2-1}}$

$\therefore \int_2^7 \frac{1}{\sqrt{x^2-1}}dx$

$= \int_2^7 \frac{dy}{dx}dx$

$= [\log_e|x + \sqrt{x^2-1}|]_2^7$

$= \log_e(7 + \sqrt{7^2-1})$

$- \log_e(2 + \sqrt{2^2-1})$

$= \log_e(7 + \sqrt{48}) - \log_e(2 + \sqrt{3})$

$= \log_e\left(\frac{7 + 4\sqrt{3}}{2 + \sqrt{3}}\right)$

$= \log_e\left(\frac{7 + 4\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}\right)$

$= \log_e\left(\frac{14 + 8\sqrt{3} - 7\sqrt{3} - 12}{4 - 3}\right)$

$= \log_e(2 + \sqrt{3})$, as required to show.

17 a $\int \frac{1}{4 + x^2}dx = \frac{1}{2}\int \frac{2}{4 + x^2}dx$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$ $(c = 0)$

b $\dfrac{1}{4 - x^2} = \dfrac{1}{(2 + x)(2 - x)},\ x \in (-2, 2)$

$= \dfrac{A}{(2 + x)} + \dfrac{B}{(2 - x)}$

When $x = 2,\ B = \dfrac{1}{4}$

When $x = -2,\ A = \dfrac{1}{4}$

$\therefore\ frac14 - x^2 = \dfrac{1}{4(x + 2)} + \dfrac{1}{4(2 - x)}$

$\therefore\ \displaystyle\int \frac{1}{4 - x^2}dx$

$= \dfrac{1}{4}\left[\displaystyle\int \frac{1}{x + 2}dx + \int \frac{1}{2 - x}dx\right]$

$= \dfrac{1}{4}(\log_e|x + 2| - \log_e|2 - x)| + c$

$= \dfrac{1}{4}\log_e\left|\dfrac{x + 2}{2 - x}\right|\ (c = 0)$

c $\displaystyle\int \frac{4 + x^2}{x}dx$

$= \displaystyle\int \frac{4}{x}dx + \int xdx$

$= 4\log_e|x| + \dfrac{1}{2}x^2 + c$

$= 4\log_e|x| + \dfrac{1}{2}x^2\ (c = 0)$

d $\displaystyle\int \frac{x}{4 + x^2}dx$

Let $u = 4 + x^2$

$\therefore\ \dfrac{du}{dx} = 2x$

$\therefore\ \displaystyle\int \frac{x}{4 + x^2}dx = \frac{1}{2}\int \frac{1}{u}du$

$= \dfrac{1}{2}\log_e(4 + x^2) + c$

$= \dfrac{1}{2}\log_e(4 + x^2)\ (c = 0)$

e $\displaystyle\int \frac{x^2}{4 + x^2}dx$

$$\begin{array}{r|l} & 1 \\ \hline x^2 + 4 & x^2 \\ & \underline{x^2 + 4} \\ & -4 \end{array}$$

$\dfrac{x^2}{x^2 + 4} = 1 - \dfrac{4}{x^2 + 4}$

$\displaystyle\int \frac{x^2}{x^2 + 4}dx$

$= \displaystyle\int 1dx - 2\int \frac{2}{4 + x^2}dx$

$= x - 2\tan^{-1}\left(\dfrac{x}{2}\right) + c$

$= x - 2\tan^{-1}\left(\dfrac{x}{2}\right)\ (c = 0)$

f $\displaystyle\int \frac{1}{1 + 4x^2}dx = \int \frac{1}{4\left(\frac{1}{4} + x^2\right)}dx$

$= \dfrac{1}{2}\displaystyle\int \frac{\frac{1}{2}}{\frac{1}{4} + x^2}dx$

$= \dfrac{1}{2}\tan^{-1}(2x)\ (c = 0)$

g $\displaystyle\int x\sqrt{4 + x^2}dx$

Let $u = 4 + x^2$

$\therefore\ \dfrac{du}{dx} = 2x$

$\therefore\ \displaystyle\int x\sqrt{4 + x^2}dx$

$= \dfrac{1}{2}\displaystyle\int \sqrt{u}du$

$= \dfrac{1}{2} \times \dfrac{2}{3}u^{\frac{3}{2}} + c$

$= \dfrac{1}{3}\left(4 + x^2\right)^{\frac{3}{2}}\ (c = 0)$

h $\int x\sqrt{4+x}\,dx$

Let $u = x + 4$

$\therefore\ x = u - 4$ and $\dfrac{du}{dx} = 1$

$$\therefore\ \int x\sqrt{4+x}\,dx$$

$$= \int (u-4)u^{\frac{1}{2}}\,du$$

$$= \int u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}}\ (c = 0)$$

i $\displaystyle\int \frac{1}{\sqrt{4-x}}\,dx$

Let $u = 4 - x$

$\therefore\ \dfrac{du}{dx} = -1$

$$\therefore\ \int \frac{1}{\sqrt{4-x}}\,dx = -\int \frac{1}{\sqrt{u}}\,du$$

$$= -2u^{\frac{1}{2}} + c$$

$$= -2\sqrt{4-x}\ (c = 0)$$

j $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-x^2}}\,dx$

$$= \sin^{-1}\left(\frac{x}{2}\right) + c$$

$$= \sin^{-1}\left(\frac{x}{2}\right)\ (c = 0)$$

k $\displaystyle\int \frac{x}{\sqrt{4-x}}\,dx$

Let $u = 4 - x$

$\therefore\ x = 4 - u$ and $\dfrac{du}{dx} = -1$

$$\therefore\ \int \frac{x}{\sqrt{4-x}}\,dx$$

$$= -\int \frac{4-u}{\sqrt{u}}\,du$$

$$= -4\int u^{-\frac{1}{2}}\,du + \int u^{\frac{1}{2}}\,du$$

$$= -8u^{\frac{1}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c$$

$$= -8\sqrt{4-x} + \frac{2}{3}(4-x)^{\frac{3}{2}}\ (c = 0)$$

l $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

Let $u = 4 - x^2$

$\therefore\ \dfrac{du}{dx} = -2x$

$$\therefore\ \int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{2} \times 2u^{\frac{1}{2}} + c$$

$$= -\sqrt{4-x^2}\ (c = 0)$$

18 $\displaystyle\int \frac{x^3 - x + 2}{x^2 - 1}\,dx = c + \log_e d$

$$\begin{array}{r|l} & x \\ \hline x^2 - 1 & x^3 - x + 2 \\ & x^3 - x \\ \hline & \quad\quad + 2 \end{array}$$

$$\therefore\ \int_2^3 \frac{x^3 - x + 2}{x^2 - 1}\,dx$$

$$= \int_2^3 x\,dx + \int_2^3 \frac{2}{x^2-1}\,dx$$

Now $\dfrac{2}{x^2-1} = \dfrac{2}{(x+1)(x-1)}$

$$\equiv \frac{A}{x+1} + \frac{B}{x-1}$$

$\therefore\ A(x-1)+B(x+1)=2$

When $x=1,\ B=1$

When $x=-1,\ A=-1$

$\therefore\ \dfrac{2}{x^2-1}=\dfrac{-1}{x+1}+\dfrac{1}{x-1}$

$\therefore\ \displaystyle\int_2^3 \frac{x^3-x+2}{x^2-1}dx$

$= \displaystyle\int_2^3 x\,dx+\int_2^3 \frac{-1}{x+1}dx$

$+ \displaystyle\int_2^3 \frac{1}{x-1}dx$

$= \left[\dfrac{1}{2}x^2\right]_2^3+[-\log_e|x+1|]_2^3$

$+[\log_e|x-1|]_2^3$

$= \dfrac{1}{2}(3^2-2^2)-\log_e 4+\log_e 3$

$+\log_e 2-\log_e 1$

$= \dfrac{5}{2}+\log_e\left(\dfrac{3\times 2}{4\times 1}\right)$

$= \dfrac{5}{2}+\log_e\dfrac{3}{2}$

$\therefore\ c=\dfrac{5}{2}$ and $d=\dfrac{3}{2}$

19 a $f(x)=\sin(x)\cos^{n-1}(x)$

$f'(x)=\sin(x)\times(n-1)\cos^{n-2}(x)$

$\times -\sin(x)+\cos(x)$

$\times\cos^{n-1}(x)$

$=-(n-1)\sin^2(x)\cos^{n-2}(x)$

$+\cos^n(x)$

b $\therefore\ \displaystyle\int f'(x)dx$

$=-(n-1)\displaystyle\int \sin^2(x)\cos^{n-2}(x)dx$

$+\displaystyle\int\cos^n(x)dx$

$\therefore\ \sin(x)\cos^{n-1}(x)$

$=-(n-1)\displaystyle\int(1-\cos^2(x))$

$\times\cos^{n-2}(x)dx+\displaystyle\int\cos^n(x)dx$

$=-(n-1)\left(\displaystyle\int\cos^{n-2}(x)dx\right.$

$\left.-\displaystyle\int\cos^n(x)dx\right)+\displaystyle\int\cos^n(x)dx$

$=-(n-1)\displaystyle\int\cos^{n-2}(x)dx+(n-1)$

$\times\displaystyle\int\cos^n(x)dx+\int\cos^n(x)dx$

$=-(n-1)\displaystyle\int\cos^{n-2}(x)dx$

$+n\displaystyle\int\cos^n(x)dx$

$\therefore\ n\displaystyle\int\cos^n(x)dx=\sin(x)\cos^{n-1}(x)$

$+(n-1)\displaystyle\int\cos^{n-2}(x)dx,$

as required to verify.

c i $\displaystyle\int_0^{\frac{\pi}{2}}\cos^4(x)dx$

$=\dfrac{1}{4}\times 4\displaystyle\int_0^{\frac{\pi}{2}}\cos^4(x)dx$

$=\dfrac{1}{4}\Big[[\sin(x)\cos^3(x)]_0^{\frac{\pi}{2}}$

$+3\displaystyle\int_0^{\frac{\pi}{2}}\cos^2(x)\,dx\Big]$

$=\dfrac{1}{4}\Big[\sin\dfrac{\pi}{2}\cos^3\dfrac{\pi}{2}-\sin 0\cos^3 0$

$+3\left[\dfrac{1}{2}x+\dfrac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{2}}\Big]$

$$\left(\int \cos^2(x)dx = \frac{1}{2}x + \frac{1}{2}\sin(2x)\right)$$

$$= \frac{1}{4}\left[3\left(\frac{1}{2} \times \frac{\pi}{2} + \frac{1}{2}\sin\left(2 \times \frac{\pi}{2}\right)\right.\right.$$

$$\left.\left. - \frac{1}{2} \times 0 - \frac{1}{2}\sin(2 \times 0)\right)\right]$$

$$= \frac{1}{4}\left(3\left(\frac{\pi}{4} + \frac{1}{2}\sin\pi - 0 - \frac{1}{2}\sin 0\right)\right)$$

$$= \frac{3}{4}\left(\frac{\pi}{4} + 0 - 0\right)$$

$$= \frac{3\pi}{16}$$

ii $\int_0^{\frac{\pi}{2}} \cos^6(x)dx$

$$= \frac{1}{6} \times 6\int_0^{\frac{\pi}{2}} \cos^6(x)dx$$

$$= \frac{1}{6}\left[[\sin(x)\cos^5(x)]_0^{\frac{\pi}{2}}\right.$$

$$\left. + 5\int_0^{\frac{\pi}{2}} \cos^4(x)dx\right]$$

$$= \frac{1}{6}\left[0 - 0 + 5 \times \frac{3\pi}{16}\right] \text{ from } \mathbf{i}.$$

$$= \frac{5\pi}{32}$$

iii

$$\int_0^{\frac{\pi}{2}} \cos^4(x)\sin^2(x)dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^4(x)(1 - \cos^2(x))dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^4(x)dx - \int_0^{\frac{\pi}{2}} \cos^6(x)dx$$

$$= \frac{3\pi}{16} - \frac{5\pi}{32}$$

$$= \frac{\pi}{32}$$

iv $\int_0^{\frac{\pi}{4}} \sec^4(x)dx = \int_0^{\frac{\pi}{4}} \cos^{-4}(x)dx$

Now $n\int \cos^n x\,dx = \sin x \cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)dx$

$$\therefore\ -2\int_0^{\frac{\pi}{4}} \cos^{-2} x\,dx$$

$$= [\sin x \cos^{-3}(x)]_0^{\frac{\pi}{4}}$$

$$+ (-3)\int_0^{\frac{\pi}{4}} \cos^{-4}(x)dx$$

$$\therefore\ -2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= \left[\sin\frac{\pi}{4}\cos^{-3}\left(\frac{\pi}{4}\right)\right.$$

$$\left. - \sin 0\cos^{-3}(0)\right]$$

$$- 3\int_0^{\frac{\pi}{4}} \cos^{-4}(x)dx$$

$$\therefore\ -2[\tan x]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{1}{\sqrt{2}} \times (\sqrt{2})^3 - 0\right]$$

$$- 3\int_0^{\frac{\pi}{4}} \cos^{-4}(x)dx$$

$$\therefore\ -2(1 - 0)$$

$$= 2 - 3\int_0^{\frac{\pi}{4}} \cos^{-4}(x)dx$$

$$\therefore\ -4 = -3\int_0^{\frac{\pi}{4}} \cos^{-4}(x)dx$$

$$\therefore\ \int_0^{\frac{\pi}{4}} \cos^{-4}(x)dx = \frac{4}{3}$$

$$\therefore\ \int_0^{\frac{\pi}{4}} \sec^4(x)dx = \frac{4}{3}$$

20 a $\int \frac{x}{(x+1)^n} dx$

Let $u = x + 1$, $\quad \therefore \frac{du}{dx} = 1$

and $x = u - 1$

$$\therefore \int \frac{x}{(x+1)^n} dx$$

$$= \int \frac{u-1}{u^n} du$$

$$= \int u^{1-n} - u^{-n} du$$

$$= \frac{1}{2-n} u^{2-n} - \frac{1}{1-n} u^{1-n} + c$$

$$= \frac{1}{2-n}(x+1)^{2-n} - \frac{1}{1-n}(x+1)^{1-n} + c$$

b $\int_1^2 x(x-1)^n dx$

Let $u = x - 1$, $\quad \therefore x = u + 1$

and $\frac{du}{dx} = 1$

When $x = 1,\ u = 0$

When $x = 2,\ u = 1$

$$\therefore \int_1^2 x(x-1)^n dx$$

$$= \int_0^1 (u+1)u^n du$$

$$= \int_0^1 u^{n+1} + u^n du$$

$$= \left[\frac{1}{n+2} u^{n+2}\right]_0^1 + \left[\frac{1}{n+1} u^{n+1}\right]_0^1$$

$$= \frac{1}{n+2} + \frac{1}{n+1}$$

21 a $\int_0^1 (1 + ax)^2 dx$

Let $u = 1 + ax$, $\quad \therefore \dfrac{du}{dx} = a$

When $x = 0$, $u = 1$

When $x = 1$, $u = 1 + a$

$$\therefore \int_0^1 (1 + ax)^2\, dx$$

$$= \frac{1}{a}\int_1^{1+a} u^2\, du$$

$$= \frac{1}{a}\left[\frac{u^3}{3}\right]_1^{1+a}$$

$$= \frac{1}{3a}[(1 + a)^3 - 1]$$

$$= \frac{1}{3a}(1 + 3a + 3a^2 + a^3 - 1)$$

$$= 1 + a + \frac{1}{3}a^2$$

b Let $y = \dfrac{1}{3}a^2 + a + 1$

The value of y is a minimum when $\dfrac{dy}{da} = 0$

$$\therefore \frac{2}{3}a + 1 = 0$$

$$\therefore a = -\frac{3}{2}$$

22 a Let $y = \dfrac{a\sin x - b\cos x}{a\cos x + b\sin x} = \dfrac{u}{v}$

where $u = a\sin x - b\cos x$ and $v = a\cos x + b\sin x$

$\therefore \dfrac{du}{dx} = a\cos x + b\sin x$ and

$\dfrac{dv}{dx} = -a\sin x + b\cos x$

$$\therefore \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(a\cos x + b\sin x)(a\cos x + b\sin x) - (a\sin x - b\cos x)(b\cos x - a\sin x)}{(a\cos x + b\sin x)^2}$$

$$= \frac{a^2\cos^2 x + 2ab\sin x\cos x + b^2\sin^2 x - (2ab\sin x\cos x - b^2\cos^2 x - a^2\sin^2 x)}{(a\cos x + b\sin x)^2}$$

$$= \frac{a^2(\cos^2 x + \sin^2 x) + b^2(\sin^2 x + \cos^2 x)}{(a\cos x + b\sin x)^2}$$

$$= \frac{a^2 + b^2}{(a\cos x + b\sin x)^2}$$

b

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(a\cos x + b\sin x)^2}$$

$$= \frac{1}{a^2 + b^2}\int_0^{\frac{\pi}{2}} \frac{a^2 + b^2}{(a\cos x + b\sin x)^2}dx$$

$$= \frac{1}{a^2 + b^2}\left[\frac{a\sin x - b\cos x}{a\cos x + b\sin x}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{a^2 + b^2}\left[\frac{a\sin\frac{\pi}{2} - b\cos\frac{\pi}{2}}{a\cos\frac{\pi}{2} + b\sin\frac{\pi}{2}} - \frac{a\sin 0 - b\cos 0}{a\cos 0 + b\sin 0}\right]$$

$$= \frac{1}{a^2 + b^2}\left[\frac{a}{b} - \frac{-b}{a}\right]$$

$$= \frac{1}{a^2 + b^2}\left[\frac{a}{b} + \frac{b}{a}\right]$$

$$= \frac{1}{a^2 + b^2}\left(\frac{a^2 + b^2}{ab}\right)$$

$$= \frac{1}{ab}$$

23 $U_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$
where $n \in Z$ and $n > 1$

a $U_n + U_{n-2}$

$$= \int_0^{\frac{\pi}{4}} \tan^n x\,dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x(\tan^2 x + 1)dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\,dx$$

Let $w = \tan x$

$\therefore \dfrac{dw}{dx} = \sec^2 x$

When $x = 0,\ w = 0$

When $x = \dfrac{\pi}{4},\ w = 1$

$\therefore U_n + U_{n-2}$

$$= \int_0^1 w^{n-2} dw$$

$$= \left[\frac{1}{n-1} w^{n-1}\right]_0^1$$

$$= \frac{1}{n-1}[1-0]$$

$$= \frac{1}{n-1}$$

b $U_n + U_{n-2} = \dfrac{1}{n-1}$

$\therefore U_n = \dfrac{1}{n-1} - U_{n-2}$

$U_6 = \dfrac{1}{5} - U_4$

$$= \frac{1}{5} - \left(\frac{1}{3} - U_2\right)$$

$$= \frac{1}{5} - \frac{1}{3} + (1 - U_0)$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - U_0$$

where $U_0 = \displaystyle\int_0^{\frac{\pi}{4}} \tan^0 x\,dx$

$$= \int_0^{\frac{\pi}{4}} 1dx$$

$$= [x]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4}$$

$\therefore U_6 = \dfrac{1}{5} - \dfrac{1}{3} + 1 - \dfrac{\pi}{4}$

$$= \frac{13}{15} - \frac{\pi}{4}$$

as required to show

24 a $\dfrac{1}{1+\tan x} + \dfrac{1}{1+\cot x}$

$$= \frac{1 + \cot x + 1 + \tan x}{(1+\tan x)(1+\cot x)}$$

$$= \frac{2 + \tan x + \cot x}{1 + \tan x + \cot x + \tan x \cot x}$$

$$= \frac{2 + \tan x + \cot x}{2 + \tan x + \cot x}$$

(since $\tan x \cot x = 1$)

$= 1$

b Consider $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\tan\theta}$

Let $\phi = \frac{\pi}{2} - \theta \quad \therefore \frac{d\phi}{d\theta} = -1$

$\therefore\ \theta = \frac{\pi}{2} - \phi$

When $\theta = 0,\ \phi = \frac{\pi}{2}$

When $\theta = \frac{\pi}{2},\ \phi = 0$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\tan\theta}$$

$$= -\int_{\frac{\pi}{2}}^{0} \frac{d\phi}{1+\tan\left(\frac{\pi}{2}-\phi\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\cot\phi}$$

$\left(\text{since } \tan\left(\frac{\pi}{2}-\phi\right) = \cot\phi\right)$

as required to show.

c $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\tan\theta}$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\cot\phi}$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+\tan\phi} + \frac{1}{1+\cot\phi}\right) d\phi$$

$$- \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\tan\phi}$$

$$= \int_0^{\frac{\pi}{2}} 1\,d\phi - \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\tan\phi}$$

Now $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\tan\theta} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\tan\phi}$

$\therefore\ 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\tan\theta} = \int_0^{\frac{\pi}{2}} 1\,d\phi$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\tan\theta} = \frac{1}{2}[\phi]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{4}$$

25 a We first let $u = \sqrt{x}$ so that

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}.$$

Therefore

$$\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du.$$

b By integrating by parts we now find that

$$\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du.$$

$$= \int 2u\frac{d}{du}e^u\,du.$$

$$= 2ue^u - \int e^u \frac{d}{du}2u\,du$$

$$= 2ue^u - \int 2e^u\,du$$

$$= 2ue^u - 2e^u + c$$

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$$

$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + c.$$

26 a The reduction formula is $\int \sin^n(x)\,dx = \frac{n-1}{n}\int \sin^{n-2} x\,dx - \frac{\sin^{n-1} x\cos x}{n}$.
We use this twice to find that

$$\int \sin^4(x)\,dx = \frac{3}{4}\int \sin^2 x\,dx - \frac{\sin^3 x\cos x}{4}$$

$$= \frac{3}{4}\left(\frac{1}{2}\int 1\,dx - \frac{\sin x\cos x}{2}\right) - \frac{\sin^3 x\cos x}{4} + c$$

$$= \frac{3}{4}\left(\frac{x}{2} - \frac{\sin x\cos x}{2}\right) - \frac{\sin^3 x\cos x}{4} + c$$

$$= \frac{3x}{8} - \frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + c.$$

b Now using the double angle formula $\cos 2x = 1 - 2\sin^2 x$, we find that $\sin^2 x = \frac{1-\cos 2x}{2}$ so that

$$\int \sin^4(x)\,dx = \int\left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2(2x)\,dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1+\cos 4x}{2}, dx$$

$$= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}, dx$$

$$= \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right) + c$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{\sin 4x}{32} + c$$

c Equating the two answers gives

$$-\frac{1}{4}\sin 2x + \frac{\sin 4x}{32} + k = -\frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x$$

For some constant k. We can find k by letting $x = 0$ to give $k = 0$.
Therefore

$$\frac{\sin 4x}{32} = \frac{1}{4}\sin 2x - \frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x$$

$$\sin 4x = 8\sin 2x - 12\sin x\cos x - 8\sin^3 x\cos x$$

$$= 16\sin x\sin x - 12\sin x\cos x - 8\sin^3 x\cos x$$

$$= 4\sin x\cos x - 8\sin^3 x\cos x.$$

Solutions to Technology-free questions

1 a $\int \cos^3 2x dx = \int (1 - \sin^2 2x) \times \cos 2x dx$

Let $\sin 2x = u$

then $\frac{du}{dx} = 2\cos 2x$

$\therefore \int \left(1 - \sin^2 2x\right) \cos 2x \, dx$

$= \int (1 - u^2) \frac{du}{dx} dx$

$= \frac{1}{2} \int (1 - u^2) du$

$= \frac{1}{2}u - \frac{1}{6}u^3 + c$

$= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + c$

b

$\int \frac{2x + 3}{4x^2 + 1} dx$

$= \int \frac{du}{4u + 1} + 3\int \frac{dx}{4x^2 + 1}$

where $u = x^2$, $\frac{du}{dx} = 2x$

$= \frac{1}{4}\log_e |4u + 1| + \frac{3}{2}\tan^{-1} 2x + c$

$= \frac{1}{4}\log_e |4x^2 + 1|$

$+ \frac{3}{2}\tan^{-1} 2x + c$

$= \frac{1}{4}\log_e (4x^2 + 1) + \frac{3}{2}\tan^{-1} 2x + c,$

since $4x^2 + 1 > 0$

c $\frac{1}{1 - 4x^2} = \frac{A}{1 - 2x} + \frac{B}{1 + 2x}$

$\therefore 1 = A(1 + 2x) + B(1 - 2x)$

When $x = \frac{1}{2}$, $A = \frac{1}{2}$

When $x = -\frac{1}{2}$, $B = \frac{1}{2}$

$\int \frac{dx}{1 - 4x^2}$

$= \frac{1}{2}\int \frac{dx}{1 - 2x} + \frac{1}{2}\int \frac{dx}{1 + 2x}$

$= -\frac{1}{4}\log_e |1 - 2x|$

$+ \frac{1}{4}\log_e |1 + 2x| + c$

$= \frac{1}{4}\log_e \left|\frac{1 + 2x}{1 - 2x}\right| + c$

d Let $1 - 4x^2 = u$

then $\frac{du}{dx} = -8x$

$\therefore \int \frac{x \, dx}{\sqrt{1 - 4x^2}} = -\frac{1}{8}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{4}\sqrt{u} + c$

$= -\frac{1}{4}\sqrt{1 - 4x^2} + c$

e

$\frac{x^2}{1 - 4x^2} = -\frac{1}{4} + \frac{1}{4\left(1 - 4x^2\right)}$

$\therefore \int \frac{x^2 dx}{1 - 4x^2} = -\frac{x}{4} + \frac{1}{16}\log_e \left|\frac{1 + 2x}{1 - 2x}\right| + c$

see **1 c** above

f Let $1 - 2x^2 = u$

then $\frac{du}{dx} = -4x$

$\therefore \int x\sqrt{1 - 2x^2} dx$

$= -\frac{1}{4}\int \sqrt{u} du$

$= -\frac{1}{4} \times \frac{2}{3}u^{\frac{3}{2}} + c$

$= -\frac{1}{6}\sqrt{\left(1 - 2x^2\right)^3} + c$

g $\sin^2\left(x - \frac{\pi}{3}\right)$

$$= \frac{1 - \cos\left(2x - \frac{2\pi}{3}\right)}{2}$$

$$\therefore \int \sin^2\left(x - \frac{\pi}{3}\right)dx$$

$$= \frac{1}{2}\int 1dx$$

$$- \frac{1}{2}\int \cos\left(2x - \frac{2\pi}{3}\right)dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin\left(2x - \frac{2\pi}{3}\right) + c$$

h Let $x^2 - 2 = u$

then $\frac{du}{dx} = 2x$

$$\therefore \int \frac{x\,dx}{\sqrt{x^2 - 2}} = \frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= \sqrt{u} + c$$

$$= \sqrt{x^2 - 2} + c$$

i $$\int \sin^2 3x dx = \int \frac{1 - \cos 6x}{2}dx$$

$$= \frac{1}{2}x - \frac{1}{12}\sin 6x + c$$

j Let $\cos 2x = u$

then $-2\sin 2x = \frac{du}{dx}$

$\sin^3 2x = (1 - \cos^2 2x)\sin 2x$

$$\int \sin^3 2x\,dx$$

$$= -\frac{1}{2}\int (1 - u^2)du$$

$$= -\frac{1}{2}\left(u - \frac{u^3}{3}\right) + c$$

$$= \frac{1}{6}\cos 2x\left(\cos^2 2x - 3\right) + c$$

k Let $u = x + 1$

then $\frac{du}{dx} = 1$ and $x = u - 1$

$$\int x\sqrt{x + 1}dx$$

$$= \int (u - 1)u^{\frac{1}{2}}du$$

$$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x + 1)^{\frac{5}{2}} - \frac{2}{3}(x + 1)^{\frac{3}{2}} + c$$

$$= 2(x + 1)^{\frac{3}{2}}\left(\frac{1}{5}(x + 1) - \frac{1}{3}\right) + c$$

l $1 + \cos 2x = 2\cos^2 x$

$$\int \frac{dx}{1 + \cos 2x} = \frac{1}{2}\int \frac{dx}{\cos^2 x}$$

$$= \frac{1}{2}\sec^2 x + c$$

$$= \frac{1}{2}\tan x + c$$

m $$\int \frac{e^{3x} + 1}{e^{3x+1}}dx$$

$$= \int (e^{-1} + e^{-3x-1})dx \text{ by division}$$

$$= \frac{x}{e} + \frac{-1}{3e^{3x+1}} + c$$

n Let $x^2 - 1 = u$

then $\frac{du}{dx} = 2x$

$$\int \frac{x\,dx}{x^2 - 1} = \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\log_e |u| + c$$

$$= \frac{1}{2}\log_e |x^2 - 1| + c$$

o $\sin^2 x \cos^2 x$

$= \frac{1}{4}\sin^2 2x$

$= \frac{1 - \cos 4x}{8}$

$\int \sin^2 x \cos^2 x \, dx$

$= \frac{1}{8}\int 1dx - \frac{1}{8}\int \cos 4x dx$

$= \frac{x}{8} - \frac{1}{32}\sin 4x + c$

p $\frac{x^2}{1 + x} = x - 1 + \frac{1}{x + 1}$

by division

$\int \frac{x^2 dx}{1 + x} = \frac{x^2}{2} - x + \log_e |x + 1| + c$

2 a Let $1 - x^2 = u$
then $\frac{du}{dx} = -2x$
When $x = 0$, $u = 1$ and when
$x = \frac{1}{2}$, $u = \frac{3}{4}$

$\therefore \int_0^{\frac{1}{2}} (1 - x^2)^{\frac{1}{2}} x \, dx = -\frac{1}{2}\int_1^{\frac{3}{4}} u^{\frac{1}{2}} du$

$= -\frac{1}{3}\left[u^{\frac{3}{2}}\right]_1^{\frac{3}{4}}$

$= \frac{1}{3}\left(1 - \frac{3\sqrt{3}}{8}\right)$

$= \frac{1}{3} - \frac{\sqrt{3}}{8}$

b $\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}$

$1 = A(1 + x) + B(1 - x)$

When $x = 1$, $A = \frac{1}{2}$ and when

$x = -1$, $B = \frac{1}{2}$

$\therefore \int_0^{\frac{1}{2}} \frac{dx}{1 - x^2} = \left[-\frac{1}{2}\log_e |1 - x| + \frac{1}{2}\log_e |1 + x|\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\log_e 3$

c $\int_0^{\frac{1}{2}} x(1 + x^2)^{\frac{1}{2}} dx$

$= \frac{1}{2}\int_1^{\frac{5}{4}} u^{\frac{1}{2}} du$

where $u = 1 + x^2$, $\frac{du}{dx} = 2x$

$= \frac{1}{2} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{\frac{5}{4}}$

$= \frac{1}{3}\left(\frac{5\sqrt{5}}{8} - 1\right)$

$= \frac{5\sqrt{5} - 8}{24}$

d $\frac{1}{x(x + 6)} = \frac{A}{x} + \frac{B}{x + 6}$
$1 = A(x + 6) + Bx$
When $x = 0$, $A = \frac{1}{6}$ and when
$x = -6$, $B = -\frac{1}{6}$

$\therefore \int_0^2 \frac{dx}{x(x + 6)}$

$= \left[\frac{1}{6}\log_e |x| - \frac{1}{6}\log_e |x + 6|\right]_1^2$

$= \frac{1}{6}\left[\log_e \left|\frac{x}{x + 6}\right|\right]_1^2$

$= \frac{1}{6}\log_e\left(\frac{2}{8} \div \frac{1}{7}\right)$

$= \frac{1}{6}\log_e \frac{7}{4}$

e $\dfrac{2x^2+3x+2}{x^2+3x+2} = 2 - \dfrac{3x+2}{(x+1)(x+2)}$ by division

$= 2 - \dfrac{A}{x+1} - \dfrac{B}{x-2}$

$\therefore\ A(x+2) + B(x+1) = 3x+2$

When $x = -1,\ A = -1$ and when $x = -2,\ B = 4$

$$\therefore \int_0^1 \frac{2x^2+3x+2}{x^2+3x+2}dx$$

$$= \int_0^1 2 + \frac{1}{x+1} - \frac{4}{x+2}dx$$

$$= \Big[2x + \log_e|x+1| - 4\log_e|x+2|\Big]_0^1$$

$$= 2 + \log_e 2 - 4\log_e 3 + 4\log_e 2$$

$$= 2 + \log_e \frac{32}{81}$$

f Let $4 - 3x = u$

then $-3 = \dfrac{du}{dx}$

When $x = 0,\ u = 4$ and when $x = 1,\ u = 1$

$$\int_0^1 \frac{dx}{\sqrt{4-3x}} = -\frac{1}{3}\int_4^1 \frac{1}{\sqrt{u}}du$$

$$= -\frac{2}{3}\Big[u^{\frac{1}{2}}\Big]_4^1$$

$$= \frac{2}{3}(2-1)$$

$$= \frac{2}{3}$$

g $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$$= \frac{\pi}{6}$$

h $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 2x\,dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 4x)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

i $\displaystyle\int_{-\pi}^{\pi} \sin^2 x\cos^2 x dx$

$$= \left[\frac{x}{8} - \frac{1}{32}\sin 4x\right]_{-\pi}^{\pi}$$ (see **1 o**)

$$= \frac{\pi}{4}$$

j Using **1 o** again with $2x = t$

$$\frac{1}{2}\int_0^{\pi} \sin^2 t\cos^2 t dt$$

$$= \frac{1}{2}\left[\frac{t}{8} - \frac{1}{32}\sin 4t\right]_0^{\pi}$$

$$= \frac{\pi}{16}$$

k Let $u = 2\sin x + \cos x$

$\dfrac{du}{dx} = 2\cos x - \sin x$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{2\cos x - \sin x}{2\sin x + \cos x}dx$$

$$= \int_2^{\frac{3\sqrt{2}}{2}} \frac{1}{u}du$$

$$= \Big[\log_e|u|\Big]_2^{\frac{3\sqrt{2}}{2}}$$

$$= \log_e\left(\frac{3\sqrt{2}}{2}\right)$$

1 Let $x^3 + 1 = u$

then $\dfrac{du}{dx} = 3x^2$

When $x = -1,\ u = 0$

and when $x = 2,\ u = 9$

$$\int_{-1}^{2} x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int_0^9 \sqrt{u}\,du$$
$$= \frac{2}{9}\left[u^{\frac{3}{2}}\right]_0^9$$
$$= 6$$

3 $\dfrac{1}{2}\left(\dfrac{2x+2}{x^2+2x+3}\right) - \dfrac{1}{x^2+2x+3}$

$$= \frac{x+1-1}{x^2+2x+3}$$
$$= \frac{x}{x^2+2x+3}$$

$x^2 + 2x + 3 = (x+1)^2 + 2$

Let $x + 1 = u$

then $\dfrac{du}{dx} = 1$

$$\int \frac{x\,dx}{x^2+2x+3}$$
$$= \int \frac{x}{(x+1)^2+2}dx$$
$$= \int \frac{u-1}{u^2+2}du$$
$$= \int \frac{u}{u^2+2}du - \int \frac{1}{u^2+2}du$$
$$= \frac{1}{2}\int \frac{2u}{u^2+2}du - \int \frac{1}{u^2+2}du$$
$$= \frac{1}{2}\log_e(u^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$
$$= \frac{1}{2}\log_e(x^2+2x+3) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

4 a
$$\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right) = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x(1-x)}}$$
$$\therefore \int \frac{dx}{\sqrt{x(1-x)}} = 2\sin^{-1}\sqrt{x} + c$$

b
$$\frac{d}{dx}\left(\sin^{-1}(x^2)\right) = \frac{2x}{\sqrt{1-x^4}}$$
$$\therefore \int \frac{2x}{\sqrt{1-x^4}}dx = \sin^{-1}(x^2) + c$$

5 a
$$\frac{d}{dx}\left(x\sin^{-1}x\right) = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$$
$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$
$$= x\sin^{-1}x + \frac{1}{2}\int \frac{du}{\sqrt{u}}$$

where $u = 1 - x^2$,

$$\frac{du}{dx} = -2x$$
$$= x\sin^{-1}x + \sqrt{u} + c$$
$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

b
$$\frac{d}{dx}\left(x\log_e x\right) = \log_e|x| + 1$$
$$\int \log_e x\,dx = x\log_e|x| - \int 1dx$$
$$= x(\log_e|x| - 1) + c$$

c
$$\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$$
$$\therefore \int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x\,dx}{1+x^2}$$

Let $1 + x^2 = u$, then $\dfrac{du}{dx} = 2x$

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{du}{u}$$
$$= \frac{1}{2}\log_e |1+x^2| + c$$
$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\log_e |1+x^2| + c$$

6 a $\int \sin 2x \cos 2x dx$

$= \frac{1}{2}\int \sin 4x\,dx$

$= -\frac{1}{8}\cos 4x + c$

b Let $x^3 + 1 = u$

then $\frac{du}{dx} = 3x^2$

$\int (x^3+1)^2 x^2 dx = \frac{1}{3}\int u^2 du$

$= \frac{1}{9}u^3 + c$

$= \frac{1}{9}(x^3+1)^3 + c$

c Let $3 + 2\sin\theta = u$

then $2\cos\theta = \frac{du}{d\theta}$

$\int \frac{\cos\theta d\theta}{(3+2\sin\theta)^2}$

$= \frac{1}{2}\int \frac{du}{u^2}$

$= -\frac{1}{2u} + c$

$= -\frac{1}{2(3+2\sin\theta)} + c$

d Let $1 - x^2 = u$

then $\frac{du}{dx} = -2x$

$\int xe^{1-x^2} dx = -\frac{1}{2}\int e^u du$

$= -\frac{1}{2}e^u + c$

$= -\frac{1}{2}e^{1-x^2} + c$

e $\int \tan^2(x+3)dx$

$= \int \sec^2(x+3) - 1dx$

$= \tan(x+3) - x + c$

f Let $6 + 2x^2 = u$

then $\frac{du}{dx} = 4x$

$\int \frac{2x\,dx}{\sqrt{6+2x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= \sqrt{u} + c$

$= \sqrt{6+2x^2} + c$

g Let $\tan x = u$

then $\frac{du}{dx} = \sec^2 x$

$\int \tan^2 x \sec^2 x dx = \int u^2 du$

$= \frac{u^3}{3} + c$

$= \frac{1}{3}\tan^3 x + c$

h Now $\int \sec^3 x \tan x\,dx = \int \frac{\sin x\,dx}{\cos^4 x}$

Let $\cos x = u$

then $\frac{du}{dx} = -\sin x$

$\int \frac{\sin x\,dx}{\cos^4 x} = -\int \frac{du}{u^4}$

$= \frac{1}{3u^3}$

$= \frac{1}{3}\sec^3 x$

i $\int \tan^2 3x dx = \int \sec^2 3x - 1 dx$

$= \frac{1}{3}\tan 3x - x + c$

7 a Let $\cos x = u$

then $\frac{du}{dx} = -\sin x$

When $x = 0, u = 1$ and when $x = \frac{\pi}{2},\ u = 0$

$\therefore\ \sin^5 x = \sin^4 x \sin x$

$= (1 - \cos^2 x)^2 \sin x$

$\therefore\ \int_0^{\frac{\pi}{2}} \sin^5 x dx$

$= -\int_1^0 (1 - u^2)^2 du$

$= -\int_1^0 \left(1 - 2u^2 + u^4\right)^2 du$

$= -\left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_1^0$

$= 1 - \frac{2}{3} + \frac{1}{5}$

$= \frac{8}{15}$

b Let $13 - 5x = u$

then $\frac{du}{dx} = -5$

When $x = 1,\ u = 8$ and when $x = 8,\ u = 27$

$\therefore\ \int_1^8 (13 - 5x)^{\frac{1}{3}} dx$

$= -\frac{1}{5}\int_8^{27} u^{\frac{1}{3}} du$

$= -\frac{1}{5} \times \frac{3}{4}\left[u^{\frac{4}{3}}\right]_8^{27}$

$= -\frac{3}{20}(81 - 16)$

$= -\frac{39}{4}$

c $\int_0^{\frac{\pi}{8}} \sec^2 2x\, dx = \frac{1}{2}[\tan 2x]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}$

d $\int_1^2 (3 - y)^{\frac{1}{2}} dy = -\frac{2}{3}\left[(3 - y)^{\frac{3}{2}}\right]_1^2$

$= -\frac{2}{3}(1 - 2\sqrt{2})$

$= \frac{2}{3}(2\sqrt{2} - 1)$

e $\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \frac{1 - \cos 2x}{2} dx$

$= \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\pi}$

$= \frac{\pi}{2}$

f Let $u = x^3 + 3x$

$\therefore\ \frac{du}{dx} = 3x^2 + 3$

$= 3(x^2 + 1)$

When $x = -3,\ u = -36$ and when $x = -1,\ u = -4$

$\therefore\ \int_{-3}^{-1} \frac{x^2 + 1}{x^3 + 3x} dx = \frac{1}{3}\int_{-36}^{-4} \frac{1}{u} du$

$= \frac{1}{3}[\log_e |u|]_{-36}^{-4}$

$= \frac{1}{3}(\log_e |-4|$

$- \log_e |-36|)$

$= \frac{1}{3}\log_e \frac{1}{9}$

8 $\frac{d}{dx}\left(x^2+\frac{1}{x}\right)^{\frac{1}{2}} = \frac{1}{2}\left(x^2+\frac{1}{x}\right)^{-\frac{1}{2}}(2x-x^{-2})$

$$\int_{-1}^{2} \frac{2x-x^{-2}}{\sqrt{x^2+\frac{1}{x}}}dx = 2\left[\left(x^2+\frac{1}{x}\right)^{\frac{1}{2}}\right]_{-1}^{2}$$

$$= 2\sqrt{4+\frac{1}{2}}$$

$$= 3\sqrt{2}$$

9 a $\frac{4x^2+16x}{(x-2)^2(x^2+4)} =$
$\frac{1}{x-2}+\frac{6}{(x-2)^2}-\frac{x+4}{x^2+4}$

b

$$\int_{-2}^{0} \frac{4x^2+16x}{(x-2)^2(x^2+4)}\,dx$$

$$= \int_{-2}^{0} \frac{1}{x-2}+\frac{6}{(x-2)^2}-\frac{x+4}{x^2+4}\,dx$$

$$= \int_{-2}^{0} \frac{1}{x-2}+\frac{6}{(x-2)^2}-\frac{x}{x^2+4}-\frac{4}{x^2+4}\,dx$$

$$= \left[\log_e|x-2| - \frac{6}{x-2} - \frac{1}{2}\log_e(x^2+4) - 2\tan^{-1}(\frac{x}{2})\right]_{-2}^{0}$$

$$= (\log_e 2 + 3 - \frac{1}{2}\log_e 4 - 2\tan^{-1}0)$$

$$- (\log_e 4 + \frac{3}{2} - \frac{1}{2}\log_e 8 - 2\tan^{-1}(-1))$$

$$= \frac{3-\pi+\frac{1}{2}\log_e 2}{2}$$

Therefore $c = 3$ and $d = 2$

10 a

Let $u = e^{-2x}$ and $\frac{dv}{dx} = \cos(2x+3)$

Then, $\frac{du}{dx} = -2e^{-2x}$ and $v = \frac{1}{2}\sin(2x+3)$

$$\int e^{-2x}\cos(2x+3)$$

$$= \frac{1}{2}e^{-2x}\sin(2x+3) + \frac{1}{2}\int e^{-2x}\sin(2x+3)\,dx \ldots (1)$$

Now work with $\int e^{-2x}\sin(2x+3)\,dx$

Using integration by parts again

$$\int e^{-2x}\sin(2x+3)\,dx$$

$$= -\frac{1}{2}e^{-2x}\cos(2x+3) - \int e^{-2x}\cos(2x+3)\,dx$$

Substitute in (1)

$2\int e^{-2x}\cos(2x+3) =$
$\frac{1}{2}e^{-2x}\sin(2x+3) - \frac{1}{2}e^{-2x}\cos(2x+3)$

Therefore

$\int e^{-2x}\cos(2x+3)$
$= \frac{1}{4}e^{-2x}(\sin(2x+3) - e^{-2x}\cos(2x+$
$3) + c)$

b Let $u = x$ and $\dfrac{dv}{dx} = \sec^2 x$

Then, $\dfrac{du}{dx} = 1$ and $v = \tan x$

$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx$

$= x \tan x + \log_e |\tan x| + c$

11 a $\int_1^2 x^2 \log_e x \, dx$

Let $u = \log_e x$, then $\dfrac{du}{dx} = \dfrac{1}{x}$

$\int_1^2 x^2 \log_e x \, dx$

$= \left[\dfrac{x^3}{3} \log_e x\right]_1^2 - \dfrac{1}{3} \int_1^2 x^2 \, dx$

$= \dfrac{8}{3} \log_e 2 - \left[\dfrac{x^3}{9}\right]_1^2$

$= \dfrac{8}{3} \log_e 2 - \dfrac{7}{9}$

b $\int_1^2 \dfrac{\log_e x}{x} \, dx$

Let $u = \log_e x$, then $\dfrac{du}{dx} = \dfrac{1}{x}$

$\int_1^2 \dfrac{\log_e x}{x} \, dx = \int_0^{\log_e 2} u \, du$

$= \left[\dfrac{u^2}{2}\right]_0^{\log_e 2}$

$= \dfrac{1}{2}(\log_e 2)^2 + c$

c $\int_0^1 x e^{-2x} \, dx$

Let $u = x$, then $\dfrac{du}{dx} = 1$

Let $\dfrac{dv}{dx} = e^{-2x}$, then $v = -\dfrac{1}{2} e^{-2x}$

$\int_0^1 x e^{-2x} \, dx$

$= \left[-\dfrac{1}{2} x e^{-2x}\right]_0^1 - \int_0^1 \dfrac{1}{2} e^{-2x} \, dx$

$= -\dfrac{1}{2} e^{-2} - \dfrac{1}{4} \left[e^{-2x}\right]_0^1$

$= -\dfrac{3}{4} e^{-2} + \dfrac{1}{4}$

Solutions to multiple-choice questions

1 E

Let $4 - x = u, \quad \therefore\ x = 4 - u$

Then $\dfrac{du}{dx} = -1$

$$\therefore \int x\sqrt{4 - x}\,dx$$

$$= -\int (4 - u)\sqrt{u}\,du$$

$$= \int u^{\frac{3}{2}} - 4\sqrt{u}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{5}(4 - x)^{\frac{5}{2}} - \frac{8}{3}(4 - x)^{\frac{3}{2}} \quad (c = 0)$$

2 C

$\int_0^m \tan x \sec^2 x\, dx = \dfrac{3}{2}$

Let $\tan x = u$ then $\dfrac{du}{dx} = \sec^2 x$

When $x = 0,\ u = 0$

When $x = m,\ u = \tan m$

$$\therefore \int_0^m \tan x \sec^2 x = \int_0^{\tan m} u\,du$$

$$= \left[\frac{u^2}{2}\right]_0^{\tan m}$$

$$= \frac{\tan^2 m}{2}$$

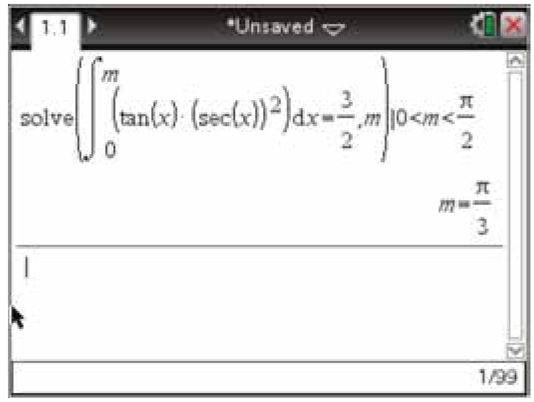
Thus,

$$\frac{\tan^2 m}{2} = \frac{3}{2}$$

$\therefore\ \tan m = \sqrt{3} \quad$ since $m \in \left(0, \dfrac{\pi}{2}\right)$

$\therefore \quad m = \dfrac{\pi}{3}$

Or using a CAS calculator with the condition that $m \in \left(0, \dfrac{\pi}{2}\right)$



3 C

$$\int \tan(2x)dx = \int \frac{\sin(2x)}{\cos(2x)}dx$$

Let $\cos 2x = u \quad \therefore\ \dfrac{du}{dx} = -2\sin 2x$

$$\therefore \int \tan(2x)dx = -\frac{1}{2}\int \frac{1}{u}du$$

$$= -\frac{1}{2}\log_e |u| + c$$

$$= -\frac{1}{2}\log_e |\cos 2x| + c$$

$$= \frac{1}{2}\log_e \left|\frac{1}{\cos 2x}\right| + c$$

$$= \frac{1}{2}\log_e |\sec 2x|$$

(where $c = 0$)

4 D

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 2x}{2 + \cos 2x}dx$$

Let $2 + \cos 2x = u$

Then $\dfrac{du}{dx} = -2\sin 2x$

When $x = \dfrac{\pi}{4},\ u = 2$

When $x = \dfrac{\pi}{2},\ u = 1$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 2x}{2+\cos 2x}dx$$

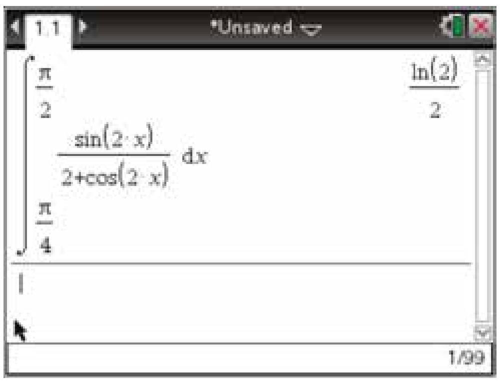
$$= -\frac{1}{2}\int_{2}^{1} \frac{1}{u}du$$

$$= \frac{1}{2}\int_{1}^{2} \frac{1}{u}du$$

$$= \left[\frac{1}{2}\log_e |u|\right]_1^2$$

$$= \frac{1}{2}\log 2$$

Using CAS



5 A

$$\int_0^{\frac{\pi}{3}} \sin x \cos^3 x \, dx$$

Let $\cos x = u$, $\therefore$ $fracdudx =$ $-\sin x$

When $x = 0$, $u = 1$

When $x = \frac{\pi}{3}$, $u = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{3}} \sin x \cos^3 x \, dx$$

$$= -\int_1^{\frac{1}{2}} u^3 du$$

$$= \int_{\frac{1}{2}}^{1} u^3 du$$

6 C $\displaystyle\int_0^2 \cos^2 x - \sin^2 x \, dx$

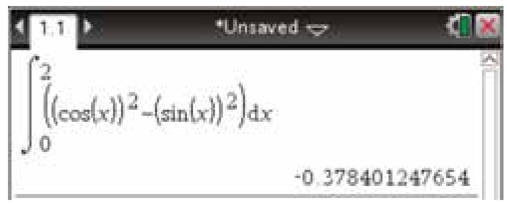
$$= \int_0^2 \cos 2x \, dx$$

$$= \left[\frac{1}{2}\sin 2x\right]_0^2$$

$$= \frac{1}{2}\sin 4$$

$$\approx -0.3784$$

Using CAS



7 D $\displaystyle\int \frac{2}{\sqrt{1-16x^2}}dx$

$$= 2\int \frac{1}{\sqrt{1-(4x)^2}}dx$$

$$= \frac{1}{2}\int \frac{4}{\sqrt{1-(4x)^2}}dx$$

$$= \frac{1}{2}\sin^{-1}(4x) + c$$

$$= \frac{1}{2}\sin^{-1}(4x) \quad (c = 0)$$

8 C $\int \frac{1}{9+4x^2}dx$

$$= \int \frac{1}{4\left(\frac{9}{4}+x^2\right)}dx$$

$$= \frac{1}{4}\int \frac{1}{\left(\frac{3}{2}\right)^2+x^2}dx$$

$$= \left(\frac{1}{4}\times\frac{2}{3}\right)\tan^{-1}\left(\frac{x}{\frac{3}{2}}\right)+c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) \quad (c=0)$$

9 A

$$\frac{d(xf(x))}{dx} = xf'(x)+f(x)$$

$$\therefore\ f(x) = \frac{d(xf(x))}{dx} - xf'(x)$$

$$\therefore\ f(x) = \frac{d(xf(x))}{dx} - \frac{1}{1+x^2}$$

$$\therefore\ \int f(x)dx = xf(x) - \int \frac{1}{1+x^2}dx$$

$$\therefore\ \int f(x)dx = xf(x) - \tan^{-1} x$$

10 D

Given $F'(x) = f(x)$

$$\therefore\ F'(3-2x) = f(3-2x)\times -2$$

$$\therefore\ -\frac{1}{2}F'(3-2x) = f(3-2x)$$

$$\therefore\ 3f(3-2x) = -\frac{3}{2}F'(3-2x)$$

$$\therefore\ \int 3f(3-2x)dx = -\frac{3}{2}F(3-2x)$$

Solutions to extended-response questions

1 a RHS $= \tan^{n-2} x \sec^2 x - \tan^{n-2} x$

$= tan^{n-2}x(1 + \tan^2 x) - \tan^{n-2} x$

$= \tan^{n-2} x + \tan^n x - \tan^{n-2} x$

$= \tan^n x$

$=$ LHS

b $I_n = \int \tan^n x\, dx$ Using the previous result:

$$\int \tan^n x\, dx = \int \tan^{n-2} x \sec^2 x - \tan^{n-2} x\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

Let $u = \tan x, \dfrac{du}{dx} = \sec^2 x$

$$\int \tan^n x\, dx = \int u^{n-2} du - I_{n-2}$$

$$= \frac{1}{n-1} u^{n-1} - I_{n-2}$$

$$= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$$

c **i** $\int \tan^2 x\, dx = \dfrac{1}{2-1} \tan x - I_0$

$= \tan x - \int 1\, dx$

$= \tan x - x$

ii $\int \tan^3 x\, dx = \dfrac{1}{3-1} \tan^2 x - I_1$

$= \dfrac{1}{2} \tan^2 x - \int \tan x\, dx$

$= \dfrac{1}{2} \tan^2 x + \log_e |\cos x|$

$(= \dfrac{1}{2} \sec^2 x - \dfrac{1}{2} + \log_e |\cos x|)$

iii $\displaystyle\int \tan^4 x\,dx = \frac{1}{4-1}\tan^3 x - I_2$

$$= \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x$$

$$(= x + \frac{1}{3} x \tan x(\tan^2 x - 3))$$

$$(= x + \frac{1}{3} x \tan x(\sec^2 x - 4))$$

iv $\displaystyle\int \tan^5 x\,dx = \frac{1}{5-1}\tan^4 x - I_3$

$$= \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$$

$$= \frac{1}{4}\tan^4 x - (\frac{1}{2}\tan^2 x + \log_e|\cos x|)$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \log_e|\cos x|)$$

$$(= \frac{1}{4}\tan^2 x(\tan^2 x - 2) - \log_e|\cos x|))$$

$$(= \frac{1}{4}(\sec^2 x - 1)(\sec^2 x - 3) - \log_e|\cos x|))$$

$$(= \frac{1}{4}(\sec^4 x - 4\sec^2 x + 3) - \log_e|\cos x|))$$

d Let $u = \sec x + \tan x$

$$\frac{du}{dx} = \frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x}$$

$$= \frac{\sin x + 1}{\cos^2 x}$$

$$= \frac{\tan x + \sec(x)}{\cos x}$$

$$= \sec x(\tan x + \sec x)$$

$$\int \sec x\,dx = \int \frac{1}{u}\,du$$

$$= \log_e u$$

$$= \log_e(\sec x + \tan x)$$

e Let $u = \sec^{n-2} x$ and $\dfrac{dv}{dx} = \sec^2 x$

Then,

$$\frac{du}{dx} = (n-2)\sec^{n-3} x \sec x \tan x \text{ and } v = \tan x$$

$$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$$
$$= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$$
$$= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$$
$$= \tan x \sec^{n-2} x - (n-2)\int \sec^n x - \sec^{n-2} x\,dx$$
$$I_n = \tan x \sec^{n-2} x - (n-2)I_n - (n-2)I_{n-2}$$
$$(n-1)I_n = \tan x \sec^{n-2} x - (n-2)I_{n-2}$$
$$I_n = \frac{1}{n-1}\tan x \sec^{n-2} x - \frac{n-2}{n-1}I_{n-2}$$

f **i** $$\int \sec^3 x\,dx = \frac{1}{2}\tan x \sec x - \frac{1}{2}I_1$$
$$= \frac{1}{2}\tan x \sec x - \frac{1}{2}\int \sec x\,dx$$
$$= \frac{1}{2}(\tan x \sec x - \log_e|\sec x + \tan x|$$

ii $$\int \sec^4 x\,dx = \frac{1}{3}\tan x \sec^2 x - \frac{2}{3}I_2$$
$$= \frac{1}{3}\tan x \sec^2 x - \frac{2}{3}\int \sec^2 x\,dx$$
$$= \frac{1}{2}\tan x \sec^2 x - \frac{2}{3}\tan x$$

iii $$\int \sec^5 x\,dx = \frac{1}{4}\tan x \sec^3 x - \frac{3}{4}I_3$$
$$= \frac{1}{4}\tan x \sec^3 x - \frac{3}{4}\int \sec^3 x\,dx$$
$$= \frac{1}{4}\tan x \sec^3 x - \frac{3}{8}(\tan x \sec x + \log_e|\sec x + \tan x|)$$

2 **a** Let $u = \dfrac{1}{(1+x^2)^n}$ and $\dfrac{dv}{dx} = 1$

Then $\dfrac{du}{dx} = -\dfrac{2xn}{(1+x^2)^{n+1}}$ and $v = x$.

$$I_n = \frac{x}{(1+x^2)^n} + \int \frac{2x^2 n}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{x}{(1+x^2)^n} + 2n\int \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

Write $\frac{x^2}{(x^2+1)^{n+1}} = \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}$

$$I_n = \frac{x}{(1+x^2)^n} + 2n\int \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}\,dx$$

$$I_n = \frac{x}{(1+x^2)^n} + 2nI_n - 2nI_{n+1}$$

$$(1-2n)I_n = \frac{x}{(1+x^2)^n} - 2nI_{n+1}$$

$$2nI_{n+1} = \frac{x}{(1+x^2)^n} + (2n-1)I_n$$

Writing this for I_n in terms of I_{n-1}

$$2(n-1)I_n = \frac{x}{(1+x^2)^{n-1}} + (2n-3)I_{n-1}$$

$$I_n = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)}I_{n-1}$$

b **i** $\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \left[\frac{x}{2(1+x^2)}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$

$$= \frac{1}{4} + \frac{1}{2}\left[\tan^{-1} x\right]_0^1$$

$$= \frac{1}{4} + \frac{\pi}{8}$$

ii $\int_0^1 \frac{1}{(1+x^2)^3}\,dx = \left[\frac{x}{4(1+x^2)^2}\right]_0^1 + \frac{3}{4}\int_0^1 \frac{1}{(x^2+1)^2}\,dx$

$$= \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right)$$

$$= \frac{1}{4} + \frac{3\pi}{32}$$

iii $\int_0^1 \frac{1}{(1+x^2)^4}\,dx = \left[\frac{x}{6(1+x^2)^3}\right]_0^1 + \frac{5}{6}\int_0^1 \frac{1}{(x^2+1)^3}\,dx$

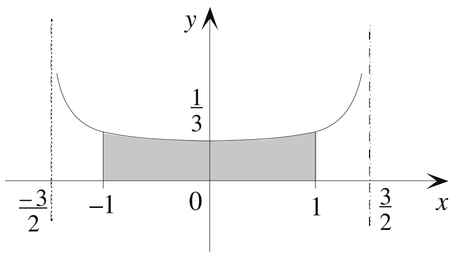
$$= \frac{1}{48} + \frac{5}{6}\left(\frac{1}{4} + \frac{3\pi}{32}\right)$$

$$= \frac{11}{48} + \frac{5\pi}{64}$$

Chapter 10 – Applications of integration

Solutions to Exercise 10A

1



By symmetry,

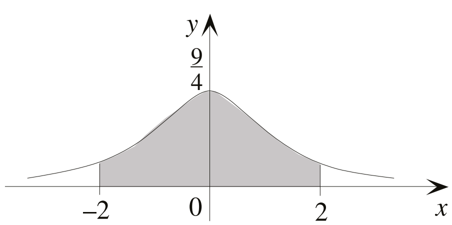
$$A = 2\int_0^1 \frac{dx}{\sqrt{9-4x^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$$

$$= \left[\sin^{-1}\frac{2x}{3}\right]_0^1$$

$$= \sin^{-1}\frac{2}{3} \text{ square units}$$

2



By symmetry,

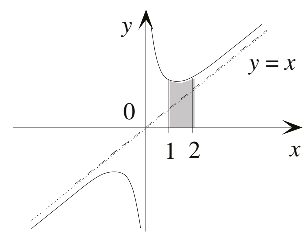
$$A = 2\int_0^2 \frac{9\,dx}{4+x^2}$$

$$= 9\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= 9\tan^{-1} 1$$

$$= \frac{9\pi}{4} \text{ square units}$$

3



$$A = \int_1^2 \left(x + \frac{2}{x}\right)dx$$

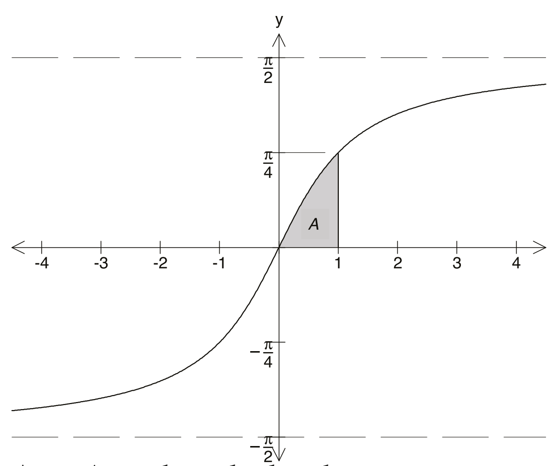
$$= \left[\frac{x^2}{2} + 2\log_e x\right]_1^2$$

$$= 2 + 2\log_e 2 - \frac{1}{2}$$

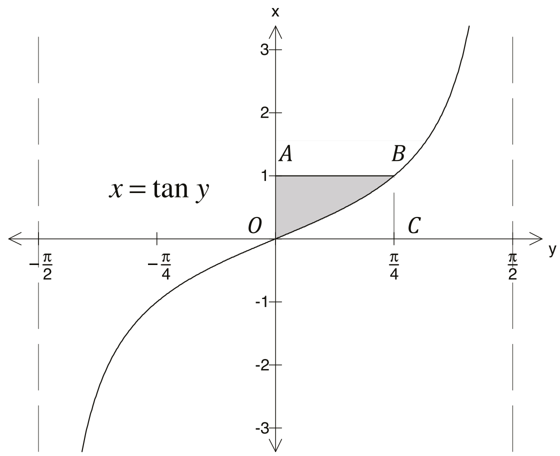
$$= \left(\frac{3}{2} + 2\log_e 2\right) \text{ square units}$$

4 **a** $y = \tan^{-1} x$.

i



Area A can be calculated as area of the rectangle $OABC$ minus area under the tangent curve.



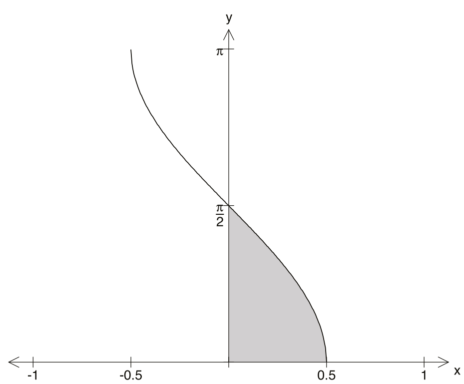
ii $A = \dfrac{\pi}{4} - \displaystyle\int_0^{\frac{\pi}{4}} \tan y \, dy$

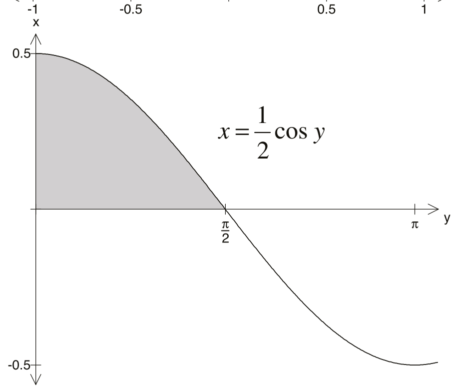
$= \dfrac{\pi}{4} + [\log_e \cos y]_0^{\frac{\pi}{4}}$

$= \left(\dfrac{\pi}{4} - \log_e \sqrt{2}\right)$ square units

b $y = \cos^{-1} 2x$

i





ii Comparing Picture 1 with Picture 2 makes it obvious that

$A = \displaystyle\int_0^{\frac{1}{2}} \cos^{-1} 2x \, dx$

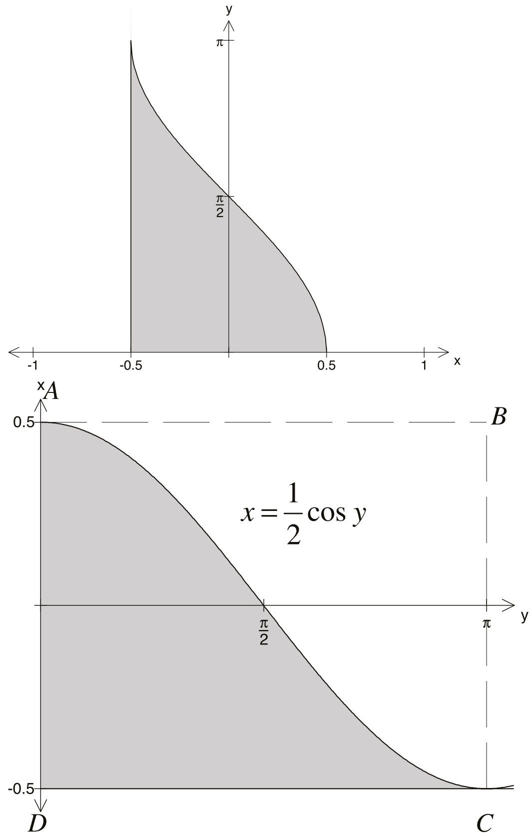
$= \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2} \cos y \, dy$

$= \dfrac{1}{2}[\sin y]_0^{\frac{\pi}{2}}$

$= \dfrac{1}{2}$ square unit

c $y = \cos^{-1} 2x$

i

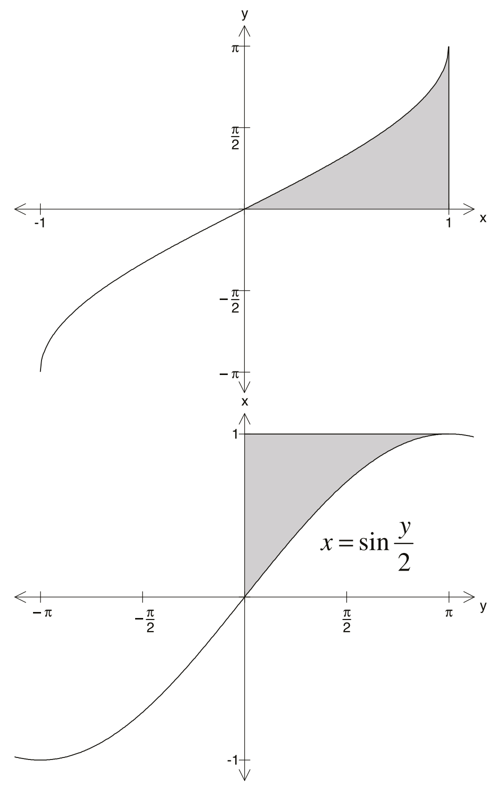


ii By symmetry, the shaded area is half that of the rectangle $ABCD$.
Area of rectangle = π square units

$$\therefore \quad A = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{-1} 2x$$
$$= \pi \times \frac{1}{2}$$
$$= \frac{\pi}{2} \text{ square units}$$

d $y = 2\sin^{-1} x$

i

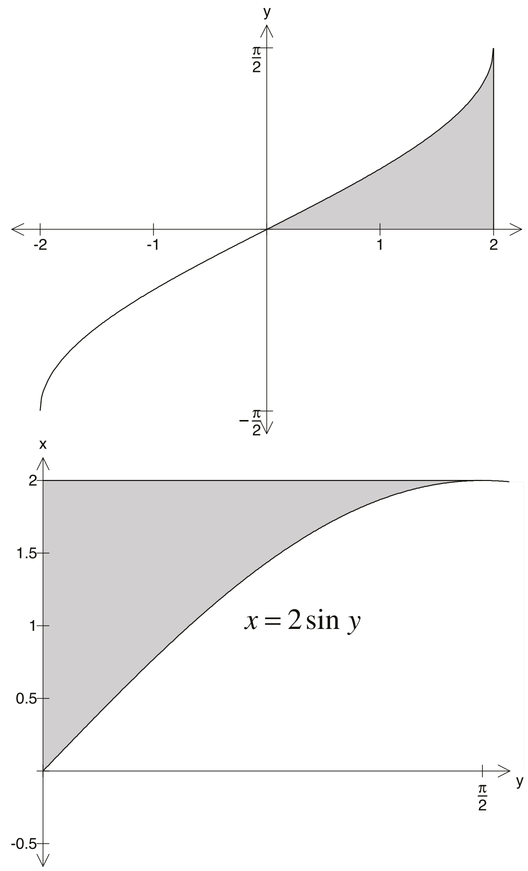


ii $\int_0^1 2\sin^{-1} x \, dy$

$$= \pi - \int_0^{\pi} \sin\frac{y}{2}\, dy$$
$$= \pi + \left[2\cos\frac{y}{2}\right]_0^{\pi}$$
$$= (\pi - 2) \text{ square units}$$

e $y = \sin^{-1}\left(\frac{x}{2}\right)$

i



ii $\int_0^2 \sin^{-1}\frac{x}{2}\,dx$

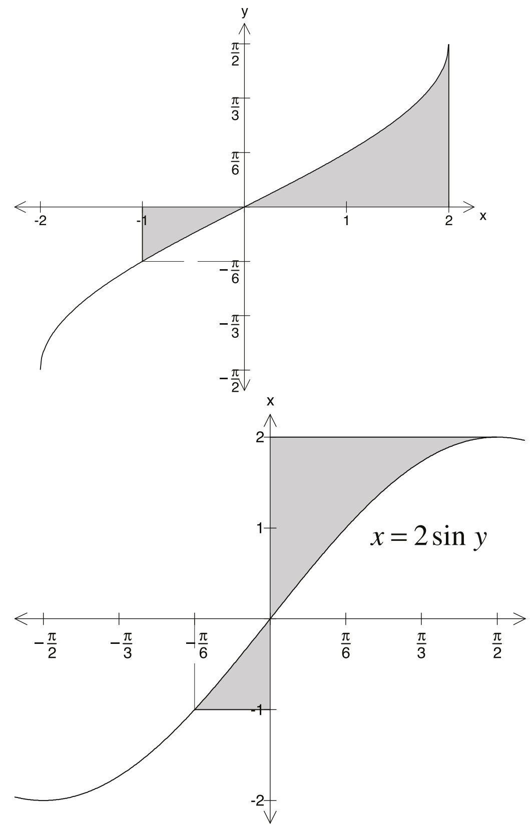
$= 2 \times \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} 2\sin y\,dy$

$= \pi + 2[\cos y]_0^{\frac{\pi}{2}}$

$= (\pi - 2)$ square units

f $y = \sin^{-1}\left(\frac{x}{2}\right).$

i



ii $\int_{-1}^{2} \sin^{-1}\frac{x}{2}dx = \int_{-1}^{0} \sin^{-1}\frac{x}{2}dx$

$+ \int_0^2 \sin^{-1}\frac{x}{2}dx$

$\int_0^2 \sin^{-1}\frac{x}{2}dx$

$= \pi - 2$ square units (see **6 e**)

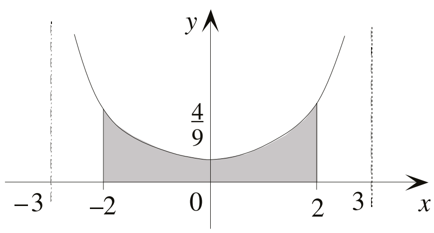
and

$$\int_{-1}^{0} \sin^{-1}\frac{x}{2}dx = -\int_{0}^{1} \sin^{-1}\frac{x}{2}\,dx$$

(by symmetry)

$$\int_{0}^{1} \sin^{-1}\frac{x}{2}dx = \frac{\pi}{6} - \int_{0}^{\frac{\pi}{6}} 2\sin y\,dy$$
$$= \frac{\pi}{6} + 2[\cos y]_{0}^{\frac{\pi}{6}}$$
$$= \frac{\pi}{6} + \sqrt{3} - 2$$
$$\therefore \int_{-1}^{2} \sin^{-1}\frac{x}{2}dx = (\pi - 2) - \left(\frac{\pi}{6} + \sqrt{3} - 2\right)$$
$$= \frac{5\pi}{6} - \sqrt{3} \text{ square units}$$

5



By symmetry,

$$A = 2\int_{0}^{2} \frac{4\,dx}{9 - x^2}$$
$$= \frac{4}{3}\int_{0}^{2} \frac{dx}{3 - x} + \frac{4}{3}\int_{0}^{2} \frac{dx}{3 + x}$$

(using partial fractions)

$$= \frac{4}{3}\left[\log_e \frac{3 + x}{3 - x}\right]_{0}^{2}$$
$$= \frac{4}{3}\log_e 5 \text{ square units}$$

6 a

$$\frac{dy}{dx} = -\frac{4x}{(x^2 + 1)^2}$$

$\therefore\ -\dfrac{4x}{(x^2 + 1)^2} = 0$ for stationary points

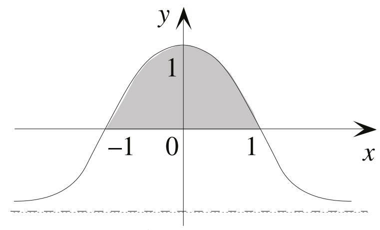
$\therefore\ x = 0, y = 1$ is a turning point (maximum)

Hence turning point occurs at $(0, 1)$

b $y = -1$ is a horizontal asymptote

c When, $y = 0$,

$$-1 + \frac{2}{x^2 + 1} = 0$$
$$\therefore\ x^2 + 1 = 2$$
$$x = \pm 1$$



$$\text{Area} = \int_{-1}^{1}\left(-1 + \frac{2}{x^2 + 1}\right)dx$$
$$= 2\int_{0}^{1}\left(-1 + \frac{2}{x^2 + 1}\right)dx$$
$$= 2[-x + 2\tan^{-1} x]_{0}^{1}$$
$$= 2\left(-1 + 2 \times \frac{\pi}{4}\right)$$
$$= (\pi - 2) \text{ square units}$$

7 a

$$x - \frac{4}{x + 3} = 0$$
$$\therefore\ x^2 + 3x - 4 = 0$$
$$\therefore\ (x + 4)(x - 1) = 0$$
$$\therefore\ x = -4 \text{ or } x = 1$$

When $x = 0,\ y = 0 - \dfrac{4}{3}$

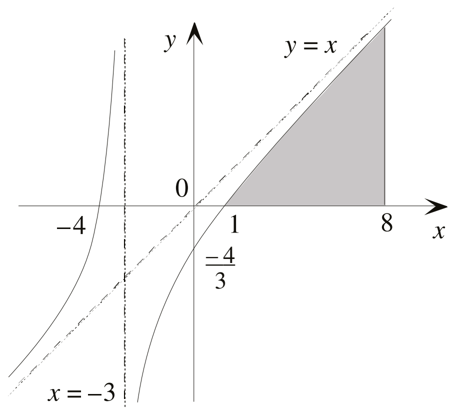
$$= -\frac{4}{3}$$

Hence intercepts with the axes are: $(-4, 0), (1, 0)$ and $\left(0, -\dfrac{4}{3}\right)$

b $y = x$ non-vertical asymptote

$x = -3$ vertical asymptote

c

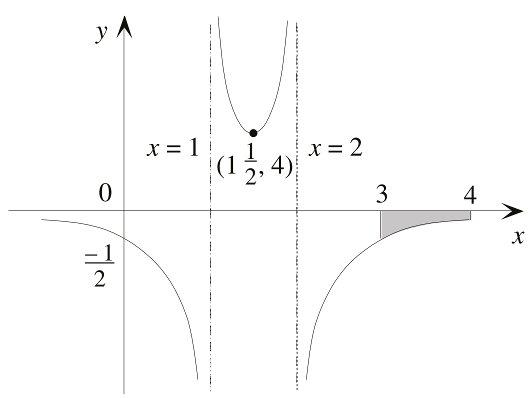


d

$$\text{Area} = \int_1^8 \left(x - \frac{4}{x+3}\right)dx$$
$$= \left[\frac{x^2}{2} - 4\log_e(x+3)\right]_1^8$$
$$= \frac{64}{2} - 4\log_e 11 - \frac{1}{2} + 4\log_e 4$$
$$= \left(31\frac{1}{2} + 4\log_e \frac{4}{11}\right) \text{ square units}$$

8 **a** $\mathbb{R}\backslash\{1, 2\}$

b



$$\frac{dy}{dx} = \frac{2x-3}{[(1-x)(x-2)]^2}$$
$$\therefore \frac{2x-3}{[(1-x)(x-2)]^2}$$
$$= 0 \text{ for stationary points}$$
$$\therefore x = \frac{3}{2}$$

$\therefore \left(\frac{3}{2}, 4\right)$ is a minimum turning point

Equations of asymptotes are:

$y = 0, \ x = 1$ and $x = 2$

c Range of $g = \mathbb{R}^- \cup [4, \infty)$

d

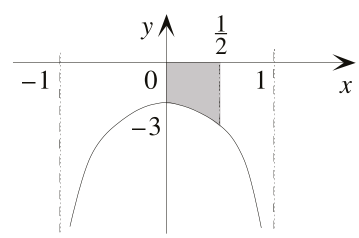
$$\text{Area} = -\int_3^4 \frac{dx}{(1-x)(x-2)}$$
$$= \int_3^4 \frac{dx}{(x-1)(x-2)}$$
$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$
$$1 = A(x-2) + B(x-1)$$

When $x = 2, \ B = 1$

When $x = 1, \ A = -1$

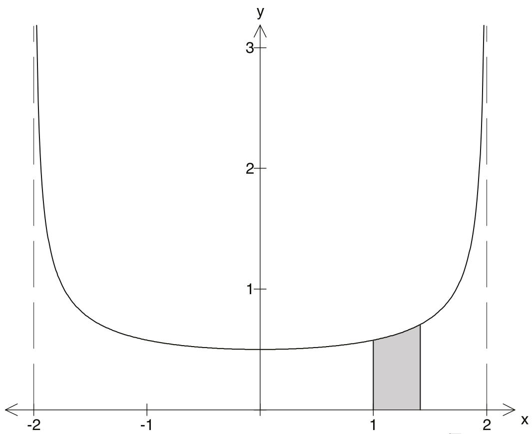
$$\text{Area} = \int_3^4 \frac{dx}{x-2} - \int_3^4 \frac{dx}{x-1}$$
$$= \left[\log_e \frac{x-2}{x-1}\right]_3^4$$
$$= \log_e \frac{2}{3} - \log_e \frac{1}{2}$$
$$= \log_e \frac{4}{3} \text{ square units}$$

9



$$\int_0^{\frac{1}{2}} \frac{-3}{\sqrt{1-x^2}}dx = -3[\sin^{-1} x]_0^{\frac{1}{2}}$$
$$= -3 \times \frac{\pi}{6}$$
$$= -\frac{\pi}{2}$$

10

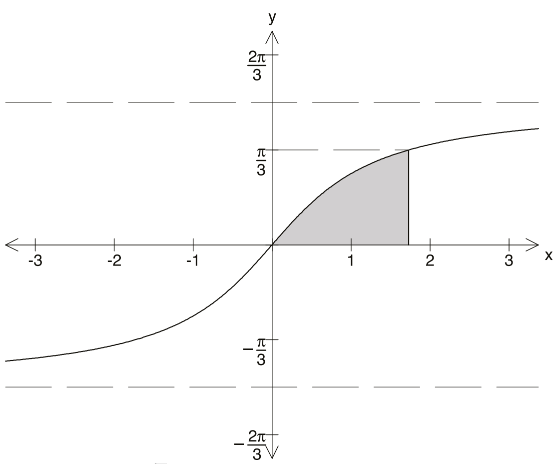


$$\text{Area} = \int_1^{\sqrt{2}} \frac{dx}{\sqrt{4 - x^2}} = \left[\sin^{-1} \frac{x}{2}\right]_1^{\sqrt{2}}$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$= \frac{\pi}{12} \text{ square units}$$

11 $y = \tan^{-1} x$



$$\text{Area} = \int_0^{\sqrt{3}} \tan^{-1} x \, dx$$

$$= \frac{\pi}{3} \times \sqrt{3} - \int_0^{\frac{\pi}{3}} \tan y \, dy$$

$$= \frac{\pi\sqrt{3}}{3} + [\log_e \cos y]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{\pi\sqrt{3}}{3} + \log_e \frac{1}{2}\right)$$

$$= \left(\frac{\pi\sqrt{3}}{3} - \log_e 2\right) \text{ square unit}$$

12 $$\text{Area} = \int_1^e \frac{2 \log_e x}{x} dx = 2 \int_0^1 u \, du$$

where $u = \log_e x, \; \frac{du}{dx} = \frac{1}{x}$

$$\therefore \text{ Area} = [u^2]_0^1$$

$$= 1 \text{ square unit}$$

13

$$\text{Area} = \int_0^{\frac{\pi}{2}} \sin^3 2x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 2x) \sin 2x \, dx$$

Let $\cos 2x = u$

Then $\frac{du}{dx} = -2 \sin 2x$

When $x = 0, \; \cos 2x = 1$

and when $x = \frac{\pi}{2}, \cos 2x = -1$

$$\therefore \text{ Area} = -\frac{1}{2} \int_1^{-1} (1 - u^2) du$$

$$= -\frac{1}{2}\left[u - \frac{u^3}{3}\right]_1^{-1}$$

$$= -\frac{1}{2}\left(-1 + \frac{1}{3}\right) + \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$= \frac{2}{3} \text{ square units}$$

14 $$\text{Area} = \int_0^{\frac{\pi}{2}} \cos^2 x \sin x \, dx$$

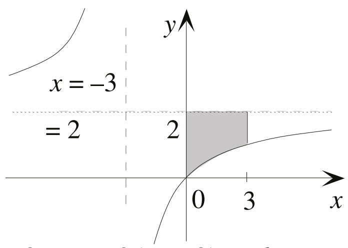
Let $u = \cos x, \; \frac{du}{dx} = -\sin x$

When $x = 0, \; \cos x = 1$

and when $x = \frac{\pi}{2}, \cos x = 0$

$$\therefore \text{ Area} = -\int_1^0 u^2 du$$
$$= \left[\frac{u^3}{3}\right]_0^1$$
$$= \frac{1}{3} \text{ square units}$$

15



$$\frac{2x}{x+3} = \frac{2(x+3)-6}{x+3}$$
$$= 2 - \frac{6}{x+3}$$
$$\text{Area} = 6 - \int_0^3 \left(2 - \frac{6}{x+3}\right)dx$$
$$= 6 - [2x - 6\log_e(x+3)]_0^3$$
$$= 6 - (6 - 6\log_e 6) + (-6\log_e 3)$$
$$= 6\log_e 2 \text{ square units}$$

16 a $\frac{dy}{dx} = \frac{3(4x-1)}{(-2x^2+x+1)^2}$

$\therefore \frac{3(4x-1)}{(-2x^2+x+1)^2}$

$= 0$ for stationary points

$\therefore 4x - 1 = 0$

$\therefore x = \frac{1}{4}$

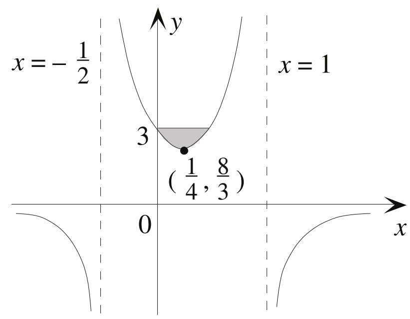
As there is only one possible value for x above, there is only one turning point.

b When $x = \frac{1}{4}$, $y = \frac{8}{3}$

x	0	0.25	0.5
$\frac{dy}{dx}$	−3	0	3
Slope	\	—	/

Hence the turning point $\left(\frac{1}{4}, \frac{8}{3}\right)$ is a minimum.

c



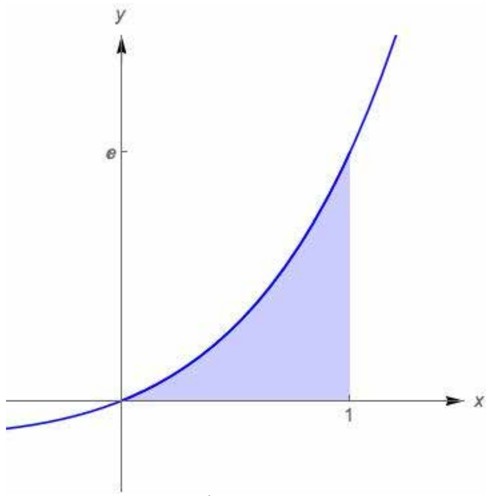
d When $y = 3$, $x = 0$ or $x = \frac{1}{2}$

$$\therefore \text{ Area} = \frac{1}{2} \times 3 - \int_0^{\frac{1}{2}} \frac{3dx}{(2x+1)(1-x)}$$

$$\frac{3}{(2x+1)(-x+1)} = \frac{A}{2x+1} + \frac{B}{-x+1}$$
$$-A + 2B = 0$$
$$A + B = 3$$
$$B = 1$$
$$A = 2$$

$$\therefore \text{ Area} = \frac{3}{2} + \int_0^{\frac{1}{2}} \frac{dx}{-x+1} + 2\int_0^{\frac{1}{2}} \frac{dx}{2x+1}$$
$$= \frac{3}{2} - \left[\log_e\left(\frac{x+\frac{1}{2}}{1-x}\right)\right]_0^{\frac{1}{2}}$$
$$= \frac{3}{2} - \log_e 2 + \log_e \frac{1}{2}$$
$$= \left(\frac{3}{2} - \log_e 4\right) \text{ square units}$$

17 a



Let $u = x$ and $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$ and $v = e^x$

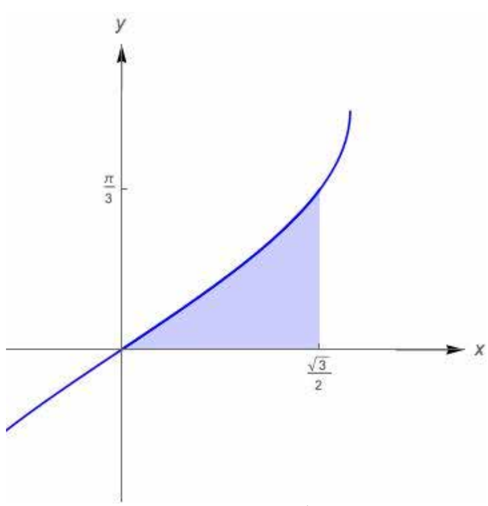
$\int_0^1 xe^x \, dx$

$= [xe^x]_0^1 - \int_0^1 e^x \, dx$

$= e - (e - 1)$

$= 1$

b



Let $u = \arcsin(x)$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}}$ and $v = x$

$\int_0^{\frac{\sqrt{3}}{2}} \arcsin x \, dx$

$= [x \arcsin x]_0^{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}} \, dx$

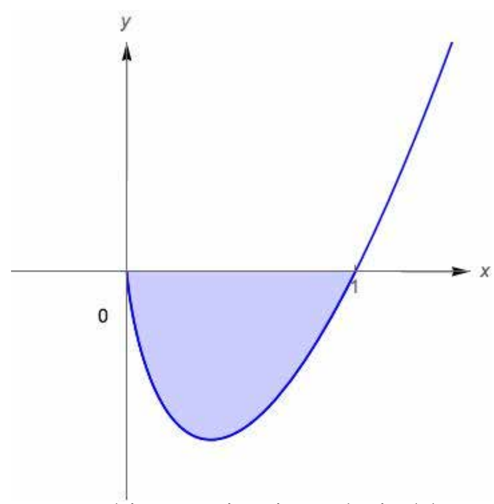
$= \frac{\sqrt{3}\pi}{6} - \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}} \, dx$

$= \frac{\sqrt{3}\pi}{6} + \left[\sqrt{1 - x^2} \right]_0^{\frac{\sqrt{3}}{2}}$

$= \frac{\sqrt{3}\pi}{6} + \frac{1}{2} - 1$

$= \frac{\sqrt{3}\pi}{6} - \frac{1}{2}$

c



Note: This question is undesirable because the function is not defined at $x = 0$

It does return a correct answer because of a limits argument which is beyond this course.

Let $u = \log_e(x)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^2}{2}$

$$\int_0^1 x\log_e x\,dx$$

$$= \left[\frac{x^2}{2}\log_e(x)\right]_0^1 - \int_0^1 \frac{1}{x}\times\frac{x^2}{2}\,dx$$

$$= 0 - \int_0^1 \frac{x}{2}\,dx$$

$$= \left[-\frac{x^2}{4}\right]_0^1$$

$$= -\frac{1}{4}$$

Area is $\frac{1}{4}$

c' Question changes to :

Find the area bounded by the x-axis, the graph of $y = x\log_e(x+1)$ and the line $x = 0$.

$\int_0^1 x\log_e(x+1)\,dx$

Let $u = \log_e(x+1)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x+1}$ and $v = \frac{x^2}{2}$

$$\int_0^1 x\log_e(x+1)\,dx$$

$$= \left[(\frac{x^2}{2})\log_e(x+1)\right]_0^1$$

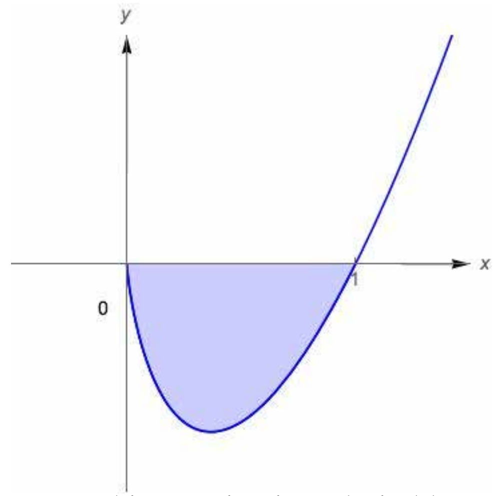
$$- \int_0^1 \frac{1}{x+1}\times(\frac{x^2}{2})\,dx$$

$$= \frac{1}{2}\log_e 2 - \int_0^1 \frac{1}{2(x+1)} + \frac{x}{2} - \frac{1}{2}\,dx$$

$$= \frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 2 + \frac{1}{4}$$

$$= \frac{1}{4}$$

d



Note: This question is undesirable because the function is not defined at $x = 0$

It does return a correct answer because of a limits argument which is beyond this course.

Let $u = \log_e(x)$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^2}{2}$

$$\int_0^1 x\log_e x\,dx$$

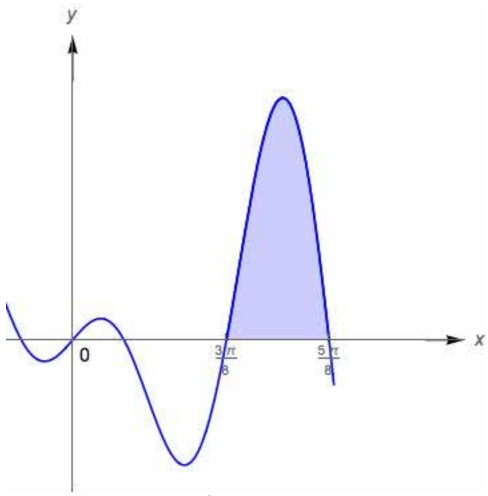
$$= \left[\frac{x^2}{2}\log_e(x)\right]_0^1 - \int_0^1 \frac{1}{x}\times\frac{x^2}{2}\,dx$$

$$= 0 - \int_0^1 \frac{x}{2}\,dx$$

$$= \left[-\frac{x^2}{4}\right]_0^1$$

$$= -\frac{1}{4}$$

e



Let $u = x$ and $\dfrac{dv}{dx} = \cos(4x)$

$\dfrac{du}{dx} = 1$ and $v = \dfrac{1}{4}\sin(4x)$

$$\int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} x\cos(4x)\,dx$$

$$= \left[\frac{x\sin 4x}{4}\right]_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} - \int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{4}\sin(4x)\,dx$$

$$= \frac{\pi}{4}$$

f $\pi^2 - 4$

18 $\int_0^a x\sqrt{a^2 - x^2}\,dx$

Let $u = a^2 - x^2$. Then $\dfrac{du}{dx} = -2x$

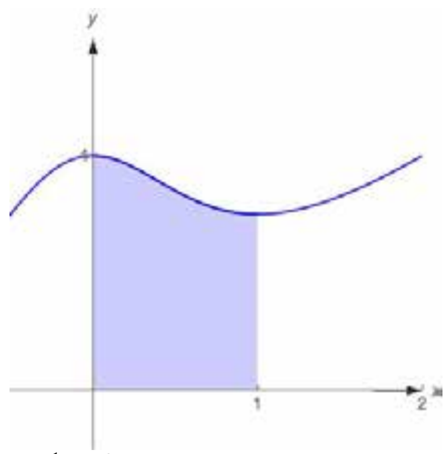
When $x = 0, u = a^2$. When $x = \dfrac{a}{2}, u = \dfrac{3a^2}{4}$

$$\int_0^{\frac{a}{2}} x\sqrt{a^2 - x^2}\,dx$$

$$= -\frac{1}{2}\int_{a^2}^{\frac{3a^2}{4}} \sqrt{u}\,du$$

$$= a^3\left(\frac{1}{3} - \frac{\sqrt{3}}{8}\right)$$

19



$$\int_0^1 \frac{2x^3 + 4}{x^2 + 1}\,dx$$

$$= \int_0^1 2x - \frac{2x}{x^2 + 1} + \frac{4}{x^2 + 1}\,dx$$

$$= [x^2 - \log_e(x^2 + 1) + 4\tan^{-1} x]_0^1$$

$$= 1 - \log_e 2 + \pi$$

Solutions to Exercise 10B

1

$$y = x^2 - 2x$$
$$y = -x^2 + 8x - 12$$
$$\therefore \quad x^2 - 2x = -x^2 + 8x - 12$$
$$2x^2 - 10x + 12 = 0$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2, \; x = 3$$
$$y = 0, \; y = 3$$

hence points of intersection are (3, 3) and (2, 0)

$$\text{Area} = \int_2^3 [(-x^2 + 8x - 12) - (x^2 - 2x)]dx$$
$$= \int_2^3 (-2x^2 + 10x - 12)dx$$
$$= \left[\frac{-2x^3}{3} + 5x^2 - 12x\right]_2^3$$
$$= -9 + \frac{28}{3}$$
$$= \frac{1}{3} \text{ square units}$$

2

$$y = -x^2$$
$$y = x^2 - 2x$$
$$\therefore \quad -x^2 = x^2 - 2x$$
$$2x^2 - 2x = 0$$
$$x = 0, \; x = 1$$

points of intersection are (0, 0) and (1, −1)

$$\text{Area} = \int_0^1 (-x^2 - x^2 + 2x)dx$$
$$= \int_0^1 (-2x^2 + 2x)dx$$
$$= \left[-\frac{2x^3}{3} + x^2\right]_0^1$$
$$= 1 - \frac{2}{3}$$
$$= \frac{1}{3} \text{ square units}$$

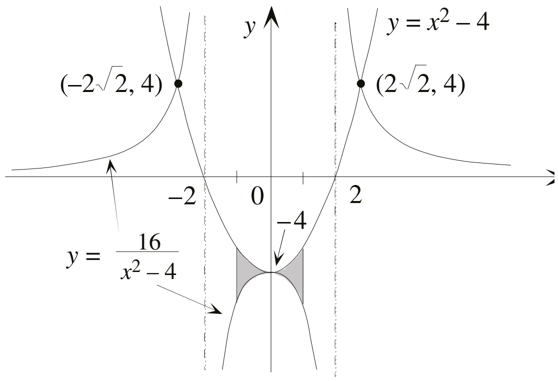
3 **a**

$$\frac{1}{x^2} = x^2$$
$$x = \pm 1$$
$$A = \int_{-1}^{\frac{1}{2}} \left(\frac{1}{x^2} - x^2\right)dx$$
$$= \left[-\frac{1}{x} - \frac{x^3}{3}\right]_{-1}^{-\frac{1}{2}}$$
$$= \left(2 + \frac{1}{24}\right) - \left(1 + \frac{1}{3}\right)$$
$$= \frac{49}{24} - \frac{4}{3}$$
$$= \frac{17}{24} \text{ square units}$$

b

$$B = \int_0^1 x^2\,dx + \int_1^2 \frac{1}{x^2}dx$$
$$= \left[\frac{x^3}{3}\right]_0^1 + \left[-\frac{1}{x}\right]_1^2$$
$$= \frac{1}{3} - \frac{1}{2} + 1$$
$$= \frac{5}{6} \text{ square units}$$

4



Intersections:

$$x^2 - 4 = \frac{16}{x^2 - 4}$$

$$x^2 - 4 = \pm 4$$

$x = 0,\ x = \pm 2\sqrt{2}$

By symmetry,

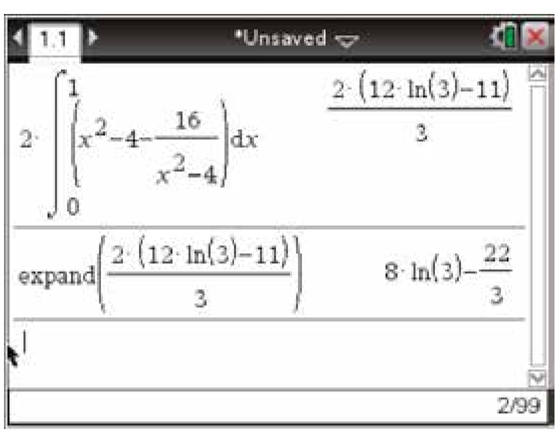
$$\text{Area} = 2\int_0^1 \left(x^2 - 4 - \frac{16}{x^2 - 4}\right)dx$$

$$= 2\left[\frac{x^3}{3} - 4x - 4\log_e\left(\frac{x-2}{x+2}\right)\right]_0^1$$

$$= 2\left(\frac{1}{3} - 4 - 4\log_e\frac{1}{3}\right)$$

$$= \left(8\log_e 3 - \frac{22}{3}\right) \text{ square units}$$

Using CAS:



5 $\text{Area} = \int_1^a \frac{12}{x}dx = [12\log_e]_1^a$

$= 12\log_e$

$\therefore 12\log_e a = 24$

$\log_e a = 2$

$a = e^2$

Using CAS:

6 $y = 4 - x^2$ has x axis intercepts at $x = \pm 2$

$\therefore$ The straight line has equation $y = 2 - x$

Intersections:

$$4 - x^2 = 2 - x$$

$$x^2 - x - 2 = 0$$

$(x + 1)(x - 2) = 0$

$x = -1,\ y = 1$ is a point where the parabola and a straight line meet.

a $A = \int_{-1}^{2}[(4 - x^2) - (2 - x)]dx$

$$= \int_{-1}^{2}(2 + x - x^2)dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(6 - \frac{8}{3}\right) - \left(-\frac{3}{2} + \frac{1}{3}\right)$$

$$= \frac{10}{3} + \frac{7}{6}$$

$$= \frac{9}{2} \text{ square units}$$

b $B = \int_{-2}^{-1} [(2 - x) - (4 - x^2)]dx$

$= \int_{-2}^{-1} (x^2 - x - 2)dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1}$

$= \frac{7}{6} + \frac{2}{3}$

$= \frac{11}{6}$ square units

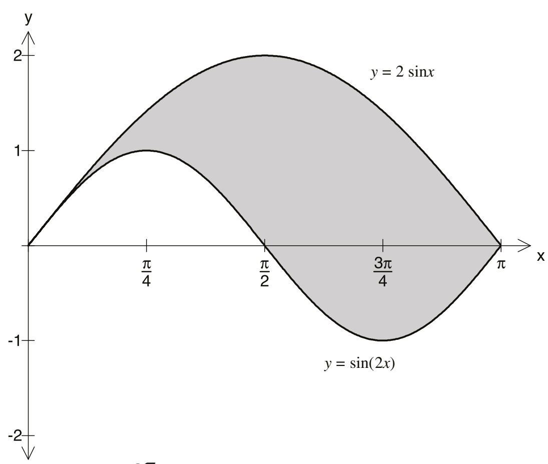
c

$C = \int_{2}^{3} [(2 - x) - (4 - x^2)]dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{3}$

$= \frac{11}{6}$ square units ($C = B$ by symmetry)

7 a



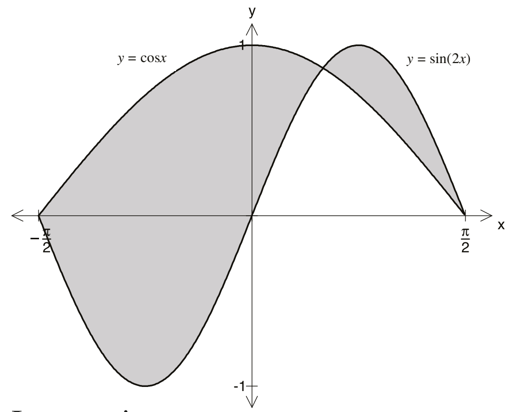
Area $= \int_{0}^{\pi} (2\sin x - \sin 2x)dx$

$= \left[-2\cos x + \frac{1}{2}\cos 2x\right]_{0}^{\pi}$

$= 2 + \frac{1}{2} + 2 - \frac{1}{2}$

$= 4$ square units

b



Intersection:

$\sin 2x = \cos x$

$2\sin x\cos x = \cos x$

$\cos x = 0$

$x = \pm\frac{\pi}{2}$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$

$y = \frac{\sqrt{3}}{2}$

Area $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (\cos x - \sin 2x)dx$

$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$

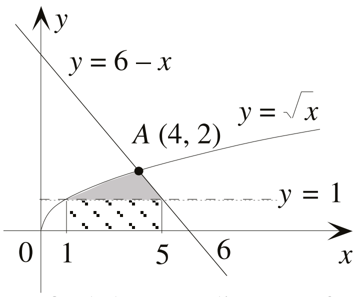
$- \left[\sin x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \frac{1}{2} + \frac{1}{4} + 1 + \frac{1}{2}$

$- 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}$

$= 2\frac{1}{2}$ square units

c



To find the coordinates of point A:

$\sqrt{x} = 6 - x$

Let $\sqrt{x} = u$

Then $u^2 + u - 6 = 0,\ u > 0$

$$(u-2)(u+3) = 0$$

$$\therefore\quad u = 2$$

$$\therefore\quad x = 4,\ y = 2$$

$$\text{Area} = \int_1^4 \sqrt{x}\,dx + \int_4^5 (6-x)dx - 4$$

(4 is the area of the rectangle)

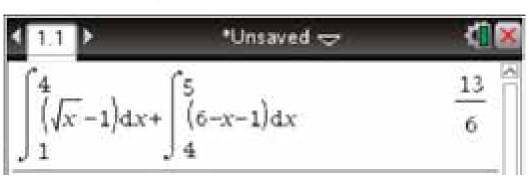
$$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^4 + \left[6x - \frac{x^2}{2}\right]_4^5 - 4$$

$$= \frac{2}{3}(8-1) + \left(30 - \frac{25}{2}\right) - (24-8) - 4$$

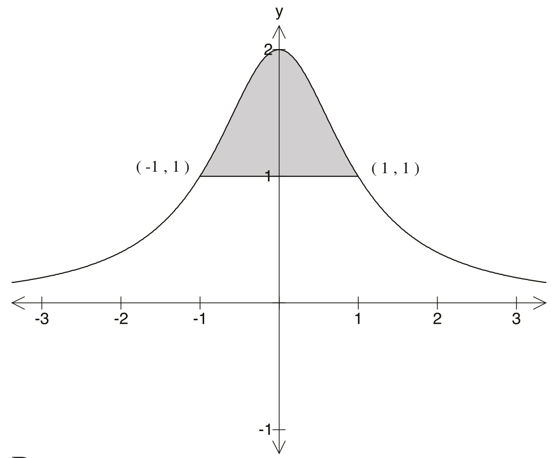
$$= \frac{14}{3} + \frac{35}{2} - 16 - 4$$

$$= 2\frac{1}{6} \text{ square units}$$

Alternatively,



d



By symmetry,

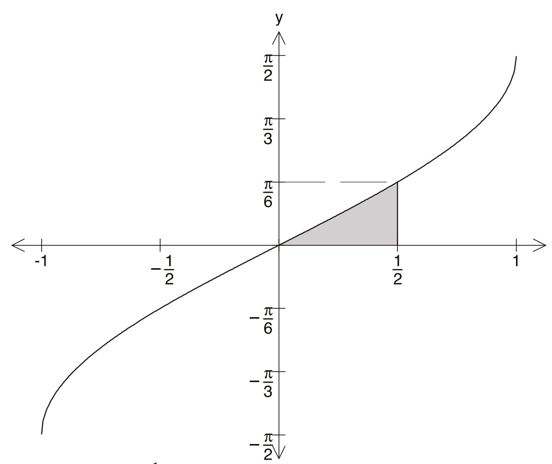
$$\text{Area} = 2\left(\int_0^1 \frac{2\,dx}{1+x^2} - 1\right)$$

$$= [4\tan^{-1} x - 2x]_0^1$$

$$= (\pi - 2) - 0$$

$$= \pi - 2 \text{ square units}$$

e



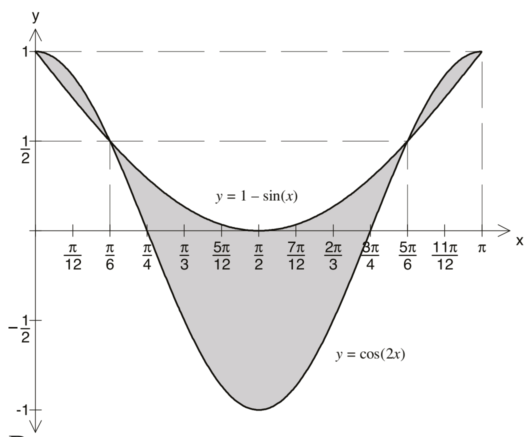
$$\text{Area} = \int_0^{\frac{1}{2}} \sin^{-1}(x)dx$$

$$= \frac{\pi}{12} - \int_0^{\frac{\pi}{6}} \sin(y)dy$$

$$= \frac{\pi}{12} + [\cos y]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right) \text{ square units}$$

f



By symmetry,

$$\text{Area} = 2\left[\int_0^{\frac{\pi}{6}} (\cos 2x - 1 + \sin x)dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \sin x - \cos 2x)dx\right]$$

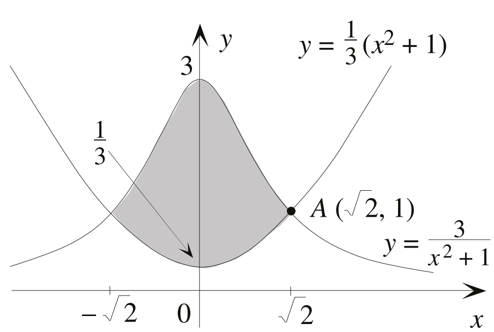
$$= 2\left[\frac{1}{2}\sin 2x - x - \cos x\right]_0^{\frac{\pi}{6}} + 2\left[-\frac{1}{2}\sin 2x + x + \cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\sqrt{3}}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{2} + \frac{\pi}{2}\right)$$

$$= 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1\right)$$

$$= \left(\frac{\pi}{3} + 2 - \sqrt{3}\right) \text{ square units}$$

g



To find the coordinates of A,

$$\frac{1}{3}(x^2 + 1) = \frac{3}{x^2 + 1}$$

$$(x^2 + 1)^2 = 9$$

$$x^2 + 1 = 3$$

$$x = \sqrt{2}$$

$$y = 1$$

By symmetry,

$$\text{Area} = 2\int_0^{\sqrt{2}}\left[\frac{3}{x^2 + 1} - \frac{1}{3}(x^2 + 1)\right]dx$$

$$= 6[\tan^{-1} x]_0^{\sqrt{2}} - \frac{2}{3}\left[\frac{x^3}{3} + x\right]_0^{\sqrt{2}}$$

$$= 6\tan^{-1}\sqrt{2} - \frac{4\sqrt{2}}{9} - \frac{2\sqrt{2}}{3}$$

$$= 6\tan^{-1}\sqrt{2} - \frac{10\sqrt{2}}{9}$$

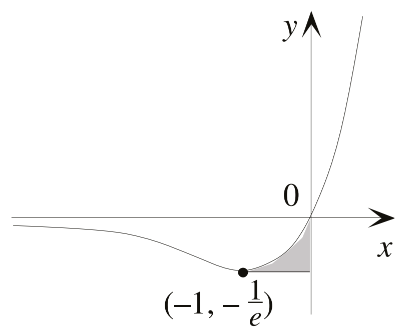
≈ 4.161 square units

8 **a** $f(x) = xe^x$

$\therefore\ f'(x) = e^x + xe^x = (x + 1)e^x$

b $f'(x) = 0$ when $x = -1 \because e^x \neq 0$

c



d $f'(-1) = 0$

Hence,

when $x = -1$, the equation of the tangent is $y = -\frac{1}{e}$

e Area $= \dfrac{1}{e} + \displaystyle\int_{-1}^{0} xe^x\,dx$

$= \dfrac{1}{e} + [xe^x - e^x]_{-1}^{0}$

$= \dfrac{1}{e} - 1 + \dfrac{1}{e} + \dfrac{1}{e}$

$= \left(\dfrac{3}{e} - 1\right)$ square units

Note: $f(x) = f'(x) - e^x$

$\Rightarrow \displaystyle\int f(x)dx = \int f'(x)dx - \int e^x dx$

$= f(x) - e^x + c$

$= xe^x - e^x + c$

9 a $f(x) = 1 + \log_e x$

$\dfrac{dy}{dx} = \dfrac{1}{x}$

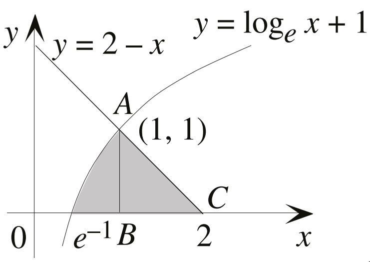
When $x = 1$, $\dfrac{dy}{dx} = 1$

$\therefore \quad m = -1$

$y - 1 = -(x - 1)$

$\therefore \quad y = -x + 2$

b



Area $= \int_{e^{-1}}^{1} (1 + \log_e x)dx + \dfrac{1}{2}$

$\left(\text{Note: Area of triangle } ABC = \dfrac{1}{2}\right)$

$\therefore$ Area $= 1 - \displaystyle\int_{0}^{1} e^{y-1}dy + \dfrac{1}{2}$

$= \dfrac{3}{2} - [e^{y-1}]_{0}^{1}$

$= \dfrac{3}{2} - 1 + e^{-1}$

$= \left(\dfrac{1}{2} + \dfrac{1}{e}\right)$ square units

10 a $(x - 1)(x - 2) = \dfrac{3(x - 1)}{x}$

$A: x = 1,\ y = 0 \Rightarrow (1, 0)$

$B: x - 2 = \dfrac{3}{x}$

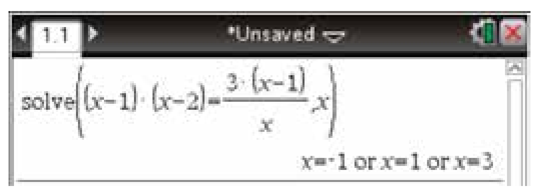
$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

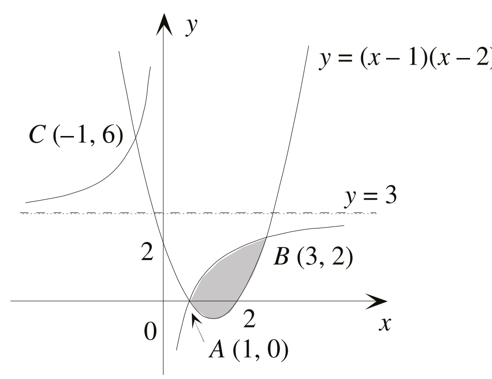
$x = 3,\ y = 2 \Rightarrow (3, 2)$

$C: x = -1,\ y = 6 \Rightarrow (-1, 6)$

Using CAS:



b



c Area $= \int_1^3 \left[\frac{3(x-1)}{x} - (x-1)(x-2)\right] dx$

$= \int_1^3 \left(3 - \frac{3}{x} - x^2 + 3x - 2\right) dx$

$= \left[3x - 3\log_e x - \frac{x^3}{3} + \frac{3x^2}{2} - 2x\right]_1^3$

$= \left(\frac{15}{2} - 3\log_e 3\right) - \left(\frac{13}{6}\right)$

$= \left(5\frac{1}{3} - 3\log_e 3\right)$ square units

11 Intersection:

$3\cos x = 4\sin x$

$\therefore \tan x = \frac{3}{4}$

$x_0 = \tan^{-1}\frac{3}{4}$

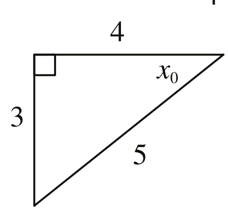
Area $= \int_0^{x_0} 4\sin x\, dx + \int_{x_0}^{\frac{\pi}{2}} 3\cos x\, dx$

$= -4[\cos x]_0^{x_0} + 3[\sin x]_{x_0}^{\frac{\pi}{2}}$

$= -4\cos x_0 + 4 + 3 - 3\sin x_0$

$= 7 - 4\cos x_0 - 3\sin x_0$

Since $\tan x_0 = \frac{3}{4} = \frac{\text{opposite}}{\text{adjacent}}$



$\therefore \sin x_0 = \frac{3}{5}, \cos x_0 = \frac{4}{5}$

$\therefore$ Area $= 7 - \frac{16}{5} - \frac{9}{5}$

$= 7 - 5$

$= 2$ as required

12 **a** The graphs of $y = 9 - x^2$ and $y = \frac{1}{\sqrt{9-x^2}}$ intersect when:

$9 - x^2 = \frac{1}{\sqrt{9-x^2}}$

$\therefore (9-x^2)^{\frac{3}{2}} = 1$

$\therefore 9 - x^2 = 1$

$\therefore x^2 = 8$

$\therefore x = \pm 2\sqrt{2}$

When $x = \pm 2\sqrt{2}$, $y = 1$

Hence the coordinates of the points of intersection are $(-2\sqrt{2}, 1)$ and $(2\sqrt{2}, 1)$

b Required area

$$= \int_{-2\sqrt{2}}^{2\sqrt{2}} 9 - x^2 - \frac{1}{\sqrt{9 - x^2}} dx$$

$$= \left[9x - \frac{1}{3}x^3 - \sin^{-1}\left(\frac{x}{3}\right)\right]_{-2\sqrt{2}}^{2\sqrt{2}}$$

$$= \left[9 \times 2\sqrt{2} - \frac{1}{3}(2\sqrt{2})^3 - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$- \left[9 \times -2\sqrt{2} - \frac{1}{3}(-2\sqrt{2})^3 - \sin^{-1}\left(\frac{-2\sqrt{2}}{3}\right)\right]$$

$$= 18\sqrt{2} - \frac{16\sqrt{2}}{3} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) + 18\sqrt{2} - \frac{16\sqrt{2}}{3} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

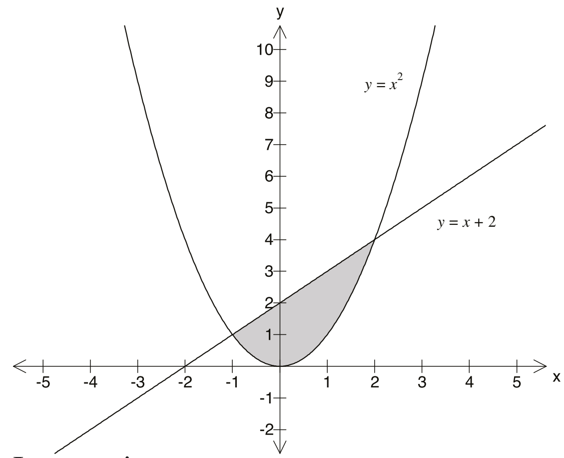
$$= 36\sqrt{2} - \frac{32\sqrt{2}}{3} - 2\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \frac{76\sqrt{2}}{3} - 2\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\approx 33.36$$

The area of the shaded region is 33.36 square units, correct to two decimal places.

13



Intersection:

$$x^2 = x + 2$$

$$\therefore\ x^2 - x - 2 = 0$$

$$\therefore\ (x - 2)(x + 1) = 0$$

$$\therefore\ x = -1 \text{ or } 2$$

Required area

$$= \int_{-1}^{2} x + 2 - x^2 dx$$

$$= \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{-1}^{2}$$

$$= \left[\frac{1}{2}(2)^2 + 2 \times 2 - \frac{1}{3}(2)^3\right] - \left[\frac{1}{2}(-1)^2 + 2 \times -1 - \frac{1}{3}(-1)^3\right]$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{9}{2}$$

The area enclosed by the graphs of $y = x^2$ and $y = x + 2$ is $\frac{9}{2}$ square units.

14 Required area

$$= \int_1^3 \frac{10}{1+x^2} - e^{x-3}dx$$

$$= [10\tan^{-1}x - e^{x-3}]_1^3$$

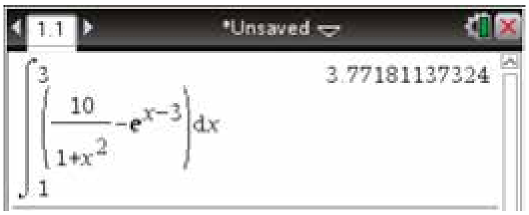
$$= (10\tan^{-1}3 - e^0) - (10\tan^{-1}1 - e^{-2})$$

$$= 10(\tan^{-1}3 - \tan^{-1}1) - 1 + e^{-2}$$

$$= 3.77181\ldots$$

The required area is 3.772 square units, correct to three decimal places.

Alternatively, CAS:



15 **a**

$$f(x) = \frac{8\sqrt{5}}{\sqrt{36-x^2}} - x$$

When $f(x) = 0$, $\dfrac{8\sqrt{5}}{\sqrt{36-x^2}} - x = 0$

$$\therefore \quad \frac{8\sqrt{5}}{\sqrt{36-x^2}} = x$$

$$\therefore \quad 8\sqrt{5} = x\sqrt{36-x^2}$$

$$\therefore \quad (8\sqrt{5})^2 = (x\sqrt{36-x^2})^2$$

$$\therefore \quad 320 = x^2(36-x^2)$$

$$\therefore \quad 320 = 36x^2 - x^4$$

$$\therefore \quad x^4 - 36x^2 + 320 = 0$$

$$\therefore \quad (x^2-16)(x^2-20) = 0$$

$$\therefore \quad x^2 = 16 \text{ or } 20$$

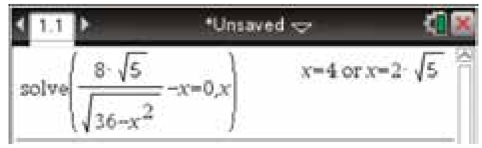
$$\therefore \quad x = \pm 4 \text{ or } \pm 2\sqrt{5}$$

but $x \geq 0$

$$\therefore \quad x = 4 \text{ or } 2\sqrt{5}$$

Therefore, $a = 4$ and $b = 2\sqrt{5}$

Using CAS:



b Total area of shaded regions

$$= \int_0^4 f(x)dx - \int_4^{2\sqrt{5}} f(x)dx$$

$$= \int_0^4 \frac{8\sqrt{5}}{\sqrt{36-x^2}} - x\,dx$$

$$- \int_4^{2\sqrt{5}} \frac{8\sqrt{5}}{\sqrt{36-x^2}} - x\,dx$$

$$= \left[8\sqrt{5}\sin^{-1}\left(\frac{x}{6}\right) - \frac{1}{2}x^2\right]_0^4$$

$$- \left[8\sqrt{5}\sin^{-1}\left(\frac{x}{6}\right) - \frac{1}{2}x^2\right]_4^{2\sqrt{5}}$$

$$= 8\sqrt{5}\sin^{-1}\left(\frac{4}{6}\right) - \frac{1}{2}(4)^2$$

$$- \left[\left(8\sqrt{5}\sin^{-1}\left(\frac{2\sqrt{5}}{6}\right)\right.\right.$$

$$\left. - \frac{1}{2}(2\sqrt{5})^2\right) - \left(8\sqrt{5}\sin^{-1}\left(\frac{4}{6}\right)\right.$$

$$\left.\left. - \frac{1}{2}(4)^2\right)\right]$$

$$= 2\left[8\sqrt{5}\sin^{-1}\left(\frac{2}{3}\right) - 8\right]$$

$$- 8\sqrt{5}\sin^{-1}\left(\frac{\sqrt{5}}{3}\right) + 10$$

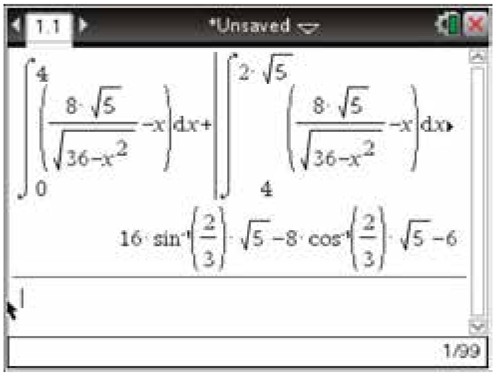
$$= 8\sqrt{5}\left[2\sin^{-1}\left(\frac{2}{3}\right)\right.$$

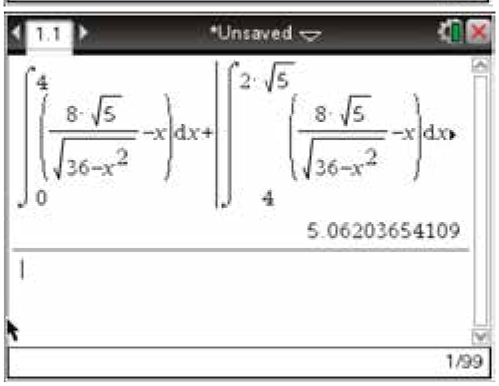
$$\left. - \sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right] - 6$$

$$= 5.06203\ldots$$

The total area of the shaded regions is 5.06 square units, correct to two decimal places.

Using CAS:





16 The graphs of $y = \cos^2 x$ and $y = \sin^2 x$ intersect where $\sin^2 x = \cos^2 x$

$\therefore\ \tan^2 x = 1$

$\therefore\ \tan x = \pm 1$

$\therefore\ x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\text{Total area of shaded regions} = 8\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x\, dx$$

$$= 8\int_0^{\frac{\pi}{4}} \cos 2x\, dx$$

$$= 8\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$

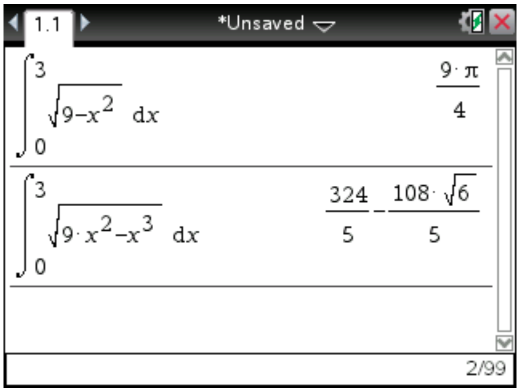
$$= 4\left[\sin\frac{\pi}{2} - \sin 0\right]$$

= 4 square units

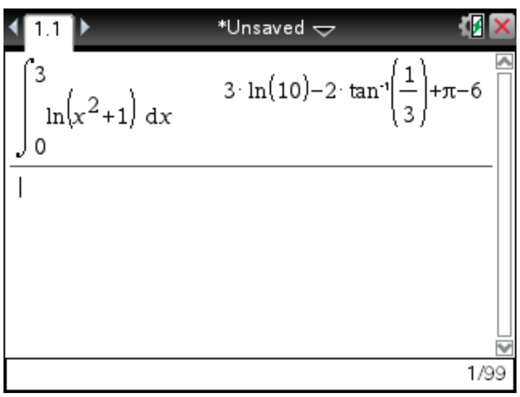
Solutions to Exercise 10C

Set your TI CAS calculators Calculation mode to **Approximate**.
Set your Casio ClassPad to **Decimal** mode.

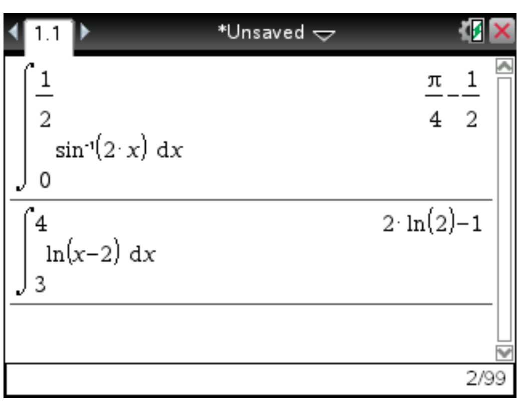
1a,b



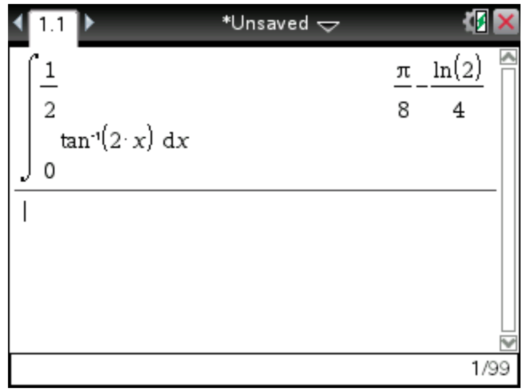
c



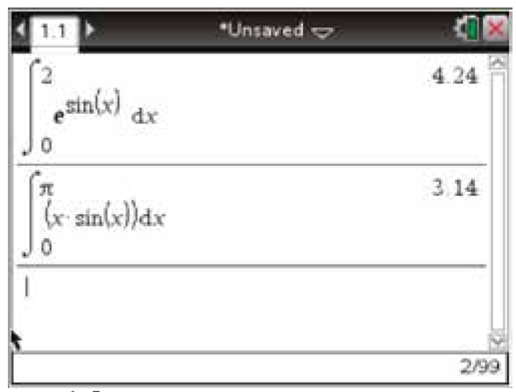
2a,b



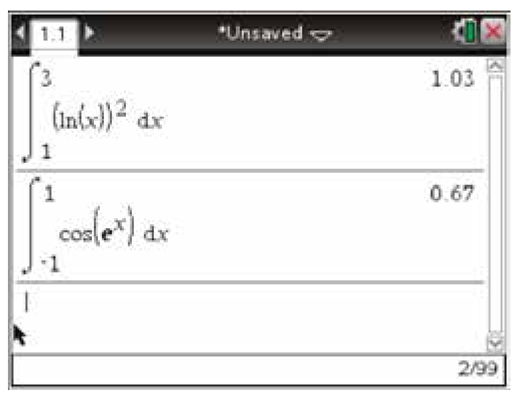
c



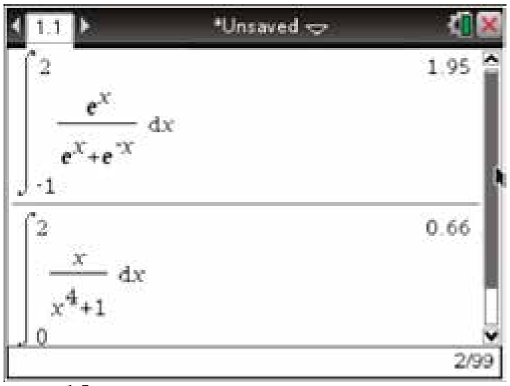
3 **a** and **b**



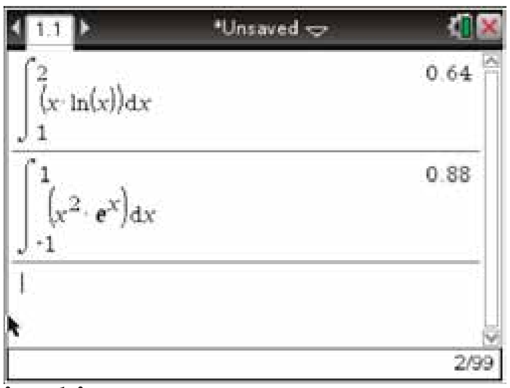
c and **d**



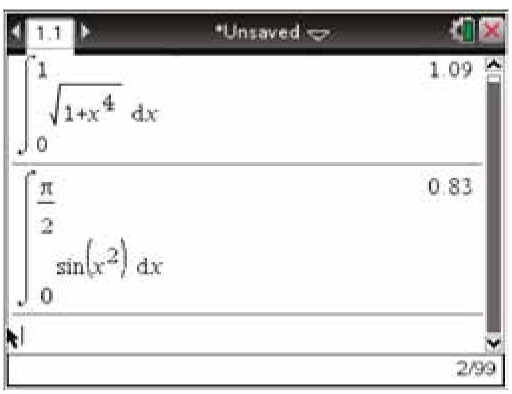
e and **f**



g and **h**



i and **j**



4 **a** $f(x) = \int_1^x \frac{1}{t}dt,\ x > 1$

$= [\log_e t]_1^x$

$= \log_e x - \log_e 1$

$= \log_e x$

b $f(x) = \int_x^1 \frac{1}{t}dt,\ 0 < x < 1$

$= [\log_e t]_x^1$

$= \log_e 1 - \log_e x$

$= -\log_e x$

c $f(x) = \int_0^x e^t dt,\ x \in R$

$= [e^t]_0^x$

$= e^x - e^0$

$= e^x - 1$

d $f(x) = \int_0^x \sin t\, dt,\ x \in R$

$= [-\cos t]_0^x$

$= -[\cos x - \cos 0]$

$= 1 - \cos x$

e $f(x) = \int_{-1}^x \frac{1}{1+t^2}dt,\ x \in R$

$= [\tan^{-1}(t)]_{-1}^x$

$= \tan^{-1}(x) - \tan^{-1}(-1)$

$= \tan^{-1}(x) - \frac{-\pi}{4}$

$= \tan^{-1}(x) + \frac{\pi}{4}$

f $f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}dt, -1 < x < 1$

$= [\sin^{-1}(t)]_0^x$

$= \sin^{-1}(x) - \sin^{-1}(0)$

$= \sin^{-1}(x)$

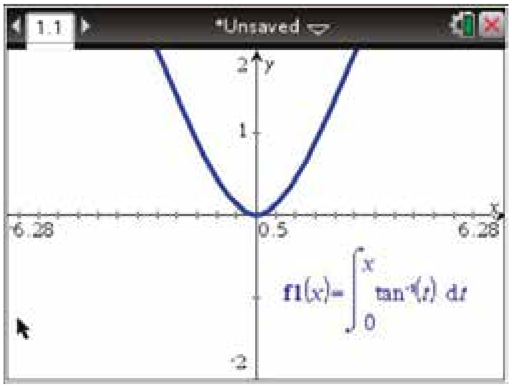
5 **TI and CP:** follow instructions given in Example 16 and 17

a Set $X_{min} = -2\pi$,

$X_{max} = 2\pi$,

$Y_{min} = -2$,

$Y_{max} = 2$

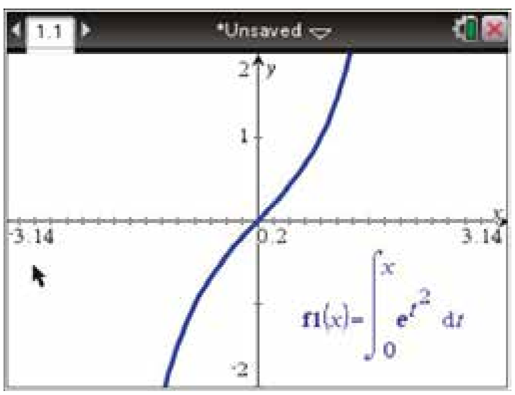


b Set $X_{min} = -\pi$,

$X_{max} = \pi$,

$Y_{min} = -2$,

$Y_{max} = 2$

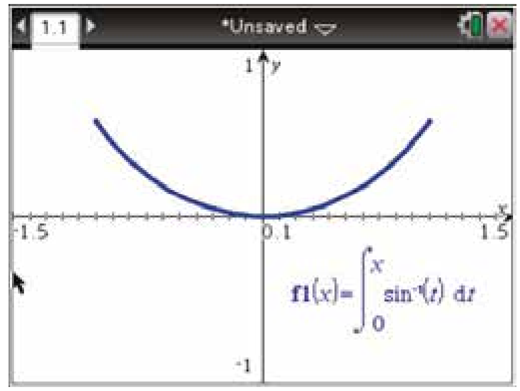


c Set $X_{min} = -1.5$.

$X_{max} = 1.5$

$Y_{min} = -1$

$Y_{max} = 1$

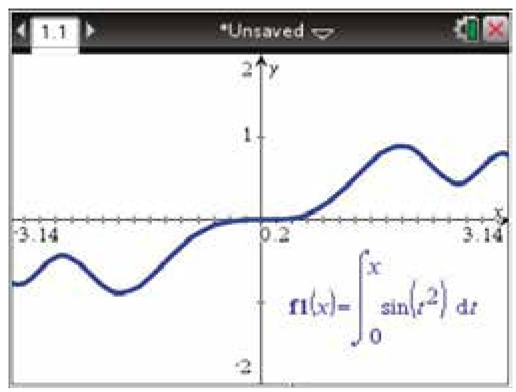


d Set $X_{min} = -\pi$,

$X_{max} = \pi$,

$Y_{min} = -2$,

$Y_{max} = 2$

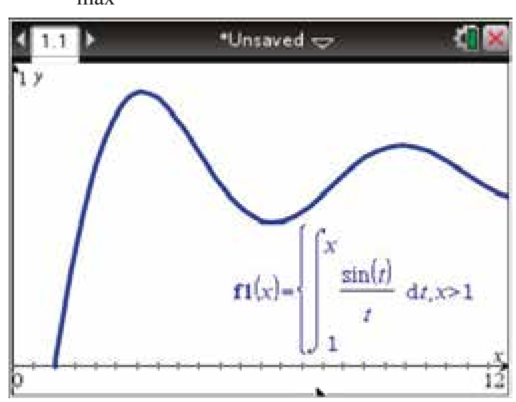


e Set $X_{min} = 0$,

$X_{max} = 12$,

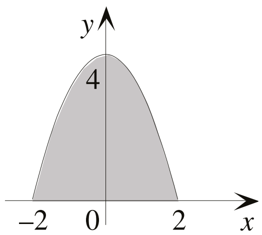
$Y_{min} = -0.2$,

$Y_{max} = 1$



Solutions to Exercise 10D

1



$$\text{Area} = 2\int_0^2 (4 - x^2)dx$$
$$= 2\left[4x - \frac{x^3}{3}\right]_0^2$$
$$= 2\left(8 - \frac{8}{3}\right)$$
$$= \frac{32}{3} \text{ square units}$$
$$\text{Volume} = \pi\int_0^4 x^2 dy$$
$$= \pi\int_0^4 (4 - y)dy$$
$$= \pi\left[4y - \frac{y^2}{2}\right]_0^4$$
$$= \pi(16 - 8)$$
$$= 8\pi \text{ cubic units}$$

2 **a** $V = \pi\int_0^4 (\sqrt{x})^2 dx$
$$= \pi\int_0^4 x\,dx$$
$$= \frac{\pi}{2}[x^2]_0^4$$
$$= 8\pi \text{ cubic units}$$

b $V = \pi\int_0^4 (2x+1)^2 dx$
$$= \frac{\pi}{2} \times \frac{1}{3}[(2x+1)^3]_0^4$$
$$= \frac{\pi}{6}(729 - 1)$$
$$= \frac{364\pi}{3} \text{ cubic units}$$

c Since $2x - 1 = 0$ when $x = \frac{1}{2}$,
$$V = \pi\int_{\frac{1}{2}}^4 (2x-1)^2 dx$$
$$= \frac{\pi}{6}[(2x-1)^3]_{\frac{1}{2}}^4$$
$$= \frac{\pi}{6}(343 - 0)$$
$$= \frac{343\pi}{6} \text{ cubic units}$$

d $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$
$$= \pi\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} dx$$
$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2}\left(\frac{\pi}{2} - 0\right)$$
$$= \frac{\pi^2}{4} \text{ cubic units}$$

e $V = \pi\int_0^2 (e^x)^2 dx$
$$= \pi\int_0^2 e^{2x} dx$$
$$= \frac{\pi}{2}[e^{2x}]_0^2$$
$$= \frac{\pi}{2}(e^4 - 1) \text{ cubic units}$$

f $V = \pi \int_{-3}^{3} (9 - x^2) dx$

$= \pi \left[9x - \frac{x^3}{3} \right]_{-3}^{3}$

$= \pi[(27 - 9) - (-27 + 9)]$

$= 36\pi$ cubic units

3 $V = \pi \int_{1}^{\sqrt{3}} (x^2 - 1) dx$

$= \pi \left[\frac{x^3}{3} - x \right]_{1}^{\sqrt{3}}$

$= \pi \left[(\sqrt{3} - \sqrt{3}) - \left(\frac{1}{3} - 1 \right) \right]$

$= \frac{2\pi}{3}$ cubic units

4 **a** $V = \pi \int_{1}^{4} \frac{1}{x^2} dx$

$= -\pi \left[\frac{1}{x} \right]_{1}^{4}$

$= -\pi \left(\frac{1}{4} - 1 \right)$

$= \frac{3\pi}{4}$ cubic units

b $V = \pi \int_{0}^{1} (x^2 + 1)^2 dx$

$= \pi \int_{0}^{1} (x^4 + 2x^2 + 1) dx$

$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_{0}^{1}$

$= \pi \left(\frac{1}{5} + \frac{2}{3} + 1 \right)$

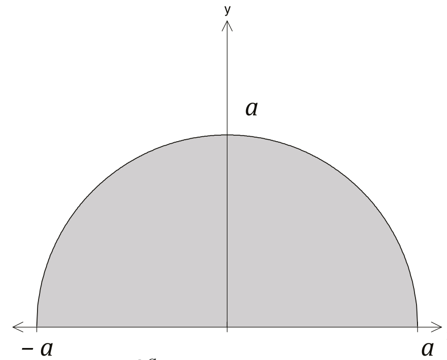
$= \frac{28\pi}{15}$ cubic units

c $V = \pi \int_{0}^{2} x\, dx$

$= \frac{\pi}{2} [x^2]_{0}^{2}$

$= 2\pi$ cubic units

d $y = \sqrt{a^2 - x^2} \Rightarrow x^2 + y^2 = a^2,\ y \geq 0$
Hence y is a semi-circle with centre $(0, 0)$ and radius a.



$V = 2\pi \int_{0}^{a} (\sqrt{a^2 - x^2})^2 dx$

$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_{0}^{a}$

$= 2\pi \left(a^3 - \frac{a^3}{3} \right)$

$= \frac{4\pi a^3}{3}$ cubic units

e Same as **d**, with $a = 3$

$\therefore\ V = \frac{4\pi(3)^3}{3}$

$= 36\pi$ cubic units

f Since $x \geq 0$ volume is the same as $\frac{1}{2}$ of **e**,
$\therefore\ V = 18\pi$ cubic units
Alternatively,

$$V = \pi \int_0^3 9 - x^2 dx$$

$$\therefore \ V = \pi\left[9x - \frac{x^3}{3}\right]_0^3$$

$$\therefore \ V = \pi(27 - 9)$$

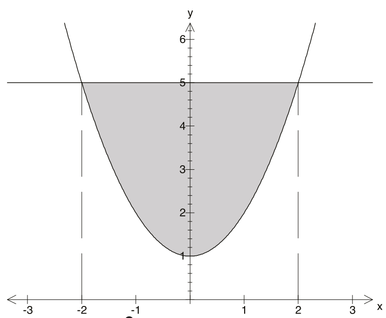
$$\therefore \ V = 18\pi \text{ cubic units}$$

5 Intersection:

$$5 = x^2 + 1$$

$$x^2 = 4$$

$$x = \pm 2$$



$$V = \pi \int_{-2}^{2} (25 - (x^2 + 1)^2) dx$$

$$= 2\pi \int_0^2 (24 - 2x^2 - x^4) dx$$

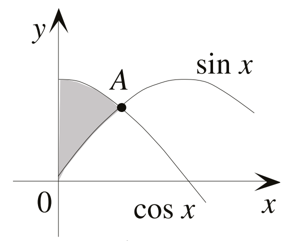
$$= \pi\left[48x - \frac{4x^3}{3} - \frac{2x^5}{5}\right]_0^2$$

$$= \pi\left[96 - \frac{32}{3} - \frac{64}{5}\right]$$

$$= 32\pi\left[3 - \frac{1}{3} - \frac{2}{5}\right]$$

$$= \frac{1088\pi}{15} \text{ cubic units}$$

6



Intersection:

$\cos x = \sin x$

$$\therefore \quad x = \frac{\pi}{4}, \ y = \frac{\sqrt{2}}{2}$$

$$\therefore \quad V = \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= \frac{\pi}{2}[\sin 2x]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}(1 - 0)$$

$$= \frac{\pi}{2} \text{ cubic units}$$

7 $V = \pi \int_1^4 \frac{16}{x^4} dx$

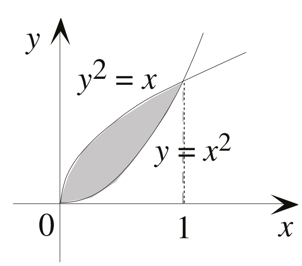
$$= -\frac{16\pi}{3}\left[\frac{1}{x^3}\right]_1^4$$

$$= -\frac{16\pi}{3}\left(\frac{1}{64} - 1\right)$$

$$= -\frac{16\pi}{3} \times -\frac{63}{64}$$

$$= \frac{21\pi}{4} \text{ cubic units}$$

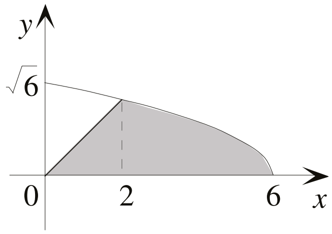
8



$$V = \pi\int_0^1 (-x^4 + x)dx$$
$$= \pi\left[-\frac{x^5}{5} + \frac{x^2}{2}\right]_0^1$$
$$= \pi\left(\frac{1}{2} - \frac{1}{5}\right)$$
$$= \frac{3\pi}{10} \text{ cubic units}$$

9 Intersection:

$$x = \sqrt{6 - x}$$
$$\therefore \quad x^2 = 6 - x,\ x > 0$$
$$\therefore \quad x^2 + x - 6 = 0,\ x > 0$$
$$\therefore \ (x - 2)(x + 3) = 0$$
$$\therefore \quad x = 2,\ x > 0$$

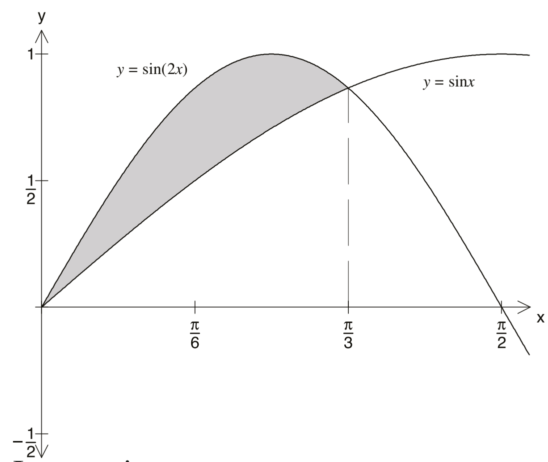


$$V = \pi\int_0^2 x^2 dx + \pi\int_2^6 (6 - x)dx$$
$$= \frac{\pi}{3}[x^3]_0^2 + \pi\left[6x - \frac{x^2}{2}\right]_2^6$$
$$= \frac{8\pi}{3} + 18\pi - 10\pi$$
$$= \frac{32\pi}{3} \text{ cubic units}$$

10 $V = \pi\int_0^{\frac{\pi}{2}} \tan^2 \frac{x}{2}dx$

$$= \pi\int_0^{\frac{\pi}{2}}\left(\sec^2 \frac{x}{2} - 1\right)dx$$
$$= 2\pi\left[\tan \frac{x}{2}\right]_0^{\frac{\pi}{2}} - \pi[x]_0^{\frac{\pi}{2}}$$
$$= 2\pi - \frac{\pi^2}{2}$$
$$= \frac{\pi}{2}(4 - \pi) \text{ cubic units}$$

11



Intersection:

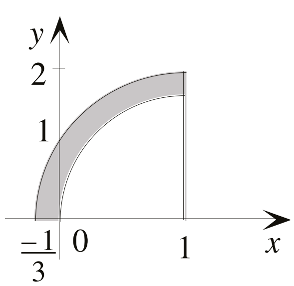
$$\sin x = \sin 2x$$
$$\therefore \ \sin x = 2\sin x\cos x$$
$$\sin x = 0,\ x = 0$$
$$\cos x = \frac{1}{2},\ x = \frac{\pi}{3}$$
$$\text{Area} = \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)dx$$
$$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1$$
$$= \frac{1}{4} \text{ square units}$$

$$V = \pi \int_0^{\frac{\pi}{3}} (\sin^2 2x - \sin^2 x)dx$$
$$= \pi \int_0^{\frac{\pi}{3}} \left(\frac{1 - \cos 4x}{2} - \frac{1 - \cos 2x}{2}\right)dx$$
$$= \frac{\pi}{2} \int_0^{\frac{\pi}{3}} (\cos 2x - \cos 4x)dx$$
$$= \frac{\pi}{4}\left[\sin 2x - \frac{1}{2}\sin 4x\right]_0^{\frac{\pi}{3}}$$
$$= \frac{\pi}{4}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right)$$
$$= \frac{3\pi\sqrt{3}}{16} \text{ cubic units}$$

Note: the following identity was used.
$\cos(2kx) = 1 - 2\sin^2(kx)$

12 $V = \pi \int_b^4 \frac{1}{x^2} dx$
$$= -\pi\left[\frac{1}{x}\right]_b^4$$
$$= \pi\left(\frac{1}{b} - \frac{1}{4}\right)$$
$$\therefore \pi\left(\frac{1}{b} - \frac{1}{4}\right) = 3\pi$$
$$\therefore \frac{1}{b} - \frac{1}{4} = 3$$
$$\therefore b = \frac{4}{13}$$

13



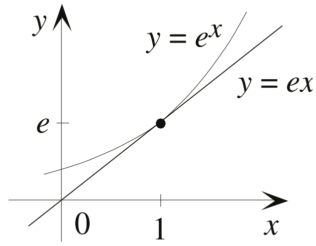
$$V = \pi\left(\int_{-\frac{1}{3}}^{1} (3x + 1)dx - \int_0^1 3x\, dx\right)$$
$$= \pi\left(\left[\frac{3x^2}{2} + x\right]_{-\frac{1}{3}}^{1} - \left[\frac{3x^2}{2}\right]_0^1\right)$$
$$= \pi\left(\frac{3}{2} + 1 - \frac{1}{6} + \frac{1}{3} - \frac{3}{2}\right)$$
$$= \frac{7\pi}{6} \text{ cubic units}$$

14 a $V = \pi \int_0^1 (4y^2 + 4)dy$
$$= 4\pi\left[\frac{y^3}{3} + y\right]_0^1$$
$$= 4\pi\left(\frac{1}{3} + 1\right)$$
$$= \frac{16\pi}{3} \text{ cubic units}$$

b Since $y = \log_e(2 - x)$,
$2 - x = e^y$ and $x = 2 - e^y$
$$V = \pi \int_0^2 (2 - e^y)^2 dy$$
$$= \pi \int_0^2 (4 - 4e^y + e^{2y})dy$$
$$= \pi\left[4y - 4e^y + \frac{1}{2}e^{2y}\right]_0^2$$
$$= \pi\left(8 - 4e^2 + \frac{1}{2}e^4 + 4 - \frac{1}{2}\right)$$
$$= \pi\left(\frac{e^4}{2} - 4e^2 + \frac{23}{2}\right) \text{ cubic units}$$

15 a $y = e^x$
$$\frac{dy}{dx} = e^x$$
When $x = 1$, $\frac{dy}{dx} = e$
Therefore, the tangent has equation
$y - e = e(x - 1)$

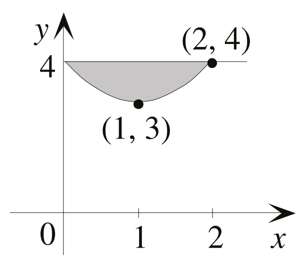
$\therefore\ y = ex$



$$\text{Area} = \int_0^1 (e^x - ex)dx$$
$$= \left[e^x - \frac{ex^2}{2}\right]_0^1$$
$$= \left[\left(e - \frac{1}{2}e\right) - (1)\right]$$
$$= \left(\frac{e}{2} - 1\right) \text{ square units}$$

b $V = \pi \int_0^1 (e^{2x} - e^2x^2)dx$
$$= \pi\left[\frac{1}{2}e^{2x} - e^2\frac{x^3}{3}\right]_0^1$$
$$= \pi\left[\left(\frac{1}{2}e^2 - \frac{1}{3}e^2\right) - \left(\frac{1}{2}\right)\right]$$
$$= \frac{\pi}{6}e^2 - \frac{\pi}{2}$$
$$= \frac{\pi}{6}(e^2 - 3) \text{ cubic units}$$

16



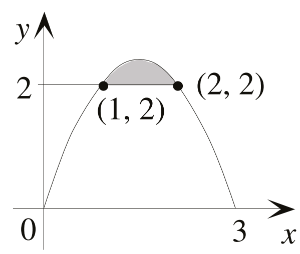
The volume generated by the given region is the same as the volume generated by the region defined by $x^2 - 2x \le y \le 0$, when it is rotated about the x axis.

$$\therefore V = \pi \int_0^2 (x^2 - 2x)^2 dx$$
$$= \pi \int_0^2 (x^4 - 4x^3 + 4x^2)dx$$
$$= \pi\left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_0^2$$
$$= \pi\left[\frac{32}{5} - 16 + \frac{32}{3}\right]$$
$$= 32\pi\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right]$$
$$= \frac{16\pi}{15} \text{ cubic units}$$

17 $V = \pi \int_0^\pi \cos^2 \frac{x}{2} dx$
$$= \pi \int_0^\pi \frac{1 + \cos x}{2} dx$$
$$= \frac{\pi}{2}[x + \sin x]_0^\pi$$
$$= \frac{\pi}{2}(\pi - 0)$$
$$= \frac{\pi^2}{2} \text{ cubic units}$$

Note: the following identity was used.
$\cos(2kx) = 2\cos^2(kx) - 1$

18



$$V = \pi\int_1^2 [(3x - x^2)^2 - 4]dx$$

$$= \pi\int_1^2 (9x^2 - 6x^3 + x^4 - 4)dx$$

$$= \pi\left[3x^3 - \frac{3x^4}{2} + \frac{x^5}{5} - 4x\right]_1^2$$

$$= \pi\left[\left(24 - 24 + \frac{32}{5} - 8\right) - \left(3 - \frac{3}{2} + \frac{1}{5} - 4\right)\right]$$

$$= \pi\left(-\frac{8}{5} + \frac{23}{10}\right)$$

$$= \frac{7\pi}{10} \text{ cubic units}$$

19 To find the intersection of two curves, solve the equation $x^2 + 3x - 4 = 0$.

$\therefore\ (x + 4)(x - 1) = 0$

$\therefore\quad x = 1,\ x = -4$

$\therefore\quad x = 1,\ \because x \geq 0$

$$\therefore\ V = \pi\left(\int_0^1 3x\,dx + \int_1^2 (4 - x^2)dx\right)$$

$$= \pi\left[\frac{3x^2}{2}\right]_0^1 + \pi\left[4x - \frac{x^3}{3}\right]_1^2$$

$$= \pi\left(\frac{3}{2} + 8 - \frac{8}{3} - 4 + \frac{1}{3}\right)$$

$$= \pi\left(\frac{3}{2} + 4 - \frac{7}{3}\right)$$

$$= \frac{19\pi}{6} \text{ cubic units}$$

20 $V = \pi\int_0^{\log_e 2} (e^x - 1)^2 dx$

$$= \pi\int_0^{\log_e 2} (e^{2x} - 2e^x + 1)dx$$

$$= \pi\left[\frac{1}{2}e^{2x} - 2e^x + x\right]_0^{\log_e 2}$$

$$= \pi\left[(2 - 4 + \log_e 2) - \left(\frac{1}{2} - 2\right)\right]$$

$$= \pi\left(\log_e 2 - 2 + 2 - \frac{1}{2}\right)$$

$$= \pi\left(\log_e 2 - \frac{1}{2}\right) \text{ cubic units}$$

21 $V = \pi\int_0^{\log_e 2} e^{-4x}dx$

$$= -\frac{\pi}{4}[e^{-4x}]_0^{\log_e 2}$$

$$= -\frac{\pi}{4}\left(\frac{1}{16} - 1\right)$$

$$= \frac{15\pi}{64} \text{ cubic units}$$

22

$$V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\tan^2 x\, dx$$

$$\therefore \quad V = 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x$$

$$- 1\, dx$$

using $\tan^2 x + 1 = \sec^2 x$

$$\therefore \quad V = 4\pi[\tan x - x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

using $\int \sec^2 x\, dx = \tan x + c$

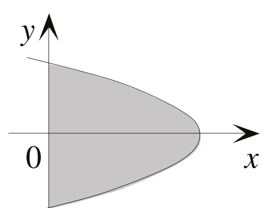
$$\therefore \quad V = 4\pi\left[\left(1 - \frac{\pi}{4}\right)\right.$$

$$\left. - \left(-1 + \frac{\pi}{4}\right)\right]$$

$$\therefore \quad V = 4\pi\left(2 - \frac{\pi}{2}\right)$$

$$\therefore \quad V = 8\pi - 2\pi^2 \text{ cubic units}$$

23 **a** $y^2 = 4(1 - x)$ rotated about the x axis (bounded by the y axis)



When $y = 0, 4(1 - x) = 0$

$$\therefore \quad 1 - x = 0$$

$$\therefore \quad x = 1$$

$$V = \int_0^1 \pi y^2\, dx$$

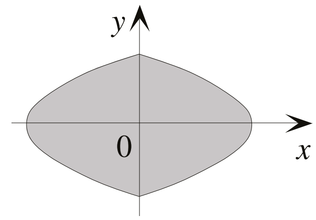
$$= 4\pi \int_0^1 1 - x\, dx$$

$$= 4\pi\left[x - \frac{1}{2}x^2\right]_0^1$$

$$= 4\pi\left(1 - \frac{1}{2}\right)$$

$$= 2\pi \text{ cubic units}$$

b $y^2 = 4(1 - x)$ rotated about the y axis



$$y^2 = 4(1 - x)$$

$$= 4 - 4x$$

$$\therefore \; 4x = 4 - y^2$$

$$\therefore \; x = 1 - \frac{y^2}{4}$$

When $x = 0$, $y^2 = 4(1 - 0)$

$$= 4$$

$$\therefore \quad y = \pm 2$$

$$V = \int_{-2}^{2} \pi x^2\, dy$$

$$= \pi \int_{-2}^{2} \left(1 - \frac{y^2}{4}\right)^2 dy$$

$$= \pi \int_{-2}^{2} 1 - \frac{y^2}{2} + \frac{y^4}{16} dy$$

$$= \pi\left[y - \frac{y^3}{6} + \frac{y^5}{80}\right]_{-2}^{2}$$

$$= \pi\left[\left(2 - \frac{8}{6} + \frac{32}{80}\right) - \left(-2 + \frac{8}{6} - \frac{32}{80}\right)\right]$$

$$= \pi\left(4 - \frac{8}{3} + \frac{4}{5}\right)$$

$$= \frac{32\pi}{15}$$

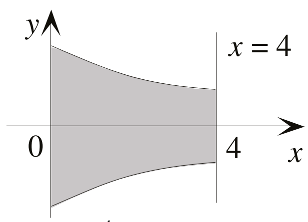
Ratio of volumes

$$= 2\pi : \frac{32\pi}{15}$$

$$= 30\pi : 32\pi$$

$= 15 : 16$, as required to show.

24 a $y = \dfrac{1}{\sqrt{x^2 + 9}}$ rotated about the x axis (bounded by the y axis and $x = 4$)



$$V = \int_0^4 \pi y^2 dx$$

$$= \pi \int_0^4 \frac{1}{x^2 + 9} dx$$

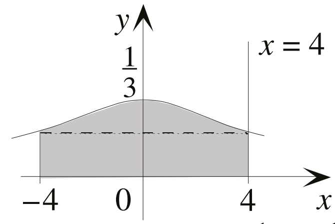
$$= \frac{\pi}{3} \int_0^4 \frac{3}{9 + x^2} dx$$

$$= \frac{\pi}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^4$$

$$= \frac{\pi}{3}\left[\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(0)\right]$$

$= \dfrac{\pi}{3} \tan^{-1}\left(\dfrac{4}{3}\right)$ cubic units

b $y = \dfrac{1}{\sqrt{x^2 + 9}}$ rotated about the y axis (bounded by the x axis and $x = 4$)



When $x = 0$, $y = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

$$y = \frac{1}{\sqrt{x^2 + 9}}$$

$$\therefore \sqrt{x^2 + 9} = \frac{1}{y}$$

$$\therefore x^2 + 9 = \frac{1}{y^2}$$

$$\therefore x^2 = \frac{1}{y^2} - 9$$

Consider the volume in two parts.

When $x = 4, y = \dfrac{1}{\sqrt{4^2 + 9}} = \dfrac{1}{5}$

$$\therefore V = \pi \times 4^2 \times \frac{1}{5} + \int_{\frac{1}{5}}^{\frac{1}{3}} \pi x^2 dy$$

$$= \frac{16\pi}{5} + \pi \int_{\frac{1}{5}}^{\frac{1}{3}} y^{-2} - 9\, dy$$

$$= \frac{16\pi}{5} + \pi[-y^{-1} - 9y]_{\frac{1}{5}}^{\frac{1}{3}}$$

$$= \frac{16\pi}{5} + \pi\left[\left(-3 - 9 \times \frac{1}{3}\right) - \left(-5 - 9 \times \frac{1}{5}\right)\right]$$

$$= \frac{16\pi}{5} + \pi\left[-6 + \frac{34}{5}\right]$$

$$= \frac{16\pi}{5} + \frac{4\pi}{5}$$

$$= \frac{20\pi}{5}$$

$= 4\pi$ cubic units

25 $V = \int_0^{40} \pi x^2\, dy$

Now $y = 40\log_e\left(\frac{x-20}{10}\right)$

$$\therefore \quad \frac{y}{40} = \log_e\left(\frac{x-20}{10}\right)$$

$$\therefore \quad e^{\frac{y}{40}} = \frac{x-20}{10}$$

$$\therefore \quad x - 20 = 10e^{\frac{y}{40}}$$

$$\therefore \quad x = 20 + 10e^{\frac{y}{40}}$$

$$\therefore \quad x^2 = \left(20 + 10e^{\frac{y}{40}}\right)^2$$

$$= 400 + 400e^{\frac{y}{40}} + 100e^{\frac{y}{20}}$$

Therefore,

$$V = \pi\int_0^{40} 400 + 400e^{\frac{y}{40}} + 100e^{\frac{y}{20}}dy$$

$$= 100\pi\int_0^{40} 4 + 4e^{\frac{y}{40}} + e^{\frac{y}{20}}dy$$

$$= 100\pi\left[4y + 160e^{\frac{y}{40}} + 20e^{\frac{y}{20}}\right]_0^{40}$$

$$= 100\pi[(160 + 160e + 20e^2) - (0 + 160 + 20)]$$

$$= 100\pi(20e^2 + 160e - 20)$$

$$= 2000\pi(e^2 + 8e - 1)$$

$$= 176\,779.371\ldots$$

If the bucket were filled to the brim, it could hold 176 779 cm^3, to the nearest cubic centimetre.

26 a

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= b^2 - \left(\frac{b}{a}\right)^2 x^2$$

$$\therefore \quad V = \int_{-a}^{a} \pi y^2\, dx$$

$$= \pi\int_{-a}^{a} b^2 - \left(\frac{b}{a}\right)^2 x^2\, dx$$

$$= \pi\left[b^2x - \frac{1}{3}\left(\frac{b}{a}\right)^2 x^3\right]_{-a}^{a}$$

$$= \pi\left[\left(b^2a - \frac{1}{3}\left(\frac{b}{a}\right)^2 a^3\right) - \left(-b^2a + \frac{1}{3}\left(\frac{b}{a}\right)^2 a^3\right)\right]$$

$$= \pi\left(2ab^2 - \frac{2}{3}ab^2\right)$$

$$= \frac{4\pi ab^2}{3} \text{ cubic units}$$

b

$$x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$

$$= a^2 - \left(\frac{a}{b}\right)^2 y^2$$

$$\therefore \quad V = \int_{-b}^{b} \pi x^2 dx$$

$$= \pi\int_{-b}^{b} a^2 - \left(\frac{a}{b}\right)^2 y^2 dy$$

$$= \pi\left[a^2y - \frac{1}{3}\left(\frac{a}{b}\right)^2 y^3\right]_{-b}^{b}$$

$$= \pi\left[\left(a^2b - \frac{1}{3}\left(\frac{a}{b}\right)^2 b^3\right) - \left(-a^2b + \frac{1}{3}\left(\frac{a}{b}\right)^2 b^3\right)\right]$$

$$= \pi\left(2a^2b - \frac{2}{3}a^2b\right)$$

$$= \frac{4\pi a^2 b}{3} \text{ cubic units}$$

27 a The equation of the line PQ is given by

$$y - 6 = \frac{2-6}{6-2}(x-2)$$

$$= \frac{-4}{4}(x-2)$$

$$= -x + 2$$

$$\therefore\ x + y = 8$$

b i $V = \pi \int_2^6 (8-x)^2 - \left(\frac{12}{x}\right)^2 dx$

$$= \pi \int_2^6 64 - 16x + x^2 - 144x^{-2} dx$$

$$= \pi\left[64x - 8x^2 + \frac{1}{3}x^3 + 144x^{-1}\right]_2^6$$

$$= \pi\left[(384 - 288 + 72 + 24) - \left(128 - 32 + \frac{8}{3} + 72\right)\right]$$

$$= \pi\left(192 - \frac{512}{3}\right)$$

$$= \frac{64\pi}{3} \text{ cubic units}$$

ii $V = \pi \int_2^6 (8-y)^2 - \left(\frac{12}{y}\right)^2 dy$

$= \frac{64\pi}{3}$ cubic units as in part **i**.

28 a When $\frac{dy}{dx} = 0$,

$$2 - \frac{9}{x^2} = 0$$

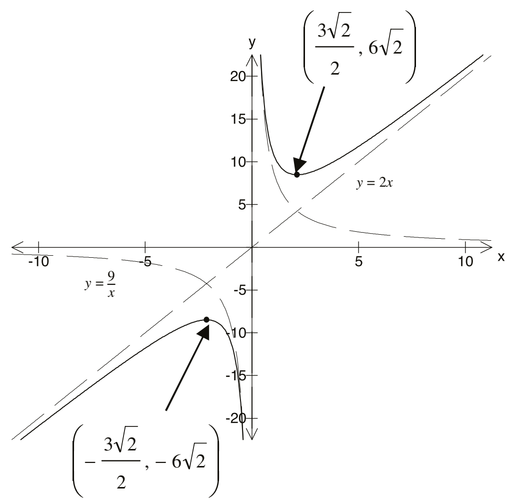
$$\therefore\ x^2 = \frac{9}{2}$$

$$\therefore\ x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$$

$$\therefore\ y = 2\left(\pm\frac{3}{\sqrt{2}}\right) + \frac{9}{\pm\frac{3}{\sqrt{2}}}$$

$$= \pm 3\sqrt{2} \pm 3\sqrt{2}$$

$$= \pm 6\sqrt{2}$$



b $V = \pi \int_1^3 y^2 dx$

$$= \pi \int_1^3 \left(2x + \frac{9}{x}\right)^2 dx$$

$$= \pi \int_1^3 4x^2 + 36 + \frac{81}{x^2} dx$$

$$= \pi\left[\frac{4}{3}x^3 + 36x - \frac{81}{x}\right]_1^3$$

$$= \pi\left[(36 + 108 - 27) - \left(\frac{4}{3} + 36 - 81\right)\right]$$

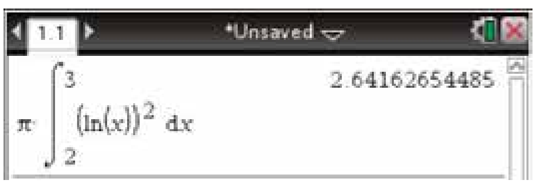
$$= \pi\left(117 + \frac{131}{3}\right)$$

$$= \frac{482\pi}{3} \text{ cubic units}$$

29 $V = \pi \int_2^3 y^2 dx$

$$= \pi \int_2^3 (\log_e x)^2 dx$$

Using CAS:



Thus the volume is 2.642 cubic units correct to three decimal places.

30 $V = \pi \int_0^{\frac{\pi}{3}} 16 - 4\sec^2 x \, dx$

$$= \pi[16x - 4\tan x]_0^{\frac{\pi}{3}}$$

$$= \frac{16\pi^2}{3} - 4\pi \tan\frac{\pi}{3}$$

$$= \frac{16\pi^2}{3} - 4\sqrt{3}$$

$$= 4\pi\left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ cubic units}$$

31 $V = \pi \int_0^{\frac{\sqrt{3}}{2}} x^2 \, dy$

$4x^2 - 1 = 4y^2 \Rightarrow x^2 = \dfrac{4y^2 + 1}{4}$

Therefore,

$$V = \pi \int_0^{\frac{\sqrt{3}}{2}} x^2 \, dy$$

$$= \pi \int_0^{\frac{\sqrt{3}}{2}} \frac{4y^2 + 1}{4} \, dy$$

$$= \pi \int_0^{\frac{\sqrt{3}}{2}} y^2 + \frac{1}{4} \, dy$$

$$= \pi\left[\frac{y^3}{3} + \frac{y}{4}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\sqrt{3}\pi}{4}$$

32 $V = \pi \int_0^2 y^2 \, dx$

$$V = \pi \int_0^2 \frac{4x^2}{(x + 2)^2} \, dx$$

Let $u = x + 2$, $x = u - 2$, $\dfrac{du}{dx} = 1$

$$= \pi \int_2^4 \frac{4(u - 2)^2}{u^2} \, du$$

$$= \pi \int_2^4 4 - \frac{16}{u} + \frac{16}{u^2} \, du$$

$$= \pi\left[4u - 16\log_e u - \frac{16}{u}\right]_2^4$$

$$= \pi(12 - 16\log_e 2)$$

$$= 4\pi(3 - 4\log_e 2)$$

33 $V = \pi \int_0^{\frac{\pi}{2}} x^2 \, dy$

$y = \sin^{-1}(2x^2 - 1)$

$\Rightarrow 2x^2 - 1 = \sin y \Rightarrow x^2 = \frac{1}{2}(\sin y + 1)$

$$V = \pi \int_0^{\frac{\pi}{2}} x^2 \, dy$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (\sin y + 1) \, dy$$

$$= \frac{\pi}{2} [-\cos y + y]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}(\frac{\pi}{2} - (-1))$$

$$= \frac{\pi^2}{4} + \frac{\pi}{2}$$

$$= \frac{\pi^2 + 2\pi}{4}$$

34 $V = \pi \int_0^{\pi} y^2 \, dx$

$$V = \pi \int_0^{\pi} y^2 \, dx$$

$$= \pi \int_0^{\pi} 2 - \cos^2 x \, dx$$

$$= \pi \left[\frac{3x}{2} - \frac{1}{4} \sin 2x \right]_0^{\pi}$$

$$= \frac{3\pi^2}{2}$$

35 $V = \pi \int_0^{\frac{3}{2}} y^2 \, dx$

$$V = \pi \int_0^{\pi} y^2 \, dx$$

$$= \pi \int_0^{\pi} \frac{6 - 4x}{4 + x^2} \, dx$$

$$= \pi \int_0^{\pi} \frac{6}{4 + x^2} - \frac{4x}{4 + x^2} \, dx$$

$$= \pi \left[\frac{3x}{2} - \frac{1}{4} \sin 2x \right]_0^{\pi}$$

$$= \pi(4 \log_e \left(\frac{4}{5}\right) + 3 \tan^{-1} \left(\frac{3}{4}\right))t$$

Solutions to Exercise 10E

1 a
$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$x = t \Rightarrow \frac{dx}{dt} = 1$$
$$\text{Now } y = 2x^{\frac{3}{2}} \Rightarrow \frac{dy}{dt} = 3t^{\frac{1}{2}}$$
$$\therefore L = \int_0^1 \sqrt{1 + 9t}\, dt$$
$$= \int_0^1 (1 + 9t)^{\frac{1}{2}}\, dt$$
$$= \left[\frac{2}{27}(1 + 9t)^{\frac{3}{2}}\right]_0^1$$
$$= \frac{2}{27}10^{\frac{3}{2}} - \frac{2}{27}$$
$$= \frac{2}{27}(10\sqrt{10} - 1)$$
$$= \frac{20\sqrt{10} - 2}{27}$$

b
$$L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$x = t \Rightarrow \frac{dx}{dt} = 1$$
$$\text{Now } y = 2t + 1 \Rightarrow \frac{dy}{dt} = 2$$
$$\therefore L = \int_0^3 \sqrt{1 + 4}\, dt$$
$$= \int_0^3 \sqrt{5}\, dt$$
$$= \left[\sqrt{5}x\right]_0^3$$
$$= 3\sqrt{5}$$

c
$$L = \int_0^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$x = t \Rightarrow \frac{dx}{dt} = 1$$
$$\text{Now } y = \frac{1}{3}(t^2 + 2)^{\frac{3}{2}} \Rightarrow \frac{dy}{dt} = t(t^2 + 2)^{\frac{1}{2}}$$
$$\therefore L = \int_0^6 \sqrt{1 + t^2(t^2 + 2)}\, dt$$
$$= \int_0^6 \sqrt{1 + t^4 + 2t^2}\, dt$$
$$= \int_0^6 t^2 + 1\, dt$$
$$= \left[\frac{t^3}{3} + t\right]_0^6$$
$$= 78$$

2 a
$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$\text{Now } x = t - 1 \Rightarrow \frac{dx}{dt} = 1$$
$$\text{Also } y = t^{\frac{3}{2}} \Rightarrow \frac{dy}{dt} = \frac{3}{2}t^{\frac{1}{2}}$$
$$\therefore L = \int_0^1 \sqrt{1 + \frac{9t}{4}}\, dt$$
$$= \int_0^1 (1 + \frac{9t}{4})^{\frac{1}{2}}\, dt$$
$$= \left[\frac{1}{27}(9t + 4)^{\frac{3}{2}}\right]_0^1$$
$$= \frac{1}{27}(13^{\frac{3}{2}} - 4^{\frac{3}{2}})$$
$$= \frac{1}{27}(13\sqrt{13} - 8)$$

b

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = t^3 + 3t^2 \Rightarrow \frac{dx}{dt} = 3t^2 + 6t$

Also $y = t^3 - 3t^2 \Rightarrow \frac{dy}{dt} = 3t^2 - 6t$

$$\therefore L = \int_0^3 \sqrt{(3t^2 + 6t)^2 + (3t^2 - 6t)^2}\, dt$$

$$= \int_0^3 \sqrt{18t^2(t^2 + 4)}\, dt$$

$$= 3\sqrt{2} \int_0^3 \sqrt{t^2(t^2 + 4)}\, dt$$

$$= 3\sqrt{2} \int_0^3 t\sqrt{(t^2 + 4)}\, dt$$

Let $u = t^2 + 4, \quad \frac{du}{dt} = 2t$

$$L = \frac{3\sqrt{2}}{2} \int_0^3 \sqrt{(u}\frac{du}{dt}\, dt$$

$$= \frac{3\sqrt{2}}{2} \int_1 3^4 \sqrt{(u}\, du$$

$$= \frac{3\sqrt{2}}{2} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_4^1 3$$

$$= 13\sqrt{26} - 8\sqrt{2}$$

c

$$L = \int_{\log_e 2}^{\log_e 3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = e^t \Rightarrow \frac{dx}{dt} = e^t$

Also $y = \frac{2}{3}e^{\frac{3t}{2}} \Rightarrow \frac{dy}{dt} = e^{\frac{3}{2}}$

$$\therefore L = \int_{\log_e 2}^{\log_e 3} \sqrt{e^{2t} + e^{3t}}\, dt$$

$$= \int_{\log_e 2}^{\log_e 3} e^t \sqrt{(1 + e^t)}\, dt$$

Let $u = 1 + e^t \Rightarrow \frac{du}{dt} = e^t$

$$L = \int_3^4 u^{\frac{1}{2}}\, du$$

$$= \left[\frac{2}{3} u^{\frac{3}{2}}\right]_3^4$$

$$= \frac{16}{3} - 2\sqrt{3}$$

d

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = \frac{1}{2}t^2 \Rightarrow \frac{dx}{dt} = t$

Also $y == \frac{1}{3}t^3 \Rightarrow \frac{dy}{dt} = t^2$

$$\therefore L = \int_0^{\sqrt{3}} \sqrt{t^2 + t^4}\, dt$$

$$= \int_0^{\sqrt{3}} t\sqrt{1 + t^2}\, dt$$

Let $u = 1 + t^2 \Rightarrow \frac{du}{dt} = 2t$

$$L = \frac{1}{2} \int_1^4 u^{\frac{1}{2}}\, du$$

$$= \frac{1}{2}\left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{2}\left(\frac{16}{3} - \frac{2}{3}\right)$$

$$= \frac{7}{3}$$

3

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = 3\sin 2t \Rightarrow \frac{dx}{dt} = 6\cos 2t$

Also $y = 6\cos 2t \Rightarrow \frac{dy}{dt} = -6\sin 2t$

$$\therefore L = \int_0^{\frac{\pi}{6}} \sqrt{36\cos^2(2t) + 36\sin^2(2t)}\, dt$$

$$= \int_0^{\frac{\pi}{6}} 6\, dt$$

$$= \pi$$

4

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = t - \sin t \Rightarrow \frac{dx}{dt} = 1 - \cos t$

Also $y = 1 - \cos t \Rightarrow \frac{dy}{dt} = \sin t$

$$\therefore L = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{1 - 1\cos t}\, dt$$

$$= 8$$

5

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = \cos^3 t \Rightarrow \frac{dx}{dt} = -3\cos^2 t \sin t$

Also $y = \sin^3 t \Rightarrow \frac{dy}{dt} = 3\sin^2 t \cos t$

$$\therefore L = 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$$

$$= 12\int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)}\, dt$$

$$= 12\int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t \sin^2 t}\, dt$$

$$= 12\int_0^{\frac{\pi}{2}} |\cos t \sin t|\, dt$$

$$= 12\int_0^{\frac{\pi}{2}} \cos t \sin t\, dt \qquad \text{since } 0 \le t \le \frac{\pi}{2}$$

$$= 12\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2t\, dt$$

$$= 6$$

6

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = e^t \sin(2t) \Rightarrow \frac{dx}{dt} = 2e^t \cos(2t) + e^t \sin(2t)$

Also $y = e^t \cos(2t) \Rightarrow \frac{dy}{dt} = e^t \cos(2t) - 2e^t \sin(2t)$

$$\therefore L = \int_0^{\pi} \sqrt{5e^{2t}}\, dt$$

$$= \int_0^{\pi} \sqrt{5}e^t\, dt$$

$$= \left[\sqrt{5}e^t\right]_0^{\pi}$$

$$= \sqrt{5}(e^{\pi} - 1)$$

7

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = 8\cos(t) + \cos(8t) \Rightarrow \dfrac{dx}{dt} = -8\sin(t) - 8\sin(8t)$

Also $y = 8\sin(t) - \sin(8t) \Rightarrow \dfrac{dy}{dt} = 8\cos(t) - 8\cos(8t)$

$$\therefore L = \int_0^{2\pi} \sqrt{128(\sin t \sin 8t - \cos t \cos 8t + 1)}\, dt$$

$$= \int_0^{2\pi} \sqrt{128(-\cos(9t) + 1)}\, dt$$

$$= \int_0^{2\pi} \sqrt{128(-(1 - 2\sin^2 \frac{9t}{2}) + 1)}\, dt$$

$$= \int_0^{2\pi} \sqrt{256 \sin^2 \frac{9t}{2}}\, dt$$

$$= \int_0^{2\pi} 16|\sin \frac{9t}{2}|\, dt$$

$$= \int_0^{2\pi} 16 \sin \frac{9t}{2}\, dt$$

$$= \frac{32}{9}\left[-\cos \frac{9t}{2}\right]_0^{2\pi}$$

$$= \frac{32}{9} \times 2$$

$$= \frac{64}{9}$$

8

$$L = \int_0^{\frac{1}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = \frac{t^3}{3} \Rightarrow \frac{dx}{dt} = t^2$

Also $y = \sin^{-1} t + t\sqrt{1 - t^2} \Rightarrow \frac{dy}{dt} = \frac{2 - 2t^2}{\sqrt{1 - t^2}}$

$$\begin{aligned}
\therefore L &= \int_0^{\frac{1}{2}} \sqrt{t^4 + 4(1 - t^2)}\, dt \\
&= \int_0^{\frac{1}{2}} \sqrt{t^4 - 4t^2 + 4)}\, dt \\
&= \int_0^{\frac{1}{2}} \sqrt{(t^2 - 2)^2}\, dt \\
&= \int_0^{\frac{1}{2}} |t^2 - 2)|\, dt \\
&= \int_0^{\frac{1}{2}} 2 - t^2\, dt \\
&= \left[2t - \frac{t^3}{3}\right]_0^{\frac{1}{2}} \\
&= 1 - \frac{1}{24} \\
&= \frac{23}{24}
\end{aligned}$$

9

$$L = \int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Now $x = 4\cos t + \cos 2t \Rightarrow \frac{dx}{dt} = -4\sin t - 2\sin 2t$

Also $y = \sin 2t + 4\sin t + 2t \Rightarrow \frac{dy}{dt} = 2\cos 2t + 4\cos t + 2$

$$\begin{aligned}
\therefore L &= \int_0^{\frac{\pi}{4}} \sqrt{16(\cos^2 t + 2\cos t + 1)}\, dt \\
&= \int_0^{\frac{\pi}{4}} 4\sqrt{(\cos t + 1)^2}\, dt \\
&= \int_0^{\frac{\pi}{4}} 4|\cos t + 1|\, dt \\
&= \int_0^{\frac{\pi}{4}} 4\cos t + 4\, dt \\
&= [4(-\sin t + t)]_0^{\frac{\pi}{4}} \\
&= 4(\frac{1}{\sqrt{2}} + \frac{\pi}{4}) \\
&= 2\sqrt{2} + \pi
\end{aligned}$$

Solutions to Exercise 10F

1 a $A = 2\pi \int_0^8 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = \frac{3}{4}x$

$$\begin{aligned}
A &= 2\pi \int_0^8 y\sqrt{1+\left(\frac{3}{4}\right)^2}\,dx \\
&= 2\pi \int_0^8 \frac{3}{4}x\sqrt{1+\left(\frac{3}{4}\right)^2}\,dx \\
&= 2\pi \int_0^8 \frac{3}{4}x\sqrt{\frac{25}{16}}\,dx \\
&= \frac{15}{8}\pi \int_0^8 x\,dx \\
&= \frac{15}{8}\pi \left[\frac{x^2}{2}\right]_0^8 \\
&= 60\pi
\end{aligned}$$

b $A = 2\pi \int_0^3 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = \frac{1}{3}x + 4$

$$\begin{aligned}
A &= 2\pi \int_0^3 y\sqrt{1+\left(\frac{1}{3}\right)^2}\,dx \\
&= 2\pi \int_0^3 (\frac{1}{3}x+4)\sqrt{\frac{10}{9}}\,dx \\
&= \frac{2\pi}{3} \int_0^3 (\frac{1}{3}x+4)\sqrt{10}\,dx \\
&= \frac{2\sqrt{10}\pi}{3} \int_0^3 (\frac{1}{3}x+4)\,dx \\
&= \frac{2\sqrt{10}\pi}{3} \left[\frac{1}{6}x^2+4x\right]_0^3 \\
&= \frac{2\sqrt{10}\pi}{3}(\frac{3}{2}+12) \\
&= \frac{2\sqrt{10}\pi}{3} \times \frac{27}{2} \\
&= 9\sqrt{10}\pi
\end{aligned}$$

c $A = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = \frac{x^3}{4}$

$A = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{3x^2}{4}\right)^2}\, dx$

$= 2\pi \int_0^1 \frac{x^3}{4} \sqrt{1 + \left(\frac{3x^2}{4}\right)^2}\, dx$

$= 2\pi \int_0^1 \frac{x^3}{4} \sqrt{\frac{16 + 9x^4}{16}}\, dx$

$= \frac{\pi}{8} \int_0^1 x^3 \sqrt{16 + 9x^4}\, dx$

Let $u = 16 + 9x^4, \frac{du}{dx} = 36x^3$

$= \frac{\pi}{8 \times 36} \int_{16}^{25} \sqrt{u}\, du$

$= \frac{\pi}{8 \times 36} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_{16}^{25}$

$= \frac{61\pi}{432}$

d

$A = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = \sqrt{4 - x^2}$

$A = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2}\, dx$

$= 2\pi \int_0^1 \sqrt{4 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2}\, dx$

$= 2\pi \int_0^1 \sqrt{4 - x^2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}}\, dx$

$= 4\pi \int_0^1 1\, dx$

$= 4\pi$

e

$A = 2\pi \int_{-1}^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = \sqrt{4 - x^2}$

$A = 2\pi \int_{-1}^1 y \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2}\, dx$

$= 2\pi \int_{-1}^1 \sqrt{4 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2}\, dx$

$= 2\pi \int_{-1}^1 \sqrt{4 - x^2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}}\, dx$

$= 4\pi \int_{-1}^1 1\, dx$

$= 8\pi$

f

$A = 2\pi \int_{\frac{1}{2}}^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = \frac{x^3}{6} + \frac{1}{2x}$

$A = 2\pi \int_{\frac{1}{2}}^1 y \sqrt{1 + \left(\frac{2x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx$

$= 2\pi \int_{\frac{1}{2}}^1 \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}\, dx$

$= 2\pi \int_{\frac{1}{2}}^1 \frac{x^4 + 3}{12x^3}(x^4 + 1)\, dx$

$= 2\pi \int_{\frac{1}{2}}^1 \frac{(x^4 + 3)(x^4 + 1)}{12x^3}\, dx$

$= \frac{263\pi}{256}$

2 **a** First $x = \frac{4}{3}y$

$$A = 2\pi \int_0^{10} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = 2\pi \int_0^{10} \frac{4}{3}y\sqrt{1 + \frac{16}{9}}\, dy$$

$$= 2\pi \int_0^{10} \frac{20}{9}y\, dy$$

$$= \frac{2000\pi}{9}$$

b First $x = -\frac{4(y-3)}{3}$

$$\frac{dx}{dy} = -\frac{4}{3}$$

$$A = 2\pi \int_0^{4} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = 2\pi \int_0^{4} -\frac{4(y-3)}{3}y\sqrt{1 + \left(-\frac{4}{3}\right)^2}\, dy$$

$$= 2\pi \int_0^{4} -\frac{20(y-3)}{9}\, dy$$

$$= \frac{160\pi}{9}$$

c First $x = \sqrt{y}$

$$\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}}$$

$$A = 2\pi \int_4^{9} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = 2\pi \int_4^{9} y^{\frac{1}{2}}\sqrt{1 + \left(\frac{1}{2}y^{-\frac{1}{2}}\right)^2}\, dy$$

$$= 2\pi \int_4^{9} y^{\frac{1}{2}}\sqrt{1 + \left(\frac{1}{4}y^{-1}\right)}\, dy$$

$$= 2\pi \int_4^{9} \frac{1}{2}\sqrt{4y+1}\, dy$$

$$= \frac{\pi}{6}(37\sqrt{37} - 17\sqrt{17})$$

d First $x = \sqrt{2y - y^2}$

$$\frac{dx}{dy} = \frac{1-y}{\sqrt{y(2-y)}}$$

$$A = 2\pi \int_0^{1} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = 2\pi \int_0^{1} \sqrt{2y - y^2}\sqrt{1 + \left(\frac{1-y}{\sqrt{y(2-y)}}\right)^2}\, dy$$

$$= 2\pi \int_0^{1} \sqrt{2y - y^2 + 1 - 2y + y^2}\, dy$$

$$= 2\pi \int_0^{1} \sqrt{1}\, dy$$

$$= 2\pi$$

e First $x = \frac{y^3}{3}$

$$\frac{dx}{dy} = y^2$$

$$A = 2\pi \int_0^{2} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = 2\pi \int_0^{2} \frac{y^3}{3}\sqrt{1 + y^4}\, dy$$

Let $u = 1 + y^4 \Rightarrow \frac{du}{dy} = 4y^3$

$$A = \frac{\pi}{6}\int_1^{17} \sqrt{u}\, du$$

$$= \frac{\pi}{9}(17\sqrt{17} - 1)$$

f We use the inverse function approach.

$y = x^2 \Rightarrow \dfrac{dy}{dx} = 2x$

$$A = 2\pi \int_1^3 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^3 x \sqrt{1 + 4x^2}\, dx$$

Let $u = 1 + 4x^2 \Rightarrow \dfrac{du}{dx} = 8x$

$$A = \frac{\pi}{4} \int_5^{37} u^{\frac{1}{2}}\, du$$

$$= \frac{\pi}{6}(37\sqrt{37} - 5\sqrt{5})$$

3 a We use the inverse function approach.

$y = x^2 \Rightarrow \dfrac{dy}{dx} = 2x$

$$A = 2\pi \int_1^2 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^2 x \sqrt{1 + 4x^2}\, dx$$

Let $u = 1 + 4x^2 \Rightarrow \dfrac{du}{dx} = 8x$

$$A = \frac{\pi}{4} \int_5^{17} u^{\frac{1}{2}}\, du$$

$$= \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$$

b First $x = \sqrt{y}$

$\dfrac{dx}{dy} = \dfrac{1}{2}y^{-\frac{1}{2}}$

$$A = 2\pi \int_1^4 x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = 2\pi \int_1^4 y^{\frac{1}{2}} \sqrt{1 + \left(\frac{1}{2}y^{-\frac{1}{2}}\right)^2}\, dy$$

$$= 2\pi \int_1^4 y^{\frac{1}{2}} \sqrt{1 + \left(\frac{1}{4}y^{-1}\right)}\, dy$$

$$= 2\pi \int_1^4 \frac{1}{2} \sqrt{4y + 1}\, dy$$

$$= \pi \int_1^4 \sqrt{4y + 1}\, dy$$

$$= \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$$

4 $$A = 2\pi \int_0^1 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$y = 1 - x^2$

$$A = 2\pi \int_0^1 x \sqrt{1 + (-2x)^2}\, dx$$

$$= 2\pi \int_0^1 x \sqrt{1 + 4x^2}\, dx$$

Let $u = 1 + 4x^2, \dfrac{du}{dx} = 8x$

$$= \frac{\pi}{4} \int_1^5 \sqrt{u}\, du$$

$$= \frac{\pi}{6}\left[u^{\frac{3}{2}}\right]_1^5$$

$$= \frac{\pi}{6}(5\sqrt{5} - 1)$$

5 $x = 4\cos 2t$ and $y = 4\sin 2t$

$\dfrac{dx}{dt} = -8\sin 2t$ and $\dfrac{dy}{dt} = 8\cos 2t$

$$A = 2\pi \int_0^{\frac{\pi}{2}} 4 \sin 2t \sqrt{64 \sin^2 2t + 64 \cos^2 2t}\, dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 32 \sin 2t\, dt$$

$$= 32\pi \Big[-\cos 2t\Big]_0^{\frac{\pi}{2}}$$

$$= 32\pi(1 + 1)$$

$$= 64\pi$$

6 **a** $x = 6 + 2t^2$ and $y = 4t$

$\frac{dx}{dt} = 4t$ and $\frac{dy}{dt} = 4$

$$A = 2\pi \int_0^4 (4t) \sqrt{16t^2 + 16}\, dt$$

$$= 32\pi \int_0^4 t \sqrt{t^2 + 1}\, dt$$

Let $u = t^2 + 1, \frac{du}{dt} = 2t$

$$A = 16\pi \int_1^{17} u^{\frac{1}{2}}\, du$$

$$A = \frac{32\pi}{3}\Big[u^{\frac{3}{2}}\Big]_1^{17}$$

$$= \frac{32\pi}{3}(17\sqrt{17} - 1)$$

b $x = 1 - t^2$ and $y = 2t$

$\frac{dx}{dt} = -2t$ and $\frac{dy}{dt} = 2$

$$A = 2\pi \int_0^1 (2t) \sqrt{4t^2 + 4}\, dt$$

$$= 8\pi \int_0^1 t \sqrt{t^2 + 1}\, dt$$

Let $u = t^2 + 1, \frac{du}{dt} = 2t$

$$A = 4\pi \int_1^2 u^{\frac{1}{2}}\, du$$

$$A = \frac{8\pi}{3}\Big[u^{\frac{3}{2}}\Big]_1^2$$

$$= \frac{8\pi}{3}(2\sqrt{2} - 1)$$

c $x = 3t - t^3$ and $y = 3t^2$

$\frac{dx}{dt} = 3 - 3t^2$ and $\frac{dy}{dt} = 6t$

$$A = 2\pi \int_0^1 (3t^2) \sqrt{(3 - 3t^2)^2 + 36t^2}\, dt$$

$$= 2\pi \int_0^1 (3t^2) \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt$$

$$= 2\pi \int_0^1 (3t^2) \sqrt{9 + 18t^2 + 9t^4}\, dt$$

$$= 2\pi \int_0^1 (3t^2) \times (3 + 3t^2)\, dt$$

$$= 2\pi \int_0^1 (9t^2 + 9t^4)\, dt$$

$$= \frac{48\pi}{5}$$

d $x = t$ and $y = t^2 - 2$

$\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 2t$

$$A = 2\pi \int_0^3 (t)\sqrt{1 + 4t^2}\, dt$$

Let $u = 4t^2 + 1$, $\frac{du}{dt} = 8t$

$$A = \frac{\pi}{4}\int_1^{37} u^{\frac{1}{2}}\, du$$

$$A = \frac{\pi}{6}\left[u^{\frac{3}{2}}\right]_1^{37}$$

$$= \frac{\pi}{6}(37\sqrt{37} - 1)$$

e $x = t + \sqrt{3}$ and $y = \frac{1}{2}t^2 + \sqrt{3}t$

$\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = t + \sqrt{3}$

$$A = 2\pi \int_0^{\sqrt{3}} (t + \sqrt{3})\sqrt{t^2 + 2\sqrt{3}t + 4}\, dt$$

Let $u = t^2 + 2\sqrt{3}t + 4$, $\frac{du}{dt} = 2t + 2\sqrt{3}$

$$A = \pi \int_1^{13} u^{\frac{1}{2}}\, du$$

$$A = \frac{2\pi}{3}\left[u^{\frac{3}{2}}\right]_1^{13}$$

$$= \frac{2\pi}{3}(13\sqrt{13} - 1)$$

f $x = 3 + 2\cos t$ and $y = 4 + 2\sin t =$

$\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = 2\cos t$

$$A = 2\pi \int_0^{\frac{\pi}{2}} (3 + 2\cos t)\sqrt{4\sin^2 t + \cos^2 t}\, dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} (3 + 2\cos t)\sqrt{4\sin^2 t + 4\cos^2 t}\, dt$$

$$= 4\pi \int_0^{\frac{\pi}{2}} (3 + 2\cos t)\, dt$$

$$= 4\pi\,[3t + 2\sin t]_0^{\frac{\pi}{2}}$$

$$= 4\pi \times (\frac{3\pi}{2} + 2)$$

$$= 6\pi^2 + 8\pi$$

g $x = 4t$ and $y = t^2 - 2\log_e t$

$\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 2t - \frac{2}{t}$

$$A = 2\pi \int_0^3 (t^2 - 2\log_e t\sqrt{4t^2 + \frac{4}{t^2} + 8}\, dt$$

$$= 4\pi \int_0^3 (t^2 - 2\log_e t) \times \frac{1}{t}\sqrt{(t^2 + 1)^2}\, dt$$

$$= 4\pi \int_0^3 (t^2 - 2\log_e t) \times \frac{1}{t} \times (t^2 + 1)\, dt$$

$$= 4\pi \int_0^3 (t^2 - 2\log_e t) \times (t + \frac{1}{t})\, dt$$

After a solid bit of algebra

$= 4\pi(28 - 9\log_e(3) - (\log_e(3))^2$

7 $x = \cos t$ and $y = 4 + \sin t$

$\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$

$$A = 2\pi \int_0^{\pi} (4 + \sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= 2\pi \int_0^{\pi} 4 + \sin t; dt$$

$$= 2\pi\left[4t - \cos t\right]_0^{\pi}$$

$$= 2\pi(4\pi + 1 - (-1))$$

$$= 4\pi(2\pi + 1)$$

8 $x = \cos^3 t$ and $y = \sin^3 t$

$$\frac{dx}{dt} = -3\cos^2 t \sin t$$
and $\frac{dy}{dt} = 3\sin^2 t \cos t$

$$A = 2\pi \int_0^{\frac{\pi}{2}} \sin^3 t \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$$
$$= 6\pi \int_0^{\frac{\pi}{2}} \sin^4 t \cos t \sqrt{\cos^2 t + \sin^2 t}\, dt$$
$$= 6\pi \int_0^{\frac{\pi}{2}} \sin^4 t \cos t\, dt$$

Let $u = \sin t$, $\frac{du}{dt} = \cos t$

$$= 6\pi \int_0^1 u^4\, du$$
$$= 6\pi \left[\frac{u^5}{5}\right]_0^1$$
$$= \frac{6\pi}{5}$$

9 $x = a\cos^2 t$ and $y = a\sin^2 t$

$\frac{dx}{dt} = -2a\cos t \sin t$ and

$\frac{dy}{dt} = 2a\sin t \cos t$

$$A = 2a^2\pi \int_0^{\frac{\pi}{2}} \sin^2 t \sqrt{8\sin^2 t \cos^2 t}\, dt$$
$$= 4\sqrt{2}a^2\pi \int_0^{\frac{\pi}{2}} sin^2 t \sin t \cos t\, dt$$
$$= 4\sqrt{2}a^2\pi \int_0^{\frac{\pi}{2}} \sin^3 t \cos t\, dt$$

Let $u = \sin t \Rightarrow \frac{du}{dt} = \cos t$

$$A = 4\sqrt{2}a^2\pi \int_0^1 u^3\, du$$
$$= \sqrt{2}\pi a^2$$

10 $A = 2\pi \int_0^h \frac{rx}{h} \sqrt{1 + \frac{r^2}{h^2}}\, dx$

$$= 2\pi \int_0^h \frac{rx}{h^2} \sqrt{h^2 + r^2}\, dx$$
$$= \frac{2\pi r}{h^2} \sqrt{h^2 + r^2} \int_0^h x\, dx$$
$$= \pi r \sqrt{h^2 + r^2}$$

11 Rotate the semicircle defined by $y = \sqrt{r^2 - x^2}$ about the x-axis. Consider the section of the semicircle between $x = a$ and $x = b$ where $a > b$

$$A = 2\pi \int_b^a \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$
$$= 2\pi \int_b^a \sqrt{r^2 - x^2 + x^2}\, dx$$
$$= 2\pi \int_b^a \sqrt{r^2}\, dx$$
$$= 2\pi \int_b^a r\, dx$$
$$= 2\pi r(a - b)$$

Let $h = a - b$

12 Calculator:

$$\frac{4\pi}{5}\left(9\sqrt{5}\cos^{-1}\left(\frac{2}{3}\right) + 10\right)$$

13 Rotating the circle $(x - R)^2 + y^2 = r^2$ about the y-axis

$y = \sqrt{r^2 - (x - R)^2}$

$\frac{dy}{dx} = -\frac{x - R}{\sqrt{r^2 - (x - R)^2}}$

Now consider,

$$A = 2\pi \int_{R-r}^{R+r} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{R-r}^{R+r} x\sqrt{1 + \frac{(x-R)^2}{r^2 - (x-R)^2}}\, dx$$

$$= 2\pi \int_{R-r}^{R+r} \frac{xr}{\sqrt{r^2 - (x-R)^2}}\, dx$$

Let$u = x - R, x = u + R,$ and $\frac{du}{dx} = 1$

$$A = 2\pi r \int_{-r}^{r} \frac{u+R}{\sqrt{r^2 - u^2}}\, dx$$

$$= 2\pi r \int_{-r}^{r} \frac{u}{\sqrt{r^2 - u^2}}\, dx + 2\pi r \int_{-r}^{r} \frac{R}{\sqrt{r^2 - u^2}}\, dx$$

$$= 2\pi r \int_{-r}^{r} \frac{R}{\sqrt{r^2 - u^2}}\, dx$$

The first integral is clearly zero

$$= 2\pi rR \left[\sin^{-1} \frac{u}{r}\right]_{-r}^{r}$$

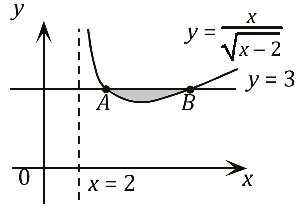
$$= 2\pi^2 rR$$

The torus has area:

$4\pi^2 rR$

Solutions to Technology-free questions

1



First find the points of intersection A and B of $y = \dfrac{x}{\sqrt{x-2}}$ and $y = 3$.

$$\frac{x}{\sqrt{x-2}} = 3$$

$$\therefore \quad x = 3\sqrt{x-2}$$

$$\therefore \quad x^2 = 9(x-2)$$

$$\therefore \quad x^2 - 9x + 18 = 0$$

$$\therefore \quad (x-3)(x-6) = 0$$

$$\therefore \quad x = 3 \text{ or } x = 6$$

Therefore, $A = (3, 3)$ and $B = (6, 3)$.

$$\text{Area} = \int_3^6 3 - \frac{x}{\sqrt{x-2}}dx$$

$$= [3x]_3^6 - \int_3^6 \frac{x}{\sqrt{x-2}}dx$$

$$= 9 - \int_3^6 \frac{x}{\sqrt{x-2}}dx$$

For $\displaystyle\int_3^6 \frac{x}{\sqrt{x-2}}dx$,

let $u = x - 2 \;\therefore\; x = u + 2$

$$\frac{du}{dx} = 1$$

$$\therefore \int_3^6 \frac{x}{\sqrt{x-2}}dx = \int_1^4 \frac{u+2}{u^{\frac{1}{2}}}du$$

$$= \int_1^4 u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right]_1^4$$

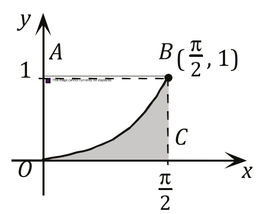
$$= \frac{2}{3} \times 8 + 8 - \left(\frac{2}{3} + 4\right)$$

$$= \frac{16}{3} + 4 - \frac{2}{3} = 8\frac{2}{3}$$

$$\therefore \quad \text{Area} = 9 - 8\frac{2}{3} = \frac{1}{3}$$

2 a $\displaystyle\int_0^{\frac{\pi}{2}} (1 - \cos x)dx = [x - \sin x]_0^{\frac{\pi}{2}}$

$$= \frac{\pi}{2} - 1$$



b $\displaystyle\int_0^1 x\,dy = |OABC| - \int_0^{\frac{\pi}{2}} y\,dx$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - 1\right) = 1$$

3 **a** $V = \pi \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

$= \pi[\tan x]_0^{\frac{\pi}{4}} = \pi$

b $V = \pi \int_0^{\frac{\pi}{4}} \sin^2 x \, dx$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} (1 - \cos 2x) dx$

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{8}(\pi - 2)$

c $V = \pi \int_0^{\frac{\pi}{4}} \cos^2 x \, dx$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} 1 + \cos 2x \, dx$

$= \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}(\pi + 2)$

d To find the intersection of the two graphs, solve the equation

$x^2 = 4x$

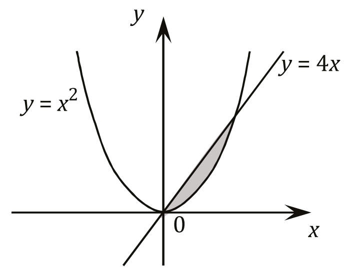
$\therefore \; x = 0$ or $x = 4$

$V = \pi \int_0^4 ((4x)^2 - (x^2))^2 \, dx$

$= \pi \int_0^4 (16x^2 - x^4) dx$

$= \pi\left[\frac{16x^3}{3} - \frac{x^5}{5}\right]_0^4$

$= \frac{2048\pi}{15}$



e $V = \pi \int_0^8 (1 + x) dx$

$= \pi\left[x + \frac{x^2}{2}\right]_0^8 = 40\pi$

4 $V = \pi \int_1^4 (1 + \sqrt{x})^2 dx$

$= \pi \int_1^4 (1 + 2\sqrt{x} + x) dx$

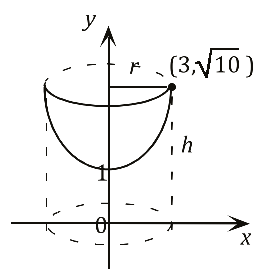
$= \pi\left[x + \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_1^4$

$= \frac{119\pi}{6}$

5 **a** $V = \pi \int_0^3 (1 + x^2) dx$

$= \pi\left[x + \frac{x^3}{3}\right]_0^3$

$= 12\pi$



b $V = \pi r^2 h - \pi \int_1^{\sqrt{10}} x^2 dy$

$= \pi\left(9\sqrt{10} - \int_1^{\sqrt{10}} (y^2 - 1)dy\right)$

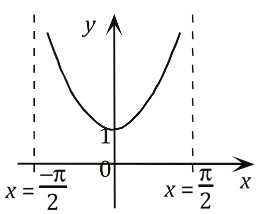
$= \pi\left(9\sqrt{10} - \left[\frac{y^3}{3} - y\right]_1^{\sqrt{10}}\right)$

$= \pi\left(\frac{20\sqrt{10}}{3} - \frac{2}{3}\right)$

6 $V = \pi \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\, dx$

$= \pi[\tan x]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$

$= 2\pi$



7 a Let $8x = (2x)^2$

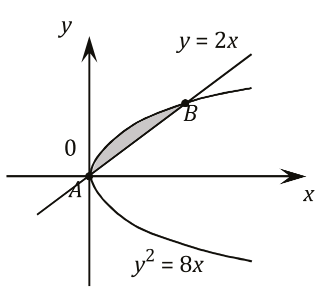
$= 4x^2$

$\therefore \quad 0 = 4x(x - 2)$

$\therefore \quad x = 0$ or $x = 2$

$\therefore \quad y = 0$ or $y = 4$

Therefore $A = (0, 0)$ and $B = (2, 4)$.

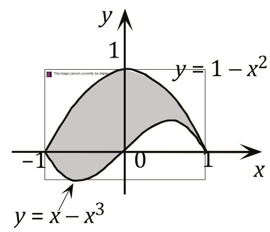


b $V = \pi \int_0^2 (8x - 4x^2)dx$

$= \pi\left[4x^2 - \frac{4x^3}{3}\right]_0^2$

$= \frac{16\pi}{3}$

8 a



b Area $= \int_{-1}^{1} (1 - x^2 - x + x^3)dx$

$= \left[x - \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4}\right]_{-1}^{1}$

$= \frac{4}{3}$

9 a

$x^2 + y^2 = 2$

$\therefore \quad x^2 = 2 - y^2$

Also, at A and B, $\quad x^2 = y$

$\therefore \quad 2 - y^2 = y$

$\therefore \quad y^2 + y - 2 = 0$

$\therefore \quad (y - 1)(y + 2) = 0$

$\therefore \quad y = -2, 1$

but $\quad y > 0$

$\therefore \quad y = 1$

When $y = 1$, $\quad x = \pm 1$

$\therefore A(-1, 1)$, $B(1, 1)$ and $C = (0, \sqrt{2})$.

b By symmetry,

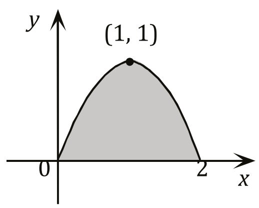
$$V = 2\pi \int_0^1 (2 - x^2 - x^4)dx$$

$$= 2\pi\left[2x - \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{44\pi}{15}$$

10 a $y = 2x - x^2$

$$= x(2 - x)$$



b Area $= \int_0^2 (2x - x^2)dx$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

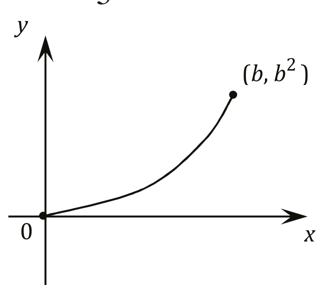
c $V = \pi \int_0^2 (4x^2 - 4x^3 + x^4)dx$

$$= \pi\left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2$$

$$= \frac{16\pi}{15}$$

11 a i $V_1 = \pi \int_0^b x^4\, dx$

$$= \frac{\pi b^5}{5}$$



ii $V_2 = \pi \int_0^{b^2} y dy$

$$= \frac{\pi b^4}{2}$$

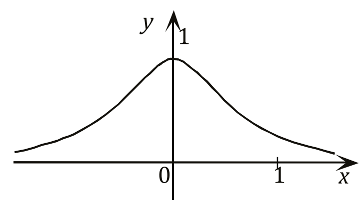
b

$V_1 = V_2$ when $\frac{b^4}{2} = \frac{b^5}{5}$

$\therefore \quad 5b^4 - 2b^5 = 0$

$\therefore \quad b = \frac{5}{2}$ since $b > 0$

12 a



b $\frac{dy}{dx} = -\frac{8x}{(4x^2 + 1)^2}$

When $x = \frac{1}{2}$, $y = \frac{1}{2}$

and $\frac{dy}{dx} = -1$

$\therefore$ equation of tangent is

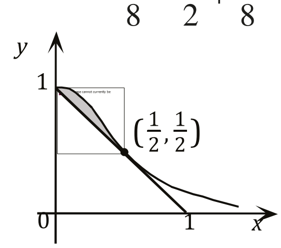
$$y - \frac{1}{2} = -\left(x - \frac{1}{2}\right)$$

$$y = -x + 1$$

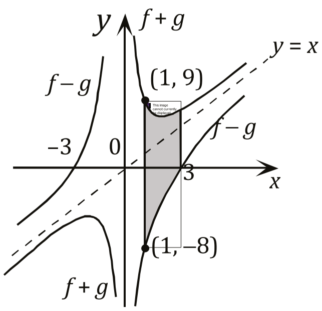
c Area $= \int_0^{\frac{1}{2}} \left(\frac{1}{4x^2 + 1} - (1 - x)\right)dx$

$$= \left[\frac{1}{2}\tan^{-1}(2x) - x + \frac{x^2}{2}\right]_0^{\frac{1}{2}}$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{1}{8} = \frac{\pi - 3}{8}$$



13 **a**



b Area $= \int_1^3 (f + g - f + g)dx$

$= 2\int_1^3 g(x)dx$

$= 2\int_1^3 \frac{9}{x}dx$

$= 18\int_1^3 \frac{1}{x}dx$

$= 18[\log_e |x|]_1^3$

$= 18\log_e 3$

14 Find the x-axis intercepts for

$y = \dfrac{x^2 - 5x + 4}{x}$

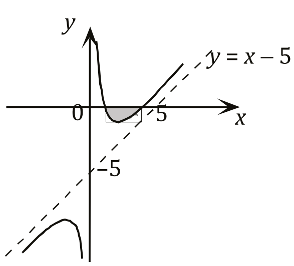
$y = 0$ when $x = 1$and $x = 4$

$$\text{Area} = -\int_1^4 \left(x - 5 + \frac{4}{x}\right)dx$$

$$= -\left[\frac{x^2}{2} - 5x + 4\log_e |x|\right]_1^4$$

$$= -\left(7\frac{1}{2} - 15 + 4\log_e 4\right)$$

$$= 7.5 - 4\log_e 4$$



15 The graph can be drawn as a reciprocal to the graph of $y = 2 + x - x^2$

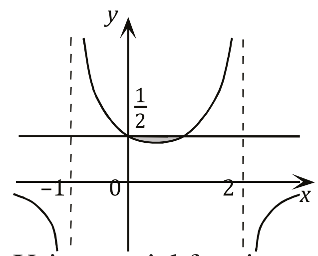
$= -(x - 2)(x + 1)$

Asymptotes are $x = -1,\ x = 2,\ y = 0$

y-axis intercept $\frac{1}{2}$. $y = \frac{1}{2}$ also when $x = 1$

$$\text{Area} = \int_0^1 \left(\frac{1}{2} - \frac{1}{2 + x - x^2}\right)dx$$

$$= \frac{1}{2} + \int_0^1 \frac{1}{2} - \frac{1}{(x - 2)(x + 1)}dx$$



Using partial fractions,

$$\frac{1}{(x - 2)(x + 1)} = \frac{A}{x - 2} + \frac{B}{x + 1}$$

$\therefore \quad 1 = A\,(x + 1) + B(x - 2)$

When $x = 2,\ A = \frac{1}{3}$ and when $x = -1,\ B = -\frac{1}{3}$

$$\therefore \text{ Area} = \frac{1}{2} + \frac{1}{3}\int_0^1 \frac{dx}{x - 2} - \frac{1}{3}\int_0^1 \frac{dx}{x + 1}$$

$$= \frac{1}{2} + \frac{1}{3}\left[\log_e\left(\frac{x - 2}{x + 1}\right)\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{3}\log_e \tfrac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{3}\log_e 4$$

16 $V = \pi \int_0^{\frac{\pi}{4}} x \sin^2(2x)\ dx$

Integrating by parts..

Let $u = x$ and $\dfrac{dv}{dx} = \sin^2 2x = \dfrac{1 - \cos 4x}{2}$

$\dfrac{du}{dx} = 1$and $v = \dfrac{x}{2} - \dfrac{1}{8}\sin 4x$

$$\pi \int_0^{\frac{\pi}{4}} x \sin^2(2x)\ dx$$

$$= \pi\left(\left[\frac{x^2}{2} - \frac{x}{8}\sin 4x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{x}{2} - \frac{1}{8}\sin 4x\ dx\right)$$

$$= \pi\left(\frac{\pi^2}{32} - \int_0^{\frac{\pi}{4}} \frac{x}{2} - \frac{1}{8}\sin 4x\ dx\right)$$

$$= \pi\left(\frac{\pi^2}{32} - \left[\frac{x^2}{4} + \frac{1}{32}\cos 4x\right]_0^{\frac{\pi}{4}}\right)$$

$$= \pi\left(\frac{\pi^2}{32} - (\frac{\pi^2}{64} - \frac{1}{32} - \frac{1}{32}\right)$$

$$= \pi\left(\frac{\pi^2}{64} + \frac{4}{64}\right)$$

$$= \frac{\pi(\pi^2 + 4)}{64}$$

17 $x = 2t - 2\sin t$ and $y = 2 - 2\cos t$

$\dfrac{dx}{dt} = 2 - 2\cos t$ and $\dfrac{dy}{dt} = 2\sin t$

$$L = \int_{\frac{\pi}{3}}^{2\pi} \sqrt{(2 - 2\cos t)^2 + (2\sin t)^2}\ dt$$

$$= \int_{\frac{\pi}{3}}^{2\pi} \sqrt{4 - 8\cos t + 4\cos^2 t + 4\sin^2 t}\ dt$$

$$= \int_{\frac{\pi}{3}}^{2\pi} \sqrt{8 - 8\cos t}\ dt$$

$$= 2\sqrt{2}\int_{\frac{\pi}{3}}^{2\pi} \sqrt{1 - \cos t}\ dt$$

$$= 2\sqrt{2}\int_{\frac{\pi}{3}}^{2\pi} \sqrt{1 - (1 - 2\sin^2\left(\frac{t}{2}\right))}\ dt$$

$$= 2\sqrt{2}\int_{\frac{\pi}{3}}^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\ dt$$

$$= 4\int_{\frac{\pi}{3}}^{2\pi} \sin\left(\frac{t}{2}\right)\ dt$$

$$= -8\left[\cos\frac{t}{2}\right]_{\frac{\pi}{3}}^{2\pi}$$

$$= -8(-1 - \frac{\sqrt{3}}{2})$$

$$= 8 + 4\sqrt{3}$$

18 $x = \cos^3 t$ and $y = \sin^3 t$

$\dfrac{dx}{dt} = 3\cos^2 t \sin t$ and $\dfrac{dy}{dt} = 3\cos t \sin^2 t$

$$L = 3\int_0^{\frac{\pi}{4}} \sqrt{(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t}\ dt$$

$$= 3\int_0^{\frac{\pi}{4}} \cos^2 t \sin^2 t \sqrt{\cos^2 t + \sin^2 t}\ dt$$

$$= 3\int_0^{\frac{\pi}{4}} \cos t \sin t\ dt$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{4}} \sin 2t\ dt$$

$$= \frac{3}{4}$$

19 $V = \pi \int_0^1 x^2 e^2 x \, dx = \frac{(e^2 - 1)\pi}{4}$ Using integration by parts twice.

20 $x = a\cos\ \theta$ and $y = b\sin\ \theta$

$\frac{dx}{d\theta} = -a\sin(\theta)$ and $\frac{dy}{d\theta} = b\cos(\theta)$

$$L = \int_0^{2\pi} \sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta}\, dt$$

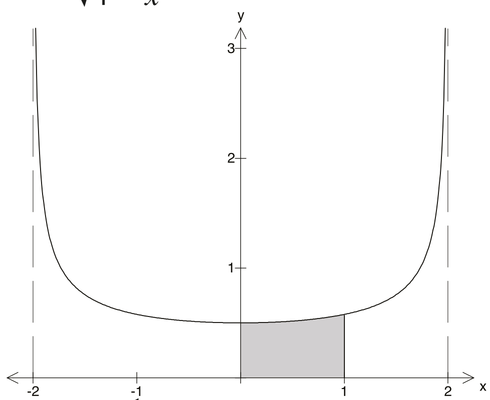
$$= \int_0^{2\pi} \sqrt{a^2(1 - \cos^2 \theta) + b^2 \cos^2 \theta}\, dt$$

$$= a \int_0^{2\pi} \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\cos^2 \theta}\, dt$$

$$= a \int_0^{2\pi} \sqrt{1 - e^2 \cos^2 \theta}\, dt$$

Solutions to multiple-choice questions

1 C $y = \dfrac{1}{\sqrt{4 - x^2}}$



$$V = \pi \int_0^1 y^2 dx$$

$$\therefore \quad V = \pi \int_0^1 \frac{1}{4 - x^2} dx$$

$$\therefore \quad V = -\pi \int_0^1 \frac{1}{x^2 - 4} dx$$

$$\therefore \quad V = -\pi \int_0^1 \frac{1}{(x + 2)(x - 2)} dx$$

Using partial fractions,

$$\frac{1}{(x + 2)(x - 2)} = \frac{1}{4(x - 2)} - \frac{1}{4(x + 2)}$$

$$\therefore \quad V = \frac{\pi}{4} \int_0^1 \frac{1}{x + 2} - \frac{1}{x - 2} dx$$

$$\therefore \quad V = \frac{\pi}{4}\Big[\log_e(x + 2) - \log_e(x - 2)\Big]_0^1$$

$$\therefore \quad V = \frac{\pi}{4}\left[\log_e\left(\frac{x + 2}{x - 2}\right)\right]_0^1$$

$$\therefore \quad V = \frac{\pi}{4}(\log_e(-3) - \log_e(-1))$$

$$\therefore \quad V = \frac{\pi}{4}\left(\log_e\left(\frac{-3}{-1}\right)\right)$$

$$\therefore \quad V = \frac{\pi}{4} \log_e(3)$$

2 D Let the upper curve of the shaded region be $f(x)$. Let the lower curve of the shaded region be $g(x)$.
Since the region is rotated about the x-axis, the rule for determining the volume of the solid of revolution is given by:

$V = \pi \int_0^2 ([f(x)]^2 - [g(x)]^2) dx$

Since $f(x) = \dfrac{6}{\sqrt{5 + x^2}}$ and $g(x) = 2$

$$\therefore \quad V = \pi \int_0^2 \left(\frac{6}{\sqrt{5 + x^2}}\right)^2 - (2)^2 dx$$

$$\therefore \quad V = \pi \int_0^2 \left(\frac{6}{\sqrt{5 + x^2}}\right)^2 - 4\, dx$$

3 B The points of intersection occur when,

$$\sin^2 x = \frac{1}{2} \cos^2 x$$

$$\therefore \quad \sin^2 x = \frac{1}{2}(1 - 2\sin^2 x)$$

$$\therefore \quad \sin^2 x = \frac{1}{2} - \sin^2 x$$

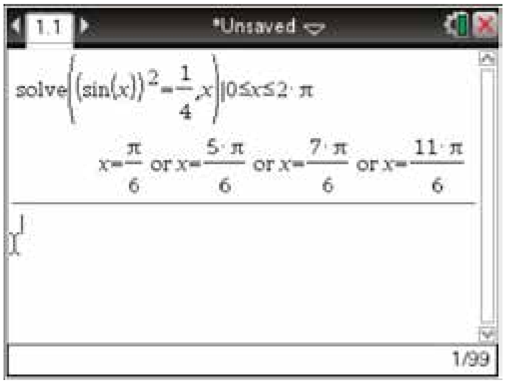
$$\therefore \quad 2\sin^2 x = \frac{1}{2}$$

$$\therefore \quad \sin^2 x = \frac{1}{4}$$

$$\therefore \quad \sin x = \pm\frac{1}{2}$$

$$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Using CAS:



Observations from the given graph:

i The blue line is $y = \sin^2 x$

ii The red line is $y = \dfrac{1}{2}\cos 2x$

iii There are 4 lots of the shaded region over the interval $x \in \left[0, \dfrac{\pi}{6}\right]$

iv There are 2 lots of the shaded region over the interval $x \in \left[\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right]$

Hence, the total area of the shaded region is:

$$A = 4\int_0^{\frac{\pi}{6}} \frac{1}{2}\cos(2x) - \sin^2 x\, dx$$

$$+ 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x - \frac{1}{2}\cos(2x)\, dx$$

4 C For a rotation about the x-axis use,

$V = \pi \int y^2 dx$

$\therefore \quad V = \pi \int_{e^2}^{e^3} [\log_e(x)]^2 dx$

5 C $A = 2\int_{\pi-a}^{\pi} \sin x\, dx$

$\therefore \quad A = 2\left[-\cos x\right]_{\pi-a}^{\pi}$

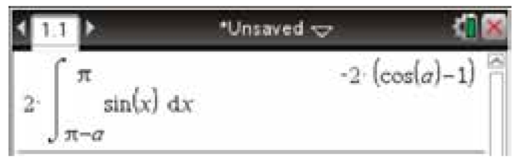
$\therefore \quad A = 2[-\cos\pi - (-\cos(\pi - a))]$

$\therefore \quad A = 2[1 + \cos(\pi - a)]$

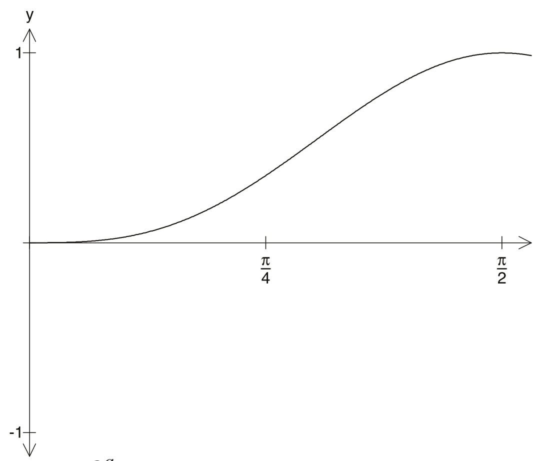
$\therefore \quad A = 2[1 - \cos a]$

since $\cos(\pi - \theta) = -\cos(\theta)$

Using CAS:



6 E $y = \sin^3 x$



$$A = \int_0^a \sin^3 x\, dx$$

$$\therefore \quad A = \int_0^a \sin x . \sin^2 x\, dx$$

$$\therefore \quad A = \int_0^a \sin x(1 - \cos^2 x) dx$$

let $u = \cos x$

then $\dfrac{du}{dx} = -\sin x$

and when $x = 0,\ u = 1$

$x = a,\ u = \cos a$

$$\therefore \quad A = \int_1^{\cos a} (u^2 - 1)du$$

$$\therefore \quad A = \left[\frac{1}{3}u^3 - u\right]_1^{\cos a}$$

$$\therefore \quad A = \left(\frac{1}{3}\cos^3 a - \cos a\right) - \left(\frac{1}{3} - 1\right)$$

$$\therefore \quad A = \frac{2}{3} - \cos a + \frac{1}{3}\cos^3 a$$

7 B For a rotation about the x-axis use,

$$V = \pi \int y^2 dx$$

$$\therefore \quad V = \pi \int_0^1 \frac{x^2}{4 - x^2} dx$$

By long division,

$$\frac{x^2}{4 - x^2} = \frac{1}{x+2} - \frac{1}{x-2} - 1$$

$$\therefore \quad V = \pi \int_0^1 \frac{1}{x+2} + \frac{1}{2-x} - 1dx$$

$$\therefore \quad V = \pi\Big[\log_e(x+2) - \log_e(2-x) - x\Big]_0^1$$

$$\therefore \quad V = \pi\left[\log_e\left(\frac{x+2}{2-x}\right) - x\right]_0^1$$

$$\therefore \quad V = \pi[(\log_e(3) - 1) - (\log_e(1) - 0]$$

$$\therefore \quad V = \pi(\log_e(3) - 1)$$

8 D By close inspection response D is a false statement.

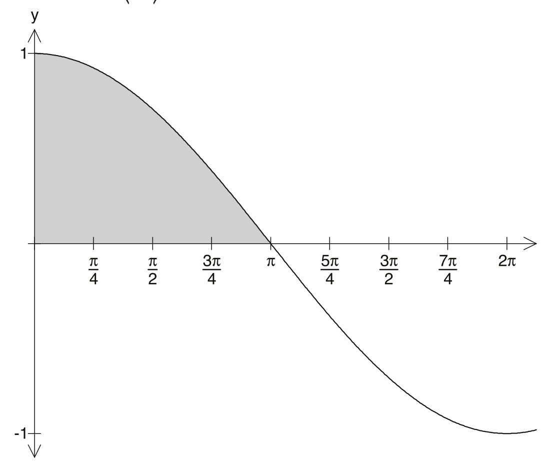
Note that $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f.

However,

$\int_a^b [f(x)]^2 dx \neq [F(b)]^2 - [F(a)]^2$

This is because the square needs to be absorbed into $f(x)$ **before** integrating.

9 C $y = \cos\left(\frac{x}{2}\right)$



$$A = \int_0^{\pi} \cos\left(\frac{x}{2}\right)dx$$

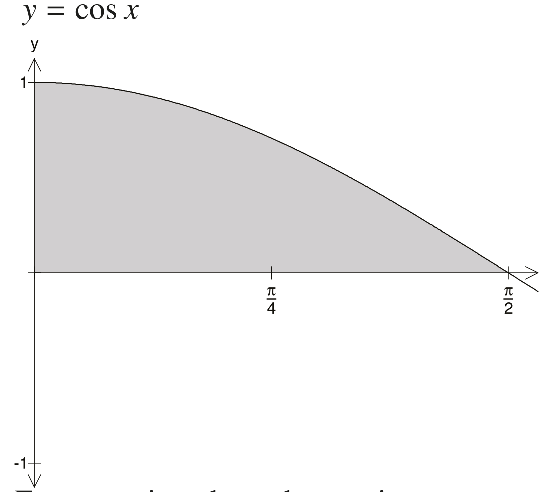
$$\therefore \quad A = \left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$\therefore \quad A = 2\sin\left(\frac{\pi}{2}\right) - 0$$

$$\therefore \quad A = 2(1)$$

$$\therefore \quad A = 2$$

10 E $y = \cos x$



For a rotation about the y-axis use,

$$V = \pi \int_{y=b}^{y=a} x^2 dy$$

$$y = \cos x \therefore \quad x = \cos^{-1} y$$

$$\therefore \quad V = \pi \int_0^1 (\cos^{-1} y)^2 dy$$

11 C If $y = \frac{1}{x}$ then $\frac{dy}{dx} = -\frac{1}{x^2}$

$$A = 2\pi \int_1^a \frac{1}{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^a \frac{1}{x} \sqrt{1 + \left(\frac{1}{x^2}\right)^2}\, dx$$

$$= 2\pi \int_1^a \frac{1}{x} \sqrt{\left(\frac{x^4 + 1}{x^4}\right)}\, dx$$

$$= 2\pi \int_1^a \frac{\sqrt{x^4 + 1}}{x^3}\, dx$$

12 A $\int_0^{\frac{\pi}{4}} 2x^2 \sec^2 x \tan x \, dx$.

We start with integration by parts.

Let $u = 2x^2$ and $\frac{dv}{dx} = \sec^2 x \tan x$

Then $\frac{du}{dx} = 4x$ and take $v = \frac{1}{2}\sec^2(x)$

$$\int_0^{\frac{\pi}{4}} 2x^2 \sec^2 x \tan x \, dx$$

$$= \left[2x^2 \times \frac{1}{2}\sec^2 x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2x \sec^2 x \, dx$$

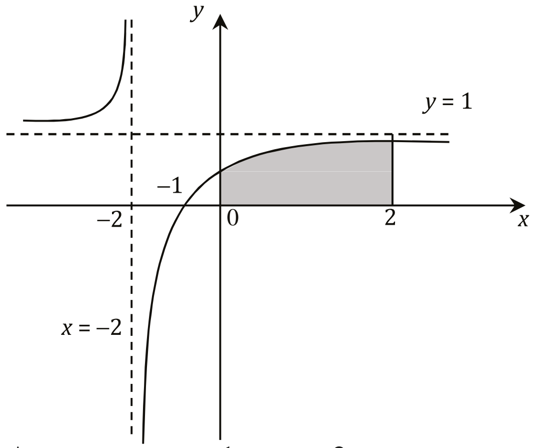
$$= \left[x^2 \times \sec^2 x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2x \sec^2 x \, dx$$

$$= \frac{\pi^2}{16} \times 2 - \int_0^{\frac{\pi}{4}} 2x \sec^2 x \, dx$$

$$= \frac{\pi^2}{8} - \int_0^{\frac{\pi}{4}} 2x \sec^2 x \, dx$$

Solutions to extended-response questions

1 a



Asymptotes $y = 1,\ x = -2$

Axis intercepts $y = \frac{1}{2}$ and $x = -1$

b $$\text{Area} = \int_0^2 \left(1 - \frac{1}{x+2}\right) dx$$
$$= [x - \log_e |x+2|]_0^2$$
$$= 2 - \log_e 2$$

c $$\text{Volume} = \pi \int_0^2 \left(1 - \frac{1}{x+2}\right)^2 dx$$
$$= \pi \int_0^2 \left(1 - \frac{2}{x+2} + \frac{1}{(x+2)^2}\right) dx$$
$$= \pi \left[x - 2\log_e |x+2| - \frac{1}{x+2}\right]_0^2$$
$$= \pi\left(\frac{9}{4} - 2\log_e 2\right)$$

2 a $f(x) = x\tan^{-1}(x)$

$$f'(x) = x \times \frac{1}{1+x^2} + \tan^{-1}(x)$$
$$= \frac{x}{1+x^2} + \tan^{-1}(x)$$

b $\int_0^1 \tan^{-1}(x)dx = \int_0^1 f'(x)dx - \int_0^1 \frac{x}{1+x^2}dx$

$= [x\tan^{-1}(x)]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$

$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx$

Let $u = 1 + x^2$.

Therefore, $\frac{du}{dx} = 2x$ and when $x = 0,\ u = 1$ and when $x = 1,\ u = 2$.

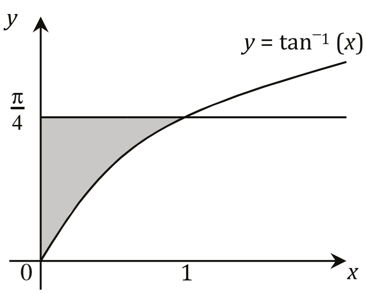
$\therefore \quad \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_1^2 \frac{1}{u}d_u$

$= \frac{1}{2}[\log_e u]_1^2$

$= \frac{1}{2}(\log_e 2 - \log_e 1) = \frac{1}{2}\log_e 2$

Therefore, $\int_0^1 \tan^{-1}(x)dx = \frac{\pi}{4} - \frac{1}{2}\log_e 2$.

c At the point of intersection, $\tan^{-1}(x) = \frac{\pi}{4}$. Therefore, $x = 1$.



Required area $= \frac{\pi}{4} \times 1 - \int_0^1 \tan^{-1}(x)dx$

$= \frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2}\log_e 2\right) = \frac{1}{2}\log_e 2$

d **i** $g(x) = (\tan^{-1}(x))^2$

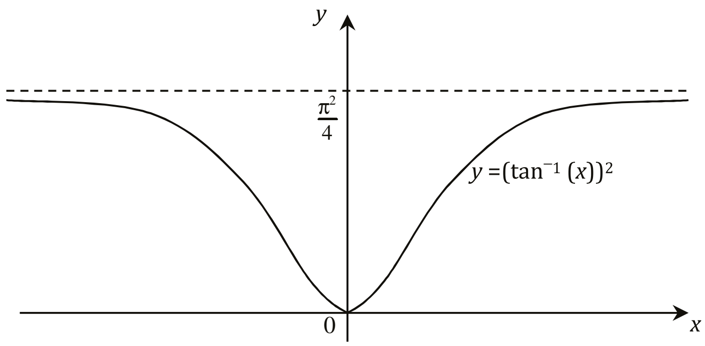
$g'(x) = 2\tan^{-1}(x) \times \frac{1}{1+x^2}$

$= \frac{2\tan^{-1}[x]}{1+x^2}$

ii When $x > 0,\ \tan^{-1}(x) > 0$ and $1 + x^2 > 0$

Therefore, $g'(x) > 0$

iii



e $y = \tan^{-1}(x)$

$\therefore \quad x = \tan y$

$\therefore \quad x^2 = \tan^2 y$

and when $x = 1$, $y = \tan^{-1}(1)$

$= \frac{\pi}{4}$

$$V = \pi \int_0^{\frac{\pi}{4}} (1 - x^2) dy$$

$$= \pi \int_0^{\frac{\pi}{4}} (1 - \tan^2 y) dy$$

$$= \pi \int_0^{\frac{\pi}{4}} (2 - \sec^2 y) dy$$

Now

$$= \pi[2y - \tan y]_0^{\frac{\pi}{4}}$$

$$= \pi\left[\left(\frac{\pi}{2} - \tan\frac{\pi}{4}\right) - (0 - \tan 0)\right]$$

$$= \pi\left(\frac{\pi}{2} - 1\right)$$

The required volume is $\pi\left(\frac{\pi}{2} - 1\right)$ cubic units.

3 a i $\frac{d}{dx}(x \log_e x) = \log_e x + 1$ product rule

$$\therefore \int \log_e x dx = x \log_e x - \int 1\, dx$$

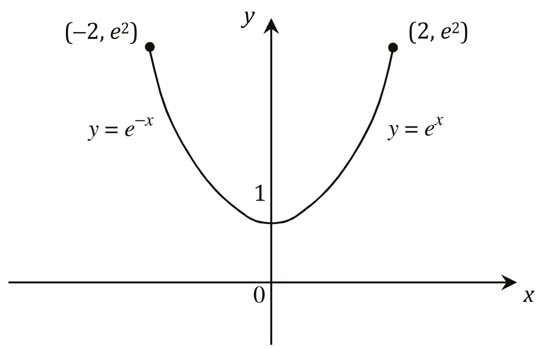
$$= x \log_e x - x + c$$

ii $\frac{d}{dx}(x(\log_e x)^2) = (\log_e x)^2 + 2\log_e x$

$\therefore \int (\log_e x)^2 dx = x(\log_e x)^2 - 2\int \log_e x\, dx$

$= x(\log_e x)^2 - 2x\log_e x + 2x + c$ using **a** above

b



c Rearrange $y = e^x,\ x = \log_e y$

When $x = 0,\ y = 1$ and when $x = 2,\ y = e^2$

$V = \pi \int_1^{e^2} (\log_e y)^2 dy$

$= \pi[y(\log_e y)^2 - 2y\log_e y + 2y]_1^{e^2}$

$= \pi(4e^2 - 4e^2 + 2e^2 - 2)$

$= 2\pi(e^2 - 1)$

$\approx 40\text{ cm}^3$

A full glass contains approximately 40 mL of liquid.

4 a $V = \pi \int_0^1 y\,dy$

$= \frac{\pi}{2}$ cubic units

b Given $\frac{dV}{dt} = R,\ y = \frac{1}{4}$.

Now $V = \pi\frac{y^2}{2}$ when depth is y units.

$\therefore \frac{dV}{dy} = \pi y$

$\therefore \frac{dy}{dt} = \frac{dy}{dV} \times \frac{dV}{dt}$

$= \frac{1}{\pi y} \times R$

$= \frac{R}{\pi y}$

When $y = \frac{1}{4}, \frac{dy}{dt} = \frac{4R}{\pi}$.

So rate of increase of the depth is $\frac{4R}{\pi}$ units/s.

c **i** $y = \frac{1}{2}$ $\quad V = \frac{\pi}{2}\left(\frac{1}{2}\right)^2$

$$= \frac{\pi}{8}$$

The volume of liquid is $\frac{\pi}{8}$ cubic units.

ii $\pi\frac{y^2}{2} = \frac{\pi}{4}$, since half full is $\frac{\pi}{4}$ cubic units

$$\therefore\ y^2 = \frac{1}{2}$$

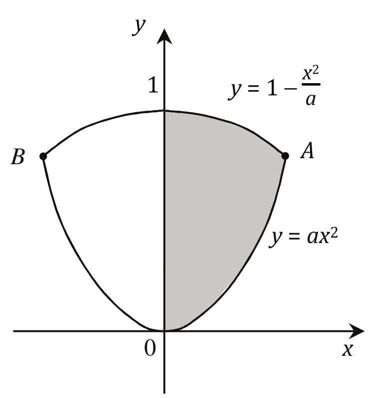
$$\therefore\ y = \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

The depth of liquid is $\frac{\sqrt{2}}{2}$ units.

5 **a** By symmetry, the whole area equals twice the shaded area.

To find the x coordinate of the point A, first solve the equation



$$ax^2 = 1 - \frac{x^2}{a}$$

$$\therefore\ x^2\left(a + \frac{1}{a}\right) = 1$$

$$\therefore\ x^2 = \frac{a}{a^2 + 1}$$

$$\therefore\ x = \sqrt{\frac{a}{a^2 + 1}} \text{ since } x > 0$$

$$\text{Area} = 2\int_0^{\sqrt{\frac{a}{a^2+1}}}\left(1 - \frac{x^2}{a} - ax^2\right)dx$$

$$= 2\left[x - \frac{x^3}{3a} - \frac{ax^3}{3}\right]_0^{\sqrt{\frac{a}{a^2+1}}}$$

$$= \frac{2\sqrt{\frac{a}{a^2+1}}}{3}\left(3 - \left(a + \frac{a}{a^2+1}\right)\left(\frac{1}{a} + a\right)\right)$$

$$= \frac{4}{3}\sqrt{\frac{a}{a^2+1}}$$

b **i** The maximum area A and the maximum of A^2 occur at the same value, so use $A^2 = \frac{16}{9}\left(\frac{a}{a^2+1}\right)$

$$\frac{d[A^2]}{da} = \frac{16}{9}\left(\frac{a^2+1-2a^2}{(a^2+1)^2}\right)$$

$$= \frac{16}{9}\left(\frac{1-a^2}{(a^2+1)^2}\right)$$

Area is a maximum when $\frac{d[A^2]}{da} = 0$

$\therefore\ a = 1$ since $a > 0$

ii When $a = 1$, Area $= \frac{4}{3}\sqrt{\frac{1}{1^2+1}}$

$$= \frac{4}{3\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{3}$$

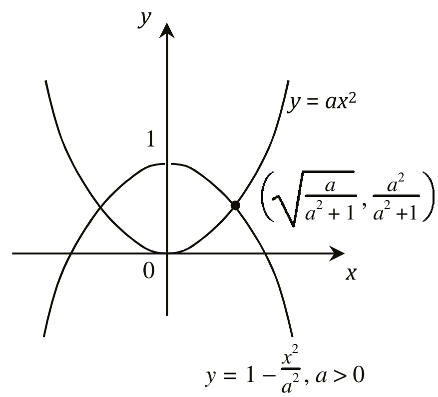
Maximum area is $\frac{2\sqrt{2}}{3}$ square units.

c When $x = 0$, $1 - \frac{x^2}{a} = 1$

For $y = ax^2$, $x^2 = \frac{y}{a}$

For $y = 1 - \frac{x^2}{a}$, $\frac{x^2}{a} = 1 - y$

$\therefore$ $x^2 = a(1 - y)$



$$V = \int_0^{\frac{a^2}{a^2+1}} \pi \frac{y}{a} dy + \int_{\frac{a^2}{a^2+1}}^1 \pi a(1-y)dy$$

$$= \frac{\pi a}{2}[y^2]_0^{\frac{a^2}{a^2+1}} + \frac{\pi a}{2}[2y - y^2]_{\frac{a^2}{a^2+1}}^1$$

$$= \frac{\pi a^3}{2(a^2+1)^2} + \frac{\pi a}{2(a^2+1)^2}$$

$$= \frac{\pi a[a^2+1]}{2(a^2+1)^2}$$

$$= \frac{\pi a}{2(a^2+1)}$$

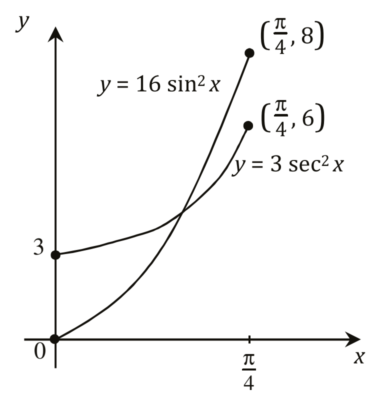
The volume of the solid is $\frac{\pi a}{2(a^2+1)}$ cubic units.

6 a When $x = 0$, $16\sin^2 x = 0$

$3\sec^2 2x = 3$

When $x = \frac{\pi}{4}$, $16\sin^2 x = 8$

$3\sec^2 x = 6$



b At the point of intersection, $16\sin^2 x = 3\sec^2 x$

$$= \frac{3}{\cos^2 x}$$

$\therefore$ $$\sin^2 x(1 - \sin^2 x) = \frac{3}{16}$$

$\therefore$ $$\sin^2 x - \sin^4 x = \frac{3}{16}$$

$\therefore$ $$\sin^4 x - \sin^2 x + \frac{3}{16} = 0$$

$\therefore$ $$\left(\sin^2 x - \frac{1}{4}\right)\left(\sin^2 x - \frac{3}{4}\right) = 0$$

$\therefore \quad \sin^2 x = \frac{1}{4} \text{ or } \frac{3}{4}$

$\therefore \quad \sin x = \pm\frac{1}{2} \text{ or } \pm \frac{\sqrt{3}}{2}$

$\therefore \quad \sin x = \frac{1}{2} \text{ since } 0 \le x \le \frac{\pi}{4}$

$\therefore \quad x = \frac{\pi}{6}$

When $x = \frac{\pi}{6'}$ $\quad y = 16 \sin^2\left(\frac{\pi}{6}\right)$

$$= 16 \times \left(\frac{1}{2}\right)^2$$

$$= 4$$

The point of intersection is $\left(\frac{\pi}{6}, 4\right)$.

c Area $= \int_0^{\frac{\pi}{6}} 3\sec^2 x - 16\sin^2 x \, dx$

$$= 3\int_0^{\frac{\pi}{6}} \sec^2 x \, dx - 16\int_0^{\frac{\pi}{6}} \sin^2 x \, dx$$

$$= 3[\tan x]_0^{\frac{\pi}{6}} - 8\int_0^{\frac{\pi}{6}} 1 - \cos 2x \, dx$$

$$= 3\tan\frac{\pi}{6} - 8\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$$

$$= 3 \times \frac{1}{\sqrt{3}} - 8\left(\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) - (0 - 0)\right)$$

$$= \sqrt{3} - 8\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$= 3\sqrt{3} - \frac{4\pi}{3}$$

The area of the region is $3\sqrt{3} - \frac{4\pi}{3}$ square units.

7 **a**

$$f(x) = \log_e(x - a) + c$$

$$\log_e(2 - a) + c = 1 \qquad ①$$

$$\log_e(1 + e^{-1} - a) + c = 0 \qquad ②$$

Therefore, $\log_e\left(\dfrac{2 - a}{1 + e^{-1} - a}\right) = 1$ using ① − ②.

and $\dfrac{2 - a}{1 + e^{-1} - a} = e$

Solving for a, $2 - a = e + 1 - ae$

$$a(e - 1) = e - 1$$

$$a = 1$$

Hence, $f(x) = \log_e(x - 1) - \log_e(1 + e^{-1} - 1)$

$$= \log_e(x - 1) + 1$$

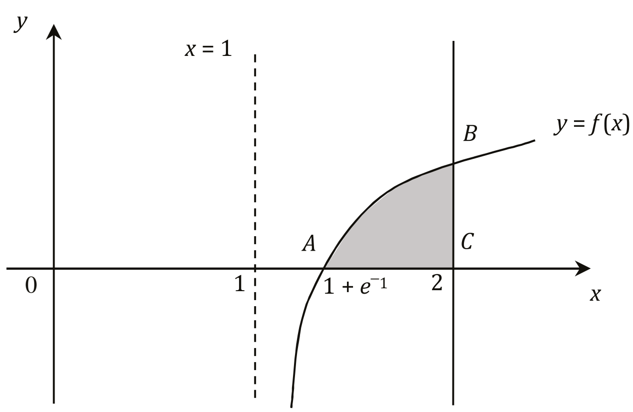
b Asymptote at $x = 1$.

x-axis intercept when $y = 0$

$\therefore \quad \log_e(x - 1) + 1 = 0$

$\therefore \quad x - 1 = e^{-1}$

$\therefore \quad x = e^{-1} + 1$



c For the inverse $\quad x = \log_e(y - 1) + 1$

$\therefore \quad y - 1 = e^{x-1}$

$\therefore \quad y = e^{x-1} + 1$

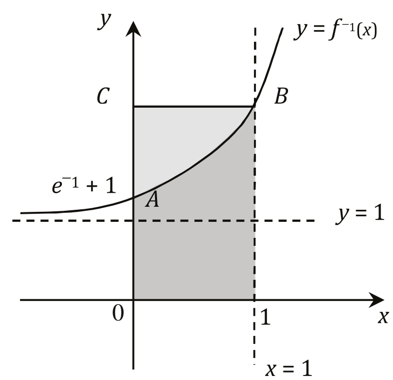
$\therefore \quad f^{-1}(x) = e^{x-1} + 1$

Domain $f^{-1} = \mathbb{R}$, range $f^{-1} = (1, \infty)$.

d Area $= \int_0^1 (e^{x-1} + 1)dx$

$= [e^{x-1} + x]_0^1$

$= 2 - e^{-1}$



e The area *ABC* in **b** is equal to the area *ABC* in **d**

$\int_{e^{-1}+1}^{2} f(x)dx = 2 - (2 - e^{-1})$

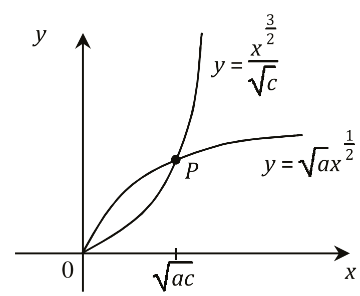
$= e^{-1}$

8 To find the coordinates of *P*

consider $\frac{x^3}{c} = ax$

As $x \neq 0$, $x^2 = ac$

Therefore, $x = \sqrt{ac}$



$A_1 = \int_0^{\sqrt{ac}} \frac{x^{\frac{3}{2}}}{\sqrt{c}} dx$

$= \frac{2}{5}\left[\frac{x^{\frac{5}{2}}}{c^{\frac{1}{2}}}\right]_0^{\sqrt{ac}}$

$= \frac{2}{5} a^{\frac{5}{4}} c^{\frac{3}{4}}$

$A_2 = \int_0^{\sqrt{ac}} \sqrt{a} x^{\frac{1}{2}} dx$

$= \frac{2}{3}\left[a^{\frac{1}{2}} x^{\frac{3}{2}}\right]_0^{\sqrt{ac}}$

$= \frac{2}{3} a^{\frac{5}{4}} c^{\frac{3}{4}}$

Hence $A_1 : A_2 = \frac{2}{5} : \frac{2}{3} = 3 : 5$

$V_1 = \pi \int_0^{\sqrt{ac}} \frac{x^3}{c} dx$

$$= \frac{\pi}{4c}[x^4]_0^{\sqrt{ac}}$$
$$= \frac{\pi}{4}a^2c$$
$$V_2 = \pi \int_0^{\sqrt{ac}} ax\,dx$$
$$= \frac{\pi a}{2}[x^2]_0^{\sqrt{ac}}$$
$$= \frac{\pi}{2}a^2c$$

Hence, $V_1 : V_2 = \frac{\pi}{4} : \frac{\pi}{2} = 1 : 2$

9 a The domain of $\cos x$ is $[0, \pi]$ for an inverse function to exist.
Therefore, $a = 2\pi$ in this case.
The largest value of a is 2π.

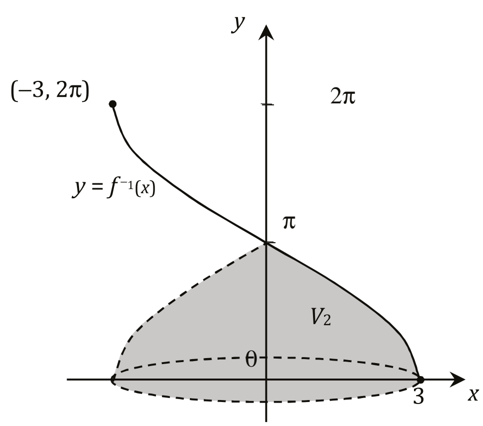
b i Domain $f^{-1} = [-3, 3]$, range $f^{-1} = [0, 2\pi]$.

ii Consider $x = 3\cos\frac{1}{2}y$

With y the subject, $y = 2\cos^{-1}\left(\frac{x}{3}\right)$

Therefore $f^{-1}(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

iii



c $\frac{dy}{dx} = -\frac{2}{\sqrt{1 - \frac{x^2}{9}}} \times \frac{1}{3}$

$$= \frac{-2}{\sqrt{9 - x^2}}$$

When $x = 0$, $\frac{dy}{dx} = -\frac{2}{3}$.

d The shaded volumes are equal, i.e., $V_1 = V_2$

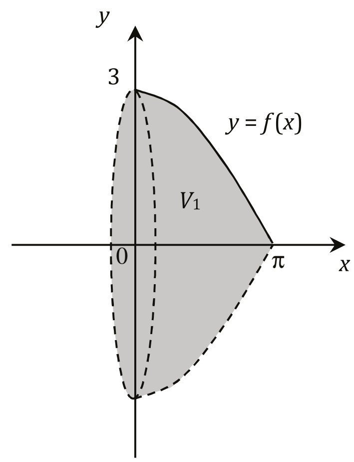
$$V_1 = \pi \int_0^{\pi} 9\cos^2\left(\frac{x}{2}\right)dx$$

$$= \frac{9\pi}{2}\int_0^{\pi}(\cos x + 1)dx$$

$$= \frac{9\pi}{2}[\sin x + x]_0^{\pi}$$

$$= \frac{9\pi^2}{2}$$

Note: $\cos^2\left(\frac{x}{2}\right) = \frac{\cos x + 1}{2}$

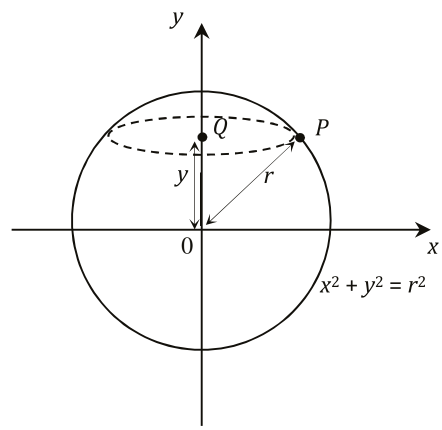


10 a $QP = \sqrt{0P^2 - OQ^2}$

$$= \sqrt{r^2 - y^2}$$

Area $= \pi(QP)^2$

$$= \pi(r^2 - y^2)$$



b $V = \pi \int_{\frac{3r}{4}}^{r}(r^2 - y^2)dx$

$$= \pi\left[r^2y - \frac{y^3}{3}\right]_{\frac{3r}{4}}^{r}$$

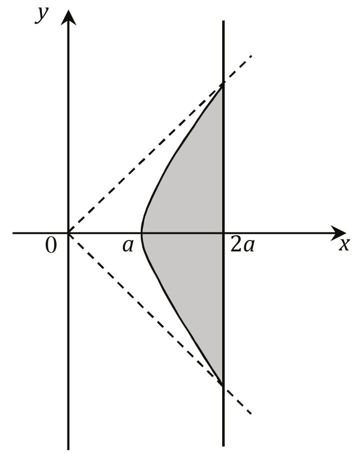
$$= \pi\left(\frac{2}{3}r^3 - \frac{3}{4}r^3 + \frac{9}{64}r^3\right)$$

$$= \frac{11\pi r^3}{192},$$ as required.

11 a $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ a \le x \le 2a,\ y > 0$

Therefore $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$

$$V = \int_a^{2a} \pi y^2 dx$$
$$= \pi \int_a^{2a} b^2\left(\frac{x^2}{a^2} - 1\right)dx$$
$$= \pi b^2\left[\frac{x^3}{3a^2} - x\right]_a^{2a}$$
$$= \pi b^2\left(\left(\frac{8a^3}{3a^2} - 2a\right) - \left(\frac{a^3}{3a^2} - a\right)\right)$$
$$= \pi b^2\left(\frac{2}{3}a + \frac{2}{3}a\right)$$
$$= \frac{4\pi ab^2}{3}$$



Volume when rotated about the x axis is $\frac{4\pi ab^2}{3}$ cubic units.

b $$x^2 = a^2\left(1 + \frac{y^2}{b^2}\right)$$

When $x = 2a$, $y^2 = b^2\left(\frac{4a^2}{a^2} - 1\right) = 3b^2$

Therefore $y = \pm\sqrt{3}b$

$$V = \pi \times (2a)^2 \times 2\sqrt{3}b - \int_{-\sqrt{3}b}^{\sqrt{3}b} \pi x^2\, dy$$
$$= 8\sqrt{3}\pi a^2 b - 2\pi a^2 \int_0^{\sqrt{3}b}\left(1 + \frac{y^2}{b^2}\right)dy$$
$$= 8\sqrt{3}\pi a^2 b - 2\pi a^2\left[y + \frac{y^3}{3b^2}\right]_0^{\sqrt{3}b}$$
$$= 8\sqrt{3}\pi a^2 b - 2\pi a^2\left(\sqrt{3}b + \frac{3\sqrt{3}b^3}{3b^2}\right)$$
$$= 8\sqrt{3}\pi a^2 b - 2\pi a^2 \times 2\sqrt{3}b = 4\sqrt{3}\pi a^2 b$$

Volume when rotated about the y axis is $4\sqrt{3}\pi a^2 b$ cubic units.

12 a Let $$\frac{3x}{2} = \frac{1}{\sqrt{1 - x^2}}$$

$\therefore \quad 3x\sqrt{1 - x^2} = 2$

$\therefore \quad 9x^2(1 - x^2) = 4$

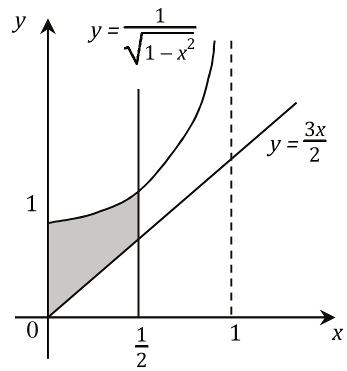
$\therefore \quad 9x^4 - 9x^2 + 4 = 0$

This is a quadratic equation in x^2 with $\Delta = (-9)^2 - 4 \times 9 \times 4$

$$= 81 - 144 = -63$$

Since $\Delta < 0$ there are no points of intersection between $y = \frac{3x}{2}$ and $y = \frac{1}{\sqrt{1-x^2}}$.

b $y = \frac{1}{\sqrt{1-x^2}}$



$$\text{Area} = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} - \frac{3x}{2} dx$$

$$= \left[\sin^{-1} x - \frac{3x^2}{4}\right]_0^{\frac{1}{2}}$$

$$= \left(\sin^{-1}\left(\frac{1}{2}\right) - \frac{3}{4} \times \left(\frac{1}{2}\right)^2\right) - (0 - 0)$$

$$= \frac{\pi}{6} - \frac{3}{16}$$

c $\therefore$ Volume $= \int_0^{\frac{1}{2}} \pi\left(\frac{1}{\sqrt{1-x^2}}\right)^2 dx - \int_0^{\frac{1}{2}} \pi\left(\frac{3x}{2}\right)^2 dx$

$$= \pi \int_0^{\frac{1}{2}} \frac{1}{1-x^2} dx - \pi \int_0^{\frac{1}{2}} \frac{9x^2}{4} dx$$

Now $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$

$\therefore$ $A(1+x) + B(1-x) = 1$

When $x = -1, 2B = 1$

$\therefore$ $B = \frac{1}{2}$

When $x = 1,\ 2A = 1$

$\therefore$ $A = \frac{1}{2}$

$\therefore$ $\frac{1}{1-x^2} = \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$

$$\text{Volume} = \pi \int_0^{\frac{1}{2}} \frac{1}{2(1-x)} + \frac{1}{2(1+x)} - \frac{9}{4}x^2 dx$$

$$= \frac{\pi}{2} \int_0^{\frac{1}{2}} \frac{1}{1-x} + \frac{1}{1+x} - \frac{9}{2}x^2 dx$$

$$= \frac{\pi}{2}\left[-\log_e|1-x| + \log_e|1+x| - \frac{9}{6}x^3\right]_0^{\frac{1}{2}} \quad -1 < x < 1$$

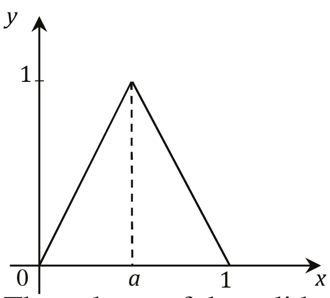
$$= \frac{-\pi}{2}\left[\log_e\left|\frac{1-x}{1+x}\right| + \frac{9}{6}x^3\right]_0^{\frac{1}{2}}$$

$$= \frac{-\pi}{2}\left(\log_e \tfrac{1}{3} + \frac{9}{48} - \log_e 1\right)$$

$$= \frac{-\pi}{2}\left(\frac{9}{48} + \log_e \frac{1}{3}\right)$$

$$= \frac{\pi}{2}\left(\log_e 3 - \frac{3}{16}\right)$$

13 a



The volume of the solid of revolution, V, equals the sum of the volumes of two cones (one has height a and base radius 1 and the other has height $1 - a$ and base radius 1).

$$\therefore\ V = \frac{1}{3}\pi \times 1^2 \times a + \frac{1}{3}\pi \times 1^2 \times (1-a)$$

$$= \frac{\pi a}{3} + \frac{\pi(1-a)}{3}$$

$$= \frac{\pi}{3}$$

So $\frac{\pi}{3}$ is the volume of the solid of revolution.

(Alternatively, find the rules for each straight line and use integration.)

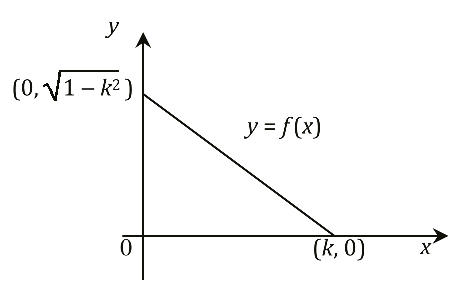
b $f(x) = \frac{\sqrt{1-k^2}}{-k}(x - k),\ 0 \le x \le k$

Volume of cone with base radius $\sqrt{1-k^2}$ and height k.

So $V = \frac{1}{3}\pi \times (1-k^2) \times k$

$$\therefore\ V = \frac{\pi}{3}(k - k^3)$$

$$\frac{dV}{dk} = \frac{\pi}{3}(1 - 3k^2)$$

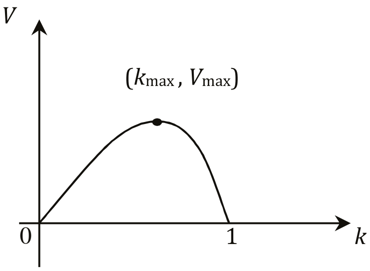


When $\frac{dV}{dk} = 0,\ 1 - 3k^2 = 0$

$$\therefore \quad 3k^2 = 1$$

$$\therefore \quad k^2 = \frac{1}{3}$$

$$\therefore \quad k = \frac{1}{\sqrt{3}} \text{ as } 0 \le k \le 1$$



When $k = \frac{1}{\sqrt{3}},\ V = \dfrac{\pi \times \frac{1}{\sqrt{3}} \times \left(1 - \frac{1}{3}\right)}{3}$

$$= \frac{2\pi}{9\sqrt{3}} = \frac{2\pi\sqrt{3}}{27}$$

Volume is a maximum of $\frac{2\pi\sqrt{3}}{27}$ cubic units when $k = \frac{\sqrt{3}}{3}$.

14 a i Using (0, 0) $\quad d = 0$

Using (5, 1) $\quad 125a + 25b + 5c = 1 \quad$ ①

Using (10, 2.5) $\quad 1000a + 100b + 10c = 2.5 \quad$ ②

Using (30, 10) $\quad 27\,000a + 900b + 30c = 10 \quad$ ③

ii ② −2 × 1 yields

$750a + 50b = 0.5 \quad$ ④

③ −3 × 2 yields

$24\,000a + 600b = 2.5 \quad$ ⑤

⑤ −12 × 4 yields

$15\,000a = -3.5$

$$\therefore \quad a = \frac{-7}{30\,000}$$

Substituting $a = \frac{-7}{30\,000}$ into ④

$$750 \times \frac{-7}{30\,000} + 50b = 0.5$$

$$\therefore \quad \frac{-7}{40} + 50b = 0.5$$

$$\therefore \quad 50b = \frac{27}{40}$$

$$\therefore \quad b = \frac{27}{2000}$$

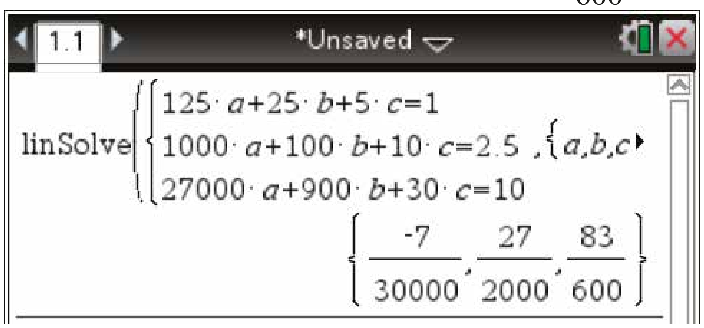
Substituting $a = \frac{-7}{30\,000}$ and $b = \frac{27}{2000}$ into ①

$$125 \times \frac{-7}{30\,000} + 25 \times \frac{27}{2000} + 5c = 1$$

$$\therefore \quad \frac{-7}{240} + \frac{27}{80} + 5c = 1$$

$$\therefore \quad 5c = \frac{83}{120}$$

$$\therefore \quad c = \frac{83}{600}$$



b $$f(x) = \frac{-7}{30\,000}x^3 + \frac{27}{2000}x^2 + \frac{83}{600}x$$

$$= \frac{1}{30\,000}(-7x^3 + 405x^2 + 4150x)$$

$$\text{Area} = \int_0^{30} \frac{1}{30\,000}(-7x^3 + 405x^2 + 4150x)dx$$

$$= \frac{1}{30\,000}\left[\frac{-7}{4}x^4 + 135x^3 + 2075x^2\right]_0^{30}$$

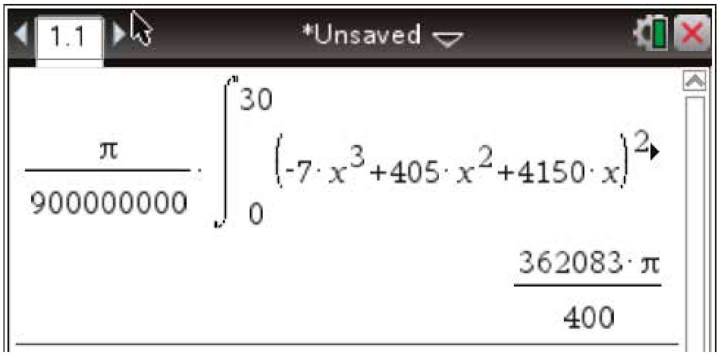
$$= \frac{1}{30\,000}(-1\,417\,500 + 3\,645\,000 + 1\,867\,500 - 0)$$

$$= \frac{273}{2}$$

c **i** $$V = \int_0^{30} \pi(f(x))^2 dx$$

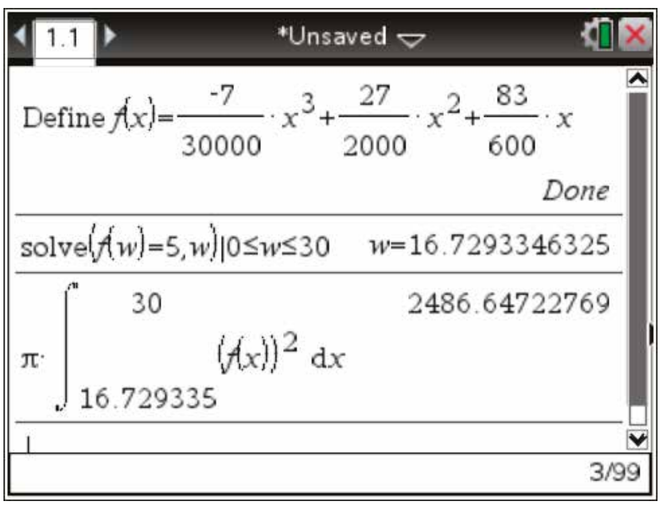
$$= \frac{\pi}{900\,000\,000}\int_0^{30}(-7x^3 + 405x^2 + 4150x)^2 dx$$

ii



d **i** Using a CAS calculator, the point of intersection between $f(x)$ and $y = 5$ is (16.729335,5)
$\therefore\ w = 16.729335$

ii New volume = $\int_{16.729335}^{30} \pi(f(x))^2 dx$
Using a CAS calculator as in **c ii**, the volume is 2487 cubic units, correct to four significant figures.



e

$$f'(x) = \frac{1}{30\,000}(-21x^2 + 810x + 4150)$$

$$f''(x) = \frac{1}{30\,000}(-42x + 810)$$

$$= \frac{1}{5000}(-7x + 135)$$

Now $f''(p) = \dfrac{1}{5000}(-7p + 135)$

and when $f''(p) = 0$,

$$\frac{1}{5000}(-7p + 135) = 0$$

$$\therefore \quad 7p = 135$$

$$\therefore \quad p = \frac{135}{7}$$

$$\text{and } f(p) = \frac{1}{30\,000}\left(-7\left(\frac{135}{7}\right)^3 + 405\left(\frac{135}{7}\right)^2 + 4150\left(\frac{135}{7}\right)\right)$$

$$= \frac{1}{30\,000}\left(-\frac{2\,460\,375}{49} + \frac{7\,381\,125}{49} + \frac{3\,921\,750}{49}\right)$$

$$= \frac{1179}{196}$$

Therefore $(p, f(p)) = \left(\frac{135}{7}, \frac{1179}{196}\right)$.

15 a The line segment AB is described by the function

$$y = \frac{H}{b-a}(x-a), \ a \le x \ \le b$$

$$\therefore \ \frac{b-a}{H}y + a = x$$

$$\therefore \ x^2 = \left(\frac{b-a}{H}y + a\right)^2$$

$$= \frac{[b-a]^2}{H^2}y^2 + \frac{2a[b-a]}{H}y + a^2$$

$$V = \pi \int_0^H x^2 dy$$

$$= \pi \int_0^H \left(\frac{[b-a]^2}{H^2}y^2 + \frac{2a[b-a]}{H}y + a^2\right)dy$$

$$= \pi\left[\frac{[b-a]^2}{3H^2}y^3 + \frac{a[b-a]}{H}y^2 + a^2y\right]_0^H$$

$$= \pi\left(\frac{[b-a]^2H^3}{3H^2} + \frac{a[b-a]H^2}{H} + a^2H\right)$$

$$= \pi\left(\frac{[b-a]^2H}{3} + \frac{3a[b-a]H}{3} + \frac{3a^2H}{3}\right)$$

$$= \frac{\pi H}{3}(b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2)$$

$$= \frac{\pi H}{3}(a^2 + ab + b^2)$$

The capacity of the bowl is $\frac{\pi H}{3}(a^2 + ab + b^2)$ cubic centimetres.

b Volume of water $= \pi\int_0^{\frac{H}{2}} x^2 dy$

$$= \pi\left[\frac{[b-a]^2}{3H^2}y^3 + \frac{a[b-a]}{H}y^2 + a^2y\right]_0^{\frac{H}{2}}$$

$$= \pi\left(\frac{[b-a]^2}{3H^2} \times \frac{H^3}{8} + \frac{a[b-a]}{H} \times \frac{H^2}{4} + \frac{a^2H}{2}\right)$$

$$= \pi H\left(\frac{[b-a]^2}{24} + \frac{6a[b-a]}{24} + \frac{12a^2}{24}\right)$$

$$= \frac{\pi H}{24}\left(b^2 - 2ab + a^2 + 6ab - 6a^2 + 12a^2\right)$$

$$= \frac{\pi H}{24}(7a^2 + 4ab + b^2)$$

The volume of water is $\frac{\pi H}{24}(7a^2 + 4ab + b^2)$ cubic centimetres.

c When $x = r,\ y = \frac{H[r-a]}{b-a}$

$$V = \pi\int_0^{\frac{H[r-a]}{b-a}} x^2 dy$$

$$= \pi\left[\frac{[b-a]^2}{3H^2}y^3 + \frac{a[b-a]}{H}y^2 + a^2y\right]_0^{\frac{H[r-a]}{b-a}}$$

$$= \pi\left(\frac{[b-a]^2H^3[r-a]^3}{3H^2(b-a)^3} + \frac{a[b-a]H^2[r-a]^2}{H(b-a)^2} + \frac{a^2H[r-a]}{b-a}\right)$$

$$= \frac{\pi H[r-a]}{3(b-a)}((r-a)^2 + 3a(r-a) + 3a^2)$$

$$= \frac{\pi H[r-a][r^2+ar+a^2]}{3(b-a)}$$

$$= \frac{\pi H}{3(b-a)}(r^3+ar^2+a^2r-ar^2-a^2r-a^3)$$

$$= \frac{\pi H}{3(b-a)}(r^3-a^3)$$

d **i** $\frac{dV}{dr} = \frac{\pi H r^2}{b-a}$

ii $h = \frac{H[r-a]}{(b-a)}$

e **i** If $a = 10,\ b = 20$ and $H = 20$ then $\frac{dV}{dr} = \frac{\pi \times 20 \times r^2}{20-10}$

$$= 2\pi r^2$$

ii $\frac{dV}{dt} = 3 \therefore \quad \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$$= \frac{1}{2\pi r^2} \times 3$$

$$= \frac{3}{2\pi r^2}$$

When $r = 12,\ \frac{dr}{dt} = \frac{3}{2\pi \times 12^2}$

$$= \frac{1}{96\pi}$$

Now $\frac{dh}{dt} = \frac{dh}{dr} \times \frac{dr}{dt}$ and $\frac{dh}{dr} = \frac{H}{b-a}$

$$= \frac{20}{20-10} = 2$$

$$\therefore \quad \frac{dh}{dt} = 2 \times \frac{3}{2\pi r^2}$$

$$= \frac{3}{\pi r^2}$$

When $r = 12,\ \frac{dh}{dt} = \frac{3}{\pi \times 12^2}$

$$= \frac{1}{48\pi}$$

Chapter 11 – Differential equations

Solutions to Exercise 11A

1 a If $y = Ae^{2t} - 2$ then $\frac{dy}{dt} = 2Ae^{2t}$

Given $\frac{dy}{dt} = 2y + 4$

LHS $= 2Ae^{2t}$

$$\begin{aligned}\text{RHS} &= 2(Ae^{2t} - 2) + 4\\ &= 2Ae^{2t} - 4 + 4\\ &= 2Ae^{2t}\end{aligned}$$

$\therefore\ y = Ae^{2t} - 2$

is a solution of

$\frac{dy}{dt} = 2y + 4$

Substituting $y(0) = 2$ into $y = Ae^{2t} - 2$ gives:

$$\begin{aligned}2 &= Ae^{2\times 0} - 2\\ &= A - 2\end{aligned}$$

$\therefore A = 4$

$\therefore y = 4e^{2t} - 2$ is the particular solution.

b If $y = x\log_e|x| - x + c$

$$\begin{aligned}\text{then } \frac{dy}{dx} &= \log_e|x| + 1 - 1\\ &= \log_e|x|\end{aligned}$$

$\therefore\ y = x\log_e|x| - x + c$

is a solution of

$\frac{dy}{dx} = \log_e|x|$

Substituting $y(1) = 3$ into

$y = x\log_e|x| - x + c$ gives:

$$\begin{aligned}3 &= 1\log_e|1| - 1 + c\\ &= -1 + c\end{aligned}$$

$\therefore c = 4$

$\therefore\ y = x\log_e|x| - x + 4$ is the particular solution.

c If $y = \sqrt{2x + c}$

$$\begin{aligned}\text{then } \frac{dy}{dx} &= \frac{1}{2\sqrt{2x + c}} \times 2\\ &= \frac{1}{\sqrt{2x + c}}\\ &= \frac{1}{y}\end{aligned}$$

$\therefore\ y = \sqrt{2x + c}$ is a solution of

$\frac{dy}{dx} = \frac{1}{y}$

Substituting $y(1) = 9$ into

$y = \sqrt{2x + c}$ gives:

$9 = \sqrt{2 \times 1 + c}$

$81 = 2 + c$

$\therefore c = 79$

$\therefore y = \sqrt{2x + 79}$

is the particular solution.

d If $y - \log_e|y + 1| = x + c$

$$\begin{aligned}\text{then } \frac{dx}{dy} &= 1 - \frac{1}{y + 1}\\ &= \frac{y + 1 - 1}{y + 1}\\ &= \frac{y}{y + 1}\end{aligned}$$

$\therefore\ \frac{dy}{dx} = \frac{y + 1}{y}$

$\therefore\ y - \log_e|y + 1| = x + c$ is a solution of $\frac{dy}{dx} = \frac{y + 1}{y}$

Substituting $y(3) = 0$ into

$y - \log_e |y + 1| = x + c$ gives:

$0 - \log_e |0 + 1| = 3 + c$

$0 = 3 + c$

$\therefore \quad c = -3$

$\therefore \quad y - \log_e |y + 1| = x - 3$ is the particular solution.

e If $y = \dfrac{x^4}{2} + Ax + B$

then $\dfrac{dy}{dx} = 2x^3 + A$

and $\dfrac{d^2y}{dx^2} = 6x^2$

$\therefore y = \dfrac{x^4}{2} + Ax + b$ is a solution of

$\dfrac{d^2y}{dx^2} = 6x^2$

Substituting $y(0) = 2$ and $y(1) = 2$ into $y = \dfrac{x^4}{2} + Ax + B$ gives:

$2 = \dfrac{0^4}{2} + A \times 0 + B$

$\therefore \quad B = 2$

and

$2 = \dfrac{1^4}{2} + A \times 1 + B$

$= \dfrac{1}{2} + A \times 1 + 2$

$\therefore A = -\dfrac{1}{2}$

$\therefore y = \dfrac{x^4}{2} - \dfrac{x}{2} + 2$ is the particular solution.

f If $y = Ae^{2x} + Be^{-2x}$

then $\dfrac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$

and $\dfrac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$

$= 4(Ae^{2x} + Be^{-2x})$

$= 4y$

$\therefore \quad y = Ae^{2x} + Be^{-2x}$

is a solution of

$\dfrac{d^2y}{dx^2} = 4y$

Substituting $y(0) = 3$ and $y(\log_e 2) = 9$ into

$y = Ae^{2x} + Be^{-2x}$ gives:

$3 = Ae^{2\times 0} + Be^{-2\times 0}$

$\therefore \; 3 = A + B$ ①

and

$\therefore \; 9 = Ae^{2\log_e 2} + Be^{-2\log_e 2}$

$= Ae^{\log_e 4} + Be^{-\log_e 4}$

$\therefore \; 9 = 4A + \dfrac{1}{4}B$ ②

4 × ① − ② gives

$3 = 0 + \dfrac{15}{4}B$

$\therefore \; B = \dfrac{12}{15} = \dfrac{4}{5}$

Substituting $B = \dfrac{4}{5}$ in ① gives

$3 = A + \dfrac{4}{5}$

$\therefore \; A = \dfrac{11}{5}$

$\therefore \; y = \dfrac{11}{5}e^{2x} + \dfrac{4}{5}e^{-2x}$

is the particular solution.

g If $x = A\sin 3t + B\cos 3t + 2$

then $\dfrac{dx}{dt} = 3A\cos 3t - 3B\sin 3t$

and $\dfrac{d^2x}{dt^2} = -9A\sin 3t - 9B\cos 3t$

$= -9(A\sin 3t + B\cos 3t)$

Given $\dfrac{d^2x}{dt^2} + 9x = 18$

LHS $= -9(A\sin 3t + B\cos 3t)$

$+ 9(A\sin 3t + B\cos 3t + 2)$

$= -9A\sin 3t - 9B\cos 3t$

$+ 9A\sin 3t + 9B\cos 3t + 18$

$= 18$

= RHS

$\therefore$ $x = A\sin 3t + B\cos 3t + 2$ is a solution of $\dfrac{d^2x}{dt^2} + 9x = 18$

Now $x(0) = 4$

$\therefore$ $4 = B + 2$

$\therefore$ $B = 2$

and $x\left(\dfrac{\pi}{2}\right) = -1$

$\therefore$ $-1 = -A + 2$

$\therefore$ $A = 3$

$\therefore$ $x = 3\sin 3t + 2\cos 3t + 2$

is the particular solution.

2 a $y = 4e^{2x}$

$\dfrac{dy}{dx} = 8e^{2x}$

$\therefore$ $\dfrac{dy}{dx} = 2y$

b $y = \dfrac{1}{2}x^{-2}$

$\dfrac{dy}{dx} = -x^{-3} = -\dfrac{1}{x^3}$

$-4xy^2 = -4x\left(\dfrac{1}{2x^2}\right)^2 = -\dfrac{4x}{4x^4} = -\dfrac{1}{x^3}$

$\therefore$ $\dfrac{dy}{dx} = -4xy^2$

c $y = x\log_e|x| + x$

$\dfrac{dy}{dx} = \log_e|x| + \dfrac{x}{x} + 1 = \log_e|x| + 2$

$\dfrac{y}{x} + 1 = \log_e|x| + 1 + 1 = \log_e|x| + 2$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{y}{x} + 1$

d $y = (3x^2 + 27)^{\frac{1}{3}}$

$\dfrac{dy}{dx} = \dfrac{1}{3}(3x^2 + 27)^{-\frac{2}{3}} \cdot (6x)$

$= \dfrac{2x}{\sqrt[3]{(3x^2 + 27)^2}}$

$\dfrac{2x}{y^2} = \dfrac{2x}{\sqrt[3]{(3x^2 + 27)^2}}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{2x}{y^2}$

e $y = e^{-2x} + e^{3x}$

$\dfrac{dy}{dx} = -2e^{-2x} + 3e^{3x}$

$\dfrac{d^2y}{dx^2} = 4e^{-2x} + 9e^{3x}$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 4e^{-2x} + 9e^{3x}$$
$$- (-2e^{-2x} + 3e^{3x})$$
$$- 6(e^{-2x} + e^{3x})$$
$$= (-6 + 2 + 4)e^{-2x}$$
$$+ (-6 - 3 + 9)e^{3x}$$
$$= 0$$
$$\therefore \frac{d^2y}{d^2x} - \frac{dy}{dx} - 6y = 0$$

f
$$y = e^{4x}(x + 1)$$
$$= xe^{4x} + e^{4x}$$
$$\frac{dy}{dx} = 4xe^{4x} + e^{4x}$$
$$+ 4e^{4x}$$
$$= 4xe^{4x} + 5e^{4x}$$
$$\frac{d^2y}{dx^2} = 16xe^{4x} + 4e^{4x}$$
$$+ 20e^{4x}$$
$$= 16xe^{4x} + 24e^{4x}$$
$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = (16 - 32$$
$$+ 16)xe^{4x}$$
$$+ (16 - 40$$
$$+ 24)e^{4x}$$
$$= 0$$
$$\therefore \frac{d^2y}{d^2x} - 8\frac{dy}{dx} + 16y = 0$$

g
$$y = a\sin(nx)$$
$$\frac{dy}{dx} = na\cos(nx)$$
$$\therefore \frac{d^2y}{dx^2} = -n^2a\sin(nx) = -n^2y$$

h
$$y = e^{nx} + e^{-nx}$$
$$\frac{dy}{dx} = ne^{nx} - ne^{-nx}$$
$$\therefore \frac{d^2y}{dx^2} = n^2e^{nx} + n^2e^{-nx}$$
$$= n^2(e^{nx} + e^{-nx})$$
$$= n^2y$$

i
$$y = \frac{x + 1}{1 - x}$$
$$\frac{dy}{dx} = \frac{(1 - x) - (-1)(1 + x)}{(1 - x)^2}$$
$$= \frac{2}{(1 - x)^2}$$
$$\frac{1 + y^2}{1 + x^2} = \frac{1 + \frac{(x + 1)^2}{(x - 1)^2}}{1 + x^2}$$
$$= \frac{(x - 1)^2 + (x + 1)^2}{(1 + x^2)(x - 1)^2}$$
$$= \frac{2(x^2 + 1)}{(x^2 + 1)(x - 1)^2}$$
$$= \frac{2}{(x - 1)^2}$$
$$\therefore \frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$$

j
$$y = \frac{4}{x + 1} = 4(x + 1)^{-1}$$
$$\frac{dy}{dx} = \frac{-4}{(x + 1)^2}$$
$$\frac{d^2y}{dx^2} = \frac{8}{(x + 1)^3}$$
$$y\frac{d^2y}{dx^2} = \frac{4}{x + 1} \times \frac{8}{(x + 1)^3} = \frac{32}{(x + 1)^4}$$
$$2\left(\frac{dy}{dx}\right)^2 = \frac{2 \times 16}{(x + 1)^4} = \frac{32}{(x + 1)^4}$$
$$\therefore y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2$$

3 If $y = ax^n$

then $\dfrac{dy}{dx} = nax^{n-1}$

and $\dfrac{d^2y}{dx^2} = n(n-1)ax^{n-2}$

Therefore,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 10y = n(n-1)ax^n - 2anx^n - 10ax^n = 0$$

$\therefore\ n(n-1) - 2n - 10 = 0$

$\therefore\ n^2 - 3n - 10 = 0$

$\therefore\ (n-5)(n+2) = 0$

$\therefore\ n = -2$ or $n = 5$

4 If $y = a + bx + cx^2$

then $\dfrac{dy}{dx} = b + 2cx$

$\dfrac{d^2y}{dx^2} = 2c$

as $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = 4x^2$

$\Rightarrow 2c + 2(b + 2cx) + 4(a + bx + cx^2) = 4x^2$

$\therefore\ 4c = 4 \qquad c = 1$

$4b + 4c = 0 \qquad b = -1$

$2c + 2b + 4a = 0 \qquad a = 0$

$\therefore\ a = 0, b = -1$ and $c = 1$

5 If $x = t(a\cos 2t + b\sin 2t)$

then

$$\frac{dx}{dt} = a\cos 2t + b\sin 2t + t(-2a\sin 2t + 2b\cos 2t) = (a + 2bt)\cos 2t + (b - 2at)\sin 2t$$

$$\frac{d^2x}{dt^2} = 2b\cos 2t - 2(a + 2bt)\sin 2t - 2a\sin 2t + 2(b - 2at)\cos 2t = (4b - 4at)\cos 2t - (4a + 4bt)\sin 2t$$

$\therefore\ \dfrac{d^2x}{dt^2} + 4x = 4b\cos 2t - 4a\sin 2t$

and since $\dfrac{d^2x}{dt^2} + 4x = 2\cos 2t$

$\Rightarrow 4b\cos 2t - 4a\sin 2t = 2\cos 2t$

$\therefore\ 4b = 2 \Rightarrow b = \dfrac{1}{2}$

and $a = 0$

$\therefore\ a = 0, b = \dfrac{1}{2}$

6 If $y = ax^3 + bx^2 + cx + d$,

then $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$

$\dfrac{d^2y}{dx^2} = 6ax + 2b$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = ax^3 + (b + 6a)x^2 + (c + 4b + 6a)x + (d + 2c + 2b)$$

and since $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^3$

$\therefore\ a = 1$

$b + 6a = 0 \ \therefore\ b = -6$

$c + 4b + 6a = 0 \ \therefore\ c = 18$

$d + 2c + 2b = 0 \ \therefore\ d = -24$

$\therefore\ a = 1, b = -6, c = 18, d = -24$

Solutions to Exercise 11B

1 a $\dfrac{dy}{dx} = x^2 - 3x + 2$

$\therefore\ y = \dfrac{1}{3}x^3 - \dfrac{3}{2}x^2 + 2x + c,$

$c \in \mathbb{R}$, is the general solution.

b $\dfrac{dy}{dx} = \dfrac{x^2 + 3x - 1}{x},\ x \neq 0$

$= x + 3 - \dfrac{1}{x}$

$\therefore\ y = \dfrac{1}{2}x^2 + 3x - \log_e |x| + c,$

$c \in \mathbb{R}$, is the general solution.

c $\dfrac{dy}{dx} = (2x + 1)^3$

$= 8x^3 + 12x^2 + 6x + 1$

$\therefore\ y = 2x^4 + 4x^3 + 3x^2 + x + c,$

$c \in \mathbb{R}$, is the general solution.

d $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}},\ x > 0$

$= x^{-\frac{1}{2}}$

$\therefore\ y = 2x^{\frac{1}{2}} + c$

$\therefore\ y = 2\sqrt{x} + c,$

$c \in \mathbb{R}$, is the general solution.

e $\dfrac{dy}{dt} = \dfrac{1}{2t - 1},\ t \neq \dfrac{1}{2}$

$\therefore\ y = \dfrac{1}{2}\log_e |2t - 1| + c,$

$c \in \mathbb{R}$, is the general solution.

f $\dfrac{dy}{dt} = \sin(3t - 2)$

$\therefore\ y = -\dfrac{1}{3}\cos(3t - 2) + c,$

$c \in \mathbb{R}$, is the general solution.

g $\dfrac{dy}{dt} = \tan(2t)$

$= \dfrac{\sin(2t)}{\cos(2t)}$

Let $u = \cos(2t)$

$\therefore\ \dfrac{du}{dt} = -2\sin(2t)$

$\therefore\ y = \displaystyle\int -\frac{1}{2u}\frac{du}{dt}dt$

$= -\dfrac{1}{2}\displaystyle\int \frac{1}{u}du$

$= -\dfrac{1}{2}\log_e |u| + c,$

$\therefore\ y = -\dfrac{1}{2}\log_e |\cos(2t)| + c,$

$c \in \mathbb{R}$, is the general solution.

h $\dfrac{dx}{dy} = e^{-3y}$

$\therefore\ x = -\dfrac{1}{3}e^{-3y} + c,$

$c \in \mathbb{R}$, is the general solution.

i $\dfrac{dx}{dy} = \dfrac{1}{\sqrt{4 - y^2}}$

$= \dfrac{1}{\sqrt{2^2 - y^2}}$

$\therefore\ x = \sin^{-1}\left(\dfrac{y}{2}\right) + c,$

$c \in \mathbb{R}$, is the general solution

j $\dfrac{dx}{dy} = -\dfrac{1}{(1 - y)^2}$

Let $u = 1 - y$, then $\dfrac{du}{dy} = -1$

$$\therefore\quad x = -\int u^{-2}(-du)$$
$$= \int u^{-2}du$$
$$= -u^{-1} + c$$
$$\therefore\quad x = -\frac{1}{1-y} + c$$
$$\therefore\quad x = \frac{1}{y-1} + c,$$
$c \in \mathbb{R}$, is the general solution.

2 **a** $\dfrac{d^2y}{dx^2} = 5x^3$

$$\therefore\quad \frac{dy}{dx} = \frac{5}{4}x^4 + c$$
$$\therefore\quad y = \frac{1}{4}x^5 + cx + d,$$
where $c,\ d \in \mathbb{R}$, is the general solution.

b $\dfrac{d^2y}{dx^2} = \sqrt{1-x}$

Let $u = 1 - x$, then $\dfrac{du}{dx} = -1$

$$\therefore\quad \frac{dy}{dx} = \int u^{\frac{1}{2}}(-du)$$
$$= -\int u^{\frac{1}{2}}du$$
$$= -\frac{2}{3}u^{\frac{3}{2}} + c$$

$$\therefore\quad y = \int -\frac{2}{3}u^{\frac{3}{2}}dx + \int c\,dx$$
$$= \int -\frac{2}{3}u^{\frac{3}{2}}(-du) + \int c\,dx$$
$$= \frac{2}{3}\int u^{\frac{3}{2}}\frac{du}{dx}dx + \int c\,dx$$
$$= \frac{2}{3}\int u^{\frac{3}{2}}du + \int c\,dx$$
$$= \frac{4}{15}u^{\frac{5}{2}} + cx + d$$
$$= \frac{4}{15}(1-x)^{\frac{5}{2}} + cx + d,$$
where $c,\ d \in R$

$$\therefore\quad y = \frac{4}{15}(1-x)^{\frac{5}{2}} + cx + d$$
is the general solution.

c $\dfrac{d^2y}{dx^2} = \sin\left(2x + \dfrac{\pi}{4}\right)$

$$\therefore\quad \frac{dy}{dx} = -\frac{1}{2}\cos\left(2x + \frac{\pi}{4}\right) + c$$
$$\therefore\quad y = -\frac{1}{4}\sin\left(2x + \frac{\pi}{4}\right) + cx + d,$$
where $c,\ d \in \mathbb{R}$, is the general solution.

d $\dfrac{d^2y}{dx^2} = e^{\frac{x}{2}}$

$$\therefore\quad \frac{dy}{dx} = 2e^{\frac{x}{2}} + c$$
$$\therefore\quad y = 4e^{\frac{x}{2}} + cx + d,$$
where $c,\ d \in \mathbb{R}$, is the general solution.

e $\frac{d^2y}{dx^2} = \frac{1}{\cos^2 x}$

$= \sec^2 x$

$\therefore \quad \frac{dy}{dx} = \tan x + c$

$= \frac{\sin x}{\cos x} + c$

$\therefore \quad y = \int \frac{\sin x}{\cos x} + cdx$

Let $u = \cos x$ $\therefore \quad \frac{du}{dx} = -\sin x$

$\therefore \quad y = \int -\frac{1}{u}\frac{du}{dx}dx + \int cdx$

$= -\int \frac{1}{u}du + \int cdx$

$= -\log_e |u| + cx + d,$

$= -\log_e |\cos x| + cx + d,$

where $c,\ d \in R$

$\therefore \quad y = -\log_e |\cos x| + cx + d,$

is the general solution.

f $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

$= (x+1)^{-2}$

$\therefore \quad \frac{dy}{dx} = -(x+1)^{-1} + c$

$= \frac{-1}{x+1} + c$

$\therefore \quad y = -\log_e |x+1| + cx + d,$

where $c,\ d \in \mathbb{R}$ is the general solution.

3 a $\frac{dy}{dx} = \frac{1}{x^2}$

$\therefore \quad y = \int \frac{dx}{x^2}$

$\therefore \quad y = -\frac{1}{x} + c$

Initial condition: $y(4) = \frac{3}{4}$

$\therefore \quad \frac{3}{4} = -\frac{1}{4} + c$

$\Rightarrow c = 1$

$\therefore \quad y = -\frac{1}{x} + 1 = \frac{x-1}{x}$

b $\frac{dy}{dx} = e^{-x}$

$\therefore \quad y = \int e^{-x}dx$

$\therefore \quad y = -e^{-x} + c$

Initial condition: $y(0) = 0$

$\therefore \quad 0 = -1 + c$

$\Rightarrow c = 1$

$\therefore \quad y = 1 - e^{-x}$

c $\frac{dy}{dx} = \frac{x^2 - 4}{x}$

$\therefore \quad y = \int \frac{x^2-4}{x}dx$

$= \int \left(x - \frac{4}{x}\right)dx$

$= \frac{x^2}{2} - 4\log_e |x| + c$

Initial condition: $y(1) = \frac{3}{2}$

$\therefore \quad \frac{3}{2} = \frac{1}{2} + c$ (Note: $\log_e 1 = 0$)

$\Rightarrow c = 1$

$\therefore \quad y = \frac{x^2}{2} - 4\log_e |x| + 1$

d Using partial fractions

$\frac{x}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$

$\therefore \quad x = A(x+2) + B(x-2)$

When $x = 2, A = \frac{1}{2}$

$$\therefore \quad y = \int \frac{xdx}{x^2 - 4}$$

$$= \frac{1}{2}\int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x+2}$$

$$= \frac{1}{2}\log_e |x^2 - 4| + c$$

$$\therefore \quad y = \frac{1}{2}\log_e |x^2 - 4| + c$$

Initial condition: $y(2\sqrt{2}) = \log_e 2$

$$\therefore \log_e 2 = \log_e \sqrt{((2\sqrt{2})^2 - 4)} + c$$

$$\therefore \log_e 2 = \log_e \sqrt{4} + c$$

$$\therefore \log_e 2 = \log_e 2 + c$$

$$\Rightarrow c = 0$$

$$\therefore \quad y = \frac{1}{2}\log_e |x^2 - 4|$$

e $\quad \dfrac{dy}{dx} = x\sqrt{x^2 - 4}$

Let $x^2 - 4 = u$, then $\dfrac{du}{dx} = 2x$

$$y = \int x\sqrt{x^2 - 4}dx$$

$$= \int \frac{1}{2}\sqrt{u}\,du$$

$$\therefore \quad y = \frac{1}{3}u^{\frac{3}{2}} + c$$

Initial condition: $y(4) = \dfrac{1}{4\sqrt{3}}$

But when $x = 4, u = 12$

$$\therefore \quad \frac{1}{4\sqrt{3}} = \frac{1}{3}(12\sqrt{12}) + c$$

$$\therefore \quad c = \frac{\sqrt{3}}{12} - 4\sqrt{12}$$

$$\therefore \ c = \frac{\sqrt{3}}{12} - 8\sqrt{3}$$

$$\therefore \ c = \frac{-95\sqrt{3}}{12}$$

$$\therefore \ y = \frac{1}{3}(x^2 - 4)^{\frac{3}{2}} - \frac{95\sqrt{3}}{12}$$

f $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{4 - x^2}}$

$$\therefore y = \int \frac{dx}{\sqrt{4 - x^2}}$$

$$\therefore y = \sin^{-1}\left(\frac{x}{2}\right) + c$$

Initial condition: $y(1) = \dfrac{\pi}{3}$

When $x = -2, B = \dfrac{1}{2}$

$$\therefore \quad \frac{\pi}{3} = \sin^{-1}\frac{1}{2} + c$$

$$\Rightarrow c = \frac{\pi}{6}$$

$$\therefore \quad y = \sin^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{6}$$

g $\quad \dfrac{dy}{dx} = \dfrac{1}{4 - x^2}$

$$= \frac{1}{(2-x)(2+x)}$$

$$= \frac{1}{4}\left(\frac{1}{2-x} + \frac{1}{2+x}\right)$$

using partial fractions.

$$\therefore y = \frac{1}{4}(-\log_e |2 - x| + \log_e |2 + x| + c)$$

$$= \frac{1}{4}\log_e\left|\frac{2+x}{2-x}\right| + c$$

Initial condition: $y(0) = 2$

$$\therefore \ 2 = \frac{1}{4}\log_e |1| + c$$

$$\therefore \ c = 2$$

$$\therefore \ y = \frac{1}{4}\log_e\left|\frac{2+x}{2-x}\right| + 2$$

h $\frac{dy}{dx} = \frac{1}{4 + x^2}$

$\therefore \quad y = \int \frac{dx}{4 + x^2}$

$\therefore \quad y = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

Initial conditions: $y(2) = \frac{3\pi}{8}$

$\therefore \quad \frac{3\pi}{8} = \frac{1}{2}\tan^{-1} 1 + c$

$\therefore \quad \frac{3\pi}{8} = \frac{\pi}{8} + c$

$\therefore \quad c = \frac{\pi}{4}$

$\therefore \quad y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{4}$

i

$\frac{dy}{dx} = x\sqrt{4 - x}$

Let $u = 4 - x$, $\frac{du}{dx} = -1$

$x = 4 - u$

$\therefore \quad y = \int x\sqrt{4 - x}\,dx$

$= -\int (4 - u)u^{\frac{1}{2}}\,du$

$= -\int 4u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= -\left(\frac{8}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + c$

$= -\left(\frac{8}{3}(4 - x)^{\frac{3}{2}} - \frac{2}{5}(4 - x)^{\frac{5}{2}}\right) + c$

$= -\frac{8\sqrt{(4 - x)^3}}{3} + \frac{2\sqrt{(4 - x)^5}}{5} + c$

Initial conditions: $y(0) = -\frac{8}{15}$

$\therefore \quad -\frac{8}{15} = -\frac{8\sqrt{4^3}}{3} + \frac{2\sqrt{4^5}}{5} + c$

$\therefore \quad -\frac{8}{15} = -\frac{64}{3} + \frac{64}{5} + c$

$\therefore \quad -\frac{8}{15} = -\frac{128}{15} + c$

$\therefore \quad c = 8$

$\therefore \quad y = -\frac{8\sqrt{(4 - x)^3}}{3} + \frac{2\sqrt{(4 - x)^5}}{5} + 8$

j $\frac{dy}{dx} = \frac{e^x}{e^x + 1}$

Let $u = e^x + 1$

$\frac{du}{dx} = e^x$

$\therefore \quad y = \int \frac{1}{u}du$

$\therefore \quad y = \log_e(e^x + 1) + c$ (as $e^x > 0$)

Initial condition: $y(0) = 0$

$\therefore \quad 0 = \log_e 2 + c$

$\therefore \quad c = -\log_e 2$

$\therefore \quad y = \log_e(e^x + 1) - \log_e 2$

$\therefore \quad y = \log_e\left(\frac{e^x + 1}{2}\right)$

4 a $\frac{d^2y}{dx^2} = e^{-x} - e^x$

$\frac{dy}{dx} = \int (e^{-x} - e^x)dx$

$\frac{dy}{dx} = -e^{-x} - e^x + c_1$

Initial condition: $\frac{dy}{dx} = 0,\ x = 0$

$\therefore\ 0 = -1 - 1 + c_1$

$\Rightarrow c_1 = 2$

$\therefore\ \frac{dy}{dx} = -e^{-x} - e^x + 2$

$y = \int(-e^{-x} - e^x + 2)dx$

$y = e^{-x} - e^x + 2x + c_2$

Initial condition: $y(0) = 0$

$\therefore\ 0 = 1 - 1 + 0 + c_2$

$\Rightarrow c_2 = 0$

$\therefore\ y = e^{-x} - e^x + 2x$

b $\frac{d^2y}{dx^2} = 2 - 12x$

$\frac{dy}{dx} = \int(2 - 12x)dx$

$\frac{dy}{dx} = 2x - 6x^2 + c_1$

Initial condition: $\frac{dy}{dx} = 0,\ x = 0$

$\Rightarrow c_1 = 0$

$\therefore\ \frac{dy}{dx} = 2x - 6x^2$

$y = \int(2x - 6x^2)dx$

$y = x^2 - 2x^3 + c_2$

Initial condition: $y(0) = 0$

$\Rightarrow c_2 = 0$

$\therefore\ y = x^2 - 2x^3$

c $\frac{d^2y}{dx^2} = 2 - \sin 2x$

$\frac{dy}{dx} = \int(2 - \sin 2x)dx$

$\frac{dy}{dx} = 2x + \frac{1}{2}\cos 2x + c_1$

Initial condition: $\frac{dy}{dx} = \frac{1}{2},\ x = 0$

$\therefore\ \frac{dy}{dx} = 2x - \frac{1}{2}\cos 2x$

$y = \int\left(2x + \frac{1}{2}\cos 2x\right)dx$

$y = x^2 + \frac{1}{4}\sin 2x + c_2$

Initial condition: $y(0) = -1$

$\Rightarrow c_2 = -1$

$\therefore\ y = x^2 + \frac{1}{4}\sin 2x - 1$

d $\frac{d^2y}{dx^2} = 1 - \frac{1}{x^2}$

$\frac{dy}{dx} = \int\left(1 - \frac{1}{x^2}\right)dx$

$= x + \frac{1}{x} + c_1$

Initial condition: $\frac{dy}{dx} = 0,\ x = 0$

$\therefore\ 0 = 1 + 1 + c_1$

$\Rightarrow c_1 = -2$

$\therefore\ \frac{dy}{dx} = x + \frac{1}{x} - 2$

$y = \int\left(x + \frac{1}{x} - 2\right)dx$

$y = \frac{x^2}{2} + \log_e|x| - 2x + c_2$

Initial condition: $y(1) = \frac{3}{2}$

$\therefore\ \frac{3}{2} = \frac{1}{2} - 2 + c_2$

(Note: $\log_e 1 = 0$)

$\Rightarrow c_2 = 3$

$\therefore\ y = \frac{x^2}{2} + \log_e|x| - 2x + 3$

e $\frac{d^2y}{dx^2} = \frac{2x}{(1+x^2)^2}$

$$\frac{dy}{dx} = \int \frac{2x\,dx}{(1+x^2)^2}$$

$$= \int \frac{dw}{w^2},\ \text{where } w = 1 + x^2$$

$$= -\frac{1}{w} + c_1$$

$$= -\frac{1}{1+x^2} + c_1$$

Initial condition: $\frac{dy}{dx} = 0,\ x = 0$

$\Rightarrow c_1 = 1$

$\therefore\ \frac{dy}{dx} = -\frac{1}{1+x^2} + 1$

$\Rightarrow c_1 = 0$

$$y = -\int \frac{dx}{1+x^2} + \int 1\,dx$$

$$= -\tan^{-1} x + x + c_2$$

Initial condition: $y(1) = 1$

$\therefore\ 1 = -\frac{\pi}{4} + 1 + c_2$

$\Rightarrow c_2 = \frac{\pi}{4}$

$\therefore\ y = x - \tan^{-1} x + \frac{\pi}{4}$

f $\frac{d^2y}{dx^2} = 24(2x+1)$

$$\frac{dy}{dx} = \int 24(2x+1)dx$$

$$= 24(x^2 + x) + c_1$$

Initial condition: $\frac{dy}{dx} = 6,\ x = -1$

$\Rightarrow c_1 = 6$

$\therefore\ \frac{dy}{dx} = 24(x^2 + x) + 6$

$$y = 24\left(\frac{x^3}{3} + \frac{x^2}{2}\right) + 6x + c_2$$

$$= 8x^3 + 12x^2 + 6x + c_2$$

Initial condition: $y(-1) = -2$

$\therefore\ -2 = -8 + 12 - 6 + c_2$

$\Rightarrow c_2 = 0$

$\therefore\ y = 8x^3 + 12x^2 + 6x$

g $\frac{d^2y}{dx^2} = \frac{x}{(4-x^2)^{\frac{3}{2}}}$

Let $4 - x^2 = u$

then $\frac{du}{dx} = -2x$

$\therefore\ \frac{dy}{dx} = \int \frac{x}{(4-x^2)^{\frac{3}{2}}}dx$

$$= -\frac{1}{2}\int \frac{1}{u^{\frac{3}{2}}}du$$

$$= u^{-\frac{1}{2}} + c_1$$

$$= \frac{1}{\sqrt{4-x^2}} + c_1$$

Initial condition: $\frac{dy}{dx} = \frac{1}{2},\ x = 0$

$\therefore\ \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$

$\therefore\ y = \sin^{-1}\frac{x}{2} + c_2$

Initial condition: $y(-2) = -\frac{\pi}{2}$

$\therefore\ -\frac{\pi}{2} = \sin^{-1}(-1) + c_2$

$\Rightarrow c_2 = 0$

$\therefore\ y = \sin^{-1}\frac{x}{2}$

5 a $\frac{dy}{dx} = 3x + 4$

$\therefore\ y = \frac{3}{2}x^2 + 4x + c,$

$c \in \mathbb{R}$, represents the family of curves.

b $\frac{d^2y}{dx^2} = -2x$

$\therefore \quad \frac{dy}{dx} = -x^2 + c$

$\therefore \quad y = -\frac{1}{3}x^3 + cx + d,$

$c, \ d \in \mathbb{R}$, represents the family of curves.

c $\frac{dy}{dx} = \frac{1}{x-3}$

$\therefore \quad y = \log_e |x-3| + c,$

$c \in \mathbb{R}$, represents the family of curves.

6 a $\frac{dy}{dx} = 2 - e^{-x}$

$\therefore \quad y = 2x + e^{-x} + c$

Now $y(0) = 1$

$\therefore \quad 1 = 2 \times 0 + e^{-0} + c$

$\therefore \quad c = 0$

$\therefore \quad y = 2x + e^{-x}$

$\Rightarrow c_1 = 0$

b $\frac{dy}{dx} = x + \sin 2x$

$\therefore \quad y = \frac{1}{2}x^2 - \frac{1}{2}\cos 2x + c$

Now $y(0) = 4$

$\therefore \quad 4 = \frac{1}{2}(0)^2 - \frac{1}{2}\cos(2 \times 0) + c$

$= 0 - \frac{1}{2} + c$

$\therefore \quad c = \frac{9}{2}$

$\therefore \quad y = \frac{1}{2}x^2 - \frac{1}{2}\cos 2x + \frac{9}{2}$

c $\frac{dy}{dx} = \frac{1}{2-x}$

$\therefore \quad y = -\log_e |2-x| + c$

Now $y(3) = 2$

$\therefore \quad 2 = -\log_e |2-3| + c$

$= -0 + c$

$\therefore \quad c = 2$

$\therefore \quad y = 2 - \log_e |2-x|$

7 $\frac{dx}{dy} \propto \frac{1}{y}$

$\Rightarrow \frac{dx}{dy} = \frac{k}{y} \ (y > 0)$

$\therefore \quad x = k\log_e y + c$

Substituting $y(0) = 2$ and $y(2) = 4$

$0 = k\log_e 2 + c$ ①

and

$2 = k\log_e 4 + c$

$\therefore \quad 2 = 2k\log_e 2 + c$ ②

② − ①

$k\log_e 2 = 2$

$\therefore \quad k = \frac{2}{\log_e 2}$

Substituting into ① gives

$c = -2$

$\therefore \quad x = \frac{2\log_e y}{\log_e 2} - 2$

When $x = 3$,

$3 = \frac{2\log_e y}{\log_e 2} - 2$

$\therefore \quad \log_e y = \frac{5}{2}\log_e 2$

$\therefore \quad y = 4\sqrt{2}$

Solutions to Exercise 11C

1 a

$$\frac{dy}{dx} = 3y - 5$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{3y - 5}$$

$$\therefore \quad x = \frac{1}{3}\log_e |3y - 5| + c, \quad c \in R,$$

$$\therefore \quad x - c = \frac{1}{3}\log_e |3y - 5|$$

$$\therefore \quad 3(x - c) = \log_e |3y - 5|$$

$$\therefore \quad |3y - 5| = e^{3(x-c)}$$

$$\therefore \quad |3y - 5| = Ae^{3x} \text{ where } A = e^{-3c}$$

$$\therefore \quad 3y - 5 = Ae^{3x} \text{ or } 3y - 5 = -Ae^{3x}$$

$$\therefore \quad y = \frac{1}{3}(Ae^{3x} + 5) \text{ or}$$

$$y = \frac{1}{3}(5 - Ae^{3x})$$

$$\therefore \quad y = \frac{1}{3}(Ae^{3x} + 5) \text{ for } y > \frac{5}{3}$$

$$\text{or } y = \frac{1}{3}(5 - Ae^{3x}) \text{ for } y < \frac{5}{3}$$

b

$$\frac{dy}{dx} = 1 - 2y$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{1 - 2y}$$

$$= \frac{-1}{2y - 1}$$

$$\therefore \quad x = -\frac{1}{2}\log_e |2y - 1| + c,$$

$$c \in R,$$

$$\therefore \quad x - c = -\frac{1}{2}\log_e |2y - 1|$$

$$\therefore \quad -2(x - c) = \log_e |2y - 1|$$

$$\therefore \quad |2y - 1| = e^{-2(x-c)}$$

$$\therefore \quad |2y - 1| = Ae^{-2x} \text{ where } A = e^{2c}$$

$$\therefore \quad 2y - 1 = Ae^{-2x} \text{ or } 2y - 1$$

$$= -Ae^{-2x}$$

$$\therefore \quad y = \frac{1}{2}(Ae^{-2x} + 1) \text{ or}$$

$$y = \frac{1}{2}(1 - Ae^{-2x})$$

$$\therefore \quad y = \frac{1}{2}(Ae^{-2x} + 1) \text{ for } y > \frac{1}{2}$$

$$\text{or } y = \frac{1}{2}(1 - Ae^{-2x}) \text{ for } y < \frac{1}{2}$$

c

$$\frac{dy}{dx} = e^{2y-1}$$

$$\therefore \quad \frac{dx}{dy} = e^{1-2y}$$

$$\therefore \quad x = -\frac{1}{2}e^{1-2y} + c, \quad c \in R$$

$$\therefore \quad x - c = -\frac{1}{2}e^{1-2y}$$

$$\therefore \quad -2(x - c) = e^{1-2y}$$

$$\therefore \quad 1 - 2y = \log_e |-2(x - c)|$$

$$\therefore \quad 2y = 1 - \log_e |-2(x - c)|$$

$$\therefore \quad y = \frac{1}{2}(1 - \log_e |-2(x - c)|)$$

$$\therefore \quad y = \frac{1}{2} - \frac{1}{2}\log_e |2c - 2x|$$

d

$$\frac{dy}{dx} = \cos^2 y$$

$$\therefore \quad \frac{dx}{dy} = \sec^2 y$$

$$\therefore \quad x = \tan y + c$$

$$\therefore \quad x - c = \tan y$$

$$\therefore \quad y = \tan^{-1}(x - c)$$

e $\frac{dy}{dx} = \cot y$

$\therefore \quad \frac{dx}{dy} = \tan y$

$= \frac{\sin y}{\cos y}$

Let $u = \cos y \quad \therefore \quad \frac{du}{dy} = -\sin y$

$\therefore \quad \frac{dx}{dy} = -\frac{1}{u}\frac{du}{dy}$

$\therefore \quad x = -\int \frac{1}{u}\frac{du}{dy}dy$

$= -\int \frac{1}{u}du$

$= -\log_e |u| + c, \ c \in R$

$= -\log_e |\cos y| + c$

$\therefore \quad x - c = -\log_e |\cos y|$

$\therefore \quad -(x - c) = \log_e |\cos y|$

$\therefore \quad |\cos y| = e^{c-x}$

$\therefore \quad y = \cos^{-1}(e^{c-x})$ for $\cos y > 0$

or $y = \cos^{-1}(-e^{c-x})$ for $\cos y < 0$

f $\frac{dy}{dx} = y^2 - 1$

$\therefore \quad \frac{dx}{dy} = \frac{1}{y^2 - 1}$

$= \frac{1}{(y+1)(y-1)}$

Let $\frac{1}{(y+1)(y-1)} \equiv \frac{A}{y+1} + \frac{B}{y-1}$

$\therefore \quad A(y-1) + B(y+1) = 1$

When $y = -1$,

$-2A = 1$

$\therefore \quad A = -\frac{1}{2}$

When $y = 1$,

$2B = 1$

$\therefore \quad B = \frac{1}{2}$

$\therefore \quad \frac{1}{(y+1)(y-1)} \equiv \frac{1}{2(y-1)} - \frac{1}{2(y+1)}$

$\therefore \quad \frac{dx}{dy} = \frac{1}{2(y-1)} - \frac{1}{2(y+1)}$

$\therefore \quad x = \frac{1}{2}\log_e |y-1|$

$- \frac{1}{2}\log_e |y+1| + c,$

$c \in R,$

$= \frac{1}{2}\log_e\left|\frac{y-1}{y+1}\right| + c$

$\therefore \quad x - c = \frac{1}{2}\log_e\left|\frac{y-1}{y+1}\right|$

$\therefore \quad 2(x - c) = \log_e\left|\frac{y-1}{y+1}\right|$

$\therefore \quad \left|\frac{y-1}{y+1}\right| = e^{2(x-c)}$

$\therefore \quad \left|\frac{y-1}{y+1}\right| = Ae^{2x}$ where $A = e^{-2c}$

For $y > 1$ or $y < -1$:

$\therefore \quad y - 1 = Ae^{2x}(y+1)$

$\therefore \quad y - Aye^{2x} = Ae^{2x} + 1$

$\therefore \quad y(1 - Ae^{2x}) = Ae^{2x} + 1$

$\therefore \quad y = \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$

For $-1 < y < 1$:

$y = \frac{1 - Ae^{2x}}{1 + Ae^{2x}}$

g $\dfrac{dy}{dx} = 1 + y^2$

$\therefore \dfrac{dx}{dy} = \dfrac{1}{1 + y^2}$

$\therefore x = \tan^{-1} y + c, \; c \in \mathbb{R}$

$\therefore x - c = \tan^{-1} y$

$\therefore y = \tan(x - c)$

h $\dfrac{dy}{dx} = \dfrac{1}{5y^2 + 2y}$

$\therefore \dfrac{dx}{dy} = 5y^2 + 2y$

$\therefore x = \dfrac{5}{3}y^3 + y^2 + c, \; c \in \mathbb{R}$

i $\dfrac{dy}{dx} = \sqrt{y} = y^{\frac{1}{2}}$

$\therefore \dfrac{dx}{dy} = y^{-\frac{1}{2}}$

$\therefore x = 2y^{\frac{1}{2}} + c, \; c \in R$

$\therefore x - c = 2y^{\frac{1}{2}}$

$\therefore \dfrac{1}{2}(x - c) = y^{\frac{1}{2}}$

$\therefore y = \dfrac{1}{4}(x - c)^2$

j $\dfrac{dy}{dx} = y^2 + 4y$

$\therefore \dfrac{dx}{dy} = \dfrac{1}{y^2 + 4y}$

$= \dfrac{1}{y(y + 4)}$

Let $\dfrac{1}{y(y + 4)} \equiv \dfrac{A}{y} + \dfrac{B}{y + 4}$

$\therefore A(y + 4) + By = 1$

When $y = -4$,

$-4B = 1$

$\therefore B = -\dfrac{1}{4}$

When $y = 0$,

$4A = 1$

$\therefore A = \dfrac{1}{4}$

$\therefore \dfrac{1}{y(y + 4)} \equiv \dfrac{1}{4y} - \dfrac{1}{4(y + 4)}$

$\therefore \dfrac{dx}{dy} = \dfrac{1}{4y} - \dfrac{1}{4(y + 4)}$

$\therefore x = \dfrac{1}{4}\log_e |y| - \dfrac{1}{4}\log_e |y + 4|$

$+ c, c \in \mathbb{R}$

$\therefore x = \dfrac{1}{4}\log_e\left|\dfrac{y}{y + 4}\right| + c$

$\therefore x - c = \dfrac{1}{4}\log_e\left|\dfrac{y}{y + 4}\right|$

$\therefore 4(x - c) = \log_e\left|\dfrac{y}{y + 4}\right|$

$\therefore e^{4(x-c)} = \left|\dfrac{y}{y + 4}\right|$

$\therefore \left|\dfrac{y}{y + 4}\right| = Ae^{4x}$ where $A = e^{-4c}$

For $y > 0$ or $y < -4$:

$\therefore y = Ae^{4x}(y + 4)$

$\therefore y(1 - Ae^{4x}) = 4Ae^{4x}$

$\therefore y = \dfrac{4Ae^{4x}}{1 - Ae^{4x}}$

For $-4 < y < 0$:

$y = -\dfrac{4Ae^{4x}}{1 + Ae^{4x}}$

2 **a**

$$\frac{dy}{dx} = y$$

$\therefore \quad x = \int \frac{dy}{y}$

$\therefore \quad x + c = \log_e |y|$

When $\quad x = 0, y = e$

$\Rightarrow \quad c = 1$

$\therefore \quad x + 1 = \log_e |y|$

$\therefore \quad e^{x+1} = |y|$

$\therefore \quad y = e^{x+1}$ for $y > 0$

b

$$\frac{dy}{dx} = y + 1$$

$\therefore \quad x = \int \frac{dy}{y+1}$

$\therefore \quad \log_e |y + 1| = x + c$

When $\quad x = 4, y = 0$

$\Rightarrow \quad c = -4$

$\therefore \quad |y + 1| = e^{x-4}$

$\therefore \quad y = e^{x-4} - 1$ for $y > 1$

c

$$\frac{dy}{dx} = 2y$$

$\therefore \quad x = \frac{1}{2}\int \frac{dy}{y}$

$\therefore \quad \log_e |y| = 2x + c$

When $\quad x = 1, y = 1$

$\Rightarrow \quad c = -2$

$\therefore \quad |y| = e^{2x-2}$

$\therefore \quad y = e^{2x-2}$ for $y > 0$

d

$$\frac{dy}{dx} = 2y + 1$$

$\therefore \quad x = \int \frac{dy}{2y+1}$

$\therefore \quad x = \frac{1}{2}\log_e |2y + 1| + c$

When $\quad x = 0, y = -1$

$\Rightarrow \quad c = 0$

$\therefore \quad 2x = \log_e |2y + 1|$

$\therefore \quad |2y + 1| = e^{2x}$

$y = -\frac{1}{2}(e^{2x} + 1)$ for $y < -\frac{1}{2}$

e $\quad \frac{dy}{dx} = \frac{e^y}{e^y + 1}$

$\therefore \quad x = \int \frac{e^y + 1}{e^y} dy$

$\therefore \quad x = \int (1 + e^{-y}) dy$

$\therefore \quad x = y - e^{-y} + c$

When $\quad x = 0, y = 0$

$\Rightarrow \quad c = 1$

$\therefore \quad x = y - e^{-y} + 1$

f $\quad \frac{dy}{dx} = \sqrt{9 - y^2}$

$\therefore \quad x = \int \frac{dy}{\sqrt{9 - y^2}}$

$\therefore \quad x = \sin^{-1}\frac{y}{3} + c$

When $\quad x = 0, \; y = 3$

$\Rightarrow \quad c = -\frac{\pi}{2}$

$\therefore \quad y = 3\sin\left(x + \frac{\pi}{2}\right)$

$\therefore \quad y = 3\cos x$

Also $-\frac{\pi}{2} < x + \frac{\pi}{2} < \frac{\pi}{2}$

$\therefore\ -\pi < x < 0 \therefore\ y = 3\cos x,\ -\pi < x < 0$

g

$$\frac{dy}{dx} = 9 - y^2$$

$$\therefore\quad x = \int \frac{1}{9 - y^2}dy$$

$$\frac{1}{9 - y^2} \equiv \frac{A}{3 - y} + \frac{B}{3 + y}$$

$$\therefore\quad 1 = A(3 + y) + B(3 - y)$$

When $y = -3,\ B = \frac{1}{6}$

When $y = 3,\ A = \frac{1}{6}$

$$\therefore\quad x = \frac{1}{6}\int \frac{dy}{3 - y} + \frac{1}{6}\int \frac{dy}{3 + y}$$

$$\therefore\quad x = -\frac{1}{6}\log_e |3 - y| + \frac{1}{6}\log_e |3 + y| + c$$

$$\therefore\quad x = \frac{1}{6}\log_e\left|\frac{3 + y}{3 - y}\right| + c$$

When $x = \frac{7}{6},\ y = 0$

$$\Rightarrow\quad c = \frac{7}{6}$$

$$\therefore\quad 6\left(x - \frac{7}{6}\right) = \log_e\left|\frac{3 + y}{3 - y}\right|$$

$$\therefore\quad \left|\frac{3 + y}{3 - y}\right| = e^{6x-7}$$

For $-3 < y < 3$:

$$\therefore\quad 3 + y = e^{6x-7}(3 - y)$$

$$\therefore\quad y(1 + e^{6x-7}) = 3e^{6x-7} - 3$$

$$\therefore\quad y = \frac{3(e^{6x-7} - 1)}{e^{6x-7} + 1}$$

h

$$\frac{dy}{dx} = 1 + 9y^2$$

$$\therefore\quad x = \int \frac{1}{1 + 9y^2}dy$$

$$\therefore\quad x = \int \frac{1}{1 + (3y)^2}dy$$

$$\therefore\quad x = \frac{1}{3}\tan^{-1} 3y + c$$

When $x = -\frac{\pi}{12},\ y = -\frac{1}{3}$

$$\therefore\quad \frac{1}{3}\tan^{-1}(-1) + c = -\frac{\pi}{12}$$

$$\therefore\quad c - \frac{\pi}{12} = -\frac{\pi}{12}$$

$$\therefore\quad c = 0$$

$$\therefore\quad \tan^{-1} 3y = 3x$$

$$\therefore\quad 3y = \tan 3x$$

$$\therefore\quad y = \frac{1}{3}\tan 3x$$

Also $-\frac{\pi}{2} < 3x < \frac{\pi}{2}$

$\therefore\quad -\frac{\pi}{6} < x < \frac{\pi}{6}$

$\therefore\quad y = \frac{1}{3}\tan 3x, -\frac{\pi}{6} < x < \frac{\pi}{6}$

i

$$\frac{dy}{dx} = \frac{y^2 + 2y}{2}$$

$$\therefore\quad x = \int \frac{2}{y^2 + 2y}dy$$

$$\therefore\quad x = \int \frac{2}{y(y + 2)}dy$$

$$\frac{2}{y^2 + 2y} = \frac{A}{y} + \frac{B}{y + 2}$$

$$\therefore\quad 2 = A(y + 2) + By$$

When $y = 0,\ A = 1$

When $y = -2,\ B = -1$

$$x = \int \frac{2dy}{y^2 + 2y}$$

$$\therefore \quad x = \int \frac{dy}{y} - \int \frac{dy}{y+2}$$

$$\therefore \quad x = \log_e\left|\frac{y}{y+2}\right| + c$$

When $x = 0,\ y = -4$

$$\Rightarrow \quad c = -\log_e 2$$

$$\therefore \quad x = \log_e\left|\frac{y}{2(y+2)}\right|$$

For $y < -2$

$$y = 2e^x(y+2)$$

$$\therefore \quad y(1 - 2e^x) = 4e^x$$

$$\therefore \quad y = \frac{4e^x}{1 - 2e^x}$$

$$\therefore \quad y = \frac{4e^x}{1 - 2e^x} \times \frac{e^{-x}}{e^{-x}}$$

$$\therefore \quad y = \frac{4}{e^{-x} - 2}$$

3 a $\frac{dy}{dx} = y + 3$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{y+3}$$

$$x = \log_e(y+3)$$

$$y + 3 = e^x$$

$$y = e^x - 3$$

is the equation for the family of curves.

b $\frac{dy}{dx} = 2y - 1$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{2y-1}$$

$$\therefore \quad x = \frac{1}{2}\log_e|2y-1| + c,\ c \in R,$$

$$\therefore \quad x - c = \frac{1}{2}\log_e|2y-1|$$

$$\therefore \quad 2(x-c) = \log_e|2y-1|$$

$$\therefore \quad |2y-1| = e^{2(x-c)}$$

$$\therefore \quad |2y-1| = Ae^{2x} \text{ where } A = e^{-2c}$$

$$\therefore \quad y = \frac{1}{2}(Ae^{2x} + 1)$$

c

$$\frac{dy}{dx} = y(y+1)$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{y(y+1)}$$

$$\frac{dx}{dy} = \frac{1}{y} - \frac{1}{y+1}$$

$$x + c = \log_e\left|\frac{y}{y+1}\right|$$

$$Ae^x = \left|\frac{y}{y+1}\right|$$

$$|Ae^x(y+1)| = |y|$$

$$y = Ae^x(y+1) \text{ or } y = -Ae^x(y+1)$$

$$y(1 - Ae^x) = Ae^x \text{ or } y(1 + Ae^x) = -Ae^x$$

$$y = \frac{Ae^x}{(1 - Ae^x)} \text{ or } y = \frac{-Ae^x}{(1 + Ae^x)}$$

These can be written as:

$$y = \frac{Be^x}{(1 - Be^x)}$$

The solution is $y = \frac{Be^x}{(1 - Be^x)}$ or $y = -1$

d

$$\frac{dy}{dx} = (y-3)(y-4)$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{(y-3)(y-4)}$$

$$\frac{dx}{dy} = \frac{1}{y-4} - \frac{1}{y-3}$$

$$x + c = \log_e \left|\frac{y-4}{y-3}\right|$$

$$Ae^x = \left|\frac{y-4}{y-3}\right|$$

$$|Ae^x(y-3)| = |y-4|$$

$$y - 4 = Ae^x(y-3) \text{ or } y - 4 = -Ae^x(y-3)$$

$$y(1 - Ae^x) = 4 - 3Ae^x \text{ or } y(1 + Ae^x) = 4 + 3Ae^x$$

These can be written as:

$$y = \frac{4 - 3Be^x}{(1 - Be^x)}$$

The solution is $y = \dfrac{3Be^x - 4}{(Be^x - 1)}$ or $y = 3$

Solutions to Exercise 11D

1 a From the table,

$$\frac{dx}{dt} = 2t + 1$$

$$\therefore \quad x = t^2 + t + c$$

Now $x(0) = 3$

$$\therefore \quad 3 = c$$

$$\therefore \quad x = t^2 + t + 3$$

b From the table,

$$\frac{dx}{dt} = 3t - 1$$

$$\therefore \quad x = \frac{3}{2}t^2 - t + c$$

Now $x(1) = 1$

$$\therefore \quad 1 = \frac{3}{2} - 1 + c$$

$$\therefore \quad c = \frac{1}{2}$$

$$\therefore \quad x = \frac{3}{2}t^2 - t + \frac{1}{2}$$

c From the table,

$$\frac{dx}{dt} = -2t + 8$$

$$\therefore \quad x = -t^2 + 8t + c$$

Now $x(2) = -3$

$$\therefore \quad -3 = -(2)^2 + 8(2) + c$$

$$= -4 + 16 + c$$

$$= 12 + c$$

$$\therefore \quad c = -15$$

$$\therefore \quad x = -t^2 + 8t - 15$$

2 a $\dfrac{dy}{dx} = \dfrac{1}{y}, \ y \neq 0$

b $\dfrac{dy}{dx} = \dfrac{1}{y^2}, \ y \neq 0$

c $\dfrac{dN}{dt} \propto \dfrac{1}{N^2}, \ N \neq 0$

$\therefore \quad \dfrac{dN}{dt} = \dfrac{k}{N^2}, \ N \neq 0$ and $k > 0$ since the population is increasing.

d $\dfrac{dx}{dt} \propto \dfrac{1}{x}, \ x \neq 0$

$\therefore \quad \dfrac{dx}{dt} = \dfrac{k}{x}, \ x \neq 0$ and $k > 0$

e

$$\frac{dm}{dt} \propto -m$$

$$\therefore \quad \frac{dm}{dt} = -km, \ k > 0$$

or alternatively,

$$\frac{dm}{dt} = km, \ k < 0$$

f The gradient of the tangent at the point (x, y) is $\dfrac{y}{x}$. Three times this is, $\dfrac{3y}{x}$

Therefore the gradient of the normal at the point (x, y) is

$$\frac{dy}{dx} = \frac{-1}{\frac{3y}{x}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-x}{3y}, \ y \neq 0$$

3 a i $\dfrac{dP}{dt} \propto P$

$$\therefore \quad \frac{dP}{dt} = kP, \ k > 0$$

ii $\dfrac{dt}{dP} = \dfrac{1}{kP}$

$$\therefore \quad t = \frac{1}{k}\int \frac{1}{P}dP$$

$$\therefore \quad t = \frac{1}{k}\log_e P + c, \ P > 0$$

b **i**

When $t = 0,\ P = 1000$

$\therefore \quad 0 = \frac{1}{k}\log_e 1000 + c \quad ①$

When $t = 2,\ P = 1100$

$\therefore \quad 2 = \frac{1}{k}\log_e 1100 + c \quad ②$

② − ① yields

$$2 = \frac{1}{k}\log_e 1100 - \frac{1}{k}\log_e 1000$$
$$= \frac{1}{k}\log_e\left(\frac{1100}{1000}\right)$$
$$= \frac{1}{k}\log_e 1.1$$

$\therefore \quad k = \frac{1}{2}\log_e 1.1 \quad ③$

Substituting ③ in ① yields

$$0 = \frac{1}{\frac{1}{2}\log_e 1.1}\log_e 1000 + c$$
$$= \frac{2\log_e 1000}{\log_e 1.1} + c$$

$\therefore \quad c = \frac{-2\log_e 1000}{\log_e 1.1}$

$\therefore \quad t = \frac{1}{\frac{1}{2}\log_e 1.1}\log_e P + \frac{-2\log_e 1000}{\log_e 1.1}$

$$= \frac{2}{\log_e 1.1}(\log_e P - \log_e 1000)$$

$\therefore \quad t = \frac{2}{\log_e 1.1}\log_e\left(\frac{P}{1000}\right)$

Rearranging to make P the subject of the formula:

$$\frac{\log_e(1.1)t}{2} = \log_e\left(\frac{P}{1000}\right)$$

$\therefore \quad P = 1000e^{\frac{1}{2}t\log_e(1.1)}$

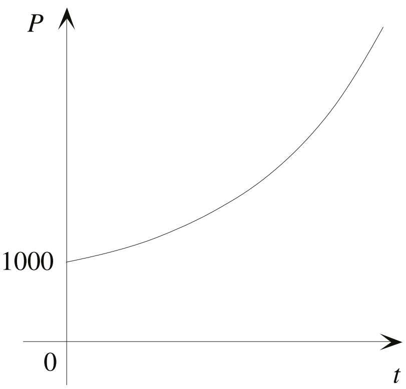
$\therefore \quad P = 1000(1.1)^{\frac{t}{2}}$

When $t = 5$,

$$P = 1000(1.1)^{\frac{5}{2}}$$
$$= 1269.05870\ldots$$

After five years, the population is 1269.

ii $P = 1000(1.1)^{\frac{t}{2}},\ t \geq 0$



4 **a** **i** $\frac{dP}{dt} \propto -\sqrt{P},\ P > 0$

$\therefore \quad \frac{dP}{dt} = k\sqrt{P},\ k < 0$ and $P > 0$

ii $\frac{dt}{dP} = \frac{1}{k\sqrt{P}}$

$\therefore \quad t = \frac{1}{k}\int P^{-\frac{1}{2}}dP$

$$= \frac{1}{k} \times 2P^{\frac{1}{2}} + c$$

$\therefore \quad t = \frac{2\sqrt{P}}{k} + c,\ k < 0$

b **i**

When $t = 0,\ P = 15\,000$

$\therefore \quad 0 = \dfrac{2\sqrt{15\,000}}{k} + c$

$\therefore \quad 0 = \dfrac{100\sqrt{6}}{k} + c \quad ①$

When $t = 5,\ P = 13\,500$

$5 = \dfrac{2\sqrt{13\,500}}{k} + c$

$5 = \dfrac{20\sqrt{135}}{k} + c \quad ②$

② − ① yields

$5 = \dfrac{20\sqrt{135}}{k} - \dfrac{100\sqrt{6}}{k}$

$\therefore \quad 5 = \dfrac{20}{k}(\sqrt{135} - 5\sqrt{6})$

$\therefore \quad k = 4\sqrt{135} - 20\sqrt{6} \quad ③$

Substituting ③ in ① yields

$$0 = \frac{100\sqrt{6}}{4\sqrt{135} - 20\sqrt{6}} + c$$

$$= \frac{25\sqrt{6}}{\sqrt{135} - 5\sqrt{6}} + c$$

$$\therefore \quad c = -\frac{25\sqrt{6}}{\sqrt{135} - 5\sqrt{6}}$$

$$\therefore \quad t = \frac{2\sqrt{P}}{4\sqrt{135} - 20\sqrt{6}} - \frac{25\sqrt{6}}{\sqrt{135} - 5\sqrt{6}}$$

$$= \frac{\sqrt{P}}{2(\sqrt{135} - 5\sqrt{6})} - \frac{2(25\sqrt{6})}{2(\sqrt{135} - 5\sqrt{6})}$$

$$= \frac{\sqrt{P} - 50\sqrt{6}}{2(\sqrt{135} - 5\sqrt{6})} \quad ④$$

Rearranging to make P the subject of the formula:

$2t(\sqrt{135} - 5\sqrt{6}) + 50\sqrt{6} = \sqrt{P}$

$\therefore \quad P = [2t(\sqrt{135} - 5\sqrt{6}) + 5\sqrt{6}]^2$

When $t = 10$,

$$P = (20\sqrt{135} - 100\sqrt{6} + 50\sqrt{6})^2$$

$$= (20\sqrt{135} - 50\sqrt{6})^2$$

$$= 400 \times 135 + 2500 \times 6 - 2 \times 20\sqrt{135} \times 50\sqrt{6}$$

$$= 69\,000 - 2000\sqrt{810}$$

$$= 69\,000 - 18\,000\sqrt{10}$$

$$= 12\,079.00212\ldots$$

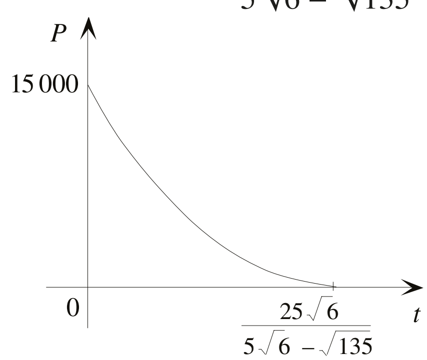
The population after 10 years is 12 079.

ii $P = [2t(\sqrt{135} - 5\sqrt{6}) + 50\sqrt{6}]^2,$

$t \geq 0$

From ④ in part **b i**.

When $P = 0,\ t = \dfrac{25\sqrt{6}}{5\sqrt{6} - \sqrt{135}}$



5 **a** **i** $\dfrac{dP}{dt} \propto \dfrac{1}{P}$

$\dfrac{dP}{dt} = \dfrac{k}{P},\ k > 0 \text{ and } P > 0$

ii $\dfrac{dt}{dP} = \dfrac{P}{k}$

$\therefore \quad t = \dfrac{1}{k}\displaystyle\int P\,dP$

$= \dfrac{1}{k} \times \dfrac{1}{2}P^2 + c$

$\therefore \quad t = \dfrac{1}{2k}P^2 + c$

b **i** When $t = 0,\ P = 1\,000\,000$

$\therefore \quad 0 = \dfrac{1}{2k}(1\,000\,000)^2 + c \quad ①$

When $t = 4,\ P = 1\,100\,000$

$\therefore \quad 4 = \dfrac{1}{2k}(1\,100\,000)^2 + c \quad ②$

② − ① yields

$4 = \dfrac{1}{2k}(1\,100\,000)^2 - \dfrac{1}{2k}(1\,000\,000)^2$

$= \dfrac{1}{2k}((1\,100\,000)^2 - (1\,000\,000)^2)$

$= \dfrac{1}{2k}(2.1 \times 10^{11})$

$= \dfrac{1.05 \times 10^{11}}{k}$

$\therefore \quad k = \dfrac{1}{4}(1.05 \times 10^{11})$

$= 2.625 \times 10^{10} \quad ③$

Substituting ③ in ① yields

$0 = \dfrac{1}{2(2.625 \times 10^{10})}(1\,000\,000)^2 + c$

$= \dfrac{1 \times 10^{12}}{5.25 \times 10^{10}} + c$

$= \dfrac{100}{5.25}$

$= \dfrac{400}{21} + c$

$\therefore \quad c = -\dfrac{400}{21}$

$\therefore \quad t = \dfrac{1}{2(2.625 \times 10^{10})}P^2 - \dfrac{400}{21}$

$= \dfrac{1}{5.25 \times 10^{10}}P^2 - \dfrac{400}{21}$

Rearranging to make P the subject of the formula:

$P^2 = 5.25 \times 10^{10}\left(t + \dfrac{400}{21}\right)$

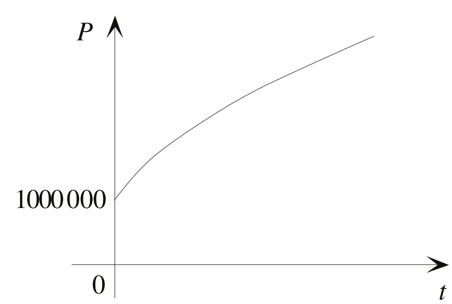
$\therefore \quad P = \sqrt{5.25 \times 10^{10}\left(t + \dfrac{400}{21}\right)}$

(The negative square root is not appropriate as $P \geq 0$.)

$= 50\,000\sqrt{21\left(t + \dfrac{400}{21}\right)}$

$\therefore \quad P = 50\,000\sqrt{21t + 400},\ t \geq 0$

ii $P = 50\,000\sqrt{21t + 400},\ t \geq 0$



6 $\frac{dy}{dx} = \frac{1}{10}y$

$$x = 10\int \frac{1}{y}dy$$

$\therefore \quad x = 10\log_e y + c, \ y > 0$

$y = 10$ when $x = 0$

$\therefore \quad c = -10\log_e 10$

$\therefore \quad x = 10\log_e \frac{y}{10}$

$\therefore \quad y = 10e^{\frac{x}{10}}$

7

$$\frac{dT}{dt} = -k(T - 20)$$

$$\frac{dt}{dT} = -\frac{1}{T - 20}$$

$$t = -\frac{1}{k}\log_e(T - 20) + c, \quad T > 20$$

When $t = 0, T = 80$

$\therefore c = \frac{1}{k}\log_e 60$

$$\therefore t = \frac{1}{k}\log_e\left(\frac{60}{T - 20}\right)$$

When $t = 20, T = 60$

$$20 = \frac{1}{k}\log_e\left(\frac{60}{60 - 20}\right)$$

$$\frac{1}{k} = \frac{20}{\log_e\left(\frac{3}{2}\right)}$$

$$\therefore t = \frac{20}{\log_e\left(\frac{3}{2}\right)}\log_e\left(\frac{60}{T - 20}\right)$$

When $t = 40$

$$40 = \frac{20}{\log_e\left(\frac{3}{2}\right)}\log_e\left(\frac{60}{T - 20}\right)$$

$$2\log_e\left(\frac{3}{2}\right) = \log_e\left(\frac{60}{T - 20}\right)$$

$$\log_e\left(\frac{9}{4}\right) = \log_e\left(\frac{60}{T - 20}\right)$$

$$\frac{9}{4} = \frac{60}{T - 20}$$

$$9T - 180 = 240$$

$$T = \frac{140}{3}$$

8 $\frac{d\theta}{dt} = 0.01\theta$

$\therefore \quad \frac{dt}{d\theta} = \frac{100}{\theta}$

$\therefore \quad t = 100\log_e \theta + c, \ \theta > 0$

$\theta = 300$ when $t = 0$

$\therefore \quad c = -100\log_e 300$

$\therefore \quad t = 100\log_e\left(\frac{\theta}{300}\right)$

$\therefore \quad \theta = 300e^{0.01t}$

When $t = 10$,

$$\theta = 300e^{0.1}$$

$$\approx 331.55°\text{K}$$

9 $\frac{dQ}{dt} = -kQ$

$\therefore \quad \frac{dt}{dQ} = \frac{1}{-kQ}$

$\therefore \quad t = \frac{-1}{k}\log_e Q + c, \ Q > 0$

When $t = 0, \ Q = 50$

$\therefore \quad c = \frac{1}{k}\log_e 50$

When $t = 10,\ Q = 25$

$$\therefore\quad 10 = -\frac{1}{k}\log_e 25 + \frac{1}{k}\log_e 50$$

$$\therefore\quad 10 = \frac{1}{k}\log_e 2$$

$$\therefore\quad k = \frac{1}{10}\log_e 2$$

$$\therefore\quad t = \frac{10}{\log_e 2}\log_e\left(\frac{50}{Q}\right)$$

$$\therefore\quad \frac{t}{10}\log_e 2 = \log_e \frac{50}{Q}$$

When $Q = 10$,

$$\therefore\quad t = \frac{10\log_e 5}{\log_e 2} \approx 23.22$$

10

$$\frac{dm}{dt} = -km$$

$$\frac{dt}{dm} = \frac{1}{-km}$$

$$\therefore\quad t = -\frac{1}{k}\log_e m + c,\ m > 0$$

Let $m = m_0$ initially.

$$\therefore\quad c = \frac{1}{k}\log_e m_0$$

$$\therefore\quad t = \frac{1}{k}\log_e \frac{m_0}{m}$$

When $m = \frac{m_0}{2},\ t = \frac{1}{k}\log_e 2$

11 a

$$\frac{dx}{dt} = \frac{20 - 3x}{30}$$

$$\therefore\quad \frac{dt}{dx} = \frac{30}{20 - 3x}$$

$$\therefore\quad t = -\frac{30}{3}\log_e(20 - 3x) + c,\ x < \frac{20}{3}$$

$$= -10\log_e(20 - 3x) + c$$

When $t = 0,\ x = 2$

$$\therefore\quad c = 10\log_e(14)$$

$$\therefore\quad t = 10\log_e\left(\frac{14}{20 - 3x}\right)$$

$$\therefore\quad \log_e \frac{14}{20 - 3x} = \frac{t}{10} \qquad ①$$

$$\therefore\quad \frac{14}{20 - 3x} = e^{\frac{t}{10}}$$

$$\therefore\quad \frac{20 - 3x}{14} = e^{-\frac{t}{10}}$$

$$\therefore\quad 20 - 3x = 14e^{-\frac{t}{10}}$$

$$\therefore\quad 3x = 20 - 14e^{-\frac{t}{10}}$$

$$\therefore\quad x = \frac{20 - 14e^{\frac{-t}{10}}}{3}$$

$$\therefore\quad x = \frac{1}{3}\left(20 - 14e^{-\frac{t}{10}}\right)$$

b From ①,

$$t = 10\log_e \frac{14}{20 - 3x}$$

Therefore, when $x = 6$,

$$t = 10\log_e \frac{14}{20 - 18}$$

$$\therefore\quad t = 10\log_e 7 \approx 19\text{ min}$$

12 $\frac{dy}{dx} = 10 - \frac{y}{10}$

$\frac{dy}{dx} = \frac{100 - y}{10}$

$\therefore \quad \frac{dx}{dy} = \frac{10}{100 - y}$

$\therefore \quad x = -10\log_e(100 - y) + c,$

$y < 100$

When $x = 0,\ y = 10$

$\therefore \quad c = 10\log_e(90)$

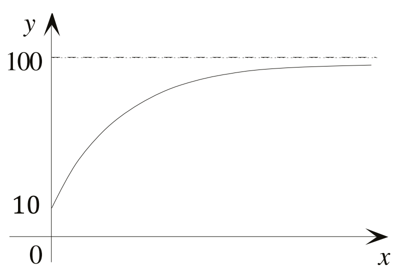
$\therefore \quad x = 10\log_e\left(\frac{90}{100 - y}\right)$

$\therefore \quad \frac{90}{100 - y} = e^{\frac{x}{10}}$

$\therefore \quad \frac{100 - y}{90} = e^{\frac{-x}{10}}$

$\therefore \quad 100 - y = 90e^{\frac{-x}{10}}$

$\therefore \quad y = 100 - 90e^{\frac{-x}{10}}$



13 $\frac{dn}{dt} = kn$

$\therefore \quad \frac{dt}{dn} = \frac{1}{kn}$

$\therefore \quad t = \frac{1}{k}\log_e n + c,\ n > 0$

When $t = 0,\ n = 4000$

$\therefore \quad c = -\frac{1}{k}\log_e 4000$

When $t = 4,\ n = 8000$

$\therefore \quad 4 = \frac{1}{k}\log_e 8000 - \frac{1}{k}\log_e 4000$

$\therefore \quad 4 = \frac{1}{k}\log_e 2$

$\therefore \quad k = \frac{1}{4}\log_e 2$

$\therefore \quad t = \frac{4}{\log_e 2}\log_e \frac{n}{4000}$

After 3 more days, $t = 7$

$\therefore \quad \frac{7}{4}\log_e 2 = \log_e\left(\frac{n}{4000}\right)$

$\therefore \quad \log_e n = \frac{7}{4}\log_e 2 + \log_e 4000$

$\therefore \quad n = 4000 \times 2^{\frac{7}{4}}$

$\approx 13\,454$

$\approx 13\,500$

(to the nearest hundred)

14 a $\frac{dN}{dt} \propto N$

$\therefore \quad \frac{dN}{dt} = kN,\ k > 0$

$\therefore \quad \frac{dt}{dN} = \frac{1}{kN}$

$\therefore \quad t = \frac{1}{k}\log_e N + c,\ N > 0$

Let year 1990 be $t = 0$, then 2000 is $t = 10$ and 2010 is $t = 20$.

When $t = 0,\ N = 10\,000$:

$\therefore \quad c = -\frac{1}{k}\log_e 10\,000$

When $t = 10,\ N = 12\,000$:

$\therefore \quad 10 = \frac{1}{k}\log_e 12\,000$

$- \frac{1}{k}\log_e 10\,000$

$\therefore \quad 10 = \frac{1}{k}\log_e \frac{6}{5}$

$\therefore \quad k = \frac{1}{10}\log_e \frac{6}{5}$

$\therefore \quad t = \frac{10}{\log_e 1.2}\log_e \frac{N}{10\,000}$

$\therefore \quad N = 10\,000 e^{0\cdot 1 t \log_e 1\cdot 2}$

$= 10\,000(1.2)^{0.1t}$

For $t = 20$,

$\therefore \quad N = 10\,000 \times (1.2)^2$

$= 14\,400$

b $\quad \frac{dN}{dt} \propto \frac{1}{N}$

$\therefore \quad \frac{dN}{dt} = \frac{k}{N},\ k > 0$

$\therefore \quad \frac{dt}{dN} = \frac{N}{k}$

$\therefore \quad t = \frac{N^2}{2k} + c$

When $t = 0,\ N = 10\,000$

$\therefore \quad c = \frac{-10^8}{2k}$

When $t = 10,\ N = 12\,000$

$\therefore \quad 10 = \frac{1}{k}\left(\frac{144 \times 10^6}{2} - \frac{10^8}{2}\right)$

$\therefore \quad k = \left(\frac{144 \times 10^6}{2} - \frac{10^8}{2}\right) \div 10$

$= 22 \times 10^5$

$\therefore \quad t = \frac{N^2}{44 \times 10^5} - \frac{10^8}{44 \times 10^5}$

For $t = 20$,

$\therefore \quad N = \sqrt{20(44 \times 10^5) + 10^8}$

$\approx 13\,711$

c $\quad \frac{dN}{dt} \propto \sqrt{N}$

$\therefore \quad \frac{dN}{dt} = k\sqrt{N},\ k > 0$

$\therefore \quad \frac{dt}{dN} = \frac{1}{k\sqrt{N}} = \frac{1}{k}N^{\frac{-1}{2}}$

$\therefore \quad t = \frac{2}{k}N^{\frac{1}{2}} + c$

When $t = 0,\ N = 10\,000$

$\therefore \quad c = \frac{-200}{k}$

$\therefore \quad t = \frac{2}{k}N^{\frac{1}{2}} - \frac{200}{k}$

When $t = 10,\ N = 12\,000$

$\therefore \quad 10 = \frac{2}{k}\sqrt{12\,000} - \frac{200}{k}$

$\therefore \quad 10 = \frac{1}{k}(2\sqrt{12\,000} - 200)$

$\therefore \quad k = \frac{1}{5}\sqrt{12\,000} - 20$

$= 4\sqrt{30} - 20$

For $t = 20$,

$\therefore \quad N^{\frac{1}{2}} = \frac{1}{2}\left[(4\sqrt{30} - 20)\left(20 + \frac{200}{4\sqrt{30} - 20}\right)\right]$

$= \frac{1}{2}(80\sqrt{30} - 400 + 200)$

$= 40\sqrt{30} - 100$

$\therefore \quad N = (40\sqrt{30} - 100)^2$

$\approx 14\,182$

15 a rate of inflow = 0.3

rate of outflow = $0.2\sqrt{V}$

$\therefore \quad \frac{dV}{dt} = 0.3 - 0.2\sqrt{V},$

$V > 0$

b

rate of inflow $= 5 \times 10 = 50$

rate of outflow $= \dfrac{m}{\text{volume}} \times 12$

$$\frac{dV}{dt} = \text{rate in} - \text{rate out}$$
$$= 10 - 12$$
$$= -2$$

$\therefore \quad V = -2t + c,\ c$ is a constant

When $t = 0,\ V = 200$:

$\Rightarrow \quad c = 200$

$\therefore \quad V = 200 - 2t$

$\therefore$ rate of outflow $= \dfrac{12m}{200 - 2t}$

$$= \frac{6m}{100 - t}$$

$$\frac{dm}{dt} = 50 - \frac{6m}{100 - t},$$

$0 \le t < 100$

c

rate of inflow $= 0 \times 6$

$= 0$

rate of outflow $= \dfrac{x}{\text{volume}} \times 5$

$$\frac{dV}{dt} = \text{rate in} - \text{rate out}$$
$$= 6 - 5$$
$$= 1$$

$\therefore \quad V = t + c,\ c$ is a constant

When $t = 0,\ V = 200$:

$\Rightarrow \quad c = 200$

$\therefore \quad V = 200 + t$

$\therefore$ rate of outflow $= \dfrac{5x}{200 + t}$

$$\frac{dx}{dt} = 0 - \frac{5x}{200 + t}$$
$$= -\frac{5x}{200 + t}$$

where $t \ge 0$

16 a $\dfrac{dm}{dt} = 0.25 - m = \dfrac{1}{4}(1 - 4m)$

b $\dfrac{dt}{dm} = \dfrac{4}{1 - 4m}$

$t = -\log_e(1 - 4m) + c$

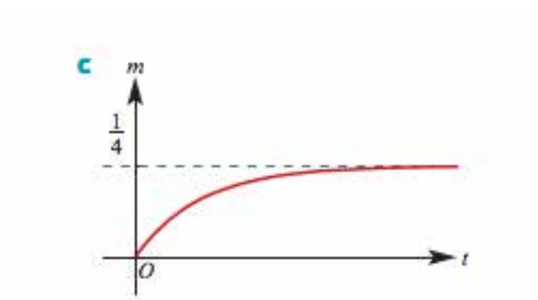
When $t = 0, m = 0$

$\therefore -t = \log_e(1 - 4m)$

$\therefore e^{-t} = 1 - 4m$

$\therefore m = \dfrac{1}{4}(1 - e^{-t})$

c



d When $t = 2, m = \dfrac{1}{4}(1 - e^{-2})$

17 a rate of outflow $= \dfrac{m}{\text{volume}} \times$ rate out

$$= \frac{m}{100} \times 1$$
$$= \frac{m}{100}$$

The sugar is being removed at $\dfrac{m}{100}$ kg/ min at time t minutes.

b rate of inflow $= 0 \times 1 = 0$

$$\frac{dm}{dt} = \text{rate of inflow} - \text{rate of outflow}$$
$$= 0 - \frac{m}{100}$$
$$\therefore \quad \frac{dm}{dt} = -\frac{m}{100}$$

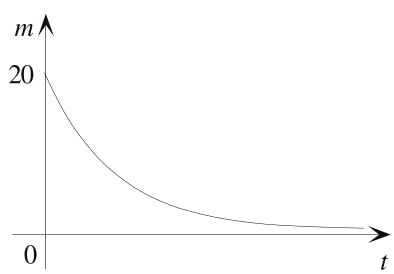
c $\dfrac{dt}{dm} = \dfrac{-100}{m}$

$$\therefore \quad t = \int \frac{-100}{m} dm$$
$$= -100 \log_e m + c, \ m > 0$$

When $t = 0, \ m = 20$:

$$\therefore \quad 0 = -100 \log_e 20 + c$$
$$\therefore \quad c = 100 \log_e 20$$
$$\therefore \quad t = -100 \log_e m + 100 \log_e 20$$
$$\therefore \quad t = 100 \log_e \left(\frac{20}{m}\right)$$
$$\therefore \quad \frac{t}{100} = \log_e \left(\frac{20}{m}\right)$$
$$\therefore \quad \frac{20}{m} = e^{\frac{t}{100}}$$
$$\therefore \quad m = 20e^{\frac{-t}{100}}, \ t \geq 0$$

d $m = 20e^{\frac{-t}{100}}, \ t \geq 0$



18 a rate of inflow $= 0.25 \times 1$

$$= 0.25$$

The sugar is being added at a rate of 0.25 kg/ min at time t.

b rate of outflow $= \dfrac{m}{100} \times 1$

$$= \frac{m}{100}$$

The sugar is being removed at a rate of $\dfrac{m}{100}$ kg/ min at time t.

c $\dfrac{dm}{dt} = \text{rate of inflow} - \text{rate of outflow}$

$$= 0.25 - \frac{m}{100}$$

d $\dfrac{dm}{dt} = \dfrac{25 - m}{100}$

$$\therefore \quad \frac{dt}{dm} = \frac{100}{25 - m}$$
$$\therefore \quad t = 100 \int \frac{1}{25 - m} dm$$
$$= -100 \log_e (25 - m) + c,$$

where $0 < m < 25$

When $t = 0, \ m = 0$

$$\therefore \quad 0 = -100 \log_e 25 + c$$
$$\therefore \quad c = 100 \log_e 25$$
$$\therefore \quad t = -100 \log_e (25 - m) + 100 \log_e 25$$
$$\therefore \quad t = 100 \log_e \left(\frac{25}{25 - m}\right) \quad ①$$
$$\therefore \quad \frac{t}{100} = \log_e \left(\frac{25}{25 - m}\right)$$
$$\therefore \quad \frac{25}{25 - m} = e^{\frac{t}{100}}$$
$$\therefore \quad 25 = (25 - m)e^{\frac{t}{100}}$$
$$= 25e^{\frac{t}{100}} - me^{\frac{t}{100}}$$
$$\therefore \quad me^{\frac{t}{100}} = 25e^{\frac{t}{100}} - 25$$
$$= 25\left(e^{\frac{t}{100}} - 1\right)$$
$$\therefore \quad m = 25\left(e^{\frac{t}{100}} - 1\right)e^{\frac{-t}{100}}$$
$$\therefore \quad m = 25\left(1 - e^{\frac{-t}{100}}\right), \ t \geq 0$$

e When the concentration is 0.1 kg per litre,

$m = 0.1 \times 100 = 10$

Substitute $m = 10$ in ① from **d**.

$$\therefore \quad t = 100\log_e\left(\frac{25}{25-10}\right)$$
$$= 100\log_e\left(\frac{25}{15}\right)$$
$$= 100\log_e\left(\frac{5}{3}\right)$$
$$\therefore \quad t = 51.08256\ldots$$

It will take 51 minutes (to the nearest minute) for the concentration in the tank to reach 0.1 kg per litre.

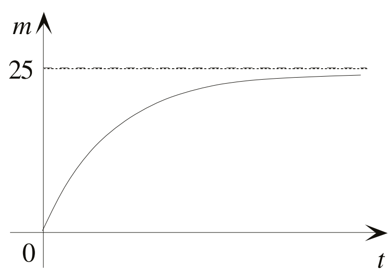
f $$m = 25\left(1 - e^{\frac{-t}{100}}\right), \; t \geq 0$$

When $t = 0$, $m = 25(1 - e^0)$

$$= 25(1-1)$$
$$= 0$$

As $t \to \infty$, $e^{\frac{-t}{100}} \to 0 \;\therefore\; m \to 25$

$m = 25$ is a horizontal asymptote.



19 a If x L is the amount pure serum in the tank at time t, then with 2 L of solution drawn off we lose $0.02x$ L pure serum, but we add at the same time 0.2 L each minute.

$$\therefore \quad \frac{dx}{dt} = 0.2 - 0.02x$$
$$\therefore \quad \frac{dx}{dt} = \frac{20-2x}{100} = \frac{10-x}{50}$$

b $$\frac{dt}{dx} = \frac{50}{10-x} = -\frac{50}{x-10}$$
$$\therefore \quad t = -50\log_e(x-10) + c$$

When $t = 0$, $x = 20$

$$\therefore \quad c = 50\log_e 10$$
$$\therefore \quad t = 50\log_e\left(\frac{10}{x-10}\right)$$

When $x = 18$,

$$\therefore \quad t = 50\log_e\frac{10}{8} \approx 11.16 \text{ min}$$

20 a $\frac{dx}{dt} = 0.4 - \frac{2x}{400}$, where 0.4 is a constant rate added and $\frac{2x}{400}$ is the rate of solution drawn off.

$$\therefore \quad \frac{dx}{dt} = 0.4 - \frac{x}{200}$$
$$= \frac{80-x}{200}$$
$$\therefore \quad \frac{dt}{dx} = \frac{200}{80-x}$$
$$\therefore \quad t = -200\log_e(80-x) + c$$

Initially, $x = 10$

$$\therefore \quad c = 200\log_e 70$$
$$\therefore \quad t = 200\log_e\left(\frac{70}{80-x}\right)$$
$$\therefore \quad \frac{70}{80-x} = e^{\frac{t}{200}}$$
$$\therefore \quad 70 = e^{\frac{t}{200}}(80-x)$$
$$\therefore \quad xe^{\frac{t}{200}} = 80e^{\frac{t}{200}} - 70$$
$$\therefore \quad x = 80 - 70e^{\frac{-t}{200}}$$

b

rate of outflow $= \dfrac{x}{\text{volume}} \times 1$

$\dfrac{dV}{dt} = 2 - 1 = 1$

$\therefore \quad V = t + c,$

c is a constant

When $t = 0,\ V = 400$:

$\Rightarrow \quad c = 400$

$\therefore \quad V = 400 + t$

$\therefore$ rate of outflow $= \dfrac{x}{400 + t}$

$\therefore \quad \dfrac{dx}{dt} = 0.4 - \dfrac{x}{400 + t}$

t

21 a rate of inflow $= 0 \times 2 = 0$

rate of outflow $= \dfrac{x}{20} \times 2 = \dfrac{x}{10}$

$\therefore \quad \dfrac{dx}{dt} = 0 - \dfrac{x}{10} = -\dfrac{x}{10}$

b $\dfrac{dt}{dx} = -\dfrac{10}{x}$

$\therefore \quad t = -10 \log_e(x) + c$

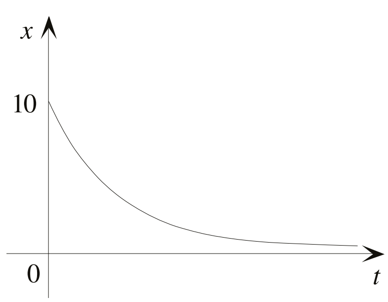
When $t = 0,\ t = 10$:

$\Rightarrow \quad c = 10 \log_e 10$

$\therefore \quad t = 10 \log_e\left(\dfrac{10}{x}\right)$ ①

$\therefore \quad x = 10e^{\frac{-t}{10}},\ t \geq 0$

c



d From ① in part **b**

When $x = 5$,

$\therefore \quad t = 10 \log_e 2 \approx 6.93$ min

22 a

$\dfrac{dN}{dt} = 0.1N - 5000$

$\therefore \quad \dfrac{dt}{dN} = \dfrac{1}{0.1N - 5000}$

$\therefore \quad t = \displaystyle\int \dfrac{1}{0.1N - 5000} dN$

$= \dfrac{1}{0.1} \log_e(0.1N - 5000) + c,$

where $N > 50\,000$

$= 10 \log_e(0.1N - 5000) + c$

When $t = 0,\ N = 5\,000\,000$:

$\therefore \quad 0 = 10 \log_e(0.1 \times 5\,000\,000$

$- 5000) + c$

$= 10 \log_e(495\,000) + c$

$\therefore \quad c = -10 \log_e(495\,000)$

$\therefore \quad t = 10 \log_e\left(\dfrac{0.1N - 5000}{495\,000}\right)$ ①

$\therefore \quad \dfrac{t}{10} = \log_e\left(\dfrac{0.1N - 5000}{495000}\right)$

$\therefore \quad e^{\frac{t}{10}} = \dfrac{0.1N - 5000}{495\,000}$

$\therefore \quad 0.1N = 495\,000e^{\frac{t}{10}} + 5000$

$\therefore \quad N = 4\,950\,000e^{\frac{t}{10}} + 50\,000$

$\therefore \quad N = 50\,000\left(99e^{\frac{t}{10}} + 1\right),\ t \geq 0$

b From ① in **a**, when $N = 10\,000\,000$

$\therefore \quad t = 10 \log_e\left(\dfrac{0.1 \times 10\,000\,000 - 5000}{495\,000}\right)$

$= 10 \log_e\left(\dfrac{199}{99}\right)$

$= 6.98184\ldots$

The country will have a population of 10 million at the end of 2016.

Solutions to Exercise 11E

1

$$\frac{dP}{dt} = P(1-P)$$

$$\therefore \frac{dt}{dP} = \frac{1}{P(1-P)}$$

$$\therefore \frac{dt}{dP} = \frac{1}{P} + \frac{1}{1-P}$$

$$\therefore t = \ln P - \ln(1-P) + c$$

$$\therefore t - c = \ln\left(\frac{P}{1-P}\right)$$

$$\therefore e^{t-c} = \frac{P}{1-P}$$

Let $A = e^{-c}$

$$\therefore Ae^{t} = \frac{P}{1-P}$$

$$P(0) = 2$$

$$\therefore A = -2$$

$$\therefore -2(1-P)e^{t} = P$$

$$\therefore P = \frac{2e^{t}}{2e^{t}-1}$$

2 **a**

$$\frac{dP}{dt} = \frac{1}{50}P(1 - \frac{P}{500})$$

$$\therefore \frac{dt}{dP} = \frac{25000}{P(500-P)}$$

$$\therefore \frac{dt}{dP} = \frac{35000}{500}\left(\frac{1}{P} + \frac{1}{1-P}\right)$$

$$\therefore t = 50(\ln P - \ln(1-P)) + c$$

$$\therefore Ae^{0.02t} = \frac{P}{500-P}$$

$$P(0) = 100$$

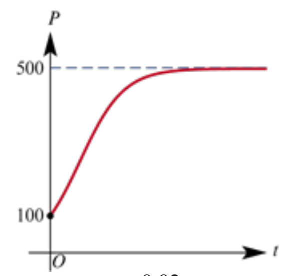
$$\therefore A = \frac{1}{4}$$

$$\therefore \frac{1}{4}(500-P)e^{0.02t} = P$$

$$\therefore P = \frac{\frac{500}{4}e^{0.02t}}{1 + \frac{1}{4}e^{0.02t}}$$

$$\therefore P = \frac{500e^{0.02t}}{4 + e^{0.02t}}$$

b



$$P = \frac{500e^{0.02t}}{4 + e^{0.02t}} = \frac{500}{4e^{-0.02t} + 1}$$

$$\lim_{t\to\infty} P = 500$$

c $\dfrac{dP}{dt} = \dfrac{1}{50}P(1 - \dfrac{P}{500})$

Maximum occurs at $P = 250$ (quadatic in P)

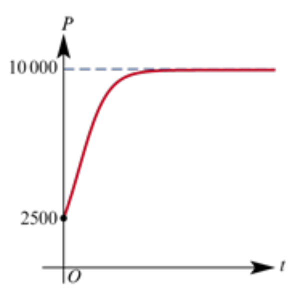
3 **a** $r = 0.3$ and $K = 10000$

$$\frac{dP}{dt} = \frac{3P}{100}\left(1 - \frac{P}{10\,000}\right)$$

$$\therefore \frac{dP}{dt} = \frac{3P(10\,000 - P)}{1\,000\,000}$$

b
$$\therefore \frac{dt}{dP} = \frac{1\,000\,000}{3P(10\,000 - P)}$$
$$\therefore \frac{dt}{dP} = \frac{100}{3}\left(\frac{1}{P} + \frac{1}{10\,000 - P}\right)$$
$$\therefore t = \frac{100}{3}(\ln P - \ln(10\,000 - P)) + c$$
$$\therefore Ae^{0.03t} = \frac{P}{10\,000 - P}$$
$$P(0) = 2500$$
$$\therefore A = \frac{1}{3}$$
$$\therefore \frac{1}{3}(10\,000 - P)e^{0.03t} = P$$
$$\therefore P = \frac{\frac{10\,000}{3}e^{0.03t}}{1 + \frac{1}{3}e^{0.03t}}$$
$$\therefore P = \frac{10\,000e^{0.03t}}{3 + e^{0.03t}}$$

c



$$P = \frac{10\,000e^{0.03t}}{3 + e^{0.03t}} = \frac{10\,000}{3e^{-0.03t} + 1}$$
$$\lim_{t\to\infty} P = 10\,000$$

d $P(5) = 5990$

e
$$P(t) = 5000$$
$$\frac{10\,000e^{0.03t}}{3 + e^{0.03t}} = 5000$$
$$\frac{2e^{0.03t}}{3 + e^{0.03t}} = 1$$
$$2e^{0.03t} = 3 + e^{0.03t}$$
$$e^{0.03t} = 3$$
$$t = \frac{100\ln(3)}{3}$$

4
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$
$$= \frac{P}{10} \times \left(1 - \frac{P}{500}\right)$$
$$= \frac{P(500 - P)}{5000}$$

The form is quadratic with negative coefficient of P^2

Maximum growth occurs when $P = 250$.

Maximum rate of growth
$$= \frac{250(500 - 250)}{5000} = 12.5$$
12.5 wasps per month

5

$$\frac{dP}{dt} = \frac{P}{20} \times \left(1 - \frac{P}{1000}\right)$$
$$= \frac{P(1000 - P)}{20\,000}$$
$$\therefore \frac{dt}{dP} = \frac{20000}{P(1000 - P)}$$
$$= \frac{20}{P} + \frac{20}{1000 - P}$$
$$= 20\left(\frac{1}{P} + \frac{1}{1000 - P}\right)$$
$$t = 20(\log_e |P| - \log_e |1000 - P| + c$$
$$= 20 \log_e \left|\frac{P}{1000 - P}\right| + c$$
$$= 200 \log_e \left(\frac{P}{1000 - P}\right) + c, 0 < P < 1000$$

Therefore

$$e^{\frac{t-c}{20}} = \frac{P}{1000 - P}$$
$$Ae^{\frac{t}{20}} = \frac{P}{1000 - P}$$

When $t = 0, P = 300$

$$A = \frac{300}{1000 - 1000} = \frac{3}{7}$$

Therefore,

$$P = \frac{3000e^{0.05t}}{7 + 3e^{0.05t}}$$

6 **a** $P(0) = \dfrac{2000}{400} = 5$

b As $t \to \infty, P \to \dfrac{2000}{5} = 400$

c $P'(t) = \dfrac{25280e^{\frac{4t}{5}}}{\left(e^{\frac{4t}{5}} + 79\right)^2}$

$$P''(t) = \frac{20244e^{\frac{4t}{5}}\left(79 - e^{\frac{4t}{5}}\right)}{\left(e^{\frac{4t}{5}} + 79\right)^3}$$
$$P''(t) = 0 \Rightarrow t = \frac{5}{4}\log_e(79)$$

d $t = \dfrac{5}{4}\log_e(79) \Rightarrow e^{\frac{4t}{5}} = 79$

$$P'(t) = \frac{25280 \times 79}{(79 + 79)^2} = 80$$

80 cases per week

e When $P = 300, t = \dfrac{5}{4}\log_e 237$

$$\Rightarrow e^{\frac{4t}{5}} = 237$$
$$P'(t) = \frac{25280 \times 237}{(237 + 79)^2} = 60$$

60 cases per week

7

$$\frac{dP}{dt} = \frac{P}{100}\left(1 - \frac{P}{1000}\right)$$
$$\therefore \frac{dP}{dt} = \frac{P(1000 - P)}{100\,000}$$
$$\therefore \frac{dt}{dP} = \frac{100\,000}{P(1000 - P)}$$
$$\therefore \frac{dt}{dP} = 100\left(\frac{1}{P} + \frac{1}{1000 - P}\right)$$
$$\therefore t = 100(\ln |P| - \ln |1000 - P|) + c$$
$$\therefore t = 100 \ln \left|\frac{P}{1000 - P}\right|$$

a If $P > 1000$ and $P(0) = 1500$

$$t = 100\ln\left|\frac{P}{1000 - P}\right| + c$$

$$t - c = 100\ln\left(\frac{P}{P - 1000}\right)$$

$$Ae^{0.01t} = \frac{P}{P - 1000}$$

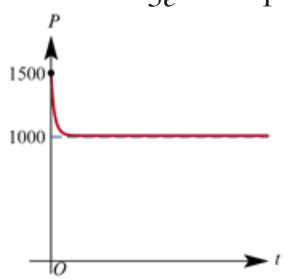
$$A(P - 1000)e^{0.01t} = P$$

$$P(0) = 1500$$

$$\therefore 500A = 1500$$

$$\therefore A = 3$$

$$\therefore P = \frac{3000e^{0.01t}}{3e^{0.01t} - 1}$$



b

c If $0 < P < 1000$ and $P(0) = 200$

$$t = 100\ln\left|\frac{P}{1000 - P}\right| + c$$

$$t - c = 100\ln\left(\frac{P}{1000 - P}\right)$$

$$Ae^{0.01t} = \frac{P}{1000 - P}$$

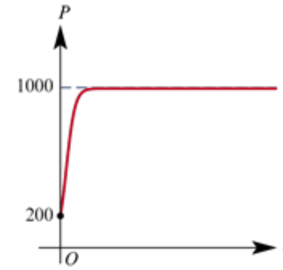
$$A(1000 - P)e^{0.01t} = P$$

$$P(0) = 200$$

$$\therefore 800A = 200$$

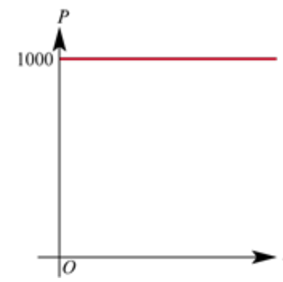
$$\therefore A = \frac{1}{4}$$

$$\therefore P = \frac{1000e^{0.01t}}{e^{0.01t} + 4}$$



d If $P = 1000$, $\dfrac{dP}{dt} = 0$

Hence $P = 1000$



8 a $P = \dfrac{50\,000e^{0.1t}}{23 + 2e^{0.1t}}$

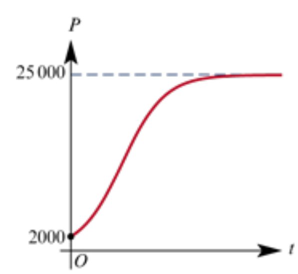
b **i** 3419

ii 24 307

c 24 months

d 38 months

e



9 $\dfrac{dy}{dx} = -\left(1 - \dfrac{y}{K_1}\right)\left(1 - \dfrac{y}{K_2}\right)$

$K_1 = 5, K_2 = 10$

$$\frac{dy}{dx} = -\left(1 - \frac{y}{5}\right)\left(1 - \frac{y}{10}\right)$$
$$= -\frac{1}{50}(5 - y)(10 - y)$$
$$\therefore \frac{dx}{dy} = -\frac{50}{(5 - y)(10 - y)}$$
$$\therefore \frac{dx}{dy} = -10\left(\frac{1}{y - 10} - \frac{1}{y - 5}\right)$$
$$\therefore -x = 10 \ln|y - 10| - \ln|y - 5| + c$$
$$\therefore -x = 10 \ln\left(\frac{|y - 10|}{|y - 5|}\right) + c$$

a $y(0) = 20, y > 10$

$$-x = 10 \ln\left(\frac{|y - 10|}{|y - 5|}\right) + c$$
$$-x - c = 10 \ln\left(\frac{(y - 10)}{(y - 5)}\right)$$
$$Ae^{-0.1x} = \frac{(y - 10)}{(y - 5)}$$
$$\therefore A = \frac{2}{3} \qquad y(0) = 20$$
$$\frac{2}{3}e^{-0.1x} = \frac{y - 10}{y - 5}$$
$$y = \frac{30 - 10e^{-0.1x}}{3 - 2e^{-0.1x}}$$

b $y(0) = 8, 5 < y < 10$

$$-x = 10 \ln\left(\frac{|y - 10|}{|y - 5|}\right) + c$$
$$-x - c = 10 \ln\left(\frac{(10 - y)}{(y - 5)}\right)$$
$$Ae^{-0.1x} = \frac{(10 - y)}{(y - 5)}$$
$$\therefore A = \frac{2}{3} \qquad y(0) = 8$$
$$\frac{2}{3}e^{-0.1x} = \frac{10 - y}{y - 5}$$
$$y = \frac{30 + 10e^{-0.1x}}{3 + 2e^{-0.1x}}$$

c $y(0) = 8, 0 < y < 5$

$$-x = 10 \ln\left(\frac{|y - 10|}{|y - 5|}\right) + c$$
$$-x - c = 10 \ln\left(\frac{(10 - y)}{(5 - y)}\right)$$
$$Ae^{-0.1x} = \frac{(10 - y)}{(5 - y)}$$
$$\therefore A = \frac{7}{2} \qquad y(0) = 3$$
$$\frac{7}{2}e^{-0.1x} = \frac{10 - y}{5 - y}$$
$$y = \frac{20 - 35e^{-0.1x}}{2 - 7e^{-0.1x}}$$

Solutions to Exercise 11F

1 a $$\frac{dy}{dx} = yx$$

$$\therefore \int \frac{1}{y}\,dy = \int x\,dx$$

$$\therefore \log_e |y| = \frac{1}{2}x^2 + c$$

$$|y| = e^{\frac{1}{2}x^2+c}$$

$$y = Ae^{\frac{x^2}{2}}$$

Notice that when we divide both sides of the equation by y we have assumed $y \neq 0$ but it is clear that $y = 0$ is a solution of the equation.

b $$\frac{dy}{dx} = \frac{x}{y}$$

$$\therefore \int y\,dy = \int x\,dx$$

$$\therefore \frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

$$y^2 = x^2 + c$$

Here $y \neq 0$

c $$\frac{4}{x^2}\frac{dy}{dx} = y$$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{1}{4}x^2\,dx$$

$$\therefore \log_e |y| = \frac{1}{12}x^3 + c$$

$$y = Ae^{\frac{x^3}{12}}$$

Notice that when we divide both sides of the equation by y we have assumed $y \neq 0$ but it is clear that $y = 0$ is a solution of the equation.

d $$\frac{dy}{dx} = \frac{1}{xy}$$

$$\therefore \int y\,dy = \int \frac{1}{x}\,dx$$

$$\therefore \frac{1}{2}y^2 = \log_e |x| + c$$

$$y^2 = 2\log_e |x| + c$$

2 $$\frac{dy}{dx} = xy^2$$

If $y = 0$, $\frac{dy}{dx} = 0$. Hence $y = 0$ is a solution. We need to check if it is covered by the general solution or not.

$$\frac{dy}{dx} = xy^2$$

$$\int \frac{1}{y^2}\,dy = \int x\,dx$$

$$-\frac{1}{y} = x^2 + c$$

$$y = -\frac{1}{x^2 + c}$$

The solution is $y = 0$ or $y = -\frac{1}{x^2 + c}$

3 $$\frac{dy}{dx} = y\sin x - \sin x$$

If $y = 1$, $\frac{dy}{dx} = 0$. Hence $y = 1$ is a solution. We need to check if it is covered by the general solution or not.

$$\frac{dy}{dx} = (y - 1)\sin x$$

$$\int \frac{1}{y - 1}\,dy = \int \sin x\,dx$$

$$\log_e(y - 1) = -\cos x + c$$

$$y - 1 = e^{-\cos x + c}$$

$$y = Ae^{-\cos x} + 1$$

The solution is $y = Ae^{-\cos x} + 1$

4 $\dfrac{dy}{dx} = 2x(1-y)^2$

If $y = 1$, $\dfrac{dy}{dx} = 0$. Hence $y = 1$ is a solution. We need to check if it is covered by the general solution or not.

$$\frac{dy}{dx} = 2x(1-y)^2$$

$$\int \frac{1}{(y-1)^2}\,dy = \int 2x\,dx$$

$$\frac{1}{y-1} = -(x^2+c)$$

$$y - 1 = -\frac{1}{x^2+c}$$

$$y = 1 - \frac{1}{x^2+c}$$

The solution is $y == 1 - \dfrac{1}{x^2+c}$ or $y = 1$

5 a

$$\frac{dy}{dx} = -\frac{x}{y} \text{ and } y(1) = 1$$

$$\therefore \int y\,dy = \int -x\,dx$$

$$\therefore \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$

We have $y(1) = 1$

$$\therefore \frac{1}{2} = -\frac{1}{2} + c$$

$$\therefore c = 1$$

$$\therefore y^2 = 2 - x^2, y > 0 \text{ or } y = \sqrt{2-x^2}$$

b

$$\frac{dy}{dx} = \frac{y}{x} \text{ and } y(1) = 1$$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\therefore \log_e|y| = \log_e|x| + c$$

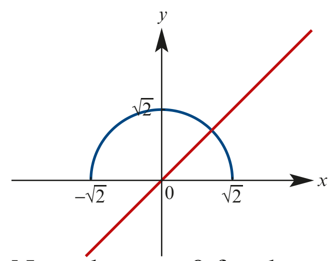
We have $y(1) = 1$

$$\therefore \log_e|y| = \log_e|x|$$

$$\therefore |y| = |x| \qquad (y(1) = 1)$$

$$\therefore y = x$$

c



Note that $y > 0$ for the semicircle and $x \neq 0$ for $y = x$. There is an open circle at (0,0) and open circles at the end points of the semicircle(not shown).

6 $(1+x^2)\dfrac{dy}{dx} = 4xy$ and $y(1) = 2$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4x}{1+x^2}$$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{4x}{1+x^2}\,dx$$

$$\therefore \log_e |y| = 2\log_e(1+x^2) + c$$

We have $y(1) = 2$

$$\log_e 2 = 2\log_e 2 + c$$

$$\therefore c = -\log_e 2$$

$$\therefore \log_e |y| = 2\log_e(1+x^2) - \log_e 2$$

$$\therefore \log_e |y| = \log_e\left(\frac{(1+x^2)^2}{2}\right)$$

$$\therefore |y| = \frac{(1+x^2)^2}{2}$$

$$\therefore y = \frac{(1+x^2)^2}{2} \qquad (y(1) = 2)$$

7 $\dfrac{dy}{dx} = \dfrac{x}{y}$ and $y(2) = 3$

$$\therefore \int y\,dy = \int x\,dx$$

$$\therefore \frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

We have $y(2) = 3$

$$\therefore c = \frac{5}{2}$$

$$\therefore y^2 = x^2 + 5$$

$$\therefore y = \sqrt{x^2+5} \qquad y(2) = 3$$

8 $\dfrac{dy}{dx} = \dfrac{x+1}{3-y}$

$$\therefore \int 3 - y\,dy = \int x + 1\,dx$$

$$\therefore 3y - \frac{1}{2}y^2 = \frac{1}{2}x^2 + x + c$$

$$\therefore 6y - y^2 = x^2 + 2x + c_1$$

$$\therefore -(y^2 - 6y + 9) = x^2 + 2x + 1 + d$$

$$\therefore (x+1)^2 + (y-3)^2 = d$$

Notice that when we divide both sides of the equation by $3 - y$ we have assumed $y \neq 3$.

The result is a family of circles with centre $(-1, 3)$ for $d > 0$

9 $y^2\dfrac{dy}{dx} = \dfrac{1}{x^3}$ and $x \neq 0$

$$\therefore \int y^2\,dy = \int \frac{1}{x^3}\,dx$$

$$\therefore \frac{1}{3}y^3 = -\frac{1}{2}x^{-2} + c$$

$$\therefore y^3 = -\frac{3}{2x^2} + d$$

10 $x^3\dfrac{dy}{dx} = y^2(x-3)$

$$\therefore \int \frac{1}{y^2}\,dy = \int \frac{x-3}{x^3}\,dx$$

$$\therefore -\frac{1}{y} = -\frac{1}{x} + \frac{3}{2x^2} + c$$

$$\therefore \frac{1}{y} = \frac{1}{x} - \frac{3}{2x^2} + d$$

$$\therefore y = -\frac{2x^2}{3-2x} + d$$

11 a

$$\frac{dy}{dx} = y(1 + e^x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + e^x \therefore \int \frac{1}{y}\, dy = \int 1 + e^x\, dx$$

$$\therefore \log_e |y| = x + e^x + c$$

$$\therefore |y| = e^{x+e^x+c}$$

$$\therefore y = Ae^{x+e^x}$$

Notice that when we divide both sides of the equation by y we have assumed $y \neq 0$ but it is clear that $y = 0$ is a solution of the equation.

b

$$\frac{dy}{dx} = 9x^2y$$

$$\frac{1}{y}\frac{dy}{dx} = 9x^2 \therefore \int \frac{1}{y}\, dy = \int 9x^2\, dx$$

$$\therefore \log_e |y| = 3x^3 + c$$

$$\therefore |y| = e^{3x^3+c}$$

$$\therefore y = Ae^{3x^3}$$

Notice that when we divide both sides of the equation by y we have assumed $y \neq 0$ but it is clear that $y = 0$ is a solution of the equation.

c

$$\frac{4}{y^3}\frac{dy}{dx} = \frac{1}{x} \qquad x \neq 0, y \neq 0$$

$$\therefore \int \frac{4}{y^3}\, dy = \int \frac{1}{x}\, dx$$

$$\therefore -\frac{2}{y^2} = \log_e |x| + c$$

$$\therefore \frac{2}{y^2} = -\log_e |x| + d$$

$$\therefore y^2 = \frac{2}{d - \log_e |x|}$$

d

$$\frac{dy}{dx} = \frac{\log_e x}{yx}$$

$$\therefore \int y\, dy = \int \frac{\log_e x}{x}\, dx$$

Let $u = \log_e x$. Then $\frac{du}{dx} = \frac{1}{x}$

$$\therefore \int y\, dy = \int u\, du$$

$$\frac{y^2}{2} = \frac{u^2}{2} + c$$

$$\therefore \frac{y^2}{2} = \frac{(\log_e x)^2}{2} + c$$

Therefore, $y^2 = (\log_e x)^2 + d$

e $\frac{dy}{dx} = yxe^{x^2}$

$y = 0$ is a solution

$$\frac{dy}{dx} = yxe^{x^2}$$

$$\int \frac{1}{y}\, dy = \int xe^{x^2}\, dx$$

Let $u = x^2$. Then $\frac{du}{dx} = 2x$

$$\int \frac{1}{y}\, dy = \frac{1}{2}\int e^u\, du$$

$$\log_e |y| = \frac{1}{2}e^u + c$$

$$\log_e |y| = \frac{1}{2}e^{x^2} + c$$

Solution is $\log_e |y| = \frac{1}{2}e^{x^2} + c$ or $y = 0$

f $\dfrac{dy}{dx} = 2y^2x\sqrt{1-x^2}$

$y = 0$ is a solution

$$\frac{dy}{dx} = 2y^2x\sqrt{1-x^2}$$

$$\int \frac{1}{y^2}\,dy = 2\int x\sqrt{1-x^2}\,dx$$

Let $u = 1 - x^2$.

Then $\dfrac{du}{dx} = -2x$

$$\int \frac{1}{y^2}\,dy = -\int u^{\frac{1}{2}}\,du$$

$$-\frac{1}{y} = -\frac{2}{3}u^{\frac{3}{2}} + c$$

$$-\frac{1}{y} = -\frac{2}{3}(1-x^2)^{\frac{3}{2}} + c$$

$$y = \frac{3}{2(1-x^2)^{\frac{3}{2}} + d}$$

12 a $y\dfrac{dy}{dx} = 1 + x^2 \qquad y(0) = 1$

$$\therefore \int y\,dy = \int 1 + x^2\,dx$$

$$\therefore \frac{1}{2}y^2 = 1 + \frac{1}{3}x^3 + c$$

$$\therefore y^2 = 2x + \frac{2}{3}x^3 + c$$

$$y(0) = 1, \quad \therefore c = 1$$

$$\therefore y^2 = 2x + \frac{2}{3}x^3 + 1$$

b $x^2\dfrac{dy}{dx} = \cos^2 y \qquad y(1) = \dfrac{\pi}{4}$

$$\therefore \int \sec^2 y\,dy = \int \frac{1}{x^2}\,dx$$

$$\therefore \frac{1}{2}y^2 = 1 + \frac{1}{3}x^3 + c$$

$$\therefore \tan y = -\frac{1}{x} + c$$

$$y(1) = \frac{\pi}{4}, \quad \therefore c = 2$$

$$\therefore \tan y = -\frac{1}{x} + 2$$

13 $\dfrac{dy}{dx} = \dfrac{x^2 - x}{y^2 - y}$

$$\therefore \int y^2 - y\,dy = \int x^2 - x\,dx$$

$$\therefore \frac{1}{3}y^3 - \frac{1}{2}y^2 = \frac{1}{3}x^3 - \frac{1}{2}x^2 + c$$

$$\therefore 2y^3 - 3y^2 = 2x^3 - 3x^2 + c$$

14 a $\frac{dx}{dt} = \text{ inflow rate} - \text{ outflow rate}$

Amount in tank after t minutes

$= 50 - 2t$ litres

The concentration

$= \frac{x}{50 - 2t}$

$\therefore \frac{dx}{dt} = 2 \times 0 - \frac{4x}{50 - 2t}$

$\therefore \frac{dx}{dt} = -\frac{4x}{50 - 2t}$

$= -\frac{2x}{25 - t}$ with $x(0) = 50$

b $-\frac{1}{2x}\frac{dy}{dx} = \frac{1}{25 - t} \qquad x(0) = 50$

$\therefore \int -\frac{1}{2x}\, dx = \int \frac{1}{25 - t}\, dt$

$\therefore -\frac{1}{2}\log_e |x| = -\log_e |25 - t| + c$

$\therefore \log_e |x| = 2\log_e |(25 - t)| + c$

$0 < x \leq 50$ and $0 \leq t \leq 25$

$\therefore \log_e x = \log_e (25 - t)^2 + c$

$\therefore x = A(25 - t)^2$

c When $t = 0, x = 50$

$50 = A(25)^2$

$A = \frac{2}{25}$

$\therefore x = \frac{2}{25}(25 - t)^2$

When $t = 10, x = \frac{2}{25}(15)^2 = 18$

Fraction remaining is $\frac{9}{25}$

15 a $\frac{dN}{dt} = \text{rate in} - \text{rate out}$

$\frac{dN}{dt} = kN - \frac{2N}{100 - 2t}$

$\therefore \frac{dN}{dt} = kN - \frac{N}{50 - t}, 0 \leq t \leq 50$

b $k = 0.6, \frac{dN}{dt} = 0.6N - \frac{N}{50 - t}$

When $t = 0, N = N_0$

$\frac{dN}{dt} = \frac{3N}{5} - \frac{N}{50 - t}$

$= N\left(\frac{145 - 3t}{5(50 - t)}\right)$

$\therefore \frac{1}{N}\frac{dN}{dt} = \frac{145 - 3t}{5(50 - t)}$

$\therefore \int \frac{1}{N}\, dN = \int \frac{145 - 3t}{5(50 - t)}\, dt$

$\log_e N = \frac{3t}{5} + \log_e(50 - t) + c$

When $t = 0, N = N_0$

$\therefore c = \log_e\left(\frac{N_0}{50}\right)$

$\therefore \log_e N = \frac{3t}{5} + \log_e(50 - t) + \log_e\left(\frac{N_0}{50}\right)$

$\therefore \log_e\left(\frac{50N}{N_0(50 - t)}\right) = \frac{3t}{5}$

$\frac{50N}{N_0(50 - t)} = e^{\frac{3t}{5}}$

When $t = 24, \frac{50N}{26N_0} = e^{14.4}$

$\therefore N = 0.52 \times N_0 e^{14.4}$

Number of bacteria present after 24 hours

$= \frac{13N_0}{25} e^{\frac{72}{5}}$

16

$$x\frac{dy}{dx} = y + x^2y \qquad y(1) = 2\sqrt{e}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1 + x^2}{x}$$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{1 + x^2}{x}\,dx$$

$$\therefore \log_e |y| = \log_e |x| + x + c$$

When $x = 1, y = 2\sqrt{e}$

$$\log_e(2\sqrt{e}) = \log_e 1 + \frac{1}{2} + c$$

$$c = \log_e 2 + \frac{1}{2} - \frac{1}{2}$$

$$\therefore c = \log_e 2$$

$$\log_e\left(\frac{y}{2x}\right) = \frac{1}{2}x^2$$

$$\frac{y}{2x} = e^{\frac{1}{2}x^2}$$

$$\therefore y = 2xe^{\frac{1}{2}x^2}$$

Note $y = 0$ is a solution

17

$$\frac{dy}{dx} = (1 + y)^2 \sin^2 x \cos x \qquad y(0) = 2$$

$$\frac{1}{(1 + y)^2}\frac{dy}{dx} = \sin^2 x \cos x$$

$$\therefore \int \frac{1}{(1 + y)^2}\,dy = \int \sin^2 x \cos x\,dx$$

$$\therefore -\frac{1}{(1 + y)} = \frac{1}{3}\sin^3 x + c$$

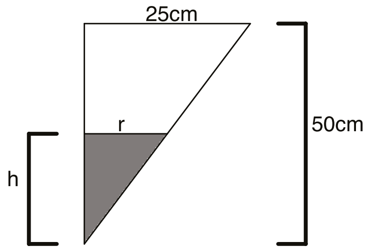
When $x = 0, y = 2, \therefore c = -\frac{1}{3}$

$$\therefore -\frac{1}{(1 + y)} = \frac{1}{3}\sin^3 x - -\frac{1}{3}$$

$$\therefore y = \frac{3}{1 - \sin^3 x} - 1$$

Solutions to Exercise 11G

1 a



Let V cm^3 be the volume at time t minutes.

$$0.5 \text{ L/min} = 0.5 \times 1000 \text{ cm}^3\text{/min} = 500 \text{ cm}^3\text{/min}$$

$$\therefore \frac{dV}{dt} = -500$$

Volume of cone $V = \frac{1}{3}\pi r^2 h$

Using similar triangles $\frac{r}{25} = \frac{h}{50}$

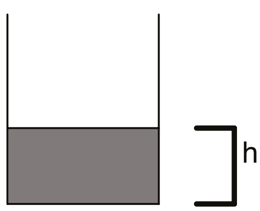
$$\Rightarrow r = \frac{h}{2}$$

$$\therefore V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

So, $\frac{dV}{dh} = \frac{\pi h^2}{4}$

$$\therefore \frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{4}{\pi h^2}\times -500 = -\frac{2000}{\pi h^2},\ h > 0$$

b



$\frac{dV}{dt}$ = rate in − rate out $= Q - c\sqrt{h}$

Volume of tank

$V = A \times h$, where A is the area

$$\therefore \frac{dV}{dh} = A$$

And so,

$$\frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{1}{A}(Q - c\sqrt{h}),\ \text{where } h > 0$$

c

$$\frac{dV}{dt} = \text{rate in} - \text{rate out} = 0.3 - 0.2\sqrt{V}$$

Volume of tank $V = 6\pi h$

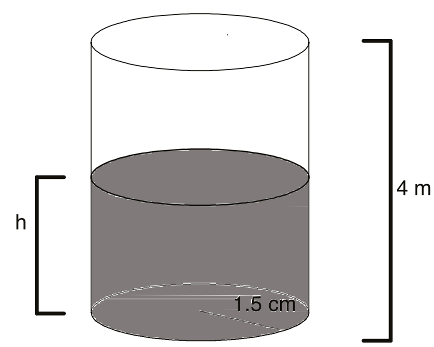
$$\therefore \frac{dV}{dh} = 6\pi$$

Hence,

$$\frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{1}{6\pi}(0.3 - 0.2\sqrt{V}) = \frac{1}{60\pi}(3 - 2\sqrt{V}),\ \text{where } V > 0$$

$$= \frac{1}{60\pi}(3 - 2\sqrt{6\pi h}),\ \text{where } h > 0$$

d



$$\frac{dV}{dt} = -\sqrt{h}$$

Volume of cylinder $V = \pi r^2 h$

With $r = 1.5$;

$$V = \pi\left(\frac{3}{2}\right)^2 h = \frac{9\pi h}{4}$$

$$\therefore \frac{dV}{dh} = \frac{9\pi}{4}$$

And so,

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{4}{9\pi} \times -\sqrt{h}$$

$$= -\frac{4\sqrt{h}}{9\pi}, \text{ where } h > 0$$

2 a $\frac{dx}{dt} = \sin t$ and $y = 5x$

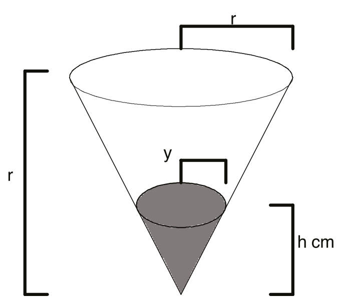
Therefore, $\frac{dy}{dx} = 5$

and, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = 5 \sin t$

b $\frac{dy}{dt} = 5 \sin t$

$y = -5 \cos t + c$

3 a



$$\frac{dV}{dt} = -5\sqrt{h}$$

Volume of cone $\quad V = \frac{1}{3}\pi y^2 h$

Using similar triangles; $\quad \frac{h}{r} = \frac{y}{r}$

$\Rightarrow \quad y = h$

Hence,

$$V = \frac{1}{3}\pi h^3$$

$$\therefore \frac{dV}{dh} = \pi h^2$$

And so,

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{\pi h^2} \times -5\sqrt{h}$$

$$= -\frac{5}{\pi\sqrt{h^3}}, \text{ where } h > 0$$

$$\therefore \frac{dt}{dh} = -\frac{\pi\sqrt{h^3}}{5}$$

$$\therefore t = \int -\frac{\pi\sqrt{h^3}}{5} dh$$

$$\therefore t = -\frac{\pi}{5}\int h^{\frac{3}{2}} dh$$

$$\therefore t = -\frac{\pi}{5}\left(\frac{2}{5}h^{\frac{5}{2}}\right) + c$$

$$\therefore t = -\frac{2\pi\sqrt{h^5}}{25} + c$$

When $t = 0,\ h = 25$:

$$\therefore c = \frac{2\pi(3125)}{25}$$

$$\therefore c = 250\pi$$

$$\therefore t = 250\pi - \frac{2\pi\sqrt{h^5}}{25} \quad h > 0$$

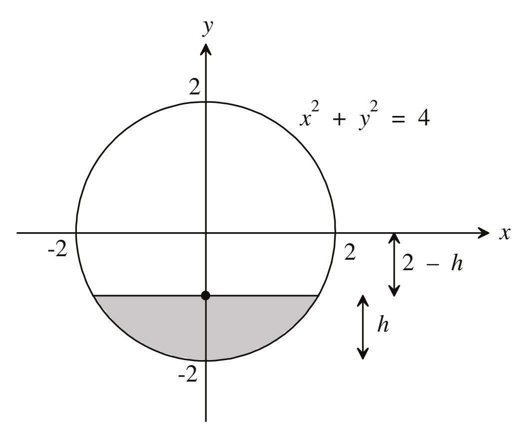
b When $h = 0,\ t = 250\pi$

$\therefore \quad t \approx 785.40$ min

$\therefore \quad t \approx 13$ hrs 5 min

Tank is empty when $h = 0$:

4 a



When $y = 2 - h$,

$$x^2 + (2 - h)^2 = 4$$

$$\therefore \quad x^2 + h^2 - 4h + 4 = 4$$

$$\therefore \quad x = \pm\sqrt{4h - h^2}$$

Therefore the width of the rectangular water surface $= 2\sqrt{4h - h^2}$

Thus the area of rectangular water surface

$$= 6 \times 2\sqrt{4h - h^2}$$

$$= 12\sqrt{4h - h^2}$$

Changing h to x, the area of the rectangular water surface

$$A = 12\sqrt{4x - x^2}$$

$$\therefore \quad \frac{dx}{dt} = \frac{-0.025\sqrt{x}}{12\sqrt{4x - x^2}}$$

$$= -\frac{\sqrt{x}}{480\sqrt{4x - x^2}} \times \frac{\sqrt{\frac{1}{x}}}{\sqrt{\frac{1}{x}}}$$

$$\therefore \quad \frac{dx}{dt} = -\frac{1}{480\sqrt{4 - x}}$$

b $$\frac{dt}{dx} = -480\sqrt{4 - x}$$

$$\therefore \quad t = \int -480(4 - x)^{\frac{1}{2}}dx$$

$$\therefore \quad t = \frac{-480(4 - x)^{\frac{3}{2}}}{\frac{3}{2} \times -1} + c$$

$$\therefore \quad t = 320(4 - x)^{\frac{3}{2}} + c$$

When $t = 0,\ x = 4$:

$$\Rightarrow \quad c = 0$$

$$\therefore \quad t = 320(4 - x)^{\frac{3}{2}}$$

c Tank is empty when $x = 0$.

So,

$$t = 320(4)^{\frac{3}{2}} = 320 \times 8 = 2560 \text{ min}$$

$$= 42\frac{2}{3} \text{ hrs}$$

$$\therefore \quad t = 42 \text{ hrs } 40 \text{ min}$$

5 a Given: $\dfrac{dV}{dt} = -2A^2$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

Volume of sphere $V = \dfrac{4}{3}\pi r^3$

$$\therefore \quad \frac{dV}{dr} = 4\pi r^2$$

And so,

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times -2A^2$$

$$= \frac{-2A}{A}$$

where $4\pi r^2$ is the surface area of a sphere.

$$\therefore \quad \frac{dr}{dt} = -2A$$

$$\therefore \quad \frac{dr}{dt} = -2 \times 4\pi r^2$$

b $$\frac{dt}{dr} = -\frac{1}{8\pi r^2}$$

$$\therefore \quad t = \int -\frac{1}{8\pi r^2} dr$$

$$\therefore \quad t = -\frac{1}{8\pi} \int \frac{1}{r^2} dr$$

$$\therefore \quad t = -\frac{1}{8\pi} \int r^{-2} dr$$

$$\therefore \quad t = \frac{1}{8\pi} \times r^{-1} + c$$

$$\therefore \quad t = \frac{1}{8\pi r} + c$$

When $t = 0,\ r = 2$:

$$\Rightarrow \quad c = -\frac{1}{16\pi}$$

$$\therefore \quad t = \frac{1}{8\pi r} - \frac{1}{16\pi}$$

$$\therefore \quad \frac{1}{8\pi r} = t + \frac{1}{16\pi}$$

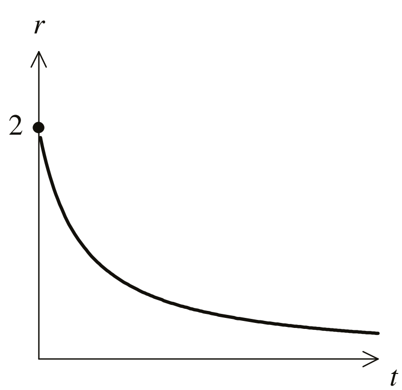
$$\therefore \quad \frac{1}{r} = 8\pi t + \frac{1}{2}$$

$$\therefore \quad \frac{1}{r} = \frac{16\pi t + 1}{2}$$

$$\therefore \quad r = \frac{2}{16\pi t + 1}$$

$$\therefore \quad \frac{dr}{dt} = -8\pi r^2$$

c radius-time graph



surface area-time graph

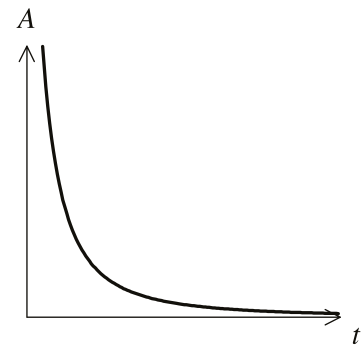
Surface area, $A = 4\pi r^2$

$$\Rightarrow r = +\sqrt{\frac{A}{4\pi}}$$

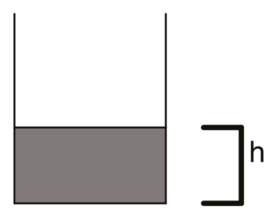
$$\therefore \quad \sqrt{\frac{A}{4\pi}} = \frac{2}{16\pi t + 1}$$

$$\therefore \quad \frac{A}{4\pi} = \frac{4}{(16\pi t + 1)^2}$$

$$\therefore \quad A = \frac{16\pi}{(16\pi t + 1)^2}$$



6 a



Let V cm^3 be the volume at time t minutes.

$$\frac{dV}{dt} = \text{rate in} - \text{rate out}$$

$$= Q - kh$$

$(Q - kh)$ L/min $= (Q - kh) \times 1000$ cm^3/min

$= 1000(Q - kh)$ cm^3/min

$$\therefore \quad \frac{dV}{dt} = 1000(Q - kh) \text{ cm}^3\text{/min}$$

Volume of tank

$V = A \times h,$ where A is the area

$$\therefore \quad \frac{dV}{dh} = A$$

And so,

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{1000}{A}(Q - kh), \text{ where } h > 0$$

b
$$\frac{dt}{dh} = \frac{A}{1000(Q - kh)}$$
$$\therefore \quad t = \frac{A}{1000} \int \frac{1}{Q - kh}\, dh$$
$$\therefore \quad t = -\frac{A}{1000k} \log_e(Q - kh) + c$$

When $t = 0,\ h = h_0$:

$$\Rightarrow c = \frac{A}{1000k} \log_e(Q - kh_0),$$
$$(Q > kh_0)$$
$$\therefore \quad t = \frac{A}{1000k} \log_e\left(\frac{Q - kh_0}{Q - kh}\right),$$
$$Q > kh_0$$

c When $h = \dfrac{Q + kh_0}{2k}$,

$$t = \frac{A}{1000k} \log_e\left(\frac{Q - kh_0}{Q - k\frac{Q+kh_0}{2k}}\right)$$
$$\therefore \quad t = \frac{A}{1000k} \log_e\left(\frac{Q - kh_0}{Q - \frac{Q+kh_0}{2}}\right)$$
$$\therefore \quad t = \frac{A}{1000k} \log_e\left(\frac{Q - kh_0}{\frac{Q-kh_0}{2}}\right)$$
$$\therefore \quad t = \frac{A}{1000k} \log_e\left((Q - kh_0) \times \frac{2}{Q - kh_0}\right)$$
$$\therefore \quad t = \left(\frac{A}{1000k} \log_e 2\right) \text{ minutes}$$

7 a $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi h^3$

$$\frac{dV}{dh} = \pi h^2$$

i
$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{\pi h^2} \times 0.1$$
$$= \frac{1}{10\pi h^2}$$

ii
$$\frac{dh}{dt} = \frac{1}{10\pi h^2}$$

therefore,

$$\frac{dt}{dh} = 10\pi h^2$$
$$t = \frac{10\pi}{3} h^3 + c$$
$$t = 0 \Rightarrow h = 0 \Rightarrow c = 0$$
$$h^3 = \frac{3t}{10\pi}$$
$$h = \sqrt[3]{\frac{3t}{10\pi}}$$

b
$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= -\frac{\sqrt{t}}{10} \times \frac{1}{\pi h^2}$$
$$\int 10\pi h^2\, dh = -\int \sqrt{t}\, dt$$
$$\frac{10}{3}\pi h^3 = -\frac{2}{3} t^{\frac{3}{2}} + c$$

When $t = 0, h = 1$

$$\therefore c = \frac{10\pi}{3}$$

$$10\pi h^3 = -2t^{\frac{3}{2}} + 10\pi$$

$$h^3 = -\frac{1}{5\pi}t^{\frac{3}{2}} + 1$$

$$h = \sqrt[3]{-\frac{1}{5\pi}t^{\frac{3}{2}} + 1}$$

Solutions to Exercise 11H

1 a $y = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin(2x)\, dx + 2$

$= \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} + 2$

$= -\frac{1}{2} + 2$

b $y = \int_1^2 e^{2x}\, dx + 3$

$= \left[\frac{1}{2}e^{2x}\right]_1^2 + 3$

$= \frac{e^4}{2} - \frac{e^2}{2} + 3$

c $y = \int_1^{\frac{3}{2}} \frac{2}{4 - x^2}\, dx + 2$

$= \left[-\frac{1}{2}\log_e\left|\frac{x-2}{x+2}\right|\right]_1^{\frac{3}{2}} + 2$

$= \frac{1}{2}\log_e\left(\frac{7}{3}\right) + 2$

2 TI: Set Calculation Mode to Approximate and Display Digits to Fix4

CP: Set to Decimal mode and change the Number Format to Fix4

a $\frac{dy}{dx} = \sqrt{\cos x},\ y(0) = 1$

Find y when $x = \frac{\pi}{4}$.

On your calculator type:

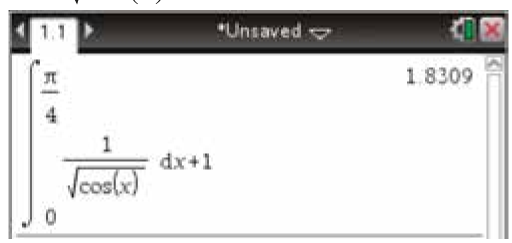
$\int_0^{\frac{\pi}{4}} \sqrt{\cos(x)}\, dx + 1$

b $\frac{dy}{dx} = \frac{1}{\sqrt{\cos x}},\ y(0) = 1$

Find y when $x = \frac{\pi}{4}$.

On your calculator type:

$\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\cos(x)}}\, dx + 1$



c $\frac{dy}{dx} = \log_e(x^2),\ y(1) = 2$

Find y when $x = e$.

On your calculator type:

$\int_1^{e^1} \ln(x^2)\, dx + 2$

d $\frac{dy}{dx} = \sqrt{\log_e x},\ y(1) = 2$

Find y when $x = e$.

On your calculator type:

$\int_1^{e^1} \sqrt{\ln(x)}\, dx + 2$

Solutions to Exercise 11I

1 a $\dfrac{dy}{dx} = \cos x$ and $y(0) = 1$

Using Euler's method:

$$y_{n+1} = y_n + 0.1[\cos(x_n)] \quad ①$$

with $x_0 = 0,\ y_0 = 1,\ h = 0.1$

Put $n = 0$ into ①:

$\therefore\ y_1 = y_0 + 0.1\cos(x_0)$

$= 1 + 0.1 \times 1$

$\therefore\ y_1 = 1.1$ and $x_1 = 0 + 0.1 = 0.1$

Put $n = 1$ into ①:

$\therefore\ y_2 = y_1 + 0.1[\cos(x_1)]$

$= 1.1 + 0.1 \times \cos(0.1)$

$= 1.19950041\ldots$

$\therefore\ y_2 = 1.1995$ and

$x_2 = 0.1 + 0.1 = 0.2$

Put $n = 2$ into ①:

$\therefore\ y_3 = y_2 + 0.1[\cos(x_2)]$

$= 1.1995 + 0.1 \times \cos(0.2)$

$= 1.29750707\ldots$

$\therefore\ y_3 = 1.2975$ and $x_3 = 0.3$

b $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ and $y(1) = 0$

Using Euler's method:

$$y_{n+1} = y_n + 0.01\left[\frac{1}{x_n^2}\right] \quad ②$$

with $x_0 = 1,\ y_0 = 0,\ h = 0.01$

Put $n = 0$ into ②:

$\therefore\ y_1 = y_0 + 0.01\left(\dfrac{1}{x_0^2}\right)$

$= 0 + 0.01(1)$

$\therefore\ y_1 = 0.01$ and

$x_1 = 1 + 0.01 = 1.01$

① Put $n = 1$ into ②:

$\therefore\ y_2 = y_1 + 0.01\left[\dfrac{1}{x_1^2}\right]$

$= 0.01 + 0.01\left[\dfrac{1}{1.01^2}\right]$

$= \dfrac{20201}{1020100}$

$= 0.01980296\ldots$

$\therefore\ y_2 = \dfrac{20201}{1020100}$ and $x_2 = 1.02$

Put $n = 2$ into ②:

$\therefore\ y_3 = y_2 + 0.01\left[\dfrac{1}{x_2^2}\right]$

$= \dfrac{20201}{1020100} + 0.01\left(\dfrac{1}{1.02^2}\right)$

$= \dfrac{78045301}{2653280100}$

$= 0.029414648\ldots$

$\therefore\ y_3 = \dfrac{78045301}{2653280100}$ and $x_3 = 1.03$

Put $n = 3$ into ②:

$\therefore\ y_4 = \dfrac{78045301}{2653280100}$

$+ 0.01\left(\dfrac{1}{1.03^2}\right)$

$= 0.038840607\ldots$

$\therefore\ y_4 = 0.0388$ and $x_4 = 1.04$

c $\frac{dy}{dx} = \sqrt{x}$ and $y(1) = 1$

Using Euler's method:

$$y_{n+1} = y_n + 0.1[\sqrt{x_n}] \qquad ③$$

with $x_0 = 1,\ y_0 = 1,\ h = 0.1$

Put $n = 0$ into ③:

$\therefore \quad y_1 = y_0 + 0.1 \times \sqrt{x_0}$

$= 1 + 0.1 \times 1$

$\therefore \quad y_1 = 1.1$ and $x_1 = 1.1$

Put $n = 1$ into ③:

$\therefore \quad y_2 = y_1 + 0.1 \times \sqrt{x_1}$

$= 1.1 + 0.1 \times \sqrt{1.1}$

$= 1.20488088\ldots$

$\therefore \quad y_2 = 1.20488\ldots$ and $x_2 = 1.2$

Put $n = 2$ into ③:

$\therefore \quad y_3 = y_2 + 0.1 \times \sqrt{x_2}$

$= 1.20488\ldots + 0.1 \times \sqrt{1.2}$

$= 1.31442539\ldots$

$\therefore \quad y_3 = 1.3144$ and $x_3 = 1.3$

d $\frac{dy}{dx} = \frac{1}{x^2 + 3x + 2}$ and $y(0) = 0$

Using Euler's method:

$$y_{n+1} = y_n + 0.01 \times \left[\frac{1}{x_n^2 + 3x_n + 2}\right] \qquad ④$$

with $x_0 = 0,\ y_0 = 0,\ h = 0.01$

Put $n = 0$ into ④:

$\therefore \quad y_1 = y_0 + 0.01\left(\frac{1}{x_0^2 + 3x_0 + 2}\right)$

$= 0 + 0.01\left(\frac{1}{2}\right)$

$\therefore \quad y_1 = 0.005$ and $x_1 = 0.01$

Put $n = 1$ into ④:

$\therefore \quad y_2 = 0.005 + 0.01\left(\frac{1}{2.0301}\right)$

$\therefore \quad y_2 = 0.009925865\ldots$ and

$x_2 = 0.02$

Put $n = 2$ into ④:

$\therefore \quad y_3 = 0.009925\ldots + 0.01\left(\frac{1}{2.0604}\right)$

$= 0.01477929\ldots$

$\therefore \quad y_3 = 0.0148$ and $x_3 = 0.03$

2 a i $\frac{dy}{dx} = \cos x$ with $y(0) = 1$

$\therefore \quad y = \sin x + c$

When $x = 0,\ y = 1$:

$\Rightarrow \quad c = 1$

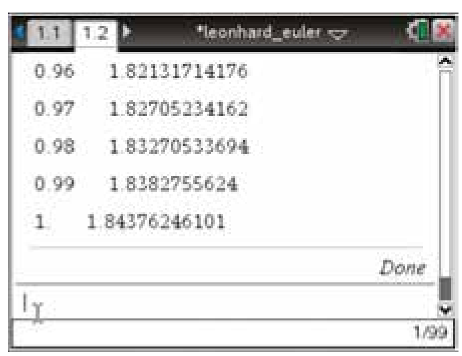
$\therefore \quad y = \sin x + 1$

When $x = 1$,

$y = \sin(1) + 1 \approx 1.8415$

ii **TI:** Use the leonhard_euler program as outlied in the textbook.

CP: Use the Spreadsheet instructions as outlied in the textbook.



$\therefore \quad y(1) = 1.8438$ using Euler

b **i** $\frac{dy}{dx} = \frac{1}{x^2}$ with $y(1) = 0$

$\therefore \quad y = \int x^{-2} dx$

$\therefore \quad y = -\frac{1}{x} + c$

When $x = 1,\ y = 0$:

$\Rightarrow \quad c = 1$

$\therefore \quad y = 1 - \frac{1}{x}$

When $x = 2$,

$\therefore \quad y = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$

ii

$\therefore \quad y(2) = 0.5038$ using Euler

c **i**

$\frac{dy}{dx} = \sqrt{x}$ with $y(1) = 1$

$\therefore \quad y = \int x^{\frac{1}{2}}\, dx$

$\therefore \quad y = \frac{2}{3}x^{\frac{3}{2}} + c$

When $x = 1,\ y = 1$:

$\Rightarrow \quad c = \frac{1}{3}$

$\therefore \quad y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}$

When $x = 2$,

$\therefore \quad y = \frac{2}{3}(2\sqrt{2}) + \frac{1}{3} = 2.2190$

ii

$\therefore \quad y(2) = 2.2169$ using Euler

d **i** $\frac{dy}{dx} = \frac{1}{x^2 + 3x + 2}$ with $y(0) = 0$

$= \frac{1}{(x+1)(x+2)}$

$\frac{1}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$

$\therefore \quad 1 = A(x+2)$

$+ B(x+1)$

When $x = -2$,

$\therefore \quad B = -1$

When $x = -1$,

$\therefore \quad A = 1$

$\therefore \quad y = \int \frac{1}{x^2 + 3x + 2}\, dx$

$= \int \frac{1}{x+1} - \frac{1}{x+2}\, dx$

$\therefore \quad y = \log_e(x+1) - \log_e(x+2) + c$

$\therefore \quad y = \log_e\left(\frac{x+1}{x+2}\right) + c$

When $x = 0,\ y = 0$:

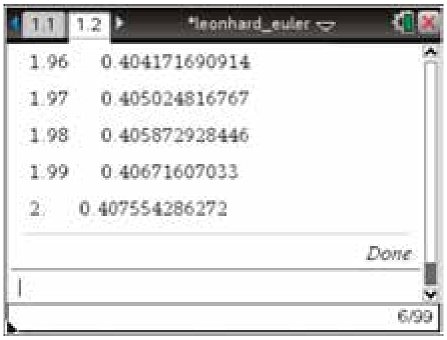
$\Rightarrow \quad c = -\log_e\left(\frac{1}{2}\right) = \log_e 2$

$\therefore \quad y = \log_e\left(\frac{x+1}{x+2}\right) + \log_e 2$

$\therefore \quad y = \log_e\left(\frac{2x+2}{x+2}\right)$

When $x = 2$,

$$\therefore \quad y = \log_e \frac{3}{2} = 0.4055$$

ii 

$\therefore \quad y(2) = 0.4076$ using Euler

3 $\dfrac{dy}{dx} = \sec^2 x$ with $y(0) = 2$

a Recall: $\displaystyle\int \sec^2 kx = \frac{1}{k}\tan kx + c$

$$\therefore \quad y = \int \sec^2 x dx$$

$$\therefore \quad y = \tan x + c$$

When $\quad x = 0,\ y = 2,$

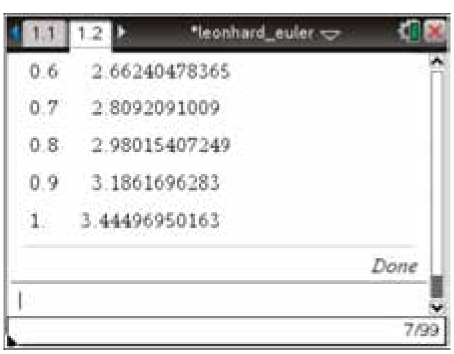
$$\Rightarrow \quad c = 2$$

$$\therefore \quad y = \tan x + 2$$

When $\quad x = 1,$

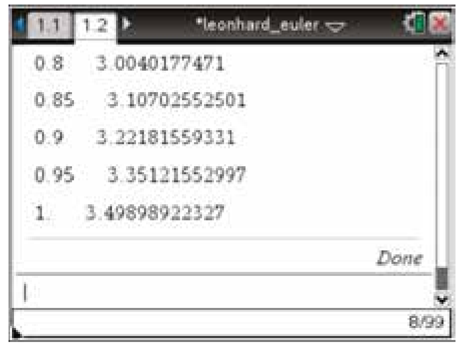
$$\therefore \quad y = \tan(1) + 2\ (\approx 3.5574)$$

b **i** step size = 0.1



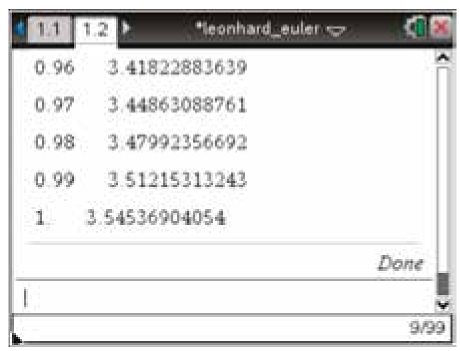
Therefore the solution at $x = 1$ using the Euler program and a step size of 0.1 is 3.444969502 correct to 9 decimal places.

ii step size = 0.05



Therefore the solution at $x = 1$ using the Euler program and a step size of 0.05 is 3.498989223 correct to 9 decimal places.

iii step size = 0.01



Therefore the solution at $x = 1$ using the Euler program and a step size of 0.01 is 3.545369041 correct to 9 decimal places.

4 $\dfrac{dy}{dx} = y^3$, with $y(0) = 1$

Answer: 2.205

5 $\dfrac{dy}{dx} = y^2 + 1)$, with $y(0) = 1$

Answer: 30.69

6 $\dfrac{dy}{dx} = xy$, with $y(0) = 1$

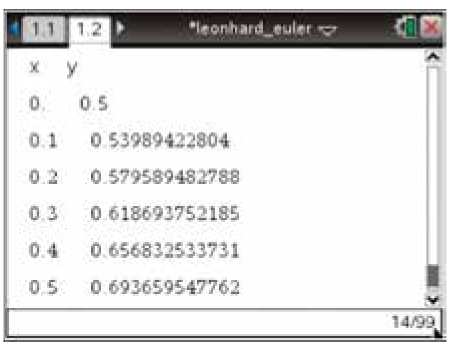
. Answer: 1.547

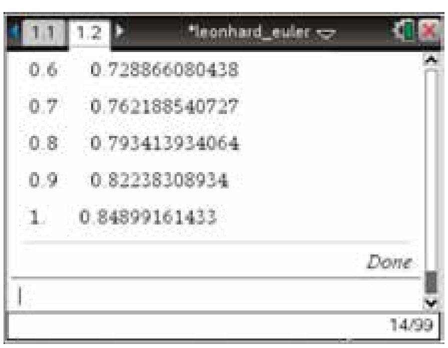
7

step	x	y
0	0	0.5
1	0.1	0.55
2	0.2	0.595
4	0.3	0.6345
⋮	⋮	⋮
⋮	⋮	⋮
10	1	0.7031...

8 $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, with $y(0) = \dfrac{1}{2}$

a For Euler's method the method described in **2a(ii.)** will be used.

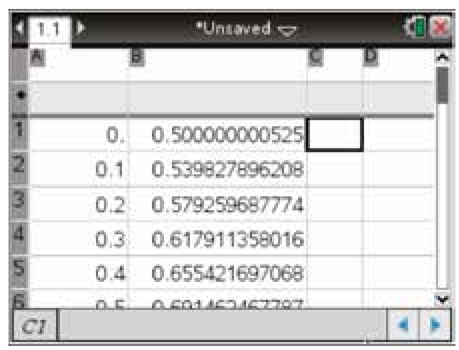




Tabulating these results (correct to 8 decimal places) gives:

	$\Pr(Z < z)$
z	**Euler's Method**
0	0.5
0.1	0.53989423
0.2	0.57958948
0.3	0.61869375
0.4	0.65683253
0.5	0.69365955
0.6	0.72886608
0.7	0.76218854
0.8	0.79341393
0.9	0.82238309
1	0.84899161

b **TI:** In a Lists & Spreadsheet page, input the numbers $0, 0.1, 0.2, 0.3, \ldots, 1$ into column A. In cell B1 type = **normCdf**($-\infty$, **A1**, **0**, **1**). Press **Menu** → **3:Data** → **Fill** and scroll down to cell 11 to copy the formula into the remaining cells.

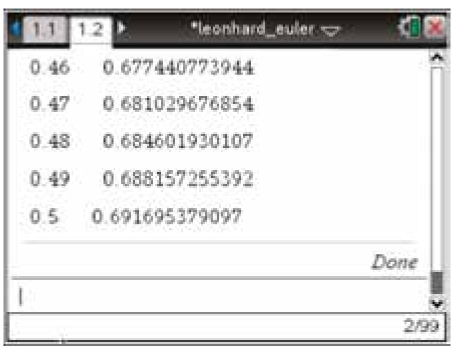


Use the down arrow key to view all results.

CP: In a Spreadsheet page, input the numbers $0, 0.1, 0.2, 0.3, \ldots, 1$ into column A. In cell B1 type = **normCDf**($-\infty$, **Al**, **1**, **0**). Select cell Bl through to B11. Tap **Edit** → **Fill Range** then OK. Tabulating these results against Euler's method gives:

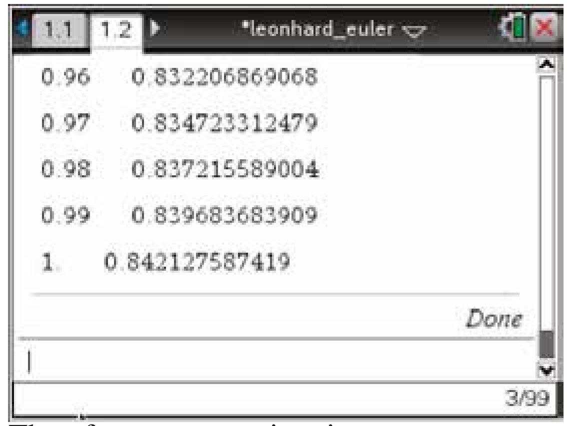
	$\Pr(Z < z)$	
z	**Euler's method**	**From tables**
0	0.5	0.5
0.1	0.53989423	0.53983
0.2	0.57958948	0.57926
0.3	0.61869375	0.61791
0.4	0.65683253	0.65542
0.5	0.69365955	0.69146
0.6	0.72886608	0.72575
0.7	0.76218854	0.75804
0.8	0.79341393	0.78814
0.9	0.82238309	0.81594
1	0.84899161	0.84134

c **i** For Euler's method the method described in **2a**(**ii.**) will be used.



Therefore an approximation to $\Pr(Z \le 0.5)$ using the Euler program and a step size of 0.01 is 0.69169538 correct to 8 decimal places.

ii For Euler's method the method described in **2a**(**ii.**) will be used.



Therefore an approximation to $\Pr(Z \le 1)$ using the Euler program and a step size of 0.01 is 0.84212759 correct to 8 decimal places.

Solutions to Exercise 11J

1 **a** $\dfrac{dy}{dx} = 3x^2$ with $y(1) = 0$

$\therefore \quad y = \int 3x^2\,dx$

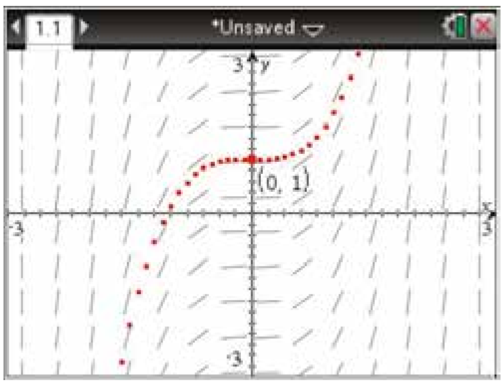
$\therefore \quad y = x^3 + c$

Using $y(1) = 0$:

$0 = 1^3 + c$

$\Rightarrow \quad c = -1$

$\therefore \quad y = x^3 - 1$



b $\dfrac{dy}{dx} = \sin(x)$ with $y(0) = 0$

$\therefore \quad y = \int \sin(x)\,dx$

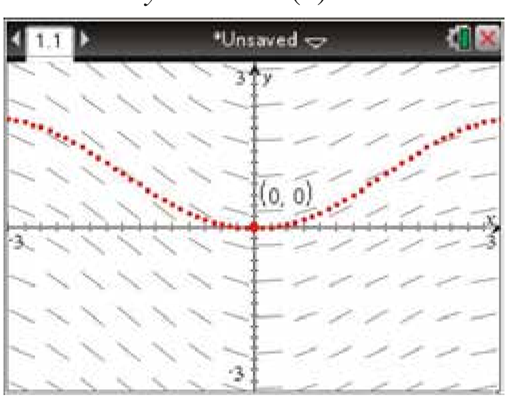
$\therefore \quad y = -\cos(x) + c$

using $y(0) = 0$:

$0 = -\cos(0) + c$

$\Rightarrow \quad c = 1$

$\therefore \quad y = 1 - \cos(x)$



c $\dfrac{dy}{dx} = e^{-2x}$ with $y(0) = 1$

$\therefore \quad y = \int e^{-2x}\,dx$

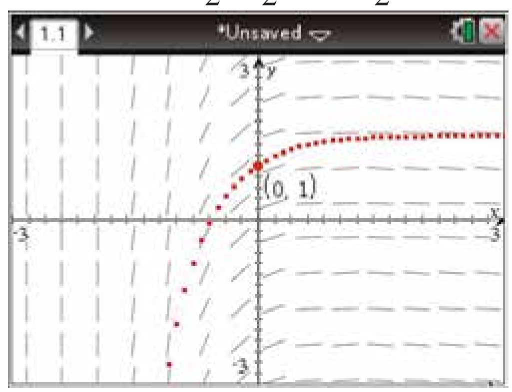
$\therefore \quad y = -\dfrac{1}{2}e^{-2x} + c$

using $y(0) = 1$:

$1 = -\dfrac{1}{2}e^{0} + c$

$\Rightarrow \quad c = \dfrac{3}{2}$

$\therefore \quad y = \dfrac{3}{2} - \dfrac{1}{2}e^{-2x} = \dfrac{1}{2}(3 - e^{-2x})$



d $\dfrac{dy}{dx} = y^2$ with $y(1) = 1$

$\therefore \quad \dfrac{dx}{dy} = \dfrac{1}{y^2}$

$\therefore \quad x = \displaystyle\int \frac{1}{y^2}\,dy$

$\therefore \quad x = \displaystyle\int y^{-2}\,dy$

$\therefore \quad x = -\dfrac{1}{y} + c$

using $y(1) = 1$:

$1 = \dfrac{-1}{1} + c$

$\Rightarrow \quad c = 2$

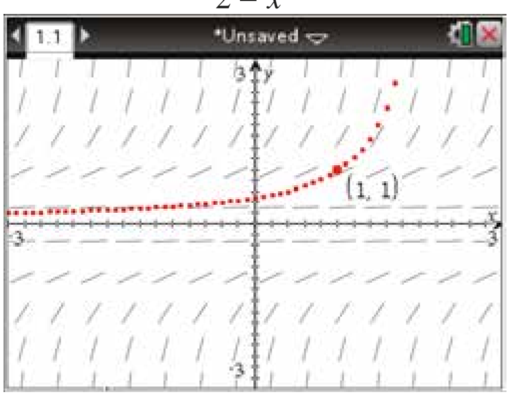
$\therefore \quad x = -\dfrac{1}{y} + 2$

$\therefore \quad x - 2 = -\dfrac{1}{y}$

$\therefore \quad y = -\dfrac{1}{x-2}, \; x < 2$

$\therefore \quad y = \dfrac{1}{-(x-2)}$

$\therefore \quad y = \dfrac{1}{2-x}, \; x < 2$



e $\dfrac{dy}{dx} = y^2$ with $y(1) = -1$

$\therefore \quad \dfrac{dx}{dy} = \dfrac{1}{y^2}$

$\therefore \quad x = \displaystyle\int \frac{1}{y^2}\,dy$

$\therefore \quad x = \displaystyle\int y^{-2}\,dy$

$\therefore \quad x = -\dfrac{1}{y} + c$

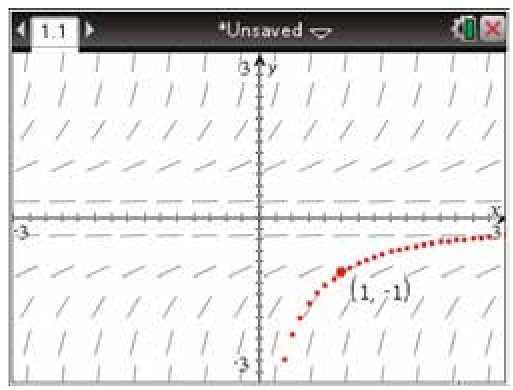
using $y(1) = -1$:

$1 = \dfrac{-1}{-1} + c$

$\Rightarrow \quad c = 0$

$\therefore \quad x = -\dfrac{1}{y} \Leftrightarrow y = -\dfrac{1}{x}$ and

$x > 0$



f

$$\frac{dy}{dx} = y(y-1) \text{ with } y(0) = -1$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{y(y-1)}$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{y-1} - \frac{1}{y}$$

using partial fractions

$$\therefore \quad x = \int \left(\frac{1}{y-1} - \frac{1}{y} \right) dy$$

$$\therefore \quad x = \log_e |y-1| - \log_e |y| + c$$

$$\therefore \quad x = \log_e \left| \frac{y-1}{y} \right| + c$$

using $y(0) = -1$:

$$0 = \log_e \left| \frac{-1-1}{-1} \right| + c$$

$$\Rightarrow \quad c = -\log_e(2)$$

$$\therefore \quad x = \log_e \left(\frac{y-1}{y} \right) - \log_e(2),$$

$$y > 1$$

$$\therefore \quad x = \log_e \left(\frac{y-1}{2y} \right)$$

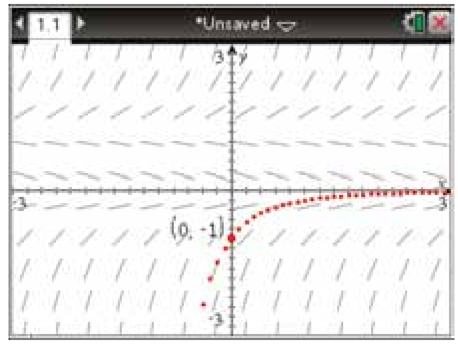
$$\therefore \quad e^x = \frac{y-1}{2y}$$

$$\therefore \quad e^x = \frac{1}{2} - \frac{1}{2y}$$

$$\therefore \quad e^x - \frac{1}{2} = -\frac{1}{2y}$$

$$\therefore \quad 1 - 2e^x = \frac{1}{y}$$

$$\therefore \quad y = \frac{1}{1 - 2e^x}$$



g

$$\frac{dy}{dx} = y(y-1) \text{ with } y(0)$$

$= 2$ from part **f**.

$$x = \log_e \left| \frac{y-1}{y} \right| + c$$

using $y(0) = 2$:

$$0 = \log_e \left| \frac{2-1}{2} \right| + c$$

$$\Rightarrow \quad c = -\log_e \left(\frac{1}{2} \right) = \log_e(2)$$

$$\therefore \quad x = \log_e \left(\frac{y-1}{y} \right) + \log_e(2),$$

$$y > 1$$

$$\therefore \quad x = \log_e \left(\frac{2(y-1)}{y} \right)$$

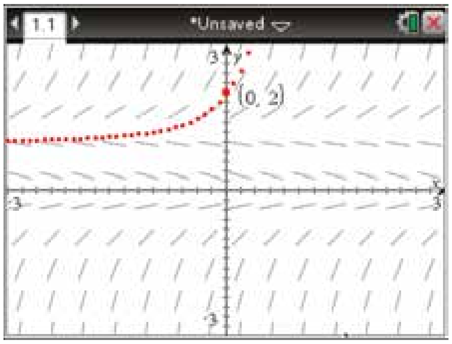
$$\therefore \quad e^x = \frac{2(y-1)}{y}$$

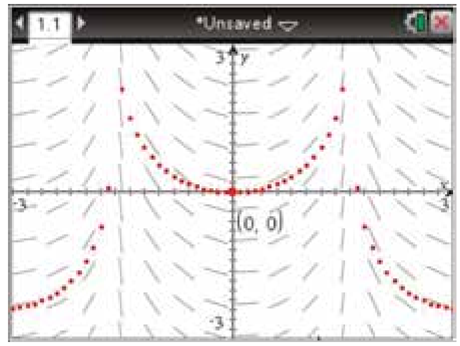
$$\therefore \quad e^x = 2 - \frac{2}{y}$$

$$\therefore \quad e^x - 2 = -\frac{2}{y}$$

$$\therefore \quad y = -\frac{2}{e^x - 2}$$

$$\therefore \quad y = \frac{2}{2 - e^x}$$





h

$$\frac{dy}{dx} = \tan(x) \text{ with } y(0) = 0$$

$$\therefore \quad y = \int \tan(x)\, dx$$

$$\therefore \quad y = \int \frac{\sin x}{\cos x}\, dx$$

$$\text{Let} \quad u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$$

$$\therefore \quad y = \int \frac{-\frac{du}{dx}}{u}\, dx$$

$$\therefore \quad y = \int -\frac{1}{u}\, du$$

$$\therefore \quad y = -\log_e(u) + c, \; u > 0$$

$$\therefore \quad y = -\log_e(\cos x) + c$$

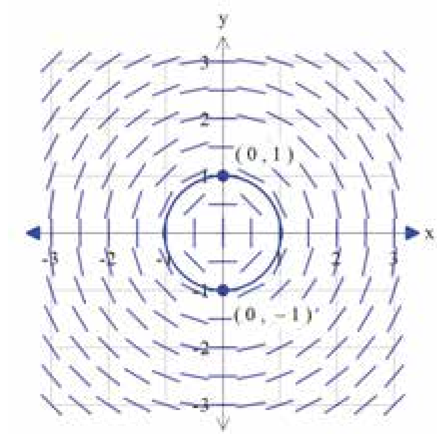
using $y(0) = 0$:

$$0 = -\log_e(1) + c$$

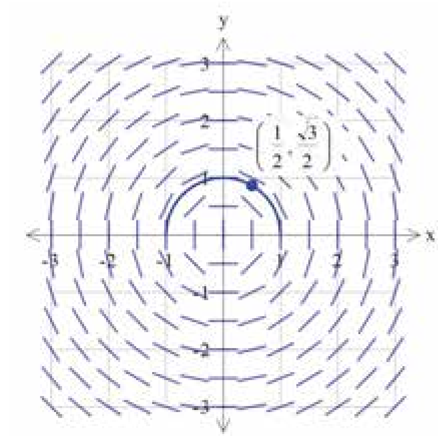
$$\Rightarrow \quad c = 0$$

$$\therefore \quad y = -\log_e(\cos x)$$

2 a



b



Solutions to Technology-free questions

1 a $\frac{dy}{dx} = \frac{x^2+1}{x^2}$

$= 1 + \frac{1}{x^2}$

$\therefore \quad y = \int (1 + x^{-2})dx$

$= x - x^{-1} + c$

$= x - \frac{1}{x} + c$

b $\frac{1}{y} \times \frac{dy}{dx} = 10$

$\therefore \quad \frac{dy}{dx} = 10y$

$\therefore \quad \frac{dx}{dy} = \frac{1}{10y}$

$\therefore \quad x = \int \frac{1}{10y}dy$

$= \frac{1}{10}\log_e y + d, \text{ since } y > 0$

$\therefore \quad e^{10(x-d)} = y$

$\therefore \quad y = ce^{10x}$

c $\frac{d^2y}{dt^2} = \frac{1}{2}(\sin 3t + \cos 2t), \ t \geq 0$

$\therefore \quad \frac{dy}{dt} = \frac{1}{2}\int(\sin 3t + \cos 2t)dt$

$= -\frac{1}{6}\cos 3t + \frac{1}{4}\sin 2t + c_1$

$\therefore \quad y = \int\left(-\frac{1}{6}\cos 3t + \frac{1}{4}\sin 2t + c_1\right)dt$

$= -\frac{1}{18}\sin 3t - \frac{1}{8}\cos 2t + c_1 t + c_2$

d $\frac{d^2y}{dx^2} = e^{-3x} + e^{-x}$

$\therefore \quad \frac{dy}{dx} = \int(e^{-3x} + e^{-x})dx$

$= -\frac{1}{3}e^{-3x} - e^{-x} + c_1$

$\therefore \quad y = \int\left(-\frac{1}{3}e^{-3x} - e^{-x} + c_1\right)dx$

$= \frac{1}{9}e^{-3x} + e^{-x} + c_1 x + c_2$

e $\frac{dy}{dx} = \frac{3-y}{2}$

$\therefore \quad \frac{dx}{dy} = \frac{2}{3-y}$

$\therefore \quad x = \int \frac{2}{3-y}dy$

$\therefore \quad x = -2\log_e(3-y) + c, \ y < 3$

$\therefore \quad 3 - y = e^{-\frac{x-c}{2}}$

$\therefore \quad y = 3 - e^{-\frac{x-c}{2}}$

f $\frac{dy}{dx} = \frac{3-x}{2}$

$\therefore \quad y = \int \frac{3-x}{2}dx$

$= \frac{3}{2}x - \frac{x^2}{4} + c$

2 a

$$\frac{dy}{dx} = \pi\cos(2\pi x)$$

$$\therefore \quad y = \pi\int\cos(2\pi x)dx$$

$$= \frac{\pi}{2\pi}\sin(2\pi x) + c$$

When $y = -1,\ x = \frac{5}{2},$ and

$$-1 = \frac{1}{2}\sin 5\pi + c$$

$$\therefore \quad c = -1$$

$$\therefore \quad y = \frac{1}{2}\sin(2\pi x) - \frac{1}{2}$$

b

$$\frac{dy}{dx} = \frac{\cos 2x}{\sin 2x}$$

Let $\sin 2x = u,$ then $\frac{du}{dx} = 2\cos 2x$

$$\therefore \quad y = \frac{1}{2}\int\frac{du}{u}$$

$$= \frac{1}{2}\log_e|u| + c$$

$$= \frac{1}{2}\log_e|\sin 2x| + c$$

When $y = 0,\ x = \frac{\pi}{4},$ and

$$0 = \frac{1}{2}\log_e\left(\sin\frac{\pi}{2}\right) + c$$

$$\therefore \quad 0 = \frac{1}{2}\log_e 1 + c$$

$$\therefore \quad c = 0$$

$$\therefore \quad y = \frac{1}{2}\log_e|\sin 2x|$$

c

$$\frac{dy}{dx} = \frac{1 + x^2}{x^2}$$

$$y = \int\left(\frac{1}{x} + x\right)dx$$

$$= \log_e|x| + \frac{x^2}{2} + c$$

When $y = 0,\ x = 1,$ and

$$0 = \log_e 1 + \frac{1}{2} + c$$

$$\therefore \quad c = -\frac{1}{2}$$

$$\therefore \quad y = \log_e|x| + \frac{x^2}{2} - \frac{1}{2}$$

d

$$\frac{dy}{dx} = \frac{x}{1 + x^2}$$

Let $1 + x^2 = u,\ 2x = \frac{du}{dx}$

$$\therefore \quad y = \frac{1}{2}\int\frac{du}{u}$$

$$= \frac{1}{2}\log_e|u| + c$$

$$= \frac{1}{2}\log_e(1 + x^2) + c$$

When $y = 1,\ x = 0,$ and

$$1 = \frac{1}{2}\log_e 1 + c$$

$$\therefore \quad c = 1$$

$$\therefore \quad y = \frac{1}{2}\log_e(1 + x^2) + 1$$

e $\dfrac{dy}{dx} = -\dfrac{1}{2}y$

$\therefore \quad \dfrac{dx}{dy} = -\dfrac{2}{y}$

$\therefore \quad x = \int -\dfrac{2}{y}dy$

$= -2\log_e|y| + c$

When $y = e^{-1}$, $x = 2$, and

$2 = -2 \times -1 + c$

$\therefore \quad c = 0$

$\therefore \quad x = -2\log_e y$

$\therefore \quad y = e^{-\frac{x}{2}}$

f $\dfrac{d^2x}{dt^2} = -10$

$\dfrac{dx}{dt} = -10t + c_1$

Since $\dfrac{dx}{dt} = 4$ when $x = 0$,

$4 = -10 \times 0 + c_1$

$\therefore \quad c_1 = 4$

$\therefore \quad \dfrac{dx}{dt} = -10t + 4$

$\therefore \quad x = \int -10t + 4dt$

$= -5t^2 + 4t + c_2$

When $x = 0$, $t = 4$, and

$0 = -5 \times 16 + 16 + c_2$

$\therefore \quad c_2 = 64$

$\therefore \quad x = 64 + 4t - 5t^2$

3 a

$\dfrac{dy}{dx} = \sin x + x\cos x$

product rule

$\therefore \quad \dfrac{d^2y}{dx^2} = \cos x + \cos x - x\sin x$

$= 2\cos x - x\sin x$

$\therefore \quad x^2\dfrac{d^2y}{dx^2} = 2x^2\cos x - x^3\sin x$

and $\quad kx\dfrac{dy}{dx} = kx\sin x + kx^2\cos x$

and $\quad (x^2 - m)y = (x^2 - m)x\sin x$

since $y = x\sin x$

$= x^3\sin x - mx\sin x$

$\therefore \quad x^2\dfrac{d^2y}{dx^2} - kx\dfrac{dy}{dx} + (x^2 - m)y = 0$

becomes $2x^2\cos x - x^3\sin x$

$-kx\sin x - kx^2\cos x$

$+x^3\sin x - mx\sin x = 0$

$(2x^2 - kx^2)\cos x + (-kx - mx)\sin x = 0$

$(2 - k)x^2\cos x + (-k - m)x\sin x = 0$

Equating coefficients, $\quad 2 - k = 0$

$\therefore \quad k = 2$

and $\quad -k - m = 0$

$\therefore \quad m = -k$

$= -2$

So $k = 2$ and $m = -2$.

b $\dfrac{dy}{dx} = e^{2x} + 2xe^{2x}$

$\therefore \quad \dfrac{d^2y}{dx^2} = 2e^{2x} + 2e^{2x} + 4xe^{2x}$

$= 4e^{2x} + 4xe^{2x}$

$\therefore \quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 3e^{2x} = 4e^{2x} + 4xe^{2x} - e^{2x}$

$- 2xe^{2x} - 3e^{2x}$

$= 2xe^{2x}$, as required.

4 a

$$f''(x) = 2\sec^2 x$$

$$\therefore f'(x) = 2\tan x + c$$

$$f'(\left(\frac{\pi}{4}\right) = 1$$

$$\therefore c = -1.$$

$$\therefore f'(x) = 2\tan\frac{\pi}{6} - 1$$

$$f'\left(\frac{\pi}{6}\right) = 2\tan\left(\frac{\pi}{6}\right) - 1$$

$$= \frac{2\sqrt{3}}{3} - 1$$

Gradient is $\frac{2\sqrt{3}}{3} - 1$

b $f''\left(\frac{\pi}{6}\right) = 2\sec^2\left(\frac{\pi}{6}\right) = \frac{8}{3}$

5

$$y = e^{nx}$$

$$\therefore \frac{dy}{dx} = ne^{nx}$$

$$\therefore \frac{d^2y}{dx^2} = n^2e^{nx}$$

Hence

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15e^{nx} = 0$$

$$\Leftrightarrow n^2 - 2n - 15 = 0$$

$$\Leftrightarrow (n-5)(n+3) = 0$$

$$\Leftrightarrow n = 5 \text{ or } n = -3$$

6 a

$$\frac{dy}{dx} = (y+4)^2 + 9 \text{ and } y(0) = 0$$

$$\therefore \frac{dy}{dx} = \frac{1}{(y+4)^2 + 9}$$

$$\therefore x = \frac{1}{3}\tan^{-1}\left(\frac{y+4}{3}\right) + c$$

When $x = 0, y = 0$

$$\therefore c = -\frac{1}{3}\tan^{-1}\frac{4}{3}$$

$$\therefore x = \frac{1}{3}\tan^{-1}\left(\frac{y+4}{3}\right) - \frac{1}{3}\tan^{-1}\frac{4}{3}$$

$$\therefore \frac{1}{3}\tan^{-1}\left(\frac{y+4}{3}\right) = x + \frac{1}{3}\tan^{-1}\frac{4}{3}$$

b

$$y_0 = 0, x_0 = 0$$

$$y_1 = y_0 + hf(x_0)$$

$$\therefore y_1 = 0 + 0.2 \times ((0.14)^2 + 9)$$

$$\therefore y_1 = 5$$

7 a

$$y_0 = \frac{1}{2}, x_0 = 1$$

$$y_1 = y_0 + hf(x_0)$$

$$\therefore y_1 = \frac{1}{2} + 0.1 \times 1$$

$$\therefore y_1 = \frac{3}{5}$$

$$y_2 = \frac{3}{5} + 0.1 \times \frac{1}{1.1^2}$$

$$\therefore y_2 = 0.6816$$

b $\frac{dy}{dx} = \frac{1}{x^2}, x = 1, y = \frac{1}{2}$

$$\therefore y = -\frac{1}{x} + c$$

$$c = \frac{3}{2}$$

$$y = \frac{3x-2}{2x}$$

c When $x = 1.2, y = 0.667$

8 a Use calculator

b

$$\frac{dy}{dx} = (y+4)^2 \text{ and } y(2) = -1$$

$$\therefore \frac{dx}{dy} = \frac{1}{(y+4)^2}$$

$$\therefore x = \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + c$$

When $x = 2, y = -1$

$$\therefore c = 2 - \frac{1}{2}\tan^{-1} - \frac{1}{2}$$

$$\therefore x = \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + 4 - \tan^{-1}\left(-\frac{1}{2}\right)$$

$$y = 2\tan\left(2x - 4 + \tan^{-1}\left(-\frac{1}{2}\right)\right)$$

9 a

$$\frac{dT}{dt} = -k(T-25)$$

$$\therefore \frac{dt}{dT} = -\frac{1}{k}\frac{1}{T-25}$$

$$\therefore t = -\frac{1}{k}\log_e(T-25) + c$$

When $t = 0, T = 100$

$$\therefore 0 = -\frac{1}{k}\log_e(75) + c$$

$$\therefore c = \frac{1}{k}\log_e\left(\frac{75}{T-25}\right)$$

When $t = 10, T = 85$

$$10 = \frac{1}{k}\log_e\left(\frac{75}{60}\right)$$

$$k = \frac{1}{10}\log_e\left(\frac{5}{4}\right)$$

b

$$t = \frac{10}{\log_e\left(\frac{5}{4}\right)}\log_e\left(\frac{75}{T-25}\right)$$

When $t = 15$

$$15 = \frac{10}{\log_e\left(\frac{5}{4}\right)}\log_e\left(\frac{75}{T-25}\right)$$

$$\frac{3}{2}\log_e\left(\frac{5}{4}\right) = \log_e\left(\frac{75}{T-25}\right)$$

$$\log_e\left(\frac{5}{4}\right)^{\frac{3}{2}} = \log_e\left(\frac{75}{T-25}\right)$$

$$\left(\frac{5}{4}\right)^{\frac{3}{2}} = \frac{75}{T-25}$$

$$T - 25 = 75\left(\frac{4}{5}\right)^{\frac{3}{2}}$$

$$T = 25 + 75\left(\frac{4}{5}\right)^{\frac{3}{2}}$$

$$T = 25 + 24\sqrt{5} \quad (\approx 78.67)$$

10

$$\frac{dy}{dx} = 2x\sqrt{25 - x^2}, y(4) = 25$$

$$\therefore y = \int 2x\sqrt{25 - x^2}\, dx$$

$$= -\int \sqrt{u}\frac{du}{dx}\, dx$$

$$= -\int u^{\frac{1}{2}} du$$

$$= -\frac{2}{3}u^{\frac{3}{2}} + c$$

$$= -\frac{2}{3}(2 - x^2)^{\frac{3}{2}} + c$$

$$x = 4, y = 25 \therefore 25 = -\frac{2}{3}(9)^{\frac{3}{2}} + c$$

$$\therefore c = 43$$

$$\therefore y = -\frac{2}{3}(25 - x^2)^{\frac{3}{2}} + 43$$

11

$y = e^x \sin x$

$\dfrac{dy}{dx} = e^x \sin x + e^x \cos x$

$= e^x(cos x + \sin x)$

$\dfrac{d^2y}{dx^2} = e^x(\cos x - \sin x) + e^x(\sin x + \cos x)$

$= 2e^x \cos x$

$\dfrac{d^2y}{dx^2} + k\dfrac{dy}{dx} + y = e^x \cos x$

$\Leftrightarrow 2e^x \cos x$

$+ ke^x(\sin x + \cos x) + e^x \sin x) = e^x \cos x$

$\Leftrightarrow e^x \cos x + ke^x(\sin x + \cos x) + e^x \sin x = 0$

$\Leftrightarrow \cos x(1 + k) + \sin x(1 + k) = 0$

This is to be true for all x. Therefore

$k = -1$

12 $\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dt}$

Given $\dfrac{dV}{dt} = 3 \text{ cm}^3/\text{s}$, $\dfrac{dx}{dt} = 3 \div \dfrac{dV}{dx}$

Now $\dfrac{dV}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{3}(18x^2 - x^3)\right)$

$= \dfrac{\pi}{3}(36x - 3x^2) = \pi(12x - x^2)$

$\therefore \quad \dfrac{dx}{dt} = \dfrac{3}{\pi(12x - x^2)}$

$= \dfrac{3}{\pi x(12 - x)}$

13 Now $C = 2\pi r$,

$\therefore \quad r = \dfrac{C}{2\pi}$

Also $A = \pi r^2$,

$\therefore \quad A = \pi\left(\dfrac{C}{2\pi}\right)^2$

$= \dfrac{C^2}{4\pi}$

$\therefore \quad \dfrac{dA}{dC} = \dfrac{C}{2\pi}$

Now $\dfrac{dA}{dt} = \dfrac{dA}{dC} \times \dfrac{dC}{dt}$

Given $\dfrac{dA}{dt} = 4$, $\dfrac{dC}{dt} = 4 \div \dfrac{C}{2\pi}$

$= \dfrac{8\pi}{C}$

14 $\dfrac{dx}{dt} = -\dfrac{x}{100}$

$\therefore \quad \dfrac{dt}{dx} = -\dfrac{100}{x}$

$\therefore \quad t = \displaystyle\int -\dfrac{100}{x}dx$

$= -100 \log_e x + c, \ x > 0$

When $t = 0$, $x = x_0$ and

$0 = -100 \log_e x_0 + c$

$\therefore \quad c = 100 \log_e x_0$

$\therefore \quad t = 100 \log_e\left(\dfrac{x_0}{x}\right)$

When $x = \dfrac{x_0}{2}$, $t = 100 \log_e 2 \approx 69$

It takes approximately 69 days.

15 Each minute, the amount of soap in the solution decreases by the same proportion, i.e. $\dfrac{40}{1000} = \dfrac{1}{25}$, since the volume of water remains constant.

$$\therefore \quad \frac{dS}{dt} = -\frac{S}{25}$$

$$\therefore \quad \frac{dt}{dS} = -\frac{25}{S}$$

$$\therefore \quad t = \int -\frac{25}{S} dS$$

$$= -25 \log_e S + c, \; S > 0$$

When $t = 0, \; S = 3,$ and

$$0 = -25 \log_e 3 + c$$

$$\therefore \quad c = 25 \log_e 3$$

$$\therefore \quad t = -25 \log_e S + 25 \log_e 3$$

$$\therefore \quad = 25 \log_e \frac{3}{S}$$

$$\therefore \quad \frac{t}{25} = \log_e \frac{3}{S}$$

$$\therefore \quad S = 3e^{-\frac{t}{25}}$$

16 **a** $$\frac{d\theta}{dt} = \frac{30 - \theta}{20}$$

$$\therefore \quad \frac{dt}{d\theta} = \frac{20}{30 - \theta}$$

$$\therefore \quad t = \int \frac{20}{30 - \theta} d\theta$$

$$= -20 \log_e$$

$$(30 - \theta) + c, \theta < 30$$

At $t = 0, \; \theta = 10, \; 0 = -20 \log_e 20 + c$

$$\therefore \quad c = 20 \log_e 20$$

$$\therefore \quad t = 20 \log_e \left(\frac{20}{30 - \theta} \right)$$

$$\therefore \quad \theta = 30 - 20e^{-\frac{t}{20}}$$

$$\therefore \quad e^{\frac{t}{20}} = \frac{20}{30 - \theta}$$

b $\theta = 30 - 20e^{-\frac{60}{20}}$

≈ 29

So temperature is approximately 29°C .

c $$t = 20 \log_e \left(\frac{20}{30 - \theta} \right)$$

At $\theta = 20, \; t = 20 \log_e 2 \approx 14$

So it takes approximately14 minutes.

17 **a** The rate of change is a constant proportion of the area, 2%,

$$\therefore \quad \frac{dA}{dt} = 0.02A$$

b $$\frac{dA}{dt} = 0.02A$$

$$= \frac{A}{50}$$

$$\therefore \quad \frac{dt}{dA} = \frac{50}{A}$$

$$\therefore \quad t = \int \frac{50}{A} dA$$

$$= 50 \log_e A + c, \; A > 0$$

When $t = 0, \; A = \frac{1}{2},$ and

$$0 = 50 \log_e \frac{1}{2} + c$$

$$\therefore \quad c = -50 \log_e \frac{1}{2}$$

$$= 50 \log_e 2$$

$$\therefore \quad t = 50 \log_e (2A)$$

$$\therefore \quad 2A = e^{0.02t}$$

$$\therefore \quad A = \frac{1}{2} e^{0.02t}$$

After 10 hours, the area is

$A = \frac{1}{2} e^{0.2} \approx 0.61$ hectares.

c $3 = \frac{1}{2}e^{0.02t}$

$\therefore \quad 6 = e^{0.02t}$

$\therefore \quad \log_e 6 = 0.02t$

$\therefore \quad t = \frac{\log_e 6}{0.02}$

≈ 89.59

So 3 hectares have been covered at $89\frac{1}{2}$ hours.

18 $\frac{dy}{dx} = \frac{1}{16}\int (L - 3x)dx$

$= \frac{Lx}{16} - \frac{3x^2}{32} + c$

The rate of change of the deflection is zero at the point of the support,

i.e., $\frac{dy}{dx} = 0$ at $x = 0$.

$\therefore \quad c = 0$

To find where the deflection has its greatest magnitude, we need to find x for which $\frac{dy}{dx} = 0$ $(x > 0)$.

$\therefore \quad \frac{Lx}{16} - \frac{3x^2}{32} = 0$

$\therefore \quad x = \frac{2L}{3}$

Now $\quad y = \int\left(\frac{Lx}{16} - \frac{3x^2}{32}\right)dx$

$= \frac{Lx^2}{32} - \frac{x^3}{32} + d$

The deflection itself is zero at the point of the support, i.e., $y = 0$ when $x = 0$,

$\therefore \quad d = 0$

$\therefore \quad y = \frac{Lx^2}{32} - \frac{x^3}{32}$

When $x = \frac{2L}{3}$, $y = \frac{L \times 4L^2}{32 \times 9} - \frac{8L^3}{32 \times 27}$

$= \frac{L^3}{72} - \frac{L^3}{108}$

$= \frac{L^3}{216}$

So the magnitude is greater at $x = \frac{2L}{3}$ where the deflection is $\frac{L^3}{216}$.

19 $r = h\tan 30°$

$= \frac{\sqrt{h}}{\sqrt{3}}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

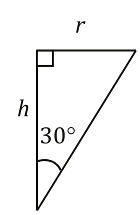
$= \frac{\pi h^3}{9}$

$\therefore \quad \frac{dV}{dh} = \frac{\pi h^2}{3}$ and $\frac{dV}{dt} = 2 - 0.05\sqrt{h}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{3}{\pi h^2}(2 - 0.05\sqrt{h})$

$= \frac{6 - 0.15\sqrt{h}}{\pi h^2}$



Solutions to multiple-choice questions

1 C $a = \sin(2t)$, when $t = 0,\ v = 4$

$\therefore\ v = \int_0^t \sin(2x)dt + 4$

2 D

$f'(x) = x^2 - 1$ and $f(1) = 3$

Using Euler's Method:

$y_{n+1} = y_n + 0.2[(x_n)^2 - 1]$ ①

with $x_0 = 1,\ y_0 = 3$

Put $n = 0$ into ①:

$\therefore\ y_1 = y_0 + 0.2[(x_0)^2 - 1]$

$= 3 + 0.2[1^2 - 1]$

$\therefore\ y_1 = 3$ and $x_1 = 1.2$

Put $n = 1$ into 1:

$\therefore\ y_2 = y_1 + 0.2[(x_1)^2 - 1]$

$= 3 + 0.2[(1.2)^2 - 1]$

$= 3 + 0.088$

$\therefore\ y_2 = 3.088$ and $x = 1.4$

3 B

$\dfrac{dy}{dx} = x\log_e(x)$ and $y(2) = 2$

Using Euler's Method:

$y_{n+1} = y_n + 0.1[x_n \log_e(x_n)]$ ②

with $x_0 = 2,\ y_0 = 2$

Put $n = 0$ into ②:

$\therefore\ y_1 = y_0 + 0.1[x_0 \log_e(x_0)]$

$= 2 + 0.1[2\log_e(2)]$

$\therefore\ y_1 = 2 + 0.2\log_e(2)$ and $x = 2.1$

Put $n = 1$ into ②:

$\therefore\ y_2 = y_1 + 0.1[x_1 \log_e(x_1)]$

$= 2 + 0.2\log_e(2)$

$+ 0.1[2.1\log_e(2.1)]$

$= 2 + 0.2\log_e(2)$

$+ 0.21\log_e(2.1)$

$= 2.294436\ldots$

$\therefore\ y_2 \approx 2.294$ and $x = 2.2$

4 A

$\dfrac{dy}{dx} = \dfrac{2-y}{4}$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{4}{2-y}$

$\therefore\ x = \displaystyle\int_1^{\frac{1}{2}} \frac{4}{2-t}dt + 3$

5 E $\dfrac{dy}{dx} = \dfrac{2x+1}{4},\ y(2) = 0$

$\therefore\ y = \dfrac{1}{4}\displaystyle\int (2x+1)dx$

$\therefore\ y = \dfrac{1}{4}(x^2 + x) + c$

When $x = 2,\ y = 0$:

$\Rightarrow c = -\dfrac{3}{2}$

$\therefore\ y = \dfrac{1}{4}(x^2 + x) - \dfrac{3}{2}$

$\therefore\ y = \dfrac{1}{4}(x^2 + x - 6)$

6 C

$$\frac{dy}{dx} = \frac{(y-1)^2}{5},\ y(0) = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{5}{(y-1)^2} = 5(y-1)^{-2}$$

$$\therefore \quad x = 5\int (y-1)^{-2}dy$$

$$\therefore \quad x = -\frac{5}{y-1} + c$$

When $x = 0,\ y = 0:$

$$\Rightarrow \quad c = -5$$

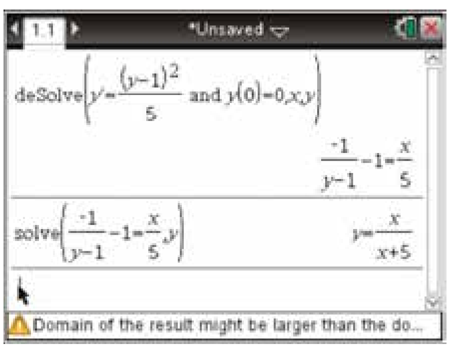
$$\therefore \quad x + 5 = -\frac{5}{y-1}$$

$$\therefore \quad y - 1 = -\frac{5}{x+5}$$

$$\therefore \quad y = 1 - \frac{5}{x+5}$$

$$\therefore \quad y = \frac{x}{x+5}$$

Using CAS:



7 D

$$\frac{dy}{dx} = e^{-x^2},\ y(1) = 4$$

$$\therefore \quad y = \int_1^x e^{-u^2}du + 4$$

8 E

$$y = 2xe^{2x}$$

then $$\frac{dy}{dx} = 2e^{2x} + 4xe^{2x}$$

and $$\frac{d^2y}{dx^2} = 4e^{2x} + 4e^{2x} + 8xe^{2x}$$

$$= 8e^{2x} + 8xe^{2x}$$

Response A:

$$\frac{dy}{dx} - 2y = 2e^{2x} \neq 0$$

Response B:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 4e^{2x} \neq 0$$

Response C:

$$\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$= 8xe^{4x}(2x+1) + e^{2x}(4x+2) \neq 0$$

Response D:

$$\frac{d^2y}{dx^2} - 4y = 8e^{2x} \neq e^{2x}$$

Response E:

$$\frac{d^2y}{dx^2} - 4y = 8e^{2x} = \text{RHS}$$

9 A Given: $$\frac{dV}{dt} = -\frac{5\sqrt{h}}{2h+45}$$

$$V = \pi(15h^2 + 225h)$$

$$\therefore \quad \frac{dV}{dh} = \pi(10h + 225)$$

$$= 5\pi(2h + 45)$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\therefore \quad \frac{dh}{dt} = \frac{1}{5\pi(2h+45)} \times \frac{-5\sqrt{h}}{2h+45}$$

$$= \frac{-\sqrt{h}}{\pi(2h+45)^2}$$

10 C $\frac{dy}{dx} = y$

$\Rightarrow \frac{dx}{dy} = \frac{1}{y}$

$\therefore \quad x = \log_e(y) + c, \text{ for } y > 0$

When $x = 0,\ y = 2$:

$\Rightarrow \quad c = -\log_e(2)$

$\therefore \quad x = \log_e\left(\frac{y}{2}\right)$

$\therefore \quad e^x = \frac{y}{2}$

$\therefore \quad y = 2e^x, \text{ for } y > 0$

11 A

12 E

13 D

14 E

15 C

16 C

Solutions to extended-response questions

1 a i $\dfrac{dx}{dt} = -kx,\ k > 0$

ii Now $\dfrac{dt}{dx} = -\dfrac{1}{k} \times \dfrac{1}{x}$

$\therefore\quad t = -\dfrac{1}{k} \times \displaystyle\int \dfrac{1}{x} dx$

$= -\dfrac{1}{k} \log_e x + c,\ x > 0$

$x = 100$ when $t = 0$, since the initial amount counts as 100%.

$\therefore\quad c = \dfrac{1}{k} \log_e 100$

$\therefore\quad t = \dfrac{1}{k} \log_e \dfrac{100}{x}$

$\therefore\quad e^{-kt} = \dfrac{x}{100}$

$\therefore\quad x = 100e^{-kt}$

Now $x = 50$ when $t = 5760$,

$\therefore\quad k = \dfrac{1}{5760} \log_e 2$

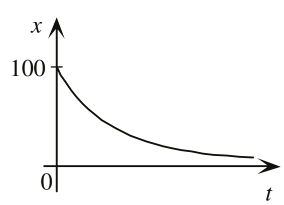
$\therefore\quad x = 100e^{\frac{-t}{5760} \log_e 2}$

b $t = \dfrac{5760}{\log_e 2} \log_e \dfrac{100}{45.1}$

≈ 6617 years

The eruption occurred 6617 years ago.

c $x = 100e^{\frac{-t \log_e 2}{5760}},\ t \geq 0$



2 **a** Unreacted amount of A after x minutes $= 2 - \frac{1}{4}x$

$$= \frac{8 - x}{4}$$

Unreacted amount of B after x minutes $= 3 - \frac{3}{4}x$

$$= \frac{3(4 - x)}{4}\Big)$$

$$\therefore \quad \frac{dx}{dt} = \frac{3k(8 - x)(4 - x)}{16}$$

b Now $\qquad \frac{dt}{dx} = \frac{16}{3k} \times \frac{1}{(8 - x)(4 - x)}$

and $\qquad \frac{1}{(8 - x)(4 - x)} = \frac{A}{8 - x} + \frac{B}{4 - x}$

$\therefore \quad A(4 - x) + B(8 - x) = 1$

When $\qquad x = 4,\ 4B = 1,$

$\therefore \qquad B = \frac{1}{4}$

When $\qquad x = 8,\ -4A = 1,$

$\therefore \qquad A = -\frac{1}{4}$

$$\therefore \qquad t = \frac{16}{3k} \times \frac{1}{4} \int \frac{1}{4 - x} - \frac{1}{8 - x} dx$$

$$= \frac{4}{3k}(\log_e |8 - x| - \log_e |4 - x| + c)$$

$$= \frac{4}{3k} \log_e \left|\frac{8 - x}{4 - x}\right| + c$$

$x = 0$ when $t = 0$ (no reaction yet) ,

$$\therefore \qquad c = -\frac{4}{3k} \log_e 2$$

$$\therefore \qquad t = \frac{4}{3k} \log_e \left|\frac{8 - x}{2(4 - x)}\right|$$

$x = 1$ when $t = 1$,

$$\therefore \quad 1 = \frac{4}{3k}\log_e\left|\frac{8-1}{2(4-1)}\right|$$

$$\therefore \quad k = \frac{4}{3}\log_e\frac{7}{6}$$

$$\therefore \quad t = \frac{1}{\log_e\frac{7}{6}}\log_e\left|\frac{8-x}{2(4-x)}\right|$$

c At $x = 2$, $t = \dfrac{1}{\log_e\frac{7}{6}}\log_e\dfrac{6}{4}$

$$\approx 2.633$$

$$\approx 2\,\text{min}\,38\text{ s}$$

It takes 2 minutes 38 seconds to form 2 kg of X.

d

$$\frac{8-x}{2(4-x)} = e^{t\log_e\frac{7}{6}}$$

$$x\left(2e^{t\log_e\frac{7}{6}} - 1\right) = 8\left(e^{t\log_e\frac{7}{6}} - 1\right)$$

$$x = \frac{8\left(\left(\frac{7}{6}\right)^t - 1\right)}{2\left(\frac{7}{6}\right)^t - 1}$$

When $t = 2$, $x = \dfrac{8\left(\frac{49}{36} - 1\right)}{2\times\frac{49}{36} - 1}$

$$= \frac{52}{31}$$

The mass of X formed after two minutes is $\dfrac{52}{31}$ kg.

3 a $\dfrac{dT}{dt} = k(T - T_s),\ k < 0$

b $\dfrac{dT}{dt} = k(T - 22)$

$$\therefore \quad \frac{dt}{dT} = \frac{1}{k}\times\frac{1}{T-22}$$

$$\therefore \quad t = \frac{1}{k}\times\int\frac{1}{T-22}dT$$

$$= \frac{1}{k}\log_e(T-22) + c,\ T > 22$$

When $T = 72,\ t = 0,$

$$\therefore \quad 0 = \frac{1}{k}\log_e(72 - 22) + c$$

$$\therefore \quad c = -\frac{1}{k}\log_e 50$$

$$\therefore \quad t = \frac{1}{k}\log_e(T - 22) - \frac{1}{k}\log_e 50$$

$$= \frac{1}{k}\log_e\left(\frac{T - 22}{50}\right)$$

i $\quad \therefore \quad k = \frac{1}{t}\log_e\left(\frac{T - 22}{50}\right)$

When $T = 65,\ t = 5,$

$$\therefore \quad k = \frac{1}{5}\log_e\left(\frac{65 - 22}{50}\right)$$

$$= \frac{1}{5}\log_e 0.86$$

$$\therefore \quad t = \frac{5}{\log_e 0.86}\log_e\left(\frac{T - 22}{50}\right)$$

When $T = 50,\ t = \frac{5}{\log_e 0.86}\log_e\left(\frac{50 - 22}{50}\right)$

$$= \frac{5\log_e 0.56}{\log_e 0.86}$$

$$\approx 19.2$$

The coffee remains drinkable for 19.2 minutes.

ii $\quad$ Now at $t = 30,\ 30 = \frac{5}{\log_e 0.86}\log_e\left(\frac{T - 22}{50}\right)$

$$\therefore \quad \frac{30}{5}\log_e 0.86 = \log_e\left(\frac{T - 22}{50}\right)$$

$$\therefore \quad \log_e(0.86)^6 = \log_e\left(\frac{T - 22}{50}\right)$$

$$\therefore \quad \frac{T - 22}{50} = (0.86)^6$$

$$\therefore \quad T = 50(0.86)^6 + 22$$

$$\approx 42.2$$

The temperature of the coffee at the end of 30 minutes is 42.2°C.

4 a $\frac{dp}{dt}$ = rate of increase–rate of decrease

$= kp - 1000,\ k > 0$

b $\frac{dt}{dp} = \frac{1}{kp - 1000}$

$\therefore\quad t = \int \frac{1}{kp - 1000}\,dp$

$= \frac{1}{k}\log_e(kp - 1000) + c,\ kp - 1000 > 0$

When $t = 0$, $p = 5000$,

$\therefore\quad 0 = \frac{1}{k}\log_e(5000k - 1000) + c$

$\therefore\quad c = -\frac{1}{k}\log_e(5000k - 1000)$

$\therefore\quad t = \frac{1}{k}\log_e(kp - 1000) - \frac{1}{k}\log_e(5000k - 1000)$

$= \frac{1}{k}\log_e\left(\frac{kp - 1000}{5000k - 1000}\right)$

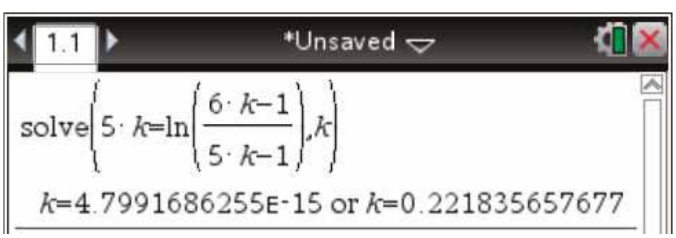
c i When $t = 5$, $p = 6000$,

$\therefore\quad 5 = \frac{1}{k}\log_e\left(\frac{6000k - 1000}{5000k - 1000}\right)$

$\therefore\quad 5k = \log_e\left(\frac{1000(6k - 1)}{1000(5k - 1)}\right)$

$\therefore\quad 5k = \log_e\left(\frac{6k - 1}{5k - 1}\right)$

ii TI: Type **solve(5 × k = ln((6 × k − 1)/(5 × k − 1)), k)**



Interpreting these results gives $k = 0$ or $k = 0.22183565\ldots$

CP: Sketch the graphs of **y1** = **5*x*** and **y2** = **ln**((**6x − 1**)/(**5x − 1**)). Tap **Analysis** → **G-Solve** → **Interset**

Thus an approximation for the value of k of 0.221 835 66.

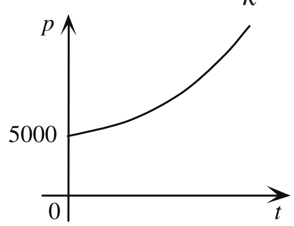
d $$t = \frac{1}{k}\log_e\left(\frac{kp - 1000}{5000k - 1000}\right)$$

$$\therefore \quad kt = \log_e\left(\frac{kp - 1000}{5000k - 1000}\right)$$

$$\therefore \quad \frac{kp - 1000}{5000k - 1000} = e^{kt}$$

$$\therefore \quad kp - 1000 = e^{kt}(5000k - 1000)$$

$$\therefore \quad p = \frac{1}{k}(e^{kt}(5000k - 1000) + 1000)$$



5 a $\frac{dN}{dt} = 100 - kN,\ k > 0$

b $$\frac{dt}{dN} = \frac{1}{100 - kN}$$

$$\therefore \quad t = \int \frac{1}{100 - kN}dN$$

$$= -\frac{1}{k}\log_e(100 - kN) + c,\ 100 - kN > 0$$

When $t = 0,\ N = 1000$,

$$\therefore \quad 0 = -\frac{1}{k}\log_e(100 - 1000k) + c$$

$$\therefore \quad c = \frac{1}{k}\log_e(100 - 1000k)$$

$$\therefore \quad t = -\frac{1}{k}\log_e(100 - kN) + \frac{1}{k}\log_e(100 - 1000k)$$

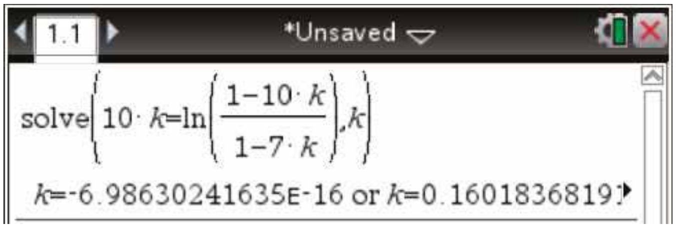
$$= \frac{1}{k}\log_e(\frac{100 - 1000k}{100 - kN})$$

c When $t = 10,\ N = 700$,

$$\therefore \quad 10 = \frac{1}{k}\log_e\left(\frac{100 - 1000k}{100 - 700k}\right)$$

$$\therefore \quad 10k = \log_e\left(\frac{1 - 10k}{1 - 7k}\right)$$

TI: Type **solve** (**10** × **k** = **ln**((**1** − **10** × **k**)/(**1** − **7** × **k**)), **k**)



Interpreting these results gives $k = 0$ or $k = 0.16018368\ldots$

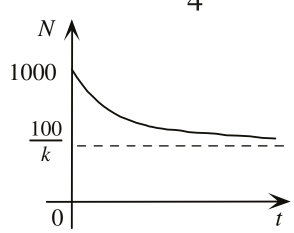
CP: Sketch the graphs of **y1** = **10*x*** and **y2** = **ln**((**1** − **10*x***)/(**1** − **7*x***)). Tap **Analysis** → **G-Solve** → **Intersect**

Thus an approximation for the value of k of 0.160 183 68.

d
$$t = \frac{1}{k}\log_e\left(\frac{100 - 1000k}{100 - kN}\right)$$
$$\therefore \quad kt = \log_e\left(\frac{100 - 1000k}{100 - kN}\right)$$
$$\therefore \quad e^{kt} = \frac{100 - 1000k}{100 - kN}$$
$$\therefore \quad 100 - kN = e^{-kt}(100 - 1000k)$$
$$\therefore \quad N = \frac{1}{k}(100 - e^{-kt}(100 - 1000k))$$

When $k \approx 0.16,\ N \approx \dfrac{1}{0.16}(100 - e^{-0.16t}(100 - 1000 \times 0.16))$

$$\approx \frac{25}{4}(100 + 60^{e-0.16t})$$



e As $t \to +\infty,\ N \to \dfrac{100}{k}$

The eventual trout population in the lake will be $\dfrac{100}{k}$.

When $k \approx 0.16, \dfrac{100}{k} \approx 625$

So the trout population approaches 625.

6 a $\dfrac{dy}{dx} = \dfrac{9}{40L^2}\displaystyle\int (3x - L)dx$

$$= \frac{9}{40L^2}\left(\frac{3x^2}{2} - LX\right) + c$$

When $x = 0$ (at A), $\dfrac{dy}{dx} = 0$, $\therefore \quad c = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{9}{40L^2}\left(\frac{3x^2}{2} - Lx\right)$$

$$\frac{dy}{dx} = 0 \text{ when } \frac{3x^2}{2} = Lx(x \neq 0)$$

$$\therefore \quad x = \frac{2L}{3}$$

The maximum deflection occurs $\dfrac{2L}{3}$ cm from the end A.

b $$y = \frac{9}{40L^2}\int\left(\frac{3x^2}{2} - LX\right)dx$$

$$= \frac{9}{40L^2}\left(\frac{x^3}{2} - \frac{Lx^2}{2}\right) + c$$

When $x = 0$, $y = 0$, $\therefore \quad c = 0$

$$\therefore \quad y = \frac{9x^2}{80L^2}(x - L)$$

$$\text{when } x = \frac{2L}{3},\ y = \frac{9}{80L^2} \times \left(\frac{2L}{3}\right)^2 \times \left(\frac{2L}{3} - L\right)$$

$$= \frac{9 \times 4L^2 \times (-L)}{80L^2 \times 9 \times 3}$$

$$= -\frac{L}{60}$$

The maximum deflection is $\dfrac{L}{60}$ cm downwards.

7 a $$\frac{dT}{dt} = 2 - k(T - T_0)$$

When $T = 60$, $\dfrac{dT}{dt} = -1$, $\therefore \quad -1 = -k(60 - T_0)$

$$\therefore \quad k = \frac{1}{60 - T_0}$$

$$\therefore \quad \frac{dT}{dt} = 2 - \frac{T - 20}{60 - T_0}$$

Given $T_0 = 20$, $\dfrac{dT}{dt} = 2 - \dfrac{T - 20}{40}$

$$= \frac{100 - T}{40}$$

b $$\frac{dt}{dT} = \frac{40}{100 - T}$$

$$t = \int \frac{40}{100 - T} dT$$

$$= -40 \log_e(100 - T) + c, \ T < 100$$

When $t = 0, \ T = 20,$

$$\therefore \quad c = 40 \log_e 80$$

$$\therefore \quad t = 40 \log_e\left(\frac{80}{100 - T}\right)$$

$$\therefore \quad e^{\frac{t}{40}} = \frac{80}{100 - T}$$

$$\therefore \quad 100 - T = 80e^{-\frac{t}{40}}$$

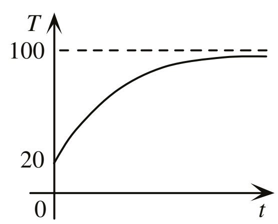
$$\therefore \quad T = 100 - 80e^{-\frac{t}{40}}$$

c When $t = 30, \ T = 100 - 80e^{-\frac{3}{4}}$

$$= 62.210\ldots$$

The temperature is 62.2°C after 30 minutes.

d



8 a i $$\frac{dW}{dt} = 0.04W$$

$$\therefore \quad \frac{dt}{dW} = \frac{1}{0.04W}$$

$$= \frac{25}{W}$$

$$\therefore \quad t = \int \frac{25}{W} dW$$

$$= 25 \log_e W + c, \ W > 0$$

When $t = 0$, $W = 350$,

$\therefore \quad 0 = 25 \log_e 350 + c$

$\therefore \quad c = -25 \log_e 350$

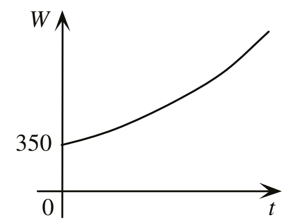
$\therefore \quad t = 25 \log_e W - 25 \log_e 350$

$= 25 \log_e \left(\dfrac{W}{350} \right)$

ii $\dfrac{t}{25} = \log_e \left(\dfrac{W}{350} \right)$

$\therefore \quad \dfrac{W}{350} = e^{\frac{t}{25}}$

$\therefore \quad W = 350e^{\frac{t}{25}}$



iii When $t = 50$, $W = 350_e^{\frac{50}{25}}$

$= 350e^2 \approx 2586$

b If $\dfrac{dW}{dt} = kW$ and the population remains constant then $\dfrac{dW}{dt} = 0$.

$\therefore \quad k = 0$ since $W > 0$

c **i** $\dfrac{dW}{dt} = (0.04 - 0.00005W)W$

$\therefore \quad \dfrac{dt}{dW} = \dfrac{1}{(0.04 - 0.00005W)W} = \dfrac{20\,000}{(800 - W)W}$

Now $\dfrac{20\,000}{(800 - W)W} = \dfrac{A}{800 - W} + \dfrac{B}{W}$

$\therefore \quad AW + B(800 - W) = 20\,000$

When $W = 0$, $800B = 20\,000$, $\therefore \quad B = 25$

When $W = 800$, $800A = 20\,000$, $\therefore \quad A = 25$

$$\therefore \quad \frac{20\,000}{(800-W)W} = \frac{25}{800-W} + \frac{25}{W}$$

$$\therefore \quad \frac{dt}{dW} = \frac{25}{800-W} + \frac{25}{W}$$

$$\therefore \quad t = \int \frac{25}{800-W} + \frac{25}{W} dW$$

$$= -25\log_e(800-W) + 25\log_e W + c, \; 0 < W < 800$$

$$= 25\log_e\left(\frac{W}{800-W}\right) + c$$

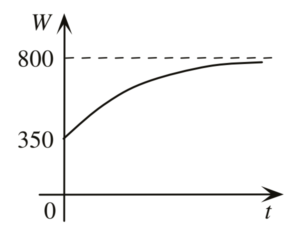
$$t = 0, \; W = 350, \; \therefore \quad 0 = 25\log_e\left(\frac{350}{450}\right) + c$$

$$\therefore \quad c = -25\log_e \frac{7}{9}$$

$$\therefore \quad t = 25\log_e\left(\frac{W}{800-W}\right) - 25\log_e \frac{7}{9}$$

$$= 25\log_e\left(\frac{9W}{7(800-W)}\right)$$

ii $$\frac{t}{25} = \log_e\left(\frac{9W}{7(800-W)}\right)$$



$$\therefore \quad \frac{9W}{7(800-W)} = e^{\frac{t}{25}}$$

$$\therefore \quad 9W = 7(800-W)e^{\frac{t}{25}}$$

$$= 5600e^{\frac{t}{25}} - 7We^{\frac{t}{25}}$$

$$\therefore \quad 9W + 7We^{\frac{t}{25}} = 5600e^{\frac{t}{25}}$$

$$\therefore \quad W\left(9 + 7e^{\frac{t}{25}}\right) = 5600e^{\frac{t}{25}}$$

$$\therefore \quad W = \frac{5600e^{\frac{t}{25}}}{9 + 7e^{\frac{t}{25}}}$$

iii When $t = 50$, $W = \dfrac{5600e^{\frac{50}{25}}}{9 + 7e^{\frac{50}{25}}}$

$$= \frac{5600e^2}{9 + 7e^2}$$

$$= 681.429\,55\ldots$$

The population after 50 years is approximately 681 iguanas.

9 **a** **i** $\dfrac{dx}{dt}$ = rate of input − rate of output

$= R - kx,\ k > 0$

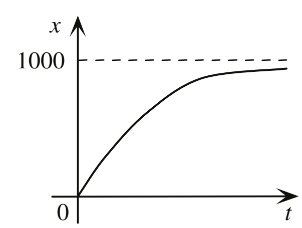
ii
$$\frac{dt}{dx} = \frac{1}{R - kx}$$
$$\therefore\quad t = \int \frac{1}{R - kx}dx$$
$$\therefore\quad t = -\frac{1}{k}\log_e(R - kx) + c,\ R - kx > 0$$

When $t = 0,\ x = 0,$

$$\therefore\quad 0 = -\frac{1}{k}\log_e R + c$$
$$\therefore\quad c = \frac{1}{k}\log_e R$$
$$\therefore\quad t = -\frac{1}{k}\log_e(R - kx) + \frac{1}{k}\log_e R$$
$$= \frac{1}{k}\log_e\left(\frac{R}{R - kx}\right)$$
$$\therefore\quad kt = \log_e\left(\frac{R}{R - kx}\right)$$
$$\therefore\quad e^{kt} = \frac{R}{R - kx}$$
$$\therefore\quad (R - kx)e^{kt} = R$$
$$\therefore\quad kxe^{kt} = R(e^{kt} - 1)$$
$$\therefore\quad x = \frac{R(e^{kt} - 1)}{ke^{kt}}$$
$$= \frac{R}{k}(1 - e^{-kt})$$

b **i** If $R = 50$ and $k = 0.05$,

$$x = \frac{50}{0.05}(1 - e^{0.05t})$$
$$= 1000\left(1 - e^{-\frac{t}{20}}\right)$$



ii $t = \frac{1}{k}\log_e\left(\frac{R}{R-kx}\right)$

When $R = 50$ and $k = 0.05$,

$$t = 20\log_e\left(\frac{50}{50-0.05x}\right)$$

$$= 20\log_e\left(\frac{1000}{1000-x}\right)$$

When $x = 200, t = 20\log_e\left(\frac{1000}{1000-200}\right)$

$$= 20\log_e\frac{5}{4} = 4.4628\ldots$$

There are 200 mg of the dmg in the patient after 4.46 hours, correct to two decimal places.

c i When $t > 20\log_e\frac{5}{4}$, $\frac{dx}{dt} = -kx$ and $k = 0.05 = \frac{1}{20}$,

$$\therefore \quad \frac{dx}{dt} = \frac{-x}{20}$$

$$\therefore \quad \frac{dt}{dx} = \frac{-20}{x}$$

$$\therefore \quad t = \int \frac{-20}{x}dx$$

$$= -20\log_e x + c, \ x > 0$$

When $t = 20\log_e\frac{5}{4}$, $x = 200$,

$$\therefore \quad 20\log_e\frac{5}{4} = -20\log_e 200 + c$$

$$\therefore \quad c = 20\log_e\frac{5}{4} + 20\log_e 200$$

$$\therefore \quad t = 20\log_e\frac{5}{4} + 20\log_e 200 - 20\log_e x$$

$$= 20\log_e\frac{250}{x}$$

When $x = 100$, $t = 20\log_e\frac{5}{2}$

$$= 18.325\,81\ldots$$

The amount of dmg falls to 100 mg after 18.33 hours, correct to two decimal places, a further 13.86 hours after the drip was disconnected.

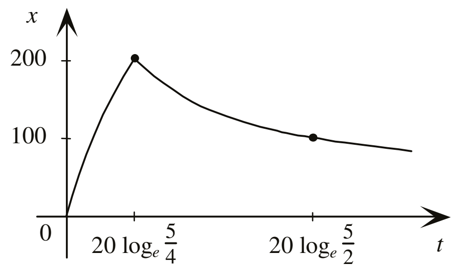
ii $t = 20\log_e \dfrac{250}{x}$

$\therefore \quad \dfrac{t}{20} = \log_e \dfrac{250}{x}$

$\therefore \quad e^{\frac{t}{20}} = \dfrac{250}{x}$

$\therefore \quad x = 250e^{-\frac{t}{20}}$

$$\therefore \quad x = \begin{cases} 1000\left(1 - e^{-\frac{t}{20}}\right) & 0 \le t \le 20\log_e \dfrac{5}{4} \\ 250\, e^{-\frac{t}{20}} & t > 20\log_e \dfrac{5}{4} \end{cases}$$



Chapter 12 – Kinematics

Solutions to Exercise 12A

1 $x = 3t - t^2$

a

t	0	1	2	3	4
x	0	2	2	0	−4

Need to find the times when velocity = 0.

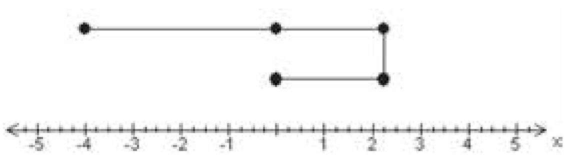
$v = \dfrac{dx}{dt} = 3 - 2t$

So,

$3 - 2t = 0 \Rightarrow t = \dfrac{3}{2} = 1.5$ s

$\therefore$ the particle is at rest after 1.5 seconds and this occurs when $x = 2.25$ m

Hence, the motion of the particle can be illustrated by:



b When $t = 5$, $x = 15 - 25 = -10$

$\therefore$ displacement $= -10 - (-4)$

$= -6$ m

c average velocity $= \dfrac{-4 - (0)}{4 - 0}$

$= -1$ m/s

d $v = \dfrac{dx}{dt} = 3 - 2t$

e When $t = 2.5$, $v = 3 - 2(2.5)$

$= -2$ m/s

f Particle changes direction when $v = 0$.

From part **a** this occurs when $t = \dfrac{3}{2}$ s and where $x = \dfrac{9}{4}$ m from O.

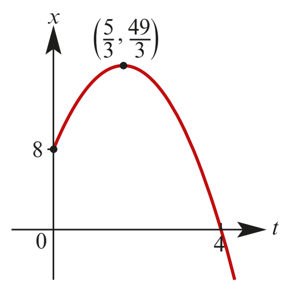
g Distance travelled

$= 2.25 + 2.25 + 4 = 8.5 = \dfrac{17}{2}$ m

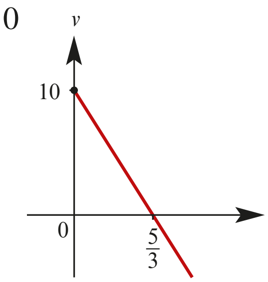
h Average speed $= \dfrac{\text{distance travelled}}{t_2 - t_1}$

$= \dfrac{\frac{17}{2}}{4} = \dfrac{17}{8}$ m/s

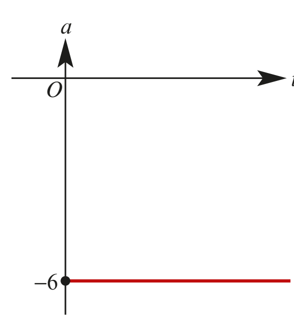
2 a

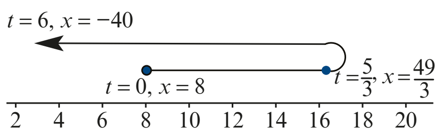


b $v(t) = -6t + 10$



c $a(t) = -6$



d 

e $x(3) - x(2) = 11 - 16 = -5$

f Changes direction when $x = \dfrac{5}{3}$

$x(0) = 8\ x\left(\dfrac{5}{3}\right) = \dfrac{49}{3}$

$x(3) = 11$

Therefore distance travelled

$= \dfrac{49}{3} - 8 + \dfrac{49}{3} - 11 = \dfrac{41}{3}$

3 $x = t^3 - 9t^2 + 24t$

a Instantaneously at rest when $v = 0$.

$v = \dfrac{dx}{dt} = 3t^2 - 18t + 24$

So,

$3t^2 - 18t + 24 = 0$

$\therefore\ t^2 - 6t + 8 = 0$

$\therefore\ (t - 4)(t - 2) = 0$

$\therefore\ t = 2$ s or $t = 4$ s

b $a = \dfrac{dv}{dt} = 6t - 18$

$\therefore$ when $t = 5,\ a = 12$ m/s^2

c Average velocity $= \dfrac{20 - 0}{2 - 0} = 10$ m/s

d

t	0	1	2	3	4
x	0	16	20	18	16

Average speed $= \dfrac{20 + 4}{4 - 0} = \dfrac{24}{4} = 6$ m/s

4 $x = t(t - 3)^2$

a Using the product rule to differentiate:

$v = \dfrac{dx}{dt} = 1 \times (t - 3)^2$

$+ t \times 2(t - 3)$

$\therefore\ v = (t - 3)^2 + 2t(t - 3)$

$\therefore\ v = (t - 3)(3t - 3)$

$\therefore$ when $t = 2,\ v = -3$ m/s

b Instantaneously at rest when $v = 0$.

$\therefore\ (t - 3)(3t - 3) = 0$

$\therefore\ t = 1$ s or $t = 3$ s

c $a = \dfrac{dv}{dt} = 1 \times (3t - 3) + (t - 3) \times 3$

$\therefore\ a = 6t - 12$

$\therefore$ when $t = 4,\ a = 12$ m/s^2

Using CAS:

5 $x = 2t^3 - 4t^2 - 100$

$v = \dfrac{dx}{dt} = 6t^2 - 8t$

Zero velocity when $v = 0$:

$\therefore\ 6t^2 - 8t = 0$

$\therefore\ 2t(3t - 4) = 0$

$\therefore\ t = 0$ s or $t = \dfrac{4}{3}$ s

6 $v = 4 + 3t - t^2$

a Maximum value of velocity requires the equation $v' = 0$ to be solved.

$v' = 3 - 2t$

Solving $v' = 0$ gives $t = \dfrac{3}{2}$ s

$\Rightarrow$ maximum velocity occurs when $t = \dfrac{3}{2}$ s

$\therefore\ v = 4 + 3\left(\dfrac{3}{2}\right) - \left(\dfrac{3}{2}\right)^2 = \dfrac{25}{4}$ m/s is the maximum velocity

b $x = \int v\, dt$

$\therefore\ x = \int 4 + 3t - t^2\, dt$

$\therefore\ x = 4t + \frac{3}{2}t^2 - \frac{1}{3}t^3 + c$

Passes through $(0, 0) \Rightarrow c = 0$

$\therefore\ x = 4t + \frac{3}{2}t^2 - \frac{1}{3}t^3$

when $t = 4,\ x = \frac{56}{3}$ m

7 $v = 3t^2 - 30t + 72$

a $a = \frac{dv}{dt} = 6t - 30$

when $t = 0,\ a = -30\ \text{m/s}^2$

b Instantaneously at rest when $v = 0$.

$\therefore\ 3t^2 - 30t + 72 = 0$

$\therefore\ 3(t^2 - 10t + 24) = 0$

$\therefore\ 3(t - 6)(t - 4) = 0$

$\therefore\ t = 4$ s or $t = 6$ s

c $x = \int 3t^2 - 30t + 72$

$dt = t^3 - 15t^2 + 72t + c$

Passes through $(0, 0) \Rightarrow c = 0$

$\therefore\ x = t^3 - 15t^2 + 72t$

when $t = 6,\ x = 108$ m

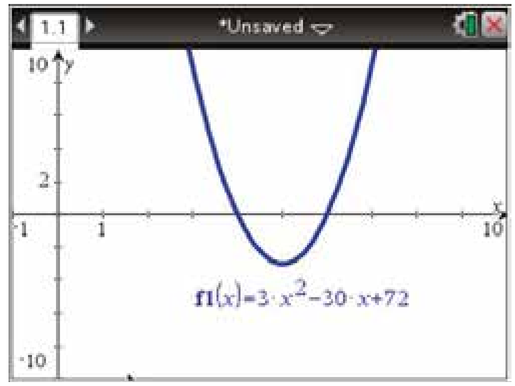
when $t = 4,\ x = 112$ m

$\therefore$ distance travelled $= 112 - 108$

$= 4$ m

Alternatively, distance travelled is equal to the area under the velocity-time graph

The graph of $v = 3t^2 - 30t + 72$ is



Since the area under the curve between $t = 4$ and $t = 6$ is **under** the t-axis,

Distance travelled

$= -\int_4^6 3t^2 - 30t + 72\, dt$

$= -[t^3 - 15t^2 + 72t]_4^6$

$= -[(216 - 540 + 432)$

$\quad - (64 - 240 + 288)]$

$= -[108 - 112]$

$= 4$ m

d when $t = 0,\ x = 0$ m

when $t = 4,\ x = 112$ m

when $t = 6,\ x = 108$ m

when $t = 7,\ x = 112$ m

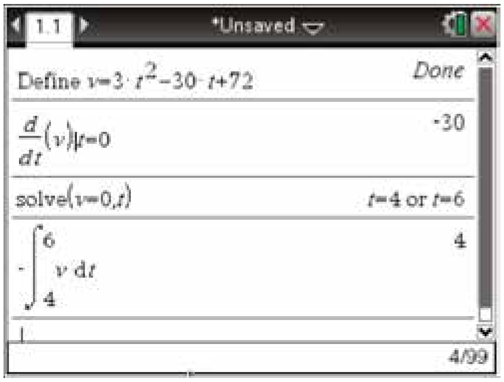
$\therefore$ distance travelled $= 112 + 4 + 4$

$= 120$ m

Alternatively, this distance can be calculated by evaluating the following definite integral:

$\int_0^4 3t^2 - 30t + 72\, dt - \int_4^6 3t^2 - 30t + 72\, dt + \int_6^7 3t^2 - 30t + 72\, dt$

Using CAS for question **6** parts **a**, **b** and **c** we have:



8 $a = 12 - 6t$

a $a = 12 - 6t$, and when $t = 0,\ v = 8$.

$$v = \int a\,dt$$

$$\therefore\ v = \int 12 - 6t\,dt$$

$$\therefore\ v = 12t - 3t^2 + c$$

when $t = 0,\ v = 8$:

$$\Rightarrow c = 8$$

$$\therefore\ v = 12t - 3t^2 + 8$$

$\therefore$ when $t = 2,\ v = 20$ m/s

b

$$x = \int v\,dt$$

$$\therefore\ x = \int 12t - 3t^2 + 8\,dt$$

$$\therefore\ x = 6t^2 - t^3 + 8t + c$$

when $t = 0,\ x = 0$:

$$\Rightarrow c = 0$$

$$\therefore\ x = 6t^2 - t^3 + 8t$$

$\therefore$ when $t = 2,\ x = 32$ m

9 $a = 13 - 6t$

a

$$v = \int a\,dt$$

$$\therefore\ v = \int 13 - 6t\,dt$$

$$\therefore\ v = 13t - 3t^2 + c$$

when $t = 0,\ v = 30$:

$$\Rightarrow c = 30$$

$$\therefore\ v = 13t - 3t^2 + 30$$

$\therefore$ when $t = 3,\ v = 42$ m/s

b For maximum distance solve

$x' = 0 \leftrightarrow v = 0$

$$\therefore\ -3t^2 + 13t + 30 = 0$$

$$\therefore\ -(t - 6)(3t + 5) = 0$$

$$\therefore\ t = -\frac{5}{3}\text{ s or } t = 6\text{ s}$$

$$\therefore\ t = 6\text{ s} \qquad \because t \geq 0$$

c

$$x = \int v\,dt$$

$$\therefore\ x = \int -3t^2 + 13t + 30\,dt$$

$$\therefore\ x = -t^3 + \frac{13}{2}t^2 + 30t + c$$

when $t = 0,\ x = 0$:

$$\Rightarrow c = 0$$

$$\therefore\ x = -t^3 + \frac{13}{2}t^2 + 30t$$

$\therefore$ when $t = 6,\ x = 198$ m

10 $a = 9.8 \text{ m/s}^2$

a **i** $v = \int a \, dt$

$\therefore \; v = \int 9.8 \, dt$

$\therefore \; v = 9.8t + c$

Initially, the object is at rest

$\Rightarrow$ when $t = 0, \; v = 0$:

$\therefore \; c = 0$

$\therefore \; v = 9.8t$

ii $x = \int v \, dt$

$\therefore \; x = \int 9.8t \, dt$

$\therefore \; x = 4.9t^2 + c$

Initially, the object starts from O

$\Rightarrow$ when $t = 0, \; x = 0$:

$\therefore \; c = 0$

$\therefore \; x = 4.9t^2$

b The object takes two seconds to reach the bottom.

$\therefore$ when $t = 2, \; x = 4.9 \times 4 = 19.6$ m

Hence, the depth of the well is 19.6 m

c when $t = 2, \; v = 9.8 \times 2 = 19.6$ m/s

speed $= |19.6| = 19.6$ m/s

11 $v = \cos\left(\frac{1}{2}t\right), \; t \in [0, 4\pi]$

a $x = \int \cos\left(\frac{1}{2}t\right) dt$

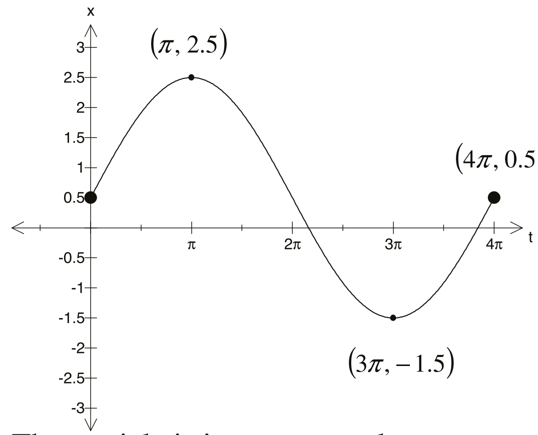
$\therefore \; x = 2\sin\left(\frac{1}{2}t\right) + c$

when $t = 0, \; x = 0.5$:

$\Rightarrow c = 0.5$

$\therefore \; x = 2\sin\left(\frac{1}{2}t\right) + 0.5$

b



The particle is instantaneously at rest at $t = \pi$ and $t = 3\pi$.

c $a = \frac{dv}{dt}$

$\therefore \; a = -\frac{1}{2}\sin\left(\frac{1}{2}t\right)$

d **i** We have: $x = 2\sin\left(\frac{1}{2}t\right) + 0.5$ and

$a = -\frac{1}{2}\sin\left(\frac{1}{2}t\right)$

$\Rightarrow x = -4a + 0.5$

ii We have: $x = 2\sin\left(\frac{1}{2}t\right) + 0.5$ and

$v = \cos\left(\frac{1}{2}t\right)$

Using the Pythagorean identity we have:

$$\cos^2\left(\frac{1}{2}t\right) + \sin^2\left(\frac{1}{2}t\right) = 1$$

$$\therefore\ v^2 + \sin^2\left(\frac{1}{2}t\right) = 1$$

$$\therefore\ \sin^2\left(\frac{1}{2}t\right) = 1 - v^2$$

$$\therefore\ \sin\left(\frac{1}{2}t\right) = \pm\sqrt{1 - v^2}$$

$$\therefore\ 2\sin\left(\frac{1}{2}t\right) = \pm 2\sqrt{1 - v^2}$$

$$\therefore\ 2\sin\left(\frac{1}{2}t\right) + 0.5 = \pm 2\sqrt{1 - v^2} + 0.5$$

$$\therefore\ x = \pm 2\sqrt{1 - v^2} + 0.5$$

iii We have: $v = \cos\left(\frac{1}{2}t\right)$ and

$$a = -\frac{1}{2}\sin\left(\frac{1}{2}t\right)$$

$$\therefore\ a^2 = \frac{1}{4}\sin^2\left(\frac{1}{2}t\right)$$

$$\therefore\ 4a^2 = \sin^2\left(\frac{1}{2}t\right)$$

Using the Pythagorean identity we have:

$$\cos^2\left(\frac{1}{2}t\right) + \sin^2\left(\frac{1}{2}t\right) = 1$$

$$\therefore\ v^2 = 1 - \sin^2\left(\frac{1}{2}t\right)$$

$$\therefore\ v^2 = 1 - 4a^2$$

$$\therefore\ v = \pm\sqrt{1 - 4a^2}$$

12 $x = t^3 - \frac{15}{2}t^2 + 12t + 10$

a $v = \frac{dx}{dt} = 3t^2 - 15t + 12$

Solving $v = 0$ gives:

$$3t^2 - 15t = 12 = 0$$

$$\therefore\ 3(t - 4)(t - 1) = 0$$

$$\therefore\ t = 1 \text{ s and } t = 4 \text{ s}$$

when $t = 1,\ x = 15.5$ m

when $t = 4,\ x = 2$ m

b when $t = 2,\ x = 12$ m

when $t = 3,\ x = 5.5$ m

$$\therefore\ \text{average velocity} = \frac{5.5 - 12}{3 - 2} = -6.5 \text{ m/s}$$

c when $t = 2,\ v = 3(2)^2 - 15(2) + 12$

$= -6$ m/s

d

t	0	1	2
x	10	15.5	12

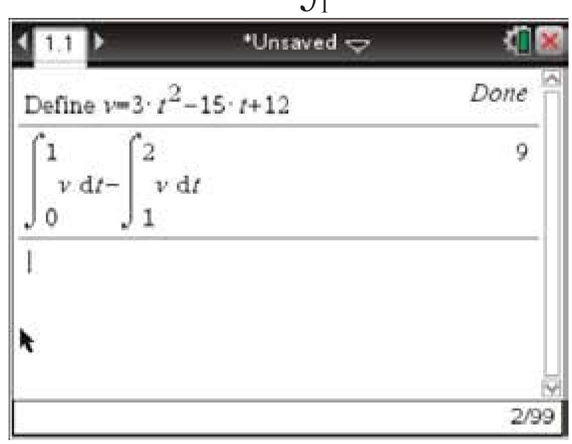
$\therefore$ distance travelled $= 5.5 + 3.5$

$= 9$ m

Alternatively, distance travelled can also be calculated by determining the area under the velocity-time graph between $t = 0$ and $t = 2$.

i.e.

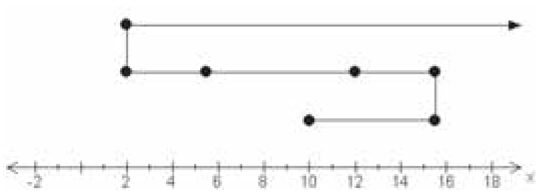
$$\text{distance travelled} = \int_0^1 3t^2 - 15t + 12\,dt - \int_1^2 3t^2 - 15t + 12\,dt$$



e Using the table below, a sketch of the particle's motion can be produced.

t	0	1	2
x	10	15.5	12

t	3	4	$t \to \infty$
x	5.5	2	$x \to \infty$



Hence, the closest the particle comes to O is 2 metres.

13 $\ddot{x} = 2\sin\left(\frac{1}{2}t\right)$

a $v = \dot{x} = \int \ddot{x}\, dt$

$\therefore\ v = \int 2\sin\left(\frac{1}{2}t\right) dt$

$\therefore\ v = -4\cos\left(\frac{1}{2}t\right) + c$

when $t = 0,\ v = 1$:

$\Rightarrow c = 5$

$\therefore\ v = -4\cos\left(\frac{1}{2}t\right) + 5$

The range of the velocity function is [1, 9], hence the maximum velocity is 9 m/s.

b Time taken to reach maximum velocity occurs when $v = 9$.

$\therefore\ -4\cos\left(\frac{1}{2}t\right) + 5 = 9$

$\therefore\ \cos\left(\frac{1}{2}t\right) = -1$

$\therefore\ \frac{1}{2}t = (2k+1)\pi,\ k \in Z$

$\therefore\ t = 2(2k+1)\pi,\ k \in Z$

The first time the velocity reaches a maximum will occur when $k = 0$.

$\therefore$ the velocity reaches a maximum when $t = 2\pi$ s

14 a Take upwards as positive.

Before the stone was dropped it followed the motion of the balloon, i.e. it moved upwards with a speed of 10 m/s. $\therefore$ at the moment when it was dropped $v = +10$.

The acceleration of the stone was due to gravity (downward) and

$\therefore$ the acceleration is downward,

$\therefore\ a = -9.8$ m/s^2.

$v = \int -9.8\, dt$

$\therefore\ v = -9.8t + c.$

when $t = 0,\ v = +10$:

$\Rightarrow c = 10$

$\therefore\ v = -9.8t + 10.$

$x = \int v\, dt$

$\therefore\ x = \int -9.8t + 10\, dt$

$\therefore\ x = -4.9t^2 + 10t + d$

When $t = 0,\ x = 0$: i.e. the starting point was where the stone was dropped.

$\Rightarrow d = 0$

$\therefore\ x = -4.9t^2 + 10t$

when $t = 12,\ x = -4.9(12)^2 + 10(12)$

$= -585.6$ m.

So the stone was 585.6 m **below** (− sign) its starting point.

Hence. the height of the balloon when the stone was dropped was 585.6 m.

b When the stone reached its highest point the velocity must equal zero.

$\therefore\ -9.8t + 10 = 0$

$$\therefore\ t = \frac{50}{49}$$

When

$$t = \frac{50}{49},\ x = -4.9\left(\frac{50}{49}\right)^2 + 10\left(\frac{50}{49}\right)$$

$$= \frac{250}{49}$$

$$\approx 5.1 \text{ m}$$

i.e. the stone was 5.1 m **above** (+ sign) its starting point.

Hence the greatest height reached by the stone $= 585.6 + 5.1 = 590.7$ m.

15

$$\ddot{x} = \frac{1}{(2t+3)^2} = (2t+3)^{-2}$$

$$\dot{x} = \int \ddot{x}\, dt$$

$$\therefore\ \dot{x} = \int (2t+3)^{-2}\, dt$$

$$\therefore\ \dot{x} = \frac{(2t+3)^{-1}}{-1 \times 2} + c$$

$$\therefore\ \dot{x} = -\frac{1}{2(2t+3)} + c$$

when $t = 0, \dot{x}$ (or v) $= 0$:

$$\Rightarrow c = \frac{1}{6}$$

$$\therefore\ \dot{x} = -\frac{1}{2(2t+3)} + \frac{1}{6}$$

and so,

$$x = \int \dot{x}\, dt$$

$$\therefore\ x = \int -\frac{1}{2(2t+3)} + \frac{1}{6}\, dt$$

$$\therefore\ x = -\frac{1}{2}\int \frac{1}{2t+3}\, dt + \int \frac{1}{6}\, dt$$

$$\therefore\ x = -\frac{1}{4}\log_e(2t+3) + \frac{1}{6}t + c$$

when $t = 0,\ x = 0$:

$$\Rightarrow c = \frac{1}{4}\log_e(3)$$

$$\therefore\ x = -\frac{1}{4}\log_e(2t+3) + \frac{1}{6}t + \frac{1}{4}\log_e(3)$$

$$\therefore\ x = \frac{1}{4}\log_e\left(\frac{3}{2t+3}\right) + \frac{1}{6}t$$

$$\therefore\ x = -\frac{1}{4}\log_e\left(\frac{2t+3}{3}\right) + \frac{1}{6}t$$

16

$$\ddot{x} = \frac{2t}{(1+t^2)^2}$$

$$\therefore\ \dot{x} = \int \frac{2t}{(1+t^2)^2}\, dt$$

Let $u = 1 + t^2,\ \Rightarrow \dfrac{du}{dx} = 2t$

Hence,

$$\dot{x} = \int \frac{2t}{(1+t^2)^2}\, dt = \int \frac{1}{u^2}\, du$$

$$\therefore\ \dot{x} = -\frac{1}{u} + c$$

$$\therefore\ \dot{x} = -\frac{1}{1+t^2} + c$$

when $t = 0, \dot{x} = 0.5$:

$$\Rightarrow c = 1.5$$

$$\therefore\ \dot{x} = -\frac{1}{1+t^2} + \frac{3}{2}$$

and so,

$$x = \int -\frac{1}{1+t^2} + \frac{3}{2}\,dt$$

$$\therefore\ x = -\tan^{-1}(t) + \frac{3}{2}t + c$$

when $t = 0,\ x = 0$:

$$\Rightarrow c = 0$$

$$\therefore\ x = -\tan^{-1}(t) + \frac{3}{2}t$$

$\dot{x} \neq 0$ for $t \geq 0$ and when
$t = 0,\ x = 0$, and when

$$t = \sqrt{3},\ x = -\tan^{-1}(\sqrt{3}) + \frac{3\sqrt{3}}{2}$$

$$\therefore\ x = -\frac{\pi}{3} + \frac{3\sqrt{3}}{2}$$

$$\therefore\ \text{distance travelled} = \left(\frac{3\sqrt{3}}{2} - \frac{\pi}{3}\right)\text{ m}$$

Alternatively, distance travelled is equal to the **area** under the velocity-time graph.

17 $\dot{x} = \dfrac{t}{(1+t^2)}$

a when $t = 0, \dot{x} = 0$ m/s

b Maximum velocity occurs when
$\ddot{x} = 0$

$$\therefore\ \ddot{x} = \frac{(1+t^2)\times 1 - t\times 2t}{(1+t^2)^2}$$

$$\therefore\ \ddot{x} = \frac{1+t^2-2t^2}{(1+t^2)^2}$$

$$\therefore\ \ddot{x} = \frac{1-t^2}{(1+t^2)^2}$$

So,

$$\therefore\ \frac{1-t^2}{(1+t^2)^2} = 0$$

$$\therefore\ 1 - t^2 = 0$$

$$\therefore\ t^2 = 1$$

$$\therefore\ t = \pm 1$$

$$\therefore\ t = 1 \qquad \because t \geq 0$$

and when $t = 1, \dot{x} = \frac{1}{2}$ m/s

$\therefore$ maximum velocity is $\frac{1}{2}$ m/s

c A CAS calculator can be used to determine the distance travelled in the third second by evaluating

$$\int_2^3 \frac{t}{1+t^2}\,dt$$

Hence, distance travelled in the third second $= \frac{1}{2}\log_e(2)$ m

Alternatively, using the answer to part **d** we have;

when $t = 2,\ x = \frac{1}{2}\log_e(5)$

when $t = 3,\ x = \frac{1}{2}\log_e(10)$

$$\therefore\ \text{distance travelled} = \frac{1}{2}\log_e(10) - \frac{1}{2}\log_e(5) = \frac{1}{2}\log_e(2)\text{ m}$$

d Let $u = 1 + t^2, \Rightarrow \dfrac{du}{dx} = 2t$

$$\therefore\ t = \frac{1}{2}\frac{du}{dx}$$

$$x = \int \dot{x}\, dt$$

$$\therefore\ x = \frac{1}{2}\int \frac{1}{u}\, du$$

$$\therefore\ x = \frac{1}{2}\log_e(u) + c$$

$$\therefore\ x = \frac{1}{2}\log_e(1 + t^2) + c$$

When $t = 0,\ x = 0$:

$$\Rightarrow c = 0$$

$$\therefore\ x = \frac{1}{2}\log_e(1 + t^2)$$

e As calculated in part **b**

$$\therefore\ \ddot{x} = \frac{(1 + t^2) \times 1 - t \times 2t}{(1 + t^2)^2}$$

$$\therefore\ \ddot{x} = \frac{1 - t^2}{(1 + t^2)^2}$$

f when $t = 2, \dot{x} = \dfrac{2}{5}$ m/s

when $t = 3, \dot{x} = \dfrac{3}{10}$ m/s

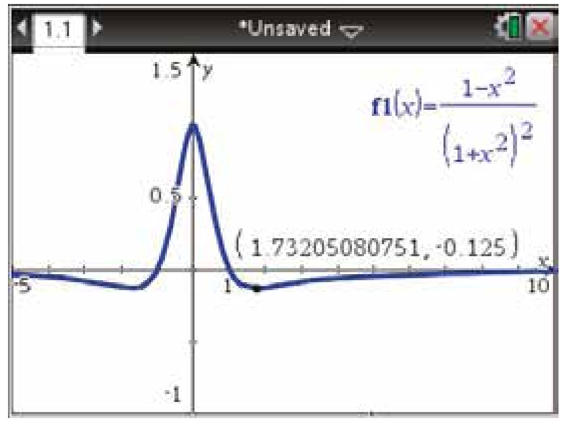
$\therefore$ average acceleration

$$= \frac{\frac{3}{10} - \frac{2}{5}}{3 - 2} = -\frac{1}{10} = -0.1 \text{ m/s}^2$$

g The minimum acceleration can be found by solving $\dddot{x} = 0$ by hand, using a CAS calculator to solve $\dddot{x} = 0$ or a graphical approach.

1. Graphically

Sketch the graph of $\ddot{x}$ and find the minimum.



$$\therefore\ \text{Minimum acceleration} = -0.125 = -\frac{1}{8} \text{ m/s}^2$$

2. By hand

If $\ddot{x} = \dfrac{1 - t^2}{(1 + t^2)^2}$ then,

$$\dddot{x} = \frac{(1 + t^2)^2 \times -2t - (1 - t^2) \times 2(1 + t^2) \times 2t}{(1 + t^2)^4}$$

$$\therefore\ \dddot{x} = \frac{-2t(1 + t^2)^2 - 4t(1 - t^2)(1 + t^2)}{(1 + t^2)^4}$$

$$\therefore\ \dddot{x} = \frac{-2t(1 + t^2) - 4t(1 - t^2)}{(1 + t^2)^3}$$

For minimum solve $\dddot{x} = 0$:

$$\therefore\ \frac{-2t(1 + t^2) - 4t(1 - t^2)}{(1 + t^2)^3} = 0$$

$$\therefore\ -2t(1 + t^2) - 4t(1 - t^2) = 0$$

$$\therefore\ -2t[(1 + t^2) + 2(1 - t^2)] = 0$$

$$\therefore\ -2t(-t^2 + 3) = 0$$

$$\therefore\ -2t = 0 \text{ or}$$

$$-t^2 + 3 = 0$$

$$\therefore\ t = 0 \text{ or}$$

$$t = \pm\sqrt{3}$$

Since we are concerned with $t \geq 0$, we must examine the points when $t = 0$ and $t = \sqrt{3}$. i.e. check which t value gives a minimum.

when $t = 0$:

t	**−1**	**0**	**1**
$\dddot{x}$	0.5	0	−0.5
Slope	/	–	\

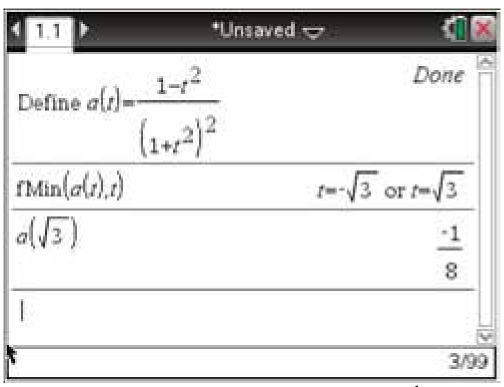
Hence when $t = 0$, acceleration is a maximum.
when $t = \sqrt{3}$:

t	**0.5**	$\sqrt{3}$	**2**
$\ddot{x}$	−1.408	0	0.032
Slope	\	–	/

Hence when $t = \sqrt{3}$, acceleration is a minimum.
So the minimum acceleration can be found by substituting $t = \sqrt{3}$ into $\ddot{x}$

$\therefore$ Minimum acceleration $= \dfrac{1-3}{(1+3)^2} = \dfrac{-2}{16} = -\dfrac{1}{8}$ m/s^2

3. CAS
The **fMin** command.



$\therefore$ Minimum acceleration $= -\dfrac{1}{8}$ m/s^2

18

$$x = 2 + \sqrt{t+1}$$
$$= 2 + (t+1)^{\frac{1}{2}}$$
$$v = \frac{dx}{dt}$$
$$\therefore\ v = \frac{1}{2}(t+1)^{-\frac{1}{2}}$$
$$a = \frac{dv}{dt}$$
$$\therefore\ a = -\frac{1}{4}(t+1)^{-\frac{3}{2}}$$

When $a = -0.016$ m/s^2 :

$$-\frac{1}{4}(t+1)^{-\frac{3}{2}} = -0.016$$
$$\therefore\ (t+1)^{-\frac{3}{2}} = \frac{8}{125}$$
$$\therefore\ \left[(t+1)^{-\frac{3}{2}}\right]^{-\frac{2}{3}} = \left(\frac{8}{125}\right)^{-\frac{2}{3}}$$
$$\therefore\ t+1 = \left(\frac{125}{8}\right)^{\frac{2}{3}}$$
$$\therefore\ t+1 = \left(\frac{\sqrt[3]{125}}{\sqrt[3]{8}}\right)^2$$
$$\therefore\ t+1 = \left(\frac{5}{2}\right)^2$$
$$\therefore\ t+1 = \frac{25}{4}$$
$$\therefore\ t = \frac{21}{4} = 5.25 \text{ s}$$

19 $x = 2\sin t + \cos t,\ t \geq 0$
Instantaneously at rest when $v = 0$.

$$v = \frac{dx}{dt}$$

$$\therefore\ v = 2\cos t - \sin t$$

For $v = 0$:

$$2\cos t - \sin t = 0$$

$$\therefore\ 2\cos t = \sin t$$

$$\therefore\ 2 = \frac{\sin t}{\cos t}$$

$$\therefore\ \tan t = 2$$

$$\therefore\ t = \tan^{-1}(2)$$

$$\therefore\ t = 1.1 \text{ s}$$

(using a calculator to obtain answer)

20 $$\frac{d^2x}{dt^2} = 8 - e^{-t}$$

$$v = \frac{dx}{dt} = \int \frac{d^2x}{dt^2}\, dt$$

$$\therefore\ v = \int 8 - e^{-t}\, dt$$

$$\therefore\ v = 8t + e^{-t} + c$$

when $t = 0,\ v = 3$:

$$\Rightarrow c = 2$$

$$\therefore\ v = 8t + e^{-t} + 2$$

and when $t = 2,\ v = 18 + \frac{1}{e^2} = 18.14 \text{ m/s}$

Solutions to Exercise 12B

1 $u = 15$ m/s, $v = 48$ m/s, $t = 11$ s

Using $v = u + at$ we have:

$$48 = 15 + 11a$$

$$\therefore\ 11a = 33$$

$$\therefore\ a = 3 \text{ m/s}^2$$

2 a $u = 5$ km/h, $v = 41$ km/h, $t = 10$ s

Units must be compatible

$$\Rightarrow t = 10 \text{ s} = \frac{10}{60 \times 60} \text{ h} = \frac{1}{360} \text{ h}$$

Using $v = u + at$ we have:

$$41 = 5 + \frac{1}{360}a$$

$$\therefore\ a = 36 \times 360$$

$$\therefore\ a = 12\,960 \text{ km/h}^2$$

b $$12\,960 \text{ km/h}^2 = 12\,960 \times \frac{1}{12\,960} \text{ m/s}^2 = 1 \text{ m/s}^2$$

$$\therefore\ a = 1 \text{ m/s}^2$$

3 a $v = 25, u = 10, t = 5$

$$v = u + at$$

$$\therefore 25 = 10 + 5a$$

$$a = 3$$

The acceleration is 3 m/s^2

b $v = 35, u = 10, t = 5$

$$s = ut + \frac{1}{2}at^2$$

$$\therefore s = 87.5$$

The distance is 87.5 m

c Now $s = 100$

$$s = ut + \frac{1}{2}at^2$$

$$\therefore 100 = 25t + \frac{1}{2} \times 3 \times t^2$$

$$\therefore t = \frac{10(\sqrt{7} - 1)}{3}, t \geq 0$$

Time taken ≈ 5.485seconds

4 $u = 20, v = 0, s = 40$

Now $v^2 = u^2 + 2as$

$$0 = 20^2 + 80a$$

$$\therefore a = -\frac{20^2}{80}$$

$$= -5$$

Acceleration is -5 m/s^2

5 a $u = -10$ m/s, $a = 4$ m/s^2, $t = 6$ s

Using $s = ut + \frac{1}{2}at^2$ we have:

$s = -60 + 2(6)^2 = 12$ m

b Using $v = u + at$ we have:

$v = -10 + 24 = 14$ m/s

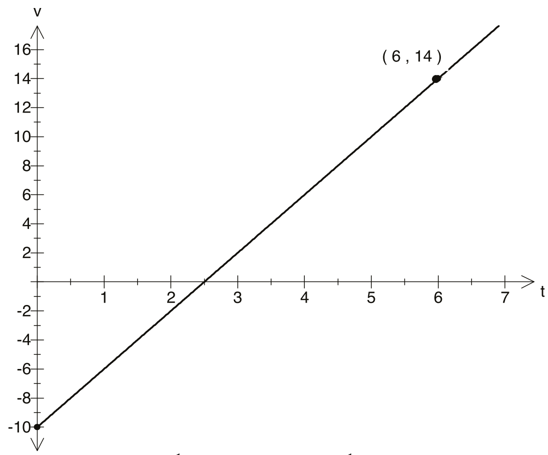
c Using $v = u + at$ we have:

$$0 = -10 + 4t$$

$$\therefore\ t = \frac{10}{4} = 2.5 \text{ s}$$

d $v = 4t - 10$

Sketching a velocity-time graph gives:



Distance $= \frac{1}{2}(2.5)(10) + \frac{1}{2}(3.5)(14)$

$= 37$ m

6 $t = 2$ s, $u = 21$ m/s, $a = -9.8$ m/s^2

a **i** Using $s = ut + \frac{1}{2}at^2$ we have:

$s = 21 \times 2 + \frac{1}{2} \times -9.8 \times 4$

$\therefore \quad s = 22.4$ m

ii $v = 21 - 9.8t$

Maximum height occurs when $v = 0$.

$\therefore \quad 0 = 21 - 9.8t$

$\therefore \quad t = \frac{15}{7}$ s

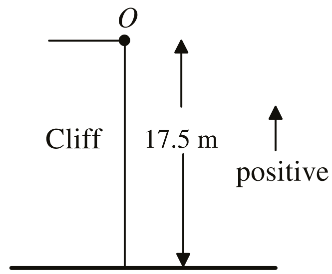
Hence the maximum height reached occurs when $t = \frac{15}{7}$ s

So,

$s = \frac{1}{2}(21 + 0)\frac{15}{7} = 22.5$ m

Therefore the maximum height reached by the stone is 22.5 m.

b **i**



Take the origin at the top of the cliff. $s = -17.5$, $u = 21$, $a = -9.8$

Using $s = ut + \frac{1}{2}at^2$ we have:

$-17.5 = 21t + \frac{1}{2} \times -9.8 \times t^2$

$\therefore \quad 4.9t^2 - 21t - 17.5 = 0$

Using the quadratic formula

$$t = \frac{21 \pm \sqrt{(-21)^2 - 4 \times 4.9 \times -17.5}}{2 \times 4.9}$$

$\therefore \quad t = -\frac{5}{7}$ or $t = 5$

But $t \geq 0$. Therefore, it takes 5 seconds for the stone to reach the bottom of the cliff.

ii Using $v = u + at$

$v = 21 - 9.8 \times 5 = -28$ m/s

Therefore, the stone has a velocity of -28 m/s when it hits the ground.

7 **a** $u = 14$, $a = -9.8$, $v = 0$

Using $v = u + at$

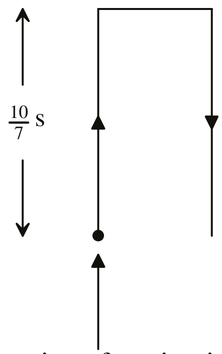
$0 = 14 - 9.8t$

$\therefore \quad t = \frac{10}{7}$ s

Therefore, it takes $\frac{10}{7}$ s for the ball to reach maximum height.

b $s = \frac{1}{2}(14 + 0)\frac{10}{7} = 10$ m

c



point of projection

Time to return to point of projection $= 2 \times \frac{10}{7} = \frac{20}{7}$ s

8 a $a = -0.1,\ u = 20,\ v = 0$

Using $v = u + at$

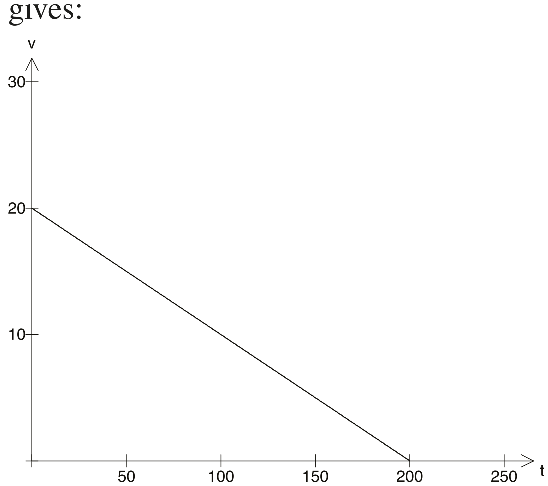
$0 = 20 - 0.1t$

$\therefore,\ \ t = 200$ s

Therefore, it takes 200 seconds for the particle to come to rest.

b $v = 20 - 0.1t$

Sketching a velocity-time graph gives:



distance $= \frac{1}{2}(200)(20) = 2000$ m $= 2$ km

9 a $s = 100,\ a = 9.8,\ u = 0$

Using $s = ut + \frac{1}{2}at^2$ we have:

$4.9t^2 = 100$

$\therefore\ t^2 = \frac{1000}{49}$

$\therefore\ \ t = \sqrt{\frac{1000}{49}}$ since $t \geq 0$

$\therefore\ \ t = \frac{10\sqrt{10}}{7}$

Hence, it takes $\frac{10\sqrt{10}}{7}$ seconds for the particle

b Using $v = u + at$

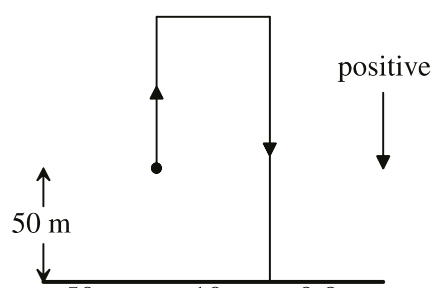
$v = 0 + 9.8\left(\frac{10\sqrt{10}}{7}\right)$

$\therefore\ \ v = \frac{49}{5}\left(\frac{10\sqrt{10}}{7}\right)$

$\therefore\ \ v = 14\sqrt{10}$ m/s

Therefore, the object has a velocity of $14\sqrt{10}$ m/s when it hits the ground.

10 a



$s = 50,\ u = -10,\ a = 9.8$

Using $s = ut + \frac{1}{2}at^2$ we have:

$50 = -10t + 4.9t^2$

$\therefore\ \ 4.9t^2 - 10t - 50 = 0$

Using the quadratic formula

$t = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 4.9 \times -50}}{2 \times 4.9}$

$\therefore\ \ t = \frac{50 \pm 30\sqrt{30}}{49}$

$\therefore\ \ t = -2.33$ or $t = 4.37$

But $t \geq 0$. Therefore, it takes 4.37 seconds for the object to reach the ground.

b Taking upwards as the positive direction.

$$t = \frac{50 + 30\sqrt{30}}{49}$$

Using $v = u + at$

$$v = 10 - 9.8\left(\frac{50 + 30\sqrt{30}}{49}\right)$$

$$\therefore\ v = 10 - \frac{49}{5}\left(\frac{50 + 30\sqrt{30}}{49}\right)$$

$$\therefore\ v = 10 - 6\sqrt{30} - 10$$

$$\therefore\ v = -6\sqrt{30}\ \text{m/s}$$

11 a $a = -0.8,\ u = 1,\ v = 0$

Using $v = u + at$

$$0 = 1 - 0.8t$$

$$\therefore\ t = \frac{5}{4}\ \text{s}$$

$$\therefore\ t = 1.25\ \text{s}$$

Hence it takes 1.25 seconds for the book to stop.

b Using $s = ut + \frac{1}{2}at^2$ we have:

$$s = \frac{1}{2}(1 + 0) \times 1.25$$

$$\therefore\ s = 0.625\ \text{m}$$

$$\therefore\ s = 62.5\ \text{cm}$$

12 a $u = 1.2,\ v = 0,$

$s = 3.2,\ a = -a$

Using $v^2 = u^2 + 2as$

$$\therefore\ 0 = (1.2)^2 + 2(-a)(3.2)$$

$$\therefore\ -\frac{36}{25} = -6.4a$$

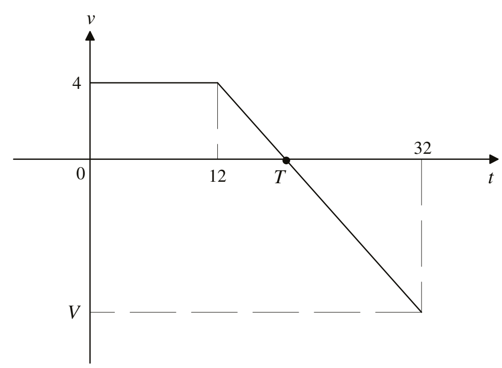
$$\therefore\ a = \frac{9}{40} \approx 0.23\ \text{m/s}^2$$

b Using $v = u + at$

$$0 = 1.2 - \frac{9}{40}t$$

$$\therefore\ t = \frac{16}{3} = 5\frac{1}{3}\ \text{s}$$

13 Sketch a velocity-time graph of the situation.



a Acceleration (gradient) $= \dfrac{\text{rise}}{\text{run}}$

$$\therefore\ -\frac{4}{T - 12} = \frac{V}{32 - T}$$

$$\therefore\ V = -\frac{4(32 - T)}{T - 12} \quad ①$$

After 32 seconds the particle has returned to its original position. This implies that after 32 seconds the displacement is zero.

Area between graph and t-axis $=$

$$\frac{1}{2}(12 + T) \times 4 + \frac{1}{2}(32 - T)V$$

$$\therefore\ 2(12+T)+\frac{1}{2}(32-T)V=0$$

$$\therefore\ (32-T)V=-4(12+T)$$

$$\therefore\ V=\frac{-4(12+T)}{32-T}$$

Equating equations:

$$-\frac{4(32-T)}{T-12}=-\frac{4(12+T)}{32-T}$$

$$\therefore\ (32-T)^2=(12+T)(T-12)$$

$$\therefore\ T^2-64T+1024=T^2-144$$

$$\therefore\ 64T=1168$$

$$\therefore\ T=\frac{73}{4}$$

$$\begin{aligned}\text{acceleration} &= -\frac{4}{T-12}\\ &= -\frac{4}{\frac{73}{4}-12}\\ &= -\frac{16}{25}\\ &= -0.64\end{aligned}$$

$$\therefore\ a=-0.64\ \text{m/s}^2$$

b Travelling back towards the original position from $t = T$ to $t = 32$.

Time taken $= 32 - T = \frac{55}{4}$ s.

②

14 a $u = 0,\ s = 4,\ v = 2$

using $s=\frac{1}{2}(u+v)t$

$$\therefore\ 4=\frac{1}{2}(0+2)t$$

$$\therefore\ t=4\ \text{s}$$

Hence it takes the child 4 seconds to go down the slide.

b Average acceleration $=\frac{v_2-v_1}{t_2-t_1}$

$$=\frac{2-0}{4-0}$$

$$=\frac{1}{2}\ \text{m/s}^2$$

Solutions to Exercise 12C

1 a i The particle travels with constant velocity of 6 m/s for 10 seconds.

ii distance = area under curve

$= 6 \times 10$

$= 60$ m

b i The particle accelerates uniformly for 5 seconds by which time it has reached 8 m/s.

ii distance $= \frac{1}{2}(5)(8) = 20$ m

c i The particle accelerates uniformly for 4 seconds by which time it has reached 6 m/s. It then decelerates uniformly until it comes to rest after 10 seconds.

ii distance $= \frac{1}{2}(10)(6) = 30$ m

d i The particle travels with constant velocity of 5 m/s for 7 seconds. It then decelerates uniformly until it comes to rest after 15 seconds.

ii distance $= 5 \times 7 + \frac{1}{2}(8)(5)$

$= 35 + 20$

$= 55$ m

e i The particle travels with constant velocity of 4 m/s for 6 seconds. It then decelerates uniformly until it comes to rest after 8 seconds before changing direction and continuing to decelerate uniformly for a further 4 seconds until it reaches a velocity of −8 m/s.

ii distance $= 4 \times 6 + \frac{1}{2}(2)(4)$

$+ \frac{1}{2}(4)(8)$

$= 24 + 4 + 16$

$= 44$ m

f i The particle accelerates uniformly for 1 second by which time it has reached 7 m/s. It then decelerates uniformly until it comes to rest after 2.5 seconds before changing direction and continuing to decelerate uniformly for a further 2.5 seconds until it reaches a velocity of $-11\frac{2}{3}$ m/s.

ii distance $= \frac{1}{2}(1)(7) + \frac{1}{2}\left(\frac{3}{2}\right)(7)$

$+ \frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{35}{3}\right)$

$= \frac{7}{2} + \frac{21}{4} + \frac{175}{12}$

$= \frac{70}{3}$ m

g i The particle travels with constant velocity of 10 m/s for 1 second. It then decelerates uniformly until it comes to rest after 3 seconds before changing direction and continuing to decelerate uniformly for a further 5 seconds until it reaches a velocity of −25 m/s.

ii $\text{distance} = \frac{1}{2}(1+3)\times 10$

$+ \frac{1}{2}(5)(25)$

$= 20 + \frac{125}{2}$

$\therefore \text{ distance } = \frac{165}{2}$ m

h **i** An object starting at −4 m/s accelerates uniformly until it comes to rest after 3 seconds before changing direction and continuing to accelerate uniformly for a further 3 seconds by which time it has reached 4 m/s.
The particle then decelerates uniformly until it comes to rest after 10 seconds before changing direction and continuing to decelerate uniformly for a further 3 seconds until it reaches a velocity of −3 m/s.

ii $\text{distance} = \frac{1}{2}(3)(14) + \frac{1}{2}(7)(4)$

$+ \frac{1}{2}(3)(3)$

$= 6 + 14 + \frac{9}{2}$

$= \frac{49}{2}$

$= 24.5$ m

2 **a** **i** By observation, the equation of the line is given by $v = -\frac{1}{2}t + 5$

ii $a = \frac{dv}{dt} = -\frac{1}{2}$

iii $x = \int -\frac{1}{2}t + 5\, dt$

$\therefore\ x = -\frac{1}{4}t^2 + 5t + c$
Passes through (0, 0)
$\Rightarrow c = 0$

$\therefore\ x = -\frac{1}{4}t^2 + 5t$

b **i** $v = at^2 + b$
Passes through (0, 10)
$\therefore\ v = at^2 + 10$
Passes through (5, 0)
$\therefore\ a = -\frac{2}{5}$

$\therefore\ v = -\frac{2}{5}t^2 + 10$

ii $a = \frac{dv}{dt} = -\frac{4}{5}t$

iii $x = \int -\frac{2}{5}t^2 + 10\, dt$

$\therefore\ x = -\frac{2}{15}t^3 + 10t + c$
Passes through (0, 0)
$\Rightarrow c = 0$

$\therefore\ x = -\frac{2}{15}t^3 + 10t$

c **i** By observation, the equation of the line is given by $v = 2t - 10$

ii $a = \frac{dv}{dt} = 2$

iii $x = \int 2t - 10\, dt$

$\therefore\ x = t^2 - 10t + c$
Passes through (0, 0)
$\Rightarrow c = 0$

$\therefore\ x = t^2 - 10t$

d **i** $v = at^2 + bt + c$
t-intercepts are 1 and 5.

$\therefore\ v = a(t-1)(t-5)$

Passes through (0, 30)

$\Rightarrow a = 6$

$\therefore\ v = 6(t-1)(t-5)$

$\therefore\ v = 6t^2 - 36t + 30$

ii $a = \dfrac{dv}{dt} = 12t - 36 = 12(t-3)$

iii $x = \displaystyle\int 6t^2 - 36t + 30\, dt$

$\therefore\ x = 2t^3 - 18t^2 + 30t + d$

Passes through (0, 0)

$\Rightarrow d = 0$

$\therefore\ x = 2t^3 - 18t^2 + 30t$

e i $v = a \sin bt + c$

amplitude = 10

$\therefore\ a = 10$

period = 20

$\Rightarrow \dfrac{2\pi}{b} = 20$

$\therefore\ b = \dfrac{\pi}{10}$

vertical shift upwards of 10 units

$\therefore\ c = 10$

$\therefore\ v = 10 \sin\left(\dfrac{\pi}{10}t\right) + 10$

ii $a = \dfrac{dv}{dt} = \pi \cos\left(\dfrac{\pi}{10}t\right)$

iii $x = \displaystyle\int 10 \sin\left(\dfrac{\pi}{10}t\right) + 10\, dt$

$\therefore\ x = -\dfrac{100}{\pi} \cos\left(\dfrac{\pi}{10}t\right) + 10t + d$

Passes through (0, 0)

$\Rightarrow d = \dfrac{100}{\pi}$

$\therefore\ x = -\dfrac{100}{\pi} \cos\left(\dfrac{\pi}{10}t\right)$
$+ 10t + \dfrac{100}{\pi}$

$\therefore\ x = 10\left(t + \dfrac{10}{\pi} - \dfrac{10}{\pi} \cos\left(\dfrac{\pi}{10}t\right)\right)$

f i $v = ae^{bt}$

Passes through (0, 10)

$\therefore\ a = 10$

Passes through $(\log_e 2, 40)$

$\therefore\ 40 = 10e^{(\log_e 2)^b}$

$\therefore\ 4 = 2^b$

$\therefore\ 2^2 = 2^b$

$\therefore\ b = 2$

$\therefore\ v = 10e^{2t}$

ii $a = \dfrac{dv}{dt} = 20e^{2t}$

iii $x = \displaystyle\int 10e^{2t}\, dt$

$\therefore\ x = 5e^{2t} + c$

Passes through (0, 0)

$\Rightarrow c = -5$

$\therefore\ x = 5e^{2t} - 5$

3 Distance travelled in the first 15 seconds
$= \dfrac{1}{2} \times 15 \times 100 \times \dfrac{5}{18} = \dfrac{625}{3}$ m

Distance travelled for the next 12 seconds $= 120 * \dfrac{5}{18} \times 100 = \dfrac{10\,000}{3}$ m

Using the formula $v = u + at$

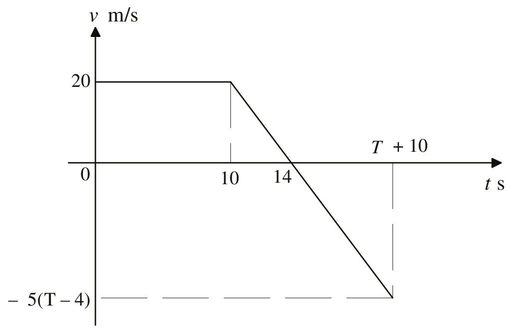
$0 = 120 \times \dfrac{5}{18} - 8t$

$t = \dfrac{25}{6}$

Therefore distance travelled in period of deceleration
$= \dfrac{1}{2} \times \dfrac{25}{6} \times 100 \times \dfrac{5}{18} = \dfrac{3124}{54}$ m

Total distance travelled ≈ 3599.2 m

4 a



b As the particle returns to its original position this implies that the forward displacement is equal to the backward displacement.

forward displacement

$= \text{backwards displacement}$

$= \frac{1}{2}(10 + 14) \times 20$

$= 240$ m

So,

$$s = -240,\ u = 0,\ a = -5,$$

$$t = T + 10 - 14 = T - 4$$

Using $s = u + \frac{1}{2}a\,t^2$

$\therefore\ -240 = \frac{1}{2}(-5)(T - 4)^2$

$\therefore\ 96 = (T - 4)^2$

$\therefore\ T - 4 = 4\sqrt{6}$

$\therefore\ T = 4\sqrt{6} + 4$

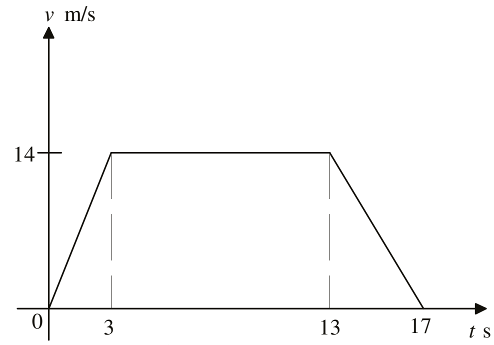
$\therefore\ T = 4(\sqrt{6} + 1)$

$\therefore\ T \approx 13.80$ s

But we want $T + 10$

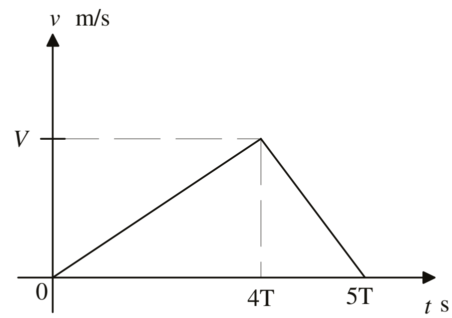
$\therefore$ It takes 23.80 seconds for the particle to return to its original position.

5 The velocity-time graph is



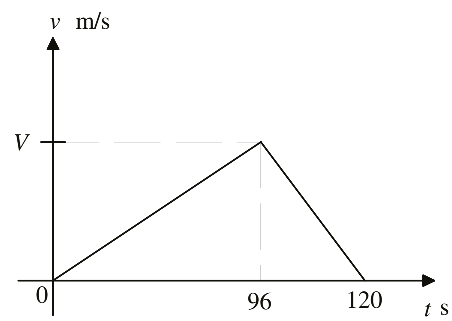
distance $= \frac{1}{2}(17 + 10) \times 14 = 189$ m

6



$5T = 120\text{ s} \Rightarrow T = 24$

Thus,



As the distance between stop A and stop B is 500 metres this implies that the area under the graph and the t-axis is equal to 500.

$\therefore\ \frac{1}{2}(96)V + \frac{1}{2}(24)V = 500$

$\therefore\ \frac{1}{2}V(120) = 500$

$\therefore\ 60V = 500$

$\therefore\ V = \frac{50}{6}$

$\therefore\ V = \frac{25}{3}$ m/s

$\therefore\ V_{\text{max}} = 8\frac{1}{3}$ m/s

Hence, the maximum velocity reached by the tram is $8\frac{1}{3}$ m/s

$a = \frac{\text{rise}}{\text{run}}$

$\therefore\ a = \frac{V}{96}$

$\therefore\ a = \frac{8}{3 \times 96}$

$\therefore\ a = \frac{25}{288}$ m/s^2

7 All units must be compatible, so we need to convert everything into *metres* and *seconds*.

1 km = 1000 m

60 km/h $= 60 \times \frac{5}{18} = \frac{50}{3}$ m/s

$\therefore$ Maximum velocity is $\frac{50}{3}$ m/s

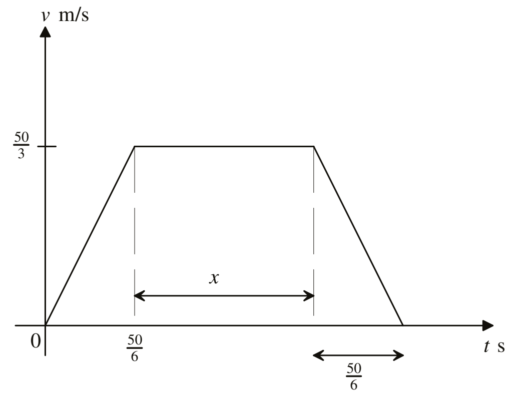
Maximum rate of acceleration and deceleration is 2 m/s^2. The word 'rate' refers to derivative or gradient.

Since $a = \frac{dv}{dt}$ the information means that the maximum gradient of acceleration and deceleration is 2.

Thus, if the maximum velocity is $\frac{50}{3}$ m/s then the time taken to reach that speed must be $\frac{50}{6}$ s because

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{\frac{50}{3}}{\frac{50}{6}} = 2$$

Hence the velocity-time graph is:



Since the distance travelled is 1000 m this implies that the area under the graph and the t-axis is equal to 1000.

Area of first triangle $= \frac{1}{2}\left(\frac{50}{6}\right)\left(\frac{50}{3}\right)$

$= \frac{625}{9}$

$\therefore$ Area of both triangles $= 2 \times \left(\frac{625}{9}\right)$

$= \frac{1250}{9}$

Distance left to travel $= 1000 - \frac{1250}{9}$

$= \frac{7750}{9}$

$\Rightarrow$ The area of the rectangle must equal $\frac{7750}{9}$

$\therefore\ x \times \frac{50}{3} = \frac{7750}{9}$

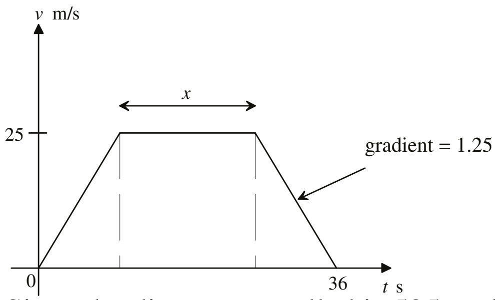
$\therefore\ x = \frac{155}{3}$ s

Therefore total time to travel between the bus stops =

$\frac{50}{6} + \frac{155}{3} + \frac{50}{6} = \frac{205}{3}$ s $= 68\frac{1}{3}$ s

8 $90 \text{ km/h} = 90 \times \frac{5}{18} \text{ m/s} = 25 \text{ m/s}$

The velocity-time graph is:



Since the distance travelled is 525 m this implies that the area under the graph and the t-axis is equal to 525.

$\therefore \frac{1}{2}(x + 36) \times 25 = 525$

$\therefore x + 36 = 42$

$\therefore x = 6 \text{ s}$

Therefore, the distance covered when travelling 90 km/h = $6 \times 25 = 150$ m

For the deceleration phase:

$\frac{\text{rise}}{\text{run}} = 1.25$

$\therefore \frac{25}{\text{run}} = 1.25$

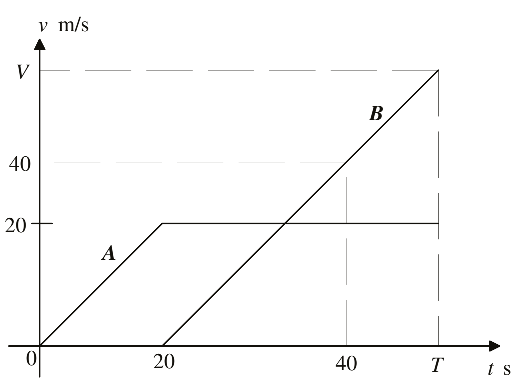
$\therefore \text{run} = \frac{25}{1.25} = 20 \text{ s}$

Hence, it takes 20 seconds for the deceleration phase.

Thus, the time taken in the acceleration phase = $36 - 20 - 6 = 10$ s

Therefore, the acceleration phase takes 10 seconds.

9



For car B:

$s = ut + \frac{1}{2}at^2$

$\therefore s = \frac{1}{2}(2)(T - 20)^2$

$\therefore s = (T - 20)^2$

For car A:

$s = \frac{1}{2}(20)(20) + 20(T - 20)$

$\therefore s = 200 + 20T - 400$

$\therefore s = 20(T - 10)$

The cars draw level with each other when their displacements are the same.

$\therefore (T - 20)^2 = 20(T - 10)$

$\therefore T^2 - 40T + 400 = 20T - 200$

$\therefore T^2 - 60T + 600 = 0$

$\therefore T = -10\sqrt{3} + 30$ or

$T = 10\sqrt{3} + 30$

$\therefore T = 10\sqrt{3} + 30$

(practical solution)

Time taken by $B = T - 20$

$= (10\sqrt{3} + 30) - 20$

$= 10\sqrt{3} + 10$

$= 10(\sqrt{3} + 1)$ s

Therefore, the time taken by car B to catch car A is $10(\sqrt{3} + 1)$ seconds

Distance travelled by $B = (T - 20)^2$

$\therefore (T - 20)^2 = 100(\sqrt{3} + 1)^2$

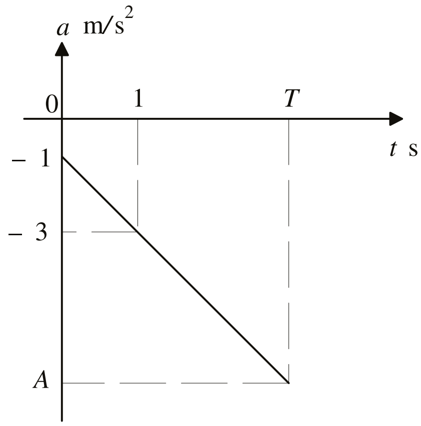
$\therefore (T - 20)^2 = 100(3 + 2\sqrt{3} + 1)$

$\therefore (T - 20)^2 = 400 + 200\sqrt{3}$

$\therefore (T - 20)^2 = 200(2 + \sqrt{3})$ m

Therefore, the distance travelled by car B to catch car A is $200(2 + \sqrt{3})$ m

10 a The acceleration-time graph is:



The object comes to rest at $t = T$ and $a = A$. Uniform deceleration means same gradient throughout.

$$\therefore \quad \frac{A+1}{T} = \frac{-3+1}{1}$$

$$\therefore \quad A = -2T - 1 \qquad ①$$

The area between the graph and the t-axis is final velocity − initial velocity

$$\therefore \quad 0 - 6 = \frac{1}{2}(T)(A - 1)$$

$$\therefore \quad T(A - 1) = -12 \qquad ②$$

Substituting ① into ② gives

$$T(-2T - 2) = -12$$

$$\therefore \quad -2T^2 - 2T = -12$$

$$\therefore \quad 2T^2 + 2T - 12 = 0$$

$$\therefore \quad T = 2$$

(practical solution)

Therefore it takes 2 seconds for the object to come to rest.

b From the $a - t$ graph the relationship between the two variables is $a = -2t - 1$

Thus,

$$v = \int -2t - 1 \, dt$$

$$\therefore \quad v = -t^2 - t + c$$

Initial velocity is 6 m/s.

$\Rightarrow c = 6$

$$\therefore \quad v = -t^2 - t + 6$$

Since $v \geq 0$ for $t \in [0, 2]$ the distance travelled can be calculated by evaluating the following definite integral.

$$\int_0^2 -t^2 - t + 6 \, dt$$

$$= \left[-\frac{1}{3}t^3 - \frac{1}{2}t^2 + 6t \right]_0^2$$

$$= -\frac{8}{3} - 2 + 12$$

$$= \frac{22}{3}$$

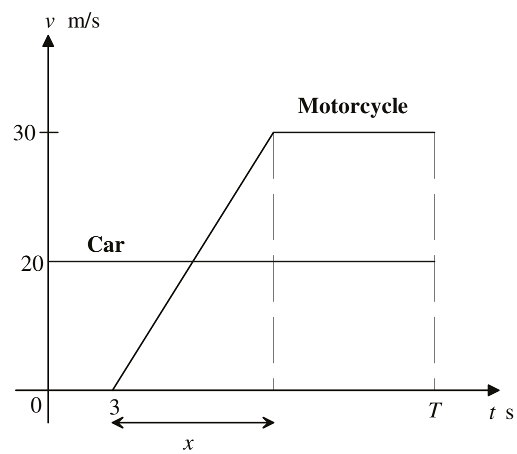
$$= 7\frac{1}{3}$$

Therefore the distance travelled by the object is $7\frac{1}{3}$ m

11 $72 \text{ km/h} = 72 \times \frac{5}{18} \text{ m/s} = 20 \text{ m/s}$

$108 \text{ km/h} = 108 \times \frac{5}{18} \text{ m/s} = 30 \text{ m/s}$

The velocity-time graph is:



Acceleration phase for the motorcycle:

$$\frac{1}{2}(x)(30) = 300$$

$$\therefore \quad x = 20 \text{ s}$$

Hence it takes 20 seconds for the motorcycle to reach a speed of 108 km/h.

Car:

$s = 20T$

Motorcycle:

$$s = 300 + 30(T - 23)$$

$$\therefore \quad s = 30T - 390$$

Equating displacements:

$30T - 390 = 20T$

$\therefore\ 10T = 390$

$\therefore\ T = 39$

It takes the motorcycle $(T - 3)$ seconds to catch the car.

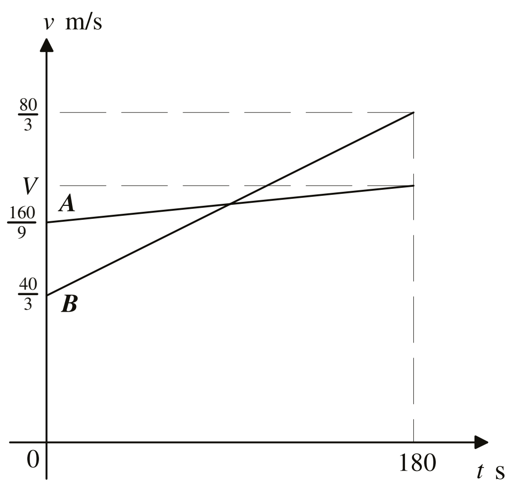
Therefore it takes 36 seconds for the motorcycle to catch the car.

12 $64 \text{ km/h} = 64 \times \dfrac{5}{18} \text{ m/s} = \dfrac{160}{9} \text{ m/s}$

$48 \text{ km/h} = 48 \times \dfrac{5}{18} \text{ m/s} = \dfrac{40}{3} \text{ m/s}$

$96 \text{ km/h} = 96 \times \dfrac{5}{18} \text{ m/s} = \dfrac{80}{3} \text{ m/s}$

The velocity-time graph is:



a distance travelled by A

= distance traveled by B

= area under each graph

$= \dfrac{1}{2}\left(\dfrac{40}{3} + \dfrac{80}{3}\right) \times 180$

= 3600 m

Also,

$\dfrac{1}{2}\left(\dfrac{160}{9} + V\right) \times 180 = 3600$

$\therefore\ \dfrac{160}{9} + V = 40$

$\therefore\ V = \dfrac{200}{9} \text{ m/s}$

$\therefore\ V = \dfrac{200}{9} \times \dfrac{18}{5} \text{ km/h}$

$\therefore\ V = 80 \text{ km/h}$

Therefore, the distance travelled by A and B is 3600 metres and the speed of A is 80 km/h.

b By close inspection it can be seen that the two triangles formed by the two graphs are congruent. Hence the point of intersection between the two graphs occurs at

$\dfrac{1}{2} \times 180 = 90 \text{ s}$

Therefore, the two cars are moving with the same speed 90 seconds after A passed B.

The distance between them at this instant

$= \dfrac{1}{2}\left(\dfrac{160}{9} - \dfrac{40}{3}\right) \times 90$

= 200 m

13 $\ddot{y} = ke^{-t},\ k < 0$

a $v = \dot{y} = \displaystyle\int ke^{-t}\, dt$

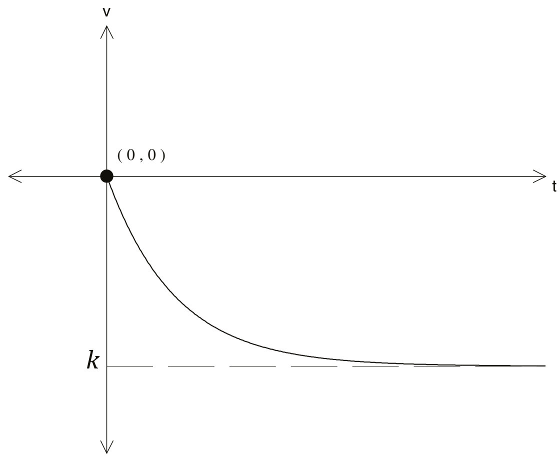
$\therefore\ \dot{y} = -ke^{-t} + c$

When $\dot{y} = 0,\ t = 0$

$\Rightarrow c = k$

$\therefore\ \dot{y} = -ke^{-t} + k$

$\therefore\ \dot{y} = k(1 - e^{-t}),\ k < 0$



b The particle decelerates exponentially with terminal velocity k m/s.

Solutions to Exercise 12D

1 $\dot{x} = \dfrac{1}{2x-4},\ x > 2$

a
$$\frac{dx}{dt} = \frac{1}{2x-4}$$
$$\therefore \frac{dt}{dx} = 2x - 4$$
$$\therefore t = \int 2x - 4dx$$
$$\therefore t = x^2 - 4x + c$$

When $t = 0,\ x = 3$:

$$\Rightarrow c = 3$$
$$\therefore t = x^2 - 4x + 3$$
$$\therefore t = x^2 - 4x + 4 - 4 + 3$$
$$\therefore t = (x-2)^2 - 1$$
$$\therefore \sqrt{t+1} = x - 2 \text{ since } x > 2$$
$$\therefore x = \sqrt{t+1} + 2$$

$\therefore$ When $t = 24,\ x = 7$ m

b As $x = \sqrt{t+1} + 2$ is an increasing function, the distance travelled in the first 24 seconds is

$$x(24) - x(0) = (\sqrt{25} + 2) - (\sqrt{1} + 2)$$
$$= 4 \text{ m}$$

Therefore, the distance travelled in the first 24 seconds is 4 metres.

2 $v = 1 + e^{-2x}$

a
$$\frac{dx}{dt} = 1 + e^{-2x}$$
$$\therefore \frac{dx}{dt} = \frac{e^{2x}}{e^{2x}} + \frac{1}{e^{2x}}$$
$$\therefore \frac{dx}{dt} = \frac{1 + e^{2x}}{e^{2x}}$$
$$\therefore \frac{dt}{dx} = \frac{e^{2x}}{1 + e^{2x}}$$
$$\therefore t = \int \frac{e^{2x}}{1 + e^{2x}}dx$$
$$\therefore t = \frac{1}{2}\log_e(1 + e^{2x}) + c$$

When $x = 0,\ t = 0$:

$$\Rightarrow c = -\frac{1}{2}\log_e(2)$$
$$\therefore t = \frac{1}{2}\log_e\left(\frac{1 + e^{2x}}{2}\right)$$
$$\therefore e^{2t} = \frac{1 + e^{2x}}{2}$$
$$\therefore 2e^{2t} = 1 + e^{2x}$$
$$\therefore 2e^{2t} - 1 = e^{2x}$$
$$\therefore 2x = \log_e(2e^{2t} - 1)$$
$$\therefore x = \frac{1}{2}\log_e(2e^{2t} - 1)$$

b To find the acceleration when $t = \log_e(5)$ we need to evaluate $\left.\dfrac{d^2x}{dt^2}\right|_{t = \log_e(5)}$.

$$v = \frac{dx}{dt} = \frac{1}{2}\left(\frac{4e^{2t}}{2e^{2t} - 1}\right)$$
$$= \frac{4e^{2t}}{4e^{2t} - 2}$$
$$a = \frac{dv}{dt}$$
$$= \frac{(4e^{2t} - 2) \times 8e^{2t} - 4e^{2t} \times 8e^{2t}}{(4e^{2t} - 2)^2}$$
$$\therefore a = \frac{-16e^{2t}}{(4e^{2t} - 2)^2}$$

When $t = \log_e(5)$:

$$a = \frac{-16e^{2\log_e(5)}}{(4e^{2\log_e(5)} - 2)^2}$$

$$= \frac{-16e^{\log_e(25)}}{(4e^{\log_e(25)} - 2)^2}$$

$$= -\frac{400}{(98)^2}$$

$$= -\frac{400}{9604}$$

$$= -\frac{100}{2401}$$

Therefore when $t = \log_e(5)$,
$a = -\frac{100}{2401}$

3 $a = 3 + v$

a $\frac{dv}{dt} = 3 + v$

$$\therefore \frac{dt}{dv} = \frac{1}{3+v}$$

$$\therefore t = \int \frac{1}{3+v}\, dv$$

$$\therefore t = \log_e(3+v) + c$$

When $t = 0,\ v = 0$:

$$\Rightarrow c = -\log_e(3)$$

$$\therefore t = \log_e\left(\frac{3+v}{3}\right)$$

$$\therefore 3e^t = 3 + v$$

$$\therefore v = 3e^t - 3$$

$$\therefore v = 3(e^t - 1)$$

b If $v = 3e^t - 3$ then
$a = \frac{dv}{dt} = 3e^t$

c $x = \int v\, dt$

$$\therefore x = \int 3e^t - 3\, dt$$

$$\therefore x = 3e^t - 3t + c$$

When $t = 0,\ x = 0$:

$$\Rightarrow c = -3$$

$$\therefore x = 3e^t - 3t - 3$$

$$\therefore x = 3(e^t - t - 1)$$

4 $a = g - kv,\ k > 0$

a $\frac{dv}{dt} = g - kv$

$$\therefore \frac{dt}{dv} = \frac{1}{g - kv}$$

$$\therefore t = \int \frac{1}{g-kv}\, dv$$

$$\therefore t = -\frac{1}{k}\log_e(g - kv) + c$$

When $t = 0,\ v = 0$:

$$\Rightarrow c = \frac{1}{k}\log_e(g)$$

$$\therefore t = \frac{1}{k}\log_e\left(\frac{g}{g-kv}\right)$$

$$\therefore e^{kt} = \frac{g}{g-kv}$$

$$\therefore g - kv = \frac{g}{e^{kt}}$$

$$\therefore g - kv = ge^{-kt}$$

$$\therefore kv = g - ge^{-kt}$$

$$\therefore v = \frac{1}{k}(g - ge^{-kt})$$

$$\therefore v = \frac{g}{k}(1 - e^{-kt}),\ k > 0$$

For terminal velocity let $t \to \infty$

$\therefore \quad v = \frac{g}{k}(1 - e^{-\infty})$

$\therefore \quad v = \frac{g}{k}(1 - 0)$ since $e^{-\infty} = 0$

$\therefore \quad v = \frac{g}{k}$

Therefore, the terminal velocity is $\frac{g}{k}$.

5 $a = -0.3(v^2 + 1)$

a

$$\frac{dv}{dt} = -\frac{3(v^2+1)}{10}$$

$$\therefore \quad \frac{dt}{dv} = -\frac{10}{3(v^2+1)}$$

$$\therefore \quad t = -\frac{10}{3}\int \frac{1}{v^2+1}\,dv$$

$$\therefore \quad t = -\frac{10}{3}\tan^{-1}(v) + c$$

When $t = 0,\ v = \sqrt{3}$:

$$\Rightarrow c = \frac{10\pi}{9}$$

$$\therefore \quad t = -\frac{10}{3}\tan^{-1}(v) + \frac{10\pi}{9}$$

$$\therefore \quad \frac{10\pi}{9} - t = \frac{10}{3}\tan^{-1}(v)$$

$$\therefore \quad \frac{\pi}{3} - \frac{3}{10}t = \tan^{-1}(v)$$

$$\therefore \quad v = \tan\left(\frac{\pi}{3} - \frac{3}{10}t\right)$$

b

$$x = \int \tan\left(\frac{\pi}{3} - \frac{3}{10}t\right) dt$$

$$\therefore \quad x = \int \frac{\sin\left(\frac{\pi}{3} - \frac{3}{10}t\right)}{\cos\left(\frac{\pi}{3} - \frac{3}{10}t\right)}\,dt$$

Note that:

$$\frac{d}{dx}\left[\cos\left(\frac{\pi}{3} - \frac{3}{10}t\right)\right] = \frac{3}{10}\sin\left(\frac{\pi}{3} - \frac{3}{10}t\right)$$

So,

$$x = \frac{10}{3}\int \frac{\frac{3}{10}\sin\left(\frac{\pi}{3} - \frac{3}{10}t\right)}{\cos\left(\frac{\pi}{3} - \frac{3}{10}t\right)}\,dt$$

We now have an integral of the form:

$$\int \frac{f'(x)}{f(x)}dx = \log_e(f(x))$$

Thus,

$$\therefore \quad x = \frac{10}{3}\log_e\left(\cos\left(\frac{\pi}{3} - \frac{3}{10}t\right)\right) + c$$

When $x = 0,\ t = 0$:

$$\Rightarrow c = \frac{10}{3}\log_e(2)$$

$$\therefore \quad x = \frac{10}{3}\log_e\left(2\cos\left(\frac{\pi}{3} - \frac{3}{10}t\right)\right)$$

6

$$a = \frac{450 - v}{50},\ v < 450$$

$$\therefore \quad \frac{dv}{dt} = \frac{450 - v}{50}$$

$$\therefore \quad \frac{dt}{dv} = \frac{50}{450 - v}$$

$$\therefore \quad t = 50\int \frac{1}{450 - v}\,dv$$

$$\therefore \quad t = -50\log_e(450 - v) + c$$

When $t = 0,\ v = 0$:

$$\Rightarrow = c = 50\log_e(450)$$

$$\therefore \quad t = 50\log_e\left(\frac{450}{450 - v}\right)$$

$$\therefore \quad e^{\frac{t}{50}} = \frac{450}{450 - v}$$

$$\therefore \quad 450 - v = \frac{450}{e^{\frac{t}{50}}}$$

$$\therefore \quad 450 - v = 450e^{-\frac{t}{50}}$$

$$\therefore \quad v = 450 - 450e^{-\frac{t}{50}}$$

$$\therefore \quad v = 450\left(1 - e^{-\frac{t}{50}}\right)$$

7

$$a = -0.4\sqrt{225 - v^2}$$

$$\therefore \frac{dv}{dt} = -\frac{2\sqrt{225 - v^2}}{5}$$

$$\therefore \frac{dv}{dt} = -\frac{5}{2\sqrt{225 - v^2}}$$

$$\therefore t = \frac{5}{2}\int -\frac{1}{15^2 - v^2}\,dv$$

$$\therefore t = \frac{5}{2}\cos^{-1}\left(\frac{v}{15}\right) + c$$

When $t = 0,\ v = 12$:

$$\Rightarrow c = -\frac{5}{2}\cos^{-1}\left(\frac{4}{5}\right)$$

$$\therefore t = \frac{5}{2}\cos^{-1}\left(\frac{v}{15}\right) - \frac{5}{2}\cos^{-1}\left(\frac{4}{5}\right)$$

$$\therefore t + \frac{5}{2}\cos^{-1}\left(\frac{4}{5}\right) = \frac{5}{2}\cos^{-1}\left(\frac{v}{15}\right)$$

$$\therefore \frac{2}{5}t + \cos^{-1}\left(\frac{4}{50}\right) = \cos^{-1}\left(\frac{v}{15}\right)$$

$$\therefore \frac{v}{15} = \cos\left(\frac{2}{5}t + \cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$\therefore v = 15\cos\left(\frac{2}{5}t + \cos^{-1}\left(\frac{4}{5}\right)\right)$$

8

$$v \propto x$$

$$\Rightarrow v = kx$$

When $x = 5,\ v = 2$:

$$\therefore 2 = 5k$$

$$\therefore k = \frac{2}{5}$$

$$\therefore v = \frac{2}{5}x$$

a

$$\frac{dx}{dt} = \frac{2}{5}x$$

$$\therefore \frac{dt}{dx} = \frac{5}{2x}$$

$$\therefore t = \frac{5}{2}\int \frac{1}{x}\,dx$$

$$\therefore t = \frac{5}{2}\log_e(x) + c$$

When $t = 0,\ x = 5$:

$$\Rightarrow c = -\frac{5}{2}\log_e(5)$$

$$\therefore t = \frac{5}{2}\log_e\left(\frac{x}{5}\right)$$

$$\therefore \frac{2}{5}t = \log_e\left(\frac{x}{5}\right)$$

$$\therefore \frac{x}{5} = e^{\frac{2}{5}t}$$

$$\therefore x = 5e^{\frac{2}{5}t}$$

b When $t = 10$,

$$x = 5e^4 \approx = 273 \text{ m}$$

9 $a = \frac{1}{50}(500 - v),\ 0 \le v < 500$

a

$$\frac{dv}{dt} = \frac{500 - v}{50}$$

$$\therefore \frac{dt}{dv} = \frac{50}{500 - v}$$

$$\therefore t = 50\int \frac{1}{500 - v}\,dv$$

$$\therefore t = -50\log_e(500 - v) + c$$

When $t = 0,\ v = 0$:

$$\Rightarrow c = 50\log_e(500)$$

$$\therefore t = 50\log_e\left(\frac{500}{500 - v}\right)$$

b from part **a**:

$$e^{\frac{t}{50}} = \frac{500}{500 - v}$$

$$\therefore\ 500 - v = \frac{500}{e^{\frac{t}{50}}}$$

$$\therefore\ 500 - v = 500e^{-\frac{t}{50}}$$

$$\therefore\ v = 500 - 500e^{-\frac{t}{50}}$$

$$\therefore\ v = 500\left(1 - e^{-\frac{t}{50}}\right)$$

10 $a = -k(2u - v)$

$$\therefore\ \frac{dv}{dt} = -k(2u - v)$$

$$\therefore\ \frac{dt}{dv} = -\frac{1}{k(2u - v)}$$

$$\therefore\ t = \frac{1}{k}\int -\frac{1}{2u - v}\,dv$$

$$\therefore\ t = \frac{1}{k}\log_e(2u - v) + c$$

When $t = 0,\ v = u$:

$$\Rightarrow c = -\frac{1}{k}\log_e(u)$$

$$\therefore\ t = \frac{1}{k}\log_e\left(\frac{2u - v}{u}\right)$$

The particle will come to rest when $v = 0$.

$$\therefore\ t = \frac{1}{k}\log_e\left(\frac{2u}{u}\right)$$

$$\therefore\ t = \frac{1}{k}\log_e(2)$$

Therefore it takes $\frac{1}{k}\log_e(2)$ seconds for particle to come to rest.

11 $\frac{dv}{dt} = -\frac{v}{5}$

$$\frac{dt}{dv} = -\frac{5}{v}$$

$$\therefore\ t = -5\int \frac{1}{v}\,dv$$

$$\therefore\ t = -5\log_e(v) + c$$

When $t = 0,\ v = 8$:

$$\Rightarrow c = 5\log_e(8)$$

$$\therefore\ t = 5\log_e\left(\frac{8}{v}\right)$$

$$\therefore\ e^{\frac{t}{5}} = \frac{8}{v}$$

$$\therefore\ v = 8e^{-\frac{t}{5}}$$

When $t = 4$,

$$v = 8e^{-\frac{4}{5}} \approx 3.59 \text{ m/s}$$

When $t = 10$,

12 $a = -kv^2$

When $t = 0,\ a = -20$ and $v = 30$:

$$\therefore\ -20 = 900k,\ \text{so } k = \frac{1}{45}$$

a $a = -\frac{v^2}{45}$

$$\therefore\ \frac{dv}{dt} = -\frac{v^2}{45}$$

$$\therefore\ \frac{dt}{dv} = -\frac{45}{v^2}$$

$$\therefore\ t = -45\int \frac{1}{v^2}\,dv$$

$$\therefore\ t = \frac{45}{v} + c$$

When $t = 0,\ v = 30$:

$$\Rightarrow c = -\frac{3}{2}$$

$$\therefore \quad t = \frac{45}{v} - \frac{3}{2}$$

$$\therefore \quad t + \frac{3}{2} = \frac{45}{v}$$

$$\therefore \quad v = \frac{45}{t + \frac{3}{2}}$$

$$\therefore \quad v = \frac{2 \times 45}{2\left(t + \frac{3}{2}\right)}$$

$$\therefore \quad v = \frac{90}{2t + 3}$$

b From part **a**:

$$\frac{dx}{dt} = \frac{90}{2t + 3}$$

$$\therefore \quad x = 90 \int \frac{1}{2t + 3}\, dt$$

$$\therefore \quad x = 45 \int \frac{2}{2t + 3}\, dt$$

$$\therefore \quad x = 45 \log_e(2t + 3) + c$$

When $t = 0,\ x = 0$:

$$\Rightarrow c = -45 \log_e(3)$$

$$\therefore \quad x = 45 \log_e\left(\frac{2t + 3}{3}\right)$$

$$x = 45 \log_e\left(\frac{23}{3}\right) \approx 91.66 \text{ m}$$

Solutions to Exercise 12E

1 $v^2 = 9 - x^2$

Now $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore a = \frac{d}{dx}\left(\frac{9}{2} - \frac{1}{2}x^2\right) = -x$

At $x = 2, a = -2$

The acceleration is -2 m/s^2.

2 **a**

$$a = -x$$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -x$$

$$\therefore \frac{1}{2}v^2 = \int -x\,dx$$

$$\therefore \frac{1}{2}v^2 = -\frac{1}{2}x^2 + c$$

When $v = 0,\ x = 4$:

$$\Rightarrow c = 8$$

$$\therefore \frac{1}{2}v^2 = -\frac{1}{2}x^2 + 8$$

$$v^2 = -x^2 + 16$$

When $x = 0,\ v = \pm 4$ m/s

b

$$a = 2 - v$$

$$\therefore \frac{dv}{dt} = 2 - v$$

$$\therefore \frac{dt}{dv} = \frac{1}{2 - v}$$

$$\therefore t = \int \frac{1}{2 - v}\,dv$$

$$\therefore t = -\log_e(2 - v) + c$$

When $v = 0,\ t = 0$:

$$\Rightarrow c = \log_e(2)$$

$$\therefore t = \log_e\left(\frac{2}{2 - v}\right)$$

When $v = -2$,

$$t = \log_e\left(\frac{1}{2}\right)$$

$$\therefore t = \log_e(2^{-1})$$

$$\therefore t = -\log_e(2)$$

c $a = 2 - v$

$\therefore\ v\dfrac{dv}{dx} = 2 - v$

$\therefore\ \dfrac{dv}{dx} = \dfrac{2 - v}{v}$

$\therefore\ \dfrac{dx}{dv} = \dfrac{v}{2 - v} = \dfrac{2}{2 - v} - 1$

$\therefore\ x = \displaystyle\int \frac{2}{2 - v} - 1\, dv$

$\therefore\ x = -2\log_e(2 - v) - v + c$

When $v = 0,\ x = 0$:

$\Rightarrow c = 2\log_e(2)$

$\therefore\ x = 2\log_e\left(\dfrac{2}{2 - v}\right) - v$

When $v = -2$,

$x = 2\log_e\left(\dfrac{1}{2}\right) + 2$

$\therefore\ x = -2\log_e(2) + 2$

$\therefore\ x = 2(1 - \log_e(2))$

3 a $a = -v^3$

$\therefore\ v\dfrac{dv}{dx} = -v^3$

$\therefore\ \dfrac{dv}{dx} = -v^2$

$\therefore\ x = \displaystyle\int -\frac{1}{v^2}\, dv$

$\therefore\ x = \dfrac{1}{v} + c$

When $v = 1,\ x = 0$:

$\Rightarrow c = -1$

$\therefore\ x = \dfrac{1}{v} - 1$

$\therefore\ x + 1 = \dfrac{1}{v}$

$\therefore\ v = \dfrac{1}{x + 1}$

b $v = x + 1$

i $\dfrac{dx}{dt} = x + 1$

$\therefore\ t = \displaystyle\int \frac{1}{x + 1}\, dx$

$\therefore\ t = \log_e(x + 1) + c$

When $x = 0,\ t = 0$:

$\Rightarrow c = 0$

$\therefore\ t = \log_e(x + 1)$

$\therefore\ x + 1 = e^t$

$\therefore\ x = e^t - 1$

ii As $x = e^t - 1$

$v = \dfrac{dx}{dt} = e^t$

$\therefore\ a = \dfrac{d^2x}{dt^2} = e^t$

iii From part **i**. $t = \log_e(x + 1)$

$\therefore\ a = e^{\log_e(x+1)}$

$\therefore\ a = x + 1$

and since $v = x + 1$

$\therefore\ a = v$

Alternatively, $a = v\dfrac{dv}{dx}$

$v = x + 1$ so $\dfrac{dv}{dx} = 1$

So $a = v \times 1 = v$

4 $a = -g - 0.2v^2$

$\therefore\ v\dfrac{dv}{dx} = -g - 0.2v^2$

$\therefore\ \dfrac{dv}{dx} = \dfrac{-g - 0.2v^2}{v}$

$\therefore\ \dfrac{dx}{dv} = \dfrac{v}{-g - 0.2v^2}$

$\therefore\ x = \displaystyle\int \frac{v}{-g - 0.2v^2}\,dv$

$\therefore\ x = -\displaystyle\int \frac{v}{g + 0.2v^2}\,dv$

$\therefore\ x = -\dfrac{1}{0.4}\displaystyle\int \frac{0.4v}{g + 0.2v^2}\,dv$

Using the fact that

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log e(f(x))$

$\therefore\ x = -\dfrac{1}{0.4}\log_e(g + 0.2v^2) + c$

$\therefore\ x = -\dfrac{5}{2}\log_e(g + 0.2v^2) + c$

When $x = 0,\ v = 100$:

$\Rightarrow c = \dfrac{5}{2}\log_e(g + 2000)$

$\therefore\ x = \dfrac{5}{2}\log_e\left(\dfrac{g + 2000}{g + 0.2v^2}\right)$

or equivalently,

$x = -\dfrac{5}{2}\log_e\left(\dfrac{g + 0.2v^2}{g + 2000}\right)$

Maximum height occurs when $v = 0$.

$\therefore\ x_{\max} = \dfrac{5}{2}\log_e\left(\dfrac{g + 2000}{g}\right)$

5 $v = 2\sqrt{1 - x^2}$

a $\dfrac{dx}{dt} = 2\sqrt{1 - x^2}$

$\therefore\ \dfrac{dt}{dx} = \dfrac{1}{2\sqrt{1 - x^2}}$

$\therefore\ t = \dfrac{1}{2}\displaystyle\int \frac{1}{\sqrt{1 - x^2}}dx$

$\therefore\ t = \dfrac{1}{2}\sin^{-1}(x) + c$

When $t = 0,\ x = 1$:

$\Rightarrow c = -\dfrac{\pi}{4}$

$\therefore\ t = \dfrac{1}{2}\sin^{-1}(x) - \dfrac{\pi}{4}$

$\therefore\ 2t + \dfrac{\pi}{2} = \sin^{-1}(x)$

$x = \sin\left(2t + \dfrac{\pi}{2}\right)$

$x = \cos(2t)$

b $x = \cos(2t)$

$\therefore\ v = \dfrac{dx}{dt} = -2\sin(2t)$

$\therefore\ a = \dfrac{dv}{dt} = -4\cos(2t)$

$\therefore\ a = -4x$ since

$x = \cos(2t)$

Alternatively, $v^2 = 4(1 - x^2)$

$\dfrac{1}{2}v^2 = 2(1 - x^2)$

$a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$= 2 \times (-2x)$

$= -4x$

6 a $a = \dfrac{1}{1 + t}$

$\therefore\ \dfrac{dv}{dt} = \dfrac{1}{1 + t}$

$\therefore\ v = \displaystyle\int \frac{1}{1 + t}\,dt$

$\therefore\ v = \log_e(1 + t) + c$

When $v = 0,\ t = 0$:

$\Rightarrow c = 0$

$\therefore\ v = \log_e(1 + t)$

b $a = \frac{1}{1+x}, \ x > 1$

$\therefore \ \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{1+x}$

$\therefore \ \frac{1}{2}v^2 = \int \frac{1}{1+x} dx$

$\therefore \ \frac{1}{2}v^2 = \log_e(1+x) + c$

When $v = 0, \ x = 0$:

$\Rightarrow c = 0$

$\therefore \ \frac{1}{2}v^2 = \log_e(1+x)$

$\therefore \ v^2 = 2\log_e(1+x)$

c $a = \frac{1}{1+v}$

$\therefore \ \frac{dv}{dt} = \frac{1}{1+v}$

$\therefore \ \frac{dt}{dv} = 1 + v$

$\therefore \ t = \int 1 + v \, dv$

$\therefore \ t = v + \frac{1}{2}v^2 + c$

When $t = 0, \ v = 0$:

$\Rightarrow c = 0$

$\therefore \ t = \frac{1}{2}v^2 + v$

$\therefore \ 2t = v^2 + 2v$

$\therefore \ 2t = v^2 + 2v + 1 - 1$

$\therefore \ 2t + 1 = (v+1)^2$

$\therefore \ \pm\sqrt{2t+1} = v + 1$

$\therefore \ v + 1 = \sqrt{2t+1}$

as $v = 0$ when $t = 0$

$\therefore \ v = \sqrt{2t+1} - 1$

7 $a = (2+x)^{-2}$

$\therefore \ \frac{d}{dx}\left(\frac{1}{2}v^2\right) = (2+x)^{-2}$

$\therefore \ \frac{1}{2}v^2 = \int (2+x)^{-2} \, dx$

$\therefore \ \frac{1}{2}v^2 = -\frac{1}{2+x} + c$

When $x = 0, \ v = 0$:

$\Rightarrow c = \frac{1}{2}$

$\therefore \ \frac{1}{2}v^2 = -\frac{1}{2+x} + \frac{1}{2}$

$\therefore \ v^2 = -\frac{2}{2+x} + 1$

$\therefore \ v^2 = \frac{2+x}{2+x} - \frac{2}{2+x}$

$\therefore \ v^2 = \frac{x}{2+x}$

8 **a** $a = 1 + 2x$

$\therefore \ \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 + 2x$

$\therefore \ \frac{1}{2}v^2 = \int 1 + 2x \, dx$

$\therefore \ \frac{1}{2}v^2 = x + x^2 + c$

When $x = 0, \ v = 2$:

$\Rightarrow c = 2$

$\therefore \ \frac{1}{2}v^2 = x^2 + x + 2$

$\therefore \ v^2 = 2x^2 + 2x + 4$

$\therefore \ v = \sqrt{2x^2 + 2x + 4}$

(as $x = 0$ when $v = 0$)

$\therefore$ When $x = 2, \ v = \sqrt{16} = 4$

b $a = 2 - v$

$\therefore\ v\dfrac{dv}{dx} = 2 - v$

$\therefore\ \dfrac{dv}{dx} = \dfrac{2 - v}{v}$

$\therefore\ x = \displaystyle\int \frac{v}{2 - v}\, dv$

Using long division:

$\therefore\ x = \displaystyle\int \frac{2}{2 - v} - 1\, dv$

$\therefore\ x = -2\log_e(2 - v) - v + c$

When $v = 0,\ x = 0$:

$\Rightarrow c = 2\log_e(2)$

$\therefore\ x = 2\log_e\left(\dfrac{2}{2 - v}\right) - v$

$\therefore$ When $v = 1,\ x = 2\log_e(2) - 1$

9 $a = -\dfrac{1}{5}(v^2 + 50)$

a $\therefore\ v\dfrac{dv}{dx} = -\dfrac{1}{5}(v^2 + 50)$

$\therefore\ \dfrac{dv}{dx} = -\dfrac{1}{5v}(v^2 + 50)$

$\therefore\ \dfrac{dv}{dx} = -\dfrac{v^2 + 50}{5v}$

$\therefore\ \dfrac{dx}{dv} = -\dfrac{5v}{v^2 + 50}$

$\therefore\ x = -5\displaystyle\int \frac{v}{v^2 + 50}\, dv$

$\therefore\ x = -\dfrac{5}{2}\displaystyle\int \frac{2v}{v^2 + 50}\, dv$

$\therefore\ x = -\dfrac{5}{2}\log_e(v^2 + 50) + c$

When $x = 0,\ v = 50$:

$\Rightarrow c = \dfrac{5}{2}\log_e(2550)$

$$\therefore \quad x = \frac{5}{2}\log_e\left(\frac{2550}{v^2 + 50}\right)$$

Maximum height occurs when $v = 0$.

$$\therefore \quad x_{\max} = \frac{5}{2}\log_e(51) \approx 9.83 \text{ m}$$

Therefore the maximum height reached by the particle is 9.83 metres.

b

$$a = -\frac{1}{5}(v^2 + 50)$$

$$\therefore \quad \frac{dv}{dt} = -\frac{1}{5}(v^2 + 50)$$

$$\therefore \quad \frac{dt}{dv} = -\frac{5}{v^2 + 50}$$

$$\therefore \quad t = -5\int \frac{1}{v^2 + 50}\, dv$$

$$\therefore \quad t = -\frac{5}{\sqrt{50}}\int \frac{\sqrt{50}}{v^2 + (\sqrt{50})^2}\, dv$$

We now have an integral of the form:

$$\int \frac{a}{x^2 + a^2}\, dx = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad t = -\frac{5}{\sqrt{50}}\tan^{-1}\left(\frac{v}{\sqrt{50}}\right) + c$$

$$\therefore \quad t = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{v}{5\sqrt{2}}\right) + c$$

When $t = 0,\ v = 50$:

$$\Rightarrow c = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{10}{\sqrt{2}}\right)$$

$$\therefore \quad t = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{v}{5\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{10}{\sqrt{2}}\right)$$

Maximum height occurs when $v = 0$.

$$\therefore \quad t = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{10}{\sqrt{2}}\right) \approx 1.01$$

Therefore, it takes the particle 1.01 seconds to reach maximum height.

Solutions to Technology-free questions

1 a For $x = t^2 - 7t + 10$

the velocity $v = \dfrac{dx}{dt}$

$= 2t - 7$

For the velocity to be 0, $2t - 7 = 0$

which implies $t = \dfrac{7}{2}$

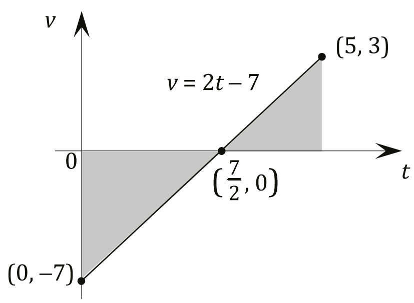
The velocity is zero after 3.5 seconds.

b acceleration $= \dfrac{dv}{dt}$

$= 2$

The acceleration is 2 m/s^2.

c



Distance travelled is the total area of the shaded regions

$= \dfrac{1}{2} \times \dfrac{7}{2} \times 7 + \dfrac{1}{2} \times \dfrac{3}{2} \times 3$

$= \dfrac{49}{4} + \dfrac{9}{4}$

$= \dfrac{58}{4}$

$= 14.5$ metres

d $v = -2$

implies $2T - 7 = -2$

Hence $t = \dfrac{5}{2}$

At $t = \dfrac{5}{2}$, $x = \left(\dfrac{5}{2}\right)^2 - 7\left(\dfrac{5}{2}\right) + 10$

$= -\dfrac{5}{4}$

So when $v = -2$, $t = 2.5$ and $x = -1.25$, i.e. the particle is 1.25 m to the left of O.

2

The acceleration, $a = 2t - 3$

$\dfrac{dv}{dt} = 2t - 3$

Antidifferentiating gives $v = t^2 - 3t + c$

When $t = 0$, $v = 3$ hence $v = t^2 - 3t + 3$

This can be written as $\dfrac{dx}{dt} = t^2 - 3t + 3$

Hence $x = \dfrac{t^3}{3} - \dfrac{3t^2}{2} + 3t + 2$

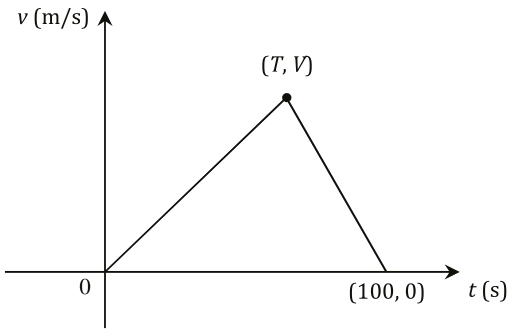
When $t = 10$, $x = \dfrac{1000}{3} - 150 + 30 + 2$

$= \dfrac{646}{3}$

When $t = 10$, $v = 100 - 30 + 3$

$= 73$

So, after 10 seconds, the body is $215\frac{1}{3}$ m from O moving at 73 m/s.

3 

a Distance travelled is 800 m.
Total time taken is 100 seconds.
Area under graph gives the distance travelled.
Hence, $\frac{1}{2} \times 100 \times V = 800$
where V m/s is the maximum velocity reached after T seconds.
Therefore, $50V = 800$

and $V = \frac{800}{50}$

$= 16$ m/s

$= 57.6$ km/h

b Initial segment has gradient $= a$.
Therefore, using $V = 16$, $\frac{16}{T} = a$
where T is the time at which maximum velocity is reached and, using the gradient of the other line segment,

$$\frac{16}{T - 100} = -2a$$

Therefore $-\frac{32}{T} = \frac{16}{T - 100}$

and $-32T + 3200 = 16T$

Therefore, $T = 66\frac{2}{3}$

Maximum speed occurs after 1 minute $6\frac{2}{3}$ seconds, when the brakes are applied.

c Substituting $T = 66\frac{2}{3}$ in $\frac{16}{T} = a$ gives $a = 0.24$ m/s^2.

4 **a** Deceleration $= \frac{150 - 125}{0.003}$

$= \frac{25\,000}{3}$ m/s^2

b Distance travelled

$=$ area under $v - t$ graph

$= \frac{0.003}{2}(150 + 125)$

$= 0.4125$

Therefore, thickness is 0.4125 metres.

c Deceleration in wood

$= \frac{125 - 75}{0.008 - 0.003}$

$= 10\,000$ m/s^2

d Distance travelled in wood

$= \frac{0.005}{2}(125 + 75)$

$= 0.5$ metre

e Deceleration in brick $= \frac{75}{0.002}$

$= 37\,500$ m/s^2

f Distance travelled in brick

$= \frac{0.002}{2} \times 75$

$= 0.075$ metre

5 **a** Average velocity

$= \frac{h(2) - h(0)}{2}$

$= \frac{110 + 55 \times 2 - 5.5 \times 4 - 110}{2}$

$= 44$ m/s

The average velocity is 44 m/s.

b $v = \frac{dh}{dt}$

$= 55 - 11t$

c $55 - 11 = 44$

The velocity at 1 second is 44 m/s.

d $55 - 11t = 0$

$\therefore \quad t = 5$

So it takes 5 seconds to reach zero velocity.

e $h(5) = 110 + 55 \times 5 - 5.5 \times 25 = 247.5$

Maximum height reached is 247.5 metres.

6 Using $v = u + at,\ v = 8 - 2t$

$\therefore$ at $v = 0,\ t = 4.$

Using $x = ut + \frac{1}{2}at^2,\ x = 8 \times 4 - 4^2 = 16$

The golf ball will roll a distance of 16 metres.

7 **a** $\sqrt{5} = \sqrt{9 - t^2}$

$5 = 9 - t^2$

$t^2 = 4$

$t = 2,$ since $t \geq 0$

So the displacement is $\sqrt{5}$ metres after 2 seconds.

b $v = \frac{dx}{dt}$

$= \frac{-2t}{2\sqrt{9 - t^2}}$

using the chain rule

$v = \frac{-t}{\sqrt{9 - t^2}}$

Now $a = \frac{dv}{dt}$

$= \frac{\sqrt{9 - t^2} - t\left(\frac{-t}{\sqrt{9 - t^2}}\right)}{9 - t^2}$

$= -\frac{9 - t^2 + t^2}{(9 - t^2)\sqrt{9 - t^2}}$

$\therefore \quad a = \frac{-9}{(9 - t^2)^{\frac{3}{2}}}$

c The maximum magnitude of the displacement from O is 3 metres.

d Now $v = 0, \quad \therefore \quad \frac{-t}{\sqrt{9 - t^2}} = 0$

$\therefore \quad t = 0$

So velocity is zero at $t = 0$.

8 **a** $v = \int_0^2 (12 - 6t)\, dt + 8$

$= [12t - 3t^2]_0^2 + 8$

$= 12 + 8$

$\therefore$ velocity $= 20$ m/s

b displacement $= \int_0^2 (12t - 3t^2 + 8)\, dt$

$= [6t^2 - t^3 + 8t]_0^2$

$= 24 - 8 + 16$

$= 32$ m

9 **a** Distance travelled is area under $v - t$

graph

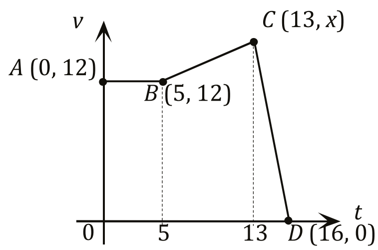
$$= 60 + 4(12 + x) + \frac{3}{2}x$$

$$= 108 + \frac{11}{2}x$$

Now $108 + \frac{11}{2}x = 218$

$\therefore \quad \frac{11}{2}x = 110$

$\therefore \quad x = 20$



b Average speed $= \frac{\text{distance}}{\text{time}}$

$$= \frac{218}{5 + 8 + 3}$$

$$= \frac{218}{16}$$

$$= \frac{109}{8}$$

$= 13.625$ m/s

10 **a** The deceleration of the ball due to gravity is g m/s^2.
Using $v = u + at$, $v(t) = 35 - gt$

i $v(3) = 35 - g \times 3$

$= 35 - 3g$

So velocity is $35 - 3g$ m/s. (Note: $v(3) > 0$.)

ii $v(5) = 35 - g \times 5$

$= 35 - 5g$

So velocity is $35 - 5g$ m/s. (Note: $v(5) < 0$.)

b The total distance is double the distance up to the point where $v(t) = 0$.

Now at $v = 0$,

$35 - gt = 0$

$\therefore \quad t = \frac{35}{g}$

$$\therefore \quad \text{total distance} = 2\left(ut + \frac{1}{2}at^2\right)$$

$$= 2\left(35 \times \frac{35}{g} + \frac{1}{2} \times (-g) \times \left(\frac{35}{g}\right)^2\right)$$

$$= 2\left(\frac{1225}{g} - \frac{1225}{2g}\right)$$

$$= \frac{1225}{g} \text{ m}$$

c $v\left(\frac{70}{g}\right) = 35 - g \times \frac{70}{g}$

$= -35$

So velocity at $x = 0$ is -35 m/s.

11 Distance is area under $v - t$ graph.

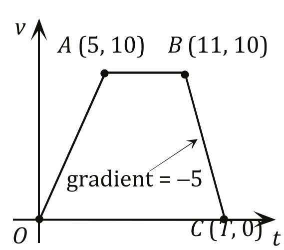
Now at C, $\frac{10}{11 - T} = -5$

$\therefore \quad 10 = 5T - 55$

$\therefore \quad T = 13$

So, distance (area) $= \frac{10}{2}(13 + 6)$

$= 95$ m



12 $v = \frac{dx}{dt}$

$\therefore \quad v = \frac{4}{t-1}$

$a = \frac{dv}{dt}$

$\therefore \quad a = -\frac{4}{(t-1)^2}$

13 We know that $h = -\frac{gt^2}{2} + ut$ ①
where h is height, t is time of movement, u is the initial velocity and g is the gravitational constant.

a $u = \frac{h}{t} + \frac{gt}{2}$

$= \frac{64}{0.8} + \frac{0.8g}{2}$

$= 80 + 0.4g$

So the initial velocity is $(80 + 0.4g)$ m/s.

b The height is the greatest when velocity is zero.
Using $v = u + at$,
$u - gt = 0$

$t = \frac{u}{g}$

$= \frac{80}{g} + 0.4$ ②

So the time taken to reach the greatest height is $\left(\frac{80}{g} + 0.4\right)$ seconds.

c Substitute ② into ①

$h = -\frac{g}{2}\left(\frac{80}{g} + 0.4\right)^2$

$+ (80 + 0.4g)\left(\frac{80}{g} + 0.4\right)$

$= \frac{(80 + 0.4g)^2}{2g}$ m

So the greatest height is $\frac{(80 + 0.4g)^2}{2g}$ metres.

d Length of time above the top of the tower on the way up

$= \frac{80}{g} + 0.4 - 0.8$

$= \left(\frac{80}{g} - 0.4\right)$ seconds.

So the total length time above the top of the tower

$= 2 \times \left(\frac{80}{g} - 0.4\right)$

$= \left(\frac{160}{g} - 0.8\right)$ seconds.

Solutions to multiple-choice questions

1 A $x(t) = t^3 - 9t^2 + 24t - 1$

$\therefore \quad x(3) = 27 - 81 + 72 - 1 = 17\text{ m}$

2 C $x(t) = t^3 - 9t^2 + 24t - 1$

$$\text{Average velocity} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x(2) - x(0)}{2 - 0} = \frac{19 - (-1)}{2} = \frac{20}{2} = 10\text{ m/s}$$

3 A $u = 30,\ a = -10,\ t = 2$

Use $v = u + at$

$\therefore \quad v = 30 + (-10)(2)$

$\therefore \quad v = 10\text{ m/s}$

4 D

$(0, 0)$ and $\left(5, \dfrac{125}{9}\right)$

Where t is in seconds and v is in m/s.

$$\text{acceleration} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\frac{125}{9} - 0}{5 - 0} = \frac{125}{45} = \frac{25}{9}\text{ m/s}^2$$

5 B $v = 5 - \dfrac{2}{t + 2}$

$\therefore \quad v = 5 - 2(t + 2)^{-1}$

$$a = \frac{dv}{dt} = 2(t + 2)^{-2} = \frac{2}{(t + 2)^2}$$

Initial acceleration occurs when $t = 0$.

$\Rightarrow a = \dfrac{1}{2}\text{ m/s}^2$

6 C Deceleration phase:

$(80, 20)$ and $(180, -10)$

$$v = -\frac{30}{100}t + c$$

$$\therefore \quad v = -\frac{3}{10}t + c$$

Passes through $(80, 20)$:

$\Rightarrow c = 44$

$$\therefore \quad v = -\frac{3}{10}t + 44$$

When $v = 0$:

$$\frac{3}{10}t = 44$$

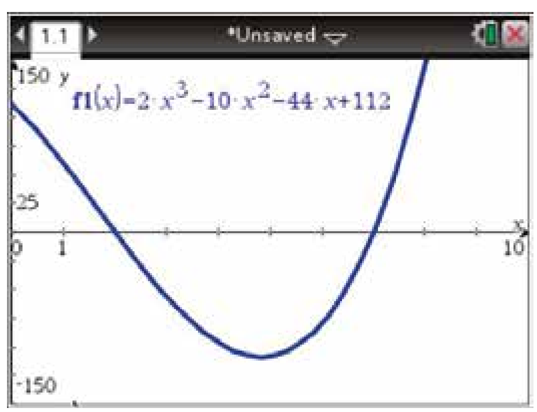
$$\therefore \quad t = \frac{440}{3}$$

$$\therefore \quad t = 146\frac{2}{3}$$

Which is closest to 147.

7 C $x = 2t^3 - 10t^2 - 44t + 112$

Sketch x for $0 \le t \le 10$ and look for the number of zeros.



There are two zeros.

8 C $a = -x, \; -\sqrt{3} \le x \le \sqrt{3}$

$$\therefore \; \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -x$$

$$\therefore \; \frac{1}{2}v^2 = \int -x\,dx$$

$$\frac{1}{2}v^2 = -\frac{1}{2}x^2 + c$$

When $x = 0, \; v = \sqrt{3}$:

$$\Rightarrow c = \frac{3}{2}$$

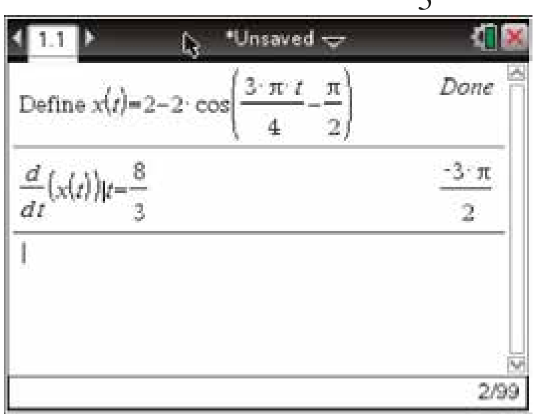
$$\therefore \; \frac{1}{2}v^2 = -\frac{1}{2}x^2 + \frac{3}{2}$$

$$\therefore \; v^2 = -x^2 + 3$$

$$\therefore \; v = \pm\sqrt{3 - x^2}$$

9 A $x = 2 - 2\cos\left(\frac{3\pi}{4}t - \frac{\pi}{2}\right)$

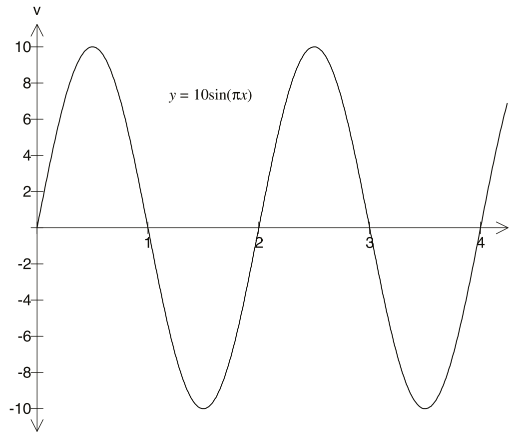
Using a CAS calculator we can determine the velocity when $t = \frac{8}{3}$ by differentiating x with respect to t and making the substitution $t = \frac{8}{3}$



Therefore the velocity at time $t = \frac{8}{3}$ s is $-\frac{3\pi}{2}$ m/s

10 E $v = 10\sin(\pi t)$

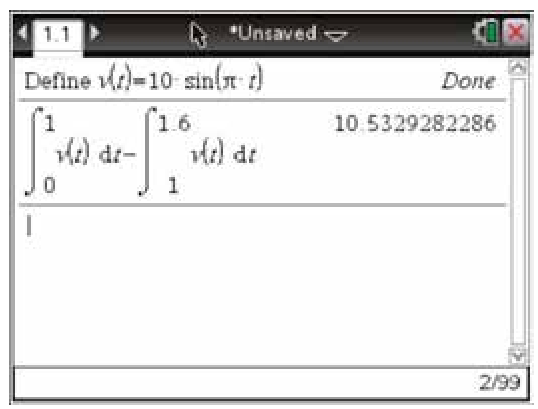
The velocity-time graph is:



The distance the object has travelled in the first 1.6 seconds can be calculated by evaluating the following definite integral.

$\int_0^1 10\sin(\pi t)\,dt - \int_1^{1.6} 10\sin(\pi t)\,dt$

Using CAS to compute the above we have



Therefore the distance travelled by the object in the first 1.6 seconds is 10.53 correct to two decimal places.

Solutions to extended-response questions

1 a

$$5\frac{dv}{dt} + v = 50$$

By definition, $a = \dfrac{dv}{dt}$, $\quad \therefore \; 5a + v = 50$

$$a = \frac{50 - v}{5}$$

When $t = 0$, $v = 0$, $\quad \therefore \; a = \dfrac{50}{5}$

$$\therefore \; \text{acceleration} = 10 \text{ m/s}^2$$

b

$$5\frac{dv}{dt} = 50 - v$$

$$\therefore \; \frac{dv}{dt} = \frac{50 - v}{5}$$

$$\therefore \; \frac{dt}{dv} = \frac{5}{50 - v}$$

$$\therefore \; t = \int \frac{5}{50 - v}\, dv$$

$$= -5\log_e(50 - v) + c, \; 50 - v > 0$$

When $t = 0$, $v = 0$,

$$\therefore \; 0 = -5\log_e 50 + c$$

$$\therefore \; c = 5\log_e 50$$

$$\therefore \; t = 5\log_e 50 - 5\log_e(50 - v)$$

$$= 5\log_e\left(\frac{50}{50 - v}\right)$$

$$\therefore \; \frac{t}{5} = \log_e\left(\frac{50}{50 - v}\right)$$

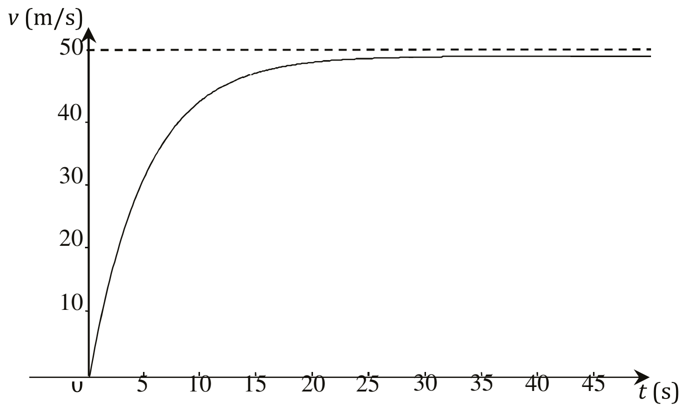
$$\therefore \; e^{\frac{t}{5}} = \frac{50}{50 - v}$$

$$\therefore \; (50 - v)e^{\frac{t}{5}} = 50$$

$$\therefore \; 50 - v = 50e^{-\frac{t}{5}}$$

$$\therefore \; v = 50 - 50e^{-\frac{t}{5}} = 50\left(1 - e^{-\frac{t}{5}}\right)$$

c **i**



ii

$$v = 47.5$$

$$\therefore \quad e^{\frac{t}{5}} = \frac{50}{50 - 47.5}$$

$$= 20$$

$$\therefore \quad \frac{t}{5} = \log_e 20$$

$$\therefore \quad t = \log_e 20$$

$$\approx 14.98$$

Alternatively, use a CAS calculator to solve the equation $50\left(1 - e^{\frac{t}{5}}\right) = 47.5$.
This gives $t = 14.98$, correct to 2 decimal places.

d **i**

$$v = \frac{dx}{dt} = 50\left(1 - e^{-\frac{t}{5}}\right)$$

$$\therefore \quad x = \int 50\left(1 - e^{-\frac{t}{5}}\right) dt$$

$$= 50\left(t + 5e^{-\frac{t}{5}}\right) + c$$

When $t = 0,\ x = 0$,

$$\therefore \quad 0 = 50(0 + 5e^0)\ c$$

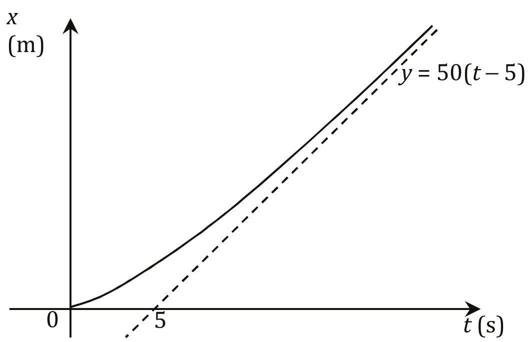
$$= 250 + c$$

$$\therefore \quad c = -250$$

$$\therefore \quad x = 50\left(t + 5e^{-\frac{t}{5}}\right) - 250$$

$$\therefore \quad x = 50\left(t + e^{-\frac{t}{5}} - 5\right),\ t \geq 0$$

ii



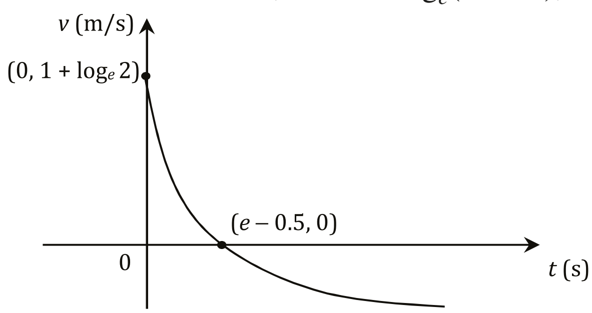
iii Use a CAS calculator to solve the equation

$50\left(t + 5e^{-\frac{t}{5}} - 5\right) = 8$ for $t > 0$.

This gives $t = 1.32$ correct to 2 decimal places.

2 a i $$v = A - \log_e(t + B)$$

If $A = 1$ and $B = 0.5$, $v = 1 - \log_e(t + 0.5)$, $t \geq 0$



ii When $t = 3$,

position of the particle $= \int_0^3 1 - \log_e(t + 0.5)\, dt$

$= 1.268\,756$ (CAS calculator)

$= 1.27$, correct to two decimal places.

The particle is 1.27 metres from 0 after three seconds.

iii Distance $= \int_0^{e-0.5} 1 - \log_e(t + 0.5)\, dt - \int_0^{e-0.5} 1 - \log_e(t + 0.5)\, dt$

$= 1.371\,708\,2 - (-0.102\,952\,2)$

$= 1.474\,660\,4$

$= 1.47$, correct to two decimal places.

The particle travels 1.47 metres in the three seconds after passing 0.

b $$v = A - \log_e(t + B)$$
$$\therefore \quad a = \frac{dv}{dt}$$
$$= \frac{-1}{t + B}$$

When $t = 10,\ a = \frac{-1}{20}$, $\quad \therefore \quad \frac{-1}{20} = \frac{-1}{10 + B}$
$$\therefore \quad B = 10$$
$$\therefore \quad v = A - \log_e(t + 10)$$

When $t = 100,\ v = 0$, $\quad \therefore \quad 0 = A - \log_e(110)$
$$\therefore \quad A = \log_e(110)$$
$$= 4.700\,48\ldots$$
$$= 4.70, \text{ correct to two decimal places.}$$

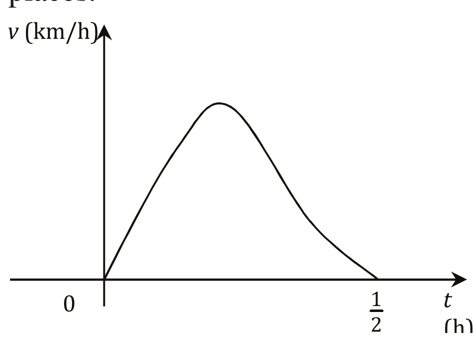
3 a $$v = kt(1 - \sin(\pi t))$$

When $v = 0,\ t = 0$ or $1 - \sin(\pi t) = 0$
$$\therefore \quad \sin(\pi t) = 1$$
$$\therefore \quad \pi t = \frac{\pi}{2} \text{ first value only required}$$
$$\therefore \quad t = \frac{1}{2}$$

It takes half an hour for the train to travel from A to B.

b **i** $a = \frac{dv}{dt}$
$$= kt(-\pi\cos(\pi t)) + k(1 - \sin(\pi t))$$
$$= -k(\sin(\pi t) + \pi t\ \cos(\pi t) - 1)$$

ii v is increasing from $t = 0$ to the value of t where v is a maximum, i.e., $t = 0.176\,979\,8$ from a CAS calculator. Thus the interval of time for which the velocity is increasing is $[0, 0.18]$, or the first 0.18 hours, correct to two decimal places.



c To find k when $\displaystyle\int_0^{0.5} kt(1 - \sin(\pi t))\ dt = 20,$

$$\int_0^{0.5} t(1 - \sin(\pi t))\ dt = 0.023\,678\,82 \text{ using a CAS calculator}$$

$$\therefore \int_0^{0.5} kt(1 - \sin(\pi t))\ dt = k\int_0^{0.5} t(1 - \sin(\pi t))\ dt$$

$$= k \times 0.023\,678\,82$$

$$k = \frac{20}{0.023\,678\,82}$$

$$= 844.636\,81\ldots$$

$$= 845, \text{ to three significant figures.}$$

4 a i $x = 28 + 4t - 5t^2 - t^3$

$$v = \frac{dx}{dt}$$

$$= 4 - 10t - 3t^2$$

ii $a = \dfrac{dv}{dt}$

$$= -10 - 6t$$

iii When $v = 0$, $-3t^2 - 10t + 4 = 0$

$$\therefore \quad t = \frac{10 \pm \sqrt{100 + 48}}{-6}$$

$$= \frac{10 - \sqrt{148}}{-6} \text{ since } t \geq 0$$

$$= \frac{-5 + \sqrt{37}}{3}$$

$$\approx 0.36$$

iv When $x = 28$, $28 + 4t - 5t^2 - t^3 = 28$

$$t^3 + 5t^2 - 4t = 0$$

$$t(t^2 + 5t - 4) = 0$$

$$\therefore \quad t = 0 \text{ or } \frac{-5 \pm \sqrt{25 + 16}}{2}$$

$$\text{As } t \geq 0,\ t = 0 \text{ or } \frac{\sqrt{41} - 5}{2}$$

$$= 0 \text{ or } 0.70, \text{ to two decimal places.}$$

v Use a CAS calculator to solve $28 + 4t - 5t^2 - t^3 = -28$. This gives $t = 2.92$ correct to 2 decimal places. Therefore the particle is 28 m to the left of 0 $t = 2.92$, correct to two decimal places.

b **i** For particle B, $a = 2 - 6t$

$$\therefore \quad v = \int 2 - 6t \, dt$$

$$= 2t - 3t^2 + c$$

When $t = 0$, $v = 2$ $\quad \therefore \quad c = 2$

$$\therefore \quad v = 2t - 3t^2 + 2$$

Now the position of B is $x = \int v \, dt$

$$= \int 2t - 3t^2 + 2 \, dt$$

$$= t^2 - t^3 + 2t + d$$

When $t = 0$, $x = 0$ $\quad \therefore \quad d = 0$

$\therefore \quad x = t^2 - t^3 + 2t$ is the position of B at time t.

ii When A and B collide,

$$28 + 4t - 5t^2 - t^3 = t^2 - t^3 + 2t$$

$$\therefore \quad 28 + 2t = 6t^2$$

$$\therefore \quad 3t^2 - t - 14 = 0$$

$$\therefore \quad (3t - 7)(t + 2) = 0$$

$$\therefore \quad t = \frac{7}{3} \text{ since } t \geq 0$$

A and B collide after $2\frac{1}{3}$ seconds.

iii Velocity of $A = v_A = 4 - 10t - 3t^2$

Velocity of $B = v_B = 2t - 3t^2 + 2$

When $t = \frac{7}{3}$, $v_A = 4 - 10 \times \frac{7}{3} - 3 \times \left(\frac{7}{3}\right)^2$

$$= \frac{-107}{3}$$

$$v_B = 2 \times \frac{7}{3} - 3 \times \left(\frac{7}{3}\right)^2 + 2$$

$$= \frac{-29}{3}$$

Yes, both particles are travelling to the left at the time of collision.

5 **a** **i** $x = 5\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$ ①

$v = \frac{dx}{dt} = \frac{-5\pi}{4}\sin\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$ ②

ii $a = \frac{dv}{dt} = \frac{-5\pi^2}{16}\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$ ③

b **i** Now from ② $v^2 = \frac{25\pi^2}{16}\sin^2\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

$\therefore \quad \sin^2\left(\frac{\pi}{4}t + \frac{\pi}{3}\right) = \frac{16v^2}{25\pi^2}$

and from ① $x^2 = 25\cos^2\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

$= 25\left(1 - \sin^2\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)\right)$

$= 25\left(1 - \frac{16v^2}{25\pi^2}\right)$

$= 25 - \frac{16v^2}{\pi^2}$

$\therefore \quad 25 - x^2 = \frac{16v^2}{\pi^2}$

$\therefore \quad v^2 = \frac{\pi^2}{16}(25 - x^2)$

$\therefore \quad v = \pm\frac{\pi}{4}\sqrt{25 - x^2}$

ii From ① and ③ $a = -\frac{\pi^2}{16}x$

c $v = \pm\frac{\pi}{4}\sqrt{25 - x^2}$

When $x = -2.5$, $v \approx \pm 3.400\,87$
The speed is 3.4 cm/s, correct to one decimal place.

d Now $a = \dfrac{-5\pi^2}{16}\cos\left(\dfrac{\pi}{4}t + \dfrac{\pi}{3}\right)$

When $t = 0$, $a = \dfrac{-5\pi^2}{16}\cos\left(\dfrac{\pi}{3}\right)$

$= \dfrac{-5\pi^2}{32}$

$= -1.542\,12\ldots$

The acceleration is -1.54 cm/s^2, correct to two decimal places.

e **i** Distance is modelled by a periodic circular function of amplitude 5. The maximum distance from 0 is 5 cm.

ii Velocity is modelled by a periodic circular function of amplitude $\dfrac{5\pi}{4}$. The maximum speed of the particle is $\dfrac{5\pi}{4}$ cm/s.

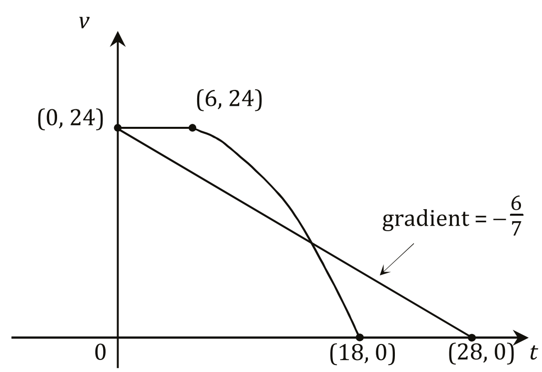
iii Acceleration is modelled by a periodic circular function, amplitude $\dfrac{5\pi^2}{16}$. The maximum magnitude of acceleration for the particle is $\dfrac{5\pi^2}{16}$ cm/s^2.

6 For the second lift,

$$a = -\frac{1}{3}(t - 6)$$
$$= -\frac{1}{3}t + 2$$
$$\therefore\ v = -\frac{t^2}{6} + 2t + c$$

When $t = 6$, $v = 24$

$$\therefore\ 24 = -\frac{36}{6} + 12 + c$$
$$\therefore\ 24 = 6 + c$$
$$\therefore\ c = 18$$
$$\therefore\ v = -\frac{t^2}{6} + 2t + 18$$



t-axis intercepts : $v = 0$

$$\therefore \quad -\frac{t^2}{6} + 2t + 18 = 0$$

$$\therefore \quad -\frac{1}{6}(t - 18)(t + 6) = 0$$

$$\therefore \quad t = 18 (t \geq 0)$$

For the first lift, $a = -\frac{6}{7}$

$$\therefore \quad v = 24 - \frac{6}{7}t$$

When $t = t_1$, $v = 0$,

$$\therefore \quad t_1 = 24 \times \frac{7}{6}$$

$$= 28$$

Distance travelled by first lift $= \frac{1}{2} \times 24 \times 28$

$$= 336 \text{ m}$$

Distance travelled by second lift $= 24 \times 6 + \int_6^{18} -\frac{t^2}{6} + 2t + 18 \; dt$

$$= 144 + \left[-\frac{t^3}{18} + t^2 + 18t\right]_6^{18}$$

$$= 144 + (324 - 132)$$

$$= 336 \text{ m}$$

The difference between the heights of the lifts when both have come to rest is zero.

7 a $a = -30(v + 110)^2, \; v \geq 0$

This can be written as $\frac{dv}{dt} = -30(v + 110)^2$

$$\therefore \quad \frac{dt}{dv} = -\frac{1}{30(v + 110)^2}$$

$$\therefore \quad t = -\frac{1}{30} \int \frac{1}{(v + 110)^2} \; dv$$

$$= -\frac{1}{30} \int (v + 110)^{-2} \; dv$$

$$= \frac{1}{30(v + 110)} + c$$

When $t = 0,\ v = 300,$

$$\therefore\quad 0 = \frac{1}{30(300 + 110)} + c$$

$$\therefore\quad c = \frac{-1}{12\,300}$$

$$t = \frac{1}{30(v + 110)} - \frac{1}{12\,300}$$

$$\text{and } t + \frac{1}{12\,300} = \frac{1}{30(v + 110)}$$

$$\therefore\quad \frac{12\,300t + 1}{12\,300} = \frac{1}{30(v + 110)}$$

$$\therefore\quad \frac{410}{12\,300t + 1} = v + 110$$

$$\therefore\quad v = \frac{410}{12\,300t + 1} - 110$$

$$= \frac{410 - 110(12\,300t + 1)}{12\,300t + 1}$$

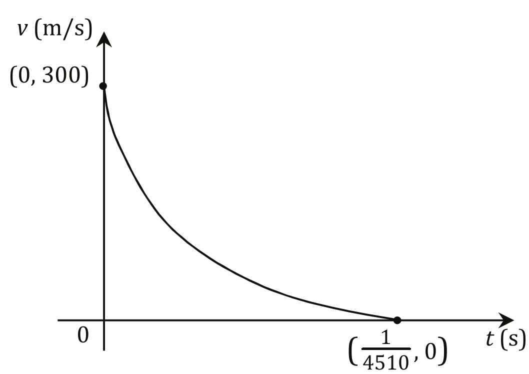
$$= \frac{300(1 - 4510t)}{12\,300t + 1}$$

When $v = 0,\ \dfrac{300(1 - 4510t)}{12\,300t + 1} = 0$

Solving for $t,\ t = \dfrac{1}{4510}$

$$\therefore\quad v = \frac{300(1 - 4510t)}{12\,300t + 1},\ 0 \le t \le \frac{1}{4510}$$

b



c **i** Now $\dfrac{dx}{dt} = \dfrac{300(1 - 4510t)}{12\,300t + 1}$

$$= -110 + \frac{410}{12\,300t + 1}$$

$$\therefore \quad x = \int -110 + \frac{410}{12\,300t + 1}\ dt$$

$$= -110t + \frac{1}{30}\log_e(12\,300t + 1) + c,\ t > 0$$

When $t = 0,\ x = 0$ and therefore $c = 0$

$$\therefore \quad x = -110t + \frac{1}{30}\log_e(12\,300t + 1)$$

ii Now $a = -30(v + 110)^2,\ v > 0$

$$\therefore \quad v\frac{dv}{dx} = -30(v + 110)^2$$

$$\frac{dv}{dx} = \frac{-30[v + 110]^2}{v}$$

$$\therefore \quad \frac{dx}{dv} = \frac{-v}{30(v + 110)^2}$$

$$\therefore \quad x = \int \frac{-v}{30(v + 110)^2}\ dv$$

$$= -\frac{1}{30}\int \frac{v}{(v + 110)^2}\ dv$$

Let $w = v + 110$, $\quad \therefore \quad \dfrac{dw}{dv} = 1$

$$\text{and } x = -\frac{1}{30}\int \frac{(w - 110)}{w^2}\,dw$$

$$= -\frac{1}{30}\int \frac{1}{w} - \frac{110}{w^2}\,dw$$

$$= -\frac{1}{30}\log_e(w) - \frac{11}{3w} + c,\ w > 0$$

$$= -\frac{1}{30}\log_e(v + 110) - \frac{11}{3(v + 110)} + c$$

When $x = 0,\ v = 300$

$$\therefore \quad 0 = -\frac{1}{30}\log_e(410) - \frac{11}{3(410)} + c$$

$$c = \frac{1}{30}\left(\log_e(410) + \frac{11}{41}\right)$$

$$x = \frac{1}{30}\left(\log_e\left(\frac{410}{v + 110}\right) - \frac{110}{v + 110} + \frac{11}{41}\right)$$

iii When $v = 0$, $x = \frac{1}{30}\left(\log_e\left(\frac{41}{11}\right) - 1 + \frac{11}{41}\right)$

$$= \frac{1}{30}\log_e\left(\frac{41}{11}\right) - \frac{1}{41}$$

$$= 0.019\,46\ldots$$

The bullet penetrates the shield by 0.19 m or 19 mm, to the nearest millimetre.

d **i**

$$a = -30(v^2 + 11\,000), \; v \ge 0$$

$$\therefore \quad \frac{dv}{dt} = -30(v^2 + 11\,000)$$

$$\therefore \quad \frac{dt}{dv} = \frac{-1}{30(v^2 + 11\,000)}$$

$$\therefore \quad t = \frac{-1}{30}\int \frac{1}{v^2 + 11\,000}\,dv$$

$$= \frac{-1}{30\sqrt{11\,000}}\int \frac{\sqrt{11\,000}}{(v^2 + 11\,000)}\,dv$$

$$= \frac{-1}{30\sqrt{11\,000}}\tan^{-1}\left(\frac{v}{10\sqrt{110}}\right) + c$$

When $t = 0$, $v = 300$, $\quad \therefore \quad 0 = \frac{-1}{30\sqrt{11\,000}}\tan^{-1}\left(\frac{30}{\sqrt{110}}\right) + c$

$$\therefore \quad c = \frac{1}{300\sqrt{110}}\tan^{-1}\left(\frac{30}{\sqrt{110}}\right)$$

$$\therefore \quad t = \frac{1}{300\sqrt{110}}\left(-\tan^{-1}\left(\frac{v}{10\sqrt{110}}\right) + \tan^{-1}\left(\frac{30}{\sqrt{110}}\right)\right)$$

ii Solving for v, $\quad 300\sqrt{110}\,t = -\tan^{-1}\left(\frac{v}{10\sqrt{110}}\right) + \tan^{-1}\left(\frac{30}{\sqrt{110}}\right)$

$$\tan^{-1}\left(\frac{v}{10\sqrt{110}}\right) = \tan^{-1}\left(\frac{30}{\sqrt{110}}\right) - 300\sqrt{110}t$$

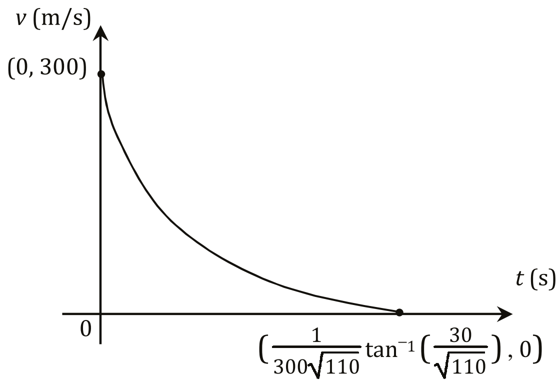
$$\therefore \quad v = 10\sqrt{110}\tan\left(\tan^{-1}\left(\frac{30}{\sqrt{110}}\right) - 300\sqrt{110}t\right)$$

When $v = 0$, $t = \frac{1}{300\sqrt{110}}\tan^{-1}\left(\frac{30}{\sqrt{110}}\right)$,

$$\therefore \quad 0 \le t \le \frac{1}{300\sqrt{110}}\tan^{-1}\left(\frac{30}{\sqrt{110}}\right)$$

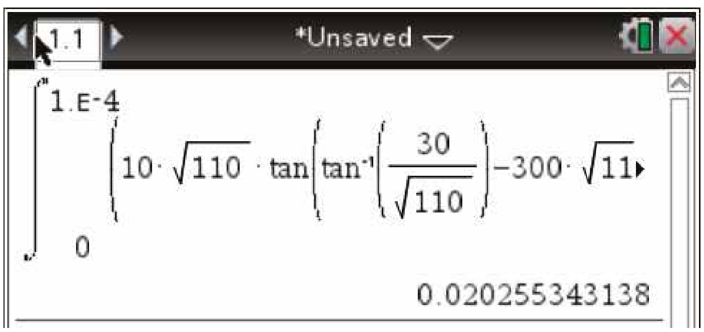
$$\text{or } 0 \le t \le \frac{\sqrt{110}}{33\,000}\tan^{-1}\left(\frac{3\sqrt{110}}{11}\right)$$

iii



iv Use a CAS calculator to find the area under the graph in **d iii**, given by $\int_0^{0.0001} v\,dt$ where v is given in **ii**.

The value is 0.020255 ≈ 0.020



The distance travelled in the first 0.0001 seconds is 20 mm, to the nearest millimetre.

8 a

$$v(t) = -\frac{3}{10}\left(t^3 - 21t^2 + \frac{364}{3}t - \frac{1281}{6}\right)$$

$$\therefore\ v(10) = -\frac{3}{10}\left((10)^3 - 21(10)^2 + \frac{364}{3}(10) - \frac{1281}{6}\right)$$

$$= \frac{601}{20}$$

$$= 30.05$$

b i $\frac{dv}{dt} = -\frac{3}{10}\left(3t^2 - 42t + \frac{364}{3}\right)$, where $4 \le t \le 10$

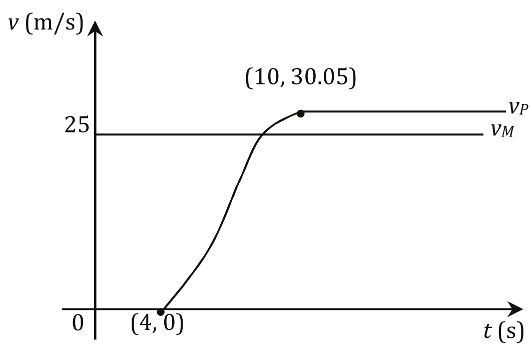
ii
$$a = \frac{dv}{dt}$$
$$= -\frac{3}{10}\left(3t^2 - 42t + \frac{364}{3}\right), \text{ where } 4 \le t \le 10$$

a is a maximum when $\frac{da}{dt} = 0$ (concave-down parabola)
$$\frac{da}{dt} = -\frac{3}{10}(6t - 42)$$
When $\frac{da}{dt} = 0, \quad t = 7$

The policeman's acceleration is a maximum when he has been chasing for three seconds

c For the policeman,
$$v_P = \begin{cases} 0 & 0 \le t < 4 \\ -\frac{3}{10}\left(t^3 - 21t^2 + \frac{364}{3}t - \frac{1281}{6}\right) & 4 \le t \le 10 \\ 30.05 & t > 10 \end{cases}$$

For the motorist, $v_M = 25$ for $t \ge 0$.



d **i** Distance travelled $= \int_4^{10} -\frac{3}{10}\left(t^3 - 21t^2 + \frac{364}{3}t - \frac{1281}{6}\right) dt$
$$= \left[-\frac{3}{10}\left(\frac{1}{4}t^4 - 7t^3 + \frac{182}{3}t^2 - \frac{1281}{6}t\right)\right]_4^{10}$$
$$= 90.3$$
The policeman travelled 90.3 m to reach his maximum speed.

ii Let x_P be the distance travelled by the policeman.
For $4 \le t \le 10$,
$$x_P = -\frac{3}{10}\left(\frac{1}{4}t^4 - 7t^3 + \frac{182}{3}t^2 - \frac{1281}{6}t\right) + c$$

When $t = 4$, $x_P = 0$,

$$\therefore \quad c = \frac{-401}{5}$$

$$\text{and } x_P = -\frac{3}{10}\left(\frac{1}{4}t^4 - 7t^3 + \frac{182}{3}t^2 - \frac{1281}{6}t\right) - \frac{401}{5},\ 4 \le t \le 10$$

$$= -\frac{3}{40}t^4 + \frac{21}{10}t^3 - \frac{91}{5}t^2 + \frac{1281}{20}t - \frac{401}{5},\ 4 \le t \le 10$$

For $t > 10$,

$$x_P = \frac{601}{20}t + d$$

When $t = 10$, $x_P = \dfrac{903}{10}$,

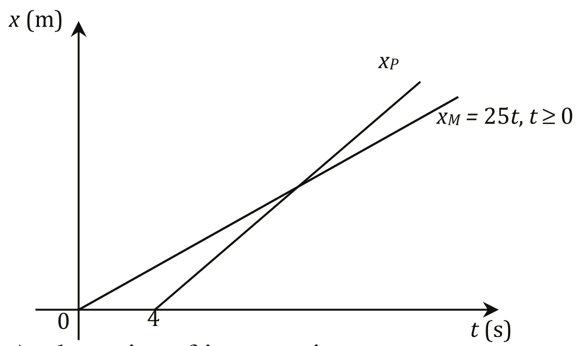
$$\therefore \quad \frac{903}{10} = \frac{601}{20} \times 10 + d$$

$$\text{and } d = -\frac{1051}{5}$$

$$\therefore \quad x_P = \frac{601}{20}t - \frac{1051}{5},\ t > 10$$

In summary:

$$x_P = \begin{cases} 0 & 0 \le t < 4 \\ -\dfrac{3}{40}t^4 + \dfrac{21}{10}t^3 - \dfrac{91}{5}t^2 + \dfrac{1281}{20}t - \dfrac{401}{5} & 4 \le t \le 10 \\ \dfrac{601}{20}t - \dfrac{1051}{5} & t > 10 \end{cases}$$

e 

At the point of intersection,

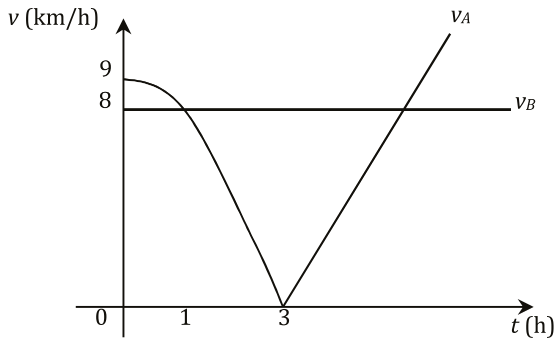
$$25t = \frac{601}{20}t - \frac{1051}{5}$$

$$\therefore \quad t = \frac{4204}{101}$$

$$= 41.623\,76\ldots$$

The policeman draws level with the motorist 41.62 seconds after the motorist passed him, correct to two decimal places.

9 **a**



b When $v_A = v_B$, $9 - t^2 = 8$

$$\therefore \quad t^2 = 1$$

which implies $t = 1$ $(t > 0)$

and $2t - 6 = 8$

$$\therefore \quad 2t = 14$$

$$\therefore \quad t = 7$$

The cyclists have the same speed after one hour and again after seven hours.

c **i** Let x_A and x_B be the distance of the cyclists A and B from the stationary point after T hours.

$x_B = 8T$

For x_A, first consider $0 \le T \le 3$.

$$x_A = \int_0^T 9 - t^2 \, dt$$

$$= 9T - \frac{1}{3}T^3$$

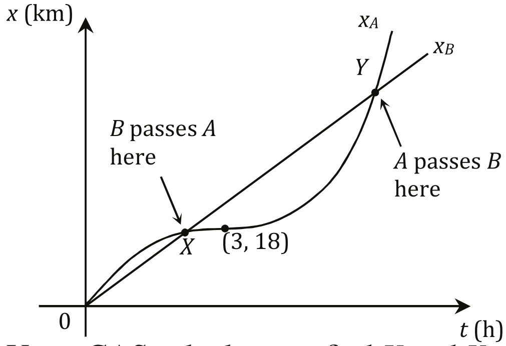
For $T > 3$, $x_A = \int_0^3 9 - t^2 \, dt + \int_3^T 2t - 6 \, dt$

$$= 18 + T^2 - 6T + 9$$

$$= T^2 - 6T + 27$$

Now that the integration has been completed we will change back to t.

$$x_A = \begin{cases} 9t - \frac{1}{3}t^3 & 0 \le t \le 3 \\ t^2 - 6t + 27 & t > 3 \end{cases}$$

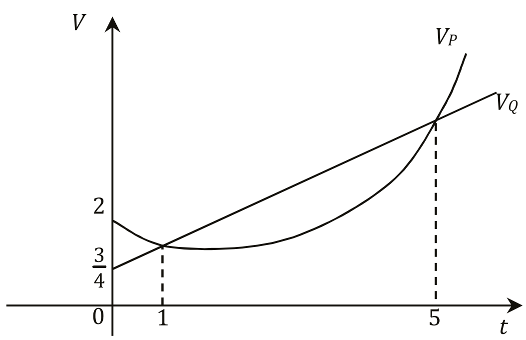


Use a CAS calculator to find X and Y.
At X, $t = 1.73$.
At Y, $t = 11.69$.
Therefore, A passes B 11.7 hours after the start of the race, correct to one decimal places.

ii B passes A 1.7 hours after the start of the race, correct to one decimal place.

10 a i When $V_P = V_Q$,
$$2 - t + \frac{1}{4}t^2 = \frac{3}{4} + \frac{1}{2}t$$
$$\therefore \quad \frac{1}{4}t^2 - \frac{3}{2}t + \frac{5}{4} = 0$$
$$\therefore \quad t^2 - 6t + 5 = 0$$
$$\therefore \quad (t-1)(t-5) = 0$$
$$\therefore \quad t = 1 \text{ or } t = 5$$
The velocities of P and Q are the same at $t = 1$ or $t = 5$.

ii



b i Let X_P and X_Q be the displacements of particles P and Q from the origin.
$$X_P = \int 2 - t + \frac{1}{4}t^2 \, dt$$
$$= 2t - \frac{1}{2}t^2 + \frac{1}{12}t^3 + c$$
When $t = 0$, $X_P = 0$ and thus $c = 0$.

$$\therefore \quad X_P = 2t - \frac{1}{2}t^2 + \frac{1}{12}t^3$$

Now $X_Q = \int \frac{3}{4} + \frac{1}{2}t \, dt$

$$= \frac{3}{4}t + \frac{1}{4}t^2 + d$$

When $t = 0$, $X_Q = 0$ and thus $d = 0$.

$$\therefore \quad X_Q = \frac{3}{4}t + \frac{1}{4}t^2$$

When $X_P = X_Q$,

$$2t - \frac{1}{2}t^2 + \frac{1}{12}t^3 = \frac{3}{4}t + \frac{1}{4}t^2$$

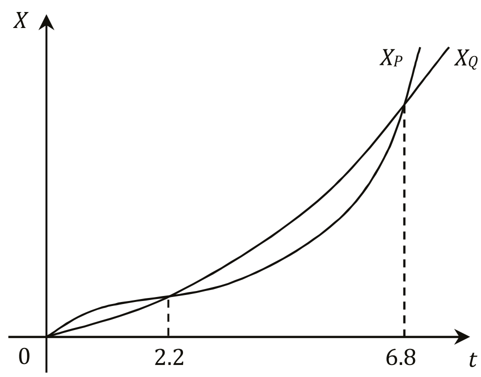
$$\therefore \quad t^3 - 9t^2 + 15t = 0$$

$$\therefore \quad t(t^2 - 9t + 15) = 0$$

$$\therefore \quad t = 0 \text{ or } \frac{9 \pm \sqrt{9^2 - 4 \times 15}}{2}$$

$$= 2.208\,71\ldots \text{ or } 6.791\,28\ldots$$

P and Q meet again when $t = 2.2$, correct to one decimal place.



ii P is further than Q from the starting point for $0 < T < 2.2$ and $t > 6.8$.

11 a i Choose vertically downwards to be the positive direction.

$a = 9.8$, $u = 0$, $s = 1.2$

Use the constant acceleration formula

$$v^2 = u^2 + 2as$$

$$= 0^2 + 2 \times 9.8 \times 1.2$$

$$= 23.52$$

$\therefore \quad v = 4.849\,74\ldots$ since $v > 0$

Annabelle's velocity when she hits the ground is 4.85 m/s, correct to two decimal places.

ii Use $v = u + at$

$$\therefore \quad t = \frac{v-u}{a}$$

$$= \frac{\sqrt{23.52} - 0}{9.8}$$

$$= 0.49487\ldots$$

It takes 0.49 seconds, correct to two decimal places, for Annabelle to hit the ground.

b **i**

$$a = 9.8 - t$$

$$\therefore \quad \frac{dv}{dt} = 9.8 - t$$

$$\therefore \quad v = \int 9.8 - t\,dt$$

$$= 9.8t - \frac{1}{2}t^2 + c$$

When $t = 0$, $v = 0$, $\quad \therefore \quad c = 0$

$$\therefore \quad v = 9.8t - \frac{1}{2}t^2$$

ii

$$\frac{dx}{dt} = 9.8t - \frac{1}{2}t^2$$

$$\therefore \quad x = \int 9.8t - \frac{1}{2}t^2\,dt$$

$$= 4.9t^2 - \frac{1}{6}t^3 + d$$

When $t = 0$, $x = 0$,

and therefore $d = 0$

$$x = 4.9t^2 - \frac{1}{6}t^3$$

iii Use a CAS calculator to solve $4.9t^2 - \frac{1}{6}t^3 = 1.2$. This gives $t = 0.499$.
So Annabelle hits the ground after 0.50 seconds correct to two decimal places.

c **i** Choose vertically upwards to be the positive direction, $\therefore \quad a = -4.9$

$$v = \int a\,dt$$

$$= \int -4.9\,dt$$

$$= -4.9t + c$$

When $t = 0,\ v = 0,$

$$\therefore \quad c = 0$$

$$\therefore \quad v = -4.9t$$

$$\therefore \quad x = \int v\, dt$$

$$= \int -4.9t\, dt$$

$$= -2.45t^2 + d$$

When $t = 0,\ x = 1.2,$

$$\therefore \quad d = 1.2$$

$$\therefore \quad x = 1.2 - 2.45t^2$$

ii For Annabelle, $a = -9.8,$

$$\therefore \quad v = -9.8t + c,\ t \geq 0.45$$

When $t = 0.45,\ v = 1.4,$

$$\therefore \quad 1.4 = -9.8 \times 0.45 + c$$

$$= -4.41 + c$$

$$\therefore \quad c = 5.81$$

$$\therefore \quad v = 5.81 - 9.8t,\ t \geq 0.45$$

$$\therefore \quad x = \int 5.81 - 9.8t\, dt$$

$$\therefore \quad x = 5.81t - 4.9t^2 + d,\ t \geq 0.45$$

When $t = 0.45,\ x = 0,$

$$\therefore \quad 0 = 5.81 \times 0.45 - 4.9 \times 0.45^2 + d$$

$$= 2.6145 - 0.992\,25 + d$$

$$\therefore \quad d = -1.622\,25$$

$$\therefore \quad x = 5.81t - 4.9t^2 - 1.622\,25,\ t \geq 0.45$$

Use a CAS calculator to solve the equation $1.2 - 2.45t^2 = 5.81t - 4.9t^2 - 1.622\,25$. This gives $t = 0.68175041$ and substituting gives $x = 0.06128014$. Thus the collision between Annabelle and Cuthbert occurs at a distance of 0.06 m, or 6 cm, above the ground, correct to the nearest centimetre.

12 a Acceleration $= \dfrac{\text{change in velocity}}{\text{change in time}}$

$$\therefore \quad 2 = \frac{6}{t}$$

$$\therefore \quad t = 3$$

The car is accelerating for three seconds.

b For $t \geq 13$,

$$a = -(v+2)$$

$$\therefore \quad \frac{dv}{dt} = -(v+2)$$

$$\therefore \quad \frac{dt}{dv} = \frac{-1}{v+2}$$

$$\therefore \quad t = \int \frac{-1}{v+2}\,dv$$

$$= -\log_e(v+2) + c, \ v \geq 0$$

When $t = 13$, $v = 6$,

$$\therefore \quad 13 = -\log_e 8 + c$$

$$\therefore \quad c = 13 + \log_e 8$$

$$\therefore \quad t = 13 + \log_e 8 - \log_e(v+2)$$

$$= 13 + \log_e\left(\frac{8}{v+2}\right)$$

$$\therefore \quad t - 13 = \log_e\left(\frac{8}{v+2}\right)$$

$$\therefore \quad e^{t-13} = \frac{8}{v+2}$$

$$\therefore \quad v = 8e^{13-t} - 2, \ t \geq 13$$

When $v = 0$, $8e^{13-t} = 2$,

$$\therefore \quad 13 - t = \log_e \frac{1}{4}$$

$$\therefore \quad t = 13 + \log_e 4$$

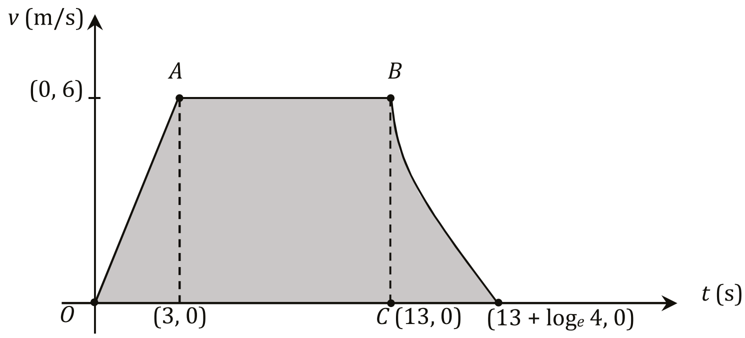
$$\therefore \quad v = \begin{cases} 2t & 0 \leq t \leq 3 \\ 6 & 3 < t \leq 13 \\ 8e^{13-t} - 2 & 13 < t \leq 13 + \log_e 4 \end{cases}$$

c When $v = 0$, $t = 13 + \log_e 4$

$$= 14.386\,29\ldots$$

The car is in motion for 14.4 seconds, to the nearest tenth of a second.

d



e Total distance = area of trapezium $OABC + \int_{13}^{13+\log_e 4} 8e^{13-t} - 2\,dt$

$$= 3(10 + 13) + [-8e^{13-t} - 2t]_{13}^{13+\log_e 4}$$

$$= 69 + (-8e^{13-(13+\log_e 4)} - 2(13 + \log_e 4) - (-8e^{13-13} - 2 \times 13)$$

$$= 69 + \left(-8e^{\log_e\left(\frac{1}{4}\right)} - 26 - 2\log_e 4\right) - (-8e^0 - 26)$$

$$= 69 - 8 \times \frac{1}{4} - 26 - 2\log_e 4 + 8 + 26$$

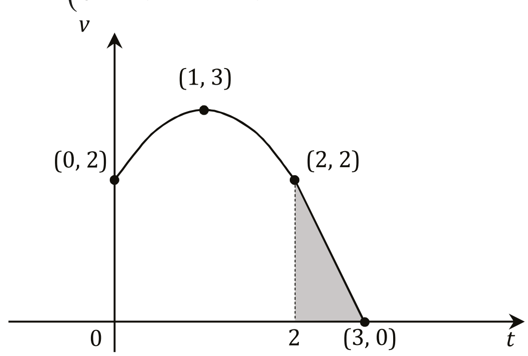
$$= 69 - 2 - 2\log_e 4 + 8$$

$$= 75 - 2\log_e 4$$

$$= 72.22741\ldots$$

The total distance travelled by the car is 72.2 m, to the nearest tenth of a metre.

13 a $v = \begin{cases} 3 - (t-1)^2 & 0 \le t \le 2 \\ 6 - 2t & t > 2 \end{cases}$



b The particle comes to rest when $v = 0$,

i.e. $6 - 2t = 0$

$\therefore \quad t = 3$

$$\text{Distance travelled} = \int_0^2 3 - (t-1)^2 \, dt + \frac{1}{2} \times 1 \times 2$$

$$= \left[3t - \frac{1}{3}(t-1)^3\right]_0^2 + 1$$

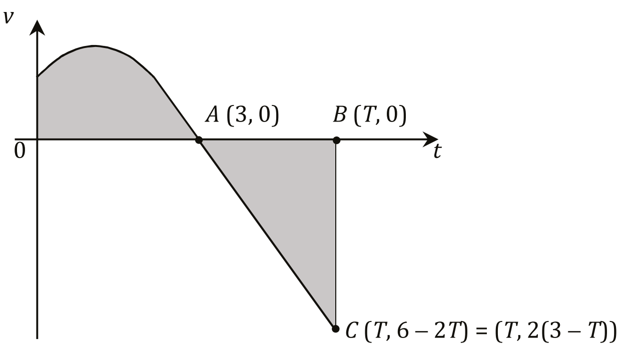
$$= \left(3 \times 2 - \frac{1}{3}(2-1)^3\right) - \left(0 - \frac{1}{3}(0-1)^3\right) + 1$$

$$= \left(6 - \frac{1}{3}\right) - \frac{1}{3} + 1$$

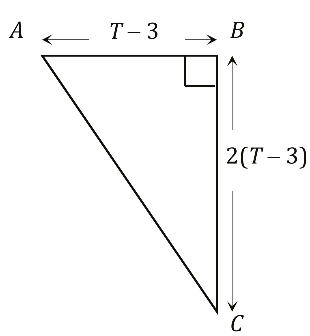
$$= \frac{19}{3}$$

The distance travelled when the particle first comes to rest is $\frac{19}{3}$ units.

c For return to original position, the areas on either side of the t axis are equal.



Consider the triangle ABC.



$$\text{Area of } \triangle ABC = \frac{1}{2}(T-3) \times 2(T-3)$$

$$= (T-3)^2$$

$$\text{Now } (T-3)^2 = \frac{19}{3}$$

$$\therefore \quad T - 3 = \sqrt{\frac{19}{3}} \text{ since } T - 3 > 0$$

$$\therefore \quad T = 3 + \sqrt{\frac{19}{3}}$$

$$= 5.516\,61\ldots$$

$$= 5.52, \text{ correct to two decimal places.}$$

Chapter 13 – Vector functions

Solutions to Exercise 13A

1 a $\boldsymbol{r}(t) = t\,\boldsymbol{i} + 2t\,\boldsymbol{j}$

Let (x, y) be a point on the curve.

$\therefore\ x = t$

$y = 2t$

$\therefore\ y = 2x$ is the cartesian equation

Domain is $\mathbb{R}$, range is $\mathbb{R}$

b $\boldsymbol{r}(t) = 2\,\boldsymbol{i} + 5t\,\boldsymbol{j}$

$\therefore\ x = 2$

$y = 5t$

$\therefore\ x = 2$ is the cartesian equation

Domain is $\{x : x = 2\}$, range is $\mathbb{R}$

c $\boldsymbol{r}(t) = -t\,\boldsymbol{i} + 7\,\boldsymbol{j}$

$\therefore\ x = -t$

$y = 7$

$\therefore\ y = 7$ is the cartesian equation

Domain is $\mathbb{R}$, range is $\{y : y = 7\}$

d $\boldsymbol{r}(t) = (2 - t)\,\boldsymbol{i} + (t + 7)\,\boldsymbol{j}$

Let (x, y) be a point on the curve.

$\therefore\ x = 2 - t \quad ①$

$y = t + 7 \quad ②$

Substitute ① into ②

$y = (2 - x) + 7$

$y = 9 - x$

Domain is $\mathbb{R}$, range is $\mathbb{R}$

e $\boldsymbol{r}(t) = t^2\,\boldsymbol{i} + (2 - 3t)\,\boldsymbol{j}$

$\therefore\ x = t^2 \quad ①$

$y = 2 - 3t$

$\therefore\ t = \dfrac{2 - y}{3}$

Substitute into ①

$\therefore\ x = \left(\dfrac{2 - y}{3}\right)^2$

$\therefore\ 9x = 4 - 4y + y^2$

$\therefore\ x = \dfrac{1}{9}(2 - y)^2$

Domain is $[0, \infty)$, range is $\mathbb{R}$

f $\boldsymbol{r}(t) = (t - 3)\,\boldsymbol{i} + (t^3 + 1)\,\boldsymbol{j}$

$\therefore\ x = t - 3 \quad ①$

$\therefore\ y = t^3 + 1 \quad ②$

From ① : $t = x + 3 \quad ③$

Substitute ③ into ②

$\therefore\ y = (x + 3)^3 + 1$

Domain is R, range is $\mathbb{R}$

g $\boldsymbol{r}(t) = (2t + 1)\,\boldsymbol{i} + 3^t\,\boldsymbol{j}$

$\therefore\ x = 2t + 1$

$\therefore\ t = \dfrac{x - 1}{2}$

$y = 3^t$

$\therefore\ y = 3^{\left(\frac{x-1}{2}\right)}$

Domain is $\mathbb{R}$, range is $(0, \infty)$

h $\boldsymbol{r}(t) = \left(t - \dfrac{\pi}{2}\right)\boldsymbol{i} + \cos 2t\,\boldsymbol{j}$

$x = t - \dfrac{\pi}{2} \quad ①$

$y = \cos 2t \quad ②$

From ① : $t = x + \dfrac{\pi}{2} \quad ③$

Substitute ③ into ②

$\therefore\ y = \cos(2x + \pi)$

$= -\cos 2x$

Domain is $\mathbb{R}$, range is $[-1, 1]$

i $\boldsymbol{r}(t) = \dfrac{1}{t+4}\boldsymbol{i} + (t^2+1)\boldsymbol{j},\ t \neq -4$

$\therefore \quad x = \dfrac{1}{t+4}$

$y = t^2 + 1 \qquad ①$

$\therefore \quad t = \dfrac{1}{x} - 4$

Substitute in ①

$y = \left(\dfrac{1}{x} - 4\right)^2 + 1$

Domain is $\mathbb{R}\backslash\{0\}$, range is $[1, \infty)$

j $\boldsymbol{r}(t) = \dfrac{1}{t}\boldsymbol{i} + \dfrac{1}{t+1}\boldsymbol{j}$

$x = \dfrac{1}{t} \qquad ①$

$y = \dfrac{1}{t+1} \qquad ②$

From ① : $\quad t = \dfrac{1}{x} \qquad ③$

Substitute ③ into ②

$y = \dfrac{1}{\dfrac{1}{x} + 1}$

$\therefore \quad y = \dfrac{x}{x+1},\ x \neq 0, -1$

Domain is $\mathbb{R}\backslash\{-1, 0\}$, range is $\mathbb{R}\backslash\{0, 1\}$

2 a Let $x = 2\cos(t)$ and $y = 3\sin(t), t \in \mathbb{R}$

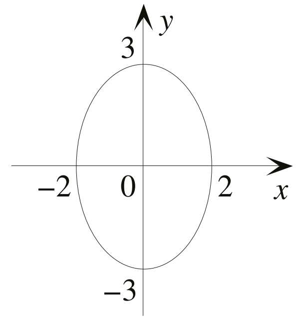
$\therefore \quad \dfrac{x}{2} = \cos(t)$ and $\dfrac{y}{3} = \sin(t)$

Squaring each and adding yields

$\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = \cos^2(t) + \sin^2(t)$

$\therefore \quad \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

The relation represents an ellipse with centre $(0, 0)$. The domain of the relation is $[-2, 2]$ and the range is $[-3, 3]$.



b Let $x = 2\cos^2(t)$ and $y = 3\sin^2(t)$, $t \in \mathbb{R}$

$\therefore \quad \dfrac{x}{2} = \cos^2(t)$ and $\dfrac{y}{3} = \sin^2(t)$

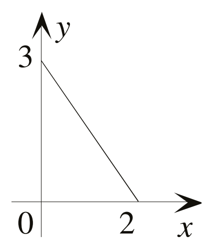
Adding yields

$\dfrac{x}{2} + \dfrac{y}{3} = \cos^2(t) + \sin^2(t)$

$\therefore \quad \dfrac{x}{2} + \dfrac{y}{3} = 1$

$\therefore \quad 3x + 2y = 6$

The relation is a straight line. The domain is $[0, 2]$ and the range is $[0, 3]$.

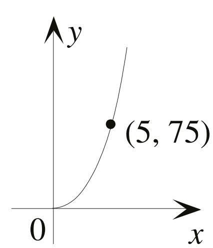


c Let $x = t$ and $y = 3t^2,\ t \geq 0$

$\therefore \quad y = 3x^2$

The relation is a parabola.

The domain is $R^+ \cup \{0\}$ and the range is $R^+ \cup \{0\}$.

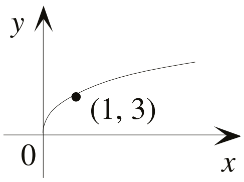


d Let $x = t^3$ and $y = 3t^2,\ t \geq 0$

$\therefore \quad t = x^{\frac{1}{3}}$ and $y = 3\left(x^{\frac{1}{3}}\right)^2$

$\therefore \quad y = 3x^{\frac{2}{3}}$

The domain is $\mathbb{R}^+ \cup \{0\}$ and the range is $\mathbb{R}^+ \cup \{0\}$.



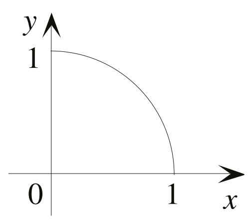
e Let $x = \cos(\lambda)$ and $y = \sin(\lambda)$, $\lambda \in \left[0, \frac{\pi}{2}\right]$

Squaring each and adding yields

$x^2 + y^2 = \cos^2(\lambda) + \sin^2(\lambda)$

$\therefore \quad x^2 + y^2 = 1$

The relation represents a circle with centre $(0, 0)$ and radius 1. The domain is $[0, 1]$ and the range is $[0, 1]$.



f Let $x = 3\sec(\lambda)$ and $y = 2\tan(\lambda)$, $\lambda \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \frac{x}{3} = \sec(\lambda)$ and $\frac{y}{2} = \tan(\lambda)$

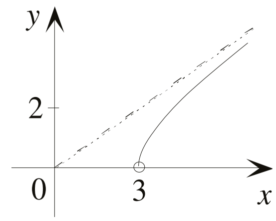
$\therefore \quad \left(\frac{x}{3}\right)^2 = \sec^2(\lambda)$ and $\left(\frac{y}{2}\right)^2 = \tan^2(\lambda)$

Since $\sec^2(\lambda) = \tan^2(\lambda) + 1$

then $\frac{x^2}{9} = \frac{y^2}{4} + 1$

$\therefore \quad \frac{x^2}{9} - \frac{y^2}{4} = 1$

The relation represents a hyperbola with centre $(0, 0)$ and asymptotes $y = \pm\frac{2x}{3}$. The domain is $(3, \infty)$ and the range is $(0,\ \infty)$.



g Let $x = 4\cos(2t)$ and $y = 4\sin(2t)$, $t \in \left[0, \frac{\pi}{2}\right]$

$\therefore \quad \frac{x}{4} = \cos(2t)$ and $\frac{y}{4} = \sin(2t)$

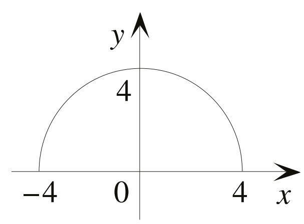
Squaring each and adding yields

$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2(2t) + \sin^2(2t)$

$\therefore \quad \frac{x^2}{16} + \frac{y^2}{16} = 1$

$\therefore \quad x^2 + y^2 = 16$

The relation represents a circle with centre $(0, 0)$ and radius 4. The domain is $[-4, 4]$ and the range is $[0, 4]$.



h Let $x = 3\sec^2(\lambda)$ and $y = 2\tan^2(\lambda)$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \frac{x}{3} = \sec^2(\lambda)$ and $\frac{y}{2} = \tan^2(\lambda)$

Since $\sec^2(\lambda) = \tan^2(\lambda) + 1$

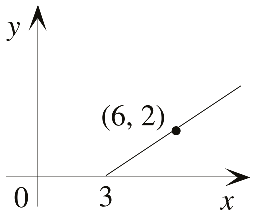
then $\frac{x}{3} = \frac{y}{2} + 1$

$\therefore \quad 2x = 3y + 6$

$\therefore \quad 3y = 2x - 6$

The relation is a straight line. The

domain is $[3, \infty)$ and the range is $[0, \infty)$.

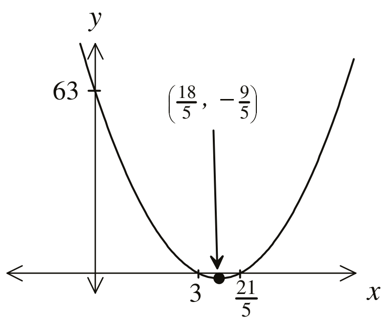


i Let $x = 3 - t$ and $y = 5t^2 + 6t,\ t \in R$

$\therefore \quad t = 3 - x$

$$\therefore \quad y = 5(3 - x)^2 + 6(3 - x)$$
$$= 5(x^2 - 6x + 9) + 18 - 6x$$
$$= 5x^2 - 30x + 45 + 18 - 6x$$
$$\therefore \quad y = 5x^2 - 36x + 63$$

The relation represents a parabola. The domain is $\mathbb{R}$ and the range is $\left[\frac{-9}{5}, \infty\right)$.



3 Let $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

a $y = 3 - 2x$

Let $x = t,\ t \in \mathbb{R}$

$\therefore \quad y = 3 - 2t$

$\therefore \boldsymbol{r}(t) = t\boldsymbol{i} + (3 - 2t)\boldsymbol{j},\ t \in \mathbb{R}$

b $x^2 + y^2 = 4$

$\therefore \quad y^2 = 4 - x^2$

Let $x = 2\cos(t)$

$$\therefore \quad y^2 = 4 - 4\cos^2(t)$$
$$= 4\sin^2(t)$$
$$\therefore \quad y = 2\sin(t)$$

$\therefore \quad \boldsymbol{r}(t) = 2\cos(t)\boldsymbol{i} + 2\sin(t)\boldsymbol{j},\ t \in \mathbb{R}$

c $(x - 1)^2 + y^2 = 4$

$\therefore \quad y^2 = 4 - (x - 1)^2$

Let $x = 2\cos(t) + 1$

$$\therefore \quad y^2 = 4 - (2\cos(t) + 1 - 1)^2$$
$$= 4 - (2\cos(t))^2$$
$$= 4 - 4\cos^2(t)$$
$$= 4\sin^2(t)$$
$$\therefore \quad y = 2\sin(t)$$

$\therefore \quad \boldsymbol{r}(t) = (2\cos(t) + 1)\boldsymbol{i} + 2\sin(t)\boldsymbol{j},\ t \in R$

d $x^2 - y^2 = 4$

$\therefore \quad y^2 = x^2 - 4$

Let $x = 2\sec(t)$

$$\therefore \quad y^2 = 4\sec^2(t) - 4$$
$$= 4\tan^2(t)$$
$$\therefore \quad y = 2\tan(t)$$

$\therefore \quad \boldsymbol{r}(t) = 2\sec(t)\boldsymbol{i} + 2\tan(t)\boldsymbol{j},$

$t \in R \backslash \left\{(2n + 1)\frac{\pi}{2},\ n \in \mathbb{Z}\right\}$

e $y = (x-3)^2 + 2(x-3)$

Let $x = t,\ t \in R$

$\therefore\quad y = (t-3)^2 + 2(t-3)$

$\therefore\quad \boldsymbol{r}(t) = t\,\boldsymbol{i} + ((t-3)^2 + 2(t-3))\boldsymbol{j},$

$t \in R$

f $2x^2 + 3y^2 = 12$

$\therefore\quad 3y^2 = 12 - 2x^2$

$\therefore\quad y^2 = 4 - \dfrac{2}{3}x^2$

Let $\dfrac{2}{3}x^2 = 4\cos^2(t)$

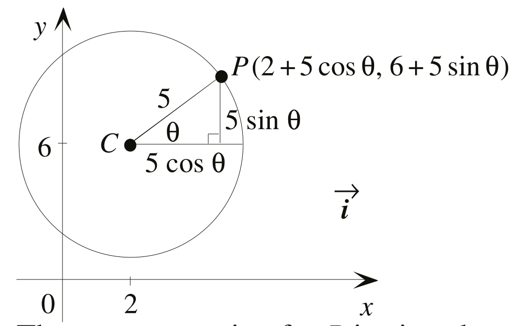
$\therefore\quad x^2 = 6\cos^2(t)$ and $y^2 = 4 - 4\cos^2(t)$

$\therefore\quad x = \sqrt{6}\cos(t) = 4\sin^2(t)$

$\therefore\quad y = 2\sin(t)$

$\therefore\quad \boldsymbol{r}(t) = \sqrt{6}\cos(t)\,\boldsymbol{i} + 2\sin(t)\boldsymbol{j},\ t \in R$

4 a



The vector equation for P is given by $\boldsymbol{r}(\theta) = (2 + 5\cos\theta)\,\boldsymbol{i} + (6 + 5\sin\theta)\boldsymbol{j}$

b The cartesian equation for P is given by $(x-2)^2 + (y-6)^2 = 25$

Solutions to Exercise 13B

1 $\boldsymbol{r}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}$

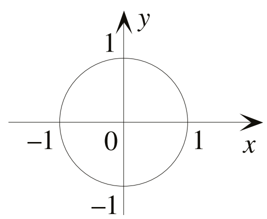
a $x = \cos t$

$y = \sin t$

$\therefore \quad x^2 + y^2 = \cos^2 t + \sin^2 t$

$\therefore \quad x^2 + y^2 = 1$

b



c $x = 0$ so $\cos t = 0$

$\therefore \quad t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \ldots$

i.e. $(2n-1)\dfrac{\pi}{2},\ n \in N$

2 a i $\boldsymbol{r}(t) = (t^2 - 9)\,\boldsymbol{i} + 8t\,\boldsymbol{j}$

$x = t^2 - 9 \qquad (1)$

$y = 8t \qquad (2)$

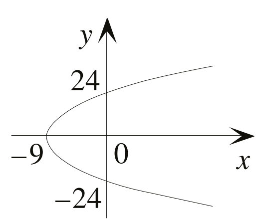
From (2): $t = \dfrac{y}{8} \qquad (3)$

Substitute (3) into (1)

$x = \left(\dfrac{y}{8}\right)^2 - 9$

$\therefore \quad x = \dfrac{y^2}{64} - 9$

ii



iii The path crosses the y axis when $x = 0$

$\therefore \quad t = -3$ and 3

b i $\boldsymbol{r}(t) = (t+1)\,\boldsymbol{i} + \dfrac{1}{t+2}\boldsymbol{j},\ t > -2$

$\therefore \quad x = t + 1,\ x > -1$

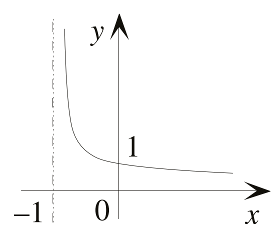
$\therefore \quad t = x - 1$

$y = \dfrac{1}{t+2}$

$\therefore \quad y = \dfrac{1}{x - 1 + 2}$

$\therefore \quad y = \dfrac{1}{x+1},\ x > -1$

ii



iii $x = 0$

$\therefore \quad t = -1$ (since $x = t + 1$)

c i $\boldsymbol{r}(t) = \dfrac{t-1}{t+1}\boldsymbol{i} + \dfrac{2}{t+1}\boldsymbol{j},$

$t > -1$

$x = \dfrac{t-1}{t+1} \qquad (1)$

$y = \dfrac{2}{t+1} \qquad (2)$

From (2): $t = \dfrac{2}{y} - 1 \qquad (3)$

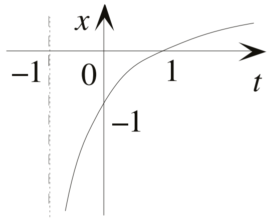
Substitute (3) into (1)

$$x = \frac{\left(\frac{2}{y} - 1\right) - 1}{\left(\frac{2}{y} - 1\right) + 1}$$

$x = 1 - y$

$y = 1 - x$

Graph $x = \frac{t-1}{t+1}$, $t > -1$ to find range of $y = 1 - x$

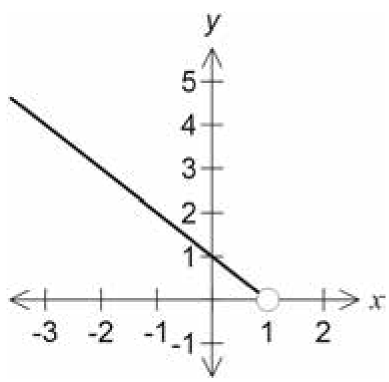


$$x = \frac{t-1}{t+1}$$

$$= 1 - \frac{2}{t+1}, \; t > -1$$

So for $y = 1 - x$, $x < 1$

ii



iii $x = 0$

$\therefore \; \frac{t-1}{t+1} = 0$

$\therefore \; t = 1$

3 a $\boldsymbol{r}_1(t) = (3t - 5)\boldsymbol{i} + (8 - t^2)\boldsymbol{j}$

$\boldsymbol{r}_2(t) = (3 - t)\boldsymbol{i} + 2t\boldsymbol{j}$

$\boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$

$(3t - 5)\boldsymbol{i} + (8 - t^2)\boldsymbol{j} = (3 - t)\boldsymbol{i} + 2t\boldsymbol{j}$

$3t - 5 = 3 - t$ ①

$8 - t^2 = 2t$ ②

From ①: $t = 2$

From ② $t^2 + 2t - 8 = 0$

$t = 2, -4$

$\therefore \; t = 2$

So the two particles collide at (1, 4) when $\boldsymbol{r} = \boldsymbol{i} + 4\boldsymbol{j}$

b $\boldsymbol{r}_1(t) = (3t - 5)\boldsymbol{i} + (8 - t^2)\boldsymbol{j}$

$\boldsymbol{r}_2(s) = (3 - s)\boldsymbol{i} + 2s\boldsymbol{j}$

$\boldsymbol{r}_1(t) = \boldsymbol{r}_2(s)$

$3t - 5 = 3 - s$ ①

$8 - t^2 = 2s$ ②

From ①: $s = 8 - 3t$ ③

Substitute ③ into ②

$t^2 + 2(8 - 3t) - 8 = 0$

$t^2 - 6t + 8 = 0$

$(t - 4)(t - 2) = 0$

$\therefore \; t = 2, 4$

$\therefore \; s = 2, \; -4$

Paths cross at (1, 4) and (7, −8)

c When $t = 3$,

$\boldsymbol{r}_1(t) = 4\boldsymbol{i} - \boldsymbol{j}$

$\boldsymbol{r}_2(t) = 6\boldsymbol{j}$

The displacement vector is

$\boldsymbol{r}_1(3) - \boldsymbol{r}_2(3) = 4\boldsymbol{i} - 7\boldsymbol{j}$

$\therefore$ the distance between the two particles

$= \sqrt{4^2 + (-7)^2}$

$= \sqrt{65}$

4 a $\boldsymbol{r}_1(t) = (2t^2 + 4)\boldsymbol{i} + (t - 2)\boldsymbol{j}$

$\boldsymbol{r}_2(t) = 9t\boldsymbol{i} + 3(t - 1)\boldsymbol{j}$

$\boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$

$(2t^2 + 4)\boldsymbol{i} + (t - 2)\boldsymbol{j} = 9t\boldsymbol{i} + 3(t - 1)\boldsymbol{j}$

$\therefore \quad 2t^2 + 4 = 9t$

$\therefore \quad 2t^2 - 9t + 4 = 0$

$\therefore \quad (2t - 1)(t - 4) = 0$

$\therefore \quad t = \frac{1}{2} \quad \text{or} \quad t = 4$

Now $t - 2 = 3(t - 1)$

$\therefore \quad t - 2 = 3t - 3$

$\therefore \quad 2t = 1$

$\therefore \quad t = \frac{1}{2}$

$\therefore \quad t = \frac{1}{2}$ at collision

$t = \frac{1}{2},$

$$\boldsymbol{r}_2\left(\frac{1}{2}\right) = \frac{9}{2}\boldsymbol{i} + 3\left(\frac{1}{2} - 1\right)\boldsymbol{j}$$

$\therefore \quad \left(\frac{9}{2}, -\frac{3}{2}\right)$ are the coordinates of the point where they collide.

b

$\boldsymbol{r}_1(t) = \boldsymbol{r}_2(s)$

$(2t^2 + 4)\boldsymbol{i} + (t - 2)\boldsymbol{j} = 9s\boldsymbol{i} + 3(s - 1)\boldsymbol{j}$

$\therefore \quad 2t^2 + 4 = 9s \quad ①$

$t - 2 = 3(s - 1)$

$\therefore \quad t = 3s - 3 + 2$

$\therefore \quad t = 3s - 1$

Substitute into ①

$2(3s - 1)^2 + 4 = 9s$

$\therefore \quad 2(9s^2 - 6s + 1) + 4 - 9s = 0$

$\therefore \quad 18s^2 - 21s + 6 = 0$

$\therefore \quad (9s - 6)(2s - 1) = 0$

$\therefore \quad s = \frac{2}{3} \quad \text{or} \quad s = \frac{1}{2}$

$s = \frac{2}{3}, \ x = 9 \times \frac{2}{3} = 6$

$y = 3\left(\frac{2}{3} - 1\right) = -1$

$\therefore \quad$ paths cross at $(6, -1)$ and $\left(\frac{9}{2}, -\frac{3}{2}\right)$

c $t = 3, \ \boldsymbol{r}_1(3) = 22\boldsymbol{i} + \boldsymbol{j}$

$\boldsymbol{r}_2(3) = 27\boldsymbol{i} + 6\boldsymbol{j}$

$\therefore \boldsymbol{r}_2(3) - \boldsymbol{r}_1(3) = 5\boldsymbol{i} + 5\boldsymbol{j}$

$\therefore \quad$ distance $= \sqrt{5^2 + 5^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$

5 a $\boldsymbol{r}(t) = (1 + t)\boldsymbol{i} + (3t + 2)\boldsymbol{j}$

When $t = 3$,

$\boldsymbol{r}(3) = 4\boldsymbol{i} + 11\boldsymbol{j}$

$\therefore \quad$ distance from origin $= \sqrt{(4)^2 + (11)^2}$

$= \sqrt{137}$

b $1 = \sqrt{(1 + t)^2 + (3t + 2)^2}$

$\therefore \quad (1 + t)^2 + (3t + 2)^2 = 1$

$t^2 + 2t + 1 + 9t^2 + 12t + 4 = 1$

$10t^2 + 14t + 4 = 0$

$(5t + 2)(t + 1) = 0$

$\therefore \quad t = -\frac{2}{5} \quad \text{or} \quad t = -1$

6 a $\boldsymbol{r}(t) = t\,\boldsymbol{i} + 2t\boldsymbol{j} - 3\boldsymbol{k}$

$\boldsymbol{r}(3) = 3\,\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$

$\therefore \quad A = (3, 6, -3)$

b distance $= \sqrt{9 + 36 + 9}$

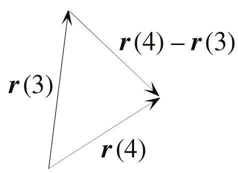
$= \sqrt{54}$

$= 3\sqrt{6}$

c $\boldsymbol{r}(4) = 4\,\boldsymbol{i} + 8\boldsymbol{j} - 3\boldsymbol{k}$

$\therefore \quad B = (4, 8, -3)$

d



displacement $= \boldsymbol{r}(4) - \boldsymbol{r}(3)$

$= 4\,\boldsymbol{i} + 8\boldsymbol{j} - 3\boldsymbol{k}$

$\quad - (3\,\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k})$

$= \boldsymbol{i} + 2\boldsymbol{j}$

7 a $\boldsymbol{r}(t) = (t + 1)\,\boldsymbol{i} + (3 - t)\boldsymbol{j} + 2t\boldsymbol{k}$

When $t = 2$,

$\boldsymbol{r}(2) = 3\,\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$

b Distance from the point $(4, -1, 1)$

Let $\boldsymbol{a} = 4\,\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

So $\boldsymbol{r}(2) - \boldsymbol{a} = (3 - 4)\,\boldsymbol{i} + (1 + 1)\boldsymbol{j}$

$\quad + (4 - 1)\boldsymbol{k}$

$= -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore \quad$ distance $= \sqrt{(-1)^2 + (2)^2 + (3)^2}$

$= \sqrt{14}$

8 $\boldsymbol{r}(t) = at^2\,\boldsymbol{i} + (b - t)\boldsymbol{j}$

$\boldsymbol{r}(3) = 9a\,\boldsymbol{i} + (b - 3)\boldsymbol{j}$

$\therefore \quad 9a = 6$

$a = \dfrac{2}{3}$

$b - 3 = 4$

$\therefore \quad b = 7$

$\therefore \quad a = \dfrac{2}{3},\ b = 7$

9 a $\boldsymbol{r}(t) = 3\cos t\,\boldsymbol{i} + 2\sin t\,\boldsymbol{j}$

$= x\,\boldsymbol{i} + y\boldsymbol{j}$

$x = 3\cos t \quad ①$

$y = 2\sin t \quad ②$

Squaring ① and ② gives

$x^2 = 9\cos^2 t \quad ③$

$y^2 = 4\sin^2 t \quad ④$

From ③ : $\dfrac{x^2}{9} = \cos^2 t \quad ⑤$

From 4 : $\dfrac{y^2}{4} = \sin^2 t \quad ⑥$

Adding ⑤ and ⑥

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

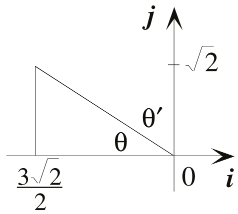
b When $t = 0$,

$\boldsymbol{r}(0) = 3\cos(0)\,\boldsymbol{i} + 2\sin(0)\boldsymbol{j} = 3\,\boldsymbol{i}$

c $t = \frac{3\pi}{4}$

i $\boldsymbol{r}(t) = 3\cos\left(\frac{3\pi}{4}\right)\boldsymbol{i} + 2\sin\left(\frac{3\pi}{4}\right)\boldsymbol{j}$

$= \frac{-3\sqrt{2}}{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{j}$



$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{2}}{\frac{3\sqrt{2}}{2}}$

$\theta = \tan^{-1}\left(\frac{2}{3}\right)$

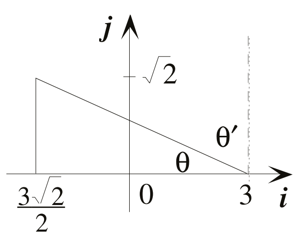
$\approx 33.69°$

$\therefore \quad \theta' = 90° - 33.69°$

$\approx 56.31°$

So the bearing of the particle from the origin is $360° - 56.31° = 303.69°$

ii Initial position is $\boldsymbol{r}(t) = 3\boldsymbol{i}$



$\tan\theta = \frac{\sqrt{2}}{\frac{3\sqrt{2}}{2} + 3}$

$\theta \approx 15.44°$

$\theta' \approx 90° - 15.44°$

$= 74.56°$

$\therefore$ the bearing of the particle from the initial position is $360° - 74.56° = 285.44°$

10 $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$

a $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$x = e^t$ ①

$y = e^{-t}$ ②

From ①: $t = \log_e x$ ③

Substitute ③ into ②

$y = \frac{1}{e^t}$

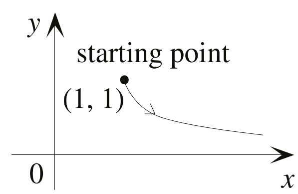
$= \frac{1}{e^{\log_e x}}$

$= \frac{1}{x}, \quad x \geq 1 \quad \text{if} \quad t \geq 0$

b When $t = 0$,

$\boldsymbol{r}(0) = e^0\boldsymbol{i} + e^{-0}\boldsymbol{j} = \boldsymbol{i} + \boldsymbol{j}$

c



11 $\boldsymbol{r}(t) = (e^t + e^{-t})\boldsymbol{i} + (e^t - e^{-t})\boldsymbol{j}$

a When $t = 0$,

$\boldsymbol{r}(0) = 2\boldsymbol{i}$

b When $t = \log_e 2$,

$\boldsymbol{r}(\log_e 2) = (e^{\log_e 2} + e^{-\log_e 2})\boldsymbol{i} + (e^{\log_e 2} - e^{-\log_e 2})\boldsymbol{j}$

$\boldsymbol{r}(\log_e 2) = \left(2 + \frac{1}{2}\right)\boldsymbol{i} + \left(2 - \frac{1}{2}\right)\boldsymbol{j}$

$= \frac{5}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j}$

c $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$x = e^t + e^{-t}$ ①

$y = e^t - e^{-t}$ ②

Squaring ① and ②

$x^2 = e^{2t} + 2 + e^{-2t}$ ③

$y^2 = e^{2t} - 2 + e^{-2t}$ ④

Subtracting ④ from ③

$x^2 - y^2 = 4$

12 $\boldsymbol{r}(t) = 100t\,\boldsymbol{i} + (100\sqrt{3}t - 5t^2)\boldsymbol{j}$,

$0 \le t \le 20\sqrt{3}$

a $t = 0$ (initial position)

$$\boldsymbol{r}(0) = 100(0)\,\boldsymbol{i} + (100\sqrt{3}(0) - 5(0)^2)\boldsymbol{j} = \boldsymbol{0}$$

$t = 20\sqrt{3}$ (final position)

$$\begin{aligned}\boldsymbol{r}(20\sqrt{3}) &= 100(20\sqrt{3})\,\boldsymbol{i} + [100\sqrt{3}(20\sqrt{3}) - 5(20\sqrt{3})^2]\boldsymbol{j} \\ &= 2000\sqrt{3}\,\boldsymbol{i} + (6000 - 6000)\boldsymbol{j} \\ &= 2000\sqrt{3}\,\boldsymbol{i}\end{aligned}$$

b $\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

$x = 100t$ ①

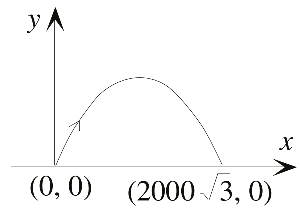
$y = 100\sqrt{3}t - 5t^2$ ②

From ① : $t = \dfrac{x}{100}$ ③

Substitute ③ into ②

$$y = 100\sqrt{3}\left(\frac{x}{100}\right) - 5\left(\frac{x}{100}\right)^2 = \sqrt{3}x - \frac{x^2}{2000},\ 0 \le x \le 2000\sqrt{3}$$

c



13 $\boldsymbol{r}_A(t) = 6t^2\,\boldsymbol{i} + (2t^3 - 18t)\boldsymbol{j}$

$\boldsymbol{r}_B(t) = (13t - 6)\,\boldsymbol{i} + (3t^2 - 27)\boldsymbol{j}$

For $\boldsymbol{i}$ direction:

$$6t^2 = 13t - 6$$

$$6t^2 - 13t + 6 = 0$$

$$(3t - 2)(2t - 3) = 0$$

$$\therefore \quad t = \frac{2}{3}, \frac{3}{2}$$

In the $\boldsymbol{j}$ direction:

$$2t^3 - 18t = 3t^2 - 27$$

$$2t^3 - 3t^2 - 18t + 27 = 0$$

$(t - 3)$ is a factor, leaving

$$(t - 3)(2t^2 + 3t - 9) = 0$$

$$\therefore \quad (t - 3)(2t - 3)(t + 3) = 0$$

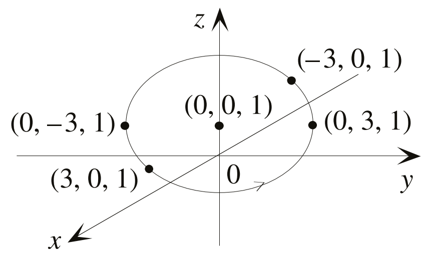
$$t = 3, \frac{3}{2} \text{ and } -3$$

Therefore, the particles collide at $t = \dfrac{3}{2}$

and $$\boldsymbol{r}_A\left(\frac{3}{2}\right) = 6\left(\frac{3}{2}\right)^2\boldsymbol{i} + \left(2\left(\frac{3}{2}\right)^2 - 18\left(\frac{3}{2}\right)\right)\boldsymbol{j} = \frac{27}{2}\boldsymbol{i} - \frac{81}{4}\boldsymbol{j}$$

14 $\boldsymbol{r}(t) = 3\cos t\,\boldsymbol{i} + 3\sin t\,\boldsymbol{j} + \boldsymbol{k}$

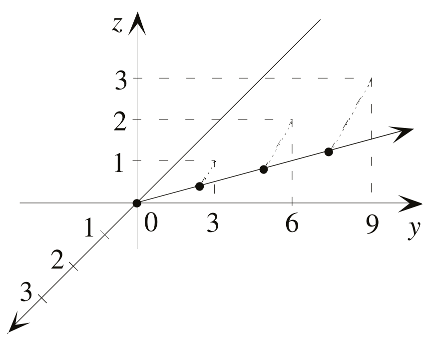
The particle is moving along a circular path, centred on (0, 0, 1) with radius length 3, starting at (3, 0, 1) and moving anticlockwise, always a distance of 1 'above' the x–y plane. It takes 2π units of time to complete one circle.



15 $\boldsymbol{r}(t) = t\boldsymbol{i} + 3t\boldsymbol{j} + t\boldsymbol{k}$

t	$\boldsymbol{r}(t)$	Point
0	$\boldsymbol{0}$	(0, 0, 0)
1	$\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$	(1, 3, 1)
2	$2\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}$	(2, 6, 2)
3	$3\boldsymbol{i} + 9\boldsymbol{j} + 3\boldsymbol{k}$	(3, 9, 3)

The particle is moving along a linear path, starting at (0, 0, 0) and moving 'forward' one, 'across' three and 'up' one at each step.



16

$$\boldsymbol{r}(t) = (1 - 2\cos 2t)\boldsymbol{i} + (3 - 5\sin 2t)\boldsymbol{j}$$

where $t \geq 0$

a Let $x = 1 - 2\cos 2t$ and $y = 3 - 5\sin 2t,\ t \geq 0$

$\therefore \quad \dfrac{x-1}{-2} = \cos 2t$ and $\dfrac{y-3}{-5} = \sin 2t$

Squaring and adding yields

$$\left(\frac{x-1}{-2}\right)^2 + \left(\frac{y-3}{-5}\right)^2 = \cos^2 2t + \sin^2 2t$$

$$\therefore \quad \frac{(x-1)^2}{4} + \frac{(y-3)^2}{25} = 1$$

The cartesian equation represents an ellipse with centre (1, 3).

The domain is [−1, 3] and the range is [−2, 8].

b **i** When $t = 0$,

$x = 1 - 2\cos 0$ and $y = 3 - 5\sin 0$

$= 1 - 2 \qquad = 3 - 0$

$= -1 \qquad = 3$

The position of the particle at $t = 0$ is (−1, 3).

ii When $t = \dfrac{\pi}{4}$,

$x = 1 - 2\cos\dfrac{\pi}{2}$ and $y = 3 - 5\sin\dfrac{\pi}{2}$

$= 1 - 0 \qquad = 3 - 5$

$= 1 \qquad = -2$

The position of the particle at $t = \dfrac{\pi}{4}$ is (1, −2).

iii When $t = \dfrac{\pi}{2}$,

$x = 1 - 2\cos\pi$ and $y = 3 - 5\sin\pi$

$= 1 + 2 \qquad = 3 - 0$

$= 3 \qquad = 3$

The position of the particle at $t = \dfrac{\pi}{2}$ is (3, 3).

c The particle moves along the ellipse with a period of π, i.e. it takes π units of time to complete one circuit.

d The particle moves in an anti-clockwise direction along the curve.

t

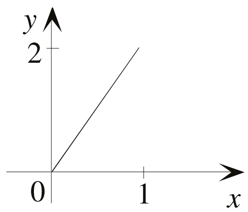
17 **a** $\boldsymbol{r}(t) = \cos^2(3\pi t)\,\boldsymbol{i} + 2\cos^2(3\pi t)\,\boldsymbol{j}$

i Let $x = \cos^2(3\pi t)$ and

$y = 2\cos^2(3\pi t),\ t \geq 0$

$\therefore\quad y = 2x,\ 0 \leq x \leq 1$

ii



iii The particle starts at (1, 2) and moves along a linear path towards the origin. When it reaches (0, 0) it reverses direction and heads towards (1, 2). It continues indefinitely in this pattern. It takes $\dfrac{2\pi}{6\pi} = \dfrac{1}{3}$ units of time to complete one cycle, i.e. to return to either end point.

b $\boldsymbol{r}(t) = \cos(2\pi t)\,\boldsymbol{i} + \cos(4\pi t)\,\boldsymbol{j}$

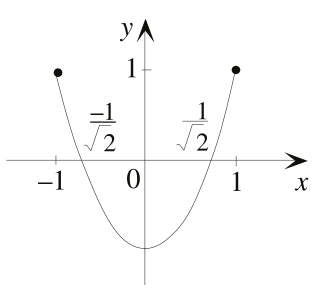
i Let $x = \cos(2\pi t)$ and

$y = \cos(4\pi t), t \geq 0$

$\therefore\quad y = 2\cos^2(2\pi t) - 1$

$\therefore\quad y = 2x^2 - 1,\ -1 \leq x \leq 1$

ii



iii

t	Point
0	(1, 1)
$\frac{1}{4}$	(0, −1)
$\frac{1}{2}$	(−1, 1)
$\frac{3}{4}$	(0, −1)
1	(1, 1)

The particle is moving along a parabolic path, starting at (1, 1) and reversing direction at (−1, 1). It takes $\dfrac{2\pi}{2\pi} = 1$ unit of time for one cycle.

c $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + e^{-2t}\,\boldsymbol{j}$

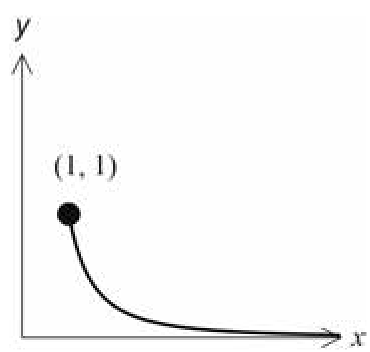
i Let $x = e^t$ and $y = e^{-2t},\ t \geq 0$

$\therefore\quad y = (e^t)^{-2}$

$= x^{-2}$

$\therefore\quad y = \dfrac{1}{x^2},\ x \geq 1$

ii



iii starting at (1, 1) and moving to the 'right' indefinitely.

Solutions to Exercise 13C

1 a $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} - e^{-t}\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$

b $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2t\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = 2\boldsymbol{j}$

c $\boldsymbol{r}(t) = \frac{1}{2}t\boldsymbol{i} + t^2\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = \frac{1}{2}\boldsymbol{i} + 2t\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = 2\boldsymbol{j}$

d $\boldsymbol{r}(t) = 16t\boldsymbol{i} - 4(4t-1)^2\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = 16\boldsymbol{i} - 32(4t-1)\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = -128\boldsymbol{j}$

e $\boldsymbol{r}(t) = \sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = \cos t\,\boldsymbol{i} - \sin t\,\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = -\sin t\,\boldsymbol{i} - \cos t\,\boldsymbol{j}$

f $\boldsymbol{r}(t) = (3+2t)\boldsymbol{i} + 5t\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = 2\boldsymbol{i} + 5\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = \boldsymbol{0}$

g $\boldsymbol{r}(t) = 100t\boldsymbol{i} + (100\sqrt{3}t - 4.9t^2)\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = 100\boldsymbol{i} + (100\sqrt{3} - 9.8t)\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$

h $\boldsymbol{r}(t) = \tan t\,\boldsymbol{i} + \cos^2 t\,\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = \sec^2 t\,\boldsymbol{i} - 2\cos t\sin t\,\boldsymbol{j}$

$= \sec^2 t\,\boldsymbol{i} - \sin 2t\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = 2\sec^2 t\tan t\,\boldsymbol{i} - 2\cos 2t\boldsymbol{j}$

2 a $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j},\ t_0 = 0$

$\boldsymbol{r}(t_0):\quad \boldsymbol{r}(0) = e^0\boldsymbol{i} + e^{-0}\boldsymbol{j} = \boldsymbol{i} + \boldsymbol{j}$

$\dot{\boldsymbol{r}}(t_0):\quad \dot{\boldsymbol{r}}(0) = e^0\boldsymbol{i} - e^{-0}\boldsymbol{j} = \boldsymbol{i} - \boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t_0):\quad \ddot{\boldsymbol{r}}(0) = e^0\boldsymbol{i} + e^{-0}\boldsymbol{j} = \boldsymbol{i} + \boldsymbol{j}$

Cartesian equation:

$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$x = e^t \quad ①$

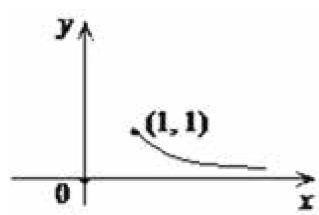
$y = e^{-t} \quad ②$

From ① : $t = \log_e x \quad ③$

Substitute ③ into ②

$y = \frac{1}{x}$

$t \geq 0 : x \geq 1,\ 0 < y \leq 1$



b $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j},\ t_0 = 1$

$\boldsymbol{r}(t_0):\quad \boldsymbol{r}(1) = \boldsymbol{i} + \boldsymbol{j}$

$\dot{\boldsymbol{r}}(t_0):\quad \dot{\boldsymbol{r}}(1) = \boldsymbol{i} + 2\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t_0):\quad \ddot{\boldsymbol{r}}(1) = 2\boldsymbol{j}$

Cartesian equation

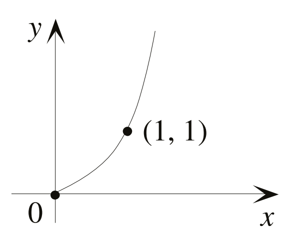
$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$x = t \quad ①$

$y = t^2 \quad ②$

Substitute ① into ②

$y = x^2 \quad t \geq 0 : x \geq 0,\ y \geq 0$



c $\boldsymbol{r}(t) = \sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j},\ t_0 = \frac{\pi}{6}$

$\boldsymbol{r}(t_0) : \boldsymbol{r}\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)\boldsymbol{i} + \cos\left(\frac{\pi}{6}\right)\boldsymbol{j}$

$= \frac{1}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t_0) : \dot{\boldsymbol{r}}(t) = \cos t\,\boldsymbol{i} - \sin t\,\boldsymbol{j}$

$\dot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)\boldsymbol{i} - \sin\left(\frac{\pi}{6}\right)\boldsymbol{j}$

$= \frac{\sqrt{3}}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t_0) : \ddot{\boldsymbol{r}}(t) = -\sin t\,\boldsymbol{i} - \cos t\,\boldsymbol{j}$

$\ddot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)\boldsymbol{i} - \cos\left(\frac{\pi}{6}\right)\boldsymbol{j}$

$= -\frac{1}{2}\boldsymbol{i} - \frac{\sqrt{3}}{2}\boldsymbol{j}$

Cartesian equation:

$\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

$x = \sin t$ ①

$y = \cos t$ ②

Square ① and ②

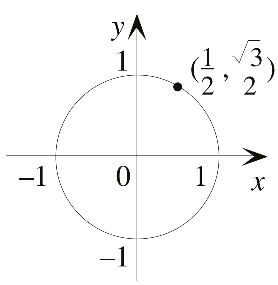
$x^2 = \sin^2 t$ ③

$y^2 = \cos^2 t$ ④

Add ③ and ④

$x^2 + y^2 = \sin^2 t + \cos^2 t$

$\therefore\ x^2 + y^2 = 1$



d $\boldsymbol{r}(t) = 16t\,\boldsymbol{i} - 4(4t - 1)^2\boldsymbol{j},$

$t_0 = 1$

$\boldsymbol{r}(t_0) : \boldsymbol{r}(1) = 16\boldsymbol{i} - 36\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t_0) : \dot{\boldsymbol{r}}(t) = 16\boldsymbol{i} - 32(4t - 1)\boldsymbol{j},$

$\dot{\boldsymbol{r}}(1) = 16\boldsymbol{i} - 96\boldsymbol{j},$

$\ddot{\boldsymbol{r}}(t_0) : \ddot{\boldsymbol{r}}(t) = -128\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(1) = -128\boldsymbol{j}$

Cartesian equation:

$\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

$x = 16t$ ①

$y = -4(4t - 1)^2$ ②

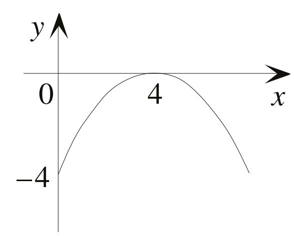
From ① : $t = \frac{x}{16}$ ③

Substitute ③ into ②

$y = -4\left(\frac{x}{4} - 1\right)^2$

$\therefore\ y = -\frac{1}{4}(x - 4)^2$

$t \geq 0 : x \geq 0,\ y \geq 0$



e

$\boldsymbol{r}(t) = \frac{1}{t + 1}\boldsymbol{i} + (t + 1)^2\boldsymbol{j},\ t_0 = 1$

$\boldsymbol{r}(t_0) : \boldsymbol{r}(1) = \frac{1}{2}\boldsymbol{i} + 4\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t_0) : \dot{\boldsymbol{r}}(t) = \frac{-1}{(t + 1)^2}\boldsymbol{i} + 2(t + 1)\boldsymbol{j}$

$\dot{\boldsymbol{r}}(1) = -\frac{1}{4}\boldsymbol{i} + 4\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t_0) : \ddot{\boldsymbol{r}}(t) = \frac{2}{(t + 1)^3}\boldsymbol{i} + 2\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(1) = \frac{1}{4}\boldsymbol{i} + 2\boldsymbol{j}$

Cartesian equation:

$$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$$

$$x = \frac{1}{t+1} \quad (1)$$

$$y = (t+1)^2 \quad (2)$$

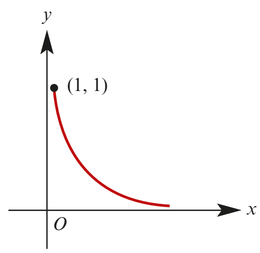
From (1): $t = \frac{1}{x} - 1 \quad (3)$

Substitute (3) into (2)

$$y = \left(\frac{1}{x} - 1 + 1\right)^2$$

$$\therefore \quad y = \frac{1}{x^2}$$

$t \geq 0 : 0 < x \leq 1, \ y \geq 1$



3 a $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j}; \ t = \frac{\pi}{4}$

Let $x = \cos(t)$ and $y = \sin(t)$

$$\therefore \quad \frac{dx}{dt} = -\sin(t) \quad \text{and} \quad \frac{dy}{dt} = \cos(t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \cos(t) \times \frac{1}{-\sin(t)}$$

$$= -\cot(t)$$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = -\cot\left(\frac{\pi}{4}\right)$

$$= -1$$

The gradient of the curve at $t = \frac{\pi}{4}$ is -1.

b $\boldsymbol{r}(t) = \sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j}; \ t = \frac{\pi}{2}$

Let $x = \sin(t)$ and $y = \cos(t)$

$$\therefore \quad \frac{dx}{dt} = \cos(t) \quad \text{and} \quad \frac{dy}{dt} = -\sin(t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -\sin(t) \times \frac{1}{\cos(t)}$$

$$= -\tan(t)$$

When $t = \frac{\pi}{2}$, $\frac{dy}{dx} = -\tan\left(\frac{\pi}{2}\right)$ which is undefined.

The gradient of the curve at $t = \frac{\pi}{2}$ is undefined.

c $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-2t}\boldsymbol{j}; \ t = 1$

Let $x = e^t$ and $y = e^{-2t}$

$$\therefore \quad \frac{dx}{dt} = e^t \quad \text{and} \quad \frac{dy}{dt} = -2e^{-2t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -2e^{-2t} \times \frac{1}{e^t}$$

$$= -2e^{-3t}$$

When $t = 1$, $\frac{dy}{dx} = -2e^{-3}$

The gradient of the curve at $t = 1$ is $-2e^{-3}$.

d $\boldsymbol{r}(t) = 2t^2\boldsymbol{i} + 4t\boldsymbol{j}; \ t = 2$

Let $x = 2t^2$ and $y = 4t$

$$\therefore \quad \frac{dx}{dt} = 4t \quad \text{and} \quad \frac{dy}{dt} = 4$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 4 \times \frac{1}{4t}$$

$$= \frac{1}{t}$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$

The gradient of the curve at $t = 2$ is $\frac{1}{2}$.

e $\boldsymbol{r}(t) = (t+2)\boldsymbol{i} + (t^2 - 2t)\boldsymbol{j};\ t = 3$

Let $x = t + 2$ and $y = t^2 - 2t$

$$\therefore \quad \frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = 2t - 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= (2t - 2) \times 1$$
$$= 2t - 2$$

When $t = 3$, $\frac{dy}{dx} = 2(3) - 2 = 4$

The gradient of the curve at $t = 3$ is 4.

f

$$\boldsymbol{r}(t) = \cos(\pi t)\,\boldsymbol{i} + \cos(2\pi t)\boldsymbol{j};\ t = \frac{1}{4}$$

Let $x = \cos(\pi t)$ and $y = \cos(2\pi t)$

$$\therefore \quad \frac{dx}{dt} = -\pi \sin(\pi t)$$

and $\frac{dy}{dt} = -2\pi \sin(2\pi t)$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= -2\pi \sin(2\pi t) \times \frac{1}{-\pi \sin(\pi t)}$$
$$= \frac{2\sin(2\pi t)}{\sin(\pi t)}$$
$$= \frac{2 \times 2 \sin(\pi t)\cos(\pi t)}{\sin(\pi t)}$$
$$= 4\cos(\pi t)$$

When $t = \frac{1}{4}$, $\frac{dy}{dx} = 4\cos\left(\frac{\pi}{4}\right)$

$$= \frac{4}{\sqrt{2}}$$
$$= 2\sqrt{2}$$

The gradient of the curve at $t = \frac{1}{4}$ is $2\sqrt{2}$.

4 a $\dot{\boldsymbol{r}}(t) = 4\boldsymbol{i} + 3\boldsymbol{j}$

$\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

$\boldsymbol{r}(t) = 4t\boldsymbol{i} + 3t\boldsymbol{j} + \boldsymbol{c}_1$

$\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

$\therefore \quad \boldsymbol{c}_1 = \boldsymbol{i} - \boldsymbol{j}$

$\therefore \quad \boldsymbol{r}(t) = (4t + 1)\boldsymbol{i} + (3t - 1)\boldsymbol{j}$

b $\dot{\boldsymbol{r}}(t) = 2t\boldsymbol{i} + 2\boldsymbol{j} - 3t^2\boldsymbol{k}$

$\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

$\boldsymbol{r}(t) = t^2\boldsymbol{i} + 2t\boldsymbol{j} - t^3\boldsymbol{k} + \boldsymbol{c}_1$

$\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

$\therefore \quad \boldsymbol{c}_1 = \boldsymbol{i} - \boldsymbol{j}$

$\therefore \quad \boldsymbol{r}(t) = (t^2 + 1)\boldsymbol{i} + (2t - 1)\boldsymbol{j} - t^3\boldsymbol{k}$

c $\dot{\boldsymbol{r}}(t) = e^{2t}\boldsymbol{i} + 2e^{0.5t}\boldsymbol{j}$

$\boldsymbol{r}(0) = \frac{1}{2}\boldsymbol{i}$

$\therefore \quad \boldsymbol{r}(t) = \frac{1}{2}e^{2t}\boldsymbol{i} + 4e^{0.5t}\boldsymbol{j} + \boldsymbol{c}_1$

$\boldsymbol{r}(0) = \frac{1}{2}\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{c}_1$

$\frac{1}{2}\boldsymbol{i} = \frac{1}{2}\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{c}_1$

$\therefore \quad \boldsymbol{c}_1 = -4\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}(t) = \frac{1}{2}e^{2t}\boldsymbol{i} + 4(e^{0.5t} - 1)\boldsymbol{j}$

d $\ddot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2t\boldsymbol{j}$

$\dot{\boldsymbol{r}}(0) = \boldsymbol{i}$

$\boldsymbol{r}(0) = \boldsymbol{0}$

$\dot{\boldsymbol{r}}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + \boldsymbol{c}_1$

$\dot{\boldsymbol{r}}(0) = \boldsymbol{c}_1$

$\therefore \quad \boldsymbol{c}_1 = \boldsymbol{i}$

$\dot{\boldsymbol{r}}(t) = (t+1)\boldsymbol{i} + t^2\boldsymbol{j}$

$\boldsymbol{r}(t) = \left(\frac{1}{2}t^2 + t\right)\boldsymbol{i} + \frac{1}{3}t^3\boldsymbol{j} + \boldsymbol{c}_2$

$\boldsymbol{r}(0) = \boldsymbol{c}_2$

$\therefore \quad \boldsymbol{c}_2 = \boldsymbol{0}$

$\therefore \quad \boldsymbol{r}(t) = \left(\frac{1}{2}t^2 + t\right)\boldsymbol{i} + \frac{1}{3}t^3\boldsymbol{j}$

e $\ddot{\boldsymbol{r}}(t) = \sin 2t\,\boldsymbol{i} - \cos\frac{1}{2}t\boldsymbol{j}$

$\dot{\boldsymbol{r}}(0) = -\frac{1}{2}\boldsymbol{i}$

$\boldsymbol{r}(0) = 4\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = -\frac{1}{2}\cos 2t\,\boldsymbol{i} - 2\sin\frac{1}{2}t\boldsymbol{j} + \boldsymbol{c}_1$

$\dot{\boldsymbol{r}}(0) = -\frac{1}{2}\boldsymbol{i} + \boldsymbol{c}_1$

$-\frac{1}{2}\boldsymbol{i} = -\frac{1}{2}\boldsymbol{i} + \boldsymbol{c}_1$

$\therefore \quad \boldsymbol{c}_1 = \boldsymbol{0}$

$\therefore \quad \boldsymbol{r}(t) = -\frac{1}{4}\sin 2t\,\boldsymbol{i} + 4\cos\frac{1}{2}t\boldsymbol{j} + \boldsymbol{c}_2$

$\boldsymbol{r}(0) = 4\boldsymbol{j} + \boldsymbol{c}_2$

$4\boldsymbol{j} = 4\boldsymbol{j} + \boldsymbol{c}_2$

$\therefore \quad \boldsymbol{c}_2 = \boldsymbol{0}$

$\therefore \quad \boldsymbol{r}(t) = -\frac{1}{4}\sin 2t\,\boldsymbol{i} + 4\cos\frac{1}{2}t\boldsymbol{j}$

5 $\boldsymbol{r}(t) = \sin t\,\boldsymbol{i} + t\boldsymbol{j} + \cos t\boldsymbol{k}$

$\dot{\boldsymbol{r}}(t) = \cos t\,\boldsymbol{i} + \boldsymbol{j} - \sin t\boldsymbol{k}$

$\ddot{\boldsymbol{r}}(t) = -\sin t\,\boldsymbol{i} - \cos t\boldsymbol{k}$

For $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ to be perpendicular

$\dot{\boldsymbol{r}}(t)\cdot\ddot{\boldsymbol{r}}(t) = 0$

and

$(\cos t \times -\sin t) + (-\sin t \times -\cos t)$

$= -\sin t \times \cos t + \sin t \times \cos t$

$= 0$

6 $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + 16t^2(3-t)\boldsymbol{j}$

a $\dot{\boldsymbol{r}}(t) = 2\boldsymbol{i} + (96t - 48t^2)\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = (96 - 96t)\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are perpendicular when

$\dot{\boldsymbol{r}}(t)\cdot\ddot{\boldsymbol{r}}(t) = 0$

$\therefore \quad (96t - 48t^2)(96 - 96t) = 0$

$\therefore \quad t = 0$ or 1 or 2

But when $t = 1, \ddot{\boldsymbol{r}}(1) = 0$

So $t = 0$ or 2

b $\dot{\boldsymbol{r}}(0) = 2\boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(0) = 96\boldsymbol{j}$

$\dot{\boldsymbol{r}}(2) = 2\boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(2) = -96\boldsymbol{j}$

7 $\boldsymbol{r}(t) = at\,\boldsymbol{i} + \frac{a^2t^2}{4}\boldsymbol{j},\ a > 0$

a Cartesian equation:

$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$x = at$ ①

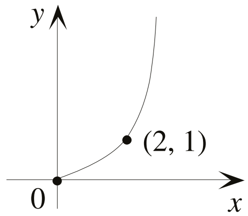
$y = \frac{a^2f^2}{4}$ ②

From ① : $t = \frac{x}{a}$ ③

Substitute ③ into ②

$$y = \frac{a^2}{4}\left(\frac{x}{a}\right)^2$$

$$\therefore \quad y = \frac{x^2}{4}, t \geq 0 \text{ so } x \geq 0$$



b

$$\ddot{\boldsymbol{r}}(t) = a\,\boldsymbol{i} + \frac{a^2}{2}t\boldsymbol{j}$$

$$\ddot{\boldsymbol{r}}(t) = \frac{a^2}{2}\boldsymbol{j}$$

Using $\cos\theta^\circ = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$

where $\boldsymbol{a} = \dot{\boldsymbol{r}}(t)$ and $\boldsymbol{b} = \ddot{\boldsymbol{r}}(t)$

$$\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= (a \times 0) + \left(\frac{a^2 t}{2} \times \frac{a^2}{2}\right) \\
&= \frac{a^4 t}{4} \\
|\boldsymbol{a}|\,|\boldsymbol{b}| &= \sqrt{a^2 + \frac{a^4 t^2}{4}} \times \sqrt{\frac{a^4}{4}} \\
&= \sqrt{\frac{4a^2 + a^4 t^2}{4}} \times \frac{a^2}{2} \\
&= \sqrt{\frac{a^2}{4}}(\sqrt{4 + a^2 t^2}) \times \frac{a^2}{2} \\
&= \frac{a^3}{4}(\sqrt{4 + a^2 t^2})
\end{aligned}$$

$$\begin{aligned}
\cos 45^\circ &= \frac{a^4 t}{4} \times \frac{4}{a^3\sqrt{4 + a^2 t^2}} \\
\frac{1}{\sqrt{2}} &= \frac{at}{\sqrt{4 + a^2 t^2}} \\
\sqrt{2}at &= \sqrt{4 + a^2 t^2} \\
2a^2 t^2 &= 4 + a^2 t^2 \\
a^2 t^2 &= 4 \\
t^2 &= \frac{4}{a^2} \\
\therefore \quad t &= \frac{2}{a} \quad (t \geq 0)
\end{aligned}$$

8 $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + (t^2 - 4)\boldsymbol{j}$

a Cartesian equation:

$$\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$$

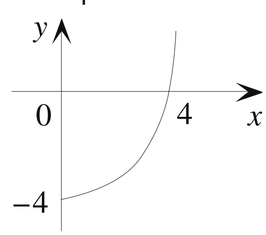
$$x = 2t \qquad ①$$

$$y = t^2 - 4 \qquad ②$$

From ①: $\quad t = \dfrac{x}{2} \qquad ③$

Substitute ③ into ②

$$y = \frac{x^2}{4} - 4, \ t \geq 0 \text{ so } x \geq 0$$



b $\dot{\boldsymbol{r}}(t) = 2\boldsymbol{i} + 2t\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = 2\boldsymbol{j}$

At $t = 1$, $\dot{\boldsymbol{r}}(1) = 2\boldsymbol{i} + 2\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(1) = 2\boldsymbol{j}$

$$\cos\theta^\circ = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

where $\boldsymbol{a} = \dot{\boldsymbol{r}}(1)$ and $\boldsymbol{b} = \ddot{\boldsymbol{r}}(1)$

$\boldsymbol{a}\cdot\boldsymbol{b} = 4$

$|\boldsymbol{a}|\,|\boldsymbol{b}| = \sqrt{8}\times 2$

$$\cos\theta^\circ = \frac{4}{2\sqrt{8}}$$

$$\cos\theta^\circ = \frac{1}{\sqrt{2}}$$

$\therefore\quad \theta = 45^\circ$

c $$\cos\theta^\circ = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

where $\boldsymbol{a} = \dot{\boldsymbol{r}}(t)$ and $\boldsymbol{b} = \ddot{\boldsymbol{r}}(t)$

$$\cos 30^\circ = \frac{4t}{\sqrt{4+4t^2}\times 2}$$

$$\frac{\sqrt{3}}{2} = \frac{t}{\sqrt{1+t^2}}$$

$$\sqrt{1+t^2} = \frac{2t}{\sqrt{3}}$$

$$1+t^2 = \frac{4t^2}{3}$$

$$1 = \frac{t^2}{3}$$

$\therefore\quad t = \sqrt{3}\quad (t \geq 0)$

At $t = \sqrt{3}$ seconds the magnitude of the angle between $\dot{r}(t)$ and $\ddot{r}(t)$ is 30°

9 $\boldsymbol{r} = 3t\,\boldsymbol{i} + \frac{1}{3}t^3\boldsymbol{j} + t^3\boldsymbol{k}$

a $\dot{\boldsymbol{r}} = 3\boldsymbol{i} + t^2\boldsymbol{j} + 3t^2\boldsymbol{k}$

b $|\dot{\boldsymbol{r}}| = \sqrt{(3)^2 + (t^2)^2 + (3t^2)^2}$

$= \sqrt{9+10t^4}$

c $\ddot{\boldsymbol{r}} = 2t\boldsymbol{j} + 6t\boldsymbol{k}$

d $|\ddot{\boldsymbol{r}}| = \sqrt{(2t)^2 + (6t)^2}$

$= \sqrt{4t^2 + 36t^2}$

$= \sqrt{40t^2}$

$= 2\sqrt{10}t\quad$ (assuming $t \geq 0$)

e $|\ddot{\boldsymbol{r}}| = 16$

$\therefore\quad 16 = 2\sqrt{10}t$

$$\therefore\quad t = \frac{8}{\sqrt{10}} = \frac{4\sqrt{10}}{5}$$

10 $\boldsymbol{r} = (V\cos\alpha)t\,\boldsymbol{i} + \left((V\sin\alpha)t - \frac{1}{2}gt^2\right)\boldsymbol{j}$

a $\dot{\boldsymbol{r}} = V\cos\alpha\,\boldsymbol{i} + (V\sin\alpha - gt)\boldsymbol{j}$

b $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$

c $\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}} = 0$

$(V\sin\alpha - gt)(-g) = 0$

$g^2t - gV\sin\alpha = 0$

$$t = \frac{V\sin\alpha}{g}$$

$\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are perpendicular when $t = \dfrac{V\sin\alpha}{g}$

d with $t = \dfrac{V\sin\alpha}{g}$,

$$\boldsymbol{r} = (V\cos\alpha)\left(\frac{V\sin\alpha}{g}\right)\boldsymbol{i} + \left((V\sin\alpha)\left(\frac{V\sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{V\sin\alpha}{g}\right)^2\right)\boldsymbol{j}$$

$$\therefore \quad \boldsymbol{r} = \left(\frac{V^2\cos\alpha\sin\alpha}{g}\right)\boldsymbol{i} + \left(\frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}\right)\boldsymbol{j}$$

Using the trigonometric identity:

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$\therefore \quad \frac{\sin 2\alpha}{2} = \sin\alpha\cos\alpha$$

$$\therefore \quad \boldsymbol{r} = \left(\frac{V^2\sin 2\alpha}{2g}\right)\boldsymbol{i} + \left(\frac{V^2\sin^2\alpha}{2g}\right)\boldsymbol{j}$$

Solutions to Exercise 13D

1 $\boldsymbol{r}(t) = t^2\boldsymbol{i} - (1+2t)\boldsymbol{j}$

a $\boldsymbol{v}(t) = \dot{\mathbf{r}}(t) = 2t\boldsymbol{i} - 2\boldsymbol{j}$, the velocity at time t.

b $\boldsymbol{a}(t) = \ddot{\mathbf{r}}(t) = 2\boldsymbol{i}$, the acceleration at time t.

c $\boldsymbol{r}(2) = 4\boldsymbol{i} - 5\boldsymbol{j},\ \boldsymbol{r}(0) = -\boldsymbol{j}$

Average velocity for the first two seconds

$$= \frac{\boldsymbol{r}(2) - \boldsymbol{r}(0)}{2}$$

$$= \frac{4\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{j}}{2}$$

$$= 2\boldsymbol{i} - 2\boldsymbol{j}$$

2 $\ddot{\boldsymbol{r}}(t) = -g\boldsymbol{j}$

a $\boldsymbol{v}(t) = \dot{\mathbf{r}}(t)$

$$= \int \ddot{\mathbf{r}}(t)\,dt$$

$$= -9.8t\boldsymbol{j} + \boldsymbol{c}_1$$

where $\boldsymbol{c}_1$ is a constant vector.

$\dot{\mathbf{r}}(0) = 2\boldsymbol{i} + 6\boldsymbol{j}$

$\therefore\ \boldsymbol{c}_1 = 2\boldsymbol{i} + 6\boldsymbol{j}$

$\therefore\ \boldsymbol{v}(t) = 2\boldsymbol{i} + (6 - 9.8t)\boldsymbol{j}$,

the velocity at time t.

b $\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\,dt$

$$= \int 2\boldsymbol{i} + (6 - 9.8t)\boldsymbol{j}\,dt$$

$$= \int 2dt\,\boldsymbol{i} + \int 6 - 9.8t\,dt\boldsymbol{j}$$

$$= 2t\boldsymbol{i} + (6t - 4.9t^2)\boldsymbol{j} + \boldsymbol{c}_2$$

where $\boldsymbol{c}_2$ is a constant vector.

$\boldsymbol{r}(0) = 0\boldsymbol{i} + 6\boldsymbol{j}$

$\therefore\ \boldsymbol{c}_2 = 0\boldsymbol{i} + 6\boldsymbol{j}$

$\therefore\ \boldsymbol{r}(t) = 2t\boldsymbol{i} + (6t - 4.9t^2 + 6)\boldsymbol{j}$,

the displacement at time t.

3 $\dot{\boldsymbol{r}}(t) = 3\boldsymbol{i} + 2t\boldsymbol{j} + (1 - 4t)\boldsymbol{k}$

a $\boldsymbol{a}(t) = \ddot{\mathbf{r}}(t) = 2\boldsymbol{j} - 4\boldsymbol{k}$, the acceleration at time t.

b $\boldsymbol{r}(t) = \int \dot{\mathbf{r}}(t)\,dt$

$$= \int 3\boldsymbol{i} + 2t\boldsymbol{j} + (1 - 4t)\boldsymbol{k}\,dt$$

$$= \int 3dt\,\boldsymbol{i} + \int 2t\,dt\boldsymbol{j} + \int 1 - 4t\,dt\,\boldsymbol{k}$$

$$= 3t\boldsymbol{i} + t^2\boldsymbol{j} + (t - 2t^2)\boldsymbol{k} + \boldsymbol{c}$$

where $\boldsymbol{c}$ is a constant vector.

$\boldsymbol{r}(0) = \boldsymbol{j} + \boldsymbol{k}$

$\therefore\ \boldsymbol{c} = \boldsymbol{j} + \boldsymbol{k}$

$\therefore\ \boldsymbol{r}(t) = 3t\boldsymbol{i} + (t^2 + 1)\boldsymbol{j} + (t - 2t^2 + 1)\boldsymbol{k}$,

the position at time t.

c $|\dot{\mathbf{r}}(t)| = \sqrt{3^2 + (2t)^2 + (1 - 4t)^2}$

$$= \sqrt{9 + 4t^2 + 1 - 8t + 16t^2}$$

$$= \sqrt{20t^2 - 8t + 10},$$

the speed at time t.

d **i** Minimum speed occurs when $20t^2 - 8t + 10$ is a minimum.

$20t^2 - 8t + 10$

$$= 20\left[t^2 - \frac{2}{5}t + \frac{1}{2}\right]$$

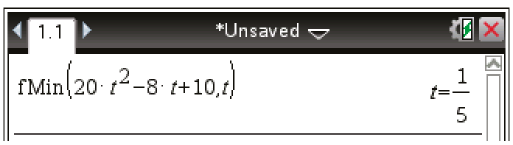
$$= 20\left[t^2 - \frac{2}{5}t + \frac{1}{25} + \frac{1}{2} - \frac{1}{25}\right]$$

$$= 20\left[\left(t - \frac{1}{5}\right)^2 + \frac{23}{50}\right]$$

$$= 20\left(t - \frac{1}{5}\right)^2 + \frac{46}{5}$$

$\therefore$ minimum speed occurs when $t = \frac{1}{5}$ seconds.

Using CAS:



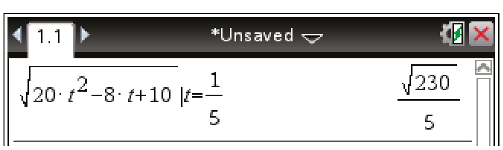
ii When $t = \frac{1}{5}$,

$$\text{Speed} = \sqrt{20\left(\frac{1}{5}\right)^2 - 8\left(\frac{1}{5}\right) + 10}$$

$$= \sqrt{\frac{1}{25}(20 - 40 + 250)}$$

$$= \frac{1}{5}\sqrt{230} \text{ m/s}$$

Using CAS



4 $\ddot{\boldsymbol{r}}(t) = 10\boldsymbol{i} - g\boldsymbol{k}$

a

$$\boldsymbol{v}(t) = \dot{\mathbf{r}}(t)$$

$$= \int \ddot{\mathbf{r}}(t)\,dt$$

$$= \int 10\boldsymbol{i} - g\boldsymbol{k}\,dt$$

$$= \int 10dt\,\boldsymbol{i} - \int 9.8dt\,\boldsymbol{k} \text{ since } g = 9.8$$

$$= 10t\boldsymbol{i} - 9.8t\boldsymbol{k} + \boldsymbol{c}_1,$$

where $\boldsymbol{c}_1$ is a constant vector.

$\dot{\mathbf{r}}(0) = 20\boldsymbol{i} - 20\boldsymbol{j} + 40\boldsymbol{k}$

$\therefore \quad \boldsymbol{c}_1 = 20\boldsymbol{i} - 20\boldsymbol{j} + 40\boldsymbol{k}$

$\therefore \quad \boldsymbol{v}(t) = (10t + 20)\boldsymbol{i} - 20\boldsymbol{j} + (40 - 9.8t)\boldsymbol{k}$,

the velocity at time t.

b $\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\,dt$

$$= \int (10t + 20)\boldsymbol{i} - 20\boldsymbol{j} + (40 - 9.8t)\boldsymbol{k}\,dt$$

$$= \int 10t + 20dt\,\boldsymbol{i} - \int 20dt\,\boldsymbol{j} + \int 40 - 9.8t\,dt\,\boldsymbol{k}$$

$$= (5t^2 + 20t)\boldsymbol{i} - 20t\boldsymbol{j} + (40t - 4.9t^2)\boldsymbol{k} + \boldsymbol{c}_2,$$

where $\boldsymbol{c}_2$ is a constant vector.

$\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$

$\therefore \quad \boldsymbol{c}_2 = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$

$\therefore \quad \boldsymbol{r}(t) = (5t^2 + 20t)\boldsymbol{i} - 20t\boldsymbol{j} + (40t - 4.9t^2)\boldsymbol{k}$,

the displacement at time t.

5

$\boldsymbol{r}(t) = 5\cos(1+t^2)\,\boldsymbol{i} + 5\sin(1+t^2)\,\boldsymbol{j}$

$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$

$= -10t\sin(1+t^2)\,\boldsymbol{i} + 10t\cos(1+t^2)\,\boldsymbol{j}$

Speed:

$|\boldsymbol{v}(t)| = \sqrt{100t^2\sin^2(1+t^2) + 100t^2\cos^2(1+t^2)}$

$= 10t\sqrt{\sin^2(1+t^2) + \cos^2(1+t^2)}$

$= 10t$

6 $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + (t^2 - 4)\,\boldsymbol{j}$

$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = 2\,\boldsymbol{i} + 2t\,\boldsymbol{j}$

$\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t) = 2\,\boldsymbol{j}$

When $t = 1$,

$\boldsymbol{v}(1) = 2\,\boldsymbol{i} + 2\,\boldsymbol{j}$

$\cos\theta^\circ = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$

where $\boldsymbol{a} = \boldsymbol{v}(1)$ and $\boldsymbol{b} = \boldsymbol{a}(1)$

$\boldsymbol{a}\cdot\boldsymbol{b} = 4$

$|\boldsymbol{a}|\cdot|\boldsymbol{b}| = \sqrt{8}\times\sqrt{4}$

$\therefore \quad \cos\theta = \dfrac{2}{\sqrt{8}}$

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 45^\circ$

7 $\boldsymbol{r}(t) = 12\sqrt{t}\,\boldsymbol{i} + t^{\frac{3}{2}}\,\boldsymbol{j}$

$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$

$= 6t^{-\frac{1}{2}}\,\boldsymbol{i} + \dfrac{3}{2}t^{\frac{1}{2}}\,\boldsymbol{j}$

$= \dfrac{6}{t^{\frac{1}{2}}}\boldsymbol{j} + \dfrac{3}{2}t^2\boldsymbol{j}$

$= \dfrac{1}{t^{\frac{1}{2}}}\left(6\,\boldsymbol{i} + \dfrac{3}{2}t\boldsymbol{j}\right)$

$|\dot{\boldsymbol{r}}(t)| = \sqrt{\dfrac{1}{t}\left(36 + \dfrac{9}{4}t^2\right)}$ ①

Let $S = \dfrac{36}{t} + \dfrac{9}{4}t$

$\dfrac{dS}{dt} = \dfrac{-36}{t^2} + \dfrac{9}{4}$

Let $\dfrac{dS}{dt} = 0$

$\therefore \quad t^2 = 16 \Rightarrow t = 4\ (t \geq 0)$

Substitute $t = 4$ into ①

$\therefore \quad |\dot{\boldsymbol{r}}(t)| = 3\sqrt{2}$

Position: $\boldsymbol{r}(4) = 12\sqrt{4}\,\boldsymbol{i} + 4^{\frac{3}{2}}2\boldsymbol{j}$

$= 24\,\boldsymbol{i} + 8\,\boldsymbol{j}$

Therefore the minimum speed is $3\sqrt{2}$ m/s and the position of the particle when at this speed is $24\,\boldsymbol{i} + 8\,\boldsymbol{j}$.

8 $\boldsymbol{r}(t) = 400t\,\boldsymbol{i} + (300t - 4.9t^2)\,\boldsymbol{j}$

a $\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

$x = 400t$ ①

$y = 300t - 4.9t^2$ ②

From ① : $t = \dfrac{x}{400}$ ③

Substitute ③ into ②

$$y = \frac{3}{4}x - \frac{4.9x^2}{160\,000}$$

$$0 = \frac{3}{4}x - \frac{4.9x^2}{160\,000}$$

$$120\,000x = 4.9x^2$$

$$\therefore \quad x = \frac{1200\,000}{49}$$

$$\therefore \quad t = \frac{1200\,000}{49 \times 400}$$

$$\therefore \quad t = 61\frac{11}{49} \text{ seconds}$$

Therefore, it takes $61\frac{11}{49}$ seconds for the projectile to reach the ground.

b $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = 400\boldsymbol{i} + (300 - 9.8t)\boldsymbol{j}$

$$\text{Speed} = \left|\boldsymbol{v}\left(\frac{3000}{49}\right)\right|$$

$$= \sqrt{400^2 + \left[300 - 9.8\left(\frac{3000}{49}\right)\right]^2}$$

$$= \sqrt{160\,000 + 90\,000}$$

$$= 500 \text{ m/s}$$

Therefore the object hits the ground at 500 m/s.

c

$$y = \frac{3}{4}x - \frac{4.9x^2}{160\,000}$$

$$y' = \frac{3}{4} - \frac{9.8x}{160\,000}$$

$$\text{For maximum: } 0 = \frac{3}{4} - \frac{9x}{80\,000}$$

$$60\,000 = 4.9x$$

$$\therefore \quad x = \frac{600\,000}{49}$$

$$y = \frac{3}{4}\left(\frac{600\,000}{49}\right) - \frac{4.9\left(\frac{600\,000}{49}\right)^2}{160\,000}$$

$$= \frac{225\,000}{49}$$

Therefore the maximum height reached is $\frac{225\,000}{49}$ metres.

d Initial speed: $t = 0$

$$\boldsymbol{v}(0) = 400\boldsymbol{i} + 300\boldsymbol{j}$$

$$|\boldsymbol{v}(0)| = \sqrt{400^2 + 300^2}$$

$$= 500 \text{ m/s}$$

Therefore, the initial speed is 500 m/s.

e The initial angle is found with the velocity vector when $t = 0$.

$\boldsymbol{v}(0) = 400\boldsymbol{i} + 300\boldsymbol{j}$

$$\therefore \quad \tan\theta^\circ = \frac{3}{4}$$

$$\therefore \quad \theta = 36.87^\circ$$

9 $\boldsymbol{r}''(t) = -3(\sin 3t\,\boldsymbol{i} + \cos 3t\,\boldsymbol{j})$

a

$$\boldsymbol{r}'(t) = \cos 3t\,\boldsymbol{i} - \sin 3t\,\boldsymbol{j} + \boldsymbol{c}_1$$

$$\boldsymbol{r}'(0) = \boldsymbol{j}$$

$$,\boldsymbol{i} = \boldsymbol{i} + \boldsymbol{c}_1$$

$$\therefore \quad \boldsymbol{c}_1 = \boldsymbol{0}$$

$$\therefore \quad \boldsymbol{r}'(t) = \cos 3t\,\boldsymbol{i} - \sin 3t\,\boldsymbol{j}$$

$$\boldsymbol{r}(t) = \frac{1}{3}\sin 3t\,\boldsymbol{i} + \frac{1}{3}\cos 3t\,\boldsymbol{j} + \boldsymbol{c}_2$$

$$\boldsymbol{r}(0) = -3\boldsymbol{i} + 3\boldsymbol{j}$$

$$\therefore \quad -3\boldsymbol{i} + 3j = \frac{1}{3}\boldsymbol{j} + \boldsymbol{c}_2$$

$$\therefore \quad \boldsymbol{r}(t) = \left(\frac{1}{3}\sin 3t - 3\right)\boldsymbol{i} + \left(\frac{1}{3}\cos 3t + \frac{8}{3}\right)\boldsymbol{j}$$

b $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$$x = \frac{1}{3}\sin 3t - 3 \quad ①$$

$$y = \frac{1}{3}\cos 3t + \frac{8}{3} \quad ②$$

$$x + 3 = \frac{1}{3}\sin 3t \quad ③$$

$$y - \frac{8}{3} = \frac{1}{3}\cos 3t \quad ④$$

Squaring ③ and ④

$$(x+3)^2 = \frac{1}{9}\sin^2 3t \quad ⑤$$

$$\left(y - \frac{8}{3}\right)^2 = \frac{1}{9}\cos^2 3t \quad ⑥$$

Adding ⑤ and ⑥

$$(x+3)^2 + \left(y - \frac{8}{3}\right)^2 = \frac{1}{9}(\sin^2 3t + \cos^2 3t)$$

$$\therefore \quad (x+3)^2 + \left(y - \frac{8}{3}\right)^2 = \frac{1}{9}$$

The position of the centre of the circular path is $\left(-3, \frac{8}{3}\right)$.

c $\boldsymbol{v}(t) = \boldsymbol{r}'(t) = \cos 3t\,\boldsymbol{i} - \sin 3t\,\boldsymbol{j}$

$\boldsymbol{a}(t) = \boldsymbol{r}''(t) = -3(\sin 3t\,\boldsymbol{i} + \cos 3t\,\boldsymbol{j})$

Required to show $\boldsymbol{a}(t) \cdot \boldsymbol{v}(t) = 0$

$$\begin{aligned}\boldsymbol{a}(t) \cdot \boldsymbol{v}(t) &= (\cos 3t \times -3\sin 3t) \\ &\quad + (-\sin 3t \times -3\cos 3t) \\ &= -3\sin 3t \cdot \cos 3t \\ &\quad + 3\sin 3t \cdot \cos 3t \\ &= 0\end{aligned}$$

$\therefore$ $\boldsymbol{a}(t)$ is perpendicular to $\boldsymbol{v}(t)$.

10 $\boldsymbol{r}(t) = 2\cos t\,\boldsymbol{i} + 4\sin t\,\boldsymbol{j} + 2t\,\boldsymbol{k}$

$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = -2\sin t\,\boldsymbol{i} + 4\cos t\,\boldsymbol{j} + 2\boldsymbol{k}$

$$\begin{aligned}|\boldsymbol{v}(t)| &= \sqrt{4\sin^2 t + 16\cos^2 t + 4} \\ &= \sqrt{4} \times \sqrt{\sin^2 t + 4\cos^2 t + 1} \\ &= 2\sqrt{1 - \cos^2 t + 4\cos^2 t + 1}\end{aligned}$$

$\therefore \quad |\boldsymbol{v}(t)| = 2\sqrt{2 + 3\cos^2 t}$

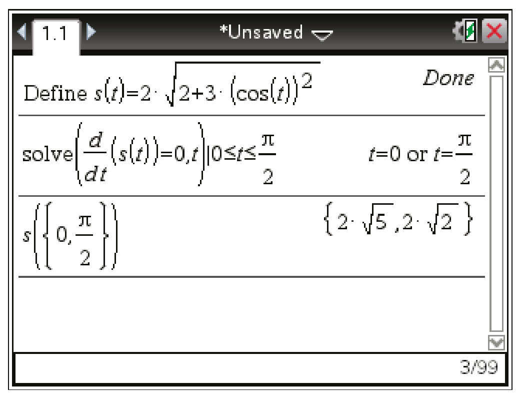
$t = 0: |\boldsymbol{v}| = 2\sqrt{5}$

$t = \frac{\pi}{2}: |\boldsymbol{v}| = 2\sqrt{2}$

$\therefore \quad \boldsymbol{c}_2 = -3\boldsymbol{i} + \frac{8}{3}\boldsymbol{j}$

Therefore the minimum speed of the particle is $2\sqrt{2}$ and the maximum speed is $2\sqrt{5}$

Using CAS:



11 $\boldsymbol{v}(t) = (2t+1)^2\boldsymbol{i} + \dfrac{1}{\sqrt{2t+1}}\boldsymbol{j}$

a $\boldsymbol{a}(t) = \boldsymbol{v}'(t) = 4(2t+1)\boldsymbol{i} - \dfrac{1}{(2t+1)^{\frac{3}{2}}}\boldsymbol{j}$

When $t = 1$,

$$\boldsymbol{a}(1) = 12\boldsymbol{i} - \frac{1}{3\sqrt{3}}\boldsymbol{j}$$

$$|\boldsymbol{a}(1)| = \sqrt{12^2 + \frac{1}{27}} = \sqrt{144 + \frac{1}{27}} = \sqrt{\frac{3889}{27}} = \sqrt{\frac{1}{81}(3 \times 3889)} = \frac{\sqrt{11667}}{9}\text{ m/s}^2$$

$$\hat{\mathbf{a}}(1) = \frac{9}{\sqrt{11667}}\left(12\boldsymbol{i} - \frac{1}{3\sqrt{3}}\right) = \frac{1}{\sqrt{11667}}\left(108\boldsymbol{i} - \frac{3}{\sqrt{3}}\boldsymbol{j}\right) = \frac{1}{\sqrt{11667}}(108\boldsymbol{i} - \sqrt{3}\boldsymbol{j})$$

Therefore the direction of the acceleration after 1 second is $\dfrac{1}{\sqrt{11667}}(108\boldsymbol{i} - \sqrt{3}\boldsymbol{j})$ and the magnitude is $\dfrac{\sqrt{11667}}{9}$ m/s^2.

b $\boldsymbol{r}(t) = \dfrac{1}{6}(2t+1)^3\boldsymbol{i} + (2t+1)^{\frac{1}{2}}\boldsymbol{j} + \boldsymbol{c}_1$

$\boldsymbol{r}(0) = \boldsymbol{0}$

$\therefore \quad \boldsymbol{0} = \dfrac{1}{6}\boldsymbol{j} + \boldsymbol{j} + \boldsymbol{c}_1$

$\therefore \quad \boldsymbol{c}_1 = -\dfrac{1}{6}\boldsymbol{i} - \boldsymbol{j}$

So

$$\boldsymbol{r}(t) = \frac{1}{6}((2t+1)^3 - 1)\boldsymbol{i} + \left((2t+1)^{\frac{1}{2}} - 1\right)\boldsymbol{j}$$
$$= \left(\frac{4t^3}{3} + 2t^2 + t\right)\boldsymbol{i} + (\sqrt{2t+1} - 1)\boldsymbol{j}$$

12 a $\boldsymbol{a}(t) = -g\boldsymbol{j}$

$\boldsymbol{v}(t) = V\cos\alpha\,\boldsymbol{i} + (V\sin\alpha - gt)\boldsymbol{j}$

$\boldsymbol{r}(t) = V\cos\alpha t\,\boldsymbol{i} + \left(V\sin\alpha t - \dfrac{gt^2}{2}\right)\boldsymbol{j}$

b $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$x = V\cos\alpha t \qquad ①$

$y = V\sin\alpha t - \dfrac{gt^2}{2} \qquad ②$

From ① : $t = \dfrac{x}{V\cos\alpha} \qquad ③$

Substitute ③ into ②

$$y = V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) - \frac{g}{2}\left(\frac{x^2}{V^2\cos^2\alpha}\right)$$

$\therefore \quad y = x\tan\alpha - \dfrac{gx^2}{2V^2}\sec^2\alpha$

As required to prove.

13 a $\boldsymbol{r}_A'(t) = \boldsymbol{j} + 2\boldsymbol{j},\ \boldsymbol{r}_B'(t) = 2\boldsymbol{i} + 3\boldsymbol{j}$

$\boldsymbol{r}_A(2) = 3\boldsymbol{i} + 4\boldsymbol{j},\ \boldsymbol{r}_B(3) = \boldsymbol{j} + 3j$

$\boldsymbol{r}_A(t) = t\boldsymbol{i} + 2t\boldsymbol{j} + \boldsymbol{c}_1$

$3\boldsymbol{i} + 4\boldsymbol{j} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{c}_1$

$\therefore \quad \boldsymbol{c}_1 = \boldsymbol{i}$

$\boldsymbol{r}_A(t) = (t+1)\boldsymbol{i} + 2t\boldsymbol{j}$

$\boldsymbol{r}_B(t) = 2t\boldsymbol{i} + 3t\boldsymbol{j} + \boldsymbol{c}_2$

$\boldsymbol{i} + 3\boldsymbol{j} = 6\boldsymbol{i} + 9\boldsymbol{j} + \boldsymbol{c}_2$

$\therefore \quad \boldsymbol{c}_2 = -5\boldsymbol{i} - 6\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_B(t) = (2t-5)\boldsymbol{i} + (3t-6)\boldsymbol{j}$

For the same position:

$(t+1)\boldsymbol{i} + 2t\boldsymbol{j} = (2t-5)\boldsymbol{i} + (3t-6)\boldsymbol{j}$

$t + 1 = 2t - 5$ ①

$2t = 3t - 6$ ②

From ① : $t = 6$

From ② : $t = 6$

$\therefore$ the particles collide when $t = 6$.

b $\boldsymbol{r}_A(6) = (6+1)\boldsymbol{i} + 2(6)\boldsymbol{j}$

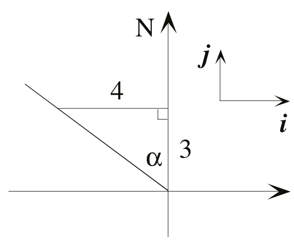
$= 7\boldsymbol{i} + 12\boldsymbol{j}$

$\boldsymbol{r}_B(6) = (2(6)-5)\boldsymbol{i} + (3(6)-6)\boldsymbol{j}$

$= 7\boldsymbol{i} + 12\boldsymbol{j}$

$\therefore$ the position vector of the point of collision at $t = 6$ is $7\boldsymbol{i} + 12\boldsymbol{j}$

14 a



$|\dot{\mathbf{r}}(t)| = 20$

Let $\dot{\mathbf{r}}(t) = -a\boldsymbol{i} + b\boldsymbol{j},\ a,\ b \in R^+$

$\therefore \quad |\dot{\mathbf{r}}(t)| = \sqrt{(-a)^2 + b^2}$

$= \sqrt{a^2 + b^2}$

$= 20$

$\therefore \quad a^2 + b^2 = 400$ ①

$\boldsymbol{r}(t) = \int \dot{\mathbf{r}}(t)\,dt$

$= \int -a\boldsymbol{i} + b\boldsymbol{j}\,dt$

$= 1 - a\,dt\,\boldsymbol{i} + \int b\,dt\boldsymbol{j}$

$= -at\boldsymbol{i} + bt\boldsymbol{j} + \boldsymbol{c},$

where $\boldsymbol{c}$ is a constant vector.

Since $\boldsymbol{r}(0) = \mathbf{0},\ \boldsymbol{c} = \mathbf{0}$

$\therefore \quad \boldsymbol{r}(t) = -at\boldsymbol{i} + bt\boldsymbol{j}$

Now $\tan\alpha = \frac{4}{3}$ and also $\tan\alpha = \frac{at}{bt}$

$\therefore \quad \frac{at}{bt} = \frac{4}{3}$

$\therefore \quad 3a = 4b$

$\therefore \quad b = \frac{3a}{4}$ ②

Substituting in ① yields

$a^2 + \left(\frac{3a}{4}\right)^2 = 400$

$\therefore \quad a^2 + \frac{9a^2}{16} = 400$

$\therefore \quad \frac{25a^2}{16} = 400$

$\therefore \quad a^2 = \frac{16 \times 400}{25}$

$= 256$

$\therefore \quad a = 16$

Substituting in ② yields

$b = \frac{3 \times 16}{4}$

$= 12$

$\therefore \quad \dot{\mathbf{r}}(t) = -16\boldsymbol{i} + 12\boldsymbol{j},$

the velocity at time t.

b $\boldsymbol{r}(t) = -16t\boldsymbol{i} + 12t\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}(5) = (-16 \times 5)\boldsymbol{i} + (12 \times 5)\boldsymbol{j}$

$= -80\boldsymbol{i} + 60\boldsymbol{j},$

the position after five seconds.

15 $\boldsymbol{r} = 4\sin(2t)\boldsymbol{i} + 4\cos(2t)\boldsymbol{j}$

a $\boldsymbol{v}(t) = \dot{\mathbf{r}}(t)$

$= 8\cos 2t\boldsymbol{i} - 8\sin 2t\boldsymbol{j},\ t \geq 0,$

the velocity at time t.

b The speed at time t is given by

$$|\boldsymbol{v}(t)| = \sqrt{(8\cos 2t)^2 + (-8\sin 2t)^2}$$
$$= \sqrt{64\cos^2 2t + 64\sin^2 2t}$$
$$= \sqrt{64(\cos^2 2t + \sin^2 2t)}$$
$$= \sqrt{64}$$
$$= 8$$

c $\boldsymbol{a}(t) = \ddot{\mathbf{r}}(t)$

$$= -16\sin 2t\,\boldsymbol{i} - 16\cos 2t\,\boldsymbol{j}$$
$$= -4(4\sin 2t\,\boldsymbol{i} + 4\cos 2t\,\boldsymbol{j})$$
$$= -4\boldsymbol{r},$$

the acceleration in terms of $\boldsymbol{r}$.

16 $\dot{\mathbf{r}}(t) = (2t - 5)\,\boldsymbol{i};\ t \geq 0$

a $\boldsymbol{r}(t) = \int \dot{\mathbf{r}}(t)\,dt$

$$= \int 2t - 5dt\,\boldsymbol{i}$$
$$= (t^2 - 5t)\,\boldsymbol{i} + \boldsymbol{c},$$

where $\boldsymbol{c}$ is a constant vector.

$\boldsymbol{r}(0) = -2\,\boldsymbol{i} + 2\,\boldsymbol{j}$

$\therefore\quad \boldsymbol{c} = -2\,\boldsymbol{i} + 2\,\boldsymbol{j}$

$\therefore\quad \boldsymbol{r}(t) = (t^2 - 5t - 2)\,\boldsymbol{i} + 2\,\boldsymbol{j},$

the position at time t.

b Find $\boldsymbol{r}(t)$ when $\boldsymbol{v}(t) = 0$

When $\boldsymbol{v}(t) = 0$,

$$(2t - 5)\,\boldsymbol{i} = 0$$
$$\therefore\quad 2t - 5 = 0$$
$$\therefore\quad t = \frac{5}{2}$$

$$\boldsymbol{r}\left(\frac{5}{2}\right) = \left(\left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) - 2\right)\boldsymbol{i} + 2\boldsymbol{j}$$
$$= \left(\frac{25}{4} - \frac{25}{2} - 2\right)\boldsymbol{i} + 2\boldsymbol{j}$$
$$= -\frac{33}{4}\,\boldsymbol{i} + 2\boldsymbol{j},$$

the position of the particle when it is instantaneously at rest.

c Let $\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

$\therefore\quad x = t^2 - 5t - 2$ and $y = 2$

$\therefore\quad y = 2$ is the cartesian equation of the path.

Note: $x \geq -\frac{33}{4}$

17 $\boldsymbol{r}(t) = 6\sec(t)\,\boldsymbol{i} + 4\tan(t)\,\boldsymbol{j};\ t \geq 0$

a Let $\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

$\therefore\quad x = 6\sec(t)$ and

$y = 4\tan(t),\ t \geq 0$

$\therefore\quad x \geq 6$ or $x \leq -6,\ y \in R$

$\therefore\quad x^2 = 36\sec^2(t)$ and

$y^2 = 16\tan^2(t)$

$$\therefore\quad \frac{x^2}{36} = \sec^2(t) \text{ and } \frac{y^2}{16} = \tan^2(t)$$
$$= \tan^2(t) + 1$$
$$= \frac{y^2}{16} + 1$$
$$\therefore\quad \frac{x^2}{36} - \frac{y^2}{16} = 1,$$

a hyperbola with asymptotes at $y = \pm\frac{2x}{3}$

b

$$\begin{aligned} \boldsymbol{v}(t) &= \dot{\mathbf{r}}(t) \\ &= \frac{d}{dt}(6\sec(t))\,\boldsymbol{i} \\ &\quad + \frac{d}{dt}(4\tan(t))\,\boldsymbol{j} \end{aligned}$$

Now let $y_1 = 6\sec(t)$

$$\begin{aligned} &= \frac{6}{\cos t} \\ &= 6u^{-1} \text{ where } u = \cos(t) \end{aligned}$$

$$\frac{dy_1}{du} = -6u^{-2} \text{ and}$$

$$\frac{du}{dt} = -\sin(t)$$

$$= \frac{-6}{\cos^2(t)}$$

$$\therefore \quad \frac{dy_1}{dt} = \frac{dy_1}{du} \cdot \frac{du}{dt}$$

$$= \frac{-6}{\cos^2(t)} \cdot -\sin(t)$$

$$= 6\tan(t)\sec(t)$$

Also let

$$\begin{aligned} y_2 &= 4\tan(t) \\ &= \frac{4\sin(t)}{\cos(t)} \\ &= \frac{u}{v} \text{ where } u = 4\sin(t) \text{ and} \end{aligned}$$

$$v = \cos(t)$$

$$\therefore \quad \frac{du}{dt} = 4\cos(t) \text{ and } \frac{dv}{dt} = -\sin(t)$$

$$\therefore \quad \frac{dy_2}{dt} = \frac{v \cdot \frac{du}{dt} - u \cdot \frac{dv}{dt}}{v^2}$$

$$= \frac{\cos(t) \cdot 4\cos(t) - 4\sin(t)[-\sin(t)]}{[\cos(t)]^2}$$

$$= \frac{4\cos^2(t) + 4\sin^2(t)}{\cos^2(t)}$$

$$= \frac{4[\cos^2(t) + \sin^2(t)]}{\cos^2(t)}$$

$$= 4\sec^2(t)$$

$\therefore$ $\boldsymbol{v}(t) = 6\tan(t)\sec(t)\,\boldsymbol{i} + 4\sec^2(t)\,\boldsymbol{j},\ t \geq 0$, the velocity at time t.

18 $\boldsymbol{r}(t) = 4\cos(t)\,\boldsymbol{i} + 3\sin(t)\,\boldsymbol{j};\ 0 \leq t \leq 2\pi$

a Let $\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

$$\therefore \quad x = 4\cos t \text{ and } y = 3\sin t$$

$$\therefore \quad x^2 = 16\cos^2 t \text{ and } y^2 = 9\sin^2 t$$

$$\therefore \quad \frac{x^2}{16} = \cos^2 t \text{ and } \frac{y^2}{9} = \sin^2 t$$

$$\therefore \quad \frac{x^2}{16} + \frac{y^2}{9} = \cos^2 t + \sin^2 t$$

$$\therefore \quad \frac{x^2}{16} + \frac{y^2}{9} = 1$$

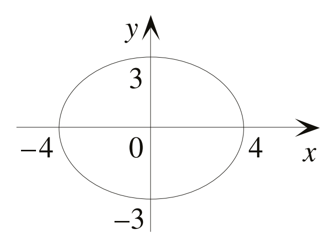
an ellipse with centre $(0, 0)$

For $0 \leq t \leq 2\pi$:

Range $x = [-4, 4]$ and Range $y = [-3, 3]$

$\Rightarrow$ Dom(Cartesian eqn) $= [-4, 4]$

Ran(Cartesian eqn) $= [-3, 3]$



b **i** $v(t) = \dot{\mathbf{r}}(t)$

$= -4 \sin t\, \boldsymbol{i} + 3 \cos t\, \boldsymbol{j},$

$0 \le t \le 2\pi$

$\boldsymbol{v}(t) \cdot \boldsymbol{r}(t) = (-4 \sin t\, \boldsymbol{i} + 3 \cos t\, \boldsymbol{j})$

$\cdot\, (4 \cos t\, \boldsymbol{i} + 3 \sin t\, \boldsymbol{j})$

$= -16 \sin t \cos t$

$+ 9 \sin t \cos t$

$= -7 \sin t \cos t$

$= \frac{-7}{2}(2 \sin t \cos t)$

$= \frac{-7}{2} \sin 2t, \ 0 \le 2t \le 4\pi$

The velocity of the particle is perpendicular to its position vector when

$\boldsymbol{v}(t) \cdot \boldsymbol{r}(t) = 0$

i.e. $\frac{-7}{2} \sin 2t = 0$

$\therefore \quad \sin 2t = 0$

$\therefore \quad 2t = 0, \pi, 2\pi, 3\pi, 4\pi$

$\therefore \quad t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

ii $\boldsymbol{r}(0) = 4\boldsymbol{i}$

$\boldsymbol{r}\left(\frac{\pi}{2}\right) = 3\boldsymbol{j}$

$\boldsymbol{r}(\pi) = -4\boldsymbol{i}$

$\boldsymbol{r}\left(\frac{3\pi}{2}\right) = -3\boldsymbol{j}$

$\boldsymbol{r}(2\pi) = 4\boldsymbol{i}$

c **i** $|\boldsymbol{v}(t)| = \sqrt{(-4 \sin t)^2 + (3 \cos t)^2}$

$= \sqrt{16 \sin^2 t + 9 \cos^2 t}$

$= \sqrt{16 \sin^2 t + 9(1 - \sin^2 t)}$

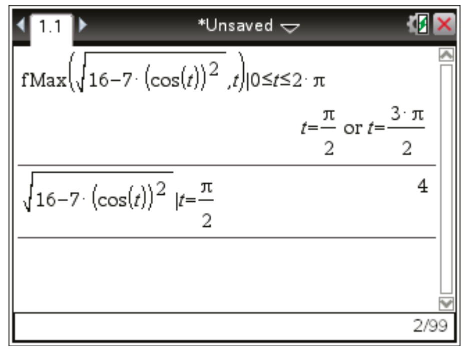
$= \sqrt{9 + 7 \sin^2 t},$

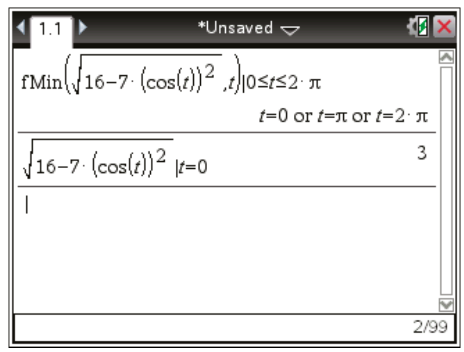
the speed of the particle at time t.

ii $|\boldsymbol{v}(t)| = \sqrt{9 + 7(1 - \cos^2 t)}$

$= \sqrt{16 - 7 \cos^2 t}$

iii The maximum and minimum speeds are 4 and 3 respectively. Using CAS:





19 $\boldsymbol{r}(t) = (t + 2)\boldsymbol{i} + (6t + 1)\boldsymbol{j}$

$x = t + 2$ and $y = 6t + 1$

$\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 6$

Hence distance travelled

$= \int_1^3 \sqrt{1 + 36}\, dt$

$= \left[\sqrt{37}t\right]_1^3$

$= 2\sqrt{37}$

20 $\boldsymbol{r}(t) = \cos(2t)\boldsymbol{i} + \sin(2t)\boldsymbol{j}$

$x = \cos(2t)$ and $y = \sin(2t)$

$\frac{dx}{dt} = -2\sin(2t)$ and $\frac{dy}{dt} = 2\cos(2t)$

Hence distance travelled

$$= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2(2t) + 4\cos^2(2t)}\, dt$$

$$= \int_0^{\frac{\pi}{4}} 2\, dt$$

$$= \Big[2t\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}$$

21 **a** $\boldsymbol{r}(t) = \sqrt{t}\boldsymbol{i} + (2t + 4)\boldsymbol{j}$

$x = \sqrt{t}$ and $y = 2t + 4$

$\frac{dx}{dt} = \frac{1}{2}x^{-\frac{1}{2}}$ and $\frac{dy}{dt} = 2$

Hence distance travelled

$$= \int_1^4 \sqrt{\frac{1}{4}x^{-1} + 4}\, dt$$

$$= \int_1^4 \sqrt{\frac{1}{4}x^{-1} + 4}\, dt$$

$$\approx 6.086$$

b

$$\boldsymbol{r}(4) = 2\boldsymbol{i} + 12\boldsymbol{j}$$

$$\boldsymbol{r}(1) = \boldsymbol{i} + 6\boldsymbol{j}$$

$$\boldsymbol{r}(4) - \boldsymbol{r}(1) = \boldsymbol{i} + 6\boldsymbol{j}$$

$$|\boldsymbol{r}(4) - \boldsymbol{r}(1)| = \sqrt{37}$$

$$\approx 6.083$$

22 **a** $\boldsymbol{r}(t) = 4\cos(t)\boldsymbol{i} + 3\sin(t)\boldsymbol{j}$

$x = 4\cos(t)$ and $y = 3\sin(t)$

$\frac{dx}{dt} = -4\sin(t)$ and $\frac{dy}{dt} = 3\cos(t)$

Hence distance travelled

$$= \int_0^{\frac{\pi}{4}} \sqrt{16\sin^2(t) + 9\cos^2(t)}\, dt$$

$$\approx 2.514$$

b $\boldsymbol{r}(\frac{\pi}{4}) - \boldsymbol{r}(0) = 4\cos(\frac{\pi}{4})\boldsymbol{i} + 3\sin(\frac{\pi}{4})\boldsymbol{j} - 4\boldsymbol{i}$

$|\boldsymbol{r}(\frac{\pi}{4}) - \boldsymbol{r}(0)| \approx 2.423$

Solutions to Technology-free questions

1 a $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + (t^2 - 4)\boldsymbol{j}$

$$\text{velocity} = \dot{\mathbf{r}}(t) = 2\,\boldsymbol{i} + 2t\boldsymbol{j}$$

$$\text{acceleration} = \ddot{\mathbf{r}}(t) = 2\boldsymbol{j}$$

When $t = 2$, $\dot{\mathbf{r}}(2) = 2\,\boldsymbol{i} + 4\boldsymbol{j}$

$$\ddot{\mathbf{r}}(2) = 2\boldsymbol{j}$$

b

For $\boldsymbol{r}(t) = x\,\boldsymbol{i} + y\boldsymbol{j}$

$x(t) = 2t$ and $y(t) = t^2 - 4$

$\Rightarrow t = \dfrac{x}{2}$ $\qquad y = \dfrac{x^2}{4} - 4$

or $4y = x^2 - 16$

2 a $\boldsymbol{r}(t) = 2t^2\,\boldsymbol{i} + 4t\boldsymbol{j} + 8\boldsymbol{k}$

$\dot{\mathbf{r}}(t) = 4t\,\boldsymbol{i} + 4\boldsymbol{j}$

$\ddot{\mathbf{r}}(t) = 4\,\boldsymbol{i}$

b $\boldsymbol{r} = 4\sin t\,\boldsymbol{i} + 4\cos t\boldsymbol{j} + t^2\boldsymbol{k}$

$\dot{\mathbf{r}}(t) = 4\cos t\,\boldsymbol{i} - 4\sin t\boldsymbol{j} + 2t\boldsymbol{k}$

$\ddot{\mathbf{r}}(t) = -4\sin t\,\boldsymbol{i} - 4\cos t\boldsymbol{j} + 2\boldsymbol{k}$

3 Position vector is given by
$\boldsymbol{r} = 6t\,\boldsymbol{i} + (t^2 + 4)\boldsymbol{j}$
$\dot{\mathbf{r}}(t) = 6\,\boldsymbol{i} + 2t\boldsymbol{j}$ is a vector along a tangent to the path

$\therefore$ $\dot{\mathbf{r}}(4) = 6\,\boldsymbol{i} + 8\boldsymbol{j}$ is the vector along the tangent at $t = 4$

magnitude $|\dot{\mathbf{r}}(4)| = \sqrt{36 + 64}$

$= 10$

$\therefore$ the unit vector along this tangent is given by

$$\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{6\,\boldsymbol{i} + 8\boldsymbol{j}}{10}$$

$$= 0.6\,\boldsymbol{i} + 0.8\boldsymbol{j}$$

4 $\boldsymbol{r}(t) = 10\sin 2t\,\boldsymbol{i} + 5\cos 2t\boldsymbol{j}$

a $\boldsymbol{r}\left(\dfrac{\pi}{6}\right) = 10\sin\dfrac{\pi}{3}\,\boldsymbol{i} + 5\cos\dfrac{\pi}{3}\boldsymbol{j}$

$$= 5\sqrt{3}\,\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$$

b $\dot{\mathbf{r}}(t) = 20\cos 2t\,\boldsymbol{i} - 10\sin 2t\boldsymbol{j}$

At $t = 0$, $\dot{\mathbf{r}}(0) = 20\,\boldsymbol{i}$

At $t = \dfrac{\pi}{6}$, $\dot{\mathbf{r}}\left(\dfrac{\pi}{6}\right) = 20\cos\dfrac{\pi}{3}\,\boldsymbol{i} - 10\sin\dfrac{\pi}{3}\boldsymbol{j}$

$$= 10\,\boldsymbol{i} - 5\sqrt{3}\boldsymbol{j}$$

To find the angle between these two vectors,

use $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Now $\boldsymbol{r}(0)\cdot\boldsymbol{r}\left(\dfrac{\pi}{6}\right) = 200$

$$\left|\boldsymbol{r}\left(\frac{\pi}{6}\right)\right| = \sqrt{100 + 75} = 5\sqrt{7}$$

$$\therefore \quad \cos\theta = \frac{200}{20 \times 5\sqrt{7}} = \frac{2}{\sqrt{7}}$$

5 $\boldsymbol{r} = (\cos t + t\sin t)\,\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j}$

$\dot{\mathbf{r}}(t) = (-\sin t + t\cos t + \sin t)\,\boldsymbol{i}$

$\qquad + (\cos t + t\sin t - \cos t)\boldsymbol{j}$

$\qquad = t\cos t\,\boldsymbol{i} + t\sin t\boldsymbol{j}$

Now, $\dot{\mathbf{r}}$ is a vector along the tangent to the curve, therefore find the unit vector in this direction.

$$|\dot{\mathbf{r}}(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2(\cos^2 t + \sin^2 t)}$$

$$= \sqrt{t^2} = t$$

$$\therefore \quad \hat{\boldsymbol{r}} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{t\cos t\,\boldsymbol{i} + t\sin t\boldsymbol{j}}{t}$$

$= \cos t\,\boldsymbol{i} + \sin t\boldsymbol{j}$ is the unit vector.

6 $\boldsymbol{r} = 5(\cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j})$

a $\dot{\mathbf{r}}(t) = 5(-\sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j})$ is the velocity at time t.

b Speed is the magnitude of velocity

$\therefore$ speed $= |\dot{\mathbf{r}}(t)| = \sqrt{(-5\sin t)^2 + 5(\cos t)^2} = 5$

c $\ddot{\mathbf{r}}(t) = 5(-\cos t\,\boldsymbol{i} - \sin t\,\boldsymbol{j})$

$= -5(\cos t\,\boldsymbol{i} + \sin\theta\boldsymbol{j})$

gives the acceleration

d The dot product (scalar product) of $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ is

$$\begin{aligned}\dot{\mathbf{r}}(t)\cdot\ddot{\mathbf{r}}(t) &= (-5\sin t\,\boldsymbol{i} + 5\cos t\,\boldsymbol{j}) \\ &\quad\cdot(-5\cos t\,\boldsymbol{i} - 5\sin t\,\boldsymbol{j}) \\ &= 25\sin t\cos t \\ &\quad - 25\cos t\sin t \\ &= 0\end{aligned}$$

$\Rightarrow$ acceleration $\ddot{\mathbf{r}}(t)$ is at right angles to the velocity $\dot{\mathbf{r}}(t)$

7 Velocity of particle A is given by $\boldsymbol{V}_A = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}$

Take the antiderivative to get the position vector.

$$\text{i.e. } \boldsymbol{r}_A = (\sin t)\boldsymbol{i} + (-\cos t)\boldsymbol{j} + \boldsymbol{c}$$

Given that $t = 0$, $\boldsymbol{r}_A = \boldsymbol{i}$

$$\begin{aligned}&\Rightarrow -\boldsymbol{j} + \boldsymbol{c} = \boldsymbol{i} \\ &\therefore\quad \boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} \\ &\therefore\quad \boldsymbol{r}_A = (\sin t + 1)\boldsymbol{i} \\ &\qquad\qquad + (-\cos t + 1)\boldsymbol{j}\end{aligned}$$

Similarly, $\boldsymbol{V}_B = \sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j}$

$$\therefore\quad \boldsymbol{r}_B = (-\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + \boldsymbol{c}_1$$

At $t = 0$, $\boldsymbol{r}_B = \boldsymbol{j}$

$$\begin{aligned}&\Rightarrow \boldsymbol{c}_1 + (-\cos 0)\boldsymbol{i} + (\sin 0)\boldsymbol{j} = \boldsymbol{j} \\ &\Rightarrow -\boldsymbol{i} + \boldsymbol{c}_1 = \boldsymbol{j} \\ &\therefore\quad \boldsymbol{c}_1 = \boldsymbol{i} + \boldsymbol{j} \\ &\therefore\quad \boldsymbol{r}_B = (-\cos t + 1)\boldsymbol{i} \\ &\qquad\qquad + (\sin t + 1)\boldsymbol{j}\end{aligned}$$

If A and B collide, then at some time, $t, \boldsymbol{r}_A = \boldsymbol{r}_B$

i.e. $(\sin t + 1)\boldsymbol{i} + (-\cos t + 1)\boldsymbol{j} = (-\cos t + 1)\boldsymbol{i} + (\sin t + 1)\boldsymbol{j}$

$$\begin{aligned}\text{Equating gives }\ \sin t + 1 &= -\cos t + 1 \\ \text{or }\ \sin t &= -\cos t \\ \therefore\quad t &= \frac{3\pi}{4} + k\pi,\ k \in \mathbb{Z}\end{aligned}$$

Note: Equating $\boldsymbol{i}$ and $\boldsymbol{j}$ components gives the same result in this case.

The particles will collide when $t = \dfrac{3\pi}{4} + k\pi,\ k \in \mathbb{Z}$.

8 $\boldsymbol{r}(t) = (1 + \sin t)\boldsymbol{i} + (1 - \cos t)\boldsymbol{j}$ is the position vector.

a

$$\begin{aligned}\text{velocity } \dot{\mathbf{r}}(t) &= \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j} \\ \text{acceleration } \ddot{\mathbf{r}}(t) &= -\sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j} \\ \text{For any } t,\ |\dot{\mathbf{r}}(t)| &= \sqrt{\cos^2 t + \sin^2 t} \\ &= 1 \\ \text{and } |\ddot{\mathbf{r}}(t)| &= \sqrt{\sin^2 t + \cos^2 t} \\ &= 1\end{aligned}$$

Therefore, the magnitudes of velocity and acceleration are constants.

b Let (x, y) represent any point on the cartesian graph.
Then $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$$= (1 + \sin t)\boldsymbol{i} + (1 - \cos t)\boldsymbol{j}$$

$\therefore \quad 1 + \sin t = x$ and

or $\quad \sin t = x - 1$

$1 - \cos t = y$

$\cos t = 1 - y$

Now $\sin^2 t + \cos^2 t = 1$

$\therefore \quad (x - 1)^2 + (1 - y)^2 = 1$

or $(x - 1)^2 + (y - 1)^2 = 1$
is the cartesian equation of the path of the particle.

c If displacement is perpendicular to velocity,

then $\boldsymbol{r}(t) \cdot \dot{\mathbf{r}}(t) = 0$

i.e. $(1 + \sin t)(\cos t) + (1 - \cos t)(\sin t) = 0$

$\Rightarrow \cos t + \sin t = 0$

or $\cos t = -\sin t$

$\therefore \quad t = \dfrac{3\pi}{4}$

9 $\boldsymbol{V}_A = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{V}_B = 3\boldsymbol{i} - 4\boldsymbol{j}$
Integrating with respect to t gives

$$\boldsymbol{r}_A = (2t)\boldsymbol{i} + (3t)\boldsymbol{j} + \boldsymbol{c}$$

Given that $\boldsymbol{r}_A(0) = \boldsymbol{i} - \boldsymbol{j}$

$\therefore \quad \boldsymbol{i} - \boldsymbol{j} = \boldsymbol{c}$

$\therefore \quad \boldsymbol{r}_A = (2t + 1)\boldsymbol{i} + (3t - 1)\boldsymbol{j}$

The particles collide after 3 seconds,
$\Rightarrow \boldsymbol{r}_A(3) = \boldsymbol{r}_B(3)$

$\boldsymbol{r}_A(3) = (2(3) + 1)\boldsymbol{i} + (9 - 1)\boldsymbol{j}$

$= 7\boldsymbol{i} + 8\boldsymbol{j}$

Now $\boldsymbol{V}_B = 3\boldsymbol{i} - 4\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_B = (3t)\boldsymbol{i} - (4t)\boldsymbol{j} + \boldsymbol{c}_1$

At $t = 3, \boldsymbol{c}_1 + (9)\boldsymbol{i} - (12)\boldsymbol{j} = 7\boldsymbol{i} + 8\boldsymbol{j}$

$\therefore \quad \boldsymbol{c}_1 = -2\boldsymbol{i} + 20\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_B(t) = (3t - 2)\boldsymbol{i} - (4t - 20)\boldsymbol{j}$

The initial position vector of particle B is
$\boldsymbol{r}_B(0) = -2\boldsymbol{i} + 20\boldsymbol{j}$

10 Let the particles be A and B.

a $\boldsymbol{V}_A = t\boldsymbol{i} + \boldsymbol{j}$
Integrating with respect to t gives

$$\boldsymbol{r}_A = \left(\frac{t^2}{2}\right)\boldsymbol{i} + (t)\boldsymbol{j} + \boldsymbol{c}$$

At $t = 0$, $\boldsymbol{r}_A(0) = \boldsymbol{i} - 2\boldsymbol{j}$

$\therefore \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_A = \left(\dfrac{t^2}{2} + 1\right)\boldsymbol{i} + (t - 2)\boldsymbol{j}$

b $\quad \boldsymbol{r}_A = \left(\dfrac{t^2}{2} + 1\right)\boldsymbol{i} + (t - 2)\boldsymbol{j}$

and $\boldsymbol{r}_B = (s - 4)\boldsymbol{i} + 3\boldsymbol{j}$ (given)
If their paths cross, then both paths pass through a point,

$\therefore \quad \left(\dfrac{t^2}{2} + 1\right)\boldsymbol{i} + (t - 2)\boldsymbol{j} = (s - 4)\boldsymbol{i} + 3\boldsymbol{j}$

$\therefore \quad \dfrac{t^2}{2} + 1 = s - 4 \qquad$ and $t - 2 = 3$

$\therefore \quad t = 5$

$\therefore \quad \dfrac{5^2}{2} + 1 = s - 4$

$s = \dfrac{25}{2} + 5$

$= 17.5$

At $t = 5$, $\boldsymbol{r}_A = \left(\dfrac{25}{2} + 1\right)\boldsymbol{i} + 3\boldsymbol{j}$

i.e., A is at the point $\left(\dfrac{27}{2}, 3\right)$ or $(13.5, 3)$.
At $s = 17.5$, $\boldsymbol{r}_B = (17.5 - 4)\boldsymbol{i} + 3\boldsymbol{j}$

$= 13.5\boldsymbol{i} + 3\boldsymbol{j}$

i.e., B is at the point (13.5, 3).
Thus their paths cross at (13.5, 3).

c If the particles actually collide, then they share the same position at the same time, say t.

$\therefore$ the only possibility is at $t = 5$ as points where paths cross include collisions.
At $t = 5$, we know that A is at (13.5, 3).
B reaches the same point when $s = 17.5$.
Thus, we can conclude that if the particles actually collide, then B should start 12.5 seconds before A, so that both reach (13.5, 3) at the same time.

11 acceleration $\ddot{\mathbf{r}}(t) = \boldsymbol{i} + 2\boldsymbol{j}$ (given)

a Integrating with respect to t gives
$\dot{\mathbf{r}}(t) = (t)\boldsymbol{i} + (2t)\boldsymbol{j} + \boldsymbol{c}$
At $t = 2$, the particle is travelling at $2\boldsymbol{i} - \boldsymbol{j}$

$$\therefore \quad \dot{\mathbf{r}}(2) = (2)\boldsymbol{i} + (4)\boldsymbol{j} + \boldsymbol{c}$$
$$= 2\boldsymbol{i} - \boldsymbol{j}$$
$$\Rightarrow \boldsymbol{c} = -5\boldsymbol{j}$$

$\therefore$ velocity $\dot{\boldsymbol{r}}(t) = t\boldsymbol{i} + (2t - 5)\boldsymbol{j}$

b Integrating again with respect to t gives
$r = \left(\frac{t^2}{2}\right)\boldsymbol{j} + \left(\frac{t^2}{2} - 5t\right)\boldsymbol{j} + \boldsymbol{c}$ (position vector)
At $t = 2$, the particle passes through $\boldsymbol{i}$

$$\therefore \quad \boldsymbol{r}(2) = \left(\frac{4}{2}\right)\boldsymbol{i} + (4 - 10)\boldsymbol{j} + \boldsymbol{c}$$
$$= \boldsymbol{i}$$
$$\therefore \quad 2\boldsymbol{i} - 6\boldsymbol{j} + \boldsymbol{c} = \boldsymbol{i}$$
$$\therefore \quad \boldsymbol{c} = -\boldsymbol{i} + 6\boldsymbol{j}$$
$$\therefore \quad \boldsymbol{r}(t) = \left(\frac{t^2}{2} - 1\right)\boldsymbol{i} + (t^2 - 5t + 6)\boldsymbol{j}$$

c initial position $\boldsymbol{r}(0) = -\boldsymbol{i} + 6\boldsymbol{j}$
initial velocity $\dot{\mathbf{r}}(0) = 0\boldsymbol{i} - 5\boldsymbol{j} = -5\boldsymbol{j}$

12 a i Acceleration of the second particle is
$\ddot{\boldsymbol{r}}_2 = 2\boldsymbol{i} + \boldsymbol{j}$
Integrating with respect to t gives velocity $\dot{\boldsymbol{r}}_2 = 2t\boldsymbol{i} + t\boldsymbol{j} + \boldsymbol{c}$
Given that $\dot{\boldsymbol{r}}_2(0) = -4\boldsymbol{i}$,

$$\therefore \quad -4\boldsymbol{i} = \boldsymbol{c}$$
$$\therefore \quad \dot{\boldsymbol{r}}_2 = (2t - 4)\boldsymbol{i} + t\boldsymbol{j}$$

ii Acceleration of the first particle is
$$\ddot{\boldsymbol{r}}_1(t) = \boldsymbol{i} - \boldsymbol{j}$$

$\therefore$ velocity $\dot{\boldsymbol{r}}_1(t) = t\boldsymbol{i} - t\boldsymbol{j} + \boldsymbol{c}_1$
Now $\dot{\boldsymbol{r}}_1(0) = k\boldsymbol{j}$,
$\therefore \quad \boldsymbol{c}_1 = k\boldsymbol{j}$

$$\therefore \quad \dot{\boldsymbol{r}}_1(t) = t\boldsymbol{i} - (t - k)\boldsymbol{j}$$
$$= t\boldsymbol{i} + (k - t)\boldsymbol{j}$$

b At some time t, $\dot{\boldsymbol{r}}_1(t) = \dot{\boldsymbol{r}}_2(t)$

i
$$\Rightarrow t\boldsymbol{i} + (k - t)\boldsymbol{j} = (2t - 4)\boldsymbol{i} + t\boldsymbol{j}$$
$$\Rightarrow t = 2t - 4 \text{ and } k - t = t$$
$$\Rightarrow t = 4$$

ii $\Rightarrow k - 4 = 4$
$$k = 8$$

iii

$\dot{\boldsymbol{r}}_1(4) = 4\boldsymbol{i} - (8-4)\boldsymbol{j}$

$= 4(\boldsymbol{i} + \boldsymbol{j})$is the common velocity.

Check with $\dot{\boldsymbol{r}}_2$: $\dot{\boldsymbol{r}}_2(4) = (8-4)\boldsymbol{i} + 4\boldsymbol{j}$

$= 4(\boldsymbol{i} + \boldsymbol{j})$

$\Rightarrow 8e^{2t} = 12e^t$

$\Rightarrow e^t = \dfrac{12}{8}$

$= 1.5$

$\therefore \quad t = \log_e 1.5$

13 $\boldsymbol{r}(t) = e^t\boldsymbol{i} + 4e^{2t}\boldsymbol{j},\ t \geq 0$

a Let $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$\Rightarrow e^t = x$ and $4e^{2t} = y$

$\therefore \quad y = 4(e^t)^2$

$= 4x^2$

or $f(x) = 4x^2$

Now $t \geq 0 \Rightarrow e^t \geq 1$
The path of the particle is given by $f(x)$ whose domain is $[1, \infty)$
i.e., $f : [1, \infty) \to R,\ f(x) = 4x^2$

b $\boldsymbol{r}(t) = e^t\boldsymbol{i} + 4e^{2t}\boldsymbol{j}$

i $\therefore \quad \boldsymbol{v}(t) = e^t\boldsymbol{i} + 8e^{2t}\boldsymbol{j}$

ii $v(0) = e^0\boldsymbol{i} + 8e^0\boldsymbol{j}$

$= \boldsymbol{i} + 8\boldsymbol{j}$

iii Now $\boldsymbol{v}(t) = e^t\boldsymbol{i} + 8e^{2t}\boldsymbol{j}$
If it is parallel to $1 + 12\boldsymbol{j}$, then
$e^t\boldsymbol{i} + 8e^{2t}\boldsymbol{j} = k(1 + 12\boldsymbol{j})$

for some constant k.

$\Rightarrow e^t = k$ and $8e^{2t} = 12\boldsymbol{k}$

14 velocity $\dot{\mathbf{r}}(t) = (t-3)\boldsymbol{j},\ t > 0$

a All changes position described by the velocity are parallel to $\boldsymbol{j}$, therefore the path is linear.

b $\boldsymbol{r}(t) = \left(\dfrac{t^2}{2} - 3t\right)\boldsymbol{j} + \boldsymbol{c}$
Now $\boldsymbol{r}(0) = 2\boldsymbol{i} + \boldsymbol{j}$,
$\Rightarrow \boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j}$

i Therefore the particle moves in the direction of $\boldsymbol{j}$, with

$x = 2$ and $\quad y = \dfrac{t^2}{2} - 3t + 1,\ t > 0$

$\therefore \quad x = 2$ and $\quad y \geq -3.5$
(minimum value of y against t graph)

ii The particle is at rest when $\boldsymbol{v} = \boldsymbol{0}$,
i.e., $\dot{\mathbf{r}}(t) = \mathbf{0}$

$\Rightarrow t - 3 = 0$

This is when $t = 3$

The point is $x = 2$ and $y = \dfrac{9}{2} - 9 + 1$

$= -3.5$

$\therefore \quad$ the point is $(2, -3.5)$.

Solutions to multiple-choice questions

1 E $\mathbf{r} = 2t^2\,\boldsymbol{i} + (3t-1)\boldsymbol{j}$

$\dot{\mathbf{r}} = 4t\,\boldsymbol{i} + 3\boldsymbol{j}$

$\therefore\quad \ddot{\mathbf{r}} = 4i$

2 E

$$\begin{aligned}\mathbf{r} &= \sin(3t)\,\boldsymbol{i} - 2\cos(t)\boldsymbol{j}\\ \dot{\mathbf{r}} &= 3\cos(3t)\,\boldsymbol{i} + 2\sin(t)\boldsymbol{j}\\ \dot{\mathbf{r}}(\pi) &= -3\,\boldsymbol{i}\\ \text{Speed} &= |\dot{\mathbf{r}}(\pi)|\\ &= \sqrt{(-3)^2}\\ &= 3\end{aligned}$$

3 B

$$\begin{aligned}\dot{\mathbf{r}}(t) &= 5\,\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}\\ \mathbf{r}(0) &= 3\,\boldsymbol{i} - 6\boldsymbol{k}\\ \mathbf{r}(t) &= 5t\,\boldsymbol{i} - 4t\boldsymbol{j} + 2t\boldsymbol{k} + \mathbf{c}_1\\ \mathbf{r}(0) &= \mathbf{c}_1\\ \therefore\quad \mathbf{c}_1 &= 3\,\boldsymbol{i} - 6\boldsymbol{k}\\ \therefore\quad \mathbf{r}(t) &= (5t+3)\,\boldsymbol{i} - 4\boldsymbol{j} + (2t-6)\boldsymbol{k}\end{aligned}$$

4 E

$$\begin{aligned}\mathbf{r}(t) &= (2t^3-1)\,\boldsymbol{i} + (2t^2+3)\boldsymbol{j} + 6t\boldsymbol{k}\\ \dot{\mathbf{r}}(t) &= 6t^2\,\boldsymbol{i} + 4t\boldsymbol{j} + 6\boldsymbol{k}\\ \ddot{\mathbf{r}} &= 12t\,\boldsymbol{i} + 4\boldsymbol{j}\\ \therefore\quad \ddot{\mathbf{r}}(1) &= 12\,\boldsymbol{i} + 4\boldsymbol{j}\end{aligned}$$

5 C

$$\begin{aligned}\mathbf{r}(t) &= (t^2-4t)(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})\\ \mathbf{r}(0) &= \mathbf{0}\\ \mathbf{r}(4) &= \mathbf{0}\end{aligned}$$

Note: do not automatically assume that the distance travelled is zero. Since the position vector is quadratic, further investigation is needed.

$\dot{\mathbf{r}} = (2t-4)(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

To determine when the particle comes to rest solve $\dot{\mathbf{r}} = 0$.

$\Rightarrow 2t - 4 = 0 \;\therefore\quad t = 2$

Therefore, the particle comes to rest after 2 seconds.

$\mathbf{r}(2) = -4(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = -4\,\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$

Hence, the particle initially begins at $(0, 0, 0)$ and after traveling 2 seconds stops at $(-4, 4, -4)$. After a further 2 seconds the particle is back at $(0, 0, 0)$.

distance travelled

$$\begin{aligned}&= 2 \times |\mathbf{r}(2) - \mathbf{r}(0)|\\ &= 2 \times |(-4, 4, -4)|\\ &= 2 \times \sqrt{16+16+16} = 2\sqrt{48}\\ &= 8\sqrt{3}\text{ m}\end{aligned}$$

6 C

$$\begin{aligned}\ddot{\mathbf{r}}(t) &= 2\,\boldsymbol{i} - \boldsymbol{j}\\ \dot{\mathbf{r}}(0) &= 2\boldsymbol{j}\\ \mathbf{r}(0) &= 3\,\boldsymbol{i}\\ \dot{\mathbf{r}}(t) &= 2t\,\boldsymbol{i} - t\boldsymbol{j} + \mathbf{c}_1\\ \dot{\mathbf{r}}(0) &= \mathbf{c}_1\\ \therefore\quad \mathbf{c}_1 &= 2\boldsymbol{j}\\ \therefore\quad \dot{\mathbf{r}}(t) &= 2t\,\boldsymbol{i} + (2-t)\boldsymbol{j}\\ \mathbf{r}(t) &= t^2\,\boldsymbol{i} + \left(2t - \frac{1}{2}t^2\right)\boldsymbol{j} + \mathbf{c}_2\\ \mathbf{r}(0) &= \mathbf{c}_2\\ \therefore\quad \mathbf{c}_2 &= 3\,\boldsymbol{i}\\ \therefore\quad \mathbf{r}(t) &= (t^2+3)\,\boldsymbol{i} + \left(2t - \frac{1}{2}t^2\right)\boldsymbol{j}\end{aligned}$$

7 C

$$\mathbf{r}(0) = \boldsymbol{i} - 5\boldsymbol{j} + 2k$$

$$\mathbf{r}(3) = 7\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}$$

$$\text{Avg. velocity} = \frac{\mathbf{r}(3) - \mathbf{r}(0)}{3}$$

$$= \frac{1}{3}(6\boldsymbol{i} + 12\boldsymbol{j} - 6\boldsymbol{k})$$

$$= 2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$$

8 E

$$\dot{\mathbf{r}}(t) = 2t\boldsymbol{i} + 3\boldsymbol{j}$$

$$\mathbf{r}(0) = 3\boldsymbol{i} + \boldsymbol{j}$$

$$\mathbf{r}(t) = t^2\boldsymbol{i} + 3t\boldsymbol{j} + \mathbf{c}_1$$

$$\mathbf{r}(0) = \mathbf{c}_1$$

$$\therefore \quad \mathbf{c}_1 = 3\boldsymbol{i} + \boldsymbol{j}$$

$$\therefore \quad \mathbf{r}(t) = (t^2 + 3)\boldsymbol{i} + (3t + 1)\boldsymbol{j}$$

9 C

$$\dot{\mathbf{r}}(t) = t\boldsymbol{i} + e^t\boldsymbol{j}$$

$$\mathbf{r}(0) = 3\boldsymbol{i}$$

$$\mathbf{r}(t) = \frac{1}{2}t^2\boldsymbol{i} + e^t\boldsymbol{j} + \mathbf{c}_1$$

$$\mathbf{r}(0) = \boldsymbol{j} + \mathbf{c}_1$$

$$\therefore \quad \mathbf{c}_1 = 3\boldsymbol{i} - \boldsymbol{j}$$

$$\therefore \quad \mathbf{r}(t) = \left(\frac{1}{2}t^2 + 3\right)\boldsymbol{i} + (e^t - 1)\boldsymbol{j}$$

10 E $\mathbf{r}(t) = 2\cos(\pi t)\boldsymbol{i} + 3\sin(\pi t)\boldsymbol{j}$

At $(\sqrt{3}, 1.5)$:

$2\cos(\pi t) = \sqrt{3}$ and $3\sin(\pi t) = 1.5$

$$\therefore \quad \cos(\pi t) = \frac{\sqrt{3}}{2} \text{ and } \sin(\pi t) = \frac{1}{2}$$

$$\therefore \quad \pi t = \frac{\pi}{6}$$

$$\therefore \quad t = \frac{1}{6}$$

$x = 2\cos(\pi t)$ and

$y = 3\sin(\pi t)$

$$\therefore \quad \frac{dx}{dt} = -2\pi\sin(\pi t) \text{ and}$$

$$\frac{dy}{dt} = 3\pi\cos(\pi t)$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= 3\pi\cos(\pi t)\cdot\frac{1}{-2\pi\sin(\pi t)}$$

$$= -\frac{3}{2}\cot(\pi t)$$

When $t = \frac{1}{6}$,

$$\frac{dy}{dx} = -\frac{3}{2}\cot\left(\frac{\pi}{6}\right)$$

$$= -\frac{3}{2}\left(\frac{1}{\tan\left(\frac{\pi}{6}\right)}\right)$$

$$= -\frac{3}{2}(\sqrt{3})$$

$$= -\frac{3\sqrt{3}}{2}$$

Solutions to extended-response questions

1 a Let $\boldsymbol{v}_P = 9\boldsymbol{i} + 6\boldsymbol{j}$ and $\boldsymbol{v}_Q = 5\boldsymbol{i} + 4\boldsymbol{j}$

$\therefore \quad |\boldsymbol{v}_P| = \sqrt{9^2 + 6^2}$ and $|\boldsymbol{v}_Q| = \sqrt{5^2 + 4^2}$

$= \sqrt{81 + 36} \qquad = \sqrt{25 + 16}$

$= \sqrt{117} \qquad = \sqrt{41}$

The speeds of particles P and Q are $3\sqrt{13}$ m/s and $\sqrt{41}$ m/s respectively.

b i $\boldsymbol{r}_P(t) = \int \boldsymbol{v}_P \, dt$

$= \int 9\boldsymbol{i} + 6\boldsymbol{j} \, dt$

$= \int 9dt\,\boldsymbol{i} + \int 6dt\,\boldsymbol{j}$

$= 9t\boldsymbol{i} + 6t\boldsymbol{j} + \boldsymbol{c}_1$, where $\boldsymbol{c}_1$ is a constant vector.

Now $\boldsymbol{r}_P(4) = 96\boldsymbol{i} + 44\boldsymbol{j}$,

$\therefore \quad 96\boldsymbol{i} + 44\boldsymbol{j} = 36\boldsymbol{i} + 24\boldsymbol{j} + \boldsymbol{c}_1$

$\therefore \quad \boldsymbol{c}_1 = 60\boldsymbol{i} + 20\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_P(t) = (9t + 60)\boldsymbol{i} + (6t + 20)\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_P(0) = 60\boldsymbol{i} + 20\boldsymbol{j}$, the position vector of P at time $t = 0$.

$\boldsymbol{r}_Q(t) = \int \boldsymbol{v}_Q \, dt$

$= \int 5\boldsymbol{i} + 4\boldsymbol{j} \, dt$

$= \int 5dt\,\boldsymbol{i} + \int 4dt\,\boldsymbol{j}$

$= 5t\boldsymbol{i} + 4t\boldsymbol{j} + \boldsymbol{c}_2$, where $\boldsymbol{c}_2$ is a constant vector.

Now $\boldsymbol{r}_Q(4) = 100\boldsymbol{i} + 96\boldsymbol{j}$,

$\therefore \quad 100\boldsymbol{i} + 96\boldsymbol{j} = 20\boldsymbol{i} + 16\boldsymbol{j} + \boldsymbol{c}_2$

$\therefore \quad \boldsymbol{c}_2 = 80\boldsymbol{i} + 80\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_Q(t) = (5t + 80)\boldsymbol{i} + (4t + 80)\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_Q(0) = 80\boldsymbol{i} + 80\boldsymbol{j}$, the position vector of Q at time $t = 0$.

ii $\overrightarrow{PQ} = \boldsymbol{r}_Q(t) - \boldsymbol{r}_P(t)$

$= (5t + 80)\boldsymbol{i} + (4t + 80)\boldsymbol{j} - ((9t + 60)\boldsymbol{i} + (6t + 20)\boldsymbol{j})$

$= (20 - 4t)\boldsymbol{i} + (60 - 2t)\boldsymbol{j}$

c $|\overrightarrow{PQ}| = \sqrt{(20 - 4t)^2 + (60 - 2t)^2}$

$= \sqrt{400 - 160t + 16t^2 + 3600 - 240t + 4t^2}$

$= \sqrt{20(t^2 - 20t + 200)}$

$= 2\sqrt{5(t^2 - 20t + 200)}$

P and Q are nearest to each other when $|\overrightarrow{PQ}|$ is a minimum, i.e., when $t^2 - 20t + 200$ is a minimum.

$$\text{Let } y = t^2 - 20t + 200$$

$$\therefore \quad \frac{dy}{dt} = 2t - 20$$

When $\dfrac{dy}{dt} = 0$, $2t - 20 = 0$

$$\therefore \quad t = 10$$

Since $t^2 - 20t + 200$ is a concave-up parabola, P and Q are nearest to each other when $t = 10$ seconds.

When $t = 10$, $|\overrightarrow{PQ}| = 2\sqrt{5(10^2 - 20 \times 10 + 200)}$

$= 2\sqrt{500}$

$= 2\sqrt{5}$ metres

2 a Let $\boldsymbol{v}_A = -3\boldsymbol{i} + 29\boldsymbol{j}$ and $\boldsymbol{v}_B = v(\boldsymbol{i} + 7\boldsymbol{j}) = v\boldsymbol{i} + 7v\boldsymbol{j}$

$\therefore \quad \boldsymbol{r}_A(t) = \int \boldsymbol{v}_A\, dt$ $\qquad \therefore \quad \boldsymbol{r}_B(t) = \int \boldsymbol{v}_B\, dt$

$= -3t\boldsymbol{i} + 29t\boldsymbol{j} + \boldsymbol{c}_1$ $\qquad = vt\boldsymbol{i} + 7vt\boldsymbol{j} + \boldsymbol{c}_2$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are constant vectors.

$$\overrightarrow{AB} = \boldsymbol{r}_B(t) - \boldsymbol{r}_A(t)$$

$$= vt\boldsymbol{i} + 7vt\boldsymbol{j} + \boldsymbol{c}_2 - (-3t\boldsymbol{i} + 29t\boldsymbol{j} + \boldsymbol{c}_1)$$

$$= (v + 3)t\boldsymbol{i} + (7v - 29)t\boldsymbol{j} + \boldsymbol{c}_2 - \boldsymbol{c}_1$$

When $t = 0$, $\overrightarrow{AB} = -56\boldsymbol{i} + 8\boldsymbol{j}$

$$\therefore \quad \boldsymbol{c}_2 - \boldsymbol{c}_1 = -56\boldsymbol{i} + 8\boldsymbol{j}$$

$$\therefore \quad \overrightarrow{AB} = ((v + 3)t - 56)\boldsymbol{i} + ((7v - 29)t + 8)\boldsymbol{j}$$

b When the particles collide,

$$\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$$

$$\therefore \quad \boldsymbol{r}_B(t) - \boldsymbol{r}_A(t) = \boldsymbol{0}$$

$$\therefore \quad \overrightarrow{AB} = \boldsymbol{0}$$

$$\therefore \quad ((v + 3)t - 56)\boldsymbol{i} + ((7v - 29)t + 8)\boldsymbol{j} = \boldsymbol{0}$$

$\therefore \quad (v + 3)t - 56 = 0$ and $(7v - 29)t + 8 = 0 \qquad ①$

$\therefore \quad vt + 3t = 56$

$\therefore \quad vt = 56 - 3t$

$\therefore \quad v = \dfrac{56 - 3t}{t} \qquad ②$

Substituting ② into ①:

$$\left(7\left(\frac{56 - 3t}{t}\right) - 29\right)t + 8 = 0$$

$\therefore \quad 7(56 - 3t) - 29t + 8 = 0$

$\therefore \quad 400 - 50t = 0$

$\therefore \quad t = 8$

Substituting $t = 8$ into ② : $v = \dfrac{56 - 3 \times 8}{8}$

$= 4$

The particles collide when $v = 4$.

c **i** If $v = 3$, $\overrightarrow{AB} = ((3 + 3)t - 56)\boldsymbol{i} + ((7 \times 3 - 29)t + 8)\boldsymbol{j}$

$= (6t - 56)\boldsymbol{i} + (8 - 8t)\boldsymbol{j}$

ii When the particles are closest, $|\overrightarrow{AB}|$ is a minimum.

$\therefore \quad |\overrightarrow{AB}| = \sqrt{(6t - 56)^2 + (8 - 8t)^2}$

$= \sqrt{36t^2 - 672t + 3136 + 64 - 128t + 64t^2}$

$= \sqrt{100t^2 - 800t + 3200}$

$= 10\sqrt{t^2 - 8t + 32}$

$|\overrightarrow{AB}|$ is a minimum when $t^2 - 8t + 32$ is a minimum.

Let $y = t^2 - 8t + 32$

$\therefore \quad \dfrac{dy}{dt} = 2t - 8$

When $\dfrac{dy}{dt} = 0$, $2t - 8 = 0$

$\therefore \quad t = 4$

As $t^2 - 8t + 32$ is a concave-up parabola, A and B are closest after 4 s.

3 **a** $\overrightarrow{BF} = \overrightarrow{CF} - \overrightarrow{CB}$

$= (7\boldsymbol{i} + 8\boldsymbol{j}) - (10\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$

$= -3\boldsymbol{i} + 6\boldsymbol{j} - 6\boldsymbol{k}$

b $BF = |\overrightarrow{BF}|$

$= \sqrt{(-3)^2 + 6^2 + (-6)^2}$

$= \sqrt{81}$

$= 9$

c Speed of the bee $= \dfrac{9}{3} = 3$ m/s.

d Velocity of the bee, $\boldsymbol{v}_B = 3 \times \dfrac{1}{9}(-3\boldsymbol{i} + 6\boldsymbol{j} - 6\boldsymbol{k})$

$= (-\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$ m/s

e Let the position of the bee relative to the child be $\boldsymbol{r}_B(t)$.

$$\boldsymbol{r}_B(t) = \int \boldsymbol{r}_B\, dt$$

$$= \int -\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}\, dt$$

$= -t\boldsymbol{i} + 2t\boldsymbol{j} - 2tk + \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector.

Now $\boldsymbol{r}_B(0) = \overrightarrow{CB}$, $\boldsymbol{c} = 10\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$

$\therefore \quad \boldsymbol{r}_B(t) = (10 - t)\boldsymbol{i} + (2t + 2)\boldsymbol{j} + (6 - 2t)\boldsymbol{k}$

Check: $\boldsymbol{r}_B(3) = (10 - 3)\boldsymbol{i} + (2 \times 3 + 2)\boldsymbol{j} + (6 - 2 \times 3)\boldsymbol{k}$

$= 7\boldsymbol{i} + 8\boldsymbol{j}$

$= \overrightarrow{CF}$

Now $|\boldsymbol{r}_B(t)| = \sqrt{(10 - t)^2 + (2t + 2)^2 + (6 - 2t)^2}$

$= \sqrt{100 - 20t + t^2 + 4t^2 + 8t + 4 + 36 - 24t + 4t^2}$

$= \sqrt{9t^2 - 36t + 140}$

The bee is closest to the child when $|\boldsymbol{r}_B(t)|$ is a minimum,
i.e., when $9t^2 - 36t + 140$ is a minimum.

Let $y = 9t^2 - 36t + 140$

$\therefore \quad \dfrac{dy}{dt} = 18t - 36$

When $\dfrac{dy}{dt} = 0$, $18t - 36 = 0$

$\therefore \quad t = 2$

Since $9t^2 - 36t + 140$ is a concave-up parabola, the bee is closest to the child after two seconds.

$$|\boldsymbol{r}_B(2)| = \sqrt{9 \times 2^2 - 36 \times 2 + 140}$$
$$= \sqrt{36 - 72 + 140}$$
$$= \sqrt{104}$$
$$= 2\sqrt{26}$$

The shortest distance between the bee and the child is $2\sqrt{26}$ metres.

4 a i Let $\overrightarrow{JM}$ and $\overrightarrow{JB}$ be the position vectors of the motor boat and the police boat, respectively, with respect to the jetty, J.

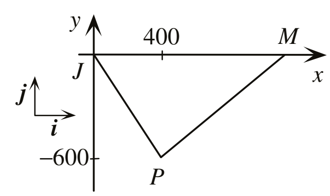
$$\overrightarrow{JM} = \int 6\boldsymbol{i}\,dt$$
$$= 6t\boldsymbol{i} + \boldsymbol{c}_1, \text{ where } \boldsymbol{c}_1 \text{ is a constant vector.}$$

When $t = 0$, $\overrightarrow{JM} = \boldsymbol{0}$

$$\therefore \quad \boldsymbol{c}_1 = \boldsymbol{0}$$
$$\therefore \quad \overrightarrow{JM} = 6t\boldsymbol{i}$$
$$\overrightarrow{JB} = \int u(8\boldsymbol{i} + 6\boldsymbol{j})\,dt$$
$$= u(8t\boldsymbol{i} + 6t\boldsymbol{j}) + \boldsymbol{c}_2, \text{ where } \boldsymbol{c}_2 \text{ is a constant vector.}$$

When $t = 0$, $\overrightarrow{JB} = 400\boldsymbol{i} - 600\boldsymbol{j}$

$$\therefore \quad \boldsymbol{c}_2 = 400\boldsymbol{i} - 600\boldsymbol{j}$$
$$\therefore \quad \overrightarrow{JB} = (400 + 8ut)\boldsymbol{i} + (6ut - 600)\boldsymbol{j}$$



When the motor boat and police boat meet at the point M,

$$\overrightarrow{JM} = \overrightarrow{JB}$$
$$\therefore \quad 6t\boldsymbol{i} = (400 + 8ut)\boldsymbol{i} + (6ut - 600)\boldsymbol{j}$$

$\therefore \quad 6t = 400 + 8ut$ and $6ut = 600$

$= 400 + 8\left(\frac{100}{t}\right)t$ $\qquad \therefore \quad u = \frac{100}{t}, \ t \neq 0$

$= 1200$

$\therefore \quad t = 200$

ii When $t = 200$, $u = \frac{100}{200}$

$$= \frac{1}{2}$$

iii Velocity of police boat $= u(8\boldsymbol{i} + 6\boldsymbol{j})$

$$= \frac{1}{2}(8\boldsymbol{i} + 6\boldsymbol{j})$$

$$= 4\boldsymbol{i} + 3\boldsymbol{j}$$

Speed of police boat $= \sqrt{4^2 + 3^2}$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$= 5$$

Speed of police boat is 5 m/s.

iv When $t = 200$, $\overrightarrow{JM} = 6 \times 200\boldsymbol{i}$

$$= 1200\boldsymbol{i}$$

The position of the point where the boats meet is (1200, 0).

b The position of the police boat relative to J is

$$\overrightarrow{JB} = (4t + 400)\boldsymbol{i} + (3t - 600)\boldsymbol{j}$$

$$|\overrightarrow{JB}| = \sqrt{(4t + 400)^2 + (3t - 600)^2}$$

$$= \sqrt{16t^2 + 3200t + 160\,000 + 9t^2 - 3600t + 360\,000}$$

$$= \sqrt{25(t^2 - 16t + 20\,800)}$$

$$= 5\sqrt{t^2 - 16t + 20\,800}$$

The police boat is closest to J when $|\overrightarrow{JB}|$ is a minimum,
i.e., when $t^2 - 16t + 20\,800$ is a minimum.

Let $y = t^2 - 16t + 20\,800$

$$\therefore \quad \frac{dy}{dt} = 2t - 16$$

When $\frac{dy}{dt} = 0$, $2t - 16 = 0$

$$\therefore \quad t = 8$$

Since $t^2 - 16t + 20\,800$ is a concave-up parabola, the police boat is closest to J when $t = 8$

$$|\overrightarrow{JB}| = 5\sqrt{8^2 - 16 \times 8 + 20\,800}$$

$$= 5\sqrt{64 - 128 + 20\,800}$$

$$= 5\sqrt{20\,736}$$

$$= 5 \times 144$$

$$= 720$$

After eight seconds, the police boat is closest to J at a distance of 720 metres.

5 **a** **i** $\overrightarrow{OA} = \int 6\boldsymbol{i} + 3\boldsymbol{j}dt$

$= 6t\boldsymbol{i} + 3t\boldsymbol{j} + \boldsymbol{c}_1$, where $\boldsymbol{c}_1$ is a constant vector.

When $t = 0, \overrightarrow{OA} = -\boldsymbol{i} + 2\boldsymbol{j}$,

$\therefore \quad \boldsymbol{c} = -\boldsymbol{i} + 2\boldsymbol{j}$

$\therefore \quad \overrightarrow{OA} = (6t - 1)\boldsymbol{i} + (3t + 2)\boldsymbol{j}$

ii $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= (6t - 1)\boldsymbol{i} + (3t + 2)\boldsymbol{j} - (2\boldsymbol{i} + \boldsymbol{j})$

$\therefore \quad \overrightarrow{BA} = (6t - 3)\boldsymbol{i} + (3t + 1)\boldsymbol{j}$

b $|\overrightarrow{BA}| = \sqrt{(6t - 3)^2 + (3t + 1)^2}$

$= \sqrt{36t^2 - 36t + 9 + 9t^2 + 6t + 1}$

$= \sqrt{45t^2 - 30t + 10}$

$= \sqrt{5(9t^2 - 6t + 2)}$

When $|\overrightarrow{BA}| = 5$,

$\sqrt{5(9t^2 - 6t + 2)} = 5$

$\therefore \quad 5(9t^2 - 6t + 2) = 25$

$\therefore \quad 9t^2 - 6t + 2 = 5$

$\therefore \quad 9t^2 - 6t - 3 = 0$

$\therefore \quad 3t^2 - 2t - 1 = 0$

$\therefore \quad (3t + 1)(t - 1) = 0$

$\therefore \quad t = -\frac{1}{3}$ or $t = 1$

Assuming $t \geq 0$, $|\overrightarrow{BA}| = 5$ after one second.

c **i** When $t = 1$, $\overrightarrow{BA} = (6 \times 1 - 3)\boldsymbol{i} + (3 \times 1 + 1)\boldsymbol{j}$

$= 3\boldsymbol{i} + 4\boldsymbol{j}$

$\boldsymbol{c} = \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|}$

$= \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

ii $\boldsymbol{d} = \frac{1}{5}(4\boldsymbol{i} - 3\boldsymbol{j})$

iii $p\boldsymbol{c} + q\boldsymbol{d} = p \times \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j}) + q \times \frac{1}{5}(4\boldsymbol{i} - 3\boldsymbol{j})$

$= \frac{3p + 4q}{5}\boldsymbol{i} + \frac{4p - 3q}{5}\boldsymbol{j}.$

Since $6\boldsymbol{i} + 3\boldsymbol{j} = p\boldsymbol{c} + q\boldsymbol{d}$

then $\frac{3p + 4q}{5} = 6$ and $\frac{4p - 3q}{5} = 3$

$\therefore \quad 3p + 4q = 30 \quad ① \qquad 4p - 3q = 15 \quad ②$

$4 \times ① - 3 \times ②$ yields $25q = 75$

$\therefore \quad q = 3$

Substituting $q = 3$ in ① : $3p + 4 \times 3 = 30$

$\therefore \quad 3p = 18$

$\therefore \quad p = 6$

$\therefore \quad 6\boldsymbol{i} + 3\boldsymbol{j} = 6\boldsymbol{c} + 3\boldsymbol{d}$

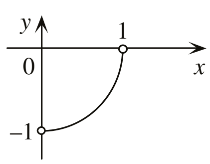
6 a Let $\boldsymbol{r}(\theta) = x\boldsymbol{i} + y\boldsymbol{j}$

$\therefore \quad x = \cos\theta$ and $y = -\sin\theta,\ 0 < \theta < \frac{\pi}{2}$

$\therefore \quad x^2 = \cos^2\theta \qquad y^2 = \sin^2\theta$

$\therefore \quad x^2 + y^2 = \cos^2\theta + \sin^2\theta$

$\therefore \quad x^2 + y^2 = 1,\ 0 < x < 1,\ -1 < y < 0$



b i At $t = 0$, P is at the point $(16, 0)$ and $\overrightarrow{OP} = a\boldsymbol{i}$

$\therefore \quad a = 16$

ii At $t = \frac{\pi}{4}$, P is at the point $(0, -16)$ and $\overrightarrow{OP} = 16\cos\frac{n\pi}{4}\boldsymbol{i} + b\sin\frac{n\pi}{4}\boldsymbol{j}$

$\therefore \quad 16\cos\frac{n\pi}{4} = 0$ and $b\sin\frac{n\pi}{4} = -16$

The $\boldsymbol{i}$ component equals zero for the first time,

$\therefore \quad \cos\frac{n\pi}{4} = 0$

$\Rightarrow \frac{n\pi}{4} = \pm\frac{\pi}{2}$

$\therefore \quad n = \pm 2$

iii Assuming $n = 2$, then $b = -16$.
(For $n = -2,\ b = 16$.)

iv Let $\boldsymbol{v}(t)$ and $\boldsymbol{a}(t)$ be the velocity and acceleration of P at time t.

$$\begin{aligned}\text{Then } \boldsymbol{v}(t) &= \frac{d}{dt}\overrightarrow{OP} \\ &= \frac{d}{dt}(16\cos(2t)\,\boldsymbol{i} - 16\sin(2t)\,\boldsymbol{j}) \\ &= -32\sin(2t)\,\boldsymbol{i} - 32\cos(2t)\,\boldsymbol{j} \\ \text{and } \boldsymbol{a}(t) &= \boldsymbol{v}'(t) \\ &= -64\cos(2t)\,\boldsymbol{i} + 64\sin(2t)\,\boldsymbol{j}\end{aligned}$$

c **i**
$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= (8\sin(t)\,\boldsymbol{i} + 8\cos(t)\,\boldsymbol{j}) - (16\cos(2t)\,\boldsymbol{i} - 16\sin(2t)\,\boldsymbol{j}) \\ &= (8\sin(t) - 16\cos(2t))\,\boldsymbol{i} + (8\cos(t) + 16\sin(2t))\,\boldsymbol{j} \\ &= 8((\sin(t) - 2\cos(2t))\,\boldsymbol{i} + (\cos(t) + 2\sin(2t))\,\boldsymbol{j})\end{aligned}$$

ii
$$\begin{aligned}|\overrightarrow{PQ}|^2 &= 8^2((\sin(t) - 2\cos(2t))^2 + (\cos(t) + 2\sin(2t))^2) \\ &= 64(\sin^2(t) - 4\sin(t)\cos(2t) + 4\cos^2(2t) + \cos^2(t) \\ &\quad + 4\cos(t)\sin(2t) + 4\sin^2(2t)) \\ &= 64(\sin^2(t) + \cos^2(t) + 4(\cos^2(t) + \sin^2(2t)) \\ &\quad + 4(\sin(2t)\cos(t) - \cos(2t)\sin(t))) \\ &= 64(1 + 4 + 4(\sin(2t - t))) \\ &= 64(5 + 4\sin(t))\end{aligned}$$

d The minimum distance between P and Q is when $|\overrightarrow{PQ}|^2$ is a minimum, i.e., when $5 + 4\sin(t)$ is a minimum.

$$\therefore \quad \sin(t) = -1$$

$$\therefore \quad t = \frac{3\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$$

$$\begin{aligned}\therefore \quad \text{minimum}|\overrightarrow{PQ}|^2 &= 64\left(5 + 4\sin\left(\frac{3\pi}{2} + 2k\pi\right)\right) \\ &= 64(5 + 4 \times -1) \\ &= 64 \times 1 \\ &= 64\end{aligned}$$

$$\therefore \quad \overrightarrow{PQ} = 8$$

The minimum distance between P and Q is eight centimetres.

7 **a** Let $\boldsymbol{r}(t)$ be the position of the particle at time t.

$$\boldsymbol{r}(t) = \int \boldsymbol{v}\,dt$$
$$= \int (2\cos t)\boldsymbol{i} - (4\sin t\cos t)\boldsymbol{j}\,dt$$
$$= 2\int \cos t\,dt\,\boldsymbol{i} - 2\int \sin(2t)\,dt$$
$$= 2\sin t\,\boldsymbol{i} + \cos(2t)\boldsymbol{j} + \boldsymbol{c},\ \text{where } \boldsymbol{c} \text{ is a constant vector.}$$

Now $\boldsymbol{r}(0) = 3\boldsymbol{j}$,

$$\therefore\quad 3\boldsymbol{j} = \boldsymbol{j} + \boldsymbol{c}$$
$$\therefore\quad \boldsymbol{c} = 2\boldsymbol{j}$$
$$\therefore\quad \boldsymbol{r}(t) = 2\sin t\,\boldsymbol{i} + (\cos(2t) + 2)\boldsymbol{j},\ t \geq 0$$

b The particle comes to rest when $|\boldsymbol{v}| = 0$.

$$|\boldsymbol{v}| = \sqrt{(2\cos t)^2 + (4\sin t\cos t)^2}$$
$$= \sqrt{4\cos^2 t + 16\sin^2 t\cos^2 t}$$
$$= \sqrt{4\cos^2 t(1 + 4\sin^2 t)}$$
$$= 2\cos t\sqrt{1 + 4\sin^2 t}$$

When $|\boldsymbol{v}| = 0, 2\cos t\sqrt{1 + 4\sin^2 t} = 0$

$$\therefore\quad \cos t = 0,\ \text{since } 1 + 4\sin^2 t > 0$$
$$\therefore\quad t = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$$

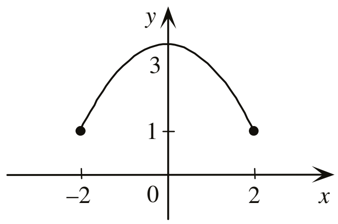
The particle first comes to rest when $k = 0$ at $t = \dfrac{\pi}{2}$,

$$\boldsymbol{r}\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right)\boldsymbol{i} + (\cos(\pi) + 2)\boldsymbol{j}$$
$$= 2\boldsymbol{i} + \boldsymbol{j}$$

c **i** Let $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$

$$\therefore\quad x = 2\sin t \qquad \text{and } y = \cos(2t) + 2,\ t \geq 0$$
$$\therefore\quad \frac{x}{2} = \sin t \qquad = 1 - 2\sin^2 t + 2$$
$$\therefore\quad \frac{x^2}{4} = \sin^2 t \qquad = 3 - 2\sin^2 t$$
$$\therefore\quad \frac{x^2}{2} = 2\sin^2 t \qquad = 3 - \frac{x^2}{2},\ -2 \leq x \leq 2, 1 \leq y \leq 3$$

ii



d $|\boldsymbol{v}|^2 = 4\cos^2 t(1 + 4\sin^2 t)$

$= 4\cos^2 t(1 + 4(1 - \cos^2 t))$

$= 4\cos^2 t(5 - 4\cos^2 t)$

$= -16\cos^4 t + 20\cos^2 t$

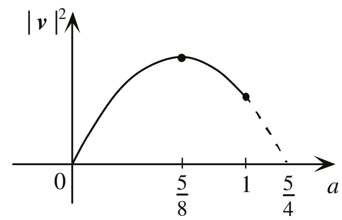
$= -16a^2 + 20a, \text{ where } a = \cos^2 t, \ 0 \le a \le 1$

To find the a-axis intercepts,

let $-16a^2 + 20a = 0$

$\therefore \quad 4a(5 - 4a) = 0$

$\therefore \quad a = 0 \text{ or } \dfrac{5}{4}$



By symmetry, the maximum speed of the particle occurs when $a = \dfrac{5}{8}$.

When $a = \dfrac{5}{8}$, $|\boldsymbol{v}|^2 = -16 \times \left(\dfrac{5}{8}\right)^2 + 20 \times \left(\dfrac{5}{8}\right)$

$= \dfrac{-16 \times 25}{64} + \dfrac{20 \times 5}{8}$

$= \dfrac{25}{4}$

$\therefore \quad |\boldsymbol{v}| = \dfrac{5}{2}$

The maximum speed of the particle is $\dfrac{5}{2}$.

e The particle is at rest when $t = \dfrac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$.

The particle comes to rest for the second time when $k = 1$, i.e., $t = \dfrac{3\pi}{2}$.

f **i** $d^2 = |\boldsymbol{r}(t)|^2$

$= (2\sin t)^2 + (\cos(2t) + 2)^2$

$= 4\sin^2 t + \cos^2(2t) + 4\cos(2t) + 4$

$= 4\sin^2 t + \cos^2(2t) + 4(\cos^2 t - \sin^2 t) + 4$

$= \cos^2(2t) + 4\cos^2 t + 4$

$= \cos^2(2t) + 2(\cos(2t) + 1) + 4$

$= \cos^2(2t) + 2\cos(2t) + 6$, as required.

ii The particle is closest to the origin when d^2 is a minimum,
i.e., when $\cos^2(2t) + 2\cos(2t) + 6$ is a minimum,
i.e., when $d^2 = c^2 + 2c + 6$ is a minimum, where $c = \cos(2t)$.

Now $\dfrac{d}{dc}(d^2) = 2c + 2$

When $\dfrac{d}{dc}(d^2) = 0,\ 2c + 2 = 0$

$\therefore\quad c = -1$

$\therefore\quad \cos(2t) = -1$

$\therefore\quad 2t = \pi + 2k\pi,\ k \in Z$

$\therefore\quad t = \dfrac{\pi}{2} + k\pi,\ k \in Z$

The particle is closest to the origin when $t = \dfrac{\pi}{2} + k\pi,\ k \in Z$

8 **a** $\boldsymbol{v}(t) = \displaystyle\int 2\boldsymbol{j} - 10\boldsymbol{k}\,dt$

$= 2t\boldsymbol{j} - 10t\boldsymbol{k} + \boldsymbol{c}_1$, where $\boldsymbol{c}_1$ is a constant vector.

Now $\boldsymbol{v}(0) = a\boldsymbol{i} + b\boldsymbol{j} + 20\boldsymbol{k}$,

$\therefore\quad \boldsymbol{c}_1 = a\boldsymbol{i} + b\boldsymbol{j} + 20\boldsymbol{k}$

$\therefore\quad \boldsymbol{v}(t) = a\boldsymbol{i} + (b + 2t)\boldsymbol{j} + (20 - 10t)\boldsymbol{k}$, the velocity of the ball at time t.

b $\boldsymbol{r}(t) = \displaystyle\int \boldsymbol{v}(t)\,dt$

$= \displaystyle\int a\boldsymbol{i} + (b + 2t)\boldsymbol{j} + (20 - 10t)\boldsymbol{k}\,dt$

$= at\boldsymbol{i} + (bt + t^2)\boldsymbol{j} + (20t - 5t^2)\boldsymbol{k} + \boldsymbol{c}_2$, where $\boldsymbol{c}_2$ is a constant vector

Now $\boldsymbol{r}(0) = \boldsymbol{0}$,

$\therefore\quad \boldsymbol{c}_2 = \boldsymbol{0}$

$\therefore\quad \boldsymbol{r}(t) = at\boldsymbol{i} + (bt + t^2)\boldsymbol{j} + (20t - 5t^2)\boldsymbol{k}$, position vector of the ball at time$t$.

c Assuming the ground is flat in the x–y plane, the ball will return to the ground when the $\boldsymbol{k}$ component is zero.

$20t - 5t^2 = 0$

$\therefore \quad t(20 - 5t) = 0$

$\therefore \quad t = 0$ or $t = 4$

The time of flight of the ball is four seconds.

d When $t = 4$, $\boldsymbol{r}(4) = 4a\boldsymbol{i} + (4b + 16)\boldsymbol{j}$

Also $\boldsymbol{r}(4) = 100\boldsymbol{i}$,

hence $4a = 100$

$\therefore \quad a = 25$

and $4b + 16 = 0$

$\therefore \quad b = -4$

e The angle of projection is between vectors $25\boldsymbol{i} - 4\boldsymbol{j} + 20\boldsymbol{k}$ (the flight of the ball) and $25\boldsymbol{i} - 4\boldsymbol{j}$ (the vector directly below the flight on the ground).

Using $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$, where $\boldsymbol{a} = 25\boldsymbol{i} - 4\boldsymbol{j} + 20\boldsymbol{k}$ and $\boldsymbol{b} = 25\boldsymbol{i} - 4\boldsymbol{j}$ and θ is the angle of projection,

$$|\boldsymbol{a}| = \sqrt{625 + 16 + 400}$$

$$= \sqrt{1041}$$

$$|\boldsymbol{b}| = \sqrt{625 + 16}$$

$$= \sqrt{641}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = 625 + 16$$

$$= 641$$

Then $641 = \sqrt{641} \times \sqrt{1041} \times \cos\theta$

$$\therefore \quad \cos\theta = \frac{\sqrt{641}}{\sqrt{1041}}$$

$\therefore \quad \theta = 38.307\,06$

The angle of projection is 38.3°, correct to one decimal place.

9 a i Let $\boldsymbol{p}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k},\ 0 \le t \le 2\pi$

$$= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$\therefore \quad x = \cos t, y = \sin t,\ z = -1,\ -1 \le x \le 1,\ -1 \le y \le 1$

$$\therefore \quad x^2 + y^2 = \cos^2 t + \sin^2 t$$

$$= 1$$

The particle P is moving along a circular path centred on $(0, 0, -1)$ with radius length one.

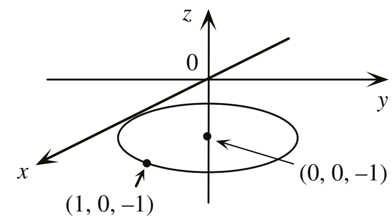
$$\boldsymbol{p}(0) = \boldsymbol{i} - \boldsymbol{k}$$

$$\boldsymbol{p}\left(\frac{\pi}{2}\right) = \boldsymbol{j} - \boldsymbol{k}$$

$$\boldsymbol{p}(\pi) = -\boldsymbol{i} - \boldsymbol{k}$$

$$\boldsymbol{p}\left(\frac{3\pi}{2}\right) = -\boldsymbol{j} - \boldsymbol{k}$$

$$\boldsymbol{p}(2\pi) = \boldsymbol{i} - \boldsymbol{k}$$



As t increases, x decreases from 1 to -1, hence the particle starts at $(1, 0, -1)$ and moves 'anticlockwise' around the circular path a distance of one 'below' the x–y (horizontal) plane. The particle finishes at $(1, 0, -1)$ after one revolution.

ii $|\boldsymbol{p}(t)| = \sqrt{(\cos t)^2 + (\sin t)^2 + (-1)^2}$

$$= \sqrt{\cos^2 t + \sin^2 t + 1}$$

$$= \sqrt{1 + 1}$$

$$= \sqrt{2}$$

The distance of P from the origin at time t is $\sqrt{2}$ units.

iii Let $\dot{\boldsymbol{p}}(t)$ be the velocity of particle P at time t.

$\dot{\boldsymbol{p}}(t) = (-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j},\ 0 \le t \le 2\pi$

iv $((\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}) \cdot \dot{\boldsymbol{p}}(t) = ((\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}) \cdot ((-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j})$

$$= -\cos t \sin t + \sin t \cos t$$

$$= 0$$

Hence $(\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}$ is perpendicular to $\dot{\boldsymbol{p}}(t)$ for any value of t.

v $\ddot{\boldsymbol{p}}(t) = (-\cos t)\boldsymbol{i} - (\sin t)\boldsymbol{j},\ 0 \le t \le 2\pi$

b **i** $\overrightarrow{PQ} = \boldsymbol{q}(t) - \boldsymbol{p}(t)$

$$= \left((\cos 2t)\boldsymbol{i} - (\sin 2t)\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}\right) - ((\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k})$$

$$= (\cos 2t - \cos t)\boldsymbol{i} + (-\sin t - \sin 2t)\boldsymbol{j} + \frac{3}{2}\boldsymbol{k}$$

ii The distance between P and Q at time t is given by $|\overrightarrow{PQ}|$

$$|\overrightarrow{PQ}| = \sqrt{(\cos 2t - \cos t)^2 + (-\sin t - \sin 2t)^2 + \left(\frac{3}{2}\right)^2}$$

$$= \sqrt{\cos^2 2t - 2\cos 2t \cos t + \cos^2 t + \sin^2 t + 2\sin t \sin 2t + \sin^2 2t + \frac{9}{4}}$$

$$= \sqrt{(\cos^2 2t + \sin^2 2t) + (\cos^2 t + \sin^2 t) - 2(\cos 2t \cos t - \sin 2t \sin t) + \frac{9}{4}}$$

$$= \sqrt{1 + 1 + \frac{9}{4} - 2\cos 3t}$$

$$= \sqrt{\frac{17}{4} - 2\cos 3t}, \text{ as required.}$$

iii The maximum distance between the particles occurs when $|\overrightarrow{PQ}|$ is a maximum, i.e., when $\frac{17}{4} - 2\cos 3t$ is a maximum.

$$-1 \leq \cos 3t \leq 1$$

$$\therefore \quad -2 \leq -2\cos 3t \leq 2$$

$$\therefore \quad \frac{9}{4} \leq \frac{17}{4} - 2\cos 3t \leq \frac{25}{4}$$

$$\therefore \quad \frac{3}{2} \leq \sqrt{\frac{17}{4} - 2\cos 3t} \leq \frac{5}{2}$$

The maximum distance between the particles is $\frac{5}{2}$ units.

iv $\frac{17}{4} - 2\cos 3t = \frac{25}{4}, \; 0 \leq t \leq 2\pi$

$$\therefore \quad 2\cos 3t = -2$$

$$\therefore \quad \cos 3t = -1$$

$$\therefore \quad 3t = \pi, 3\pi, 5\pi \text{ since } 0 \leq 3t \leq 6\pi$$

$$\therefore \quad t = \frac{\pi}{3}, \pi, \frac{5\pi}{3} \text{ when the particles are furthest apart.}$$

v The minimum distance between the particles is $\frac{3}{2}$ units.

vi $\frac{17}{4} - 2\cos 3t = \frac{9}{4},\ 0 \le t \le 2\pi$

$\therefore\quad 2\cos 3t = 2$

$\therefore\quad \cos t = 1$

$\therefore\quad 3t = 0, 2\pi, 4\pi, 6\pi$ since $0 \le 3t \le 6\pi$

$\therefore\quad t = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ when the particles are closest together.

c **i** $\boldsymbol{p}(t)\cdot\boldsymbol{q}(t) = ((\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k})\cdot\left((\cos 2t)\boldsymbol{i} - (\sin 2t)\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}\right)$

$= \cos t\cos 2t - \sin t\sin 2t - \frac{1}{2}$

$= \cos(3t) - \frac{1}{2}$, as required.

ii $\boldsymbol{p}(t)\cdot\boldsymbol{q}(t) = |\boldsymbol{p}(t)|\,|\boldsymbol{q}(t)|\cos POQ$

From **c i**, $\boldsymbol{p}(t)\cdot\boldsymbol{q}(t) = \cos(3t) - \frac{1}{2}$

From **a ii**, $|\boldsymbol{p}(t)| = \sqrt{2}$

$$|\boldsymbol{q}(t)| = \sqrt{(\cos 2t)^2 + (-\sin 2t)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\cos^2 2t + \sin^2 2t + \frac{1}{4}}$$

$$= \sqrt{\frac{5}{4}}$$

$$= \frac{\sqrt{5}}{2}$$

$$\therefore\quad \cos POQ = \frac{\boldsymbol{p}(t)\cdot\boldsymbol{q}(t)}{|\boldsymbol{p}(t)|\,|\boldsymbol{q}(t)|}$$

$$= \frac{\cos(3t) - \frac{1}{2}}{\sqrt{2}\times\frac{\sqrt{5}}{2}}$$

$$= \frac{2}{\sqrt{10}}\left(\cos(3t) - \frac{1}{2}\right)$$

$$= \frac{\sqrt{10}}{5}\left(\cos(3t) - \frac{1}{2}\right)$$

iii $\quad -1 \le \cos(3t) \le 1$

$$\therefore \quad \frac{-3}{2} \le \cos 3t - \frac{1}{2} \le \frac{1}{2}$$

$$\therefore \quad \frac{-3\sqrt{10}}{10} \le \frac{\sqrt{10}}{5}\left(\cos(3t) - \frac{1}{2}\right) \le \frac{\sqrt{10}}{10}$$

$$\therefore \quad \frac{-3\sqrt{10}}{10} \le \cos POQ \le \frac{\sqrt{10}}{10}$$

Angle POQ has greatest magnitude when $\cos POQ = \dfrac{-3\sqrt{10}}{10}$

$$\therefore \quad POQ = \cos^{-1}\left(\frac{-3\sqrt{10}}{10}\right)$$

$$= (161.565\,05\ldots)^\circ$$

The greatest angle of POQ is 162°, correct to the nearest degree.

10 a $\boldsymbol{v}_B(t) = \boldsymbol{r}'_B$

$$= -4\alpha\cos(\alpha t)\boldsymbol{i} - 4\alpha\sin(\alpha t)\boldsymbol{j},$$

so the speed of B in terms of α

$$= \sqrt{16\alpha^2\cos^2(\alpha t) + 16\alpha^2\sin^2(\alpha t)}$$

$$= 4\alpha$$

b Let $\boldsymbol{r}_A = x\boldsymbol{i} + y\boldsymbol{j}$

$\therefore \quad x = 2t \qquad$ and $y = t$

$\therefore \quad \dfrac{x}{2} = t \qquad \therefore \quad y = \dfrac{x}{2},\ x \ge 0$

Let $\boldsymbol{r}_B = x\boldsymbol{i} + y\boldsymbol{j}$

$\therefore \quad x = 4 - 4\sin(\alpha t) \qquad$ and $y = 4\cos(\alpha t)$

$\therefore \quad x - 4 = -4\sin(\alpha t) \qquad \therefore \quad \dfrac{y}{4} = \cos(\alpha t)$

$$\therefore \quad \frac{-(x-4)}{4} = \sin(\alpha t)$$

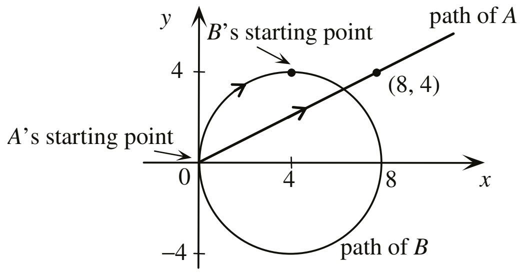
$\therefore \quad \dfrac{(x-4)^2}{16} = \sin^2(\alpha t) \qquad \therefore \quad \dfrac{y^2}{16} = \cos^2(\alpha t)$

$$\therefore \quad \frac{(x-4)^2}{16} + \frac{y^2}{16} = \sin^2(\alpha t) + \cos^2(\alpha t)$$

$$= 1$$

$$\therefore \quad (x-4)^2 + y^2 = 16,\ 0 \le x \le 8,\ -4 \le y \le 4$$

c



d To find the point of intersection of the paths of A and B, substitute $y = \frac{x}{2}$ into $(x-4)^2 + y^2 = 16$

$$\therefore \quad (x-4)^2 + \left(\frac{x}{2}\right)^2 = 16$$

$$\therefore \quad (x-4)^2 + \frac{x^2}{4} = 16$$

$$\therefore \quad 4(x^2 - 8x + 16) + x^2 = 64$$

$$\therefore \quad 5x^2 - 32x + 64 = 64$$

$$\therefore \quad x(5x - 32) = 0$$

$$\therefore \quad x = 0 \text{ or } \frac{32}{5}$$

When $x = 0, y = 0$.

When $x = \frac{32}{5}$, $y = \frac{16}{5}$.

The points of intersection are $(0, 0)$ and $\left(\frac{32}{5}, \frac{16}{5}\right)$.

e From graphs it is clear that the paths of A and B cross at $(0, 0)$ at different times.

At $\left(\frac{32}{5}, \frac{16}{5}\right)$, $t = \frac{x}{2} = \frac{16}{5}$ for A.

For B, when $t = \frac{16}{5}$,

$x = 4 - 4\sin(\alpha t)$ becomes $x = 4 - 4\sin\left(\frac{16\alpha}{5}\right)$

and $y = 4\cos(\alpha t)$ becomes $y = 4\cos\left(\frac{16\alpha}{5}\right)$

Now $x = \frac{32}{5}$ and $y = \frac{16}{5}$

$\therefore \quad \frac{32}{5} = 4 - 4\sin\left(\frac{16\alpha}{5}\right)$ and $\frac{16}{5} = 4\cos\left(\frac{16\alpha}{5}\right)$

$\therefore \quad 4\sin\left(\frac{16\alpha}{5}\right) = \frac{-12}{5}$ $\qquad \therefore \quad \cos\left(\frac{16\alpha}{5}\right) = \frac{4}{5}$

$\therefore \quad \sin\left(\frac{16\alpha}{5}\right) = \frac{-3}{5}$

so $\frac{16\alpha}{5}$ is in the fourth quadrant.

$\therefore \quad \frac{16\alpha}{5} = -\sin^{-1}\left(\frac{3}{5}\right) + 2k\pi, \ k \in Z$

The minimum value of $\alpha (\alpha > 0)$ is given when $k = 1$

$\therefore \quad \frac{16\alpha}{5} = -\sin^{-1}\left(\frac{3}{5}\right) + 2\pi$.

$\therefore \quad \alpha = \frac{-5}{16}\sin^{-1}\left(\frac{3}{5}\right) + \frac{5\pi}{8}$

$= 1.762\,40\ldots$

$= 1.76$, correct to two decimal places.

11 a i $\boldsymbol{a}(t) = -9.8\boldsymbol{j}$

ii $\boldsymbol{v}(t) = -9.8t\boldsymbol{j} + \boldsymbol{c}_1$, where $\boldsymbol{c}_1$ is a constant vector.

Now $\boldsymbol{v}(0) = 2\boldsymbol{i}$,

$\therefore \quad 2\boldsymbol{i} = \boldsymbol{c}_1$

$\therefore \quad \boldsymbol{v}(t) = 2\boldsymbol{i} - 9.8t\boldsymbol{j}$

iii $\boldsymbol{r}(t) = 2t\boldsymbol{i} - 4.9t^2\boldsymbol{j} + \boldsymbol{c}_2$, where $\boldsymbol{c}_2$ is a constant vector.

Now $\boldsymbol{r}(0) = \boldsymbol{0}$,

$\therefore \quad \boldsymbol{0} = \boldsymbol{c}_2$

$\therefore \quad \boldsymbol{r}(t) = 2t\boldsymbol{i} - 4.9t^2\boldsymbol{j}$, the position of the glass with respect to the edge of the table at time t seconds.

b i When the glass hits the floor,

$-4.9t^2 = -0.8$

$\therefore \quad t^2 = \frac{8}{49}$

$\therefore \quad t = \frac{2\sqrt{2}}{7}$

It takes $\frac{2\sqrt{2}}{7}$ seconds for the glass to hit the floor.

ii $\boldsymbol{r}\left(\frac{2\sqrt{2}}{7}\right) = 2 \times \frac{2\sqrt{2}}{7}\boldsymbol{i} - 0.8\boldsymbol{j}$

$= \frac{4\sqrt{2}}{7}\boldsymbol{i} - 0.8\boldsymbol{j}$

The glass hits the floor at a horizontal distance from the table of $\frac{4\sqrt{2}}{7}$ metres.

12 a i $\overrightarrow{OL} = 6\boldsymbol{i} - 3\boldsymbol{j}$

ii $\hat{\overrightarrow{OL}} = \dfrac{\overrightarrow{OL}}{|\overrightarrow{OL}|}$

$= \dfrac{1}{\sqrt{45}}(6\boldsymbol{i} - 3\boldsymbol{j})$

$= \dfrac{\sqrt{5}}{5}(2\boldsymbol{i} - \boldsymbol{j})$

b The vector resolute of $\overrightarrow{OY}$ in the direction of $\overrightarrow{OL}$ is given by

$$\frac{\overrightarrow{OY} \cdot \overrightarrow{OL}}{|\overrightarrow{OL}|}\hat{\overrightarrow{OL}} = \frac{(7\boldsymbol{i} + 4\boldsymbol{j}) \cdot (6\boldsymbol{i} - 3\boldsymbol{j})}{3\sqrt{5}} \times \frac{\sqrt{5}}{5}(2\boldsymbol{i} - \boldsymbol{j})$$

$$= \frac{42 - 12}{15}(2\boldsymbol{i} - \boldsymbol{j})$$

$$= 4\boldsymbol{i} - 2\boldsymbol{j}$$

The coordinates of the point on the shore closest to the yacht at noon are (4, −2).

c **i** $\overrightarrow{LP} = \overrightarrow{OP} - \overrightarrow{OL}$

$= \boldsymbol{r}(t) - \overrightarrow{OL}$

$= \left(7 - \dfrac{7}{2}t\right)\boldsymbol{i} + (4 - 2t)\boldsymbol{j} - (6\boldsymbol{i} - 3\boldsymbol{j})$

$= \left(1 - \dfrac{7}{2}t\right)\boldsymbol{i} + (7 - 2t)\boldsymbol{j}$

ii Find $|\overrightarrow{LP}|$

$|\overrightarrow{LP}| = \sqrt{\left(1 - \dfrac{7}{2}t\right)^2 + (7 - 2t)^2}$

$= \sqrt{1 - 7t + \dfrac{49}{4}t^2 + 49 - 28t + 4t^2}$

$= \dfrac{1}{2}\sqrt{65t^2 - 140t + 200}$

The yacht is closest to the navigation sign when $|\overrightarrow{LP}|$ is at a minimum, i.e., when $65t^2 - 140t + 200$ is a minimum.

Let $y = 65t^2 - 140t + 200$

$\therefore \quad \dfrac{dy}{dt} = 130t - 140$

When $\dfrac{dy}{dt} = 0$, $130t - 140 = 0$

$\therefore \quad t = \dfrac{14}{13}$, a minimum since $65t^2 - 140t + 200$ is a concave-up parabola.

The yacht is closest to the navigation sign after $\frac{14}{13}$ hours, i.e., at 1:05 p.m.

iii When $t = \frac{14}{13}$, $|\overrightarrow{LP}| = \frac{1}{2}\sqrt{65 \times \left(\frac{14}{13}\right)^2 - 140 \times \frac{14}{13} + 200}$

$$= \frac{1}{2}\sqrt{\frac{1620}{13}}$$

$$= \frac{9\sqrt{65}}{13}$$

The closest distance between the sign and the yacht is $\frac{9\sqrt{65}}{13}$ kilometres.

Chapter 14 – Revision of Chapters 8 to 13

Solutions to Exercise 14A

1 **a** Let $y = \dfrac{1}{\arcsin x}$.

Let $u = \arcsin x$

Then, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$$\frac{dy}{dx} = -\frac{1}{(\arcsin x)^2\sqrt{1-x^2}}$$

b Let $y = \dfrac{1}{\arctan x}$.

Let $u = \arctan x$

Then, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$$\frac{dy}{dx} = -\frac{1}{(\arctan x)^2(1+x^2)}$$

c Let $y = \dfrac{1}{(\arcsin x)^2}$.

Let $u = \arcsin x$

Then, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$$\frac{dy}{dx} = -\frac{2}{(\arcsin x)^3\sqrt{1-x^2}}$$

2 **a** $\dfrac{dQ}{dt} = -\dfrac{Q}{20+2t}\times 2$

$= -\dfrac{Q}{10+t}$

b $\dfrac{dQ}{dt} = -\dfrac{Q}{10+t}$

$$\int \frac{1}{Q}\,dQ = -\int \frac{1}{1+t}\,dt$$

$\log_e Q = -\log_e(10+t) + c$

When $t = 0, Q = 1 \Rightarrow c = \log_e 10$

$\therefore \log_e Q = -\log_e(10+t) + \log_e 10$

$\log_e Q = \log_e \dfrac{10}{10+t}$

$Q = \dfrac{10}{10+t}$

3 $\dfrac{dy}{dx} = -\dfrac{x}{4+x^2}$

Let $u = 4 + x^2$. Then, $\dfrac{du}{dx} = 2x$

$y = \displaystyle\int -\frac{x}{4+x^2}\,dx$

$= -\dfrac{1}{2}\displaystyle\int \frac{1}{u}\,du$

$= -\dfrac{1}{2}\log_e u + c$

$= -\dfrac{1}{2}\log_e(x^2+4) + c$

When $x = 1, y = 2$

$2 = -\dfrac{1}{2}\log_e(5) + c$

$c = 2 + \dfrac{1}{2}\log_e 5$

Therefore,

$y = -\dfrac{1}{2}\log_e(x^2+4) + 2 + \log_e 5$

$y = \dfrac{1}{2}\log_e \dfrac{5}{x^2+4} + 4$

4 a $\int_{-2}^{2} 3\arccos\left(\frac{x}{2}\right) dx$

$= \int_{0}^{3\pi} 2\cos\left(\frac{x}{3}\right) + 2\ dx$

$= \left[-6\sin\left(\frac{x}{3}\right) + 2x\right]_0^{3\pi}$

$= (-6\sin\pi + 6\pi) - (-6\sin 0 + 0)$

$= 6\pi$

b $V = \pi\int_0^{3\pi} x^2\ dy$

$= \pi\int_0^{3\pi} 4\cos^2\left(\frac{y}{3}\right) dy$

$= \pi\int_0^{3\pi} 2 + 2\cos\left(\frac{2y}{3}\right) dy$

$= \pi\left[2y + 3\sin\left(\frac{2y}{3}\right)\right]_0^{3\pi}$

$= 6\pi^2$

5 a

$$5x^2 + 2xy + y^2 = 13$$

$$\therefore 10x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx}(2x + 2y) = -10x - 2y$$

$$\therefore \frac{dy}{dx} = \frac{-10x - 2y}{2x + 2y}$$

$$= \frac{-5x - y}{x + y}$$

When $x = 1$

$$5 + 2y + y^2 = 13$$

$$\therefore y^2 + 2y - 8 = 0$$

$$\therefore y = 2 \text{ or } y = -4$$

Therefore, gradient of tangents at $x = 1$ are $-\frac{7}{3}$ and $\frac{1}{3}$

b When $x = 1, y = 2$ or $y = -4$

Point is in the first quadrant.

Therefore, consider the point $(1, 2)$.

Gradient of tangent

$= -\frac{-5 \times 1 - 2}{1 + 2} = -\frac{7}{3}$

Gradient of the normal is $\frac{3}{7}$

Therefore equation is:

$$y - 2 = \frac{3}{7}(x - 1)$$

$$y = \frac{3}{7}x + \frac{11}{7}$$

or $3x - 7y = -11$

6 $\dfrac{dx}{dt} = x^2 \sin(2t)$

$$\int \frac{1}{x^2}\,dx = \int \sin(2t)\,dt$$

$$-\frac{1}{x} = -\frac{1}{2}\cos(2t) + c$$

When $t = 0$, $x = \dfrac{1}{2}$

$$-2 = -\frac{1}{2} + c$$

$$c = -\frac{3}{2}$$

$$-\frac{1}{x} = -\frac{1}{2}\cos(2t) - \frac{3}{2}$$

$$\frac{1}{x} = \frac{1}{2}(\cos(2t) + 3$$

$$x = \frac{2}{\cos(2t) + 3}$$

7 $y = \dfrac{4 - x^3}{3x^2}$

$$= -\frac{1}{3}x + \frac{4}{3x^2}$$

Asymptotes $y = -\dfrac{x}{3}$, $x = 0$;

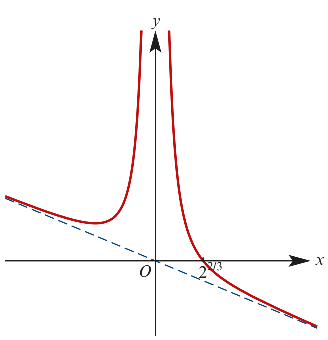
Axis intercept $(\sqrt[3]{4}, 0)$;

$$\frac{dy}{dx} = \frac{-(x^3 + 8)}{3x^3}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = -2$$

Stationary point $(-2, 1)$

$\dfrac{d^2y}{dx^2} = \dfrac{8}{x^4} > 0$ for all x. In particular it indicates a local minimum at $(-2, 1)$



8 **a** $y = \dfrac{1 + x^2}{4 - x^2}$

$$= -1 + \frac{5}{4 - x^2}$$

$$= -1 + \frac{5}{4(x+2)} - \frac{5}{4(2-x)}$$

b

$$\text{Area} = \int_{-1}^{1} \frac{1 + x^2}{4 - x^2}\,dx$$

$$= 2\int_{0}^{1} \frac{1 + x^2}{4 - x^2}\,dx$$

$$= 2\int_{0}^{1} -1 + \frac{5}{4(x+2)} - \frac{5}{4(2-x)}\,dx$$

$$= 2\Big[-x + \frac{5}{4}\log_e\Big|\frac{x+2}{x-2}\Big|\Big]_0^1$$

$$= \Big[-1 + \frac{5}{4}\log_e 3\Big]$$

$$= \frac{5}{2}\log_e(3) - 1$$

$$= \frac{5\log_e(3) - 4}{2}$$

9 **a** $x = 2\cos^2(\theta), y = \sin(2\theta)$

$\frac{dx}{d\theta} = -4\cos\theta\sin\theta, \frac{dy}{d\theta} = 2\cos 2\theta,$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{(-4\cos\theta\sin\theta)^2 + (2\cos 2\theta)^2}\ d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \sqrt{\sin^2(2\theta) + \cos^2 2\theta}\ d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} 1\ d\theta$$

$$= \frac{\pi}{2}$$

b $x = 2\cos^2(\theta), y = \sin(2\theta)$

$y^2 = 4\sin^2\theta\cos^2\theta$

$\cos^2\theta = \frac{x}{2}$ and $\sin^2\theta = 1 - \frac{x}{2}$

Therefore

$y^2 = 4 \times \frac{x}{2}(1 - \frac{x}{2})$

$y^2 = x(2 - x)$

$y = \sqrt{x(2 - x)}$ since $y \geq 0$

Domain= $[1, 2]$ and range= $[0, 1]$

c $x = 2\cos^2\theta$ and $y = \sin 2\theta$

$\frac{dx}{d\theta} = -4\sin\theta\cos\theta$ and $\frac{dy}{d\theta} = 2\cos 2\theta$

$$A = 2\pi\int_0^{\frac{\pi}{4}} \sin 2\theta\sqrt{(-2\sin 2\theta)^2 + (2\cos 2\theta)^2}\ d\theta$$

$$= 4\pi\int_0^{\frac{\pi}{4}} \sin 2\theta\ d\theta$$

$$= 4\pi\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\pi$$

10 $\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx$

a Let $u = x$ and $\frac{dv}{dx} = \sec^2 2x$

$\frac{du}{dx} = 1$ and $v = \frac{1}{2}\tan 2x$

$$\int x\sec^2 2x\ dx = \frac{x}{2}\tan 2x - \int\frac{1}{2}\tan 2x\ dx$$

$$= \frac{x}{2}\tan 2x - \frac{1}{4}\log_e|\sec 2x| + c$$

$$= \frac{1}{2}x\tan 2x + \frac{1}{4}\log_e|\cos 2x| + c$$

b Let $u = \log_e(x+5)$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x+5}$ and $v = x + 5$

$$\int x \log_e(x+5)\, dx$$

$$= (x+5)\log_e(x+5) - \int \frac{x+5}{x+5}; dx$$

$$= (x+5)\log_e(x+5) - x + c$$

c Let $u = e^{2x}$ and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2e^{2x}$ and $v = -\cos x$

$$\int e^{2x} \sin x\, dx$$

$$= -e^{2x}\cos x + \int 2e^{2x}\cos x; dx \ldots (1)$$

Now integrate $2e^{2x}\cos x$ by parts

$$\int 2e^{2x}\cos x; dx$$

$$= 2e^{2x}\sin x - \int 4e^{2x}\sin x\, dx$$

Substitute in (1)

$$\int e^{2x}\sin x\, dx$$

$$= -e^{2x}\cos x + 2e^{2x}\sin x - \int 4e^{2x}\sin x\, dx$$

$$\therefore \int 5e^{2x}\sin x\, dx$$

$$= 2e^{2x}\sin x - e^{2x}\cos x$$

$$\therefore \int e^{2x}\sin x\, dx$$

$$= \frac{1}{5}e^{2x}(\sin x - \cos x)$$

11 $\frac{dy}{dx} = e^{2y}\sin 2x$ and $y(0) = 0$

$$\therefore e^{-2y}\frac{dy}{dx} = \sin(2x)$$

$$\therefore \int e^{-2y}\, dy = \int \sin 2x\, dx$$

$$\therefore -\frac{1}{2}e^{-2y} = -\frac{1}{2}\cos 2x + c$$

When $x = 0$, $y = 0$ and therefore $c = 0$

$$\therefore -\frac{1}{2}e^{-2y} = -\frac{1}{2}\cos 2x$$

$$\therefore e^{-2y} = \cos 2x$$

$$\therefore y = -\frac{1}{2}\log_e(\cos(2x))$$

12 $(1+x^2)\frac{dy}{dx} = 2xy$ and $y(0) = 2$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{2x}{1+x^2}$$

$$\therefore \int \frac{1}{y}\, dy = \int \frac{2x}{1+x^2}\, dx$$

$$\therefore \log_e|y| = \log_e(1+x^2) + c$$

Also $y(0) = 2$ and $\therefore c = \log_e 2$

$$\therefore \log_e|y| = \log_e(x^2+1) + \log_e 2$$

But $y > 0$

$$\therefore \log_e y = \log_e(2(1+x^2))$$

$$\therefore y = 2(1+x^2)$$

13 $f(x) = \arcsin(4x^2 - 3)$

For maximal domain

$-1 \le 4x^2 - 3 \le 1$

$2 \le 4x^2 \le 4$

$\frac{1}{2} \le x^2 \le 1$

Maximal domain =

$$\left[-1, -\frac{\sqrt{2}}{2}\right] \cup \left[\frac{\sqrt{2}}{2}, 1\right]$$

14

$$y = \frac{4x^2 + 5}{x^2 + 1}$$

$$\therefore y = 4 + \frac{1}{1 + x^2}$$

$\therefore$ asymptote $y = 4$

$$\text{Now}, \frac{dy}{dx} = -\frac{2x}{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = 0$$

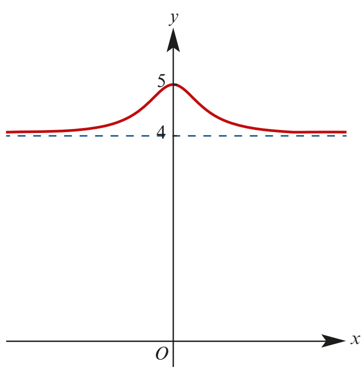
y-intercept :$(0, 5)$

The second derivative

$$\frac{d^2y}{dx^2} = \frac{2(3x^2 - 1)}{(x^2 + 1)^3}$$

When $x = 0 \frac{d^2y}{dx^2} < 0$

Therefore local maximum at $(0, 5)$.



15

$$x = 2 \sin t + 1$$

$$\therefore \frac{dx}{dt} = 2 \cos t$$

$$y = 2 \cos t - 3$$

$$\therefore \frac{dy}{dt} = -2 \sin t.$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$= \frac{-2 \sin t}{2 \cos t}$$

$$= -\tan t.$$

When $t = \frac{\pi}{4}, \frac{dy}{dx} = -1$

16 $x = e^t - t$ and $y = 4e^{\frac{t}{2}}$

$\frac{dx}{dt} = e^t - 1$ and $\frac{dy}{dt} = 2e^{\frac{t}{2}}$

$$L = \int_0^1 \sqrt{(e^t - 1)^2 + \left(2e^{\frac{t}{2}}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} + 2e^t + 1}\, dt$$

$$= \int_0^1 \sqrt{(e^t + 1)^2}\, dt$$

$$= \left[e^t + t\right]_0^1$$

$$= e + 1 - e^0$$

$$= e$$

17 **a** $x = at^2, y = 2at$

Hence $t = \frac{y}{2a}$

and substituting gives,

$x = a\left(\frac{y^2}{4a^2}\right)$ and rearranging we have

$y^2 = 4ax.$

When $t = \sqrt{3}, x = 3a$ and when

$t = 0, x = 0$

$$V = \pi \int_0^{3a} y^2 dx$$

$$= \pi \int_0^{3a} 4ax\, dx$$

$$= \pi \left[2ax^2\right]_0^{3a}$$

$$= 18a^2\pi$$

b $x = at^2, y = 2at$

$\dfrac{dx}{dt} = 2at$ and $\dfrac{dy}{dt} = 2a$

$$A = 2\pi \int_0^{\sqrt{3}} 2at\sqrt{4a^2t^2 + 4a^2}\, dt$$

$$= 8a^2\pi \int_0^{\sqrt{3}} t\sqrt{t^2 + 1}\, dt$$

Let $u = t^2 + 1 \Rightarrow \dfrac{du}{dx} = 2t$

$$A = 4a^2\pi \int_1^4 u^{\frac{1}{2}}\, du$$

$$= \frac{8a^2\pi}{3}\left[u^{\frac{3}{2}}\right]_1^4$$

$$= \frac{56a^2\pi}{3}$$

18 a $$\int_0^1 e^{2x}\cos(e^2x)\, dx$$

Let $u = e^{2x}$ then $\dfrac{du}{dx} = 2e^{2x}$

$$\therefore \int_0^1 e^{2x}\cos(e^2x)\, dx$$

$$= \frac{1}{2}\int_0^1 \cos u\frac{du}{dx}\, dx$$

$$= \frac{1}{2}\int_1^{e^2} \cos u\, du$$

$$= \frac{1}{2}\left[\sin u\right]_1^{e^2}$$

$$= \frac{1}{2}(\sin e^2 - \sin 1)$$

b $$\int_1^2 (x-1)\sqrt{2-x}\, dx$$

Let $u = 2 - x$ then $\dfrac{du}{dx} = -1$

$x = 2 - u$

$$\therefore \int_1^2 (x-1)\sqrt{2-x}\, dx$$

$$= \frac{1}{2}\int_1^2 -(1-u)\sqrt{u}\frac{du}{dx}\, dx$$

$$= \int_1^0 (u-1)\sqrt{u}\, du$$

$$= \int_1^0 (u^{3/2} - u^{1/2})\, du$$

$$= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^0$$

$$= \frac{4}{15}$$

c $$\int_0^1 \frac{x-2}{x^2 - 7x + 12}\, dx$$

$$= \int_0^1 \frac{x-2}{(x-3)(x-4)}\, dx$$

$$= \int_0^1 \frac{2}{x-4} - \frac{1}{x-3}\, dx$$

$$= \left[2\log_e|x-4| - \log_e|x-3|\right]_0^1$$

$$= \log_e\left(\frac{9}{2}\right) - \log_e\left(\frac{16}{3}\right)$$

$$= \log_e\left(\frac{27}{32}\right)$$

d $\int_3^5 \frac{6}{x^2 - 6x + 4}\, dx$

$= \int_3^5 \frac{6}{(x-3)^2 - 5}\, dx$

Let $u = x - 3$

$= \int_0^2 \frac{6}{u^2 - 5}\, du$

$= -\frac{3\sqrt{5}}{5} \int_0^2 \frac{1}{u + \sqrt{5}} - \frac{1}{u + \sqrt{5}}\, du$

$= -\frac{3\sqrt{5}}{5} \left[\log_e \left|\frac{u + \sqrt{5}}{u - \sqrt{5}}\right|\right]_0^2$

$= -\frac{3\sqrt{5}}{5} \left(\log_e \left|\frac{2 + \sqrt{5}}{2 - \sqrt{5}}\right|\right)$

$= -\frac{3\sqrt{5}}{5} \left(\log_e \left|-4\sqrt{5} - 9\right|\right)$

$= -\frac{3\sqrt{5}}{5} \left(\log_e(4\sqrt{5} + 9)\right)$

This is equivalent to the answer in the back of the book.
That is obtained by noticing that $\sqrt{4\sqrt{5} + 9} = 2 + \sqrt{5}$

e $\int_2^7 \frac{2 + x}{\sqrt{2 + x}}\, dx = \int_2^7 \sqrt{2 + x}\, dx$

Let $u = 2 + x \Rightarrow \frac{du}{dx} = 1$

$\int_2^7 \sqrt{2 + x}\, dx = \int_4^9 u^{\frac{1}{2}}\, du$

$= \frac{2}{3}\left[u^{\frac{3}{2}}\right]_4^9$

$= \frac{2}{3} \times (27 - 80)$

$= \frac{38}{3}$

f $\int_0^{\frac{\pi}{4}} \sec^3 x \tan x\, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^4 x}\, dx$

Let $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$\int_0^{\frac{\pi}{4}} \sec^3 x \tan x\, dx = -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u^4}\, du$

$= \frac{1}{3}\left[u^{-3}\right]_1^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{3}(2\sqrt{2} - 1)$

19 $\dfrac{dy}{dx} = -2x^2$ and $y(1) = 2$

$x_0 = 1, y_0 = 2, h = 0.1$

First step:

$y_1 = y_0 + hf(x_0)$

$\therefore y_1 = 2 + 0.1 \times (-2)$

$\therefore y_1 = 1.8$

Also, $x_1 = x_0 + h$

$\therefore x_1 = 1.1$

Second step

$y_2 = y_1 + hf(x_1)$

$\therefore y_2 = 1.8 + 0.1 \times (-2) \times 1.1^2$

$\therefore y_2 = 1.558$

Also, $x_2 = 1.2$

Third step

$y_3 = y_2 + hf(x_2)$

$\therefore y_3 = 1.558 + 0.1 \times (-2) \times 1.2^2$

$\therefore y_3 = 1.270$

20 $V = \pi \displaystyle\int_{y_1}^{y_2 i} x^2 \, dy$

$= \pi \displaystyle\int_0^a 16a^4 - 16^3 y \, dy$

$= \pi \Big[16a^4 y - 8a^3 y^2\Big]_0^a$

$= \pi[16a^5 - 8a^5]$

$= 8a^5\pi$

21 $a = -(1 + v^2)$

$\therefore v\dfrac{dv}{dx} = -(1 + v^2)$

$\therefore \dfrac{dv}{dx} = -\dfrac{1 + v^2}{v}$

$\therefore \dfrac{dx}{dv} = -\dfrac{v}{1 + v^2}$

$\therefore x = \displaystyle\int -\frac{v}{1 + v^2} \, dv$

$\therefore x = -\dfrac{1}{2} \displaystyle\int \frac{2v}{1 + v^2} \, dv$

$\therefore x = -\dfrac{1}{2} \log_e(1 + v^2)$

When $x = 0, v = u$

$\therefore c = \dfrac{1}{2} \log_e(1 + u^2)$

$\therefore x = \dfrac{1}{2} \log_e\left(\dfrac{1 + u^2}{1 + v^2}\right)$

When at rest $v = 0$

$\therefore x = \dfrac{1}{2} \log_e(1 + u^2)$

22 $a = g - 0.4v$

$\dfrac{dv}{dt} = g - 0.4v$

$\therefore \dfrac{dt}{dv} = \dfrac{1}{g - 0.4v}$

$\therefore t = -\dfrac{5}{2} \displaystyle\int \frac{-0.4}{g - 0.4v} \, dv$

$\therefore t = -\dfrac{5}{2} \log_e |g - 0.4v| + c$

Since $g - 0.4v > 0$

$$t = -\frac{5}{2}\log_e(g - 0.4v) + c$$

When $t = 0, v = 0$

$$\therefore c = \frac{5}{2}log_e g$$

$$\therefore t = \frac{5}{2}log_e g - \frac{5}{2}\log_e(g - 0.4v)$$

$$= \frac{5}{2}\log_e\left(\frac{g}{g - 0.4v}\right)$$

$$\therefore e^{\frac{2t}{5}} = \frac{g}{g - 0.4v}$$

$$\therefore ge^{-\frac{2t}{5}} = g - 0.4v$$

$$\therefore v = \frac{5g}{2}(1 - e^{\frac{2t}{5}})$$

$$\therefore A = \frac{5g}{2} \text{ and } B = \frac{2}{5}$$

23

$$a = -(1 + \frac{v}{100})$$

$$\therefore v\frac{dv}{dx} = -(1 + \frac{v}{100})$$

$$\therefore v\frac{dv}{dx} = -(1 + \frac{v}{100})$$

$$\therefore \frac{dv}{dx} = -\frac{100 + v}{100v}$$

$$\therefore \frac{dx}{dv} = -\frac{100v}{100 + v}$$

$$\therefore x = \int -100 + \frac{10\,000}{v + 100}\,dv$$

$$\therefore x = -100v + 10\,000\log_e(v + 100) + c$$

When $x = 0, v = 20$

$$\therefore c = 2000 - 10\,000\log_e 120$$

$$\therefore x = 2000 - 100v + 10\,000\log_e\left(\frac{v + 100}{120}\right)$$

At rest $v = 0$

$$x = 2000 + 10\,000\log_e\left(\frac{5}{6}\right)$$

Therefore,

$A = 10\,000, B = \frac{5}{6}, C = 2000$

24 a Use the quotient rule

$$f(x) = \frac{-2(x^2 - 1)}{(x^2 + 1)^2}$$

b

$$\frac{d^2y}{dx^2} = \frac{4x(x^2 - 3)}{(x^2 + 1)^3}$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0 \text{ or } x \pm \sqrt{3}$$

The sign of $\frac{d^2y}{dx^2}$ has to be checked to the left and right of each of these points.

Points of inflection are

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right), (0, 0) \text{ and } \left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$$

25 a $\boldsymbol{v}(t) = \cos t\,\boldsymbol{i} + \cos(2t)\boldsymbol{j}$

b $\boldsymbol{a}(t) = -\sin t\,\boldsymbol{i} - 2\sin(2t)\boldsymbol{j}$

c

$$|\boldsymbol{r}(t)| = \sqrt{\sin^2 t + \frac{1}{4}\sin^2 2t}$$

$$= \sqrt{\sin^2 t + \sin^2 t\cos^2 t}$$

$$= |\sin t|\sqrt{1 + \cos^2 t}$$

$$= |\sin t|\sqrt{2 - \sin^2 t}$$

d

$$|\boldsymbol{r}(t)| = \sqrt{\cos^2 t + \cos^2 2t}$$

$$= \sqrt{\cos^2 t + 1 - \sin^2 2t}$$

$$= \sqrt{\cos^2 t + 1 - 4\sin^2 t\cos^2 t}$$

$$= \sqrt{\cos^2 t + 1 - 4\sin^2 t(1 - \sin^2 t)}$$

$$= \sqrt{2 - 5\sin^2 t + 4\sin^4 t}$$

e $x = \sin t$ and $y = \sin t\cos t$

$x^2 = \sin^2 t$

and

$y^2 = \sin^2 t\cos^2 t = \sin^2 t(1 - \sin^2 t)$

$\therefore y^2 = x^2(1 - x^2)$

26 a $\boldsymbol{r}(t) = 2\sec t\boldsymbol{i} + \frac{1}{2}\tan t\boldsymbol{j}$

$\therefore x = 2\sec t$ and $y = \frac{1}{2}\tan t$

$\therefore \frac{x}{2} = \sec t$ and $2y = \tan t$

$\therefore 1 + 4y^2 = \frac{x^2}{4}$

$\frac{x^2}{4} - 4y^2 = 1 \quad x \geq 2, y \geq 0$

b $\boldsymbol{v}(t) = 2\tan t\ \sec t\,\boldsymbol{i} + 0.5\sec^2 t\,\boldsymbol{j}$

c $\boldsymbol{v}\left(\frac{\pi}{3}\right) = 2\tan\left(\frac{\pi}{3}\right)\boldsymbol{i} + \frac{1}{2}\sec^2\left(\frac{\pi}{3}\right)\boldsymbol{j}$

$= 4\sqrt{3}\boldsymbol{i} + \frac{4}{2}\boldsymbol{j}$

$\therefore |\boldsymbol{v}\left(\frac{\pi}{3}\right)|^2 = 52$

$\therefore |\boldsymbol{v}\left(\frac{\pi}{3}\right)| = 2\sqrt{13}$

27

$\boldsymbol{v}(t) = e^{2t}\boldsymbol{i} - e^{-2t}\boldsymbol{k}$

$\boldsymbol{r}(t) = \frac{1}{2}e^{2t}\boldsymbol{i} + \frac{1}{2}e^{-2t}\boldsymbol{k} + \boldsymbol{c}$

$\boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

$\therefore \boldsymbol{r}(t) = \frac{1}{2}(e^{2t} + 1)\boldsymbol{i} + \boldsymbol{j} + \frac{1}{2}(e^{-2t} - 5)\boldsymbol{k}$

$\therefore \boldsymbol{r}(\log_e 2) = \frac{5}{2}\boldsymbol{i} + \boldsymbol{j} - \frac{19}{8}\boldsymbol{k}$

28 a $\boldsymbol{a}(t) = -\boldsymbol{j}$ t

$\boldsymbol{v}(t) = -gt\boldsymbol{j} + \boldsymbol{c}$

$\boldsymbol{v}(0) = 10\boldsymbol{i} + 10\sqrt{3}\boldsymbol{j}$

$\therefore \boldsymbol{v}(t) = 10\boldsymbol{i} + (10\sqrt{3} - gt)\boldsymbol{j}$

b $\boldsymbol{r}(t) = 10t\boldsymbol{i} + (10\sqrt{3}t - \frac{1}{2}gt^2)\boldsymbol{j}$

$\therefore x = 10t$ and $y = 10\sqrt{3}t - \frac{1}{2}gt^2$

$\therefore y = \sqrt{3}x - \frac{g}{200}x^2$

29 a $\boldsymbol{v}(t) = -\sin(2t)\boldsymbol{i} + 2\cos(2t)\boldsymbol{j}$

$\boldsymbol{r}(t) = \cos(2t)\boldsymbol{i} + \sin(2t)\boldsymbol{j} + \boldsymbol{c}$

It is given that $\boldsymbol{r}(0) = 2\boldsymbol{i} - \boldsymbol{j}$

$\therefore \boldsymbol{r}(t) = (\cos(2t) + 1)\boldsymbol{i} + (\sin(2t) - 1)\boldsymbol{j}$

b $\boldsymbol{r}(t) = (\cos(2t) + 1)\boldsymbol{i} + (\sin(2t) - 1)\boldsymbol{j}$

$x - 1 = \cos 2t$ and $y + 1 = \sin 2t$

$\therefore (x - 1)^2 + (y + 1)^2 = 1$

c For this $2\cos 2t = 0$ and $-\sin 2t < 0$

$\cos 2t = 0 \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}$ and $\frac{5\pi}{4}$

But only two of these satisfy

$-\sin 2t < 0 \therefore t = \frac{\pi}{4}, \frac{5\pi}{4}$

30 $\boldsymbol{r}(t) = 14\sqrt{3}t\boldsymbol{i} + (14t - \frac{gt^2}{2})\boldsymbol{j}$

a $t14t - \frac{gt^2}{2} = 0$

$t(14 - \frac{gt}{2} = 0$

$t = 0$ or $t = \frac{28}{g}$

The particle reaches the ground in $\frac{28}{g}$ seconds

b $x = 14\sqrt{3}t$

$y = 14t - \frac{gt^2}{2}$

$\therefore y = \frac{\sqrt{3}}{3}x - \frac{g}{1176}x^2$

c Reaches maximum height when $t = \frac{14}{g}$

$$\text{Maximum height} = 14 \times \frac{14}{g} - \frac{g}{2} \times \left(\frac{14}{g}\right)^2$$

$$= \frac{196}{g} - \frac{196}{2g}$$

$$= \frac{98}{g}$$

Maximum height $= \frac{98}{g} = 10$ metres

31 a

$$\frac{dy}{dx} = y(1 + y)(1 - x)$$

$$\int \frac{1}{y(1 + y)}\, dy = \int 1 - x\, dx$$

$$\int \frac{1}{y} - \frac{1}{y + 1}\, dy = \int 1 - x\, dx$$

$$\log_e y - \log_e(y + 1) = x - \frac{1}{2}x^2 + c$$

$$\log_e\left(\frac{y}{y + 1}\right) = x - \frac{1}{2}x^2 + c$$

$y = 1$ when $x = 1$

$$\therefore c = \log_e\left(\frac{1}{2}\right) - 1 + \frac{1}{2} = \log_e\left(\frac{1}{2}\right) - \frac{1}{2}$$

$$\therefore \log_e\left(\frac{y}{y + 1}\right) = x - \frac{1}{2}x^2 + \log_e\left(\frac{1}{2}\right) - \frac{1}{2}$$

$$\log_e\left(\frac{2y}{y + 1}\right) = x - \frac{1}{2}x^2 - \frac{1}{2}$$

$$\frac{2y}{y + 1} = e^{x - \frac{1}{2}x^2 - \frac{1}{2}}$$

$$\frac{2y}{y + 1} = e^{\frac{1}{2}(2x - x^2 - 1}$$

$$\frac{2y}{y + 1} = e^{-\frac{1}{2}(x - 1)^2}$$

$$2y = e^{-\frac{1}{2}(x - 1)^2}(y + 1)$$

$$y(2 - e^{-\frac{1}{2}(x - 1)^2}) = e^{-\frac{1}{2}(x - 1)^2}$$

$$y = \frac{e^{-\frac{1}{2}(x - 1)^2}}{2 - e^{-\frac{1}{2}(x - 1)^2}} \text{ or } y = 0 \text{ or } y = -1$$

b

$$\frac{dy}{dx} = 0$$

$$y(1 + y)(1 - x) = 0$$

$$y = 0, -1 \text{or } x = 1$$

Only $x = 1$ is possible.

When $x = 1, y = 1$

32 $x = t$ and $y = \frac{1}{2}(e^x + e^{-x})$

$$\frac{dx}{dt} = 1 \text{ and } \frac{dy}{dt} = \frac{1}{2}(e^x - e^{-x})$$

$$L = \int_{-1}^{1} \sqrt{1 + \frac{1}{4}(e^t - e^{-t})^2}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} \sqrt{4 + (e^{2t} - 2 + e^{-2t})}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} \sqrt{e^{2t} + 2 + e^{-2t})}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} \sqrt{(e^t + e^{-t})^2}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^t + e^{-t}\, dt$$

$$= \frac{1}{2}(2e - 2e^{-1})$$

$$= e - e^{-1}$$

34 **a** Consider $\int_0^a f(a - x)\, dx$

Let $u = a - x$.

Then, $\frac{du}{dx} = -1$

$$\int_0^a f(a - x)\, dx = -\int_a^0 f(u)\, du$$

Therefore,

$$\int_0^a f(a - x)\, dx = \int_0^a f(x)\, dx$$

b

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

Hence by adding the two forms

$$2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \frac{\pi}{4}$$

Solutions to multiple-choice questions

1 D $\dfrac{x^2 + x + 2}{x} = x + 1 + \dfrac{2}{x}$

Horizontal asymptote at $x = 0$.

Non-vertical asymptote at $y = x + 1$.

2 D $\dfrac{d^2y}{dx^2} = 2\cos x + 1$

$\therefore\ \dfrac{dy}{dx} = 2\sin x + x + c$

$\therefore\ y = -2\cos x + \dfrac{x^2}{2} + cx + d$

where $c,\ d \in R$

Putting $c = 1$ and $d = 0$

$y = -2\cos x + \dfrac{x^2}{2} + x$

3 B $y = 2x^3$

Gradient of tangent at any point:

$\dfrac{dy}{dx} = 6x^2$

Gradient perpendicular to the tangent (i.e. the gradient of the normal):

$\dfrac{dy}{dx} = -\dfrac{1}{6x^2}$

4 C Since car Q accelerates at the same rate as car P their gradients are the same i.e. they are parallel to each other. Thus responses A, B and E are incorrect.

Since car Q accelerates to a speed of 15 m/s this implies response D is incorrect.

Therefore, response C is correct.

5 A Passes through the point (1, 1) and the gradient at any point is twice the reciprocal of the x-coordinate.

$\therefore\ \dfrac{dy}{dx} = \dfrac{2}{x},\ x(1) = 1$

$\therefore\ x\dfrac{dy}{dx} = 2,\ x(1) = 1$

6 C $\dfrac{dV}{dt} = 2 - 2 = 0$

$\dfrac{dQ_{in}}{dt} = 3 \times 2 = 6$ g/min

$\dfrac{dQ_{out}}{dt} = \dfrac{Q}{20} \times 2 = \dfrac{Q}{10}$

$\therefore\ \dfrac{dQ}{dt} = 6 - \dfrac{Q}{10}$

7 C $\dfrac{dy}{dx} = 3x^2 + 1$ with $y(1) = 3$

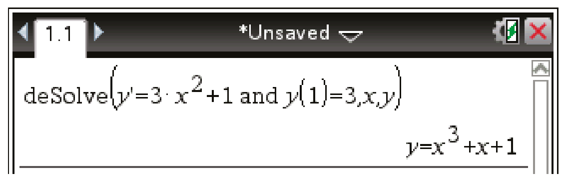
$\therefore\ y = x^3 + x + c$

When $x = 1,\ y = 3$:

$\Rightarrow c = 1$

$\therefore\ y = x^3 + x + 1$

Using CAS



8 A For option A $g''(x) = 15(x - 2)^2$

There will be no change in sign.

So the graph of $y = g(x)$ has no inflection points.

9 E $\dfrac{d^2y}{dx^2} = e^{3x}$

$\dfrac{dy}{dx} = \dfrac{1}{3}e^{3x} + c$

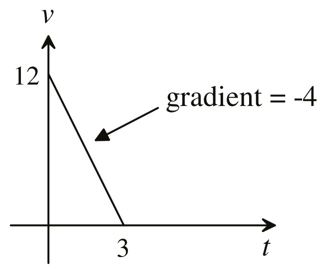
$y = \dfrac{1}{9}e^{3x} + cx + d$

where $c,\ d \in R$

Putting $c = 1$ and $d = 0$

$\therefore\ y = \dfrac{1}{9}e^{3x} + x$

10 B



The body takes 3 seconds to come to rest.

distance $= \frac{1}{2} \times 3 \times 12 = 18$

$\therefore\ t = 3$ and $s = 18$

11 A $\frac{dx}{dt} = -\frac{x}{50 + 2t} \times 8$

$\frac{dx}{dt} = -\frac{4x}{25 + t}$

Using separation of variables

$4 \int \frac{1}{4x}\, dx = -\int \frac{1}{25 + t}\, dt$

12 B $x = 2\sin^2 y = 2(\sin y)^2$

Using the chain rule

$\frac{dx}{dy} = 4\sin y \times \cos y = 2\sin 2x$

$\therefore\ \frac{dy}{dx} = \frac{1}{2\sin 2x} = \frac{1}{2}\text{cosec } 2x$

13 E $\frac{dx}{dt} = -kx$

$\therefore\ \frac{dt}{dx} = -\frac{1}{k\kappa}$

$\therefore\ t = -k\log_e(x) + c \quad$ for $\quad x > 0$

When $t = 0,\ x = 20$:

$\Rightarrow c = k\log_e 20$

$\therefore\ t = k\log_e\left(\frac{20}{x}\right)$

When $t = 20,\ x = 5$:

$\Rightarrow k = \frac{20}{\log_e 4}$

$\therefore\ t = \frac{20}{\log_e 4}\log_e\left(\frac{20}{x}\right)$

When $x = 2$,

$$t = \frac{20}{\log_e 4}\log_e 10 \approx 33.22$$

14 A $\int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x\, dx$

Let $u = \tan x \qquad \therefore\ \frac{du}{dx} = \sec^2 x$

When $x = 0,\ u = 0$

When $x = \frac{\pi}{3},\ u = \sqrt{3}$

$$\therefore\ \int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x\, dx = \int_0^{\sqrt{3}} u^2\, du$$

$$= \frac{1}{3}[u^3]_0^{\sqrt{3}}$$

$$= \frac{1}{3} \times 3\sqrt{3}$$

$$= \sqrt{3}$$

15 A $\ddot{y} = e^x + e^{-2x}$

$\dot{y} = e^x - \frac{1}{2}e^{-2x} + c$

When $x = 0,\ \dot{y} = \frac{1}{2}$:

$\Rightarrow c = 0$

$\therefore\ \dot{y} = e^x - \frac{1}{2}e^{-2x}$

$y = e^x + \frac{1}{4}e^{-2x} + d$

When $x = 0,\ y = 0$:

$\Rightarrow c = -\frac{5}{4}$

$\therefore\ y = e^x = \frac{1}{4}e^{-2x} - \frac{5}{4}$

16

A $\dfrac{dy}{dx} = 2y + 1$

$\therefore\ \dfrac{dx}{dy} = \dfrac{1}{2y+1}$

$\therefore\ x = \dfrac{1}{2}\log_e(2y+1) + c \quad \text{for} \quad y > -\dfrac{1}{2}$

When $x = 0,\ y = 3$:

$\Rightarrow c = -\dfrac{1}{2}\log_e 7$

$\therefore\ x = \dfrac{1}{2}\log_e\left(\dfrac{2y+1}{7}\right)$

$\therefore\ e^{2x} = \dfrac{2y+1}{7}$

$\therefore\ y = \dfrac{7e^{2x} - 1}{2}$

17 C $\dfrac{d}{dx}[x\tan^{-1} x] = \dfrac{x}{1+x^2} + \tan^{-1} x$

$\therefore\ x\tan^{-1} x = \displaystyle\int\left(\frac{x}{1+x^2} + \tan^{-1} x\right)dx$

$\therefore\ \displaystyle\int \tan^{-1} x = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$

Let $u = 1 + x^2$, $\quad \therefore\ \dfrac{du}{dx} = 2x$

$\therefore\ \displaystyle\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$

$= \dfrac{1}{2}\log_e |u| \quad (c = 0)$

$= \dfrac{1}{2}\log_e(1+x^2)$

$\therefore\ \displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x$

$- \dfrac{1}{2}\log_e(1+x^2)$

$= x\tan^{-1} x$

$- \log_e \sqrt{1+x^2}$

18 C distance $= \dfrac{1}{2}(240 + 360) \times 10$

$= 5 \times 600$

$= 3000$

19 B $\dfrac{dy}{dx} = 1 - e^{-x}$

$\therefore\ y = x + e^{-x} + c$

When $x = 0,\ y = 6$:

$\Rightarrow c = 5$

$\therefore\ y = x + e^{-x} + 5$

20 D The graph of response A does have asymptotes at $x = 1$ and $x = 2$, however when $x = 0,\ y = \dfrac{1}{2}$ but the given graph does not have this property. Thus response A is incorrect.

The graph of response B has the property that when $x = 0,\ y = 0$. The given graph does not have this property. Thus response B is incorrect.

The graph of response C should have an asymptote at $x = 0$, however the given graph does not have this property. Thus response C is also incorrect.

When $x = 1.5$,

For response D : $y = -8$

For response E : $y = 8$

The given graph is negative for $x \in (1, 2)$, thus response D is correct.

21 B $y = e^{mx}$

$\dfrac{dy}{dx} = me^{mx} \quad \text{and} \quad \dfrac{d^2y}{dx^2} = m^2e^{mx}$

For $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0$

$\therefore\ m^2e^{mx} - 2me^{mx} - 3e^{mx} = 0$

$\therefore\ e^{mx}(m^2 - 2m - 3) = 0$

$\therefore\ m^2 - 2m - 3 = 0$ (since $e^{mx} \neq 0$)

$\therefore\ (m+1)(m-3) = 0$

$\therefore\ m = -1 \quad \text{or} \quad m = 3$

22 B The particle should initially have a velocity of 20 m/s. Thus responses C and D are incorrect. Since there are no other forces acting on the body, except gravity, the acceleration of the particle should be constant. Therefore, response E is incorrect. Since the particle returns to the point of projection the displacement must be equal to zero. Response A has a non-zero displacement. Response B has a zero displacement.

23 B

$$y = e^{3x}$$

$$\frac{d^2y}{dx^2} = 9e^{3x}$$

$$\frac{d^2y}{dx^2} - 9y = 9e^{3x} - 9e^{3x} = 0$$

Thus response B is correct.

24 E $a = 6t^2 + 5t - 3$

$$v = \int a\,dt = 2t^3 + \frac{5}{2}t^2 - 3t + c$$

When $t = 0,\ v = 3$:

$$\Rightarrow c = 3$$

$$\therefore\ v = 2t^3 + \frac{5}{2}t^2 - 3t + 3$$

When $t = 2,\ v = 23$ m/s

25 E

$$v = 4\sin 2t$$

$$\therefore s = \int v\,dt = -2\cos 2t + c$$

When $t = 0,\ s = 0$:

$$\Rightarrow c = 2$$

$$\therefore\ s = 2 - 2\cos 2t$$

26 E For a rotation about the y-axis

$$V = \pi \int_{y=b}^{y=a} x^2 dy$$

$$y = e^{2x}$$

$$\therefore\ x = \frac{1}{2}\log_e y$$

Thus the shaded region rotated about the y-axis is given by:

$$V = \pi \int_1^2 \left(\frac{1}{2}\log_e y\right)^2 dy$$

$$\therefore\ V = \pi \int_1^2 \frac{1}{4}(\log_e y)^2 dy$$

27 A The area of the shaded region is given by

$$-\int_{-2}^{0} f(x)\,dx + \int_0^1 f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_0^{-2} f(x)\,dx$$

28 C For P:

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 2x)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{4} \approx 0.785$$

For Q:

$$\int_0^{\frac{\pi}{4}} \cos^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(\cos 2x + 1)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$= \frac{1}{4} + \frac{\pi}{8} \approx\ 0.643$$

For R:

$$\int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 2x)\,dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$
$$= \frac{\pi}{8} - \frac{1}{4} \approx 0.143$$

In ascending order we have: $R,\ Q,\ P$

29 B Area of region is given by

$$\int_{-3}^{-3} ((9-x^2)-(x^2-9))\,dx$$
$$= \int_{-3}^{3} (18-2x^2)\,dx$$

30 A Volume is given by

$$\pi \int_0^1 (2e^{2x})^2 - 2^2\,dx$$
$$= \pi \int_0^1 4e^{4x} - 4\,dx$$

31 D $\dfrac{d^2x}{dt^2} = 4 - e^{-t}$

$$v = \frac{dx}{dt} = 4t + e^{-t} + c$$

When $t = 0,\ v = 3$:

$\Rightarrow c = 2$

$\therefore\ v = 4t + e^{-t} + 2$

When $t = 2,\ v = 10 + e^{-2}$

32 C distance = area under graph

$$= \frac{1}{2}(4+6)\times 10$$
$$+ \frac{1}{2}\times 2\times 10$$
$$= 50 + 10$$
$$= 60$$

33 C $V = \dfrac{\log_e t}{t} = t^{-1}\log_e(t)$

$$\frac{dV}{dt} = -t^{-2}\log_e(t) + \frac{1}{t^2}$$
$$= \frac{1}{t^2}(-\log_e t + 1)$$
$$\frac{dV}{dt} = 0 \Rightarrow -\log_e t + 1 = 0$$
$$\therefore t = e$$
$$\frac{d^2V}{dt^2} = -e^{-3} < 0 \text{ when } t = e.$$

Therefore local maximum at $t = e$

34 B

$$\frac{dy}{dx} = -\frac{x}{ye^{x^2}}$$
$$\int y\,dy = -\int xe^{-x^2}\,dx$$
$$\frac{y^2}{2} = -\int xe^{-x^2}\,dx$$

Let $u = -x^2 \Rightarrow \dfrac{du}{x} = -2x$

$$\frac{y^2}{2} = \frac{1}{2}\int e^u\,du$$
$$\frac{y^2}{2} = \frac{1}{2}e^{-x^2} + c$$

When $x = 0, y = 2 \Rightarrow c = \dfrac{3}{2}$

$$\frac{y^2}{2} = \frac{1}{2}e^{-x^2} + \frac{3}{2}$$
$$y^2 = e^{-x^2} + 3$$

35 C $g(x)$ is the upper curve for the interval $x \in [0,\ b]$. $f(x)$ is the upper curve for the interval $x \in [b,\ c]$. Thus the total area is given by

$$\int_0^b g(x) - f(x)\,dx + \int_b^c f(x) - g(x)\,dx$$
$$= \int_b^c f(x) - g(x)\,dx + \int_0^b -(f(x) - g(x))\,dx$$
$$= \int_b^c f(x) - g(x)\,dx + \int_b^0 f(x) - g(x)\,dx$$

36 A $\dfrac{1}{(2x+6)(x-4)} \equiv \dfrac{A}{2x+6} + \dfrac{B}{x-4}$

$\therefore\ 1 = A(x-4) + B(2x+6)$

When $x = 4,\ B = \dfrac{1}{14}$

When $x = -3,\ A = -\dfrac{1}{7}$

$\therefore\ \dfrac{1}{(2x+6)(x-4)} = \dfrac{1}{14(x-4)} - \dfrac{1}{7(2x+6)}$

$$\therefore \int \frac{1}{(2x+6)(x-4)}\,dx = \frac{1}{14}\int \frac{1}{x-4}\,dx - \frac{1}{7}\int \frac{1}{2x+6}\,dx$$

$$= \frac{1}{14}\log_e(x-4) - \frac{1}{14}\log_e(2x+6) + c$$

$$= \frac{-\frac{1}{7}}{2}\log_e(2x+6) + \frac{1}{14}\log_e(x-4)\ (c=0)$$

$\therefore\ a = -\dfrac{1}{7}$ and $b = \dfrac{1}{14}$

37 E $\displaystyle\int_0^1 x\sqrt{2x+1}\,dx$

Let $u = 2x+1$, $\quad \therefore\ \dfrac{du}{dx} = 2$

and $x = \dfrac{1}{2}(u-1)$

When $x = 0,\ u = 1$

When $x = 1,\ u = 3$

$$\therefore \int_0^1 x\sqrt{2x+1}\,dx$$

$$= \int_1^3 \left(\frac{1}{2}(u-1)\right)\sqrt{u}\left(\frac{1}{2}du\right)$$

$$= \frac{1}{4}\int_1^3 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

38 D Let $u = e^x$

$$\int_0^1 \frac{e^x}{(1+e^x)^2} = \int_1^e \frac{1}{(1+u)^2}\,du$$

39 D $\displaystyle\int_0^{\frac{\pi}{6}} \sin^n x\cos x\,dx = \frac{1}{64}$

Let $u = \sin x$, $\quad \therefore\ \dfrac{du}{dx} = \cos x$

When $x = 0,\ u = 0$

When $x = \dfrac{\pi}{6},\ u = \dfrac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^n x\cos x\,dx = \int_0^{\frac{1}{2}} u^n du$$

$$= \left[\frac{1}{n+1}u^{n+1}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{n+1}\left[\left(\frac{1}{2}\right)^{n+1}\right]$$

$$\therefore \frac{1}{n+1}\times\left(\frac{1}{2}\right)^{n+1} = \frac{1}{64}$$

$$\therefore \left(\frac{1}{2}\right)^{n+1} = \frac{n+1}{64}$$

$$\therefore \frac{1}{2^{n+1}} = \frac{n+1}{2^6}$$

$$\therefore \frac{2^6}{2^{n+1}} = n+1$$

$$\therefore 2^{5-n} = n+1$$

When $x = 3$:

$2^2 = 3 + 1 \Rightarrow 4 = 4$

Which is a true statement

40 A $\int x^2 \sin x \, dx = g(x) + \int 2x \cos x \, dx$

Let $u = x^2$ and $\frac{dv}{dx} = \sin x$

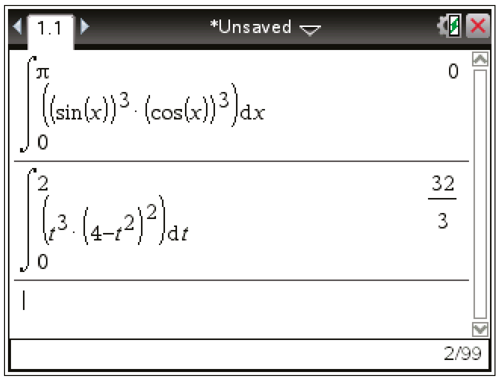
Then $\frac{du}{dx} = 2x$ and $v = -\cos x$

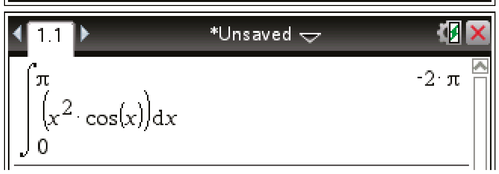
$\int x^2 \sin x \, dx$

$= (-x^2 \cos x) - \int 2x(-\cos x) \, dx$

$= (-x^2 \cos x) + \int 2x \cos x \, dx$

41 E Using CAS:





Order is: 0 + −

42 A $\int_0^{\frac{\pi}{4}} \cos 2x \, dx = \left[\frac{1}{2} \sin 2x\right]_0^{\frac{\pi}{4}} = \frac{1}{2}$

Using a CAS calculator to evaluate all of the responses yields that

$\frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2x \, dx = \frac{1}{2}$

$\therefore \int_0^{\frac{\pi}{4}} \cos 2x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2x \, dx$

43 C $\int_{-a}^{a} \tan x \, dx = \int_{-a}^{a} \frac{\sin x}{\cos x} dx$

$= [-\log_e(\cos x)]_{-a}^{a}$

Using the substitution $u = \cos x$ and for $\cos x > 0$.

When $x = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ $\cos x = 0$

Thus $\log_e(\cos x)$ is undefined.

When $x = \pi$, $\cos \pi = -1$

Thus $\log_e(-1)$ is undefined.

$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\log_e\left(\frac{\sqrt{2}}{2}\right)$ is defined

44 B $e^{f(x)} = x^2 + 9$

For convenience we right $e^y = x^2 + 9$

Using implicit differentiation.

$e^y \frac{dy}{dx} = 2x$

Therefore,

$f'(x) = 2xe^{-y}$

$= 2xe^{-\log_e(x^2+9)}$

$= \frac{2x}{x^2 + 9}$

45 B Consider $\int x^3 \sin(3x) \, dx$

We apply integration by parts.

Let $u = x^3$ and $\frac{dv}{dx} = \sin 3x$

$\frac{du}{dx} = 3x^2$ and $v = -\frac{1}{3} \cos 3x$

$\int x^3 \sin(3x) \, dx = \left[-\frac{x^3}{3} \cos 3x\right] + \int x^2 \cos 3x \, dx$

46 D

$y = \log_e(x^2 + 5)$

$\frac{dy}{dx} = \frac{2x}{x^2 + 5}$

$\frac{d^2y}{dx^2} = \frac{-2x^2 + 10}{(x^2 + 5)^2}$

Concave up for $\frac{d^2y}{dx^2} > 0$

$-2x^2 + 10 > 0 \Leftrightarrow -\sqrt{5} < x < \sqrt{5}$

47 C Volume of revolution is equal to

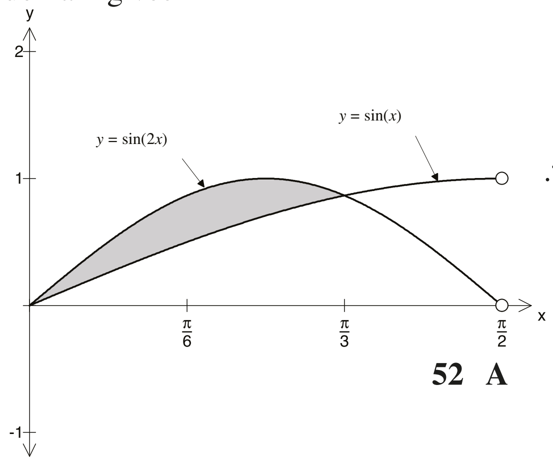
$\pi \int_0^{\frac{\pi}{4}} (2 \sin x - 1)^2 dx$

$= \pi \int_0^{\frac{\pi}{4}} (1 - 2 \sin x)^2 dx$

48 B $f:\left[0,\frac{\pi}{2}\right)\to R,\ f(x)=\sin x$

$g:\left[0,\frac{\pi}{2}\right)\to R,\ g(x)=\sin 2x$

Sketching the two functions over the given domain gives



Area bounded by the two graphs f and g is equal to

$$\int_0^{\frac{\pi}{3}} \sin 2x - \sin x\,dx$$

49 D

$\frac{dN}{dt}=4000e^{0.4t}$

$N=10\,000e^{0.4t}+c$

When $t=0, N=10\,000 \Rightarrow c=0$

$N=10\,000e^{0.4t}$

When $t=10, N=10\,000e^{4}$

50 C Response A is clearly true.

Response B is also true.

Response D and E are correct statements.

Response C is false because if the region shown is rotated about the y-axis the lower limit should be $y=0$ not $y=f(0)$

Note: $f(0)\neq 0$

51 D $\frac{dy}{dx}+y=1$

$\therefore \frac{dy}{dx}=1-y$

$\therefore \frac{dx}{dy}=\frac{1}{1-y}$

$\therefore \frac{dx}{dy}=\frac{-1}{y-1}$

$\therefore x=-\log_e(y-1)+c$ for $y>1$

$\therefore e^{c-x}=y-1$

$\therefore y=1+e^{c-x}$

$\therefore y=1+Pe^{-x}$ where $P=e^{c}$

52 A Given $\frac{dV}{dt}=-2\text{ m}^3/\text{s}$

$V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

and surface area $S=4\pi r^2$

$\therefore \frac{dS}{dr}=8\pi r$

$\frac{dS}{dt}=\frac{dS}{dV}\times\frac{dV}{dt}$

where $\frac{dS}{dV}=\frac{dS}{dr}\times\frac{dr}{dV}$

$=8\pi r\times\frac{1}{4\pi r^2}$

$=\frac{2}{r}$

$\therefore \frac{dS}{dt}=\frac{2}{r}\times -2=-\frac{4}{r}$

When $r=5$, $\frac{dS}{dt}=-\frac{4}{5}$

Therefore the surface area is decreasing at a rate of $\frac{4}{5}\text{ m}^2/\text{s}$

53 **E** $x = \sin^2 2t, y = \cos 3t$

$\dfrac{dx}{dt} 4\sin 2t \cos 2t$ and $\dfrac{dy}{dt} = -3\sin 3t$

L

$= \int_0^{\frac{\pi}{2}} \sqrt{16\sin^2 2t\cos^2 t + 9\sin^2 3t}\, dt$

We know $\sin 4t = 2\sin 2t\cos 2t$ and

$\sin^2 4t = 4\sin^2 2t\cos^2 2t$

Therefore,

$L = \int_0^{\frac{\pi}{2}} \sqrt{4\sin^2 4t + 9\sin^2 3t}\, dt$

54 **D**

$$\frac{dr}{dt} = 2 \text{ cm/s}$$

$$V = \frac{4}{3}\pi r^3 \text{ and } \frac{dV}{dr} = 4\pi r^2$$

$$\therefore \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$= 4\pi r^2 \times 2$$

$$= 8\pi r^2$$

When $r = 10$, $\dfrac{dV}{dt} = 800\pi$

55 **A** Let $g(x) = \sec\theta + \tan\theta$

$$= (\cos\theta)^{-1} + \tan\theta$$

Then $g'(x) = -(\cos\theta)^{-2} \times -\sin\theta$

$$+ \sec^2\theta$$

$$= \frac{\sin\theta}{\cos^2\theta} + \sec^2\theta$$

$$= \tan\theta\ \sec\theta\ + \sec^2\theta$$

$$= \sec\theta\ (\sec\theta\ + \tan\theta)$$

It is known that

$$\frac{d}{dx}[\log_e(g(x))] = \frac{g'(x)}{g(x)}$$

$$\therefore \frac{d}{dx}[\log_e (\sec\theta\ + \tan\theta)]$$

$$= \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}$$

$$= \sec\theta$$

56 **E**

$$x = 3\cos 2t$$

$$v = \frac{dx}{dt} = -6\sin 2t$$

$$a = \frac{d^2x}{dt^2} = -12\cos 2t$$

When $t = \dfrac{\pi}{2}$, $a = 12$

57 **C**

$$\boldsymbol{r}(t) = 2t^2\boldsymbol{i} + t^3\boldsymbol{j}$$

$$\dot{\boldsymbol{r}}(t) = 4t\boldsymbol{i} + 3t^2\boldsymbol{j}$$

When $t = 1$

$$\dot{\boldsymbol{r}}(1) = 4\boldsymbol{i} + 3\boldsymbol{j}$$

Speed $= |\dot{\boldsymbol{r}}(1)|$

$$= \sqrt{4^2 + 3^2}$$

$$= 5 \text{ m/s}$$

58 **B** Average velocity

$$= \frac{(4, -1, 4) - (2, 5, 2)}{2}$$

$$= \frac{(2, -6, 2)}{2}$$

$$= (1, -3, 1)$$

$$= \boldsymbol{j} - 3\boldsymbol{j} + \boldsymbol{k}$$

59 B $\ddot{\boldsymbol{x}}(t) = 2\boldsymbol{i} + t\boldsymbol{j}$

$$\dot{\boldsymbol{x}}(t) = 2t\boldsymbol{i} + \frac{t^2}{2}\boldsymbol{j} + c$$

When $t = 0, \dot{\boldsymbol{x}}(t) = 2\boldsymbol{i}$

$$\Rightarrow \boldsymbol{c} = 2\boldsymbol{i}$$

$$\therefore\ \dot{\boldsymbol{x}}(t) = (2\mathrm{t} + 2)\boldsymbol{i} + \frac{t^2}{2}\boldsymbol{j}$$

60 D $\dot{\boldsymbol{r}}(t) = 2t\boldsymbol{i} + 3\boldsymbol{j}$

$$\therefore\ \boldsymbol{r}(t) = t^2\boldsymbol{i} + 3t\boldsymbol{j} + \boldsymbol{c}$$

$$\boldsymbol{r}(0) = 3\boldsymbol{i} + \boldsymbol{j}$$

$$\Rightarrow \boldsymbol{c} = 3\boldsymbol{i} + \boldsymbol{j}$$

$$\therefore\ \boldsymbol{r}(t) = (t^2 + 3)\boldsymbol{i} + (3t + 1)\boldsymbol{j}$$

61 C $\boldsymbol{r}(t) = 4t\boldsymbol{i} - \frac{1}{3}t^2\boldsymbol{j}$

Average velocity in the third second is

$$= \boldsymbol{r}(3) - \boldsymbol{r}(2)$$

$$= (12\boldsymbol{i} - 3\boldsymbol{j}) - \left(12\boldsymbol{i} - \frac{4}{3}\boldsymbol{j}\right)$$

$$= 4\boldsymbol{i} - \frac{5}{3}\boldsymbol{j}$$

Average speed in the third second is

$$= |\boldsymbol{r}(3) - \boldsymbol{r}(2)|$$

$$= \sqrt{16 + \frac{25}{9}}$$

$$= \sqrt{\frac{169}{9}}$$

$$= \frac{13}{3}$$

$$= 4\frac{1}{3}\text{ m/s}$$

62 D $\boldsymbol{r}(t) = (t^2 - 2t)(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

Particle initially begins at the origin.
Particle is at rest when

$\dot{\boldsymbol{r}}(t) = 0$

$\dot{\boldsymbol{r}}(t) = (2t - 2)(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) \quad = 0$

$\boldsymbol{r}(t) = 0$ when $t = 1$

Thus, the particle begins at the origin and after 1 second is instantaneously at rest at the point $(-1, 2, -2)$. After a further second the particle returns to the origin. Therefore, the distance travelled by the particle in the first two seconds is equal to twice the distance covered in the first second.

Distance travelled

$$= 2 \times |\boldsymbol{r}(1) - \boldsymbol{r}(0)|$$

$$= 2|\boldsymbol{r}(1)| \text{ since } \boldsymbol{r}(0) = \boldsymbol{0}$$

$$= 2 \times \sqrt{(-1)^2 + 2^2 + (-2)^2}$$

$$= 2 \times 3$$

$$= 6\text{ m}$$

63 C

$\int_0^3 4xf''(x)\,dx$

We apply integration by parts:

Let $u = 4x$ and $\frac{dv}{dx} = f''(x)$

Then, $\frac{du}{dx} = 4$ and $v = f'(x)$

$$\int_0^3 4xf''(x)\,dx = [4xf'(x)]_0^3 - 4\int_0^3 f'(x)\,dx$$

$$= 12 \times f'(3) - 4\,[f(x)]_0^3$$

$$= 132 - 4(6 - 3)$$

$$= 120$$

64 C $\boldsymbol{r}(t) = \left(\frac{1}{3}t^3 - 4t^2 + 15t\right)\boldsymbol{i}$

$+ \left(t^3 - \frac{15}{2}t^2\right)\boldsymbol{j}$

$\dot{\boldsymbol{r}}(t) = (t^2 - 8t - 15)\boldsymbol{i} + (3t^2 - 15t)\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = (2t - 8)\boldsymbol{i} + (3t - 15)\boldsymbol{j}$

Instantaneously at rest when $\dot{\boldsymbol{r}}(t) = 0$

$t^2 - 8t + 15 = 0$ and $3t^2 - 15t = 0$

$(t - 5)(t - 3) = 0$ and $3t(t - 5) = 0$

$t = 3$ or 5 and $t = 0$ or 5

$\therefore\ t = 5$

Therefore, the particle is instantaneously at rest when $t = 5$.

$\ddot{\boldsymbol{r}}(5) = 2\boldsymbol{i} + 15\boldsymbol{j}$

65 E $\boldsymbol{r}(t) = (3t^3 - t)\boldsymbol{i} + (2t^2 + 1)\boldsymbol{j} + 5\boldsymbol{k}$

$\dot{\boldsymbol{r}}(t) = (9t^2 - t)\boldsymbol{i} + 4t\boldsymbol{j} + 5\boldsymbol{k}$

$\ddot{\boldsymbol{r}}(t) = 18t\boldsymbol{i} + 4\boldsymbol{j}$

$\ddot{\boldsymbol{r}}\left(\frac{1}{2}\right) = 9\boldsymbol{i} + 4\boldsymbol{j}2$

$\left|\ddot{\boldsymbol{r}}\left(\frac{1}{2}\right)\right| = \sqrt{81 + 16} = \sqrt{16}$

66 E

$\dot{\boldsymbol{r}}(t) = \sin t\boldsymbol{i} + \cos 2t\boldsymbol{j}$

$\therefore\ \boldsymbol{r}(t) = -\cos t\boldsymbol{i} + \frac{1}{2}\sin 2t\boldsymbol{j} + \boldsymbol{c}$

$\boldsymbol{r}(0) = 6\boldsymbol{i} - 4\boldsymbol{j}$

$\Rightarrow \boldsymbol{c} = 7\boldsymbol{i} - 4\boldsymbol{j}$

$\therefore\ \boldsymbol{r}(t) = (7 - \cos t)\boldsymbol{i} + \left(\frac{1}{2}\sin 2t - 4\right)\boldsymbol{j}$

67 D $\ddot{\boldsymbol{r}}(t) = \boldsymbol{i} - \boldsymbol{j}$

$\therefore\ \dot{\boldsymbol{r}}(t) = t\boldsymbol{i} - t\boldsymbol{j} + \boldsymbol{c}$

$\dot{\boldsymbol{r}}(0) = 3\boldsymbol{j}$

$\Rightarrow \boldsymbol{c} = 3\boldsymbol{j}$

$\therefore\ \dot{\boldsymbol{r}}(t) = t\boldsymbol{i} - (3 - t)\boldsymbol{j}$

$\boldsymbol{r}(t) = \frac{1}{2}t^2\boldsymbol{i} + \left(3t + \frac{1}{2}t^2\right)\boldsymbol{j} + \boldsymbol{d}$

$\boldsymbol{r}(0) = 2\boldsymbol{i}$

$\Rightarrow \boldsymbol{d} = 2\boldsymbol{i}$

$\therefore\ \boldsymbol{r}(t) = \left(2 + \frac{1}{2}t^2\right)\boldsymbol{i} + \left(3t - \frac{1}{2}t^2\right)\boldsymbol{j}$

$= \left(2 + \frac{1}{2}t^2\right)\boldsymbol{i} + \frac{t}{2}(6 - t)\boldsymbol{j}$

Solutions to extended-response questions

1 a Volume $= \int_0^{25} \pi x^2 dy$

$= \int_0^{25} 4\pi y dy \ (x^2 = 4y)$

$= [2\pi y^2]_0^{25}$

$= 1250\pi$

b i $\dfrac{dV}{dt} = -kh, \ k > 0$

but $V = 2\pi h^2 \left(= \int_0^h \pi x^2 dy \right)$

$\dfrac{dV}{dh} = 4\pi h$

$\therefore \dfrac{dh}{dt} = \dfrac{dh}{dv} \times \dfrac{dV}{dt}$

$= \dfrac{1}{4\pi h} \times -kh$

$= \dfrac{-k}{4\pi}, \ k > 0$

ii $h = \dfrac{-kt}{4\pi} + c$ is the solution of the differential equation.

When $t = 0, \ h = 25,$

$\therefore \ c = 25$

When $h = 0, \ t = 30,$

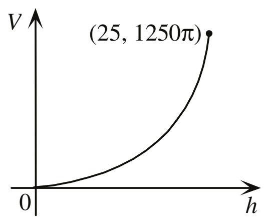
$\therefore 0 = \dfrac{-30k}{4\pi} + 25$

$\therefore k = \dfrac{100\pi}{30}$

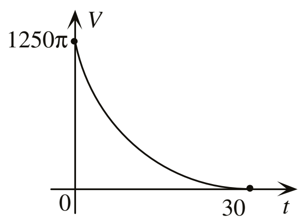
iii From above $h = 25 - \dfrac{5t}{6}$

iv $V = 2\pi\left(25 - \dfrac{5t}{6}\right)^2$

c **i**



ii



2 **a** Let $u = \tan^{-1} x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{1 + x^2}$

Let $\dfrac{dv}{dx} = x^n \Rightarrow v = \dfrac{x^{n+1}}{n + 1}$

$$I_n = \int_0^1 x^n \tan^{-1} x \, dx$$

$$= \left[\frac{x^{n+1}}{n + 1} \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x^{n+1}}{(n + 1)(1 + x^2)} \, dx$$

$$= \frac{\pi}{4(n + 1)} - \frac{1}{n + 1} \int_0^1 \frac{x^{n+1}}{1 + x^2} \, dx$$

$$\therefore (n + 1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1 + x^2} \, dx$$

b $I_0 = \dfrac{\pi}{4} - \displaystyle\int_0^1 \frac{x}{1 + x^2} \, dx$

Let $u = 1 + x^2 \Rightarrow \dfrac{du}{dx} = 2x$

$$I_0 = \frac{\pi}{4} - \frac{1}{2} \int_1^2 \frac{1}{u} \, du$$

$$= \frac{\pi}{4} - \frac{1}{2} \log_e 2$$

$$I_1 = \frac{\pi}{4} - \int_0^1 \frac{x^2}{1 + x^2} \, dx$$

$$\frac{\pi}{4} - \int_0^1 1 - \frac{1}{x^2 + 1} \, dx$$

$$I_0 = \frac{\pi}{4} - \frac{1}{2} \int_0^1 x - \tan^{-1} x \, dx$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

c

$$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{(1+x^2)}\,dx \ldots (1)$$

$$(n+1)I_{n+1} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{(1+x^2)}\,dx \ldots (2)$$

Therefore,

$$(n+3)I_{n+2} + (n+1)I_{n+1} = \frac{\pi}{2} - \int_0^1 \frac{x^{n+3}}{(1+x^2)} - \frac{x^{n+1}}{(1+x^2)}\,dx$$

$$= \frac{\pi}{2} - \int_0^1 x^{n+1}\,dx$$

$$= \frac{\pi}{2} - \left[\frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \frac{\pi}{2} - \frac{1}{n+2}$$

d **i** Carefully substitute $n = 0$ in **c** to obtain the result and similarly for the other parts.

3 **a** Let $\frac{du}{dv} = 1 \Rightarrow u = x + 1$ and $v = \log_e(x+1) \Rightarrow \frac{dv}{dx} = \frac{1}{x+1}$

$$I_0 = [(x+1)\log_e(x+1)]_0^1 - \int_0^1 1\,dx$$

$$= 2\log_e 2 - 1$$

b Let $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$ and $\frac{dv}{dx} = \log_e(x+1) \Rightarrow v = (x+1)\log_e(x+1) - x$

$$I_n = [x^n[(x+1)\log_e(x+1) - x]]_0^1 - \int_0^1 nx^{n-1}[(x+1)\log_e(x+1) - x]\,dx$$

$$= 2\log_e 2 - \int_0^1 nx^{n-1}[(x+1)\log_e(x+1) - x]\,dx$$

$$= 2\log_e 2 - \int_0^1 nx^{n-1}(x+1)\log_e(x+1)\,dx - \int_0^1 nx^n\,dx$$

$$= 2\log_e 2 - n\int_0^1 nx^n\log_e(x+1)\,dx - n\int_0^1 nx^{n-1}\log_e(x+1)\,dx - \int_0^1 nx^n\,dx$$

$$= 2\log_e 2 - nI_n - nI_{n-1} - \frac{1}{n+1}$$

$$\therefore (n+1)I_n = 2\log_e 2 - nI_{n-1} - \frac{1}{n+1}$$

c $I_0 = 2\log_e 2 - 1$

$2I_1 = 2\log_e(2) - \frac{1}{2} - (2\log_e(2) - 1)$

$2I_1 = 1 - \frac{1}{2}$

$3I_2 = 2\log_e(2) - 2I_1 - \frac{1}{3} = 2\log_e(2) - (1 - \frac{1}{2}) - \frac{1}{3}$

$4I_3 = 2\log_e(2) - 3I_2 - \frac{1}{4} = 2\log_e(2) - (2\log_e(2) - (1 - \frac{1}{2}) - \frac{1}{3}) - \frac{1}{4}$

$4I_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

We present an informal induction argument.

For k odd assume $(k+1)I_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{1}{k} + \frac{1}{k+1}$

$k + 1$ is even

$(k+2)I_{k+1} = 2\log_e 2 - (k+1)I_k - \frac{1}{k+2}$

$= 2\log_e 2 - (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{1}{k} + \frac{1}{k+1}) - \frac{1}{k+2}$

$k + 2$ is the next odd number

$(k+3)I_{k+2} = 2\log_e 2 - (k+2)I_{k+1} - \frac{1}{k+3}$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{1}{k} + \frac{1}{k+1} - \frac{1}{k+2} - \frac{1}{k+3}$

4 a When $x = 0$,

$2y = y^2 \Rightarrow y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$

When $y = 0$ $2x = x^2 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$

Intercepts: $(0, 0), (0, 2), (2, 0)$

b If $y = x, 4x = 0 \Rightarrow x = 0$

Intersect at $(0, 0)$

c Interchanging x and y does not change the rule.

d i

$$\frac{d}{dx}(2(x+y) - (x-y)^2) = 0$$

$$2 + \frac{dy}{dx} - \frac{d}{dx}(x^2 - 2xy + y^2) = 0$$

$$2 + 2\frac{dy}{dx} - (2x - (2y + 2x\frac{dy}{dx}) + 2y\frac{dy}{dx}) = 0$$

$$(2 - 2x + 2y)\frac{dy}{dx} = -2 + 2x - 2y$$

$$\frac{dy}{dx} = \frac{x - y - 1}{x - y + 1}$$

ii $\dfrac{x-y-1}{x-y+1}=0$

$x-y-1=0$

$y=x-1$

Substitute in $2(x+y)-(x-y)^2=0$

$2(x+x-1)-(x-(x-1))^2=0\ 4x-2-1=0$

$x=\dfrac{3}{4}, y=-\dfrac{1}{4}$

iii $\dfrac{x-y+1}{x-y-1}=0$

$x-y+1=0$

$y=x+1$

Substitute in $2(x+y)-(x-y)^2=0$

$2(x+x+1)-(x-(x+1))^2=0\ 4x+2-1=0$

$x=-\dfrac{3}{4}, y=\dfrac{1}{4}$

e Let $P(x_1, y_1)$ be a point on the curve and $A(a, a)$ the point on the line $y = x$ such that PA has gradient -1

M is the point on the line $y=-x-\dfrac{1}{2}$ such that PM has gradient 1.

The line $y = x$ intersects the line $y=-x-\dfrac{1}{2}$ at the point $M'\left(-\dfrac{1}{4},-\dfrac{1}{4}\right)$

Clearly $PM = AM'$

$$AM' = \sqrt{\left(a+\frac{1}{4}\right)^2+\left(a+\frac{1}{4}\right)^2}$$

Now $\dfrac{a-y_1}{a-x_1}=-1$. Hence $a=\dfrac{x_1+y_1}{2}$

We can write

$$PM = AM' = \frac{\sqrt{2}}{4}(2x_1+2y_1+1)$$

$$PF = \sqrt{\left(x_1-\frac{1}{4}\right)^2+\left(y_1-\frac{1}{4}\right)^2}$$

$$= \frac{\sqrt{2}}{4}\sqrt{8x_1^2-4x_1+8y_1^2-4y_1+1}$$

We complete the proof by showing

$$(2x_1+2y_1+1)^2 = 8x_1^2-4x_1+8y_1^2-4y_1+1$$

$(2x_1 + 2y_1 + 1)^2 = 8x_1^2 - 4x_1 + 8y_1^2 - 4y_1 + 1$

$\Leftrightarrow 4x_1^2 + 8x_1y_1 + 4x_1 + 4y_1^2 + 4y_1 + 1 = 8x_1^2 - 4x_1 + 8y_1^2 - 4y_1 + 1$

$\Leftrightarrow 4x_1^2 - 8x_1 + 4y_1^2 - 8y_1 - 8x_1y_1 = 0$

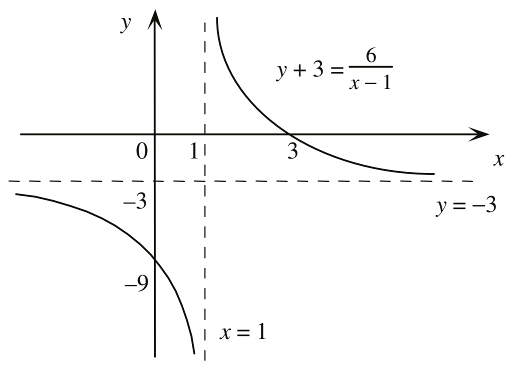
$\Leftrightarrow 4x_1^2 - 8x_1y_1 + 4y_1^2 - 8(x_1 + y_1) = 0$

$\Leftrightarrow x_1^2 - 2x_1y_1 + y_1^2 = 2(x_1 + y_1)$

$\Leftrightarrow (x_1 - y_1)^2 = 2(x_1 + y_1)$

We know that (x_1, y_1) is on the curve so the last statement is true. Read back through the equivalences.

5 a



b Now $y + 3x = 9$

$\therefore y = 9 - 3x$

and also $y + 3 = \dfrac{6}{x-1}$

$\therefore 9 - 3x + 3 = \dfrac{6}{x-1}$

$\therefore (4 - x)(x - 1) = 2$

$\therefore -4 + 5x - x^2 = 2$

$\therefore x^2 - 5x + 6 = 0$

$\therefore x = 2 \text{ or } 3$

The points of intersection are (2, 3) and (3, 0).

c $$\int_2^3 9 - 3x - \frac{6}{x-1} + 3\,dx = \int_2^3 12 - 3x - \frac{6}{x-1}dx$$

$$= \left[12x - \frac{3x^2}{2} - 6\log_e(x-1)\right]_2^3$$

$$\therefore \text{Area} = 12 - \frac{15}{2} - 6\log_e 2 = \frac{9}{2} - 6\log_e 2$$

d $$\frac{dy}{dx} = -\frac{6}{(x-1)^2}$$

Now gradient of line = −3,

$$\therefore -\frac{6}{(x-1)^2} = -3$$

$$\therefore (x-1)^2 = 2$$

$$\therefore x - 1 = \pm\sqrt{2}$$

$$\therefore x = 1 \pm \sqrt{2}$$

Tangents are at points $\left(1+\sqrt{2},\ -3+\frac{6}{\sqrt{2}}\right), \left(1-\sqrt{2},\ -3-\frac{6}{\sqrt{2}}\right)$

i.e., $(1+\sqrt{2},\ -3(1-\sqrt{2}))$, $(1-\sqrt{2},\ -3(1+\sqrt{2}))$

Equations of tangents

$$y + 3(1-\sqrt{2}) = -3(x - 1 - \sqrt{2})$$

$$\therefore y + 3 - 3\sqrt{2} = -3x + 3 + 3\sqrt{2}$$

$$\Rightarrow\ y = -3x + 6\sqrt{2}$$

or $y + 3(1+\sqrt{2}) = -3(x - 1 + \sqrt{2})$

$$\therefore y + 3 + 3\sqrt{2} = -3x + 3 - 3\sqrt{2}$$

$$\Rightarrow\ y = -3x - 6\sqrt{2}$$

6 a $$\int_0^6 2\pi k(6-x)^{\frac{1}{2}}x^2\,dx = 600\,000$$

Let $u = 6 - x$,

$$\therefore -\int_6^0 2\pi k u^{\frac{1}{2}}(6-u)^2\,du = 600\,000$$

$$\therefore -2\pi k\int_6^0 36u^{\frac{1}{2}} - 12u^{\frac{3}{2}} + u^{\frac{5}{2}}\,du = 600\,000$$

$$-2\pi k\left[\frac{36u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{12u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{7}{2}}}{\frac{7}{2}}\right]_6^0 = 600\,000$$

$$\therefore k\left(24\times 6^{\frac{3}{2}} - 4.8\times 6^{\frac{5}{2}} + \frac{2}{7}\times 6^{\frac{7}{2}}\right) = \frac{600\,000}{2\pi}$$

$$\therefore k = 1184.435\ldots$$

$$= 1180,\ \text{correct to 3 significant figures.}$$

b $\int_0^3 2\pi k(6-x)^{\frac{1}{2}}x^2dx = -2\pi k\int_6^3 36u^{\frac{1}{2}} - 12u^{\frac{3}{2}} + u^{\frac{5}{2}}du$

$$= -2\pi k\left[\left(24\times 3^{\frac{3}{2}} - 4.8\times 3^{\frac{5}{2}} + \frac{2}{7}\times 3^{\frac{7}{2}}\right) - \left(24\times 6^{\frac{3}{2}} - 4.8\times 6^{\frac{5}{2}} + \frac{2}{7}\times 6^{\frac{7}{2}}\right)\right]$$

$$= -470\,668 + 600\,000$$

$$= 129\,332$$

129 332 people live within 3 km of the city centre.

7 **a** Intercepts with the x axis at $x = 10$ and $x = -10$,

$\therefore$ parabola is of the form $y = k(x^2 - 100)$

When $x = 20, y = 36,$

$$\therefore 36 = k(20^2 - 100)$$

$$\therefore k = \frac{36}{300}$$

$$= 0.12$$

$$\therefore y = 0.12x^2 - 12$$

b Volume $= \int_0^{36} \pi x^2 dy$

$$= \pi\int_0^{36} \frac{y+12}{0.12}dy$$

$$= \frac{\pi}{0.12}\left[\frac{y^2}{2} + 12y\right]_0^{36}$$

$$= \frac{\pi}{0.12}\left(\frac{36^2}{2} + 12\times 36\right)$$

$$= \frac{\pi}{0.01}\ (54 + 36)$$

$$= 9000\pi = 9\pi \text{ litres}$$

c $A = \pi x^2,$ where $h = 0.12x^2 - 12$

$$\therefore A = \frac{\pi}{0.12}(h + 12)$$

$$\therefore \frac{dv}{dt} = \frac{-\sqrt{h}}{\frac{\pi}{0.12}(h + 12)}$$

$$= \frac{-0.12\sqrt{h}}{\pi(h + 12)}$$

$$= \frac{-3\sqrt{h}}{25\pi(h + 12)}, \text{ as required.}$$

d From **b**, when $y = h$,

$$\text{Volume} = \pi \int_0^h \frac{y + 12}{0.12} dy$$

$$= \pi \int_0^h \frac{25y}{3} + 100 dy$$

e **i** From **d**, $v = \pi\left(\frac{25h^2}{6} + 100h\right)$

$$\therefore \frac{dv}{dh} = \pi\left(\frac{25h}{3} + 100\right)$$

ii $\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$

$$= \frac{1}{\pi\left(\frac{25h}{3} + 100\right)} \times \frac{-3\sqrt{h}}{25\pi(h + 12)}$$

$$= \frac{-9\sqrt{h}}{625\pi^2(h + 12)^2}$$

f $$\frac{dh}{dt} = \frac{-9}{625\pi^2} \times \frac{\sqrt{h}}{h^2 + 24h + 144}$$

$$\therefore \frac{dt}{dh} = \frac{-625\pi^2}{9}\left(h^{\frac{3}{2}} + 24h^{\frac{1}{2}} + 144h^{\frac{-1}{2}}\right)$$

$$\therefore t = \frac{-625\pi^2}{9}\left(\frac{2h^{\frac{5}{2}}}{5} + 16h^{\frac{3}{2}} + 288h^{\frac{1}{2}}\right) + c$$

When $t = 0,\ h = 36$,

$$\therefore c = \frac{625\pi^2}{9}\left(\frac{2 \times 6^5}{5} + 16 \times 216 + 288 \times 6\right)$$

$$c = 8294.4 \times \frac{625\pi^2}{9}$$

Time, when $h = 0$, is given by

$t = c$

$\therefore t = 5\,684\,892.135$ seconds

≈ 65.8 days

It takes approximately 65 days 19 hours for the bucket to empty.

8 $\frac{dV}{dt} = -kh,\ k > 0$

a **i** Volume (to height h) $= \int_{-a}^{h-a} \pi x^2 dy$

$$= \left[\pi a^2 y - \frac{\pi y^3}{3}\right]_{-a}^{h-a}$$

$$= \left(\pi a^2(h-a) - \frac{\pi(h-a)^3}{3}\right) - \left(-\pi a^3 + \frac{\pi a^3}{3}\right)$$

$$= \pi a^2 h - \frac{\pi h^3}{3} + \pi a h^2 - \pi a^2 h$$

$$\therefore V = \pi h^2\left(\frac{-h}{3} + a\right)$$

or $V = \pi\left(ah^2 - \frac{h^3}{3}\right)$, for $0 < h \le a$

ii If $a = 10$, and $V = 1\text{ L} = 1000\text{ cm}^3$,

then $1000 = \pi\left(10h^2 - \frac{h^3}{3}\right)$

Use a CAS calculator to solve this equation for h. This gives $h = 6.3550081$ So the depth water is 6.355 cm, correct to three decimal places.

b $$\frac{dV}{dh} = \pi(2ah - h^2)$$

$$\text{as } \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\text{then } \pi(2ah - h^2)\frac{dh}{dt} = -kh,\ k > 0 \text{ from above} \quad ①$$

c ① leads to $\pi(2a - h)\frac{dh}{dt} = -k$

$$\therefore \frac{dh}{dt} = \frac{-k}{\pi(2a - h)}$$

$$\therefore \frac{dt}{dh} = \frac{\pi(2a - h)}{-k}$$

$$\therefore t = \frac{\pi}{-k}\int (2a - h)dh$$

$$\therefore t = \frac{\pi}{-k}\left(2ah - \frac{h^2}{2}\right) + c$$

When $h = a,\ t = 0$,

$$\therefore c = \frac{\pi}{k}\left(\frac{3a^2}{2}\right)$$

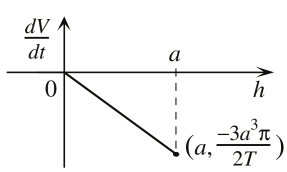
$$\text{and } t = \frac{\pi}{k}\left(\frac{3a^2}{2} - \left(2ah - \frac{h^2}{2}\right)\right)$$

When $h = 0,\ t = T$,

$$\therefore k = \frac{3a^2\pi}{2T}$$

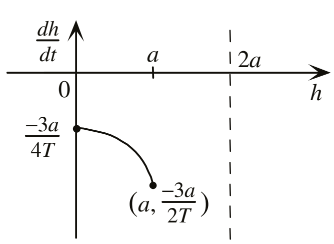
d $T = 30$ and $a = 10,\ k = 5\pi \approx 15.7$.

e **i**



$$\frac{dV}{dt} = \frac{-3a^2\pi}{2T}h$$

ii



$$\frac{dh}{dt} = \frac{-3a^2}{2T(2a - h)}$$

f **i** When $h = \frac{a}{2}$, $\frac{dh}{dt} = \frac{-3a^2}{2T\left(2a - \frac{a}{2}\right)}$

$$= \frac{-3a^2}{3aT}$$

$$= \frac{-a}{T}$$

ii When $h = \frac{a}{4}$, $\frac{dh}{dt} = \frac{-3a^2}{2T\left(2a - \frac{a}{4}\right)}$

$$= \frac{-6a^2}{7aT}$$

$$= \frac{-6a}{7T}$$

g When $a = 10$, $T = 30$ and $V = 1000$, $k = 5\pi$ and $h = 6.355$.

Now $\frac{dh}{dt} = \frac{-k}{\pi(2a - h)}$

$$= \frac{-5\pi}{\pi(2 \times 10 - 6.355)}$$

$$= -0.36643\ldots$$

9

10 **a** $f(x) = \frac{1}{ax^2 + bx + c}$

$\therefore f'(x) = \frac{-(2ax + b)}{(ax^2 + bx + c)^2}$

$f'(x) = 0$ at stationary points

b For the turning point of $f(x)$, $x = \frac{-b}{2a}$, provided $b^2 - 4ac \neq 0$.

Coordinates of turning point are $\left(\frac{-b}{2a}, \frac{4a}{4ac - b^2}\right)$

i $a > 0$

x	$\frac{-b}{2a} - 1$	$\frac{-b}{2a}$	$\frac{-b}{2a} + 1$
$f'(x)$	$\frac{2a}{(\ldots)^2} = +\text{ve}$	0	$\frac{-2a}{(\ldots)^2} = -\text{ve}$
	/	—	\

∴ maximum

ii $a < 0$

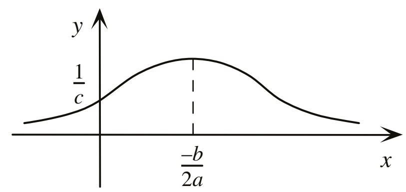
x	$\frac{-b}{2a} - 1$	$\frac{-b}{2a}$	$\frac{-b}{2a} + 1$
$f'(x)$	$\frac{2a}{(\ldots)^2} = -\text{ve}$	0	$\frac{-2a}{(\ldots)^2} = +\text{ve}$
	\	—	/

∴ minimum

c **i** $b^2 - 4ac < 0,\ a > 0$

no vertical asymptote for $f(x) = \dfrac{1}{ax^2 + bx + c}$

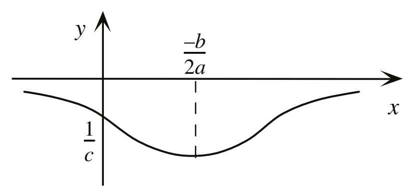
asymptote: $y = 0$



ii $b^2 - 4ac < 0,\ a < 0$

no vertical asymptote for $f(x) = \dfrac{1}{ax^2 + bx + c}$

asymptote: $y = 0$

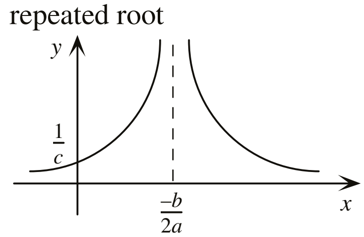


d **i** $b^2 - 4ac = 0,\ a > 0$

repeated root

asymptotes: $y = 0$ and $x = \dfrac{-b}{2a}$

repeated root

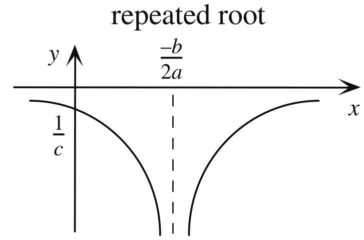


ii $b^2 - 4ac = 0,\ a < 0$

repeated root

asymptotes: $y = 0$ and $x = \dfrac{-b}{2a}$

repeated root

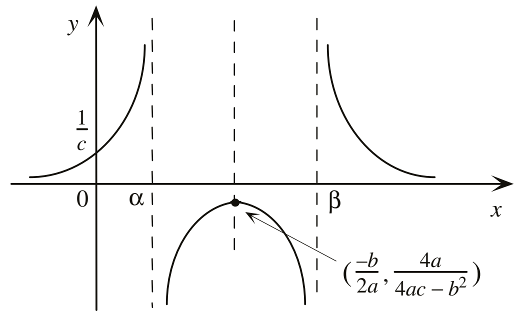


e $b^2 - 4ac > 0,\ a > 0$

Let $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

vertical asymptotes: $x = \alpha$ and $x = \beta$

horizontal asymptote: $y = 0$



11 $y = ax^2 + \dfrac{b}{x^2},\ a,\ b \in R^+$

a $\dfrac{dy}{dx} = 2ax - \dfrac{2b}{x^3} \left(= \dfrac{2a}{x^3}\left(x^4 - \dfrac{b}{a}\right)\right)$

b stationary points where $2ax - \dfrac{2b}{x^3} = 0$

$$\therefore x^4 = \frac{b}{a}$$

$$x = \pm\sqrt[4]{\frac{b}{a}} \text{ i.e. } \left(\pm\sqrt[4]{\frac{b}{a}},\ 2(ab)^{\frac{1}{2}}\right)$$

For $x = \sqrt[4]{\dfrac{b}{a}}$

x	$\sqrt[4]{\frac{b}{a}} - h$	$\sqrt[4]{\frac{b}{a}}$	$\sqrt[4]{\frac{b}{a}} + h$
$f'(x)$	−ve	0	+ve

$h > 0$

$\therefore$ minimum

For $x = -\sqrt[4]{\dfrac{b}{a}}$

x	$-\sqrt[4]{\frac{b}{a}} - h$	$-\sqrt[4]{\frac{b}{a}}$	$-\sqrt[4]{\frac{b}{a}} + h$
$f'(x)$	−ve	0	+ve

$h < 0$

$\therefore$ minimum

c $y = ax^2 + \dfrac{1}{x^2}$ $\quad\therefore\ b = 1$

Turning points are $\left(\dfrac{1}{\sqrt[4]{a}},\ 2\sqrt{a}\right), \left(-\dfrac{1}{\sqrt[4]{a}},\ \sqrt[2]{a}\right)$.

12 a $f'(x) = -e^{-x}\sin x + e^{-x}\cos x$

When $f'(x) = 0$,

$e^{-x}(\cos x - \sin x) = 0$

$\therefore \cos x = \sin x$

$\therefore \tan x = 1$

$\therefore x = \frac{\pi}{4} + n\pi,\ n \in Z$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$, as dom $f = [0, 4\pi]$

b $f(a + 2\pi) = e^{-2\pi - a}\sin(2\pi + a)$

$= e^{-2\pi}(e^{-a}\sin a)$

$\therefore f(a + 2\pi) : f(a) = e^{-2\pi} : 1$

c Stationary points found in **a** are at $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

Nature of stationary points

x	0	$\frac{\pi}{4}$	π	$\frac{5\pi}{4}$	2π	$\frac{9\pi}{4}$	3π	$\frac{13\pi}{4}$	4π
$f'(x)$	+ve	0	−ve	0	+ve	0	−ve	0	+ve

f has local maximum at $x = \frac{\pi}{4}$ and $\frac{9\pi}{4}$, i.e., $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}e^{\frac{-\pi}{4}}\right), \left(\frac{9\pi}{4}, \frac{\sqrt{2}}{2}e^{\frac{-9\pi}{4}}\right)$

f has local minimum at $x = \frac{5\pi}{4}$ and $\frac{13\pi}{4}$, i.e., $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}e^{\frac{-5\pi}{4}}\right), \left(\frac{13\pi}{4}, -\frac{\sqrt{2}}{2}e^{\frac{-13\pi}{4}}\right)$

d $\frac{d}{dx}\left(-\frac{1}{2}e^{-x}(\cos x + \sin x)\right) = \frac{1}{2}(e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x))$

$= \frac{1}{2}e^{-x} \times 2\sin x$

$= e^{-x}\sin x$

$\therefore \int_0^{\pi} e^{-x}\sin x\,dx = \left[-\frac{1}{2}e^{-x}(\cos x + \sin x)\right]_0^{\pi}$

$= -\frac{1}{2}e^{-\pi}(-1 + 0) - \left(-\frac{1}{2}e^{0}(1 + 0)\right)$

$= \frac{1}{2}(e^{-\pi} + 1)$

$= \frac{1 + e^{\pi}}{2e^{\pi}}$

e From **b** $f(x+2\pi) = f(x) \times e^{-2\pi}$

$$\therefore \int_0^{\pi} f(x + 2\pi)dx = \int_0^{\pi} f(x) \times e^{-2\pi} dx$$

Let $y = x + 2\pi$,

$$\therefore \int_{2\pi}^{3\pi} f(y)dy = e^{-2\pi} \times \frac{1}{2}(e^{-\pi} + 1)$$

$$= \frac{e^{-2\pi}}{2}(e^{-\pi} + 1)$$

$$= \frac{1 + e^{\pi}}{2e^{3\pi}}$$

13 a $$\int_0^{\frac{\pi}{4}} \tan^4\theta \sec^2\theta \, d\theta = \left[\frac{\tan^5\theta}{5}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{5}$$

b $$\int_0^{\frac{\pi}{4}} \tan^6\theta \, d\theta = \int_0^{\frac{\pi}{4}} \tan^4\theta(\sec^2\theta - 1)\, d\theta$$

$$= -\int_0^{\frac{\pi}{4}} \tan^6\theta \, d\theta + \int_0^{\frac{\pi}{4}} \tan^4\theta \sec^2\theta \, d\theta$$

$$= \frac{1}{5} - \int_0^{\frac{\pi}{4}} \tan^4\theta \, d\theta$$

c $$\int_0^{\frac{\pi}{4}} \tan^4\theta \, d\theta = \int_0^{\frac{\pi}{4}} -\tan^2\theta + \tan^2\theta\sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 1 - \sec^2\theta + \tan^2\theta \sec^2\theta \, d\theta$$

$$= \left[\theta - \tan\theta + \frac{\tan^3\theta}{3}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - 1 + \frac{1}{3}$$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^6\theta \, d\theta = \frac{1}{5} - \left(\frac{\pi}{4} - 1 + \frac{1}{3}\right)$$

$$= \frac{13}{15} - \frac{\pi}{4}$$

14 a i At time t, p is the proportion of the population which has the disease.

$$\frac{dp}{dt} \propto p(1-p)$$

$$\Rightarrow \frac{dp}{dt} = kp(1-p) \text{ for constant } k$$

$$\therefore \frac{dt}{dp} = \frac{1}{kp(1-p)}$$

$$\text{and } t = \frac{1}{k}\int \frac{1}{p} + \frac{1}{1-p}dp$$

$$\therefore t = \frac{1}{k}\log_e \frac{p}{1-p} + c \text{ (since } 0 < p < 1)$$

When $t = 0$, $p = \frac{1}{10}$

$$\therefore c = -\frac{1}{k}\log_e \frac{1}{9}$$

$$\text{and } t = \frac{1}{k}\log_e \frac{9p}{1-p}$$

When $t = 2$, $p = \frac{1}{5}$

$$\therefore 2k = \log_e \frac{9}{4}$$

$$\therefore k = \frac{1}{2}\log_e \frac{9}{4}$$

$$= \log_e \frac{3}{2}$$

ii $$\therefore t\log_e \frac{3}{2} = \log_e \frac{9p}{1-p}$$

$$\therefore \frac{9p}{1-p} = e^{t\log_e \frac{3}{2}}$$

$$= \left(e^{\log_e \frac{3}{2}}\right)^t$$

$$= \left(\frac{3}{2}\right)^t$$

b When $t = 4$, $\frac{9p}{1-p} = \left(\frac{3}{2}\right)^4$

$$\frac{p}{1-p} = \frac{9}{16}$$

$$\therefore p = \frac{9}{25}$$

c $$\frac{9p}{1-p} = \left(\frac{3}{2}\right)^t$$

$$\therefore \frac{1-p}{9p} = \left(\frac{2}{3}\right)^t$$

$$\therefore \frac{1}{p} - 1 = 9\left(\frac{2}{3}\right)^t$$

$$\therefore \frac{1}{p} = 9\left(\frac{2}{3}\right)^t + 1$$

$$\therefore p = \frac{1}{9\left(\frac{2}{3}\right)^t + 1}$$

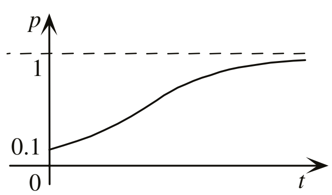
d $$\left(9 + \left(\frac{3}{2}\right)^t\right)p = \left(\frac{3}{2}\right)^t$$

$$p > \frac{1}{2} \text{ implies } \left(\frac{3}{2}\right)^t > \frac{1}{2}\left(9 + \left(\frac{3}{2}\right)^t\right)$$

$$\text{and } \left(\frac{3}{2}\right)^t > 9$$

$$\Rightarrow t > \frac{\log_e 9}{\log_e 1.5},\ t > 5.419$$

e



15 a

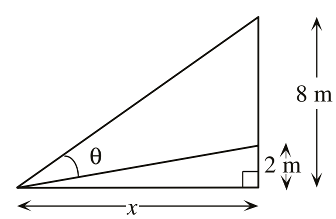
$$v\frac{dv}{dx} = \frac{p}{v} - kv^2$$
$$= \frac{p - kv^3}{v}$$
$$\therefore \frac{dv}{dx} = \frac{p - kv^3}{v^2}$$
$$\text{and } x = \int \frac{v^2}{p - kv^3}dv$$
$$\therefore x = \frac{-1}{3k}\log_e(p - kv^3) + c, \ p - kv^3 > 0$$

When $v = 0$, $x = 0$

$$\therefore c = \frac{1}{3k}\log_e p$$
$$\therefore x = \frac{-1}{3k}\log_e\left(\frac{p - kv^3}{p}\right)$$
$$e^{-3kx} = \frac{p - kv^3}{p}$$
$$\therefore v^3 = \frac{p}{k}(1 - e^{-3kx})$$

b $\lim_{x\to\infty} v = 3\sqrt{\frac{p}{k}}$

16 a $\theta = \tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right), \ x > 0$



b

$$\therefore \frac{d\theta}{dx} = \frac{-8}{x^2} \times \frac{1}{1 + \left(\frac{8}{x}\right)^2} - \left(\frac{-2}{x^2}\right) \times \frac{1}{1 + \left(\frac{2}{x}\right)^2}$$
$$= \frac{-8}{x^2 + 64} + \frac{2}{x^2 + 4}$$

c $\frac{d\theta}{dx} = 0$ when $\frac{-8(x^2 + 4) + 2(x^2 + 64)}{(x^2 + 64)(x^2 + 4)} = 0$

implies $-8x^2 - 32 + 2x^2 + 128 = 0$

$$-6x^2 + 96 = 0$$
$$\therefore x^2 = 16$$
$$x = 4 \text{ (positive value taken)}$$

Test for maximum

x	< 4	4	> 4
$\frac{d\theta}{dx}$	+ve	0	−ve
	/	—	\

$\therefore$ θ is a maximum when observer stands 4 metres from the screen.

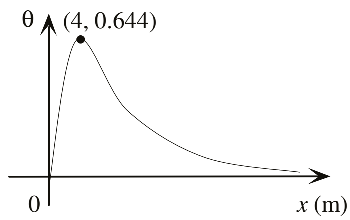
When $x = 4$, $\theta = \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right)$

$= 0.643\,50\ldots$

$\therefore$ allowable values for θ are $0 < \theta < 0.644$.

(Note that in Question **12 b ii** below, we find that $\tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{3}{4}\right)$, so we could write $0 < \theta < \tan^{-1}\left(\frac{3}{4}\right)$.)

d



e θ is a minimum for $1 \le x \le 25$ at either side of the two ends.

When $x = 1$, $\theta = \tan^{-1}(8) - \tan^{-1}(2)$

$= 0.339\,29\ldots$

When $x = 25$, $\theta = \tan^{-1}\left(\frac{8}{25}\right) - \tan^{-1}\left(\frac{2}{25}\right)$

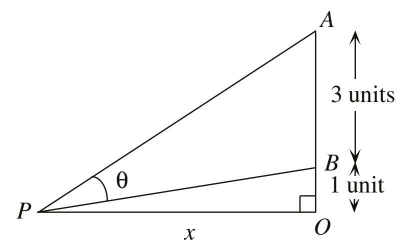
$= 0.229\,87\ldots$

Minimum $\theta = \tan^{-1}\left(\frac{8}{25}\right) - \tan^{-1}\left(\frac{2}{25}\right) \approx 0.23$

17 a $\theta = \angle OBP - \angle OAP$ (exterior angle of triangle theorem)

$= \tan^{-1}(x) - \tan^{-1}\left(\frac{x}{4}\right)$ where $x = OP$

b **i**



$$\frac{d\theta}{dx} = \frac{1}{1+x^2} - \frac{4}{4^2+x^2}$$

$$\frac{d\theta}{dx} = 0$$

$$\Rightarrow \frac{16 + x^2 - 4(1+x^2)}{(1+x^2)(16+x^2)} = 0$$

$$\Rightarrow 16 + x^2 - 4 - 4x^2 = 0$$

$$\therefore 3x^2 = 12$$

$$\therefore x^2 = 4$$

$\therefore x = 2$ taking positive value

x	1	2	3
$\frac{d\theta}{dx}$	$\frac{1}{2} - \frac{4}{17} = +\text{ve}$	0	$\frac{1}{10} - \frac{4}{25} = -\text{ve}$
	/	—	\

$\therefore$ maximum when $x = 2$

ii $\therefore$ maximum value for $\theta = \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right)$

Let $\tan^{-1}(2) = \alpha$ and $\tan^{-1}\left(\frac{1}{2}\right) = \beta$

$$\therefore \tan\alpha = 2 \text{ and } \tan\beta = \frac{1}{2}$$

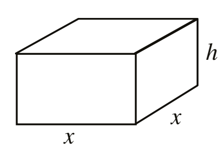
$$\text{Now } \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{2 - \frac{1}{2}}{1+1}$$

$$= \frac{3}{4}$$

$$\therefore \theta = \alpha - \beta = \tan^{-1}\left(\frac{3}{4}\right) \text{ as } \theta,\ \alpha,\ \beta < \frac{\pi}{2}$$

18 **a** Let h be the height (in metres) of the tank.

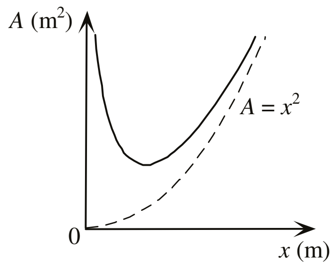


$V = x^2h$ and $V = 4000$

$$\therefore x^2h = 4000$$

$$\therefore h = \frac{4000}{x^2}$$

$$A = x^2 + 4xh$$
$$= x^2 + 4x \times \frac{4000}{x^2}$$
$$= x^2 + \frac{16\,000}{x}, \text{ as required.}$$

b 

c Use a CAS calculator to solve $x^2 + \dfrac{16\,000}{x} = 2500$.
This gives $x \approx 6.5103$ and $x \approx 46.4259$
When 2500 m^2 of sheet metal is used, x may be 6.51 m or 46.43 m, correct to two decimal places.

d From a CAS calculator, the minimum is located at (20, 1200). Therefore A is a minimum when $x = 20$.
Using calculus, $\dfrac{dA}{dx} = 2x - \dfrac{16\,000}{x^2}$
and $\dfrac{dA}{dx} = 0$ where A is a minimum,
$$\therefore 2x - \frac{16\,000}{x^2} = 0$$
$$\therefore 2x = \frac{16\,000}{x^2}$$
$$\therefore 2x^3 = 16\,000$$
$$\therefore x^3 = 8000$$
$$\therefore x = 20$$

19 Dimensions of box are $x \times 3x \times l$.
Volume $= 3x^2 l = 288$
$$\therefore l = \frac{96}{x^2}$$

The surface area, $A = 2(x \times 3x + x \times l + 3x \times l)$

$$= 2(3x^2 + 4xl)$$

$$= 6x^2 + 8x \times \frac{96}{x^2} = 6x^2 + \frac{768}{x}$$

Minimum where $\frac{dA}{dx} = 0$,

i.e. $12x - \frac{768}{x^2} = 0$

$$\therefore x^3 = 64$$

$$\therefore x = 4$$

x	3	4	5
$\frac{dA}{dx}$	$36 - \frac{256}{3} = -\text{ve}$	0	$60 - 30.72 = +\text{ve}$
	\	—	/

$\therefore$ minimum when $x = 4$

The minimum surface area of box $= 6 \times 16 + \frac{768}{4}$

$$= 96 + 192 = 288 \text{ cm}^2$$

20 **a** $y = kx^2$

Where $x = 10, y = 40$, $\therefore\ k = \frac{2}{5}$

$$y = \frac{2}{5}x^2,\ -10 \le x \le 10$$

b $\text{Volume} = 60 \times 2 \times \int_0^y x\, dy$

$$= 60 \times \sqrt{10} \times \int_0^y y^{\frac{1}{2}} dy$$

$$= 60\sqrt{10} \times \frac{2}{3}y^{\frac{3}{2}} + c$$

$$= 40\sqrt{10}y^{\frac{3}{2}} + c$$

$V = 0$ when $y = 0$, $\therefore\ c = 0$

$$\therefore \text{Volume} = 40\sqrt{10}y^{\frac{3}{2}}$$

c $\text{Full volume} = 40^{\frac{5}{2}}\sqrt{10}$

When half full $\frac{1}{2} \times 40^{\frac{5}{2}}\sqrt{10} = 40\sqrt{10}y^{\frac{3}{2}}$

$$\therefore y^{\frac{3}{2}} = \frac{40^{\frac{3}{2}}}{2}$$

$$\therefore y \approx 25.198\ldots$$

∴ Depth is 252 mm.

d

$$\frac{dV}{dt} = 60$$

$$\frac{dV}{dy} = 60\sqrt{10}y^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dt} = \frac{dy}{dv} \times \frac{dv}{dt}$$

$$= \frac{1}{60\sqrt{10}y^{\frac{1}{2}}} \times 60$$

$$= \frac{1}{\sqrt{10y}}$$

$$\therefore \frac{dt}{dy} = \sqrt{10}y^{\frac{1}{2}}$$

$$\therefore t = \frac{2\sqrt{10}}{3}y^{\frac{3}{2}} + c$$

When $y = 0$, $t = 0$,

$$\therefore c = 0$$

$$\text{and } t = \frac{2\sqrt{10}}{3}y^{\frac{3}{2}}$$

e **i** When $y = 20$, $t = \frac{2\sqrt{10}}{3} \times 20^{\frac{3}{2}}$

$$= 188.56$$

Time taken is approximately 189 seconds, or 3 minutes and 9 seconds.

ii When $y = 40$, $t = \frac{2\sqrt{10}}{3} \times 40^{\frac{3}{2}}$

$$\approx 533.33$$

$533.33 - 188.56 = 344.77$

Further time taken is 345 seconds, or 5 minutes and 45 seconds.

21 a For A, $\dfrac{dv}{dt} = -\dfrac{1}{400}v^3$

$$\therefore \frac{dt}{dv} = \frac{-400}{v^3}$$

$$\therefore t = -400 \int \frac{1}{v^3} dv$$

$$= \frac{-400}{-2v^2} + c_1$$

When $t = 0$, $v = 20$,

$$\therefore c_1 = -\frac{1}{2}$$

$$\text{i.e. } t = \frac{200}{v^2} - \frac{1}{2}$$

$$\therefore \frac{2t+1}{2} = \frac{200}{v^2}$$

$$\frac{1}{v^2} = \frac{1}{400}(2t+1)$$

$$v_A = \frac{20}{\sqrt{2t+1}}$$

For B, $\dfrac{dv}{dt} = -\dfrac{1}{100}v^2$

$$\therefore \frac{dt}{dv} = \frac{-100}{v^2}$$

$$\therefore t = \frac{100}{v} + c_2$$

When $t = 0$, $v = 10$,

$$\therefore c_2 = -10$$

$$\text{i.e. } t = \frac{100}{v} - 10$$

$$\therefore t + 10 = \frac{100}{v}$$

$$v_b = \frac{100}{t+10}$$

b Let position be defined by x_A and x_b.

$$x_A = \int \frac{20}{\sqrt{2t+1}} dt$$

$$= 20\sqrt{2t+1} + c_3$$

When $t = 0,\ x_A = 0,$

$$\therefore c_2 = -20$$

$$\text{i.e. } x_A = 20(\sqrt{2t+1} - 1)$$

$$\text{Now } x_b = \int \frac{100}{t+10}\,dt$$

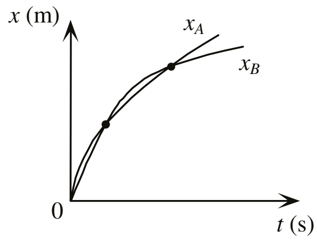
$$= 100\log_e(t+10) + c_4,\ t + 10 > 0$$

When $t = 0,\ x_b = 0,$

$$\therefore c_4 = -100\log_e 10$$

$$\text{i.e. } x_b = 100\log_e\left(\frac{t+10}{10}\right)$$

c



d Use a CAS calculator to solve $x_A = x_b$ for t. This gives $t \approx 14.3515$ and $t \approx 43.8514$. (If your CAS calculator gives just one solution, use the graph to specify limits on t in order to find the second solution. Therefore object B passes object A after 14 seconds, and object A passes object B after 44 seconds, to the nearest second.

22 a $5\dfrac{dv}{dt} + v = 50$

$$\therefore \frac{dv}{dt} = \frac{50 - v}{5}$$

$$\text{and } \frac{dt}{dv} = \frac{5}{50 - v}$$

$$\therefore t = \int \frac{5}{50 - v}\,dt$$

$$= -5\log_e(50 - v) + c,\ 50 - v > 0$$

When $t = 0,\ v = 0,$

and therefore $c = -5\log_e 50$

$$\therefore t = 5\log_e\left(\frac{50}{50 - v}\right)$$

$$\text{and } e^{\frac{t}{5}} = \frac{50}{50 - v}$$

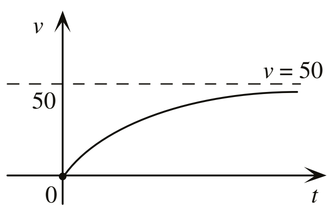
$$50 - v = 50_e^{\frac{-t}{5}}$$

$$v = 50(1 - e^{\frac{-t}{5}})$$

b When $t = 47.5,\ v = 50(1 - e^{-9.5})$

$$= 49.9963$$

c



d **i** $$x = 50(t + 5e^{\frac{-t}{5}}) + c$$

$t = 0,\ x = 0,\ c = -250$

$$\therefore x = 50(t + 5e^{\frac{-t}{5}} - 5)$$

ii When $t = 6,\ x = 50(1 + 5e^{-1.2})$

$$= 125.2986$$

23 a $$\frac{dy}{dt} = \frac{2y(N-y)}{N}$$

$$\therefore \frac{1}{2}\int \frac{N}{y(N-y)}dy = t$$

$$\therefore \frac{1}{2}\int \frac{1}{y} + \frac{1}{N-y}dy = t$$

$$\therefore c + \log_e\left(\frac{y}{N-y}\right) = 2t$$

When $t = 0, y = \frac{N}{4}$,

$$\therefore c = -\log_e \frac{1}{3}$$

$$= \log_e 3$$

$$\therefore \log_e\left(\frac{3y}{N-y}\right) = 2t$$

$$3y = (N-y)e^{2t}$$

$$\therefore y = \frac{Ne^{2t}}{3+e^{2t}}$$

$$\frac{dy}{dt} = \frac{(3+e^{2t}) \times 2Ne^{2t} - Ne^{2t} \times 2e^{2t}}{(3+e^{2t})^2}$$

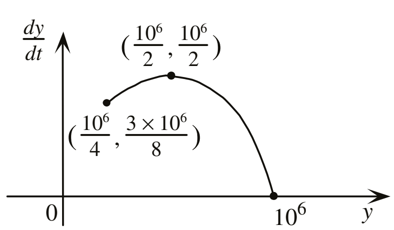
$$\therefore \frac{dy}{dt} = \frac{6Ne^{2t}}{(3+e^{2t})^2}$$

b $t \to \infty, y \to N$ since $y = N - \frac{3N}{e^{2t}+3}$

c $\frac{dy}{dt} = \frac{6Ne^{2t}}{(3+e^{2t})^2}$ which is always positive, hence the population is always increasing.

d When $y = \frac{N}{2}$, $\left(\frac{dy}{dt}\text{ maximum, i.e., } y(N-y)\text{ maximum}\right)$.

e **i**



ii $\dfrac{N}{2} = \dfrac{Ne^{2t}}{3 + e^{2t}}$

$3 + e^{2t} = 2e^{2t}$

$$t = \frac{1}{2}\log_e 3 \approx 0.549306$$

24 $a = \dfrac{-gR^2}{x^2}$

a i $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{-gR^2}{x^2}$$

$$\therefore \frac{1}{2}v^2 = \frac{gR^2}{x} + C$$

When $x = R$ and $v = u$ (initially),

$$c = \frac{1}{2}u^2 - gR$$

$$\therefore v^2 = \frac{2gR^2}{x} + u^2 - 2gR$$

ii $v = 0$ implies $\dfrac{2gR^2}{x} + u^2 - 2gR = 0$

$$\therefore \frac{2gR^2}{x} = 2gR - u^2$$

$$\therefore x = \frac{2gR^2}{2gR - u^2}$$

iii When $u \geq \sqrt{2gR}$.

b For $v \neq 0$, we need $u^2 \geq 2gR$

so, minimum value of $u = \sqrt{2gR}$

When $g = 9.8$ and $R = 6.4 \times 10^6$,

$$u = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

$$= 11\,200 \text{ m/s}$$

$$= 11\,200 \times \frac{3600}{1000} \text{ km/h}$$

$$= 40\,320$$

Escape velocity is 40 320 km/h.

25 $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

a $f(0) = 0$

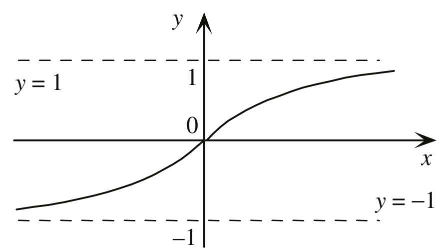
b $\lim\limits_{x\to\infty} f(x) = 1$ as $e^{-x} \to 0$

c $\lim\limits_{x\to-\infty} f(x) = -1$ as $e^{x} \to 0$

d $f'(x) = \dfrac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$

$= \dfrac{4}{(e^x + e^{-x})^2}$ always positive.

e $f'(x) > 0,\ x \in R$



f Let $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\therefore ye^x + ye^{-x} = e^x - e^{-x}$

$(y - 1)e^x = -(1 + y)e^{-x}$

$\therefore e^{2x} = \dfrac{-(1 + y)}{y - 1}$

$= \dfrac{1 + y}{1 - y}$

$x = \dfrac{1}{2}\log_e\left(\dfrac{1 + y}{1 - y}\right)$

$\therefore f^{-1}(x) = \dfrac{1}{2}\log_e\left(\dfrac{1 + x}{1 - x}\right)$ for $-1 < x < 1$

g $g'(x) = \frac{1}{2} \times \frac{1-x}{1+x} \times \frac{(1-x)-(1+x)\times -1}{(1-x)^2}$

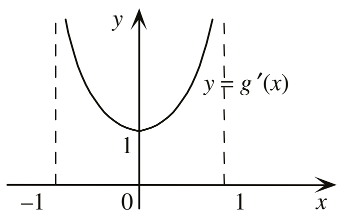
$= \frac{1}{2} \times \frac{1-x}{1+x} \times \frac{2}{(1-x)^2}$

$= \frac{1}{1-x^2}$

or $g(x) = \frac{1}{2}(\log_e(1+x) - \log_e(1-x))$

$\therefore g'(x) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)$

h



Required to prove $\int_0^{\frac{1}{2}} g'(x)\,dx = \log_e \sqrt{3}$

Now $\int_0^{\frac{1}{2}} g'(x)\,dx = [g(x)]_0^{\frac{1}{2}}$

$= \frac{1}{2}\log_e\left(\frac{1.5}{0.5}\right) - \frac{1}{2}\log_e 1$

$= \frac{1}{2}\log_e 3$

$= \log_e 3^{\frac{1}{2}}$

26 a i $y = 2r\sin\left(\frac{1}{2}\theta\right)$

ii $\cos\theta = \frac{r}{r+h}$

b **i**
$$\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dh} \times \frac{dh}{dt}$$
$$\text{where } \frac{dy}{d\theta} = r\cos\left(\frac{1}{2}\theta\right)$$
$$\text{and } h = \frac{r}{\cos\theta} - r$$
$$\text{which implies } \frac{dh}{d\theta} = \frac{r\sin\theta}{\cos^2\theta}$$
$$\text{i.e. } \frac{d\theta}{dh} = \frac{\cos^2\theta}{r\sin\theta}$$
$$\therefore \frac{dy}{dt} = r\cos\left(\frac{1}{2}\theta\right) \times \frac{\cos^2\theta}{r\sin\theta} \times r\sin t$$
$$= \frac{r\cos\left(\frac{1}{2}\theta\right)\cos^2\theta\ \sin t}{\sin\theta}$$

ii
$$h = -r\cos t + c$$
$$t = 0,\ h = 0,$$
$$\therefore c = r$$
$$\therefore h = r - r\cos t$$
$$\therefore \text{ when } t = \frac{\pi}{2},\ h = r$$

The height is 6000 km.

iii When $t = \frac{\pi}{2},\ \theta = \frac{\pi}{3}$ and $r = 6000$,
$$\therefore \frac{dy}{dt} = \frac{6000\cos\frac{\pi}{6}\cos^2\frac{\pi}{3}\sin\frac{\pi}{2}}{\sin\frac{\pi}{3}}$$
$$= 1500 \text{ km/h}$$

27 **a**
$$f(x) = e^{-x}x^n$$
$$\therefore f'(x) = nx^{n-1}e^{-x} - e^{-x}x^n$$
$$\therefore e^{-x}x^n = \int nx^{n-1}e^{-x}dx - \int e^{-x}x^n dx$$
$$\text{i.e. } \int e^{-x}x^n dx = n\int x^{n-1}e^{-x}dx - e^{-x}x^n$$

b $g : R^+ \to R,\ g(n) = \int_0^\infty e^{-x}x^n dx$

i $g(0) = \int_0^\infty e^{-x}x^0 dx$

$= \int_0^\infty e^{-x} dx$

$= \lim\limits_{b\to\infty} [-e^{-x}]_0^b$

$= \lim\limits_{b\to\infty} (-e^{-b} + e^0)$

$= 1$, as $\lim\limits_{b\to\infty} e^{-b} = 0$

ii From above, $\int_0^\infty f'(x)\,dx = \int_0^\infty nx^{n-1}e^{-x}dx - \int_0^\infty e^{-x}x^n dx$

$\lim\limits_{b\to\infty} [e^{-x}x^n]_0^b = ng(n-1) - g(n)$

$\lim\limits_{b\to\infty} e^{-x}x^n = 0$

$\therefore ng(n-1) - g(n) = 0$ or $g(n) = ng(n - l)$

iii If $g(n) = ng(n-1)$,
then $g(n) = n \times (n-1)g(n-2)$

$= n(n-1)(n-2)g(n-3)$

$= n(n-1) \times \ldots \times 2 \times 1g(0)$

$= n! \times 1$ since $g(O) = 1$

$= n!$

28 a $V_{\text{cone}} = \frac{1}{3}\pi r^2 \times 2r \quad V_{\text{hemi}} = \frac{2}{3}\pi r^3$

$\therefore$ total volume $V = \frac{4}{3}\pi r^3,\ r \geq 2$

b $\frac{dV}{dt} = -t^2$ (m^3/min)

$\frac{dV}{dr} = 4\pi r^2$ and $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\therefore 4\pi r^2 \frac{dr}{dt} = -t^2$

c $\int 4\pi r^2 \frac{dr}{dt} dt = \int -t^2 dt$

$\therefore \int 4\pi r^2 dr = \int -t^2 dt$

$\therefore \frac{4}{3}\pi r^3 = \frac{-t^3}{3} + c$

$t = 0$ when $r = 10$,

$\therefore c = \frac{4000\pi}{3}$

i.e. $\frac{4}{3}\pi r^3 = \frac{4000\pi - t^3}{3}$

or $4\pi r^3 = 4000\pi - t^3$

$\therefore r = \left(\frac{4000\pi - t^3}{4\pi}\right)^{\frac{1}{3}}$

d When $r = 2$, $t^3 = 4000\pi - 32\pi$

$= 3968\pi$

$\therefore t = (3968\pi)^{\frac{1}{3}}$

≈ 23.2

The time taken is 23.2 minutes.

29 **a** average velocity $= \frac{\text{displacement}}{\text{time taken}}$

$\boldsymbol{r}_1(0) = -2\boldsymbol{j}$

$\boldsymbol{r}_1(10) = 20\boldsymbol{i} - 102\boldsymbol{j}$

displacement $= \boldsymbol{r}_1(10) - \boldsymbol{r}_1(0)$

$= 20\boldsymbol{i} - 102\boldsymbol{j} + 2\boldsymbol{j}$

$= 20\boldsymbol{i} - 100\boldsymbol{j}$

$\therefore$ average velocity $= (20\boldsymbol{i} - 100\boldsymbol{j}) \div 10$

$= 2\boldsymbol{i} - 10\boldsymbol{j}$

b $\dot{\boldsymbol{r}}_1(t) = 2\boldsymbol{i} - 2t\boldsymbol{j}$

c When $t = 3$, $\dot{\boldsymbol{r}}_1(t) = 2\boldsymbol{i} - 6\boldsymbol{j}$

$= 2(\boldsymbol{i} - 3\boldsymbol{j})$

i.e., direction is determined by the vector $\boldsymbol{i} - 3\boldsymbol{j}$

d speed $= \sqrt{4 + 4t^2}$

Speed is minimum when $4 + 4t^2$ is a minimum.
Minimum when $t = 0$.

e average velocity $= \dfrac{\text{displacement}}{\text{time}}$

$= 2\boldsymbol{i} - 10\boldsymbol{j}$

When $2\boldsymbol{i} - 2t\boldsymbol{j} = 2\boldsymbol{i} - 10\boldsymbol{j}$

i.e., when $t = 5$,
the velocity is equal to the average velocity.

f if the particles are coincident,

$(t^3 - 4)\boldsymbol{i} - 3t\boldsymbol{j} = 2t\boldsymbol{i} - (t^2 + 2)\boldsymbol{j}$

i.e. $t^3 - 4 = 2t$ and $-3t = -t^2 - 2$

$\therefore\ t^3 - 2t - 4 = 0$ and $t^2 - 3t + 2 = 0$

$\therefore\ (t - 2)(t^2 + 2t + 2) = 0$ and $(t - 1)(t - 2) = 0$

$\therefore\ t = 2$ is a solution.

Note: $t = 1$ is not a solution of $t^2 + 2t + 2 = 0$.

30 $\ddot{\boldsymbol{r}}(t) = -16(\cos 4t\boldsymbol{i} + \sin 4t\boldsymbol{j})$

a $\dot{\boldsymbol{r}}(t) = 4(\sin 4t\boldsymbol{i} + \cos 4t\boldsymbol{j}) + \boldsymbol{c}$

When $t = 0, \dot{\boldsymbol{r}}(0) = 4\boldsymbol{j}$,

$\therefore\ 4\boldsymbol{j} = 4\boldsymbol{j} + \boldsymbol{c}$

$\therefore\ \boldsymbol{c} = \boldsymbol{0}$

$\therefore\ \dot{\boldsymbol{r}}(t) = -\sin 4t\boldsymbol{i} + \cos 4t\boldsymbol{j}$

$\therefore\ \boldsymbol{r}(t) = \cos 4t\boldsymbol{i} + \sin 4t\boldsymbol{j} + \boldsymbol{d}$

Now $\boldsymbol{r}(0) = \boldsymbol{j}$,

$\therefore\ \boldsymbol{j} = \boldsymbol{i} + \boldsymbol{d}$

$\therefore\ \boldsymbol{j} - \boldsymbol{i} = \boldsymbol{d}$

$\therefore\ \boldsymbol{r}(t) = (\cos 4t - 1)\boldsymbol{i} + (\sin 4t + 1)\boldsymbol{j}$

b $x = \cos 4t - 1$ and $y = \sin 4t + 1$

$x + 1 = \cos t$ $\quad y - 1 = \sin 34t$

$\therefore (x + 1)^2 + (y - 1)^2 = 1$

Position vector of the centre is $-\boldsymbol{i} + \boldsymbol{j}$

c $\ddot{\boldsymbol{r}}.\dot{\boldsymbol{r}} = 0$

Therefore, $\ddot{\boldsymbol{r}}$ is perpendicular to $\dot{\boldsymbol{r}}$, i.e., acceleration is always perpendicular to velocity.

31 $\boldsymbol{r} = 18\cos\left(\frac{t}{3}\right)\boldsymbol{i} + 13.5\sin\left(\frac{t}{3}\right)\boldsymbol{j}$

a When $18\boldsymbol{i} = \boldsymbol{r}$, $18\boldsymbol{i} = 18\cos\left(\frac{t}{3}\right)\boldsymbol{i} + 13.5\sin\left(\frac{t}{3}\right)\boldsymbol{j}$

$$\therefore\ 18 = 18\cos\left(\frac{t}{3}\right)$$

$$\Rightarrow \cos\left(\frac{t}{3}\right) = 1$$

$$\Rightarrow \frac{t}{3} = 0, 2\pi, 4\pi, \ldots$$

$$\Rightarrow t = 0, 6\pi,\ 12\pi, \ldots$$

The skater takes 6π seconds to go around the rink once.

b **i**

$$\dot{\boldsymbol{r}} = -6\sin\left(\frac{t}{3}\right)\boldsymbol{i} + 4.5\cos\left(\frac{t}{3}\right)\boldsymbol{j}$$

When $t = 2\pi$, $\dot{\boldsymbol{r}} = -6\sin\left(\frac{2\pi}{3}\right)\boldsymbol{i} + 4.5\cos\left(\frac{2\pi}{3}\right)\boldsymbol{j}$

$$= -6\left(\frac{\sqrt{3}}{2}\right)\boldsymbol{i} + \frac{9}{2}\times\left(\frac{-1}{2}\right)\boldsymbol{j}$$

$\therefore$ for $t = 2\pi$, velocity $= -3\sqrt{3}\boldsymbol{i} - \frac{9}{4}\boldsymbol{j}$

ii

$$\ddot{\boldsymbol{r}} = -2\cos\left(\frac{t}{3}\right)\boldsymbol{i} - 1.5\sin\left(\frac{t}{3}\right)\boldsymbol{j}$$

When $t = 2\pi$, $\ddot{\boldsymbol{r}}(2\pi) = -2\cos\left(\frac{2\pi}{3}\right)\boldsymbol{i} - 1.5\sin\left(\frac{2\pi}{3}\right)\boldsymbol{j}$

$$= \boldsymbol{i} - \frac{3\sqrt{3}}{4}\boldsymbol{j}$$

c **i** speed $= \sqrt{36\sin^2\left(\frac{t}{3}\right) + \frac{81}{4}\cos^2\left(\frac{t}{3}\right)}$

$= \sqrt{36\sin^2\left(\frac{t}{3}\right) + \frac{81}{4}\left(1 - \sin^2\left(\frac{t}{3}\right)\right)}$

$= \sqrt{\frac{81}{4} + \frac{144 - 81}{4}\sin^2\left(\frac{t}{3}\right)}$

$= \frac{1}{2}\sqrt{81 + 63\sin^2\left(\frac{t}{3}\right)}$

$= \frac{3}{2}\sqrt{9 + 7\sin^2\left(\frac{t}{3}\right)}$

ii Maximum when $\sin^2\left(\frac{t}{3}\right) = 1$, $\left(\text{maximum} = \frac{3}{2} \times 4 = 6\right)$.

i.e., when $\sin\left(\frac{t}{3}\right) = \pm 1$

$$\frac{t}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\text{i.e. } t = \frac{3\pi}{2}, \frac{9\pi}{2}, \frac{15\pi}{2}, \ldots$$

$$= 3\left(\frac{\pi}{2} + n\pi\right),\ n \in N \cup \{0\}$$

The skater's speed is greatest when $t = 3\left(\frac{\pi}{2} + n\pi\right)$, $n \in N \cup \{0\}$.

d

$$\ddot{\boldsymbol{r}} = -2\cos\left(\frac{t}{3}\right)\boldsymbol{i} + 1.5\sin\left(\frac{t}{3}\right)\boldsymbol{j}$$

$$= -\frac{1}{9}\boldsymbol{r}$$

acceleration magnitude $= \sqrt{4\cos^2\left(\frac{t}{3}\right) + \frac{9}{4}\sin^2\left(\frac{t}{3}\right)}$

$= \sqrt{4\left(1 - \sin^2\left(\frac{t}{3}\right)\right) + \frac{9}{4}\sin^2\left(\frac{t}{3}\right)}$

$= \sqrt{4 - \frac{7}{4}\sin^2\left(\frac{t}{3}\right)}$

$\therefore$ acceleration is a maximum when $\sin\left(\frac{t}{3}\right) = 0$

which implies $t = 0, 3\pi,\ 6\pi, \ldots$

$$t = 3n\pi,\ n \in\ N \cup \{0\}$$

32 a i $\dot{\boldsymbol{r}}_1(t) = 3\cos 2t\boldsymbol{i} + 4\sin 2t\boldsymbol{j}$

$$\boldsymbol{r}_1(t) = \frac{3}{2}\sin 2t\boldsymbol{i} - 2\cos 2t\boldsymbol{j} + \boldsymbol{c}$$

As $\boldsymbol{r}_1(0) = -2\boldsymbol{j},\ \boldsymbol{c} = 0$

$$\therefore\ \boldsymbol{r}_1(t) = \frac{3}{2}\sin 2t\boldsymbol{i} - 2\cos 2t\boldsymbol{j}$$

ii $\ddot{\boldsymbol{r}}_1(t) = -6\sin 2t\boldsymbol{i} + 8\cos 2t\boldsymbol{j}$

iii If displacement and velocity vectors are perpendicular

$\boldsymbol{r}_1 . \dot{\boldsymbol{r}}_1 = 0$ implies

$$\left(\frac{3}{2}\sin 2t\boldsymbol{i} - 2\cos 2t\boldsymbol{j}\right).(3\cos 2t\boldsymbol{i} + 4\sin 2t\boldsymbol{j}) = 0$$

$$\Rightarrow \frac{9}{2}\sin 2t\cos 2t - 8\cos 2t\sin 2t = 0$$

$$\Rightarrow \frac{-7}{2}\sin 2t\cos 2t = 0$$

$$\Rightarrow \sin 4t = 0$$

$$\therefore\ t = \frac{\pi n}{4},\ n \in\ N \cup \{0\}$$

The displacement and velocity vectors are perpendicular when $t = \dfrac{\pi n}{4}, n \in\ N \cup \{0\}$.

iv $x = \dfrac{3}{2}\sin 2t$ and $y = -2\cos 2t$

$$\therefore\ \frac{2x}{3} = \sin 2t \quad \text{and } \frac{y}{-2} = \cos 2t$$

$$\therefore\ \frac{4x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \quad 16x^2 + 9y^2 = 36$$

b $$r_1(t) = \frac{3}{2}\sin 2t\boldsymbol{i} - 2\cos 2t\boldsymbol{j}$$
$$r_2(t) = \frac{3}{2}\sin 2t\boldsymbol{i} + 2\cos 2t\boldsymbol{j} + (a-t)\boldsymbol{k}$$

For $\boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$

$$2\cos 2t = -2\cos 2t, \text{ since } a - t = 0$$
$$\therefore\ 4\cos 2t = 0$$
$$\therefore\ \cos 2t = 0$$
$$\Rightarrow 2t = (2n+1)\frac{\pi}{2},\ n \in N \cup \{0\}$$
$$\therefore\ t = (2n+1)\frac{\pi}{4},\ n \in N \cup \{0\}$$

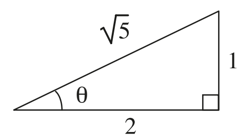
Therefore, $a = (2n+1)\frac{\pi}{4},\ n \in N \cup \{0\}$, for the particles to collide.

33 a 

where $\tan\theta = \frac{1}{2}$

Consider the triangle

$\therefore\ \cos\theta = \frac{2}{\sqrt{5}}, \sin\theta = \frac{1}{\sqrt{5}}$



The displacement vector in km, at time t seconds, is given by.

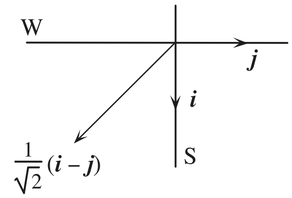
$$\boldsymbol{r} = \left(\frac{225\sqrt{5}}{3600}\cos\theta\right)t\boldsymbol{i} + \left(\frac{225\sqrt{5}}{3600}\sin\theta\right)t\boldsymbol{k}$$
$$= \left(\frac{225\sqrt{5}}{3600} \times \frac{2}{\sqrt{5}}\right)t\boldsymbol{i} + \left(\frac{225\sqrt{5}}{3600} \times \frac{1}{\sqrt{5}}\right)t\boldsymbol{k}$$
$$= \frac{t}{16}(2\boldsymbol{i} + \boldsymbol{k})$$

b i $\boldsymbol{V} = 720\sqrt{2}$ km/h

$= \frac{\sqrt{2}}{5}$ km/s

The unit vector velocity is as shown in the diagram.

$$\therefore\ \boldsymbol{V} = \frac{\sqrt{2}}{5\sqrt{2}}(\boldsymbol{i}-\boldsymbol{j})$$
$$= 0.2(\boldsymbol{i}-\boldsymbol{j})$$



$$\text{i.e. } \frac{d\boldsymbol{r}_2}{dt} = 0.2(\boldsymbol{i}-\boldsymbol{j})$$
$$\text{and } \boldsymbol{r}_2 = 0.2t(\boldsymbol{i}-\boldsymbol{j}) + \boldsymbol{c}$$

When $t = 0$, $\boldsymbol{r}_2 = -1.2\boldsymbol{i} + 3.2\boldsymbol{j} + \boldsymbol{k}$

$$\therefore\ \boldsymbol{r}_2 = 0.2t(\boldsymbol{i}-\boldsymbol{j}) - 1.2\boldsymbol{i} + 3.2\boldsymbol{j} + \boldsymbol{k}$$
$$= (0.2t - 1.2)\boldsymbol{i} + (3.2 - 0.2t)\boldsymbol{j} + \boldsymbol{k}$$

ii $\boldsymbol{r}_2 = \boldsymbol{r}_1$

$$\Rightarrow 0.2t - 1.2 = \frac{2t}{16} \quad ①$$
$$\text{and } \frac{t}{16} = 1 \quad ②$$
$$\text{and } 3.2 - 0.2t = 0 \quad ③$$

When $t = 16$, LHS of ① $= 2$, RHS of ① $= 2$

$\therefore$ LHS = RHS

LHS of ② $= 1$, RHS of ② $= 1$

$\therefore$ LHS = RHS

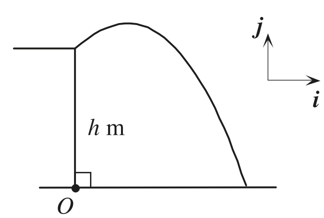
LHS of ③ $= 0$, RHS of ③ $= 0$

$\therefore$ LHS = RHS

Therefore $t = 16$ is a solution of all three equations.

$\therefore$ collision takes place after 16 seconds.

34 a i $h\boldsymbol{j}$, for $0\boldsymbol{i} + 0\boldsymbol{j}$ at the foot of the cliff.



ii initial velocity $= V\cos\alpha\boldsymbol{i} + V\sin\alpha\boldsymbol{j}$

b i $\boldsymbol{a} = -g\boldsymbol{j}$ $\quad \boldsymbol{v} = -gt\boldsymbol{j} + c$

but $\boldsymbol{v} = V\cos\alpha\boldsymbol{i} + V\sin\alpha\boldsymbol{j}$ when $t = 0$

$\therefore\ \boldsymbol{v} = V\cos\alpha\boldsymbol{i} + (V\sin\alpha - gt)\boldsymbol{j}$

ii $\boldsymbol{x} = (V\cos\alpha)t\boldsymbol{i} + \left(h + V(\sin\alpha)t\frac{1}{2} - gt^2\right)\boldsymbol{j}$ is the position vector.

c Highest point is reached when the velocity $\boldsymbol{j}$ component is 0,
i.e. $V\sin\alpha - gt = 0$

$$\therefore\ t = \frac{V\sin\alpha}{g}$$

d When it hits the sea, the position vector $\boldsymbol{j}$ component is equal to zero.
i.e. $h + V(\sin\alpha)t - \frac{1}{2}gt^2 = 0$

$\therefore\ \frac{1}{2}gt^2 - V(\sin\alpha)t - h = 0$

$$\text{i.e. } t = \frac{V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}}{g} \text{ as } t \geq 0$$

35 a For $\boldsymbol{r}(t) = (t-1)\boldsymbol{i} + (t^2-1)\boldsymbol{j}$

Let $x = t - 1$ and $y = t^2 - 1$

Then $t = x + 1$ and $y = (x+1)^2 - 1$

$= x^2 + 2x$

b $\boldsymbol{r}(t) = (e^{-t} - 1)\boldsymbol{i} + (e^{-2t} - 1)\boldsymbol{j}$

Let $x = e^{-t} - 1$ and $y = e^{-2t} - 1$

Then $e^{-t} = x + 1$ and $y = (x+1)^2 - 1$

$= x^2 + 2x$

c i $\boldsymbol{r}_1(0) = -\boldsymbol{i} - \boldsymbol{j}$

and $\boldsymbol{r}_2(0) = 0\boldsymbol{i} + 0\boldsymbol{j}$

ii $\boldsymbol{r}_1'(t) = \boldsymbol{i} + 2t\boldsymbol{j}$

and $\boldsymbol{r}_2'(t) = -e^{-t}\boldsymbol{i} - 2e^{-2t}$

iii $\boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$

$(t-1)\boldsymbol{i} + (t^2-1)\boldsymbol{j} = (e^{-t}-1)\boldsymbol{i} + (e^{-2t}-1)\boldsymbol{j}$
For collision

$t - 1 = e^{-t} - 1$ and $t^2 - 1 = e^{-2t} - 1$

By trial and error, $t = 0.57$,

$\therefore\ \boldsymbol{r}_1(0.57) = -0.43\boldsymbol{i} - 0.68\boldsymbol{j}$

and $\boldsymbol{r}_2(0.57) = -0.43\boldsymbol{i} - 0.68\boldsymbol{j}$

36 a i $\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j}$

ii

$$\dot{\boldsymbol{r}}(t) = 10\boldsymbol{i} + (10\sqrt{3} - 9.8t)\boldsymbol{j}$$

When $t = 0$, $\dot{\boldsymbol{r}}(0) = 10\boldsymbol{i} + 10\sqrt{3}\boldsymbol{j}$

$$\therefore\ |\dot{\boldsymbol{r}}(0)| = \sqrt{100 + 300} = 20$$

Direction given by θ where

$$\tan\theta = \frac{10\sqrt{3}}{10} = \sqrt{3}$$

i.e. $\theta = 60°$ (60° to the horizontal).

iii $\ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$

b i time taken $= \dfrac{x}{10}$

ii position vector, $\boldsymbol{r}\left(\dfrac{x}{10}\right) = x\boldsymbol{i} + \dfrac{x}{10}\left(10\sqrt{3} - 4.9 \times \dfrac{x}{10}\right)\boldsymbol{j}$

$$= x\boldsymbol{i} + (x\sqrt{3} - 0.049x^2)\boldsymbol{j}$$

iii velocity when $t = \dfrac{x}{10}$

$$\dot{\boldsymbol{r}}\left(\frac{x}{10}\right) = 10\boldsymbol{i} + \left(10\sqrt{3} - \frac{9.8x}{10}\right)\boldsymbol{j}$$

$$= 10\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$$

iv velocity, $\dot{\boldsymbol{r}}(t) = -0.8 \times 10\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$

$$= -8\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$$

c i

$$\ddot{\boldsymbol{r}}(t_1) = -9.8\boldsymbol{j}$$

$$\therefore\ \dot{\boldsymbol{r}}(t_1) = -9.8t_1\boldsymbol{j} + \boldsymbol{c}$$

When $t_1 = 0$, $\dot{\boldsymbol{r}}(0) = -8\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$

$$\therefore\ \dot{\boldsymbol{r}}(t_1) = -9.8t_1\boldsymbol{j} + -8\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$$

$$= -8\boldsymbol{i} + (10\sqrt{3} - 0.98x - 9.8t_1)\boldsymbol{j}$$

ii $\boldsymbol{r}(t_1) = -8t_1\boldsymbol{i} + (10\sqrt{3}\,t_1 - 0.98xt_1 - 4.9t_1^2)\boldsymbol{j} + \boldsymbol{c}$

When $t_1 = 0$, $\boldsymbol{r}(t_1) = x\boldsymbol{i} + (x\sqrt{3} - 0.049x^2)\boldsymbol{j}$

$\therefore\ \boldsymbol{r}(t_1) = (x - 8t_1)\boldsymbol{i} + (x\sqrt{3} - 0.049x^2 + 10\sqrt{3}\,t_1 - 0.98xt_1 - 4.9t_1^2)\boldsymbol{j}$

d It hits the ground when the $\boldsymbol{j}$ component is zero,
i.e., when $x\sqrt{3} - 0.049x^2 + 10\sqrt{3}t_1 - 0.98xt_1 - 4.9t_1^2 = 0$

$$\Rightarrow\ 4.9t_1^2 + (0.98x - 10\sqrt{3})t_1 + 0.049x^2 - x\sqrt{3} = 0$$

$$\Rightarrow\ t_1 = \frac{10\sqrt{3} - 0.98x \pm \sqrt{(0.98x - 10\sqrt{3})^2 - 4(0.049x^2 - x\sqrt{3})4.9}}{9.8}$$

$$= \frac{10\sqrt{3} - 0.98x \pm \sqrt{0.9604x^2 - 19.6\sqrt{3}x + 300 - 0.9604x^2 + 19.6x\sqrt{3}}}{9.8}$$

$$= \frac{10\sqrt{3} - 0.98x \pm \sqrt{300}}{9.8}$$

$$= \frac{10\sqrt{3} - 0.98x \pm 10\sqrt{3}}{9.8}$$

$$\therefore\ t_1 = \frac{20\sqrt{3} - 0.98x}{9.8},\ \text{since } t_1 > 0$$

e It will hit the same position if the $\boldsymbol{i}$ component is 0

$$\therefore\ x = 8t_1$$

$$\text{i.e. } x = 8\left(\frac{20\sqrt{3} - 0.98x}{9.8}\right)$$

$$\therefore\ 9.8x + 7.84x = 160\sqrt{3}$$

$$\therefore\ 17.64x = 160\sqrt{3}$$

$$\therefore\ x = \frac{160\sqrt{3}}{17.64}$$

$$\approx 15.71 \text{ metres}$$

37 **a** $\boldsymbol{r}(0) = 5i$

b **i** $\boldsymbol{r}(t_1) = (5 - 3t_1)\boldsymbol{i} + 2t_1\boldsymbol{j} + t_1\boldsymbol{k}$

$\boldsymbol{r}(t_2) = (5 - 3t_2)\boldsymbol{i} + 2t_2\boldsymbol{j} + t_2\boldsymbol{k}$

ii $\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1) = 3(t_1 - t_2)\boldsymbol{i} + 2(t_2 - t_1)\boldsymbol{j} + (t_2 - t_1)\boldsymbol{k}$

$= (t_2 - t_1)(-3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

c The displacement between $\boldsymbol{r}(t_1)$ and $\boldsymbol{r}(t_2)$ is a scalar multiple of $-3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

d **i** $\boldsymbol{i}(-3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = -3 = \sqrt{14}\cos\theta$

$$\therefore \cos\theta = \frac{-3}{\sqrt{14}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-3}{\sqrt{14}}\right) = 143.30°$$

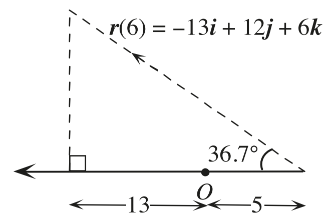
$\therefore$ the acute angle is 36.70°.

ii When $t = 6$,

$$\boldsymbol{r}(6) = -13\boldsymbol{i} + 12\boldsymbol{j} + 6\boldsymbol{k}$$

When $t = 0$,

$$\text{now } r(0) = 5\boldsymbol{i}$$



$$\text{Shortest distance } = 18 \tan 36.7° \text{ km}$$
$$= 13.42 \text{ km}$$

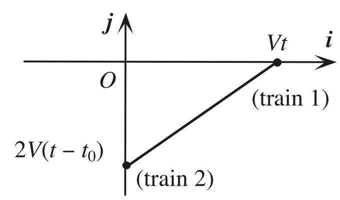
38 $\boldsymbol{r}_1 = Vt\boldsymbol{i}$ and $\boldsymbol{r}_2 = 2V(t - t_0)\boldsymbol{j}$

a **i** When $t = 0$,

$\boldsymbol{r}_1 = \boldsymbol{0}$ and $\boldsymbol{r}_2 = -2Vt_0\boldsymbol{j}$

i.e., train 1 is at O at time $t = 0$

T_1 goes through O first.



ii Train 2 goes through O when

$$\boldsymbol{r}_2 = \boldsymbol{0}$$

i.e. $2V(t - t_0) = 0$

$$\Rightarrow t = t_0$$

Train 2 goes through O, t_0 units of time after train 1.

b **i**
$$\text{distance apart } = \sqrt{V^2t^2 + 4V^2(t - t_0)^2}$$
$$\text{Let } x = V^2t^2 + 4V^2(t - t_0)^2$$

then $\dfrac{dx}{dt} = 0$ implies

$$2tV^2 + 8V^2(t - t_0) = 0$$
$$2t + 8t - 8t_0 = 0$$
$$\therefore 10t = 8t_0$$
$$t = \frac{4}{5}t_0$$

and this is a minimum because $V^2t^2 + 4V^2(t - t_0)^2$ is a quadratic with positive coefficient of t^2.

ii distance apart, $x = \sqrt{V^2\left(\frac{4}{5}t_0\right)^2 + 4V^2\left(\frac{4}{5}t_0 - t_0\right)^2}$

$= \sqrt{\frac{16}{25}t_0^2V^2 + \frac{4t_0^2V^2}{25}}$

$= \sqrt{\frac{20t_0^2V^2}{25}}$

$= \frac{2t_0V\sqrt{5}}{5} = \frac{2\sqrt{5}}{5}t_0V$

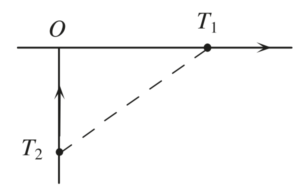
i.e., the minimum distance between the trains is $\frac{2\sqrt{5}}{5}t_0V$ units.

iii When $t = \frac{4}{5}t_0$

$\boldsymbol{r}_1 = \frac{4}{5}t_0V\boldsymbol{i}$

and $\boldsymbol{r}_2 = 2V\left(\frac{4}{5}t_0 - t_0\right)\boldsymbol{j}$

$= \frac{-2}{5}Vt_0\boldsymbol{j}$



39 a $\boldsymbol{r}_1(t) = (2 - t)\boldsymbol{i} + (2t + 1)\boldsymbol{j}$

$x = 2 - t$ $\quad$ $y = 2t + 1$

$\therefore\ t = 2 - x$ $\quad$ and $\quad$ $y = 2(2 - x) + 1$

$= 4 - 2x + 1$

$= 5 - 2x,\ x \leq 2$ as $2 - t \leq 2$ for $t \geq 0$

b **i** $\boldsymbol{r}_1(t) = 2\boldsymbol{i} + \boldsymbol{j} + t(-\boldsymbol{i} + 2\boldsymbol{j})$

$= \boldsymbol{a} + t\boldsymbol{b}$

ii When $t = 0$, $\boldsymbol{r}_1(0) = \boldsymbol{a}$
and $\boldsymbol{b}$ is a vector parallel to the path of the particle, i.e., representing the velocity of the particle.

c **i** $\boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$

$(2 - t)\boldsymbol{i} + (2t + 1)\boldsymbol{j} = \boldsymbol{c} + t(2\boldsymbol{i} + \boldsymbol{j})$ when $t = 5$

i.e. $-3\boldsymbol{i} + 11\boldsymbol{j} = \boldsymbol{c} + 10\boldsymbol{i} + 5\boldsymbol{j}$

$\therefore\ \boldsymbol{c} = -13\boldsymbol{i} + 6\boldsymbol{j}$

ii $|\boldsymbol{a} - \boldsymbol{c}| = |-13\boldsymbol{i} + 6\boldsymbol{j} - (2\boldsymbol{i} + \boldsymbol{j})|$

$= |-15\boldsymbol{i} + 5\boldsymbol{j}|$

$= \sqrt{250}$

Distance $= 5\sqrt{10}$ metres

40 **a** $\boldsymbol{r}_1(1) = 13\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$

b $\boldsymbol{r}_1(t) = 16\boldsymbol{i} + 3\boldsymbol{k} + t(-3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$\boldsymbol{r}_2(t) = 3\boldsymbol{i} + \boldsymbol{j} + 11\boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

Unit vector parallel to path of first plane $= \dfrac{1}{\sqrt{14}}(-3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

and, to the second $= \dfrac{1}{\sqrt{6}}(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

c $(-3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = \sqrt{14}\sqrt{6}\cos\theta$

$-6 + 1 - 2 = \sqrt{14}\sqrt{6}\cos\theta$

$\dfrac{-7}{\sqrt{14 \times 6}} = \cos\theta$

$\therefore$ acute angle magnitude is 40.20°.

d Paths cross (note: planes do not meet) since $\boldsymbol{r}_1(t) \neq \boldsymbol{r}_2(t)$ for any t.

Now $\boldsymbol{r}_1(t) = \boldsymbol{r}_2(s)$

$16 - 3t = 3 + 2s$ ①

$t = 1 + s$ ②

$3 + 2t = 11 - s$ ③

Substitute ② into ①

$16 - 3(1 + s) = 3 + 2s$

$16 - 3 - 3s = 3 + 2s$

$13 - 3 = 5s$

$\therefore\ s = 2$ and $t = 3$.

Check in ③

LHS $= 3 + 6 = 9$, RHS $= 11 - 2 = 9$

$\therefore$ paths cross at $7\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k}$

e $\boldsymbol{r}_2(t) - \boldsymbol{r}_1(t) = (16 - 3t - 3 - 2t)\boldsymbol{i} + (t - 1 - t)\boldsymbol{j} + (3 + 2t - 11 + t)\boldsymbol{k}$

$= (13 - 5t)\boldsymbol{i} - \boldsymbol{j} + (3t - 8)\boldsymbol{k}$

f distance $= \sqrt{(13 - 5t)^2 + 1 + (3t - 8)^2}$ km

minimum when $-10(13 - 5t) + 6(3t - 8) = 0$

$\Rightarrow -130 + 50t + 18t - 48 = 0$

$\Rightarrow -178 + 68t = 0$

$\Rightarrow t = \dfrac{89}{34}$

$\therefore$ minimum distance $= \dfrac{\sqrt{1190}}{34}$ km

Algorithms and pseudocode

1 **a** **i** Desk Check

ii
```
define f(x):
  return abs(cos x − sin 2x)

a ← 0
b ← 2π
n ← 12
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    area ← 1/2(f(x) + f(x + h)) × h
    sum ← sum + area
    x ← x + h
end for
print sum
```

b **i**
```
define f(x):
  return sin^(−1) x

a ← 0
b ← 0.5
n ← 5
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    volume ← π * [f(x)]^2 × h
    sum ← sum + volume
    x ← x + h
end for
print sum
```

ii Deskcheck

iii
```
define f(x):
  return x

define g(x):
```

```
  return x^2

a ← 0
b ← 1
n ← 5
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    volume ← (π ∗ [f(x)]^2 × h) − (π ∗ [g(x)]^2 × h)
    sum ← sum + volume
    x ← x + h
end for
print sum
```

c **i** The length of the curve $y = x^2$ from $x = 0$ to $x = 5$ by taking 5 line segments.

ii Desk check

iii

```
define f(x):
  return x^3

a ← 0
b ← 2
n ← 10
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    length ← √(h^2 + (f(x + h) − f(x))^2)
    sum ← sum + length
    x ← x + h
end for
print sum
```

d **i**

```
define f(x):
  return x^2

a ← 0
b ← 1
n ← 5
```

```
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    area ← π × (f(x) + f(x + h)) × √(h² + (f(x + h) − f(x))²)
    sum ← sum + area
    x ← x + h
end for
print sum
```

ii
```
define f(x):
  return √x

a ← 0
b ← 1
n ← 5
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    area ← π × (f(x) + f(x + h)) × √(h² + (f(x + h) − f(x))²)
    sum ← sum + area
    x ← x + h
end for
print sum
```

2 a
```
define g(x, y):
  return x²y

define euler(x0, y0, 0.1, 20):
  x ← x0
  y ← y0
  for i from 1 to n
      x ← x + h
      y ← y + h × g(x, y)
      print i, (x, y)
  end for
  return
```

b Code for trapezoidal estimate:

```
define f(x):
  return √(sin x)

a ← 0
b ← π/4
n ← 10
h ← (b − a)/n
left ← a
sum ← 0
for i from 1 to n
    strip ← f(left) × h
    sum ← sum + strip
    left ← left + h
end for
print sum
```

3 **a** The rectangle considered has vertices $(0,0), (\pi,0), (\pi,1), (0,1)$

```
define f(x):
  return sin³ x

count ← 0
for i from 1 to 10⁶
    x ← π × random()
    y ← random()
    if y < f(x) then
        count ← count + 1
    end if
end for
area ← (count / 10⁶) × π
print area
```

b The rectangle considered has vertices $(0,0), (\pi,0), (\pi,1), (0,1)$

```
define f(x):
  return √(sin x)

count ← 0
for i from 1 to 10⁶
    x ← π × random()
    y ← random()
    if y < f(x) then
```

```
        count ← count + 1
    end if
end for
area ← (count / 10^6) × π
print area
```

Chapter 15 – Linear combinations of random variables and the sample mean

Solutions to Exercise 15A

1 a $C = 450 + 0.5X$

b

c	950	1200	1450	1700	1950	2450
$\Pr(C = c)$	0.05	0.15	0.35	0.25	0.15	0.05

c $\Pr(C > 2000) = 0.05$

2 a $W = 2.5X - 5$

b

w	-5	−2.5	0	2.5
$\Pr(W = w)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

c $\Pr(W > 2) = \frac{1}{8}$

3 a $\Pr(X < 0.3) = \int_0^{0.3} 3x^2\,dx$

$= \left[x^3\right]_0^{0.3}$

$= (0.3)^3$

$= 0.027$

b $Y = X + 1$

$\Pr(Y \le 1.5) = \Pr(X \le 0.5)$

$= \int_0^{0.5} 3x^2\,dx$

$= \left[x^3\right]_0^{0.5}$

$= 0.125$

4 a $\Pr(X < 0.5) = \int_0^{0.5} \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right)dx$

$= \left[\sin\left(\frac{\pi x}{4}\right)\right]_0^{0.5}$

$= \sin\left(\frac{\pi}{8}\right)$

$= 0.3827$

b $\Pr(Y > 2) = \Pr(3X - 1 > 2)$

$= \Pr(X > 1)$

$= \left[\sin\frac{\pi x}{4}\right]_1^2$

$= 0.2929$

5 a $\Pr(X < 2.5) = \int_0^{2.5} \frac{x+2}{16}\,dx$

$= \frac{1}{16}\left[\frac{x^2}{2} + 2x\right]_0^{0.5}$

$= 0.5078$

b $\Pr(Y > 2) = \Pr(4X + 2 > 2)$

$= \Pr(X > 0)$

$= 0.5$

6 a $Y = 3X + 2$

$E(X) = 3 \times 25 + 2 = 77$

$Var(X) = 9 \times 9 = 81$

b $U = 5 - 2X$

$E(U) = 5 - 2 \times 25 = -45$

$Var(U) = 4 \times 9 = 36$

$Sd(U) = 36$

c $V = 4 - 0.5X$
$E(V) = 4 - 0.5 \times 25 = -8.5$
$Var(V) = 0.25 \times 9 = 2.25$

7 a $E(Y) = m \times E(X) + n$
$45 = 25m + n$ ①
$var(Y) = m^2 \times var(X)$
$64 = m^2 \times 16$ ②
From ②, $m^2 = 4 \therefore m = 2$.
Substituting in ①, $n = -5$.

b $Y = 40 - 5 = 35$

8 a $E(X) = \int_{-1}^{0} 0.2x \, dx + \int_{0}^{1} 0.2x + 1.2x^2 \, dx$
$= \left[0.1x^2\right]_{-1}^{0} + \left[0.1x^2 + 0.4x^3\right]_{0}^{1}$
$= 0.4$

b $E(X^2) = \int_{-1}^{0} 0.2x^2 \, dx + \int_{0}^{1} 0.2x^2 + 1.2x^3 \, dx$
$= \left[\frac{0.2x^3}{3}\right]_{-1}^{0} + \left[\frac{0.2x^3}{3} + 0.3x^4\right]_{0}^{1}$
$= \frac{1.3}{3} = 0.4333\ldots$
$Var(X) = \frac{1.3}{3} - (0.4)^2 = 0.27333$

c $E(4X + 2) = 4 \times E(X) + 2 = 3.6$
$Var(4X + 2) = 16 \times Var(X) =$
$4.3733 \therefore sd(4X + 2) = 2.0913$

9 a $V = \pi r^2 h$
$\therefore V = 9\pi X$
$E(V) = 9\pi E(X) = 15 \times 9\pi$
$= 135\pi = 424.1$ mL

b $Var(V) = (9\pi)^2 Var(X) = 81\pi^2 \times 0.04$
$= 31.978$ mL

10 $A = 100\,000 + 5000X$

a $E(A) = 100\,000 + 2000 \times E(X) =$
\$110 000

b $sd(A) = 5000 \times sd(X) = \$1000$

11 $P = 1000 + 5000X$

a $E(P) = 1000 + 5000 \times E(X)$
$E(X) = 0.93$
$E(P) = \$5650$

b $sd(A) = 5000 \times sd(X)$
$sd(X) = \sqrt{0.8651}$
$sd(P) = \$4650$

c $E(P) = 1000 + c \times E(X) = 6150$
$\therefore c = 5537.63$
Increase by \$537.63

Solutions to Exercise 15B

1

$\Pr(X_1 + X_2 > 3)$

$= \Pr(X_1 = 1, X_2 = 3) + \Pr(X_1 = 1, X_2 = 4) + \ldots$

$= 0.45$

2 a $E(X_1) = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{5}{5} = 3$

b $E(X_1^2) = \frac{1}{5} + \frac{4}{5} + \frac{9}{5} + \frac{16}{5} + \frac{25}{5} = 11$

$\mathrm{Var}(X_1) = 11 - 3^2 = 2$

c $E(X_1 + X_2) = E(X_1) + E(X_2) = 6$

d $E(X_1 - X_2) = E(X_1) + E(X_2) = 4$

3 a $E(X_1 + X_2) = E(X_1) + E(X_2) = 20$

b $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 18$

c $\mathrm{sd}(X_1 + X_2) = \sqrt{\mathrm{Var}(X_1) + \mathrm{Var}(X_2)} = 4.243$

4 a $E(X_1 + \cdots + X_5)$

$= E(X_1) + \ldots + E(X_5) = 35$

b $\mathrm{Var}(X_1 + \cdots + X_5)$

$= \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_5) = 20$

c $\mathrm{sd}(X_1 + \ldots + X_5) = \sqrt{20} = 4.472$

5 a $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_0^1 2x^4 - x^2 + x\,dx$

$= \left[\frac{2}{5}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1$

$= \frac{2}{5} - \frac{1}{3} + \frac{1}{2}$

$= \frac{12 - 10 + 15}{30}$

$= \frac{17}{30}$

$E(X_1 + X_2 + X_3) = 3 \times \frac{17}{30} = 1.7$

b $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_0^1 2x^5 - x^3 + x^2\,dx$

$= \frac{5}{12}$

$\mathrm{Var}(X) = \frac{43}{450}$

$\mathrm{Var}(X_1 + X_2 + X_3) = 3 \times \frac{43}{450} = 0.2867$

c $\mathrm{Var}(X_1 + X_2 + X_3) = 0.535$

6 a

s	3	4	5	6	7
$\Pr(S = s)$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{7}{18}$	$\frac{2}{9}$	$\frac{1}{9}$

b 5

c $\frac{2}{3}$

7 a

$\Pr(X_1 - X_2 = 0)$

$= \Pr(X_1 = 1, X_2 = 1) + \Pr(X_1 = 2, X_2 = 2) + \ldots$

$= 6 \times \frac{1}{6} \times \frac{1}{6}$

$= \frac{1}{6}$

b $\Pr(X_1 + 3X_2 = 6)$

$= \Pr(X_1 = 3, X_2 = 1)$

$= \frac{1}{6} \times \frac{1}{6}$

$= \frac{1}{36}$

8 a $E(S) = 5$

b $E(S^2) = 26.4444$
$\text{Var}(S) = 26.4444 - 25 = 1.4444$
$\text{sd}(S) = 1.202$

9 $E(X + Y) = 17 + 32 = 49$ minutes
$\text{sd}(X + Y) = \sqrt{(\text{Var}X + \text{Var}Y} = \sqrt{(4.9^2 + 7^2)} = 8.5446$ minutes

10 a $E(C + M) = 50 + 145 = 195$ mL
$\text{sd}(X + Y) = \sqrt{(\text{Var}C + \text{Var}M} = \sqrt{(25 + 100)} = 11.1803$ mL

b \$10 per litre = 1 cent per mL
\$4 per litre = 0.4 cents per mL Cost
$= C + 0.4M$
$\text{E(Cost)} = E(C) + 0.4E(M)$
$= 50 + 0.4 \times 145 = \$1.08$
$\text{Var(Cost)} = \text{Var}(C) + 0.16 \times \text{Var}(M)$
$= 25 + 0.16 \times 100 = 6.40$ cents

11 $E(aX + bY) = 2a + 3b = 19$ ①
$\text{Var}(aX + bY) = 3a^2 + 4b^2 = 111$ ②
$a = \frac{19 - 3b}{2}$ from ①
Substituting in ②
$\frac{3(19 - 3b)^2}{4} + 4b^2 = 111$
$b = 3$
Substituting in ①
$a = 5$

12 a $C = 0.45A + 0.3B$
$E(C) = 0.45 \times E(A) + 0.3 \times E(B)$
$= 675$ cents $= \$6.75$
$\text{Var}(C) = 0.45^2 \times \text{Var}(A) + 0.3^2 \times \text{Var}(B)$
$= 0.45^2 \times 50 + 0.3^2 \times 25$
$= 12.375$ cents
$\text{sd}(C) = 3.52$ cents
$= 675$ cents $= \$6.75$

b $T = A_1 + A_2 + B_1 + B_2 + B_3$
$E(T) = 3 \times 750 + 2 \times 1000 = 4250$ grams
$\text{sd}(T) = \sqrt{(2 \times 50 + 3 \times 25)} = 13.23$ grams

Solutions to Exercise 15C

1 $E(P + C) = 12 + 14 = 26$
$sd(P + C) = \sqrt{(3^2 + 3^2)} = \sqrt{18}$
$Pr(X > 30) = 0.1729$

2 $E(A + B) = 5 + 8 = 13$
$sd(A + B) = \sqrt{(0.0225 + 0.04)} = 0.25$
$Pr(X > 13.4) = 0.0548$

3 $Pr(M > E) = Pr(M - E > 0)$
$E(M - E) = 63 - 68 = -5$
$sd(M - E) = \sqrt{(100 + 49)} = 12.2066$
$Pr(M > E) = 0.3410$

4 $Pr(A > B) = Pr(A - B > 0)$
$E(A - B) = 0.425 - 0.428 = -0.003$
$sd(A - B) = \sqrt{(0.0001 + 0.0004)}$
$= 0.02236$
$Pr(A > B) = 0.4466$

5 Let X represent student 1, and Y represent student 2.
$E(X - Y) = 0$
$sd(X - Y) = \sqrt{(2 \times 3000^2)} = 4242.64$
$Pr(X - Y < -7500) + Pr(X - Y > 7500) = 0.0771$

6 $T = X_1 + \cdots + X_6$
$E(T) = 1080$, $sd(T) = \sqrt{6 \times 20^2} = \sqrt{2400}$
$Pr(T < 1000) = 0.0512$

7 $\mu = 82, \sigma = 9$
Try 8 people: $\mu = 8 \times 82 = 656, \sigma = \sqrt{(881)} = 25.4558$
Pr(weight < 680) = 0.8271
Try 7 people: $\mu = 7 \times 82 = 574, \sigma = \sqrt{(781)} = 23.8118$
Pr(weight < 680) = 0.9999 . . .
Answer: 7 people

8 $\mu_{20} = 140, \sigma_{20} = 2.236$
Pr(total > 145) = 0.0127

9 **a** $\mu_3 = 900, \sigma_3 = 17.3205$
Pr(total > 1000) = 0.0019

b $\mu = 1200, \sigma_4 = 20$
Pr(total > 1250) = 0.0062

10 $\mu = 10 - 12 = -2, \sigma = \sqrt{(9 + 16)} = 5$
$Pr(X - Y < 0) = 0.6554$

Solutions to Exercise 15D

1 $E(\bar{X}) = \mu = 74$

$sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{8}{\sqrt{3}} = 4.6188$

2 $E(\bar{X}) = \mu = 25.025$

$sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{.003}{\sqrt{5}} = 0.0013$

3 **a** $\Pr(X > 80) = 0.0478$

b $E(\bar{X}) = \mu = 70$

$sd(\bar{X}) = \frac{6}{\sqrt{2}}$

$\Pr(\bar{X} > 80) = 0.0092$

c Much smaller probability for the mean than for an individual

4 **a** $\Pr(X > 120) = 0.0912$

b $E(\bar{X}) = \mu = 100$

$sd(\bar{X}) = \frac{15}{\sqrt{3}}$

$\Pr(\bar{X} > 120) = 0.0105$

c Much smaller probability for the mean than for an individual

5 $E(\bar{X}) = \mu = 266$

$sd(\bar{X}) = \frac{16}{\sqrt{7}}$

$\Pr(\bar{X} > 280) = 0.0103$

6 $E(\bar{X}) = \mu = 100$

$sd(\bar{X}) = \frac{6000}{\sqrt{10}}$

$\Pr(\bar{X} < 38\,000) = 0.0089$

7 $E(\bar{X}) = \mu = 42\,500$

$sd(\bar{X}) = \frac{15}{\sqrt{25}} = 3$

$\Pr(\bar{X} > 105) = 0.0478$

8 $E(\bar{X}) = \mu = 1.00$

$sd(\bar{X}) = \frac{0.03}{\sqrt{20}}$

$\Pr(\bar{X} < 0.98) = 0.0014$

9 $E(\bar{X}) = \mu = 10$

$sd(\bar{X}) = \frac{0.5}{\sqrt{50}}$

$\Pr(\bar{X} > 10.1) = 0.0786$

10 $E(X_1 + \ldots + X_{20}) = 20\mu = 70$

$sd(X_1 + \ldots + X_{20}) = \sqrt{20\sigma^2} = \sqrt{20}$

$\Pr(X_1 + \ldots + X_{20} < 60 = 0.0127$

Solutions to Exercise 15E

1 Answers will vary.

2 Answers will vary.

3 **a** Answers will vary.

b $\mathrm{E}(\bar{X}) = \mu = 1$

$\mathrm{sd}(\bar{X}) = \dfrac{0.01}{\sqrt{25}} = 0.002$

Solutions to Exercise 15F

1 **a** $\Pr(X > 10.1) = \int_{10.0}^{10.1} 5\, dx$

$= \Big[5x\Big]_{10.0}^{10.1}$

$= 0.5$

b $E(X) = 10.1$

$E(X^2) = \int_{10.0}^{10.2} 5x\, dx$

$= \Big[5 \times x^{2/3}\Big]_{10.0}^{10.1}$

$= 102.0133$

$Var(X) = 102.0133 - 10.12$

$= 0.0033$

$sd(X) = 0.0577$

$E(\bar{X}) = 10.1,$

$sd(\bar{X}) = \dfrac{0.0577}{\sqrt{3}}$

$\Pr(\bar{X} > 10.12) = 0.0288$

2 $E(\bar{X}) = 0.25$

$sd(\bar{X}) = \dfrac{0.1}{\sqrt{52}}$

$\Pr(\bar{X}) < 0.22) = 0.0153$

3 $E(\bar{X}) = 1200,$

$sd(\bar{X}) = \dfrac{200}{\sqrt{64}}$

$\Pr(\bar{X}) < 1150) = 0.0228$

4 **a** $\Pr(X > 0.5) = 0.7292$

b $E(X) = \int_0^1 \frac{4}{9}x^2(5 - x^2)\, dx$

$= 0.65185$

$E(X^2) = \int_0^1 \frac{4}{9}x^3(5 - x^2)\, dx$

$= 0.48148$

$Var(X) = 0.05637,$

$sd(X) = 0.2378$

$\Pr(\bar{X} > 0.5) = 0.9998$

5 $E(X) = 8, E(X^2) = 65.8$

$Var(X) = 1.8, sd(X) = \sqrt{1.8}$

$E(\bar{X}) = 8, sd(\bar{X}) = \sqrt{\dfrac{1.8}{40}}$

$\Pr(\bar{X} < 7.5) = 0.0092$

6 $p = 0.55, n = 100$

$E(X) = 100 \times 0.55$

$= 55,$

$Var(X) = 100 \times 0.55 \times 0.45$

$= 24.75,$

$sd(X) = \sqrt{24.75}$

$\Pr(X > 50) = 0.8426$

7 $p = 0.15, n = 1000\ E(X) = 150,$

$Var(X) = 1000 \times 0.15 \times 0.85 = 127.5$

$\Pr(X > 200) = 0.000005$

8 **a** $\Pr(0.85 < X < 1.10) = 0.7745$

b $E(X) = 154.91, sd(X) = 5.910$

$\Pr(140 < X < 160) = 0.7997$

Solutions to Technology-free questions

1 $\mu = 15, \sigma^2 = 25$

a $Y = 2X + 1$
$E(Y) = 2E(X) + 1 = 31$
$Var(Y) = 4Var(X) = 100$

b $U = 10 - 3X$
$E(U) = 10 - 3E(X) = -35$
$Var(U) = 9Var(X) = 225$
$sd(U) = 15$

c $V = Y + 2U$
$= 2X + 1 + 2(10 - 3X)$
$= -4X + 21$
$E(V) = -39, Var(V) = 400$

2 **a** $\Pr(X < 1.6) = \int_1^{1.6} 2(1 - \frac{1}{x^2})\, dx$
$= 2\Big[x + 1/x\Big]_1^{1.6}$
$= 0.45$

b $\Pr(Y \le 3.5) = \Pr(X \le 1.75)$
$= 2\Big[x + \frac{1}{x}\Big]_1^{1.75}$
$= \frac{9}{14}$

3 $V = 3 \times 3 \times X = 9X$

a $E(V) = 9 \times 3 = 27$ cm^3

b $var(V) = 81 \times var(X) = 81 \times 0.0001$ cm^6
$= 0.0081$

c TSA= $18 + 12X$
E(TSA)= $18 + 12 \times 3 = 54$ cm^2

4 **a** T=C+M
E(T)=E(C)+E(M)=200 mL
var(T)= var(C) + var(M)
= 25 + 144 = 169
sd(T)= 13

b Cost= $0.012C + 0.003M$
E(Cost)= $0.012 \times 60 + 0.003 \times 140 =$ \$1.14

5 $E(aX - bY) = 3a - 2b = 7$ ①
$Var(aX - bY) = 5a^2 + 4b^2 = 49$ ②
$a = \frac{7 + 2b}{3}$ from ①
Substituting in ②
$5\left(\frac{7 + 2b}{3}\right)^2 + 4b^2 = 49$
$b = 1$
Substituting in ①
$a = 3$

6 **a** $T_O = O_1 + O_2 + O_3 + O_4$
$E(T_O) = 4\mu_O = 500$
$\mu_O = 125$ gm
$var(T_O) = 4\sigma_O^2 = 25$
$\sigma_O^2 = 6.25$
$\sigma_O = \sqrt{6.25} = 2.5$ gm

b $T_L = L_1 + L_2 \ldots + L_9$
$E(T_L) = 9\mu_L = 585$
$\mu_L = 65$ gm
$var(T_L) = 9\sigma_L^2 = 36$
$\sigma_L^2 = 4$
$\sigma_L = \sqrt{4} = 2$ gm

c Total = $T_{O1} + T_{O2} + T_{O3} + T_{L1} + T_{L2} + T_{L3}$
E(Total)= $500 \times 3 + 585 \times 3 =$ 3255gm
var(Total)= $25 \times 3 + 36 \times 3 = 183$gm^2

7 $E(X) = \int_{-1}^{0} 0.2x\,dx + \int_{0}^{1} 0.2x + 1.2x^2\,dx$
$= 0.4$
$E(V) = 4 \times 0.4 = 1.6$

8 $E(\bar{X}) = 65$
$sd(\bar{X}) = \dfrac{7}{\sqrt{25}} = 1.4$

9 $sd(\bar{X}) = \dfrac{\sigma}{\sqrt{n}} < 0.00075$
$\sqrt{n} > \dfrac{0.003}{0.00075}$
$\sqrt{n} > 4$
$n > 16$

Solutions to multiple-choice questions

1 D $E(R) = 20 - 10 + 2 = 12$

$Var(R) = 4 \times 4 + 4 \times 1 = 20$

$Pr(R > 6) = Pr\left(Z > \frac{6 - 12}{\sqrt{20}}\right)$

$= Pr\left(Z > \frac{-3}{\sqrt{5}}\right)$

2 C $T = X_1 + X_2 + \cdots + X_{100}$

$E(T) = 100 \times E(X) = 8000$

$Var(T) = 100 \times Var(X) = 1000$

$Pr(T > 10000) = 0$

3 E $\mu = 10, \sigma = 1.5$

$E(T) = 12 \times 10 = 120$

$sd(T) = \sqrt{(12 \times 1.52)} = 5.196$

$Pr(T > 130) = 0.0271$

4 A $T = M_1 + \cdots + M_4 + P_1 + P_2$

$E(T) = 4 \times 150 + 2 \times 1000 = 2600$

$Var(T) = 4 \times 4 + 2 \times 36 = 88$

$sd(T) = \sqrt{88} = 2\sqrt{22}$

5 A

$\mu = 3.6, \sigma^2 = 1.44$

$Y = 3 - 4X$

$E(Y) = -11.4$

$Var(Y) = 16Var(X) = 23.04$

$sd(Y) = 4.8$

6 B $E(Y) = 2m + n = 2.1$

$Var(Y) = 0.5m^2 = 0.32$

$m^2 = 0.64 \Rightarrow m = 0.8$

$2 \times 0.8 + n = 2.1 \Rightarrow n = 0.5$

$Y = 0.8 \times 2.2 + 0.5 = 2.26$

7 B $E(\bar{X}) = 2489$

$sd(\bar{X}) = \frac{554}{\sqrt{100}} = 55.4$

8 B $E(\bar{X}) = 40$

$sd(\bar{X}) = \frac{4}{\sqrt{25}} = 0.8$

$Pr(\bar{X} > 42) = 0.0062$

9 D

$Pr(A > J) = Pr(A - J > 0)$

$E(A - J) = 140 - 150 = -10$

$var(A - J) = \sqrt{25^2 + 20^2} = 32.016$

$Pr(A - J > 0) = 0.3774 \approx 38\%$

Solutions to extended-response questions

1 $E(X) = 60, \text{sd}(X) = 20$

a $\Pr(X < 54) = 0.3821$

b $\Pr(a < X < b) = 0.95$
$a = 1.96 \times 20 + 60 = 20.8$
$b = 1.96 \times 20 + 60 = 99.2$

c **i** $n = 5, E(\bar{X}) = 60, \text{sd}(\bar{X}) = \dfrac{20}{\sqrt{5}}$
$\Pr(\bar{X} < 54) = 0.2512$

ii $E(T) = 300, \text{sd}(T) = 20 \times \sqrt{5}$
$\Pr(T < 270) = 0.2512$

iii Binomial with $p = 0.3821, n = 5$
$\Pr(X > 2) = 0.2870$

d $\Pr(c < \bar{X} < d) = 0.95$
$c = -1.96 \times \dfrac{20}{\sqrt{5}} + 60 = 42.47$
$b = 1.96 \times \dfrac{20}{\sqrt{5}} + 60 = 77.53$

2 $\Pr(X > 10.2) = 0.05 \Rightarrow \dfrac{10.2 - \mu}{\sigma} = 1.6449$
$\Pr(\bar{X} < 6.1) = 0.025 \Rightarrow \dfrac{6.1 - \mu}{\sigma/\sqrt{7}} = -1.9600$
Solving gives $\mu = 7.37, \sigma = 1.72$

3 **a** $E(T_x) = 8000, \text{sd}(T_x) = 200$ $\Pr(T_x > 8500) = 0.00621$

b $E(T_y) = 2700, \text{sd}(T_y) = 40$ $\Pr(T_y > 2850) = 0.000088$

c $E(T_w) = 800, \text{sd}(T_w) = 25$ $\Pr(T_w > 900) = 0.000032$

d $E(T_x + T_y + T_w) = 11500$
$\text{sd}(T_x + T_y + T_w) = \sqrt{(2002 + 402 + 252)} = 205.487$
$\Pr(\text{all} > 12000) = 0.0075$

4 a

$$E(T) = \int_0^{\infty} \frac{t}{250} e^{-\frac{t}{250}}\, dt$$
$$= 250$$
$$E(T^2) = \int_0^{\infty} \frac{t^2}{250} e^{-\frac{t}{250}}\, dt$$
$$= 125000$$
$$\text{Var}(T) = 62\,500$$
$$\text{sd}(T) = 250$$
$$E(\bar{T}) = 250$$
$$\text{sd}(\bar{T}) = \frac{250}{10} = 25$$
$$\Pr(\bar{T} < 220) = 0.1151$$

b $\Pr(\bar{T} > 220) = 0.8$

$$\Pr\left(Z > \frac{220 - 250}{\frac{250}{\sqrt{n}}}\right)$$
$$\Rightarrow \frac{220 - 250}{\frac{250}{\sqrt{n}}} = 0.8416$$
$$\sqrt{n} \geq 7.013$$
$$n = 50$$

Chapter 16 – Hypothesis testing for the mean

Solutions to Exercise 16A

1 $\mathbf{H}_0$: $\mu = 2.4$
$\mathbf{H}_1$: $\mu < 2.4$

2 $\mathbf{H}_0$: $\mu = 2.66$ $\mathbf{H}_1$: $\mu > 2.66$

3 $\mathbf{H}_0$: $\mu = 60$
$\mathbf{H}_1$: $\mu > 60$
$\sigma = 4.50, n = 25, \bar{x} = 65.80$
p-value= 0.00002

4 $\mathbf{H}_0$: $\mu = 34$
$\mathbf{H}_1$: $\mu < 34$
$\sigma = 8, n = 50, \bar{x} = 32.5$
p-value = 0.0924

5 **a** good evidence against $\mathbf{H}_0$

b insufficient evidence against $\mathbf{H}_0$

c strong evidence against $\mathbf{H}_0$

d strong evidence against $\mathbf{H}_0$

e very strong evidence against $\mathbf{H}_0$

6 Reject $\mathbf{H}_0$ at the 5% level of significance. There is good evidence that the mean is less than 50.

7 Do not reject $\mathbf{H}_0$ at the 5% level of significance. There is insufficient evidence that the mean is greater than 10.

8 Reject $\mathbf{H}_0$ at the 5% level of significance. There is good evidence that the mean is less than 40.

9 **a** $\mathbf{H}_0$: $\mu = 2.9$ $\mathbf{H}_1$: $\mu > 2.9$

b p -value = 0.003

c Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the average monthly weight gain has increased with the new diet.

10 **a** $\mathbf{H}_0$: $\mu = 3.6$ $\mathbf{H}_1$: $\mu < 3.6$

b $\sigma = 1.2, n = 11, \bar{x} = 2.6$
p -value = 0.003

c Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the mean number of residents has decreased.

11 **a** $\mathbf{H}_0$: $\mu = 42150$ $\mathbf{H}_1$: $\mu < 42150$

b $\sigma = 10,000, n = 20, \bar{x} = 39,500$
p -value = 0.118

c No, since the p-value is greater than 0.05 we can cannot reject $\mathbf{H}_0$ and conclude that the mean income for this town is the same as the rest of the state.

12 **a** $\mathbf{H}_0$: $\mu = 10$ $\mathbf{H}_1$: $\mu < 10$

b $\sigma = 0.5, n = 50, \bar{x} = 10.2$

p -value = 0.002

c Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the mean tar content of the cigarettes has decreased.

13 a $\mathbf{H}_0$: $\mu = 3.5$ $\mathbf{H}_1$: $\mu > 3.5$

b $\sigma = 1.5, n = 50, \bar{x} = 4.0$
p -value = 0.009

c Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the average service time has increased.

14 $\mathbf{H}_0$: $\mu = 20$ $\mathbf{H}_1$: $\mu > 20$
$\sigma = 3, n = 12, \bar{x} = 23$
p -value = 0.0003
Yes, since the p-value is less than 0.01 we can reject $\mathbf{H}_0$ and conclude that the students who sleep 8 hours score higher on the test.

Solutions to Exercise 16B

1 **a** $\mathbf{H}_0$: $\mu = 0.5$
$\mathbf{H}_1$: $\mu \neq 0.5$

b $\sigma = 0.04, n = 25, \bar{x} = 0.52$
p -value = 0.012

c Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the mean diameter of the ball bearings has changed.

2 $\mathbf{H}_0$: $\mu = 2$ $\mathbf{H}_1$:$\mu \neq 2$
$\sigma = 0.02, n = 20, \bar{x} = 1.99$
p -value = 0.025
Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the average weight of the bags has changed.

3 $\mathbf{H}_0$: $\mu = 40$ $\mathbf{H}_1$: $\mu \neq 40$
$\sigma = 10, n = 56, \bar{x} = 43$
p -value = 0.025
Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the average stay in this hospital is different from the others.

4 $\mathbf{H}_0$: $\mu = 484$ $\mathbf{H}_1$: $\mu \neq 484$
$\sigma = 42, n = 30, \bar{x} = 456$
p -value = 0.0003
Yes, since the p-value is less than 0.01 we can reject $\mathbf{H}_0$ and conclude that the average number of visitors to the museum has changed.

5 $\mathbf{H}_0$: $\mu = 2$ $\mathbf{H}_1$: $\mu \neq 2$
$\sigma = 1.2, n = 19, \bar{x} = 2.85$
p -value = 0.0015
Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the average hours children watch TV in this town has changed.

6 $\mathbf{H}_0$: $\mu = 60$ $\mathbf{H}_1$: $\mu \neq 60$
$\sigma = 10, n = 30, \bar{x} = 65$
p -value = 0.0062
Yes, since the p-value is less than 0.05 we can reject $\mathbf{H}_0$ and conclude that the mean life of batteries has changed after the new process.

7 **a** $\mathbf{H}_0$: $\mu = 9$ $\mathbf{H}_1$: $\mu \neq 9$
$\sigma = 2, n = 20, \bar{x} = 8.5$
p -value = 0.2636
No, since the p-value is greater than 0.05 we cannot reject $\mathbf{H}_0$. There is insufficient evidence that the mean number of hours children sleep has changed.

b (7.6235, 9.3765)

c Also leads us to not reject $\mathbf{H}_0$ since the hypothesised value (9) is within the interval.

8 $\mathbf{H}_0$: $\mu = 55000$ $\mathbf{H}_1$: $\mu \neq 55000$
$\sigma = 5000, n = 50, \bar{x} = 53445$

a p -value = 0.0279
Yes, since the p-value is less than 0.05 we reject $\mathbf{H}_0$ We can conclude that theaverage starting salary for graduates of this university diffrs from the rest of the state.

b (52059, 54831)

c Leads us to reject $\mathbf{H}_0$ since the hypothesised value (55000) is not within the interval.

Solutions to Exercise 16C

1 a $\Pr(|Z| > 1) = \Pr(Z > 1) + \Pr(Z < -1)$

$= 2\Pr(Z < -1)$

$= 0.3173$

b $\Pr(|Z| \le 0.5) = \Pr(-0.5 \le Z \le 0.5)$

$= 2\Pr(Z \le 0.5) - 1$

$= 0.3829$

c

$\Pr(|Z| \ge 1.75) = \Pr(Z \ge 1.75) + \Pr(Z \le -1.75)$

$= 2\Pr(Z \le -1.75)$

$= 0.0801$

d $\Pr(|Z| \le 2.1) = \Pr(-2.1 \le Z \le 2.1)$

$= 2\Pr(Z \le 2.1) - 1$

$= 0.9643$

e

$\Pr(|Z| \ge 0.995) = \Pr(Z \ge 0.995) + \Pr(Z \le -0.995)$

$= 2\Pr(Z \le -0.995)$

$= 0.3198$

2 $\Pr(|X - \mu| \ge 5) = \Pr(X \le \mu - 5) + \Pr(X \ge \mu + 5)$

$= \Pr(X \le 0) + \Pr(X \ge 10)$

$= 0.3173$

3

$\Pr(|X - \mu| \ge 8.5) = \Pr(X \le \mu - 8.5) + \Pr(X \ge \mu + 8.5)$

$= \Pr(X \le 39) + \Pr(X \ge 56)$

$= 0.1841$

4

$\Pr(|X - \mu| \ge 23) = \Pr(X \le \mu - 23) + \Pr(X \ge \mu + 23)$

$= \Pr(X \le 597) + \Pr(X \ge 643)$

$= 0.2145$

5 $\Pr(|\bar{X} - \mu| \ge 1) = \Pr\left(|Z| \ge \frac{\sqrt{20}}{5}\right)$

$= \Pr(|Z| \ge 0.8944)$

$= 2\Pr(Z \le -0.8944)$

$= 0.3711$

6 a

$\Pr(|\bar{X} - \mu| \ge |x_0 - \mu|) = \Pr\left(|Z| \ge \frac{|2.52 - 2.56|}{0.09/\sqrt{30}}\right)$

$= \Pr(|Z| \ge 2.4343)$

$= 2\Pr(Z \le -2.4343)$

$= 0.0149$

b

$\Pr(|\bar{X} - \mu| \ge |x_0 - \mu|) = \Pr\left(|Z| \ge \frac{|2.57 - 2.56|}{0.09/\sqrt{30}}\right)$

$= \Pr(|Z| \ge 0.6086)$

$= 2\Pr(Z \le -0.6086)$

$= 0.5428$

7 a

$\Pr(|\bar{X} - \mu| \ge |x_0 - \mu|) = \Pr\left(|Z| \ge \frac{|25\,450 - 27\,583|}{13525/10}\right)$

$= \Pr(|Z| \ge 1.5771)$

$= 2\Pr(Z \le -1.5771)$

$= 0.0149$

b

$$\Pr(|\bar{X} - \mu| \geq |x_0 - \mu|) = \Pr\left(|Z| \geq \frac{|30\,000 - 27\,583|}{13525/10}\right)$$
$$= \Pr(|Z| \geq 1.7871)$$
$$= 2\Pr(Z \leq -1.7871)$$
$$= 0.0739$$

8 $\Pr(|\bar{X} - \mu| \geq 3) = \Pr\left(|Z| \geq \frac{15}{7}\right)$
$$= \Pr(|Z| \geq 2.1429)$$
$$= 2\Pr(Z \leq -2.1429)$$
$$= 0.0321$$

9 $\Pr(|\bar{X} - \mu| \geq 0.25) = \Pr\left(|Z| \geq \frac{0.25 \times \sqrt{10}}{0.5}\right)$
$$= \Pr(|Z| \geq 1.58114)$$
$$= 2\Pr(Z \leq -1.58114)$$
$$= 0.1138$$

10 **a** $\Pr(|\bar{X} - \mu| \geq 2) = \Pr\left(|Z| \geq \frac{2 \times \sqrt{20}}{5}\right)$
$$= \Pr(|Z| \geq 1.78885)$$
$$= 2\Pr(Z \leq -1.78885)$$
$$= 0.0736$$

b $\mathbf{H}_0: \mu = 15$; $\mathbf{H}_1: \mu \neq 15$. Do not reject $\mathbf{H}_0$, since 0.0736 is greater than 0.05.

c You want the p-value to be less than 0.05.

$$\Pr(|\bar{X} - \mu| \geq a) = \Pr\left(|Z| \geq \frac{a \times \sqrt{20}}{5}\right)$$
$$2\Pr\left(Z \geq \frac{-a \times \sqrt{20}}{5}\right) \leq 0.05$$
$$\Pr\left(Z \geq \frac{-a \times \sqrt{20}}{5}\right) \leq 0.025$$
$$\frac{-a \times \sqrt{20}}{5} \leq -1.95996$$

$a \geq 2.191..$

More than 2.19 minutes

Solutions to Exercise 16D

1 $\mathbf{H}_0$: weight game the same
$\mathbf{H}_1$: weight gain is higher

a A Type I error would be concluding the weight gain is higher on the special diet when it fact it is the same.

b A Type II error would be concluding the weight gain is the same on the special diet when it fact it is higher.

2 $\mathbf{H}_0$: test scores have not improved under the new program
$\mathbf{H}_1$: test scores have improved under the new programs

a A Type I error would be concluding the test scores have improved under the new program when it fact they have not.

b A Type II error would be concluding the test scores have not improved under the new program when in fact they have.

3 $\mathbf{H}_0$: test shows no TB
$\mathbf{H}_1$: test shows TB

a A false positive is a Type I error. i.e., test is positive when in fact the person does not have TB .

b A false negative is a Type II error. i.e., test is negative when the person does have TB.

Solutions to Technology-free questions

1 **a** $\mathbf{H}_0$: $\mu = 70$ $\mathbf{H}_1$: $\mu \neq 70$

b Type I error: Concluding the pulse rate changes after exercise for one minutes when it doesn't.

c Type II error: Concluding the pulse rate doesn't change after exercise for one minutes when it does.

2 **a** **i** do not reject $\mathbf{H}_0$

ii do not reject $\mathbf{H}_0$

b **i** reject $\mathbf{H}_0$

ii do not reject $\mathbf{H}_0$

c **i** reject $\mathbf{H}_0$

ii reject $\mathbf{H}_0$

d **i** reject $\mathbf{H}_0$

ii reject $\mathbf{H}_0$

3 **a** $\mathbf{H}_0$: time is the same $\mathbf{H}_1$: time is reduced when there is no noise

b p-value = 0.02. Yes, since the p-value is less than 0.05 we reject $\mathbf{H}_0$. We can conclude that the time to solve the puzzle is less when there is no background noise.

c 2% of the time.

4 Yes the teaching method has been effective. If the sample mean is 5 standard deviations away from the hypothesised mean than the p-value will be very small (< 0.00001).

5 **a** $\mathbf{H}_0$: $\mu = 4$ $\mathbf{H}_1$: $\mu > 4$

b Type I error: Concluding that praise does increase happiness when it doesn't.

c Type II error: Concluding praise does not increase happiness when it does.

6 When calculating a p-value we calculate a z-value using: $z = \dfrac{\bar{x} - \mu}{\sqrt{n}}$ An increase in the value of the z-value will result in a decrease in the p-value. Thus

a decrease

b decrease

c no effect

d increase

7 **a** 18 or 22

b p-value = 0.044

c Reject $\mathbf{H}_0$ and conclude that the population mean is not 20

8 **a**

$$\begin{aligned}\Pr(|Z| > 1.5) &= \Pr(Z > 1.5) + \Pr(Z < -1.5) \\ &= 2 \times 0.0608\end{aligned}$$

b 0.9108

Solutions to multiple-choice questions

1 A

2 B

3 B

4 C $\mathbf{H}_0$: $\mu = 70$ $\mathbf{H}_1$: $\mu \neq 70$
$\sigma = 10, n = 25, \bar{x} = 76.5$
p -value = 0.0012

5 A We would want the strongest evidence possible

6 E $\Pr(\bar{X} \leq 96|\mu = 100) = \Pr(Z \leq \dfrac{96 - 100}{21\sqrt{50}}$
$= \Pr(Z \leq -1.347)$
$= 0.0890$

7 A Since a Type I error means rejecting the null hypothesis when it is in fact true it can only happen if the null hypothesis is rejected.

8 C We are looking for options which do not contain words similar to 'more' or 'less'

9 E
$\mathbf{H}_0$: $\mu = 8$ $\mathbf{H}_1$: $\mu < 8$
$\sigma = 2, n = 25, \bar{x} = 7.5$
p -value = 0.1056

10 D Increasing the sample size will decrease the p-value

Solutions to extended-response questions

1 **a** $\mathbf{H}_0$: $\mu = 42$ $\mathbf{H}_1$: $\mu < 42$

b **i** 4/100 = 0.04

ii Good evidence

iii Reject $\mathbf{H}_0$ and conclude the bookcase assembly time is reduced.

c p-value = 0.0368

d These answers are very similar.

e **i** For the two tail test p-value = 10/100 = 0.1

ii Do not reject as p-value > 0.05

iii Theoretical p-value = 0.0736. A bit different but would lead to the same conclusion.

2 **a** $\mathbf{H}_0$: $\mu = 70$ $\mathbf{H}_1$: $\mu > 70$

b Answers will vary

c **i,ii** Answers will vary, but if the p-value is < 0.05 then the null hypothesis should be rejected.

d Theoretical p-value = 0.0062

e Answers should be similar

f Concluding that the batteries last longer when they don't.

g Concluding that the batteries don't last longer when they do.

h The answer should be approx double the answer to **ci** p-value = 0.0124

3 **a** **i** $\mathbf{H}_0$: $\mu = 32$ $\mathbf{H}_1$: $\mu \neq 32$

ii p-value = 0.0059

iii Reject $\mathbf{H}_0$ and conclude the age of marriage for males has changed.

b **i** $\mathbf{H}_0$: $\mu = 29$ $\mathbf{H}_1$: $\mu \neq 29$

ii p-value = 0.0029

iii Reject $\mathbf{H}_0$ and conclude the age of marriage for females has changed.

c **i** males: (32.58, 35.42)

ii does not contain 32 as expected

d **i** females: (29.69, 32.32)

ii does not contain 29 as expected

Chapter 17 – Revision of Chapters 15-16

Solutions to Technology-free questions

1 $\mu = 5, \sigma^2 = 16$

a $E(Y) = 3\mu - 1 = 14$
$Var(Y) = 9\sigma^2 = 144$

b $E(U) = 3\mu - 1 = -7$
$sd(U) = \sqrt{4\sigma^2} = 8$

c $E(V) = 3E(Y) - 2E(U) = 28 + 14 = 42$
$Var(V) = 4Var(Y) + 4VarU = 832$

2 $\pi r^2 = 1.5, V = \frac{1}{3}\pi r^2 X = 0.5X$

a $E(V) = 0.5 \times 2 = 1$

b $Var(V) = 0.5^2 \times (0.02)^2 = .0001$

3 $\mu_X = 5, \sigma_X = 0.2$
$\mu_Y = 20, \sigma_X = 0.1$
$E(X_1 + X_2 + Y) = 10 + 20 = 30$
$Var(X_1 + X_2 + Y) = 0.04 + 0.04 + 0.01$
$= 0.09$
$sd(X_1 + X_2 + Y) = \sqrt{0.09} = 0.3$

4 $E(aX + bY) = 2a - 2b = -2$ ①
$Var(aX - bY) = 3a^2 + 3b^2 = 74$ ②
$a = b - 1$ from ①
Substituting in ②
$3(b-1)^2 + 3b^2 = 75$
$b = 4$
Substituting in ①
$a = 3$

5 $\mu_M = 500, \sigma^2_M = 25$
$\mu_P = 160, \sigma^2_P = 10$

a $M = X_1 + \ldots + X_5$
$E(M) = 5E(X) \Rightarrow \mu_X = 100$
$Var(M) = 5Var(X) \Rightarrow \sigma^2_X = 5$

b $P = Y_1 + \ldots + Y_8$
$E(P) = 8E(Y) \Rightarrow \mu_X = 20$
$Var(P) = 8Var(Y) \Rightarrow \sigma^2_X = 1.25$

c $T = M_1 + M_2 + P_1 + P_2 + P_3$
$E(T) = 2E(M) + 3E(P)$
$= 2 \times 500 + 3 \times 160 = 1480$
$Var(T) = 2Var(M) + 3Var(P)$
$= 2 \times 25 + 3 \times 10 = 80$

6 **a**
$$E(X) = \int_0^1 2x(1-x)\,dx$$
$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$
$$= \frac{1}{3}$$
$$E(X^2) = \int_0^1 2x^2(1-x)\,dx$$
$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1$$
$$= \frac{1}{6}$$
$$Var(X) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$
$$E(Y) = 3 \times \frac{1}{3} = 1$$
$$Var(Y) = 9 \times \frac{1}{18} = \frac{1}{2}$$

b $E(V) = 3 \times E(X) = 1$
$Var(V) = 3 \times Var(X) = \frac{1}{6}$

7 $\bar{X}$ is normally distributed with mean

$\mu = 68$ and standard deviation
$\frac{\sigma}{\sqrt{n}} = 2$

8 $\frac{\sigma}{\sqrt{n}} < 25$

$\sqrt{n} > \frac{200}{25} = 8$

$n > 64$

9 $\bar{X} \pm 1.96 \times \frac{\sigma}{\sqrt{n}} = 84 \pm 1.96 \times 2$

$= 84 \pm 3.92$

$(= (80.08, 87.92))$

10 **a** $\bar{x} = 440, \sigma = 25$

$\frac{2\sigma}{\sqrt{10}} = 10$

$n = 25$

b $\frac{2\sigma}{\sqrt{10}} = 2$

$n = 625$

11 **a** Since the p-value is less than 0.05, we would reject the null hypothesis, and conclude that the mean is less than 20.

b A two tail test will have a p-value which is double that of a one tail test.

i Thus, for the two tail test p-value $= 0.09$.

ii Since the p-value is not less than 0.05, we would not reject the null hypothesis, and conclude that the mean is still 20.

12 **a** $H_0 : \mu = 95 \qquad H_1 : \mu < 95$

b p-value $= \Pr(\bar{X} \leq 89)$

$= \Pr(Z \leq 2) = \frac{1 - 0.9545}{2} = 0.023$

c Since the p-value is less than 0.05, we would reject the null hypothesis, and conclude that students who meditate for 20 minutes complete the puzzle more quickly.

Solutions to multiple-choice questions

1 A $E(X) = 2 \times 4 - 2 \times 3 + 3 = 5$

$\text{Var}(X) = 4 \times 2 + 4 \times 5 = 28$

$\Pr(X > 4) = \Pr\left(Z > \dfrac{4-5}{\sqrt{28}}\right)$

$= \Pr\left(Z > \dfrac{-1}{\sqrt{28}}\right)$

2 B $X - Y$ is normally distributed.

$E(X - Y) = 58 - 52 = 6$

$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y)$

$= 8^2 + 6^2$

$= 100$

$\text{sd}(X - Y) = 10$

$\Pr(X - Y < 0) = 0.2743$

3 A $\bar{X}$ is normally distributed.

Mean $= 42$, sd $= \dfrac{4.5}{\sqrt{20}}$

$\Pr(38 < \bar{X} < 43) = 0.8398$

4 A A new random variable Z

$E(Z) = 4300$

$\text{sd}(Z)$

$= \sqrt{(5 \times 30^2 + 10 \times 10^2)}$

$= 10\sqrt{55}$

5 B $\mu_C = \$2.84 \Rightarrow \mu_{\bar{C}} = \$2.84$

$\sigma_C = \$0.88 \Rightarrow \sigma_{\bar{C}} = \dfrac{0.88}{4} = \$0.22$

6 A $\mu_X = \$3.68 \Rightarrow \mu_{\bar{X}} = \$3.68$

$\sigma_X = \$1.05 \Rightarrow \sigma_{\bar{X}} = \dfrac{1.05}{4} = \$0.21$

$\Pr(\bar{X} < 3.60) = 0.3516$

7 D $n = \left(2.5758 \times \dfrac{1.365}{0.3}\right)^2 \approx 138$

8 D $X + Y$ is normally distributed

$E(X + Y) = 13$

$\text{sd}(X + Y) = \sqrt{(1^2 + 1.5^2)} \approx 1.803$

$\dfrac{c - 13}{1.803} = 2.326$

$c = 17.2$

9 E $\bar{x} = 38.5$

$2.3263 \times \dfrac{\sigma}{\sqrt{21}} = 6.5$

$\sigma = 12.8$

10 B $n = 25, \bar{x} = 4.5, \sigma = 1.5$

p-value$= 0.0956$

11 D $H_0\colon \mu = 30$ $H_1\colon \mu > 30$

$\sigma = 7, n = 15, \bar{x} = 36.2$

p-value$= 0.0003$

Yes, since the p-value is less than 0.05 we reject H_0

12 D

13 D $\dfrac{\sigma}{\sqrt{n_1}} = \dfrac{1}{4}\dfrac{\sigma}{\sqrt{n_2}}$

$n_2 = 16n_1$

14 E

15 B

Solutions to extended-response questions

1 **a** $\Pr(X = 4) = 0.4^3 \times 0.6 = 0.0384$

b $\Pr(X > 4) = 1 - [\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4)]$
$= 0.0256$

c $\mathrm{E}(Y) = \mathrm{E}(X_1 + X_2 + \cdots + X_{30}) = 30 \times \frac{5}{3} = 50$

d $\mathrm{Var}(Y) = \mathrm{Var}(X_1 + X_2 + \cdots + X_{30}) = 30 \times \frac{10}{9} = \frac{100}{3}$

e Y is approximately normally distributed.
$\Pr(Y > 60) = 0.0416$

2 $\Pr(X > 1.02) = 0.057$

$$\frac{1.02 - \mu}{\sigma} = 1.5805$$

$$\mu = 1.5805\sigma + 1.02 \ldots \text{Equation1}$$

$\Pr(\bar{X} > 1.01) = 0.033$

$$\frac{1.01 - \mu}{\sigma/\sqrt{6}} = 1.8384$$

$$\mu = 0.7505\sigma + 1.01 \ldots \text{Equation 2}$$

Solving gives $\mu = 1.001,\ \sigma = 0.012$

3 **a** $\Pr(k_1 < X < k_2) = 0.95$

$$\frac{k_1 - 80}{20} = -1.96 \Rightarrow k_1 = 40.8$$

$$\frac{k_2 - 80}{20} = 1.96 \Rightarrow k_2 = 119.2$$

b $\Pr(c_1 < \bar{X} < c_2) = 0.95$

$$\frac{c_1 - 80}{20/\sqrt{20}} = -1.96 \Rightarrow c_1 = 71.2$$

$$\frac{c_2 - 80}{20/\sqrt{20}} = 1.96 \Rightarrow c_2 = 88.8$$

c Using calculator confidence interval is (76.2, 93.8)

4 **a** **i** $\mathrm{H_0}$: $\mu = 1000$ $\quad$ $\mathrm{H_1}$: $\mu > 1000$

ii $n = 10, \bar{x} = 1000.3, \sigma = 1.75$
p-value$= 0.2939$

iii No, since the p-value is more than 0.05 we cannot reject H_0 and conclude that machine does not need adjustment.

b $\Pr(\bar{X} \geq c) < 0.05 = \Pr(Z \geq \frac{c - 1000}{0.5534}) < 0.05$

$\frac{c - 1000}{0.5534} > 1.6449 \Rightarrow c > 1000.91$

c $\Pr(\bar{X} < 1000.91 | \mu = 10) = 0.0244$

5 a i H_0: $\mu = 55$ H_1: $\mu < 55$

ii $n = 10, \bar{x} = 50, \sigma = 5$

p -value= 0.0008

iii No, since the p-value is less than 0.05 we reject H_0 and conclude that the riding time has decreased.

b $\Pr(\bar{X} \leq c) < 0.05 = \Pr(Z \geq \frac{c - 55}{1.5811}) < 0.05$

$\frac{c - 55}{1.5811} < -1.6449 \Rightarrow c < 52.399$

c $\Pr(\bar{X} > 52.399 | \mu = 10) = 0.400$

Chapter 18 – Revision of Chapters 1-17

Solutions to technology-free questions

1 a To prove

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^n = \begin{bmatrix}1 & n\\0 & 1\end{bmatrix} \text{ for } n \in \mathbb{N}$$

P(1) is obviously true.

Assume $P(k)$ is true. That is,

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^k = \begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$$

For $P(k+1)$

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^{k+1} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^k \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & k\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & k+1\\0 & 1\end{bmatrix}$$

$P(k+1)$ is true.

Hence by the principle of mathematical induction the result is true for all n.

b To prove

$$\begin{bmatrix}5 & -8\\2 & -3\end{bmatrix}^n = \begin{bmatrix}4n+1 & -8n\\2n & 1-4n\end{bmatrix} \text{ for}$$

$n \in \mathbb{N}$

P(1) is obviously true.

Assume $P(k)$ is true. That is,

$$\begin{bmatrix}5 & -8\\2 & -3\end{bmatrix}^k = \begin{bmatrix}4k+1 & -8k\\2k & 1-4k\end{bmatrix}$$

For $P(k+1)$

$$\begin{bmatrix}5 & -8\\2 & -3\end{bmatrix}^{k+1} = \begin{bmatrix}5 & -8\\2 & -3\end{bmatrix}^k \begin{bmatrix}5 & -8\\2 & -3\end{bmatrix}$$

$$= \begin{bmatrix}4k+1 & -8k\\2k & 1-4k\end{bmatrix}\begin{bmatrix}5 & -8\\2 & -3\end{bmatrix}$$

$$= \begin{bmatrix}20k+5-16k & -32k-8+24k\\10k+2-8k & -16k-3+12k\end{bmatrix}$$

$$= \begin{bmatrix}4k+5 & -8-8k\\2k+2 & -4k-3\end{bmatrix}$$

$$= \begin{bmatrix}4(k+1)k+1 & -8(k+1)\\2(k+1) & -4(k+1)+1\end{bmatrix}$$

$P(k+1)$ is true.

Hence by the principle of mathematical induction the result is true for all n

2 a $\overrightarrow{OA} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

$\overrightarrow{OB} = -2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$

$\overrightarrow{OC} = \boldsymbol{i} + 2\boldsymbol{k}$

$\overrightarrow{CA} = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{CB} = -3\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\2 & 2 & 2\\-3 & 3 & 3\end{vmatrix} = 0\boldsymbol{i} - 12\boldsymbol{j} + 12\boldsymbol{k}$$

$= 12(-\boldsymbol{j} + \boldsymbol{k})$

b Area of triangle ABC

$= \frac{1}{2}|12(-\boldsymbol{j} + \boldsymbol{k})|$

$= 6\sqrt{2}$

c Area of parallelogram$= 12\sqrt{2}$

3 Line

$\boldsymbol{r} = -8\boldsymbol{i} + 4\boldsymbol{j} + 10\boldsymbol{k} + t(\boldsymbol{i} + 7\boldsymbol{j} - 2\boldsymbol{k})$

$\therefore\ \boldsymbol{r} = (-8+t)\boldsymbol{i} + (4+7t)\boldsymbol{j} + (10-2t)\boldsymbol{k}$

Plane $\quad 12x - 2y - z = 17$

$\therefore\ \boldsymbol{r} \cdot (12\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 17$

At intersection

$\therefore\ 12(-8+t) - 2(4+7t) - 1(10-2t) = 17$

$\therefore\ -96 + 12t - 8 - 14t - 10 + 2t = 17$

$\therefore\ -114 = 17$

$\therefore$ No intersection

4 Line $\boldsymbol{r} = (2-3t)\boldsymbol{i} + (-2+9t)\boldsymbol{j} + (1+t)\boldsymbol{k}$

x-z plane $y = 0$

$\therefore\ -2 + 9t = 0$

$\therefore\ t = \dfrac{2}{9}$

$\therefore\ \boldsymbol{r}_1 = \dfrac{4}{3}\boldsymbol{i} + \dfrac{11}{9}\boldsymbol{k}$

$\therefore$ Point A is $\left(\dfrac{4}{3}, 0, \dfrac{11}{9}\right)$

On $y - z$ plane $x = 0$

$\therefore\ 2 - 3t = 0$

$\therefore\ t = \dfrac{2}{3}$

$\therefore\ \boldsymbol{r}_2 = 4\boldsymbol{j} + \dfrac{5}{3}\boldsymbol{k}$

$\therefore$ Point B is $\left(0, 4, \dfrac{5}{3}\right)$

$$\therefore\ d(A, B) = \sqrt{\left(\frac{4}{3}\right)^2 + 4^2 + \left(\frac{4}{9}\right)^2}$$

$$= \sqrt{\frac{16}{9} + 16 + \frac{16}{81}}$$

$$= \sqrt{\frac{1456}{9}}$$

$$= \frac{4\sqrt{91}}{9}$$

5 $\begin{cases} 3x - y + 2z = 100 & (1) \\ x + 3y = 45 & (2) \end{cases}$

$\therefore \begin{cases} 9x - 3y + 6z = 300 & (1) \\ x + 3y = 45 & (2) \end{cases}$

From (1) + (2) $\quad \therefore\ 10x + 6z = 345$

Let $z = \lambda \quad \therefore\ x = \dfrac{345 - 6\lambda}{10}$

and $y = \dfrac{35 + 2\lambda}{10}$

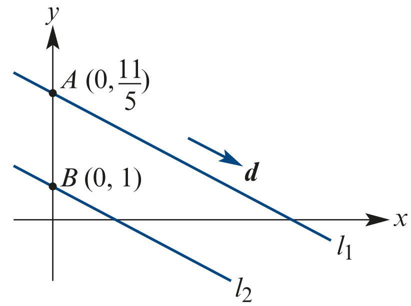
$\therefore$ Line of intersection is

$$\boldsymbol{r} = \frac{345 - 6\lambda}{10}\boldsymbol{i} + \frac{35 + 2\lambda}{10}\boldsymbol{j} + \lambda\boldsymbol{k}$$

6 $\begin{cases} 5x + 5y - 11 = 0 & (1) \quad l_1 \\ x + y - 1 = 0 & (2) \quad l_2 \end{cases}$

$\therefore$ Point on $l_1 = \left(0, \dfrac{11}{5}\right)$

Point on $l_2 = (0, 1)$



Direction vector $\boldsymbol{d}$ for both lines is $\boldsymbol{i} - \boldsymbol{j}$

Now $\overrightarrow{AB} = -\dfrac{6}{5}j$

$\therefore$ Scalar resolute of $\overrightarrow{AB}$ parallel $\boldsymbol{d}$

$$= \frac{\overrightarrow{AB} \cdot d}{|d|} = \frac{\frac{6}{5}}{\sqrt{2}}$$

$\therefore$ Distance between l_1 and $l_2 = \dfrac{3\sqrt{2}}{5}$

7 **a** If n is a perfect square it has an odd number of factors.

b If n is not a perfect square then it haseven number of factors

c n is a natural number with an odd number of factors and n is not a perfect square.

8 To prove $2^{n+2} + 3^{2n+1}$ is divisible by 7 for each positive integer n.

$P(1) : 2^3 + 3^3 = 8 + 27 = 35$ which is divisible by 7. $P(1)$ is true.

Assume $P(k)$ is true. That is $2^{k+2} + 3^{2k+1}$ is divisible by 7.

To prove $P(k + 1)$ is true:

$2^{(k+1)+2} + 3^{2(k+1)+1}$

$= 2^{k+3} + 3^{2k+3}$

$= 2 \times 2^{k+} + 9 \times 3^{2k+1}$

$= 2 \times 2^{k+} + 2 \times 3^{2k+1} + 7 \times 3^{2k+1}$

$= 2(2^{k+2} + 3^{2k+1}) + 7 \times 3^{2k+1}$

Both terms are divisible by 7 and so $P(k)$ is true

Hence by the principle of mathematical induction the result is true for all n.

9 a $\frac{d}{dx}(2y^2 - xy^3) = 0$ When

$4y\frac{dy}{dx} - [3xy^2\frac{dy}{dx} + y^3] = 0$

$\frac{dy}{dx}[4y - 3xy^2] = y^3$

$\frac{dy}{dx} = \frac{y^3}{4y - 3xy^2}$

$\frac{dy}{dx} = \frac{y^2}{4 - 3xy},\ y \neq 0$

$y = -1, 2 + x = 8 \Rightarrow x = 6$

$\frac{dy}{dx} = \frac{1}{4 + 18} = \frac{1}{22}$

b $x = 3\sin 2t \Rightarrow \frac{dx}{dt} = 6\cos 2t$

$y = -3\cos 2t \Rightarrow \frac{dy}{dt} = 6\sin 2t$

$L = \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sqrt{36\cos^2 2t + 36\sin^2 2t}\, dt$

$= \left[6\right]_{\frac{\pi}{6}}^{\frac{2\pi}{3}}$

$= 3\pi$

10 a $-1 \leq 2x - 1 \leq 1$

$0 \leq 2x \leq 2$

$0 \leq x \leq 1$

The interval $[0, 1]$

b $[0, 4\pi]$

c $f\left(\frac{1}{2}\right) = 4\arccos 0 = 2\pi$

d $4\arccos(2a - 1) = 3\pi$

$2a - 1 = \cos\left(\frac{3\pi}{4}\right)$

$2a = -\frac{1}{\sqrt{2}} + 1$

$a = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$

e

$f'(x) = -\frac{4}{x - x^2}$

$f'\left(\frac{1}{2}\right) = -8$

Use $y - y_1 = m(x - x_1)$

$y - 2\pi = -8\left(x - \frac{1}{2}\right)y = -8x + 4 + 2\pi$

11 a Resultant rate= 4 − 6 = −2 litres per minute.
Therefore, time to empty is 20 minutes

b

$$\frac{dm}{dt} = \text{rate of inflow} - \text{rate of outflow}$$
$$= -\frac{m}{40-2t} \times 6$$
$$= -\frac{3m}{20-t}$$

$m(0) = 10$

c

$$\frac{dm}{dt} = -\frac{3m}{20-t}$$
$$\int \frac{1}{3m}\, dm = -\int \frac{1}{20-t}\, dt$$
$$\frac{1}{3}\log_e m = \log_e(20-t) + c_1$$
$$\log_e m = \log_e(20-t)^3 + c$$
$$\log_e 10 = \log_e 20^3 + c$$
$$m(0) = 10$$
$$\therefore c = \log_e \frac{1}{800}$$
$$\therefore \log_e m = \log_e(20-t)^3 + \log_e \frac{1}{800}$$
$$\therefore m = \frac{(20-t)^3}{800}$$

d At time t, concentration $= \dfrac{m}{40-2t}$

$$\frac{m}{40-2t} = 0.2$$
$$\frac{(20-t)^2}{1600} = \frac{1}{5}$$
$$20 - t = \pm 40 \times \frac{1}{\sqrt{5}}$$
$$t = 20 - 8\sqrt{5} \text{ minutes}$$

12 **a** $y = 0$

b $$f'(x) = -\frac{x^2 + 6x - 3}{(x^2 + 3)^2}$$

$$x^2 + 6x - 3 = 0$$

$$x^2 + 6x + 9 - 12 = 0$$

$$(x + 3)^2 = 12$$

$$x = -3 \pm 2\sqrt{3}$$

$$\left(-3 - 2\sqrt{3}, \frac{-2\sqrt{3}}{3 + (-3 - 2\sqrt{3})^2}\right),$$

$$\left(-3 + 2\sqrt{3}, \frac{2\sqrt{3}}{3 + (-3 + 2\sqrt{3})^2}\right)$$

c

$$\int_0^3 f(x)\, dx$$

$$= \int_0^3 \frac{x + 3}{x^2 + 3}\, dx$$

$$= \int_0^3 \frac{x}{x^2 + 3} + \frac{3}{x^2 + 3}\, dx$$

$$= \left[\frac{1}{2}\log_e(x^2 + 3) + \sqrt{3}\arctan\left(\frac{\sqrt{3}}{3}x\right)\right]_0^3$$

$$= \frac{1}{2}\log_e(12) + \sqrt{3}\arctan(\sqrt{3}) - \log_e 3$$

$$= \log_e 2 + \frac{\pi}{\sqrt{3}}$$

13 **a** Let $x = t$ and $y = 3t^{\frac{3}{2}} - 1$

Then $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = \frac{9}{2}t^{\frac{1}{2}}$

Length of arc

$$= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{1 + \frac{81}{4}t}\, dt$$

$$= \left[\frac{(81t + 4)^{\frac{3}{2}}}{243}\right]_0^1$$

$$= \frac{85\sqrt{85} - 8}{243}$$

b Length of segment = $\sqrt{1^2 + 3^2} + \sqrt{10}$

14 **a** **i** $(5 + i)(4 + i) = 19 + 9i$

ii $(\sqrt{3} + i)(-2\sqrt{3} + i) = -7 - \sqrt{3}i$

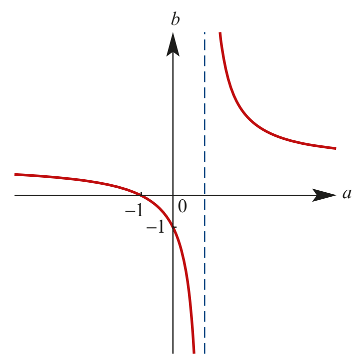
iii $\left(\frac{1}{2} + i\right)\left(-\frac{3}{4} + i\right) = -\frac{11}{8} - \frac{i}{4}$

iv $(1.2 - i)(0.4 + i) = 1.48 + 0.8i$

b **i** $(ab - 1) + (a + b)i$

ii $b = \frac{a + 1}{a - 1}$

iii



15

y	-2	-1	0	1	2
$\Pr(Y = y)$	$\frac{1}{36}$	$\frac{1}{6}$	$\frac{13}{36}$	$\frac{1}{3}$	$\frac{1}{9}$

a $\Pr(Y = 2) = \frac{1}{9}$

b $\Pr(Y = 0) = \frac{13}{36}$

c $\Pr(Y = 1) = \frac{1}{3}$

d $E(Y) = \frac{1}{3}$

16 $\mu_M = 80, \sigma_M = 12\ \mu_F = 70, \sigma_F = 10$

$T = M_1 + \cdots + M_5 + W_1 + \ldots + W_5$

$E(T) = 5 \times 80 + 5 \times 70 = 750$

$Var(T) = 5 \times 144 + 5 \times 100 = 1220$

$\Rightarrow sd(T) = \sqrt{1220}$

17 **a**

$$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = e$$

$\therefore$ Stationary point $P\left(e, \frac{1}{e}\right)$

$Q(1, 0)$

b Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$$\int_1^e \frac{\ln x}{x}\, dx = \int_0^1 u\, du$$

$$= \left[\frac{u^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

18 $\mu_V = 502, \sigma_V = 1$

$$\frac{\sigma}{\sqrt{n}} < 0.2$$

$$\frac{1}{\sqrt{n}} < 0.2$$

$$n > 25$$

19 **a**

$$\frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = e^x e^y$$

$$\int e^{-y}\, dy = \int e^x\, dx$$

$$e^{-y} = -e^x + c$$

$$y(1) = 1 \Rightarrow c = e^{-1} + e^1$$

$$y = -\log_e(e + e^{-1} - e^x)$$

b $(-\infty, \log_e(e + e^{-1}))$

c

$$\frac{dy}{dx} = e^{x+y}$$

When $x = 0, y = -\log_e(e + e^{-1} - 1)$

and $\therefore \frac{dy}{dx} = e^{-\log_e(e+e^{-1}-1} = \frac{1}{e + e^{-1} - 1}$

$\therefore$Equation of tangent

$$y = \frac{x}{e + e^{-1} - 1} - \log_e(e + e^{-1} - 1)$$

20 **a**

$$\frac{dy}{dx} = x(4 + y^2)$$

$$\int \frac{1}{4 + y^2}\, dy = \int x\, dx$$

$$\frac{1}{2}\tan^{-1}\frac{y}{2} = \frac{1}{2}x^2 + c_1$$

$$\tan^{-1}\frac{y}{2} = x^2 + c$$

$$y(0) = 2 \Rightarrow c = \frac{\pi}{4}$$

$$\therefore y = 2\tan\left(x^2 + \frac{\pi}{4}\right)$$

b $-\frac{\pi}{2} < x^2 + \frac{\pi}{4} < \frac{\pi}{2}$

$\Leftrightarrow x^2 + \frac{\pi}{4} < \frac{\pi}{2}$

$\Leftrightarrow x \in \left(-\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}\right)$

c $y = -\frac{x}{8}\sqrt{\frac{3}{\pi}} + 2\sqrt{3} + \frac{1}{16}$

21 a $\frac{x}{(1-x)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2}$

$x = a(1-x) + b$

$\Rightarrow a = 1$ and $b = -1$

$\Rightarrow \frac{x}{(1-x)^2} = \frac{1}{(1-x)^2} - \frac{1}{1-x}$

b $\int_2^4 \frac{x}{(1-x)^2}\,dx$

$= \int_2^4 \frac{1}{(1-x)^2} - \frac{1}{1-x}\,dx$

$= \left[\frac{1}{1-x} + \log_e(1-x)\right]_2^4$

$= -\frac{1}{3} + \log_e|-3| + 1 - \log_e|-1|$

$= \frac{2}{3} + \log_e 3$

22 a RHS$= \sqrt{x-1} + \frac{1}{\sqrt{x-1}}$

$= \frac{x-1+1}{\sqrt{x-1}} = \frac{x}{\sqrt{x-1}}$

b

Volume $= \pi\int_2^a [f(x)]^2\,dx$

$= \pi\int_2^a \frac{x^2}{x-1}\,dx$

$= \pi\int_2^a \frac{1}{x-1} + x + 1\,dx$

$= \pi\left[\log_e(x-1) + \frac{1}{2}x^2 + x\right]_2^a$

$= \pi(\log_e(a-1) + \frac{1}{2}a^2 + a - (2+2)$

$= \pi(\log_e(a-1) + \frac{1}{2}a^2 + a - 4)$

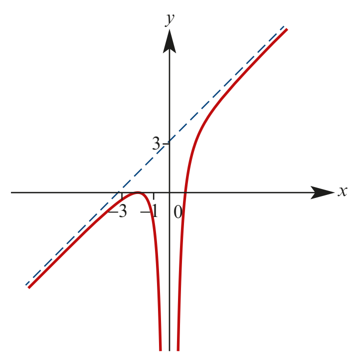
23 $y = x + 3 - \frac{4}{x^2} = \frac{(x-1)(x+2)^2}{x^2}$

$\frac{dy}{dx} = 1 + \frac{8}{x^3}$ and $\frac{dy}{dx} = 0 \Rightarrow x = -2$

Axis intercepts $(-2, 0)$, $(1, 0)$;

Asymptotes $x = 0$, $y = x + 3$;

Local maximum $(-2, 0)$



24 a Gradient of the line $x + y = 1$ is-1

Let $\boldsymbol{c} = a\boldsymbol{i} + b\boldsymbol{i}$ be a vector parallel to the line. Therefore $a = -b$

The magnitude of $\boldsymbol{c} = a\boldsymbol{i} + b\boldsymbol{i} = \sqrt{2b^2}$

$2b^2 = 1 \Rightarrow b = \pm\frac{1}{\sqrt{2}}$

Therefore, unit vectors $= \pm\frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

b The $\overrightarrow{OP'} = \boldsymbol{j}$ where $P'(0, 1)$ is a point on the line. The vector $P'P = m\boldsymbol{i} + (n-1)\boldsymbol{j}$ is parallel to the line. Therefore $\frac{n-1}{m} = -1$

Therefore, $n = -m + 1$. $m + n = 1$,

$\overrightarrow{OP} = m\boldsymbol{i} + (1 - m)\boldsymbol{j}$

c $(m\boldsymbol{i} + (1 - m)\boldsymbol{j}) \cdot (\boldsymbol{i} - \boldsymbol{j}) = \frac{\sqrt{2}}{2}\sqrt{2m^2 - 2m + 1}$

$m - (1 - m) = \frac{\sqrt{2}}{2}\sqrt{2m^2 - 2m + 1}$

$6m^2 - 6m + 1 = 0$

$m = \frac{3 \pm \sqrt{3}}{6}$

25 a $\overrightarrow{AC} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

$\boldsymbol{r} = (1 + 2t)\boldsymbol{i} + (t + 2)\boldsymbol{j} + (2t - 1)\boldsymbol{k}$

$\boldsymbol{r} \cdot \overrightarrow{AC} = 9t + 2$

$\boldsymbol{r}$ perpendicular to $\overrightarrow{AC} \Rightarrow t = -\frac{2}{9}$

b $\overrightarrow{AB} = \boldsymbol{i} + (m - 2)\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 + m - 2 + 4$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 0 \Rightarrow m = -4$

26 a $a = 1, b = 1$

b $c = 3, d = 2$

27 $P(z) = z^2 - (m + 2i)z + n(1 + i)$

Note that the coefficients are not real

$P(1 + 3i)$

$= (1 + 3i)^2 - (m + 2i)(1 + 3i) + n(1 + i)$

$= -8 + 6i - (m - 6 + (3m + 2)i) + n(1 + i)$

$Re(P(1 + 3i)) = -8 - m + 6 + n = -2 + n - m$

$Im(P(1 + 3i)) = 6 - (3m + 2) + n = 4 - 3m + n$

$Re(P(1 + 3i)) = 0$ and $Im(P(1 + 3i)) = 0$

$n - m = 2$

$3m - n = 4$

$\therefore 2m = 6$

$m = 3$ and $n = 5$

28 a $\int (2x - 6) \cdot e^x dx$

$= \int v\frac{du}{dx} \cdot dx$

Where $v = 2x - 6$, $\frac{du}{dx} = e^x$

$\therefore \frac{du}{dx} = 2$, $u = e^x$

$\int (2x - 6)e^x dx$

$= uv - \int u\frac{dv}{dx} \cdot dx$

$= e^x(2x - 6) - \int 2e^x dx$

$= e^x(2x - 6) - 2e^x$

$= 2xe^x - 6e^x - 2e^x$

$= 2xe^x - 8e^x$

$= 2e^x(x - 4)$

b $\int x \cdot \ln(2x)dx = \int v\frac{du}{dx} \cdot dx$

Where $v = \ln(2x)$ and $\frac{du}{dx} = x$

$\therefore \frac{dv}{dx} = \frac{1}{x}$ and $u = \frac{1}{2}x^2$

$\therefore \int \ln x(2x)dx$

$= \frac{1}{2}x^2 \cdot \ln(2x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}dx$

$= \frac{1}{2}x^2 \ln(2x) - \int \frac{1}{2}xdx$

$= \frac{1}{2}x^2 \ln(2x) - \frac{1}{4}x^2$

c $\int x \cdot \sec^2(3x)dx = \int v\frac{du}{dx}dx$

Where $v = x$ and $\frac{du}{dx} = \sec^2(3x)$

$\therefore \frac{dv}{dx} = 1$ and $u = \frac{1}{3}\tan(3x)$

$\therefore \int x\sec^2(3x)dx$

$= \frac{1}{3}x\tan(3x) - \int \frac{1}{3}\tan(3x)dx$

$= \frac{1}{3}x\tan(3x) - \frac{1}{3}\int \frac{\sin(3x)}{\cos(3x)}dx$

$= \frac{1}{3}x\tan(3x) + \frac{1}{9}\log_e(\cos(3x))$

d $\int x\tan^2 x\, dx = \int v\frac{du}{dx} \cdot dx$

Where $v = x$ and $\frac{du}{dx} = \tan^2 x = \sec^2 x - 1$

$\therefore \frac{du}{dx} = 1$ and $u = \tan x - x$

$\therefore \int x\tan^2 xdx = x(\tan x - x) - \int (\tan x - x)dx$

$= x\tan x - x^2 - \int \left(\frac{\sin x}{\cos x} - x\right)dx$

$= x\tan x - x^2 + \log_e(\cos x) + \frac{1}{2}x^2$

$= x\tan x + \log_e(\cos x) - \frac{1}{2}x^2$

29 $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \le \frac{4n+3}{6}\sqrt{n}$ For

$n = 1$

$P(1) : 1 \le \frac{7}{6}\sqrt{1}$

$P(1)$ is true.

Assume that $P(k)$ is true. That is,

$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{k} \le \frac{4k+3}{6}\sqrt{k}$

To prove $P(k)$ is true.

$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{k} + \sqrt{k+1} \le$

$\frac{4k+3}{6}\sqrt{k} + \sqrt{k+1}$

We need to prove

$\frac{4k+3}{6}\sqrt{k} + \sqrt{k+1} \le$

$\frac{4(k+1)+3}{6}\sqrt{k+1}$

Consider

$\frac{4(k+1)+3}{6}\sqrt{k+1} - (\frac{4k+3}{6}\sqrt{k} + \sqrt{k+1})$

We show that this is always ≥ 0

$= \frac{4k+3}{6}\sqrt{k+1} + \frac{7}{6}\sqrt{k+1} - \frac{4k+3}{6}\sqrt{k} - \sqrt{k+1}$

$= \frac{4k+3}{6}(\sqrt{k+1} - \sqrt{k}) + \frac{1}{6}\sqrt{k+1}$

Clearly both terms are positive. So

$P(k+1)$ is true

30 **a** Let **M** be a 2×2 matrix and assume that

$$\mathbf{M} = \mathbf{A}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\mathbf{A}^{-1}$$

where **A** is an invertible 2×2 matrix and $a, b \in \mathbb{R}$. Prove by induction that

$$\mathbf{M}^n = \mathbf{A}\begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}\mathbf{A}^{-1} \quad \text{for all } n \in \mathbb{N}$$

$P(1)$ is clearly true. Assume $P(k)$ is true.

$$\mathbf{M}^k = \mathbf{A}\begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}\mathbf{A}^{-1}$$

To prove $P(k + 1)$ is true.

$$\mathbf{M}^{k+1} = \mathbf{A}\begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}\mathbf{A}^{-1} \times \mathbf{M}$$

But $\mathbf{M} = \mathbf{A}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\mathbf{A}^{-1}$ Hence,

$$\mathbf{M}^{k+1} = \mathbf{A}\begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}\mathbf{A}^{-1} \times \mathbf{A}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\mathbf{A}^{-1}$$

$$= \mathbf{A}\begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\mathbf{A}^{-1}$$

$$= \begin{bmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{bmatrix}$$

That is $P(k + 1)$ is true and by the principle of mathematical induction it is true for all $n \in \mathbb{N}$

b $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} 2 & 6 \\ 6 & 21 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 \\ -21 & 9 \end{bmatrix}$$

Hence from **a**

$$\begin{bmatrix} -4 & 2 \\ -21 & 9 \end{bmatrix}^n = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 7 \times 2^n - 6 \times 3^n & 2 \times 3^n - 2 \times 2^n \\ 21 \times 2^n - 21 \times 3^n & 7 \times 3^n - 6 \times 2^n \end{bmatrix}$$

31 $\bar{x} = 1.70, \sigma = 0.5, n = 25$

a \$1.70

b $2 \times \frac{0.5}{\sqrt{25}} = 0.2$
$\Rightarrow (1.50, 1.90)$

32 $\bar{x} = 8.9, \sigma = 2.4, n = 36$

a $\mathbf{H}_0$: $\mu = 8.3$ $\mathbf{H}_1$: $\mu > 8.3$

b $\frac{\sigma}{\sqrt{n}} = \frac{2.4}{6} = 0.4$

$\Pr(\bar{X} \geq 8.9) = \Pr\left(Z \geq \frac{8.9 - 8.3}{0.4}\right)$

$= \Pr(Z \geq 1.5) = 0.0668$

c Since the p-value is greater than 0.05 we do not reject $\mathbf{H}_0$, and conclude that the new batteries do not last longer than the old batteries.

33 **a** Let $u = (1 - x^2)^n$ and $\frac{dv}{dx} = 1$

Then $\frac{du}{dx} = -2x(1 - x^2)^{n-1}$ and $v = x$

$$I_n = \int_0^1 (1 - x^2)^n \, dx$$
$$= \left[x(1 - x^2)^n\right]_0^1 + 2n \int_0^1 x^2(1 - x^2)^{n-1} \, dx$$
$$= 2n \int_0^1 x^2(1 - x^2)^{n-1} \, dx$$
$$= 2n \int_0^1 (1 - (1 - x^2))(1 - x^2)^{n-1} \, dx$$
$$= 2n \int_0^1 ((1 - x^2)^{n-1} - (1 - x^2)^n) \, dx$$
$$= 2nI_{n-1} - 2I_n$$

$\therefore (2n + 1)I_n = 2nI_{n-1}$

$$I_n = \frac{2n}{2n + 1} I_{n-1}$$

b **To establish true for** $n = 1$

$I_1 = \int_0^1 (1 - x^2) \, dx = \frac{2}{3}$

When $n = 1$, $\frac{2^{2n}(n!)^2}{(2n + 1)!} = \frac{2}{3}$

Therefore true for $n = 1$

Assume true for $n = k$

$I_k = \int_0^1 (1 - x^2)^k \, dx = \frac{2^{2k}(k!)^2}{(2k + 1)!}$

To prove true for $n = k + 1$

From **a**

$$I_{k+1} = \frac{2k + 2}{2n + 3} I_k$$
$$= \frac{2k + 2}{2k + 3} \times \frac{2^{2k}(k!)^2}{(2k + 1)!}$$
$$= \frac{2^2(k + 1)^2}{(2k + 3)(2k + 2)} \times \frac{2^{2k}(k!)^2}{(2k + 1)!}$$
$$= \frac{2^{2k+2}((k + 1)!)^2}{(2k + 3)!}$$

Solutions to multiple-choice questions

1 A Now $8x + 6y - 3z = 12$
Can be written as
$\boldsymbol{r} \cdot (8\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}) = -12$
$\therefore \boldsymbol{n} = 8\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$

2 C Remember that $p \Rightarrow q$ is logically equivalent to $\neg q \Rightarrow \neg p$

3 D For the negation change 'for all' to 'there exists and negate the rest of the statement.
Remember that the negation of $p \Rightarrow q$ is $p \wedge \neg q$

4 C

5 B For stationary points solve $f'(x) = 0$.
Use the quotient rule to find $f'(x)$.

$$f(x) = \frac{2x^2 - x + 1}{x - 1}$$

$$f'(x) = \frac{(x-1)(4x-1) - (2x^2 - x + 1)(1)}{(x-1)^2}$$

$$= \frac{2x^2 - 4x}{(x-1)^2}$$

$$= \frac{2x(x-2)}{(x-1)^2}$$

Thus $f'(x) = 0$ if $2x(x-2) = 0$, i.e.
$x = 0$
or $x = 2$.

6 D For inflexion points solve $f''(x) = 0$.
Either use the quotient rule twice or first simplify the expression.

$$f(x) = \frac{x^2 - 3x + 2}{x^2}$$

$$= 1 - 3x^{-1} + 2x^{-2}$$

$$f'(x) = 3x^{-2} - 4x^{-3}$$

$$f''(x) = -6x^{-3} + 12x^{-4}$$

$$= \frac{-6x + 12}{x^4}$$

Thus, $f''(x) = 0$ if $-6x + 12 = 0$, i.e. $x = 2$.

7 B Use implicit differentiation (as a general expression for the derivative is not required, substitute before simplifying).

$$x^3 + y^3 + 3xy = 1$$

$$3x^2 + 3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$

Substitute $(2, -1)$:

$$12 + 3\frac{dy}{dx} - 3 + 6\frac{dy}{dx} = 0$$

$$9\frac{dy}{dx} = -9$$

$$\frac{dy}{dx} = -1$$

8 E For a maximum gradient, find where the derivative of the gradient is zero, i.e. solve $f''(x) = 0$. Use the product rule for differentiation.

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \sin x + e^x \cos x$$

$$= e^x(\sin x + \cos x)$$

$$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$$

$$= 2e^x \cos x$$

Thus, $f''(x) = 0$ if $\cos x = 0$, i.e.
$x = \dfrac{\pi}{2}$
since $0 \le x \le \pi$.
Then $f'\left(\dfrac{\pi}{2}\right) = e^{\frac{\pi}{2}}(1 + 0) = e^{\frac{\pi}{2}}$.

To be sure that this gives the maximum gradient, check the endpoints: the gradients are $f'(0) = 1$ and $f'(\pi) = -e^{\pi}$.

Since each of these is smaller than $e^{\frac{\pi}{2}}$ and as there is just the one stationary point on the interval, this stationary point is a maximum.

9 B Using a CAS.
The syntax will be something like:
solve $\left(\int_0^k xe^{-x}\,dx = 0.5,\ k\right)$.
This gives 1.7, correct to one dp.

10 A Use CAS.
Use a definite integral:

$$y(x) = \int_2^x x\log_e x\,dx + 2,$$

as $y(2) = 2$, so

$$\begin{aligned} y(3) &= \int_2^3 x\log_e x\,dx + 2 \\ &= 4.30746 \quad \text{(CAS)} \\ &\approx 4.31 \end{aligned}$$

11 C

$$\begin{aligned} A &= 2\pi\int_1^b \frac{4}{x}\sqrt{1+\frac{16}{x^4}}\,dx \\ &= 8\pi\int_1^b \frac{\sqrt{x^4+16}}{x^3}\,dx \end{aligned}$$

9 B Change to sin and cos and simplify:
$\tan(2x) = \dfrac{\sin(2x)}{\cos(2x)},\ \sec(2x) = \dfrac{1}{\cos(2x)}$.
Multiply above and below by $\cos(2x)$:

$$\begin{aligned} \frac{\tan(2x)}{1+\sec(2x)} &= -\sqrt{3} \\ \frac{\sin(2x)}{\cos(2x)+1} &= -\sqrt{3} \\ \frac{2\sin x\cos x}{2\cos^2 x} &= -\sqrt{3} \\ \tan x &= -\sqrt{3} \\ x &= \frac{2\pi}{3} \text{ as } 0 \le x \le \pi \end{aligned}$$

10 C If $f(x) = g(x)$, then
$\sec x = \operatorname{cosec}(2x)$.
Change to sin and cos and simplify:

$$\frac{1}{\cos x} = \frac{1}{\sin(2x)}$$

$$2\sin x\cos x = \cos x$$

$$\sin x = \frac{1}{2} \text{ as } \cos x \ne 1$$

This has 2 solutions for $-\pi \le x \le \pi$.
(Alternatively graph f, g with a CAS.)

12 D Now $\cot\left(\frac{\theta}{2}\right) \le 0$ on $[-\pi,\ 0)$, $\cot(0)$ is undefined and as $\theta \to 0^+$, $\cot\left(\frac{\theta}{2}\right) \to \infty$.
Now solve $\cot\left(\frac{\theta}{2}\right) = \sqrt{3}$ or equivalently $\tan\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{3}}$. The solution on $(0, \pi]$ is $\frac{\theta}{2} = \frac{\pi}{6}$, i.e. $\theta = \frac{\pi}{3}$.
So $\cot\left(\frac{\theta}{2}\right) \ge \sqrt{3}$ on $\left(0, \frac{\pi}{3}\right]$.

13 A Since the velocity is positive for $t \ge 0$, the distance travelled is given by

$$\begin{aligned} \int_0^{10} \frac{4t}{1+t^2}\,dt &= [2\log_e(1+t^2)]_0^{10} \\ &= 2\log_e(101) - 2\log_e(1) \\ &= 2\log_e(101) \\ &\approx 9.23 \end{aligned}$$

14 D The highest point is reached when $v = 0$.
Since $a = \frac{dv}{dt}$, the acceleration equation can be re-written as
$\frac{dv}{dt} = -\frac{20+v^2}{50}$ or equivalently
$\frac{dt}{dv} = -\frac{50}{20+v^2}$.
To find the time, use a definite integral:

$$t = -50\int_{200}^{0} \frac{1}{20 + v^2}\, dv$$

$$= \frac{50}{\sqrt{20}} \int_{0}^{200} \frac{\sqrt{20}}{20 + v^2}\, dv$$

$$= \frac{50}{\sqrt{20}}\left[\tan^{-1}\left(\frac{v}{\sqrt{20}}\right)\right]_0^{200}$$

$$= \frac{50}{\sqrt{20}} \tan^{-1}\left(\frac{200}{\sqrt{20}}\right) \approx 17.3121$$

So the time taken is about 17 seconds.

15 D One method is to try two values for x.

Let $x = 0$:

$$-\sec b = \operatorname{cosec}\frac{\pi}{3}$$
$$= \frac{2}{\sqrt{3}}$$
$$\cos b = -\frac{\sqrt{3}}{2}$$
$$b = \frac{7\pi}{6}$$

Let $x = \frac{\pi}{6}$:

$$-\sec\left(\frac{a\pi}{6} + \frac{7\pi}{6}\right) = \operatorname{cosec}\frac{\pi}{2}$$
$$= 1$$
$$\cos\frac{(a+7)\pi}{6} = -1$$
$$\frac{(a+7)\pi}{6} = \pi$$
$$a + 7 = 6$$
$$a = -1$$

Thus, $a = -1$ and $b = \frac{7\pi}{6}$.

(Alternatively, you could use a CAS to plot the graphs of the cosec function and each possible sec function to see if any match the cosec function.)

16 B Using the product rule:

$$\frac{d(x\log_e y)}{dx} = \log_e y + x \times \frac{1}{y}\frac{dy}{dx}$$
$$= \log_e y + \frac{x}{y}\frac{dy}{dx}$$

It follows that:

$$\frac{d(x\log_e y)}{dx} - \frac{x}{y}\frac{dy}{dx} = \log_e y$$

17 D $x + 2 + 3\sec(t),\ y = 1 + 2\tan(t)$.

Since $\sec^2(t) = 1 + \tan^2(t)$,

then: $\frac{(x-2)^2}{9} = 1 + \frac{(y-1)^2}{4}$ or

equivalently $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = 1$

Asymptotes are given by:

$$\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = 0$$
$$\frac{y-1}{2} = \pm\frac{(x-2)}{3}$$
$$y = 1 \pm \frac{2(x-2)}{3}$$
$$y = \frac{2}{3}x - \frac{1}{3} \text{ or}$$
$$y = -\frac{2}{3}x + \frac{7}{3}$$

Now check the restrictions on t.

On $\left[0, \frac{\pi}{2}\right)$, $x \geq 5$ and $y \geq 1$, corresponding to the 'top half' of the branch of the hyperbola to the right of the centre (2, 1).

On $\left(\frac{\pi}{2}, \pi\right]$, $x \leq -1$ and $y \leq 1$, corresponding to the 'bottom half' of the branch of the hyperbola to the left of the centre (2, 1).

Each of these two sections of the complete hyperbola is asymptotic to just $y = \frac{2}{3}x - \frac{1}{3}$. So the graph has just the one asymptote.

18 A Equating coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ gives the simultaneous equations:

$$2m + p = 3 \quad ①$$
$$3m + n - 2p = 0 \quad ②$$
$$-m - 3n - 2p = 0 \quad ③$$

② − ③ :

$4m + 4n = 0$, i.e. $m + n = 0$ ④

2① + ② : $7m + n = 6$ ⑤

⑤ − ④ : $6m = 6$, so $m = 1$.

Substituting gives $n = -1$ and $p = 1$. However, this is not enough to determine whether the vectors are linearly dependent or independent. Check whether one of the given vectors can be expressed as a linear combination of the other two. In particular, as $\boldsymbol{b}$ and $\boldsymbol{c}$ are clearly not parallel, do there exist constants k and l such that $\boldsymbol{a} = k\boldsymbol{b} + l\boldsymbol{c}$?
Equating coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ gives the simultaneous equations:

$$l = 2 \quad ⑥$$

$$k - 2l = 3 \quad ⑦$$

$$-3k - 2l = -1 \quad ⑧$$

From ⑥, $l = 2$. Substituting in ⑦ gives $k = 7$. Then $-3k - 2l = -25 \neq -1$, so constants k and l do not exist and the three vectors are linearly independent.

19 E Since $\int \cos^2(2x)dx \neq \frac{1}{6}\int \cos^3(2x)$, the expression in alternative **E** is not equal to the given definite integral. (The other alternatives come about by re-expressions.
Alternative **A** comes from the substitution $u = 2x$.
Alternative **B** comes from the identity $\cos^2(2x) = 1 - \sin^2(2x)$, so

$$\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \cos^2(2x)dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} (1 - \sin^2(2x))dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} 1dx - \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin^2(2x)dx$$

$$= [x]_{\frac{\pi}{6}}^{\frac{2\pi}{3}} - \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin^2(2x)dx$$

$$= \left(\frac{2\pi}{3} - \frac{\pi}{6}\right) - \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin^2(2x)dx$$

$$= \frac{\pi}{2} - \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin^2(2x)dx$$

Alternative **C** comes from the direct use of a double angle formula.
Alternative **D** comes from the identity

$$\cos(2x) = \sin\left(\frac{\pi}{2} - 2x\right), \text{ so}$$

$$\cos^2(2x) = \sin^2\left(\frac{\pi}{2} - 2x\right)$$

$$= \sin^2 \frac{1}{2}(\pi - 4x))$$

20 D From the given equation:

$$z^2 = \frac{2 - 2i}{1 + i}$$

$$= \frac{2(1 - i)}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{2(1 - i)^2}{2}$$

$$= (1 - i)^2$$

$$z = 1 - i \text{ or } -1 + i$$

So one value of z could be $-1 + i$.

21 B When $x = 0$, $y = e^0 = 1$. To make it easy to use implicit differentiation, rewrite the equation in logarithmic form.

$$y = e^{xy}$$

$$\log_e y = xy$$

$$\frac{1}{y}\frac{dy}{dx} = y + x\frac{dy}{dx}$$

Substitute (0, 1) :

$$\frac{dy}{dx} = 1 + 0 = 1$$

22 A $|\mathbf{a}| = \sqrt{p^2 + q^2 + 1}$;

$|\mathbf{b}| = \sqrt{1 + 4 + 4} = 3$.

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\frac{2}{3} = \frac{1}{3}(p - 2q + 2)$.

Simplifying gives $p = 2q$.

The scalar resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$2 = \frac{1}{\sqrt{p^2 + q^2 + 1}}(p - 2q + 2)$.

Substitute $p = 2q$ and solve:

$$\frac{2}{\sqrt{4q^2 + q^2 + 1}} = 2$$

$$\sqrt{5q^2 + 1} = 1$$

$$5q^2 + 1 = 1$$

$$5q^2 = 0 \Rightarrow q = 0$$

Thus $p = 2q = 0$.

23 E Let $y = a\cos(x + c)$.

Check the range of f before proceeding.

$$f(\pi - c) = a\cos\pi = -a$$

$$f\left(\frac{3\pi}{2} - c\right) = a\cos\frac{3\pi}{2} = 0$$

Since $a \geq 0$, the range is $[-a, 0]$.

So the inverse has domain $[-a, 0]$ and range $\left[\pi - c, \frac{3\pi}{2} - c\right]$

For the inverse rule, interchange x and y and solve for y:

$$x = a\cos(y + c)$$

$$\cos(y + c) = \frac{x}{a} \qquad ①$$

Does ① mean that $y+c = \cos^{-1}\left(\frac{x}{a}\right)$?

Check the endpoints of the domain:

$x = -a, y = \pi - c; x = 0, y = \frac{\pi}{2} - c$.

But this is inconsistent with the range of the inverse. So the rule is a little trickier. ① can also be written $\cos(2\pi - (y + c)) = \frac{x}{a}$.

Solving this for y:

$$2\pi - (y + c) = \cos^{-1}\left(\frac{x}{a}\right)$$

$$y + c = 2\pi - \cos^{-1}\left(\frac{x}{a}\right)$$

$$y = 2\pi - c - \cos^{-1}\left(\frac{x}{a}\right)$$

Now check the endpoints:

$x = -a, y = 2\pi - c - \pi = \pi - c$;

$x = 0,\ y = 2\pi - c - \frac{\pi}{2} = \frac{3\pi}{2} - c$.

These are the correct end-points of the range. So

$f^{-1}(x) = 2\pi - c - \cos^{-1}\left(\frac{x}{a}\right)$.

(Note: this is rather involved. An alternative is to note that from ①, one of alternatives **C**, **D** or **E** is correct. Checking $x = 0$ gives $\frac{\pi}{2} - c$ in **C**, $\frac{3\pi}{2} + c$ in **D** and $\frac{3\pi}{2} - c$ in **E**, so **E** must be correct.)

24 C $x = \frac{a}{t + 1} \Rightarrow t = \frac{a}{x} - 1$.

Substitute for y :

$$y = 1 + t^2 = 1 + \left(\frac{a}{x} - 1\right)^2$$

$$= \frac{a^2 - 2ax + 2x^2}{x^2}$$

So one of alternatives **B** or **C** is correct.

$t = 0,\ x = a;\ t \to \infty,\ x \to 0.$
Thus, $x \in (0,\ a]$.

25 E Note that

$$\frac{d}{dx}(x^3 - 3x^2 + 4) = 3x^2 - 6x$$
$$= 3x(x - 2)$$
$$= -3x(2 - x)$$

so use the substitution
$u = x^3 - 3x^2 + 4$.
$x = 1,\ u = 2;\ x = 2,\ u = 0.$
Then:

$$\int_1^2 x(2 - x)(x^3 - 3x^2 + 4)\,dx$$
$$= -\frac{1}{3}\int_2^0 u\,du$$

26 C The slopes shown are negative and approaching zero as x increases. Only the curve given by $y = \frac{1}{x}$ has this property for x between 0 and 2. The curves given in all the other alternatives have positive slopes for this domain.

27 E The gradients at $x = \pm\frac{\pi}{2}$ are zero, which eliminates alternatives **A** and **C**; the gradients at $x = 0$ are positive, which eliminates alternatives **B** and **D**. Only alternative **E** satisfies both of these.

28 D The gradients shown are non-negative for all values of x and y. None of the curves given in alternatives **A**, **B**, **C** or **E** satisfy this fact. Checking their derivatives shows that they have negative gradients for $x < 0$.
For $x = -\frac{1}{y},\ \frac{dx}{dy} = \frac{1}{y^2} \Rightarrow \frac{dy}{dx} = y^2$, so the gradients are positive for all values of y. In addition, for any two values of y, the gradients are the same, which matches the slope field given.

29 E The gradients are zero when $x = 0$, so this eliminates alternatives **B** and **C**. Alternative **A** can be eliminated since the gradients in this case are all non-negative.
The slope field suggests vertical slopes when $y = 0$, which agrees with alternatives **D** and **E**. Now in the first quadrant, where $x > 0$ and $y > 0$, the slopes are negative. This is consistent with alternative **E** but not **D**.

30 A

Now $\boldsymbol{r}(t) = v\cos\alpha^\circ t\,\boldsymbol{i} + \left(v\sin\alpha^\circ t - \frac{g}{2}t^2\right)\boldsymbol{j}$

$\therefore\ \dot{\boldsymbol{r}}(t) = v\cos\alpha^\circ\,\boldsymbol{i} + (v\sin\alpha^\circ - gt)\,\boldsymbol{j}$

But $\dot{\boldsymbol{r}}(0) = 40\cos\left(\arctan\frac{3}{4}\right)\boldsymbol{i} + \left(40\ \sin\left(\arctan\frac{3}{4}\right)\right)\boldsymbol{j}$.

$\therefore\ \dot{\boldsymbol{r}}(0) = 32\boldsymbol{i} + 24\boldsymbol{j}$

$\therefore\ \dot{\boldsymbol{r}}(t) = 32\boldsymbol{i} + (24 - 10t)\boldsymbol{j}$

$\therefore\ \dot{\boldsymbol{r}}(2) = 32\boldsymbol{i} + 4\boldsymbol{j}$

$\therefore\ \tan\theta = \frac{4}{32} = \frac{1}{8}$

$\therefore\ \theta = \arctan\left(\frac{1}{8}\right)$

31 B $\boldsymbol{n} = -2\boldsymbol{i} + \boldsymbol{j} -$
$2\boldsymbol{k}$ is normal for Π_1 & Π_2. P is given point, Q is on plane
For A (4, 3, 4) $\overrightarrow{OP} = 4\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$
and $\overrightarrow{OQ} = x\boldsymbol{i} + y\boldsymbol{j} + 2\boldsymbol{k}$
$\therefore\ \overrightarrow{PQ} = (x - 4)\boldsymbol{i} + (y - 3)\boldsymbol{j} + (z - 4)\boldsymbol{k}$
Now scalar resolute of
$\overrightarrow{PQ}$ parallel $\boldsymbol{n} = \mathrm{d}$

Where $d = \dfrac{\overrightarrow{PQ} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$

$= \dfrac{-2(x-4)+(y-3)-2(z-4)}{3}$

$= \dfrac{-2x+8+y-3-2z+8}{3}$

$= \dfrac{-2x+y-2z+13}{3}$

$\therefore$ For π_1 $d = \dfrac{-3+13}{3} = \dfrac{10}{3}$

and for π_2 $d = \dfrac{-21+13}{3} = -\dfrac{8}{3}$

$\therefore$ Not equidistant.

For $B(2,4,6)$ $\overrightarrow{OP} = 2\boldsymbol{i}+4\boldsymbol{j}+6\boldsymbol{k}$.

$\therefore$ $\overrightarrow{PQ} = (x-2)\boldsymbol{i}+(y-4)\boldsymbol{j}+(z-6)\boldsymbol{k}$

$\therefore$ scalar resolute of

$\overrightarrow{PQ}$ parallel $\boldsymbol{n} = d$.

Where

$d = \dfrac{\overrightarrow{PQ} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$

$= \dfrac{-2(x-2)+(y-4)-2(z-6)}{3}$

$= \dfrac{-2x+4+y-4-2z+12}{3}$

$= \dfrac{-2x+y-2z+12}{3}$

For $\therefore$ π_1 $d = \dfrac{-3+12}{3} = 3$

And for π_2 $d = \dfrac{-21+12}{3} = -3$

$\therefore$ $(2,4,6)$ is equidistant

32 B $\sigma = 15.6$

$\Pr(|\bar{X}-\mu| < 2.0) = 0.99$

$\Pr(-2.0 < \bar{X}-\mu < 2.0) = 0.99$

$\dfrac{2.0}{\frac{15.6}{\sqrt{n}}} = 2.5758$

$\Rightarrow n = 404$

33 B $\mu = 40\,000$, $n = 30$, $\bar{x} = 33\,500$,

$\sigma = 3000$

$\Pr(\bar{X} \leq 38\,500) = 0.0031$

Solutions to extended-response questions

1 **a** **i** Solve the simultaneous equations $x + y = a$ and $x - y = b$ to find $x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$. Since a and b are even both $a + b$ and $a - b$ are even. Therefore x and y are integers.

ii Since a and b are even odd $a + b$ and $a - b$ are even. Therefore x and y are integers.

iii If a is even and b odd both $a + b$ and $a - b$ are odd. THere do not exist integers x and y.

b **i** Let a, b be even integers. Then from **a** there exists integers x and y such that $ab = (x + y)(x - y) = x^2 - y^2$.That is the product of two even integers can be expressed as a difference of perfect squares.

ii Let a, b be odd integers. Then from **a** there exists integers x and y such that $ab = (x + y)(x - y) = x^2 - y^2$.That is the product of two odd integers can be expressed as a difference of perfect squares.

c **i** Let a be an odd integer. Then $a = a \times 1$. That is, a is a product of odd integers, Then by **b** every odd integer is a difference of perfect squares.

ii If a is a multiple of 4 it can be written as $a = 2 \times c \times 2 \times d$ where c and d are integers. Then $2c$ and $2d$ are even integers and therefore a can be expressed as a difference of perfect squares.

2 **a** Consider $x^2 + y^2 + 2x + 4y = 20$

Implicitly differentiating,

$$2x + 2y\frac{dy}{dx} + 2 + 4\frac{dy}{dx} = 0$$

$$(2y + 4)\frac{dy}{dx} = -2x - 2$$

$$\frac{dy}{dx} = -\frac{2x + 2}{2y + 4}$$

When $x = 1$ and $y = 3$

$$\frac{dy}{dx} = -\frac{2}{5}$$

b $y - 3 = -\frac{2}{5}(x - 1)$

$5y - 15 = -2x + 2$

$5y + 2x = 17$

c $$x^2 + y^2 + 2x + 4y = 24$$

$$x^2 + 2x + 1 + y^2 + 4y + 4 = 29$$

$$(x + 1)^2 + (y + 2)^2 = 29$$

$$\frac{(x + 1)^2}{29} + \frac{(y + 1)^2}{29} = 1$$

Now from the parametric equations,

$$\frac{(x + 1)^2}{b^2} + \frac{(y + 1)^2}{d^2} = 1$$

So, $b = d = \sqrt{29}$

d **i** $x = -1 + b\cos t \Rightarrow \dfrac{dx}{dt} = -\sqrt{29}\sin(t)$ $y = -2 + d\sin t \Rightarrow \dfrac{dy}{dt} = \sqrt{29}\cos(t)$

$$A = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(-2 + \sqrt{29}\sin t)\sqrt{29}\,dt$$

$$= 2\pi\sqrt{29}\left[-2t - \sqrt{29}\cos t\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 2\pi\sqrt{29}\left(-\frac{2\pi}{3} - \frac{\sqrt{29}}{2} - \left(-\frac{\pi}{3} - \frac{\sqrt{3\times 29}}{2}\right)\right)$$

$$= 29\pi(\sqrt{3} - 1) - \frac{2\pi^2\sqrt{29}}{3}$$

ii $$A = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(-1 + \sqrt{29}\cos t)\sqrt{29}\,dt$$

$$= 2\pi\sqrt{29}\left[-t + \sqrt{29}\sin t\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 2\pi\sqrt{29}\left(-\frac{\pi}{3} + \frac{\sqrt{29\times 3}}{2} - \left(-\frac{\pi}{6} - \frac{\sqrt{29}}{2}\right)\right)$$

$$= 29\pi(\sqrt{3} - 1) - \frac{\sqrt{29}\pi^2}{3}$$

e $$\frac{dx}{dt} = \frac{dx}{dy}\times\frac{dy}{dt}$$

$= -\dfrac{5}{2}\times 2 = -5$ units per second

3 **a** $P(n)$ is the statement

$$\sum_{r=1}^{n} r(r^2+1) = \frac{1}{4}n(n+1)(n^2+n+2)$$

Step 1

To prove $P(1)$ is true.

$$\begin{aligned}\text{LHS} &= 1 \times (1+1)\\ &= 2\\ \text{LHS} &= \frac{1}{4} \times 1 \times 2 \times 4\\ &= 2\end{aligned}$$

$\therefore$ LHS = RHS

Step 2

Assuume $P(k)$ is true. That is

$$\sum_{r=1}^{k} r(r^2+1) = \frac{1}{4}k(k+1)(k^2+k+2)$$

Step 3

Prove $P(k+1)$ is true

$$\begin{aligned}\sum_{r=1}^{k+1} r(r^2+1) &= \sum_{r=1}^{k} r(r^2+1) + (k+1)((k+1)^2+1)\\ &= \frac{1}{4}k(k+1)(k^2+k+2) + (k+1)((k+1)^2+1)\\ &= (k+1)\left(\frac{1}{4}k(k^2+k+2) + ((k+1)^2+1\right)\\ &= \frac{(k+1)}{4}\left(k(k^2+k+2) + 4((k+1)^2+1\right)\\ &= \frac{(k+1)}{4}\left(k^3+k^2+2k+4(k^2+2k+1+1)\right)\\ &= \frac{(k+1)}{4}\left(k^3+5k^2+10k+8\right)\\ &= \frac{1}{4}(k+1)(k+2)(k^2+3k+4)\\ &= \frac{1}{4}(k+1)(k+1+1)((k+1)^2+(k+1)+2)\end{aligned}$$

By the principle of mathematical induction $P(n)$ is true for all n

b $n \leftarrow 0$

$sum \leftarrow 0$

while $sum < 10\,000$

  $n \leftarrow n + 1$

```
    sum ← n^3(n^2 + 1)
end while
print n
```

4 **a** Now $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) - (2\boldsymbol{i} + \boldsymbol{j} + 2k)$

$= \boldsymbol{j} - \boldsymbol{k}$

Now B is on line $(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$\therefore\ \boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k} + \mathrm{t}(\boldsymbol{j} - \boldsymbol{k})$

b Perpendicular to $\overrightarrow{OA} \therefore\ \overrightarrow{OA} = \boldsymbol{n}$

$\therefore \boldsymbol{n} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

Now $\boldsymbol{r} \cdot \boldsymbol{n} = k$

$\therefore\ \overrightarrow{OA} \cdot \boldsymbol{n} = k$

But $(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) \cdot (\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = 9$

$\therefore$ Equation of plane π is

$\boldsymbol{r} \cdot (\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = 9$

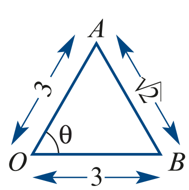
c Now $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \boldsymbol{j} - \boldsymbol{k}$

Now $(\boldsymbol{j} - \boldsymbol{k}) \cdot (\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = 0$

$\therefore\ \boldsymbol{j} - \boldsymbol{k}$ is perpendicular to $\boldsymbol{n}$

$\therefore\ \boldsymbol{j} - \boldsymbol{k}$ is parallel to π.

d



Now $\overrightarrow{OA} = 3 = \overrightarrow{OB}$ = radius length.

Now $AB = \sqrt{1^2 + 1^2} = \sqrt{2}$

$(\sqrt{x})^2 = 3^2 + 3^2 - 2 \times 3 \times 3 \times \cos\theta$

$\therefore\ 2 = 18 - 18\cos\theta$

$\therefore\ \cos\theta = \dfrac{16}{18} = \dfrac{8}{9}$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{8}{9}\right)$

Now $\quad l = r\theta$

$\therefore\ l = 3\cos^{-1}\left(\dfrac{8}{9}\right) = 1.42765$

$\therefore$ length of minor arc $\simeq 1.43$

e **Verification** $\overrightarrow{OA} \cdot (2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$

$$= (\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) \cdot (2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$$

$$= 2 + 4 - 6$$

$$= 0$$

$\therefore\ \overrightarrow{OA} \perp 2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$

And $\overrightarrow{OB} \cdot (2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$

$$= (2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) \cdot (2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$$

$$= 4 + 2 - 6$$

$$= 0$$

$\therefore\ \overrightarrow{OB} \perp 2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$

Now vector perpendicular to OAC plane is

smallskip

$\overrightarrow{OA} \times \overrightarrow{OC} = -2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

or $2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

angle between planes = angle between normals

$\boldsymbol{n}_1 = 2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k},\ \boldsymbol{n}_2 = -2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

$\therefore\ \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = -4 + 6 + 6 = 8$

$|\boldsymbol{n}_1| = \sqrt{17} = |\boldsymbol{n}_2|$

$$\therefore\ 8 = \sqrt{17} \cdot \sqrt{17} \cos\theta$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{8}{17}\right)$$

$\therefore$ angle $= 61.9^\circ$

5 **a** Now $l_1 \quad \boldsymbol{r} = (4 - t)\boldsymbol{i} + (2t - 3)\boldsymbol{j} + (t + 7)\boldsymbol{k}$

And $\pi_1 \quad 3x + 2y - z = -1.$

From $l_1 \quad x = 4 - t, y = 2t - 3, z = t + 7$

$\therefore 3x + 2y - z = 3(4 - t) + 2(2t - 3) - (t + 7)$

$$= 12 - 3t + 4t - 6 - t - 7$$

$$= -1$$

$\therefore\ l_1$ is on Π_1

b For $l_2 \quad \boldsymbol{r} = t\boldsymbol{i} + (10 + 3t)\boldsymbol{j} + (7 + 2t)\boldsymbol{k}$

$\therefore$ At point of intersection of l_2 and Π_1

$3t + 2(10 + 3t) - (7 + 2t) = -1$

$\therefore t = -2$

$\therefore$ Point of intersection $\boldsymbol{r} = -2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k} = (-2, 4, 3) = A.$

c Π_2 passes through $(-2, 4, 3) \perp l_1$

Now for l_1 $\boldsymbol{r} = 4\boldsymbol{i} - 3\boldsymbol{j} + 7\boldsymbol{k}\, t + (-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

$\therefore\ \boldsymbol{n} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ since l_1 is normal to Π_2

$\therefore\ \Pi_2$ is $\boldsymbol{r} \cdot (-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = k$

Using a $\boldsymbol{r} = -2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$

$\therefore\ (-2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}) \cdot (-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = k$

$\therefore\ k = 13$

$\therefore$ Equation of π_2 is $\boldsymbol{r} \cdot (-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = 13$

or $-x + 2y + z = 13$

d At point of intersection of l_1 and π_2

$-(4 - t) + 2(2t - 3) + (t + 7) = 13$

$$\therefore t = \frac{8}{3}$$

Now at $t = \frac{8}{3}$ l_1 $= (4 - \frac{8}{3})\boldsymbol{i} + (2 \times \frac{8}{3} - 3)\boldsymbol{j} + (\frac{8}{3} + 7)\boldsymbol{k}$

$\therefore\ l_1 = \frac{4}{3}\boldsymbol{i} + \frac{7}{3}\boldsymbol{j} + \frac{29}{3}\boldsymbol{k}$

$\therefore$ Point of intersection $= \left(\frac{4}{3}, \frac{7}{3}, \frac{29}{3}\right)$

e Now for $l_1, \boldsymbol{d}_1 = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

Assume for $l_3, \boldsymbol{d}_3 = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$

$\therefore$ For l_3, $\boldsymbol{r} = -2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k} + t\,\boldsymbol{d}_3$

$= -2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k} + t(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})$

$= (-2 + at)\boldsymbol{i} + (4 + bt)\boldsymbol{j} + (3 + ct)\boldsymbol{k}$

But $\boldsymbol{d}_3 \cdot \boldsymbol{d}_1 = 0$ (since $l_3 \perp l_1$)

$\therefore -a + 2b + 3c = 0$ (1)

Also $\boldsymbol{r}$ is in Π_1

$\therefore\ 3(-2 + at) + 2(4 + bt) - (3 + ct) = -1$

$\therefore -6 + 3at + 8 + 2bt - 3 - ct = -1$

$\therefore 3at + 2bt - ct = 0$

$\therefore 3a + 2b - c = 0$ (2)

From (1) & (2) $2a + 4b = 0$

$\therefore\ a + 2b = 0$

Now if $a = 1, 1 + 2b = 0, 3 + 2b - c = 0$

$\therefore\ a = 1, b = -\frac{1}{2}, c = 2$

$\therefore\ \boldsymbol{d}_3 = \boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore$ For l_3, $\quad \boldsymbol{r} = -2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k} + t_1(\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + 2\boldsymbol{k})$

or $\boldsymbol{r} = -2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k} + t_2(2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$

6 a π_1 is given by $\boldsymbol{r} \cdot (2\boldsymbol{i} + 3\boldsymbol{j}) = -6$

$\therefore \ \boldsymbol{n} = 2\boldsymbol{i} + 3\boldsymbol{j}$

If line passes through $P(2, 1, 4)$, vector equation is $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k} + t(2\boldsymbol{i} + 3\boldsymbol{j})$

b Now $\boldsymbol{r} = (2 + 2t)\boldsymbol{i} + (1 + 3t)\boldsymbol{j} + 4\boldsymbol{k}$.

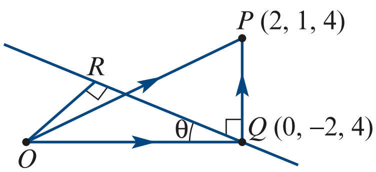
Now at $Q \qquad 2x + 3y = -6$

$\therefore 2(2 + 2t) + 3(1 + 3t) = -6$

$t = -1$

$\therefore$ At $Q \qquad \boldsymbol{r} = -2\boldsymbol{j} + 4\boldsymbol{k} \ \therefore \ Q = (0, -2, 4)$

c



$\overrightarrow{QP} = \boldsymbol{n} = 2\boldsymbol{i} + 3\boldsymbol{j}$

Now $\overrightarrow{OR} = \lambda(2\boldsymbol{i} + 3\boldsymbol{j}) = 2\lambda\boldsymbol{i} + 3\lambda\boldsymbol{j}$

and $\overrightarrow{OQ} = -2\boldsymbol{j} + 4\boldsymbol{k}$ so $\left|\overrightarrow{OQ}\right| = \sqrt{20} = 2\sqrt{5}$

But R is on $\Pi_1 \ \therefore \ (2\lambda\boldsymbol{i} + 3\lambda\boldsymbol{j}) \cdot (2\boldsymbol{i} + 3\boldsymbol{j}) = -6$

$\therefore \ 4\lambda + 9\lambda = -6$

$\therefore \ \lambda = -\frac{6}{13}$

$\therefore \ \overrightarrow{OR} = -\frac{12}{13}\boldsymbol{i} - \frac{18}{13}\boldsymbol{j}$

$$\therefore \ \text{In } \Delta ORQ \ \sin\theta = \frac{OR}{OQ} = \frac{\sqrt{\left(\frac{-12}{13}\right)^2 + \left(\frac{-18}{13}\right)^2}}{2\sqrt{5}}$$

$\therefore \ \sin\theta = \frac{3}{\sqrt{65}} = 0.372$

$\therefore$ angle between OQ and $\Pi_1 = 21.85°$

d For $\Pi_1 \qquad 2x + 3y = -6 \qquad (1)$

For $\Pi_2 \qquad x + y + z = 5 \qquad (2)$

For $\Pi_3 \qquad x = 0 \qquad (3)$

$\therefore \ x = 0, y = -2, z = 7$

$\therefore$ Point of intersection at $(0, -2, 7)$

7 a $S - R$ is a random variable of a normal distributionwith $\mathrm{E}(S - R) = 12 - 10 = 2$ and $\mathrm{Var}\,(S - R) = 9 + 16 = 25$

$\therefore \mu = 2$ and $\sigma = 5$

$\Pr(R < S) = \Pr(S > R) = \Pr(S - R > 0) = 0.6554\ldots.$

b $2R - S_1 - S_2$ is normally distributed with $\mathrm{E}(2R - S_1 - S_2) = 20 - 24 = -4$ and $\mathrm{Var}\,(2R - S_1 - S_2) = 4 \times 9 + 16 + 16 = 68$

$\therefore \mu = -4$ and $\sigma = \sqrt{68}$

$\Pr(2R > S_1 + S_2) = \Pr(2R - S_1 - S_2 > 0) = 0.3138\ldots$

8 a Let $T = \sum_1^8 T_i$ where the T_i are independent random variables of the thickness of each piece of paper.

$E(T) = 8 \times 0.1 = 0.8$ mm and $\mathrm{Var}(T) = 8 \times 0.005^2$. Therefore, $\mathrm{sd}(T) = 0.0143\ldots$ mm.

b Let T_1 be the random variable of the thickness of one sheet of paper. $\mathrm{E}(8T_1) = 8 \times 0.1 = 0.8$ mm and $\mathrm{Var}(8T_1) = 64 \times 0.005^2$. Therefore, $\mathrm{sd}(8T_1) = 0.04$ mm

9 a Let L_1, L_2, L_3, L_4 be identical independent random variables that each give the length of a tile.

$\mathrm{E}(L_1 + L_2 + L_3 + L_4 = 20 \times 4 = 80$

$\Pr(L_1 + L_2 + L_3 + L_4 > 80) = 0.5$

b $\frac{1}{2}L - W$ is the random variable of a normal distributionwith $\mathrm{E}(\frac{1}{2}L - W) = 0$

$\Pr(\frac{1}{2}L - W) > 0) = 0.5$

c $\mathrm{E}(S - T) = 50 \times 20 - 80 \times 10 = 200$ cm

and $\mathrm{Var}(S - T) = 50 \times 0.01 + 80 \times 0.01 = 1.3\ \mathrm{cm}^2$

10 a For $x > 0,\ f'(x) = \log_e x + x \times \frac{1}{x} - 3 = \log_e x - 2$ (using the product rule).

b For $x > 0$, solve $f(x) = 0$: $x(\log_e x - 3) = 0$

$$\log_e x = 3 \Rightarrow x = e^3$$

So the coordinates of A are $(e^3, 0)$.

c At A, $x = e^3$ and the gradient of the tangent is $f'(e^3) = 1og_e e^3 - 2 = 3 - 2 = 1$.

So the equation of the tangent is $y - 0 = 1(x - e^3)$, i.e. $y = x - e^3$.

d For x values between O and A, the graph lies below the x axis (check with a CAS or simply note that $f(1) = -3$). The tangent cuts the y axis when $x = 0$, i.e. at $y = -e^3$.

So the area bounded by the tangent and the coordinate axes is $\frac{1}{2} \times e^3 \times e^3 = \frac{1}{2}e^6$

(area of triangle formula).

The area bounded by the graph of $y = f(x)$ and the x axis is

$$-\int_0^{e^3} f(x)\,dx = -\int_0^{e^3} (x\log_e x - 3x)\,dx$$

$$= \frac{1}{4}e^6 \text{ (using a CAS since the integrand is non-standard)}$$

This is half the previous area, so the required ratio is 2:1.

11 **a** $y = \dfrac{a + b\sin x}{b + a\sin x},\ 0 < a < b$

i Use the quotient rule:

$$\frac{dy}{dx} = \frac{(b + a\sin x)(b\cos x) - (a + b\sin x)(a\cos x)}{(b + a\sin x)^2}$$

$$= \frac{b^2\cos x + ab\sin x\cos x - a^2\cos x - ab\sin x\cos x}{(b + a\sin x)^2}$$

$$= \frac{(b^2 - a^2)\cos x}{(b + a\sin x)^2}$$

ii The derivative is zero when $\cos x = 0$. Since this occurs at multiples of $\dfrac{\pi}{2}$, it follows that the corresponding values of $\sin x$ are 1 and −1. Now when $\sin x = 1,\ y = \dfrac{a + b}{b + a} = 1$; when $\sin x = -1,\ y = \dfrac{a - b}{b - a} = -1$. So these are the maximum and minimum values of y.

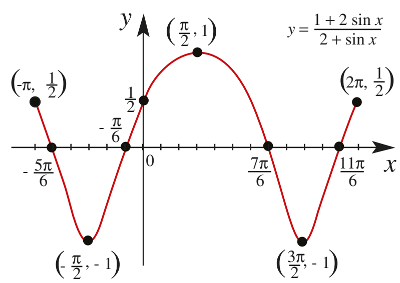
b $y = \dfrac{1 + 2\sin x}{2 + \sin x}, -\pi \le x \le 2\pi$

i $x = 0,\ y = \dfrac{1}{2}$, so the y intercept has coordinates $\left(0,\ \dfrac{1}{2}\right)$.

ii $y = 0,\ \sin x = -\dfrac{1}{2} \Rightarrow x = -\dfrac{5\pi}{6},\ -\dfrac{\pi}{6},\ \dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$ on $[-\pi,\ 2\pi]$. So the x intercepts have coordinates $\left(-\dfrac{5\pi}{6},\ 0\right), \left(-\dfrac{\pi}{6}, 0\right), \left(\dfrac{7\pi}{6},\ 0\right)$ and $\left(\dfrac{11\pi}{6},\ 0\right)$.

iii On $[-\pi,\ 2\pi],\ \sin x = -1$ at $x = -\dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$; $\sin x = 1$ at $x = \dfrac{\pi}{2}$. So the stationary points have coordinates $\left(-\dfrac{\pi}{2},\ -1\right), \left(\dfrac{\pi}{2},\ 1\right)$ and $\left(\dfrac{3\pi}{2},\ -1\right)$.

iv



v The required area, using area between two curves, is

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{1 + 2\sin x}{2 + \sin x} - (-1)\right) dx = 2\pi(3 - \sqrt{3}) \text{ using a CAS to do the integration}$$

(since it is not a standard type).

12 a

$$r\cos(x - a) = \cos x + \sqrt{3}\sin x$$

$$r\cos x\cos a + r\sin x\sin a = \cos x + \sqrt{3}\sin x$$

$$r\cos a = 1 \quad ①$$

$$r\sin a = \sqrt{3} \quad ②$$

$①^2 + ②^2$ gives $r^2 = 4$, so $r = 2$.

Then from ①, $\cos a = \dfrac{1}{2} \Rightarrow a = \dfrac{\pi}{3}$ since $0 < a < \dfrac{\pi}{2}$.

Thus $\cos x + \sqrt{3}\sin x = 2\cos\left(x - \dfrac{\pi}{3}\right)$.

b Since $-1 \le \cos \le 1$, the range is $[-2, 2]$.

c $x = 0$, $y = 1$, so the y-intercept has coordinates $(0, 1)$.

d $y = 0$, $\cos x + \sqrt{3}\sin x = 0$. This can be re-arranged and solved on $[0, 2\pi]$:

$$\sqrt{3}\sin x = -\cos x$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6}, \frac{11\pi}{6}$$

(Alternatively, solve the equation in the equivalent form $2\cos\left(x - \dfrac{\pi}{3}\right) = 0$.)

So the coordinates are $\left(\dfrac{5\pi}{6}, 0\right)$ and $\left(\dfrac{11\pi}{6}, 0\right)$.

e Using the equivalent form gives:

$$2\cos\left(x-\frac{\pi}{3}\right) = \sqrt{2}$$

$$\cos\left(x-\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$$

$$x-\frac{\pi}{3} = -\frac{\pi}{4}, \frac{\pi}{4}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}$$

f $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sqrt{3}\sin x}\,dx = \frac{\log_e(3+2\sqrt{3})}{2}$ using a CAS to do the integration (since it is not a standard type). Alternatively, a by-hand solution using the equivalent form can be found with some effort.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2\cos\left(x-\frac{\pi}{3}\right)}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos\left(x-\frac{\pi}{3}\right)}{\cos^2\left(x-\frac{\pi}{3}\right)}\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos\left(x-\frac{\pi}{3}\right)}{1-\sin^2\left(x-\frac{\pi}{3}\right)}\,dx$$

$$= \frac{1}{2}\int_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{1-u^2}\,du$$

using the substitution $u = \sin\left(x-\frac{\pi}{3}\right)$.

Now partial fractions gives $\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)$.

So the integral becomes:

$$\frac{1}{4}\int_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)du = \frac{1}{4}[-\log_e|1-u|+\log_e|1+u|]_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{4}\left[\log_e\left|\frac{1+u}{1-u}\right|\right]_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{4}\left(\log_e 3 - \log_e \frac{2-\sqrt{3}}{2+\sqrt{3}}\right)$$

$$= \frac{1}{4}\log_e\left(\frac{3(2+\sqrt{3})}{2-\sqrt{3}}\right)$$

$$= \frac{1}{4}\log_e\left(\frac{3(2+\sqrt{3})}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)$$

$$= \frac{1}{4}\log_e 3(2+\sqrt{3})^2$$

$$= \frac{1}{4}\log_e 3(7+4\sqrt{3})$$

$$= \frac{\log_e(21+12\sqrt{3})}{4}$$

The answer appears different from the CAS answer, but they are in fact the same:

$$\frac{\log_e(3+2\sqrt{3})}{2} = \frac{2\log_e(3+2\sqrt{3})}{4}$$

$$= \frac{\log_e(3+2\sqrt{3})^2}{4}$$

$$= \frac{\log_e(9+12+12\sqrt{3})}{4}$$

$$= \frac{\log_e(21+12\sqrt{3})}{4}$$

(Of course, a CAS is very quick, but it is instructive to see the by-hand solution.)

g Since the graph starts at (0, 1) and cuts the x axis at $\frac{5\pi}{6}$, the volume V is given by

$$V = \int_0^{\frac{5\pi}{6}} \pi(f(x))^2\,dx$$

$$= \pi\int_0^{\frac{5\pi}{6}} 4\cos^2\left(x-\frac{\pi}{3}\right)dx$$

$$= 2\pi\int_0^{\frac{5\pi}{6}}\left(\cos 2\left(x-\frac{\pi}{3}\right)+1\right)dx$$

using the alternative form and a double angle substitution. Thus:

$$V = \pi \int_0^{\frac{5\pi}{6}} \left(2\cos 2\left(x - \frac{\pi}{3}\right) + 2\right) dx$$

$$= \pi\left[\sin 2\left(x - \frac{\pi}{3}\right) + 2x\right]_0^{\frac{5\pi}{6}}$$

$$= \pi\left(\sin \pi + \frac{5\pi}{3} - \sin\left(-\frac{2\pi}{3}\right)\right)$$

$$= \pi\left(\frac{5\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{6}(10\pi + 3\sqrt{3})$$

13 a $\frac{dv}{dt} = -\frac{v}{50}(1 + v^2),\ t > 0$, initial velocity of 10 m/s.

i Inverting gives $\frac{dt}{dv} = -\frac{50}{v(1 + v^2)}$, so the time taken to go from 10 m/s to 5 m/s is given by $-\int_{10}^{5} \frac{50}{v(1 + v^2)}\, dv$.

ii Using a CAS for evaluation gives $25 \log_e\left(\frac{104}{101}\right)$ seconds.

(Alternatively, the substitution $u = v^2$ eventually leads to this result.)

b i

$$\frac{dv}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

$$= \frac{dv}{dt} \div \frac{dx}{dt}$$

$$= -\frac{v}{50}(1 + v^2) \div v$$

$$= \frac{-(1 + v^2)}{50}$$

ii

$$\frac{dx}{dv} = -\frac{50}{1 + v^2}$$

$$x = -50\tan^{-1} v + c$$

When $x = 0,\ v = 10.$

$$0 = -50\tan^{-1} 10 + c$$

$$c = 50\tan^{-1} 10$$

$$x = 50(\tan^{-1} 10 - \tan^{-1} v)$$

iii $\dfrac{x}{50} = \tan^{-1} 10 - \tan^{-1} v$

$$\tan^{-1} v = \tan^{-1} 10 - \frac{x}{50}$$

$$v = \tan\left(\tan^{-1} 10 - \frac{x}{50}\right)$$

$$= \frac{10 - \tan\left(\frac{x}{50}\right)}{1 + 10\tan\left(\frac{x}{50}\right)} \text{ (using the formula for } \tan(A - B))$$

iv From part **iii**, when $v = 0$, $x = 50\tan^{-1} 10 \approx 73.56$. So the displacement of the particle when it first comes to rest is 74 metres, correct to the nearest metre.

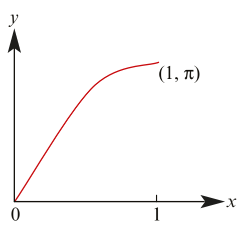
14

$f(x) = \sin(\pi x) + px$

a i $f'(x) = \pi\cos(\pi x) + p$, so $f'(1) = \pi\cos(\pi) + p = -\pi + p$. Then $f'(1) = 0$ if $p = \pi$.

ii Substituting $p = \pi$ gives $f'(x) = \pi\cos(\pi x) + \pi = \pi(1 + \cos(\pi x))$.
Since the least value of $\cos(\pi x)$ is -1, then $1 + \cos(\pi x) \geq 0$, so $f'(x) \geq 0$.

b



c $V = \int_0^1 \pi(f(x))^2 \, dx$

$$= \pi \int_0^1 (\sin(\pi x) + \pi x)^2 \, dx$$

$$= \pi \int_0^1 (\pi^2 x^2 + 2\pi x \sin(\pi x) + \sin^2(\pi x)) \, dx$$

$$= \int_0^1 \left(\pi^3 x^2 + 2\pi^2 x \sin(\pi x) + \frac{\pi}{2}(1 - \cos(2\pi x))\right) dx$$

$$= \left[\frac{\pi^3 x^3}{3} + 2\sin(\pi x) - 2\pi x \cos(\pi x) + \frac{\pi}{2}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right)\right]_0^1 \text{ (using part } \mathbf{aii}\text{)}$$

$$= \left(\frac{\pi^3}{3} + 2\pi + \frac{\pi}{2}\right) - 0$$

$$= \frac{\pi^3}{3} + \frac{5\pi}{2}$$

$$= \frac{(2\pi^2 + 15)\pi}{6}$$

d $f(x) = \sin(\pi x) + \pi x$, so $f(1) = \sin(\pi) + \pi = \pi$.

$g(x) = k \arcsin(x)$, so $g(1) = k \arcsin(1) = k \times \dfrac{\pi}{2}$.

So $f(1) = g(1)$ if $k = 2$.

e A quick plot of the graph of $y = g(x) = 2\arcsin(x)$ shows that it lies beneath the graph of $y = f(x)$ on $(0, 1)$. So the required area, using area between two curves, is

$$\int_0^1 (f(x) - g(x)) \, dx = \int_0^1 (\sin(\pi x) + \pi x - 2\arcsin(x)) \, dx$$

$$= 1.0658$$

using a CAS to do the integration (since it is not a standard type).

Thus, the area of the region enclosed by the two graphs is 1.066, correct to 3 decimal places.

f From above, $y(x) = f(x) - g(x) \geq 0$ on $[0, 1]$ and $y(0) = y(1) = 0$. So there will be a maximum on $(0, 1)$.

$y'(x) = f'(x) - g'(x)$

$$= \pi \cos(\pi x) + \pi - \frac{2}{\sqrt{1 - x^2}}$$

$$y'(x) = 0 \Rightarrow \pi \cos(\pi x) + \pi - \frac{2}{\sqrt{1 - x^2}} = 0$$

Use the 'solve' command of a CAS to find the value of x on $(0, 1)$.
This gives $x = 0.57189\ldots$, so the value of a is 0.572, correct to 3 dp.

15 a

16 a Distance of A From O is $\sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$.

Distance of B from O is $m\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}m$.
So if the points are equidistant from O, then $m = \sqrt{3}$.

b **i** $\overrightarrow{OA} = a$, $\overrightarrow{OC} = -a = -\overrightarrow{OA}$, so A, O and C are collinear. Thus AC is a diameter of the circle. Also, note that from part **a**, the radius of the circle is 3 units.

ii
$$\overrightarrow{AB} = (\sqrt{3} - 2)\boldsymbol{i} + (\sqrt{3} + 1)\boldsymbol{j} + (-\sqrt{3} - 2)\boldsymbol{k}$$
$$\overrightarrow{CB} = (\sqrt{3} + 2)\boldsymbol{i} + (\sqrt{3} - 1)\boldsymbol{j} + (-\sqrt{3} + 2)\boldsymbol{k}$$
$$\overrightarrow{AB} \cdot \overrightarrow{CB} = (3 - 4) + (3 - 1) + (3 - 4) = 0$$
As the vectors are non-zero, they are perpendicular. So $\angle ABC = 90°$.
(Alternatively, you can do this without writing the vectors in component form:
$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a},\ \overrightarrow{CB} = \boldsymbol{b} + \boldsymbol{a}$$
$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{CB} &= (\boldsymbol{b} - \boldsymbol{a})\cdot(\boldsymbol{b} + \boldsymbol{a})\\ &= \boldsymbol{b} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{a}\\ &= b^2 - a^2 = 9 - 9 = 0\end{aligned}$$
since $a = b = 3$, the radius of the circle.)

c **i** If D is a point on the circle with position vector $\boldsymbol{d}$, then $\boldsymbol{d} \cdot \boldsymbol{d} = d^2 = 9$ (since the radius of the circle is 3). Thus:
$$(k\boldsymbol{a} + l\boldsymbol{b})\cdot(k\boldsymbol{a} + l\boldsymbol{b}) = 9$$
$$k^2a^2 + 2kl\boldsymbol{a}\cdot\boldsymbol{b} + l^2b^2 = 9$$
Now $a^2 = b^2 = 9$ and $\boldsymbol{a}\cdot\boldsymbol{b} = \sqrt{3}(2 - 1 - 2) = -\sqrt{3}$, so substituting gives
$9k^2 - 2\sqrt{3}kl + 9l^2 = 9$ [*]

ii Substitute $k = 1$ into [*] and solve for l.

$$9l^2 - 2\sqrt{3}l + 9 = 9$$

$$l(9l - 2\sqrt{3}) = 0$$

$$l = 0,\ \frac{2\sqrt{3}}{9}$$

$$k = 1,\ l = 0: \boldsymbol{d} = \boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

$$k = 1,\ l = \frac{2\sqrt{3}}{9}: \boldsymbol{d} = \boldsymbol{a} + \frac{2\sqrt{3}}{9}\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} + \frac{2}{3}(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) == \frac{8}{3}\boldsymbol{i} - \frac{1}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$$

d Let P have position vector $\boldsymbol{d}$ for some values of k and l. As OP bisects AB, then OP is also perpendicular to AB (a radius bisects any chord at right angles). Thus:

$$(k\boldsymbol{a} + l\boldsymbol{b})\cdot(\boldsymbol{b} - \boldsymbol{a}) = 0$$

$$lb^2 - ka^2 + (k - l)\boldsymbol{a}\cdot\boldsymbol{b} = 0$$

From part **ci**, $a^2 = b^2 = 9$ and $\boldsymbol{a}\cdot\boldsymbol{b} = -\sqrt{3}$, so this gives $9(l - k) + \sqrt{3}(l - k) = 0$.

The solution to this equation is $l = k$. Substitute into [*]:

$(18 - 2\sqrt{3})k^2 = 9 \Rightarrow k = \dfrac{3}{\sqrt{18 - 2\sqrt{3}}} = l$ (+ve $\sqrt{\ }$ so P on arc AB)

Then $\boldsymbol{d} = k\boldsymbol{a} + l\boldsymbol{b}$

$$= \frac{3}{\sqrt{18 - 2\sqrt{3}}}(\boldsymbol{a} + \boldsymbol{b})$$

$$= \frac{3}{\sqrt{18 - 2\sqrt{3}}}((\sqrt{3} + 2)\boldsymbol{i} + (\sqrt{3} - 1)\boldsymbol{j} + (-\sqrt{3} + 2)\boldsymbol{k})$$

e $k\boldsymbol{a} + l\boldsymbol{b} = (2k + \sqrt{3}l)\boldsymbol{i} + (-k + \sqrt{3}l)\boldsymbol{j} + (2k - \sqrt{3}l)\boldsymbol{k}$

$$\mathbf{r} = (5 - t)\boldsymbol{i} + (2 + t)\boldsymbol{j} + (t - 3)\boldsymbol{k}$$

Equate components:

$2k + \sqrt{3}l = 5 - t$ ①

$-k + \sqrt{3}l = 2 + t$ ②

$2k - \sqrt{3}l = t - 3$ ③

① + ③: $4k = 2,\ \text{so } k = \frac{1}{2}$.

② + ③: $k = 2t - 1,\ \text{so } t = \frac{3}{4}$.

Subst. in ①: $1 + \sqrt{3}l = \frac{17}{4} \Rightarrow l = \frac{13}{4\sqrt{3}} = \frac{13\sqrt{3}}{12}$.

f $\boldsymbol{r}\left(\frac{3}{4}\right) = \frac{17}{4}\boldsymbol{i} + \frac{11}{4}\boldsymbol{j} - \frac{9}{4}\boldsymbol{k}$

$\left|\boldsymbol{r}\left(\frac{3}{4}\right)\right| = \frac{1}{4}\sqrt{17^2 + 11^2 + 9^2} = \frac{\sqrt{491}}{4} \approx 5.54$

This is greater than 3, the radius of the circle, so the particle lies outside the circle at this time.

17 **a** $x = 3\sin(t),\ y = 6\cos(t) - a,\ 0 < a < 6.$

i Using $\sin^2(t) + \cos^2(t) = 1$ gives $\frac{x^2}{9} + \frac{(y+a)^2}{36} = 1$.

ii Substitute $y = 0$ and solve for x:

$$\frac{x^2}{9} + \frac{a^2}{36} = 1$$

$$\frac{x^2}{9} = 1 - \frac{a^2}{36}$$

$$= \frac{36 - a^2}{36}$$

$$x^2 = \frac{36 - a^2}{4}$$

$$x = \pm\frac{\sqrt{36 - a^2}}{2}$$

b $\frac{(y+a)^2}{36} = 1 - \frac{x^2}{9}$

$$= \frac{9 - x^2}{9}$$

$(y+a)^2 = 4(9 - x^2)$

$y + a = 2\sqrt{9 - x^2}$ (negative square root gives curve below x axis)

$y = 2\sqrt{9 - x^2} - a$

So the required function f has rule $f(x) = 2\sqrt{9 - x^2} - a$.

c $\frac{d}{dx}(x\sqrt{9 - x^2}) = \sqrt{9 - x^2} + x \times \frac{1}{2}(-2x)(9 - x^2)^{-\frac{1}{2}}$

$$= \sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}}$$

d **i** $$\frac{A}{\sqrt{9-x^2}} - \sqrt{9-x^2} = \frac{A-(9-x^2)}{\sqrt{9-x^2}}$$
$$= \frac{x^2+A-9}{\sqrt{9-x^2}}$$
$$= \frac{x^2}{\sqrt{9-x^2}} \quad \text{if } A = 9$$

ii $$\sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}} = \sqrt{9-x^2} - \left(\frac{9}{\sqrt{9-x^2}} - \sqrt{9-x^2}\right) \text{ from part } \mathbf{i}$$
$$= 2\sqrt{9-x^2} - \frac{9}{\sqrt{9-x^2}}$$

e $$\int\left(\sqrt{9-x^2} - \frac{x^2}{\sqrt{9-x^2}}\right)dx = x\sqrt{9-x^2} \text{ from part } \mathbf{c}$$
$$\int\left(2\sqrt{9-x^2} - \frac{9}{\sqrt{9-x^2}}\right)dx = x\sqrt{9-x^2} \text{ using the result of part } \mathbf{d\ ii}$$
$$2\int\sqrt{9-x^2}\,dx = \int\frac{9}{\sqrt{9-x^2}}\,dx + x\sqrt{9-x^2}$$
$$= 9\sin^{-1}\left(\frac{x}{3}\right) + x\sqrt{9-x^2}$$
$$\int\sqrt{9-x^2}\,dx = \frac{1}{2}\left(x\sqrt{9-x^2} + 9\sin^{-1}\left(\frac{x}{3}\right)\right)$$

f Using symmetry together with the results in parts **a ii**, **b** and **e**, the required area A is given by

$$A = 2\int_0^{\frac{\sqrt{36-a^2}}{2}}\left(2\sqrt{9-x^2} - a\right)dx$$
$$= 2\left[x\sqrt{9-x^2} + 9\sin^{-1}\left(\frac{x}{3}\right) - ax\right]_0^{\frac{\sqrt{36-a^2}}{2}}$$
$$= 2\left(\frac{\sqrt{36-a^2}}{2} \times \sqrt{9 - \frac{36-a^2}{4}} + 9\sin^{-1}\left(\frac{\sqrt{36-a^2}}{6}\right) - \frac{a\sqrt{36-a^2}}{2}\right)$$
$$= 2\left(\frac{a\sqrt{36-a^2}}{4} + 9\sin^{-1}\left(\frac{\sqrt{36-a^2}}{6}\right) - \frac{a\sqrt{36-a^2}}{2}\right)$$
$$= 18\sin^{-1}\left(\frac{\sqrt{36-a^2}}{6}\right) - \frac{a\sqrt{36-a^2}}{2}$$

g From the result of part **a i**, the curve is an ellipse.
From part **f** with $a = 0$, $A = 18\sin^{-1}(1) = 9\pi$.

Now $a = 0$ corresponds to half the ellipse lying above the x axis.
So the area of the region enclosed by the curve is 18π.

h Use $a = 0$ and rotate the curve with Cartesian equation $y = f(x) = 2\sqrt{9 - x^2}$ about the y axis to find the required volume V. Thus V is given by

$$
\begin{aligned}
V &= 2\int_0^3 \pi(f(x))^2\,dx \\
&= 2\pi\int_0^3 (2\sqrt{9 - x^2})^2\,dx \\
&= 2\pi\int_0^3 4(9 - x^2)\,dx \\
&= 2\pi\left[36x - \frac{4}{3}x^3\right]_0^3 \\
&= 2\pi(108 - 36) \\
&= 144\pi
\end{aligned}
$$

18 $x = t^2,\ y = \frac{1}{3}t^3 - t$

a
$$\begin{aligned} y &= \frac{1}{3}t^3 - t \\ &= t\left(\frac{1}{3}t^2 - 1\right) \\ y^2 &= t^2\left(\frac{1}{3}t^2 - 1\right)^2 \\ &= x\left(\frac{x}{3} - 1\right)^2 \end{aligned}$$

So $g(x) = x\left(\frac{x}{3} - 1\right)^2$.

b $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$= \frac{t^2 - 1}{2t}$

Then stationary points occur for $t^2 - 1 = 0$, i.e. $t = \pm 1$.

$t = 1 : x = 1,\ y = -\frac{2}{3};\ t = -1 : x = 1,\ y = \frac{2}{3}$.

So the stationary points are $\left(1, \pm\frac{2}{3}\right)$.

c $g(x) = x\left(\frac{x}{3} - 1\right)^2 = 0$ when $x = 0, 3$. From this and the stationary points in part **c**, the curve has a loop to the right of the y axis (you can confirm this by plotting the curve with a CAS). So the area A of the region enclosed by the curve is given by

$$A = 2\int_0^3 y\,dx$$

$$= 2\int_0^{-\sqrt{3}} y\frac{dx}{dt}\,dt$$

$$= 2\int_0^{-\sqrt{3}} \left(\frac{1}{3}t^3 - t\right)(2t)\,dt$$

$$= 4\int_0^{-\sqrt{3}} \left(\frac{1}{3}t^4 - t^2\right)dt$$

$$= 4\left[\frac{1}{15}t^5 - \frac{1}{3}t^3\right]_0^{-\sqrt{3}}$$

$$= 4\left(-\frac{3\sqrt{3}}{5} + \sqrt{3}\right)$$

$$= \frac{8\sqrt{3}}{5}$$

(Note: when the parameter is introduced, $x = 3$ corresponds to either $t = \sqrt{3}$ or $t = -\sqrt{3}$. But the formula for y shows that between $t = 0$ and $t = \sqrt{3}$, y is negative while between $t = 0$ and $t = -\sqrt{3}$, y is positive; so the upper terminal should be $t = -\sqrt{3}$ or you will get a negative result.)

d $V = \int^3 \pi(f(x))^2\,dx$

$$= \pi\int_0^3 x\left(\frac{x}{3} - 1\right)^2 dx$$

$$= \pi\int_0^3 \left(\frac{x^3}{9} - \frac{2x^2}{3} + x\right)dx$$

$$= \pi\left[\frac{x^4}{36} - \frac{2x^3}{9} + \frac{x^2}{2}\right]_0^3$$

$$= \pi\left(\frac{9}{4} - 6 + \frac{9}{2}\right)$$

$$= \frac{3}{4}\pi$$

19 $x = \sin t,\ y = \sin 4t,\ 0 \le t \le 2\pi.$

a $y = \sin 4t$

$= 2\sin 2t \cos 2t$

$= 4\sin t \cos t(1 - 2\sin^2 t)$

$= 4x\cos t(1 - 2x^2)$

$y^2 = 16x^2\cos^2 t(1 - 2x^2)^2$

$= 16x^2(1 - \sin^2 t)(1 - 2x^2)^2$

$= 16x^2(1 - x^2)(1 - 2x^2)^2$

b $\dfrac{dx}{dt} = \cos t,\ \dfrac{dy}{dt} = 4\cos 4t$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

$= \dfrac{4\cos 4t}{\cos t}$

c **i** $\dfrac{dy}{dx} = 0$ when $\cos 4t = 0$, so $4t = (2n+1)\dfrac{\pi}{2} \Rightarrow t = (2n+1)\dfrac{\pi}{8},\ n = 0, 1, \ldots$.

In the given domain, this gives solutions $\dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}, \dfrac{9\pi}{8}, \dfrac{11\pi}{8}, \dfrac{13\pi}{8}, \dfrac{15\pi}{8}$.

ii $x^2 = \sin^2 t = \dfrac{1}{2}(1 - \cos 2t)$; for the t values in part **i**, the corresponding values of $\cos 2t$ are $\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}$.

Then x^2 takes just two values, $\dfrac{2 - \sqrt{2}}{4}$ and $\dfrac{2 + \sqrt{2}}{4}$, so x takes the four values

$-\dfrac{\sqrt{2 - \sqrt{2}}}{2}, -\dfrac{\sqrt{2 + \sqrt{2}}}{2}, \dfrac{\sqrt{2 - \sqrt{2}}}{2}$ and $\dfrac{\sqrt{2 - \sqrt{2}}}{2}$.

iii For the t values in part **i**, $y = \sin 4t$ alternates $1, -1, 1, -1, \ldots$. So the coordinates of the stationary points are

$\left(-\dfrac{\sqrt{2 - \sqrt{2}}}{2}, 1\right), \left(-\dfrac{\sqrt{2 - \sqrt{2}}}{2}, -1\right), \left(-\dfrac{\sqrt{2 + \sqrt{2}}}{2}, 1\right), \left(-\dfrac{\sqrt{2 + \sqrt{2}}}{2}, -1\right),$

$\left(\dfrac{\sqrt{2 - \sqrt{2}}}{2}, 1\right), \left(\dfrac{\sqrt{2 - \sqrt{2}}}{2}, -1\right), \left(\dfrac{\sqrt{2 + \sqrt{2}}}{2}, 1\right), \left(\dfrac{\sqrt{2 + \sqrt{2}}}{2}, -1\right)$

iv At $x = \frac{1}{\sqrt{2}}$, $t = \frac{\pi}{4}, \frac{3\pi}{4}$ so $\frac{dy}{dx} = \frac{4\cos 4t}{\cos t} = -4\sqrt{2}, 4\sqrt{2}$.

At $x = -\frac{1}{\sqrt{2}}$, $t = \frac{5\pi}{4}, \frac{7\pi}{4}$ so $\frac{dy}{dx} = \frac{4\cos 4t}{\cos t} = 4\sqrt{2}, -4\sqrt{2}$.

At $x = 0$ and $y = 0$, $t = 0, \pi, 2\pi$ so $\frac{dy}{dx} = \frac{4\cos 4t}{\cos t} = 4, -4$.

v At $x = -1$, $t = \frac{3\pi}{2}$ and then $\cos t = 0$, so $\frac{dy}{dx}$ is undefined.

At $x = 1$, $t = \frac{\pi}{2}$ and then $\cos t = 0$, so $\frac{dy}{dx}$ is undefined.

d There are 4 identical regions of one type and 4 of a second type.

Find the area A_1 bounded by the curve in the 1st quadrant from $x = 0$ to $x = \frac{1}{\sqrt{2}}$.

$$
\begin{aligned}
A_1 &= \int_0^{\frac{1}{\sqrt{2}}} y\, dx \\
&= \int_0^{\frac{\pi}{4}} y \frac{dx}{dt}\, dt \\
&= \int_0^{\frac{\pi}{4}} \sin 4t \cos t\, dt \\
&= \int_0^{\frac{\pi}{4}} 2\sin 2t \cos 2t \cos t\, dt \\
&= 4\int_0^{\frac{\pi}{4}} \sin t \cos t(2\cos^2 t - 1)\cos t\, dt \\
&= 4\int_0^{\frac{\pi}{4}} \cos^2 t(2\cos^2 t - 1)\sin t\, dt
\end{aligned}
$$

Now use the substitution $u = \cos t$, so $\frac{du}{dt} = -\sin t$.

$t = 0,\ u = 1; t = \frac{\pi}{4},\ u = \frac{1}{\sqrt{2}}.$

$$A_1 = 4\int_1^{\frac{1}{\sqrt{2}}} u^2(2u^2 - 1)(-1)\,du$$
$$= 4\int_1^{\frac{1}{\sqrt{2}}} (u^2 - 2u^4)\,du$$
$$= 4\left[\frac{1}{3}u^3 - \frac{2}{5}u^5\right]_1^{\frac{1}{\sqrt{2}}}$$
$$= 4\left(\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{20}\right) - 4\left(\frac{1}{3} - \frac{2}{5}\right)$$
$$= \frac{2(\sqrt{2} + 2)}{15}$$

Find the area A_2 bounded by the curve in the 4th quadrant from $x = \frac{1}{\sqrt{2}}$ to $x = 1$.

$\left(\text{Use this quadrant since the curve here corresponds to } t = \frac{\pi}{4} \text{ to } t = \frac{\pi}{2}.\right)$

Using the same method as for A_1, and noting that this region lies below the x-axis, the required integral after substitution becomes

$$A_2 = -4\int_{\frac{1}{\sqrt{2}}}^{0} u^2(2u^2 - 1)(-1)\,du$$
$$= 4\int_{\frac{1}{\sqrt{2}}}^{0} (2u^4 - u^2)\,du$$
$$= 4\left[\frac{2}{5}u^5 - \frac{1}{3}u^3\right]_{\frac{1}{\sqrt{2}}}^{0}$$
$$= 0 - 4\left(\frac{\sqrt{2}}{20} - \frac{\sqrt{2}}{12}\right)$$
$$= \frac{2\sqrt{2}}{15}$$

So the area of the enclosed region is $4(A_1 + A_2) = \frac{16(\sqrt{2} + 1)}{15}$.

e The volume V can be found without the use of the parameter t (though it is not wrong to find it with the parameter). It is given by

$$
\begin{aligned}
V &= 2\int_0^1 \pi(f(x))^2\,dx \\
&= 2\pi\int_0^1 16x^2(1-x^2)(1-2x^2)^2\,dx \\
&= 32\pi\int_0^1 (x^2-x^4)(1-4x^2+4x^4)\,dx \\
&= 32\pi\int_0^1 (x^2-4x^4+4x^6-x^4+4x^6-4x^8)\,dx \\
&= 32\pi\int_0^1 (x^2-5x^4+8x^6-4x^8)\,dx \\
&= 32\pi\left[\frac{1}{3}x^3-x^5+\frac{8}{7}x^7-\frac{4}{9}x^9\right]_0^1 \\
&= 32\pi\left(\frac{1}{3}-1+\frac{8}{7}-\frac{4}{9}\right) \\
&= \frac{64\pi}{63}
\end{aligned}
$$

20 $f(x) = \dfrac{x^3}{x^2+a},\ a > 0$

a

$$
\begin{aligned}
f'(x) &= \frac{3x^2(x^2+a) - x^3 \times 2x}{(x^2+a)^2} \\
&= \frac{x^4+3ax^2}{(x^2+a)^2} \\
f''(x) &= \frac{(4x^3+6ax)(x^2+a)^2 - (x^4+3ax^2)\times 2(x^2+a)\times 2x}{(x^2+a)^4} \\
&= \frac{(4x^3+6ax)(x^2+a) - 4x(x^4+3ax^2)}{(x^2+a)^3} \\
&= \frac{-2ax^3+6a^2x}{(x^2+a)^3}
\end{aligned}
$$

b $f'(x) = 0$ if $x^2(x^2+3a) = 0$. Since $a > 0$, the only solution is $x = 0$.
Since $x^2(x^2+3a) > 0$ for all non-zero values of x, it follows that the stationary point is an inflexion point (the gradient is positive both sides of $x = 0$).
$f(0) = 0$, so $(0, 0)$ is a stationary point of inflexion.

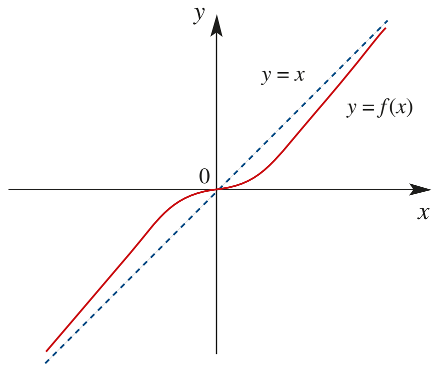
c $f''(x) = 0$ if $2ax(-x^2+3a) = 0$.
From part **b**, $x = 0$ gives a stationary point of inflexion.
So non-stationary points of inflexion are given by $x = \pm\sqrt{3a}$. Then:

$f(\pm\sqrt{3a}) = \pm\dfrac{3a\sqrt{3a}}{4a} = \pm\dfrac{3\sqrt{3a}}{4}$

So the coordinates of the non-stationary points of inflexion are $\left(\pm\sqrt{3a}, \pm\dfrac{3\sqrt{3a}}{4}\right)$.

d Division shows that $f(x)$ can be rewritten as $f(x) = x - \dfrac{ax}{x^2 + a}$.
As $x \to \pm\infty$, the fractional term approaches zero, i.e. $f(x) \to 0$.
So $y = x$ is an asymptote.

e



f The enclosed area A is given by

$$\begin{aligned} A &= \int_0^a \left(x - \frac{x^3}{x^2 + a}\right) dx \\ &= \int_0^a \frac{ax}{x^2 + a}\, dx \\ &= \frac{a}{2}\int_0^a \frac{2x}{x^2 + a}\, dx \\ &= \frac{a}{2}\Big[\log_e(x^2 + a)\Big]_0^a \\ &= \frac{a}{2}\Big(\log_e(a^2 + a) - \log_e(a)\Big) \\ &= \frac{a}{2}\log_e(a + 1) \end{aligned}$$

Since the area is to be $\dfrac{1}{2}\log_e 2$, it is evident that $a = 1$.

21 $f(x) = \dfrac{x^3}{x^2 - a}$, $a > 0$

a $f'(x) = \dfrac{3x^2(x^2 - a) - x^3 \times 2x}{(x^2 - a)^2}$

$= \dfrac{x^4 - 3ax^2}{(x^2 - a)^2}$

$f''(x) = \dfrac{(4x^3 - 6ax)(x^2 - a)^2 - (x^4 - 3ax^2) \times 2(x^2 - a) \times 2x}{(x^2 - a)^4}$

$= \dfrac{(4x^3 - 6ax)(x^2 - a) - 4x(x^4 - 3ax^2)}{(x^2 - a)^3}$

$= \dfrac{2ax^3 + 6a^2x}{(x^2 - a)^3}$

b $f'(x) = 0$ if $x^2(x^2 - 3a) = 0$. This has solutions $x = 0$ and $x = \pm\sqrt{3a}$. To test the nature of the stationary points, check the sign of the derivative. Since the denominator is positive, only the numerator needs to be considered.
Now $x^2(x^2 - 3a) = x^2(x + \sqrt{3a})(x - \sqrt{3a})$, so:

$x < -\sqrt{3a},\ f'(x) > 0$

$-\sqrt{3a} < x < 0,\ f'(x) < 0$

$0 < x < \sqrt{3a},\ f'(x) < 0$

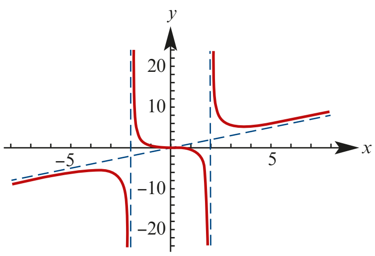
$x > \sqrt{3a},\ f'(x) > 0$

Then it follows that $x = -\sqrt{3a}$ gives a maximum, $x = 0$ gives a stationary point of inflexion and $x = \sqrt{3a}$ gives a minimum.
$f(\pm\sqrt{3a}) = \pm\dfrac{3a\sqrt{3a}}{2a} = \pm\dfrac{3\sqrt{3a}}{2}$ and $f(0) = 0$.
So $\left(-\sqrt{3a},\ -\dfrac{3\sqrt{3a}}{2}\right)$ is a local maximum, $\left(\sqrt{3a},\ \dfrac{3\sqrt{3a}}{2}\right)$ is a local minimum and $(0, 0)$ is a stationary point of inflexion.

c $f''(x) = 0$ if $2ax(x^2 + 3a) = 0$.
The only solution is $x = 0$ and from part **b**, $(0, 0)$ is a stationary point of inflexion.

d Division shows that $f(x)$ can be rewritten as $f(x) = x + \dfrac{ax}{x^2 - a}$.
As $x \to \pm\infty$, the fractional term approaches zero, i.e. $f(x) \to 0$.
So $y = x$ is an asymptote.
Also as $x^2 - a \to 0$, $f(x) \to \pm\infty$, so $x = \sqrt{a}$ and $x = -\sqrt{a}$ are vertical asymptotes.

e



f If there is a stationary point where $x = 4\sqrt{3}$, then from part **b**, it must correspond to $x = \sqrt{3a}$, i.e. $\sqrt{3a} = 4\sqrt{3} \Rightarrow a = 16$.

22 $f(x) = x \arcsin(x),\ g(x) = \arcsin(x),\ -1 \le x \le 1.$

a $f'(x) = \arcsin(x) + \dfrac{x}{\sqrt{1-x^2}}$

Observe that $f'(0) = 0$.

For $-1 < x < 0$, $f'(x) < 0$ (since both terms are negative); for $0 < x < 1$, $f'(x) > 0$ (since both terms are positive).

Thus, $x = 0$ gives a minimum, and the test also shows that there are no further stationary points.

As $f(0) = 0$, then $(0, 0)$ is a minimum turning point.

b
$$
\begin{aligned}
f''(x) &= \frac{1}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2} - x \times \frac{1}{2}(1-x^2)^{\frac{1}{2}} \times (-2x)}{1-x^2} \\
&= \frac{1}{\sqrt{1-x^2}} + \frac{1-x^2+x^2}{(1-x^2)^{\frac{3}{2}}} \\
&= \frac{1-x^2}{(1-x^2)^{\frac{3}{2}}} + \frac{1}{(1-x^2)^{\frac{3}{2}}} \\
&= \frac{2-x^2}{(1-x^2)^{\frac{3}{2}}}
\end{aligned}
$$

But $-1 < x < 1$, so the second derivative is positive for all values of x in this interval. Thus there are no points of inflexion.

c If $x < 0, \arcsin(x) < 0$ and so $x \arcsin(x) > 0$ (product oftwo negative numbers).

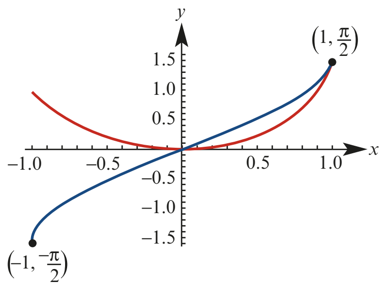
If $x > 0, \arcsin(x) > 0$ and so $x \arcsin(x) > 0$.

Also, $f(0) = 0$, so $f(x) \ge 0$ for all values of x in the domain.

d $(fx) = g(x)$, then $f(x) - g(x) = 0$, i.e. $\arcsin(x)(x - 1) = 0$.

Then either $x = 1$ or $\arcsin(x) = 0$, i.e. $x = 0$.

e



f The required area A is given by $A = \int_0^1 \arcsin(x)(1-x)dx$.
This is a non-standard integrand, which suggests the use of a CAS.
This gives $A = \dfrac{3\pi}{8} - 1$.

23 $\dfrac{dx}{dt} = -3y,\ \dfrac{dy}{dt} = \sin 2t,\ t = 0: y = -\dfrac{1}{2},\ x = 0.$

Note that the first equation involves three variables, whereas the second involves two, just y and t. Thus to find x and y in terms of t, you need to first solve the second equation for y and then substitute its solution into the first equation.

a

$$\frac{dy}{dt} = \sin 2t$$

$$y = -\frac{1}{2}\cos 2t + c$$

$$t = 0,\ y = -\frac{1}{2}$$

$$-\frac{1}{2} = -\frac{1}{2} + c \Rightarrow\ c = 0$$

$$\text{So } y = -\frac{1}{2}\cos 2t.$$

Substitute and solve for x:

$$\frac{dx}{dt} = \frac{3}{2}\cos 2t$$

$$x = \frac{3}{4}\sin 2t + d$$

$$t = 0,\ x = 0$$

$$0 = 0 + d \Rightarrow\ d = 0$$

$$\text{So } x = \frac{3}{4}\sin 2t.$$

b $\sin^2 2t + \cos^2 2t = 1$

$$\left(\frac{4x}{3}\right)^2 + (-2y)^2 = 1$$

$$\frac{16x^2}{9} + 4y^2 = 1$$

c $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

$$= \sin 2t \div \left(\frac{3}{2}\cos 2t\right)$$

$$= \frac{2}{3}\tan 2t$$

d The x and y coordinates at a general point t are the solutions found in part **a**. So the equation of the tangent is

$$y - \left(-\frac{1}{2}\cos 2t\right) = \frac{2}{3}\tan 2t\left(x - \frac{3}{4}\sin 2t\right)$$

$$y = \left(\frac{2}{3}\tan 2t\right)x - \frac{1}{2}\frac{\sin^2 2t}{\cos 2t} - \frac{1}{2}\cos 2t$$

$$= \left(\frac{2}{3}\tan 2t\right)x - \frac{1}{2}\frac{\sin^2 2t + \cos^2 2t}{\cos 2t}$$

$$\text{i.e. } y = \left(\frac{2}{3}\tan 2t\right)x - \frac{1}{2\cos 2t} \qquad ①$$

For the x intercept, let $y = 0$ in ①:

$$\left(\frac{2}{3}\tan 2t\right)x - \frac{1}{2\cos 2t} = 0$$

$$\left(\frac{2}{3}\tan 2t\right)x = \frac{1}{2\cos 2t}$$

$$x = \frac{1}{2\cos 2t} \times \frac{3}{2}\frac{\cos 2t}{\sin 2t}$$

$$= \frac{3}{4}\operatorname{cosec} 2t$$

For the y-intercept, let $x = 0$ in ①:

$$y = -\frac{1}{2\cos 2t} = -\frac{1}{2}\sec 2t$$

e Using $\frac{1}{2}$ base $\times$ height and noting that the values of the intercepts will be negative for some values of t, the area is given by

$$\frac{1}{2}\left|\frac{3}{4}\operatorname{cosec} 2t \times \left(-\frac{1}{2}\sec 2t\right)\right| = \left|\frac{3}{16} \times \frac{1}{\sin 2t \cos 2t}\right|$$
$$= \left|\frac{3}{8} \times \frac{1}{\sin 4t}\right|$$
$$= \left|\frac{3}{8}\operatorname{cosec} 4t\right|$$

Now the minimum value of $|\operatorname{cosec} 4t|$ is 1 when $4t = \pm\frac{\pi}{2} + 2k\pi \Rightarrow t = \pm\frac{\pi}{8} + \frac{k\pi}{2}$;
thus $t = \ldots, -\frac{3\pi}{8}, -\frac{\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}, \ldots$.

f From part **a** or **b**, the x axis intercepts of the curve are $\left(\pm\frac{3}{4}, 0\right)$, so the circle has equation $x^2 + y^2 = \frac{9}{16}$.
There are many possible answers, but keeping to the '$2t$' parameter setup used for the ellipse, one answer would be $x = \frac{3}{4}\sin 2t,\ y = \frac{3}{4}\cos 2t$.

g Using the notation as in part **g**, the volume V is given by $V = 2\int_0^{\frac{3}{4}} (y_1^2 - y_2^2)\, dx$.
In this case, there is no need to use parametric form since no square roots are involved. Thus:

$$V = 2\pi \int_0^{\frac{3}{4}} \left(\left(\frac{9}{16} - x^2\right) - \frac{1}{4}\left(1 - \frac{16}{9}x^2\right)\right) dx$$
$$= 2\pi \int_0^{\frac{3}{4}} \left(\frac{5}{16} - \frac{5}{9}x^2\right) dx$$
$$= 2\pi \left[\frac{5}{16}x - \frac{5}{27}x^3\right]_0^{\frac{3}{4}}$$
$$= 2\pi\left(\frac{15}{64} - \frac{5}{64}\right) = \frac{5\pi}{16}$$

24 **a** Let X be the riding time.
$\Pr(X < 48) = 0.0808$

b $\Pr(k_1 < X < k_2) = 0.95$ Assume that k_1 and k_2 are equal distance fom the mean. Let k be this distance in the standard normal.
$\Pr(-k < Z < k) = 0.95$
$2\Pr(Z < k) - 1 = 0.95$
$\Pr(Z < k) = 0.975$
$k = 1.95996$
Therefore $k_1 = 5 \times 1.95996 + 55$ and $k_2 = -5 \times 1.95996 + 55$

Therefore $k_1 = 64.7998$ and $k_2 = 45.2002$

c **i** $E(\bar{X}) = 55$ and $sd(\bar{X}) = \dfrac{5}{\sqrt{10}} \approx 1.5811$

$\Pr(\bar{X} < 50) = 0.000782\ldots$

ii $E(X_1 + X_2 + \cdots X_{10}) = 55 \times 10 = 550$

and $sd(X_1 + X_2 + \cdots X_{10}) = \sqrt{10 \times 25} \approx 15.811388\ldots$

$\Pr(X > 580) = 0.0289$

iii Binomial with $n = 10$ and $p = \Pr(X < 50) = 0.158655\ldots$

Let Y be the number of times per week that the ride takes less than 50 minutes.

$\Pr(Y \geq 4) = 0.0598$

d Using the result from **b**

$c_1 = 55 - 1.5811 \times 1.95996 = 51.9011$ and $c_2 = 55 + 1.5811 \times 1.95996 = 58.0989$

25 **a** $E(X) = \int_0^b \frac{x}{b}\,dx = \frac{b}{2}$

$E(X^2) = \int_0^b \frac{x^2}{b}\,dx = \frac{b^2}{3}$

$\mathrm{Var}(X) = \frac{b^2}{3} - \left(\frac{b^2}{4}\right) = \frac{b^2}{12}$

$sd(X) = \frac{b}{\sqrt{12}} = \frac{b}{2\sqrt{3}}$

b $E(\bar{X}) = \int_0^b \frac{x}{b}\,dx = \frac{b}{2}$

$sd(\bar{X}) = \frac{b}{2\sqrt{3n}}$

c $E(\bar{X}) = 2.4$ and $sd(\bar{X}) = \frac{b}{\sqrt{12 \times 50}} = \frac{b}{10\sqrt{6}}$

90% Confidence interval

$(2.4 - 1.644 \times \frac{b}{10\sqrt{6}}, 2.4 - 1.644 \times \frac{b}{10\sqrt{6}}) = (2.4 - 0.0672b, 2.4 + 0.0672b)$

d $2.4 - 0.0672b < \mu < 2.4 + 0.067b \Rightarrow -0.0672b < \mu - 2.4 < 0.0672b$

$\mu - 2.4 < 0.0672b$

$\Rightarrow \frac{b}{2} - 2.4 < 0.0672b$

$\Rightarrow b < 5.54$

$\mu - 2.4 > -0.0672b$

$\Rightarrow \frac{b}{2} - 2.4 > -0.0672b$

$\Rightarrow b > 4.23$

26 **a**

$$z + \frac{4}{z} = k$$

$$z^2 + 4 = kz$$

Let $z = x + iy, \quad x, y \in \mathbb{R}$

$x^2 + 2ixy - y^2 + 4 = k(x + iy)$

Equating real and imaginary parts

$$x^2 - y^2 + 4 = kx \ldots (1)$$

$$2xy = ky \ldots (2)$$

From(2)

$$y(2x - k) = 0$$

$$\Rightarrow y = 0 \text{ or } 2x = k$$

If $2x = k$, substituting in (1)

gives $x^2 - y^2 + 4 = 2x^2$

$$\Rightarrow x^2 + y^2 = 4$$

b Suppose that $y = 0$

From (1)

$x^2 + 4 = kx$

$k = x + \frac{4}{x}$ The graph of k against x has a local minimum at $(2, 4)$ and a local maximum at $(-2, -4)$. From this it can be shown that $k > 4$ for $x > 0$ and $k < -4$ for $x < 0$. Hence $|k| \geq 4$

c If $x^2 + y^2 = 4$ then $y^2 = 4 - x^2$. This implies $x^2 \leq 4$ or equivalently $-2 \leq x \leq 2$. Also again from (1), $x^2 - (4 - x^2) + 4 = kx \Rightarrow 2x^2 = kx \Rightarrow x = 0$ or $k = 2x$ Therefore $-2 \leq \frac{k}{2} \leq 2 \Rightarrow -4 \leq k \leq 4$. That is $|k| \leq 4$

27 **a** Let X be the diameter of nails from machine A.

$\mathrm{E}(\bar{X}) = 3$ and $\mathrm{sd}(\bar{X}) = \frac{0.03}{\sqrt{30}} = 0.005477\ldots$

$\Pr(\bar{X}) < 2.99 \text{ or } > 3.01) = 1 - \Pr(2.99 \leq \bar{X} \leq 3.01) = 1 - 0.9321\cdots = 0.0678890\ldots$

b Let Y be the diameter of nails from machine B.

$\mathrm{E}(\bar{Y}) = 3.01$ and $\mathrm{sd}(\bar{Y}) = \frac{0.02}{\sqrt{30}} = 0.00365\ldots$

$\Pr(\bar{Y}) < 2.99 \text{ or } > 3.01) = 1 - \Pr(2.99 \leq \bar{Y} \leq 3.01) = 1 - 0.499999\cdots = 0.50000\ldots$

c **i** 95% confidence interval $(2.991, 3.009)$

ii Machine A since mean of X lies in this interval but mean of Y does not.

28 a $\dfrac{dP}{dt} = k(P - 0.5P_0) \qquad k \in \mathbb{R}^+$

b
$$\frac{dP}{dt} = k(P - 500)$$
$$\frac{dt}{dP} = \frac{1}{k(P - 500)}$$
$$t = \frac{1}{k}\ln(P - 500) + c$$

When $t = 0, P = 1000$

$$0 = \frac{1}{k}\ln(1000 - 500) + c$$
$$c = -\frac{1}{k}\ln(500)$$
$$\therefore t = \frac{1}{k}\ln\left(\frac{P - 500}{500}\right)$$

When $t = 1, P = 1100$

$$1 = \frac{1}{k}\ln\left(\frac{6}{5}\right)$$
$$\therefore k = \ln\left(\frac{6}{5}\right)$$
$$t = \frac{1}{\ln\left(\frac{6}{5}\right)}\ln\left(\frac{P - 500}{500}\right)$$
$$t\ln\left(\frac{6}{5}\right) = \ln\left(\frac{P - 500}{500}\right)$$
$$\ln\left(\frac{6}{5}\right)^t = \ln\left(\frac{P - 500}{500}\right)$$
$$P = 500\left(\frac{6}{5}\right)^t + 500 = 500\left(\left(\frac{6}{5}\right)^t + 1\right)$$

c $P(3) - P(2) = 500(\left(\frac{6}{5}\right)^3 - \left(\frac{6}{5}\right)^2) = 500 \times \frac{36}{25} \times \frac{1}{5} = 144$

d $t = \dfrac{1}{\ln\left(\frac{6}{5}\right)}\ln\left(\dfrac{2000 - 500}{500}\right)$

$= \dfrac{1}{\ln\left(\frac{6}{5}\right)}\ln 3 \approx 6.03$ years

29 **a**

$$\ddot{\boldsymbol{r}}_A = -g\boldsymbol{j}$$

$$\dot{\boldsymbol{r}}_A = 60\cos 30°\boldsymbol{i} + (60\sin 30° - gt)\boldsymbol{j}$$

$$\boldsymbol{r}_A = 60t\cos 30°\boldsymbol{i} + (60t\sin 30° - \frac{1}{2}gt^2)\boldsymbol{j}$$

$$\ddot{\boldsymbol{r}}_B = -g\boldsymbol{j}$$

$$\dot{\boldsymbol{r}}_B = -50\cos\beta°\boldsymbol{i} + (50\sin\beta° - gt)\boldsymbol{j}$$

$$\boldsymbol{r}_B = (-50t\cos\beta° + 100)\boldsymbol{i} + (50t\sin\beta° - \frac{1}{2}gt^2)\boldsymbol{j}$$

Therefore

$$\boldsymbol{r}_A = 30\sqrt{3}t\boldsymbol{i} + (30t - \frac{1}{2}gt^2)\boldsymbol{j}$$

and

$$\boldsymbol{r}_B = (-50t\cos\beta° + 100)\boldsymbol{i} + (50t\sin\beta° - \frac{1}{2}gt^2)\boldsymbol{j}$$

b At time t_1 the particles collide.

$$\boldsymbol{r}_A(t_1) = \boldsymbol{r}_B(t_1)$$

$$\therefore 30\sqrt{3}t_1 = -50t_1\cos\beta° + 100 \ldots (1)$$

and

$$30t_1 - \frac{1}{2}gt_1^2 = 50t_1\sin\beta° - \frac{1}{2}gt_1^2 \ldots (2)$$

From (2)

$$30 = 50\sin\beta$$

$$\beta = \sin^{-1}\frac{3}{5} \approx 36.87^{circ}$$

c Substitute in (1)

$$30\sqrt{3}t_1 = -50t_1 \times \frac{4}{5} + 100$$

$$30\sqrt{3}t_1 + 40t_1 = 100$$

$$t_1 = \frac{100}{30\sqrt{3} + 40} \approx 1.09 \text{ seconds}$$

d

$$\boldsymbol{r}_A = 30\sqrt{3}t\boldsymbol{i} + (30t - \frac{1}{2}gt^2)\boldsymbol{j}$$

$$\boldsymbol{r}_A(t_1) = 30\sqrt{3} \times (1.087...)\boldsymbol{i} + (30 \times 1.087... - \frac{1}{2}g(1.087...)^2)\boldsymbol{j}$$

$$\boldsymbol{r}_A(t_1) = (56.5035\ldots)\boldsymbol{i} + (26.828\ldots)\boldsymbol{j}$$

Giving values to two decimal places, the particles collide at.

$$(56.50)\boldsymbol{i} + (26.83)\boldsymbol{j}$$

30 **a** $\Pr(X > 100) = 0.90 \Rightarrow \mu = 112.8$

b **i** $\mathbf{H}_0$: $\mu = 112.8$ $\mathbf{H}_1$: $\mu < 112.8$

ii p-value = 0.0729

iii Since the p-value is greater than 0.05 we do not reject $\mathbf{H}_0$. There is insufficient evidence to reject the manufacturer's claim.