CAMBRIDGE

SPECIALIST MATHEMATICS

VCE UNITS 3 & 4

CAMBRIDGE SENIOR MATHEMATICS **VCE**
SECOND EDITION

MICHAEL EVANS | DAVID TREEBY | KAY LIPSON | JOSIAN ASTRUC
NEIL CRACKNELL | GARETH AINSWORTH | DANIEL MATHEWS

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Contents

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Introduction and overview

Cambridge Specialist Mathematics VCE Units 3&4 Second Edition provides a complete teaching and learning resource for the VCE Study Design **to be first implemented in 2023**. It has been written with understanding as its chief aim, and with ample practice offered through the worked examples and exercises. The work has been trialled in the classroom, and the approaches offered are based on classroom experience and the helpful feedback of teachers to earlier editions.

Specialist Mathematics Units 3 and 4 provide a study of elementary functions, algebra, calculus, and probability and statistics and their applications in a variety of practical and theoretical contexts. This book has been carefully prepared to meet the requirements of the new Study Design.

The book begins with a review of some topics from Specialist Mathematics Units 1 and 2, including algorithms and pseudocode, circular functions and proof.

The concept of proof now features more strongly throughout the course. To account for this, we have a specially written Proof chapter that involves topics such as divisibility; inequalities; graph theory; combinatorics; sequences and series, including partial sums and partial products and related notations; complex numbers; matrices; vectors and calculus. Other chapters also feature exercises aimed to further develop your students' skills in mathematical reasoning.

In addition to the online appendices on the general use of calculators, there are three online appendices for using **both the programming language Python and the inbuilt capabilities of students' CAS calculators**.

The four revision chapters provide technology-free, multiple-choice and extended-response questions. Each of the first three revision chapters contain a section on algorithms and pseudocode.

The TI-Nspire calculator examples and instructions have been completed by Peter Flynn, and those for the Casio ClassPad by Mark Jelinek, and we thank them for their helpful contributions.

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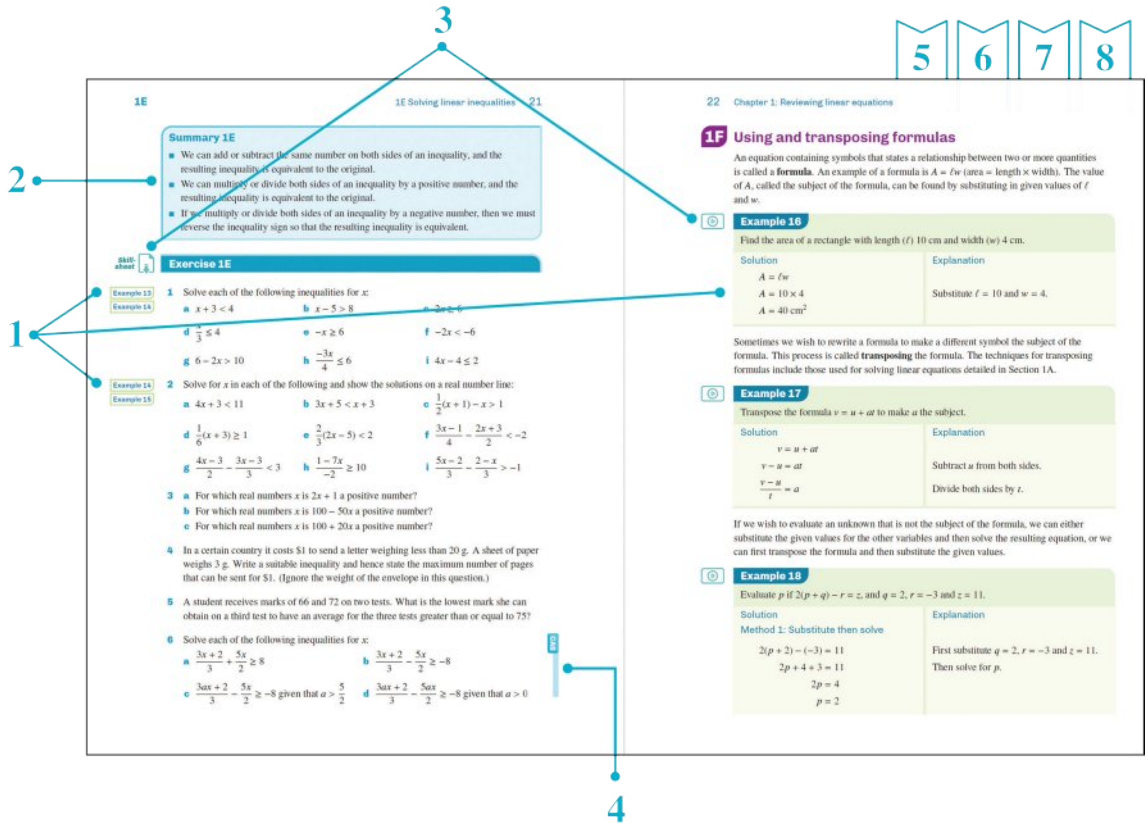
Overview of the print book

1. Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercise questions.
2. Section summaries provide important concepts in boxes for easy reference.
3. Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.
4. Questions that suit the use of a CAS calculator to solve them are identified within exercises.
5. Chapter reviews contain a chapter summary and technology-free, multiple-choice, and extended-response questions.
6. Revision chapters provide comprehensive revision and preparation for assessment, including new practice Investigations.
7. The glossary includes page numbers of the main explanation of each term.
8. In addition to coverage within chapters, print and online appendices provide additional support for learning and applying algorithms and pseudocode, including the use of Python and TI-Nspire and Casio ClassPad for coding.

Numbers refer to descriptions above.



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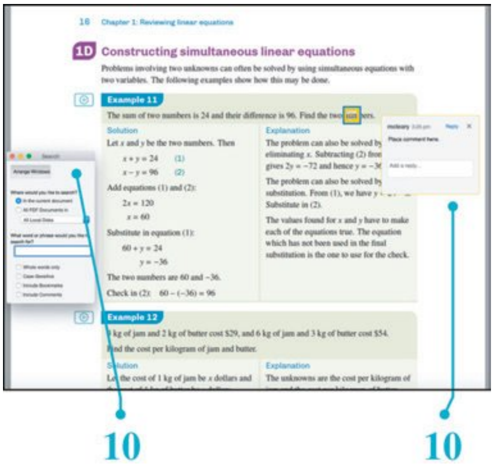
Overview of the downloadable PDF textbook

9 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

10 PDF annotation and search features are enabled.



Overview of the Interactive Textbook

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

11 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

12 **Workspaces** for all questions, which can be enabled or disabled by the teacher, allow students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing done on paper.

13 **Self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.

14 All worked examples have **video versions** to encourage independent learning.

15 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

16 An expanded and revised set of **Desmos interactives** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

17 The **Desmos graphics calculator, scientific calculator**, and **geometry tool** are also embedded for students to use for their own calculations and exploration.

18 **Revision of prior knowledge** is provided with links to diagnostic tests and Year 10 HOTmaths lessons.

19 **Quick quizzes** containing automarked multiple-choice questions have been thoroughly expanded and revised, enabling students to check their understanding.

20 **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

21 Messages from the teacher assign tasks and tests.

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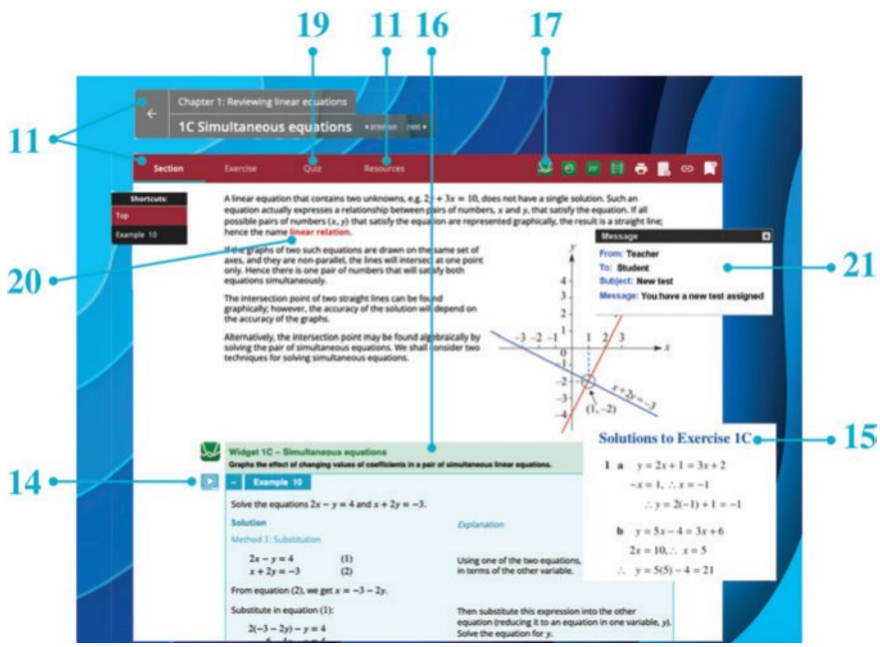
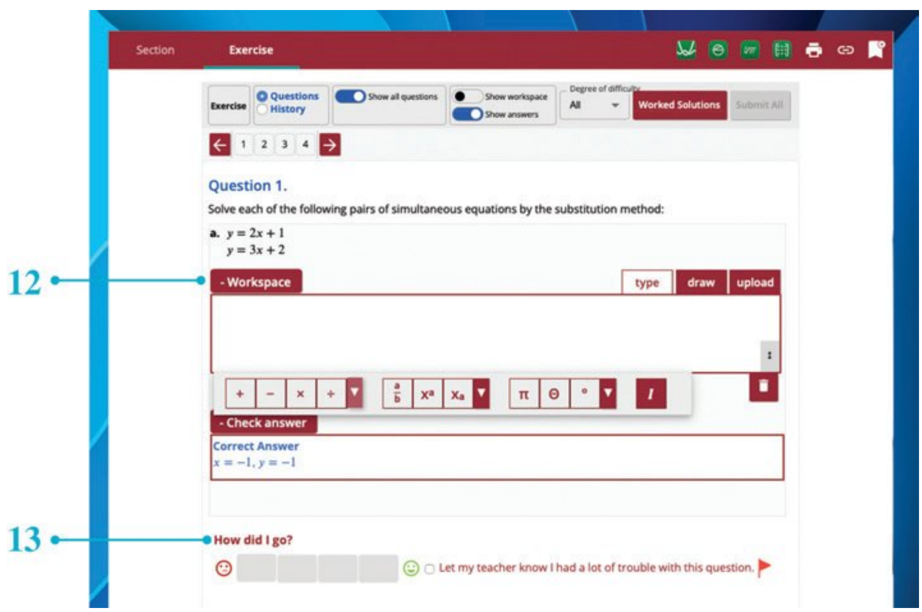
INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

A selection of features is shown. Numbers refer to the descriptions on pages xi–xii. HOTmaths platform features are updated regularly



WORKSPACES AND SELF-ASSESSMENT



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Overview of the Online Teaching Suite powered by the HOTmaths platform

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

22 The HOTmaths learning management system with class and student analytics and reports, and communication tools.

23 Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had diffculty.

24 A HOTmaths-style test generator.

25 An expanded and revised suite of chapter tests, assignments and sample investigations.

26 Editable curriculum grids and teaching programs.

27 A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite includes a comprehensive bank of VCAA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- VCAA marking scheme
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

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1 Preliminary topics

Objectives

- To revise the properties of **sine**, **cosine** and **tangent**.
- To revise the **sine rule** and the **cosine rule**.
- To revise **arithmetic sequences and series**.
- To revise **geometric sequences and series**.
- To revise **infinite geometric series**.
- To revise sequences defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where r and d are constants.
- To revise the **modulus function**.
- To sketch graphs of **circles**, **ellipses** and **hyperbolas** from their Cartesian equations.
- To revise the use of **parametric equations** to describe curves in the plane.
- To revise the use of **pseudocode** to describe algorithms.

In this chapter, we revise some of the knowledge and skills from Specialist Mathematics Units 1 & 2 that will be required in this course. We start by revising basic trigonometry, including the sine and cosine rules. There is further revision of trigonometry in Chapter 3.

We also revise sequences and series, the modulus function and the description of circles, ellipses and hyperbolas in the plane by Cartesian equations and by parametric equations. Finally, we revise the use of pseudocode to describe algorithms. For further details on these topics, refer to the relevant chapters of Specialist Mathematics Units 1 & 2.

We will be building on the introduction to parametric equations given in Section 1G in several new contexts in later chapters of this book.

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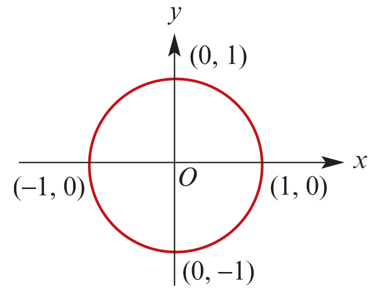
1A Circular functions

Defining sine, cosine and tangent

The unit circle is a circle of radius 1 with centre at the origin. It is the graph of the relation $x^2 + y^2 = 1$.

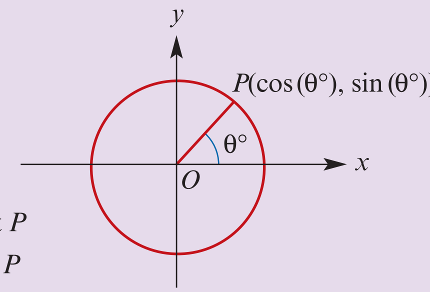


We can define the sine and cosine of any angle by using the unit circle.

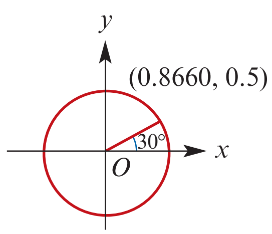
Definition of sine and cosine

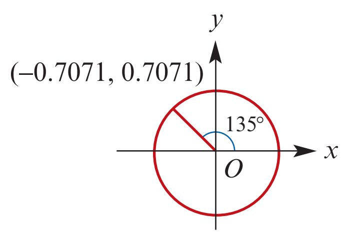
For each angle $\theta°$, there is a point P on the unit circle as shown. The angle is measured anticlockwise from the positive direction of the x-axis.

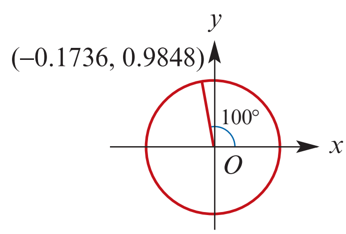
- $\cos(\theta°)$ is defined as the x-coordinate of the point P
- $\sin(\theta°)$ is defined as the y-coordinate of the point P



For example:







$\sin 30° = 0.5$ (exact value)

$\cos 30° = \dfrac{\sqrt{3}}{2} \approx 0.8660$

$\sin 135° = \dfrac{1}{\sqrt{2}} \approx 0.7071$

$\cos 135° = \dfrac{-1}{\sqrt{2}} \approx -0.7071$

$\sin 100° \approx 0.9848$

$\cos 100° \approx -0.1736$

Definition of tangent

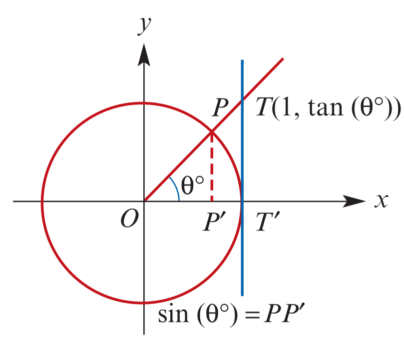
$$\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$$

The value of $\tan(\theta°)$ can be illustrated geometrically through the unit circle.

By considering similar triangles OPP' and OTT', it can be seen that

$$\frac{TT'}{OT'} = \frac{PP'}{OP'}$$

i.e. $$TT' = \frac{\sin(\theta°)}{\cos(\theta°)} = \tan(\theta°)$$



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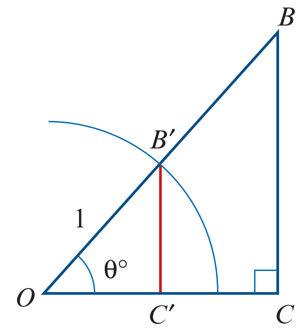
The trigonometric ratios

For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:

$$B'C' = \sin(\theta^\circ) \quad \text{and} \quad OC' = \cos(\theta^\circ)$$



As triangles OBC and $OB'C'$ are similar, we have

$$\frac{BC}{OB} = \frac{B'C'}{1} \quad \text{and} \quad \frac{OC}{OB} = \frac{OC'}{1}$$

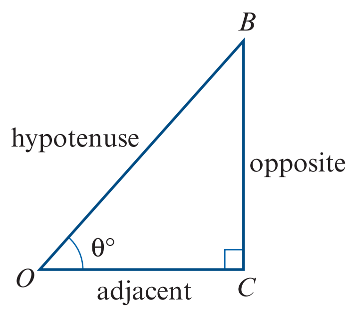
$$\therefore \quad \frac{BC}{OB} = \sin(\theta^\circ) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta^\circ)$$

This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

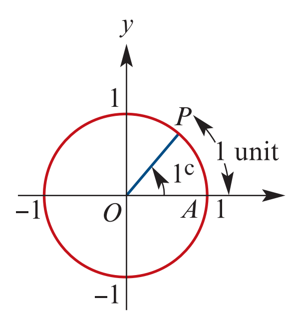
$$\cos(\theta^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta^\circ) = \frac{\text{opposite}}{\text{adjacent}}$$



Definition of a radian

In moving around the unit circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



> One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Usually the symbol for radians, ^{c}, is omitted. Any angle is assumed to be measured in radians unless indicated otherwise.

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The following table displays the conversions of some special angles from degrees to radians.

Angle in degrees	0°	30°	45°	60°	90°	180°	360°
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

Some values for the trigonometric functions are given in the following table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

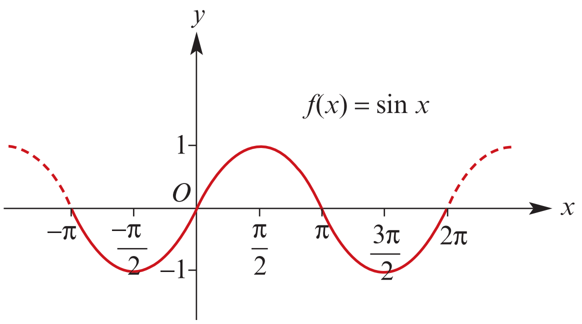
The graphs of sine and cosine

A sketch of the graph of

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$$

is shown opposite.

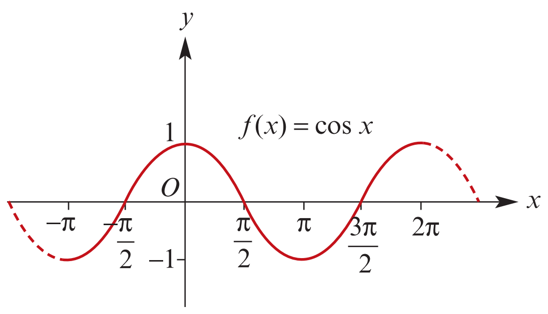
As $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$, the sine function is **periodic**. The period is 2π. The amplitude is 1.



A sketch of the graph of

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \cos x$$

is shown opposite. The period of the cosine function is 2π. The amplitude is 1.



For the graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$, where $a > 0$ and $n > 0$:

- Period $= \dfrac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a, a]$

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Symmetry properties of sine and cosine

The following results may be obtained from the graphs of the functions or from the unit-circle definitions:

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi + \theta) = -\sin\theta \qquad \cos(\pi + \theta) = -\cos\theta$$
$$\sin(2\pi - \theta) = -\sin\theta \qquad \cos(2\pi - \theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\sin(\theta + 2n\pi) = \sin\theta \qquad \cos(\theta + 2n\pi) = \cos\theta \qquad \text{for } n \in \mathbb{Z}$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

Example 1

a Convert 135° to radians. **b** Convert 1.5^c to degrees, correct to two decimal places.

Solution

a $135^\circ = \dfrac{135 \times \pi^c}{180} = \dfrac{3\pi^c}{4}$

b $1.5^c = \dfrac{1.5 \times 180^\circ}{\pi} = 85.94^\circ$ to two decimal places

Example 2

Find the exact value of:

a $\sin 150^\circ$

b $\cos(-585^\circ)$

Solution

a $\sin 150^\circ = \sin(180^\circ - 150^\circ)$
$= \sin 30^\circ$
$= \dfrac{1}{2}$

b $\cos(-585^\circ) = \cos 585^\circ$
$= \cos(585^\circ - 360^\circ)$
$= \cos 225^\circ$
$= -\cos 45^\circ$
$= -\dfrac{1}{\sqrt{2}}$

Example 3

Find the exact value of:

a $\sin\left(\dfrac{11\pi}{6}\right)$

b $\cos\left(-\dfrac{45\pi}{6}\right)$

Solution

a $\sin\left(\dfrac{11\pi}{6}\right) = \sin\left(2\pi - \dfrac{\pi}{6}\right)$
$= -\sin\left(\dfrac{\pi}{6}\right)$
$= -\dfrac{1}{2}$

b $\cos\left(-\dfrac{45\pi}{6}\right) = \cos(-7\tfrac{1}{2} \times \pi)$
$= \cos\left(\dfrac{\pi}{2}\right)$
$= 0$

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The Pythagorean identity

For any value of θ:

$$\cos^2\theta + \sin^2\theta = 1$$

Example 4

If $\sin x = 0.3$ and $0 < x < \frac{\pi}{2}$, find:

a $\cos x$

b $\tan x$

Solution

a $\cos^2 x + \sin^2 x = 1$

$\cos^2 x + 0.09 = 1$

$\cos^2 x = 0.91$

$\therefore \quad \cos x = \pm\sqrt{0.91}$

Since $0 < x < \frac{\pi}{2}$, this gives

$$\cos x = \sqrt{0.91} = \sqrt{\frac{91}{100}} = \frac{\sqrt{91}}{10}$$

b $\tan x = \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}}$

$= \frac{3}{\sqrt{91}}$

$= \frac{3\sqrt{91}}{91}$

Solution of equations involving sine and cosine

If a trigonometric equation has a solution, then it will have a corresponding solution in each 'cycle' of its domain. Such an equation is solved by using the symmetry of the graph to obtain solutions within one 'cycle' of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

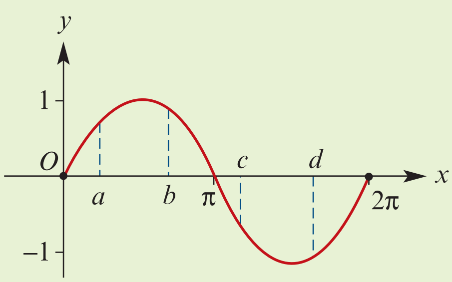
Example 5

The graph of $y = f(x)$ for

$$f\colon [0, 2\pi] \to \mathbb{R},\ f(x) = \sin x$$

is shown.

For each pronumeral marked on the x-axis, find the other x-value which has the same y-value.



Solution

For $x = a$, the other value is $\pi - a$.

For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

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Example 6

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

Solution

Let $\theta = 2x + \frac{\pi}{3}$. Note that

$$\begin{aligned} 0 \le x \le 2\pi \quad &\Leftrightarrow \quad 0 \le 2x \le 4\pi \\ &\Leftrightarrow \quad \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{13\pi}{3} \\ &\Leftrightarrow \quad \frac{\pi}{3} \le \theta \le \frac{13\pi}{3} \end{aligned}$$

To solve $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$, we first solve $\sin\theta = \frac{1}{2}$ for $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$.

Consider $\sin\theta = \frac{1}{2}$.

$$\therefore \quad \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \dots$$

The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required, as they lie outside the restricted domain for θ.

For $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$:

$$\theta = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

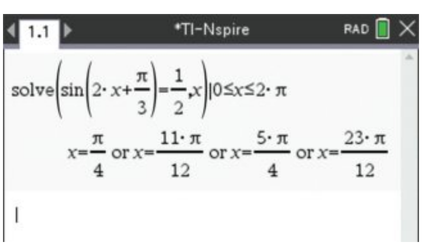
$$\therefore \quad 2x + \frac{2\pi}{6} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore \quad 2x = \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12}$$

Using the TI-Nspire

- Ensure your calculator is in radian mode. (To change the angle mode, either go to (on) > **Settings** > **Document Settings** or else hover the cursor over **RAD** or **DEG** at the top of the screen and click to toggle modes.)
- Complete as shown.



Note: The **Graph** application has its own settings, which are accessed from a **Graph** page using (menu) > **Settings**.

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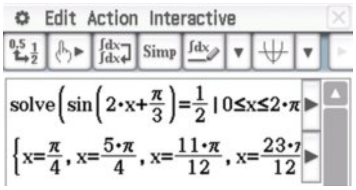
Using the Casio ClassPad

- Open the Main $\sqrt{\alpha}$ application.
- Ensure your calculator is in radian mode (with **Rad** in the status bar at the bottom of the main screen).
- Enter and highlight

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \mid 0 \le x \le 2\pi$$

- Select **Interactive** > **Equation/Inequality** > **solve**.
- Tap ► on the solution line to view the entire solution.



Transformations of the graphs of sine and cosine

The graphs of functions with rules of the form

$$f(x) = a\sin(nx + \varepsilon) + b \qquad \text{and} \qquad f(x) = a\cos(nx + \varepsilon) + b$$

can be obtained from the graphs of $y = \sin x$ and $y = \cos x$ by transformations.

Example 7

Sketch the graph of the function

$$h\colon [0, 2\pi] \to \mathbb{R},\ h(x) = 3\cos\left(2x + \frac{\pi}{3}\right) + 1$$

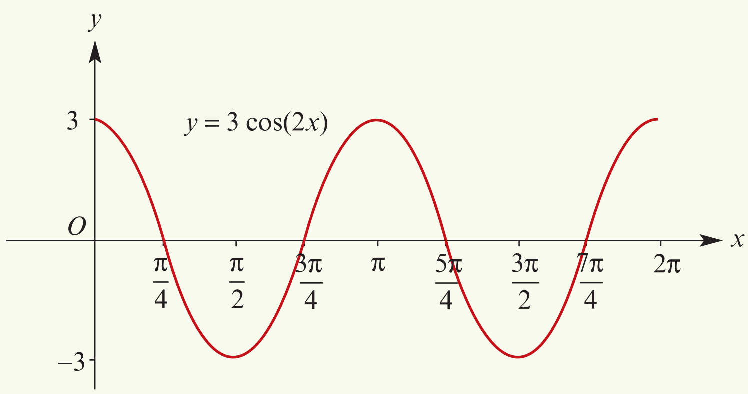
Solution

We can write $h(x) = 3\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 1$.

The graph of $y = h(x)$ is obtained from the graph of $y = \cos x$ by:

- a dilation of factor $\frac{1}{2}$ from the y-axis
- a dilation of factor 3 from the x-axis
- a translation of $\frac{\pi}{6}$ units in the negative direction of the x-axis
- a translation of 1 unit in the positive direction of the y-axis.

First apply the two dilations to the graph of $y = \cos x$.



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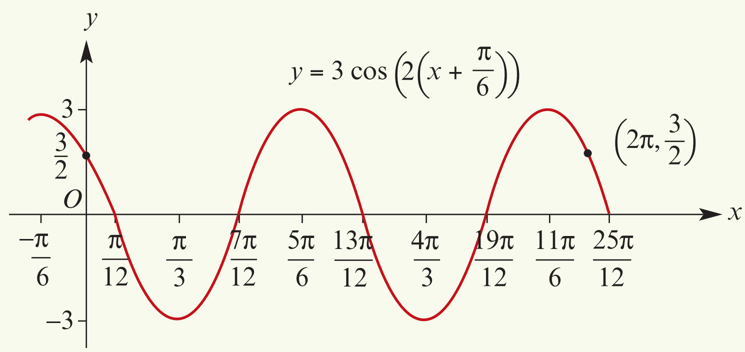
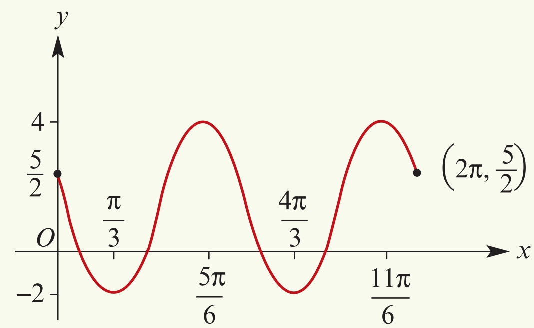
Next apply the translation $\frac{\pi}{6}$ units in the negative direction of the x-axis.

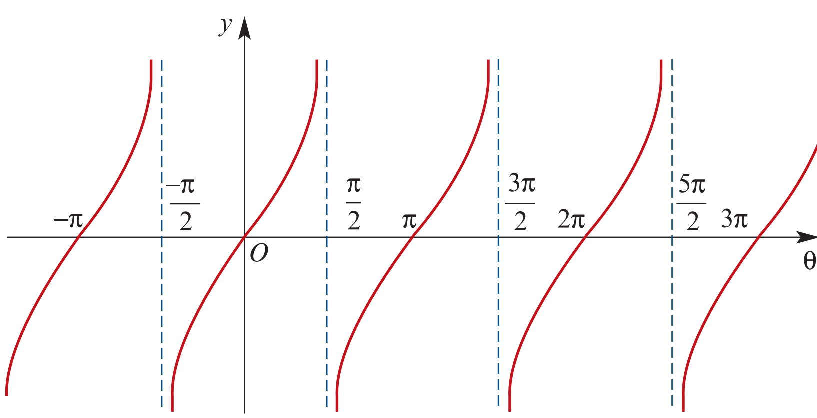


Apply the final translation and restrict the graph to the required domain.



The graph of tan

A sketch of the graph of $y = \tan\theta$ is shown below.



Notes:

- The domain of tan is $\mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$.
- The range of tan is $\mathbb{R}$.
- The graph repeats itself every π units, i.e. the period of tan is π.
- The vertical asymptotes have equations $\theta = \frac{(2n+1)\pi}{2}$, for $n \in \mathbb{Z}$.

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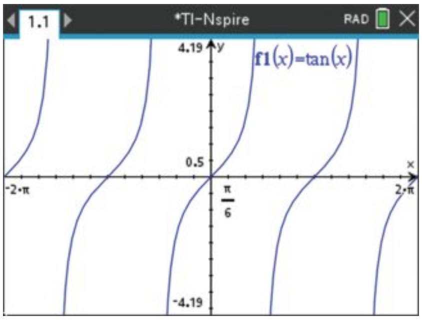
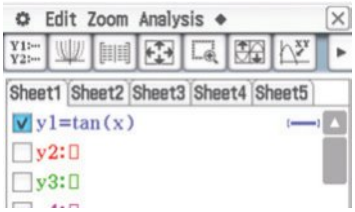
Using the TI-Nspire

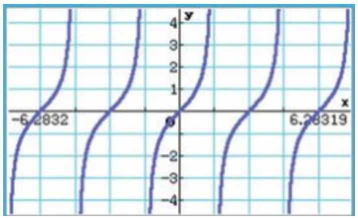
- Open a **Graphs** application and define $f1(x) = \tan(x)$.
- Press (enter) to obtain the graph.
- To change the viewing window, go to (menu) > **Window/Zoom** > **Window Settings**.



Using the Casio ClassPad

- Open the menu Menu; select **Graph & Table** Graph& Table.
- Enter $\tan(x)$ in $y1$, tick the box and tap .
- If necessary, select **Zoom** > **Quick** > **Quick Trig** or tap to manually adjust the window. In the graph shown below, the x-axis scale has been set to $\frac{\pi}{2}$.





Symmetry properties of tan

The following results are obtained from the definition of tan:

$$\tan(\pi - \theta) = -\tan\theta \qquad \tan(2\pi - \theta) = -\tan\theta$$

$$\tan(\pi + \theta) = \tan\theta \qquad \tan(-\theta) = -\tan\theta$$

Example 8

Find the exact value of:

a $\tan 330^\circ$

b $\tan\left(\frac{4\pi}{3}\right)$

Solution

a $\tan 330^\circ = \tan(360^\circ - 30^\circ)$

$= -\tan 30^\circ$

$= -\frac{1}{\sqrt{3}}$

b $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right)$

$= \tan\left(\frac{\pi}{3}\right)$

$= \sqrt{3}$

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Solution of equations involving tan

The procedure here is similar to that used for solving equations involving sin and cos, except that only one solution needs to be selected then all other solutions are one period length apart.

Example 9

Solve the following equations:

a $\tan x = -1$ for $x \in [0, 4\pi]$ **b** $\tan(2x - \pi) = \sqrt{3}$ for $x \in [-\pi, \pi]$

Solution

a $\tan x = -1$

Now $\tan\left(\frac{3\pi}{4}\right) = -1$

$$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{3\pi}{4} + \pi \text{ or } \frac{3\pi}{4} + 2\pi \text{ or } \frac{3\pi}{4} + 3\pi$$

$$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4} \text{ or } \frac{11\pi}{4} \text{ or } \frac{15\pi}{4}$$

b Let $\theta = 2x - \pi$. Then

$$-\pi \le x \le \pi \Leftrightarrow -2\pi \le 2x \le 2\pi$$
$$\Leftrightarrow -3\pi \le 2x - \pi \le \pi$$
$$\Leftrightarrow -3\pi \le \theta \le \pi$$

To solve $\tan(2x - \pi) = \sqrt{3}$, we first solve $\tan\theta = \sqrt{3}$.

$$\theta = \frac{\pi}{3} \text{ or } \frac{\pi}{3} - \pi \text{ or } \frac{\pi}{3} - 2\pi \text{ or } \frac{\pi}{3} - 3\pi$$

$$\therefore \quad \theta = \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{2} \text{ or } -\frac{8\pi}{3}$$

$$\therefore \quad 2x - \pi = \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3} \text{ or } -\frac{8\pi}{3}$$

$$\therefore \quad 2x = \frac{4\pi}{3} \text{ or } \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3}$$

$$\therefore \quad x = \frac{2\pi}{3} \text{ or } \frac{\pi}{6} \text{ or } -\frac{\pi}{3} \text{ or } -\frac{5\pi}{6}$$

Skill-sheet

Exercise 1A

Example 1

1 a Convert the following angles from degrees to exact values in radians:

i 720° **ii** 540° **iii** −450° **iv** 15° **v** −10° **vi** −315°

b Convert the following angles from radians to degrees:

i $\frac{5\pi}{4}$ **ii** $-\frac{2\pi}{3}$ **iii** $\frac{7\pi}{12}$ **iv** $-\frac{11\pi}{6}$ **v** $\frac{13\pi}{9}$ **vi** $-\frac{11\pi}{12}$

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2 Perform the correct conversion on each of the following angles, giving the answer correct to two decimal places.

a Convert from degrees to radians:

i $7°$ **ii** $-100°$ **iii** $-25°$ **iv** $51°$ **v** $206°$ **vi** $-410°$

b Convert from radians to degrees:

i 1.7^c **ii** -0.87^c **iii** 2.8^c **iv** 0.1^c **v** -3^c **vi** -8.9^c

Example 2

3 Find the exact value of each of the following:

a $\sin(135°)$ **b** $\cos(-300°)$ **c** $\sin(480°)$

d $\cos(240°)$ **e** $\sin(-225°)$ **f** $\sin(420°)$

Example 3

4 Find the exact value of each of the following:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\cos\left(-\frac{\pi}{3}\right)$

d $\cos\left(\frac{5\pi}{4}\right)$ **e** $\cos\left(\frac{9\pi}{4}\right)$ **f** $\sin\left(\frac{11\pi}{3}\right)$

g $\cos\left(\frac{31\pi}{6}\right)$ **h** $\cos\left(\frac{29\pi}{6}\right)$ **i** $\sin\left(-\frac{23\pi}{6}\right)$

Example 4

5 If $\sin x = 0.5$ and $\frac{\pi}{2} < x < \pi$, find:

a $\cos x$ **b** $\tan x$

6 If $\cos x = -0.7$ and $\pi < x < \frac{3\pi}{2}$, find:

a $\sin x$ **b** $\tan x$

7 If $\sin x = -0.5$ and $\pi < x < \frac{3\pi}{2}$, find:

a $\cos x$ **b** $\tan x$

8 If $\sin x = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, find:

a $\cos x$ **b** $\tan x$

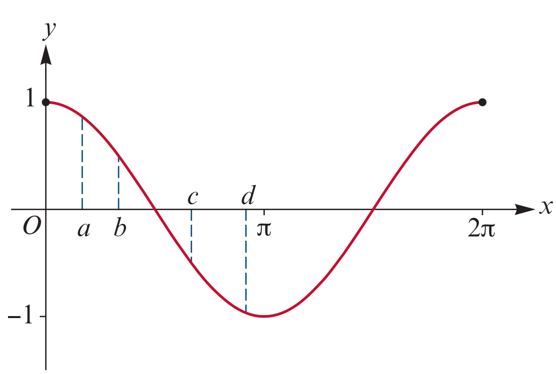
Example 5

9 The graph of $y = f(x)$ for

$f\colon [0, 2\pi] \to \mathbb{R}, f(x) = \cos x$

is shown.

For each pronumeral marked on the x-axis, find the other x-value which has the same y-value.



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Example 6 **10** Solve each of the following for $x \in [0, 2\pi]$:

a $\sin x = -\frac{\sqrt{3}}{2}$ **b** $\sin(2x) = -\frac{\sqrt{3}}{2}$

c $2\cos(2x) = -1$ **d** $\sin\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$

e $2\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -1$ **f** $2\sin\left(2x + \frac{\pi}{3}\right) = -\sqrt{3}$

Example 7 **11** Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin(2x),\ x \in [0, 2\pi]$ **b** $f(x) = \cos\left(x + \frac{\pi}{3}\right),\ x \in \left[-\frac{\pi}{3}, \pi\right]$

c $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right),\ x \in [0, \pi]$ **d** $f(x) = 2\sin(3x) + 1,\ x \in [0, \pi]$

e $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3},\ x \in [0, 2\pi]$

Example 8 **12** Find the exact value of each of the following:

a $\tan\left(\frac{5\pi}{4}\right)$ **b** $\tan\left(-\frac{2\pi}{3}\right)$ **c** $\tan\left(-\frac{29\pi}{6}\right)$ **d** $\tan 240°$

13 If $\tan x = \frac{1}{4}$ and $\pi \le x \le \frac{3\pi}{2}$, find the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(\pi - x)$

14 If $\tan x = -\frac{\sqrt{3}}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(x - \pi)$

Example 9 **15** Solve each of the following for $x \in [0, 2\pi]$:

a $\tan x = -\sqrt{3}$ **b** $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

c $2\tan\left(\frac{x}{2}\right) + 2 = 0$ **d** $3\tan\left(\frac{\pi}{2} + 2x\right) = -3$

16 Sketch the graph of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:

a $f(x) = \tan(2x)$ **b** $f(x) = \tan\left(x - \frac{\pi}{3}\right)$

c $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right)$ **d** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$

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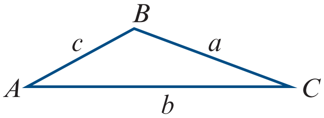
1B The sine and cosine rules

In this section, we revise methods for finding unknown quantities (side lengths or angles) in a non-right-angled triangle.

Labelling triangles

The following convention is used in the remainder of this chapter:

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.



For example, the magnitude of angle BAC is denoted by A, and the length of side BC is denoted by a.

The sine rule

The sine rule is used to find unknown quantities in a triangle in the following two situations:

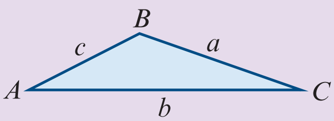
1 one side and two angles are given
2 two sides and a non-included angle are given.

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD:

$$\sin A = \frac{h}{b}$$

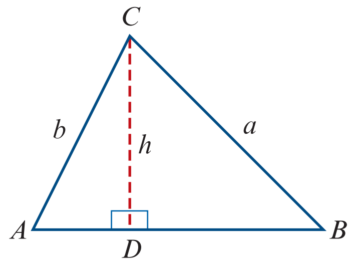
$$\therefore \quad h = b \sin A$$

In triangle BCD:

$$\sin B = \frac{h}{a}$$

$$\therefore \quad a \sin B = b \sin A$$

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B}$$



Similarly, starting with a perpendicular from A to BC would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

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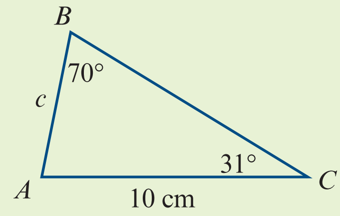
Example 10

Use the sine rule to find the length of AB.



Solution

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

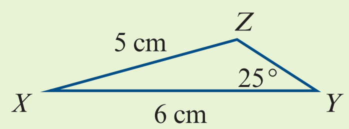
$$\therefore \quad c = \frac{10 \sin 31^\circ}{\sin 70^\circ}$$

$$= 5.4809\ldots$$

The length of AB is 5.48 cm, correct to two decimal places.

Example 11

Use the sine rule to find the magnitude of angle XZY, given that $Y = 25^\circ$, $y = 5$ and $z = 6$.



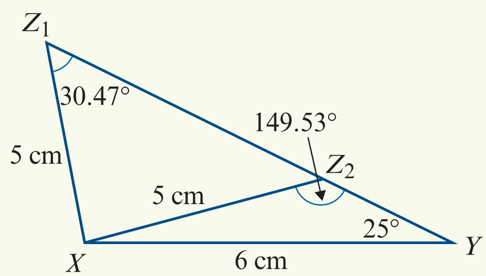
Solution

$$\frac{5}{\sin 25^\circ} = \frac{6}{\sin Z}$$

$$\frac{\sin Z}{6} = \frac{\sin 25^\circ}{5}$$

$$\sin Z = \frac{6 \sin 25^\circ}{5}$$

$$= 0.5071\ldots$$



$$\therefore \quad Z = (30.473\ldots)^\circ \quad \text{or} \quad Z = (180 - 30.473\ldots)^\circ$$

Hence $Z = 30.47^\circ$ or $Z = 149.53^\circ$, correct to two decimal places.

Notes:

- Remember that $\sin(180 - \theta)^\circ = \sin(\theta^\circ)$.
- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles. An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180°.

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The cosine rule

The cosine rule can be used to find unknown quantities in a triangle in the following two situations:

1 two sides and the included angle are given

2 three sides are given.

In each case, the triangle is uniquely defined up to congruence.

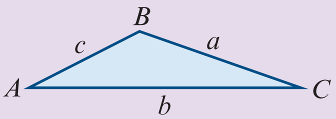
Cosine rule

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

or equivalently

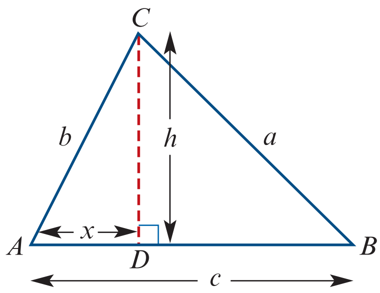
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



The symmetrical results also hold:

- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.



In triangle ACD:

$$\cos A = \frac{x}{b}$$

$$\therefore \quad x = b\cos A$$

Using Pythagoras' theorem in triangles ACD and BCD:

$$b^2 = x^2 + h^2$$

$$a^2 = (c - x)^2 + h^2$$

Expanding gives

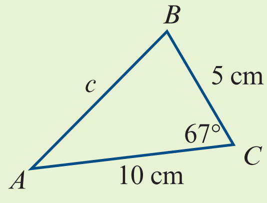
$$a^2 = c^2 - 2cx + x^2 + h^2$$

$$= c^2 - 2cx + b^2 \qquad (\text{as } b^2 = x^2 + h^2)$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A \qquad (\text{as } x = b\cos A)$$

Example 12

For triangle ABC, find the length of AB in centimetres correct to two decimal places.



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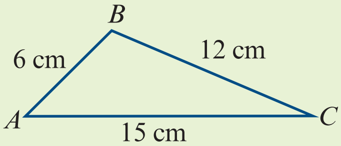
Solution

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore \quad c = 9.2696\ldots$$

The length of AB is 9.27 cm, correct to two decimal places.

Example 13

For triangle ABC, find the magnitude of angle ABC correct to two decimal places.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$

The magnitude of angle ABC is 108.21°, correct to two decimal places.

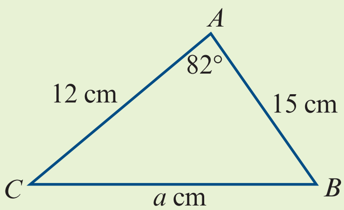
Example 14

In $\triangle ABC$, $\angle CAB = 82^\circ$, $AC = 12$ cm and $AB = 15$ cm.

Find correct to two decimal places:

a BC

b $\angle ACB$



Solution

a Find BC using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$= 12^2 + 15^2 - 2 \times 12 \times 15 \cos 82^\circ$$
$$= 144 + 225 - 360 \cos 82^\circ$$
$$= 318.8976\ldots$$
$$a = 17.8577\ldots$$

Thus $BC = a = 17.86$ cm, correct to two decimal places.

b Find $\angle ACB$ using the sine rule:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\therefore \quad \sin C = \frac{c \sin A}{a}$$
$$= \frac{15 \sin 82^\circ}{17.8577}$$

Thus $\angle ACB = 56.28^\circ$, correct to two decimal places.

Note: In part **b**, the angle $C = 123.72^\circ$ is also a solution to the equation, but it is discarded as a possible answer because it is inconsistent with the given angle $A = 82^\circ$.

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Skillsheet

Exercise 1B

Example 10

1 In triangle ABC, $\angle BAC = 73°$, $\angle ACB = 55°$ and $AB = 10$ cm. Find correct to two decimal places:

a BC **b** AC

Example 11

2 In $\triangle ABC$, $\angle ACB = 34°$, $AC = 8.5$ cm and $AB = 5.6$ cm. Find correct to two decimal places:

a the two possible values of $\angle ABC$ (one acute and one obtuse)

b BC in each case.

Example 12

3 In triangle ABC, $\angle ABC = 58°$, $AB = 6.5$ cm and $BC = 8$ cm. Find correct to two decimal places:

a AC **b** $\angle BCA$

Example 13
Example 14

4 In $\triangle ABC$, $AB = 5$ cm, $BC = 12$ cm and $AC = 10$ cm. Find:

a the magnitude of $\angle ABC$, correct to two decimal places

b the magnitude of $\angle BAC$, correct to two decimal places.

5 The adjacent sides of a parallelogram are 9 cm and 11 cm. One of its angles is 67°. Find the length of the longer diagonal, correct to two decimal places.

Example 14

6 In $\triangle ABC$, $\angle ABC = 35°$, $AB = 10$ cm and $BC = 4.7$ cm. Find correct to two decimal places:

a AC **b** $\angle ACB$

7 In $\triangle ABC$, $\angle ABC = 45°$, $\angle ACB = 60°$ and $AC = 12$ cm. Find AB.

8 In $\triangle PQR$, $\angle QPR = 60°$, $PQ = 2$ cm and $PR = 3$ cm. Find QR.

9 In $\triangle ABC$, the angle ABC has magnitude 40°, $AC = 20$ cm and $AB = 18$ cm. Find the distance BC correct to two decimal places.

10 In $\triangle ABC$, the angle ACB has magnitude 30°, $AC = 10$ cm and $AB = 8$ cm. Find the distance BC using the cosine rule.

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1C Sequences and series

The following are examples of sequences of numbers:

a $1, 3, 5, 7, 9, \dots$ **b** $10, 7, 4, 1, -2, \dots$ **c** $0.6, 1.7, 2.8, \dots, 9.4$

d $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ **e** $0.1, 0.11, 0.111, 0.1111, 0.11111, \dots$

Each sequence is a list of numbers, with order being important.

The numbers of a sequence are called its **terms**. The nth term of a sequence is denoted by the symbol t_n. So the first term is t_1, the 12th term is t_{12}, and so on.

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation**, a **recursive formula** or an **iterative rule**. For example:

- The sequence $1, 3, 5, 7, 9, \dots$ may be defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ may be defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

Example 15

Use the recurrence relation to find the first four terms of the sequence

$$t_1 = 3, \quad t_n = t_{n-1} + 5$$

Solution

$$t_1 = 3$$
$$t_2 = t_1 + 5 = 8$$
$$t_3 = t_2 + 5 = 13$$
$$t_4 = t_3 + 5 = 18$$

The first four terms are 3, 8, 13, 18.

Example 16

Find a possible recurrence relation for the following sequence:

$$9, -3, 1, -\frac{1}{3}, \dots$$

Solution

$$-3 = -\frac{1}{3} \times 9 \qquad \text{i.e. } t_2 = -\frac{1}{3}t_1$$

$$1 = -\frac{1}{3} \times -3 \qquad \text{i.e. } t_3 = -\frac{1}{3}t_2$$

The sequence is defined by $t_1 = 9$ and $t_n = -\frac{1}{3}t_{n-1}$.

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A sequence may also be defined explicitly by a rule that is stated in terms of n. For example:

- The rule $t_n = 2n$ defines the sequence $t_1 = 2, t_2 = 4, t_3 = 6, t_4 = 8, \dots$
- The rule $t_n = 2^{n-1}$ defines the sequence $t_1 = 1, t_2 = 2, t_3 = 4, t_4 = 8, \dots$
- The sequence $1, 3, 5, 7, 9, \dots$ can be defined by $t_n = 2n - 1$.
- The sequence $t_1 = \frac{1}{3}$, $t_n = \frac{1}{3}t_{n-1}$ can be defined by $t_n = \frac{1}{3^n}$.

Example 17

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

Solution

$$t_1 = 2(1) + 3 = 5$$
$$t_2 = 2(2) + 3 = 7$$
$$t_3 = 2(3) + 3 = 9$$
$$t_4 = 2(4) + 3 = 11$$

The first four terms are $5, 7, 9, 11$.

Arithmetic sequences

A sequence in which each successive term is found by adding a fixed amount to the previous term is called an **arithmetic sequence**. That is, an arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is a constant.

For example: $2, 5, 8, 11, 14, 17, \dots$ is an arithmetic sequence.

The nth term of an arithmetic sequence is given by

$$t_n = a + (n - 1)d$$

where a is the first term and d is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

Example 18

Find the 10th term of the arithmetic sequence $-4, -1, 2, 5, \dots$.

Solution

$a = -4$, $d = 3$

$$t_n = a + (n - 1)d$$
$$\therefore \quad t_{10} = -4 + (10 - 1) \times 3$$
$$= 23$$

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Arithmetic series

The sum of the terms in a sequence is called a **series**. If the sequence is arithmetic, then the series is called an **arithmetic series**.

The symbol S_n is used to denote the sum of the first n terms of a sequence. That is,

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

Writing this sum in reverse order, we have

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + d) + a$$

Adding these two expressions together gives

$$2S_n = n(2a + (n - 1)d)$$

Therefore

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big)$$

Since the last term $\ell = t_n = a + (n - 1)d$, we can also write

$$S_n = \frac{n}{2}(a + \ell)$$

Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed amount is called a **geometric sequence**. That is, a geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is a constant.

For example: $2, 6, 18, 54, \dots$ is a geometric sequence.

The nth term of a geometric sequence is given by

$$t_n = ar^{n-1}$$

where a is the first term and r is the **common ratio** of successive terms, that is, $r = \dfrac{t_k}{t_{k-1}}$, for all $k > 1$.

Example 19

Find the 10th term of the geometric sequence $2, 6, 18, \dots$.

Solution

$a = 2$, $r = 3$

$$t_n = ar^{n-1}$$

$$\therefore \quad t_{10} = 2 \times 3^{10-1} = 39\,366$$

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Geometric series

The sum of the terms in a geometric sequence is called a **geometric series**. An expression for S_n, the sum of the first n terms of a geometric sequence, can be found using a similar method to that used for arithmetic series.

Let $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad (1)$

Then $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad (2)$

Subtract (1) from (2):

$$rS_n - S_n = ar^n - a$$
$$S_n(r-1) = a(r^n - 1)$$

Therefore

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

For values of r such that $-1 < r < 1$, it is often more convenient to use the equivalent formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

which is obtained by multiplying both the numerator and the denominator by -1.

Example 20

Find the sum of the first nine terms of the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$.

Solution

$a = \frac{1}{3}$, $r = \frac{1}{3}$, $n = 9$

$$\therefore \quad S_9 = \frac{\frac{1}{3}\left(1 - (\frac{1}{3})^9\right)}{1 - \frac{1}{3}}$$
$$= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^9\right)$$
$$\approx 0.499975$$

Infinite geometric series

If a geometric sequence has a common ratio with magnitude less than 1, that is, if $-1 < r < 1$, then each successive term is closer to zero. For example, consider the sequence

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$$

In Example 20 we found that the sum of the first 9 terms is $S_9 \approx 0.499975$. The sum of the first 20 terms is $S_{20} \approx 0.49999999986$. We might conjecture that, as we add more and more terms of the sequence, the sum will get closer and closer to 0.5, that is, $S_n \to 0.5$ as $n \to \infty$.

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An infinite series $t_1 + t_2 + t_3 + \cdots$ is said to be **convergent** if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. This limit is called the **sum to infinity** of the series.

If $-1 < r < 1$, then the infinite geometric series $a + ar + ar^2 + \cdots$ is convergent and the sum to infinity is given by

$$S_\infty = \frac{a}{1-r}$$

Proof We know that

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

As $n \to \infty$, we have $r^n \to 0$ and so $\dfrac{ar^n}{1-r} \to 0$. Hence $S_n \to \dfrac{a}{1-r}$ as $n \to \infty$.

Example 21

Find the sum to infinity of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution

$a = \frac{1}{2}$, $r = \frac{1}{2}$ and therefore

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Using a CAS calculator with sequences

Example 22

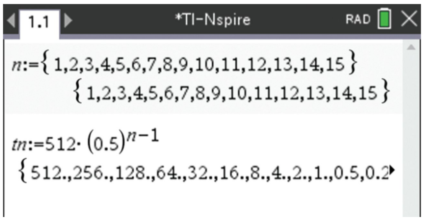
Use a calculator to generate terms of the geometric sequence defined by

$$t_n = 512(0.5)^{n-1} \quad \text{for } n = 1, 2, 3, \ldots$$

Using the TI-Nspire

Sequences defined in terms of n can be investigated in a **Calculator** application.

- To generate the first 15 terms of the sequence defined by the rule $t_n = 512(0.5)^{n-1}$, complete as shown.



Note: Alternatively, assign these values to n by entering $n := \text{seq}(k, k, 1, 15, 1)$. Assigning (storing) the resulting list as tn enables the sequence to be graphed. The lists n and tn can also be created in a **Lists & Spreadsheet** application.

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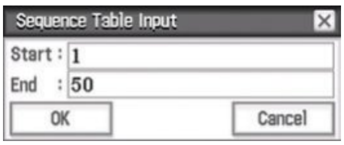
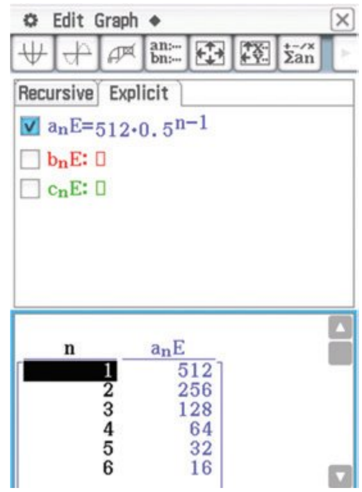
Using the Casio ClassPad

- Open the menu **Menu**; select **Sequence**.
- Ensure that the **Explicit** window is activated.
- Tap the cursor next to a_nE and enter $512 \times 0.5^{n-1}$. (The variable n can be entered by tapping on n in the toolbar.)
- Tick the box or tap EXE.
- Tap to view the sequence values.
- Tap to open the Sequence Table Input window and complete as shown below; tap OK.



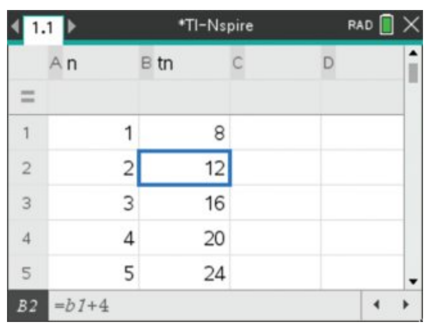


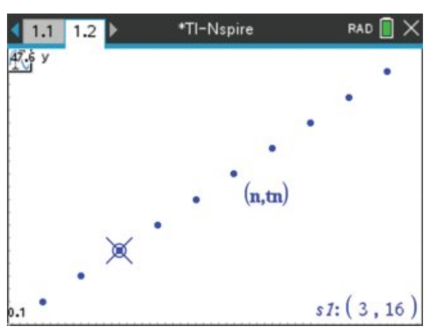
Example 23

Use a CAS calculator to plot the graph of the arithmetic sequence defined by the recurrence relation $t_n = t_{n-1} + 4$ and $t_1 = 8$.

Using the TI-Nspire

- In a **Lists & Spreadsheet** page, name the first two columns n and tn respectively.
- Enter 1 in cell A1 and enter 8 in cell B1.
- Enter $= a1 + 1$ in cell A2 and enter $= b1 + 4$ in cell B2.
- Highlight the cells A2 and B2 using shift and the arrows.
- Use menu > **Data** > **Fill** to fill down to row 10 and press enter. This generates the first 10 terms of the sequence.
- To graph the sequence, open a **Graphs** application (ctrl I > **Add Graphs**).
- Create a scatter plot using menu > **Graph Entry/Edit** > **Scatter Plot**. Enter the list variables as n and tn in their respective fields.
- Set an appropriate window using menu > **Window/Zoom** > **Zoom – Data**.





Note: It is possible to see the coordinates of the points: menu > **Trace** > **Graph Trace**. The scatter plot can also be graphed in a **Data & Statistics** page.

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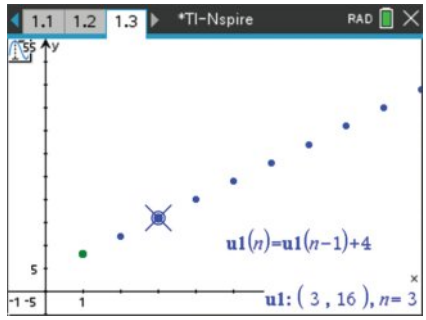
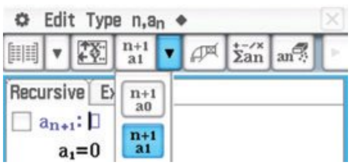
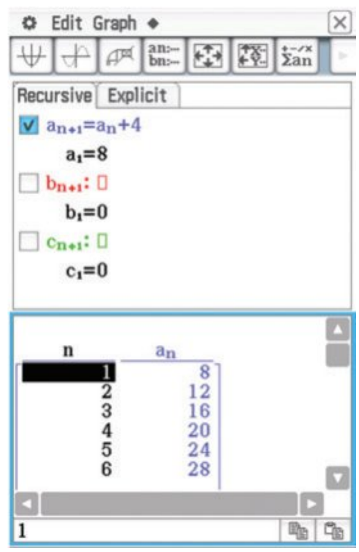
- Alternatively, the sequence can be graphed directly in the sequence plotter (menu > **Graph Entry/Edit** > **Sequence** > **Sequence**) with initial value 8.



Using the Casio ClassPad

- Open the menu Menu; select **Sequence** .
- Ensure that the **Recursive** window is activated.
- Select the setting n+1 a1 as shown below.

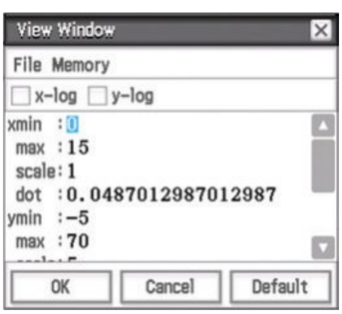


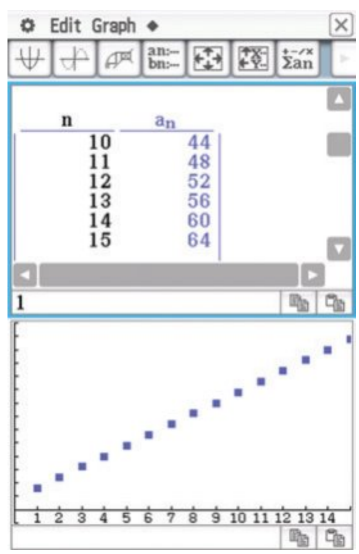


- Tap the cursor next to a_{n+1} and enter $a_n + 4$.

Note: The symbol a_n can be found in the dropdown menu n, a_n.

- Enter 8 for the value of the first term, a_1.
- Tick the box. Tap to view the sequence values.
- Tap to view the graph.
- Tap and adjust the window setting for the first 15 terms as shown below.





- Select **Analysis** > **Trace** and use the cursor ► to view each value in the sequence.

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Recurrence relations of the form $t_n = rt_{n-1} + d$

We now consider a generalisation of arithmetic and geometric sequences. We shall study sequences defined by a recurrence relation of the form

$$t_n = rt_{n-1} + d$$

where r and d are constants.

Note: The case where $r = 1$ corresponds to an arithmetic sequence.
The case where $d = 0$ corresponds to a geometric sequence.

We can establish a general formula for the nth term of the sequence.

For a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where $r \neq 1$, the nth term is given by

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

where t_1 is the first term.

Proof We can establish the formula by checking that it gives the correct first term and that it satisfies the recurrence relation.

First term Using the formula to find the first term ($n = 1$) gives

$$r^{1-1}t_1 + \frac{d(r^{1-1} - 1)}{r - 1} = 1 \times t_1 + 0 = t_1$$

which is correct.

Recurrence relation We now check that the formula satisfies the recurrence relation $t_n = rt_{n-1} + d$. Starting from the right-hand side:

$$\begin{aligned} rt_{n-1} + d &= r\left(r^{n-2}t_1 + \frac{d(r^{n-2} - 1)}{r - 1}\right) + d && \text{(using the formula for } t_{n-1}\text{)} \\ &= r^{n-1}t_1 + \frac{d(r^{n-1} - r)}{r - 1} + d \\ &= r^{n-1}t_1 + \frac{d(r^{n-1} - r)}{r - 1} + \frac{d(r - 1)}{r - 1} \\ &= r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1} \\ &= t_n && \text{(using the formula for } t_n\text{)} \end{aligned}$$

So the recurrence relation holds.

We have shown that the formula gives the correct value for t_1. Since it satisfies the recurrence relation, this means that t_2 is correct, and then this means that t_3 is correct, and so on. (This proof uses mathematical induction, which is revised in Chapter 2.)

Note: This general formula for t_n can be rewritten into a rule of the form $t_n = Ar^{n-1} + B$, for constants A and B. We use this observation in Example 25.

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Example 24

Find a formula for the nth term of the sequence defined by the recurrence relation

$$t_n = 2t_{n-1} + 1, \quad t_1 = 10$$

Solution

We will use the general formula

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1}$$

Here $r = 2$ and $d = 1$. Hence

$$\begin{aligned} t_n &= 2^{n-1} \times 10 + \frac{1 \times (2^{n-1} - 1)}{2 - 1} \\ &= 10 \times 2^{n-1} + 2^{n-1} - 1 \\ &= 11 \times 2^{n-1} - 1 \end{aligned}$$

Example 25

The sequence $5, 16, 38, \ldots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the nth term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants A and B.

Solution

From the first three terms, we have

$$t_1 = A + B = 5 \qquad (1)$$

$$t_2 = Ar + B = 16 \qquad (2)$$

$$t_3 = Ar^2 + B = 38 \qquad (3)$$

Subtract equation (1) from both (2) and (3):

$$A(r - 1) = 11 \qquad (4)$$

$$A(r^2 - 1) = 33 \qquad (5)$$

Divide (5) by (4):

$$\begin{aligned} \frac{r^2 - 1}{r - 1} &= 3 \\ \frac{(r + 1)(r - 1)}{r - 1} &= 3 \\ r + 1 &= 3 \\ r &= 2 \end{aligned}$$

Using (4) now gives $A = 11$, and using (1) gives $B = -6$.

The formula for the nth term is

$$t_n = 11 \times 2^{n-1} - 6$$

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Exercise 1C

Example 15 **1** Use the recurrence relation to find the first four terms of the sequence $t_1 = 3$, $t_n = t_{n-1} - 4$.

Example 16 **2** Find a possible recurrence relation for the sequence $-2, 6, -18, \dots$.

Example 17 **3** Find the first four terms of the sequence defined by $t_n = 2n - 3$ for $n \in \mathbb{N}$.

4 The Fibonacci sequence is given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $F_1 = F_2 = 1$. Find the first 10 terms of the Fibonacci sequence.

Example 18 **5** Find the 10th term of the arithmetic sequence $-4, -7, -10, \dots$.

Example 19 **6** Calculate the 10th term of the geometric sequence $2, -6, 18, \dots$.

7 Find the sum of the first 10 terms of an arithmetic sequence with first term 3 and common difference 4.

Example 20 **8** Find the sum of the first eight terms of a geometric sequence with first term 6 and common ratio -3.

Example 21 **9** Find the sum to infinity of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

10 The first, second and third terms of a geometric sequence are $x + 5$, x and $x - 4$ respectively. Find:

a the value of x

b the common ratio

c the difference between the sum to infinity and the sum of the first 10 terms.

11 Find the sum to infinity of the geometric sequence $a, \frac{a}{\sqrt{2}}, \frac{a}{2}, \frac{a}{2\sqrt{2}}, \dots$ in terms of a.

12 Consider the sum

$$S_n = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots + \frac{x^{n-1}}{2^{n-1}}$$

a Calculate S_{10} when $x = 1.5$.

b **i** Find the possible values of x for which the sum to infinity S_∞ exists.

ii Find the values of x for which $S_\infty = 2S_{10}$.

13 **a** Find an expression for the sum to infinity of the infinite geometric series

$$1 + \sin\theta + \sin^2\theta + \cdots$$

b Find the values of θ for which the sum to infinity is 2.

Example 23 **14** A sequence is defined recursively by $t_1 = 6$, $t_{n+1} = 3t_n - 1$. Find t_2 and t_3. Use a CAS calculator to find t_8.

15 A sequence is defined recursively by $y_1 = 5$, $y_{n+1} = 2y_n + 6$. Find y_2 and y_3. Use a CAS calculator to find y_{10} and to plot a graph showing the first 10 terms.

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Example 24 **16** For each of the following recurrence relations, determine an expression for the nth term of the sequence in terms of n:

a $t_n = 2t_{n-1} - 8, \quad t_1 = 8$ **b** $t_n = 2t_{n-1} - 2, \quad t_1 = 10$ **c** $t_{n+1} = \frac{1}{2}t_n + 6, \quad t_1 = 40$

Example 25 **17** The sequence $6, 7, 9, \dots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the nth term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants A and B.

18 The sequence $8, 23, 98, \dots$ is defined by a recurrence relation $t_n = rt_{n-1} + d$. Determine a formula for the nth term of this sequence by recognising that it can be written in the form $t_n = Ar^{n-1} + B$, for constants A and B.

1D The modulus function

The **modulus** or **absolute value** of a real number x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$. For example: $|5| = 5$ and $|-5| = 5$.

Example 26

Evaluate each of the following:

a **i** $|-3 \times 2|$ **ii** $|-3| \times |2|$

b **i** $\left|\frac{-4}{2}\right|$ **ii** $\frac{|-4|}{|2|}$

c **i** $|-6 + 2|$ **ii** $|-6| + |2|$

Solution

a **i** $|-3 \times 2| = |-6| = 6$ **ii** $|-3| \times |2| = 3 \times 2 = 6$ Note: $|-3 \times 2| = |-3| \times |2|$

b **i** $\left|\frac{-4}{2}\right| = |-2| = 2$ **ii** $\frac{|-4|}{|2|} = \frac{4}{2} = 2$ Note: $\left|\frac{-4}{2}\right| = \frac{|-4|}{|2|}$

c **i** $|-6 + 2| = |-4| = 4$ **ii** $|-6| + |2| = 6 + 2 = 8$ Note: $|-6 + 2| \neq |-6| + |2|$

Properties of the modulus function

- $|ab| = |a|\,|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \leq |a| + |b|$
- If a and b are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \geq 0$, then $|x| \leq a$ is equivalent to $-a \leq x \leq a$.
- If $a \geq 0$, then $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$.

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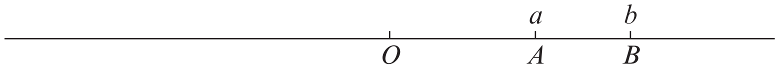
The modulus function as a measure of distance

Consider two points A and B on a number line:



On a number line, the distance between points A and B is $|a - b| = |b - a|$.

Thus $|x - 2| \leq 3$ can be read as 'the distance of x from 2 is less than or equal to 3', and $|x| \leq 3$ can be read as 'the distance of x from the origin is less than or equal to 3'. Note that $|x| \leq 3$ is equivalent to $-3 \leq x \leq 3$ or $x \in [-3, 3]$.

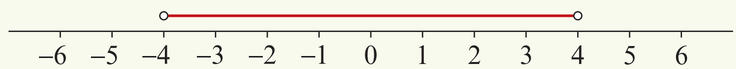
Example 27

Illustrate each set on a number line and represent the set using interval notation:

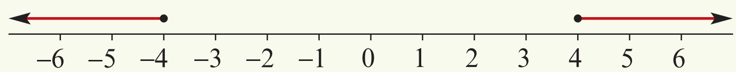
a $\{ x : |x| < 4 \}$ **b** $\{ x : |x| \geq 4 \}$ **c** $\{ x : |x - 1| \leq 4 \}$

Solution

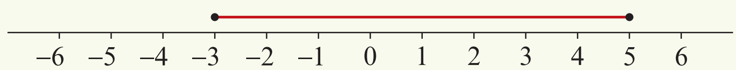
a $(-4, 4)$



b $(-\infty, -4] \cup [4, \infty)$



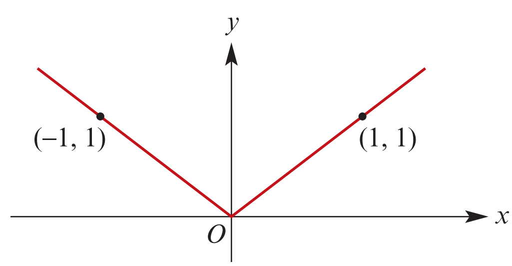
c $[-3, 5]$



The graph of $y = |x|$

The graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is shown here.

This graph is symmetric about the y-axis, since $|x| = |-x|$.



Example 28

For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 3| + 1$ **b** $f(x) = -|x - 3| + 1$

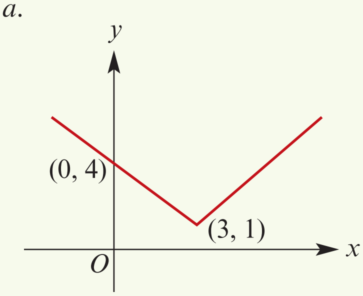
Solution

Note that $|a - b| = a - b$ if $a \geq b$, and $|a - b| = b - a$ if $b \geq a$.

a $$f(x) = |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases}$$

$$= \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases}$$

Range $= [1, \infty)$



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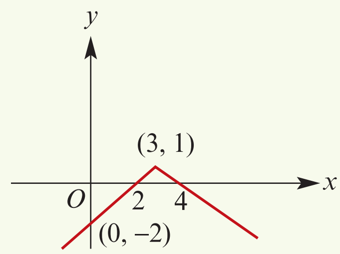
b $f(x) = -|x-3| + 1 = \begin{cases} -(x-3)+1 & \text{if } x \ge 3 \\ -(3-x)+1 & \text{if } x < 3 \end{cases}$

$= \begin{cases} -x+4 & \text{if } x \ge 3 \\ -2+x & \text{if } x < 3 \end{cases}$

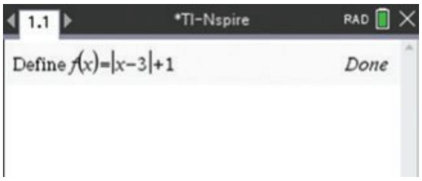
Range $= (-\infty, 1]$



Using the TI-Nspire

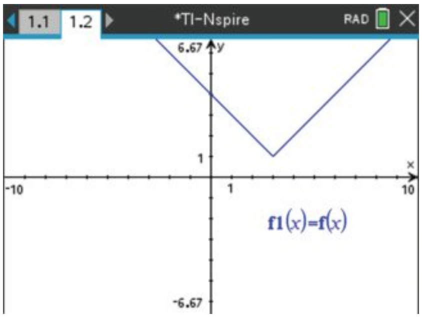
- Use menu > **Actions** > **Define** to define the function $f(x) = \text{abs}(x-3)+1$.

Note: The absolute value function can be obtained by typing **abs( )** or using the 2D-template palette.



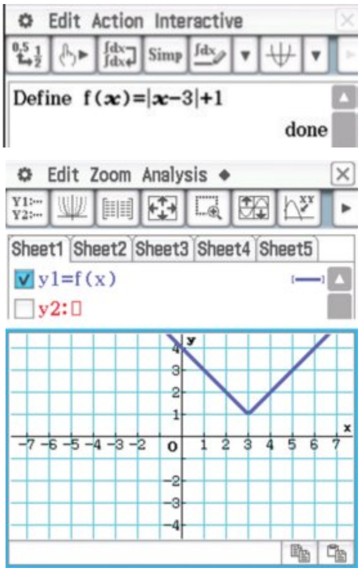
- Open a **Graphs** application (ctrl I > **Graphs**) and let $f1(x) = f(x)$.
- Press enter to obtain the graph.

Note: The expression $\text{abs}(x-3)+1$ could have been entered directly for $f1(x)$.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, define the function $f(x) = |x-3| + 1$:
  - Select Define and f from the Math3 keyboard.
  - Complete the rule for f as shown by using ▮ from the Math1 keyboard.
  - Tap EXE.
- Open the **Graph & Table** application.
- Enter $f(x)$ in $y1$. Tick the box or tap EXE.
- Tap to view the graph.



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Functions with rules of the form $y = |f(x)|$ and $y = f(|x|)$

If the graph of $y = f(x)$ is known, then we can sketch the graph of $y = |f(x)|$ using the following observation:

$$|f(x)| = f(x) \text{ if } f(x) \geq 0 \quad \text{and} \quad |f(x)| = -f(x) \text{ if } f(x) < 0$$

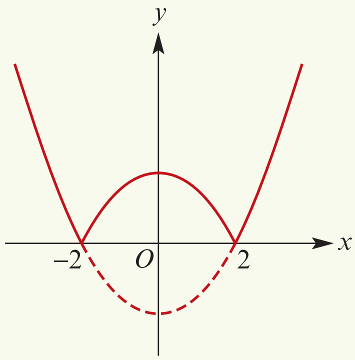
Example 29

Sketch the graph of each of the following:

a $y = |x^2 - 4|$ **b** $y = |2^x - 1|$

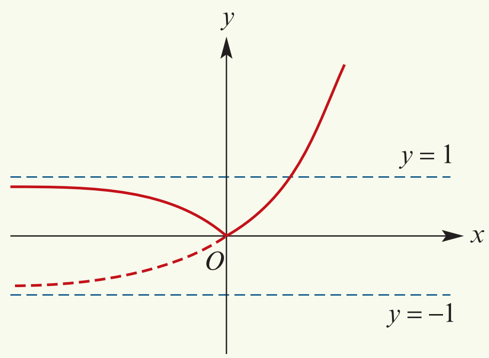
Solution

a



The graph of $y = x^2 - 4$ is drawn and the negative part reflected in the x-axis.

b



The graph of $y = 2^x - 1$ is drawn and the negative part reflected in the x-axis.

The graph of $y = f(|x|)$, for $x \in \mathbb{R}$, is sketched by reflecting the graph of $y = f(x)$, for $x \geq 0$, in the y-axis.

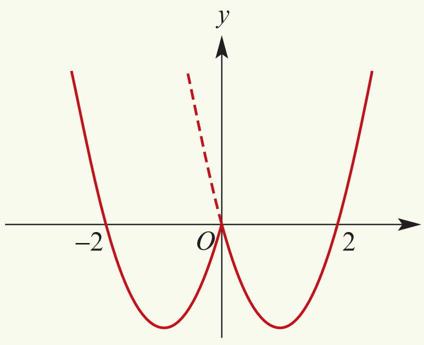
Example 30

Sketch the graph of each of the following:

a $y = |x|^2 - 2|x|$ **b** $y = 2^{|x|}$

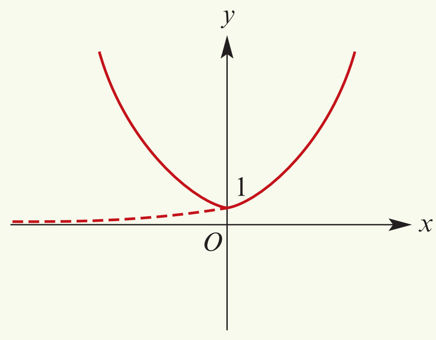
Solution

a



The graph of $y = x^2 - 2x$, $x \geq 0$, is reflected in the y-axis.

b



The graph of $y = 2^x$, $x \geq 0$, is reflected in the y-axis.

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Skill-sheet

Exercise 1D

Example 26

1 Evaluate each of the following:

a $|-5| + 3$ **b** $|-5| + |-3|$ **c** $|-5| - |-3|$

d $|-5| - |-3| - 4$ **e** $|-5| - |-3| - |-4|$ **f** $|-5| + |-3| - |-4|$

2 Solve each of the following equations for x:

a $|x - 1| = 2$ **b** $|2x - 3| = 4$ **c** $|5x - 3| = 9$ **d** $|x - 3| - 9 = 0$

e $|3 - x| = 4$ **f** $|3x + 4| = 8$ **g** $|5x + 11| = 9$

Example 27

3 For each of the following, illustrate the set on a number line and represent the set using interval notation:

a $\{ x : |x| < 3 \}$ **b** $\{ x : |x| \geq 5 \}$ **c** $\{ x : |x - 2| \leq 1 \}$

d $\{ x : |x - 2| < 3 \}$ **e** $\{ x : |x + 3| \geq 5 \}$ **f** $\{ x : |x + 2| \leq 1 \}$

Example 28

4 For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 4| + 1$ **b** $f(x) = -|x + 3| + 2$

c $f(x) = |x + 4| - 1$ **d** $f(x) = 2 - |x - 1|$

5 Solve each of the following inequalities, giving your answer using set notation:

a $\{ x : |x| \leq 5 \}$ **b** $\{ x : |x| \geq 2 \}$ **c** $\{ x : |2x - 3| \leq 1 \}$

d $\{ x : |5x - 2| < 3 \}$ **e** $\{ x : |-x + 3| \geq 7 \}$ **f** $\{ x : |-x + 2| \leq 1 \}$

6 Solve each of the following for x:

a $|x - 4| - |x + 2| = 6$ **b** $|2x - 5| - |4 - x| = 10$ **c** $|2x - 1| + |4 - 2x| = 10$

Example 29

7 Sketch the graph of each of the following:

a $y = |x^2 - 9|$ **b** $y = |3^x - 3|$ **c** $y = |x^2 - x - 12|$

d $y = |x^2 - 3x - 40|$ **e** $y = |x^2 - 2x - 8|$ **f** $y = |2^x - 4|$

Example 30

8 Sketch the graph of each of the following:

a $y = |x|^2 - 4|x|$ **b** $y = 3^{|x|}$ **c** $y = |x|^2 - 7|x| + 12$

d $y = |x|^2 - |x| - 12$ **e** $y = |x|^2 + |x| - 12$ **f** $y = -3^{|x|} + 1$

9 If $f(x) = |x - a| + b$ with $f(3) = 3$ and $f(-1) = 3$, find the values of a and b.

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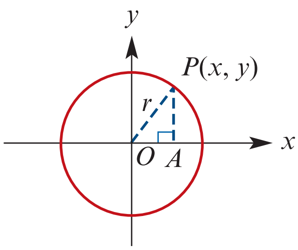
1E Circles

Consider a circle with centre at the origin and radius r.



If a point with coordinates (x, y) lies on the circle, then Pythagoras' theorem gives

$$x^2 + y^2 = r^2$$

The converse is also true. That is, a point with coordinates (x, y) such that $x^2 + y^2 = r^2$ lies on the circle.

Cartesian equation of a circle

The circle with centre (h, k) and radius r is the graph of the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

Note: This circle is obtained from the circle with equation $x^2 + y^2 = r^2$ by the translation defined by $(x, y) \to (x + h, y + k)$.

Example 31

Sketch the graph of the circle with centre $(-2, 5)$ and radius 2, and state the Cartesian equation for this circle.

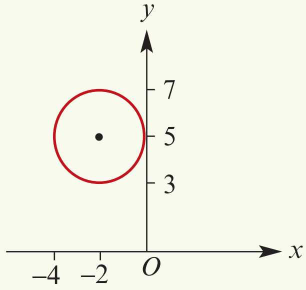
Solution

The equation is

$$(x + 2)^2 + (y - 5)^2 = 4$$

which may also be written as

$$x^2 + y^2 + 4x - 10y + 25 = 0$$



The equation $x^2 + y^2 + 4x - 10y + 25 = 0$ can be 'unsimplified' by completing the square:

$$\begin{aligned} x^2 + y^2 + 4x - 10y + 25 &= 0 \\ x^2 + 4x + 4 + y^2 - 10y + 25 + 25 &= 29 \\ (x + 2)^2 + (y - 5)^2 &= 4 \end{aligned}$$

This suggests a general form of the equation of a circle:

$$x^2 + y^2 + Dx + Ey + F = 0$$

Completing the square gives

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} + F = \frac{D^2 + E^2}{4}$$

i.e.
$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}$$

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- If $D^2 + E^2 - 4F > 0$, then this equation represents a circle.
- If $D^2 + E^2 - 4F = 0$, then this equation represents one point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$.
- If $D^2 + E^2 - 4F < 0$, then this equation has no representation in the Cartesian plane.

Example 32

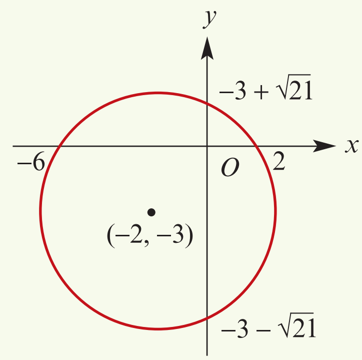
Sketch the graph of $x^2 + y^2 + 4x + 6y - 12 = 0$. State the coordinates of the centre and the radius.

Solution

Complete the square in both x and y:

$$x^2 + y^2 + 4x + 6y - 12 = 0$$
$$x^2 + 4x + 4 + y^2 + 6y + 9 - 12 = 13$$
$$(x + 2)^2 + (y + 3)^2 = 25$$

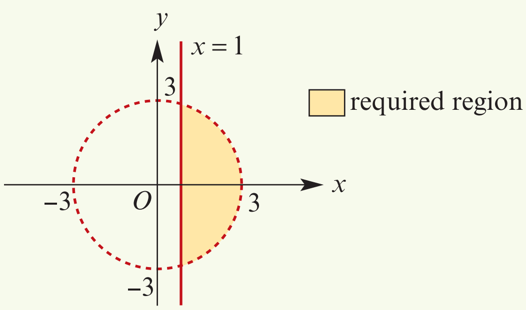
The circle has centre $(-2, -3)$ and radius 5.



Example 33

Sketch a graph of the region of the plane such that $x^2 + y^2 < 9$ and $x \geq 1$.

Solution



Exercise 1E

Example 31

1 For each of the following, find the equation of the circle with the given centre and radius:

a centre $(2, 3)$; radius 1 **b** centre $(-3, 4)$; radius 5
c centre $(0, -5)$; radius 5 **d** centre $(3, 0)$; radius $\sqrt{2}$

Example 32

2 Find the radius and the coordinates of the centre of the circle with equation:

a $x^2 + y^2 + 4x - 6y + 12 = 0$ **b** $x^2 + y^2 - 2x - 4y + 1 = 0$
c $x^2 + y^2 - 3x = 0$ **d** $x^2 + y^2 + 4x - 10y + 25 = 0$

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3 Sketch the graph of each of the following:

a $2x^2 + 2y^2 + x + y = 0$
b $x^2 + y^2 + 3x - 4y = 6$
c $x^2 + y^2 + 8x - 10y + 16 = 0$
d $x^2 + y^2 - 8x - 10y + 16 = 0$
e $2x^2 + 2y^2 - 8x + 5y + 10 = 0$
f $3x^2 + 3y^2 + 6x - 9y = 100$

Example 33

4 For each of the following, sketch the graph of the specified region of the plane:

a $x^2 + y^2 \le 16$
b $x^2 + y^2 \ge 9$
c $(x - 2)^2 + (y - 2)^2 < 4$
d $(x - 3)^2 + (y + 2)^2 > 16$
e $x^2 + y^2 \le 16$ and $x \le 2$
f $x^2 + y^2 \le 9$ and $y \ge -1$

5 The points $(8, 4)$ and $(2, 2)$ are the ends of a diameter of a circle. Find the coordinates of the centre and the radius of the circle.

6 Find the equation of the circle with centre $(2, -3)$ that touches the x-axis.

7 Find the equation of the circle that passes through $(3, 1)$, $(8, 2)$ and $(2, 6)$.

8 Consider the circles with equations

$$4x^2 + 4y^2 - 60x - 76y + 536 = 0 \quad \text{and} \quad x^2 + y^2 - 10x - 14y + 49 = 0$$

a Find the radius and the coordinates of the centre of each circle.
b Find the coordinates of the points of intersection of the two circles.

9 Find the coordinates of the points of intersection of the circle with equation $x^2 + y^2 = 25$ and the line with equation:

a $y = x$
b $y = 2x$

1F Ellipses and hyperbolas

Ellipses and hyperbolas will arise in our study of vector calculus in Chapter 13. In this section, we revise sketching graphs of ellipses and hyperbolas from their Cartesian equations.

Ellipses

For positive constants a and b, the curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is obtained from the unit circle $x^2 + y^2 = 1$ by applying the following dilations:

- a dilation of factor a from the y-axis, i.e. $(x, y) \to (ax, y)$
- a dilation of factor b from the x-axis, i.e. $(x, y) \to (x, by)$.

The result is the transformation $(x, y) \to (ax, by)$.

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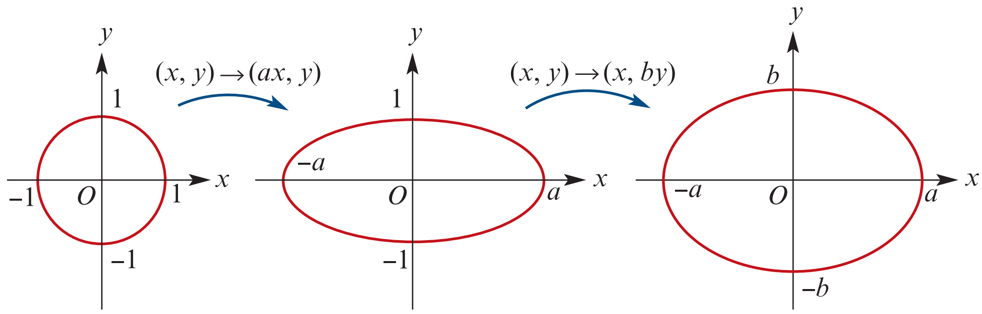


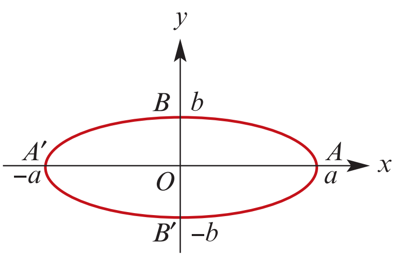
The curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is an ellipse centred at the origin with x-axis intercepts at $(-a, 0)$ and $(a, 0)$ and with y-axis intercepts at $(0, -b)$ and $(0, b)$.

If $a = b$, then the ellipse is a circle centred at the origin with radius a.

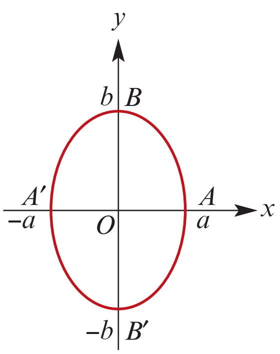
Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$



AA' is the major axis

BB' is the minor axis

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b > a$



AA' is the minor axis

BB' is the major axis

Cartesian equation of an ellipse

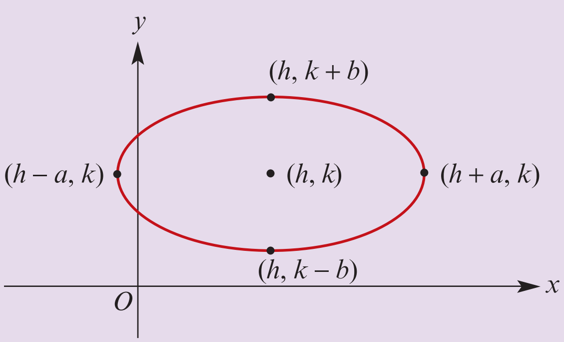
The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse with centre (h, k). It is obtained from the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

by the translation $(x, y) \to (x + h, y + k)$.



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Example 34

Sketch the graph of each of the following ellipses. Give the coordinates of the centre and the axis intercepts.

a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

b $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

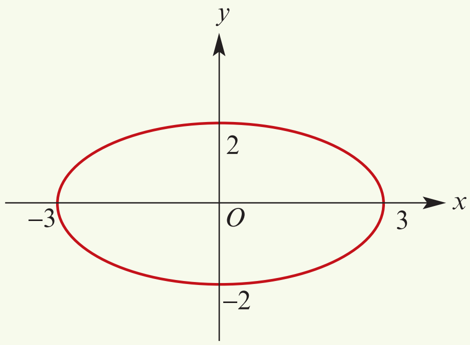
c $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{16} = 1$

d $3x^2 + 24x + y^2 + 36 = 0$

Solution

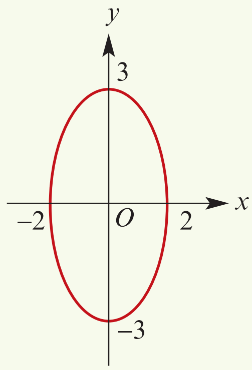
a Centre $(0, 0)$

Axis intercepts $(\pm 3, 0)$ and $(0, \pm 2)$



b Centre $(0, 0)$

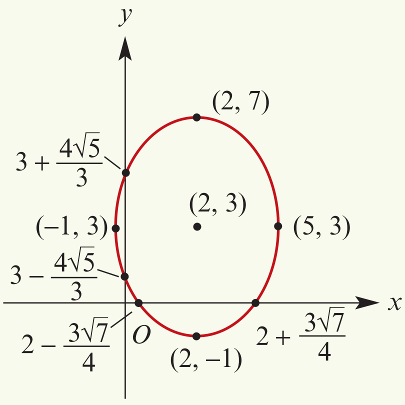
Axis intercepts $(\pm 2, 0)$ and $(0, \pm 3)$



c Centre $(2, 3)$

y-axis intercepts

When $x = 0$:
$$\frac{4}{9} + \frac{(y-3)^2}{16} = 1$$
$$\frac{(y-3)^2}{16} = \frac{5}{9}$$
$$(y-3)^2 = \frac{16 \times 5}{9}$$
$$\therefore \quad y = 3 \pm \frac{4\sqrt{5}}{3}$$



x-axis intercepts

When $y = 0$:
$$\frac{(x-2)^2}{9} + \frac{9}{16} = 1$$
$$\frac{(x-2)^2}{9} = \frac{7}{16}$$
$$(x-2)^2 = \frac{9 \times 7}{16}$$
$$\therefore \quad x = 2 \pm \frac{3\sqrt{7}}{4}$$

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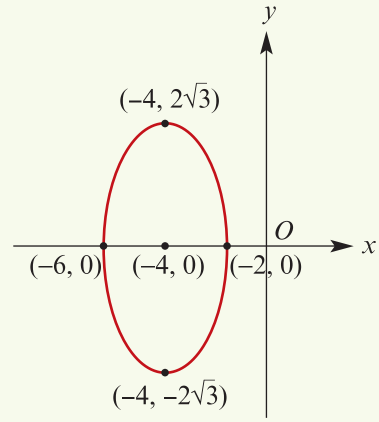
d Completing the square:

$$3x^2 + 24x + y^2 + 36 = 0$$
$$3(x^2 + 8x + 16) + y^2 + 36 - 48 = 0$$
$$3(x + 4)^2 + y^2 = 12$$

i.e. $$\frac{(x + 4)^2}{4} + \frac{y^2}{12} = 1$$

Centre $(-4, 0)$

Axis intercepts $(-6, 0)$ and $(-2, 0)$



Given an equation of the form

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

where both A and B are positive, there are three possibilities for the corresponding graph. The graph may be an ellipse (which includes the special case where the graph is a circle), the graph may be a single point, or there may be no pairs (x, y) that satisfy the equation.

Hyperbolas

The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is a hyperbola centred at the origin with axis intercepts $(a, 0)$ and $(-a, 0)$.

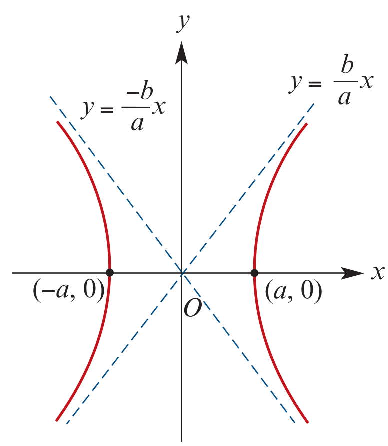
The hyperbola has asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

To see why this should be the case, we rearrange the equation of the hyperbola as follows:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \quad y^2 = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$



As $x \to \pm\infty$, we have $\frac{a^2}{x^2} \to 0$. This suggests that

$$y^2 \to \frac{b^2x^2}{a^2}$$

i.e. $$y \to \pm\frac{bx}{a}$$

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Cartesian equation of a hyperbola

The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola with centre (h, k). The asymptotes are

$$y - k = \pm\frac{b}{a}(x - h)$$

Note: This hyperbola is obtained from the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by the translation defined by $(x, y) \to (x + h, y + k)$.

Example 35

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

c $(x-1)^2 - (y+2)^2 = 1$

d $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

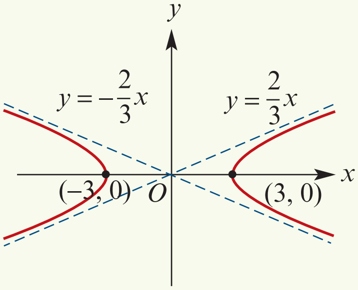
Solution

a Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The x-axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.



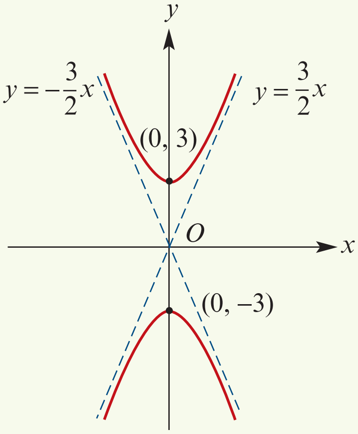
b Since $\frac{y^2}{9} - \frac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4}\left(1 + \frac{4}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{3}{2}x$.

The y-axis intercepts are $(0, 3)$ and $(0, -3)$.

The centre is $(0, 0)$.



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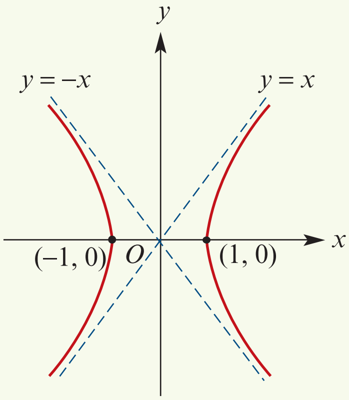
c First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This is called a **rectangular hyperbola**, as its asymptotes are perpendicular.



Now to sketch the graph of

$$(x-1)^2 - (y+2)^2 = 1$$

we apply the translation $(x, y) \to (x + 1, y - 2)$.

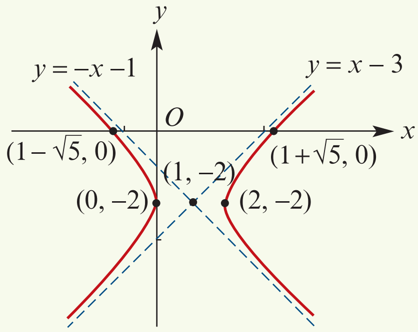
The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x - 1)$. That is, $y = x - 3$ and $y = -x - 1$.

Axis intercepts

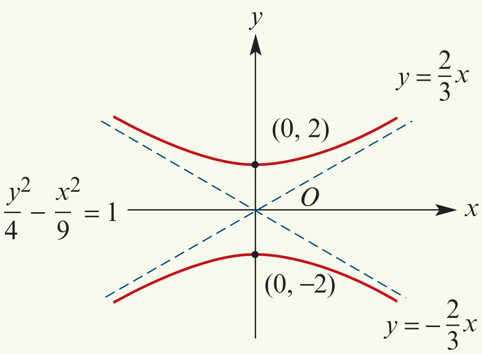
If $x = 0$, then $y = -2$.

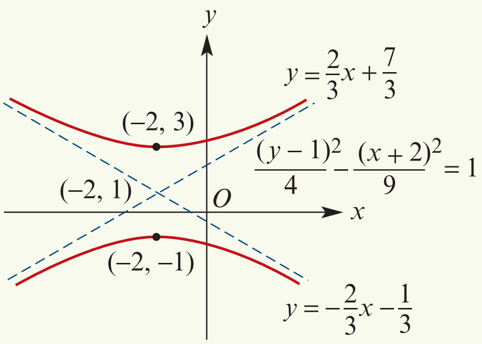
If $y = 0$, then $(x - 1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.



d The graph of $\dfrac{(y-1)^2}{4} - \dfrac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ through the translation $(x, y) \to (x - 2, y + 1)$. Its centre will be $(-2, 1)$.





The axis intercepts are $\left(0, 1 \pm \dfrac{2\sqrt{13}}{3}\right)$.

Note: The hyperbolas $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ and $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ have the same asymptotes; they are called **conjugate hyperbolas**.

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Skill-sheet

Exercise 1F

Example 34

1 Sketch the graph of each of the following. Label the axis intercepts and state the coordinates of the centre.

a $\frac{x^2}{9} + \frac{y^2}{16} = 1$

b $25x^2 + 16y^2 = 400$

c $\frac{(x-4)^2}{9} + \frac{(y-1)^2}{16} = 1$

d $x^2 + \frac{(y-2)^2}{9} = 1$

e $9x^2 + 25y^2 - 54x - 100y = 44$

f $9x^2 + 25y^2 = 225$

g $5x^2 + 9y^2 + 20x - 18y - 16 = 0$

h $16x^2 + 25y^2 - 32x + 100y - 284 = 0$

i $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

j $2(x-2)^2 + 4(y-1)^2 = 16$

Example 35

2 Sketch the graph of each of the following. Label the axis intercepts and give the equations of the asymptotes.

a $\frac{x^2}{16} - \frac{y^2}{9} = 1$

b $\frac{y^2}{16} - \frac{x^2}{9} = 1$

c $x^2 - y^2 = 4$

d $2x^2 - y^2 = 4$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$

f $9x^2 - 25y^2 - 90x + 150y = 225$

g $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{9} = 1$

h $4x^2 - 8x - y^2 + 2y = 0$

i $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

j $25x^2 - 16y^2 = 400$

3 Find the coordinates of the points of intersection of $y = \frac{1}{2}x$ with:

a $x^2 - y^2 = 1$

b $\frac{x^2}{4} + y^2 = 1$

4 Show that there is no intersection point of the line $y = x + 5$ with the ellipse $x^2 + \frac{y^2}{4} = 1$.

5 Let $a, b > 0$. Prove that the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ intersect on the vertices of a square.

6 Find the coordinates of the points of intersection of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and the line with equation $5x = 4y$.

7 On the one set of axes, sketch the graphs of $x^2 + y^2 = 9$ and $x^2 - y^2 = 9$.

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1G Parametric equations

In Chapter 13, we will study motion along a curve. A **parameter** (usually t representing time) will be used to help describe these curves. In Chapter 5, we will use a parameter to describe lines in two- or three-dimensional space.

This section gives an introduction to parametric equations of curves in the plane.

The unit circle

The unit circle can be expressed in Cartesian form as $\{ (x, y) : x^2 + y^2 = 1 \}$. We have seen in Section 1A that the unit circle can also be expressed as

$$\{ (x, y) : x = \cos t \text{ and } y = \sin t, \text{ for some } t \in \mathbb{R} \}$$

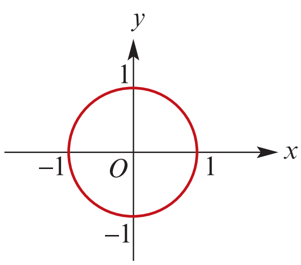
The set notation is often omitted, and we can describe the unit circle by the equations

$$x = \cos t \quad \text{and} \quad y = \sin t \quad \text{for } t \in \mathbb{R}$$

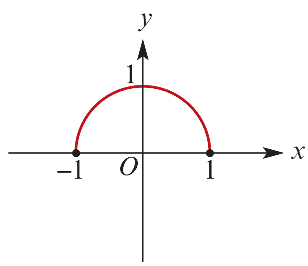
These are the **parametric equations** for the unit circle.

We still obtain the entire unit circle if we restrict the values of t to the interval $[0, 2\pi]$. The following three diagrams illustrate the graphs obtained from the parametric equations $x = \cos t$ and $y = \sin t$ for three different sets of values of t.

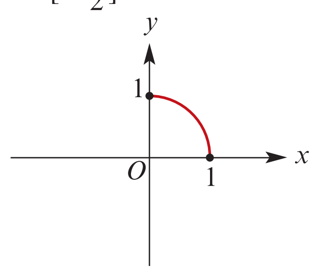
$t \in [0, 2\pi]$



$t \in [0, \pi]$



$t \in \left[0, \frac{\pi}{2}\right]$



Circles

Parametric equations for a circle centred at the origin

The circle with centre the origin and radius a is described by the parametric equations

$$x = a\cos t \quad \text{and} \quad y = a\sin t$$

The entire circle is obtained by taking $t \in [0, 2\pi]$.

Note: To obtain the Cartesian equation, first rearrange the parametric equations as

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{a} = \sin t$$

Square and add these equations to obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \cos^2 t + \sin^2 t = 1$$

This equation can be written as $x^2 + y^2 = a^2$, which is the Cartesian equation of the circle with centre the origin and radius a.

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The domain and range of the circle can be found from the parametric equations:

- **Domain** The range of the function with rule $x = a\cos t$ is $[-a, a]$.
  Hence the domain of the relation $x^2 + y^2 = a^2$ is $[-a, a]$.
- **Range** The range of the function with rule $y = a\sin t$ is $[-a, a]$.
  Hence the range of the relation $x^2 + y^2 = a^2$ is $[-a, a]$.

Example 36

A circle is defined by the parametric equations

$$x = 2 + 3\cos\theta \quad \text{and} \quad y = 1 + 3\sin\theta \quad \text{for } \theta \in [0, 2\pi]$$

Find the Cartesian equation of the circle, and state the domain and range of this relation.

Solution

Domain The range of the function with rule $x = 2 + 3\cos\theta$ is $[-1, 5]$. Hence the domain of the corresponding Cartesian relation is $[-1, 5]$.

Range The range of the function with rule $y = 1 + 3\sin\theta$ is $[-2, 4]$. Hence the range of the corresponding Cartesian relation is $[-2, 4]$.

Cartesian equation

Rewrite the parametric equations as

$$\frac{x-2}{3} = \cos\theta \quad \text{and} \quad \frac{y-1}{3} = \sin\theta$$

Square both sides of each of these equations and add:

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{9} = \cos^2\theta + \sin^2\theta = 1$$

i.e. $(x-2)^2 + (y-1)^2 = 9$

Parametric equations for a circle

The circle with centre (h, k) and radius a is described by the parametric equations

$$x = h + a\cos t \quad \text{and} \quad y = k + a\sin t$$

The entire circle is obtained by taking $t \in [0, 2\pi]$.

Parametric equations in general

A **parametric curve** in the plane is defined by a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable t is called the **parameter**. Each value of t gives a point $(f(t), g(t))$ in the plane. The set of all such points will be a curve in the plane.

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Note: If $x = f(t)$ and $y = g(t)$ are parametric equations for a curve C and you eliminate the parameter t between the two equations, then each point of the curve C lies on the curve represented by the resulting Cartesian equation.

Example 37

A curve is defined parametrically by the equations

$$x = at^2 \quad \text{and} \quad y = 2at \quad \text{for } t \in \mathbb{R}$$

where a is a positive constant. Find:

a the Cartesian equation of the curve

b the equation of the line passing through the points where $t = 1$ and $t = -2$

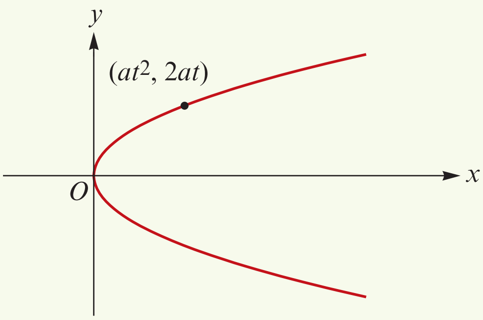
c the length of the chord joining the points where $t = 1$ and $t = -2$.

Solution

a The second equation gives $t = \dfrac{y}{2a}$.

Substitute this into the first equation:

$$x = at^2 = a\left(\frac{y}{2a}\right)^2 = a\left(\frac{y^2}{4a^2}\right) = \frac{y^2}{4a}$$



This can be written as $y^2 = 4ax$.

b At $t = 1$, $x = a$ and $y = 2a$. This is the point $(a, 2a)$.

At $t = -2$, $x = 4a$ and $y = -4a$. This is the point $(4a, -4a)$.

The gradient of the line is

$$m = \frac{2a - (-4a)}{a - 4a} = \frac{6a}{-3a} = -2$$

Therefore the equation of the line is

$$y - 2a = -2(x - a)$$

which simplifies to $y = -2x + 4a$.

c The chord joining $(a, 2a)$ and $(4a, -4a)$ has length

$$\sqrt{(a - 4a)^2 + (2a - (-4a))^2} = \sqrt{9a^2 + 36a^2} = \sqrt{45a^2} = 3\sqrt{5}\,a \qquad (\text{since } a > 0)$$

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Ellipses

Parametric equations for an ellipse

The ellipse with the Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be described by the parametric equations

$$x = a\cos t \quad \text{and} \quad y = b\sin t$$

The entire ellipse is obtained by taking $t \in [0, 2\pi]$.

Note: We can rearrange these parametric equations as

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

Square and add these equations to obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

The domain and range of the ellipse can be found from the parametric equations:

- **Domain** The range of the function with rule $x = a\cos t$ is $[-a, a]$.
  Hence the domain of the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $[-a, a]$.
- **Range** The range of the function with rule $y = b\sin t$ is $[-b, b]$.
  Hence the range of the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $[-b, b]$.

Example 38

Find the Cartesian equation of the curve with parametric equations

$$x = 3 + 3\sin t \quad \text{and} \quad y = 2 - 2\cos t \quad \text{for } t \in \mathbb{R}$$

and describe the graph.

Solution

We can rearrange the two equations as

$$\frac{x-3}{3} = \sin t \quad \text{and} \quad \frac{2-y}{2} = \cos t$$

Now square both sides of each equation and add:

$$\frac{(x-3)^2}{9} + \frac{(2-y)^2}{4} = \sin^2 t + \cos^2 t = 1$$

Since $(2-y)^2 = (y-2)^2$, this equation can be written more neatly as

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

This is the equation of an ellipse with centre $(3, 2)$ and axis intercepts at $(3, 0)$ and $(0, 2)$.

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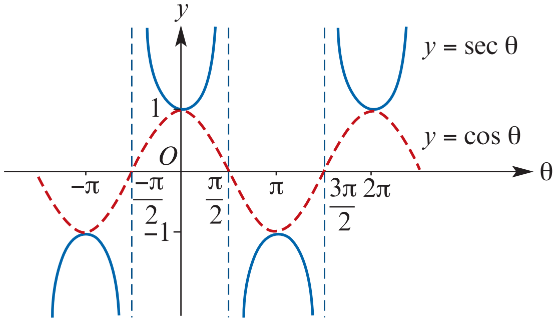
Hyperbolas

In order to give parametric equations for hyperbolas, we will be using the **secant function**, which is defined by

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{if } \cos\theta \neq 0$$

The graphs of $y = \sec\theta$ and $y = \cos\theta$ are shown here on the same set of axes. The secant function is studied further in Chapter 3.



We will also use an alternative form of the Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Dividing both sides by $\cos^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$

We will use this identity in the form

$$\sec^2\theta - \tan^2\theta = 1$$

Parametric equations for a hyperbola

The hyperbola with the Cartesian equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be described by the parametric equations

$$x = a\sec t \quad \text{and} \quad y = b\tan t \quad \text{for } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Note: We can rearrange these parametric equations as

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

Square and subtract these equations to obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

The domain and range of the hyperbola can be determined from the parametric equations.

- **Domain** There are two cases, giving the left and right branches of the hyperbola:
  - For $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the range of the function with rule $x = a\sec t$ is $[a, \infty)$.
    The domain $[a, \infty)$ gives the right branch of the hyperbola.
  - For $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the range of the function with rule $x = a\sec t$ is $(-\infty, a]$.
    The domain $(-\infty, a]$ gives the left branch of the hyperbola.
- **Range** For both sections of the domain, the range of the function with rule $y = b\tan t$ is $\mathbb{R}$.

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Example 39

Find the Cartesian equation of the curve with parametric equations

$$x = 3\sec t \quad \text{and} \quad y = 4\tan t \quad \text{for } t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Describe the curve.

Solution

Rearrange the two equations:

$$\frac{x}{3} = \sec t \quad \text{and} \quad \frac{y}{4} = \tan t$$

Square both sides of each equation and subtract:

$$\frac{x^2}{9} - \frac{y^2}{16} = \sec^2 t - \tan^2 t = 1$$

The Cartesian equation of the curve is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

The range of the function with rule $x = 3\sec t$ for $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is $(-\infty, -3]$. Hence the domain for the graph is $(-\infty, -3]$.

The curve is the left branch of a hyperbola centred at the origin with x-axis intercept at $(-3, 0)$. The equations of the asymptotes are $y = \frac{4x}{3}$ and $y = -\frac{4x}{3}$.

Finding parametric equations for a curve

When converting from a Cartesian equation to a pair of parametric equations, there are many different possible choices.

Example 40

Give parametric equations for each of the following:

a $x^2 + y^2 = 9$

b $\frac{x^2}{16} + \frac{y^2}{4} = 1$

c $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} = 1$

Solution

a One possible solution is $x = 3\cos t$ and $y = 3\sin t$ for $t \in [0, 2\pi]$.

Another solution is $x = -3\cos(2t)$ and $y = 3\sin(2t)$ for $t \in [0, \pi]$.

Yet another solution is $x = 3\sin t$ and $y = 3\cos t$ for $t \in \mathbb{R}$.

b One possible solution is $x = 4\cos t$ and $y = 2\sin t$.

c One possible solution is $x - 1 = 3\sec t$ and $y + 1 = 2\tan t$.

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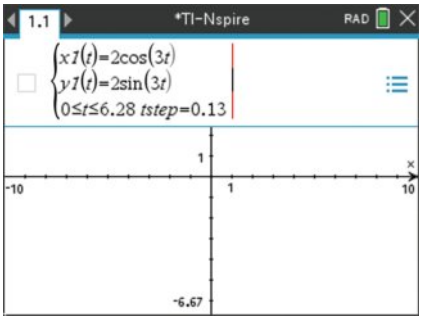
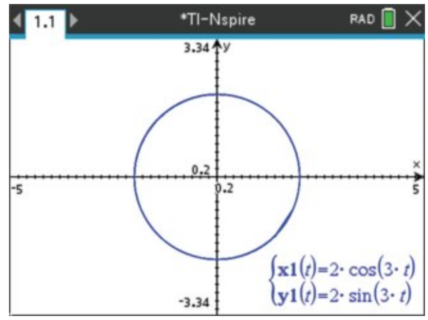
Using the TI-Nspire

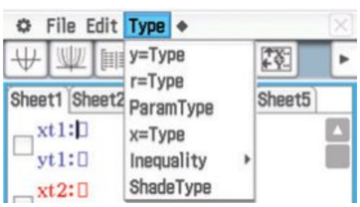
- Open a **Graphs** application ([⌂ on] > **New** > **Add Graphs**).
- Use [menu] > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown.



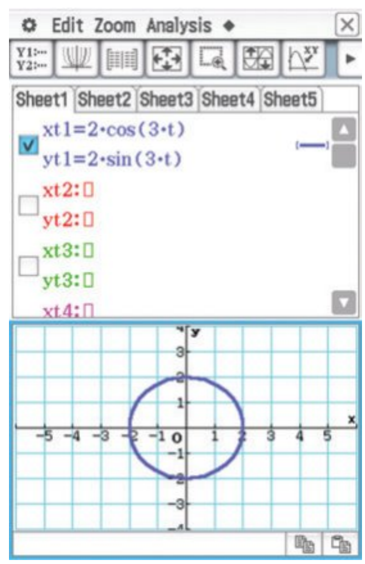


Using the Casio ClassPad

- Open the **Graph & Table** application [Graph& Table].
- From the toolbar, select **Type** > **ParamType**.



- Use the [Trig] keyboard to enter the equations as shown on the right.
- Tick the box and tap [graph icon].
- Use [window icon] to adjust the window.



Skill-sheet

Exercise 1G

Example 36

1 Find the Cartesian equation of the curve with parametric equations $x = 2\cos(3t)$ and $y = 2\sin(3t)$, and determine the domain and range of the corresponding relation.

Example 37

2 A curve is defined parametrically by the equations $x = 4t^2$ and $y = 8t$ for $t \in \mathbb{R}$. Find:

a the Cartesian equation of the curve

b the equation of the line passing through the points where $t = 1$ and $t = -1$

c the length of the chord joining the points where $t = 1$ and $t = -3$.

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Example 38

3 Find the Cartesian equation of the curve with parametric equations $x = 2 + 3\sin t$ and $y = 3 - 2\cos t$ for $t \in \mathbb{R}$, and describe the graph.

Example 39

4 Find the Cartesian equation of the curve with parametric equations $x = 2\sec t$ and $y = 3\tan t$ for $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and describe the curve.

5 Find the corresponding Cartesian equation for each pair of parametric equations:

a $x = 4\cos(2t)$ and $y = 4\sin(2t)$

b $x = 2\sin(2t)$ and $y = 2\cos(2t)$

c $x = 4\cos t$ and $y = 3\sin t$

d $x = 4\sin t$ and $y = 3\cos t$

e $x = 2\tan(2t)$ and $y = 3\sec(2t)$

f $x = 1 - t$ and $y = t^2 - 4$

g $x = t + 2$ and $y = \frac{1}{t}$

h $x = t^2 - 1$ and $y = t^2 + 1$

i $x = t - \frac{1}{t}$ and $y = 2\left(t + \frac{1}{t}\right)$

6 For each of the following pairs of parametric equations, determine the Cartesian equation of the curve and sketch its graph:

a $x = \sec t,\ y = \tan t,\ t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

b $x = 3\cos(2t),\ y = -4\sin(2t)$

c $x = 3 - 3\cos t,\ y = 2 + 2\sin t$

d $x = 3\sin t,\ y = 4\cos t,\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

e $x = \sec t,\ y = \tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

f $x = 1 - \sec(2t),\ y = 1 + \tan(2t),\ t \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

7 A circle is defined by the parametric equations

$$x = 2\cos(2t) \quad \text{and} \quad y = -2\sin(2t) \quad \text{for } t \in \mathbb{R}$$

a Find the coordinates of the point P on the circle where $t = \frac{4\pi}{3}$.

b Find the equation of the tangent to the circle at P.

Example 40

8 Give parametric equations corresponding to each of the following:

a $x^2 + y^2 = 16$

b $\frac{x^2}{9} - \frac{y^2}{4} = 1$

c $(x-1)^2 + (y+2)^2 = 9$

d $\frac{(x-1)^2}{9} + \frac{(y+3)^2}{4} = 9$

9 A circle has centre $(1, 3)$ and radius 2. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where a, b, c and d are positive constants, state the values of a, b, c and d.

10 An ellipse has x-axis intercepts $(-4, 0)$ and $(4, 0)$ and y-axis intercepts $(0, 3)$ and $(0, -3)$. State a possible pair of parametric equations for this ellipse.

11 The circle with parametric equations $x = 2\cos(2t)$ and $y = 2\sin(2t)$ is dilated by a factor of 3 from the x-axis. For the image curve, state:

a a possible pair of parametric equations

b the Cartesian equation.

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12 The ellipse with parametric equations $x = 3 - 2\cos\left(\frac{t}{2}\right)$ and $y = 4 + 3\sin\left(\frac{t}{2}\right)$ is translated 3 units in the negative direction of the x-axis and 2 units in the negative direction of the y-axis. For the image curve, state:

a a possible pair of parametric equations **b** the Cartesian equation.

13 Sketch the graph of the curve with parametric equations $x = 2 + 3\sin(2\pi t)$ and $y = 4 + 2\cos(2\pi t)$ for:

a $t \in [0, \frac{1}{4}]$ **b** $t \in [0, \frac{1}{2}]$ **c** $t \in [0, \frac{3}{2}]$

For each of these graphs, state the domain and range.

1H Algorithms and pseudocode

An **algorithm** is a finite, unambiguous sequence of instructions for performing a specific task. An algorithm can be described using step-by-step instructions, illustrated by a flowchart, or written out in pseudocode.

You have seen many examples of algorithms in Year 11 and you will meet several new algorithms throughout this book. This section gives a summary of writing algorithms in pseudocode.

Note: The Interactive Textbook includes online appendices that provide an introduction to coding using the language *Python* and also to coding using the TI-Nspire and the Casio ClassPad.

Assigning values to variables

A **variable** is a string of one or more letters that acts as a placeholder that can be assigned different values. For example, the notation

$$x \leftarrow 3$$

means 'assign the value 3 to the variable x'.

Controlling the flow of steps

The steps of an algorithm are typically carried out one after the other. However, there are constructs that allow us to control the flow of steps.

- **If–then blocks** This construct provides a means of making decisions within an algorithm. Certain instructions are only followed if a condition is satisfied.

```
if condition then
    follow these instructions
end if
```

- **For loops** This construct provides a means of repeatedly executing the same set of instructions in a controlled way. In the template on the right, this is achieved by performing one iteration for each value of i in the sequence $1, 2, 3, \dots, n$.

```
for i from 1 to n
    follow these instructions
end for
```

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- **While loops** This construct provides another means of repeatedly executing the same set of instructions in a controlled way. This is achieved by performing iterations indefinitely, as long as some condition remains true.

```
while condition
    follow these instructions
end while
```

In the following example, we construct a table of values to demonstrate the algorithm. This is called a **desk check**. In general, we carry out a desk check of an algorithm by carefully following the algorithm step by step, and constructing a table of the values of all the variables after each step.

Example 41

Consider the sequence defined by the rule

$$x_{n+1} = 3x_n - 2, \qquad \text{where } x_1 = 3$$

Write an algorithm that will determine the smallest value of n for which $x_n > 1000$. Show a desk check to test the operation of the algorithm.

Solution

We use a `while` loop, since we don't know how many iterations will be required.

The variable x is used for the current term of the sequence, and the variable n is used to keep track of the number of iterations.

```
n ← 1
x ← 3
while x ≤ 1000
    n ← n + 1
    x ← 3x − 2
end while
print n
```

n	x
1	3
2	7
3	19
4	55
5	163
6	487
7	1459

Note: The output is 7.

Functions

A **function** takes one or more input values and returns an output value. Functions can be defined and then used in other algorithms.

```
define function(input):
    follow these instructions
    return output
```

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Example 42

Construct a function that inputs a natural number n and outputs the value of

$$n! = 1 \times 2 \times \cdots \times n$$

Solution

```
define factorial(n):
   product ← 1
   for i from 1 to n
        product ← product × i
   end for
   return product
```

For example, calling *factorial*(4) will return the value 24.

Lists

In programming languages, a finite sequence is often called a **list**. We will write lists using square brackets. For example, we can define a list A by

$$A \leftarrow [2, 3, 5, 7, 11]$$

The notation $A[n]$ refers to the nth entry of the list. So $A[1] = 2$ and $A[5] = 11$.

We can add an entry to the end of a list using append. For example, the instruction

```
append 9 to A
```

would result in $A = [2, 3, 5, 7, 11, 9]$.

Example 43

Write a function that returns a list of the first n square numbers for a given natural number n.

Solution

```
define squares(n):
   A ← [ ]
   for i from 1 to n
        append i^2 to A
   end for
   return A
```

For example, calling *squares*(5) will return the list [1, 4, 9, 16, 25].

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Nested loops

The next example illustrates how we can use loops within loops.

Example 44

Using pseudocode, write an algorithm to find the positive integer solutions of the equation

$$11x + 19y + 13z = 200$$

Solution

We use three loops to run through all the possible positive integer values of x, y and z. We first note that

$$200 \div 11 \approx 18.2, \qquad 200 \div 19 \approx 10.5, \qquad 200 \div 13 \approx 15.4$$

Therefore we know that we will find all the solutions from the following nest of three loops.

```
for x from 1 to 18
    for y from 1 to 10
        for z from 1 to 15
            if 11x + 19y + 13z = 200 then
                print (x, y, z)
            end if
        end for
    end for
end for
```

This algorithm prints the five solutions $(2, 8, 2)$, $(3, 4, 7)$, $(6, 5, 3)$, $(7, 1, 8)$ and $(10, 2, 4)$.

Skill-sheet

Exercise 1H

Example 41

1 Consider the sequence defined by the rule

$$x_{n+1} = 2x_n + 3, \qquad \text{where } x_1 = 3$$

Write an algorithm that will determine the smallest value of n for which $x_n > 100$. Show a desk check to test the operation of the algorithm.

Example 42

2 Construct a function that inputs a natural number n and outputs the product of the first n even natural numbers.

Example 43

3 Write a function that returns a list of the first n powers of 2 for a given natural number n.

Example 44

4 For each of the following, use pseudocode to describe an algorithm that will find all the positive integer solutions of the equation:

a $x^2 + y^2 + z^2 = 500$ **b** $x^3 + y^3 + z^3 = 1\,000\,000$

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5 The sine function can be given by an infinite sum:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The following pseudocode function evaluates the sum of the first n terms for a given value of x. (The code uses the factorial function from Example 42.)

```
define sinsum(x, n):
  sum ← 0
  for k from 1 to n
      sum ← sum + (−1)^(k+1) × x^(2k−1) / factorial(2k − 1)
  end for
  return sum
```

a Perform a desk check to evaluate:

i $sinsum(0.1, 4)$ **ii** $sinsum(1, 4)$ **iii** $sinsum(2, 4)$

b Compare the values found in part **a** with the values of $\sin 0.1$, $\sin 1$ and $\sin 2$.

c Using a device, investigate the function $sinsum(x, n)$ for a range of input values.

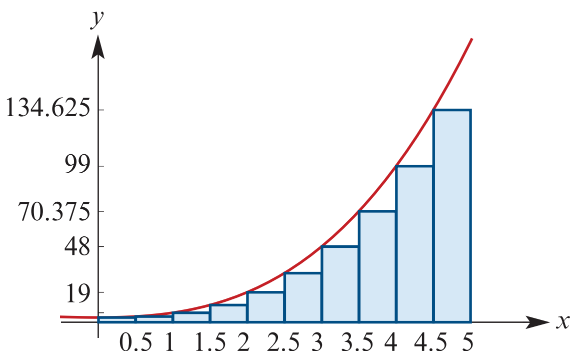
6 The cosine function can be given by an infinite sum:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

a Modify the pseudocode function $sinsum(x, n)$ from the previous question to give approximations of the cosine function.

b Find an approximation for $\cos 2$ using the first four terms of the sum.

7 Riemann sums The block of pseudocode on the right finds an approximation to the area under the curve $y = x^3 + 2x^2 + 3$ between $x = 0$ and $x = 5$. This is done by summing the areas of 10 rectangular strips, as shown in the diagram below.




```
define f(x):
  return x^3 + 2x^2 + 3

a ← 0
b ← 5
n ← 10
h ← (b − a)/n
left ← a
sum ← 0
for i from 1 to n
    strip ← f(left) × h
    sum ← sum + strip
    left ← left + h
end for
print sum
```

a Carry out a desk check for the algorithm.

b Modify the algorithm to use 50 trapezoidal strips.

Note: We will use similar algorithms for length, surface area and volume in Chapter 14.

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8 **Newton's method** You met Newton's method in Mathematical Methods Units 1 & 2. We aim to find an approximate solution to an equation of the form $f(x) = 0$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function. We start from an initial estimate $x = x_0$ and construct a sequence of approximations $x_1, x_2, x_3, \ldots$ using the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{where } n = 0, 1, 2, \ldots$$

The following algorithm uses this method for the equation $-x^3 + 5x^2 - 3x + 4 = 0$. The table shows a desk check of the algorithm.

```
define f(x):
    return −x^3 + 5x^2 − 3x + 4

define Df(x):
    return −3x^2 + 10x − 3

x ← 3.8
while |f(x)| > 10^−6
    x ← x − f(x)/Df(x)
    print x, f(x)
end while
```

	x	$f(x)$
Initial	3.8	9.928
Pass 1	4.99326923	−10.81199119
Pass 2	4.60526316	−1.44403339
Pass 3	4.53507148	−0.04308844
Pass 4	4.53284468	−0.00004266
Pass 5	4.53284247	0.00000000

a Modify this algorithm for the equation $x^3 - 2 = 0$ and the initial estimate $x_0 = 2$. Carry out a desk check to determine an approximation of $2^{\frac{1}{3}}$.

b In Mathematical Methods Units 3 & 4, you will learn that if $f(x) = \sin x$, then $f'(x) = \cos x$. Use this fact and Newton's method to find an approximation of π. Start with the initial estimate $x_0 = 3$.

Note: We will use a similar method in Chapter 11 for the numerical solution of differential equations.

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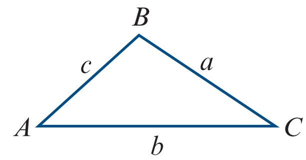
Chapter summary

Assign-ment

Nrich

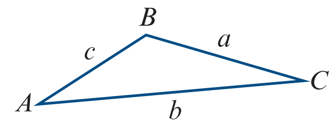
Triangles

■ Labelling triangles



- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

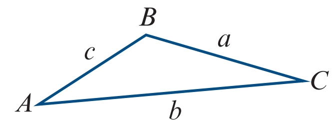
■ Sine rule



For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

■ Cosine rule



For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Sequences and series

■ The nth term of a sequence is denoted by t_n.

■ A **recurrence relation** enables each subsequent term to be found from previous terms. A sequence specified in this way is said to be defined **recursively**.

e.g. $t_1 = 1, \quad t_n = t_{n-1} + 2$

■ A sequence may also be defined by a rule that is stated in terms of n.

e.g. $t_n = 2n - 1$

■ Arithmetic sequences and series

- An **arithmetic sequence** has a rule of the form $t_n = a + (n-1)d$, where a is the first term and d is the **common difference** (i.e. $d = t_k - t_{k-1}$ for all $k > 1$).
- The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

■ Geometric sequences and series

- A **geometric sequence** has a rule of the form $t_n = ar^{n-1}$, where a is the first term and r is the **common ratio** (i.e. $r = \frac{t_k}{t_{k-1}}$ for all $k > 1$).
- For $r \neq 1$, the sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

- For $-1 < r < 1$, the sum S_n approaches a limiting value as $n \to \infty$, and the series is said to be **convergent**. This limit is called the **sum to infinity** and is given by $S_\infty = \frac{a}{1 - r}$.

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Review

- **Recurrence relations of the form $t_n = rt_{n-1} + d$**

  Let $t_1, t_2, t_3, \ldots$ be a sequence defined by a recurrence relation of the form $t_n = rt_{n-1} + d$, where r and d are constants. Then the nth term of the sequence is given by

$$t_n = r^{n-1}t_1 + \frac{d(r^{n-1} - 1)}{r - 1} \qquad \text{(provided } r \neq 1\text{)}$$

The modulus function

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

  For example: $|5| = 5$ and $|-5| = 5$.

- On the number line, the distance between two numbers a and b is given by $|a - b| = |b - a|$. For example: $|x - 2| < 5$ can be read as 'the distance of x from 2 is less than 5'.

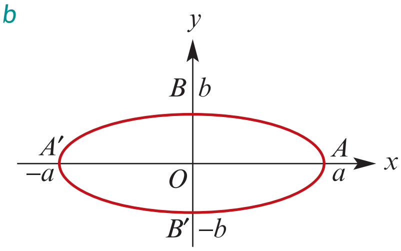
Circles

- The circle with centre at the origin and radius a has Cartesian equation $x^2 + y^2 = a^2$.
- The circle with centre (h, k) and radius a has equation $(x - h)^2 + (y - k)^2 = a^2$.

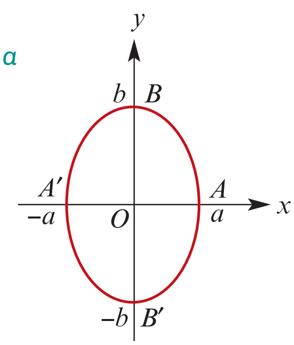
Ellipses

- The curve with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is an ellipse centred at the origin with axis intercepts $(\pm a, 0)$ and $(0, \pm b)$.

$a > b$



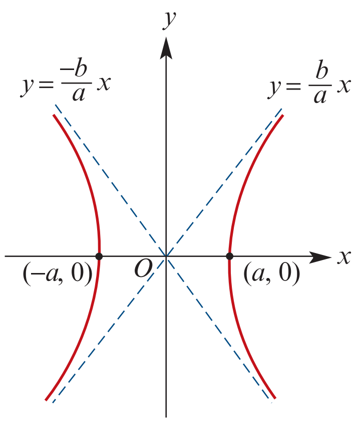
$b > a$



- The curve with equation $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$ is an ellipse with centre (h, k).

Hyperbolas

- The curve with equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is a hyperbola centred at the origin.
  - The axis intercepts are $(\pm a, 0)$.
  - The asymptotes have equations $y = \pm\dfrac{b}{a}x$.



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- The curve with equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is a hyperbola with centre (h, k). The asymptotes have equations $y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$.

Parametric equations

- A **parametric curve** in the plane is defined by a pair of functions

  $x = f(t)$ and $y = g(t)$

  where t is called the **parameter** of the curve.
- Parameterisations of familiar curves:

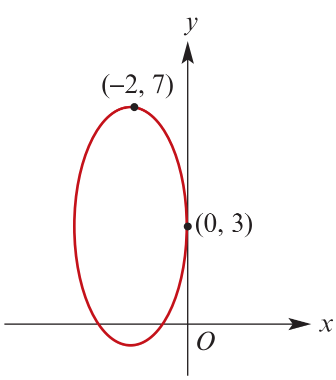
	Cartesian equation	Parametric equations
Circle	$x^2 + y^2 = a^2$	$x = a\cos t$ and $y = a\sin t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t$ and $y = b\sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec t$ and $y = b\tan t$

  Note: To obtain the entire circle or the entire ellipse using these parametric equations, it suffices to take $t \in [0, 2\pi]$.
- Translations of parametric curves: The circle with equation $(x-h)^2 + (y-k)^2 = a^2$ can also be described by the parametric equations $x = h + a\cos t$ and $y = k + a\sin t$.

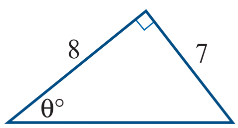
Technology-free questions

1 A sequence is defined recursively by $f_n = 5f_{n-1}$ and $f_0 = 1$. Find f_n in terms of n.

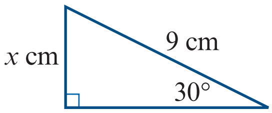
2 Write down the equation of the ellipse shown.



3 Find $\sin\theta°$.



4 Find x.



5 **a** Find the exact value of $\cos 315°$.

b Given that $\tan x° = \frac{3}{4}$ and $180 < x < 270$, find the exact value of $\cos x°$.

c Find an angle $A°$ (with $A \neq 330$) such that $\sin A° = \sin 330°$.

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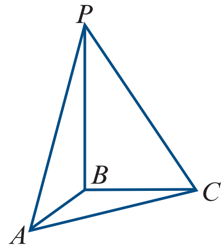
Review

6 ABC is a horizontal right-angled triangle with the right angle at B. The point P is 3 cm directly above B. The length of AB is 1 cm and the length of BC is 1 cm. Find the angle that the triangle ACP makes with the horizontal.



7 **a** Solve $2\cos(2x + \pi) - 1 = 0$ for $-\pi \le x \le \pi$.

b Sketch the graph of $y = 2\cos(2x + \pi) - 1$ for $-\pi \le x \le \pi$, clearly labelling the axis intercepts.

c Solve $2\cos(2x + \pi) < 1$ for $-\pi \le x \le \pi$.

8 The triangular base ABC of a tetrahedron has side lengths $AB = 15$ cm, $BC = 12$ cm and $AC = 9$ cm. The apex D is 9 cm vertically above C.

a Find the angle C of the triangular base.

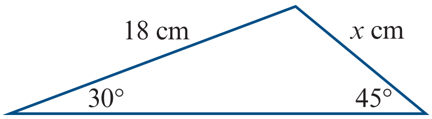
b Find the angles that the sloping edges make with the horizontal.

9 Two ships sail from port at the same time. One sails 24 nautical miles due east in 3 hours, and the other sails 33 nautical miles on a bearing of 030° in the same time.

a How far apart are the ships 3 hours after leaving port?

b How far apart would they be in 5 hours if they maintained the same bearings and constant speed?

10 Find x.



11 An airport A is 480 km due east of airport B. A pilot flies on a bearing of 225° from A to C and then on a bearing of 315° from C to B.

a Make a sketch of the situation.

b Determine how far the pilot flies from A to C.

c Determine the total distance the pilot flies.

12 Find the equations of the asymptotes of the hyperbola with rule $x^2 - \dfrac{(y-2)^2}{9} = 15$.

13 A curve is defined by the parametric equations $x = 3\cos(2t) + 4$ and $y = \sin(2t) - 6$. Give the Cartesian equation of the curve.

14 A curve is defined by the parametric equations $x = 2\cos(\pi t)$ and $y = 2\sin(\pi t) + 2$. Give the Cartesian equation of the curve.

15 **a** Sketch the graphs of $y = -2\cos x$ and $y = -2\cos\left(x - \dfrac{\pi}{4}\right)$ on the same set of axes, for $x \in [0, 2\pi]$.

b Solve $-2\cos\left(x - \dfrac{\pi}{4}\right) = 0$ for $x \in [0, 2\pi]$.

c Solve $-2\cos x < 0$ for $x \in [0, 2\pi]$.

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Review

16 Find all angles θ with $0 \le \theta \le 2\pi$, where:

a $\sin\theta = \frac{1}{2}$ **b** $\cos\theta = \frac{\sqrt{3}}{2}$ **c** $\tan\theta = 1$

17 A circle has centre $(1, 2)$ and radius 3. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where a, b, c and d are positive constants, state the values of a, b, c and d.

18 Find the centre and radius of the circle with equation $x^2 + 8x + y^2 - 12y + 3 = 0$.

19 Find the x- and y-axis intercepts of the ellipse with equation $\frac{x^2}{81} + \frac{y^2}{9} = 1$.

20 The first term of an arithmetic sequence is $3p + 5$, where p is a positive integer. The last term is $17p + 17$ and the common difference is 2.

a Find in terms of p:

i the number of terms **ii** the sum of the sequence.

b Show that the sum of the sequence is divisible by 14 only when p is odd.

21 A sequence is formed by using rising powers of 3 as follows: $3^0, 3^1, 3^2, \ldots$.

a Find the nth term.

b Find the product of the first 20 terms.

22 State the value of each of the following without using the absolute value function in your answer:

a $|-9|$ **b** $\left|-\frac{1}{400}\right|$ **c** $|9 - 5|$ **d** $|5 - 9|$ **e** $|\pi - 3|$ **f** $|\pi - 4|$

23 **a** Let $f\colon \{x : |x| > 100\} \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$. State the range of f.

b Let $f\colon \{x : |x| < 0.1\} \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$. State the range of f.

24 Let $f(x) = |x^2 - 3x|$. Solve the equation $f(x) = x$.

25 For each of the following, sketch the graph of $y = f(x)$ and state the range of f:

a $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = 2|\sin x|$

b $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = |x^2 - 4x| - 3$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 3 - |x^2 - 4x|$

Multiple-choice questions

1 The 3rd term of a geometric sequence is 4 and the 8th term is 128. The 1st term is

A 2 **B** 1 **C** 32 **D** 5 **E** none of these

2 If the numbers 5, x and y are in arithmetic sequence, then

A $y = x + 5$ **B** $y = x - 5$ **C** $y = 2x + 5$ **D** $y = 2x - 5$ **E** none of these

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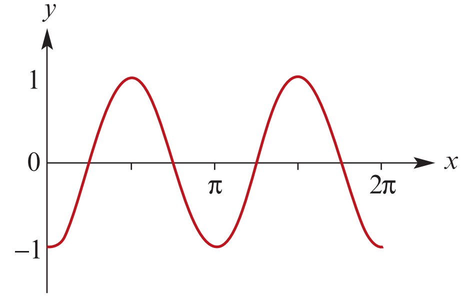
3 If $2\cos x^\circ - \sqrt{2} = 0$, then the value of the acute angle x° is

A 30° **B** 60° **C** 45° **D** 25° **E** 27.5°

4 The equation of the graph shown is



A $y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

B $y = \cos\left(x + \frac{\pi}{4}\right)$

C $y = \sin(2x)$

D $y = -2\sin(x)$

E $y = \sin\left(x + \frac{\pi}{4}\right)$

5 Which of the following recurrence relations generates the sequence $2, 6, 22, 86, 342, \ldots$?

A $t_1 = 2,\ t_{n+1} = t_n + 4$ **B** $t_1 = 2,\ t_{n+1} = 2t_n + 2$ **C** $t_1 = 2,\ t_{n+1} = 3t_n$

D $t_1 = 2,\ t_{n+1} = 4t_n - 2$ **E** $t_1 = 2,\ t_{n+1} = 5t_n - 4$

6 In a geometric sequence, $t_2 = 24$ and $t_4 = 54$. If the common ratio is positive, then the sum of the first five terms is

A 130 **B** 211 **C** 238 **D** 316.5 **E** 810

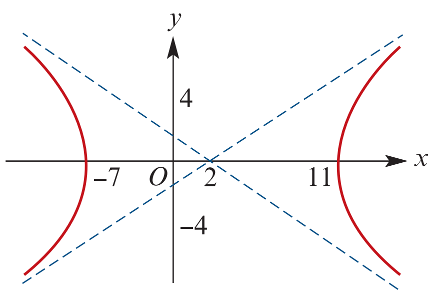
7 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of c to the nearest whole number is

A 9 **B** 10 **C** 11 **D** 81 **E** 129

8 The coordinates of the centre of the circle with equation $x^2 - 8x + y^2 - 2y = 8$ are

A $(-8, -2)$ **B** $(8, 2)$ **C** $(-4, -1)$ **D** $(4, 1)$ **E** $(1, 4)$

9 The equation of the graph shown is



A $\frac{(x+2)^2}{27} - \frac{y^2}{108} = 1$

B $\frac{(x-2)^2}{9} - \frac{y^2}{34} = 1$

C $\frac{(x+2)^2}{81} - \frac{y^2}{324} = 1$

D $\frac{(x-2)^2}{81} - \frac{y^2}{324} = 1$

E $\frac{(x+2)^2}{9} - \frac{y^2}{36} = 1$

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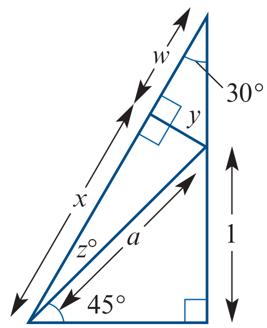
Extended-response questions

1 **a** Find the values of a, y, z, w and x.

b Hence deduce exact values for $\sin 15°$, $\cos 15°$ and $\tan 15°$.

c Find the exact values of $\sin 75°$, $\cos 75°$ and $\tan 75°$.



2 A hiker walks from point A on a bearing of 010° for 5 km and then on a bearing of 075° for 7 km to reach point B.

a Find the length of AB.

b Find the bearing of B from the start point A.

A second hiker walks from point A on a bearing of 080° for 4 km to a point P, and then walks in a straight line to B.

c **i** Find the total distance travelled by the second hiker.

ii Find the bearing on which the hiker must travel in order to reach B from P.

A third hiker also walks from point A on a bearing of 080° and continues on that bearing until he reaches point C. He then turns and walks towards B. In doing so, the two legs of the journey are of equal length.

d Find the total distance travelled by the third hiker to reach B.

3 An ellipse is defined by the rule $\dfrac{x^2}{2} + \dfrac{(y+3)^2}{5} = 1$.

a Find:

i the domain of the relation

ii the range of the relation

iii the centre of the ellipse.

An ellipse E is given by the rule $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$. The domain of E is $[-1, 3]$ and its range is $[-1, 5]$.

b Find the values of a, b, h and k.

The line $y = x - 2$ intersects the ellipse E at $A(1, -1)$ and at P.

c Find the coordinates of the point P.

A line perpendicular to the line $y = x - 2$ is drawn at P. This line intersects the y-axis at the point Q.

d Find the coordinates of Q.

e Find the equation of the circle through A, P and Q.

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Review

4 **a** Show that the circle with equation $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the x-axis and the y-axis.

b Show that every circle that touches both the x-axis and the y-axis has an equation of a similar form.

c Hence show that there are exactly two circles that pass through the point $(2, 4)$ and just touch the x-axis and the y-axis, and give their equations.

d For each of these two circles, state the coordinates of the centre and give the radius.

e For each circle, find the gradient of the line which passes through the centre and the point $(2, 4)$.

f For each circle, find the equation of the tangent to the circle at the point $(2, 4)$.

5 A circle is defined by the parametric equations $x = a\cos t$ and $y = a\sin t$. Let P be the point with coordinates $(a\cos t, a\sin t)$.

a Find the equation of the straight line which passes through the origin and the point P.

b State the coordinates, in terms of t, of the other point of intersection of the circle with the straight line through the origin and P.

c Find the equation of the tangent to the circle at the point P.

d Find the coordinates of the points of intersection A and B of the tangent with the x-axis and the y-axis respectively.

e Find the area of triangle OAB in terms of t if $0 < t < \frac{\pi}{2}$. Find the value of t for which the area of this triangle is a minimum.

6 This diagram shows a straight track through points A, S and B, where A is 10 km northwest of B and S is exactly halfway between A and B. A surveyor is required to reroute the track through P from A to B to avoid a major subsidence at S. The surveyor determines that A is on a bearing of 330° from P and that B is on a bearing of 070° from P. Assume the region under consideration is flat.

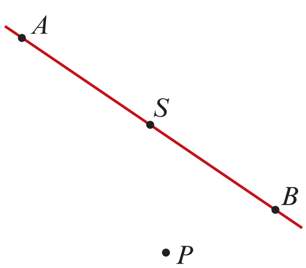
Find:

a the magnitudes of angles APB, PAB and PBA

b the distance from P to B and from P to S

c the bearing of S from P

d the distance from A to B through P, if the surveyor chooses to reroute the track along a circular arc.



7 Consider the function with rule $f(x) = |x^2 - ax|$, where a is a positive constant.

a State the coordinates of the x-axis intercepts.

b State the coordinates of the y-axis intercept.

c Find the maximum value of the function in the interval $[0, a]$.

d Find the possible values of a for which the point $(-1, 4)$ lies on the graph of $y = f(x)$.

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2 Logic and proof

Objectives

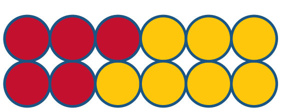
- To revise the concept of **divisibility** for integers.
- To revise basic concepts of **proof**, including:
  - conditional statements
  - equivalent statements
  - proof by contrapositive
  - proof by contradiction
  - counterexamples.
- To prove results involving **inequalities**.
- To evaluate **telescoping series**.
- To understand the **principle of mathematical induction**.
- To use mathematical induction to prove results involving:
  - divisibility
  - partial sums and products of sequences.

A **mathematical proof** is an argument that confirms the truth of a mathematical statement. It shows that a list of stated assumptions will guarantee a conclusion.

For example, consider the following simple claim involving even and odd numbers.

		Assumption		Conclusion
Claim	If	m and n are odd integers	then	$m + n$ is an even integer.

The truth of this claim is suggested by the picture on the right; an odd number of red dots can be combined with an odd number of yellow dots to give an even aggregate of dots.



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However, a more rigorous argument would proceed as follows. Assume that both m and n are odd integers. Then $m = 2a + 1$ and $n = 2b + 1$, for integers a and b. Therefore

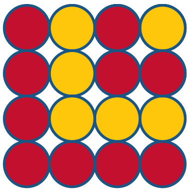
$$\begin{aligned} m + n &= (2a + 1) + (2b + 1) \\ &= 2a + 2b + 2 \\ &= 2(a + b + 1) \\ &= 2k \end{aligned}$$

where $k = a + b + 1$ is an integer. Hence $m + n$ is even.

The next claim has only one assumption.

		Assumption		Conclusion
Claim	If	n is a natural number	then	$1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

This claim is illustrated by the picture on the right. Each L-shaped configuration of dots represents a different odd number. These can be nestled perfectly into the shape of a square.



So this picture gives great insight into why the claim is true. However, the picture alone does not constitute a proof. We need to show that the equation holds for every natural number n. We can prove results like this using *mathematical induction*.

In this chapter, we first revise concepts of logic and proof from Specialist Mathematics Units 1 & 2, before moving on to other applications of proof and mathematical induction.

Note: In the Interactive Textbook, each section of this chapter includes a skillsheet to provide further practice in areas such as sequences and series, combinatorics, matrices and graph theory.

2A Revision of proof techniques

We start by revising the fundamental ideas of proof introduced in Specialist Mathematics Units 1 & 2.

Divisibility of integers

- The set of **natural numbers** is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.
- The set of **integers** is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Divisibility

Let a and b be integers. Then we say that a is **divisible** by b if there exists an integer k such that $a = kb$. In this case, we also say that b is a **divisor** of a.

Note: Alternatively, we can say that a is a **multiple** of b and that b is a **factor** of a.

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For example:

- 12 is divisible by 3, since $12 = 4 \times 3$
- -6 is divisible by 2, since $-6 = -3 \times 2$
- 0 is divisible by any integer n, since $0 = 0 \times n$.

On the other hand, the integer 14 is not divisible by 3. In this case, the best we can write is

$$14 = 4 \times 3 + 2$$

We say that 14 leaves a remainder of 2 when divided by 3. More generally, we have the following important result.

Euclidean division

If a and b are integers with $b \neq 0$, then there are unique integers q and r such that

$$a = qb + r \quad \text{where} \quad 0 \leq r < |b|$$

Note: Here q is the **quotient** and r is the **remainder** when a is divided by b.

Example 1

Let $n \in \mathbb{Z}$. Prove that $n^3 - n$ is divisible by 3.

Solution

Method 1

Note that $n^3 - n = n(n-1)(n+1)$.

When the integer n is divided by 3, the remainder must be 0, 1 or 2.
Therefore n can be written in the form $3k$, $3k + 1$ or $3k + 2$, for some integer k.

Case 1: $n = 3k$. Then $n^3 - n = n(n-1)(n+1)$

$$= 3k(3k-1)(3k+1)$$

Case 2: $n = 3k + 1$. Then $n^3 - n = n(n-1)(n+1)$

$$= (3k+1)(3k)(3k+2)$$
$$= 3k(3k+1)(3k+2)$$

Case 3: $n = 3k + 2$. Then $n^3 - n = n(n-1)(n+1)$

$$= (3k+2)(3k+1)(3k+3)$$
$$= 3(k+1)(3k+1)(3k+2)$$

In all three cases, we see that $n^3 - n$ is divisible by 3.

Method 2

Note that $n^3 - n$ is the product of the three consecutive integers $n - 1$, n and $n + 1$.

In any set of three consecutive integers, one of the integers must be a multiple of 3. (This fact, although true, actually requires its own proof!) Therefore the product of three consecutive integers must also be a multiple of 3.

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Conditional statements and direct proof

The statement proved in Example 1 can be broken down into two parts:

Statement	If	n is an integer	then	$n^3 - n$ is divisible by 3.

This is an example of a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

This can be abbreviated as $P \Rightarrow Q$, which is read as 'P **implies** Q'. We call P the **hypothesis** and Q the **conclusion**.

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 2

Show that if n is an odd integer, then it is the sum of two consecutive integers.

Solution

Assume that n is an odd integer. Then $n = 2k + 1$ for some integer k. We can write

$$\begin{aligned} n &= 2k + 1 \\ &= k + (k + 1) \end{aligned}$$

Hence n is the sum of the consecutive integers k and $k + 1$.

The negation of a statement

To **negate** a statement P we write its very opposite, which we call '**not** P'. Negation turns a true statement into a false statement, and a false statement into a true statement.

Statement	The sum of any two odd numbers is even.	(true)
Negation	There are two odd numbers whose sum is odd.	(false)

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

De Morgan's laws

not (P *and* Q) is the same as (not P) *or* (not Q)

not (P *or* Q) is the same as (not P) *and* (not Q)

For example, we can use the second law to negate the following statement about integers a and b:

Statement	a is odd	or	b is odd
Negation	a is even	and	b is even

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Note: When negating a statement involving variables, it helps to know the set that each variable takes its value from. For example, if we know that a is an integer, then the negation of 'a is odd' is 'a is even'.

Proof by contrapositive

Consider this statement about an integer n:

Statement	If $n^2 + 2n$ is odd then n is odd.

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

Contrapositive	If n is even then $n^2 + 2n$ is even.

Note that a conditional statement and its contrapositive are always logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.

- The **contrapositive** of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P).
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

Example 3

Let $n \in \mathbb{Z}$. Prove that if $n^2 + 2n$ is odd, then n is odd.

Solution

We will prove the contrapositive statement: If n is even, then $n^2 + 2n$ is even.

Assume that n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 2n &= (2k)^2 + 2(2k) \\ &= 4k^2 + 4k \\ &= 2(2k^2 + 2k) \end{aligned}$$

Hence $n^2 + 2n$ is even, since $2k^2 + 2k \in \mathbb{Z}$.

As the contrapositive is equivalent to the original statement, we have proved the claim.

Proof by contradiction

The basic outline of a **proof by contradiction** is:

1. Assume that the statement we want to prove is false.
2. Show that this assumption leads to mathematical nonsense.
3. Conclude that we were wrong to assume that the statement is false.
4. Conclude that the statement must be true.

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Example 4

Suppose x satisfies $2^x = 5$. Using a proof by contradiction, show that x is irrational.

Solution

Suppose that x is rational. Since x must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned} 2^x = 5 \quad &\Rightarrow \quad 2^{\frac{m}{n}} = 5 \\ &\Rightarrow \quad \left(2^{\frac{m}{n}}\right)^n = 5^n \qquad \text{(raise both sides to the power } n\text{)} \\ &\Rightarrow \quad 2^m = 5^n \end{aligned}$$

The left-hand side of this equation is even and the right-hand side is odd. This gives a contradiction, and so x is not rational.

The converse of a conditional statement

Consider this statement about integers m and n:

Statement	If	m and n are odd	then	$m + n$ is even.	(true)

By switching the hypothesis and the conclusion, we obtain the **converse** statement.

Converse	If	$m + n$ is even	then	m and n are odd.	(false)

The converse of a true statement may not be true.

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

Example 5

a Let n be an integer. Statement: If n^2 is divisible by 2, then n is divisible by 2.
Write the converse statement and show that it is true.

b Let S be a quadrilateral. Statement: If S is a square, then S has equal diagonals.
Write the converse statement and show that it is not true.

Solution

a Converse: If n is divisible by 2, then n^2 is divisible by 2.
Assume that n is divisible by 2. Then $n = 2k$ for some integer k. Therefore

$$n^2 = (2k)^2 = 2(2k^2)$$

which is divisible by 2.

b Converse: If S has equal diagonals, then S is a square.
The converse statement is false, since any rectangle has equal diagonals.

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Equivalent statements

Now consider the following two statements about a particular triangle:

P: The triangle has three equal sides.

Q: The triangle has three equal angles.

Both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that P and Q are **equivalent** statements. We write $P \Leftrightarrow Q$.

We can also say that P is true **if and only if** Q is true. So in the above example, we can say that a triangle has three equal sides if and only if it has three equal angles.

To prove that two statements P and Q are equivalent, you have to prove two things:

$$P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P$$

Example 6

Let $n \in \mathbb{Z}$. Prove that n is divisible by 3 if and only if n^2 is divisible by 3.

Solution

(⇒) Assume that n is divisible by 3. We want to show that n^2 is divisible by 3.

Since n is divisible by 3, there exists an integer k such that $n = 3k$. Therefore $n^2 = (3k)^2 = 3(3k^2)$. Hence n^2 is divisible by 3.

(⇐) Assume that n^2 is divisible by 3. We want to show that n is divisible by 3.

When the integer n is divided by 3, the remainder must be 0, 1 or 2. Therefore n can be written in the form $3k$, $3k + 1$ or $3k + 2$, for some integer k.

Case 1: $n = 3k$. This is the case where n is divisible by 3.

Case 2: $n = 3k + 1$. Then $n^2 = 9k^2 + 6k + 1$, which leaves remainder 1 when divided by 3. This contradicts our assumption that n^2 is divisible by 3. So this case cannot occur.

Case 3: $n = 3k + 2$. Then $n^2 = 9k^2 + 12k + 4$, which leaves remainder 1 when divided by 3. This contradicts our assumption that n^2 is divisible by 3. So this case cannot occur.

Hence n must be divisible by 3.

Skill-sheet

Exercise 2A

Divisibility of integers

1 Let n be an even integer. Prove that $n^2 + 2n$ is divisible by 4.

2 Let m and n be integers. Prove that $(2m + n)^2 - (2m - n)^2$ is divisible by 8.

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3 Assume that m is divisible by 3 and n is divisible by 5. Prove that:

a mn is divisible by 15

b m^2n is divisible by 45.

Example 1 **4** **a** Let $n \in \mathbb{Z}$. Prove that $n^2 - n$ is even by considering the cases when n is odd and n is even.

b Provide another proof by factorising $n^2 - n$.

5 Consider integers m, n, a and b. Prove that if m is a divisor of a and n is a divisor of b, then mn is a divisor of ab.

Direct proof

Example 2 **6** Show that if n is an odd integer, then $n^2 + 8n + 3$ is even.

7 Prove that if m and n are perfect cubes, then mn is a perfect cube.

8 **a** Factorise the expression $n^4 + 2n^3 - n^2 - 2n$.

b Use your factorised expression to provide a simple proof that $n^4 + 2n^3 - n^2 - 2n$ is divisible by 24 for all $n \in \mathbb{Z}$.

9 **a** Prove that if n is an odd integer, then there is an integer m such that $n^2 = 8m + 1$.

b Hence, prove that there is only one integer whose square has the form $2^k - 1$, where $k \in \mathbb{N}$.

10 Every integer n is of the form $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$, for some integer k.

a Using this fact, prove that the cube of every integer n is of the form $9m$, $9m + 1$ or $9m + 8$, for some integer m.

b Explain why there are no cubes in the sequence $92, 992, 9992, 99992, \dots$.

11 Let $n \in \mathbb{Z}$. Prove that $3n^2 + 7n + 11$ is odd.
Hint: Consider the cases when n is odd and n is even.

12 Prove that, for any two positive integers that are not divisible by 3, the difference between their squares is divisible by 3.
Hint: If an integer n is not divisible by 3, then $n = 3k + 1$ or $n = 3k + 2$ for some $k \in \mathbb{Z}$.

13 **a** Prove that every square number is of the form $5k$, $5k + 1$ or $5k + 4$, where $k \in \mathbb{Z}$.
Hint: Every natural number is of the form $5m$, $5m + 1$, $5m + 2$, $5m + 3$ or $5m + 4$, where $m \in \mathbb{Z}$.

b Hence, explain why no square number has a final digit equal to 2 or 3.

c Hence, determine how many square numbers appear in this list:

$$1!, \quad 1! + 2!, \quad 1! + 2! + 3!, \quad 1! + 2! + 3! + 4!, \quad \dots$$

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Proof by contrapositive

Example 3

14 Let $a, b \in \mathbb{Z}$. Consider the statement: If ab is even, then a is even or b is even.

a Write down the contrapositive of the statement.

b Prove that the contrapositive is true.

15 Let $m, n \in \mathbb{Z}$. Consider the statement: If $m^2 + n^2$ is even, then $m + n$ is even.

a Write down the contrapositive of the statement.

b Prove that the contrapositive is true.

16 Let $n \in \mathbb{N}$. Consider the statement: If $8^n - 1$ is prime, then n is odd.

a Write down the contrapositive of the statement.

b Prove that the contrapositive is true.

c Is there anything special about the number 8 here? Can you generalise your proof?

17 Let $n \in \mathbb{Z}$. Prove that if n is even, then n cannot be expressed as the sum of two consecutive integers.

18 Let $x \in \mathbb{R}$. Prove that if x is irrational, then $2x - 3$ is irrational.

Proof by contradiction

Example 4

19 Use proof by contradiction for each of the following:

a Prove that there is no largest natural number.

b Let $a, b \in \mathbb{R}$ such that $a + b > 100$. Show that $a > 50$ or $b > 50$.

c Let a and b be positive integers. Show that $a \leq \sqrt{ab}$ or $b \leq \sqrt{ab}$.

d Prove that $\log_2 7$ is irrational.

e Let $a, b \in \mathbb{R}$ such that a is rational and b is irrational. Show that $a + b$ is irrational.

f Prove that the product of two consecutive natural numbers is never a square number.

g Let $n \in \mathbb{N}$ and assume that n, $n + 2$ and $n + 4$ are all prime. Show that $n = 3$.

20 Define the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $f(x) = \dfrac{x}{x-1}$. Prove, by way of contradiction, that 1 does not belong to the range of f.

21 Define the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $f(x) = \dfrac{x^2}{x-1}$. Prove, by way of contradiction, that 1 does not belong to the range of f.

The converse of a conditional statement

Example 5

22 Let m and n be integers. For each of the following statements, write down the converse statement. Decide whether the converse is true or false, and explain why.

a If $3n$ is odd, then n is odd.

b If m is even and n is odd, then mn is even.

c If n is divisible by 6, then n is divisible by 2 and 3.

d If n is divisible by 24, then n is divisible by 4 and 6.

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Equivalent statements

23 Let n be an integer. Prove that n is even if and only if $n + 1$ is odd.

24 Let a, m and n be integers, where $m \neq 0$. Prove that nm is a divisor of am if and only if n is a divisor of a.

Example 6

25 **a** Let $n \in \mathbb{N}$. Prove that n is divisible by 2 if and only if n^2 is divisible by 2.
b Hence, prove that $\sqrt{2q}$ is irrational, whenever q is an odd natural number.
c Using this fact, prove that $\sqrt{2} + \sqrt{3}$ is irrational.

26 Let n be an integer. Prove that n is divisible by 3 if and only if n^3 is divisible by 9.

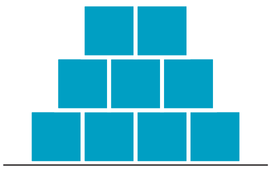
27 **a** Write the number 99 as the sum of three consecutive integers.
b Let n be an integer. Prove that n is divisible by 3 if and only if n can be written as the sum of three consecutive integers.

Mixed proof questions

28 Prove that the sum of three consecutive positive integers is a divisor of the sum of their cubes.

29 Let $k \in \mathbb{N}$. Prove that the product of k consecutive positive integers is divisible by $k!$.
Hint: Consider the binomial coefficient ${}^{n+k}C_k$.

30 We will say that a natural number n is **stackable** if it is possible to form a tower of n blocks with at least two rows in such a way that every row above the bottom row has exactly one less block than the row below. For example, the number 9 is stackable, as shown in the diagram.



Prove that no power of 2 is stackable. Hint: Use a proof by contradiction.

31 Notice that

$$1 = 1^2, \qquad 2 = -1^2 - 2^2 - 3^2 + 4^2, \qquad 3 = -1^2 + 2^2$$

a Let m be an integer. Prove that

$$(m+1)^2 - (m+2)^2 - (m+3)^2 + (m+4)^2 = 4$$

b Hence, prove that every natural number can be written in the form

$$\pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm n^2$$

for some value of n and a suitable choice of sign for each term.

32 In this question, we will give a proof that there are infinitely many prime numbers.
a Let $m \in \mathbb{N}$. Prove that if d is a divisor of both m and $m + 1$, then $d = 1$.
b Now let $n \in \mathbb{N}$ and assume that p is a prime factor of $n! + 1$. Prove that $p > n$.
Hint: Suppose that $p \leq n$.
c Why does this mean that there are infinitely many primes?

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2B Quantifiers and counterexamples

Quantification using 'for all' and 'there exists'

For all

A **universal statement** claims that a property holds for *all* members of a given set. Such a statement can be written using the quantifier '**for all**'. For example:

Statement	For all real numbers x and y, we have $x^2 + 5y^2 \geq 2xy$.

To prove that this statement is true, we need to give a general argument that applies for every choice of real numbers x and y. We will prove inequalities like this in the next section.

There exists

An **existence statement** claims that a property holds for *at least one* member of a given set. Such a statement can be written using the quantifier '**there exists**'. For example:

Statement	There exists a triple of integers (a, b, c) such that $a^2 + b^2 = c^2$.

To prove that this statement is true, we just need to give one instance. The triple $(3, 4, 5)$ provides an example, since $3^2 + 4^2 = 5^2$.

Example 7

Rewrite each statement using either 'for all' or 'there exists':

a Some real numbers are irrational.

b Every integer that is divisible by 4 is also divisible by 2.

Solution

a There exists $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$.

b For all $m \in \mathbb{Z}$, if m is divisible by 4, then m is divisible by 2.

Negation without quantifiers

We discussed negation in the previous section, and we used De Morgan's laws to negate statements involving 'and' and 'or'. It is also helpful to be able to negate statements involving 'implies'.

Consider the conditional statement 'If you study Mathematics, then you study Physics'. The only way this can be false is if you are studying Mathematics but not Physics. So the negation of the statement is 'You study Mathematics and you do not study Physics'.

Negations of basic compound statements

- not (P and Q) is equivalent to (not P) or (not Q)
- not (P or Q) is equivalent to (not P) and (not Q)
- not ($P \Rightarrow Q$) is equivalent to P and (not Q)

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Negation with quantifiers

To negate a statement involving a quantifier, we interchange 'for all' with 'there exists' and then negate the rest of the statement.

Example 8

Write down the negation of each of the following statements:

a For all natural numbers n, we have $2n \geq n + 1$.

b There exists an integer m such that $m^2 = 4$ and $m^3 = -8$.

c For all real numbers x and y, if $x < y$, then $x^2 < y^2$.

Solution

a There exists a natural number n such that $2n < n + 1$.

b For all integers m, we have $m^2 \neq 4$ or $m^3 \neq -8$.

c There exist real numbers x and y such that $x < y$ and $x^2 \geq y^2$.

Notation for quantifiers

The words 'for all' can be abbreviated using the *turned A* symbol, ∀. The words 'there exists' can be abbreviated using the *turned E* symbol, ∃. For example, the two statements considered at the start of this section can be written in symbols as follows:

- $(\forall x, y \in \mathbb{R})\ \ x^2 + 5y^2 \geq 2xy$
- $(\exists (a, b, c) \in \mathbb{Z}^3)\ \ a^2 + b^2 = c^2$

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.

Disproving universal statements

We have seen that a universal statement claims that a property holds for *all* members of a given set. For example:

Statement	For all real numbers x, the number $x^2 - x$ is positive.

So to disprove a universal statement, we simply need to give one example where the property does not hold. Such an example is called a **counterexample**.

Example 9

Disprove the statement: For all real numbers x, the number $x^2 - x$ is positive.

Solution

When $x = 0$, we obtain $x^2 - x = 0$, which is not positive.

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Example 10

A function $f: \mathbb{R} \to \mathbb{R}$ is **strictly increasing** if $a < b$ implies $f(a) < f(b)$, for all $a, b \in \mathbb{R}$.

Disprove the statement: If a function $f: \mathbb{R} \to \mathbb{R}$ is strictly increasing and differentiable, then $f'(x) > 0$ for all $x \in \mathbb{R}$.

Solution

The statement is not true, as the function $f(x) = x^3$ is a counterexample. The function f is strictly increasing and differentiable, but $f'(0) = 0$.

Note that the negation of a universal statement is an existence statement. For example:

Statement	For all $x, y \in \mathbb{R}$, if $x < y$, then $x^2 < y^2$.
Negation	There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 \geq y^2$.

Clearly, the universal statement above is false, because there exist real numbers for which the property does not hold. For example: $-1 < 0$ but $(-1)^2 \geq 0^2$.

Disproving existence statements

Consider this existence statement:

Statement	There exists $n \in \mathbb{N}$ such that $n^2 + 7n + 12$ is a prime number.

To show that such a statement is false, we prove that its negation is true:

Negation	For all $n \in \mathbb{N}$, the number $n^2 + 7n + 12$ is not a prime number.

The negation is easy to prove, since

$$n^2 + 7n + 12 = (n + 3)(n + 4)$$

is clearly a composite number for each $n \in \mathbb{N}$. As this example demonstrates, the negation of an existence statement is a universal statement.

Example 11

Disprove each of the following statements:

a There exists $n \in \mathbb{N}$ such that $n^2 + 15n + 56$ is a prime number.

b There exists some real number x such that $x^2 = -1$.

Solution

a We need to prove that, for all $n \in \mathbb{N}$, the number $n^2 + 15n + 56$ is not prime. This is true, since

$$n^2 + 15n + 56 = (n + 7)(n + 8)$$

is a composite number for each $n \in \mathbb{N}$.

b We need to prove that, for all real numbers x, we have $x^2 \neq -1$. This is true, since for every real number x, we have $x^2 \geq 0$ and so $x^2 \neq -1$.

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Skill-sheet

Exercise 2B

Example 7

1 Which of the following are universal statements ('for all') and which are existence statements ('there exists')?

a For each $n \in \mathbb{N}$, the number $(2n+1)^2$ is odd.
b There is an even prime number.
c For every integer n, the integer $n(n+1)$ is even.
d All squares have four sides.
e Some natural numbers are composites.
f At least one real number x satisfies the equation $x^2 - 2x - 5 = 0$.
g Any real number has a cube root.
h The angle sum of a quadrilateral is 360°.

Example 8

2 Write down the negation of each of the following statements:

a For all $x \in \mathbb{R}$, we have $x^2 \geq 0$.
b For every natural number n, the number $n^2 + n + 11$ is prime.
c There exist prime numbers p and q for which $p + q = 100$.
d For all $x \in \mathbb{R}$, if $x > 0$, then $x^3 > x$.
e There exist integers a, b and c such that $a^3 + b^3 = c^3$.
f For all $x, y \in \mathbb{R}$, we have $(x+y)^3 = x^3 + y^3$.
g There exist $x, y \in \mathbb{R}$ such that $x \geq y$ and $x^2 \leq y^2$.
h There exists a real number x such that $x^2 + x + 1 = 0$.
i For all natural numbers n, if n is not divisible by 3, then $n^2 + 2$ is divisible by 3.
j For every integer m, if $m > 2$ or $m < -2$, then $m^2 > 4$.
k For all integers m and n, we have that mn is even or $m + n$ is even.
l There exists a rational number a for which $\sqrt{2} \cdot a$ is rational.
m For all real numbers x, if $x \in (-1, 1)$, then $x^2 < 1$.
n There exist real numbers x and y such that $xy > 0$ and $x + y < 0$.

Example 9
Example 10

3 Provide a counterexample for each of the following statements:

a For all natural numbers n, the number $n^2 + n + 1$ is prime.
b For all real numbers x, we have $x^2 > 0$.
c For all $a, b \in \mathbb{R}$, if a and b are irrational, then $a + b$ is irrational.
d For all $a, b \in \mathbb{R}$, if a and b are irrational, then ab is irrational.
e For all real numbers a, b and c, if $ab = ac$, then $b = c$.
f For each $n \in \mathbb{N}$, if n^2 is divisible by 4, then n is divisible by 4.
g For all $a, b, c \in \mathbb{Z}$, if c is a divisor of ab, then c is a divisor of a or c is a divisor of b.
h For all integers m and n, if n^2 is a divisor of m^3, then n is a divisor of m.
i If a function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then its graph crosses the x-axis.
j If f is a differentiable function and $f'(0) = 0$, then f has a turning point at $(0, f(0))$.
k If every vertex of a graph has degree at least 1, then the graph is connected.

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4 The 2×2 identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the 2×2 zero matrix is $\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Provide a counterexample for each of the following statements:

a For all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{AB} = \mathbf{BA}$.

b For each 2×2 matrix $\mathbf{A}$, if $\mathbf{A}^2 = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$.

c For each 2×2 matrix $\mathbf{A}$, if $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} = \mathbf{I}$.

Example 11

5 Show that each of the following existence statements is false:

a There exists $n \in \mathbb{N}$ such that $25n^2 - 9$ is a prime number.

b There exists $n \in \mathbb{N}$ such that $n^2 + 11n + 30$ is a prime number.

c There exists $x \in \mathbb{R}$ such that $5 + 2x^2 = 1 + x^2$.

2C Proving inequalities

An **inequality** is a statement that orders two real numbers.

$x < y$	x is less than y
$x \le y$	x is less than or equal to y
$x > y$	x is greater than y
$x \ge y$	x is greater than or equal to y

Possibly the most important inequality is the statement that $x^2 \ge 0$ for every real number x. This simply says that the square of any real number is non-negative. It important because so many inequalities depend on this result.

Example 12

Let x and y be real numbers. Prove that $x^2 + 5y^2 \ge 2xy$.

Solution

We will prove this result by showing that

$$x^2 - 2xy + 5y^2 \ge 0$$

To show this, we can complete the square (thinking of x as the variable and y as a constant). We find that

$$\begin{aligned} x^2 - 2xy + 5y^2 &= (x^2 - 2xy + y^2) - y^2 + 5y^2 \\ &= (x - y)^2 + 4y^2 \\ &= (x - y)^2 + (2y)^2 \\ &\ge 0 \end{aligned}$$

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Example 13

Prove that the area of a rectangle is no more than the area of a square with the same perimeter.

Solution

Let x and y be the side lengths of a rectangle. Its perimeter is $2x + 2y$.

The side length of a square with the same perimeter is $\frac{1}{4}(2x + 2y) = \frac{x + y}{2}$.

Therefore

$$\begin{aligned}
\text{Area of square} - \text{Area of rectangle} &= \left(\frac{x+y}{2}\right)^2 - xy \\
&= \frac{x^2 + 2xy + y^2}{4} - \frac{4xy}{4} \\
&= \frac{x^2 - 2xy + y^2}{4} \\
&= \left(\frac{x-y}{2}\right)^2 \geq 0
\end{aligned}$$

In the above example, we see a proof of the following useful inequality.

AM–GM inequality

For $x, y \geq 0$, the arithmetic mean is greater than or equal to the geometric mean:

$$\frac{x + y}{2} \geq \sqrt{xy}$$

Note: The two means are equal if and only if $x = y$.

We can use this inequality to give quick proofs of many results.

Example 14

a Suppose $a, b > 0$ and $ab = 24$. Using the AM–GM inequality, prove that $2a + 3b \geq 24$.

b Suppose $a, b \geq 0$ and $2a + 3b = 12$. Using the AM–GM inequality, prove that $ab \leq 6$.

Solution

a Assume $ab = 24$.
Using the AM–GM inequality (with $x = 4a$ and $y = 6b$) we find

$$\begin{aligned}
2a + 3b &= \frac{4a + 6b}{2} \\
&\geq \sqrt{(4a)(6b)} \\
&= \sqrt{24ab} \\
&= \sqrt{24 \times 24} \\
&= 24
\end{aligned}$$

b Assume $2a + 3b = 12$.
Using the AM–GM inequality (with $x = 2a$ and $y = 3b$) we find

$$\begin{aligned}
ab &= \frac{1}{6}(2a)(3b) \\
&\leq \frac{1}{6}\left(\frac{2a + 3b}{2}\right)^2 \\
&= \frac{1}{6}\left(\frac{12}{2}\right)^2 \\
&= 6
\end{aligned}$$

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Skill-sheet

Exercise 2C

Example 12

1 Let $b \geq a \geq 0$. Prove that $\dfrac{b}{b+1} \geq \dfrac{a}{a+1}$.

2 Given that a and b are positive real numbers, prove that $a^3 + b^3 \geq a^2b + ab^2$.

3 **a** Prove that $11\sqrt{10} \geq 10\sqrt{11}$.

b We can prove a much more general statement. Let $a \geq b \geq 0$. Prove that $a\sqrt{b} \geq b\sqrt{a}$.

4 Let $a > 0$. Prove that $a + \dfrac{1}{a} \geq 2$.

5 **a** Prove that $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$ when $x, y \in \mathbb{R}^+$.

b Prove that $\left(\dfrac{1}{x} + \dfrac{1}{y}\right)(x + y) \geq 4$ when $x, y \in \mathbb{R}^+$.

c Prove that $\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)(x + y + z) \geq 9$ when $x, y, z \in \mathbb{R}^+$.

Example 13

6 **a** Let $x, y \geq 0$. Prove that

$$\left(\frac{x+y}{2}\right)^2 \leq \frac{x^2 + y^2}{2}$$

b Two pieces of string are used to form two squares, with sides of length x and y respectively. These two pieces of string are joined together and then cut in half. The two new pieces of string are used to form two squares of equal size. Prove that the total area has not increased.

Example 14

7 Let $a, b, c \geq 0$. Use the AM–GM inequality to prove each of the following:

a If $ab = 9$, then $a + b \geq 6$.

b If $a + b = 4$, then $ab \leq 4$.

c If $ab = 48$, then $3a + 4b \geq 24$.

d If $3a + 4b = 24$, then $ab \leq 12$.

e If $a + b + c = 1$, then $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \leq 1$.

8 Use the AM–GM inequality to prove that $(a + b)(b + c)(c + a) \geq 8abc$ for all $a, b, c \geq 0$.

9 **a** Assume $0 < a < 1$. Prove that $a > a^2$.

b Let θ be an acute angle. Using part **a**, prove that

$$\cos\theta + \sin\theta > 1$$

c Prove that

$$\cos\theta + \sin\theta \leq \sqrt{2}$$

10 Let a, b and c be real numbers.

a Prove that $a^2 + b^2 \geq 2ab$.

b Hence, prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

c Hence, prove that $3(a^4 + b^4 + c^4) \geq (a^2 + b^2 + c^2)^2$.

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2D Telescoping series

The technique demonstrated in this section is called **telescopic cancelling**. In the solution of the following example, you will notice that a sum with $2n$ terms cancels down to a sum with just two terms. The sum collapses in a similar manner to a traditional extendable telescope.

This technique can be used to find the partial sums of some sequences. In the next section, we will consider another approach to proving some of these results: mathematical induction.

Example 15

a Find constants a and b such that $\dfrac{1}{k(k+1)} = \dfrac{a}{k} + \dfrac{b}{k+1}$ for all $k \in \mathbb{N}$.

b Hence, prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

Solution

a We aim to find the partial fraction decomposition of the left-hand side. We have

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}$$

$$\therefore \quad \frac{1}{k(k+1)} = \frac{a(k+1)+bk}{k(k+1)}$$

$$\therefore \quad 1 = a(k+1) + bk$$

$$\therefore \quad 1 = a + (a+b)k$$

Equating coefficients, we find that $a = 1$ and $a + b = 0$. Therefore $b = -1$. We have found the partial fraction decomposition to be

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

b We use the result from part **a** to expand each term of the series:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \qquad \text{(regrouping)}$$

$$= 1 - \frac{1}{n+1} \qquad \text{(cancelling)}$$

$$= \frac{n}{n+1}$$

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Skill-sheet

Exercise 2D

Example 15

1 **a** Using partial fractions, find real numbers a and b such that

$$\frac{1}{k(k+2)} = \frac{a}{k} + \frac{b}{k+2}$$

b Hence, evaluate each of the following sums:

i $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{97 \cdot 99}$ **ii** $\dfrac{1}{2 \cdot 4} + \dfrac{1}{4 \cdot 6} + \cdots + \dfrac{1}{98 \cdot 100}$

2 Show that

$$\log_{10}\left(\frac{1}{2}\right) + \log_{10}\left(\frac{2}{3}\right) + \log_{10}\left(\frac{3}{4}\right) + \cdots + \log_{10}\left(\frac{99}{100}\right) = -2$$

3 **a** Show that $m \cdot m! = (m+1)! - m!$.

b Hence, prove that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

4 **a** Show that $\dfrac{m}{(m+1)!} = \dfrac{1}{m!} - \dfrac{1}{(m+1)!}$.

b Hence, prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

5 **a** Show that

$$k(k+1) = \frac{1}{3}\Big(k(k+1)(k+2) - (k-1)k(k+1)\Big)$$

b Let $n \in \mathbb{N}$. Use part **a** to prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

6 **a** Using partial fractions, find real numbers a, b and c such that

$$\frac{1}{k(k+1)(k+2)} = \frac{a}{k} + \frac{b}{k+1} + \frac{c}{k+2}$$

b Hence, prove that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

7 **a** Show that

$$\frac{\log_{10}\left(\frac{a}{b}\right)}{\log_{10}(a)\,\log_{10}(b)} = \frac{1}{\log_{10}(b)} - \frac{1}{\log_{10}(a)}$$

b Hence, evaluate

$$\frac{\log_{10}\left(\frac{2}{3}\right)}{\log_{10}(2)\,\log_{10}(3)} + \frac{\log_{10}\left(\frac{3}{4}\right)}{\log_{10}(3)\,\log_{10}(4)} + \cdots + \frac{\log_{10}\left(\frac{19}{20}\right)}{\log_{10}(19)\,\log_{10}(20)}$$

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2E Mathematical induction

In Example 15 from the previous section, we obtained the result

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

This result involves an infinite sequence of propositions, one for each natural number n:

$$P(1):\quad \frac{1}{1\cdot 2}=\frac{1}{1+1}$$

$$P(2):\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{2+1}$$

$$P(3):\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{3}{3+1}$$

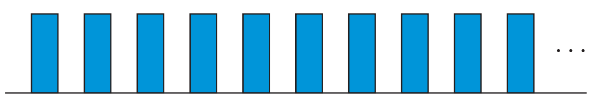
$$\vdots$$

We proved that the proposition $P(n)$ is true for every natural number n. In this section, we give an alternative proof of this result using **mathematical induction**.

The principle of mathematical induction

Imagine a row of dominoes extending infinitely to the right. Each of these dominoes can be knocked over provided two conditions are met:

1. The first domino is knocked over.
2. Each domino is sufficiently close to the next domino.



This scenario provides an accurate physical model of the following proof technique.

Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number n.

We can prove that $P(n)$ is true for every natural number n as follows:

a Show that $P(1)$ is true.

b Show that, for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1) \text{ is true} \quad\Rightarrow\quad P(2) \text{ is true} \quad\Rightarrow\quad P(3) \text{ is true} \quad\Rightarrow\quad \cdots$$

Let's see how mathematical induction is used in practice.

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Using induction for partial sums

Mathematical induction is useful for proving many results about partial sums.

Example 16

Using mathematical induction, prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Step 1 $P(1)$ is the proposition $\frac{1}{1 \cdot 2} = \frac{1}{1+1}$, that is, $\frac{1}{2} = \frac{1}{2}$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{(using } P(k)\text{)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n.

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Using induction for divisibility results

We now use mathematical induction to prove results about divisibility. You should compare the next example with Example 1 from the start of this chapter.

Example 17

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$n^3 - n$ is divisible by 3

Step 1 $P(1)$ is the proposition $1^3 - 1 = 0$ is divisible by 3. Clearly, $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$k^3 - k = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k+1)$ is true, that is, we have to prove that the number $(k+1)^3 - (k+1)$ is divisible by 3. We have

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= k^3 - k + 3k^2 + 3k\\ &= 3m + 3k^2 + 3k \qquad \text{(using } P(k))\\ &= 3(m + k^2 + k)\end{aligned}$$

Therefore $(k+1)^3 - (k+1)$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

Example 18

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$7^n - 4$ is divisible by 3

Step 1 $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. So $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

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Step 3 We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3.
We have

$$\begin{aligned} 7^{k+1} - 4 &= 7 \cdot 7^k - 4 \\ &= 7(3m+4) - 4 \qquad \text{(using } P(k)\text{)} \\ &= 21m + 28 - 4 \\ &= 21m + 24 \\ &= 3(7m+8) \end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

Notation for sums

The sum of the first n squares can be written in two different ways:

$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2$$

The notation on the right-hand side is called **sigma notation**, and is a convenient shorthand for the **expanded form** you see on the left-hand side. The notation uses the symbol Σ, which is the uppercase Greek letter *sigma*. This is the Greek equivalent to the Roman letter S, with S here standing for the word *sum*.

More generally, if m and n are integers with $m \le n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

This is read as 'the sum of the numbers a_i from $i = m$ to $i = n$'.

Example 19

Write $\sum_{i=1}^{5} 2^i$ in expanded form and evaluate.

Solution

$$\begin{aligned} \sum_{i=1}^{5} 2^i &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 2 + 4 + 8 + 16 + 32 \\ &= 62 \end{aligned}$$

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You may prefer to use this notation in induction proofs involving partial sums. The next example uses this notation to give the sum of the cubes of the first n odd numbers.

Example 20

Prove using the principle of mathematical induction that

$$\sum_{i=1}^{n} (2i-1)^3 = n^2(2n^2-1)$$

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\sum_{i=1}^{n} (2i-1)^3 = n^2(2n^2-1)$$

Step 1 First consider $P(1)$. Let $n = 1$ so that

$$\text{LHS of } P(1) = \sum_{i=1}^{1} (2i-1)^3 = (2 \cdot 1 - 1)^3 = 1$$

$$\text{RHS of } P(1) = 1^2(2 \cdot 1^2 - 1) = 1$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\sum_{i=1}^{k} (2i-1)^3 = k^2(2k^2-1)$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$\sum_{i=1}^{k+1} (2i-1)^3 = (k+1)^2(2(k+1)^2-1)$$

We have

$$\begin{aligned}
\text{LHS of } P(k+1) &= \sum_{i=1}^{k+1} (2i-1)^3 \\
&= \sum_{i=1}^{k} (2i-1)^3 + (2(k+1)-1)^3 \\
&= k^2(2k^2-1) + (2k+1)^3 \qquad \text{(using } P(k)\text{)} \\
&= 2k^4 + 8k^3 + 11k^2 + 6k + 1
\end{aligned}$$

$$\begin{aligned}
\text{RHS of } P(k+1) &= (k+1)^2(2(k+1)^2-1) \\
&= 2(k+1)^4 - (k+1)^2 \\
&= 2k^4 + 8k^3 + 11k^2 + 6k + 1
\end{aligned}$$

Therefore $P(k+1)$ is true. Hence we have shown that $P(k)$ implies $P(k+1)$, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

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Notation for products

As with finite sums of numbers, we also have an efficient shorthand for expressing finite products of numbers.

For example, the product of the first n odd numbers can be written as

$$1 \times 3 \times 5 \times \cdots \times (2n - 1) = \prod_{i=1}^{n} (2i - 1)$$

The notation on the right-hand side is called **pi notation**, and is a convenient shorthand for the **expanded form** you see on the left-hand side. The notation uses the symbol Π, which is the uppercase Greek letter *pi*. This is the Greek equivalent to the Roman letter P, with P here standing for the word *product*.

More generally, if m and n are integers with $m \le n$, then

$$\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times a_{m+2} \times \cdots \times a_n$$

This is read as 'the product of the numbers a_i from $i = m$ to $i = n$'.

Example 21

Write each of the following in expanded form and evaluate:

a $\displaystyle\prod_{i=1}^{3} i$ **b** $\displaystyle\prod_{i=1}^{5} (2i - 1)$

Solution

a $\displaystyle\prod_{i=1}^{3} i = 1 \times 2 \times 3 = 6$

b
$$\prod_{i=1}^{5} (2i - 1) = (2 \cdot 1 - 1) \times (2 \cdot 2 - 1) \times (2 \cdot 3 - 1) \times (2 \cdot 4 - 1) \times (2 \cdot 5 - 1)$$
$$= 1 \times 3 \times 5 \times 7 \times 9$$
$$= 945$$

You may prefer to use this notation in induction proofs involving partial products.

In the next example, we give a proof by induction that

$$\left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \cdots \times \left(1 + \frac{1}{n}\right) = n + 1$$

Using pi notation, this is expressed more compactly as

$$\prod_{i=1}^{n} \left(1 + \frac{1}{i}\right) = n + 1$$

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Example 22

Using the principle of mathematical induction, prove that

$$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1$$

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1$$

Step 1 First consider $P(1)$. Let $n = 1$ so that

$$\text{LHS of } P(1)=\prod_{i=1}^{1}\left(1+\frac{1}{i}\right)=1+\frac{1}{1}=2$$

$$\text{RHS of } P(1)=1+1=2$$

Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)=k+1$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)=(k+1)+1$$

We have

$$\begin{aligned}\text{LHS of } P(k+1)&=\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)\\&=\left(1+\frac{1}{k+1}\right)\times\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)\\&=\left(1+\frac{1}{k+1}\right)(k+1) \qquad \text{(using } P(k)\text{)}\\&=k+1+\frac{k+1}{k+1}\\&=(k+1)+1\\&=\text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k + 1)$ is true. Hence we have shown that $P(k)$ implies $P(k + 1)$, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

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Using induction to prove inequalities

Mathematical induction can also be used to prove various inequalities. For most induction proofs, the base case is $n = 1$. In the next example, the base case is $n = 4$.

Example 23

Prove by induction that $3^n > n^3$ for all natural numbers $n \geq 4$.

Solution

For each natural number $n \geq 4$, let $P(n)$ be the proposition:

$$3^n > n^3$$

Step 1 First consider $P(4)$. Let $n = 4$ so that

$$\text{LHS of } P(4) = 3^4 = 81$$
$$\text{RHS of } P(4) = 4^3 = 64$$

Therefore $P(4)$ is true.

Step 2 Let k be a natural number with $k \geq 4$, and assume $P(k)$ is true. That is,

$$3^k > k^3$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$3^{k+1} > (k + 1)^3$$

Note that the right-hand side of $P(k + 1)$ expands to $k^3 + 3k^2 + 3k + 1$. This is what we are aiming for in the calculation below.

We have

$$\begin{aligned}
\text{LHS of } P(k+1) &= 3^{k+1} \\
&= 3 \cdot 3^k \\
&> 3 \cdot k^3 && \text{(using } P(k)\text{)} \\
&= k^3 + k^3 + k^3 \\
&> k^3 + 3k^2 + 3k^2 && \text{(since } k \geq 4\text{)} \\
&= k^3 + 3k^2 + k^2 + k^2 + k^2 \\
&> k^3 + 3k^2 + 3k + 1 && \text{(since } k \geq 4\text{)} \\
&= (k+1)^3 \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k + 1)$ is true. Hence we have shown that $P(k)$ implies $P(k + 1)$, for each $k \in \mathbb{N}$ with $k \geq 4$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers $n \geq 4$.

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Skill-sheet

Exercise 2E

Example 16

1 Prove each of the following using mathematical induction:

a $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

b $1 + 3 + 5 + \cdots + (2n-1) = n^2$

c $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

d $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

e $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$, where $x \neq 1$

f $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

g $2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$

h $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

Example 17
Example 18

2 Prove that each of the following divisibility statements is true for all $n \in \mathbb{N}$:

a $11^n - 1$ is divisible by 10

b $7^n - 3^n$ is divisible by 4

c $8^n - (-6)^n$ is divisible by 14

d $2^{3n+1} + 5$ is divisible by 7

e $4^{2n+1} + 5^{2n+1}$ is divisible by 9

3 **a** Prove by induction that $2^{2n} - 1$ is a multiple of 3 for all $n \in \mathbb{N}$.

b Provide a different proof of this result by first writing $2^{2n} - 1$ as the difference of two squares.

Example 19

4 Write each of the following in expanded form and evaluate:

a $\sum_{i=1}^{4} i^3$

b $\sum_{k=1}^{5} 3^k$

c $\sum_{i=0}^{3} (-1)^i\, i$

d $\frac{1}{5}\sum_{i=1}^{5} i$

e $\sum_{i=1}^{4} 2i$

f $\sum_{k=1}^{4} (k-1)^2$

g $\sum_{i=1}^{3} (i-2)^2$

h $\sum_{i=1}^{4} (2i-1)^2$

i $\sum_{i=1}^{3} r^i$

j $\sum_{i=1}^{3} i \cdot 2^i$

k $\sum_{i=0}^{3} 3^{3-i}$

l $\sum_{i=1}^{2} (x-1)^i$

m $\sum_{i=3}^{3} i^2$

n $\sum_{i=-2}^{2} i$

o $\sum_{i=1}^{n} 1$

p $\sum_{i=-4}^{-2} 2i$

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Example 20

5 Prove each of the following using mathematical induction:

a $\sum_{m=1}^{n} (2m-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ **b** $\sum_{m=1}^{n} (-1)^{m+1} m^2 = (-1)^{n+1} \frac{n(n+1)}{2}$

Example 21

6 Write each of the following in expanded form and evaluate:

a $\prod_{i=1}^{4} i$ **b** $\prod_{j=1}^{3} 2j$ **c** $\prod_{k=1}^{3} k^2$ **d** $\prod_{i=0}^{2} 10^i$

e $\prod_{j=1}^{4} \frac{j}{j+1}$ **f** $\prod_{k=1}^{5} \sqrt{k}$ **g** $\prod_{i=1}^{3} \left(\frac{i+1}{i}\right)^2$ **h** $\prod_{i=0}^{4} \frac{1}{2^i}$

Example 22

7 Using mathematical induction, prove that

$$\prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}$$

for all natural numbers $n \geq 2$.

8 **a** Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$.

b Find a nicer proof involving factorisation that works for all $n \in \mathbb{Z}$.

Example 23

9 Use induction to prove each of the following. (Note that the base case is not $n = 1$.)

a $2^n > n^2$ for all $n \geq 5$ **b** $n! > 2^n$ for all $n \geq 4$

c $4^n > 2 \times 3^n$ for all $n \geq 3$ **d** $3^n > 2n + 1$ for all $n \geq 2$

10 **a** Prove, by mathematical induction, that $n^3 + 3n^2 + 2n$ is divisible by 6 for all $n \in \mathbb{N}$.

b Prove the result without mathematical induction by instead factorising the expression.

11 **a** Show that $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2 + (bx - ay)^2$.

b Write 13 as the sum of two squares.

c Hence, using mathematical induction, prove that 13^n can be written as the sum of two squares, for every natural number n.

12 Let $m \in \mathbb{N}$. Prove by induction that if m is odd, then m^n is odd, for every $n \in \mathbb{N}$.

13 The Fibonacci sequence is defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$.

a Find f_n for $n = 1, 2, \ldots, 10$.

b Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

c Evaluate $f_1 + f_3 + \cdots + f_{2n-1}$ for $n = 1, 2, 3, 4$.

d Try to find a general formula for the expression from part **c**.

e Confirm that your formula works using mathematical induction.

f Using induction, prove that f_{5n} is divisible by 5 for all $n \in \mathbb{N}$.

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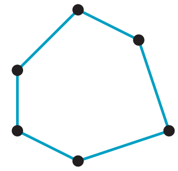
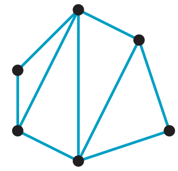
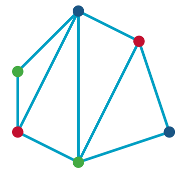
14 A polygonal area can be **triangulated** if we can add extra edges between the vertices so that the polygon is a union of non-intersecting triangles.





a Using mathematical induction, prove that for all $n \geq 3$, every convex polygonal area with n vertices can be triangulated.

b Consider a triangulation of a convex polygon with n vertices, where $n \geq 3$. Using mathematical induction, prove that we can colour the vertices using three colours, in such a way that no two adjacent vertices have the same colour.



15 Using induction, prove that

$$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \geq \sqrt{n}$$

for every natural number n.

16 The Fibonacci sequence is defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Prove that, for each $n \geq 2$, the following matrix equation holds:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

17 **a** Prove by induction that $3^n - (-2)^n$ is divisible by 5 for all $n \in \mathbb{N}$.

b Prove by induction that $4^n - (-3)^n$ is divisible by 7 for all $n \in \mathbb{N}$.

c Generalise the previous two results. Prove your generalisation using induction.

18 Prove each of the following using mathematical induction:

a $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$

b $1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n(3n+1) = n(n+1)^2$

19 Let $f\colon [0,1] \to \mathbb{R}$, $f(x) = \dfrac{x}{2-x}$.

a Determine the rule for $(f \circ f)(x) = f(f(x))$, the composition of f with itself.

b For $x \in [0,1]$ and $n \in \mathbb{N}$, let

$$F_n(x) = (\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies of } f})(x)$$

Prove by mathematical induction that

$$F_n(x) = \frac{x}{2^n - (2^n - 1)x}$$

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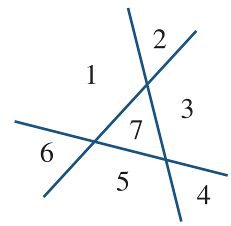
20 Suppose that we draw n lines in a plane so that no three are concurrent and no two are parallel. Let R_n be the number of regions into which these lines divide the plane.

For example, the diagram on the right illustrates that $R_3 = 7$.



a By drawing diagrams, find R_0, R_1, R_2, R_3 and R_4.

b Guess a formula for R_n in terms of n.

c Confirm that your formula is valid by using mathematical induction.

21 In this question, you will prove the **binomial theorem**, which states that

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_{n-1} a^1 b^{n-1} + {}^nC_n a^0 b^n$$

for all $n \in \mathbb{N}$.

a **Pascal's rule** is the identity

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad \text{(where } 1 \le r < n\text{)}$$

Prove this identity by using the formula ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$.

b Prove the binomial theorem by using mathematical induction and Pascal's rule.

c Using the binomial theorem, show that $2^n = {}^nC_0 + {}^nC_1 + \cdots + {}^nC_n$ for all $n \in \mathbb{N}$.

22 In Question 8, you proved that if n is an integer, then $n^2 - n$ is even. In this question, you will prove a generalisation of this result. The proof will use the binomial theorem.

a Let p be a prime number and let i be a positive integer less than p. Explain why ${}^pC_i$ is divisible by p.

b **Fermat's little theorem** states:

If p is a prime number and n is any integer, then $n^p - n$ is divisible by p.

Prove this theorem.

Hint: First prove the theorem in the case that n is positive. You can do this by using mathematical induction, the binomial theorem and part **a**.

23 Consider the sequence defined by the recurrence relation $t_n = 2t_{n-1} + 3$, where $t_1 = 3$. Prove by induction that $t_n = 3 \times 2^n - 3$.

24 Consider the sequence defined by the recurrence relation $t_n = 2t_{n-1} - n$, where $t_1 = 1$. Prove by induction that $t_n = n - 2^n + 2$.

25 Consider a cricket tournament with n teams, where each team plays every other team exactly once. Assume that there are no draws. Show that it is always possible to label the teams $T_1, T_2, \ldots, T_n$ in such a way that

$$T_1 > T_2 > \cdots > T_n$$

where the notation $T_i > T_{i+1}$ means that team T_i beat team T_{i+1}.

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Chapter summary

Assign-ment

Nrich

Basic concepts of proof

- A **conditional statement** has the form: If P is true, then Q is true. This can be abbreviated as $P \Rightarrow Q$, which is read 'P implies Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements P and Q are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- The **contrapositive** of $P \Rightarrow Q$ is (not Q) $\Rightarrow$ (not P).
- Proving the contrapositive of a statement may be easier than giving a direct proof.
- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- A **counterexample** can be used to demonstrate that a universal statement is false.

Proof by mathematical induction

- **Mathematical induction** is used to prove that a statement is true for all natural numbers.
- The basic outline of a proof by mathematical induction is:
  - **0** Define the proposition $P(n)$ for $n \in \mathbb{N}$.
  - **1** Show that $P(1)$ is true.
  - **2** Assume that $P(k)$ is true for some $k \in \mathbb{N}$.
  - **3** Show that $P(k+1)$ is true.
  - **4** Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Technology-free questions

1 A **Pythagorean triple** (a, b, c) consists of natural numbers a, b, c such that $a^2 + b^2 = c^2$.

  a Let a and d be natural numbers and assume that $(a, a+d, a+2d)$ is a Pythagorean triple. Prove that $a = 3d$.

  b Assume that (p, q, r) is a Pythagorean triple, where p is a prime number. Prove that $p = \sqrt{2q+1}$.

2 When you reverse the digits of the three-digit number 435 you obtain 534. The difference between the two numbers is divisible by 9, since

$$534 - 435 = 99 = 9 \cdot 11$$

Prove that this works for any three-digit number.

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Review

3 **a** Find real numbers a and b for which

$$\frac{1}{x(x+3)} = \frac{a}{x} + \frac{b}{x+3}$$

b Hence, evaluate

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{97 \cdot 100}$$

4 Let a and b be positive real numbers.

a Prove that $a > b$ if and only if $a^2 > b^2$.

b Hence, prove that $\sqrt{15} > \sqrt{2} + \sqrt{6}$.

c Also prove that $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$.

5 Let n be an integer with $n \geq 2$. Prove that $\log_n(n+1)$ is irrational.

6 Let n be an integer, and consider the statement:

If $n + 1$ is divisible by 3, then $n^3 + 1$ is divisible by 3.

a Prove that the statement is true.

b Write down the contrapositive of the statement.

c Write down the converse of the statement.

d Is the converse statement true or false? If it is true, give a proof. Otherwise, give a counterexample.

7 **a** Let b be a non-zero rational number. Prove by contradiction that $\sqrt{2} \cdot b$ is irrational.

Note: You may assume that $\sqrt{2}$ is irrational.

b Hence, prove that every non-zero rational number b can be written as the product of two irrational numbers.

8 Provide a counterexample for each of the following statements:

a If p is an odd prime, then $p + 2$ is also an odd prime.

b For all $n \in \mathbb{N}$, if n^3 is divisible by 8, then n is divisible by 8.

c For all real numbers a, b, c and d, if $a < b$ and $c < d$, then $ac < bd$.

d For all $n \in \mathbb{N}$, the numbers n, $n + 4$ and $n + 6$ cannot all be prime.

9 Let a, b and c be positive integers, and consider the statement:

If $a^2 + b^2 = c^2$, then at least one of a, b or c is even.

a Write down the contrapositive of the statement.

b Prove the contrapositive statement.

10 **a** Prove that the square of any integer n is of the form $3k$ or $3k + 1$, where $k \in \mathbb{Z}$.

b Explain why this means that there are no square numbers in the sequence

$$11,\ 101,\ 1001,\ 10001,\ 100001,\ \ldots$$

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Review

11 Prove by mathematical induction that:

a $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$ for all $n \in \mathbb{N}$

b $3^{-1} + 3^{-2} + \cdots + 3^{-n} = \dfrac{3^n - 1}{2(3^n)}$ for all $n \in \mathbb{N}$

12 For every natural number $n \geq 2$, prove that

$$\sum_{j=2}^{n} \frac{4}{j^2 - 1} = \frac{(n-1)(3n+2)}{n(n+1)}$$

13 **a** Prove by induction that $n^3 > 2n + 1$ for all $n \geq 2$.

b Hence, prove by induction that $n! > n^2$ for all $n \geq 4$.

14 Prove by induction that $3^n > n^2 + n$ for all $n \in \mathbb{N}$.

15 Prove each of the following divisibility results for every natural number n:

a $7^{2n-1} + 5$ is divisible by 12

b $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9

Multiple-choice questions

1 Suppose that m is odd and n is even. Which one of the following statements is true?

A $m + n$ is even
B $m^2 + n^2$ is even
C $(m + n)^2$ is even
D $3m + 2n$ is even
E $2m + 3n$ is even

2 Let n be an integer, and consider the statement: If n is odd, then n^2 is odd. The converse of this statement is

A If n is even, then n^2 is odd.
B If n is odd, then n^2 is even.
C If n^2 is odd, then n is even.
D If n^2 is even, then n is odd.
E If n^2 is odd, then n is odd.

3 Let a be an integer, and consider the statement: If $1 + a + a^2$ is odd, then a is even. The contrapositive of this statement is

A If a is even, then $1 + a + a^2$ is odd.
B If a is odd, then $1 + a + a^2$ is even.
C If $1 + a + a^2$ is even, then a is odd.
D If $1 + a + a^2$ is odd, then a is odd.
E If $1 + a + a^2$ is even, then a is even.

4 Consider the statement: There exists $n \in \mathbb{N}$ such that n is odd and n^2 is even. The negation of this statement is

A There exists $n \in \mathbb{N}$ such that n is even and n^2 is odd.
B There exists $n \in \mathbb{N}$ such that n is even or n^2 is odd.
C For all $n \in \mathbb{N}$, we have that n is odd and n^2 is even.
D For all $n \in \mathbb{N}$, we have that n is odd or n^2 is even.
E For all $n \in \mathbb{N}$, we have that n is even or n^2 is odd.

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5 Let a, b and c be real numbers such that $ca = cb$. Which one of the following statements *must* be true?

A $a = b$ or $c = 0$ **B** $a = b$ and $c = 1$ **C** $a = b$
D $c = 1$ **E** $c = 0$

6 Consider the statement:

- For every function $f : \mathbb{R} \to \mathbb{R}$, if f is strictly increasing, then the range of f is $\mathbb{R}$.

Which one of the following functions shows that this statement is false?

A $f(x) = 0$ **B** $f(x) = x$ **C** $f(x) = 2^x$ **D** $f(x) = x^2$ **E** $f(x) = x^3$

7 The sum $\sum_{i=3}^{5} i^2$ is equal to

A 12 **B** 24 **C** 48 **D** 50 **E** 60

8 If $\prod_{i=1}^{n} i = 24$, then

A $n = 2$ **B** $n = 3$ **C** $n = 4$ **D** $n = 5$ **E** $n = 6$

Extended-response questions

1 **a** Let $m \in \mathbb{N}$. By expanding the right-hand side, prove that

$$x^m - 1 = (x - 1)(1 + x + x^2 + \cdots + x^{m-1})$$

b Hence, prove the statement:

For all $n \in \mathbb{N}$, if n is not prime, then $2^n - 1$ is not prime.

Hint: If $n \in \mathbb{N} \setminus \{1\}$ and n is not prime, then $n = km$ for some $k, m \in \mathbb{N} \setminus \{1\}$.

c Now consider the converse statement:

For all $n \in \mathbb{N}$, if $2^n - 1$ is not prime, then n is not prime.

Find a counterexample to show that this statement is false.

2 Consider a Pythagorean triple (a, b, c). This means that a, b and c are natural numbers such that $a^2 + b^2 = c^2$, and therefore a, b and c are the side lengths of a right-angled triangle with hypotenuse c.

a Show that if c is odd, then exactly one of a or b is odd.

b Prove that

$$\frac{abc}{a + b + c} = \frac{c(a + b - c)}{2}$$

c Hence, prove that the product of the side lengths of the triangle, abc, is divisible by the perimeter, $a + b + c$.

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3 Consider a function $f : \mathbb{R} \to \mathbb{R}$.

- We say that f is **even** if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
- We say that f is **odd** if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

For example, the function $f(x) = x^2$ is even, since $f(-x) = (-x)^2 = x^2 = f(x)$.

a Prove that $f(x) = x^3$ is an odd function.

b Prove that the product of two even functions is even.

c Prove that the product of two odd functions is even.

d Prove that the sum of two odd functions is odd.

e Prove that the sum of two even functions is even.

f Prove that the graph of every odd function passes through the origin.

g Find the only function $f : \mathbb{R} \to \mathbb{R}$ that is both even and odd.

4 For $n \in \mathbb{N}$, the nth derivative of $f(x)$ can be written as $f^{(n)}(x)$.

Let $f(x) = \dfrac{1}{2x+1}$. Prove that $f^{(n)}(x) = (-1)^n \dfrac{2^n \cdot n!}{(2x+1)^{n+1}}$.

5 A sequence $a_1, a_2, a_3, \dots$ is given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Using mathematical induction, prove each of the following:

a $a_{n+1} > a_n$ for all $n \in \mathbb{N}$

b $a_n < 2$ for all $n \in \mathbb{N}$

6 Prove that, for any natural number $n \geq 3$, you can find a set A consisting of n natural numbers such that the sum of the numbers in A is divisible by each of the numbers in A.
Hint: This is an induction proof with the base case $n = 3$.

7 We say that a point $P(x, y)$ in the Cartesian plane is a **rational point** if both x and y are rational numbers. The unit circle $x^2 + y^2 = 1$ has infinitely many rational points. For example, the rational points $(1, 0)$ and $(\frac{3}{5}, \frac{4}{5})$ lie on the unit circle.

a Show that the curve $x^2 + y^2 = 3$ has no rational points.
Hint: This is a challenging proof by contradiction.

b Hence, prove that $\sqrt{3}$ is irrational.

c You have shown that the curve $x^2 + y^2 = 3$ has no rational points. Explain why this implies that $x^2 + y^2 = 3^k$ has no rational points, where k is an odd natural number.

d Hence, prove that $\sqrt{3^k}$ is irrational, for every odd natural number k.

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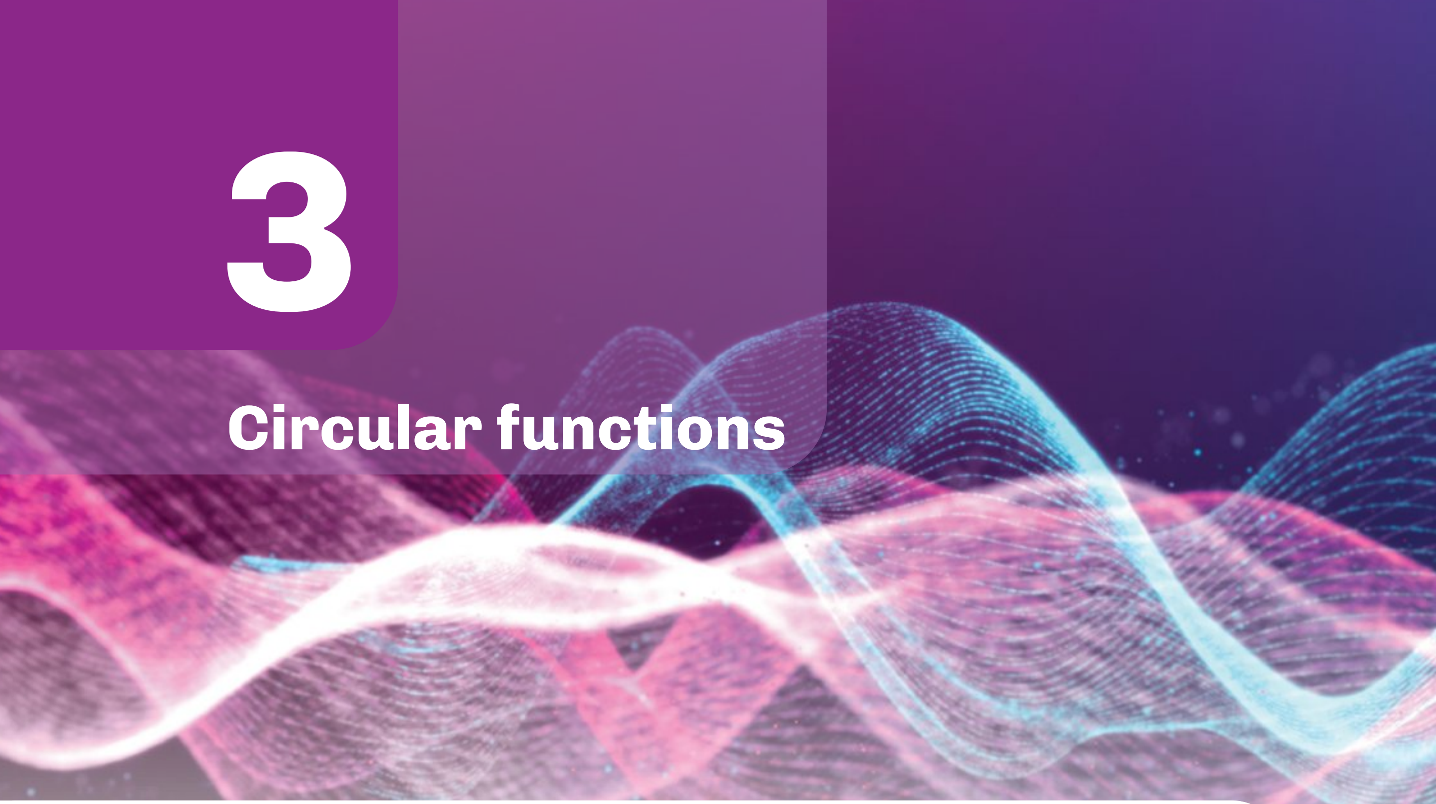

3

Circular functions

Objectives

- To revise the **reciprocal circular functions** cosecant, secant and cotangent.
- To revise the different forms of the **Pythagorean identity**.
- To revise the **compound angle formulas** and the **double angle formulas**.
- To revise the **inverse circular functions** $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.
- To understand the graphs of the inverse circular functions.
- To solve equations involving circular functions.
- To revise the trigonometric identities for products of sines and cosines expressed as sums or differences, and vice versa.

There are many interesting and useful relationships between the circular functions. The most fundamental is the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

Astronomy was the original motivation for these identities, many of which were discovered a very long time ago.

For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

They are of great importance in many areas of mathematics, including calculus.

The sine, cosine and tangent functions are discussed in some detail in Section 1A. Several new circular functions are introduced in this chapter. You have met most of the material in this chapter in Specialist Mathematics Units 1 & 2.

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3A The reciprocal circular functions

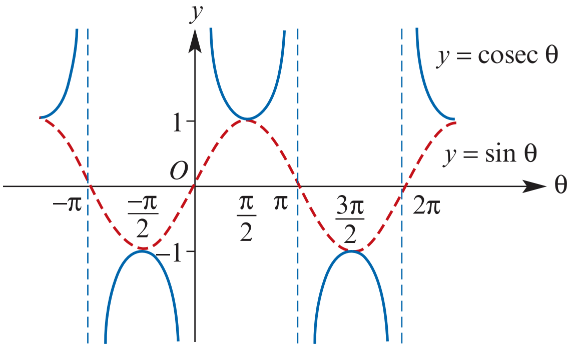
The cosecant function: $y = \operatorname{cosec}\theta$

The cosecant function is defined by

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

provided $\sin\theta \neq 0$.

The graphs of $y = \operatorname{cosec}\theta$ and $y = \sin\theta$ are shown here on the same set of axes.



- Domain As $\sin\theta = 0$ when $\theta = n\pi$, $n \in \mathbb{Z}$, the domain of $y = \operatorname{cosec}\theta$ is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$.
- Range The range of $y = \sin\theta$ is $[-1, 1]$, so the range of $y = \operatorname{cosec}\theta$ is $\mathbb{R} \setminus (-1, 1)$.
- Turning points The graph of $y = \sin\theta$ has turning points at $\theta = \frac{(2n+1)\pi}{2}$, for $n \in \mathbb{Z}$, as does the graph of $y = \operatorname{cosec}\theta$.
- Asymptotes The graph of $y = \operatorname{cosec}\theta$ has vertical asymptotes with equations $\theta = n\pi$, for $n \in \mathbb{Z}$.

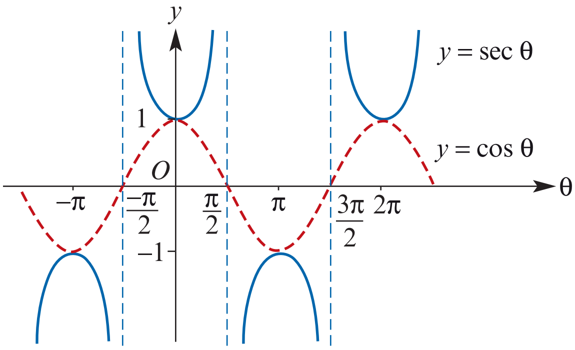
The secant function: $y = \sec\theta$

The secant function is defined by

$$\sec\theta = \frac{1}{\cos\theta}$$

provided $\cos\theta \neq 0$.

The graphs of $y = \sec\theta$ and $y = \cos\theta$ are shown here on the same set of axes.



- Domain The domain of $y = \sec\theta$ is $\mathbb{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\right\}$.
- Range The range of $y = \sec\theta$ is $\mathbb{R} \setminus (-1, 1)$.
- Turning points The graph of $y = \sec\theta$ has turning points at $\theta = n\pi$, for $n \in \mathbb{Z}$.
- Asymptotes The vertical asymptotes have equations $\theta = \frac{(2n+1)\pi}{2}$, for $n \in \mathbb{Z}$.

Since the graph of $y = \cos\theta$ is a translation of the graph of $y = \sin\theta$, the graph of $y = \sec\theta$ is a translation of the graph of $y = \operatorname{cosec}\theta$, by $\frac{\pi}{2}$ units in the negative direction of the θ-axis.

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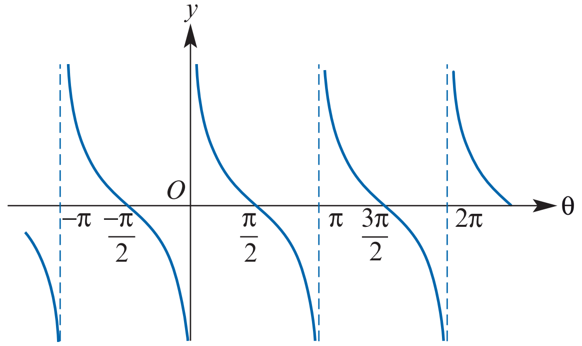
The cotangent function: $y = \cot\theta$

The cotangent function is defined by

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

provided $\sin\theta \neq 0$.



Using the complementary properties of sine and cosine, we have

$$\begin{aligned}\cot\theta &= \tan\left(\frac{\pi}{2} - \theta\right)\\ &= -\tan\left(\pi - \left(\frac{\pi}{2} - \theta\right)\right)\\ &= -\tan\left(\theta + \frac{\pi}{2}\right)\end{aligned}$$

Therefore the graph of $y = \cot\theta$, shown above, is obtained from the graph of $y = \tan\theta$ by a translation of $\frac{\pi}{2}$ units in the negative direction of the θ-axis and then a reflection in the θ-axis.

- **Domain** As $\sin\theta = 0$ when $\theta = n\pi$, $n \in \mathbb{Z}$, the domain of $y = \cot\theta$ is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$.
- **Range** The range of $y = \cot\theta$ is $\mathbb{R}$.
- **Asymptotes** The vertical asymptotes have equations $\theta = n\pi$, for $n \in \mathbb{Z}$.

Note: $\cot\theta = \dfrac{1}{\tan\theta}$ provided $\cos\theta \neq 0$

Example 1

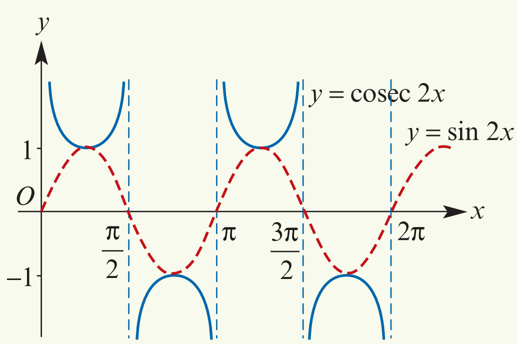
Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}(2x)$ **b** $y = \sec\left(x + \frac{\pi}{3}\right)$ **c** $y = \cot\left(x - \frac{\pi}{4}\right)$

Solution

a The graph of $y = \operatorname{cosec}(2x)$ is obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

The graph of $y = \sin(2x)$ is also shown.



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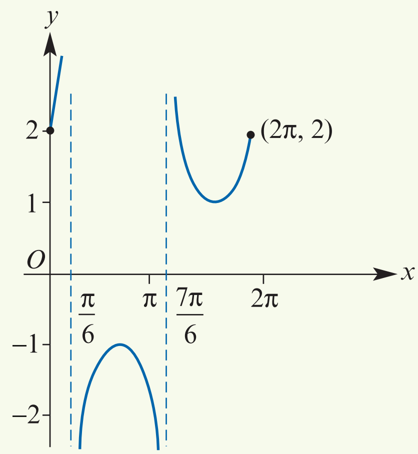
b The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ is obtained from the graph of $y = \sec x$ by a translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.

The y-axis intercept is $\sec\left(\frac{\pi}{3}\right) = 2$.

The asymptotes are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.

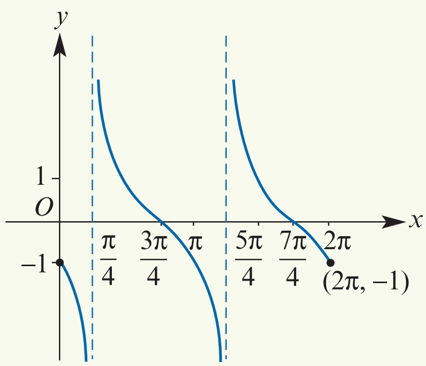


c The graph of $y = \cot\left(x - \frac{\pi}{4}\right)$ is obtained from the graph of $y = \cot x$ by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis.

The y-axis intercept is $\cot\left(-\frac{\pi}{4}\right) = -1$.

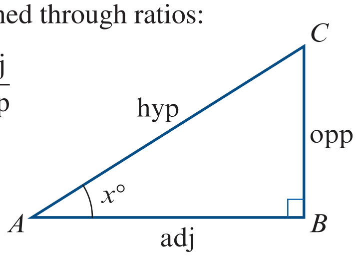
The asymptotes are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

The x-axis intercepts are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.



For right-angled triangles, the reciprocal functions can be defined through ratios:

$$\text{cosec}(x^\circ) = \frac{\text{hyp}}{\text{opp}} \qquad \sec(x^\circ) = \frac{\text{hyp}}{\text{adj}} \qquad \cot(x^\circ) = \frac{\text{adj}}{\text{opp}}$$

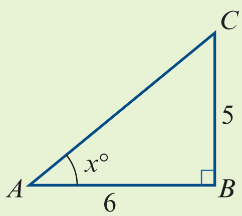


Example 2

In triangle ABC, $\angle ABC = 90^\circ$, $\angle CAB = x^\circ$, $AB = 6$ cm and $BC = 5$ cm. Find:

a AC

b the trigonometric ratios related to x°



Solution

a By Pythagoras' theorem,

$$AC^2 = 5^2 + 6^2 = 61$$

$$\therefore\ AC = \sqrt{61} \text{ cm}$$

b $\sin(x^\circ) = \frac{5}{\sqrt{61}}$ $\quad\cos(x^\circ) = \frac{6}{\sqrt{61}}$ $\quad\tan(x^\circ) = \frac{5}{6}$

$\text{cosec}(x^\circ) = \frac{\sqrt{61}}{5}$ $\quad\sec(x^\circ) = \frac{\sqrt{61}}{6}$ $\quad\cot(x^\circ) = \frac{6}{5}$

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Useful properties

The symmetry properties of sine, cosine and tangent can be used to establish the following:

$$\sec(\pi - x) = -\sec x \qquad \operatorname{cosec}(\pi - x) = \operatorname{cosec} x \qquad \cot(\pi - x) = -\cot x$$
$$\sec(\pi + x) = -\sec x \qquad \operatorname{cosec}(\pi + x) = -\operatorname{cosec} x \qquad \cot(\pi + x) = \cot x$$
$$\sec(2\pi - x) = \sec x \qquad \operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x \qquad \cot(2\pi - x) = -\cot x$$
$$\sec(-x) = \sec x \qquad \operatorname{cosec}(-x) = -\operatorname{cosec} x \qquad \cot(-x) = -\cot x$$

The complementary properties are also useful:

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x \qquad \operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$
$$\cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Example 3

Find the exact value of each of the following:

a $\sec\left(\frac{11\pi}{4}\right)$ **b** $\operatorname{cosec}\left(-\frac{23\pi}{4}\right)$ **c** $\cot\left(\frac{11\pi}{3}\right)$

Solution

a
$$\begin{aligned}\sec\left(\frac{11\pi}{4}\right) &= \sec\left(2\pi + \frac{3\pi}{4}\right)\\ &= \sec\left(\frac{3\pi}{4}\right)\\ &= \frac{1}{\cos(\frac{3\pi}{4})}\\ &= \frac{1}{-\frac{1}{\sqrt{2}}}\\ &= -\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\operatorname{cosec}\left(-\frac{23\pi}{4}\right) &= \operatorname{cosec}\left(-6\pi + \frac{\pi}{4}\right)\\ &= \operatorname{cosec}\left(\frac{\pi}{4}\right)\\ &= \frac{1}{\sin(\frac{\pi}{4})}\\ &= \frac{1}{\frac{1}{\sqrt{2}}}\\ &= \sqrt{2}\end{aligned}$$

c
$$\begin{aligned}\cot\left(\frac{11\pi}{3}\right) &= \cot\left(4\pi - \frac{\pi}{3}\right)\\ &= \cot\left(-\frac{\pi}{3}\right)\\ &= -\cot\left(\frac{\pi}{3}\right)\\ &= -\frac{1}{\tan(\frac{\pi}{3})}\\ &= -\frac{1}{\sqrt{3}}\end{aligned}$$

The Pythagorean identity

We can establish two alternative forms of the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

$$1 + \tan^2 x = \sec^2 x \qquad \text{provided } \cos x \neq 0$$
$$\cot^2 x + 1 = \operatorname{cosec}^2 x \qquad \text{provided } \sin x \neq 0$$

Proof To obtain the first identity, divide each term in the Pythagorean identity by $\cos^2 x$:

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$\therefore \quad 1 + \tan^2 x = \sec^2 x$$

For the second identity, divide each term in the Pythagorean identity by $\sin^2 x$.

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Example 4

Simplify the expression

$$\frac{\cos x - \cos^3 x}{\cot x}$$

Solution

$$\frac{\cos x - \cos^3 x}{\cot x} = \frac{\cos x \cdot (1 - \cos^2 x)}{\cot x}$$

$$= \cos x \cdot \sin^2 x \cdot \frac{\sin x}{\cos x}$$

$$= \sin^3 x$$

Example 5

If $\tan x = 2$ and $x \in \left(0, \frac{\pi}{2}\right)$, find:

a $\sec x$ **b** $\cos x$ **c** $\sin x$ **d** $\operatorname{cosec} x$

Solution

a $\sec^2 x = 1 + \tan^2 x$

$= 1 + 4$

$\therefore \sec x = \pm\sqrt{5}$

Since $x \in \left(0, \frac{\pi}{2}\right)$, we have $\sec x = \sqrt{5}$.

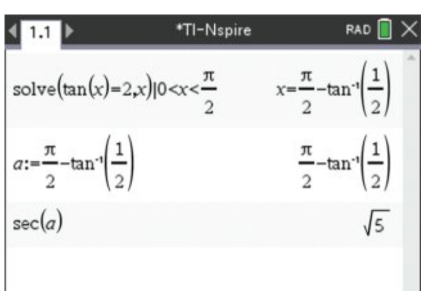
b $\cos x = \dfrac{1}{\sec x} = \dfrac{\sqrt{5}}{5}$

c $\sin x = \tan x \cdot \cos x = \dfrac{2\sqrt{5}}{5}$

d $\operatorname{cosec} x = \dfrac{1}{\sin x} = \dfrac{\sqrt{5}}{2}$

Using the TI-Nspire

- Choose **solve** from the **Algebra** menu and complete as shown.
- Assign (ctrl) or store (ctrl var) the answer as the variable a to obtain the results.



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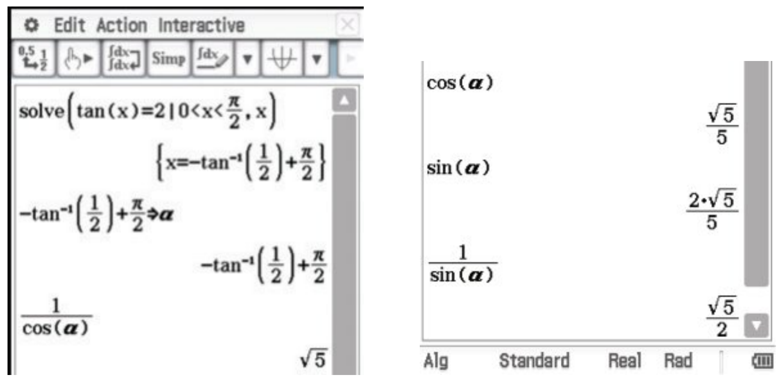
Using the Casio ClassPad

- In $\sqrt{\alpha}$ (Main), enter and highlight: $\tan(x) = 2 \mid 0 < x < \frac{\pi}{2}$
- Go to **Interactive** > **Equation/Inequality** > **solve**.
- Highlight the answer and drag it to the next entry line. Enter $\Rightarrow a$.
- The results are obtained as shown.



Exercise 3A

Example 1

1 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}\left(x + \frac{\pi}{4}\right)$ **b** $y = \sec\left(x - \frac{\pi}{6}\right)$ **c** $y = \cot\left(x + \frac{\pi}{3}\right)$

d $y = \sec\left(x + \frac{2\pi}{3}\right)$ **e** $y = \operatorname{cosec}\left(x - \frac{\pi}{2}\right)$ **f** $y = \cot\left(x - \frac{3\pi}{4}\right)$

2 Sketch the graph of each of the following over the interval $[0, \pi]$:

a $y = \sec(2x)$ **b** $y = \operatorname{cosec}(3x)$ **c** $y = \cot(4x)$

d $y = \operatorname{cosec}\left(2x + \frac{\pi}{2}\right)$ **e** $y = \sec(2x + \pi)$ **f** $y = \cot\left(2x - \frac{\pi}{3}\right)$

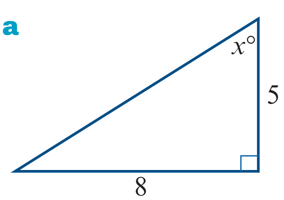
3 Sketch the graph of each of the following over the interval $[-\pi, \pi]$:

a $y = \sec\left(2x - \frac{\pi}{2}\right)$ **b** $y = \operatorname{cosec}\left(2x + \frac{\pi}{3}\right)$ **c** $y = \cot\left(2x - \frac{2\pi}{3}\right)$

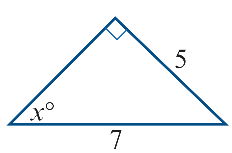
Example 2

4 Find the trigonometric ratios $\cot(x°)$, $\sec(x°)$ and $\operatorname{cosec}(x°)$ for each of the following triangles:

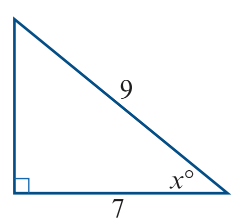
a



b



c



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Example 3

5 Find the exact value of each of the following:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\tan\left(-\frac{\pi}{4}\right)$ **d** $\operatorname{cosec}\left(\frac{\pi}{6}\right)$

e $\sec\left(\frac{\pi}{4}\right)$ **f** $\cot\left(-\frac{\pi}{6}\right)$ **g** $\sin\left(\frac{5\pi}{4}\right)$ **h** $\tan\left(\frac{5\pi}{6}\right)$

i $\sec\left(-\frac{\pi}{3}\right)$ **j** $\operatorname{cosec}\left(\frac{3\pi}{4}\right)$ **k** $\cot\left(\frac{9\pi}{4}\right)$ **l** $\cos\left(-\frac{7\pi}{3}\right)$

Example 4

6 Simplify each of the following expressions:

a $\sec^2 x - \tan^2 x$ **b** $\cot^2 x - \operatorname{cosec}^2 x$ **c** $\dfrac{\tan^2 x + 1}{\tan^2 x}$

d $\dfrac{\sin^2 x}{\cos x} + \cos x$ **e** $\sin^4 x - \cos^4 x$ **f** $\tan^3 x + \tan x$

Example 5

7 If $\tan x = -4$ and $x \in \left(-\frac{\pi}{2}, 0\right)$, find:

a $\sec x$ **b** $\cos x$ **c** $\operatorname{cosec} x$

8 If $\cot x = 3$ and $x \in \left(\pi, \frac{3\pi}{2}\right)$, find:

a $\operatorname{cosec} x$ **b** $\sin x$ **c** $\sec x$

9 If $\sec x = 10$ and $x \in \left(-\frac{\pi}{2}, 0\right)$, find:

a $\tan x$ **b** $\sin x$

10 If $\operatorname{cosec} x = -6$ and $x \in \left(\frac{3\pi}{2}, 2\pi\right)$, find:

a $\cot x$ **b** $\cos x$

11 If $\sin x^\circ = 0.5$ and $90 < x < 180$, find:

a $\cos x^\circ$ **b** $\cot x^\circ$ **c** $\operatorname{cosec} x^\circ$

12 If $\operatorname{cosec} x^\circ = -3$ and $180 < x < 270$, find:

a $\sin x^\circ$ **b** $\cos x^\circ$ **c** $\sec x^\circ$

13 If $\cos x^\circ = -0.7$ and $0 < x < 180$, find:

a $\sin x^\circ$ **b** $\tan x^\circ$ **c** $\cot x^\circ$

14 If $\sec x^\circ = 5$ and $180 < x < 360$, find:

a $\cos x^\circ$ **b** $\sin x^\circ$ **c** $\cot x^\circ$

15 Simplify each of the following expressions:

a $\sec^2\theta + \operatorname{cosec}^2\theta - \sec^2\theta\operatorname{cosec}^2\theta$ **b** $(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta)$

c $(1 - \cos^2\theta)(1 + \cot^2\theta)$ **d** $\dfrac{\sec^2\theta - \operatorname{cosec}^2\theta}{\tan^2\theta - \cot^2\theta}$

16 Let $x = \sec\theta - \tan\theta$. Prove that $x + \frac{1}{x} = 2\sec\theta$ and also find a simple expression for $x - \frac{1}{x}$ in terms of θ.

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3B Compound and double angle formulas

The compound angle formulas

The following identities are known as the compound angle formulas.

Compound angle formulas

- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

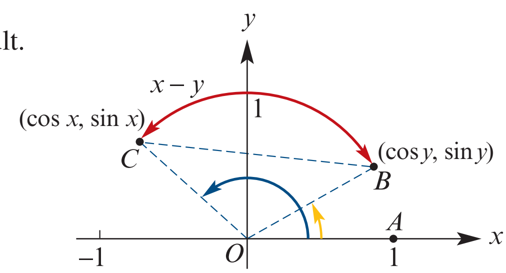
Proof of the initial identity

We start by proving the identity

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

The other identities will be derived from this result.

Consider a unit circle as shown:



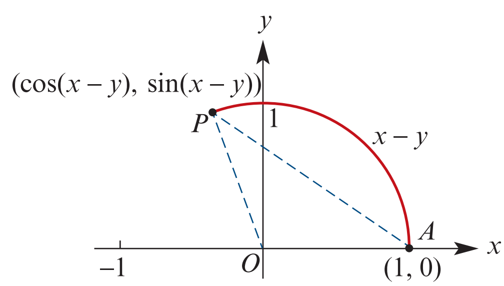
arc length $AB = y$ units

arc length $AC = x$ units

arc length $BC = x - y$ units

Rotate ΔOCB so that B is coincident with A.
Then C is moved to

$$P(\cos(x - y), \sin(x - y))$$



As the triangles CBO and PAO are congruent, we have $CB = PA$.

Using the coordinate distance formula:

$$\begin{aligned} CB^2 &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\ &= 2 - 2(\cos x \cos y + \sin x \sin y) \end{aligned}$$

$$\begin{aligned} PA^2 &= (\cos(x - y) - 1)^2 + (\sin(x - y) - 0)^2 \\ &= 2 - 2\cos(x - y) \end{aligned}$$

Since $CB = PA$, this gives

$$2 - 2\cos(x - y) = 2 - 2(\cos x \cos y + \sin x \sin y)$$

$$\therefore \quad \cos(x - y) = \cos x \cos y + \sin x \sin y$$

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Derivation of the other identities

$$\begin{aligned}\cos(x+y) &= \cos(x-(-y))\\ &= \cos x\cos(-y)+\sin x\sin(-y)\\ &= \cos x\cos y-\sin x\sin y\end{aligned}$$

$$\begin{aligned}\sin(x-y) &= \cos\left(\frac{\pi}{2}-x+y\right)\\ &= \cos\left(\frac{\pi}{2}-x\right)\cos y-\sin\left(\frac{\pi}{2}-x\right)\sin y\\ &= \sin x\cos y-\cos x\sin y\end{aligned}$$

$$\begin{aligned}\tan(x-y) &= \frac{\sin(x-y)}{\cos(x-y)}\\ &= \frac{\sin x\cos y-\cos x\sin y}{\cos x\cos y+\sin x\sin y}\end{aligned}$$

Dividing top and bottom by $\cos x \cos y$ gives

$$\begin{aligned}\tan(x-y) &= \frac{\dfrac{\sin x\cos y}{\cos x\cos y}-\dfrac{\cos x\sin y}{\cos x\cos y}}{1+\dfrac{\sin x\sin y}{\cos x\cos y}}\\ &= \frac{\tan x-\tan y}{1+\tan x\tan y}\end{aligned}$$

The derivation of the remaining two identities is left as an exercise.

Example 6

a Use $\frac{5\pi}{12}=\frac{\pi}{6}+\frac{\pi}{4}$ to evaluate $\sin\left(\frac{5\pi}{12}\right)$.

b Use $\frac{\pi}{12}=\frac{\pi}{3}-\frac{\pi}{4}$ to evaluate $\cos\left(\frac{\pi}{12}\right)$.

Solution

a $$\begin{aligned}&\sin\left(\frac{5\pi}{12}\right)\\ &= \sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{4}(1+\sqrt{3})\end{aligned}$$

b $$\begin{aligned}&\cos\left(\frac{\pi}{12}\right)\\ &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{4}(1+\sqrt{3})\end{aligned}$$

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Example 7

If $\sin x = 0.2$ and $\cos y = -0.4$, where $x \in \left[0, \frac{\pi}{2}\right]$ and $y \in \left[\pi, \frac{3\pi}{2}\right]$, find $\sin(x + y)$.

Solution

We first find $\cos x$ and $\sin y$.

$$\begin{aligned} \cos x &= \pm\sqrt{1 - 0.2^2} \quad \text{as } \sin x = 0.2 \\ &= \pm\sqrt{0.96} \\ \therefore \quad \cos x &= \sqrt{0.96} \quad \text{as } x \in \left[0, \frac{\pi}{2}\right] \\ &= \frac{2\sqrt{6}}{5} \end{aligned}$$

$$\begin{aligned} \sin y &= \pm\sqrt{1 - (-0.4)^2} \quad \text{as } \cos y = -0.4 \\ &= \pm\sqrt{0.84} \\ \therefore \quad \sin y &= -\sqrt{0.84} \quad \text{as } y \in \left[\pi, \frac{3\pi}{2}\right] \\ &= -\frac{\sqrt{21}}{5} \end{aligned}$$

Hence

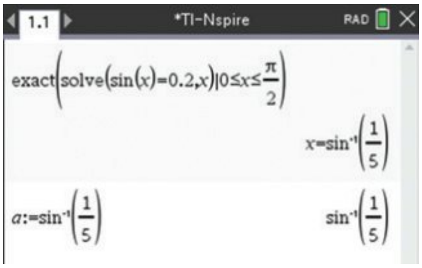
$$\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \\ &= 0.2 \times (-0.4) + \frac{2\sqrt{6}}{5} \times \left(-\frac{\sqrt{21}}{5}\right) \\ &= -0.08 - \frac{2}{25} \times 3\sqrt{14} \\ &= -\frac{2}{25}(1 + 3\sqrt{14}) \end{aligned}$$

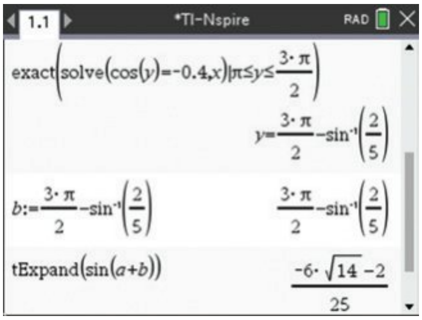
Using the TI-Nspire

- First solve $\sin(x) = 0.2$ for $0 \le x \le \frac{\pi}{2}$.
- Assign the result to a.
- Then solve $\cos(y) = -0.4$ for $\pi \le y \le \frac{3\pi}{2}$.
- Assign the result to b.

Note: If a decimal is entered, then the answer will be given in approximate form, even in **Auto** mode. To obtain an exact answer, use **exact(** at the start of the entry or write the decimal as a fraction.

- Use (menu) > **Algebra** > **Trigonometry** > **Expand** to expand the expression $\sin(a + b)$.





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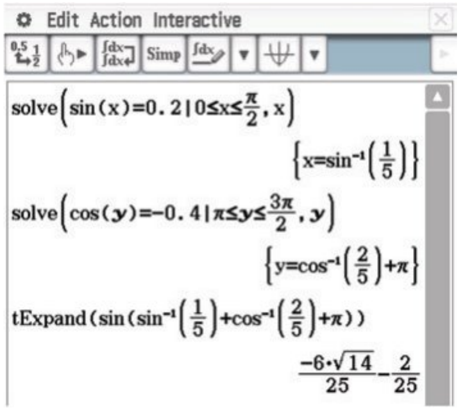
Using the Casio ClassPad

- Solve $\sin(x) = 0.2 \mid 0 \le x \le \frac{\pi}{2}$ for x.
- Solve $\cos(y) = -0.4 \mid \pi \le y \le \frac{3\pi}{2}$ for y.
  (Don't forget to specify the solution variable as y.)
- Paste the results to form the expression
  $$\sin\left(\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{2}{5}\right) + \pi\right)$$
- Highlight and go to **Interactive** > **Transformation** > **tExpand**.



The double angle formulas

Double angle formulas

- $\cos(2x) = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$
- $\sin(2x) = 2\sin x \cos x$
- $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$

Proof These formulas can be derived from the compound angle formulas. For example:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\therefore \quad \cos(x + x) = \cos x \cos x - \sin x \sin x$$
$$\therefore \quad \cos(2x) = \cos^2 x - \sin^2 x$$

The two other expressions for $\cos(2x)$ are obtained using the Pythagorean identity:

$$\cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$

and

$$\cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$$

Example 8

If $\sin\alpha = 0.6$ and $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, find $\sin(2\alpha)$.

Solution

$\cos\alpha = \pm\sqrt{1 - 0.6^2}$ since $\sin\alpha = 0.6$

$= \pm 0.8$

$\therefore \quad \cos\alpha = -0.8$ since $\alpha \in \left[\frac{\pi}{2}, \pi\right]$

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Hence

$$\begin{aligned}\sin(2\alpha) &= 2\sin\alpha\cos\alpha\\ &= 2\times 0.6\times(-0.8)\\ &= -0.96\end{aligned}$$

Example 9

If $\cos\alpha = 0.7$ and $\alpha \in \left[\frac{3\pi}{2}, 2\pi\right]$, find $\sin\left(\frac{\alpha}{2}\right)$.

Solution

We use a double angle formula:

$$\begin{aligned}\cos(2x) &= 1 - 2\sin^2 x\\ \therefore\quad \cos\alpha &= 1 - 2\sin^2\left(\frac{\alpha}{2}\right)\\ 2\sin^2\left(\frac{\alpha}{2}\right) &= 1 - 0.7\\ &= 0.3\\ \sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{0.15}\end{aligned}$$

Since $\alpha \in \left[\frac{3\pi}{2}, 2\pi\right]$, we have $\frac{\alpha}{2} \in \left[\frac{3\pi}{4}, \pi\right]$, so $\sin\left(\frac{\alpha}{2}\right)$ is positive.

Hence

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{0.15} = \frac{\sqrt{15}}{10}$$

Skill-sheet

Exercise 3B

Example 6

1 Use the compound angle formulas and appropriate angles to find the exact value of each of the following:

a $\sin\left(\frac{\pi}{12}\right)$ **b** $\tan\left(\frac{5\pi}{12}\right)$ **c** $\cos\left(\frac{7\pi}{12}\right)$ **d** $\tan\left(\frac{\pi}{12}\right)$

2 Use the compound angle formulas to expand each of the following:

a $\sin(2x - 5y)$ **b** $\cos(x^2 + y)$ **c** $\tan(x + (y + z))$

3 Simplify each of the following:

a $\sin(x)\cos(2y) - \cos(x)\sin(2y)$ **b** $\cos(3x)\cos(2x) + \sin(3x)\sin(2x)$

c $\dfrac{\tan A - \tan(A - B)}{1 + \tan A\tan(A - B)}$ **d** $\sin(A + B)\cos(A - B) + \cos(A + B)\sin(A - B)$

e $\cos(y)\cos(-2y) - \sin(y)\sin(-2y)$

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4 **a** Expand $\sin(x + 2x)$. **b** Hence express $\sin(3x)$ in terms of $\sin x$.

5 **a** Expand $\cos(x + 2x)$. **b** Hence express $\cos(3x)$ in terms of $\cos x$.

Example 7

6 If $\sin x = 0.6$ and $\tan y = 2.4$, where $x \in \left[\frac{\pi}{2}, \pi\right]$ and $y \in \left[0, \frac{\pi}{2}\right]$, find the exact value of each of the following:

a $\cos x$ **b** $\sec y$ **c** $\cos y$
d $\sin y$ **e** $\tan x$ **f** $\cos(x - y)$
g $\sin(x - y)$ **h** $\tan(x + y)$ **i** $\tan(x + 2y)$

7 If $\cos x = -0.7$ and $\sin y = 0.4$, where $x \in \left[\pi, \frac{3\pi}{2}\right]$ and $y \in \left[0, \frac{\pi}{2}\right]$, find the value of each of the following, correct to two decimal places:

a $\sin x$ **b** $\cos y$ **c** $\tan(x - y)$ **d** $\cos(x + y)$

8 Simplify each of the following:

a $\frac{1}{2}\sin x \cos x$ **b** $\sin^2 x - \cos^2 x$ **c** $\dfrac{\tan x}{1 - \tan^2 x}$

d $\dfrac{\sin^4 x - \cos^4 x}{\cos(2x)}$ **e** $\dfrac{4\sin^3 x - 2\sin x}{\cos x \cos(2x)}$ **f** $\dfrac{4\sin^2 x - 4\sin^4 x}{\sin(2x)}$

Example 8

9 If $\sin x = -0.8$ and $x \in \left[\pi, \frac{3\pi}{2}\right]$, find:

a $\sin(2x)$ **b** $\cos(2x)$ **c** $\tan(2x)$

10 If $\tan x = 3$ and $x \in \left(0, \frac{\pi}{2}\right)$, find:

a $\tan(2x)$ **b** $\tan(3x)$

Example 9

11 If $\sin x = -0.75$ and $x \in \left[\pi, \frac{3\pi}{2}\right]$, find correct to two decimal places:

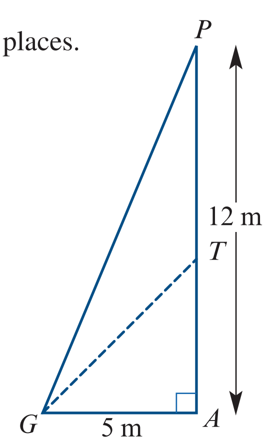
a $\cos x$ **b** $\sin(\frac{1}{2}x)$

12 Use the double angle formula for $\tan(2x)$ and the fact that $\tan\left(\frac{\pi}{4}\right) = 1$ to find the exact value of $\tan\left(\frac{\pi}{8}\right)$.

13 If $\cos x = 0.9$ and $x \in \left[0, \frac{\pi}{2}\right]$, find $\cos(\frac{1}{2}x)$ correct to two decimal places.

14 In a right-angled triangle GAP, $AP = 12$ m and $GA = 5$ m. The point T on AP is such that $\angle AGT = \angle TGP = x^c$. Without using a calculator, find the exact values of the following:

a $\tan(2x)$
b $\tan x$, by using the double angle formula
c AT



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3C The inverse circular functions

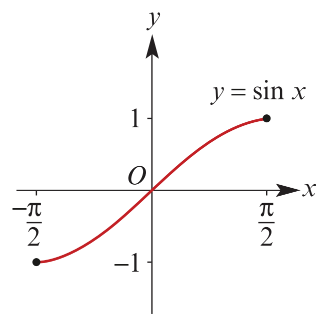
As the circular functions sine, cosine and tangent are periodic, they are not one-to-one and therefore they do not have inverse functions. However, by restricting their domains to form one-to-one functions, we can define the inverse circular functions.

The inverse sine function: $y = \sin^{-1} x$

Restricting the sine function

When the domain of the sine function is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the resulting function is one-to-one and therefore has an inverse function.

Note: Other intervals (defined through consecutive turning points of the graph) could have been used for the restricted domain, but this is the convention.



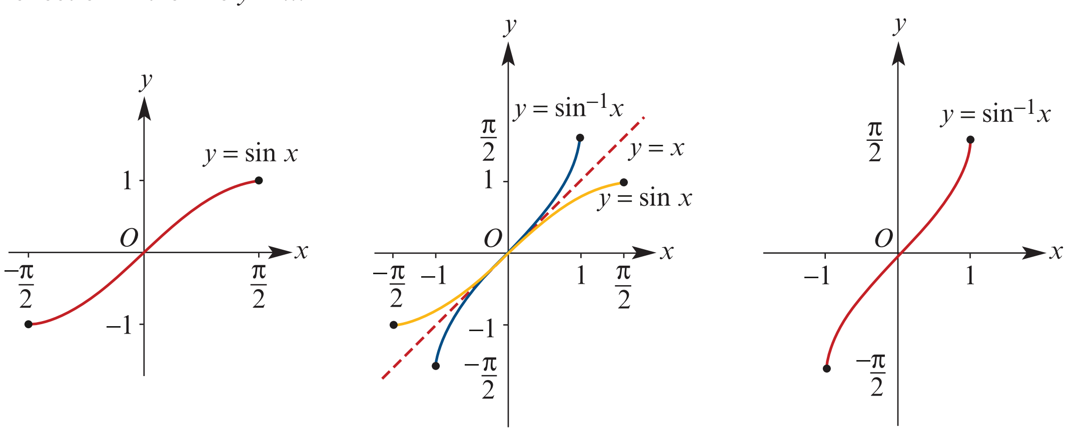
Defining the inverse function

The inverse of the restricted sine function is usually denoted by $\sin^{-1}$ or arcsin.

Inverse sine function

$\sin^{-1}: [-1, 1] \to \mathbb{R}, \ \sin^{-1} x = y, \ $ where $\sin y = x$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The graph of $y = \sin^{-1} x$ is obtained from the graph of $y = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, through a reflection in the line $y = x$.



- Domain Domain of $\sin^{-1}$ = range of restricted sine function = $[-1, 1]$
- Range Range of $\sin^{-1}$ = domain of restricted sine function = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- Inverse relationship
  - $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$
  - $\sin^{-1}(\sin x) = x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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The inverse cosine function: $y = \cos^{-1} x$

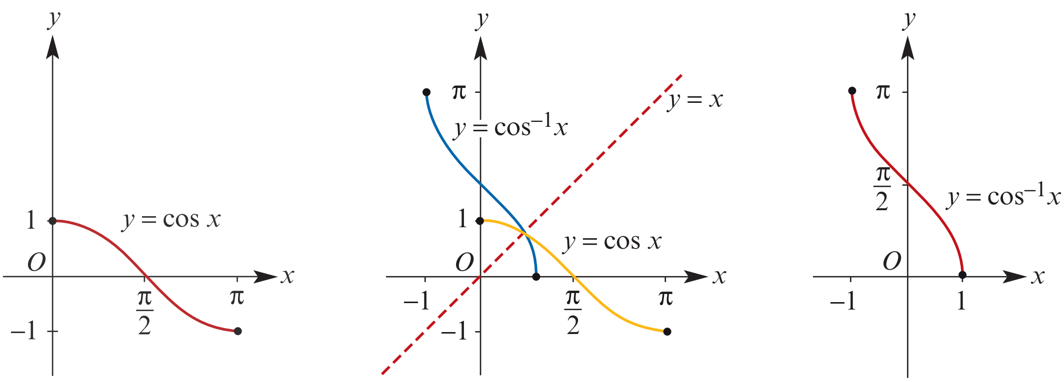
The standard domain for the restricted cosine function is $[0, \pi]$.

The restricted cosine function is one-to-one, and its inverse is denoted by $\cos^{-1}$ or arccos.

Inverse cosine function

$\cos^{-1} : [-1, 1] \to \mathbb{R}$, $\cos^{-1} x = y$, where $\cos y = x$ and $y \in [0, \pi]$

The graph of $y = \cos^{-1} x$ is obtained from the graph of $y = \cos x$, $x \in [0, \pi]$, through a reflection in the line $y = x$.



- **Domain** Domain of $\cos^{-1}$ = range of restricted cosine function = $[-1, 1]$
- **Range** Range of $\cos^{-1}$ = domain of restricted cosine function = $[0, \pi]$
- **Inverse relationship**
  - $\cos(\cos^{-1} x) = x$ for all $x \in [-1, 1]$
  - $\cos^{-1}(\cos x) = x$ for all $x \in [0, \pi]$

The inverse tangent function: $y = \tan^{-1} x$

The domain of the restricted tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The restricted tangent function is one-to-one, and its inverse is denoted by $\tan^{-1}$ or arctan.

Inverse tangent function

$\tan^{-1} : \mathbb{R} \to \mathbb{R}$, $\tan^{-1} x = y$, where $\tan y = x$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The graph of $y = \tan^{-1} x$ is obtained from the graph of $y = \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, through a reflection in the line $y = x$.

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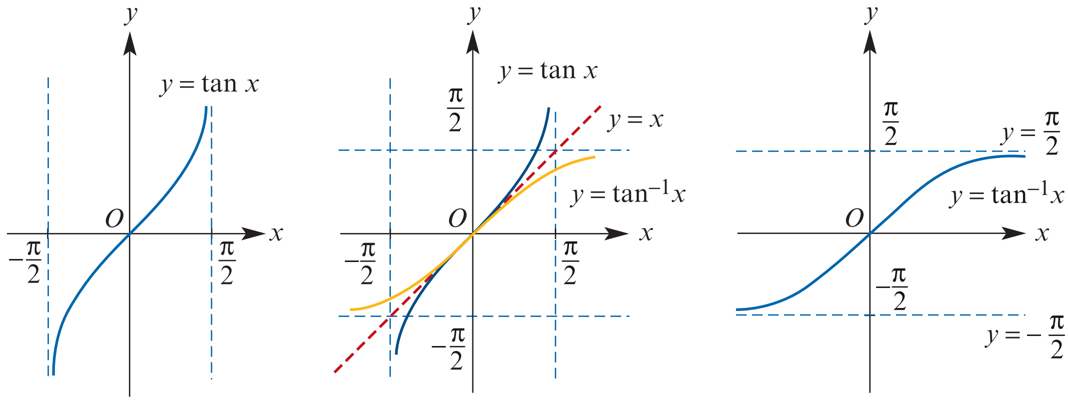


- **Domain** Domain of $\tan^{-1}$ = range of restricted tangent function = $\mathbb{R}$
- **Range** Range of $\tan^{-1}$ = domain of restricted tangent function = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- **Inverse relationship**
  - $\tan(\tan^{-1} x) = x$ for all $x \in \mathbb{R}$
  - $\tan^{-1}(\tan x) = x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 10

Sketch the graph of each of the following functions for the maximal domain:

a $y = \cos^{-1}(2 - 3x)$ **b** $y = \tan^{-1}(x + 2) + \frac{\pi}{2}$

Solution

a $\cos^{-1}(2 - 3x)$ is defined $\Leftrightarrow -1 \le 2 - 3x \le 1$

$\Leftrightarrow -3 \le -3x \le -1$

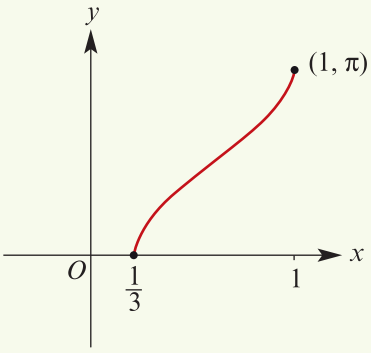
$\Leftrightarrow \frac{1}{3} \le x \le 1$

The implied domain is $\left[\frac{1}{3}, 1\right]$.

We can write $y = \cos^{-1}\left(-3\left(x - \frac{2}{3}\right)\right)$.

The graph is obtained from the graph of $y = \cos^{-1} x$ by the following sequence of transformations:

- a dilation of factor $\frac{1}{3}$ from the y-axis
- a reflection in the y-axis
- a translation of $\frac{2}{3}$ units in the positive direction of the x-axis.



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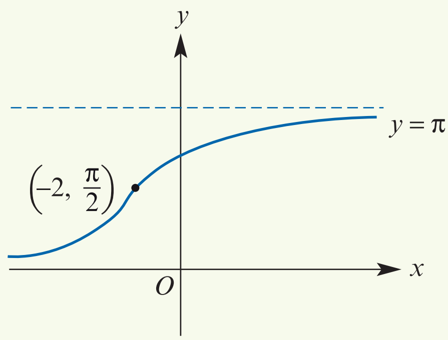
b The domain of $\tan^{-1}$ is $\mathbb{R}$.

The graph of

$$y = \tan^{-1}(x+2) + \frac{\pi}{2}$$

is obtained from the graph of $y = \tan^{-1} x$ by a translation of 2 units in the negative direction of the x-axis and $\frac{\pi}{2}$ units in the positive direction of the y-axis.



Example 11

a Evaluate $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

b Simplify:

i $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$

ii $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

iii $\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$

iv $\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$

Solution

a Evaluating $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is equivalent to solving $\sin y = -\frac{\sqrt{3}}{2}$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

b **i** Since $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by definition we have

$$\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$$

ii
$$\begin{aligned}\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) &= \sin^{-1}\left(\sin\left(\pi - \frac{5\pi}{6}\right)\right)\\ &= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)\\ &= \frac{\pi}{6}\end{aligned}$$

iii
$$\begin{aligned}\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) &= \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right)\\ &= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)\\ &= \frac{\pi}{6}\end{aligned}$$

iv
$$\begin{aligned}\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) &= \sin\left(\frac{\pi}{4}\right)\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

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Example 12

Find the implied domain and range of:

a $y = \sin^{-1}(2x - 1)$

b $y = 3\cos^{-1}(2 - 2x)$

Solution

a For $\sin^{-1}(2x - 1)$ to be defined:

$$\begin{aligned} -1 &\le 2x - 1 \le 1 \\ \Leftrightarrow \quad 0 &\le 2x \le 2 \\ \Leftrightarrow \quad 0 &\le x \le 1 \end{aligned}$$

Thus the implied domain is $[0, 1]$.

The range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b For $3\cos^{-1}(2 - 2x)$ to be defined:

$$\begin{aligned} -1 &\le 2 - 2x \le 1 \\ \Leftrightarrow \quad -3 &\le -2x \le -1 \\ \Leftrightarrow \quad \frac{1}{2} &\le x \le \frac{3}{2} \end{aligned}$$

Thus the implied domain is $\left[\frac{1}{2}, \frac{3}{2}\right]$.

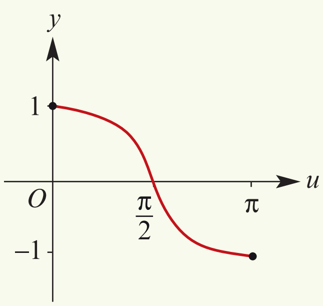
The range is $[0, 3\pi]$.

Example 13

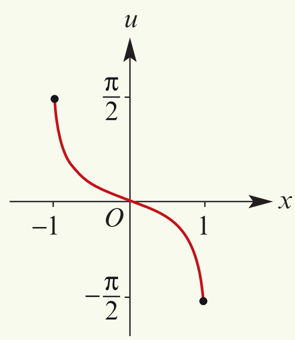
Find the implied domain and range of $y = \cos(-\sin^{-1} x)$, where cos has the restricted domain $[0, \pi]$.

Solution

Let $y = \cos u$, $u \in [0, \pi]$.



Let $u = -\sin^{-1} x$.



From the graphs, it can be seen that the function $u = -\sin^{-1} x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

But for $y = \cos u$ to be defined, the value of u must belong to the domain of $y = \cos u$, which is $[0, \pi]$. Hence the values of u must belong to the interval $\left[0, \frac{\pi}{2}\right]$.

$$\begin{aligned} 0 \le u \le \frac{\pi}{2} \quad &\Leftrightarrow \quad 0 \le -\sin^{-1} x \le \frac{\pi}{2} \qquad (\text{since } u = -\sin^{-1} x) \\ &\Leftrightarrow \quad -\frac{\pi}{2} \le \sin^{-1} x \le 0 \\ &\Leftrightarrow \quad -1 \le x \le 0 \end{aligned}$$

Hence the domain of $y = \cos(-\sin^{-1} x)$ is $[-1, 0]$. The range is $[0, 1]$.

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Skill-sheet

Exercise 3C

Example 10

1 Sketch the graphs of the following functions, stating clearly the implied domain and the range of each:

a $y = \tan^{-1}(x - 1)$ **b** $y = \cos^{-1}(x + 1)$ **c** $y = 2\sin^{-1}\left(x + \frac{1}{2}\right)$

d $y = 2\tan^{-1}(x) + \frac{\pi}{2}$ **e** $y = \cos^{-1}(2x)$ **f** $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$

Example 11a

2 Evaluate each of the following:

a $\arcsin 1$ **b** $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$ **c** $\arcsin 0.5$

d $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **e** $\cos^{-1} 0.5$ **f** $\tan^{-1} 1$

g $\tan^{-1}(-\sqrt{3})$ **h** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **i** $\cos^{-1}(-1)$

Example 11b

3 Simplify:

a $\sin(\cos^{-1} 0.5)$ **b** $\sin^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right)$ **c** $\tan\left(\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$

d $\cos(\tan^{-1} 1)$ **e** $\tan^{-1}\left(\sin\left(\frac{5\pi}{2}\right)\right)$ **f** $\tan(\cos^{-1} 0.5)$

g $\cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)$ **h** $\sin^{-1}\left(\sin\left(-\frac{2\pi}{3}\right)\right)$ **i** $\tan^{-1}\left(\tan\left(\frac{11\pi}{4}\right)\right)$

j $\cos^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$ **k** $\cos^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$ **l** $\sin^{-1}\left(\cos\left(-\frac{3\pi}{4}\right)\right)$

4 Let $f : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}$, $f(x) = \sin x$.

a Define f^{-1}, clearly stating its domain and its range.

b Evaluate:

i $f\left(\frac{\pi}{2}\right)$ **ii** $f\left(\frac{3\pi}{4}\right)$ **iii** $f\left(\frac{7\pi}{6}\right)$

iv $f^{-1}(-1)$ **v** $f^{-1}(0)$ **vi** $f^{-1}(0.5)$

Example 12

5 Given that the domains of sin, cos and tan are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $[0, \pi]$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ respectively, give the implied domain and range of each of the following:

a $y = \sin^{-1}(2 - x)$ **b** $y = \sin\left(x + \frac{\pi}{4}\right)$ **c** $y = \sin^{-1}(2x + 4)$

d $y = \sin\left(3x - \frac{\pi}{3}\right)$ **e** $y = \cos\left(x - \frac{\pi}{6}\right)$ **f** $y = \cos^{-1}(x + 1)$

g $y = \cos^{-1}(x^2)$ **h** $y = \cos\left(2x + \frac{2\pi}{3}\right)$ **i** $y = \tan^{-1}(x^2)$

j $y = \tan\left(2x - \frac{\pi}{2}\right)$ **k** $y = \tan^{-1}(2x + 1)$ **l** $y = \tan(x^2)$

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6 Simplify each of the following expressions, in an exact form:

a $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$ **b** $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$ **c** $\cos\left(\tan^{-1}\left(\frac{7}{24}\right)\right)$

d $\tan\left(\sin^{-1}\left(\frac{40}{41}\right)\right)$ **e** $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$ **f** $\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$

g $\sin(\tan^{-1}(-2))$ **h** $\cos\left(\sin^{-1}\left(\frac{3}{7}\right)\right)$ **i** $\sin(\tan^{-1} 0.7)$

7 Let $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{5}{13}$, where $\alpha \in \left[0, \frac{\pi}{2}\right]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$.

a Find:

i $\cos\alpha$ **ii** $\cos\beta$

b Use a compound angle formula to show that:

i $\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) = \sin^{-1}\left(\frac{16}{65}\right)$

ii $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Example 13

8 Given that the domains of sin and cos are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively, give the implied domain and range of each of the following:

a $y = \sin^{-1}(\cos x)$ **b** $y = \cos(\sin^{-1} x)$

c $y = \cos^{-1}(\sin(2x))$ **d** $y = \sin(-\cos^{-1} x)$

e $y = \cos(2\sin^{-1} x)$ **f** $y = \tan^{-1}(\cos x)$

g $y = \cos(\tan^{-1} x)$ **h** $y = \sin(\tan^{-1} x)$

9 **a** Use a compound angle formula to show that $\tan^{-1}(3) - \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{4}$.

b Hence show that $\tan^{-1} x - \tan^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{4}$ for $x > -1$.

10 Given that the domains of sin and cos are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively, explain why each expression cannot be evaluated:

a $\cos(\arcsin(-0.5))$ **b** $\sin(\cos^{-1}(-0.2))$ **c** $\cos(\tan^{-1}(-1))$

11 Consider the function $f(x) = \sin(\cos^{-1} x)$.

a State the maximal domain of f.

b Explain why $f(x) \geq 0$ for all x in the domain of f.

c By using the identity $\cos^2 x + \sin^2 x = 1$, show that $f(x) = \sqrt{1 - x^2}$.

12 Consider the function $f(x) = \sin^{-1}(\sin x)$.

a State the maximal domain and range of f.

b Sketch the graph of $y = f(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

c Sketch the graph of $y = f(x)$ for $-\pi \leq x \leq 2\pi$.

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3D Solution of equations

In Section 1A, we looked at the solution of equations involving sine, cosine and tangent. In this section, we introduce equations involving the reciprocal circular functions and the use of the double angle formulas. We also consider equations that are not able to be solved by analytic methods.

Example 14

Solve the equation $\sec x = 2$ for $x \in [0, 2\pi]$.

Solution

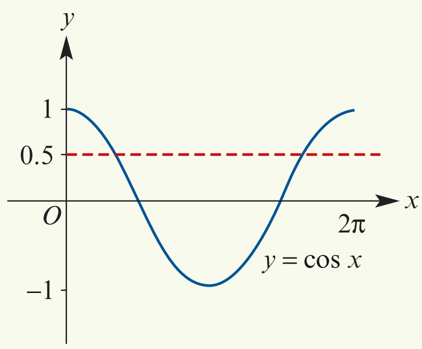
$$\sec x = 2$$

$$\therefore \quad \cos x = \frac{1}{2}$$

We are looking for solutions in $[0, 2\pi]$:

$$x = \frac{\pi}{3} \quad \text{or} \quad x = 2\pi - \frac{\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$



Example 15

Solve the equation $\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = -\frac{2\sqrt{3}}{3}$ for $x \in [0, 2\pi]$.

Solution

$$\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = -\frac{2\sqrt{3}}{3}$$

implies $\quad \sin\left(2x - \frac{\pi}{3}\right) = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$

Let $\theta = 2x - \frac{\pi}{3}$ where $\theta \in \left[-\frac{\pi}{3}, \frac{11\pi}{3}\right]$.

Then $\quad \sin\theta = -\frac{\sqrt{3}}{2}$

$$\therefore \quad \theta = -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad 2x - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad 2x = 0, \frac{5\pi}{3}, 2\pi, \frac{11\pi}{3} \text{ or } 4\pi$$

$$\therefore \quad x = 0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6} \text{ or } 2\pi$$

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General solution of trigonometric equations

We recall the following from Mathematical Methods Units 3 & 4.

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

Example 16

a Find all the values of x for which $\cot x = -1$.

b Find all the values of x for which $\sec\left(2x - \frac{\pi}{3}\right) = 2$.

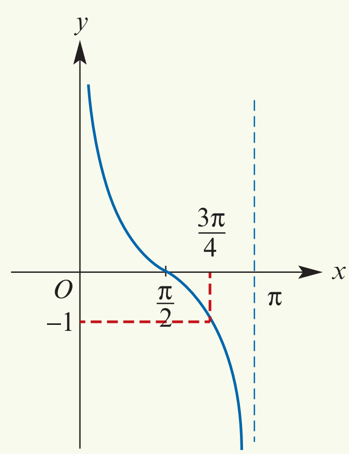
Solution

a The period of the function $y = \cot x$ is π.

The solution of $\cot x = -1$ in $[0, \pi]$ is $x = \frac{3\pi}{4}$.

Therefore the solutions of the equation are

$$x = \frac{3\pi}{4} + n\pi \quad \text{where } n \in \mathbb{Z}$$



b First write the equation as

$$\cos\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$$

We now proceed as usual to find the general solution:

$$2x - \frac{\pi}{3} = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$$

$$2x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$$

$$2x - \frac{\pi}{3} = 2n\pi + \frac{\pi}{3} \quad \text{or} \quad 2x - \frac{\pi}{3} = 2n\pi - \frac{\pi}{3}$$

$$2x = 2n\pi + \frac{2\pi}{3} \quad \text{or} \quad 2x = 2n\pi$$

$$\therefore \quad x = n\pi + \frac{\pi}{3} \quad \text{or} \quad x = n\pi \quad \text{where } n \in \mathbb{Z}$$

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Using identities to solve equations

The double angle formulas can be used to help solve trigonometric equations.

Example 17

Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sin(4x) = \sin(2x)$

b $\cos x = \sin\left(\frac{x}{2}\right)$

Solution

a

$$\sin(4x) = \sin(2x)$$

$$2\sin(2x)\cos(2x) = \sin(2x)$$

$$\sin(2x)(2\cos(2x) - 1) = 0 \qquad \text{where } 2x \in [0, 4\pi]$$

Thus $\sin(2x) = 0$ or $2\cos(2x) - 1 = 0$

i.e. $\sin(2x) = 0$ or $\cos(2x) = \frac{1}{2}$

$\therefore$ $2x = 0, \pi, 2\pi, 3\pi, 4\pi$ or $2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ or $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Hence $x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ or 2π.

b

$$\cos x = \sin\left(\frac{x}{2}\right)$$

$$1 - 2\sin^2\left(\frac{x}{2}\right) = \sin\left(\frac{x}{2}\right)$$

$$2\sin^2\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) - 1 = 0 \qquad \text{where } \frac{x}{2} \in [0, \pi]$$

Let $a = \sin\left(\frac{x}{2}\right)$. Then $a \in [0, 1]$. We have

$$2a^2 + a - 1 = 0$$

$\therefore$ $(2a - 1)(a + 1) = 0$

$\therefore$ $2a - 1 = 0$ or $a + 1 = 0$

$\therefore$ $a = \frac{1}{2}$ or $a = -1$

Thus $a = \frac{1}{2}$, since $a \in [0, 1]$. We now have

$$\sin\left(\frac{x}{2}\right) = \frac{1}{2}$$

$\therefore$ $\frac{x}{2} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\therefore$ $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

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Maximum and minimum values

We know that $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$. This can be used to find the maximum and minimum values of trigonometric functions without using calculus.

For example:

- The function $y = 2\sin x + 3$ has a maximum value of 5 and a minimum value of 1. The maximum value occurs when $\sin x = 1$ and the minimum value occurs when $\sin x = -1$.
- The function $y = \dfrac{1}{2\sin x + 3}$ has a maximum value of 1 and a minimum value of $\dfrac{1}{5}$.

Example 18

Find the maximum and minimum values of:

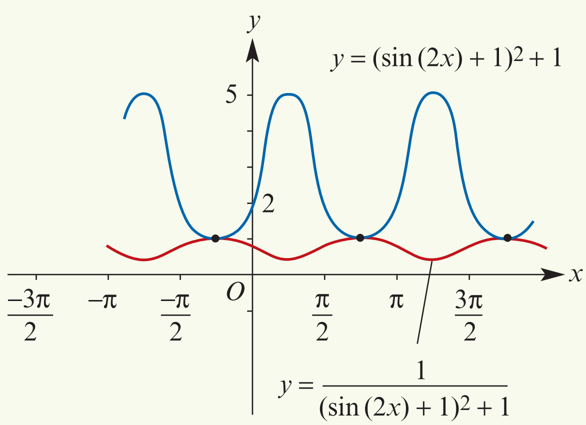
a $\sin^2(2x) + 2\sin(2x) + 2$ **b** $\dfrac{1}{\sin^2(2x) + 2\sin(2x) + 2}$

Solution

a Let $a = \sin(2x)$. Then

$$\begin{aligned}&\sin^2(2x) + 2\sin(2x) + 2\\ &= a^2 + 2a + 2\\ &= (a+1)^2 + 1\\ &= (\sin(2x)+1)^2 + 1\end{aligned}$$

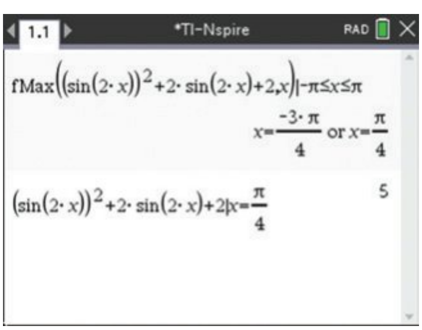
Now $-1 \le \sin(2x) \le 1$.
Therefore the maximum value is 5 and the minimum value is 1.



b Note that $\sin^2(2x) + 2\sin(2x) + 2 > 0$ for all x. Thus its reciprocal also has this property.
A local maximum for the original function yields a local minimum for the reciprocal.
A local minimum for the original function yields a local maximum for the reciprocal.
Hence the maximum value is 1 and the minimum value is $\dfrac{1}{5}$.

Using the TI-Nspire

- To find the x-values for which the maximum occurs, use menu > **Calculus** > **Function Maximum**. The restriction is chosen to give particular solutions.
- Use one of these x-values to find the maximum value of the expression.
- Similarly, to find the x-values for which the minimum occurs, use menu > **Calculus** > **Function Minimum**.



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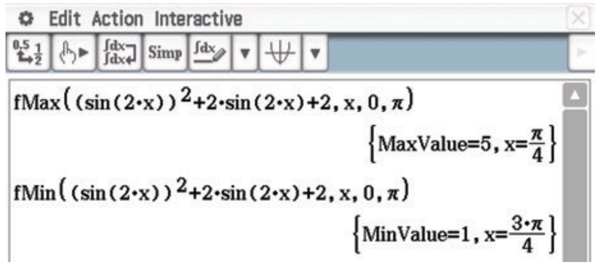
Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, enter and highlight $(\sin(2x))^2 + 2\sin(2x) + 2$.
- To find the maximum value, select **Interactive** > **Calculation** > **fMax**.
- Enter the domain: start at 0; end at π.



Note: The minimum value can be found similarly by choosing **fMin**.

Using a CAS calculator to obtain approximate solutions

Many equations involving the circular functions cannot be solved using analytic techniques. A CAS calculator can be used to solve such equations numerically.

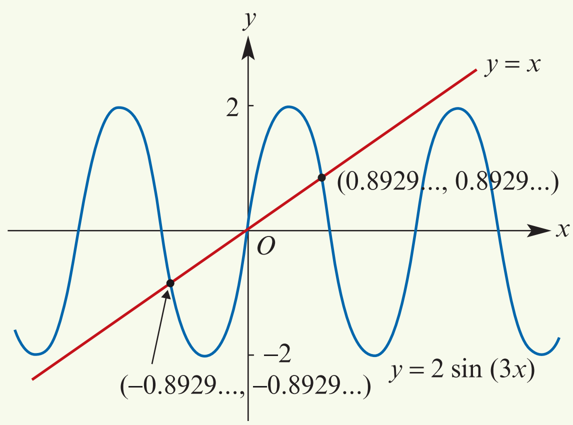
Example 19

Find the solutions of the equation $2\sin(3x) = x$, correct to three decimal places.

Solution

The graphs of $y = 2\sin(3x)$ and $y = x$ are plotted using a CAS calculator.

The solutions are $x = 0$, $x \approx 0.893$ and $x \approx -0.893$.



Skill-sheet

Exercise 3D

Example 14
Example 15

1 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\operatorname{cosec} x = -2$ **b** $\operatorname{cosec}\left(x - \frac{\pi}{4}\right) = -2$ **c** $3\sec x = 2\sqrt{3}$

d $\operatorname{cosec}(2x) + 1 = 2$ **e** $\cot x = -\sqrt{3}$ **f** $\cot\left(2x - \frac{\pi}{3}\right) = -1$

2 Solve each of the following equations, giving solutions in the interval $[0, 2\pi]$:

a $\sin x = 0.5$ **b** $\cos x = -\frac{\sqrt{3}}{2}$ **c** $\tan x = \sqrt{3}$

d $\cot x = -1$ **e** $\sec x = -2$ **f** $\operatorname{cosec} x = -\sqrt{2}$

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Example 16

3 Find all the solutions to each of the following equations:

a $\sin x = \frac{1}{\sqrt{2}}$ **b** $\sec x = 1$ **c** $\cot x = \sqrt{3}$

d $\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = 2$ **e** $\operatorname{cosec}\left(3x - \frac{\pi}{3}\right) = \frac{2\sqrt{3}}{3}$ **f** $\sec\left(3x - \frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$

g $\cot\left(2x - \frac{\pi}{6}\right) = \sqrt{3}$ **h** $\cot\left(2x - \frac{\pi}{4}\right) = -1$ **i** $\operatorname{cosec}\left(2x - \frac{\pi}{4}\right) = 1$

4 Solve each of the following in the interval $[-\pi, \pi]$, giving the answers correct to two decimal places:

a $\sec x = 2.5$ **b** $\operatorname{cosec} x = -5$ **c** $\cot x = 0.6$

Example 17

5 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\cos^2 x - \cos x \sin x = 0$ **b** $\sin(2x) = \sin x$

c $\sin(2x) = \cos x$ **d** $\sin(8x) = \cos(4x)$

e $\cos(2x) = \cos x$ **f** $\cos(2x) = \sin x$

g $\sec^2 x + \tan x = 1$ **h** $\tan x\,(1 + \cot x) = 0$

i $\cot x + 3\tan x = 5\operatorname{cosec} x$ **j** $\sin x + \cos x = 1$

Example 18

6 Find the maximum and minimum values of each of the following:

a $2 + \sin\theta$ **b** $\frac{1}{2 + \sin\theta}$ **c** $\sin^2\theta + 4$

d $\frac{1}{\sin^2\theta + 4}$ **e** $\cos^2\theta + 2\cos\theta$ **f** $\cos^2\theta + 2\cos\theta + 6$

Example 19

7 Using a CAS calculator, find the coordinates of the points of intersection for the graphs of the following pairs of functions. (Give values correct to two decimal places.)

a $y = 2x$ and $y = 3\sin(2x)$ **b** $y = x$ and $y = 2\sin(2x)$

c $y = 3 - x$ and $y = \cos x$ **d** $y = x$ and $y = \tan x,\ x \in [0, 2\pi]$

8 Let $a \in [-1, 1]$ with $a \neq -1$. Consider the equation $\cos x = a$ for $x \in [0, 2\pi]$. If q is one of the solutions, find the second solution in terms of q.

9 Let $\sin\alpha = a$ where $\alpha \in \left(0, \frac{\pi}{2}\right)$. Find, in terms of α, two values of x in $[0, 2\pi]$ which satisfy each of the following equations:

a $\sin x = -a$ **b** $\cos x = a$

10 Let $\sec\beta = b$ where $\beta \in \left(\frac{\pi}{2}, \pi\right)$. Find, in terms of β, two values of x in $[-\pi, \pi]$ which satisfy each of the following equations:

a $\sec x = -b$ **b** $\operatorname{cosec} x = b$

11 Let $\tan\gamma = c$ where $\gamma \in \left(\pi, \frac{3\pi}{2}\right)$. Find, in terms of γ, two values of x in $[0, 2\pi]$ which satisfy each of the following equations:

a $\tan x = -c$ **b** $\cot x = c$

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12 Solve, correct to two decimal places, the equation $\sin^2\theta = \dfrac{\theta}{\pi}$ for $\theta \in [0, \pi]$.

13 Find the value of x, correct to two decimal places, such that $\tan^{-1} x = 4x - 5$.

14 A curve on a light rail track is an arc of a circle of length 300 m and the straight line joining the two ends of the curve is 270 m long.

a Show that, if the arc subtends an angle of $2\theta°$ at the centre of the circle, then θ is a solution of the equation $\sin\theta° = \dfrac{\pi}{200}\theta$.

b Solve this equation for θ, correct to two decimal places.

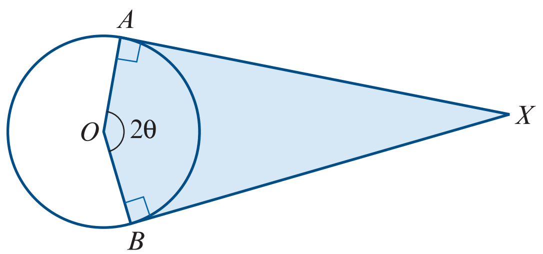
15 Solve, correct to two decimal places, the equation $\tan x = \dfrac{1}{x}$ for $x \in [0, \pi]$.

16 The area of a segment of a circle is given by the equation $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is the angle subtended at the centre of the circle. If the radius is 6 cm and the area of the segment is 18 cm^2, find the value of θ correct to two decimal places.

17 Two tangents are drawn from a point so that the area of the shaded region is equal to the area of the remaining region of the circle.

a Show that θ satisfies the equation $\tan\theta = \pi - \theta$.

b Solve for θ, giving the answer correct to three decimal places.



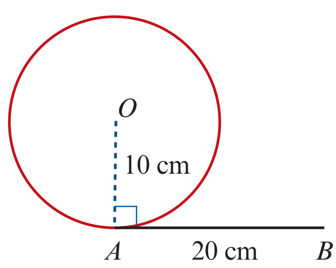
$\angle AOB = 2\theta$

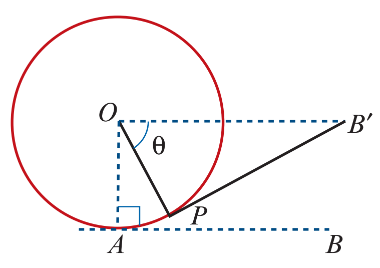
18 Two particles A and B move in a straight line. At time t, their positions relative to a point O are given by

$$x_A = 0.5\sin t \quad \text{and} \quad x_B = 0.25t^2 + 0.05t$$

Find the times at which their positions are the same, and give this position. (Distances are measured in centimetres and time in seconds.)

19 A string is wound around a disc and a horizontal length of the string AB is 20 cm long. The radius of the disc is 10 cm. The string is then moved so that the end of the string, B', is moved to a point at the same level as O, the centre of the circle. The line $B'P$ is a tangent to the circle.





a Show that θ satisfies the equation $\dfrac{\pi}{2} - \theta + \tan\theta = 2$.

b Find the value of θ, correct to two decimal places, which satisfies this equation.

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3E Sums and products of sines and cosines

In Section 3B, we derived the compound angle formulas for sine and cosine. We use them in this section to obtain new identities which allow us to rewrite products of sines and cosines as sums or differences, and vice versa.

Expressing products as sums or differences

Product-to-sum identities

$$2\cos x\cos y = \cos(x-y)+\cos(x+y)$$
$$2\sin x\sin y = \cos(x-y)-\cos(x+y)$$
$$2\sin x\cos y = \sin(x+y)+\sin(x-y)$$

Proof We use the compound angle formulas for sine and cosine:

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \quad (1)$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y \quad (2)$$
$$\sin(x+y) = \sin x\cos y + \cos x\sin y \quad (3)$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y \quad (4)$$

The first product-to-sum identity is obtained by adding (2) and (1), the second identity is obtained by subtracting (1) from (2), and the third by adding (3) and (4).

Example 20

Express each of the following products as sums or differences:

a $2\sin(3\theta)\cos(\theta)$

b $2\sin 50^\circ\cos 60^\circ$

c $2\cos\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)$

Solution

a Use the third product-to-sum identity:

$$\begin{aligned}2\sin(3\theta)\cos(\theta) &= \sin(3\theta+\theta)+\sin(3\theta-\theta)\\ &= \sin(4\theta)+\sin(2\theta)\end{aligned}$$

b Use the third product-to-sum identity:

$$\begin{aligned}2\sin 50^\circ\cos 60^\circ &= \sin 110^\circ+\sin(-10)^\circ\\ &= \sin 110^\circ-\sin 10^\circ\end{aligned}$$

c Use the first product-to-sum identity:

$$\begin{aligned}2\cos\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2}\right)+\cos(2\theta)\\ &= \cos(2\theta)\end{aligned}$$

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Expressing sums and differences as products

Sum-to-product identities

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

Proof Using the first product-to-sum identity, we have

$$\begin{aligned} 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) &= \cos\left(\frac{x+y}{2} - \frac{x-y}{2}\right) + \cos\left(\frac{x+y}{2} + \frac{x-y}{2}\right) \\ &= \cos y + \cos x \\ &= \cos x + \cos y \end{aligned}$$

The other three sum-to-product identities can be obtained similarly.

Example 21

Express each of the following as products:

a $\sin 36^\circ + \sin 10^\circ$

b $\cos 36^\circ + \cos 10^\circ$

c $\sin 36^\circ - \sin 10^\circ$

d $\cos 36^\circ - \cos 10^\circ$

Solution

a $\sin 36^\circ + \sin 10^\circ = 2\sin 23^\circ \cos 13^\circ$

b $\cos 36^\circ + \cos 10^\circ = 2\cos 23^\circ \cos 13^\circ$

c $\sin 36^\circ - \sin 10^\circ = 2\cos 23^\circ \sin 13^\circ$

d $\cos 36^\circ - \cos 10^\circ = -2\sin 23^\circ \sin 13^\circ$

Example 22

Prove that

$$\frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} = \tan(2\theta)$$

Solution

$$\begin{aligned} \text{LHS} &= \frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} \\ &= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \frac{2\sin(2\theta)\sin(\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \tan(2\theta) \\ &= \text{RHS} \end{aligned}$$

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Example 23

Solve the equation $\sin(3x) + \sin(11x) = 0$ for $x \in [0, \pi]$.

Solution

$$\begin{aligned}
& \sin(3x) + \sin(11x) = 0 \\
\Leftrightarrow \quad & 2\sin(7x)\cos(4x) = 0 \\
\Leftrightarrow \quad & \sin(7x) = 0 \quad \text{or} \quad \cos(4x) = 0 \\
\Leftrightarrow \quad & 7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi \quad \text{or} \quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\
\Leftrightarrow \quad & x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8} \text{ or } \frac{7\pi}{8}
\end{aligned}$$

Skill-sheet

Exercise 3E

Example 20

1 Express each of the following products as sums or differences:

a $2\sin(4\pi t)\cos(7\pi t)$

b $\sin 50° \cos 10°$

c $3\cos\left(\frac{\pi x}{3}\right)\sin\left(\frac{2\pi x}{3}\right)$

d $2\sin\left(\frac{A+B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)$

e $2\sin(x)\sin\left(\frac{3x}{2}\right)$

f $2\cos\left(\frac{\pi x}{4}\right)\cos\left(\frac{3\pi x}{4}\right)$

2 Express $2\sin(4\theta)\sin(\theta)$ as a difference of cosines.

3 Use a product-to-sum identity to derive the expression for $2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$ as a difference of sines.

4 Show that $\cos 75° \cos 15° = \frac{1}{4}$.

Example 21

5 Express each of the following as products:

a $\sin 66° + \sin 34°$

b $\cos 66° + \cos 34°$

c $\sin 66° - \sin 34°$

d $\cos 66° - \cos 34°$

6 Express each of the following as products:

a $\sin(8A) + \sin(2A)$

b $\cos(x) + \cos(4x)$

c $\sin(6x) - \sin(4x)$

d $\cos(5A) - \cos(3A)$

Example 22

7 Show that $\sin(A) + 2\sin(3A) + \sin(5A) = 4\cos^2(A)\sin(3A)$.

8 For any three angles α, β and γ, show that

$$\sin(\alpha+\beta)\sin(\alpha-\beta) + \sin(\beta+\gamma)\sin(\beta-\gamma) + \sin(\gamma+\alpha)\sin(\gamma-\alpha) = 0$$

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9 Show that $\cos 70° + \sin 40° = \cos 10°$.

10 Show that $\cos 20° + \cos 100° + \cos 140° = 0$.

Example 23 **11** Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\cos(5x) + \cos(x) = 0$

b $\cos(5x) - \cos(x) = 0$

c $\sin(5x) + \sin(x) = 0$

d $\sin(5x) - \sin(x) = 0$

12 Solve the equation $\sin(3x) + \sin(x) + \cos(4x) = 1$ for $0 \le x \le \pi$.

13 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\cos(2\theta) - \sin(\theta) = 0$

b $\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$

c $\sin(7\theta) - \sin(\theta) = \sin(3\theta)$

d $\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$

14 Evaluate each of the following sums:

a $\sin 1° + \sin 2° + \sin 3° + \cdots + \sin 358° + \sin 359°$

b $\cos 1° + \cos 2° + \cos 3° + \cdots + \cos 358° + \cos 359°$

15 **a** By first using the sum-to-product identities, prove that

$$\sin(\theta) + \sin(2\theta) + \cos(\theta) + \cos(2\theta) = 2\sqrt{2}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right)$$

b Hence, solve the equation $\sin(\theta) + \sin(2\theta) + \cos(\theta) + \cos(2\theta) = 0$ for $0 \le \theta \le 2\pi$.

16 Prove the following identity:

$$\frac{\sin(\theta) + \sin(3\theta) + \sin(5\theta)}{\cos(\theta) + \cos(3\theta) + \cos(5\theta)} = \tan(3\theta)$$

17 Given that $\dfrac{\sin(2A) - \cos(2B)}{\cos(2A) - \sin(2B)} = \dfrac{a}{b}$, prove that $\cot(A - B) = \dfrac{b - a}{b + a}$.

18 Let ABC be a triangle.

a Prove that if $2\cos B \sin C = \sin A$, then triangle ABC is isosceles.

b **i** Prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \cot\left(\dfrac{C}{2}\right)$.

ii Hence, prove that if $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \sin C$, then triangle ABC is right-angled at C.

19 Let ABC be a triangle. Prove that:

a $\sin A + \sin B + \sin C = 4\cos\left(\dfrac{A}{2}\right)\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)$

b $\sin(2A) + \sin(2B) + \sin(2C) = 4\sin A \sin B \sin C$

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Chapter summary

Assign-ment

Nrich

Reciprocal circular functions

- Definitions

$$\operatorname{cosec} x = \frac{1}{\sin x} \quad \text{provided } \sin x \neq 0$$

$$\sec x = \frac{1}{\cos x} \quad \text{provided } \cos x \neq 0$$

$$\cot x = \frac{\cos x}{\sin x} \quad \text{provided } \sin x \neq 0$$

- Symmetry properties

$$\sec(\pi - x) = -\sec x \qquad \operatorname{cosec}(\pi - x) = \operatorname{cosec} x \qquad \cot(\pi - x) = -\cot x$$

$$\sec(\pi + x) = -\sec x \qquad \operatorname{cosec}(\pi + x) = -\operatorname{cosec} x \qquad \cot(\pi + x) = \cot x$$

$$\sec(2\pi - x) = \sec x \qquad \operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x \qquad \cot(2\pi - x) = -\cot x$$

$$\sec(-x) = \sec x \qquad \operatorname{cosec}(-x) = -\operatorname{cosec} x \qquad \cot(-x) = -\cot x$$

- Complementary properties

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x \qquad \operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

- Pythagorean identity

$$\cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

Compound angle formulas

- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Double angle formulas

- $\cos(2x) = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$
- $\sin(2x) = 2 \sin x \cos x$
- $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$

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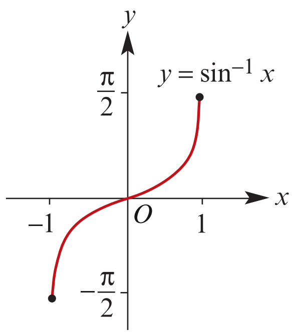
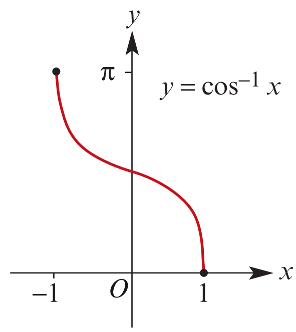
Review

Inverse circular functions

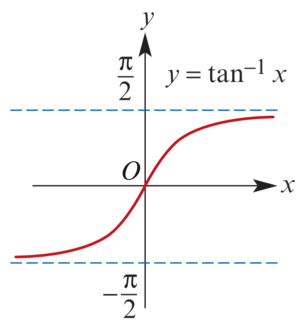
- **Inverse sine (arcsin)**
  $\sin^{-1}: [-1, 1] \to \mathbb{R},\ \sin^{-1} x = y,$
  where $\sin y = x$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



- **Inverse cosine (arccos)**
  $\cos^{-1}: [-1, 1] \to \mathbb{R},\ \cos^{-1} x = y,$
  where $\cos y = x$ and $y \in [0, \pi]$



- **Inverse tangent (arctan)**
  $\tan^{-1}: \mathbb{R} \to \mathbb{R},\ \tan^{-1} x = y,$
  where $\tan y = x$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Product-to-sum identities

- $2\cos x \cos y = \cos(x - y) + \cos(x + y)$
- $2\sin x \sin y = \cos(x - y) - \cos(x + y)$
- $2\sin x \cos y = \sin(x + y) + \sin(x - y)$

Sum-to-product identities

- $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
- $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

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Technology-free questions

1 If θ is an acute angle and $\cos\theta = \frac{4}{5}$, find:

a $\cos(2\theta)$ **b** $\sin(2\theta)$ **c** $\tan(2\theta)$ **d** $\operatorname{cosec}\theta$ **e** $\cot\theta$

2 Solve each of the following equations for $-\pi < x \le 2\pi$:

a $\sin(2x) = \sin x$ **b** $\cos x - 1 = \cos(2x)$ **c** $\sin(2x) = 2\cos x$

d $\sin^2 x \cos^3 x = \cos x$ **e** $\sin^2 x - \frac{1}{2}\sin x - \frac{1}{2} = 0$ **f** $2\cos^2 x - 3\cos x + 1 = 0$

3 Solve each of the following equations for $0 \le \theta \le 2\pi$, giving exact answers:

a $2 - \sin\theta = \cos^2\theta + 7\sin^2\theta$ **b** $\sec(2\theta) = 2$

c $\frac{1}{2}(5\cos\theta - 3\sin\theta) = \sin\theta$ **d** $\sec\theta = 2\cos\theta$

4 Find the exact value of each of the following:

a $\operatorname{cosec}\left(-\frac{5\pi}{3}\right)$ **b** $\sec\left(\frac{7\pi}{3}\right)$ **c** $\operatorname{cosec}\left(\frac{5\pi}{6}\right)$ **d** $\cot\left(-\frac{3\pi}{4}\right)$ **e** $\cot\left(-\frac{\pi}{6}\right)$

5 Given that $\tan\alpha = p$, where α is an acute angle, find each of the following in terms of p:

a $\tan(-\alpha)$ **b** $\tan(\pi - \alpha)$ **c** $\tan\left(\frac{\pi}{2} - \alpha\right)$ **d** $\tan\left(\frac{3\pi}{2} + \alpha\right)$ **e** $\tan(2\pi - \alpha)$

6 Consider the function $f(x) = \tan(\cos^{-1} x)$.

a State the maximal domain and range of f.

b Show that $f(x) = \dfrac{\sqrt{1 - x^2}}{x}$.

c Sketch the graph of f for its maximal domain.

7 Find:

a $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **b** $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$ **c** $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

d $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$ **e** $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **f** $\cos(\tan^{-1}(-1))$

8 Let a and b be real constants, with $a > 0$. State the maximal domain of each of the following functions:

a $f(x) = \sin^{-1}(\sqrt{ax + b})$ **b** $f(x) = \cos^{-1}\left(\frac{2}{ax}\right)$

c $f(x) = \sin^{-1}\left(\frac{ax}{2} - 2\right)$ **d** $f(x) = \cos^{-1}(\sqrt{2 - ax})$

9 Sketch the graph of each of the following functions, stating the maximal domain and range of each:

a $y = 2\tan^{-1} x$ **b** $y = \sin^{-1}(3 - x)$ **c** $y = 3\cos^{-1}(2x + 1)$

d $y = -\cos^{-1}(2 - x)$ **e** $y = 2\tan^{-1}(1 - x)$

10 Solve the equation $\sin(3x) = \sin(5x)$ for $0 \le x \le \pi$.

11 Prove the identity $\dfrac{\sin A + \sin B - \sin(A + B)}{\sin A + \sin B + \sin(A + B)} = \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)$.

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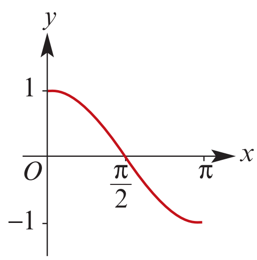
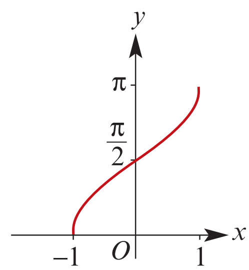
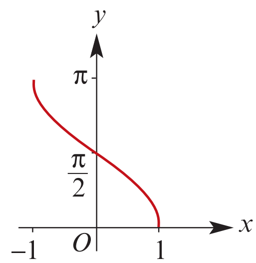
Review

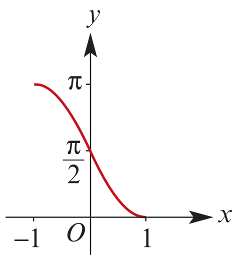
Multiple-choice questions

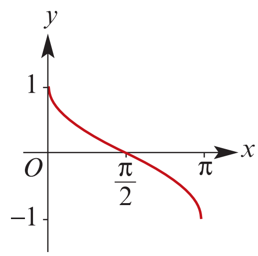
1 Which of the following is the graph of the function $y = \cos^{-1}(x)$?

A


B


C


D


E


2 If $\cos x = -\frac{2}{3}$ and $2\pi < x < 3\pi$, then the exact value of $\sin x$ is

A $2\pi + \frac{\sqrt{5}}{3}$ **B** $2\pi - \frac{\sqrt{5}}{3}$ **C** $\frac{\sqrt{5}}{3}$ **D** $-\frac{\sqrt{5}}{3}$ **E** $\frac{5}{9}$

3 Given that $\cos(x) = -\frac{1}{10}$ and $x \in \left(\frac{\pi}{2}, \pi\right)$, the value of $\cot(x)$ is

A $\frac{10}{3\sqrt{11}}$ **B** $3\sqrt{11}$ **C** $-3\sqrt{11}$ **D** $\frac{\sqrt{11}}{33}$ **E** $-\frac{\sqrt{11}}{33}$

4 The graph of the function $y = 2 + \sec(3x)$, for $x \in \left(-\frac{\pi}{6}, \frac{7\pi}{6}\right)$, has stationary points at

A $x = \frac{\pi}{3}, \pi$ **B** $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ **C** $x = \frac{\pi}{2}$

D $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$ **E** $x = 0, \frac{2\pi}{3}$

5 If $\sin x = -\frac{1}{3}$, then the possible values of $\cos x$ are

A $-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}$ **B** $-\frac{2}{3}, \frac{2}{3}$ **C** $-\frac{8}{9}, \frac{8}{9}$ **D** $-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$ **E** $-\frac{1}{2}, \frac{1}{2}$

6 The maximal domain of $y = \cos^{-1}(1 - 5x)$ is given by

A $\left[0, \frac{2}{5}\right]$ **B** $\left[\frac{1-\pi}{5}, \frac{1}{5}\right]$ **C** $[-1, 1]$ **D** $\left(0, \frac{2}{5}\right)$ **E** $\left[-\frac{1}{5}, \frac{1}{5}\right]$

7 $(1 + \tan x)^2 + (1 - \tan x)^2$ is equal to

A $2 + \tan x + 2\tan(2x)$ **B** 2 **C** $-4\tan x$ **D** $2 + \tan(2x)$ **E** $2\sec^2 x$

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8 The number of solutions of $\cos^2(3x) = \frac{1}{4}$, given that $0 \le x \le \pi$, is

A 1 **B** 2 **C** 3 **D** 6 **E** 9

9 $\dfrac{\tan(2\theta)}{1 + \sec(2\theta)}$ equals

A $\tan(2\theta)$ **B** $\tan(2\theta) + 1$ **C** $\tan\theta + 1$ **D** $\sin(2\theta)$ **E** $\tan\theta$

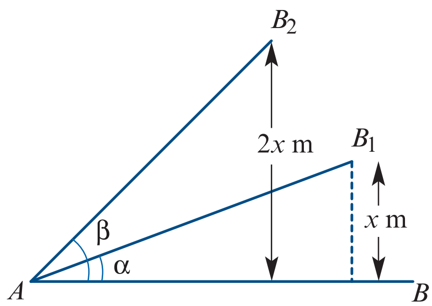
10 If $\sin A = t$ and $\cos B = t$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, then $\cos(B + A)$ is equal to

A 0 **B** $\sqrt{1 - t^2}$ **C** $2t^2 - 1$ **D** $1 - 2t^2$ **E** $-2t\sqrt{1 - t^2}$

Extended-response questions

1 A horizontal rod is 1 m long. One end is hinged at A, and the other end rests on a support B. The rod can be rotated about A, with the other end taking the two positions B_1 and B_2, which are x m and $2x$ m above the line AB respectively, where $x < 0.5$.

Let $\angle BAB_1 = \alpha$ and $\angle BAB_2 = \beta$.



a Find each of the following in terms of x:

i $\sin\alpha$ **ii** $\cos\alpha$ **iii** $\tan\alpha$ **iv** $\sin\beta$ **v** $\cos\beta$ **vi** $\tan\beta$

b Using the results of **a**, find:

i $\sin(\beta - \alpha)$ **ii** $\cos(\beta - \alpha)$ **iii** $\tan(\beta - \alpha)$
iv $\tan(2\alpha)$ **v** $\sin(2\alpha)$ **vi** $\cos(2\alpha)$

c If $x = 0.3$, find the magnitudes of $\angle B_2AB_1$ and 2α, correct to two decimal places.

2 **a** On the one set of axes, sketch the graphs of the following for $x \in (0, \pi) \cup (\pi, 2\pi)$:

i $y = \operatorname{cosec}(x)$ **ii** $y = \cot(x)$ **iii** $y = \operatorname{cosec}(x) - \cot(x)$

b **i** Show that $\operatorname{cosec} x - \cot x > 0$ for all $x \in (0, \pi)$, and hence that $\operatorname{cosec} x > \cot x$ for all $x \in (0, \pi)$.

ii Show that $\operatorname{cosec} x - \cot x < 0$ for all $x \in (\pi, 2\pi)$, and hence that $\operatorname{cosec} x < \cot x$ for all $x \in (\pi, 2\pi)$.

c On separate axes, sketch the graph of $y = \cot\left(\frac{x}{2}\right)$ for $x \in (0, 2\pi)$ and the graph of $y = \operatorname{cosec}(x) + \cot(x)$ for $x \in (0, 2\pi) \setminus \{\pi\}$.

d **i** Prove that $\operatorname{cosec}\theta + \cot\theta = \cot\left(\frac{\theta}{2}\right)$ where $\sin\theta \ne 0$.

ii Use this result to find $\cot\left(\frac{\pi}{8}\right)$ and $\cot\left(\frac{\pi}{12}\right)$.

iii Use the result $\cot^2\left(\frac{\pi}{8}\right) + 1 = \operatorname{cosec}^2\left(\frac{\pi}{8}\right)$ to find the exact value of $\sin\left(\frac{\pi}{8}\right)$.

e Use the result of **d** to show that $\operatorname{cosec}(\theta) + \operatorname{cosec}(2\theta) + \operatorname{cosec}(4\theta)$ can be expressed as the difference of two cotangents.

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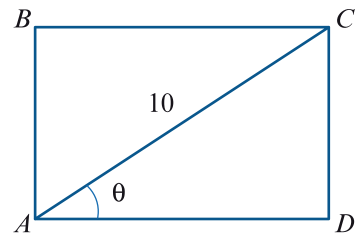
3 **a** $ABCD$ is a rectangle with diagonal AC of length 10 units.



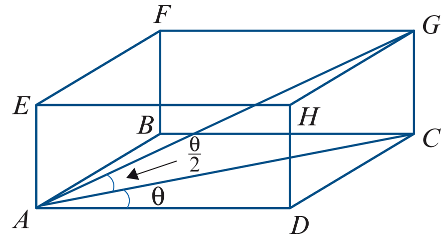
i Find the area of the rectangle in terms of θ.

ii Sketch the graph of R against θ, where R is the area of the rectangle in square units, for $\theta \in \left(0, \frac{\pi}{2}\right)$.

iii Find the maximum value of R. (Do not use calculus.)

iv Find the value of θ for which this maximum occurs.

b $ABCDEFGH$ is a cuboid with $\angle GAC = \frac{\theta}{2}$, $\angle CAD = \theta$ and $AC = 10$.



i Show that the volume, V, of the cuboid is given by

$V = 1000 \cos\theta \sin\theta \tan\left(\frac{\theta}{2}\right)$

ii Find the values of a and b such that $V = a\sin^2\left(\frac{\theta}{2}\right) + b\sin^4\left(\frac{\theta}{2}\right)$.

iii Let $p = \sin^2\left(\frac{\theta}{2}\right)$. Express V as a quadratic in p.

iv Find the possible values of p for $0 < \theta < \frac{\pi}{2}$.

v Sketch the graphs of V against θ and V against p with the help of a calculator.

vi Find the maximum volume of the cuboid and the values of p and θ for which this occurs. (Determine the maximum through the quadratic found in **b iii**.)

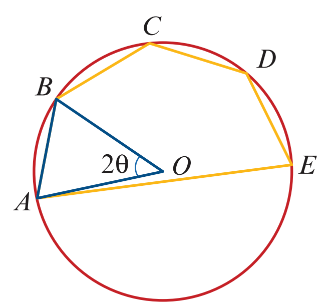
c Now assume that the cuboid satisfies $\angle CAD = \theta$, $\angle GAC = \theta$ and $AC = 10$.

i Find V in terms of θ.

ii Sketch the graph of V against θ.

iii Discuss the relationship between V and θ using the graph of **c ii**.

4 $ABCDE$ is a pentagon inscribed in a circle with $AB = BC = CD = DE = 1$ and $\angle BOA = 2\theta$. The centre of the circle is O.



a Let $p = AE$. Show that $p = \frac{\sin(4\theta)}{\sin\theta}$.

b Express p as a function of $\cos\theta$.

Let $x = \cos\theta$.

c **i** If $p = \sqrt{3}$, show that $8x^3 - 4x - \sqrt{3} = 0$.

ii Show that $\frac{\sqrt{3}}{2}$ is a solution to the equation and that it is the only real solution.

iii Find the value of θ for which $p = \sqrt{3}$.

iv Find the radius of the circle.

d Using a CAS calculator, sketch the graph of p against θ for $\theta \in \left(0, \frac{\pi}{4}\right]$.

e If $A = E$, find the value of θ.

f **i** If $AE = 1$, show that $8x^3 - 4x - 1 = 0$.

ii Hence show that $\frac{1}{4}(\sqrt{5} + 1) = \cos\left(\frac{\pi}{5}\right)$.

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5 **a** **i** Prove that $\tan x + \cot x = 2\operatorname{cosec}(2x)$ for $\sin(2x) \neq 0$.

ii Solve the equation $\tan x = \cot x$ for x.

iii On the one set of axes, sketch the graphs of $y = \tan x$, $y = \cot x$ and $y = 2\operatorname{cosec}(2x)$ for $x \in (0, 2\pi)$.

b **i** Prove that $\cot(2x) + \tan x = \operatorname{cosec}(2x)$ for $\sin(2x) \neq 0$.

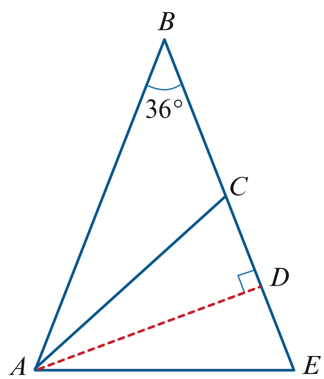
ii Solve the equation $\cot(2x) = \tan x$ for x.

iii On the one set of axes, sketch the graphs of $y = \cot(2x)$, $y = \tan x$ and $y = \operatorname{cosec}(2x)$ for $x \in (0, 2\pi)$.

c **i** Prove that $\cot(mx) + \tan(nx) = \dfrac{\cos((m-n)x)}{\sin(mx)\cos(nx)}$, for all $m, n \in \mathbb{Z}$.

ii Hence show that $\cot(6x) + \tan(3x) = \operatorname{cosec}(6x)$.

6 Triangle ABE is isosceles with $AB = BE$, and triangle ACE is isosceles with $AC = AE = 1$.



a **i** Find the magnitudes of $\angle BAE$, $\angle AEC$ and $\angle ACE$.

ii Hence find the magnitude of $\angle BAC$.

b Show that $BD = 1 + \sin 18°$.

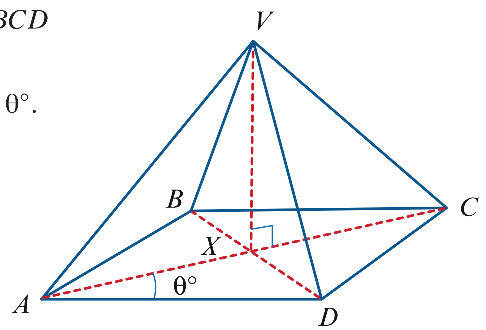
c Use triangle ABD to prove that

$$\cos 36° = \frac{1 + \sin 18°}{1 + 2\sin 18°}$$

d Hence show that $4\sin^2 18° + 2\sin 18° - 1 = 0$.

e Find $\sin 18°$ in exact form.

7 $VABCD$ is a right pyramid, where the base $ABCD$ is a rectangle with diagonal length $AC = 10$.



a First assume that $\angle CAD = \theta°$ and $\angle VAX = \theta°$.

i Show that the volume, V, of the pyramid is given by

$$V = \frac{500}{3}\sin^2(\theta°)$$

ii Sketch the graph of V against θ for $\theta \in (0, 90)$.

iii Comment on the graph.

b Now assume that $\angle CAD = \theta°$ and $\angle VAX = \dfrac{\theta°}{2}$.

i Show that the volume, V, of the pyramid is given by

$$V = \frac{1000}{3}\sin^2\left(\frac{\theta°}{2}\right)\left(1 - 2\sin^2\left(\frac{\theta°}{2}\right)\right)$$

ii State the maximal domain of the function $V(\theta)$.

iii Let $a = \sin^2\left(\dfrac{\theta°}{2}\right)$ and write V as a quadratic in a.

iv Hence find the maximum value of V and the value of θ for which this occurs.

v Sketch the graph of V against θ for the domain established in **b ii**.

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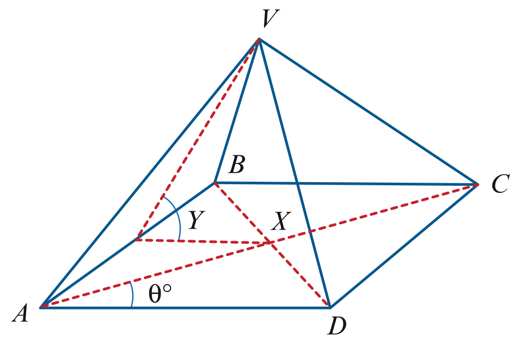
Review

8 $VABCD$ is a right pyramid, where the base $ABCD$ is a rectangle with diagonal length $AC = 10$.

Assume that $\angle CAD = \theta°$ and $AY = BY$.



a If $\angle VYX = \theta°$, find:

i an expression for the volume of the pyramid in terms of θ

ii the maximum volume and the value of θ for which this occurs.

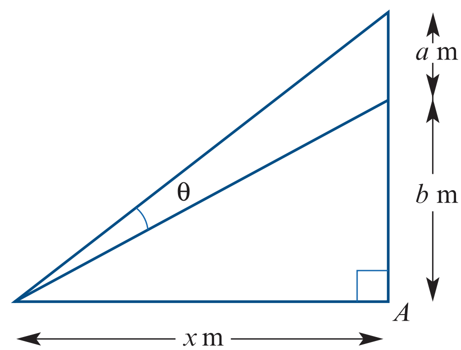
b If $\angle VYX = \dfrac{\theta°}{2}$:

i show that $V = \dfrac{500}{3}\cos^2(\theta°)\,(1 - \cos(\theta°))$

ii state the implied domain for the function.

c Let $a = \cos(\theta°)$. Then $V = \dfrac{500}{3}a^2(1 - a)$. Use a CAS calculator to find the maximum value of V and the values of a and θ for which this maximum occurs.

9 A camera is in a position x m from a point A. An object that is a metres in length is projected vertically upwards from A. When the object has moved b metres vertically up:



a Show that

$$\theta = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$$

b Use the result of **a** to show that

$$\tan\theta = \frac{ax}{x^2 + ba + b^2}$$

c If $\theta = \dfrac{\pi}{4}$, find:

i x in terms of a and b

ii x if $a = 2(1 + \sqrt{2})$ and $b = 1$

d If $a = 2(1 + \sqrt{2})$, $b = 1$ and $x = 1$, find an approximate value of θ.

e Using a CAS calculator, plot the graphs of θ against b and $\tan\theta$ against b for constant values of a and x as follows:

i $a = 1, x = 5$

ii $a = 1, x = 10$

iii $a = 1, x = 20$

f Comment on these graphs.

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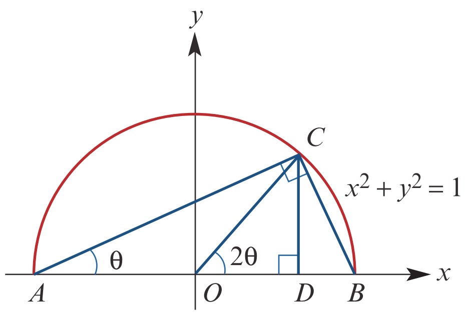
10 Points A, B and C lie on a circle with centre O and radius 1 as shown.



- **a** Give reasons why triangle ACD is similar to triangle ABC.
- **b** Give the coordinates of C in terms of circular functions applied to 2θ.
- **c** **i** Find CA in terms of θ from triangle ABC.
  ii Find CB in terms of θ from triangle ABC.
- **d** Use the results of **b** and **c** to show that $\sin(2\theta) = 2\sin\theta\cos\theta$.
- **e** Use the results of **b** and **c** to show that $\cos(2\theta) = 2\cos^2\theta - 1$.

11 **a** Prove that if $\sin(x) \neq 0$, then

$$\cos(x)\cos(2x)\cos(4x)\cos(8x) = \frac{\sin(16x)}{16\sin(x)}$$

b Prove that if $\sin(x) \neq 0$, then for all $n \in \mathbb{N}$ we have

$$\prod_{i=1}^{n} \cos(2^{i-1}x) = \frac{\sin(2^n x)}{2^n \sin(x)}$$

12 Fix an angle θ and define a sequence by $t_1 = \sin^2\theta$ and $t_n = 4t_{n-1}(1 - t_{n-1})$.

- **a** Find t_2 and t_3 in terms of θ.
- **b** Conjecture an expression for t_n and prove this result by mathematical induction.

13 Let A, B, C and D be angles in $[0, \pi]$.

a Prove that

$$\frac{\sin A + \sin B}{2} \leq \sin\left(\frac{A+B}{2}\right)$$

b Hence, prove that

$$\frac{\sin A + \sin B + \sin C + \sin D}{4} \leq \sin\left(\frac{A+B+C+D}{4}\right)$$

14 **a** Given that $x + \dfrac{1}{x} = 2\cos\theta$, show that:

i $x^2 + \dfrac{1}{x^2} = 2\cos(2\theta)$

ii $x^3 + \dfrac{1}{x^3} = 2\cos(3\theta)$

b Given that $x + \dfrac{1}{x} = 2\cos\theta$, prove by induction that

$$x^n + \frac{1}{x^n} = 2\cos(n\theta)$$

for all $n \in \mathbb{N}$.

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15 **a** Use the identity $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$ to find the exact value of $\cos\left(\frac{\pi}{8}\right)$.

b Use the identity again and the answer to part **a** to find the exact value of $\cos\left(\frac{\pi}{16}\right)$.

c Prove by mathematical induction that, for each $n \in \mathbb{N}$, we have

$$\cos\left(\frac{\pi}{2^{n+1}}\right) = \frac{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}{2}$$

where there are n square roots in total.

16 **a** Use a trigonometric identity to show that

$$2\sin(A)\cos(kA) = \sin((k+1)A) - \sin((k-1)A)$$

b Use part **a** and 'telescopic cancelling' to prove that

$$2\sin(A)\big(\cos(A) + \cos(3A) + \cdots + \cos((2n-1)A)\big) = \sin(2nA)$$

for all $n \in \mathbb{N}$.

c Use mathematical induction to give an alternative proof of the result from part **b**.

17 **a** Use a trigonometric identity to show that

$$2\sin(A)\sin(kA) = \cos((k-1)A) - \cos((k+1)A)$$

b Use part **a** and 'telescopic cancelling' to prove that

$$\sin(A) + \sin(3A) + \sin(5A) + \cdots + \sin((2n-1)A) = \sin^2(nA)\operatorname{cosec}(A)$$

for all $n \in \mathbb{N}$.

c Use mathematical induction to give an alternative proof of the result from part **b**.

18 **a** Show that

$$2\sin(A)\cos(2kA) = \sin((2k+1)A) - \sin((2k-1)A)$$

b Use part **a** and 'telescopic cancelling' to prove that

$$\cos(2A) + \cos(4A) + \cdots + \cos(2nA) = \frac{\sin(nA)\cos((n+1)A)}{\sin(A)}$$

for all $n \in \mathbb{N}$.

c Prove this result by induction.

19 Prove using mathematical induction that

$$\sum_{r=1}^{n} \sin(2rA) = \frac{\cos(A) - \cos((2n+1)A)}{2\sin(A)}$$

for every natural number n.

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4

Vectors

Objectives

- To understand the concept of a **vector** and to apply the basic operations on vectors.
- To recognise when two vectors are **parallel**.
- To understand **linear dependence** and **linear independence**.
- To use the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ to represent vectors in two dimensions.
- To use the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to represent vectors in three dimensions.
- To find the **scalar product** of two vectors.
- To use the scalar product to find the magnitude of the angle between two vectors.
- To use the scalar product to recognise when two vectors are **perpendicular**.
- To understand **vector resolutes** and **scalar resolutes**.
- To apply vector techniques to **geometric proofs** in two and three dimensions.

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

length	30 cm is the length of the page of a particular book
time	10 s is the time for one athlete to run 100 m

More is required to describe displacement, velocity or force. The direction must be recorded as well as the magnitude.

displacement	30 km in the direction north
velocity	60 km/h in the direction south-east

A quantity that has both a magnitude and a direction is called a **vector**.

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4A Introduction to vectors

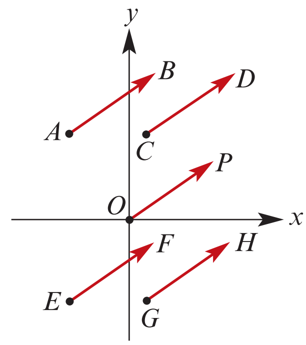
A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

Directed line segments

The five directed line segments shown all have the same length and direction, and so they are equivalent.



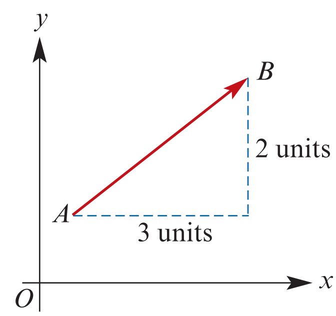
A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

For simplicity of language, this is also called vector $\overrightarrow{AB}$. That is, the set of equivalent segments can be named through one member of the set.

Note: The five directed line segments in the diagram all name the same vector: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$.

Column vectors

An alternative way to represent a vector is as a column of numbers. The column of numbers corresponds to a set of equivalent directed line segments.



For example, the column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segments which go 3 across to the right and 2 up.

Vector notation

A vector is often denoted by a single bold lowercase letter. The vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single letter, such as $\boldsymbol{v}$. We can write $\boldsymbol{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underset{\sim}{v}$.

Magnitude of vectors

The magnitude of vector $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$. Likewise, the magnitude of vector $\boldsymbol{v}$ is denoted by $|\boldsymbol{v}|$. The magnitude of a vector is represented by the length of a directed line segment corresponding to the vector.

For $\overrightarrow{AB}$ in the diagram above, we have $|\overrightarrow{AB}| = \sqrt{3^2 + 2^2} = \sqrt{13}$ using Pythagoras' theorem.

In general, if $\overrightarrow{AB}$ is represented by the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then its magnitude is given by

$$|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$$

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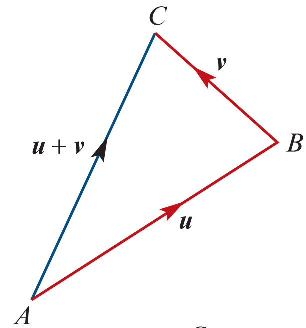
Addition of vectors

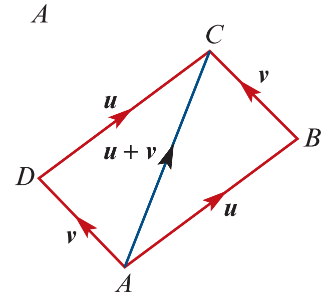
Adding vectors geometrically



Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from A to B and then a line segment representing $\boldsymbol{v}$ from B to C.

The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from A to C. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$$

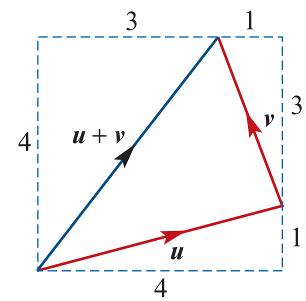


The same result is achieved if the order is reversed. This is represented in the diagram on the right:

$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{AC} \\ &= \boldsymbol{v} + \boldsymbol{u} \end{aligned}$$

Hence addition of vectors is commutative.

Adding column vectors



Two vectors can be added using column-vector notation.

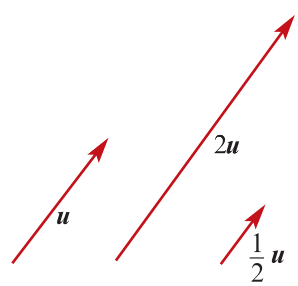
For example, if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Scalar multiplication

Multiplication by a real number (scalar) changes the length of the vector. For example:

- $2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$
- $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$

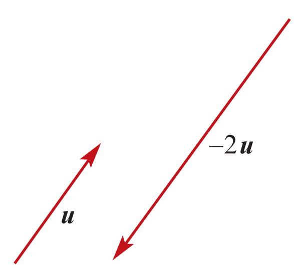


We have $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$.

In general, for $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.

When a vector is multiplied by -2, the vector's direction is reversed and the length is doubled.

When a vector is multiplied by -1, the vector's direction is reversed and the length remains the same.



If $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$. The directed line segment $-\overrightarrow{AB}$ goes from B to A.

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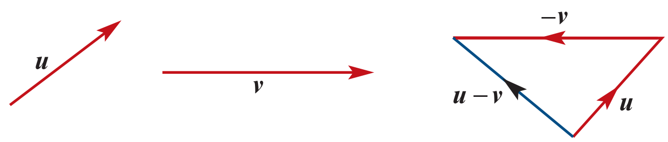
Zero vector

The **zero vector** is denoted by $\mathbf{0}$ and represents a line segment of zero length. The zero vector has no direction. The magnitude of the zero vector is 0. Note that $0\boldsymbol{a} = \mathbf{0}$ and $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$.

In two dimensions, the zero vector can be written as $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Subtraction of vectors

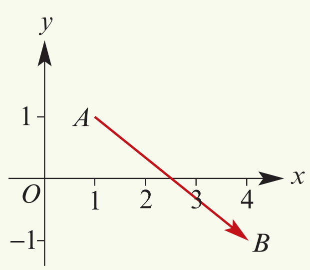
To find $\boldsymbol{u} - \boldsymbol{v}$, we add $-\boldsymbol{v}$ to $\boldsymbol{u}$.



Example 1

Draw a directed line segment representing the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and state the magnitude of this vector.

Solution



The magnitude is

$$\sqrt{3^2 + (-2)^2} = \sqrt{13}$$

Explanation

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 across to the right and 2 down'.

Note: Here the segment starts at (1, 1) and goes to (4, −1). It can start at any point.

Example 2

The vector $\boldsymbol{u}$ is defined by the directed line segment from (2, 6) to (3, 1).

If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

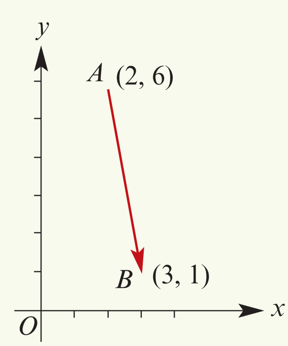
Solution

From the diagram:

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} + \boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\therefore \quad \boldsymbol{u} = \begin{bmatrix} 3-2 \\ 1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Hence $a = 1$ and $b = -5$.



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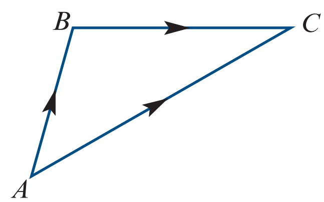
Polygons of vectors

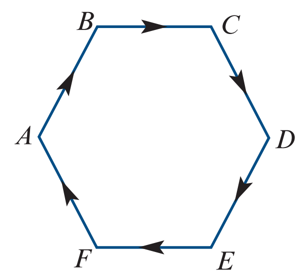
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$

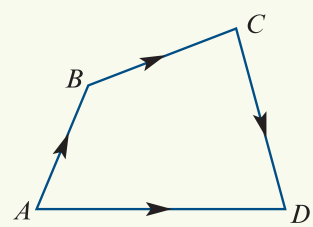


Example 3

Illustrate the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, where A, B, C and D are points in the plane.

Solution

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$



Parallel vectors

Two parallel vectors have the same direction or opposite directions.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

For example, if $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel as $\boldsymbol{v} = 3\boldsymbol{u}$.

Position vectors

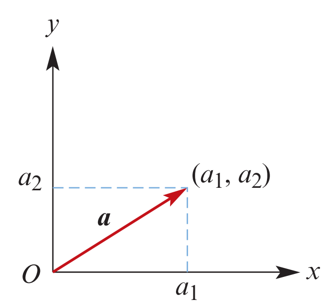
We can use a point O, the origin, as a starting point for a vector to indicate the position of a point A in space relative to O.

For a point A, the **position vector** is $\overrightarrow{OA}$.

The two-dimensional vector

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

is associated with the point (a_1, a_2). The vector $\boldsymbol{a}$ can be represented by the directed line segment from the origin to the point (a_1, a_2).



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Properties of the basic operations on vectors

The following properties can be established from our definitions of the basic operations on vectors. (In fact, these properties are used to generalise the concept of a vector, but this is beyond the scope of the course.)

commutative law for vector addition	$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$	
associative law for vector addition	$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$	
zero vector	$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$	
additive inverse	$\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$	
distributive laws	$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$	for $m \in \mathbb{R}$
	$(\ell + m)\boldsymbol{a} = \ell\boldsymbol{a} + m\boldsymbol{a}$	for $\ell, m \in \mathbb{R}$
compatibility of multiplication	$(\ell m)\boldsymbol{a} = \ell(m\boldsymbol{a})$	for $\ell, m \in \mathbb{R}$
identity of scalar multiplication	$1\boldsymbol{a} = \boldsymbol{a}$	

Thus, many of the ordinary rules of algebra apply to vectors.

Example 4

Simplify the following vector expression:

$$2(\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}) + \frac{3}{2}(\boldsymbol{b} - 4\boldsymbol{c})$$

Solution

$$2(\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}) + \frac{3}{2}(\boldsymbol{b} - 4\boldsymbol{c}) = 2\boldsymbol{a} - 2\boldsymbol{b} + 6\boldsymbol{c} + \frac{3}{2}\boldsymbol{b} - 6\boldsymbol{c}$$

$$= 2\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$$

Vectors in three dimensions

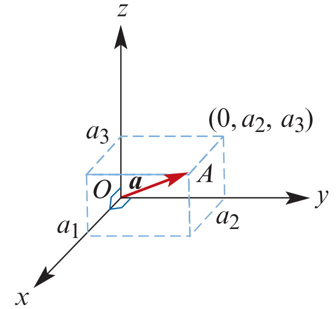
The definition of a vector is, of course, also valid in three dimensions. The properties which hold in two dimensions also hold in three dimensions.

For vectors in three dimensions, we use a third axis, denoted by z. The third axis is at right angles to the other two axes. The x-axis is drawn at an angle to indicate a direction out of the page towards you.

Vectors in three dimensions can also be written using column-vector notation:

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The vector $\boldsymbol{a}$ can be represented by the directed line segment from the origin to the point $A(a_1, a_2, a_3)$.



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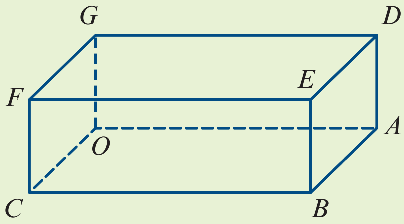
Example 5

OABCDEFG is a cuboid as shown.

Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{g} = \overrightarrow{OG}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

Find the following vectors in terms of $\boldsymbol{a}$, $\boldsymbol{g}$ and $\boldsymbol{c}$:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OF}$ **c** $\overrightarrow{GD}$ **d** $\overrightarrow{GB}$ **e** $\overrightarrow{FA}$



Solution

a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \boldsymbol{a} + \boldsymbol{c}$ (as $\overrightarrow{AB} = \overrightarrow{OC}$)

b $\overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{CF}$
$= \boldsymbol{c} + \boldsymbol{g}$ (as $\overrightarrow{CF} = \overrightarrow{OG}$)

c $\overrightarrow{GD} = \overrightarrow{OA}$
$= \boldsymbol{a}$

d $\overrightarrow{GB} = \overrightarrow{GO} + \overrightarrow{OA} + \overrightarrow{AB}$
$= -\boldsymbol{g} + \boldsymbol{a} + \boldsymbol{c}$

e $\overrightarrow{FA} = \overrightarrow{FG} + \overrightarrow{GO} + \overrightarrow{OA}$
$= -\boldsymbol{c} - \boldsymbol{g} + \boldsymbol{a}$

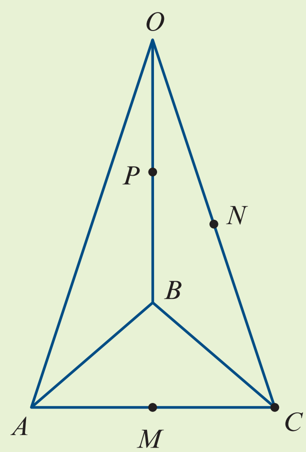
Example 6

OABC is a tetrahedron,
M is the midpoint of *AC*,
N is the midpoint of *OC*,
P is the midpoint of *OB*.

Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

Find in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

a $\overrightarrow{AC}$ **b** $\overrightarrow{OM}$ **c** $\overrightarrow{CN}$ **d** $\overrightarrow{MN}$ **e** $\overrightarrow{MP}$



Solution

a $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= -\boldsymbol{a} + \boldsymbol{c}$

b $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$
$= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{c})$
$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$

c $\overrightarrow{CN} = \frac{1}{2}\overrightarrow{CO}$
$= \frac{1}{2}(-\boldsymbol{c})$
$= -\frac{1}{2}\boldsymbol{c}$

d $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$
$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}) + \frac{1}{2}\boldsymbol{c}$
$= -\frac{1}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{c} + \frac{1}{2}\boldsymbol{c}$
$= -\frac{1}{2}\boldsymbol{a}$

(So *MN* is parallel to *AO*.)

e $\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OP}$
$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}) + \frac{1}{2}\boldsymbol{b}$
$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a} - \boldsymbol{c})$

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Linear dependence and independence

A vector $\boldsymbol{w}$ is a **linear combination** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ if it can be expressed in the form

$$\boldsymbol{w} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \dots + k_n\boldsymbol{v}_n$$

for some real numbers $k_1, k_2, \dots, k_n$.

Definition of linear dependence and linear independence

- A set of vectors is said to be **linearly dependent** if at least one of its members can be expressed as a linear combination of other vectors in the set.
- A set of vectors is said to be **linearly independent** if it is not linearly dependent. That is, a set of vectors is linearly independent if no vector in the set is expressible as a linear combination of other vectors in the set.

For example, it is easy to show that a set of two non-zero vectors is linearly dependent if and only if the two vectors are parallel.

We can give a useful alternative description of linear dependence:

- Two vectors A set of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is linearly dependent if and only if there exist real numbers k and ℓ, not both zero, such that $k\boldsymbol{a} + \ell\boldsymbol{b} = \boldsymbol{0}$.
- Three vectors A set of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is linearly dependent if and only if there exist real numbers k, ℓ and m, not all zero, such that $k\boldsymbol{a} + \ell\boldsymbol{b} + m\boldsymbol{c} = \boldsymbol{0}$.
- In general A set of n vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$ is linearly dependent if and only if there exist real numbers $k_1, k_2, \dots, k_n$, not all zero, such that $k_1\boldsymbol{a}_1 + k_2\boldsymbol{a}_2 + \dots + k_n\boldsymbol{a}_n = \boldsymbol{0}$.

Note: Any set that contains the zero vector is linearly dependent.
Any set of three or more two-dimensional vectors is linearly dependent.
Any set of four or more three-dimensional vectors is linearly dependent.

We will use the following method for checking whether three vectors are linearly dependent.

Linear dependence for three vectors

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors that are not parallel. Then vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent if and only if there exist real numbers m and n such that $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$.

This representation of a vector $\boldsymbol{c}$ in terms of two linearly independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is unique, as demonstrated in the following important result.

Linear combinations of independent vectors

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two linearly independent (i.e. not parallel) vectors. Then

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

Proof Assume that $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$. Then $(m - p)\boldsymbol{a} + (n - q)\boldsymbol{b} = \boldsymbol{0}$. As vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, it follows from the definition of linear independence that $m - p = 0$ and $n - q = 0$. Hence $m = p$ and $n = q$.

Note: This result can be extended to any finite number of linearly independent vectors.

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Example 7

Determine whether the following sets of vectors are linearly dependent:

a $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

b $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Solution

a Note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

Suppose $\quad \boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$

Then $\quad 5 = 2m + 3n$

$\quad 6 = m - n$

Solving the simultaneous equations, we have $m = \frac{23}{5}$ and $n = -\frac{7}{5}$.

This set of vectors is linearly dependent.

Note: In general, any set of three or more two-dimensional vectors is linearly dependent.

b Note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

Suppose $\quad \boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$

Then $\quad -1 = 3m + 2n$

$\quad 0 = 4m + n$

$\quad 1 = -m + 3n$

Solving the first two equations, we have $m = \frac{1}{5}$ and $n = -\frac{4}{5}$.

But these values do not satisfy the third equation, as $-m + 3n = -\frac{13}{5} \neq 1$.

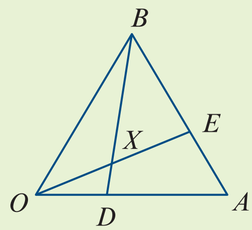
The three equations have no solution, so the vectors are linearly independent.

Example 8

Points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O.

The point D is such that $\overrightarrow{OD} = k\overrightarrow{OA}$ and the point E is such that $\overrightarrow{AE} = \ell\overrightarrow{AB}$. The line segments BD and OE intersect at X.

Assume that $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE}$ and $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB}$.



a Express $\overrightarrow{XB}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and k.

b Express $\overrightarrow{OX}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and ℓ.

c Express $\overrightarrow{XB}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and ℓ.

d Find k and ℓ.

Solution

a
$$\begin{aligned}\overrightarrow{XB} &= \frac{4}{5}\overrightarrow{DB} \\ &= \frac{4}{5}(-\overrightarrow{OD} + \overrightarrow{OB}) \\ &= \frac{4}{5}(-k\overrightarrow{OA} + \overrightarrow{OB}) \\ &= \frac{4}{5}(-k\boldsymbol{a} + \boldsymbol{b}) \\ &= -\frac{4k}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b}\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{OX} &= \frac{2}{5}\overrightarrow{OE} \\ &= \frac{2}{5}(\overrightarrow{OA} + \overrightarrow{AE}) \\ &= \frac{2}{5}(\overrightarrow{OA} + \ell\overrightarrow{AB}) \\ &= \frac{2}{5}(\boldsymbol{a} + \ell(\boldsymbol{b} - \boldsymbol{a})) \\ &= \frac{2}{5}(1 - \ell)\boldsymbol{a} + \frac{2\ell}{5}\boldsymbol{b}\end{aligned}$$

c
$$\begin{aligned}\overrightarrow{XB} &= \overrightarrow{XO} + \overrightarrow{OB} \\ &= -\overrightarrow{OX} + \overrightarrow{OB} \\ &= -\frac{2}{5}(1 - \ell)\boldsymbol{a} - \frac{2\ell}{5}\boldsymbol{b} + \boldsymbol{b} \\ &= \frac{2}{5}(\ell - 1)\boldsymbol{a} + \left(1 - \frac{2\ell}{5}\right)\boldsymbol{b}\end{aligned}$$

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d As $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent vectors, the vector $\overrightarrow{XB}$ has a unique representation in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. From parts **a** and **c**, we have

$$-\frac{4k}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b} = \frac{2}{5}(\ell - 1)\boldsymbol{a} + \left(1 - \frac{2\ell}{5}\right)\boldsymbol{b}$$

Hence

$$-\frac{4k}{5} = \frac{2}{5}(\ell - 1) \quad (1) \qquad \text{and} \qquad \frac{4}{5} = 1 - \frac{2\ell}{5} \quad (2)$$

From equation (2), we have

$$\frac{2\ell}{5} = \frac{1}{5}$$

$$\therefore \quad \ell = \frac{1}{2}$$

Substitute in (1):

$$-\frac{4k}{5} = \frac{2}{5}\left(\frac{1}{2} - 1\right)$$

$$\therefore \quad k = \frac{1}{4}$$

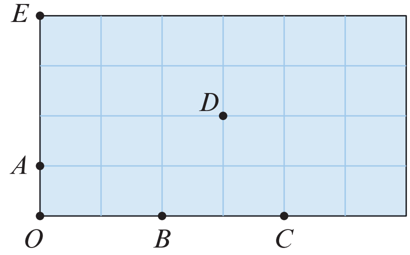
Exercise 4A

Example 1 **1** Draw a directed line segment representing the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and state the magnitude of this vector.

Example 2 **2** The vector $\boldsymbol{u}$ is defined by the directed line segment from $(-2, 4)$ to $(1, 6)$. If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

Example 3 **3** Illustrate the vector sum $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$.

4 In the diagram, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.



a Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OC}$ **ii** $\overrightarrow{OE}$ **iii** $\overrightarrow{OD}$

iv $\overrightarrow{DC}$ **v** $\overrightarrow{DE}$

b If $|\boldsymbol{a}| = 1$ and $|\boldsymbol{b}| = 2$, find:

i $|\overrightarrow{OC}|$ **ii** $|\overrightarrow{OE}|$ **iii** $|\overrightarrow{OD}|$

5 If the vector $\boldsymbol{a}$ has magnitude 3, find the magnitude of:

a $2\boldsymbol{a}$ **b** $\frac{3}{2}\boldsymbol{a}$ **c** $-\frac{1}{2}\boldsymbol{a}$

Example 4 **6** Simplify each of the following vector expressions:

a $3(\boldsymbol{a} - \boldsymbol{b} - 2\boldsymbol{c}) + \frac{5}{2}(3\boldsymbol{a} + \boldsymbol{b} - 6\boldsymbol{c})$ **b** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}) + \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a}) + \frac{1}{2}(\boldsymbol{c} + \boldsymbol{a} - \boldsymbol{b})$

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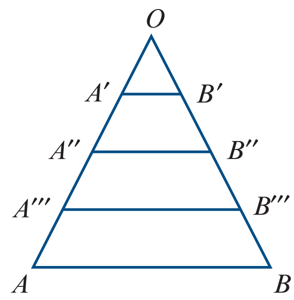
7 In the figure, $OA' = A'A'' = A''A''' = A'''A$ and $OB' = B'B'' = B''B''' = B'''B$.
If $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$, find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

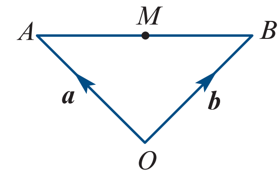
a **i** $\overrightarrow{OA'}$ **ii** $\overrightarrow{OB'}$ **iii** $\overrightarrow{A'B'}$ **iv** $\overrightarrow{AB}$

b **i** $\overrightarrow{OA''}$ **ii** $\overrightarrow{OB''}$ **iii** $\overrightarrow{A''B''}$



8 The position vectors of two points A and B are $\boldsymbol{a}$ and $\boldsymbol{b}$. The point M is the midpoint of AB. Find:

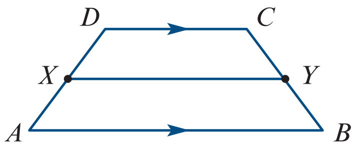
a $\overrightarrow{AB}$ **b** $\overrightarrow{AM}$ **c** $\overrightarrow{OM}$



9 Let $ABCD$ be a trapezium with AB parallel to DC. Let X and Y be the midpoints of AD and BC respectively.

a Express $\overrightarrow{XY}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{DC}$.

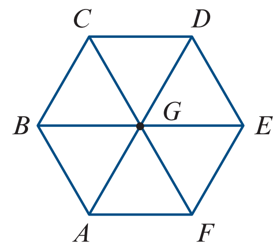
b Show that XY is parallel to AB.



10 Let $ABCDEF$ be a regular hexagon with centre G. The position vectors of A, B and C, relative to an origin O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

a Express $\overrightarrow{OG}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

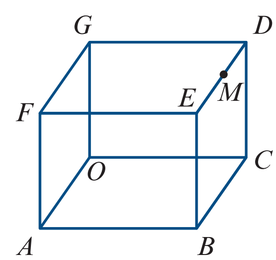
b Express $\overrightarrow{CD}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



Example 5
Example 6

11 For the cuboid shown, let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{g} = \overrightarrow{OG}$.
Let M be the midpoint of ED.
Find each of the following in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{g}$:

a $\overrightarrow{EF}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{EM}$ **d** $\overrightarrow{OM}$ **e** $\overrightarrow{AM}$



12 Let $OABCD$ be a right square pyramid with vertex O.
Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{d} = \overrightarrow{OD}$.

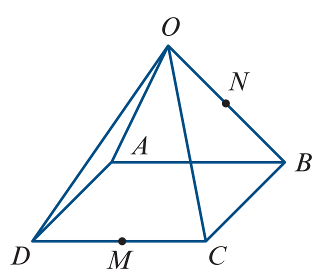
a **i** Find $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

ii Find $\overrightarrow{DC}$ in terms of $\boldsymbol{c}$ and $\boldsymbol{d}$.

iii Use the fact that $\overrightarrow{AB} = \overrightarrow{DC}$ to find a relationship between $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

b **i** Find $\overrightarrow{BC}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

ii Let M be the midpoint of DC and N the midpoint of OB. Find $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



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Example 7

13 Determine whether the following sets of vectors are linearly dependent:

a $\boldsymbol{a} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix}$

b $\boldsymbol{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$

c $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$

d $\boldsymbol{a} = \begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 9 \\ 3 \\ -5 \end{bmatrix}$

14 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors that are not parallel.

a If $k\boldsymbol{a} + \ell\boldsymbol{b} = 3\boldsymbol{a} + (1 - \ell)\boldsymbol{b}$, find the values of k and ℓ.

b If $2(\ell - 1)\boldsymbol{a} + \left(1 - \frac{\ell}{5}\right)\boldsymbol{b} = -\frac{4k}{5}\boldsymbol{a} + 3\boldsymbol{b}$, find the values of k and ℓ.

Example 8

15 Points P, Q and R have position vectors $2\boldsymbol{a} - \boldsymbol{b}$, $3\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} + 4\boldsymbol{b}$ respectively, relative to an origin O, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors. The lines OP and RQ intersect at point S, with $\overrightarrow{OS} = k\overrightarrow{OP}$ and $\overrightarrow{RS} = m\overrightarrow{RQ}$.

a Express $\overrightarrow{OS}$ in terms of:

i k, $\boldsymbol{a}$ and $\boldsymbol{b}$ **ii** m, $\boldsymbol{a}$ and $\boldsymbol{b}$

b Hence evaluate k and m.

c Hence write the position vector of S in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

16 The position vectors of points A and B, relative to an origin O, are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors. The point P is such that $\overrightarrow{OP} = 4\overrightarrow{OB}$. The midpoint of AB is the point Q. The point R is such that $\overrightarrow{OR} = \frac{8}{5}\overrightarrow{OQ}$.

a Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OQ}$ **ii** $\overrightarrow{OR}$ **iii** $\overrightarrow{AR}$ **iv** $\overrightarrow{RP}$

b Show that R lies on AP and state the ratio $AR : RP$.

c Given that the point S is such that $\overrightarrow{OS} = \lambda\overrightarrow{OQ}$, find the value of λ such that PS is parallel to BA.

17 Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Find the values of x and y for which:

a $x\boldsymbol{a} = (y - 1)\boldsymbol{b}$

b $(2 - x)\boldsymbol{a} = 3\boldsymbol{a} + (7 - 3y)\boldsymbol{b}$

c $(5 + 2x)(\boldsymbol{a} + \boldsymbol{b}) = y(3\boldsymbol{a} + 2\boldsymbol{b})$

18 Suppose that the point X lies between A and B on the line AB, with $\overrightarrow{AX} = k\overrightarrow{AB}$.

a Find $\frac{AX}{AB}$ in terms of k.

b Show that $0 < k < 1$.

c Find $\frac{AX}{XB}$ in terms of k.

d Let $m = \frac{AX}{XB}$. Express k in terms of m.

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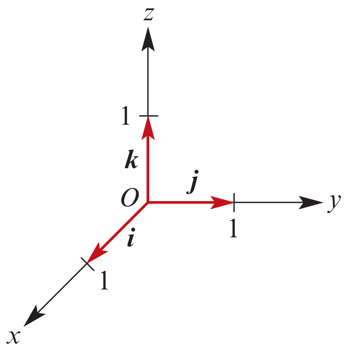
4B Resolution of a vector into rectangular components

A **unit vector** is a vector of magnitude 1. For a non-zero vector $\boldsymbol{a}$, the unit vector with the same direction as $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$ and given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

- The unit vector in the positive direction of the x-axis is $\boldsymbol{i}$.
- The unit vector in the positive direction of the y-axis is $\boldsymbol{j}$.
- The unit vector in the positive direction of the z-axis is $\boldsymbol{k}$.



In two dimensions: $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In three dimensions: $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are linearly independent. Every vector in two or three dimensions can be expressed uniquely as a linear combination of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

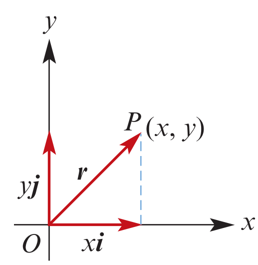
e.g. $$\boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ r_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ r_3 \end{bmatrix} = r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}$$

Two dimensions

For the point $P(x, y)$:

$$\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$$

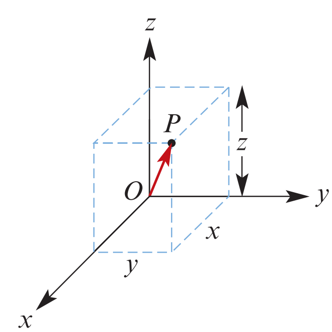


Three dimensions

For the point $P(x, y, z)$:

$$\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$



Basic operations in component form

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$.

Then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$

$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$

and $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j} + ma_3\boldsymbol{k}$ for a scalar m

Equivalence

If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Magnitude

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

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Example 9

a Using the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, give the vectors:

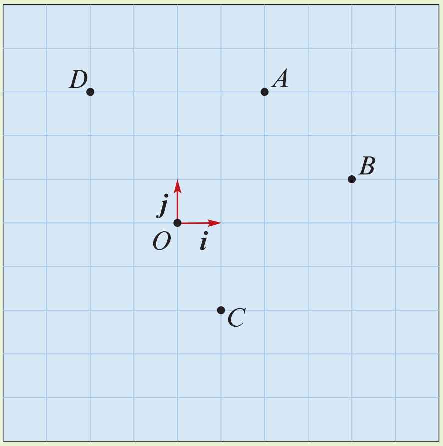
i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{OD}$

b Using the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, give the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$

c Find the magnitudes of the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$



Solution

a **i** $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ **ii** $\overrightarrow{OB} = 4\boldsymbol{i} + \boldsymbol{j}$ **iii** $\overrightarrow{OC} = \boldsymbol{i} - 2\boldsymbol{j}$ **iv** $\overrightarrow{OD} = -2\boldsymbol{i} + 3\boldsymbol{j}$

b **i**
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{i} + \boldsymbol{j}\\ &= 2\boldsymbol{i} - 2\boldsymbol{j}\end{aligned}$$

ii
$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= -4\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{i} - 2\boldsymbol{j}\\ &= -3\boldsymbol{i} - 3\boldsymbol{j}\end{aligned}$$

c **i**
$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{2^2 + (-2)^2}\\ &= \sqrt{8}\\ &= 2\sqrt{2}\end{aligned}$$

ii
$$\begin{aligned}|\overrightarrow{BC}| &= \sqrt{(-3)^2 + (-3)^2}\\ &= \sqrt{18}\\ &= 3\sqrt{2}\end{aligned}$$

Example 10

Let $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$. Find:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{a} - 2\boldsymbol{b}$ **c** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ **d** $|\boldsymbol{a}|$

Solution

a
$$\begin{aligned}\boldsymbol{a} + \boldsymbol{b} &= (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + (3\boldsymbol{i} - 2\boldsymbol{k})\\ &= 4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}\end{aligned}$$

b
$$\begin{aligned}\boldsymbol{a} - 2\boldsymbol{b} &= (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) - 2(3\boldsymbol{i} - 2\boldsymbol{k})\\ &= -5\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}\end{aligned}$$

c
$$\begin{aligned}\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} &= (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + (3\boldsymbol{i} - 2\boldsymbol{k}) + (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})\\ &= 6\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}\end{aligned}$$

d $|\boldsymbol{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

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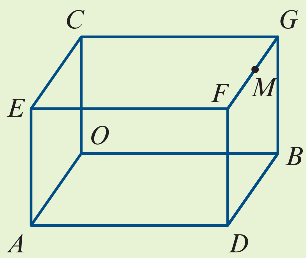
Example 11

A cuboid is labelled as shown.



$\overrightarrow{OA} = 3\boldsymbol{i}$, $\overrightarrow{OB} = 5\boldsymbol{j}$, $\overrightarrow{OC} = 4\boldsymbol{k}$

a Find in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

i $\overrightarrow{DB}$ **ii** $\overrightarrow{OD}$ **iii** $\overrightarrow{DF}$ **iv** $\overrightarrow{OF}$

b Find $|\overrightarrow{OF}|$.

c If M is the midpoint of FG, find:

i $\overrightarrow{OM}$ **ii** $|\overrightarrow{OM}|$

Solution

a **i**
$$\begin{aligned}\overrightarrow{DB} &= \overrightarrow{AO}\\ &= -\overrightarrow{OA}\\ &= -3\boldsymbol{i}\end{aligned}$$

ii
$$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{OB} + \overrightarrow{BD}\\ &= 5\boldsymbol{j} + \overrightarrow{OA}\\ &= 5\boldsymbol{j} + 3\boldsymbol{i}\\ &= 3\boldsymbol{i} + 5\boldsymbol{j}\end{aligned}$$

iii
$$\begin{aligned}\overrightarrow{DF} &= \overrightarrow{OC}\\ &= 4\boldsymbol{k}\end{aligned}$$

iv
$$\begin{aligned}\overrightarrow{OF} &= \overrightarrow{OD} + \overrightarrow{DF}\\ &= 3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}\end{aligned}$$

b
$$\begin{aligned}|\overrightarrow{OF}| &= \sqrt{9 + 25 + 16}\\ &= \sqrt{50}\\ &= 5\sqrt{2}\end{aligned}$$

c **i**
$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OD} + \overrightarrow{DF} + \overrightarrow{FM}\\ &= 3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k} + \frac{1}{2}(-\overrightarrow{GF})\\ &= 3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k} + \frac{1}{2}(-3\boldsymbol{i})\\ &= \frac{3}{2}\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}\end{aligned}$$

ii
$$\begin{aligned}|\overrightarrow{OM}| &= \sqrt{\frac{9}{4} + 25 + 16}\\ &= \frac{1}{2}\sqrt{9 + 100 + 64}\\ &= \frac{1}{2}\sqrt{173}\end{aligned}$$

Example 12

If $\boldsymbol{a} = x\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 8\boldsymbol{i} + 2y\boldsymbol{j}$ such that $\boldsymbol{a} + \boldsymbol{b} = -2\boldsymbol{i} + 4\boldsymbol{j}$, find the values of x and y.

Solution

$\boldsymbol{a} + \boldsymbol{b} = (x + 8)\boldsymbol{i} + (2y + 3)\boldsymbol{j} = -2\boldsymbol{i} + 4\boldsymbol{j}$

$\therefore \quad x + 8 = -2 \quad$ and $\quad 2y + 3 = 4$

i.e. $\quad x = -10 \quad$ and $\quad y = \frac{1}{2}$

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Example 13

Let $A = (2, -3)$, $B = (1, 4)$ and $C = (-1, -3)$. The origin is O. Find:

a **i** $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$

b F such that $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA}$

c G such that $\overrightarrow{AG} = 3\overrightarrow{BC}$

Solution

a **i** $\overrightarrow{OA} = 2\boldsymbol{i} - 3\boldsymbol{j}$

ii $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= -2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{i} + 4\boldsymbol{j}$
$= -\boldsymbol{i} + 7\boldsymbol{j}$

iii $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$
$= -\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{i} - 3\boldsymbol{j}$
$= -2\boldsymbol{i} - 7\boldsymbol{j}$

b $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}(2\boldsymbol{i} - 3\boldsymbol{j}) = \boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$

Hence $F = (1, -\frac{3}{2})$

c $\overrightarrow{AG} = 3\overrightarrow{BC} = 3(-2\boldsymbol{i} - 7\boldsymbol{j}) = -6\boldsymbol{i} - 21\boldsymbol{j}$

Therefore

$$\begin{aligned}\overrightarrow{OG} &= \overrightarrow{OA} + \overrightarrow{AG}\\ &= 2\boldsymbol{i} - 3\boldsymbol{j} - 6\boldsymbol{i} - 21\boldsymbol{j}\\ &= -4\boldsymbol{i} - 24\boldsymbol{j}\end{aligned}$$

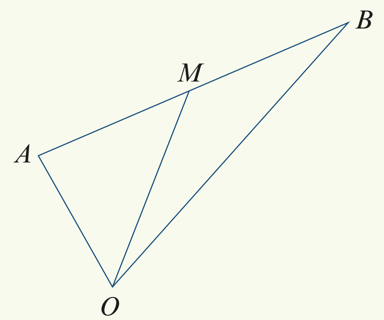
Hence $G = (-4, -24)$

Example 14

Let $A = (2, -4, 5)$ and $B = (5, 1, 7)$. Find M, the midpoint of AB.

Solution

We have $\overrightarrow{OA} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\overrightarrow{OB} = 5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}$.



Thus
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -2\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k} + 5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}\\ &= 3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}\end{aligned}$$

and so $\overrightarrow{AM} = \dfrac{1}{2}(3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k})$

Now
$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ &= 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k} + \frac{3}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + \boldsymbol{k}\\ &= \frac{7}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + 6\boldsymbol{k}\end{aligned}$$

Hence $M = \left(\dfrac{7}{2}, -\dfrac{3}{2}, 6\right)$

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Example 15

a Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ are linearly dependent.

b Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ are linearly independent.

Solution

a Vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are not parallel. We want to find constants m and n such that $\boldsymbol{a} = m\boldsymbol{b} + n\boldsymbol{c}$. Consider

$$8\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k} = m(\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) + n(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$$

This implies

$$8 = m + 2n \quad (1) \qquad 7 = -m + 3n \quad (2) \qquad 3 = 3m - n \quad (3)$$

Adding (1) and (2) gives $15 = 5n$, which implies $n = 3$.

Substitute in (1) to obtain $m = 2$.

The solution $m = 2$ and $n = 3$ must be verified for (3): $3m - n = 3 \times 2 - 3 = 3$.

Therefore

$$\boldsymbol{a} = 2\boldsymbol{b} + 3\boldsymbol{c} \quad \text{or equivalently} \quad \boldsymbol{a} - 2\boldsymbol{b} - 3\boldsymbol{c} = \boldsymbol{0}$$

Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

b Equations (1) and (2) are unchanged, and equation (3) becomes

$$3 = 3m + n \quad (3)$$

But substituting $m = 2$ and $n = 3$ gives $3m + n = 9 \neq 3$.

The three equations have no solution, so the vectors are linearly independent.

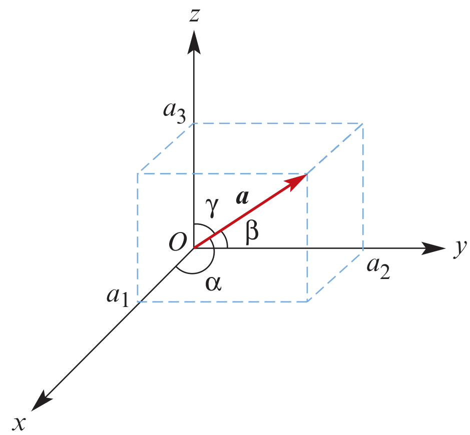
Angle made by a vector with an axis

The *direction* of a vector can be given by the angles which the vector makes with the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ directions.

If the vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ makes angles α, β and γ with the positive directions of the x-, y- and z-axes respectively, then

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

The derivation of these results is left as an exercise.



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Example 16

Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$.

For each of these vectors, find:

a its magnitude

b the angle the vector makes with the y-axis.

Solution

a $|\boldsymbol{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$|\boldsymbol{b}| = \sqrt{1^2 + 4^2 + (-3)^2} = \sqrt{26}$

b The angle that $\boldsymbol{a}$ makes with the y-axis is

$$\cos^{-1}\left(\frac{-1}{\sqrt{5}}\right) \approx 116.57°$$

The angle that $\boldsymbol{b}$ makes with the y-axis is

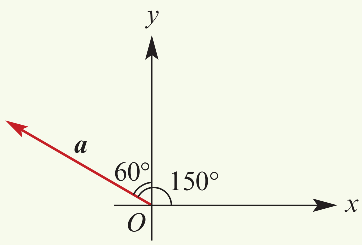
$$\cos^{-1}\left(\frac{4}{\sqrt{26}}\right) \approx 38.33°$$

Example 17

A position vector in two dimensions has magnitude 5 and its direction, measured anticlockwise from the x-axis, is 150°. Express this vector in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

Solution

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$.



The vector $\boldsymbol{a}$ makes an angle of 150° with the x-axis and an angle of 60° with the y-axis.

Therefore

$$\cos 150° = \frac{a_1}{|\boldsymbol{a}|} \quad \text{and} \quad \cos 60° = \frac{a_2}{|\boldsymbol{a}|}$$

Since $|\boldsymbol{a}| = 5$, this gives

$$a_1 = |\boldsymbol{a}| \cos 150° = -\frac{5\sqrt{3}}{2}$$

$$a_2 = |\boldsymbol{a}| \cos 60° = \frac{5}{2}$$

$$\therefore \quad \boldsymbol{a} = -\frac{5\sqrt{3}}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$$

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Example 18

Let $\boldsymbol{i}$ be a unit vector in the east direction and let $\boldsymbol{j}$ be a unit vector in the north direction, with units in kilometres.

a Show that the unit vector on a bearing of 300° is $-\frac{\sqrt{3}}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}$.

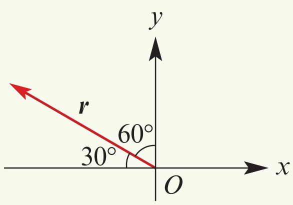
b If a car drives 3 km on a bearing of 300°, find the position vector of the car with respect to its starting point.

c The car then drives 6.5 km due north. Find:

- **i** the position vector of the car
- **ii** the distance of the car from the starting point
- **iii** the bearing of the car from the starting point.

Solution

a Let $\boldsymbol{r}$ denote the unit vector in the direction with bearing 300°.



Then $\boldsymbol{r} = -\cos 30°\boldsymbol{i} + \cos 60°\boldsymbol{j}$

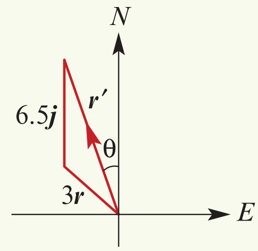
$$= -\frac{\sqrt{3}}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}$$

Note: $|\boldsymbol{r}| = 1$

b The position vector is

$$3\boldsymbol{r} = 3\left(-\frac{\sqrt{3}}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}\right) = -\frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j}$$

c Let $\boldsymbol{r}'$ denote the new position vector.



i $\boldsymbol{r}' = 3\boldsymbol{r} + 6.5\boldsymbol{j}$

$$= -\frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} + \frac{13}{2}\boldsymbol{j}$$

$$= -\frac{3\sqrt{3}}{2}\boldsymbol{i} + 8\boldsymbol{j}$$

ii $|\boldsymbol{r}'| = \sqrt{\frac{9 \times 3}{4} + 64}$

$$= \sqrt{\frac{27 + 256}{4}}$$

$$= \frac{1}{2}\sqrt{283}$$

iii Since $\boldsymbol{r}' = -\frac{3\sqrt{3}}{2}\boldsymbol{i} + 8\boldsymbol{j}$, we have

$$\tan\theta° = \frac{3\sqrt{3}}{16}$$

$$\therefore \quad \theta° = \tan^{-1}\left(\frac{3\sqrt{3}}{16}\right) \approx 18°$$

The bearing is 342°, correct to the nearest degree.

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Skill-sheet

Exercise 4B

Example 9

1 **a** Give each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

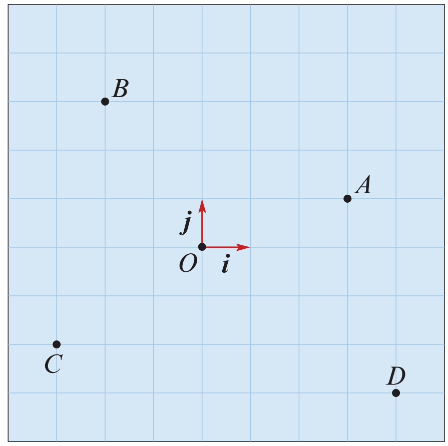
i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{OD}$

b Find each of the following:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{CD}$ **iii** $\overrightarrow{DA}$

c Find the magnitude of each of the following:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{DA}$



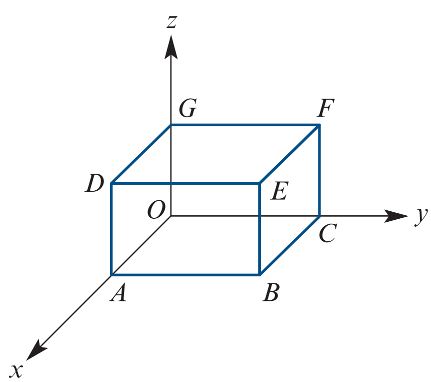
Example 10

2 Let $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 4\boldsymbol{k}$. Find:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $2\boldsymbol{a} + \boldsymbol{c}$ **c** $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}$ **d** $\boldsymbol{c} - 4\boldsymbol{a}$ **e** $|\boldsymbol{b}|$ **f** $|\boldsymbol{c}|$

Example 11

3 $OABCDEFG$ is a cuboid set on Cartesian axes with $\overrightarrow{OA} = 5\boldsymbol{i}$, $\overrightarrow{OC} = 2\boldsymbol{j}$ and $\overrightarrow{OG} = 3\boldsymbol{k}$.



a Find:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{CF}$ **iii** $\overrightarrow{AB}$

iv $\overrightarrow{OD}$ **v** $\overrightarrow{OE}$ **vi** $\overrightarrow{GE}$

vii $\overrightarrow{EC}$ **viii** $\overrightarrow{DB}$ **ix** $\overrightarrow{DC}$

x $\overrightarrow{BG}$ **xi** $\overrightarrow{GB}$ **xii** $\overrightarrow{FA}$

b Evaluate:

i $|\overrightarrow{OD}|$ **ii** $|\overrightarrow{OE}|$ **iii** $|\overrightarrow{GE}|$

c Let M be the midpoint of CB. Find:

i $\overrightarrow{CM}$ **ii** $\overrightarrow{OM}$ **iii** $\overrightarrow{DM}$

d Let N be the point on FG such that $\overrightarrow{FN} = 2\overrightarrow{NG}$. Find:

i $\overrightarrow{FN}$ **ii** $\overrightarrow{GN}$ **iii** $\overrightarrow{ON}$ **iv** $\overrightarrow{NA}$ **v** $\overrightarrow{NM}$

e Evaluate:

i $|\overrightarrow{NM}|$ **ii** $|\overrightarrow{DM}|$ **iii** $|\overrightarrow{AN}|$

Example 12

4 Find the values of x and y if:

a $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = x\boldsymbol{i} + 3y\boldsymbol{j}$, $\boldsymbol{a} + \boldsymbol{b} = 7\boldsymbol{i} - 2\boldsymbol{j}$

b $\boldsymbol{a} = x\boldsymbol{i} + 3\boldsymbol{j}$, $\boldsymbol{b} = -2\boldsymbol{i} + 5y\boldsymbol{j}$, $\boldsymbol{a} - \boldsymbol{b} = 6\boldsymbol{i} + \boldsymbol{j}$

c $\boldsymbol{a} = 6\boldsymbol{i} + y\boldsymbol{j}$, $\boldsymbol{b} = x\boldsymbol{i} - 4\boldsymbol{j}$, $\boldsymbol{a} + 2\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$

Example 13

5 Let $A = (-2, 4)$, $B = (1, 6)$ and $C = (-1, -6)$. Let O be the origin. Find:

a **i** $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$

b F such that $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA}$ **c** G such that $\overrightarrow{AG} = 3\overrightarrow{BC}$

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Example 14 **6** Let $A = (1, -6, 7)$ and $B = (5, -1, 9)$. Find M, the midpoint of AB.

7 Points A, B, C and D have position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 5\boldsymbol{i} + \boldsymbol{j} - 6\boldsymbol{k}$, $\boldsymbol{c} = 5\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{d} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ respectively.

a Find:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CD}$ **iv** $\overrightarrow{DA}$

b Evaluate:

i $|\overrightarrow{AC}|$ **ii** $|\overrightarrow{BD}|$

c Find the two parallel vectors in **a**.

8 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ respectively. The point M is on the line segment AB such that $AM : MB = 4 : 1$.

a Find:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{AM}$ **iii** $\overrightarrow{OM}$

b Find the coordinates of M.

Example 15 **9** **a** Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$ are linearly dependent.

b Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ are linearly independent.

c Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} + \frac{5}{2}\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ are linearly independent.

10 The vectors $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} + x\boldsymbol{k}$ are linearly dependent. Find the value of x.

11 Let $A = (2, 1)$, $B = (1, -3)$, $C = (-5, 2)$ and $D = (3, 5)$. Let O be the origin.

a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$ **iv** $\overrightarrow{BD}$

b Show that $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are parallel.

c What can be said about the points A, B and D?

12 Let $A = (1, 4, -4)$, $B = (2, 3, 1)$, $C = (0, -1, 4)$ and $D = (4, 5, 6)$.

a Find:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{AC}$ **iii** $\overrightarrow{BD}$ **iv** $\overrightarrow{CD}$

b Show that $\overrightarrow{OB}$ and $\overrightarrow{CD}$ are parallel.

13 Let $A = (1, 4, -2)$, $B = (3, 3, 0)$, $C = (2, 5, 3)$ and $D = (0, 6, 1)$.

a Find:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CD}$ **iv** $\overrightarrow{DA}$

b Describe the quadrilateral $ABCD$.

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14 Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Find:

a D such that $\overrightarrow{AB} = \overrightarrow{CD}$ **b** E such that $\overrightarrow{AE} = -\overrightarrow{BC}$ **c** G such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

15 $ABCD$ is a parallelogram, where $A = (2, 1)$, $B = (-5, 4)$, $C = (1, 7)$ and $D = (x, y)$.

a Find:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{AD}$ (in terms of x and y)

b Hence find the coordinates of D.

16 **a** Let $A = (1, 4, 3)$ and $B = (2, -1, 5)$. Find M, the midpoint of AB.

b Use a similar method to find M, the midpoint of XY, where X and Y have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

17 Let $A = (5, 4, 1)$ and $B = (3, 1, -4)$. Find M on line segment AB such that $AM = 4MB$.

18 Let $A = (4, -3)$ and $B = (7, 1)$. Find N such that $\overrightarrow{AN} = 3\overrightarrow{BN}$.

19 Find the point P on the line $x - 6y = 11$ such that $\overrightarrow{OP}$ is parallel to the vector $3\boldsymbol{i} + \boldsymbol{j}$.

20 The points A, B, C and D have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. Show that if $ABCD$ is a parallelogram, then $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}$.

21 Let $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{c} = 4\boldsymbol{i} + 5\boldsymbol{j}$.

a Find:

i $\frac{1}{2}\boldsymbol{a}$ **ii** $\boldsymbol{b} - \boldsymbol{c}$ **iii** $3\boldsymbol{b} - \boldsymbol{a} - 2\boldsymbol{c}$

b Find values for k and ℓ such that $k\boldsymbol{a} + \ell\boldsymbol{b} = \boldsymbol{c}$.

22 Let $\boldsymbol{a} = 5\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$, $\boldsymbol{b} = 8\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$.

a Find:

i $2\boldsymbol{a} - \boldsymbol{b}$ **ii** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ **iii** $0.5\boldsymbol{a} + 0.4\boldsymbol{b}$

b Find values for k and ℓ such that $k\boldsymbol{a} + \ell\boldsymbol{b} = \boldsymbol{c}$.

Example 16 **23** Let $\boldsymbol{a} = 5\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{d} = -\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$.

a Find:

i $|\boldsymbol{a}|$ **ii** $|\boldsymbol{b}|$ **iii** $|\boldsymbol{a} + 2\boldsymbol{b}|$ **iv** $|\boldsymbol{c} - \boldsymbol{d}|$

b Find, correct to two decimal places, the angle which each of the following vectors makes with the positive direction of the x-axis:

i $\boldsymbol{a}$ **ii** $\boldsymbol{a} + 2\boldsymbol{b}$ **iii** $\boldsymbol{c} - \boldsymbol{d}$

Example 17 **24** The table gives the magnitudes of vectors in two dimensions and the angle they each make with the x-axis (measured anticlockwise). Express each of the vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$, correct to two decimal places.

	Magnitude	Angle
$\boldsymbol{a}$	10	110°
$\boldsymbol{b}$	8.5	250°
$\boldsymbol{c}$	6	40°
$\boldsymbol{d}$	5	300°

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25 The following table gives the magnitudes of vectors in three dimensions and the angles they each make with the x-, y- and z-axes, correct to two decimal places. Express each of the vectors in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, correct to two decimal places.

	Magnitude	Angle with x-axis	Angle with y-axis	Angle with z-axis
$\boldsymbol{a}$	10	130°	80°	41.75°
$\boldsymbol{b}$	8	50°	54.52°	120°
$\boldsymbol{c}$	7	28.93°	110°	110°
$\boldsymbol{d}$	12	121.43°	35.5°	75.2°

26 **a** Show that if a vector in three dimensions makes angles α, β and γ with the x-, y- and z-axes respectively, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

b Hence, show that a vector cannot make an angle of 60° with each of the x-, y- and z-axes.

c Give an example of a vector that makes angles of 60°, 60° and 45° with the x-, y- and z-axes respectively.

27 Points A, B and C have position vectors $\boldsymbol{a} = -2\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = -2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$ respectively. Let M be the midpoint of BC.

a Show that $\triangle ABC$ is isosceles.

b Find $\overrightarrow{OM}$.

c Find $\overrightarrow{AM}$.

d Find the area of $\triangle ABC$.

28 $OABCV$ is a square-based right pyramid with V the vertex. The base diagonals OB and AC intersect at the point M. If $\overrightarrow{OA} = 5\boldsymbol{i}$, $\overrightarrow{OC} = 5\boldsymbol{j}$ and $\overrightarrow{MV} = 3\boldsymbol{k}$, find each of the following:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OM}$ **c** $\overrightarrow{OV}$ **d** $\overrightarrow{BV}$ **e** $|\overrightarrow{OV}|$

29 Points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Let M and N be the midpoints of OA and OB respectively, where O is the origin.

a Show that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$.

b Hence describe the geometric relationships between line segments MN and AB.

Example 18

30 Let $\boldsymbol{i}$ be the unit vector in the east direction and let $\boldsymbol{j}$ be the unit vector in the north direction, with units in kilometres. A runner sets off on a bearing of 120°.

a Find a unit vector in this direction.

b The runner covers 3 km. Find the position of the runner with respect to her starting point.

c The runner now turns and runs for 5 km in a northerly direction. Find the position of the runner with respect to her original starting point.

d Find the distance of the runner from her starting point.

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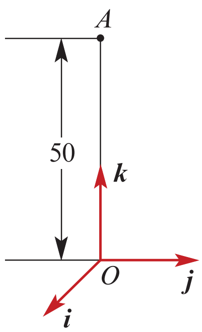
31 A hang-glider jumps from point A at the top of a 50 m cliff, as represented in the diagram.



a Give the position vector of point A with respect to O.

b After a short period of time, the hang-glider has position B given by $\overrightarrow{OB} = -80\boldsymbol{i} + 20\boldsymbol{j} + 40\boldsymbol{k}$ metres.

i Find the vector $\overrightarrow{AB}$. **ii** Find the magnitude of $\overrightarrow{AB}$.

c The hang-glider then moves 600 m in the $\boldsymbol{j}$-direction and 60 m in the $\boldsymbol{k}$-direction. Give the new position vector of the hang-glider.

32 A light plane takes off (from a point which will be considered as the origin) so that its position after a short period of time is given by $\boldsymbol{r}_1 = 1.5\boldsymbol{i} + 2\boldsymbol{j} + 0.9\boldsymbol{k}$, where $\boldsymbol{i}$ is a unit vector in the east direction, $\boldsymbol{j}$ is a unit vector in the north direction and measurements are in kilometres.

a Find the distance of the plane from the origin.

b The position of a second plane at the same time is given by $\boldsymbol{r}_2 = 2\boldsymbol{i} + 3\boldsymbol{j} + 0.8\boldsymbol{k}$.

i Find $\boldsymbol{r}_1 - \boldsymbol{r}_2$. **ii** Find the distance between the two aircraft.

c Give a unit vector which would describe the direction in which the first plane must fly to pass over the origin at a height of 900 m.

33 Jan starts at a point O and walks on level ground 200 metres in a north-westerly direction to P. She then walks 50 metres due north to Q, which is at the bottom of a building. Jan then climbs to T, the top of the building, which is 30 metres vertically above Q. Let $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ be unit vectors in the east, north and vertically upwards directions respectively. Express each of the following in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

a $\overrightarrow{OP}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{OQ}$ **d** $\overrightarrow{QT}$ **e** $\overrightarrow{OT}$

34 A ship leaves a port (at the origin O) and sails 100 km north-east to point P. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be the unit vectors in the east and north directions respectively, with units in kilometres.

a Find the position vector of point P.

b If B is the point on the shore with position vector $\overrightarrow{OB} = 100\boldsymbol{i}$, find:

i $\overrightarrow{BP}$ **ii** the bearing of P from B.

35 Given that $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + m\boldsymbol{k}$ and $\boldsymbol{c} = 3\boldsymbol{i} + n\boldsymbol{j} + \boldsymbol{k}$ are linearly dependent, express m in terms of n in simplest fraction form.

36 Let $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.

a Find $2\boldsymbol{a} - 3\boldsymbol{b}$.

b Hence find a value of m such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent, where $\boldsymbol{c} = m\boldsymbol{i} + 6\boldsymbol{j} - 12\boldsymbol{k}$.

37 Let $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = m\boldsymbol{a} + (1 - m)\boldsymbol{b}$.

a Find $\boldsymbol{c}$ in terms of m, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

b Hence find p if $\boldsymbol{c} = 7\boldsymbol{i} - \boldsymbol{j} + p\boldsymbol{k}$.

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4C Scalar product of vectors

The scalar product is an operation that takes two vectors and gives a real number.

Definition of the scalar product

We define the **scalar product** of two vectors in three dimensions $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The scalar product of two vectors in two dimensions is defined similarly.

Note: If $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{b} = \mathbf{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

The scalar product is often called the **dot product**.

Example 19

Let $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{b}$ **b** $\boldsymbol{a} \cdot \boldsymbol{a}$

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 \times (-2) + (-2) \times 3 + 3 \times 4 = 4$

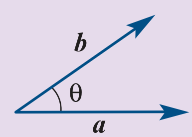
b $\boldsymbol{a} \cdot \boldsymbol{a} = 1^2 + (-2)^2 + 3^2 = 14$

Geometric description of the scalar product

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



Proof Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$. The cosine rule in $\triangle OAB$ gives

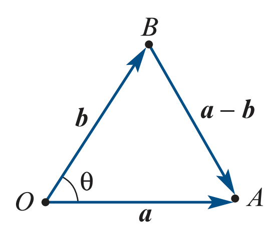
$$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta = |\boldsymbol{a} - \boldsymbol{b}|^2$$

$$(a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$$

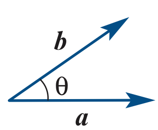
$$2(a_1b_1 + a_2b_2 + a_3b_3) = 2|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

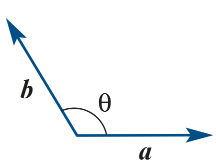
$$a_1b_1 + a_2b_2 + a_3b_3 = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

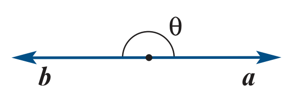
$$\therefore \quad \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$



Note: When two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are placed so that their initial points coincide, the angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is chosen as shown in the diagrams. Note that $0 \le \theta \le \pi$.







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Example 20

a If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

b If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $150°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Solution

a
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 30° \\ &= 20 \times \frac{\sqrt{3}}{2} \\ &= 10\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 150° \\ &= 20 \times \left(-\frac{\sqrt{3}}{2}\right) \\ &= -10\sqrt{3}\end{aligned}$$

Properties of the scalar product

The following properties can be established from the definition of the scalar product:

commutative law for scalar product	$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
distributive law	$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
compatibility with scalar multiplication	$k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
scalar product with zero	$\boldsymbol{a} \cdot \boldsymbol{0} = 0$

Several further properties follow from the geometric description of the scalar product:

- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in the same direction} \\ -|\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in opposite directions} \end{cases}$$

- For the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, we have $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$ and $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0$.

Example 21

a Simplify $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) - \boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c})$.

b Expand the following:

i $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$ **ii** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$

Solution

a
$$\begin{aligned}\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) - \boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) &= \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c} \\ &= \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}\end{aligned}$$

b **i**
$$\begin{aligned}(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b} \\ &= \boldsymbol{a} \cdot \boldsymbol{a} + 2(\boldsymbol{a} \cdot \boldsymbol{b}) + \boldsymbol{b} \cdot \boldsymbol{b}\end{aligned}$$

ii
$$\begin{aligned}(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) &= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{b} \\ &= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{b}\end{aligned}$$

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Example 22

Solve the equation $(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \cdot (3\boldsymbol{i} - x\boldsymbol{j} + 2\boldsymbol{k}) = 4$ for x.

Solution

$$\begin{aligned}(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \cdot (3\boldsymbol{i} - x\boldsymbol{j} + 2\boldsymbol{k}) &= 4 \\ 3 - x - 2 &= 4 \\ 1 - x &= 4 \\ \therefore \quad x &= -3\end{aligned}$$

Finding the magnitude of the angle between two vectors

The angle between two vectors can be found by using the two forms of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta \quad \text{and} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Therefore

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Example 23

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$$

Find the magnitude of $\angle ABC$, correct to one decimal place.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \cos(\angle ABC)$$

We have

$$\begin{aligned}\overrightarrow{BA} \cdot \overrightarrow{BC} &= 1 - 6 + 2 = -3 \\ |\overrightarrow{BA}| &= \sqrt{1 + 4 + 1} = \sqrt{6} \\ |\overrightarrow{BC}| &= \sqrt{1 + 9 + 4} = \sqrt{14}\end{aligned}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{-3}{\sqrt{6}\sqrt{14}}$$

Hence $\angle ABC = 109.1^\circ$, correct to one decimal place.

(Alternatively, we can write $\angle ABC = 1.9^c$, correct to one decimal place.)

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Exercise 4C

Example 19

1 Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$ **e** $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$

f $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

2 Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{a} \cdot \boldsymbol{b}$ **d** $\boldsymbol{a} \cdot \boldsymbol{c}$ **e** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b})$

Example 20

3 **a** If $|\boldsymbol{a}| = 6$, $|\boldsymbol{b}| = 7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $60°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

b If $|\boldsymbol{a}| = 6$, $|\boldsymbol{b}| = 7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $120°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Example 21

4 Expand and simplify:

a $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 2\boldsymbol{b})$ **b** $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$

c $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} + \boldsymbol{b})$ **d** $\dfrac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

Example 22

5 Solve each of the following equations:

a $(\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) \cdot (5\boldsymbol{i} + x\boldsymbol{j} + \boldsymbol{k}) = -6$ **b** $(x\boldsymbol{i} + 7\boldsymbol{j} - \boldsymbol{k}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j} + 5\boldsymbol{k}) = 10$

c $(x\boldsymbol{i} + 5\boldsymbol{k}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}) = x$ **d** $x(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) \cdot (\boldsymbol{i} + \boldsymbol{j} + x\boldsymbol{k}) = 6$

Example 23

6 If A and B are points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ respectively, find:

a $\overrightarrow{AB}$ **b** $|\overrightarrow{AB}|$ **c** the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$.

7 Let C and D be points with position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c} \cdot \boldsymbol{d} = 4$, find $|\overrightarrow{CD}|$.

8 $OABC$ is a rhombus with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

a Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$

b Find $\overrightarrow{OB} \cdot \overrightarrow{AC}$.

c Prove that the diagonals of a rhombus intersect at right angles.

9 From the following list, find three pairs of perpendicular vectors:

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{b} = -4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}, \quad \boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k},$$
$$\boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{e} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{f} = -\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$$

10 The four vertices of a regular tetrahedron have the following position vectors:

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{d} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

a Show that all the vertices are the same distance from the origin.

b Show that the angle between any two of these vectors is the same. Find this angle in degrees correct to two decimal places.

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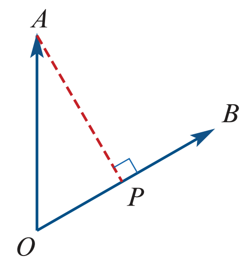
11 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$.
Let P be the point on OB such that AP is perpendicular to OB. Then $\overrightarrow{OP} = q\boldsymbol{b}$, for a constant q.



a Express $\overrightarrow{AP}$ in terms of q, $\boldsymbol{a}$ and $\boldsymbol{b}$.
b Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to find the value of q.
c Find the coordinates of the point P.

12 If $x\boldsymbol{i} + 2\boldsymbol{j} + y\boldsymbol{k}$ is perpendicular to vectors $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, find x and y.

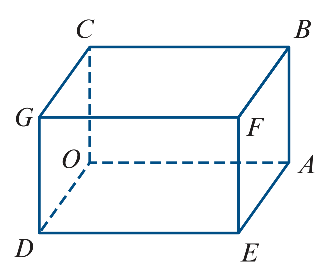
13 Find the angle, in radians, between each of the following pairs of vectors, correct to three significant figures:
a $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$
b $-2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $-2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$
c $2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$ and $4\boldsymbol{i} - 2\boldsymbol{k}$
d $7\boldsymbol{i} + \boldsymbol{k}$ and $-\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$

14 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors such that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Use the geometric description of the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

For Questions 15–18, find the angles in degrees correct to two decimal places.

15 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ respectively. Let M be the midpoint of AB. Find:
a $\overrightarrow{OM}$ **b** $\angle AOM$ **c** $\angle BMO$

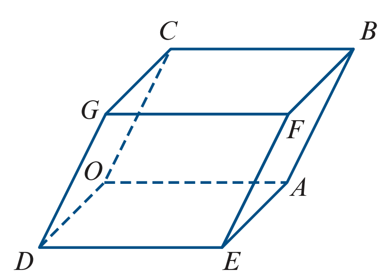
16 $OABCDEFG$ is a cuboid, set on axes at O, such that $\overrightarrow{OD} = \boldsymbol{i}$, $\overrightarrow{OA} = 3\boldsymbol{j}$ and $\overrightarrow{OC} = 2\boldsymbol{k}$. Find:



a **i** $\overrightarrow{GB}$ **ii** $\overrightarrow{GE}$
b $\angle BGE$
c the angle between diagonals $\overrightarrow{CE}$ and $\overrightarrow{GA}$

17 Let A, B and C be the points defined by the position vectors $4\boldsymbol{i}$, $5\boldsymbol{j}$ and $-2\boldsymbol{i} + 7\boldsymbol{k}$ respectively. Let M and N be the midpoints of AB and AC respectively. Find:
a **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$ **b** $\angle MON$ **c** $\angle MOC$

18 A parallelepiped is an oblique prism that has a parallelogram cross-section. It has three pairs of parallel and congruent faces, each of which is a parallelogram.



$OABCDEFG$ is a parallelepiped with $\overrightarrow{OA} = 3\boldsymbol{j}$, $\overrightarrow{OC} = -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\overrightarrow{OD} = 2\boldsymbol{i} - \boldsymbol{j}$.
Show that the diagonals DB and CE bisect each other, and find the acute angle between them.

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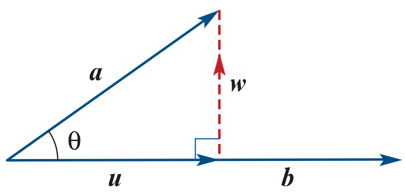
4D Vector projections

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram, it can be seen that

$$\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$$

where $\boldsymbol{u} = k\boldsymbol{b}$ and so $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$.



For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, we must have

$$\boldsymbol{w} \cdot \boldsymbol{b} = 0$$
$$(\boldsymbol{a} - k\boldsymbol{b}) \cdot \boldsymbol{b} = 0$$
$$\boldsymbol{a} \cdot \boldsymbol{b} - k(\boldsymbol{b} \cdot \boldsymbol{b}) = 0$$

Hence $k = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}$ and therefore $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

This vector $\boldsymbol{u}$ is called the **vector projection** (or **vector resolute**) of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Vector resolute

The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be expressed in any one of the following equivalent forms:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Note: The quantity $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that, from our previous calculation, we have $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

Expressing $\boldsymbol{a}$ as the sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, gives

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**.

Example 24

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$. Find the vector resolute of:

a $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 3 - 2 = -4$, $\quad \boldsymbol{b} \cdot \boldsymbol{b} = 1 + 1 + 4 = 6$

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = -\frac{4}{6}(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -\frac{2}{3}(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$$

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b $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = -4, \quad \boldsymbol{a} \cdot \boldsymbol{a} = 1 + 9 + 1 = 11$

The vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$$\frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} = -\frac{4}{11}(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$$

Example 25

Find the scalar resolute of $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ in the direction of $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{k}$.

Solution

$\boldsymbol{a} \cdot \boldsymbol{b} = -2 - 3 = -5$

$|\boldsymbol{b}| = \sqrt{1 + 9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{-5}{\sqrt{10}} = -\frac{\sqrt{10}}{2}$$

Example 26

Resolve $\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ into rectangular components, one of which is parallel to $2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$.

Solution

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - 6 + 1 = -3$$

$$\boldsymbol{b} \cdot \boldsymbol{b} = 4 + 4 + 1 = 9$$

Therefore the vector resolute is

$$\frac{-3}{9}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = -\frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$$

The perpendicular component is

$$\begin{aligned}\boldsymbol{a} - \left(-\frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})\right) &= (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) + \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) \\ &= \frac{5}{3}\boldsymbol{i} + \frac{7}{3}\boldsymbol{j} - \frac{4}{3}\boldsymbol{k} \\ &= \frac{1}{3}(5\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k})\end{aligned}$$

Hence we can write

$$\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} = -\frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) + \frac{1}{3}(5\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k})$$

Check: As a check, we verify that the second component is indeed perpendicular to $\boldsymbol{b}$. We have $(5\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}) \cdot (2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 10 - 14 + 4 = 0$, as expected.

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Skill-sheet

Exercise 4D

1 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$.

a Find $\hat{\boldsymbol{a}}$.

b Find $\hat{\boldsymbol{b}}$.

c Find $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = \overrightarrow{AB}$.

2 Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $\hat{\boldsymbol{b}}$

b Find the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

3 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors that are not parallel. In this question, we will prove that the vector $\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}$ bisects the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

a Let O be the origin, and let A' and B' be the points with position vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ respectively. Show that $OA'B'$ is an isosceles triangle.

b Show that the midpoint of line segment $A'B'$ has position vector $\frac{1}{2}(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}})$.

c Show that the vector $\frac{1}{2}(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}})$ bisects the angle between $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$.

d Hence explain why the vector $\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}$ bisects the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

4 Points A and B are defined by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{k}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $\hat{\boldsymbol{b}}$

b Find the unit vector which bisects $\angle AOB$.

Example 24

5 For each pair of vectors, find the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:

a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

b $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

c $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} - \boldsymbol{k}$

Example 25

6 For each of the following pairs of vectors, find the scalar resolute of the first vector in the direction of the second vector:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$

c $\boldsymbol{b} = 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{a} = 2\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$

d $\boldsymbol{b} = \boldsymbol{i} - \sqrt{5}\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$

7 Let $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i}$.

a Find the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

b Find the scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

c Draw a diagram illustrating the rectangular decomposition of $\boldsymbol{a}$ into components parallel and perpendicular to $\boldsymbol{b}$.

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Example 26

8 For each of the following pairs of vectors, find the resolution of the vector $\boldsymbol{a}$ into rectangular components, one of which is parallel to $\boldsymbol{b}$:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k},\ \boldsymbol{b} = 5\boldsymbol{i} - \boldsymbol{k}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{k}$

c $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k},\ \boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

9 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{j} + \boldsymbol{k}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b a unit vector through A perpendicular to OB

10 Let A and B be the points defined by the position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$

c the shortest distance from A to line OB

11 Points A, B and C have position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to line AC

d the area of triangle ABC

12 **a** Verify that vectors $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = 5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ are perpendicular to each other.

b If $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{k}$, find:

i $\boldsymbol{d}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$

ii $\boldsymbol{e}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$.

c Find $\boldsymbol{f}$ such that $\boldsymbol{c} = \boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}$.

d Hence show that $\boldsymbol{f}$ is perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

13 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be perpendicular vectors. In this question, we will show how to decompose a general vector $\boldsymbol{c}$ into three components: one parallel to $\boldsymbol{a}$, one parallel to $\boldsymbol{b}$ and the other perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. (In specific cases, some components may be zero.)

a Let $\boldsymbol{d}$ be the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$. Write $\boldsymbol{d}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b Let $\boldsymbol{e}$ be the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$. Write $\boldsymbol{e}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

c Find $\boldsymbol{f}$ such that $\boldsymbol{c} = \boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}$.

d Show that, if $\boldsymbol{f}$ is non-zero, then $\boldsymbol{f}$ is perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

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4E Collinearity

Three or more points are said to be **collinear** if they all lie on a single line.

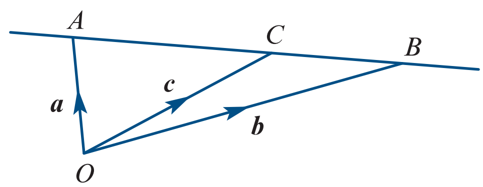
> Three distinct points A, B and C are collinear if and only if there exists a non-zero real number m such that $\overrightarrow{AC} = m\overrightarrow{AB}$ (that is, if and only if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel).

A property of collinearity

Let points A, B and C have position vectors $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Then

$$\overrightarrow{AC} = m\overrightarrow{AB} \quad \text{if and only if} \quad \boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b}$$

Proof If $\overrightarrow{AC} = m\overrightarrow{AB}$, then we have



$$\begin{aligned}\boldsymbol{c} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \overrightarrow{OA} + m\overrightarrow{AB}\\ &= \boldsymbol{a} + m(\boldsymbol{b} - \boldsymbol{a})\\ &= \boldsymbol{a} + m\boldsymbol{b} - m\boldsymbol{a}\\ &= (1-m)\boldsymbol{a} + m\boldsymbol{b}\end{aligned}$$

Similarly, we can show that if $\boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b}$, then $\overrightarrow{AC} = m\overrightarrow{AB}$.

Note: It follows from this result that if distinct points A, B and C are collinear, then we can write $\overrightarrow{OC} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB}$, where $\lambda + \mu = 1$. If C is between A and B, then $0 < \mu < 1$.

Example 27

For distinct points A and B, let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. Express $\overrightarrow{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, where C is:

a the midpoint of AB

b the point of trisection of AB nearer to A

c the point C such that $\overrightarrow{AC} = -2\overrightarrow{AB}$.

Solution

a $\overrightarrow{AC} = \frac{1}{2}\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \boldsymbol{a} + \frac{1}{2}\overrightarrow{AB}\\ &= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\\ &= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\end{aligned}$$

b $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \boldsymbol{a} + \frac{1}{3}\overrightarrow{AB}\\ &= \boldsymbol{a} + \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})\\ &= \frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}\end{aligned}$$

c $\overrightarrow{AC} = -2\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \boldsymbol{a} - 2\overrightarrow{AB}\\ &= \boldsymbol{a} - 2(\boldsymbol{b} - \boldsymbol{a})\\ &= 3\boldsymbol{a} - 2\boldsymbol{b}\end{aligned}$$

Note: Alternatively, we could have used the previous result in this example.

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Example 28

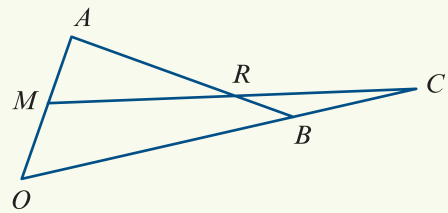
Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$, where vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.

Let M be the midpoint of OA, let C be the point such that $\overrightarrow{OC} = \frac{4}{3}\overrightarrow{OB}$ and let R be the point of intersection of lines AB and MC.

a Find $\overrightarrow{OR}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. **b** Hence find $AR : RB$.

Solution

a We have $\overrightarrow{OM} = \frac{1}{2}\boldsymbol{a}$ and $\overrightarrow{OC} = \frac{4}{3}\boldsymbol{b}$.



Since M, R and C are collinear, there exists $m \in \mathbb{R}$ with

$$\begin{aligned}\overrightarrow{MR} &= m\overrightarrow{MC}\\ &= m(\overrightarrow{MO} + \overrightarrow{OC})\\ &= m\left(-\frac{1}{2}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}\right)\end{aligned}$$

Thus
$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OM} + \overrightarrow{MR}\\ &= \frac{1}{2}\boldsymbol{a} + m\left(-\frac{1}{2}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}\right)\\ &= \frac{1-m}{2}\boldsymbol{a} + \frac{4m}{3}\boldsymbol{b}\end{aligned}$$

Since A, R and B are collinear, there exists $n \in \mathbb{R}$ with

$$\begin{aligned}\overrightarrow{AR} &= n\overrightarrow{AB}\\ &= n(\overrightarrow{AO} + \overrightarrow{OB})\\ &= n(-\boldsymbol{a} + \boldsymbol{b})\end{aligned}$$

Thus
$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OA} + \overrightarrow{AR}\\ &= \boldsymbol{a} + n(-\boldsymbol{a} + \boldsymbol{b})\\ &= (1-n)\boldsymbol{a} + n\boldsymbol{b}\end{aligned}$$

Hence, since $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, we have

$$\frac{1-m}{2} = 1 - n \quad \text{and} \quad \frac{4m}{3} = n$$

This gives $m = \frac{3}{5}$ and $n = \frac{4}{5}$. Therefore $\overrightarrow{OR} = \frac{1}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b}$.

b From part **a**, we have

$$\begin{aligned}\overrightarrow{AR} &= \overrightarrow{AO} + \overrightarrow{OR}\\ &= -\boldsymbol{a} + \frac{1}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b}\\ &= \frac{4}{5}(\boldsymbol{b} - \boldsymbol{a}) = \frac{4}{5}\overrightarrow{AB}\end{aligned}$$

Hence $AR : RB = 4 : 1$.

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Exercise 4E

Example 27

1 Points A, B and R are collinear, with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Express $\overrightarrow{OR}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, where R is:

- **a** the point of trisection of AB nearer to B
- **b** the point between A and B such that $AR : RB = 3 : 2$.

2 Let $\overrightarrow{OA} = 3\boldsymbol{i} + 4\boldsymbol{k}$ and $\overrightarrow{OB} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$. Find $\overrightarrow{OR}$, where R is:

- **a** the midpoint of line segment AB
- **b** the point such that $\overrightarrow{AR} = \frac{4}{3}\overrightarrow{AB}$
- **c** the point such that $\overrightarrow{AR} = -\frac{1}{3}\overrightarrow{AB}$.

3 The position vectors of points P, Q and R are $\boldsymbol{a}$, $3\boldsymbol{a} - 4\boldsymbol{b}$ and $4\boldsymbol{a} - 6\boldsymbol{b}$ respectively.

- **a** Show that P, Q and R are collinear.
- **b** Find $PQ : QR$.

4 In triangle OAB, $\overrightarrow{OA} = a\boldsymbol{i}$ and $\overrightarrow{OB} = x\boldsymbol{i} + y\boldsymbol{j}$. Let C be the midpoint of AB.

- **a** Find $\overrightarrow{OC}$.
- **b** If the vector $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$, find the relationship between x, y and a.

5 In parallelogram $OAUB$, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Let $\overrightarrow{OM} = \frac{1}{5}\boldsymbol{a}$ and $MP : PB = 1 : 5$, where P is on the line segment MB.

- **a** Prove that P is on the diagonal OU.
- **b** Hence find $OP : PU$.

6 $OABC$ is a square with $\overrightarrow{OA} = -4\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OC} = 3\boldsymbol{i} + 4\boldsymbol{j}$.

- **a** Find $\overrightarrow{OB}$.
- **b** Given that D is the point on AB such that $\overrightarrow{BD} = \frac{1}{3}\overrightarrow{BA}$, find $\overrightarrow{OD}$.
- **c** Given that OD intersects AC at E and that $\overrightarrow{OE} = (1 - \lambda)\overrightarrow{OA} + \lambda\overrightarrow{OC}$, find λ.

7 In triangle OAB, $\overrightarrow{OA} = 3\boldsymbol{i} + 4\boldsymbol{k}$ and $\overrightarrow{OB} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$.

- **a** Use the scalar product to show that $\angle AOB$ is an obtuse angle.
- **b** Find $\overrightarrow{OP}$, where P is:
  - **i** the midpoint of AB
  - **ii** the point on AB such that OP is perpendicular to AB
  - **iii** the point where the bisector of $\angle AOB$ intersects AB.

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4F Geometric proofs

In this section we use vectors to prove geometric results in two and three dimensions. The following properties of vectors will be useful:

Parallel vectors

- For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$.
- Three distinct points A, B and C are collinear if and only if $\overrightarrow{AC} = k\overrightarrow{AB}$ for some $k \in \mathbb{R} \setminus \{0\}$.

Scalar product

- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Linear combinations of independent vectors

- Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two linearly independent (i.e. not parallel) vectors. Then $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$ implies $m = p$ and $n = q$.

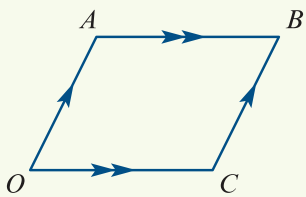
Vector proofs in two-dimensional geometry

Example 29

Prove that the diagonals of a rhombus are perpendicular.

Solution

$OABC$ is a rhombus.



Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

The diagonals of the rhombus are OB and AC.

$$\begin{aligned} \text{Now } \overrightarrow{OB} &= \overrightarrow{OC} + \overrightarrow{CB} \\ &= \overrightarrow{OC} + \overrightarrow{OA} \\ &= \boldsymbol{c} + \boldsymbol{a} \end{aligned}$$

$$\begin{aligned} \text{and } \overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= -\boldsymbol{a} + \boldsymbol{c} \end{aligned}$$

Consider the scalar product of $\overrightarrow{OB}$ and $\overrightarrow{AC}$:

$$\begin{aligned} \overrightarrow{OB} \cdot \overrightarrow{AC} &= (\boldsymbol{c} + \boldsymbol{a}) \cdot (\boldsymbol{c} - \boldsymbol{a}) \\ &= \boldsymbol{c} \cdot \boldsymbol{c} - \boldsymbol{a} \cdot \boldsymbol{a} \\ &= |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2 \end{aligned}$$

A rhombus has all sides of equal length, and therefore $|\boldsymbol{c}| = |\boldsymbol{a}|$. Hence

$$\overrightarrow{OB} \cdot \overrightarrow{AC} = |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2 = 0$$

This implies that AC is perpendicular to OB.

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Note: For many geometric proofs, there are different possible approaches that you can take. In particular, there are different ways to choose the initial vectors. Finding efficient approaches to these proofs comes with experience and practice.

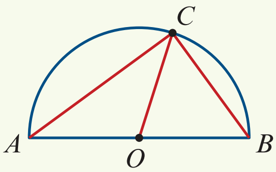
For the next example, we require the following definition:

- Suppose that we have a line segment AB and a point C not on AB. Then $\angle ACB$ is the angle **subtended** by AB at the point C.

Example 30

Prove that the angle subtended by a diameter at a point on a circle is a right angle.

Solution



Let O be the centre of the circle and let AB be a diameter.

Let C be a point on the circle (other than A or B).

We aim to show that $\angle ACB$ is a right angle.

We have $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = r$, where r is the radius.

Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Then $\overrightarrow{OB} = -\boldsymbol{a}$.

$$\text{Now } \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\boldsymbol{a} + \boldsymbol{c}$$

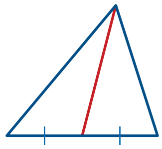
$$\text{and } \overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \boldsymbol{a} + \boldsymbol{c}$$

$$\begin{aligned} \text{Thus } \overrightarrow{AC} \cdot \overrightarrow{BC} &= (-\boldsymbol{a} + \boldsymbol{c}) \cdot (\boldsymbol{a} + \boldsymbol{c}) \\ &= -\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c} \\ &= -|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2 \\ &= -r^2 + r^2 \qquad (\text{since } |\boldsymbol{a}| = |\boldsymbol{c}| = r) \\ &= 0 \end{aligned}$$

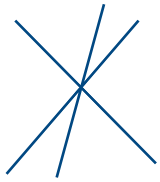
Hence $AC \perp BC$. We have shown that $\angle ACB$ is a right angle.

For the next example, we require the following two definitions:

- A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.



- Three or more lines are said to be **concurrent** if they all pass through a single point.



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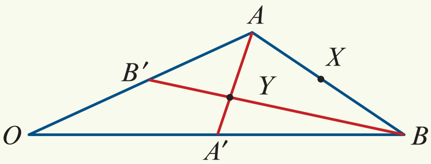
Example 31

Prove that the medians of a triangle are concurrent.

Solution

Consider triangle OAB. Let A', B' and X be the midpoints of OB, OA and AB respectively.



Let Y be the point of intersection of the medians AA' and BB'.

Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.

We start by showing that $AY : YA' = BY : YB' = 2 : 1$.

We have $\overrightarrow{AY} = \lambda\overrightarrow{AA'}$ and $\overrightarrow{BY} = \mu\overrightarrow{BB'}$, for some $\lambda, \mu \in \mathbb{R}$.

Now
$$\overrightarrow{AA'} = \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$
$$\therefore \quad \overrightarrow{AY} = \lambda\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$$
and
$$\overrightarrow{BB'} = \overrightarrow{BO} + \frac{1}{2}\overrightarrow{OA} = -\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$$
$$\therefore \quad \overrightarrow{BY} = \mu\left(-\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}\right)$$

But $\overrightarrow{BY}$ can also be obtained as follows:
$$\begin{aligned}\overrightarrow{BY} &= \overrightarrow{BA} + \overrightarrow{AY}\\ &= \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AY}\\ &= -\boldsymbol{b} + \boldsymbol{a} + \lambda\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)\end{aligned}$$
$$\therefore \quad -\mu\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{a} = (1-\lambda)\boldsymbol{a} + \left(\frac{\lambda}{2} - 1\right)\boldsymbol{b}$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent vectors, we now have
$$\frac{\mu}{2} = 1 - \lambda \quad (1) \qquad \text{and} \qquad -\mu = \frac{\lambda}{2} - 1 \quad (2)$$

Multiply (1) by 2 and add to (2):
$$0 = 2 - 2\lambda + \frac{\lambda}{2} - 1$$
$$1 = \frac{3\lambda}{2}$$
$$\therefore \quad \lambda = \frac{2}{3}$$

Substitute in (1) to find $\mu = \frac{2}{3}$. We have shown that $AY : YA' = BY : YB' = 2 : 1$.

Now, by symmetry, the point of intersection of the medians AA' and OX must also divide AA' in the ratio $2 : 1$, and therefore must be Y. Hence the three medians meet at Y.

Note: The point where the three medians meet is called the **centroid** of the triangle.

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Vector proofs in three-dimensional geometry

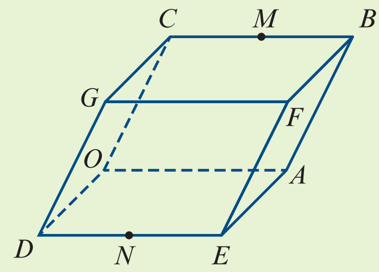
Example 32

Consider a parallelepiped $OABCDEFG$ as shown.

a Prove that the diagonals OF and CE bisect each other.

b Let M be the midpoint of CB, and let N be the midpoint of DE.

Prove that the midpoint of MN is the point where the diagonals OF and CE intersect.



Solution

Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{d} = \overrightarrow{OD}$.

a *We will show that the midpoint of OF and the midpoint of CE coincide.*

Let X be the midpoint of OF. Then

$$\begin{aligned}\overrightarrow{OX} &= \tfrac{1}{2}\overrightarrow{OF}\\ &= \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF})\\ &= \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})\end{aligned}$$

Let Y be the midpoint of CE. Then

$$\begin{aligned}\overrightarrow{OY} &= \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{OE})\\ &= \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{AE})\\ &= \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{a} + \boldsymbol{d})\\ &= \overrightarrow{OX}\end{aligned}$$

Therefore $X = Y$, and so the diagonals OF and CE bisect each other.

b We have

$$\overrightarrow{OM} = \overrightarrow{OC} + \tfrac{1}{2}\overrightarrow{CB} = \boldsymbol{c} + \tfrac{1}{2}\boldsymbol{a}$$

$$\overrightarrow{ON} = \overrightarrow{OD} + \tfrac{1}{2}\overrightarrow{DE} = \boldsymbol{d} + \tfrac{1}{2}\boldsymbol{a}$$

Let Z be the midpoint of MN. Then

$$\begin{aligned}\overrightarrow{OZ} &= \tfrac{1}{2}(\overrightarrow{OM} + \overrightarrow{ON})\\ &= \tfrac{1}{2}(\boldsymbol{c} + \tfrac{1}{2}\boldsymbol{a} + \boldsymbol{d} + \tfrac{1}{2}\boldsymbol{a})\\ &= \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})\end{aligned}$$

Therefore $Z = X$, where X is the point of intersection of OF and CE found in part **a**. Hence X is the midpoint of MN.

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Exercise 4F

Vector proofs in two-dimensional geometry

1 Prove that the diagonals of a parallelogram bisect each other.

2 Prove that if the midpoints of the sides of a rectangle are joined, then a rhombus is formed.

3 Prove that if the midpoints of the sides of a square are joined, then another square is formed.

4 Prove that the median to the base of an isosceles triangle is perpendicular to the base.

5 Prove that if the diagonals of a parallelogram are of equal length, then the parallelogram is a rectangle.

6 Prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.

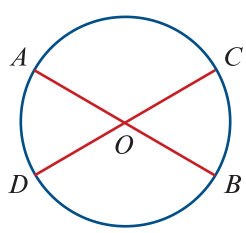
7 Prove that the sum of the squares of the lengths of the diagonals of any parallelogram is equal to the sum of the squares of the lengths of the sides.

8 Prove that if the midpoints of the sides of a quadrilateral are joined, then a parallelogram is formed.

9 Let $ABCD$ be a parallelogram, let M be the midpoint of AB and let P be the point of trisection of MD nearer to M. Prove that A, P and C are collinear and that P is a point of trisection of AC.

10 Let $ABCD$ be a parallelogram with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. The point P lies on AD and is such that $AP : PD = 1 : 2$ and the point Q lies on BD and is such that $BQ : QD = 2 : 1$. Show that PQ is parallel to AC.

11 AB and CD are diameters of a circle with centre O. Prove that $ACBD$ is a rectangle.



12 Apollonius' theorem

In triangle AOB, $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and M is the midpoint of AB.

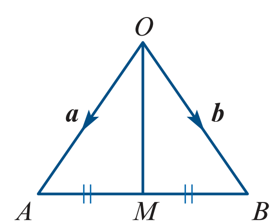
a Find:

i $\overrightarrow{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

ii $\overrightarrow{OM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

b Find $\overrightarrow{AM} \cdot \overrightarrow{AM} + \overrightarrow{OM} \cdot \overrightarrow{OM}$.

c Hence prove that $OA^2 + OB^2 = 2OM^2 + 2AM^2$.



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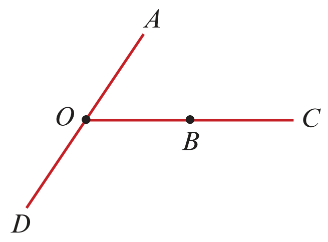
13 In the figure, O is the midpoint of AD and B is the midpoint of OC. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.
Let P be the point such that $\overrightarrow{OP} = \frac{1}{3}(\boldsymbol{a} + 4\boldsymbol{b})$.



a Prove that A, P and C are collinear.
b Prove that D, B and P are collinear.
c Find $DB : BP$.

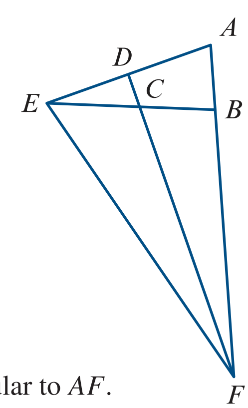
14 In triangle AOB, $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. The point P is on AB such that the length of AP is twice the length of BP. The point Q is such that $\overrightarrow{OQ} = 3\overrightarrow{OP}$.
a Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
i $\overrightarrow{OP}$ **ii** $\overrightarrow{OQ}$ **iii** $\overrightarrow{AQ}$
b Hence show that $\overrightarrow{AQ}$ is parallel to $\overrightarrow{OB}$.

15 $ORST$ is a parallelogram, U is the midpoint of RS and V is the midpoint of ST. Relative to the origin O, the position vectors of points R, S, T, U and V are $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.
a Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$.
b Express $\boldsymbol{u}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{s}$, and express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.
c Hence, or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

16 The points A, B, C, D and E shown in the diagram have position vectors

$$\boldsymbol{a} = \boldsymbol{i} + 11\boldsymbol{j} \qquad \boldsymbol{b} = 2\boldsymbol{i} + 8\boldsymbol{j} \qquad \boldsymbol{c} = -\boldsymbol{i} + 7\boldsymbol{j}$$
$$\boldsymbol{d} = -2\boldsymbol{i} + 8\boldsymbol{j} \qquad \boldsymbol{e} = -4\boldsymbol{i} + 6\boldsymbol{j}$$

respectively. The lines AB and DC intersect at F as shown.



a Show that E lies on the lines DA and BC.
b Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$.
c Find the position vector of the point F.
d Show that FD is perpendicular to EA and that EB is perpendicular to AF.
e Find the position vector of the centre of the circle through E, D, B and F.

17 Coplanar points A, B, C, D and E have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ and $\boldsymbol{e}$ respectively, relative to an origin O. The point A is the midpoint of OB and the point E divides AC in the ratio 1 : 2. If $\boldsymbol{e} = \frac{1}{3}\boldsymbol{d}$, show that $OCDB$ is a parallelogram.

18 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O. The point P divides the line segment OA in the ratio 1 : 3 and the point R divides the line segment AB in the ratio 1 : 2. Given that $PRBQ$ is a parallelogram, determine the position of Q.

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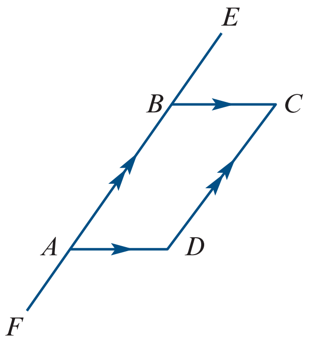
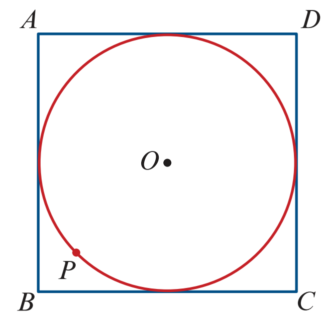
19 Let ABC be a triangle, where the points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively. Show that the centroid of triangle ABC has position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$.

20 $ABCD$ is a parallelogram, AB is extended to E and BA is extended to F such that $BE = AF = BC$. Line segments EC and FD are extended to meet at X.



- **a** Prove that the lines EX and FX meet at right angles.
- **b** If $\overrightarrow{EX} = \lambda\overrightarrow{EC}$, $\overrightarrow{FX} = \mu\overrightarrow{FD}$ and $|\overrightarrow{AB}| = k|\overrightarrow{BC}|$, find the values of λ and μ in terms of k.
- **c** Find the values of λ and μ if $ABCD$ is a rhombus.
- **d** If $|\overrightarrow{EX}| = |\overrightarrow{FX}|$, prove that $ABCD$ is a rectangle.

21 In the figure, the circle has centre O and radius r. The circle is inscribed in a square $ABCD$, and P is any point on the circle.

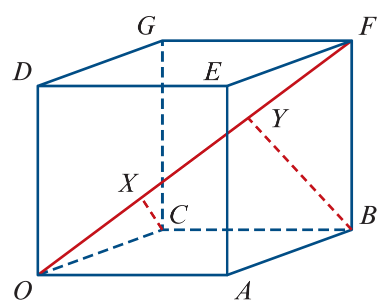


- **a** Show that $\overrightarrow{AP} \cdot \overrightarrow{AP} = 3r^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OA}$.
- **b** Hence find $AP^2 + BP^2 + CP^2 + DP^2$ in terms of r.

Vector proofs in three-dimensional geometry

22 A 'space diagonal' of a polyhedron is a line segment connecting two vertices that are not on the same face. Prove that the space diagonals of a rectangular prism are of equal length and bisect each other.

23 Consider a rectangular prism $OABCDEFG$ as shown. Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{d} = \overrightarrow{OD}$. Let $a = |\boldsymbol{a}|$, $c = |\boldsymbol{c}|$ and $d = |\boldsymbol{d}|$.



- **a** Let X be the point on diagonal OF such that CX is perpendicular to OF. Find the position vector of X in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.
- **b** Let Y be the point on diagonal OF such that BY is perpendicular to OF. Find the position vector of Y in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.
- **c** If $a = c = d = 1$, find:
  - **i** the position vectors of X and Y
  - **ii** the magnitude of $\angle CXA$
  - **iii** the magnitude of $\angle BYG$

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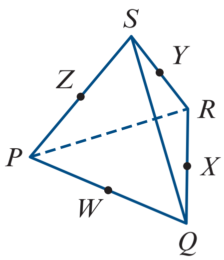
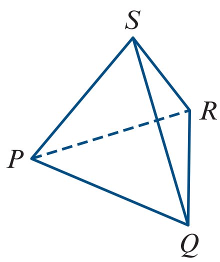
24 Let P, Q, R and S be four points in space that do not lie in the same plane. Let W, X, Y and Z be the midpoints of PQ, QR, RS and SP respectively. Relative to an origin O, denote the position vectors of points P, Q, R, S, W, X, Y, Z by $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ respectively. Prove that $WXYZ$ is a parallelogram.



25 A tetrahedron is a polyhedron with four triangular faces. In a regular tetrahedron, each face is an equilateral triangle.

Prove that, for a regular tetrahedron, the line segments joining the midpoints of opposite edges have a common midpoint.

Note: For a tetrahedron $PQRS$, the edges PQ and RS are opposite, the edges PR and QS are opposite, and the edges PS and QR are opposite.



26 Point C is a vertex of the regular tetrahedron $OABC$. Point G is the centroid of triangle OAB. Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Find $\overrightarrow{OG}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Prove that $\overrightarrow{CG}$ is perpendicular to $\overrightarrow{OG}$.

27 Prove that opposite edges of a regular tetrahedron are perpendicular.

28 Let $OABC$ be a tetrahedron. Assume that edge OA is perpendicular to edge BC, and that edge OB is perpendicular to edge AC. Prove that edge OC is perpendicular to edge AB.

29 Let $OABC$ be a tetrahedron such that opposite edges are perpendicular. Show that

$$OA^2 + BC^2 = OB^2 + AC^2 = OC^2 + AB^2$$

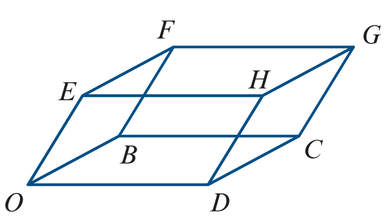
30 A regular tetrahedron $VABC$ has edges of length 4 cm.

a Let T be the point on VC such that AT is perpendicular to VC. Find the value of λ such that $\overrightarrow{VT} = \lambda\overrightarrow{VC}$.

b Prove that BT is perpendicular to VC.

c Find the magnitude of $\angle ATB$.

31 $OBCDEFGH$ is a parallelepiped. Let $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{d} = \overrightarrow{OD}$ and $\boldsymbol{e} = \overrightarrow{OE}$.



a Express each of the vectors $\overrightarrow{OG}$, $\overrightarrow{DF}$, $\overrightarrow{BH}$ and $\overrightarrow{CE}$ in terms of $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{e}$.

b Find $|\overrightarrow{OG}|^2$, $|\overrightarrow{DF}|^2$, $|\overrightarrow{BH}|^2$ and $|\overrightarrow{CE}|^2$ in terms of $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{e}$.

c Show that $|\overrightarrow{OG}|^2 + |\overrightarrow{DF}|^2 + |\overrightarrow{BH}|^2 + |\overrightarrow{CE}|^2 = 4(|\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2)$.

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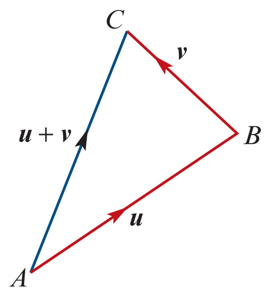
Chapter summary

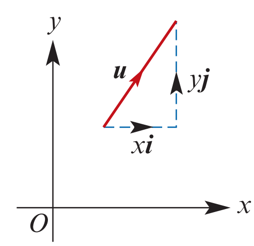
- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point A is the vector $\overrightarrow{OA}$, where O is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

Basic operations on vectors

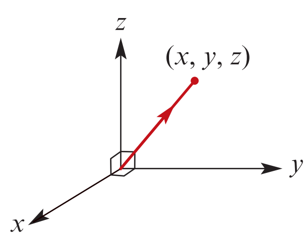


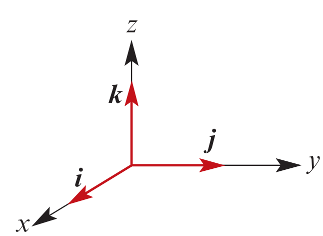
- Addition
  - The sum $\boldsymbol{u} + \boldsymbol{v}$ is obtained geometrically as shown.
  - If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.
- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - The vector $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$, but the opposite direction.
  - Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- Subtraction $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

Component form



- In two dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where:
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- In three dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors as shown.





- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.
- If the vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ makes angles α, β and γ with the positive directions of the x-, y- and z-axes respectively, then

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \qquad \cos\beta = \frac{a_2}{|\boldsymbol{a}|} \qquad \text{and} \qquad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

- The **unit vector** in the direction of vector $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

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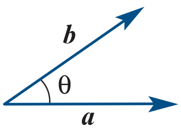
Review

Scalar product and vector projections

- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

- The scalar product is described geometrically by $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

Linear dependence and independence

- A set of vectors is said to be **linearly dependent** if at least one of its members can be expressed as a linear combination of other vectors in the set.
- A set of vectors is said to be **linearly independent** if it is not linearly dependent.
- Linear combinations of independent vectors: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two linearly independent (i.e. not parallel) vectors. Then $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$ implies $m = p$ and $n = q$.

Technology-free questions

1 $ABCD$ is a parallelogram, where A, B and C have position vectors $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $4\boldsymbol{i} - \boldsymbol{k}$ respectively. Find:

a $\overrightarrow{AD}$ **b** the cosine of $\angle BAD$

2 Points A, B and C are defined by position vectors $2\boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k}$, $-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$ respectively. Point M is on the line segment AB such that $|\overrightarrow{AM}| = |\overrightarrow{AC}|$.

a Find:

i $\overrightarrow{AM}$ **ii** the position vector of N, the midpoint of CM

b Hence show that $\overrightarrow{AN} \perp \overrightarrow{CM}$.

3 Let $\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + x\boldsymbol{k}$ and $\boldsymbol{c} = y\boldsymbol{i} + z\boldsymbol{j} - 2\boldsymbol{k}$. Find:

a x such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other

b y and z such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are mutually perpendicular

4 Let $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and let $\boldsymbol{b}$ be a vector such that the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\hat{\boldsymbol{b}}$.

a Find the cosine of the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Find $|\boldsymbol{b}|$ if the vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is $2\hat{\boldsymbol{a}}$.

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5 Let $\boldsymbol{a} = 3\boldsymbol{i} - 6\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$.

a Find $\boldsymbol{c}$, the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

b Find $\boldsymbol{d}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$.

c Hence show that $|\boldsymbol{a}|\,|\boldsymbol{d}| = |\boldsymbol{c}|^2$.

6 Points A and B have position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$. Point C has position vector $\boldsymbol{c} = 2\boldsymbol{i} + (1 + 3t)\boldsymbol{j} + (-1 + 2t)\boldsymbol{k}$.

a Find in terms of t:

i $\overrightarrow{CA}$ **ii** $\overrightarrow{CB}$

b Find the values of t for which $\angle BCA = 90^\circ$.

7 $OABC$ is a parallelogram, where A and C have position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 6\boldsymbol{j} - 3\boldsymbol{k}$ respectively.

a Find:

i $|\boldsymbol{a} - \boldsymbol{c}|$ **ii** $|\boldsymbol{a} + \boldsymbol{c}|$ **iii** $(\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

b Hence find the acute angle between the diagonals of the parallelogram.

8 $OABC$ is a trapezium with $\overrightarrow{OC} = 2\overrightarrow{AB}$. If $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$ and $\overrightarrow{OC} = 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$, find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{BC}$ **c** the cosine of $\angle BAC$.

9 The position vectors of A and B, relative to an origin O, are $6\boldsymbol{i} + 4\boldsymbol{j}$ and $3\boldsymbol{i} + p\boldsymbol{j}$.

a Express $\overrightarrow{AO} \cdot \overrightarrow{AB}$ in terms of p.

b Find the value of p for which $\overrightarrow{AO}$ is perpendicular to $\overrightarrow{AB}$.

c Find the cosine of $\angle OAB$ when $p = 6$.

10 Points A, B and C have position vectors $\boldsymbol{p} + \boldsymbol{q}$, $3\boldsymbol{p} - 2\boldsymbol{q}$ and $6\boldsymbol{p} + m\boldsymbol{q}$ respectively, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are non-zero, non-parallel vectors. Find the value of m such that the points A, B and C are collinear.

11 If $\boldsymbol{r} = 3\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$, $\boldsymbol{s} = \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$ and $\boldsymbol{t} = -2\boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k}$, find the values of λ and μ such that the vector $\boldsymbol{r} + \lambda\boldsymbol{s} + \mu\boldsymbol{t}$ is parallel to the x-axis.

12 Show that the points $A(4, 3, 0)$, $B(5, 2, 3)$, $C(4, -1, 3)$ and $D(2, 1, -3)$ form a trapezium and state the ratio of the parallel sides.

13 If $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 6\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$, show that $\boldsymbol{a} + \boldsymbol{b}$ is perpendicular to $\boldsymbol{b}$ and find the cosine of the angle between the vectors $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$.

14 O, A and B are the points with coordinates $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

a Let C be the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$. Find the coordinates of C.

b Let D be the point $(1, 24)$. If $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, find the values of h and k.

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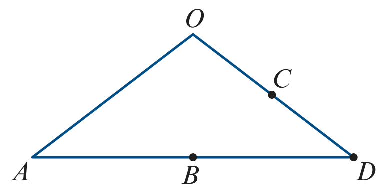
15 Relative to O, the position vectors of A, B and C are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Points B and C are the midpoints of AD and OD respectively.



a Find $\overrightarrow{OD}$ and $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

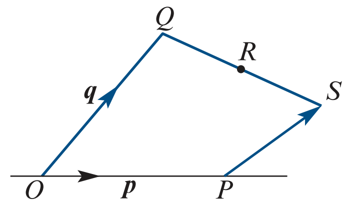
b Find $\boldsymbol{b}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

c Point E on the extension of OA is such that $\overrightarrow{OE} = 4\overrightarrow{AE}$. If $\overrightarrow{CB} = k\overrightarrow{AE}$, find the value of k.

16 $\overrightarrow{OP} = \boldsymbol{p}$ $\qquad$ $\overrightarrow{OQ} = \boldsymbol{q}$

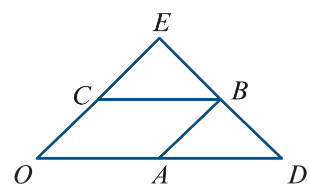
$\overrightarrow{OR} = \frac{1}{3}\boldsymbol{p} + k\boldsymbol{q}$ $\qquad$ $\overrightarrow{OS} = h\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$

Given that R is the midpoint of QS, find h and k.



17 ABC is a right-angled triangle with the right angle at B. If $\overrightarrow{AC} = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AB}$ is parallel to $\boldsymbol{i} + \boldsymbol{j}$, find $\overrightarrow{AB}$.

18 In this diagram, $OABC$ is a parallelogram with $\overrightarrow{OA} = 2\overrightarrow{AD}$. Let $\boldsymbol{a} = \overrightarrow{AD}$ and $\boldsymbol{c} = \overrightarrow{OC}$.



a Express $\overrightarrow{DB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b Use a vector method to prove that $\overrightarrow{OE} = 3\overrightarrow{OC}$.

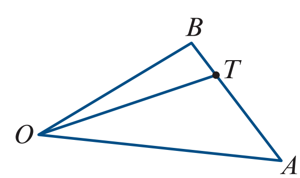
19 For a quadrilateral $OABC$, let D be the point of trisection of OC nearer O and let E be the point of trisection of AB nearer A. Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Find:

i $\overrightarrow{OD}$ $\qquad$ **ii** $\overrightarrow{OE}$ $\qquad$ **iii** $\overrightarrow{DE}$

b Hence prove that $3\overrightarrow{DE} = 2\overrightarrow{OA} + \overrightarrow{CB}$.

20 In triangle OAB, $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and T is a point on AB such that $AT = 3TB$.



a Find $\overrightarrow{OT}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b If M is a point such that $\overrightarrow{OM} = \lambda\overrightarrow{OT}$, where $\lambda > 1$, find:

i $\overrightarrow{BM}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and λ $\qquad$ **ii** λ, if $\overrightarrow{BM}$ is parallel to $\overrightarrow{OA}$.

21 Given that $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + m\boldsymbol{k}$ and $\boldsymbol{c} = -2\boldsymbol{i} + n\boldsymbol{j} + 2\boldsymbol{k}$ are linearly dependent, express m in terms of n.

22 Let $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{k}$.

a Find $\boldsymbol{v}$, the vector resolute of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

b Prove that $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.

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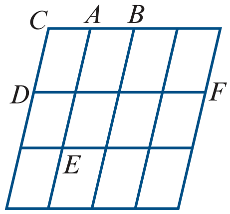
Multiple-choice questions

1 If $\overrightarrow{OX} = \boldsymbol{a} + 2\boldsymbol{b}$ and $\overrightarrow{XY} = \boldsymbol{a} - \boldsymbol{b}$, then $\overrightarrow{OY}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to

A $\boldsymbol{b}$ **B** $3\boldsymbol{b}$ **C** $2\boldsymbol{a} + \boldsymbol{b}$ **D** $2\boldsymbol{a} + 3\boldsymbol{b}$ **E** $3\boldsymbol{a} + \boldsymbol{b}$

2 The grid shown is made up of identical parallelograms. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{c} = \overrightarrow{CD}$. Then the vector $\overrightarrow{EF}$ is equal to

A $\boldsymbol{a} + 3\boldsymbol{c}$ **B** $-3\boldsymbol{a} + \boldsymbol{c}$ **C** $-3\boldsymbol{a} - \boldsymbol{c}$
D $3\boldsymbol{a} - \boldsymbol{c}$ **E** $3\boldsymbol{a} + \boldsymbol{c}$



3 $ABCD$ is a parallelogram with $\overrightarrow{AB} = \boldsymbol{u}$ and $\overrightarrow{BC} = \boldsymbol{v}$. If M is the midpoint of AB, then the vector $\overrightarrow{DM}$ expressed in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to

A $\frac{1}{2}\boldsymbol{u} + \boldsymbol{v}$ **B** $\frac{1}{2}\boldsymbol{u} - \boldsymbol{v}$ **C** $\boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$ **D** $\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$ **E** $\frac{3}{2}\boldsymbol{u} - \boldsymbol{v}$

4 If $A = (3, 6)$ and $B = (11, 1)$, then the vector $\overrightarrow{AB}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$ is equal to

A $3\boldsymbol{i} + 6\boldsymbol{j}$ **B** $8\boldsymbol{i} - 5\boldsymbol{j}$ **C** $8\boldsymbol{i} + 5\boldsymbol{j}$ **D** $14\boldsymbol{i} + 7\boldsymbol{j}$ **E** $14\boldsymbol{i} - 7\boldsymbol{j}$

5 The angle between the vectors $2\boldsymbol{i} + \boldsymbol{j} - \sqrt{2}\boldsymbol{k}$ and $5\boldsymbol{i} + 8\boldsymbol{j}$ is approximately

A $0.72°$ **B** $0.77°$ **C** $43.85°$ **D** $46.15°$ **E** $88.34°$

6 Let OAB be a triangle such that $\overrightarrow{AO} \cdot \overrightarrow{AB} = \overrightarrow{BO} \cdot \overrightarrow{BA}$ and $|\overrightarrow{AB}| \neq |\overrightarrow{OB}|$. Then triangle OAB must be

A scalene **B** equilateral **C** isosceles **D** right-angled **E** obtuse

7 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors such that $x(\boldsymbol{a} + \boldsymbol{b}) = 2y\boldsymbol{a} + (y + 3)\boldsymbol{b}$, then the values of x and y are

A $x = 3,\ y = 6$ **B** $x = -6,\ y = -3$ **C** $x = -2,\ y = -1$
D $x = 2,\ y = 1$ **E** $x = 6,\ y = 3$

8 If A and B are points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ respectively, then $|\overrightarrow{AB}|$ is equal to

A 29 **B** $\sqrt{11}$ **C** 11 **D** $\sqrt{21}$ **E** $\sqrt{29}$

9 Let $\boldsymbol{x} = 3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{y} = -5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$. The scalar resolute of $\boldsymbol{x}$ in the direction of $\boldsymbol{y}$ is

A $\dfrac{21}{\sqrt{27}}$ **B** $-\dfrac{13\sqrt{23}}{23}$ **C** $-\dfrac{13\sqrt{29}}{29}$ **D** $-\dfrac{13\sqrt{27}}{27}$ **E** $-\dfrac{13\sqrt{21}}{21}$

10 Let $ABCD$ be a rectangle such that $|\overrightarrow{BC}| = 3|\overrightarrow{AB}|$. If $\overrightarrow{AB} = \boldsymbol{a}$, then $|\overrightarrow{AC}|$ is equal to

A $2|\boldsymbol{a}|$ **B** $\sqrt{10}\,|\boldsymbol{a}|$ **C** $4|\boldsymbol{a}|$ **D** $10|\boldsymbol{a}|$ **E** $3|\boldsymbol{a}|$

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11 Vectors $\boldsymbol{a} = 2\boldsymbol{i} - 8\boldsymbol{j} + 10\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{j} + a\boldsymbol{k}$ are linearly dependent. The value of a is

A -2 **B** -4 **C** -3 **D** 2 **E** 9

12 If $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ are non-zero vectors such that $\boldsymbol{r} = \frac{1}{4}\boldsymbol{p} + \frac{3}{4}\boldsymbol{q}$, then which one of the following statements must be true?

A $\boldsymbol{p}$ and $\boldsymbol{q}$ are linearly dependent
B $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ are linearly dependent
C $\boldsymbol{p}$ and $\boldsymbol{q}$ are linearly independent
D $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ are parallel
E $\boldsymbol{r}$ is perpendicular to both $\boldsymbol{p}$ and $\boldsymbol{q}$

13 Consider the four vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{d} = 4\boldsymbol{i} - 2\boldsymbol{j}$. Which one of the following is a linearly dependent set of vectors?

A $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}$ **B** $\{\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}\}$ **C** $\{\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ **D** $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ **E** $\{\boldsymbol{a}, \boldsymbol{b}\}$

Extended-response questions

1 A spider builds a web in a garden. Relative to an origin O, the position vectors of the ends A and B of a strand of the web are $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{OB} = 3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$.

a **i** Find $\overrightarrow{AB}$. **ii** Find the length of the strand.

b A small insect is at point C, where $\overrightarrow{OC} = 2.5\boldsymbol{i} + 4\boldsymbol{j} + 1.5\boldsymbol{k}$. Unluckily, it flies in a straight line and hits the strand of web between A and B. Let Q be the point at which the insect hits the strand, where $\overrightarrow{AQ} = \lambda\overrightarrow{AB}$.

i Find $\overrightarrow{CQ}$ in terms of λ.

ii If the insect hits the strand at right angles, find the value of λ and the vector $\overrightarrow{OQ}$.

c Another strand MN of the web has endpoints M and N with position vectors $\overrightarrow{OM} = 4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{ON} = 6\boldsymbol{i} + 10\boldsymbol{j} + 9\boldsymbol{k}$. The spider decides to continue AB to join MN. Find the position vector of the point of contact.

2 The position vectors of points A and B are $2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

a **i** Find $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$. **ii** Find $\overrightarrow{AB}$.

b Let X be the midpoint of line segment AB.

i Find $\overrightarrow{OX}$. **ii** Show that $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{AB}$.

c Find the position vector of a point C such that $OACB$ is a parallelogram.

d Show that the diagonal OC is perpendicular to the diagonal AB by considering the scalar product $\overrightarrow{OC} \cdot \overrightarrow{AB}$.

e **i** Find a vector of magnitude $\sqrt{195}$ that is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

ii Show that this vector is also perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

iii Comment on the relationship between the vector found in part **e i** and the parallelogram $OACB$.

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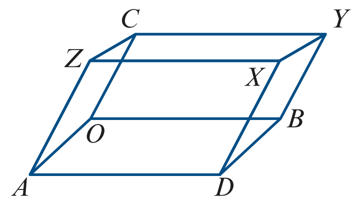
3 Points A, B and C have position vectors

$$\overrightarrow{OA} = 5\boldsymbol{i}, \quad \overrightarrow{OB} = \boldsymbol{i} + 3\boldsymbol{k}, \quad \text{and} \quad \overrightarrow{OC} = \boldsymbol{i} + 4\boldsymbol{j}$$

The parallelepiped has OA, OB and OC as three edges and remaining vertices X, Y, Z and D as shown in the diagram.



a Write down the position vectors of X, Y, Z and D in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ and calculate the lengths of OD and OY.

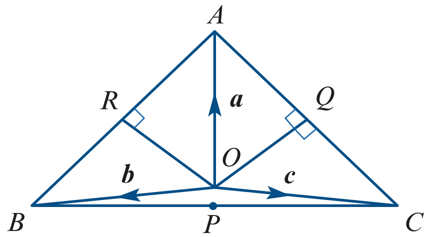
b Calculate the size of angle OZY.

c The point P divides CZ in the ratio $\lambda : 1$. That is, $CP : PZ = \lambda : 1$.

i Give the position vector of P.

ii Find λ if $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{CZ}$.

4 ABC is a triangle as shown in the diagram. The points P, Q and R are the midpoints of the sides BC, CA and AB respectively. Point O is the point of intersection of the perpendicular bisectors of CA and AB.
Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.



a Express each of the following in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

iv $\overrightarrow{OP}$ **v** $\overrightarrow{OQ}$ **vi** $\overrightarrow{OR}$

b **i** Using the fact that OR is perpendicular to AB, show that $|\boldsymbol{a}| = |\boldsymbol{b}|$.

ii Using the fact that OQ is perpendicular to AC, show that $|\boldsymbol{a}| = |\boldsymbol{c}|$.

c Prove that OP is perpendicular to BC.

d Hence prove that the perpendicular bisectors of the sides of a triangle are concurrent.

Note: The point where the perpendicular bisectors meet is called the **circumcentre** of the triangle. This point is equidistant from all three vertices.

5 The position vectors of two points B and C, relative to an origin O, are denoted by $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

a In terms of $\boldsymbol{b}$ and $\boldsymbol{c}$, find the position vector of L, the point on BC between B and C such that $BL : LC = 2 : 1$.

b Let $\boldsymbol{a}$ be the position vector of a point A such that O is the midpoint of AL. Prove that $3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c} = \boldsymbol{0}$.

c Let M be the point on CA between C and A such that $CM : MA = 3 : 2$.

i Prove that B, O and M are collinear.

ii Find the ratio $BO : OM$.

d Let N be the point on AB such that C, O and N are collinear. Find the ratio $AN : NB$.

Review

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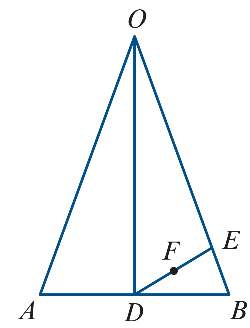
6 OAB is an isosceles triangle with $OA = OB$.
Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.



a Let D be the midpoint of AB and let E be a point on OB.
Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OD}$

ii $\overrightarrow{DE}$ if $\overrightarrow{OE} = \lambda\overrightarrow{OB}$

b If DE is perpendicular to OB, show that

$$\lambda = \frac{1}{2}\frac{(\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{b})}{\boldsymbol{b}\cdot\boldsymbol{b}}$$

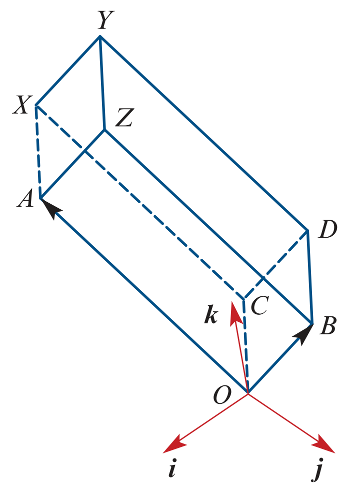
c Now assume that DE is perpendicular to OB and that $\lambda = \frac{5}{6}$.

i Show that $\cos\theta = \frac{2}{3}$, where θ is the magnitude of $\angle AOB$.

ii Let F be the midpoint of DE. Show that OF is perpendicular to AE.

7 A cuboid is positioned on level ground so that it rests on one of its vertices, O. Vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are on the ground.

$$\overrightarrow{OA} = 3\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}$$
$$\overrightarrow{OB} = 2\boldsymbol{i} + a\boldsymbol{j} + 2\boldsymbol{k}$$
$$\overrightarrow{OC} = x\boldsymbol{i} + y\boldsymbol{j} + 2\boldsymbol{k}$$



a **i** Find $\overrightarrow{OA}\cdot\overrightarrow{OB}$ in terms of a.

ii Find a.

b **i** Use the fact that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OC}$ to write an equation relating x and y.

ii Find the values of x and y.

c Find the position vectors:

i $\overrightarrow{OD}$ **ii** $\overrightarrow{OX}$ **iii** $\overrightarrow{OY}$

d State the height of points X and Y above the ground.

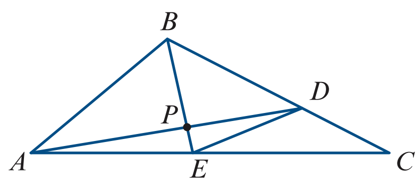
8 In the diagram, D is a point on BC with $\frac{BD}{DC} = 3$ and E is a point on AC with $\frac{AE}{EC} = \frac{3}{2}$.
Let P be the point of intersection of AD and BE. Let $\boldsymbol{a} = \overrightarrow{BA}$ and $\boldsymbol{c} = \overrightarrow{BC}$.

a Find:

i $\overrightarrow{BD}$ in terms of $\boldsymbol{c}$

ii $\overrightarrow{BE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$

iii $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$



b Let $\overrightarrow{BP} = \mu\overrightarrow{BE}$ and $\overrightarrow{AP} = \lambda\overrightarrow{AD}$.
Find λ and μ.

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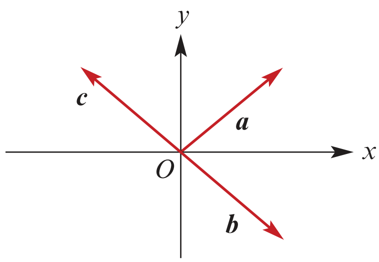
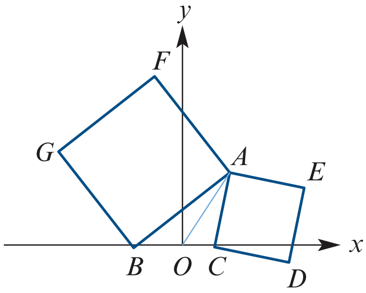
9 **a** Let $\boldsymbol{a} = p\boldsymbol{i} + q\boldsymbol{j}$. The vector $\boldsymbol{b}$ is obtained by rotating $\boldsymbol{a}$ clockwise through 90° about the origin. The vector $\boldsymbol{c}$ is obtained by rotating $\boldsymbol{a}$ anticlockwise through 90° about the origin. Find $\boldsymbol{b}$ and $\boldsymbol{c}$ in terms of p, q, $\boldsymbol{i}$ and $\boldsymbol{j}$.



b In the diagram, $ABGF$ and $AEDC$ are squares with $OB = OC = 1$. Let $\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j}$.

- **i** Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of x, y, $\boldsymbol{i}$ and $\boldsymbol{j}$.
- **ii** Use the results of **a** to find $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of x, y, $\boldsymbol{i}$ and $\boldsymbol{j}$.



c **i** Prove that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{EF}$.

- **ii** Prove that $|\overrightarrow{EF}| = 2|\overrightarrow{OA}|$.

10 Triangle ABC is equilateral and $AD = BE = CF$.

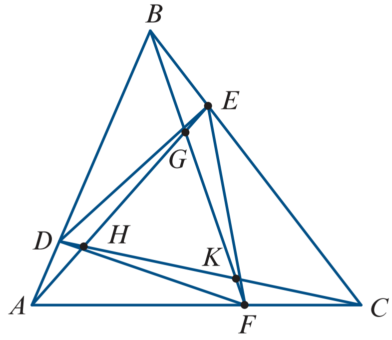
a Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be unit vectors in the directions of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively.
Let $\overrightarrow{AB} = m\boldsymbol{u}$ and $\overrightarrow{AD} = n\boldsymbol{u}$.

- **i** Find $\overrightarrow{BC}$, $\overrightarrow{BE}$, $\overrightarrow{CA}$ and $\overrightarrow{CF}$.
- **ii** Find $|\overrightarrow{AE}|$ and $|\overrightarrow{FB}|$ in terms of m and n.

b Show that $\overrightarrow{AE} \cdot \overrightarrow{FB} = \dfrac{1}{2}(m^2 - mn + n^2)$.

c Show that triangle GHK is equilateral, where:

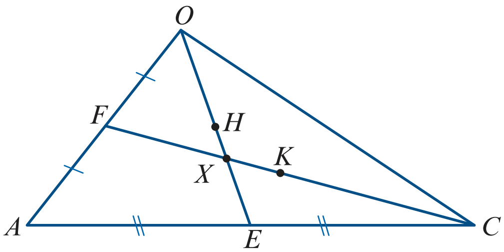
- G is the point of intersection of BF and AE
- H is the point of intersection of AE and CD
- K is the point of intersection of CD and BF.



11 AOC is a triangle. The medians CF and OE intersect at X.
Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Find $\overrightarrow{CF}$ and $\overrightarrow{OE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b **i** If $\overrightarrow{OE}$ is perpendicular to $\overrightarrow{AC}$, prove that $\triangle OAC$ is isosceles.



- **ii** If furthermore $\overrightarrow{CF}$ is perpendicular to $\overrightarrow{OA}$, find the magnitude of angle AOC, and hence prove that $\triangle AOC$ is equilateral.

c Let H and K be the midpoints of OE and CF respectively.

- **i** Show that $\overrightarrow{HK} = \lambda\boldsymbol{c}$ and $\overrightarrow{FE} = \mu\boldsymbol{c}$, for some $\lambda, \mu \in \mathbb{R} \setminus \{0\}$.
- **ii** Give reasons why $\triangle HXK$ is similar to $\triangle EXF$. (Vector method not required.)
- **iii** Hence prove that $OX : XE = 2 : 1$.

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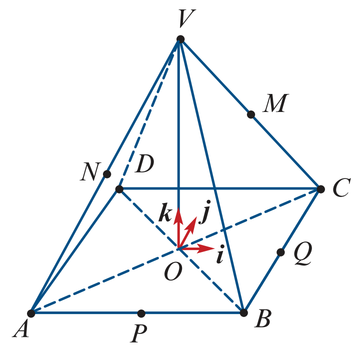
Review

12 $VABCD$ is a square-based pyramid:

- The origin O is the centre of the base.
- The unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are in the directions of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{OV}$ respectively.
- $AB = BC = CD = DA = 4$ cm
- $OV = 2h$ cm, where h is a positive real number.
- P, Q, M and N are the midpoints of AB, BC, VC and VA respectively.



a Find the position vectors of A, B, C and D relative to O.

b Find vectors $\overrightarrow{PM}$ and $\overrightarrow{QN}$ in terms of h.

c Find the position vector $\overrightarrow{OX}$, where X is the point of intersection of QN and PM.

d If OX is perpendicular to VB:

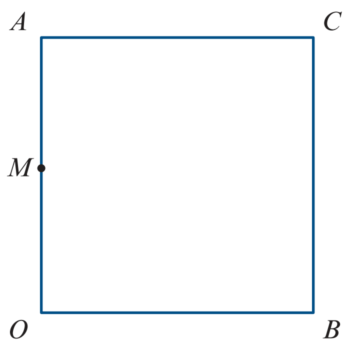
  i find the value of h

  ii find the acute angle between PM and QN, correct to the nearest degree.

e **i** Prove that $NMQP$ is a rectangle.

  ii Find h if $NMQP$ is a square.

13 $OACB$ is a square with $\overrightarrow{OA} = a\boldsymbol{j}$ and $\overrightarrow{OB} = a\boldsymbol{i}$. Point M is the midpoint of OA.



a Find in terms of a:

  i $\overrightarrow{OM}$ **ii** $\overrightarrow{MC}$

b P is a point on MC such that $\overrightarrow{MP} = \lambda\overrightarrow{MC}$. Find $\overrightarrow{MP}$, $\overrightarrow{BP}$ and $\overrightarrow{OP}$ in terms of λ and a.

c If BP is perpendicular to MC:

  i find the values of λ, $|\overrightarrow{BP}|$, $|\overrightarrow{OP}|$ and $|\overrightarrow{OB}|$

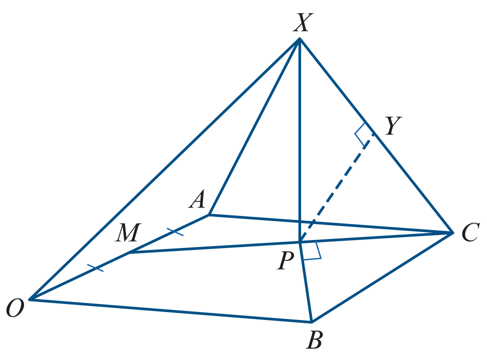
  ii evaluate $\cos\theta$, where $\theta = \angle PBO$.

d If $|\overrightarrow{OP}| = |\overrightarrow{OB}|$, find the possible values of λ and illustrate these two cases carefully.

e In the diagram:

- $\overrightarrow{OA} = a\boldsymbol{j}$ and $\overrightarrow{OB} = a\boldsymbol{i}$
- M is the midpoint of OA
- BP is perpendicular to MC
- $\overrightarrow{PX} = a\boldsymbol{k}$
- Y is a point on XC such that PY is perpendicular to XC.

Find $\overrightarrow{OY}$.



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5

Vector equations of lines and planes

Objectives

- To find a vector equation of a **line** determined by:
  - a point on the line and a direction vector
  - two points.
- To find a vector equation of the **line segment** between two given points.
- To compute the **vector product** of two vectors, and to use the vector product to find a **normal vector** for a plane.
- To find a vector equation and Cartesian equation of a **plane** determined by:
  - a point on the plane and a normal vector
  - three points.
- To determine whether two lines are skew, are parallel or intersect.
- To determine whether a line and a plane are parallel or intersect.
- To determine whether two planes are parallel or intersect.
- To find the **distance** between:
  - a point and a line
  - a point and a plane
  - two parallel planes
  - two skew lines.
- To compute the **angle** between two lines or planes.

In this chapter, we continue our study of vectors. We use them to investigate the geometric properties of lines and planes in three dimensions.

We know that a line in two-dimensional space can be described very simply by a Cartesian equation of the form $ax + by = c$. We will see that, in three-dimensional space, it is not possible to describe a line via a single linear Cartesian equation. It is simpler to describe lines in three dimensions using vector equations.

We have studied parametric equations in Section 1G. We will study vector equations more generally in Chapter 13.

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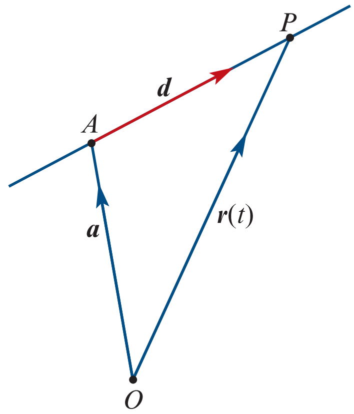
5A Vector equations of lines

Vector equation of a line given by a point and a direction

A line ℓ in two- or three-dimensional space may be described using two vectors:

- the position vector $\boldsymbol{a}$ of a point A on the line
- a vector $\boldsymbol{d}$ parallel to the line.



We can describe the line as

$$\ell = \{ P : \overrightarrow{OP} = \boldsymbol{a} + t\boldsymbol{d} \text{ for some } t \in \mathbb{R} \}$$

Usually we omit the set notation. We write $\boldsymbol{r}(t)$ for the position vector of a point P on the line, and therefore

$$\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{d}, \quad t \in \mathbb{R}$$

This is a **vector equation** of the line ℓ.

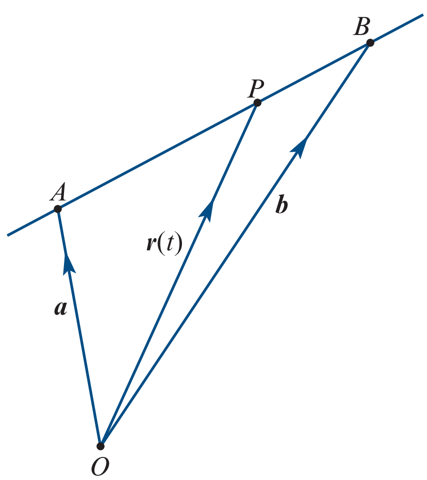
As the value of t varies over the real numbers, the position vector $\boldsymbol{r}(t)$ varies over all the points on the line ℓ. We sometimes express this idea by saying that t is a **parameter** and that $\boldsymbol{r}(t)$ is a **parameterisation** of the line ℓ.

If it is understood that t is the parameter, then we may write $\boldsymbol{r}$ instead of $\boldsymbol{r}(t)$.

Note: There is no unique vector equation of a given line. We can choose any point A as the 'starting point' on the line and any vector $\boldsymbol{d}$ parallel to the line.

Vector equation of a line given by two points

If the position vectors $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$ of two points on a line ℓ are known, then the line may be described by



$$\begin{aligned}\boldsymbol{r}(t) &= \overrightarrow{OA} + t\overrightarrow{AB} \\ &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}), \quad t \in \mathbb{R}\end{aligned}$$

This is also a vector equation of the line ℓ.

This vector equation can be rewritten as

$$\boldsymbol{r}(t) = (1 - t)\boldsymbol{a} + t\boldsymbol{b}, \quad t \in \mathbb{R}$$

In Section 4E, we derived this expression for the position vector of a point collinear with A and B.

Note: As already noted above, there is no unique vector equation of a given line. Here we can choose any two distinct points A and B on the line.

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Example 1

Verify that the point $P(-7, 4, -14)$ lies on the line represented by the vector equation

$$\boldsymbol{r}(t) = 5\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}), \quad t \in \mathbb{R}$$

Solution

The point $P(-7, 4, -14)$ has position vector $-7\boldsymbol{i} + 4\boldsymbol{j} - 14\boldsymbol{k}$.

By equating coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, we can see that the point P lies on the line if there exists $t \in \mathbb{R}$ such that

$$5 + 2t = -7$$
$$-2 - t = 4$$
$$4 + 3t = -14$$

A solution for each of these equations is $t = -6$. Hence P lies on the line.

Example 2

Find a vector equation of the line AB, where the points A and B have position vectors

$$\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} \quad \text{and} \quad \overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

Solution

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the position vectors of points A and B respectively. Then a vector equation of the line is

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})\\ &= \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t((2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) - (\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}))\\ &= \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}), \quad t \in \mathbb{R}\end{aligned}$$

Note: This can also be written as $\boldsymbol{r} = (1 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j} + (-2 + t)\boldsymbol{k}$, $t \in \mathbb{R}$.

Example 3

Find a vector equation for each of the following lines:

a the line through $A(1, 2)$ that is parallel to $2\boldsymbol{i} + 3\boldsymbol{j}$

b the line passing through the points $A(3, -5, 4)$ and $B(-4, 3, 10)$

Solution

a Point A has position vector $\boldsymbol{i} + 2\boldsymbol{j}$. So a vector equation of the line is

$$\boldsymbol{r}(t) = \boldsymbol{i} + 2\boldsymbol{j} + t(2\boldsymbol{i} + 3\boldsymbol{j}), \quad t \in \mathbb{R}$$

b The points A and B have position vectors $\boldsymbol{a} = 3\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = -4\boldsymbol{i} + 3\boldsymbol{j} + 10\boldsymbol{k}$ respectively. So a vector equation of the line is

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})\\ &= 3\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k} + t((-4\boldsymbol{i} + 3\boldsymbol{j} + 10\boldsymbol{k}) - (3\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}))\\ &= 3\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k} + t(-7\boldsymbol{i} + 8\boldsymbol{j} + 6\boldsymbol{k}), \quad t \in \mathbb{R}\end{aligned}$$

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Cartesian equation of a line in two dimensions

From a vector equation to the Cartesian equation

- For example, start with the vector equation

$$\boldsymbol{r} = \boldsymbol{i} + 5\boldsymbol{j} + t(\boldsymbol{i} + 2\boldsymbol{j}), \quad t \in \mathbb{R}$$

- Rearrange this equation as

$$\boldsymbol{r} = (1 + t)\boldsymbol{i} + (5 + 2t)\boldsymbol{j}$$

  Let $P(x, y)$ be the point on the line with position vector $\boldsymbol{r}$, so that $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$. Then, by equating coefficients of $\boldsymbol{i}$ and $\boldsymbol{j}$, we have

$$x = 1 + t \quad \text{and} \quad y = 5 + 2t$$

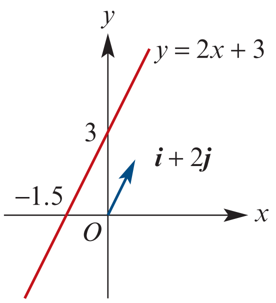
  These are parametric equations for the line.

- Now eliminate t to find y in terms of x. We have $t = x - 1$, so $y = 5 + 2(x - 1) = 2x + 3$. The Cartesian equation of the line is $y = 2x + 3$.

From the Cartesian equation to a vector equation

For example, start with the Cartesian equation $y = 2x + 3$. A point on the line is $(0, 3)$, with position vector $3\boldsymbol{j}$. The line has gradient 2, so a vector parallel to the line is $\boldsymbol{i} + 2\boldsymbol{j}$. Therefore a vector equation of the line is

$$\boldsymbol{r} = 3\boldsymbol{j} + t(\boldsymbol{i} + 2\boldsymbol{j}), \quad t \in \mathbb{R}$$



Note: For a line with equation $y = mx + c$, you can choose the point $(0, c)$ on the line and the vector $\boldsymbol{i} + m\boldsymbol{j}$ parallel to the line.

Cartesian form for a line in three dimensions

From a vector equation to Cartesian form

- For example, the line through the point $(5, -2, 4)$ that is parallel to the vector $2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ can be described by the vector equation

$$\boldsymbol{r} = 5\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}), \quad t \in \mathbb{R}$$

- Let $P(x, y, z)$ be the point on the line with position vector $\boldsymbol{r}$. Then we can write the vector equation as

$$x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (5 + 2t)\boldsymbol{i} + (-2 - t)\boldsymbol{j} + (4 + 3t)\boldsymbol{k}$$

  The corresponding parametric equations are

$$x = 5 + 2t, \quad y = -2 - t \quad \text{and} \quad z = 4 + 3t$$

- Solving each of these equations for t, we have

$$\frac{x - 5}{2} = \frac{y + 2}{-1} = \frac{z - 4}{3} = t$$

  This is in **Cartesian form**. You cannot describe a line in three dimensions using a single linear Cartesian equation.

From Cartesian form to a vector equation To convert from Cartesian form to a vector equation, we can perform these steps in the reverse order.

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We have seen that a straight line can be described by a vector equation, by parametric equations or in Cartesian form.

Lines in three dimensions

A line in three-dimensional space can be described in the following three ways, where $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is the position vector of a point A on the line, and $\boldsymbol{d} = d_1\boldsymbol{i} + d_2\boldsymbol{j} + d_3\boldsymbol{k}$ is a vector parallel to the line.

Vector equation	Parametric equations	Cartesian form
$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}, \quad t \in \mathbb{R}$	$x = a_1 + d_1 t$ $y = a_2 + d_2 t$ $z = a_3 + d_3 t$	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$

Parallel and perpendicular lines

For two lines $\ell_1 : \boldsymbol{r}_1 = \boldsymbol{a}_1 + t\boldsymbol{d}_1$, $t \in \mathbb{R}$, and $\ell_2 : \boldsymbol{r}_2 = \boldsymbol{a}_2 + s\boldsymbol{d}_2$, $s \in \mathbb{R}$:

- The lines ℓ_1 and ℓ_2 are parallel if and only if $\boldsymbol{d}_1$ is parallel to $\boldsymbol{d}_2$.
- The lines ℓ_1 and ℓ_2 are perpendicular if and only if $\boldsymbol{d}_1$ is perpendicular to $\boldsymbol{d}_2$.

Example 4

Let ℓ be the line with vector equation

$$\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + t(-\boldsymbol{i} - 3\boldsymbol{j}), \quad t \in \mathbb{R}$$

a Find a vector equation of the line through $A(1, 3, 2)$ that is parallel to the line ℓ.

b Find a vector equation of the line through $A(1, 3, 2)$ that is perpendicular to the line ℓ and parallel to the x–y plane.

Solution

a The position vector of A is $\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$, and a vector parallel to ℓ is $-\boldsymbol{i} - 3\boldsymbol{j}$.

Therefore a vector equation of the line through A parallel to ℓ is

$$\boldsymbol{r} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k} + s(-\boldsymbol{i} - 3\boldsymbol{j}), \quad s \in \mathbb{R}$$

b If a vector is parallel to the x–y plane, then its $\boldsymbol{k}$-component is zero. So we want to find a vector $\boldsymbol{d} = d_1\boldsymbol{i} + d_2\boldsymbol{j}$ that is perpendicular to $-\boldsymbol{i} - 3\boldsymbol{j}$.

Therefore we require

$$(d_1\boldsymbol{i} + d_2\boldsymbol{j}) \cdot (-\boldsymbol{i} - 3\boldsymbol{j}) = 0$$

i.e. $$-d_1 - 3d_2 = 0$$

We see that we can choose $d_1 = 3$ and $d_2 = -1$. So $\boldsymbol{d} = 3\boldsymbol{i} - \boldsymbol{j}$.

Hence a vector equation of the required line is

$$\boldsymbol{r} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k} + s(3\boldsymbol{i} - \boldsymbol{j}), \quad s \in \mathbb{R}$$

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Distance from a point to a line

We can use the scalar product to find the distance from a point to a line.

Example 5

Find the distance to the line $\boldsymbol{r}(t) = (1 - t)\boldsymbol{i} + (2 - 3t)\boldsymbol{j} + 2\boldsymbol{k}$, $t \in \mathbb{R}$, from:

a the origin **b** the point $A(1, 3, 2)$.

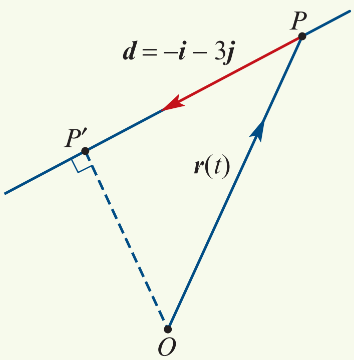
Solution

The equation of the line can be written as

$$\boldsymbol{r}(t) = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} + t(-\boldsymbol{i} - 3\boldsymbol{j}), \quad t \in \mathbb{R}$$

So the vector $\boldsymbol{d} = -\boldsymbol{i} - 3\boldsymbol{j}$ is parallel to the line.

a The required distance is $|\overrightarrow{OP'}|$, where P' is the point on the line such that OP' is perpendicular to the line.



For any point P on the line with $\overrightarrow{OP} = \boldsymbol{r}(t)$, we have

$$\begin{aligned}\overrightarrow{OP} \cdot \boldsymbol{d} &= ((1 - t)\boldsymbol{i} + (2 - 3t)\boldsymbol{j} + 2\boldsymbol{k}) \cdot (-\boldsymbol{i} - 3\boldsymbol{j}) \\ &= -(1 - t) - 3(2 - 3t) \\ &= 10t - 7\end{aligned}$$

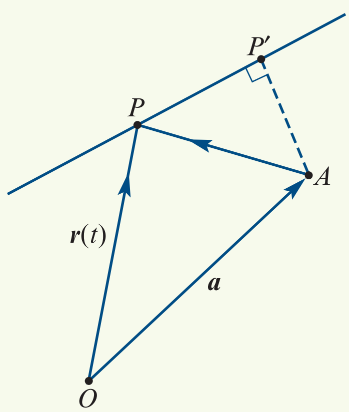
If OP' is perpendicular to the line, then

$$\overrightarrow{OP'} \cdot \boldsymbol{d} = 0 \quad \Rightarrow \quad 10t - 7 = 0 \quad \Rightarrow \quad t = \frac{7}{10}$$

Therefore $\overrightarrow{OP'} = \frac{3}{10}\boldsymbol{i} - \frac{1}{10}\boldsymbol{j} + 2\boldsymbol{k}$.

The distance from the origin to the line is $|\overrightarrow{OP'}| = \dfrac{\sqrt{410}}{10}$.

b The required distance is $|\overrightarrow{AP'}|$, where P' is the point on the line such that AP' is perpendicular to the line.



For any point P on the line with $\overrightarrow{OP} = \boldsymbol{r}(t)$, we have

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{AO} + \overrightarrow{OP} \\ &= -(\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}) + (1 - t)\boldsymbol{i} + (2 - 3t)\boldsymbol{j} + 2\boldsymbol{k} \\ &= -t\boldsymbol{i} + (-1 - 3t)\boldsymbol{j}\end{aligned}$$

$$\begin{aligned}\therefore \quad \overrightarrow{AP} \cdot \boldsymbol{d} &= (-t\boldsymbol{i} + (-1 - 3t)\boldsymbol{j}) \cdot (-\boldsymbol{i} - 3\boldsymbol{j}) \\ &= t - 3(-1 - 3t) \\ &= 10t + 3\end{aligned}$$

If $\overrightarrow{AP'} \cdot \boldsymbol{d} = 0$, then $t = -\frac{3}{10}$ and so $\overrightarrow{AP'} = \frac{3}{10}\boldsymbol{i} - \frac{1}{10}\boldsymbol{j}$.

The distance from the point A to the line is $|\overrightarrow{AP'}| = \dfrac{\sqrt{10}}{10}$.

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Describing line segments

We can use a vector equation to describe a line segment by restricting the values of the parameter. Consider a vector equation $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{d}$ with parameter t.

- As the value of t varies over $\mathbb{R}$, the position vector $\boldsymbol{r}(t)$ varies over all the points on a line.
- If the value of t only varies over an interval $[p, q]$, then the position vector $\boldsymbol{r}(t)$ only varies over the points on the line segment between $\boldsymbol{r}(p)$ and $\boldsymbol{r}(q)$.

Example 6

Points A and B have position vectors $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{k}$ respectively.

a Show that the vector equation $\boldsymbol{r}(t) = \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k})$, $t \in \mathbb{R}$, represents the line through A and B.

b Find the set of values of t which, together with this vector equation, describes the line segment AB.

c Find the set of values of t which, together with this vector equation, describes the line segment AC, where $C(4, 8, -9)$ is a point on the line AB.

Solution

a An equation of the line AB is

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})\\ &= \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}), \quad t \in \mathbb{R}\end{aligned}$$

b Taking $t = 0$ gives $\boldsymbol{r}(0) = \boldsymbol{i} - 4\boldsymbol{j} = \boldsymbol{a}$.

To find the value of t which gives $\boldsymbol{b}$, consider

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{b}\\ \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}) &= 2\boldsymbol{i} - 3\boldsymbol{k}\\ (1 + t)\boldsymbol{i} + 4(t - 1)\boldsymbol{j} - 3t\boldsymbol{k} &= 2\boldsymbol{i} - 3\boldsymbol{k}\end{aligned}$$

Therefore $t = 1$.

So the line segment AB is described by

$$\boldsymbol{r}(t) = \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}), \quad t \in [0, 1]$$

c To find the value of t which gives $\overrightarrow{OC}$, consider

$$\begin{aligned}\boldsymbol{r}(t) &= \overrightarrow{OC}\\ \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}) &= 4\boldsymbol{i} + 8\boldsymbol{j} - 9\boldsymbol{k}\\ (1 + t)\boldsymbol{i} + 4(t - 1)\boldsymbol{j} - 3t\boldsymbol{k} &= 4\boldsymbol{i} + 8\boldsymbol{j} - 9\boldsymbol{k}\end{aligned}$$

Therefore $t = 3$.

So the line segment AC is described by

$$\boldsymbol{r}(t) = \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}), \quad t \in [0, 3]$$

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Exercise 5A

Example 1

1 For each of the following, determine whether the point lies on the line:

a $(4, 2, 1)$, $\boldsymbol{r}(t) = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + t(-3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}),\ t \in \mathbb{R}$

b $(3, -3, -4)$, $\boldsymbol{r}(t) = 6\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}),\ t \in \mathbb{R}$

c $(3, -1, -1)$, $\boldsymbol{r}(t) = -\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k} + t(-\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}),\ t \in \mathbb{R}$

Example 2

2 For each of the following, find a vector equation of the line through the points A and B:

a $\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j}$, $\overrightarrow{OB} = \boldsymbol{i} + 3\boldsymbol{j}$

b $\overrightarrow{OA} = \boldsymbol{i} - 3\boldsymbol{k}$, $\overrightarrow{OB} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

c $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, $\overrightarrow{OB} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

d $\overrightarrow{OA} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{OB} = -2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Example 3

3 For each of the following, find a vector equation of the line that passes through the points A and B:

a $A(3, 1)$, $B(-2, 2)$

b $A(-1, 5)$, $B(2, -1)$

c $A(1, 2, 3)$, $B(2, 0, -1)$

d $A(1, -4, 0)$, $B(2, 3, 1)$

4 Convert each vector equation found in Question 3 into:

i parametric equations

ii Cartesian form.

5 Consider the line with equation $2x + 3y = 12$.

a Show that the point $(3, 2)$ lies on the line and that the vector $3\boldsymbol{i} - 2\boldsymbol{j}$ is parallel to the line. Hence give a vector equation for the line.

b Show that the point $(0, 4)$ lies on the line and that the vector $-9\boldsymbol{i} + 6\boldsymbol{j}$ is parallel to the line. Hence give another vector equation for the line.

c Show that the points $(6, 0)$ and $(0, 4)$ lie on the line. Hence give yet another vector equation for the line.

Example 4

6 Find a vector equation of the line through the point $A(2, 1, 0)$ that is:

a parallel to the line $\boldsymbol{r} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + t(-3\boldsymbol{i} + \boldsymbol{j}),\ t \in \mathbb{R}$

b perpendicular to the line $\boldsymbol{r} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} + t(-3\boldsymbol{i} + \boldsymbol{j}),\ t \in \mathbb{R}$, and parallel to the x–y plane.

7 Find a vector equation of the line through the origin that is:

a parallel to the vector $2\boldsymbol{j} - \boldsymbol{k}$

b perpendicular to the line $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(2\boldsymbol{j} - \boldsymbol{k}),\ t \in \mathbb{R}$, and in the y–z plane.

8 **a** Find a vector equation of the line AB, where points A and B are defined by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ respectively.

b Determine which of the following points are on this line:

i $(5, 0)$ **ii** $(0, 7)$ **iii** $(8, -3)$

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9 The line ℓ is given by the vector equation $\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k} + t(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$, $t \in \mathbb{R}$.

- **a** Find a vector equation of the line which passes through the point $(0, 1, 1)$ and is parallel to the line ℓ.
- **b** Verify that the two equations do not represent the same line ℓ.
- **c** The point $(2, m, n)$ lies on the line ℓ. Find the values of m and n.

10
- **a** Let $\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j}$. Find a vector that is perpendicular to the vector $\boldsymbol{v}$ and has the same magnitude as $\boldsymbol{v}$.
- **b** Points A and B are given by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{j}$ respectively. Find a vector equation of the line which passes through B and is perpendicular to $\overrightarrow{BA}$.
- **c** Find the x- and y-axis intercepts of this line.

11 Find parametric equations and Cartesian equations for each line:

- **a** $\boldsymbol{r} = 2\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k} + t(-3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}),\ \ t \in \mathbb{R}$
- **b** $\boldsymbol{r} = 2\boldsymbol{j} - \boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}),\ \ t \in \mathbb{R}$

Example 5

12 For each of the following, find the distance from the point to the line:

- **a** $(0, 0, 0)$, $\boldsymbol{r} = 4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k} + t(-3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}),\ \ t \in \mathbb{R}$
- **b** $(1, 10, -2)$, $\boldsymbol{r} = 4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k} + t(-3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}),\ \ t \in \mathbb{R}$
- **c** $(1, 2, 3)$, $\boldsymbol{r} = 3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}),\ \ t \in \mathbb{R}$
- **d** $(1, 1, 4)$, $\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} + t(-2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}),\ \ t \in \mathbb{R}$

Example 6

13 Points A, B and C are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{k}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 10\boldsymbol{j} + 4\boldsymbol{k}$ respectively.

- **a** Show that the vector equation $\boldsymbol{r} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$, $t \in \mathbb{R}$, represents the line through the points A and B.
- **b** Show that the point C is also on this line.
- **c** Find the set of values of t which, together with the vector equation, describes the line segment BC.

14 Find the coordinates of the point where the line through $A(3, 4, 1)$ and $B(5, 1, 6)$ crosses the x–y plane.

15 The line ℓ passes through the points $A(-1, -3, -3)$ and $B(5, 0, 6)$. Find a vector equation of the line ℓ, and find the distance from the origin to the line.

16 Find the coordinates of the nearest point to $(2, 1, 3)$ on the line given by the equation $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + t(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$, $t \in \mathbb{R}$.

17 Find a vector equation to represent the line through the point $(-2, 2, 1)$ that is parallel to the x-axis.

18 Find the distance from the origin to the line that passes through the point $(3, 1, 5)$ and is parallel to the vector $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

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19 For each of the following, give the coordinates of the endpoints of the line segment described by the vector equation:

a $\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} + t(-2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}),\ t \in [1, 3]$

b $\boldsymbol{r} = 3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}),\ t \in [-1, 2]$

20 Let ℓ be the line with vector equation $\boldsymbol{r}(t) = (3 - t)\boldsymbol{i} + (3 - t)\boldsymbol{j} + t\boldsymbol{k}$.

a Find the point on ℓ closest to the origin. (Hint: Find the point P' on ℓ such that OP' is perpendicular to ℓ.)

b Let P be a point on ℓ with position vector $\boldsymbol{r}(t)$. Show that $|\overrightarrow{OP}|^2 = 3t^2 - 12t + 18$.

c For which value of t is the quadratic function $f(t) = 3t^2 - 12t + 18$ minimised?

d Use parts **b** and **c** to find the point on ℓ closest to the origin by another method.

21 A line is given by the vector equation $\boldsymbol{r} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}),\ t \in \mathbb{R}$.

a Find the vector $\overrightarrow{OB}$ in terms of t, where B is a point on the line.

b Find $|\overrightarrow{OB}|$ in terms of t.

c Hence find the minimum value of $|\overrightarrow{OB}|$. That is, find the shortest distance from the origin to a point on the line.

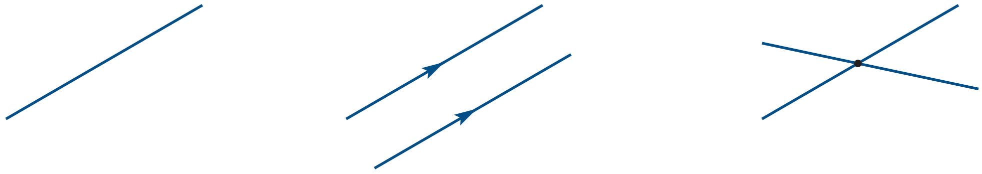
d Let A be the point $(1, 3, 2)$. Find the shortest distance from A to a point on the line.

5B Intersection of lines and skew lines

Lines in two-dimensional space

From Mathematical Methods Units 1 & 2, you know that there are three possibilities for a pair of lines in two-dimensional space:

- the lines coincide
- the lines are parallel and distinct
- the lines intersect at a point.



For example, the two lines

$$\ell_1 : \boldsymbol{r}_1(\lambda) = 2\boldsymbol{i} + 2\boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j}),\ \lambda \in \mathbb{R} \quad \text{and} \quad \ell_2 : \boldsymbol{r}_2(\mu) = 2\boldsymbol{i} + 3\boldsymbol{j} + \mu(2\boldsymbol{i} - 2\boldsymbol{j}),\ \mu \in \mathbb{R}$$

are parallel, since the direction vectors $\boldsymbol{i} - \boldsymbol{j}$ and $2\boldsymbol{i} - 2\boldsymbol{j}$ are parallel.

To check whether two parallel lines coincide, we choose a point on one line and check whether it also lies on the other line. For example, the point with position vector $2\boldsymbol{i} + 3\boldsymbol{j}$ lies on line ℓ_2. This point lies on ℓ_1 if there is a value of λ such that

$$2 + \lambda = 2 \quad \text{and} \quad 2 - \lambda = 3$$

No such λ exists, so the lines ℓ_1 and ℓ_2 are parallel and distinct.

Note: When we are considering a pair of lines, we should use different parameters for the two vector equations. (Here we used λ and μ.)

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Example 7

Find the position vector of the point of intersection of the lines

$$\boldsymbol{r}_1(\lambda) = 2\boldsymbol{i} + 2\boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j}),\ \lambda \in \mathbb{R} \quad \text{and} \quad \boldsymbol{r}_2(\mu) = -\boldsymbol{j} + \mu(3\boldsymbol{i} + 2\boldsymbol{j}),\ \mu \in \mathbb{R}$$

Solution

At the point of intersection, we have $\boldsymbol{r}_1(\lambda) = \boldsymbol{r}_2(\mu)$ and so

$$2\boldsymbol{i} + 2\boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j}) = -\boldsymbol{j} + \mu(3\boldsymbol{i} + 2\boldsymbol{j})$$

$$\therefore \quad (2 + \lambda)\boldsymbol{i} + (2 - \lambda)\boldsymbol{j} = 3\mu\boldsymbol{i} + (-1 + 2\mu)\boldsymbol{j}$$

Equate coefficients of $\boldsymbol{i}$ and $\boldsymbol{j}$:

$$2 + \lambda = 3\mu \qquad (1)$$

$$2 - \lambda = -1 + 2\mu \qquad (2)$$

Solve simultaneously by adding (1) and (2):

$$4 = -1 + 5\mu$$

Hence $\mu = 1$ and so $\lambda = 1$.

Substituting $\lambda = 1$ into the equation $\boldsymbol{r}_1(\lambda) = 2\boldsymbol{i} + 2\boldsymbol{j} + \lambda(\boldsymbol{i} - \boldsymbol{j})$ gives $\boldsymbol{r}_1(1) = 3\boldsymbol{i} + \boldsymbol{j}$.

The point of intersection has position vector $3\boldsymbol{i} + \boldsymbol{j}$.

Lines in three-dimensional space

There are four possibilities for a pair of lines in three-dimensional space: the lines may coincide, they may be parallel and distinct, they may intersect at a point and they may also be **skew**.

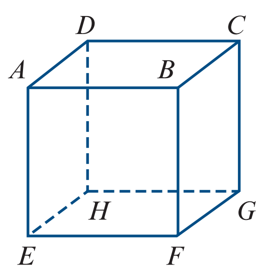
Skew lines

Two lines are **skew lines** if they do not intersect and are not parallel.

Two lines are skew if and only if they do not lie in the same plane.

For example, consider the cube *ABCDEFGH* as shown.

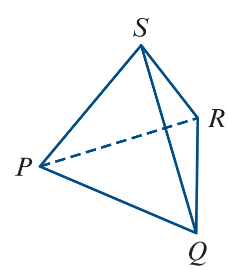
Lines *AB* and *FG* are skew. We can see that *AB* and *FG* do not lie in the same plane.



As another example, consider the tetrahedron *SPQR* as shown.

We can see three pairs of skew lines:

- lines *SP* and *RQ* are skew
- lines *SR* and *PQ* are skew
- lines *SQ* and *PR* are skew.



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Coincident and parallel lines

We can determine whether two lines in three dimensions are coincident or parallel by similar methods as in two dimensions.

Consider two lines $\ell_1 : \boldsymbol{r}_1(\lambda) = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1$ and $\ell_2 : \boldsymbol{r}_2(\mu) = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$.

- Lines ℓ_1 and ℓ_2 are parallel if and only if the direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are parallel (i.e. $\boldsymbol{d}_1 = m\boldsymbol{d}_2$ for some real number m).
- If lines ℓ_1 and ℓ_2 are parallel, then we can check whether they coincide by checking whether a point on ℓ_1 (such as the point with position vector $\boldsymbol{a}_1$) also lies on ℓ_2.

Intersecting lines

Two lines $\ell_1 : \boldsymbol{r}_1(\lambda) = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1$ and $\ell_2 : \boldsymbol{r}_2(\mu) = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$ have a point in common if there exist values of λ and μ such that $\boldsymbol{r}_1(\lambda) = \boldsymbol{r}_2(\mu)$.

Example 8

Find the point of intersection of the lines

$$\boldsymbol{r}_1(\lambda) = 5\boldsymbol{i} + 2\boldsymbol{j} + \lambda(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \quad \text{and} \quad \boldsymbol{r}_2(\mu) = -3\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$$

Solution

At the point of intersection, we have $\boldsymbol{r}_1(\lambda) = \boldsymbol{r}_2(\mu)$ and so

$$5\boldsymbol{i} + 2\boldsymbol{j} + \lambda(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = -3\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$$

Equate coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

$$5 + 2\lambda = -3 + \mu \qquad (1)$$
$$2 + \lambda = 4 - \mu \qquad (2)$$
$$\lambda = 6 - 2\mu \qquad (3)$$

From (1) and (2), we have

$$7 + 3\lambda = 1$$
$$\therefore \quad \lambda = -2$$

Substitute in (1) to find $\mu = 4$.

Now we must check that these values also satisfy equation (3):

$$\text{RHS} = 6 - 2 \times 4 = -2 = \text{LHS}$$

Hence the lines intersect where $\lambda = -2$ and $\mu = 4$.

The point of intersection has the position vector

$$\boldsymbol{r}_1(-2) = 5\boldsymbol{i} + 2\boldsymbol{j} - 2(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$
$$= \boldsymbol{i} - 2\boldsymbol{k}$$

Hence the lines intersect at the point $(1, 0, -2)$.

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Example 9

Show that the following two lines are skew lines:

$$r_1(\lambda) = i + k + \lambda(i + 3j + 4k), \quad \lambda \in \mathbb{R}$$
$$r_2(\mu) = 2i + 3j + \mu(4i - j + k), \quad \mu \in \mathbb{R}$$

Solution

We first note that the lines are not parallel, since $i + 3j + 4k \neq m(4i - j + k)$, for all $m \in \mathbb{R}$.

We now show that the lines do not meet. If they did meet, then equating coefficients of i, j and k would give

$$1 + \lambda = 2 + 4\mu \quad (1)$$
$$3\lambda = 3 - \mu \quad (2)$$
$$1 + 4\lambda = \mu \quad (3)$$

From (1) and (2), we have $\lambda = 1$ and $\mu = 0$. But this is not consistent with equation (3). So there are no values of λ and μ such that $r_1(\lambda) = r_2(\mu)$.

The two lines are skew, as they are not parallel and do not intersect.

Concurrence of three lines

A point of **concurrence** is where three or more lines meet.

Example 10

Find the point of concurrence of the following three lines:

$$\ell_1 : \quad r_1(t) = -2i + j + t(i + j), \quad t \in \mathbb{R}$$
$$\ell_2 : \quad r_2(s) = j + s(i + 2j), \quad s \in \mathbb{R}$$
$$\ell_3 : \quad r_3(u) = 8i + 3j + u(-3i + j), \quad u \in \mathbb{R}$$

Solution

The point of intersection of lines ℓ_1 and ℓ_2 can be found from the values of t and s such that $r_1(t) = r_2(s)$. Equating coefficients of i and j, we obtain

$$-2 + t = s \quad (1)$$
$$1 + t = 1 + 2s \quad (2)$$

Solving simultaneously gives $s = 2$ and $t = 4$. Taking $t = 4$ gives $r_1(4) = 2i + 5j$.

Thus lines ℓ_1 and ℓ_2 intersect at the point $(2, 5)$.

For this to be a point of concurrence, the point must also lie on ℓ_3. We must find a value of u such that

$$2i + 5j = 8i + 3j + u(-3i + j)$$

We see that $u = 2$ gives the result. The three lines are concurrent at the point $(2, 5)$.

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Angle between two lines

In Section 4C, we used the scalar product to find the angle between two vectors.

If two lines have vector equations $\boldsymbol{r}_1(\lambda) = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1$ and $\boldsymbol{r}_2(\mu) = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$, then they are in the directions of vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ respectively. The angle θ between the two vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ can be found using the scalar product:

$$\cos\theta = \frac{\boldsymbol{d}_1 \cdot \boldsymbol{d}_2}{|\boldsymbol{d}_1|\,|\boldsymbol{d}_2|}$$

The angle between the two lines is θ or $180^\circ - \theta$, whichever is in the interval $[0^\circ, 90^\circ]$.

This applies to any pair of lines, whether parallel, intersecting or skew.

The two lines are perpendicular if and only if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$.

Example 11

Find the acute angle between the following two straight lines:

$$\boldsymbol{r}_1(\lambda) = \boldsymbol{i} + 2\boldsymbol{j} + \lambda(5\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})$$
$$\boldsymbol{r}_2(\mu) = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} + \mu(-2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k})$$

Solution

The vectors $\boldsymbol{d}_1 = 5\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{d}_2 = -2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$ give the directions of the two lines.

We have $|\boldsymbol{d}_1| = \sqrt{38}$, $|\boldsymbol{d}_2| = \sqrt{38}$ and $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = -11$.

Let θ be the angle between $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$. Then

$$\cos\theta = \frac{\boldsymbol{d}_1 \cdot \boldsymbol{d}_2}{|\boldsymbol{d}_1|\,|\boldsymbol{d}_2|} = -\frac{11}{38}$$

The acute angle between the lines is 73.17°, correct to two decimal places.

Exercise 5B

Example 7

1 Find the position vector of the point of intersection of the lines with equations

$$\boldsymbol{r}_1(\lambda) = 3\boldsymbol{i} + 5\boldsymbol{j} + \lambda(2\boldsymbol{i} - \boldsymbol{j})$$
$$\boldsymbol{r}_2(\mu) = -2\boldsymbol{j} + \mu(4\boldsymbol{i} + 2\boldsymbol{j})$$

Example 8

2 Find the coordinates of the point of intersection of the lines with equations

$$\boldsymbol{r}_1(\lambda) = \boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + \lambda(-2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$$
$$\boldsymbol{r}_2(\mu) = -3\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$$

Example 9

3 Show that the following two lines are skew lines:

$$\boldsymbol{r}_1(\lambda) = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + \lambda(2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})$$
$$\boldsymbol{r}_2(\mu) = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k} + \mu(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$$

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4 For each pair of lines, answer the following questions:

i Are the lines parallel?

ii Are the lines perpendicular?

iii Do the lines coincide?

iv If they intersect at a point, what is the point of intersection?

a $\boldsymbol{r}_1(t) = \boldsymbol{i} + 2\boldsymbol{j} + t(\boldsymbol{i} + \boldsymbol{j})$
$\boldsymbol{r}_2(s) = -\boldsymbol{i} + 6\boldsymbol{j} + s(\boldsymbol{i} + 2\boldsymbol{j})$

b $\boldsymbol{r}_1(t) = -\boldsymbol{i} + \boldsymbol{j} + t(\boldsymbol{i} + 2\boldsymbol{j})$
$\boldsymbol{r}_2(s) = 3\boldsymbol{i} - \boldsymbol{j} + s(-2\boldsymbol{i} + \boldsymbol{j})$

c $\boldsymbol{r}_1(t) = 5\boldsymbol{i} + 9\boldsymbol{j} + t(-2\boldsymbol{i} - 3\boldsymbol{j})$
$\boldsymbol{r}_2(s) = \boldsymbol{i} + 3\boldsymbol{j} + s(4\boldsymbol{i} + 6\boldsymbol{j})$

d $\boldsymbol{r}_1(t) = \boldsymbol{i} - 4\boldsymbol{j} + t(2\boldsymbol{i} - \boldsymbol{j})$
$\boldsymbol{r}_2(s) = 7\boldsymbol{i} + 8\boldsymbol{j} + s(-2\boldsymbol{i} + \boldsymbol{j})$

e $\boldsymbol{r}_1(t) = 5\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$
$\boldsymbol{r}_2(s) = 4\boldsymbol{j} + \boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

f $\boldsymbol{r}_1(t) = 7\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k} + t(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$
$\boldsymbol{r}_2(s) = \boldsymbol{j} - 3\boldsymbol{k} + s(\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k})$

g $\boldsymbol{r}_1(t) = 6\boldsymbol{i} - 6\boldsymbol{j} + 5\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$
$\boldsymbol{r}_2(s) = \boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

h $\boldsymbol{r}_1(t) = 4\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k} + t(2\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k})$
$\boldsymbol{r}_2(s) = -\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k} + s(-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

i $\boldsymbol{r}_1(t) = -3\boldsymbol{i} - \boldsymbol{j} + t(3\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$
$\boldsymbol{r}_2(s) = 4\boldsymbol{i} + \boldsymbol{j} - 6\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{k})$

j $\boldsymbol{r}_1(t) = 7\boldsymbol{i} - 6\boldsymbol{j} + t(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$
$\boldsymbol{r}_2(s) = -3\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k} + s(2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$

Example 10

5 For each of the following, find the point of concurrence (if it exists) of the lines:

a $\boldsymbol{r}_1(t) = 3\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k} + t(\boldsymbol{i} - \boldsymbol{k})$
$\boldsymbol{r}_2(s) = 2\boldsymbol{i} + 3\boldsymbol{j} + s(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
$\boldsymbol{r}_3(u) = -\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k} + u(-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

b $\boldsymbol{r}_1(t) = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k} + t(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$
$\boldsymbol{r}_2(s) = 25\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k} + s(\boldsymbol{i} + 3\boldsymbol{j})$
$\boldsymbol{r}_3(u) = 5\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + u(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

c $\boldsymbol{r}_1(t) = 5\boldsymbol{i} - \boldsymbol{j} + t(\boldsymbol{i} + \boldsymbol{k})$
$\boldsymbol{r}_2(s) = 10\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k} + s(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$
$\boldsymbol{r}_3(u) = 5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k} + u(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

d $\boldsymbol{r}_1(t) = -5\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{k})$
$\boldsymbol{r}_2(s) = 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$
$\boldsymbol{r}_3(u) = 5\boldsymbol{i} + 8\boldsymbol{j} + u(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

Example 11

6 Find the acute angle between each of the following pairs of lines:

a $\boldsymbol{r}_1 = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$
$\boldsymbol{r}_2 = 5\boldsymbol{j} - 2\boldsymbol{k} + s(3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$

b $\boldsymbol{r}_1 = 4\boldsymbol{i} - \boldsymbol{j} + t(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$
$\boldsymbol{r}_2 = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} - s(2\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k})$

7 The lines ℓ_1 and ℓ_2 are given by the equations

$$\ell_1: \quad \boldsymbol{r}_1 = \boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$$
$$\ell_2: \quad \boldsymbol{r}_2 = 3\boldsymbol{i} + 3\boldsymbol{j} + 8\boldsymbol{k} + s(\boldsymbol{i} + \boldsymbol{k})$$

a Find the acute angle between the lines.

b Show that the lines are skew lines.

8 The lines ℓ_1 and ℓ_2 are given by the equations

$$\ell_1: \quad \boldsymbol{r}_1 = 3\boldsymbol{i} + \boldsymbol{j} + t(2\boldsymbol{j} + \boldsymbol{k})$$
$$\ell_2: \quad \boldsymbol{r}_2 = 4\boldsymbol{k} + s(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$$

a Find the coordinates of the point of intersection of the lines.

b Find the cosine of the angle between the lines.

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9 Three lines are represented by vector equations as follows:

$$\ell_1: \quad \boldsymbol{r}_1 = \boldsymbol{i} - 2\boldsymbol{k} + t_1(\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}), \quad t_1 \in \mathbb{R}$$
$$\ell_2: \quad \boldsymbol{r}_2 = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} + t_2(-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}), \quad t_2 \in \mathbb{R}$$
$$\ell_3: \quad \boldsymbol{r}_3 = 3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + t_3(\boldsymbol{i} - 4\boldsymbol{j}), \quad t_3 \in \mathbb{R}$$

For each pair of lines, determine whether they intersect or not. If they intersect, then find their point of intersection.

5C Vector product

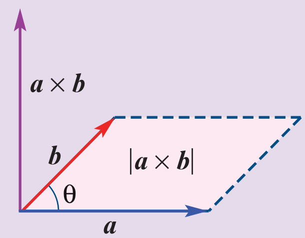
The vector product is an operation that takes two vectors and produces another vector.

Geometric definition of the vector product

Definition of the vector product

The **vector product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a} \times \boldsymbol{b}$.

- The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to $|\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, in the sense of the right-hand rule explained below.



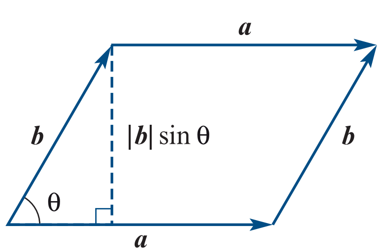
Note: The vector product is often called the **cross product**.

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$

By definition, we have

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

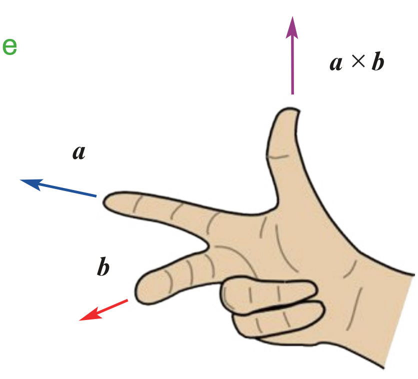


From the diagram on the right, we see that $|\boldsymbol{a} \times \boldsymbol{b}|$ is the area of the parallelogram 'spanned' by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

The direction of $\boldsymbol{a} \times \boldsymbol{b}$ using the right-hand rule

To find the direction of the vector $\boldsymbol{a} \times \boldsymbol{b}$ using your right hand:

- Point your index finger along the vector $\boldsymbol{a}$.
- Point your middle finger along the vector $\boldsymbol{b}$.
- Keep your thumb at right angles to both $\boldsymbol{a}$ and $\boldsymbol{b}$, as in the picture. The direction of your thumb gives the direction of the vector $\boldsymbol{a} \times \boldsymbol{b}$.



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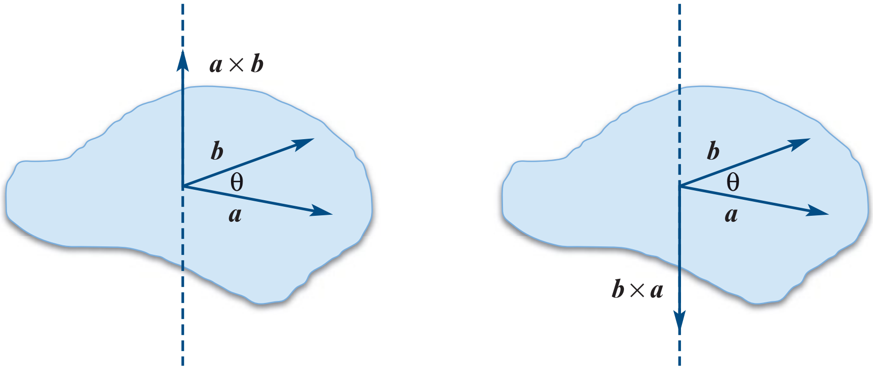
The following two diagrams show $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$.



The vector $\boldsymbol{b} \times \boldsymbol{a}$ has the same magnitude as $\boldsymbol{a} \times \boldsymbol{b}$, but the opposite direction. We can see that

$$\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b})$$

Thus the vector product is *not* commutative.

Note: The vector product is also *not* associative: in general, we have $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

Vector product of parallel vectors

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors, then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, since $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 0^\circ = 0$.

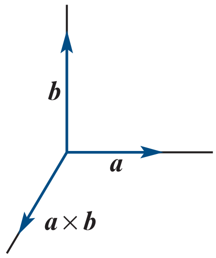
Conversely, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors such that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

Vector product of perpendicular vectors

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors, then

$$\begin{aligned} |\boldsymbol{a} \times \boldsymbol{b}| &= |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 90^\circ \\ &= |\boldsymbol{a}|\,|\boldsymbol{b}| \end{aligned}$$

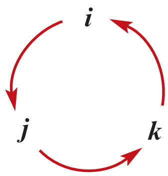
The three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ form a right-handed system of mutually perpendicular vectors, as shown in the diagram.



Vector product in component form

Using the previous observations about the vector product of parallel and perpendicular vectors:

- $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{0}$
- $\boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{0}$
- $\boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$
- $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$
- $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$
- $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$
- $\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$
- $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$
- $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$



The diagram on the right may help you to follow the pattern among these vector products.

The vector product distributes over addition. That is:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

These facts can be used to establish the following result.

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Vector product in component form

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$$

Note: In Specialist Mathematics Units 1 & 2, you may have seen how to find the determinant of a 3×3 matrix. This gives a way of 'evaluating' the vector product as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\boldsymbol{k}$$

$$= (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$$

To obtain the $\boldsymbol{i}$-component, we 'delete' the $\boldsymbol{i}$-row and the $\boldsymbol{i}$-column of the 3×3 matrix. Likewise for the $\boldsymbol{j}$- and $\boldsymbol{k}$-components.

(Here we are using $|\mathbf{A}|$ to denote the determinant of a square matrix $\mathbf{A}$.)

Example 12

Find the vector product of $\boldsymbol{a} = 3\boldsymbol{i} + 3\boldsymbol{j} + 8\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$, and hence find a unit vector that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution

The vector product can be 'evaluated' as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 3 & 8 \\ 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 8 \\ -3 & 2 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} 3 & 8 \\ 1 & 2 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix}\boldsymbol{k}$$

$$= (3 \times 2 - 8 \times (-3))\boldsymbol{i} - (3 \times 2 - 8 \times 1)\boldsymbol{j} + (3 \times (-3) - 3 \times 1)\boldsymbol{k}$$

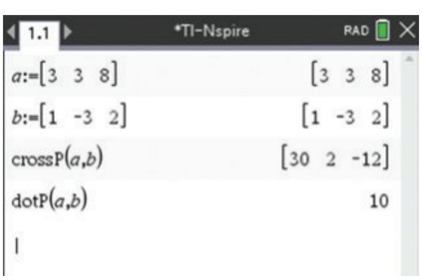
$$= 30\boldsymbol{i} + 2\boldsymbol{j} - 12\boldsymbol{k}$$

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is $\sqrt{30^2 + 2^2 + 12^2} = 2\sqrt{262}$.

Hence a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\dfrac{1}{2\sqrt{262}}(30\boldsymbol{i} + 2\boldsymbol{j} - 12\boldsymbol{k})$.

Using the TI-Nspire

- Define (assign) the vectors $\boldsymbol{a} = 3\boldsymbol{i} + 3\boldsymbol{j} + 8\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ as shown.
- Find the vector product using menu > **Matrix & Vector** > **Vector** > **Cross Product**. The vector product is $30\boldsymbol{i} + 2\boldsymbol{j} - 12\boldsymbol{k}$.
- The scalar product is found using menu > **Matrix & Vector** > **Vector** > **Dot Product**.



Note: You can enter the matrices directly into the vector commands if preferred.

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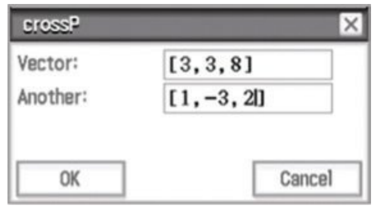
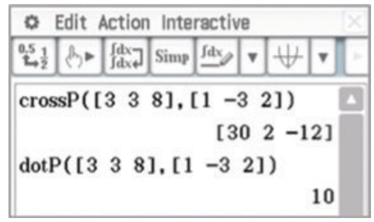
Using the Casio ClassPad

To find the vector product of two vectors:

- In Main $\sqrt{\alpha}$, go to **Interactive** > **Vector** > **crossP**.
- Tap the cursor in the first entry box.
- Select the vector icon [ ] from the Math3 keyboard.
- Enter the components of the first vector, separated by commas.
- Tap the cursor in the second entry box, enter the second vector and tap OK.

The scalar product of two vectors can be found similarly using **Interactive** > **Vector** > **dotP**.





Note: When accessing these commands via **Action** (rather than via **Interactive**), you can enter the three-dimensional vectors by tapping twice on the 1×2 matrix icon [■□] from the Math2 keyboard.

Example 13

a Simplify:

i $\boldsymbol{a} \times (\boldsymbol{a} - \boldsymbol{b})$ **ii** $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a}$

b Given that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{a}$, with $\boldsymbol{a} \neq \boldsymbol{0}$, show that $\boldsymbol{b} = -\boldsymbol{c}$ or $\boldsymbol{a} = k(\boldsymbol{b} + \boldsymbol{c})$ for some $k \in \mathbb{R}$.

Solution

a **i** Since the vector product distributes over addition, we have

$$\begin{aligned}\boldsymbol{a} \times (\boldsymbol{a} - \boldsymbol{b}) &= \boldsymbol{a} \times (\boldsymbol{a} + (-\boldsymbol{b}))\\ &= \boldsymbol{a} \times \boldsymbol{a} + \boldsymbol{a} \times (-\boldsymbol{b})\\ &= \boldsymbol{0} + \boldsymbol{a} \times (-\boldsymbol{b})\\ &= -(\boldsymbol{a} \times \boldsymbol{b})\\ &= \boldsymbol{b} \times \boldsymbol{a}\end{aligned}$$

ii Since $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$, we have $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$.

b By assumption, we have

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{c} \times \boldsymbol{a}\\ \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{c} \times \boldsymbol{a} &= \boldsymbol{0}\\ \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} &= \boldsymbol{0}\\ \therefore \quad \boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{0}\end{aligned}$$

Since $\boldsymbol{a} \neq \boldsymbol{0}$, it follows that either $\boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$ or the vectors $\boldsymbol{a}$ and $\boldsymbol{b} + \boldsymbol{c}$ are parallel. Hence we must have $\boldsymbol{b} = -\boldsymbol{c}$ or $\boldsymbol{a} = k(\boldsymbol{b} + \boldsymbol{c})$ for some $k \in \mathbb{R}$.

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Skill-sheet

Exercise 5C

Example 12

1 Use the vector product to find a vector perpendicular to the two given vectors:

a $\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and $4\boldsymbol{i} + 3\boldsymbol{j}$

b $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

c $\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{k}$

d $2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $2\boldsymbol{j}$

e $2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ and $-4\boldsymbol{i} + 3\boldsymbol{k}$

f $3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $-\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

g $-2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{i}$

h $-2\boldsymbol{i} - \boldsymbol{k}$ and $2\boldsymbol{j}$

Example 13

2 Simplify:

a $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{b}$

b $(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b})$

c $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b})$

d $(\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})) \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot ((\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a})$

f $((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}))$

3 Find a vector of magnitude 5 that is perpendicular to $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$.

4 Find a vector perpendicular to $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$.

5 A parallelogram $OABC$ has one vertex at the origin O and two other vertices at the points $A(0, 1, 3)$ and $B(0, 2, 5)$. Find the area of $OABC$.

6 Find the area of the triangle PQR with vertices $P(1, 5, -2)$, $Q(0, 0, 0)$ and $R(3, 5, 1)$.

7 The three vertices of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Show that the area of the triangle is $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}|$.

8 Let $\boldsymbol{v}$ be a vector parallel to a line ℓ, and let $\boldsymbol{u}$ be a vector from any point on the line to a point P not on the line. Show that the distance from the point P to the line ℓ is $\dfrac{|\boldsymbol{u} \times \boldsymbol{v}|}{|\boldsymbol{v}|}$.

9 In this question, we verify that the component form of the vector product has some of its desired properties. Consider vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as follows:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$
$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$
$$\boldsymbol{c} = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$$

a Verify that $\boldsymbol{c}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Verify that if we swap $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\boldsymbol{c}$ becomes $-\boldsymbol{c}$.

c From Section 4C we know that

$$a_1b_1 + a_2b_2 + a_3b_3 = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Using this result and the Pythagorean identity, verify that $|\boldsymbol{c}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta$.

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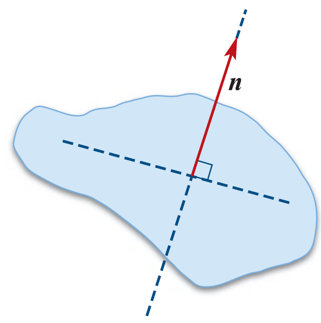
5D Vector equations of planes

Normal vectors to planes

At any point on a smooth surface, there is a line through the point that is perpendicular to the surface. For a plane, these perpendiculars are all in the same direction.



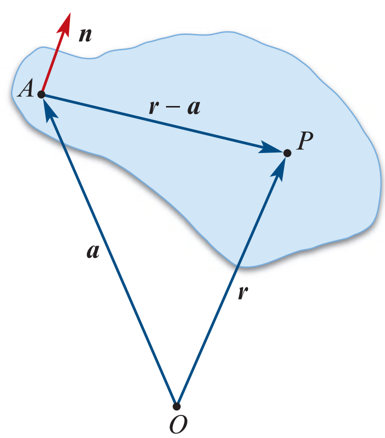
A vector that is perpendicular to a plane is called a **normal** to the plane.

Note: There is not a unique normal vector for a given plane. If the vector $\boldsymbol{n}$ is normal to the plane, then so are the vectors $k\boldsymbol{n}$ and $-k\boldsymbol{n}$, for all $k \in \mathbb{R}^+$.

Equations of planes

A plane Π in three-dimensional space may be described using two vectors:

- the position vector $\boldsymbol{a}$ of a point A on the plane
- a vector $\boldsymbol{n}$ that is normal to the plane.



Let $\boldsymbol{r}$ be the position vector of any other point P on the plane. Then the vector $\overrightarrow{AP} = \boldsymbol{r} - \boldsymbol{a}$ lies in the plane, and is therefore perpendicular to $\boldsymbol{n}$. Hence

$$(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$$

This can be written as

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

This is a **vector equation** of the plane.

If we write the position vector of the point P as $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and write the normal vector as $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$, then we obtain a **Cartesian equation** of the plane:

$$n_1x + n_2y + n_3z = \boldsymbol{a} \cdot \boldsymbol{n}$$

This is often written as

$$n_1x + n_2y + n_3z = k$$

where $k = \boldsymbol{a} \cdot \boldsymbol{n}$.

Note: We can also give parametric equations for a plane by using two parameters λ and μ. For example, if a plane has equation $n_1x + n_2y + n_3z = k$ with $n_3 \neq 0$, then it can be described by the parametric equations

$$x = \lambda, \quad y = \mu \quad \text{and} \quad z = \frac{k - n_1\lambda - n_2\mu}{n_3} \quad \text{for } \lambda, \mu \in \mathbb{R}$$

In this chapter, we usually describe planes by their vector or Cartesian equations, rather than by parameterisations.

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Planes in three dimensions

A plane in three-dimensional space can be described as follows, where $\boldsymbol{a}$ is the position vector of a point A on the plane, the vector $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$ is normal to the plane, and $k = \boldsymbol{a} \cdot \boldsymbol{n}$.

Vector equation	Cartesian equation
$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$	$n_1x + n_2y + n_3z = k$

Equations of planes can be found from various pieces of information:

- a point on the plane and a normal vector to the plane
- three points on the plane that do not line on a single line
- two lines in the plane that intersect at a point.

We now look at examples showing how to find equations of planes.

Finding the plane determined by a point and a normal vector

The following example illustrates two methods for finding an equation of a plane.

Example 14

A plane Π is such that the vector $-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$ is normal to the plane and the point A with position vector $-3\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$ is on the plane. Find a vector equation and a Cartesian equation of the plane.

Solution

Method 1: Finding a vector equation first

Using the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$, a vector equation is

$$\boldsymbol{r} \cdot (-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) = (-3\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}) \cdot (-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k})$$

i.e. $\boldsymbol{r} \cdot (-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) = 5$

For a Cartesian equation, write $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Then

$$(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \cdot (-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) = 5$$

i.e. $-x + 5y - 3z = 5$

Method 2: Finding a Cartesian equation first

The vector $\boldsymbol{n} = -\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$ is normal to the plane, so a Cartesian equation is

$$-x + 5y - 3z = k$$

for some $k \in \mathbb{R}$. Since the point $A(-3, 4, 6)$ is on the plane, we have

$$-(-3) + 5(4) - 3(6) = k$$

Therefore $k = 5$, and a Cartesian equation is $-x + 5y - 3z = 5$.

Hence a vector equation is $\boldsymbol{r} \cdot (-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) = 5$.

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Finding the plane determined by three points

Three points determine a plane provided they are not collinear.

Example 15

Consider the plane containing the points $A(0, 1, 1)$, $B(2, 1, 0)$ and $C(-2, 0, 3)$.

a Find a Cartesian equation of the plane.

b Find the axis intercepts of the plane, and hence sketch a graph of the plane.

Solution

a $\overrightarrow{AB} = 2\boldsymbol{i} - \boldsymbol{k}$ and $\overrightarrow{AC} = -2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

The vector product $\overrightarrow{AB} \times \overrightarrow{AC}$ is $-\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$.

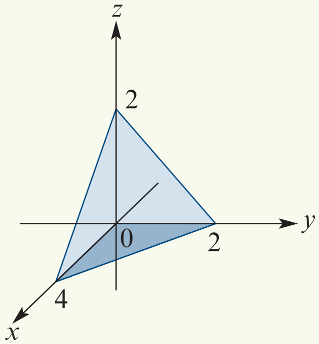
Therefore the vector $\boldsymbol{n} = -\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ is normal to the plane.

Using the point A and the normal $\boldsymbol{n}$, we can use either of the two methods to find the Cartesian equation $-x - 2y - 2z = -4$.

b We can write the Cartesian equation of the plane more neatly as $x + 2y + 2z = 4$.

x-axis intercept: Let $y = z = 0$. Then $x = 4$.
y-axis intercept: Let $x = z = 0$. Then $2y = 4$, so $y = 2$.
z-axis intercept: Let $x = y = 0$. Then $2z = 4$, so $z = 2$.

The axis intercepts of the plane are $(4, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.



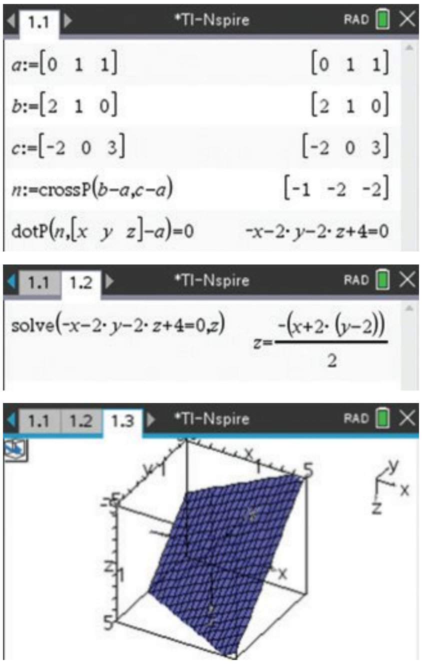
Using the TI-Nspire

To find a Cartesian equation of the plane containing $A(0, 1, 1)$, $B(2, 1, 0)$ and $C(-2, 0, 3)$:

- Define (assign) the three matrices as shown.
- Find the vector product using (menu) > **Matrix & Vector** > **Vector** > **Cross Product**.
- Display the Cartesian equation using the **Dot Product** command as shown.

To plot the Cartesian equation as a plane:

- Solve the Cartesian equation for z.
- In a **Graphs** application, use (menu) > **View** > **3D Graphing**. Enter the expression for z in $z1(x, y)$, i.e. $z1(x, y) = \dfrac{-(x + 2(y - 2))}{2}$
- To rotate the view of the plane, use (menu) > **Actions** > **Rotate** (or press (r)) and then use the arrow keys.



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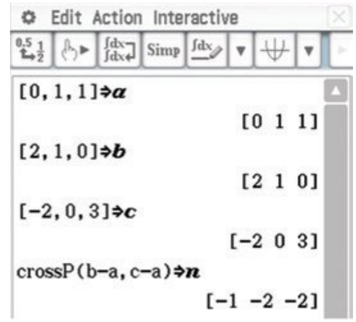
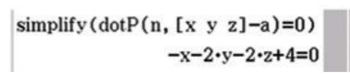
Using the Casio ClassPad

To find a Cartesian equation of the plane containing $A(0, 1, 1)$, $B(2, 1, 0)$ and $C(-2, 0, 3)$:

- Store the position vectors by assigning them to the variables $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as shown.
- Go to **Interactive** > **Vector** > **crossP**. Enter $\boldsymbol{b} - \boldsymbol{a}$ as the first vector, and $\boldsymbol{c} - \boldsymbol{a}$ as the second vector. Tap OK.
- Assign the vector product to the variable $\boldsymbol{n}$ as shown.

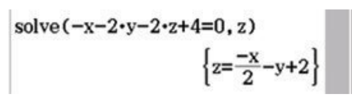


- Go to **Interactive** > **Vector** > **dotP**. Enter $\boldsymbol{n}$ as the first vector, and $[x, y, z] - \boldsymbol{a}$ as the second vector. Tap OK.
- At the end of the **dotP** expression, type = 0. Highlight and simplify to obtain the equation.

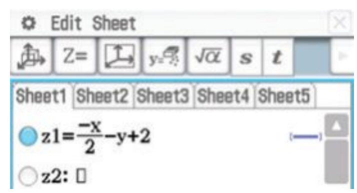


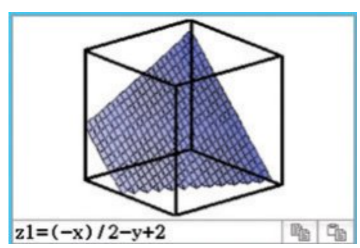
To plot the Cartesian equation as a plane:

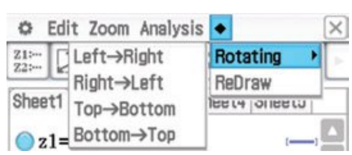
- Solve the Cartesian equation for z.



- Copy the equation. Then open the menu Menu and select **3D Graph**.
- Paste the equation in $z1$ and tap the circle.
- Tap to view the graph.
- Tap in the graph window; then tap to reveal the axes or box.
- Tap on the diamond in the menu bar to select the desired rotation option.







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Finding the plane determined by two intersecting lines

Two lines that intersect at a single point can be used to determine a plane.

Example 16

Find a vector equation and a Cartesian equation of the plane containing the lines

$$\boldsymbol{r}_1(\lambda) = 5\boldsymbol{i} + 2\boldsymbol{j} + \lambda(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

$$\boldsymbol{r}_2(\mu) = -3\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$$

Note: From Example 8, we know that these lines intersect at the point $(1, 0, -2)$.

Solution

We know that $\boldsymbol{a} = 5\boldsymbol{i} + 2\boldsymbol{j}$ is the position vector of a point on the plane.

We want to find a normal vector. It must be perpendicular to both $\boldsymbol{d}_1 = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{d}_2 = \boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$, so we can choose

$$\begin{aligned}\boldsymbol{n} &= \boldsymbol{d}_1 \times \boldsymbol{d}_2 \\ &= -\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}\end{aligned}$$

Hence a vector equation of the plane is

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

i.e. $\quad \boldsymbol{r} \cdot (-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) = 5$

The corresponding Cartesian equation is $-x + 5y - 3z = 5$.

Note: We obtain the same vector equation by choosing $\boldsymbol{a}$ to be the position vector of any point on the plane. For instance, we could have chosen the position vector $\boldsymbol{i} - 2\boldsymbol{k}$ of the point of intersection.

Skill-sheet

Exercise 5D

Example 14

1 In each of the following, a vector $\boldsymbol{n}$ normal to the plane and a point A on the plane are given. Find a vector equation and a Cartesian equation of each plane.

a $\boldsymbol{n} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\quad A(1, -2, 4)$

b $\boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{k}$, $\quad A(3, 1, 0)$

c $\boldsymbol{n} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\quad A(2, -3, -5)$

d $\boldsymbol{n} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\quad A(1, -2, 3)$

Example 15

2 Points $A = (2, 1, -1)$, $B = (1, 3, 1)$ and $C = (3, -2, 2)$ lie in a plane. Find a unit vector normal to this plane and find a vector equation of this plane.

Example 16

3 Find a vector equation and a Cartesian equation of the plane containing the lines

$$\boldsymbol{r}_1(\lambda) = \boldsymbol{i} - 10\boldsymbol{j} + 4\boldsymbol{k} + \lambda(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$$

$$\boldsymbol{r}_2(\mu) = -3\boldsymbol{i} - 2\boldsymbol{j} + \mu(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$$

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4 The point $A = (-3, 1, 1)$ and the line ℓ lie in the same plane. The line ℓ is defined by the equation $\boldsymbol{r} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$, $t \in \mathbb{R}$.

a Find a vector normal to this plane.

b Find a vector equation of the line through A that is normal to this plane.

5 Points $A = (1, 1, 3)$, $B = (1, 5, -2)$ and $C = (0, 3, -1)$ lie in a plane. Find a unit vector normal to this plane and find a vector equation of this plane.

6 A plane is defined by the vector equation $\boldsymbol{r} \cdot (2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}) = 7$. Show that each of the following is the position vector of a point on this plane:

a $\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ **b** $3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ **c** $-\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$ **d** $2\boldsymbol{j} - 3\boldsymbol{k}$

7 A plane is defined by the vector equation $\boldsymbol{r} \cdot (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 10$. Show that each of the following is a point on this plane:

a $(2, 2, -2)$ **b** $(1, 5, -2)$ **c** $(3, 4, 3)$ **d** $(2, 0, -4)$

8 Find x in each of the following:

a The point $(1, x, 2)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (-\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) = 5$.

b The point $(2, -1, 0)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (3\boldsymbol{i} + 2\boldsymbol{k}) = x$.

c The point $(1, -3, 2)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (2\boldsymbol{i} + x\boldsymbol{k}) = 8$.

d The point $(x, 1, -2)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) = 5$.

9 Find a Cartesian equation of the plane containing the three points $A(0, 3, 4)$, $B(1, 2, 0)$ and $C(-1, 6, 4)$.

10 Find a Cartesian equation of the plane that is at right angles to the line given by $x = 4 + t,\ y = 1 - 2t,\ z = 8t$ and goes through the point $P(3, 2, 1)$.

11 Find a Cartesian equation of the plane that is parallel to the plane with equation $5x - 3y + 2z = 6$ and goes through the point $P(4, -1, 2)$.

12 Find a Cartesian equation of the plane that contains the intersecting lines given by $x = 4 + t_1,\ y = 2t_1,\ z = 1 - 3t_1$ and $x = 4 - 3t_2,\ y = 3t_2,\ z = 1 + 2t_2$.

13 Find a Cartesian equation of the plane that is at right angles to the plane with equation $3x + 2y - z = 4$ and goes through the points $P(1, 2, 4)$ and $Q(-1, 3, 2)$.

14 Find a Cartesian equation of the plane that contains the two parallel lines given by $\boldsymbol{r}_1(t) = (3 + 4t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j} + t\boldsymbol{k}$ and $\boldsymbol{r}_2(s) = (5 + 4s)\boldsymbol{i} - 2s\boldsymbol{j} + (1 + s)\boldsymbol{k}$.

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5E Distances, angles and intersections

Distance from a point to a plane

The distance from a point P to a plane Π is given by

$$d = |\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}|$$

where $\hat{\boldsymbol{n}}$ is a unit vector normal to the plane and Q is any point on the plane.

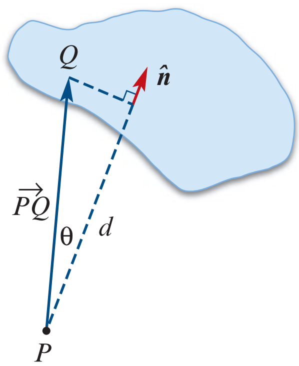
Proof For the situation shown in the diagram, we can see that the distance from P to the plane is

$$d = |\overrightarrow{PQ}| \cos \theta$$

where θ is the angle between $\overrightarrow{PQ}$ and $\hat{\boldsymbol{n}}$. Therefore

$$d = |\overrightarrow{PQ}|\,|\hat{\boldsymbol{n}}| \cos \theta = \overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}$$

The other situation is where the unit normal $\hat{\boldsymbol{n}}$ points in the opposite direction. In this case, we will obtain $d = -\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}$. Hence, in general, the distance is the absolute value of $\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}$.



Example 17

Find the distance from the point $P(1, -4, -3)$ to the plane Π: $2x - 3y + 6z = -1$.

Solution

A normal vector to the plane is $\boldsymbol{n} = 2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$. So a unit vector normal to the plane is

$$\hat{\boldsymbol{n}} = \frac{1}{7}(2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k})$$

Let $Q(x, y, z)$ be any point on the plane. Note that this implies $2x - 3y + 6z = -1$.

We want to find the projection of $\overrightarrow{PQ}$ onto $\hat{\boldsymbol{n}}$. We have

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x-1)\boldsymbol{i} + (y+4)\boldsymbol{j} + (z+3)\boldsymbol{k}$$

Therefore

$$\begin{aligned}\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}} &= \frac{1}{7}(2(x-1) - 3(y+4) + 6(z+3)) \\ &= \frac{1}{7}(2x - 3y + 6z - 2 - 12 + 18) \\ &= \frac{1}{7}(-1 + 4) \qquad \text{(since } 2x - 3y + 6z = -1\text{)} \\ &= \frac{3}{7}\end{aligned}$$

The distance from the point P to the plane Π is $\frac{3}{7}$.

Note: Alternatively, we could have chosen Q to be a specific point on the plane, such as $(1, 1, 0)$. This would give $\overrightarrow{PQ} = 5\boldsymbol{j} + 3\boldsymbol{k}$ and therefore $\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}} = \frac{3}{7}$.

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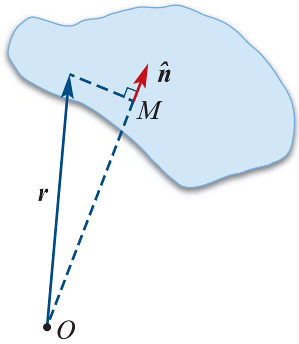
Distance of a plane from the origin

A plane that does not pass through the origin is described by a vector equation of the form $\boldsymbol{r} \cdot \boldsymbol{n} = k$, where $k \neq 0$.



The point M on the plane that is closest to the origin has a position vector of the form $\overrightarrow{OM} = m\hat{\boldsymbol{n}}$, where $|m|$ is the distance of the plane from the origin.

If $\boldsymbol{n}$ points towards the plane from the origin, then $m > 0$, and if $\boldsymbol{n}$ points away from the plane, then $m < 0$. So we can say that m is the 'signed distance' of the plane from the origin (relative to the normal vector $\boldsymbol{n}$).

Since the point M lies on the plane, we know that $(m\hat{\boldsymbol{n}}) \cdot \boldsymbol{n} = k$. But $(m\hat{\boldsymbol{n}}) \cdot \boldsymbol{n} = m(\hat{\boldsymbol{n}} \cdot \boldsymbol{n}) = m|\boldsymbol{n}|$. So we have $m|\boldsymbol{n}| = k$ and therefore $m = \dfrac{k}{|\boldsymbol{n}|}$.

For a plane with vector equation $\boldsymbol{r} \cdot \boldsymbol{n} = k$, where $k \neq 0$, the signed distance of the plane from the origin (relative to the normal vector $\boldsymbol{n}$) is given by $\dfrac{k}{|\boldsymbol{n}|}$.

Distance between two parallel planes

To find the distance between parallel planes Π_1 and Π_2, we can choose any point P on Π_1 and then find the distance from the point P to the plane Π_2.

In the following example, we use an alternative method.

Example 18

Consider the parallel planes given by the equations

$$\Pi_1 : 2x - y + 2z = 5 \quad \text{and} \quad \Pi_2 : 2x - y + 2z = -2$$

a Find the distance of each plane from the origin.

b Find the distance between the two planes.

Solution

The vector $\boldsymbol{n} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ is normal to both planes, with $|\boldsymbol{n}| = 3$.

a Relative to $\boldsymbol{n}$, the signed distance of plane Π_1 from the origin is $\dfrac{5}{|\boldsymbol{n}|} = \dfrac{5}{3}$.

So the distance of plane Π_1 from the origin is $\dfrac{5}{3}$.

Relative to $\boldsymbol{n}$, the signed distance of plane Π_2 from the origin is $\dfrac{-2}{|\boldsymbol{n}|} = -\dfrac{2}{3}$.

So the distance of plane Π_2 from the origin is $\dfrac{2}{3}$.

b Relative to the normal vector $\boldsymbol{n}$, the two planes are on different sides of the origin. So the distance between them is $\dfrac{5}{3} + \dfrac{2}{3} = \dfrac{7}{3}$.

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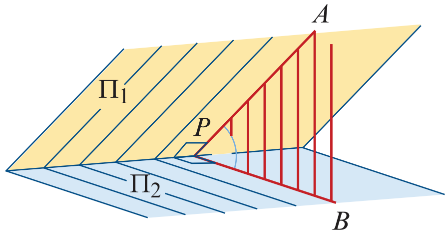
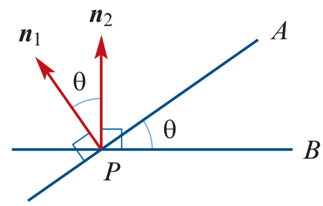
Intersections and angles

Intersection of two planes

Two planes that are not parallel will intersect in a line.



Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between planes Π_1 and Π_2.



To find the angle between the two planes, we first find the angle θ between two vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ that are normal to the two planes. The angle between the planes is θ or $180° - \theta$, whichever is in the interval $[0°, 90°]$.

- Two planes are parallel if and only if the two normal vectors are parallel.
- Two planes are perpendicular if and only if the two normal vectors are perpendicular.

Example 19

Let Π_1 and Π_2 be the planes represented by the vector equations

$$\Pi_1 : \boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = 6 \quad \text{and} \quad \Pi_2 : \boldsymbol{r} \cdot (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 4$$

a Find the angle between the planes.

b Find a vector equation of the line of intersection of the planes.

Solution

a We first find the angle between normals to the planes.

A normal to plane Π_1 is $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, and a normal to plane Π_2 is $\boldsymbol{n}_2 = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

Let θ be the angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. Then

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = |\boldsymbol{n}_1||\boldsymbol{n}_2| \cos\theta$$

$$(\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) \cdot (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = \sqrt{11}\sqrt{6} \cos\theta$$

$$\therefore \quad -2 = \sqrt{66} \cos\theta$$

Hence $\theta \approx 104.25°$. The acute angle between the planes is $180° - 104.25° = 75.75°$, correct to two decimal places.

b Consider Cartesian equations for the two planes:

$$x + y - 3z = 6 \qquad (1)$$

$$2x - y + z = 4 \qquad (2)$$

Add (1) and (2):

$$3x - 2z = 10 \qquad (3)$$

Let $x = \lambda$. Then $z = \dfrac{3\lambda - 10}{2}$, from (3), and $y = \dfrac{7\lambda - 18}{2}$, from (2).

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This gives us parametric equations for the line of intersection:

$$x = \lambda, \quad y = \frac{7\lambda - 18}{2}, \quad z = \frac{3\lambda - 10}{2}$$

These convert to the vector equation

$$\boldsymbol{r} = -9\boldsymbol{j} - 5\boldsymbol{k} + \lambda\left(\boldsymbol{i} + \frac{7}{2}\boldsymbol{j} + \frac{3}{2}\boldsymbol{k}\right), \quad \lambda \in \mathbb{R}$$

Note: Alternatively, we can use the parametric equations to find a point $A(0, -9, -5)$ on the line. A vector $\boldsymbol{d}$ parallel to the line must be perpendicular to the two normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. Hence we can choose $\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$.

Intersection of a line and a plane

A line and a plane that are not parallel intersect at a point. The angle between a line and a plane is equal to $90° - \theta$, where θ is the angle between the line and a normal to the plane.

Example 20

Consider the line represented by the equation $\boldsymbol{r} = 3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$ and the plane represented by the equation $\boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) = 2$.

a Find the point of intersection of the line and the plane.

b Find the angle between the line and the plane.

Solution

a To find the point of intersection, we want to find the value of t for which

$$\boldsymbol{r} = 3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$$

represents a point on the plane. That is,

$$(3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})) \cdot (\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) = 2$$

$$(3 + t) + (-1 + 2t) + 2(-1 - t) = 2$$

$$\therefore \quad t = 2$$

The point of intersection has position vector

$$\boldsymbol{r} = 3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + 2(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) = 5\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$$

The point of intersection is $(5, 3, -3)$.

b We first find the angle between the line and the normal to the plane.

The vector $\boldsymbol{d} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ is parallel to the line, and the vector $\boldsymbol{n} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ is normal to the plane. Let θ be the angle between $\boldsymbol{d}$ and $\boldsymbol{n}$. Then

$$\boldsymbol{d} \cdot \boldsymbol{n} = |\boldsymbol{d}|\,|\boldsymbol{n}| \cos\theta$$

$$(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) \cdot (\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) = \sqrt{6}\sqrt{6}\cos\theta$$

$$\therefore \quad 1 = 6\cos\theta$$

So $\theta = 80.4°$, correct to one decimal place. Hence the angle between the line and the plane is $90° - 80.4° = 9.6°$, correct to one decimal place.

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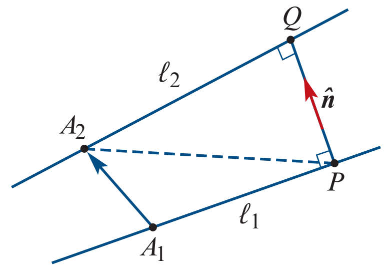
Distance between two skew lines

Given two skew lines, it can be shown that there is a unique line segment PQ joining the two lines that is perpendicular to both lines. The distance between the two lines is the length PQ.

We can find the distance between a pair of skew lines $\ell_1 : \boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda \boldsymbol{d}_1$ and $\ell_2 : \boldsymbol{r}_2 = \boldsymbol{a}_2 + \mu \boldsymbol{d}_2$ as follows.



	Steps	Explanation
1	Let P and Q be the points on ℓ_1 and ℓ_2 such that PQ is the distance between ℓ_1 and ℓ_2.	
2	A unit vector parallel to $\overrightarrow{PQ}$ is $\hat{\boldsymbol{n}} = \dfrac{\boldsymbol{d}_1 \times \boldsymbol{d}_2}{\lvert\boldsymbol{d}_1 \times \boldsymbol{d}_2\rvert}$	Vector $\overrightarrow{PQ}$ is perpendicular to both lines and thus parallel to $\boldsymbol{d}_1 \times \boldsymbol{d}_2$.
3	The distance between the skew lines is $d = \lvert(\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot \hat{\boldsymbol{n}}\rvert$	The magnitude of the projection of $\overrightarrow{PA_2}$ onto $\hat{\boldsymbol{n}}$ will give the distance, and $\overrightarrow{PA_2} \cdot \hat{\boldsymbol{n}} = (\overrightarrow{PA_1} + \overrightarrow{A_1A_2}) \cdot \hat{\boldsymbol{n}} = \overrightarrow{A_1A_2} \cdot \hat{\boldsymbol{n}}$.

Example 21

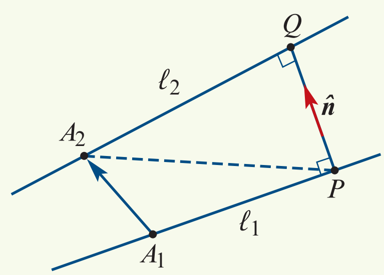
Find the distance between the two skew lines

$$\ell_1 : \boldsymbol{r}_1 = \boldsymbol{i} + \boldsymbol{j} + \lambda(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) \quad \text{and} \quad \ell_2 : \boldsymbol{r}_2 = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \mu(3\boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k})$$

Solution

Here we have

$$\boldsymbol{a}_1 = \boldsymbol{i} + \boldsymbol{j} \qquad \boldsymbol{d}_1 = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$
$$\boldsymbol{a}_2 = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \qquad \boldsymbol{d}_2 = 3\boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k}$$



Step 1 Let P and Q be the points on ℓ_1 and ℓ_2 such that PQ is the distance between ℓ_1 and ℓ_2.

Step 2 A unit vector parallel to $\overrightarrow{PQ}$ is $\hat{\boldsymbol{n}} = \dfrac{\boldsymbol{d}_1 \times \boldsymbol{d}_2}{\lvert\boldsymbol{d}_1 \times \boldsymbol{d}_2\rvert}$.

Here $\boldsymbol{d}_1 \times \boldsymbol{d}_2 = 3\boldsymbol{i} - \boldsymbol{j} - 7\boldsymbol{k}$ and $\lvert\boldsymbol{d}_1 \times \boldsymbol{d}_2\rvert = \sqrt{59}$, so

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{59}}(3\boldsymbol{i} - \boldsymbol{j} - 7\boldsymbol{k})$$

Step 3 The distance between the skew lines is $d = \lvert(\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot \hat{\boldsymbol{n}}\rvert$.
Since $\boldsymbol{a}_2 - \boldsymbol{a}_1 = \boldsymbol{i} - \boldsymbol{k}$, we have

$$d = \left\lvert(\boldsymbol{i} - \boldsymbol{k}) \cdot \frac{1}{\sqrt{59}}(3\boldsymbol{i} - \boldsymbol{j} - 7\boldsymbol{k})\right\rvert = \frac{10}{\sqrt{59}}$$

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Exercise 5E

Example 17

1 Find the distance from the point $(1, 3, 2)$ to each of the following planes:

a $\boldsymbol{r} \cdot (7\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}) = 9$ **b** $6x + 6y + 3z = 8$

Example 18

2 Find the distance between the following pair of parallel planes:

$$\Pi_1: \quad x + 2y - 2z = 4$$
$$\Pi_2: \quad x + 2y - 2z = 12$$

Example 19

3 Let Π_1 and Π_2 be the planes represented by the vector equations

$$\Pi_1: \quad \boldsymbol{r} \cdot (2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 8$$
$$\Pi_2: \quad \boldsymbol{r} \cdot (\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 6$$

a Find the angle between the planes.

b Find a vector equation of the line of intersection of the planes.

Example 20

4 Consider the line represented by the equation $\boldsymbol{r} = 3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$ and the plane represented by the equation $\boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) = 4$.

a Find the point of intersection of the line and the plane.

b Find the angle between the line and the plane.

5 Let $A = (2, 0, -1)$, $B = (1, -3, 1)$, $C = (0, -1, 2)$ and $D = (3, -2, 2)$.

a Find a vector normal to the plane containing points A, B and C.

b Find a vector normal to the plane containing points B, C and D.

c Use the two normal vectors to find the angle between these two planes.

6 In each of the following, a pair of vector equations is given that represent a line and a plane respectively. Find the point of intersection of the line and the plane and find the angle between the line and the plane, correct to two decimal places.

a $\boldsymbol{r} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k} + t(\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k})$
$\boldsymbol{r} \cdot (2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) = 7$

b $\boldsymbol{r} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k} + t(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
$\boldsymbol{r} \cdot (\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}) = 7$

c $\boldsymbol{r} = -\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k} + t(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$
$\boldsymbol{r} \cdot (-2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 4$

d $\boldsymbol{r} = -\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k} + t(2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k})$
$\boldsymbol{r} \cdot (3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) = -10$

7 The vector $\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$ is normal to a plane Π which contains the point $A(5, 4, -1)$.

a Find a vector equation of the plane.

b Find the distance of the plane from the origin.

8 **a** Find the distance from the origin to the plane $\boldsymbol{r} \cdot (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) = 7$.

b Find the vector projection of $\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ in the direction of $2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$.

c Find the magnitude of this vector projection.

d Hence find the distance from the point $(1, 1, -1)$ to the given plane.

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9 Using the method of Question 8, find the distance from the point $(2, -1, 3)$ to the plane given by the equation $\boldsymbol{r} \cdot (-\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = -3$.

10 **a** Find the point of intersection of the line $\boldsymbol{r} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{k})$ and the plane $\boldsymbol{r} \cdot (3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = 11$.

b Find the acute angle between the line and the plane, correct to one decimal place.

11 Points A, B and C have position vectors $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ respectively.

a Find a Cartesian equation of the plane containing A, B and C.

b Find the area of triangle ABC.

c Find the position vector of the foot of the perpendicular from the origin O to the plane ABC.

12 Let Π_1 and Π_2 be the planes represented by the vector equations

$$\Pi_1: \quad \boldsymbol{r} \cdot (3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}) = 3$$

$$\Pi_2: \quad \boldsymbol{r} \cdot (8\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}) = 1$$

a Find the angle between the planes.

b Find a vector equation of the line of intersection of the planes.

13 Let $A = (0, 2, -1)$, $B = (1, 1, 1)$, $C = (-1, 0, 2)$ and $D = (2, -2, 2)$.

a Find a vector normal to the plane containing points A, B and C.

b Find a vector normal to the plane containing points B, C and D.

c Use the two normal vectors to find the angle between these two planes.

14 **a** Find a vector which is perpendicular to the two lines given by

$$\boldsymbol{r}_1 = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t_1(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}), \qquad t_1 \in \mathbb{R}$$

$$\boldsymbol{r}_2 = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t_2(-\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}), \qquad t_2 \in \mathbb{R}$$

b Find a vector equation of the line which is normal to the plane containing these two lines and which passes through their point of intersection.

Example 21 **15** For each of the following, find the distance between the two skew lines:

a $\boldsymbol{r}_1 = (1 + t)\boldsymbol{i} + (1 + 6t)\boldsymbol{j} + 2t\boldsymbol{k}$

$\boldsymbol{r}_2 = (1 + 2s)\boldsymbol{i} + (5 + 15s)\boldsymbol{j} + (-2 + 6s)\boldsymbol{k}$

b $\boldsymbol{r}_1 = (1 + t)\boldsymbol{i} + (2 - t)\boldsymbol{j} + (1 + t)\boldsymbol{k}$

$\boldsymbol{r}_2 = (2 + 2s)\boldsymbol{i} + (-1 + s)\boldsymbol{j} + (-1 + 2s)\boldsymbol{k}$

c $\boldsymbol{r}_1 = (1 - t)\boldsymbol{i} + (t - 2)\boldsymbol{j} + (3 - 2t)\boldsymbol{k}$

$\boldsymbol{r}_2 = (1 + s)\boldsymbol{i} + (-1 + 2s)\boldsymbol{j} + (-1 + 2s)\boldsymbol{k}$

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Chapter summary

Assign-ment

Nrich

Lines

A line in three dimensions can be described as follows, where $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is the position vector of a point A on the line, and $\boldsymbol{d} = d_1\boldsymbol{i} + d_2\boldsymbol{j} + d_3\boldsymbol{k}$ is parallel to the line.

Vector equation	Parametric equations	Cartesian form
$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}, \quad t \in \mathbb{R}$	$x = a_1 + d_1 t$ $y = a_2 + d_2 t$ $z = a_3 + d_3 t$	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$

Planes

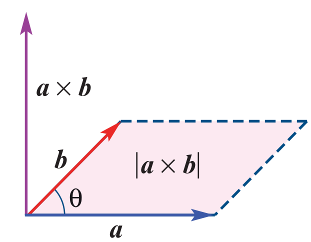
A plane in three dimensions can be described as follows, where $\boldsymbol{a}$ is the position vector of a point A on the plane, the vector $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$ is normal to the plane, and $k = \boldsymbol{a} \cdot \boldsymbol{n}$.

Vector equation	Cartesian equation
$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$	$n_1 x + n_2 y + n_3 z = k$

Vector product

- If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then

  $\boldsymbol{a} \times \boldsymbol{b} = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$
- The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to $|\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ (provided $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors and not parallel).



Distances and angles

- **Distance from a point to a line** The distance from a point P to a line ℓ is given by $|\overrightarrow{PQ}|$, where Q is the point on ℓ such that PQ is perpendicular to ℓ.
- **Distance from a point to a plane** The distance from a point P to a plane Π is given by $|\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}|$, where $\hat{\boldsymbol{n}}$ is a unit vector normal to Π and Q is any point on Π.
- **Angle between two lines** First find the angle θ between two vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ that are parallel to the two lines. The angle between the lines is θ or $180° - \theta$, whichever is in the interval $[0°, 90°]$.
- **Angle between two planes** First find the angle θ between two vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ that are normal to the two planes. The angle between the planes is θ or $180° - \theta$, whichever is in the interval $[0°, 90°]$.
- **Angle between a line and a plane** The angle between a line ℓ and a plane Π is $90° - \theta$, where θ is the angle between the line and a normal to the plane.

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Review

Technology-free questions

1 Consider two lines given by $\boldsymbol{r}_1(\lambda) = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \lambda(3\boldsymbol{i} - \boldsymbol{j})$ and $\boldsymbol{r}_2(\mu) = 4\boldsymbol{i} - \boldsymbol{k} + \mu(2\boldsymbol{i} + 3\boldsymbol{k})$. Find the position vector of the point of intersection of the lines.

2 Show that the lines $\boldsymbol{r}_1 = \boldsymbol{i} - \boldsymbol{j} + \lambda(2\boldsymbol{i} + \boldsymbol{k})$ and $\boldsymbol{r}_2 = 2\boldsymbol{i} - \boldsymbol{j} + \mu(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$ do not intersect.

3 Find a Cartesian equation of the plane through the point $(1, 2, 3)$ with normal vector $4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}$.

4 Find a vector equation of the line parallel to the x-axis that contains the point $(-2, 2, 1)$.

5 Find the coordinates of the nearest point to $(2, 1, 3)$ on the line $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + t(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$.

6 Find the distance from the origin to the line passing through the point $(3, 1, 5)$ parallel to the vector $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

7 Find the coordinates of the point of intersection of the line $\boldsymbol{r} = \boldsymbol{i} + \boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k})$, $t \in \mathbb{R}$, and the plane $\boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) = 13$.

8 Find a vector that is perpendicular to the vectors $8\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $7\boldsymbol{i} - 2\boldsymbol{j}$.

9 The line ℓ passes through the points $A(-1, -3, -3)$ and $B(5, 0, 6)$. Find a vector equation of the line ℓ. Find the point P on the line ℓ such that OP is perpendicular to the line, where O is the origin.

10 Show that the lines $\boldsymbol{r}_1 = 3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} + \lambda(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ and $\boldsymbol{r}_2 = \boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{k} + \mu(\boldsymbol{i} + \boldsymbol{k})$ are skew lines. Find the cosine of the angle between the lines.

11 Determine a Cartesian equation of the plane that contains the points $P(1, -2, 0)$, $Q(3, 1, 4)$ and $R(0, -1, 2)$.

12 Determine a Cartesian equation of the plane that contains the points $P(1, -2, 1)$, $Q(-2, 5, 0)$ and $R(-4, 3, 2)$.

13 Find an equation of the plane through $A(-1, 2, 0)$, $B(3, 1, 1)$ and $C(1, 0, 3)$ in:

a vector form **b** Cartesian form.

14 **a** Find a Cartesian equation of the plane passing through the origin O and the points $A(1, 1, 1)$ and $B(0, 1, 2)$.

b Find the area of triangle OAB.

c Show that the point $C(-2, 2, 6)$ lies on the plane and find the point of intersection of the lines OB and AC.

15 The origin O and the point $A(2, -1, -1)$ are two vertices of an equilateral triangle OAB in the plane $x + y + z = 0$. There are two possible locations for the vertex B; find the coordinates of B for both possibilities.

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Review

16 Show that the four points $(1, 0, 0)$, $(2, 1, 0)$, $(3, 2, 1)$ and $(4, 3, 2)$ are coplanar.

17 Let A, B, C and D be four points in three-dimensional space such that A, B and C are collinear. Show that A, B, C and D are coplanar.

18 For vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ such that $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$, show that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$.

Multiple-choice questions

1 The three vertices of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. The area of the triangle is equal to

A $\boldsymbol{a} \times \boldsymbol{b}$ **B** $\frac{1}{2}|\boldsymbol{b} \times \boldsymbol{c}|$ **C** $\frac{1}{2}|(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{b} - \boldsymbol{c})|$
D $(\boldsymbol{b} - \boldsymbol{c}) \times (\boldsymbol{a} - \boldsymbol{b})$ **E** $|(\boldsymbol{c} - \boldsymbol{a}) \times (\boldsymbol{a} - \boldsymbol{b})|$

2 A plane has equation $\boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = 5$. The distance from the origin to this plane is

A 5 **B** $\frac{5}{4}$ **C** $\frac{5}{6}$ **D** $\frac{5}{\sqrt{6}}$ **E** $\sqrt{5}$

3 Which of the following is a Cartesian equation of the plane that has axis intercepts at $x = 1$, $y = 2$ and $z = 3$?

A $x + 2y + 3z = 0$ **B** $x + 2y + 3z = 1$ **C** $x + 2y + 3z = 6$
D $6x + 3y + 2z = 6$ **E** $6x + 3y + 2z = 0$

4 The line given by $\boldsymbol{r} = 5\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k} + \lambda(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$, $\lambda \in \mathbb{R}$, intersects the plane given by $\boldsymbol{r} \cdot (2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = -6$ at the point with position vector

A $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ **B** $5\boldsymbol{i} - 7\boldsymbol{j} - 3\boldsymbol{k}$ **C** $3\boldsymbol{i} - 5\boldsymbol{j}$
D $\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ **E** $5\boldsymbol{i} - 4\boldsymbol{j} - 3\boldsymbol{k}$

5 The distance from the point $P(1, 5)$ to the line that passes through $(0, 0)$ and $(1, 1)$ is

A 1 **B** $\sqrt{2}$ **C** 2 **D** $2\sqrt{2}$ **E** 3

6 Which of the following vector equations does *not* describe the line passing through the points $(2, 0, 1)$ and $(3, 3, 3)$?

A $\boldsymbol{r} = (2 + t)\boldsymbol{i} + 3t\boldsymbol{j} + (1 + 2t)\boldsymbol{k}$ **B** $\boldsymbol{r} = (3 - t)\boldsymbol{i} + (3 - 3t)\boldsymbol{j} + (3 - 2t)\boldsymbol{k}$
C $\boldsymbol{r} = (2 + 3t)\boldsymbol{i} + 3t\boldsymbol{j} + (1 + 3t)\boldsymbol{k}$ **D** $\boldsymbol{r} = (1 + t)\boldsymbol{i} + (-3 + 3t)\boldsymbol{j} + (-1 + 2t)\boldsymbol{k}$
E $\boldsymbol{r} = (2 + 2t)\boldsymbol{i} + 6t\boldsymbol{j} + (1 + 4t)\boldsymbol{k}$

7 Which of the following equations describes the plane that contains the points $(0, 0, 1)$, $(1, 1, 1)$ and $(2, 0, 0)$?

A $x - y + 2z = 2$ **B** $x - y + z = 1$ **C** $x + 3y - z = 4$
D $x - y = 0$ **E** $x + y + z = 3$

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8 Which of the following vectors is parallel to the line of intersection of the planes $2x + y + z = 6$ and $x + z = 0$?

A $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ **B** $\boldsymbol{i} + \boldsymbol{k}$ **C** $3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

D $-3\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$ **E** $6\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

9 The distance from the origin to the plane $x + 2y + 2z = 5$ is

A 0 **B** $\frac{1}{5}$ **C** $\frac{1}{3}$ **D** 1 **E** $\frac{5}{3}$

10 The triangle with vertices $(0, 0, 1)$, $(1, 1, 1)$ and $(2, 3, 2)$ has area

A $\frac{1}{5}$ **B** $\frac{\sqrt{2}}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** 1 **E** $\sqrt{2}$

Extended-response questions

1 Two lines are represented by vector equations $\boldsymbol{r}_1(t_1) = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t_1(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$, $t_1 \in \mathbb{R}$, and $\boldsymbol{r}_2(t_2) = 2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k} + t_2(-\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$, $t_2 \in \mathbb{R}$.

a Show that these lines intersect and find their point of intersection, P.

b The vector equation $\boldsymbol{r}_3(t_3) = t_3(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$, $t_3 \in \mathbb{R}$, represents a line through the origin. Find the distance from the point of intersection P to this line.

2 The points A, B and C have position vectors $\overrightarrow{OA} = 5\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{OB} = -\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\overrightarrow{OC} = 3\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k}$. The plane Π_1 has vector equation $\boldsymbol{r} \cdot (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 6$.

a Show that the point C is on the plane Π_1.

b Show that the point B is the reflection in the plane Π_1 of the point A. (Hint: Show that the vector $\overrightarrow{AB}$ is normal to Π_1 and that the line segment AB is bisected by Π_1.)

c Find the length of the projection of $\overrightarrow{AC}$ onto the plane Π_1. (Hint: Let D be the projection of A onto Π_1, that is, the point D on Π_1 such that $\overrightarrow{AD}$ is normal to Π_1. Find the length of DC.)

The plane Π_2 has Cartesian equation $12x - 4y + 3z = k$, where k is a positive constant.

d Find the acute angle between planes Π_1 and Π_2.

e Given that the distance from the point C to the plane Π_2 is 3, find the value of k.

3 Consider the two lines given by

$$\ell_1: \quad \boldsymbol{r}_1 = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}), \quad t \in \mathbb{R}$$

$$\ell_2: \quad \boldsymbol{r}_2 = 16\boldsymbol{i} - 10\boldsymbol{j} + 2\boldsymbol{k} + s(3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}), \quad s \in \mathbb{R}$$

a Show that ℓ_1 and ℓ_2 are skew lines.

b Verify that both ℓ_1 and ℓ_2 are perpendicular to the vector $\boldsymbol{n} = 5\boldsymbol{i} - 7\boldsymbol{j} + \boldsymbol{k}$.

c The point $A(3, 2, 1)$ lies on line ℓ_1. Write down a vector equation of the line ℓ_3 through A in the direction of $\boldsymbol{n}$.

d Find the point of intersection, B, of the lines ℓ_2 and ℓ_3, and find the length of the line segment AB.

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4 The point O is the origin and the points A, B, C and D have position vectors

$$\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}, \quad \overrightarrow{OB} = 6\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}, \quad \overrightarrow{OC} = 9\boldsymbol{j} - 6\boldsymbol{k}, \quad \overrightarrow{OD} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

Prove that:

a the triangle OAB is isosceles

b the point D lies in the plane OAB

c the line CD is perpendicular to the plane OAB

d the line AC is inclined at an angle of $60°$ to the plane OAB.

5 A Cartesian equation of the plane Π_1 is $y + z = 0$ and a vector equation of the line ℓ is $\boldsymbol{r} = 5\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$, where $t \in \mathbb{R}$. Find:

a the position vector of the point of intersection of the line ℓ and the plane Π_1

b the length of the perpendicular from the origin to the line ℓ

c a Cartesian equation of the plane Π_2 which contains the line ℓ and the origin

d the acute angle between the planes Π_1 and Π_2, correct to one decimal place.

6 The Cartesian form of a line in three dimensions uses two linear equations in x, y and z. In this question, we show that a line in three dimensions can also be described by a single quadratic equation in x, y and z.

a Let $a, b \in \mathbb{R}$. Show that $a = b = 0$ if and only if $a^2 + b^2 = 0$.

b Show that the z-axis is described by the Cartesian equation $x^2 + y^2 = 0$.

c Show that the line given by the Cartesian equations

$$x - 3 = 2(y - 4) = z + 1$$

is also given by the single quadratic equation

$$(x - 2y + 5)^2 + (x - z - 4)^2 = 0$$

d Write a single quadratic equation for the line with Cartesian form

$$\frac{x}{2} = y - 3 = 4z + 5$$

e Write a single quadratic equation for the line with vector equation

$$\boldsymbol{r}(t) = (2 + t)\boldsymbol{i} + (3 - t)\boldsymbol{j} + 5t\boldsymbol{k}, \quad t \in \mathbb{R}$$

7 Consider a line ℓ with vector equation $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{d}$, and assume that ℓ does not pass through the origin. Let P be a point on ℓ with position vector $\boldsymbol{r}(t)$.

a Show that $|\overrightarrow{OP}|^2 = |\boldsymbol{d}|^2 t^2 + 2(\boldsymbol{a} \cdot \boldsymbol{d})t + |\boldsymbol{a}|^2$.

b Define the quadratic function $f(t) = |\boldsymbol{d}|^2 t^2 + 2(\boldsymbol{a} \cdot \boldsymbol{d})t + |\boldsymbol{a}|^2$.

Show that the minimum value of $f(t)$ occurs when $t = -\dfrac{\boldsymbol{a} \cdot \boldsymbol{d}}{|\boldsymbol{d}|^2}$.

c Hence show that the point closest to the origin on ℓ has position vector $\boldsymbol{r}\left(-\dfrac{\boldsymbol{a} \cdot \boldsymbol{d}}{|\boldsymbol{d}|^2}\right)$.

d Show that $\overrightarrow{OP} \cdot \boldsymbol{d} = 0$ if and only if $t = -\dfrac{\boldsymbol{a} \cdot \boldsymbol{d}}{|\boldsymbol{d}|^2}$.

Note: This question shows that the point closest to the origin on ℓ is the unique point P on ℓ such that OP is perpendicular to ℓ.

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8 Let A, B and C be non-collinear points, with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

a Let X be a point on the line AB. Show that X has position vector $\lambda\boldsymbol{a} + \mu\boldsymbol{b}$, for some real numbers λ and μ such that $\lambda + \mu = 1$.

b Let Y be a point on the line CX. Show that Y has position vector $\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$, for some real numbers α, β and γ such that $\alpha + \beta + \gamma = 1$.

c Let P be a point in the plane through A, B and C. Show that P has position vector $\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$, for some real numbers α, β and γ such that $\alpha + \beta + \gamma = 1$.

9 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors that are not parallel. In this question, we show that every vector $\boldsymbol{v}$ has a unique representation as a linear combination of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$.

a Show that if a vector $\boldsymbol{v}$ satisfies $\boldsymbol{v} \cdot \boldsymbol{a} = \boldsymbol{v} \cdot \boldsymbol{b} = \boldsymbol{v} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0$, then $\boldsymbol{v} = \boldsymbol{0}$.

b Show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

c Show that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ are linearly independent. (Hint: Assume that $r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$ for some $r, s, t \in \mathbb{R}$. By taking scalar products of this equation with $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ respectively, show that $r = s = t = 0$. You may need part **b**.)

d Show that for any three-dimensional vector $\boldsymbol{v}$, there are real numbers r, s and t such that $\boldsymbol{v} = r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b})$, as follows:

i Suppose that $\boldsymbol{v} = r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b})$. By taking scalar products of this equation with $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ respectively and solving simultaneous equations, show that

$$r = \frac{(\boldsymbol{v} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{v} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2}, \quad s = \frac{(\boldsymbol{v} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{a}) - (\boldsymbol{v} \cdot \boldsymbol{a})(\boldsymbol{a} \cdot \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2}, \quad t = \frac{\boldsymbol{v} \cdot (\boldsymbol{a} \times \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|^2}$$

ii For these values of r, s and t, show that the vector $r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b}) - \boldsymbol{v}$ has a scalar product of zero with $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$.

iii Conclude that these values of r, s and t satisfy $\boldsymbol{v} = r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b})$.

e Using parts **c** and **d**, show that for any three-dimensional vector $\boldsymbol{v}$, there are unique real numbers r, s and t such that $\boldsymbol{v} = r\boldsymbol{a} + s\boldsymbol{b} + t(\boldsymbol{a} \times \boldsymbol{b})$.

10 Consider two skew lines with vector equations

$$\ell_1 : \boldsymbol{r}_1(\lambda) = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1 \quad \text{and} \quad \ell_2 : \boldsymbol{r}_2(\mu) = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$$

In this question, we prove the fact asserted in Section 5E that there is a unique line segment joining ℓ_1 and ℓ_2 that is perpendicular to both lines.

a Explain why vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are non-zero and not parallel.

b Show that the line segment between the points with position vectors $\boldsymbol{r}_1(\lambda)$ and $\boldsymbol{r}_2(\mu)$ is perpendicular to both lines if and only if there exists $k \in \mathbb{R}$ such that

$$\boldsymbol{r}_2(\mu) - \boldsymbol{r}_1(\lambda) = k(\boldsymbol{d}_1 \times \boldsymbol{d}_2)$$

c Show that the equation from part **b** is equivalent to

$$\lambda\boldsymbol{d}_1 - \mu\boldsymbol{d}_2 + k(\boldsymbol{d}_1 \times \boldsymbol{d}_2) = \boldsymbol{a}_2 - \boldsymbol{a}_1$$

d Using Question 9, show that there are unique real numbers λ, μ and k which satisfy the equation in part **c**.

e Conclude that there are unique points P on ℓ_1 and Q on ℓ_2 such that the line segment PQ is perpendicular to both ℓ_1 and ℓ_2.

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11 Let ℓ_1 and ℓ_2 be two skew lines. Let P be the point on ℓ_1 and Q be the point on ℓ_2 such that PQ is perpendicular to both ℓ_1 and ℓ_2. In this question, we show that PQ is the shortest distance between any two points on ℓ_1 and ℓ_2.

a Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be the position vectors of points P and Q respectively. Show that lines ℓ_1 and ℓ_2 have vector equations given by

$$\ell_1 : \boldsymbol{r}_1(\lambda) = \boldsymbol{p} + \lambda \boldsymbol{d}_1 \quad \text{and} \quad \ell_2 : \boldsymbol{r}_2(\mu) = \boldsymbol{q} + \mu \boldsymbol{d}_2$$

where the vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ satisfy

$$(\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{d}_1 = 0 \quad \text{and} \quad (\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{d}_2 = 0$$

b Let A be a point on ℓ_1, with position vector $\boldsymbol{r}_1(\lambda)$, and let B be a point on ℓ_2, with position vector $\boldsymbol{r}_2(\mu)$. Show that

$$|\overrightarrow{AB}|^2 = |\boldsymbol{q} - \boldsymbol{p}|^2 + |\mu \boldsymbol{d}_2 - \lambda \boldsymbol{d}_1|^2$$

c Show that $|\overrightarrow{AB}|^2 \geq |\overrightarrow{PQ}|^2$, with equality if and only if $A = P$ and $B = Q$.

12 A regular tetrahedron has vertices

$$A = (1, 0, 0), \quad B = (0, 1, 0), \quad C = (0, 0, 1), \quad D = (1, 1, 1)$$

a Show that every edge of the tetrahedron has the same length. What is it?

b Show that every face of the tetrahedron is an equilateral triangle with the same area. What is it?

c Find the equations of the planes ABC and BCD. What is the angle between them?

d Show that the angle between any two faces of the tetrahedron is $\cos^{-1}\left(\frac{1}{3}\right)$.

13 A regular octahedron has vertices

$$A = (1, 0, 0), \quad B = (0, 1, 0), \quad C = (0, 0, 1),$$
$$A' = (-1, 0, 0), \quad B' = (0, -1, 0), \quad C' = (0, 0, -1)$$

a Show that every edge of the octahedron has the same length. What is it?

b Show that every face of the octahedron is an equilateral triangle with the same area. What is it?

c Show that opposite faces of the octahedron lie in parallel planes.

d Find the equations of the planes ABC and $A'BC$. What is the angle between them?

e Show that the angle between any two adjacent faces of the octahedron is $\cos^{-1}\left(-\frac{1}{3}\right)$.

Note: It follows from Questions 12 and 13 that the angle between faces in a regular tetrahedron and the angle between faces in a regular octahedron sum to 180°. In fact, the regular tetrahedron and octahedron in these questions fit together and extend in a pattern to form a tessellation of three-dimensional space!

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6

Complex numbers

Objectives

- To understand the **imaginary number** i and the set of **complex numbers** $\mathbb{C}$.
- To find the **real part** and the **imaginary part** of a complex number.
- To perform **addition**, **subtraction**, **multiplication** and **division** of complex numbers.
- To find the **conjugate** of a complex number.
- To represent complex numbers graphically on an **Argand diagram**.
- To work with complex numbers in **polar form**, and to understand the geometric interpretation of multiplication and division of complex numbers in this form.
- To understand and apply **De Moivre's theorem**.
- To **factorise** polynomial expressions over $\mathbb{C}$ and to **solve** polynomial equations over $\mathbb{C}$.
- To sketch subsets of the **complex plane**, including lines, rays and circles.

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as 'impossible', they arose in calculations to find real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano's formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.

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6A Starting to build the complex numbers

Mathematicians in the eighteenth century introduced the imaginary number i with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely i and $-i$.

By declaring that $i = \sqrt{-1}$, we can find square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)} \\ &= \sqrt{4} \times \sqrt{-1} \\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers a and b, but does not hold when both a and b are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$$

The letter z is often used to denote a complex number.

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b$$

where $\text{Re}(z)$ is called the **real part** of z and $\text{Im}(z)$ is called the **imaginary part** of z.

Note: Both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers. That is, $\text{Re}: \mathbb{C} \to \mathbb{R}$ and $\text{Im}: \mathbb{C} \to \mathbb{R}$.

Example 1

Let $z = 4 - 5i$. Find:

a $\text{Re}(z)$ **b** $\text{Im}(z)$ **c** $\text{Re}(z) - \text{Im}(z)$

Solution

a $\text{Re}(z) = 4$ **b** $\text{Im}(z) = -5$ **c** $\text{Re}(z) - \text{Im}(z) = 4 - (-5) = 9$

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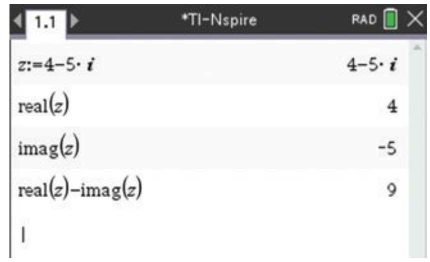
Using the TI-Nspire

- Assign the complex number z, as shown in the first line. Use (π▸) to access i.
- To find the real part, use (menu) > **Number** > **Complex Number Tools** > **Real Part**, or just type real(.
- For the imaginary part, use (menu) > **Number** > **Complex Number Tools** > **Imaginary Part**.



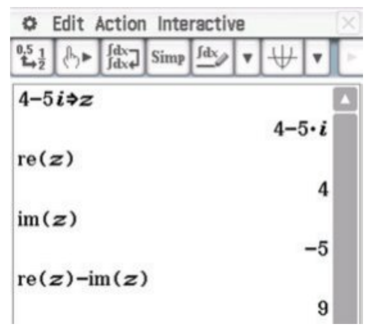
Note: You do not need to be in complex mode. If you use i in the input, then it will display in the same format.

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, tap **Real** in the status bar at the bottom of the screen to change to **Cplx** mode.
- Enter $4 - 5i \Rightarrow z$ and tap (EXE).

Note: The symbol i is found in the (Math2) keyboard.

- Enter z and highlight.
- Go to **Interactive** > **Complex** > **re**.
- Enter z and highlight.
- Go to **Interactive** > **Complex** > **im**.
- Highlight and drag the previous two entries to the next entry line and subtract as shown.



Example 2

a Represent $\sqrt{-5}$ as an imaginary number. **b** Simplify $2\sqrt{-9} + 4i$.

Solution

a
$$\begin{aligned}\sqrt{-5} &= \sqrt{5 \times (-1)} \\ &= \sqrt{5} \times \sqrt{-1} \\ &= i\sqrt{5}\end{aligned}$$

b
$$\begin{aligned}2\sqrt{-9} + 4i &= 2\sqrt{9 \times (-1)} + 4i \\ &= 2 \times 3 \times i + 4i \\ &= 6i + 4i \\ &= 10i\end{aligned}$$

Using the TI-Nspire

Enter the expression and press (enter).

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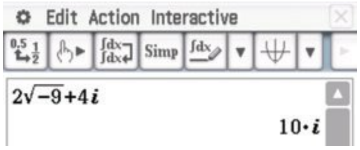
Using the Casio ClassPad

- Ensure your calculator is in complex mode (with **Cplx** in the status bar at the bottom of the main screen).
- Enter the expression and tap EXE.



Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Example 3

Solve the equation $(2a - 3) + 2bi = 5 + 6i$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Solution

If $(2a - 3) + 2bi = 5 + 6i$, then

$$2a - 3 = 5 \quad \text{and} \quad 2b = 6$$

$$\therefore \quad a = 4 \quad \text{and} \quad b = 3$$

Operations on complex numbers

Addition and subtraction

Addition of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.

If $z = a + bi$, then we define $-z = -a - bi$.

Subtraction of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$$

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z + 0 = z$
- $z + (-z) = 0$

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Multiplication by a scalar

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = k(a + bi) = ka + kbi$$

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

It is easy to check that $k(z_1 + z_2) = kz_1 + kz_2$, for all $k \in \mathbb{R}$.

Example 4

Let $z_1 = 2 - 3i$ and $z_2 = 1 + 4i$. Simplify:

a $z_1 + z_2$ **b** $z_1 - z_2$ **c** $3z_1 - 2z_2$

Solution

a $z_1 + z_2$

$= (2 - 3i) + (1 + 4i)$

$= 3 + i$

b $z_1 - z_2$

$= (2 - 3i) - (1 + 4i)$

$= 1 - 7i$

c $3z_1 - 2z_2$

$= 3(2 - 3i) - 2(1 + 4i)$

$= (6 - 9i) - (2 + 8i)$

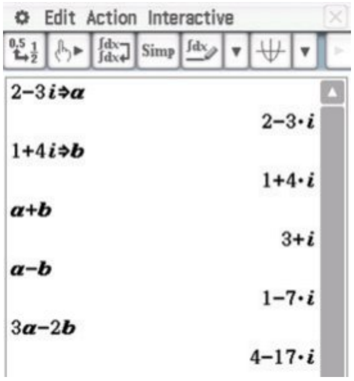
$= 4 - 17i$

Using the TI-Nspire

Enter the expressions as shown.

Using the Casio ClassPad

- Ensure your calculator is in complex mode (with **Cplx** in the status bar at the bottom of the main screen).
- Enter the expressions as shown.



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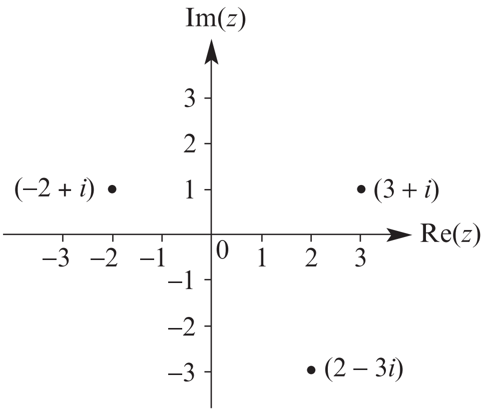
Argand diagrams

An **Argand diagram** is a geometric representation of the set of complex numbers. In a vector sense, a complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\text{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\text{Im}(z)$, for $z \in \mathbb{C}$.

Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent Cartesian axes, as shown by the examples in this figure.



A complex number written as $a + bi$ is said to be in **Cartesian form**.

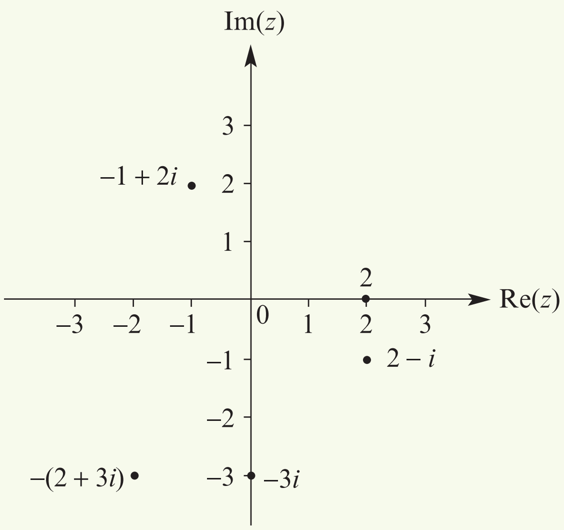
Example 5

Represent the following complex numbers as points on an Argand diagram:

a 2 **b** $-3i$ **c** $2 - i$

d $-(2 + 3i)$ **e** $-1 + 2i$

Solution



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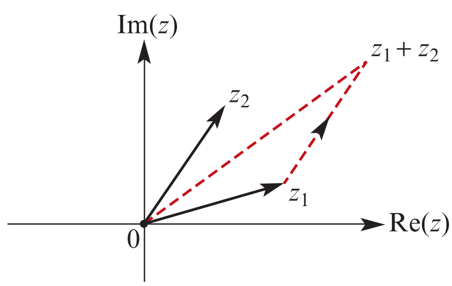
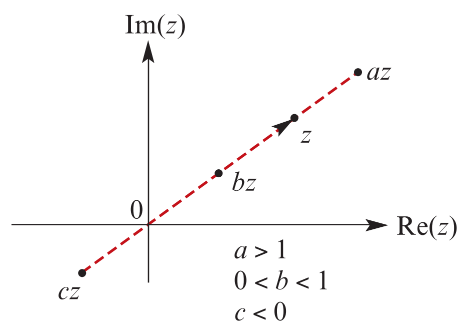
Geometric representation of the basic operations on complex numbers

Addition of complex numbers is analogous to addition of vectors. The sum of two complex numbers corresponds to the sum of their position vectors.

Multiplication of a complex number by a scalar corresponds to the multiplication of its position vector by the scalar.





The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

Example 6

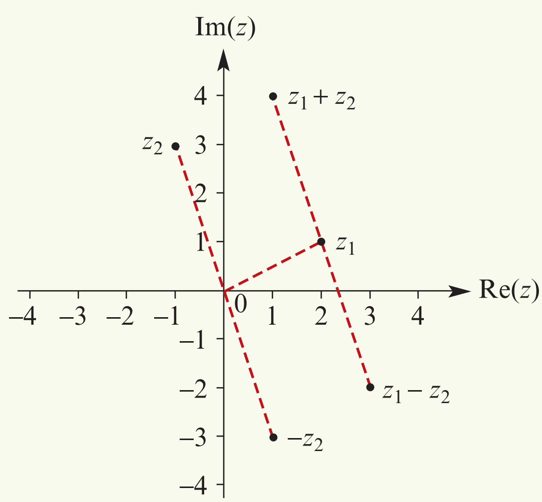
Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

Represent the complex numbers z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

Solution

$$z_1 + z_2 = (2 + i) + (-1 + 3i) = 1 + 4i$$

$$z_1 - z_2 = (2 + i) - (-1 + 3i) = 3 - 2i$$



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Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \qquad (\text{since } i^2 = -1) \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that this calculation is only being used as motivation for the following *definition* of multiplication for $\mathbb{C}$.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$. The following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z \times 1 = z$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 7

Simplify:

a $(2 + 3i)(1 - 5i)$ **b** $3i(5 - 2i)$ **c** i^3

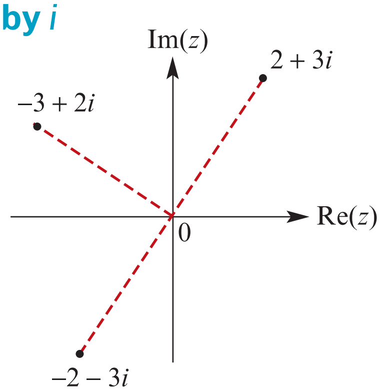
Solution

a $$\begin{aligned} (2 + 3i)(1 - 5i) &= 2 - 10i + 3i - 15i^2 \\ &= 2 - 10i + 3i + 15 \\ &= 17 - 7i \end{aligned}$$

b $$\begin{aligned} 3i(5 - 2i) &= 15i - 6i^2 \\ &= 15i + 6 \\ &= 6 + 15i \end{aligned}$$

c $$\begin{aligned} i^3 &= i \times i^2 \\ &= -i \end{aligned}$$

Geometric significance of multiplication by *i*



When the complex number $2 + 3i$ is multiplied by -1, the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.

When the complex number $2 + 3i$ is multiplied by i, we obtain

$$i(2 + 3i) = 2i + 3i^2 = -3 + 2i$$

The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by i, the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.

In general, multiplication by i gives a rotation of 90° anticlockwise about the origin.

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Powers of i

Successive multiplication by i gives the following:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \ldots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Exercise 6A

Example 1

1 Let $z = 6 - 7i$. Find:

a $\mathrm{Re}(z)$ **b** $\mathrm{Im}(z)$ **c** $\mathrm{Re}(z) - \mathrm{Im}(z)$

Example 2

2 Simplify each of the following:

a $\sqrt{-25}$ **b** $\sqrt{-27}$ **c** $2i - 7i$
d $5\sqrt{-16} - 7i$ **e** $\sqrt{-8} + \sqrt{-18}$ **f** $i\sqrt{-12}$
g $i(2 + i)$ **h** $\mathrm{Im}(2\sqrt{-4})$ **i** $\mathrm{Re}(5\sqrt{-49})$

Example 3

3 Solve the following equations for real values x and y:

a $x + yi = 5$ **b** $x + yi = 2i$
c $x = yi$ **d** $x + yi = (2 + 3i) + 7(1 - i)$
e $2x + 3 + 8i = -1 + (2 - 3y)i$ **f** $x + yi = (2y + 1) + (x - 7)i$

Example 4

4 Let $z_1 = 2 - i$, $z_2 = 3 + 2i$ and $z_3 = -1 + 3i$. Find:

a $z_1 + z_2$ **b** $z_1 + z_2 + z_3$ **c** $2z_1 - z_3$
d $3 - z_3$ **e** $4i - z_2 + z_1$ **f** $\mathrm{Re}(z_1)$
g $\mathrm{Im}(z_2)$ **h** $\mathrm{Im}(z_3 - z_2)$ **i** $\mathrm{Re}(z_2) - i\,\mathrm{Im}(z_2)$

Example 5

5 Represent each of the following complex numbers on an Argand diagram:

a $-4i$ **b** -3 **c** $2(1 + i)$
d $3 - i$ **e** $-(3 + 2i)$ **f** $-2 + 3i$

Example 6

6 Let $z_1 = 1 + 2i$ and $z_2 = 2 - i$.

a Represent the following complex numbers on an Argand diagram:

i z_1 **ii** z_2 **iii** $2z_1 + z_2$ **iv** $z_1 - z_2$

b Verify that parts **iii** and **iv** correspond to vector addition and subtraction.

Example 7

7 Simplify each of the following:

a $(5 - i)(2 + i)$ **b** $(4 + 7i)(3 + 5i)$ **c** $(2 + 3i)(2 - 3i)$
d $(1 + 3i)^2$ **e** $(2 - i)^2$ **f** $(1 + i)^3$
g i^4 **h** $i^{11}(6 + 5i)$ **i** i^{70}

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8 Solve each of the following equations for real values x and y:

a $2x + (y + 4)i = (3 + 2i)(2 - i)$
b $(x + yi)(3 + 2i) = -16 + 11i$
c $(x + 2i)^2 = 5 - 12i$
d $(x + yi)^2 = -18i$
e $i(2x - 3yi) = 6(1 + i)$

9 **a** Represent each of the following complex numbers on an Argand diagram:

i $1 + i$ **ii** $(1 + i)^2$ **iii** $(1 + i)^3$ **iv** $(1 + i)^4$

b Describe any geometric pattern observed in the position of these complex numbers.

10 Let $z_1 = 2 + 3i$ and $z_2 = -1 + 2i$. Let P, Q and R be the points defined on an Argand diagram by z_1, z_2 and $z_2 - z_1$ respectively.

a Show that $\overrightarrow{PQ} = \overrightarrow{OR}$.
b Hence find the distance PQ.

11 Evaluate $1 + i + i^2 + \cdots + i^{100}$.

6B Modulus, conjugate and division

The modulus of a complex number

Definition of the modulus

For $z = a + bi$, the **modulus** of z is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2}$$

This is the distance of the complex number from the origin.

For example, if $z_1 = 3 + 4i$ and $z_2 = -3 + 4i$, then

$$|z_1| = \sqrt{3^2 + 4^2} = 5 \quad \text{and} \quad |z_2| = \sqrt{(-3)^2 + 4^2} = 5$$

Both z_1 and z_2 are a distance of 5 units from the origin.

Properties of the modulus

- $|z_1 z_2| = |z_1|\,|z_2|$ (the modulus of a product is the product of the moduli)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)
- $|z_1 + z_2| \le |z_1| + |z_2|$ (triangle inequality)

These results will be proved in Exercise 6B.

The conjugate of a complex number

Definition of the complex conjugate

For $z = a + bi$, the **conjugate** of z is denoted by $\bar{z}$ and is defined by

$$\bar{z} = a - bi$$

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Properties of the complex conjugate

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$
- $\overline{kz} = k\overline{z}$, for $k \in \mathbb{R}$
- $z\overline{z} = |z|^2$
- $z + \overline{z} = 2\,\text{Re}(z)$

Proof The first three results will be proved in Exercise 6B. To prove the remaining two results, consider a complex number $z = a + bi$. Then $\overline{z} = a - bi$ and therefore

$$\begin{aligned} z\overline{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned} \qquad \begin{aligned} z + \overline{z} &= (a+bi) + (a-bi) \\ &= 2a \\ &= 2\,\text{Re}(z) \end{aligned}$$

It follows from these two results that if $z \in \mathbb{C}$, then $z\overline{z}$ and $z + \overline{z}$ are real numbers. We can prove a partial converse to this property of the complex conjugate:

Let $z, w \in \mathbb{C} \setminus \mathbb{R}$ such that zw and $z + w$ are real numbers. Then $w = \overline{z}$.

Proof Write $z = a + bi$ and $w = c + di$, where $b, d \neq 0$. Then

$$\begin{aligned} z + w &= (a+bi) + (c+di) \\ &= (a+c) + (b+d)i \end{aligned}$$

Since $z + w$ is real, we have $b + d = 0$. Therefore $d = -b$ and so

$$\begin{aligned} zw &= (a+bi)(c-bi) \\ &= (ac+b^2) + (bc-ab)i \end{aligned}$$

Since zw is real, we have $bc - ab = b(c - a) = 0$. As $b \neq 0$, this implies that $c = a$. We have shown that $w = a - bi = \overline{z}$.

Example 8

Find the complex conjugate of each of the following:

a 2 **b** $3i$ **c** $-1 - 5i$

Solution

a The complex conjugate of 2 is 2.

b The complex conjugate of $3i$ is $-3i$.

c The complex conjugate of $-1 - 5i$ is $-1 + 5i$.

Using the TI-Nspire

To find the complex conjugate, use menu > **Number** > **Complex Number Tools** > **Complex Conjugate**, or just type conj(.

Note: Use π▸ to access i.

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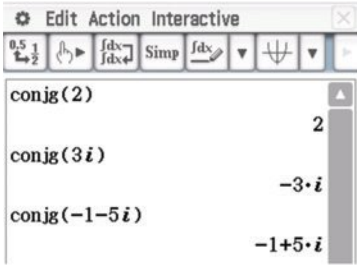
Using the Casio ClassPad

- Ensure your calculator is in complex mode.
- Enter and highlight 2.
- Go to **Interactive** > **Complex** > **conjg**.
- Repeat for $3i$ and $-1 - 5i$ as shown.



Division of complex numbers

We begin with some familiar algebra that will motivate the definition:

$$\frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

We can see that

$$(a+bi) \times \frac{a-bi}{a^2+b^2} = 1$$

Although we have carried out this arithmetic, we have not yet defined what $\dfrac{1}{a+bi}$ means.

Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$, then

$$z^{-1} = \frac{a-bi}{a^2+b^2} = \frac{\overline{z}}{|z|^2}$$

The formal definition of division in the complex numbers is via the multiplicative inverse:

Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1\overline{z_2}}{|z_2|^2} \qquad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}$$

Multiply the numerator and denominator by the conjugate of z_2:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a+bi}{c+di} \times \frac{c-di}{c-di}\\ &= \frac{(a+bi)(c-di)}{c^2+d^2}\end{aligned}$$

Complete the division by simplifying. This process is demonstrated in the next example.

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Example 9

a Write each of the following in the form $a + bi$, where $a, b \in \mathbb{R}$:

i $\dfrac{1}{3 - 2i}$ **ii** $\dfrac{4 + i}{3 - 2i}$

b Simplify $\dfrac{(1 + 2i)^2}{i(1 + 3i)}$.

Solution

a i
$$\frac{1}{3-2i} = \frac{1}{3-2i} \times \frac{3+2i}{3+2i} = \frac{3+2i}{3^2 - (2i)^2} = \frac{3+2i}{13} = \frac{3}{13} + \frac{2}{13}i$$

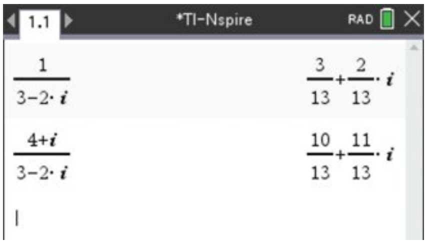
ii
$$\frac{4+i}{3-2i} = \frac{4+i}{3-2i} \times \frac{3+2i}{3+2i} = \frac{(4+i)(3+2i)}{3^2 + 2^2} = \frac{12 + 8i + 3i - 2}{13} = \frac{10}{13} + \frac{11}{13}i$$

b
$$\frac{(1+2i)^2}{i(1+3i)} = \frac{1 + 4i - 4}{-3 + i} = \frac{-3+4i}{-3+i} \times \frac{-3-i}{-3-i} = \frac{9 + 3i - 12i + 4}{(-3)^2 - i^2} = \frac{13 - 9i}{10} = \frac{13}{10} - \frac{9}{10}i$$

Note: There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

Using the TI-Nspire

Complete as shown.



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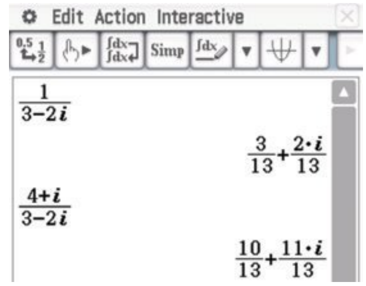
Using the Casio ClassPad

Ensure your calculator is in complex mode and complete as shown.



Exercise 6B

Example 8

1 Find the complex conjugate of each of the following complex numbers:

a $\sqrt{3}$ **b** $8i$ **c** $4 - 3i$ **d** $-(1 + 2i)$ **e** $4 + 2i$ **f** $-3 - 2i$

Example 9

2 Simplify each of the following, giving your answer in the form $a + bi$:

a $\dfrac{2 + 3i}{3 - 2i}$ **b** $\dfrac{i}{-1 + 3i}$ **c** $\dfrac{-4 - 3i}{i}$ **d** $\dfrac{3 + 7i}{1 + 2i}$ **e** $\dfrac{\sqrt{3} + i}{-1 - i}$ **f** $\dfrac{17}{4 - i}$

3 Let $z = a + bi$ and $w = c + di$. Show that:

a $\overline{z + w} = \overline{z} + \overline{w}$ **b** $\overline{zw} = \overline{z}\,\overline{w}$ **c** $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

d $|zw| = |z|\,|w|$ **e** $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$

4 Let $z = 2 - i$. Simplify the following:

a $z(z + 1)$ **b** $\overline{z + 4}$ **c** $\overline{z - 2i}$

d $\dfrac{z - 1}{z + 1}$ **e** $(z - i)^2$ **f** $(z + 1 + 2i)^2$

5 For $z = a + bi$, write each of the following in terms of a and b:

a $z\overline{z}$ **b** $\dfrac{z}{|z|^2}$ **c** $z + \overline{z}$ **d** $z - \overline{z}$ **e** $\dfrac{z}{\overline{z}}$ **f** $\dfrac{\overline{z}}{z}$

6 Prove that $|z_1 + z_2| \le |z_1| + |z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

7 Let $z \in \mathbb{C}$. Prove by mathematical induction that $(\overline{z})^n = \overline{z^n}$ for all $n \in \mathbb{N}$.

8 In Question 3, you proved that $|zw| = |z|\,|w|$, where $z = a + bi$ and $w = c + di$.

a Hence, deduce that $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$.

b The result from part **a** is called the *Brahmagupta–Fibonacci identity*; it says that the product of two sums of two squares can be written as a sum of two squares. Use this result to write 65 as the sum of two squares.

c Let n be an integer. Prove that $n^4 + 5n^2 + 4$ is the sum of two squares.

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6C Polar form of a complex number

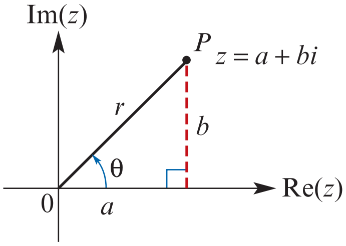
In the preceding sections, we have expressed complex numbers in Cartesian form. Another way of expressing complex numbers is using polar form.

Each complex number may be described by an angle and a distance from the origin. In this section, we will see that this is a very useful way to describe complex numbers.

Definition of polar form

The diagram shows the point P corresponding to the complex number $z = a + bi$. We see that $a = r\cos\theta$ and $b = r\sin\theta$, and so we can write

$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + (r\sin\theta)\,i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$



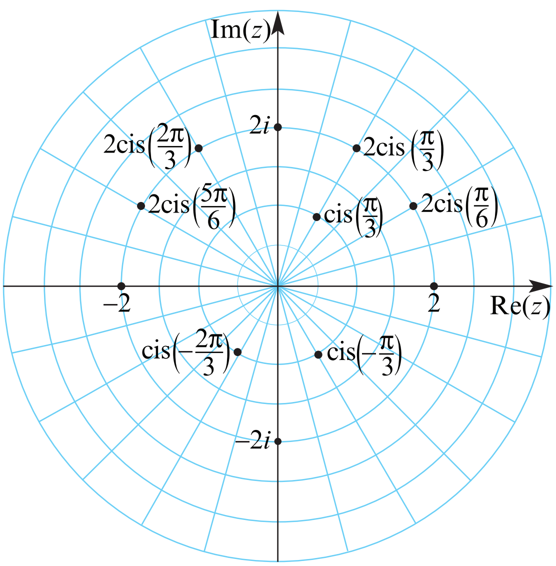
This is called the **polar form** of the complex number. The polar form is abbreviated to

$$z = r\operatorname{cis}\theta$$

- The distance $r = \sqrt{a^2 + b^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z and is denoted by $\arg z$.

Polar form for complex numbers is also called **modulus–argument form**.

This Argand diagram uses a polar grid with rays at intervals of $\frac{\pi}{12} = 15°$.



Non-uniqueness of polar form

Each complex number has more than one representation in polar form.

Since $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$, for all $n \in \mathbb{Z}$, we can write

$$z = r\operatorname{cis}\theta = r\operatorname{cis}(\theta + 2n\pi) \quad \text{for all } n \in \mathbb{Z}$$

The convention is to use the angle θ such that $-\pi < \theta \le \pi$.

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Principal value of the argument

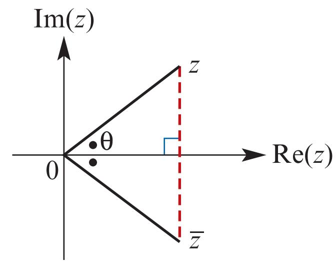
For a non-zero complex number z, the argument of z that belongs to the interval $(-\pi, \pi]$ is called the **principal value** of the argument of z and is denoted by Arg z. That is,

$$-\pi < \operatorname{Arg} z \le \pi$$

Complex conjugate in polar form

It is easy to show that the complex conjugate, $\bar{z}$, is a reflection of the point z in the horizontal axis.

Therefore, if $z = r \operatorname{cis} \theta$, then $\bar{z} = r \operatorname{cis}(-\theta)$.



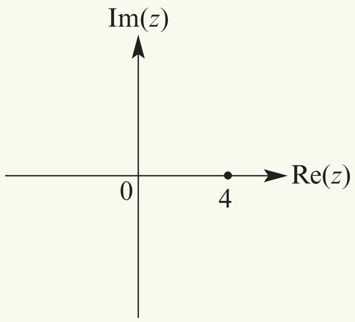
Example 10

Find the modulus and principal argument of each of the following complex numbers:

a 4 **b** $-2i$ **c** $1 + i$ **d** $4 - 3i$

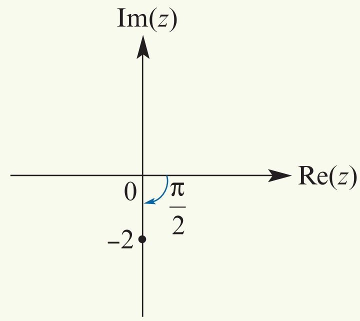
Solution

a



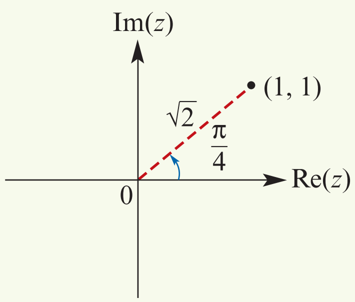
$|4| = 4, \quad \operatorname{Arg}(4) = 0$

b



$|-2i| = 2, \quad \operatorname{Arg}(-2i) = -\dfrac{\pi}{2}$

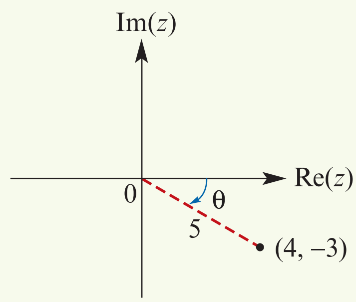
c



$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\operatorname{Arg}(1 + i) = \dfrac{\pi}{4}$

d



$|4 - 3i| = \sqrt{4^2 + (-3)^2} = 5$

$\operatorname{Arg}(4 - 3i) = -\tan^{-1}\left(\dfrac{3}{4}\right)$

≈ -0.64 radians

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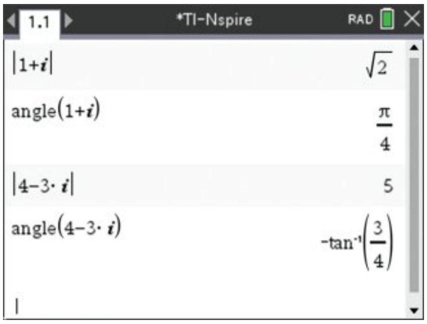
Using the TI-Nspire

- To find the modulus of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Magnitude**.
  Alternatively, use |□| from the 2D-template palette or type abs(.
- To find the principal value of the argument, use (menu) > **Number** > **Complex Number Tools** > **Polar Angle**.



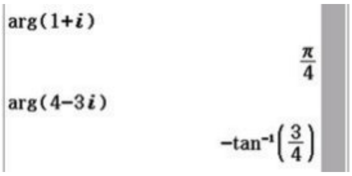
Note: Use (π▸) to access i.

Using the Casio ClassPad

Ensure your calculator is in complex mode (with **Cplx** in the status bar at the bottom of the main screen).

- To find the modulus of a complex number, tap on the modulus template in the (Math2) keyboard, then enter the expression.

- To find the principal argument of a complex number, enter and highlight the expression, then select **Interactive** > **Complex** > **arg**.



Example 11

a Find the argument of $-1 - i$ in the interval $[0, 2\pi]$.

b Find the principal argument of $-1 - i$.

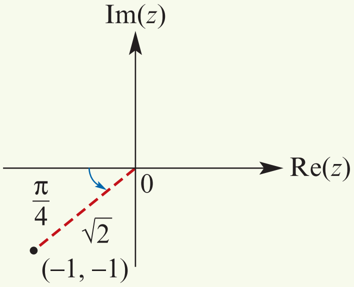
Solution

a Choosing the angle in the interval $[0, 2\pi]$ gives

$$\arg(-1 - i) = \frac{5\pi}{4}$$

b Choosing the angle in the interval $(-\pi, \pi]$ gives

$$\operatorname{Arg}(-1 - i) = -\frac{3\pi}{4}$$



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Example 12

Express $-\sqrt{3} + i$ in the form $r \operatorname{cis} \theta$, where $\theta = \operatorname{Arg}(-\sqrt{3} + i)$.

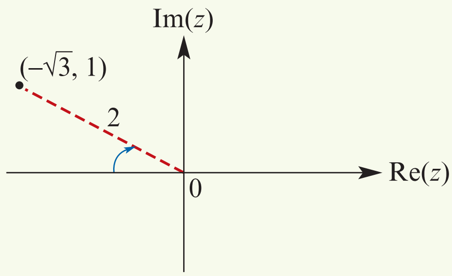
Solution

$r = |-\sqrt{3} + i|$

$= \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\theta = \operatorname{Arg}(-\sqrt{3} + i) = \dfrac{5\pi}{6}$

Therefore $-\sqrt{3} + i = 2 \operatorname{cis}\left(\dfrac{5\pi}{6}\right)$



Example 13

Express $2 \operatorname{cis}\left(-\dfrac{3\pi}{4}\right)$ in the form $a + bi$.

Solution

$a = r \cos \theta$

$= 2 \cos\left(-\dfrac{3\pi}{4}\right)$

$= -2 \cos\left(\dfrac{\pi}{4}\right)$

$= -2 \times \dfrac{1}{\sqrt{2}}$

$= -\sqrt{2}$

$b = r \sin \theta$

$= 2 \sin\left(-\dfrac{3\pi}{4}\right)$

$= -2 \sin\left(\dfrac{\pi}{4}\right)$

$= -2 \times \dfrac{1}{\sqrt{2}}$

$= -\sqrt{2}$

Therefore $2 \operatorname{cis}\left(-\dfrac{3\pi}{4}\right) = -\sqrt{2} - \sqrt{2}i$

Exercise 6C

Example 10

1 Find the modulus and principal argument of each of the following complex numbers:

a -3 **b** $5i$ **c** $i - 1$ **d** $\sqrt{3} + i$ **e** $2 - 2\sqrt{3}i$ **f** $(2 - 2\sqrt{3}i)^2$

2 Find the principal argument of each of the following, correct to two decimal places:

a $5 + 12i$ **b** $-8 + 15i$ **c** $-4 - 3i$ **d** $1 - \sqrt{2}i$ **e** $\sqrt{2} + \sqrt{3}i$ **f** $-(3 + 7i)$

Example 11

3 Find the argument of each of the following in the interval stated:

a $1 - \sqrt{3}i$ in $[0, 2\pi]$ **b** $-7i$ in $[0, 2\pi]$ **c** $-3 + \sqrt{3}i$ in $[0, 2\pi]$

d $\sqrt{2} + \sqrt{2}i$ in $[0, 2\pi]$ **e** $\sqrt{3} + i$ in $[-2\pi, 0]$ **f** $2i$ in $[-2\pi, 0]$

4 Convert each of the following arguments into principal arguments:

a $\dfrac{5\pi}{4}$ **b** $\dfrac{17\pi}{6}$ **c** $-\dfrac{15\pi}{8}$ **d** $-\dfrac{5\pi}{2}$

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Example 12

5 Convert each of the following complex numbers from Cartesian form $a + bi$ into the form $r \operatorname{cis} \theta$, where $\theta = \operatorname{Arg}(a + bi)$:

a $-1 - i$ b $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ c $\sqrt{3} - \sqrt{3}i$

d $\frac{1}{\sqrt{3}} + \frac{1}{3}i$ e $\sqrt{6} - \sqrt{2}i$ f $-2\sqrt{3} + 2i$

Example 13

6 Convert each of the following complex numbers into the form $a + bi$:

a $2 \operatorname{cis}\left(\frac{3\pi}{4}\right)$ b $5 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ c $2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

d $3 \operatorname{cis}\left(-\frac{5\pi}{6}\right)$ e $6 \operatorname{cis}\left(\frac{\pi}{2}\right)$ f $4 \operatorname{cis} \pi$

7 Let $z = \operatorname{cis} \theta$. Show that:

a $|z| = 1$ b $\frac{1}{z} = \operatorname{cis}(-\theta)$

8 Find the complex conjugate of each of the following:

a $2 \operatorname{cis}\left(\frac{3\pi}{4}\right)$ b $7 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$ c $-3 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ d $5 \operatorname{cis}\left(-\frac{\pi}{4}\right)$

6D Basic operations on complex numbers in polar form

Addition and subtraction

There is no simple way to add or subtract complex numbers in the form $r \operatorname{cis} \theta$. Complex numbers need to be expressed in the form $a + bi$ before these operations can be carried out.

Example 14

Simplify $2 \operatorname{cis}\left(\frac{\pi}{3}\right) + 3 \operatorname{cis}\left(\frac{2\pi}{3}\right)$.

Solution

First convert to Cartesian form:

$$\begin{aligned} 2 \operatorname{cis}\left(\frac{\pi}{3}\right) &= 2\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right) \\ &= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= 1 + \sqrt{3}i \end{aligned}$$

$$\begin{aligned} 3 \operatorname{cis}\left(\frac{2\pi}{3}\right) &= 3\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right) \\ &= 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= -\frac{3}{2} + \frac{3\sqrt{3}}{2}i \end{aligned}$$

Now we have

$$\begin{aligned} 2 \operatorname{cis}\left(\frac{\pi}{3}\right) + 3 \operatorname{cis}\left(\frac{2\pi}{3}\right) &= (1 + \sqrt{3}i) + \left(-\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right) \\ &= -\frac{1}{2} + \frac{5\sqrt{3}}{2}i \end{aligned}$$

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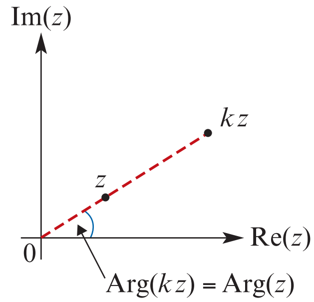
Multiplication by a scalar

Positive scalar

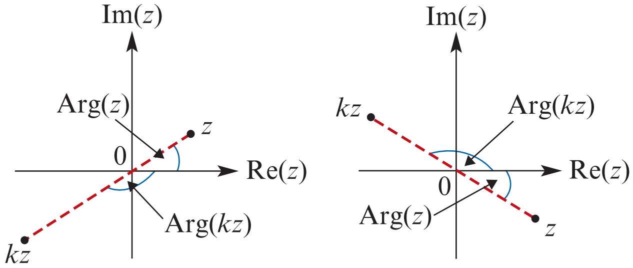
If $k \in \mathbb{R}^+$, then $\text{Arg}(kz) = \text{Arg}(z)$



Negative scalar

If $k \in \mathbb{R}^-$, then

$$\text{Arg}(kz) = \begin{cases} \text{Arg}(z) - \pi, & 0 < \text{Arg}(z) \le \pi \\ \text{Arg}(z) + \pi, & -\pi < \text{Arg}(z) \le 0 \end{cases}$$



Multiplication of complex numbers

Multiplication in polar form

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{(multiply the moduli and add the angles)}$$

Proof We have

$$\begin{aligned} z_1 z_2 &= r_1 \operatorname{cis} \theta_1 \times r_2 \operatorname{cis} \theta_2 \\ &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &= r_1 r_2 \big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\big) \end{aligned}$$

Now use the compound angle formulas from Chapter 3:

$$\begin{aligned} \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \\ \cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{aligned}$$

Hence

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \end{aligned}$$

Here are two useful properties of the modulus and the principal argument with regard to multiplication of complex numbers:

- $|z_1 z_2| = |z_1||z_2|$
- $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1

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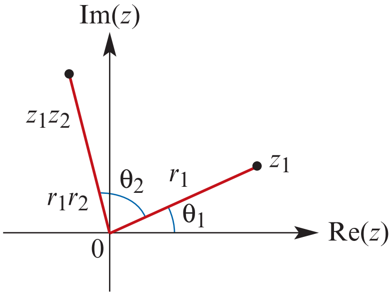
Geometric interpretation of multiplication

We have seen that:

- The modulus of the product of two complex numbers is the product of their moduli.
- The argument of the product of two complex numbers is the sum of their arguments.



Geometrically, the effect of multiplying a complex number z_1 by the complex number $z_2 = r_2 \operatorname{cis} \theta_2$ is to produce an enlargement of Oz_1, where O is the origin, by a factor of r_2 and an anticlockwise turn through an angle θ_2 about the origin.

If $r_2 = 1$, then only the turning effect will take place.

Let $z = \operatorname{cis} \theta$. Multiplication by z^2 is, in effect, the same as a multiplication by z followed by another multiplication by z. The effect is a turn of θ followed by another turn of θ. The end result is an anticlockwise turn of 2θ. This is also shown by finding z^2:

$$\begin{aligned} z^2 = z \times z &= \operatorname{cis} \theta \times \operatorname{cis} \theta \\ &= \operatorname{cis}(\theta + \theta) \qquad \text{using the multiplication rule} \\ &= \operatorname{cis}(2\theta) \end{aligned}$$

Division of complex numbers

Division in polar form

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$ with $r_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \qquad \text{(divide the moduli and subtract the angles)}$$

Proof We have already seen in Exercise 6C that $\dfrac{1}{\operatorname{cis} \theta_2} = \operatorname{cis}(-\theta_2)$.

We can now use the rule for multiplication in polar form to obtain

$$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis} \theta_1 \operatorname{cis}(-\theta_2) = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Here are three useful properties of the modulus and the principal argument with regard to division of complex numbers:

- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
- $\operatorname{Arg}\left(\dfrac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1
- $\operatorname{Arg}\left(\dfrac{1}{z}\right) = -\operatorname{Arg}(z)$, provided z is not a negative real number

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Example 15

Simplify:

a $2\operatorname{cis}\left(\frac{\pi}{3}\right) \times \sqrt{3}\operatorname{cis}\left(\frac{3\pi}{4}\right)$ **b** $\dfrac{2\operatorname{cis}\left(\frac{2\pi}{3}\right)}{4\operatorname{cis}\left(\frac{\pi}{5}\right)}$

Solution

a
$$\begin{aligned} 2\operatorname{cis}\left(\frac{\pi}{3}\right) \times \sqrt{3}\operatorname{cis}\left(\frac{3\pi}{4}\right) &= 2\sqrt{3}\operatorname{cis}\left(\frac{\pi}{3} + \frac{3\pi}{4}\right) \\ &= 2\sqrt{3}\operatorname{cis}\left(\frac{13\pi}{12}\right) \\ &= 2\sqrt{3}\operatorname{cis}\left(-\frac{11\pi}{12}\right) \end{aligned}$$

b
$$\begin{aligned} \frac{2\operatorname{cis}\left(\frac{2\pi}{3}\right)}{4\operatorname{cis}\left(\frac{\pi}{5}\right)} &= \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{3} - \frac{\pi}{5}\right) \\ &= \frac{1}{2}\operatorname{cis}\left(\frac{7\pi}{15}\right) \end{aligned}$$

Note: A solution giving the principal value of the argument, that is, the argument in the interval $(-\pi, \pi]$, is preferred unless otherwise stated.

De Moivre's theorem

De Moivre's theorem allows us to readily simplify expressions of the form z^n when z is expressed in polar form.

De Moivre's theorem

$(r\operatorname{cis}\theta)^n = r^n\operatorname{cis}(n\theta)$, where $n \in \mathbb{Z}$

Proof This result is usually proved by mathematical induction, but can be explained by a simple inductive argument.

Let $z = \operatorname{cis}\theta$

Then $z^2 = \operatorname{cis}\theta \times \operatorname{cis}\theta = \operatorname{cis}(2\theta)$ by the multiplication rule

$z^3 = z^2 \times \operatorname{cis}\theta = \operatorname{cis}(3\theta)$

$z^4 = z^3 \times \operatorname{cis}\theta = \operatorname{cis}(4\theta)$

Continuing in this way, we see that $(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta)$, for each positive integer n.

To obtain the result for negative integers, again let $z = \operatorname{cis}\theta$. Then

$$z^{-1} = \frac{1}{z} = \bar{z} = \operatorname{cis}(-\theta)$$

For $k \in \mathbb{N}$, we have

$$z^{-k} = (z^{-1})^k = (\operatorname{cis}(-\theta))^k = \operatorname{cis}(-k\theta)$$

using the result for positive integers.

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Example 16

Simplify:

a $\left(\text{cis}\left(\frac{\pi}{3}\right)\right)^9$

b $\dfrac{\text{cis}\left(\frac{7\pi}{4}\right)}{\left(\text{cis}\left(\frac{\pi}{3}\right)\right)^7}$

Solution

a
$$\begin{aligned}\left(\text{cis}\left(\frac{\pi}{3}\right)\right)^9 &= \text{cis}\left(9 \times \frac{\pi}{3}\right)\\ &= \text{cis}(3\pi)\\ &= \text{cis}\,\pi\\ &= \cos\pi + i\sin\pi\\ &= -1\end{aligned}$$

b
$$\begin{aligned}\frac{\text{cis}\left(\frac{7\pi}{4}\right)}{\left(\text{cis}\left(\frac{\pi}{3}\right)\right)^7} &= \text{cis}\left(\frac{7\pi}{4}\right)\left(\text{cis}\left(\frac{\pi}{3}\right)\right)^{-7}\\ &= \text{cis}\left(\frac{7\pi}{4}\right)\text{cis}\left(-\frac{7\pi}{3}\right)\\ &= \text{cis}\left(\frac{7\pi}{4} - \frac{7\pi}{3}\right)\\ &= \text{cis}\left(-\frac{7\pi}{12}\right)\end{aligned}$$

Example 17

Using polar form, simplify $\dfrac{(1+i)^3}{(1-\sqrt{3}i)^5}$.

Solution

First convert to polar form:

$$\begin{aligned}1 + i &= \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\\ 1 - \sqrt{3}i &= 2\,\text{cis}\left(-\frac{\pi}{3}\right)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{(1+i)^3}{(1-\sqrt{3}i)^5} &= \frac{\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^3}{\left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^5}\\ &= \frac{2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)}{32\,\text{cis}\left(-\frac{5\pi}{3}\right)} \qquad \text{by De Moivre's theorem}\\ &= \frac{\sqrt{2}}{16}\,\text{cis}\left(\frac{3\pi}{4} - \left(-\frac{5\pi}{3}\right)\right)\\ &= \frac{\sqrt{2}}{16}\,\text{cis}\left(\frac{29\pi}{12}\right)\\ &= \frac{\sqrt{2}}{16}\,\text{cis}\left(\frac{5\pi}{12}\right)\end{aligned}$$

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Skill-sheet

Exercise 6D

Example 14

1 Simplify $4\operatorname{cis}\left(\frac{\pi}{6}\right)+6\operatorname{cis}\left(\frac{2\pi}{3}\right)$.

Example 15

2 Simplify each of the following:

a $4\operatorname{cis}\left(\frac{2\pi}{3}\right)\times 3\operatorname{cis}\left(\frac{3\pi}{4}\right)$

b $\dfrac{\sqrt{2}\operatorname{cis}\left(\frac{\pi}{2}\right)}{\sqrt{8}\operatorname{cis}\left(\frac{5\pi}{6}\right)}$

c $\frac{1}{2}\operatorname{cis}\left(-\frac{2\pi}{5}\right)\times\frac{7}{3}\operatorname{cis}\left(\frac{\pi}{3}\right)$

d $\dfrac{4\operatorname{cis}\left(-\frac{\pi}{4}\right)}{\frac{1}{2}\operatorname{cis}\left(\frac{7\pi}{10}\right)}$

e $\dfrac{4\operatorname{cis}\left(\frac{2\pi}{3}\right)}{32\operatorname{cis}\left(-\frac{\pi}{3}\right)}$

Example 16

3 Simplify each of the following:

a $2\operatorname{cis}\left(\frac{5\pi}{6}\right)\times\left(\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{8}\right)\right)^4$

b $\dfrac{1}{\left(\frac{3}{2}\operatorname{cis}\left(\frac{5\pi}{8}\right)\right)^3}$

c $\left(\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^8\times\left(\sqrt{3}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^6$

d $\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{2}\right)\right)^{-5}$

e $\left(2\operatorname{cis}\left(\frac{3\pi}{2}\right)\times 3\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^3$

f $\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{8}\right)\right)^{-6}\times\left(4\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^2$

g $\dfrac{\left(6\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{\left(\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^{-5}}$

4 For each of the following, find $\operatorname{Arg}(z_1z_2)$ and $\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$ and comment on their relationship:

a $z_1=\operatorname{cis}\left(\frac{\pi}{4}\right)$ and $z_2=\operatorname{cis}\left(\frac{\pi}{3}\right)$

b $z_1=\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ and $z_2=\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

c $z_1=\operatorname{cis}\left(\frac{2\pi}{3}\right)$ and $z_2=\operatorname{cis}\left(\frac{\pi}{2}\right)$

5 Show that if $-\frac{\pi}{2}<\operatorname{Arg}(z_1)<\frac{\pi}{2}$ and $-\frac{\pi}{2}<\operatorname{Arg}(z_2)<\frac{\pi}{2}$, then

$$\operatorname{Arg}(z_1z_2)=\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)\quad\text{and}\quad\operatorname{Arg}\left(\frac{z_1}{z_2}\right)=\operatorname{Arg}(z_1)-\operatorname{Arg}(z_2)$$

6 For $z=1+i$, find:

a $\operatorname{Arg} z$

b $\operatorname{Arg}(-z)$

c $\operatorname{Arg}\left(\frac{1}{z}\right)$

7 The point (2, 3) is rotated about the origin by angle $\frac{\pi}{6}$ clockwise. By multiplying two complex numbers, find the image of the point.

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8 **a** Show that

$$\sin\theta + i\cos\theta = \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$$

b Simplify each of the following:

i $(\sin\theta + i\cos\theta)^7$ **ii** $(\sin\theta + i\cos\theta)(\cos\theta + i\sin\theta)$

iii $(\sin\theta + i\cos\theta)^{-4}$ **iv** $(\sin\theta + i\cos\theta)(\sin\varphi + i\cos\varphi)$

9 **a** Show that

$$\cos\theta - i\sin\theta = \operatorname{cis}(-\theta)$$

b Simplify each of the following:

i $(\cos\theta - i\sin\theta)^5$ **ii** $(\cos\theta - i\sin\theta)^{-3}$

iii $(\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)$ **iv** $(\cos\theta - i\sin\theta)(\sin\theta + i\cos\theta)$

10 **a** Show that

$$\sin\theta - i\cos\theta = \operatorname{cis}\left(\theta - \frac{\pi}{2}\right)$$

b Simplify each of the following:

i $(\sin\theta - i\cos\theta)^6$ **ii** $(\sin\theta - i\cos\theta)^{-2}$

iii $(\sin\theta - i\cos\theta)^2(\cos\theta - i\sin\theta)$ **iv** $\dfrac{\sin\theta - i\cos\theta}{\cos\theta + i\sin\theta}$

11 **a** Express each of the following in modulus–argument form, where $0 < \theta < \dfrac{\pi}{2}$:

i $1 + i\tan\theta$ **ii** $1 + i\cot\theta$ **iii** $\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}i$

b Hence, simplify each of the following:

i $(1 + i\tan\theta)^2$ **ii** $(1 + i\cot\theta)^{-3}$ **iii** $\dfrac{1}{\sin\theta} - \dfrac{1}{\cos\theta}i$

Example 17

12 Simplify each of the following, giving your answer in polar form $r\operatorname{cis}\theta$, with $r > 0$ and $\theta \in (-\pi, \pi]$:

a $(1 + \sqrt{3}i)^6$ **b** $(1 - i)^{-5}$ **c** $i(\sqrt{3} - i)^7$

d $(-3 + \sqrt{3}i)^{-3}$ **e** $\dfrac{(1 + \sqrt{3}i)^3}{i(1 - i)^5}$ **f** $\dfrac{(-1 + \sqrt{3}i)^4(-\sqrt{2} - \sqrt{2}i)^3}{\sqrt{3} - 3i}$

g $(-1 + i)^5\left(\dfrac{1}{2}\operatorname{cis}\left(\dfrac{\pi}{4}\right)\right)^3$ **h** $\dfrac{\left(\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{(1 - \sqrt{3}i)^2}$ **i** $\left((1 - i)\operatorname{cis}\left(\dfrac{2\pi}{3}\right)\right)^7$

13 Simplify $\dfrac{(1 + i)^{20}}{(1 - i)^{21}}$, giving your answer in Cartesian form.

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14 **a** Use De Moivre's theorem to simplify $(1+i)^{11}$.

b Let z be a complex number with $z \neq 1$. Using mathematical induction (or otherwise), prove that

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

for all $n \in \mathbb{N}$.

c Hence, simplify $1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{10}$.

6E Solving quadratic equations over the complex numbers

Quadratic equations with a negative discriminant have no real solutions. The introduction of complex numbers enables us to solve such quadratic equations.

Factorisation of quadratics

Over the complex numbers, every quadratic polynomial can be written as the product of two linear factors.

We first consider the special case where the quadratic polynomial has the form $z^2 + a^2$.

Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

Example 18

Factorise:

a $z^2 + 16$

b $2z^2 + 6$

Solution

a $\begin{aligned} z^2 + 16 &= z^2 - 16i^2 \\ &= (z + 4i)(z - 4i) \end{aligned}$

b $\begin{aligned} 2z^2 + 6 &= 2(z^2 + 3) \\ &= 2(z^2 - 3i^2) \\ &= 2(z + \sqrt{3}i)(z - \sqrt{3}i) \end{aligned}$

Note: The discriminant of $z^2 + 16$ is $\Delta = 0 - 4 \times 16 = -64$.
The discriminant of $2z^2 + 6$ is $\Delta = 0 - 4 \times 2 \times 6 = -48$.

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Example 19

Factorise:

a $z^2 + z + 3$ **b** $2z^2 - z + 1$ **c** $2z^2 - 2(3 - i)z + 4 - 3i$

Solution

a Let $P(z) = z^2 + z + 3$. Then, by completing the square, we have

$$\begin{aligned} P(z) &= \left(z^2 + z + \frac{1}{4}\right) + 3 - \frac{1}{4} \\ &= \left(z + \frac{1}{2}\right)^2 + \frac{11}{4} \\ &= \left(z + \frac{1}{2}\right)^2 - \frac{11}{4}i^2 \\ &= \left(z + \frac{1}{2} + \frac{\sqrt{11}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{11}}{2}i\right) \end{aligned}$$

b Let $P(z) = 2z^2 - z + 1$. Then

$$\begin{aligned} P(z) &= 2\left(z^2 - \frac{1}{2}z + \frac{1}{2}\right) \\ &= 2\left(\left(z^2 - \frac{1}{2}z + \frac{1}{16}\right) + \frac{1}{2} - \frac{1}{16}\right) \\ &= 2\left(\left(z - \frac{1}{4}\right)^2 + \frac{7}{16}\right) \\ &= 2\left(\left(z - \frac{1}{4}\right)^2 - \frac{7}{16}i^2\right) \\ &= 2\left(z - \frac{1}{4} + \frac{\sqrt{7}}{4}i\right)\left(z - \frac{1}{4} - \frac{\sqrt{7}}{4}i\right) \end{aligned}$$

c Let $P(z) = 2z^2 - 2(3 - i)z + 4 - 3i$. Then

$$\begin{aligned} P(z) &= 2\left(z^2 - (3 - i)z + \frac{4 - 3i}{2}\right) \\ &= 2\left(z^2 - (3 - i)z + \left(\frac{3 - i}{2}\right)^2 + \frac{4 - 3i}{2} - \left(\frac{3 - i}{2}\right)^2\right) \\ &= 2\left(z - \frac{3 - i}{2}\right)^2 + 4 - 3i - \frac{(3 - i)^2}{2} \\ &= 2\left(z - \frac{3 - i}{2}\right)^2 + \frac{8 - 6i - 9 + 6i + 1}{2} \\ &= 2\left(z - \frac{3 - i}{2}\right)^2 \end{aligned}$$

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Solution of quadratic equations

In the previous example, we used the method of completing the square to factorise quadratic expressions. This method can also be used to solve quadratic equations.

Alternatively, a quadratic equation of the form $az^2 + bz + c = 0$ can be solved by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is obtained by completing the square on the expression $az^2 + bz + c$.

Example 20

Solve each of the following equations for z:

a $z^2 + z + 3 = 0$

b $2z^2 - z + 1 = 0$

c $z^2 = 2z - 5$

d $2z^2 - 2(3 - i)z + 4 - 3i = 0$

Solution

a From Example 19 **a**:

$$z^2 + z + 3 = \left(z - \left(-\frac{1}{2} - \frac{\sqrt{11}}{2}i\right)\right)\left(z - \left(-\frac{1}{2} + \frac{\sqrt{11}}{2}i\right)\right)$$

Hence $z^2 + z + 3 = 0$ has solutions

$$z = -\frac{1}{2} - \frac{\sqrt{11}}{2}i \quad \text{and} \quad z = -\frac{1}{2} + \frac{\sqrt{11}}{2}i$$

b From Example 19 **b**:

$$2z^2 - z + 1 = 2\left(z - \left(\frac{1}{4} - \frac{\sqrt{7}}{4}i\right)\right)\left(z - \left(\frac{1}{4} + \frac{\sqrt{7}}{4}i\right)\right)$$

Hence $2z^2 - z + 1 = 0$ has solutions

$$z = \frac{1}{4} - \frac{\sqrt{7}}{4}i \quad \text{and} \quad z = \frac{1}{4} + \frac{\sqrt{7}}{4}i$$

c Rearrange the equation into the form

$$z^2 - 2z + 5 = 0$$

Now apply the quadratic formula:

$$z = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

The solutions are $1 + 2i$ and $1 - 2i$.

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d From Example 19 **c**, we have

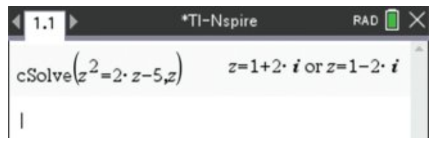
$$2z^2 - 2(3-i)z + 4 - 3i = 2\left(z - \frac{3-i}{2}\right)^2$$

Hence $2z^2 - 2(3-i)z + 4 - 3i = 0$ has solution $z = \dfrac{3-i}{2}$.

Note: In parts **a**, **b** and **c** of this example, the two solutions are conjugates of each other. We explore this further in the next section.

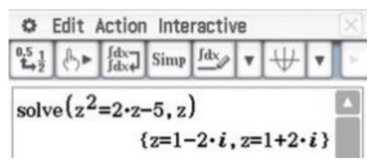
Using the TI-Nspire

To find complex solutions, use (menu) > **Algebra** > **Complex** > **Solve** as shown.



Using the Casio ClassPad

- Ensure your calculator is in complex mode.
- Enter and highlight the equation.
- Select **Interactive** > **Equation/Inequality** > **solve**.
- Ensure that the variable is z.



Every quadratic equation has two solutions over the complex numbers, if we count repeated solutions twice. For example, the equation $(z-3)^2 = 0$ has a repeated solution $z = 3$. We say that this solution has a **multiplicity** of 2.

Skill-sheet

Exercise 6E

Example 18
Example 19

1 Factorise each of the following into linear factors over $\mathbb{C}$:

a $z^2 + 16$ **b** $z^2 + 5$ **c** $z^2 + 2z + 5$
d $z^2 - 3z + 4$ **e** $2z^2 - 8z + 9$ **f** $3z^2 + 6z + 4$
g $3z^2 + 2z + 2$ **h** $2z^2 - z + 3$

Example 20

2 Solve each of the following equations over $\mathbb{C}$:

a $x^2 + 25 = 0$ **b** $x^2 + 8 = 0$
c $x^2 - 4x + 5 = 0$ **d** $3x^2 + 7x + 5 = 0$
e $x^2 = 2x - 3$ **f** $5x^2 + 1 = 3x$
g $z^2 + (1+2i)z + (-1+i) = 0$ **h** $z^2 + z + (1-i) = 0$

Hint: Show that $-3 + 4i = (1+2i)^2$.

3 Suppose that the solutions of the equation $z^2 + bz + c = 0$ are $z = \alpha$ and $z = \overline{\alpha}$. Prove that both b and c are real numbers.

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6F Solving polynomial equations over the complex numbers

You have studied polynomials over the real numbers in Mathematical Methods Units 3 & 4. We now extend this study to polynomials over the complex numbers. For $n \in \mathbb{N} \cup \{0\}$, a polynomial of degree n is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where the coefficients a_i are complex numbers and $a_n \neq 0$.

When we divide the polynomial $P(z)$ by the polynomial $D(z)$ we obtain two polynomials, $Q(z)$ the **quotient** and $R(z)$ the **remainder**, such that

$$P(z) = D(z)Q(z) + R(z)$$

and either $R(z) = 0$ or $R(z)$ has degree less than $D(z)$.

If $R(z) = 0$, then $D(z)$ is a **factor** of $P(z)$.

The remainder theorem and the factor theorem are true for polynomials over $\mathbb{C}$.

- **Remainder theorem**
  Let $\alpha \in \mathbb{C}$. When a polynomial $P(z)$ is divided by $z - \alpha$, the remainder is $P(\alpha)$.
- **Factor theorem**
  Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of a polynomial $P(z)$ if and only if $P(\alpha) = 0$.

Example 21

Factorise $P(z) = z^3 + z^2 + 4$.

Solution

Use the factor theorem to find the first factor:

$$P(-1) = -1 + 1 + 4 \neq 0$$
$$P(-2) = -8 + 4 + 4 = 0$$

Therefore $z + 2$ is a factor. We obtain $P(z) = (z + 2)(z^2 - z + 2)$ by division.

We can factorise $z^2 - z + 2$ by completing the square:

$$\begin{aligned} z^2 - z + 2 &= \left(z^2 - z + \frac{1}{4}\right) + 2 - \frac{1}{4} \\ &= \left(z - \frac{1}{2}\right)^2 - \frac{7}{4}i^2 \\ &= \left(z - \frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{7}}{2}i\right) \end{aligned}$$

Hence $\quad P(z) = (z + 2)\left(z - \frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{7}}{2}i\right)$

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Example 22

Factorise $z^3 - iz^2 - 4z + 4i$.

Solution

Factorise by grouping:

$$\begin{aligned} z^3 - iz^2 - 4z + 4i &= z^2(z - i) - 4(z - i) \\ &= (z - i)(z^2 - 4) \\ &= (z - i)(z - 2)(z + 2) \end{aligned}$$

The conjugate root theorem

Notice in Example 21 that the two complex solutions are conjugates of each other. The following theorem gives a simple condition that ensures this happens.

Conjugate root theorem

Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of the equation $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

Proof We will prove the theorem for quadratics, as it gives the idea of the general proof.

Let $P(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Assume that α is a solution of the equation $P(z) = 0$. Then $P(\alpha) = 0$. That is,

$$a\alpha^2 + b\alpha + c = 0$$

Take the conjugate of both sides of this equation and use properties of conjugates:

$$\overline{a\alpha^2 + b\alpha + c} = \overline{0}$$

$$\overline{a\alpha^2} + \overline{b\alpha} + \overline{c} = 0$$

$$a(\overline{\alpha^2}) + b\overline{\alpha} + c = 0 \qquad \text{since } a, b \text{ and } c \text{ are real numbers}$$

$$a(\overline{\alpha})^2 + b\overline{\alpha} + c = 0$$

Hence $P(\overline{\alpha}) = 0$. That is, $\overline{\alpha}$ is a solution of the equation $P(z) = 0$.

If a polynomial $P(z)$ has real coefficients, then using this theorem we can say that the complex solutions of the equation $P(z) = 0$ occur in **conjugate pairs**. (Note that this theorem does not hold without the assumption that $P(z)$ has real coefficients; see Example 22.)

Factorisation of cubic polynomials

Over the complex numbers, every cubic polynomial can be written as the product of three linear factors.

If the coefficients of the cubic are real, then at least one factor must be real (as complex factors occur in conjugate pairs). A useful method of factorisation, already demonstrated in Example 21, is to find the real linear factor using the factor theorem and then complete the square on the resulting quadratic factor. The cubic polynomial can also be factorised if one complex root is given, as shown in the next example.

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Example 23

Let $P(z) = z^3 - 3z^2 + 5z - 3$.

a Use the factor theorem to show that $z - 1 + \sqrt{2}i$ is a factor of $P(z)$.

b Find the other linear factors of $P(z)$.

Solution

a To show that $z - (1 - \sqrt{2}i)$ is a factor, we must check that $P(1 - \sqrt{2}i) = 0$.

We have

$$P(1 - \sqrt{2}i) = (1 - \sqrt{2}i)^3 - 3(1 - \sqrt{2}i)^2 + 5(1 - \sqrt{2}i) - 3 = 0$$

Therefore $z - (1 - \sqrt{2}i)$ is a factor of $P(z)$.

b Since the coefficients of $P(z)$ are real, the complex linear factors occur in conjugate pairs, so $z - (1 + \sqrt{2}i)$ is also a factor.

To find the third linear factor, first multiply the two complex factors together:

$$\begin{aligned}&\big(z - (1 - \sqrt{2}i)\big)\big(z - (1 + \sqrt{2}i)\big)\\ &= z^2 - (1 - \sqrt{2}i)z - (1 + \sqrt{2}i)z + (1 - \sqrt{2}i)(1 + \sqrt{2}i)\\ &= z^2 - (1 - \sqrt{2}i + 1 + \sqrt{2}i)z + 1 + 2\\ &= z^2 - 2z + 3\end{aligned}$$

Therefore, by inspection, the linear factors of $P(z) = z^3 - 3z^2 + 5z - 3$ are

$$z - 1 + \sqrt{2}i, \quad z - 1 - \sqrt{2}i \quad \text{and} \quad z - 1$$

Factorisation of higher degree polynomials

Polynomials of the form $z^4 - a^4$ and $z^6 - a^6$ are considered in the following example.

Example 24

Factorise:

a $P(z) = z^4 - 16$

b $P(z) = z^6 - 1$

Solution

a $$\begin{aligned}P(z) &= z^4 - 16\\ &= (z^2 + 4)(z^2 - 4) && \text{(difference of two squares)}\\ &= (z + 2i)(z - 2i)(z + 2)(z - 2) && \text{(sum and difference of two squares)}\end{aligned}$$

b $$\begin{aligned}P(z) &= z^6 - 1\\ &= (z^3 + 1)(z^3 - 1)\end{aligned}$$

We next factorise $z^3 + 1$ and $z^3 - 1$.

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We have

$$\begin{aligned} z^3 + 1 &= (z+1)(z^2 - z + 1) \\ &= (z+1)\left(\left(z^2 - z + \frac{1}{4}\right) + 1 - \frac{1}{4}\right) \\ &= (z+1)\left(\left(z - \frac{1}{2}\right)^2 - \frac{3}{4}i^2\right) \\ &= (z+1)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \end{aligned}$$

By a similar method, we have

$$\begin{aligned} z^3 - 1 &= (z-1)(z^2 + z + 1) \\ &= (z-1)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \end{aligned}$$

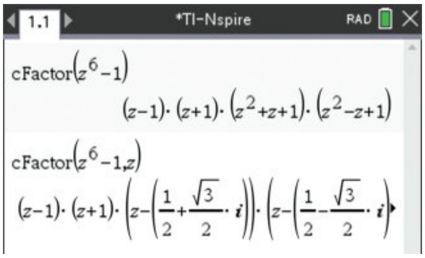
Therefore

$$z^6 - 1 = (z+1)(z-1)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

Using the TI-Nspire

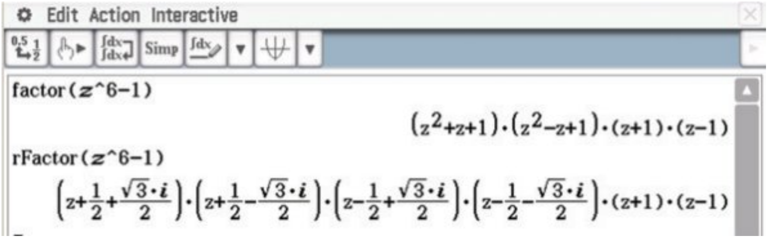
To find complex factors, use menu > **Algebra** > **Complex** > **Factor**.

The first operation shown factorises to give integer coefficients, and the second fully factorises over the complex numbers.



Using the Casio ClassPad

- Ensure your calculator is in complex mode.
- To factorise over the real numbers:
  Enter and highlight $z^6 - 1$. Select **Interactive** > **Transformation** > **factor**.
- To factorise over the complex numbers:
  Enter and highlight $z^6 - 1$. Select **Interactive** > **Transformation** > **factor** > **rFactor**.



Note: Go to **Edit** > **Clear all variables** if z has been used to store a complex expression.

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The fundamental theorem of algebra

The following important theorem has been attributed to Gauss (1799).

Fundamental theorem of algebra

Every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ of degree n, where $n \geq 1$ and the coefficients a_i are complex numbers, has at least one linear factor in the complex number system.

Given any polynomial $P(z)$ of degree $n \geq 1$, the theorem tells us that we can factorise $P(z)$ as

$$P(z) = (z - \alpha_1)Q(z)$$

for some $\alpha_1 \in \mathbb{C}$ and some polynomial $Q(z)$ of degree $n - 1$.

By applying the fundamental theorem of algebra repeatedly, it can be shown that:

A polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$:

i.e. $P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \ldots (z - \alpha_n)$, where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{C}$

A polynomial equation can be solved by first rearranging it into the form $P(z) = 0$, where $P(z)$ is a polynomial, and then factorising $P(z)$ and extracting a solution from each factor.

If $P(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n)$, then the solutions of $P(z) = 0$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$.

The solutions of the equation $P(z) = 0$ are also referred to as the **zeroes** or the **roots** of the polynomial $P(z)$.

Note: A polynomial equation may have repeated solutions. For example, the equation $(z - 2)^3 = 0$ has a repeated solution $z = 2$. This solution has a multiplicity of 3.

Example 25

Solve each of the following equations over $\mathbb{C}$:

a $z^2 + 64 = 0$ **b** $z^3 + 3z^2 + 7z + 5 = 0$ **c** $z^3 - iz^2 - 4z + 4i = 0$

Solution

a
$$z^2 + 64 = 0$$
$$(z + 8i)(z - 8i) = 0$$
$$z = -8i \text{ or } z = 8i$$

b Let $P(z) = z^3 + 3z^2 + 7z + 5$.

Then $P(-1) = 0$, so $z + 1$ is a factor, by the factor theorem.

$$\begin{aligned} P(z) &= (z + 1)(z^2 + 2z + 5) \\ &= (z + 1)(z^2 + 2z + 1 + 4) \\ &= (z + 1)((z + 1)^2 - (2i)^2) \\ &= (z + 1)(z + 1 - 2i)(z + 1 + 2i) \end{aligned}$$

If $P(z) = 0$, then $z = -1$, $z = -1 + 2i$ or $z = -1 - 2i$.

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c From Example 22:

$$z^3 - iz^2 - 4z + 4i = 0$$
$$(z - i)(z - 2)(z + 2) = 0$$
$$\therefore \quad z = i, \quad z = 2 \text{ or } z = -2$$

Skill-sheet

Exercise 6F

Example 21
Example 22

1 Factorise each of the following polynomials into linear factors over $\mathbb{C}$:

a $z^3 - 4z^2 - 4z - 5$ **b** $z^3 - z^2 - z + 10$ **c** $3z^3 - 13z^2 + 5z - 4$

d $2z^3 + 3z^2 - 4z + 15$ **e** $z^3 - (2 - i)z^2 + z - 2 + i$

Example 23

2 Let $P(z) = z^3 + 4z^2 - 10z + 12$.

a Use the factor theorem to show that $z - 1 - i$ is a linear factor of $P(z)$.

b Write down another complex linear factor of $P(z)$.

c Hence find all the linear factors of $P(z)$ over $\mathbb{C}$.

3 Let $P(z) = 2z^3 + 9z^2 + 14z + 5$.

a Use the factor theorem to show that $z + 2 - i$ is a linear factor of $P(z)$.

b Write down another complex linear factor of $P(z)$.

c Hence find all the linear factors of $P(z)$ over $\mathbb{C}$.

4 Let $P(z) = z^4 + 8z^2 + 16z + 20$.

a Use the factor theorem to show that $z - 1 + 3i$ is a linear factor of $P(z)$.

b Write down another complex linear factor of $P(z)$.

c Hence find all the linear factors of $P(z)$ over $\mathbb{C}$.

Example 24

5 Factorise each of the following into linear factors over $\mathbb{C}$:

a $z^4 - 81$ **b** $z^6 - 64$

6 For each of the following, factorise the first expression into linear factors over $\mathbb{C}$, given that the second expression is one of the linear factors:

a $z^3 + (1 - i)z^2 + (1 - i)z - i, \quad z - i$

b $z^3 - (2 - i)z^2 - (1 + 2i)z - i, \quad z + i$

c $z^3 - (2 + 2i)z^2 - (3 - 4i)z + 6i, \quad z - 2i$

d $2z^3 + (1 - 2i)z^2 - (5 + i)z + 5i, \quad z - i$

7 For each of the following, find the value of p given that:

a $z + 2$ is a factor of $z^3 + 3z^2 + pz + 12$

b $z - i$ is a factor of $z^3 + pz^2 + z - 4$

c $z + 1 - i$ is a factor of $2z^3 + z^2 - 2z + p$

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Example 25

8 Solve each of the following equations over $\mathbb{C}$:

a $x^3 + x^2 - 6x - 18 = 0$

b $x^3 - 6x^2 + 11x - 30 = 0$

c $2x^3 + 3x^2 = 11x^2 - 6x - 16$

d $x^4 + x^2 = 2x^3 + 36$

9 Let $z^2 + az + b = 0$, where a and b are real numbers. Find a and b if one of the solutions is:

a $2i$

b $3 + 2i$

c $-1 + 3i$

10 **a** $1 + 3i$ is a solution of the equation $3z^3 - 7z^2 + 32z - 10 = 0$. Find the other solutions.

b $-2 - i$ is a solution of the equation $z^4 - 5z^2 + 4z + 30 = 0$. Find the other solutions.

11 For a cubic polynomial $P(x)$ with real coefficients, $P(2 + i) = 0$, $P(1) = 0$ and $P(0) = 10$. Express $P(x)$ in the form $P(x) = ax^3 + bx^2 + cx + d$ and solve the equation $P(x) = 0$.

12 If $z = 1 + i$ is a zero of the polynomial $z^3 + az^2 + bz + 10 - 6i$, find the constants a and b, given that they are real.

13 The polynomial $P(z) = 2z^3 + az^2 + bz + 5$, where a and b are real numbers, has $2 - i$ as one of its zeroes.

a Find a quadratic factor of $P(z)$, and hence calculate the real constants a and b.

b Determine the solutions to the equation $P(z) = 0$.

14 For the polynomial $P(z) = az^4 + az^2 - 2z + d$, where a and d are real numbers:

a Evaluate $P(1 + i)$.

b Given that $P(1 + i) = 0$, find the values of a and d.

c Show that $P(z)$ can be written as the product of two quadratic factors with real coefficients, and hence solve the equation $P(z) = 0$.

15 The solutions of the quadratic equation $z^2 + pz + q = 0$ are $1 + i$ and $4 + 3i$. Find the complex numbers p and q.

16 Given that $1 - i$ is a solution of the equation $z^3 - 4z^2 + 6z - 4 = 0$, find the other two solutions.

17 Solve each of the following for z:

a $z^2 - (6 + 2i)z + (8 + 6i) = 0$

b $z^3 - 2iz^2 - 6z + 12i = 0$

c $z^3 - z^2 + 6z - 6 = 0$

d $z^3 - z^2 + 2z - 8 = 0$

e $6z^2 - 3\sqrt{2}\,z + 6 = 0$

f $z^3 + 2z^2 + 9z = 0$

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6G Using De Moivre's theorem to solve equations

Equations of the form $z^n = a$, where $a \in \mathbb{C}$, are often solved by using De Moivre's theorem.

Write both z and a in polar form, as $z = r \operatorname{cis} \theta$ and $a = q \operatorname{cis} \varphi$.

Then $z^n = a$ becomes

$$(r \operatorname{cis} \theta)^n = q \operatorname{cis} \varphi$$

$$\therefore \quad r^n \operatorname{cis}(n\theta) = q \operatorname{cis} \varphi \qquad \text{(using De Moivre's theorem)}$$

Compare modulus and argument:

$$r^n = q \qquad \operatorname{cis}(n\theta) = \operatorname{cis} \varphi$$

$$r = \sqrt[n]{q} \qquad n\theta = \varphi + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

$$\theta = \frac{1}{n}(\varphi + 2k\pi) \qquad \text{where } k \in \mathbb{Z}$$

This will provide all the solutions of the equation.

Example 26

Solve $z^3 = 1$.

Solution

Let $z = r \operatorname{cis} \theta$. Then

$$(r \operatorname{cis} \theta)^3 = 1 \operatorname{cis} 0$$

$$\therefore \quad r^3 \operatorname{cis}(3\theta) = 1 \operatorname{cis} 0$$

$$\therefore \quad r^3 = 1 \quad \text{and} \quad 3\theta = 0 + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$\therefore \quad r = 1 \quad \text{and} \quad \theta = \frac{2k\pi}{3} \quad \text{where } k \in \mathbb{Z}$$

Hence the solutions are of the form $z = \operatorname{cis}\left(\frac{2k\pi}{3}\right)$, where $k \in \mathbb{Z}$.

We start finding solutions.

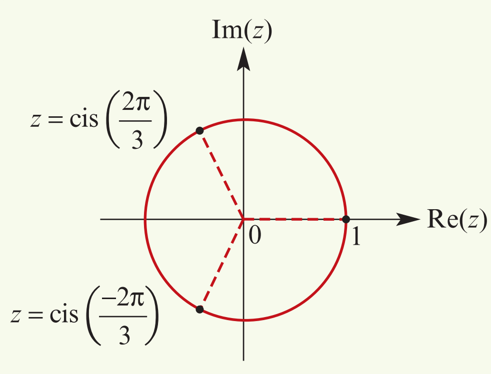
For $k = 0$: $\quad z = \operatorname{cis} 0 = 1$

For $k = 1$: $\quad z = \operatorname{cis}\left(\frac{2\pi}{3}\right)$

For $k = 2$: $\quad z = \operatorname{cis}\left(\frac{4\pi}{3}\right) = \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

For $k = 3$: $\quad z = \operatorname{cis}(2\pi) = 1$

The solutions begin to repeat.



The three solutions are 1, $\operatorname{cis}\left(\frac{2\pi}{3}\right)$ and $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$.

The solutions are shown to lie on the unit circle at intervals of $\frac{2\pi}{3}$ around the circle.

Note: An equation of the form $z^3 = a$, where $a \in \mathbb{R}$, has three solutions. Since $a \in \mathbb{R}$, two of the solutions will be conjugate to each other and the third must be a real number.

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Example 27

Solve $z^2 = 1 + i$.

Solution

Let $z = r\operatorname{cis}\theta$. Note that $1 + i = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$.

$$(r\operatorname{cis}\theta)^2 = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$\therefore \quad r^2\operatorname{cis}(2\theta) = 2^{\frac{1}{2}}\operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$\therefore \quad r = 2^{\frac{1}{4}} \quad \text{and} \quad 2\theta = \frac{\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

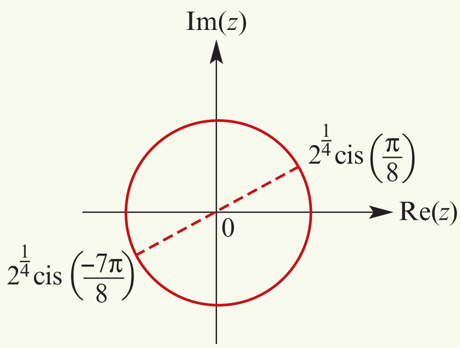
$$\therefore \quad r = 2^{\frac{1}{4}} \quad \text{and} \quad \theta = \frac{\pi}{8} + k\pi \quad \text{where } k \in \mathbb{Z}$$

Hence $z = 2^{\frac{1}{4}}\operatorname{cis}\left(\frac{\pi}{8} + k\pi\right)$, where $k \in \mathbb{Z}$.

For $k = 0$: $\quad z = 2^{\frac{1}{4}}\operatorname{cis}\left(\frac{\pi}{8}\right)$

For $k = 1$: $\quad z = 2^{\frac{1}{4}}\operatorname{cis}\left(\frac{9\pi}{8}\right)$

$$= 2^{\frac{1}{4}}\operatorname{cis}\left(-\frac{7\pi}{8}\right)$$



Note: If z_1 is a solution of $z^2 = a$, where $a \in \mathbb{C}$, then the other solution is $z_2 = -z_1$.

In Example 27, we found the two square roots of the complex number $1 + i$. More generally:

Solutions of $z^n = a$

For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, the solutions of the equation $z^n = a$ are called the ***n*th roots** of a.

- The solutions of $z^n = a$ lie on a circle with centre the origin and radius $|a|^{\frac{1}{n}}$.
- There are n solutions and they are equally spaced around the circle at intervals of $\frac{2\pi}{n}$. This observation can be used to find all solutions if one is known.

The following example shows an alternative method for finding square roots.

Example 28

Solve $z^2 = 5 + 12i$ using $z = a + bi$, where $a, b \in \mathbb{R}$. Hence factorise $z^2 - 5 - 12i$.

Solution

Let $z = a + bi$. Then
$$\begin{aligned} z^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ &= (a^2 - b^2) + 2abi \end{aligned}$$

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So $z^2 = 5 + 12i$ becomes

$$(a^2 - b^2) + 2abi = 5 + 12i$$

Equating coefficients:

$$a^2 - b^2 = 5 \quad \text{and} \quad 2ab = 12$$

$$a^2 - \left(\frac{6}{a}\right)^2 = 5 \qquad b = \frac{6}{a}$$

$$a^2 - \frac{36}{a^2} = 5$$

$$a^4 - 36 = 5a^2$$

$$a^4 - 5a^2 - 36 = 0$$

$$(a^2 - 9)(a^2 + 4) = 0$$

$$a^2 - 9 = 0$$

$$(a + 3)(a - 3) = 0$$

$$\therefore \quad a = -3 \text{ or } a = 3$$

When $a = -3$, $b = -2$ and when $a = 3$, $b = 2$.

So the solutions to the equation $z^2 = 5 + 12i$ are $z = -3 - 2i$ and $z = 3 + 2i$.

Hence $z^2 - 5 - 12i = (z + 3 + 2i)(z - 3 - 2i)$.

Roots of unity

In this section, we have used De Moivre's theorem to solve equations of the form $z^n = a$. Here we consider an important special case.

Solutions of $z^n = 1$

For $n \in \mathbb{N}$, the solutions of the equation $z^n = 1$ are called the ***n*th roots of unity**.

- The solutions of $z^n = 1$ lie on the unit circle.
- There are n solutions and they are equally spaced around the circle at intervals of $\frac{2\pi}{n}$. This observation can be used to find all solutions, since $z = 1$ is one solution.

From Example 26, we see that the cube roots of unity are 1, $\text{cis}\left(\frac{2\pi}{3}\right)$ and $\text{cis}\left(\frac{4\pi}{3}\right)$.

More generally, consider any natural number $n \geq 2$. Using De Moivre's theorem, we can show that the nth roots of unity are

$$1,\ \text{cis}\left(\frac{2\pi}{n}\right),\ \text{cis}\left(\frac{4\pi}{n}\right),\ \dots,\ \text{cis}\left(\frac{2(n-1)\pi}{n}\right)$$

So the nth roots of unity form a geometric sequence with common ratio $\omega = \text{cis}\left(\frac{2\pi}{n}\right)$.

We can list the terms of this sequence as $1, \omega, \omega^2, \dots, \omega^{n-1}$. The sum of the terms is

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \frac{\omega^n - 1}{\omega - 1} = 0$$

since $\omega^n = 1$.

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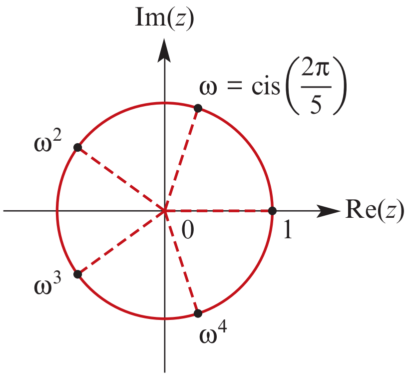
This Argand diagram shows the fifth roots of unity. Note that they are equally spaced around the unit circle.



Skill-sheet

Exercise 6G

Example 26
Example 27

1 For each of the following, solve the equation over $\mathbb{C}$ and show the solutions on an Argand diagram:

a $z^2 + 1 = 0$ **b** $z^3 = 27i$ **c** $z^2 = 1 + \sqrt{3}i$
d $z^2 = 1 - \sqrt{3}i$ **e** $z^3 = i$ **f** $z^3 + i = 0$

2 Find all the cube roots of the following complex numbers:

a $4\sqrt{2} - 4\sqrt{2}i$ **b** $-4\sqrt{2} + 4\sqrt{2}i$ **c** $-4\sqrt{3} - 4i$
d $4\sqrt{3} - 4i$ **e** $-125i$ **f** $-1 + i$

Example 28

3 Let $z = a + bi$ such that $z^2 = 3 + 4i$, where $a, b \in \mathbb{R}$.

a Find equations in terms of a and b by equating real and imaginary parts.
b Find the values of a and b and hence find the square roots of $3 + 4i$.

4 Using the method of Question 3, find the square roots of each of the following:

a $-15 - 8i$ **b** $24 + 7i$ **c** $-3 + 4i$ **d** $-7 + 24i$

5 Find the solutions of the equation $z^4 - 2z^2 + 4 = 0$ in polar form.

6 **a** Find $a, b \in \mathbb{R}$ given that $z^5 - 1 = (z - 1)(z^2 + az + 1)(z^2 + bz + 1)$.
Hint: Use $\cos\left(\frac{2\pi}{5}\right) = \frac{1}{4}(\sqrt{5} - 1)$ and the polar form of the fifth roots of unity.
b Find $a, b \in \mathbb{R}$ given that $z^6 - 1 = (z - 1)(z + 1)(z^2 + az + 1)(z^2 + bz + 1)$.
Hint: Use the value of $\cos\left(\frac{2\pi}{6}\right)$ and the polar form of the sixth roots of unity.

7 Let $n \in \mathbb{N}$ and consider the nth roots of unity $1, \omega, \omega^2, \dots, \omega^{n-1}$.

a Prove that if n is odd, then the product $1 \times \omega \times \omega^2 \times \dots \times \omega^{n-1}$ is equal to 1.
b Prove that if n is even, then the product $1 \times \omega \times \omega^2 \times \dots \times \omega^{n-1}$ is equal to -1.

8 Find the solutions of the equation $z^2 - i = 0$ in Cartesian form. Hence factorise $z^2 - i$.

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9 Find the solutions of the equation $z^8 + 1 = 0$ in polar form. Hence factorise $z^8 + 1$.

10 **a** By expanding the left-hand side, find $a, b \in \mathbb{R}$ such that $(a + bi)^2 = 1 + i$.
b Solve the equation $z^2 = 1 + i$ by using De Moivre's theorem.
c By comparing your answers to **a** and **b**, find the exact values of $\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\frac{\pi}{8}\right)$.
d Hence, solve the equation $z^4 = i$.

6H Sketching subsets of the complex plane

This section is revision of a topic from Specialist Mathematics Units 1 & 2.

Particular sets of points of the complex plane can be described by placing restrictions on z. For example:

- $\{ z : \text{Re}(z) = 6 \}$ is the straight line parallel to the imaginary axis with each point on the line having real part 6.
- $\{ z : \text{Im}(z) = 2\,\text{Re}(z) \}$ is the straight line through the origin with gradient 2.

The set of all points which satisfy a given condition is called the **locus** of the condition (plural loci). When sketching a locus, a solid line is used for a boundary which is included in the locus, and a dashed line is used for a boundary which is not included.

Example 29

On an Argand diagram, sketch the subset S of the complex plane, where

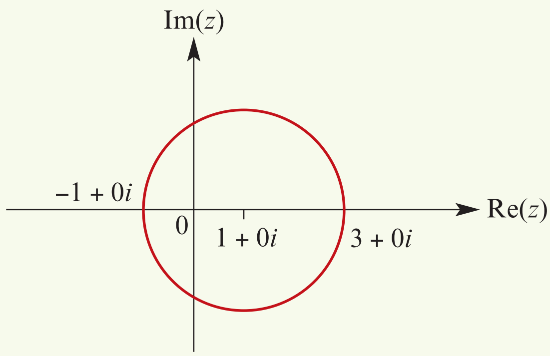
$$S = \{ z : |z - 1| = 2 \}$$

Solution

Method 1: Using algebra

Let $z = x + yi$. Then

$$|z - 1| = 2$$
$$|x + yi - 1| = 2$$
$$|(x - 1) + yi| = 2$$
$$\sqrt{(x - 1)^2 + y^2} = 2$$
$$\therefore \quad (x - 1)^2 + y^2 = 4$$



This demonstrates that S is represented by the circle with centre $1 + 0i$ and radius 2.

Method 2: Using geometry

If z_1 and z_2 are complex numbers, then $|z_2 - z_1|$ is the distance between the points on the complex plane corresponding to z_1 and z_2.

Hence $\{ z : |z - 1| = 2 \}$ is the set of all points that are distance 2 from $1 + 0i$. That is, the set S is represented by the circle with centre $1 + 0i$ and radius 2.

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Example 30

On an Argand diagram, sketch the subset S of the complex plane, where

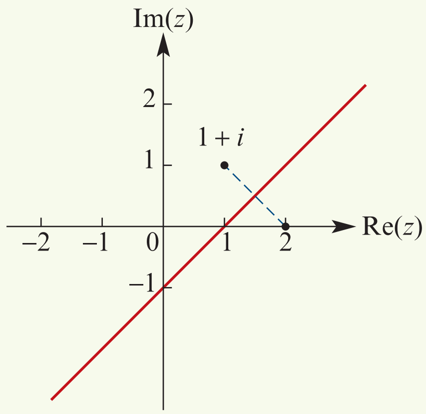
$$S = \{z : |z-2| = |z-(1+i)|\}$$

Solution

Method 1: Using algebra

Let $z = x + yi$. Then

$$|z-2| = |z-(1+i)|$$
$$|x+yi-2| = |x+yi-(1+i)|$$
$$|x-2+yi| = |x-1+(y-1)i|$$
$$\therefore \quad \sqrt{(x-2)^2+y^2} = \sqrt{(x-1)^2+(y-1)^2}$$



Squaring both sides of the equation and expanding:

$$x^2-4x+4+y^2 = x^2-2x+1+y^2-2y+1$$
$$-4x+4 = -2x-2y+2$$
$$\therefore \quad y = x-1$$

Method 2: Using geometry

The set S consists of all points in the complex plane that are equidistant from 2 and $1+i$.

In the Cartesian plane, this set corresponds to the perpendicular bisector of the line segment joining $(2, 0)$ and $(1, 1)$. The midpoint of the line segment is $(\frac{3}{2}, \frac{1}{2})$, and the gradient of the line segment is -1.

Therefore the equation of the perpendicular bisector is

$$y - \tfrac{1}{2} = 1(x - \tfrac{3}{2})$$

which simplifies to $y = x - 1$.

Example 31

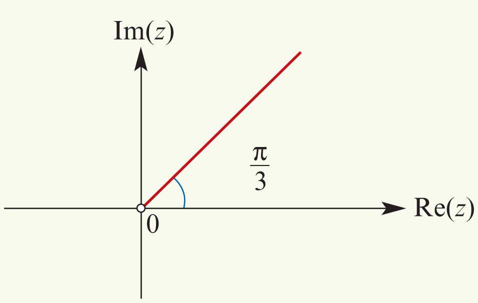
Sketch the subset of the complex plane defined by each of the following conditions:

a $\operatorname{Arg}(z) = \frac{\pi}{3}$ **b** $\operatorname{Arg}(z+3) = -\frac{\pi}{3}$ **c** $\operatorname{Arg}(z) \le \frac{\pi}{3}$

Solution

a $\operatorname{Arg}(z) = \frac{\pi}{3}$ defines a ray or a half line.

Note: The origin is not included.



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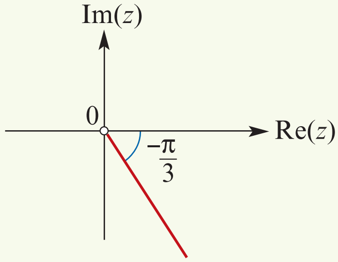
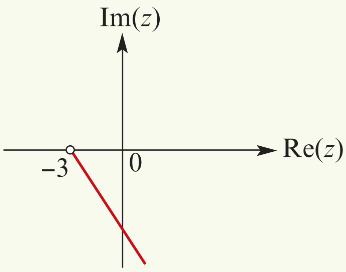
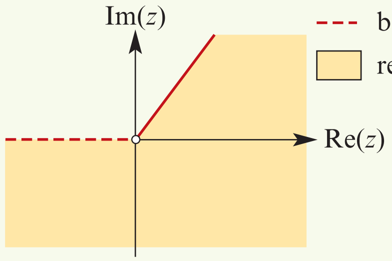
b First draw the graph of $\text{Arg}(z) = -\frac{\pi}{3}$.



The graph of $\text{Arg}(z + 3) = -\frac{\pi}{3}$ is obtained by a translation of 3 units to the left.



c Since $-\pi < \text{Arg}(z) \le \pi$ in general, the condition $\text{Arg}(z) \le \frac{\pi}{3}$ implies $-\pi < \text{Arg}(z) \le \frac{\pi}{3}$.



Example 32

Describe the locus defined by $|z + 3| = 2|z - i|$.

Solution

Let $z = x + yi$. Then

$$|z + 3| = 2|z - i|$$

$$|(x + 3) + yi| = 2|x + (y - 1)i|$$

$$\therefore \quad \sqrt{(x + 3)^2 + y^2} = 2\sqrt{x^2 + (y - 1)^2}$$

Squaring both sides gives

$$x^2 + 6x + 9 + y^2 = 4(x^2 + y^2 - 2y + 1)$$

$$0 = 3x^2 + 3y^2 - 6x - 8y - 5$$

$$5 = 3(x^2 - 2x) + 3\left(y^2 - \frac{8}{3}y\right)$$

$$\frac{5}{3} = (x^2 - 2x + 1) + \left(y^2 - \frac{8}{3}y + \frac{16}{9}\right) - \frac{25}{9}$$

$$\therefore \quad \frac{40}{9} = (x - 1)^2 + \left(y - \frac{4}{3}\right)^2$$

The locus is the circle with centre $1 + \frac{4}{3}i$ and radius $\frac{2\sqrt{10}}{3}$.

Note: For $a, b \in \mathbb{C}$ and $k \in \mathbb{R}^+ \setminus \{1\}$, the equation $|z - a| = k|z - b|$ defines a circle.

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Skill-sheet

Exercise 6H

Example 29
Example 30

1 Illustrate each of the following on an Argand diagram:

a $2\,\text{Im}(z) = \text{Re}(z)$ **b** $\text{Im}(z) + \text{Re}(z) = 1$ **c** $|z - 2| = 3$

d $|z - i| = 4$ **e** $|z - (1 + \sqrt{3}i)| = 2$ **f** $|z - (1 - i)| = 6$

2 Sketch $\{ z : z = i\bar{z} \}$ in the complex plane.

3 Describe the subset of the complex plane defined by $\{ z : |z - 1| = |z + 1| \}$.

Example 31

4 Sketch the subset of the complex plane defined by each of the following conditions:

a $\text{Arg}(z) = \dfrac{\pi}{4}$ **b** $\text{Arg}(z - 2) = -\dfrac{\pi}{4}$ **c** $\text{Arg}(z) \leq \dfrac{\pi}{4}$

5 Prove that $3|z - 1|^2 = |z + 1|^2$ if and only if $|z - 2|^2 = 3$, for any complex number z. Hence sketch the set $S = \{ z : \sqrt{3}\,|z - 1| = |z + 1| \}$ on an Argand diagram.

Example 32

6 Sketch each of the following:

a $\{ z : |z + 2i| = 2|z - i| \}$ **b** $\{ z : \text{Im}(z) = -2 \}$

c $\{ z : z + \bar{z} = 5 \}$ **d** $\{ z : z\bar{z} = 5 \}$

e $\{ z : \text{Re}(z^2) = \text{Im}(z) \}$ **f** $\left\{ z : \text{Arg}(z - i) = \dfrac{\pi}{3} \right\}$

7 On the Argand plane, sketch the curve defined by each of the following equations:

a $\left|\dfrac{z - 2}{z}\right| = 1$ **b** $\left|\dfrac{z - 1 - i}{z}\right| = 1$

8 If the real part of $\dfrac{z + 1}{z - 1}$ is zero, find the locus of points representing z in the complex plane.

9 Given that z satisfies the equation $2|z - 2| = |z - 6i|$, show that z is represented by a point on a circle and find the centre and radius of the circle.

10 On an Argand diagram with origin O, the point P represents z and Q represents $\dfrac{1}{\bar{z}}$. Prove that O, P and Q are collinear and find the ratio $OP : OQ$ in terms of $|z|$.

11 Find the locus of points described by each of the following conditions:

a $|z - (1 + i)| = 1$ **b** $|z - 2| = |z + 2i|$

c $\text{Arg}(z - 1) = \dfrac{\pi}{2}$ **d** $\text{Arg}(z + i) = \dfrac{\pi}{4}$

12 Let $w = 2z$. Describe the locus of w if z describes a circle with centre $1 + 2i$ and radius 3.

13 **a** Find the solutions of the equation $z^2 + 2z + 4 = 0$.

b Show that the solutions satisfy:

i $|z| = 2$ **ii** $|z - 1| = \sqrt{7}$ **iii** $z + \bar{z} = -2$

c On a single diagram, sketch the loci defined by the equations in **b**.

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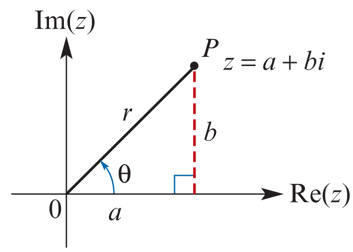
Chapter summary

Assignment

Nrich

- The imaginary number i has the property $i^2 = -1$.
- The set of **complex numbers** is $\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\text{Re}(z) = a$
  - the **imaginary part** of z is $\text{Im}(z) = b$.
- Complex numbers z_1 and z_2 are equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of z, denoted by $|z|$, is the distance from the origin to the point representing z in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The complex number $z = a + bi$ can be expressed in **polar form** as

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\,\text{cis}\,\theta$$



where $r = |z| = \sqrt{a^2 + b^2}$, $a = r\cos\theta$, $b = r\sin\theta$.
This is also called modulus–argument form.

- The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z.
- For a non-zero complex number z, the argument θ of z such that $-\pi < \theta \le \pi$ is called the **principal value** of the argument of z and is denoted by $\text{Arg}\,z$.
- The **complex conjugate** of z, denoted by $\bar{z}$, is the reflection of z in the real axis. If $z = a + bi$, then $\bar{z} = a - bi$. If $z = r\,\text{cis}\,\theta$, then $\bar{z} = r\,\text{cis}(-\theta)$. Note that $z\bar{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

- Multiplication and division in polar form:
  Let $z_1 = r_1\,\text{cis}\,\theta_1$ and $z_2 = r_2\,\text{cis}\,\theta_2$. Then

$$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$$

- De Moivre's theorem $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$, where $n \in \mathbb{Z}$
- Conjugate root theorem If a polynomial has real coefficients, then the complex roots occur in conjugate pairs.
- Fundamental theorem of algebra Every non-constant polynomial with complex coefficients has at least one linear factor in the complex number system.
- A polynomial of degree n can be factorised over $\mathbb{C}$ into a product of n linear factors.
- If z_1 is a solution of $z^2 = a$, where $a \in \mathbb{C}$, then the other solution is $z_2 = -z_1$.
- The solutions of $z^n = a$, where $a \in \mathbb{C}$, lie on the circle centred at the origin with radius $|a|^{\frac{1}{n}}$. The solutions are equally spaced around the circle at intervals of $\frac{2\pi}{n}$.
- The distance between z_1 and z_2 in the complex plane is $|z_2 - z_1|$.
  For example, the set $\{ z : |z - (1 + i)| = 2 \}$ is a circle with centre $1 + i$ and radius 2.

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Technology-free questions

1 Express each of the following in the form $a + bi$, where $a, b \in \mathbb{R}$:

a $3 + 2i + 5 - 7i$ **b** i^3 **c** $(3 - 2i)(5 + 7i)$

d $(3 - 2i)(3 + 2i)$ **e** $\dfrac{2}{3 - 2i}$ **f** $\dfrac{5 - i}{2 + i}$

g $\dfrac{3i}{2 + i}$ **h** $(1 - 3i)^2$ **i** $\dfrac{(5 + 2i)^2}{3 - i}$

2 Solve each of the following equations for z:

a $(z - 2)^2 + 9 = 0$ **b** $\dfrac{z - 2i}{z + (3 - 2i)} = 2$ **c** $z^2 + 6z + 12 = 0$

d $z^4 + 81 = 0$ **e** $z^3 - 27 = 0$ **f** $8z^3 + 27 = 0$

3 **a** Show that $2 - i$ is a solution of the equation $z^3 - 2z^2 - 3z + 10 = 0$. Hence solve the equation for z.

b Show that $3 - 2i$ is a solution of the equation $x^3 - 5x^2 + 7x + 13 = 0$. Hence solve the equation for $x \in \mathbb{C}$.

c Show that $1 + i$ is a solution of the equation $z^3 - 4z^2 + 6z - 4 = 0$. Hence find the other solutions of this equation.

4 Express each of the following polynomials as a product of linear factors:

a $2x^2 + 3x + 2$ **b** $x^3 - x^2 + x - 1$ **c** $x^3 + 2x^2 - 4x - 8$

5 If $(a + bi)^2 = 3 - 4i$, find the possible values of a and b, where $a, b \in \mathbb{R}$.

6 Pair each of the transformations given on the left with the appropriate operation on the complex numbers given on the right:

a reflection in the real axis — **i** multiply by -1

b rotation anticlockwise by 90° about O — **ii** multiply by i

c rotation through 180° about O — **iii** multiply by $-i$

d rotation anticlockwise about O through 270° — **iv** take the conjugate

7 If $(a + bi)^2 = -24 - 10i$, find the possible values of a and b, where $a, b \in \mathbb{R}$.

8 Find the values of a and b if $f(z) = z^2 + az + b$ and $f(-1 - 2i) = 0$, where $a, b \in \mathbb{R}$.

9 Express $\dfrac{1}{1 + \sqrt{3}i}$ in the form $r \operatorname{cis} \theta$, where $r > 0$ and $-\pi < \theta \le \pi$.

10 On an Argand diagram with origin O, the point P represents $3 + i$. The point Q represents $a + bi$, where both a and b are positive. If the triangle OPQ is equilateral, find a and b.

11 Let $z = 1 - i$. Find:

a $2\bar{z}$ **b** $\dfrac{1}{z}$ **c** $|z^7|$ **d** $\operatorname{Arg}(z^7)$

12 Find the value of a if $\dfrac{1}{a + 3i} + \dfrac{1}{a - 3i} = \dfrac{4}{13}$, where $a \in \mathbb{R}$.

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13 Let $w = 1 + i$ and $z = 1 - \sqrt{3}i$.

a Write down:

i $|w|$ **ii** $|z|$ **iii** $\operatorname{Arg} w$ **iv** $\operatorname{Arg} z$

b Hence write down $\left|\frac{w}{z}\right|$ and $\operatorname{Arg}(wz)$.

14 Express $\sqrt{3} + i$ in polar form. Hence find $(\sqrt{3} + i)^7$ and express in Cartesian form.

15 Consider the equation $z^4 - 2z^3 + 11z^2 - 18z + 18 = 0$. Find all real values of r for which $z = ri$ is a solution of the equation. Hence find all the solutions of the equation.

16 Express $(1 - i)^9$ in Cartesian form.

17 Consider the polynomial $P(z) = z^3 + (2 + i)z^2 + (2 + 2i)z + 4$. Find the real numbers k such that ki is a zero of $P(z)$. Hence, or otherwise, find the three zeroes of $P(z)$.

18 **a** Find the three linear factors of $z^3 - 2z + 4$.

b What is the remainder when $z^3 - 2z + 4$ is divided by $z - 3$?

19 If a and b are complex numbers such that $\operatorname{Im}(a) = 2$, $\operatorname{Re}(b) = -1$ and $a + b = -ab$, find a and b.

20 **a** Express $S = \{ z : |z - (1 + i)| \le 1 \}$ in Cartesian form.

b Sketch S on an Argand diagram.

21 Describe $\{ z : |z + i| = |z - i| \}$.

22 Let $S = \left\{ z : z = 2\operatorname{cis}\theta,\ 0 \le \theta \le \frac{\pi}{2} \right\}$. Sketch:

a S **b** $T = \{ w : w = z^2,\ z \in S \}$ **c** $U = \left\{ v : v = \frac{2}{z},\ z \in S \right\}$

23 Find the centre of the circle which passes through the points $-2i$, 1 and $2 - i$.

24 On an Argand diagram, points A and B represent $a = 5 + 2i$ and $b = 8 + 6i$.

a Find $i(a - b)$ and show that it can be represented by a vector perpendicular to $\overrightarrow{AB}$ and of the same length as $\overrightarrow{AB}$.

b Hence find complex numbers c and d, represented by C and D, such that $ABCD$ is a square.

25 Solve each of the following for $z \in \mathbb{C}$:

a $z^3 = -8$ **b** $z^2 = 2 + 2\sqrt{3}i$

26 **a** Factorise $x^6 - 1$ over $\mathbb{R}$.

b Factorise $x^6 - 1$ over $\mathbb{C}$.

c Determine all the sixth roots of unity. (That is, solve $x^6 = 1$ for $x \in \mathbb{C}$.)

27 Let z be a complex number with a non-zero imaginary part. Simplify:

a $\left|\frac{\bar{z}}{z}\right|$ **b** $\frac{i(\operatorname{Re}(z) - z)}{\operatorname{Im}(z)}$ **c** $\operatorname{Arg} z + \operatorname{Arg}\left(\frac{1}{z}\right)$

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Review

28 Let $z, w \in \mathbb{C}$. Given that $|z| = |w| = 2$ and $|z + w| = 3$, find the value of $\left|\frac{1}{z} + \frac{1}{w}\right|$.

29 If $\operatorname{Arg} z = \frac{\pi}{4}$ and $\operatorname{Arg}(z - 3) = \frac{\pi}{2}$, find $\operatorname{Arg}(z - 6i)$.

30 **a** If $\operatorname{Arg}(z + 2) = \frac{\pi}{2}$ and $\operatorname{Arg}(z) = \frac{2\pi}{3}$, find z.

b If $\operatorname{Arg}(z - 3) = -\frac{3\pi}{4}$ and $\operatorname{Arg}(z + 3) = -\frac{\pi}{2}$, find z.

31 A complex number z satisfies the inequality $\left|z + 2 - 2\sqrt{3}i\right| \le 2$.

a Sketch the corresponding region representing all possible values of z.

b **i** Find the least possible value of $|z|$.

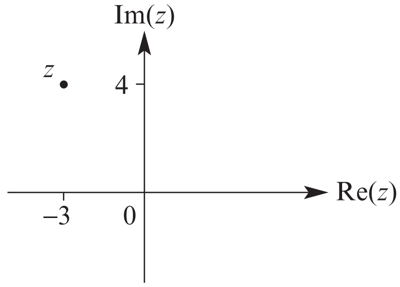
ii Find the greatest possible value of $\operatorname{Arg} z$.

Multiple-choice questions

1 If $z_1 = 5\operatorname{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2\operatorname{cis}\left(\frac{3\pi}{4}\right)$, then $z_1 z_2$ is equal to

A $7\operatorname{cis}\left(\frac{\pi^2}{4}\right)$ **B** $7\operatorname{cis}\left(\frac{13\pi}{12}\right)$ **C** $10\operatorname{cis}\left(\frac{\pi}{4}\right)$ **D** $10\operatorname{cis}\left(\frac{\pi^2}{4}\right)$ **E** $10\operatorname{cis}\left(-\frac{11\pi}{12}\right)$

2 The complex number z shown in the diagram is best represented by



A $5\operatorname{cis}(0.93)$

B $5\operatorname{cis}(126.87)$

C $5\operatorname{cis}(2.21)$

D $25\operatorname{cis}(126.87)$

E $25\operatorname{cis}(2.21)$

3 If $(x + yi)^2 = -32i$ for real values of x and y, then

A $x = 4, y = 4$ **B** $x = -4, y = 4$

C $x = 4, y = -4$ **D** $x = 4, y = -4$ or $x = -4, y = 4$

E $x = 4, y = 4$ or $x = -4, y = -4$

4 If $u = 1 - i$, then $\frac{1}{3 - u}$ is equal to

A $\frac{2}{3} + \frac{1}{3}i$ **B** $\frac{2}{5} + \frac{1}{5}i$ **C** $\frac{2}{3} - \frac{1}{3}i$ **D** $-\frac{2}{5} + \frac{1}{5}i$ **E** $\frac{2}{5} - \frac{1}{5}i$

5 The linear factors of $z^2 + 6z + 10$ over $\mathbb{C}$ are

A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$ **C** $(z + 3 + i)(z - 3 + i)$

D $(z + 3 - i)(z + 3 + i)$ **E** $(z + 3 + i)(z - 3 - i)$

6 The solutions of the equation $z^3 + 8i = 0$ are

A $\sqrt{3} - i,\ -2i,\ 2i$ **B** $\sqrt{3} - i,\ -\sqrt{3} - i,\ 2i$ **C** $-\sqrt{3} - i,\ -2,\ -2i$

D $-\sqrt{3} - i,\ \sqrt{3} - i,\ -2i$ **E** $\sqrt{3} - i,\ -8i,\ 2i$

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7 $\frac{\sqrt{6}}{2}(1+i)$ is expressed in polar form as

A $\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **B** $\sqrt{3}\operatorname{cis}\left(-\frac{7\pi}{4}\right)$ **C** $-\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

D $-\sqrt{3}\operatorname{cis}\left(-\frac{7\pi}{4}\right)$ **E** $\sqrt{3}\operatorname{cis}\left(\frac{7\pi}{4}\right)$

8 If $z = 1 + i$ is one solution of an equation of the form $z^4 = a$, where $a \in \mathbb{C}$, then the other solutions are

A $-1,\ 1,\ 0$ **B** $-1,\ 1,\ 1-i$ **C** $-1+i,\ -1-i,\ 1-i$

D $-1+i,\ -1-i,\ 1$ **E** $-1+i,\ -1-i,\ -1$

9 The square roots of $-2-2\sqrt{3}i$ in polar form are

A $2\operatorname{cis}\left(-\frac{2\pi}{3}\right),\ 2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **B** $2\operatorname{cis}\left(-\frac{\pi}{3}\right),\ 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$ **C** $4\operatorname{cis}\left(-\frac{2\pi}{3}\right),\ 4\operatorname{cis}\left(\frac{\pi}{3}\right)$

D $4\operatorname{cis}\left(-\frac{\pi}{3}\right),\ 4\operatorname{cis}\left(\frac{2\pi}{3}\right)$ **E** $4\operatorname{cis}\left(-\frac{\pi}{3}\right),\ 4\operatorname{cis}\left(\frac{\pi}{3}\right)$

10 The zeroes of the polynomial $2x^2 + 6x + 7$ are α and β. The value of $|\alpha - \beta|$ is

A $\sqrt{5}$ **B** $2\sqrt{5}$ **C** $4\sqrt{5}$ **D** $\frac{\sqrt{10}}{2}$ **E** $\frac{\sqrt{5}}{10}$

Extended-response questions

1 Let $z = 4\operatorname{cis}\left(\frac{5\pi}{6}\right)$ and $w = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$.

a Find $|z^7|$ and $\operatorname{Arg}(z^7)$.

b Show z^7 on an Argand diagram.

c Express $\frac{z}{w}$ in the form $r\operatorname{cis}\theta$.

d Express z and w in Cartesian form, and hence express $\frac{z}{w}$ in Cartesian form.

e Use the results of **d** to find an exact value for $\tan\left(\frac{7\pi}{12}\right)$ in the form $a + \sqrt{b}$, where a and b are rational.

f Use the result of **e** to find the exact value of $\tan\left(\frac{7\pi}{6}\right)$.

2 Let $v = 2 + i$ and $P(z) = z^3 - 7z^2 + 17z - 15$.

a Show by substitution that $P(2+i) = 0$.

b Find the other two solutions of the equation $P(z) = 0$.

c Let $\boldsymbol{i}$ be the unit vector in the positive Re(z)-direction and let $\boldsymbol{j}$ be the unit vector in the positive Im(z)-direction.
Let A be the point on the Argand diagram corresponding to $v = 2 + i$.
Let B be the point on the Argand diagram corresponding to $1 - 2i$.
Show that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$.

d Find a polynomial with real coefficients and with roots 3, $1 - 2i$ and $2 + i$.

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3 **a** Find the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$, writing your solutions in Cartesian form.

b **i** Plot the two solutions from **a** on an Argand diagram.

ii Find the equation of the circle, with centre the origin, which passes through these two points.

iii Find the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.

iv Let $Q(z) = (z^2 + 4)(z^2 - 2\sqrt{3}z + 4)$. Find the polynomial $P(z)$ such that $Q(z)P(z) = z^6 + 64$ and explain the significance of the result.

4 **a** Express $-4\sqrt{3} - 4i$ in exact polar form.

b Find the cube roots of $-4\sqrt{3} - 4i$.

c Carefully plot the three cube roots of $-4\sqrt{3} - 4i$ on an Argand diagram.

d **i** Show that the cubic equation $z^3 - 3\sqrt{3}iz^2 - 9z + 3\sqrt{3}i = -4\sqrt{3} - 4i$ can be written in the form $(z - w)^3 = -4\sqrt{3} - 4i$, where w is a complex number.

ii Hence find the solutions of the equation $z^3 - 3\sqrt{3}iz^2 - 9z + (3\sqrt{3} + 4)i + 4\sqrt{3} = 0$, in exact Cartesian form.

5 The points X, Y and Z correspond to the numbers $4\sqrt{3} + 2i$, $5\sqrt{3} + i$ and $6\sqrt{3} + 4i$.

a Find the vector $\overrightarrow{XY}$ and the vector $\overrightarrow{XZ}$.

b Let z_1 and z_2 be the complex numbers corresponding to the vectors $\overrightarrow{XY}$ and $\overrightarrow{XZ}$. Find z_3 such that $z_2 = z_3 z_1$.

c By writing z_3 in modulus–argument form, show that XYZ is half an equilateral triangle XWZ and give the complex number to which W corresponds.

d The triangle XYZ is rotated through an angle of $\frac{\pi}{3}$ anticlockwise about Y. Find the new position of X.

6 **a** Sketch the region T in the complex plane which is obtained by reflecting

$$S = \{z : \text{Re}(z) \le 2\} \cap \{z : \text{Im}(z) < 2\} \cap \left\{z : \frac{\pi}{6} < \text{Arg}(z) < \frac{\pi}{3}\right\}$$

in the line defined by $|z + i| = |z - 1|$.

b Describe the region T by using set notation in a similar way to that used in **a** to describe S.

7 Consider the equation $x^2 + 4x - 1 + k(x^2 + 2x + 1) = 0$. Find the set of real values k, where $k \neq -1$, for which the two solutions of the equation are:

a real and distinct

b real and equal

c complex with positive real part and non-zero imaginary part.

8 **a** If $z = \cos\theta + i\sin\theta$, prove that $\dfrac{1 + z}{1 - z} = i\cot\left(\dfrac{\theta}{2}\right)$.

b On an Argand diagram, the points O, A, Z, P and Q represent the complex numbers 0, 1, z, $1 + z$ and $1 - z$ respectively. Show these points on a diagram.

c Prove that the magnitude of $\angle POQ$ is $\frac{\pi}{2}$. Find, in terms of θ, the ratio $\dfrac{|OP|}{|OQ|}$.

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9 A regular hexagon $LMNPQR$ has its centre at the origin O and its vertex L at the point $z = 4$.

a Indicate in a diagram the region in the hexagon in which the inequalities $|z| \geq 2$ and $-\frac{\pi}{3} \leq \text{Arg}\, z \leq \frac{\pi}{3}$ are satisfied.

b Find, in the form $|z - c| = a$, the equation of the circle through O, M and R.

c Find the complex numbers corresponding to the points N and Q.

d The hexagon is rotated clockwise about the origin by 45°. Express in the form $r \operatorname{cis} \theta$ the complex numbers corresponding to the new positions of N and Q.

10 **a** A complex number $z = a + bi$ is such that $|z| = 1$. Show that $\frac{1}{z} = \bar{z}$.

b Let $z_1 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $z_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. If $z_3 = \frac{1}{z_1} + \frac{1}{z_2}$, find z_3 in polar form.

c On a diagram, show the points z_1, z_2, z_3 and $z_4 = \frac{1}{z_3}$.

11 **a** Let $P(z) = z^3 + 3pz + q$. It is known that $P(z) = (z - k)^2(z - a)$.

i Show that $p = -k^2$. **ii** Find q in terms of k. **iii** Show that $4p^3 + q^2 = 0$.

b Let $h(z) = z^3 - 6iz + 4 - 4i$. It is known that $h(z) = (z - b)^2(z - c)$. Find the values of b and c.

12 **a** Let z be a complex number with $|z| = 6$. Let A be the point representing z. Let B be the point representing $(1 + i)z$.

i Find $|(1 + i)z|$.

ii Find $|(1 + i)z - z|$.

iii Prove that OAB is an isosceles right-angled triangle.

b Let z_1 and z_2 be non-zero complex numbers satisfying $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. If $z_1 = \alpha z_2$:

i Show that $\alpha = 1 + i$ or $\alpha = 1 - i$.

ii For each of these values of α, describe the geometric nature of the triangle whose vertices are the origin and the points representing z_1 and z_2.

13 **a** Let $z = -12 + 5i$. Find:

i $|z|$ **ii** $\text{Arg}(z)$ correct to two decimal places in degrees

b Let $w^2 = -12 + 5i$ and $\alpha = \text{Arg}(w^2)$.

i Write $\cos\alpha$ and $\sin\alpha$ in exact form.

ii Using the result $r^2(\cos(2\theta) + i\sin(2\theta)) = |w^2|(\cos\alpha + i\sin\alpha)$, write r, $\cos(2\theta)$ and $\sin(2\theta)$ in exact form.

iii Use the result of **ii** to find $\sin\theta$ and $\cos\theta$.

iv Find the two values of w.

c Use a Cartesian method to find w.

d Find the square roots of $12 + 5i$ and comment on their relationship with the square roots of $-12 + 5i$.

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Review

14 **a** Find the locus defined by $2z\bar{z} + 3z + 3\bar{z} - 10 = 0$.

b Find the locus defined by $2z\bar{z} + (3 + i)z + (3 - i)\bar{z} - 10 = 0$.

c Find the locus defined by $\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$, where α, β and γ are real.

d Find the locus defined by $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

15 **a** Expand $(\cos\theta + i\sin\theta)^5$.

b By De Moivre's theorem, we know that $(\operatorname{cis}\theta)^5 = \operatorname{cis}(5\theta)$. Use this result and the result of **a** to show that:

i $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

ii $\dfrac{\sin(5\theta)}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$ if $\sin\theta \neq 0$

16 **a** If $\bar{z}$ denotes the complex conjugate of the number $z = x + yi$, find the Cartesian equation of the line given by $(1 + i)z + (1 - i)\bar{z} = -2$.

Sketch on an Argand diagram the set $\left\{ z : (1 + i)z + (1 - i)\bar{z} = -2,\ \operatorname{Arg} z \leq \dfrac{\pi}{2} \right\}$.

b Let $S = \left\{ z : \left| z - (2\sqrt{2} + 2\sqrt{2}i) \right| \leq 2 \right\}$.

i Sketch S on an Argand diagram.

ii If z belongs to S, find the maximum and minimum values of $|z|$.

iii If z belongs to S, find the maximum and minimum values of $\operatorname{Arg}(z)$.

17 The roots of the polynomial $z^2 + 2z + 4$ are denoted by α and β.

a Find α and β in modulus–argument form.

b Show that $\alpha^3 = \beta^3$.

c Find a quadratic polynomial for which the roots are $\alpha + \beta$ and $\alpha - \beta$.

d Find the exact value of $\alpha\bar{\beta} + \beta\bar{\alpha}$.

18 **a** Let $w = 2\operatorname{cis}\theta$ and $z = w + \dfrac{1}{w}$.

i Find z in terms of θ.

ii Show that z lies on the ellipse with equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = \dfrac{1}{4}$.

iii Show that $|z - 2|^2 = \left(\dfrac{5}{2} - 2\cos\theta\right)^2$.

iv Show that $|z - 2| + |z + 2| = 5$.

b Let $w = 2i\operatorname{cis}\theta$ and $z = w - \dfrac{1}{w}$.

i Find z in terms of θ.

ii Show that z lies on the ellipse with equation $\dfrac{y^2}{25} + \dfrac{x^2}{9} = \dfrac{1}{4}$.

iii Show that $|z - 2i| + |z + 2i| = 5$.

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7

Revision of Chapters 1–6

7A Technology-free questions

Logic and proof

1 Let $n \in \mathbb{Z}$. Consider the statement: If $n^2 - 6n + 5$ is even, then n is odd.

a Write down the contrapositive of the statement.

b Prove this statement by proving the contrapositive.

c Write down the converse of the statement.

d Prove the converse.

2 **a** Use a proof by contradiction to show that there are no positive integers m and n such that $m^2 - n^2 = 1$.

b Find a counterexample to the claim that there are no positive integers m and n such that $2m^2 - mn = 1$.

3 **a** Prove by mathematical induction that $(1 + i)^{4n} = (-4)^n$, where n is a natural number.

b Now give a proof of the same result by using De Moivre's theorem.

4 Prove that $\sqrt{2} + \sqrt{6} > \sqrt{14}$.

5 Prove that $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ for all $n \in \mathbb{N}$.

6 **a** Using proof by contradiction, show that both $\log_6 3$ and $\log_6 2$ are irrational.

b Hence, show that the sum of two irrational numbers may be rational.

c Let a and b be irrational. Prove by contradiction that $a + b$ or $a - b$ is irrational.

7 Prove by mathematical induction that, for every positive integer n:

a $7^n + 2$ is divisible by 3

b $5^{2n} + 3n - 1$ is divisible by 9.

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8 Provide a counterexample to each of the following statements:

a If p and q are prime numbers, then $pq + 1$ is also a prime number.

b For all $a \in \mathbb{Z}$, if a^2 is divisible by 9, then a is divisible by 9.

9 Use induction to prove that

$$1 \times 4 + 2 \times 5 + \cdots + n(n + 3) = \frac{1}{3}n(n + 1)(n + 5)$$

for each natural number n.

10 Prove by mathematical induction that, for every positive integer n:

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n + 1)!} = 1 - \frac{1}{(n + 1)!}$$

11 Prove by mathematical induction that

$$\sum_{r=1}^{n} r(2r + 1) = \frac{1}{6}n(n + 1)(4n + 5)$$

for all positive integers n.

12 Use induction to prove that $5^{2n-1} + 2^{2n-1} - 7$ is divisible by 6, for all $n \in \mathbb{N}$.

13 **a** Let $n \in \mathbb{N}$. Prove that when n^2 is divided by 4, the remainder is either 0 or 1.

b Hence, prove that $\sqrt{4a + 3}$ is irrational, where a is a natural number.

Circular functions

14 **a** Given that $\sin\left(\frac{\pi}{12}\right) = \frac{-1 + \sqrt{3}}{2\sqrt{2}}$, find $\cos^2\left(\frac{\pi}{12}\right)$.

b Given that $\cos\left(\frac{\pi}{5}\right) = \frac{1}{4}(1 + \sqrt{5})$, find:

i $\sec\left(\frac{\pi}{5}\right)$ **ii** $\tan^2\left(\frac{\pi}{5}\right)$

15 Let $f(x) = 3\arcsin(2x + 1) + 4$. State the implied domain and range of f.

16 Find the points of intersection of the graph of $y = \sec^2\left(\frac{\pi x}{3}\right)$ with the line $y = 2$ for $0 < x < 6$.

17 Find all real solutions of $4\cos x = 2\cot x$.

18 **a** Solve the equation $\sin(4x) = \cos(2x)$ for $0 \le x \le \pi$.

b Consider the graphs of $f(x) = \operatorname{cosec}(4x)$, $0 \le x \le \pi$, and $g(x) = \sec(2x)$, $0 \le x \le \pi$.

i Find the coordinates of the points of intersection of these two graphs.

ii Sketch these graphs on the same set of axes.

c On another set of axes, sketch the graph of $h(x) = 2\arccos\left(\frac{x - 2}{2}\right)$, clearly labelling the endpoints.

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19 **a** Find the maximal domain and range of the function $y = a + b\arcsin(cx + d)$, where $a, b, c, d \in \mathbb{R}^+$.

b Sketch the graph of $y = 2\pi + 4\arcsin(3x + 1)$.

Vectors

20 Consider the vectors $\boldsymbol{a} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = m\boldsymbol{i} + n\boldsymbol{j}$, where $n \neq 0$. Given that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form a linearly dependent set of vectors, find the ratio $\dfrac{m}{n}$.

21 Points $A(2, 1, 2)$, $B(-3, 2, 5)$ and $C(4, 5, -2)$ are three vertices of a parallelogram. The fourth vertex of the parallelogram is at point D. Show that there are three possible locations for the point D, and find their coordinates.

22 Resolve the vector $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ into two components: one parallel to the vector $2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and the other perpendicular to it.

23 Let O be the origin and consider points $A(2, 2, 1)$ and $B(1, 2, 1)$.

a Find $\overrightarrow{AB}$. **b** Find $\cos(\angle AOB)$. **c** Find the area of triangle AOB.

24 Consider the vectors $\boldsymbol{a} = -2\boldsymbol{i} - 3\boldsymbol{j} + m\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \dfrac{3}{2}\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

a Find the values of m for which $|\boldsymbol{a}| = \sqrt{38}$.

b Find the value of m such that $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$.

c Find $-2\boldsymbol{b} + 3\boldsymbol{c}$.

d Hence find m such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

25 Points A and B have position vectors $\overrightarrow{OA} = \boldsymbol{i} + \sqrt{3}\boldsymbol{j}$ and $\overrightarrow{OB} = 3\boldsymbol{i} - 4\boldsymbol{k}$. Point P lies on AB with $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

a Show that $\overrightarrow{OP} = (1 + 2\lambda)\boldsymbol{i} + \sqrt{3}(1 - \lambda)\boldsymbol{j} - 4\lambda\boldsymbol{k}$.

b Hence find λ if OP is the bisector of $\angle AOB$.

26 **a** Find a unit vector perpendicular to the line $2y + 3x = 6$.

b Let A be the point $(2, -5)$ and let P be the point on the line $2y + 3x = 6$ such that AP is perpendicular to the line. Find:

i $\overrightarrow{AP}$ **ii** $|\overrightarrow{AP}|$

27 Points A, B and C are defined by position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

a Let $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}$ and $\boldsymbol{c} = -4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$. Show that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent by finding values of m and n such that $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$.

b If P is a point on AB such that $\overrightarrow{OP} = \lambda\boldsymbol{c}$, find the value of λ.

28 Consider the vectors $\boldsymbol{a} = 2m\boldsymbol{i} + 3m\boldsymbol{j} + 6m\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, where $m \in \mathbb{R}^+$.

a Given that $\boldsymbol{a}$ is a unit vector, find the exact value of m.

b Using the value of m from part **a**, determine:

i $\boldsymbol{a} \cdot \boldsymbol{b}$ **ii** $\boldsymbol{a} \times \boldsymbol{b}$

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29 A parallelogram $OABC$ has one vertex at the origin O and two other vertices at the points $A(0, 2, 7)$ and $B(0, 3, 9)$. Find the area of $OABC$.

30 Find the area of the triangle PQR with vertices:

a $P(2, 2, 0)$, $Q(0, 0, 0)$ and $R(4, 3, 1)$ **b** $P(3, 2, 0)$, $Q(1, 2, 0)$ and $R(2, 3, 1)$

31 Find a if the triangle with vertices $A(a, 1, 2)$, $B(1, 0, 1)$ and $C(0, 1, 1)$ has area 1.

Vector equations of lines and planes

32 For each of the following, find a vector equation of the line through the two points:

a $(0, 0, 0),\ (3, 0, 4)$ **b** $(0, 2, 1),\ (-1, 3, 4)$ **c** $(3, 2, 4),\ (0, 4, -2)$

33 For each of the following, find a vector equation of the plane that contains the three points:

a $(0, 0, 0),\ (1, 2, 3),\ (1, 3, 5)$

b $(2, -3, 5),\ (3, -2, 6),\ (1, -2, 4)$

c $(3, 2, 4),\ (0, 4, -2),\ (3, 6, 0)$

34 **a** Find the perpendicular distance between the parallel planes with equations $2x + 2y + z = 6$ and $2x + 2y + z = 10$.

b Find the perpendicular distance from the point $P(1, 0, 1)$ to the plane with equation $x + 2y + 3z = 6$.

c Find the shortest distance between two points on the lines ℓ_1 and ℓ_2 given by

$$\ell_1: \quad \boldsymbol{r} = t\boldsymbol{i} + (t + 1)\boldsymbol{j} + (t + 2)\boldsymbol{k}, \quad t \in \mathbb{R}$$
$$\ell_2: \quad \boldsymbol{r} = 2\boldsymbol{i} + s\boldsymbol{j} + (s + 1)\boldsymbol{k}, \quad s \in \mathbb{R}$$

d Find the distance from the point $P(1, 2, 3)$ to the line given by $\dfrac{x + 1}{2} = y = z - 1$.

35 Find the point of intersection of the lines ℓ_1 and ℓ_2 given by

$$\ell_1: \quad \boldsymbol{r} = 2t\boldsymbol{i} + (2 - 2t)\boldsymbol{j} + (3 - 4t)\boldsymbol{k}, \quad t \in \mathbb{R}$$
$$\ell_2: \quad \boldsymbol{r} = (3 + s)\boldsymbol{i} + (s - 1)\boldsymbol{j} + (4s - 3)\boldsymbol{k}, \quad s \in \mathbb{R}$$

36 Find the coordinates of the point where the line through $(0, 1, 0)$ and $(1, 0, 1)$ meets the plane with equation:

a $x + y + z = 1$ **b** $x + y + z = 3$ **c** $x - y + z = 1$

37 Find the length of the perpendicular from the point with coordinates $(4, 0, 1)$ to the plane with equation $3x + 6y + 2z = -7$.

38 **a** Find the value of a for which the three planes $2x - y + 5z = 7$, $5x + 3y - z = 4$ and $3x + 4y - 6z = a$ intersect in a line.

b Find a vector equation of this line.

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39 Plane Π_1 has equation $\boldsymbol{r} \cdot (2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}) = 1$. Plane Π_2 has equation $\boldsymbol{r} \cdot (-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = 2$. These two planes intersect in a line, ℓ.

a Find the cosine of the acute angle between the planes Π_1 and Π_2.

b Find a vector equation of the line ℓ.

40 Consider the points $A(4, 1, -1)$, $B(0, 3, 3)$, $C(-4, -1, 1)$ and $D(0, -3, -3)$.

a Show that these points all lie on the plane with equation $x - 2y + 2z = 0$.

b Show that $ABCD$ is a square.

c Write a vector equation of the line through the point $P(0, 8, 5)$ that is perpendicular to the plane.

41 Find the cosine of the acute angle between each of the following pairs of planes:

a $2x + 3y - 2z = 0, \quad x - y - z = 4$ **b** $4x + 3y + 2z = 5, \quad 2x - 4y + 3z = 6$

42 Describe the line that passes through the points $A(3, 5, 9)$ and $B(1, 9, 10)$ using:

a a vector equation **b** Cartesian equations **c** parametric equations.

Complex numbers

43 Find all solutions of $z^4 - z^2 - 12 = 0$ for $z \in \mathbb{C}$.

44 Consider $z = \dfrac{\sqrt{3} - i}{1 - i}$. Find Arg z.

45 Let $P(z) = z^5 - 6z^3 - 2z^2 + 17z - 10$. Given that $P(1) = P(2) = 0$, solve the equation $P(z) = 0$ for $z \in \mathbb{C}$.

46 **a** Solve the equation $z^3 - 2z^2 + 2z - 1 = 0$ for $z \in \mathbb{C}$.

b Write the solutions in polar form.

c Show the solutions on an Argand diagram.

47 The point $(-1, 3)$ is rotated about the origin by angle $\dfrac{\pi}{4}$ anticlockwise. By multiplying two complex numbers, find the image of the point.

48 Simplify $\dfrac{\cos(2\theta) + i\sin(2\theta)}{\cos(3\theta) + i\sin(3\theta)}$, writing your answer in Cartesian form.

49 Let $z = \sqrt{3} + i$. Plot z, z^2 and z^3 on an Argand diagram.

50 **a** Show that $z - 1 - i$ is a factor of $f(z) = z^3 - (5 + i)z^2 + (17 + 4i)z - 13 - 13i$.

b Hence factorise $f(z)$.

51 Let $f(z) = z^2 + aiz + b$, where a and b are real numbers.

a Use the quadratic formula to show that the equation $f(z) = 0$ has imaginary solutions if and only if $b \geq -\dfrac{a^2}{4}$. (Imaginary solutions have no real part.)

b Hence solve each of the following:

i $z^2 + 2iz + 1 = 0$ **ii** $z^2 - 2iz - 1 = 0$ **iii** $z^2 + 2iz - 2 = 0$

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52 **a** If the equation $z^3 + az^2 + bz + c = 0$ has solutions $-1 + i$, -1 and $-1 - i$, find the values of a, b and c.

b If $\sqrt{3} + i$ and $-2i$ are two of the solutions to the equation $z^3 = w$, where w is a complex number, find the third solution.

53 Solve the equation $z^5 = 1 + i$ for z, giving your solutions in polar form. Illustrate the solutions on an Argand diagram.

54 Consider the complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = -1 - i$.

a Express z_1 and z_2 in polar form.

b Find $\dfrac{z_1}{z_2}$ in polar form.

c Find the complex conjugate of $\dfrac{z_1}{z_2}$ in polar form.

d On a single Argand diagram, sketch the graphs of:

i $|z - z_1| = 2$ **ii** $\text{Arg}(z - z_2) = \dfrac{\pi}{4}$

55 Let $z_1 = 1 + \sqrt{3}i$.

a **i** Write z_1 in polar form.

ii Write z_1^n in polar form, where n is an integer.

iii Find the integer values of n for which z_1^n is real.

iv Find the integer values of n for which z_1^n is imaginary.

b Express z_1^2 and z_1^3 in Cartesian form.

c Given that $z_1 = 1 + \sqrt{3}i$ is a solution of the equation $2z^3 + az^2 + bz + 20 = 0$, find the values of the real numbers a and b.

d For these values of a and b, solve the equation $2z^3 + az^2 + bz + 20 = 0$.

56 The vertices A, B, C and D of a square, taken anticlockwise, are drawn on an Argand diagram. The points A and B are $-1 + 4i$ and -3 respectively.

a Find the complex numbers corresponding to C and D.

b Find the complex number corresponding to the centre of the square.

57 Let $z, w \in \mathbb{C}$. Prove that $|z + w|^2 - |z - \overline{w}|^2 = 4\,\text{Re}(z)\,\text{Re}(w)$.

58 Let ω be a cube root of unity with $\omega \neq 1$, and let n be a natural number.

a Prove that if n is a multiple of 3, then $1 + \omega^n + \omega^{2n} = 3$.

b Prove that if n is not a multiple of 3, then $1 + \omega^n + \omega^{2n} = 0$.

59 A circle on an Argand diagram has equation $|z - c| = r$. The circle passes through the points $2 + i$, $2 - i$ and i. Find the values of c and r.

60 The equation $z^3 - 5z^2 + 16z + k = 0$ has a solution $z = 1 + ai$, where $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$. Find the values of a and k.

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7B Multiple-choice questions

1 Consider the curve defined by the parametric equations $x = t^2 + 2$ and $y = 6 - t^3$ for $t \in \mathbb{R}$. The point on the curve where $t = 2$ is

A $(10, 0)$ **B** $(6, 14)$ **C** $(6, -2)$ **D** $(10, -6)$ **E** $(10, -2)$

2 A curve is defined parametrically by the equations $x = 2\cos(t)$ and $y = 2\cos(2t)$. The Cartesian equation of the curve is

A $y = 2 + x^2$ **B** $y = x^2 - 2$ **C** $y = 2x$ **D** $y = x$ **E** $y = 2x^2 - 1$

3 A curve is defined parametrically by the equations $x = 2\sec t$ and $y = 3\tan t$. The point on the curve where $t = -\frac{\pi}{3}$ is

A $(4, 3\sqrt{3})$ **B** $(4, -3\sqrt{3})$ **C** $(3\sqrt{3}, -4)$ **D** $(-4, -3\sqrt{3})$ **E** $\left(4, -\frac{\sqrt{3}}{3}\right)$

4 A curve is defined parametrically by the equations $x = 2 \times 3^t + 1$ and $y = 2 \times 3^{-2t}$. The Cartesian equation of the curve is

A $y = \frac{x-1}{4}$ **B** $y = 1 - x$ **C** $y = \frac{4}{x-1}$ **D** $y = \frac{8}{(x-1)^2}$ **E** $y = \frac{8}{x-1}$

5 A curve is defined parametrically by the equations $x = t - 3$ and $y = t^2 + 5$ for $t \in \mathbb{R}$. The Cartesian equation of the curve is

A $y = x^2 - 14$ **B** $y = x^2 + 14$ **C** $y = x^2 + 6x + 14$
D $y = x^2 - 6x + 14$ **E** $y = x^2 - 6x - 14$

6 The graph of a function is described by the parametric equations $x = \sqrt{t}$ and $y = 4t + 2$ for $t \in [0, 4]$. The domain and range of this function are

A Domain $= [0, 2]$; Range $= [2, 18]$ **B** Domain $= [0, 4]$; Range $= [2, 18]$
C Domain $= [0, 2]$; Range $= [2, 10]$ **D** Domain $= [0, 4]$; Range $= [2, 10]$
E Domain $= [0, 4]$; Range $= [0, 18]$

7 Let $n \in \mathbb{Z}$. Consider the statement: If n is even, then $3n + 1$ is odd. The contrapositive of this statement is

A If $3n + 1$ is even, then n is odd. **B** If $3n + 1$ is even, then n is even.
C If $3n + 1$ is odd, then n is odd. **D** If $3n + 1$ is odd, then n is even.
E If n is odd, then $3n + 1$ is even.

8 Let $n \in \mathbb{Z}$. Consider the statement: If n is even, then $3n + 1$ is odd. The negation of this statement is

A n is even and $3n + 1$ is odd. **B** n is even or $3n + 1$ is odd.
C n is odd and $3n + 1$ is even. **D** n is odd or $3n + 1$ is even.
E n is even and $3n + 1$ is even.

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9 Let $x \in \mathbb{R}$. Consider the statement: If $x < -2$ or $x > 2$, then $x^2 > 4$.
The contrapositive of this statement is

A If $-2 \le x \le 2$, then $x^2 \le 4$.
B If $x^2 \le 4$, then $-2 \le x \le 2$.
C If $x^2 \le 4$, then $x \ge -2$ or $x \le 2$.
D If $x^2 < 4$, then $-2 < x < 2$.
E If $x^2 > 4$, then $x < -2$ or $x > 2$.

10 Let $x \in \mathbb{R}$. Consider the statement: If $x < -2$ or $x > 2$, then $x^2 > 4$.
The converse of this statement is

A If $-2 \le x \le 2$, then $x^2 \le 4$.
B If $x^2 \le 4$, then $-2 \le x \le 2$.
C If $x^2 \le 4$, then $x \ge -2$ or $x \le 2$.
D If $x^2 < 4$, then $-2 < x < 2$.
E If $x^2 > 4$, then $x < -2$ or $x > 2$.

11 Consider the claim: For all $x, y \in \mathbb{R}$, if $x^2 \le y^2$, then $x \le y$.
Which one of the following is a counterexample that disproves this claim?

A $x = 1,\ y = 2$
B $x = -1,\ y = 2$
C $x = 2,\ y = 1$
D $x = 1,\ y = -2$
E $x = 2,\ y = -1$

12 For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
The statement $P(1)$ is

A $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
B $n = 1$
C $0^2 = \dfrac{0(0+1)(2 \times 0 + 1)}{6}$
D $1^2 + 2^2 + \cdots + n^2 = \dfrac{1}{6}$
E $1^2 = \dfrac{1(1+1)(2 \times 1 + 1)}{6}$

13 For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $n^2 - n + 41$ is a prime number.
Which one of the following is correct?

A $P(1)$ is not true
B $P(2)$ is not true
C $P(3)$ is not true
D $P(5)$ is not true
E $P(41)$ is not true

14 The hyperbola $\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{16} = 1$ has asymptotes with the equations

A $y = \dfrac{3}{4}x + \dfrac{8}{3}$ and $y = \dfrac{3}{4}x + \dfrac{2}{3}$
B $y = \dfrac{3}{4}x + \dfrac{10}{3}$ and $y = \dfrac{3}{4}x + \dfrac{2}{3}$
C $y = \dfrac{4}{3}x + \dfrac{10}{3}$ and $y = -\dfrac{4}{3}x + \dfrac{2}{3}$
D $y = \dfrac{4}{3}x + \dfrac{10}{3}$ and $y = -\dfrac{4}{3}x + \dfrac{10}{3}$
E $y = \dfrac{3}{4}x - \dfrac{10}{3}$ and $y = -\dfrac{3}{4}x + \dfrac{2}{3}$

15 A circle has a diameter with endpoints at $(4, -2)$ and $(-2, -2)$. The equation of the circle is

A $(x-1)^2 + (y-2)^2 = 3$
B $(x-1)^2 + (y+2)^2 = 3$
C $(x+1)^2 + (y-2)^2 = 6$
D $(x-1)^2 + (y+2)^2 = 9$
E $(x-1)^2 + (y+2)^2 = 6$

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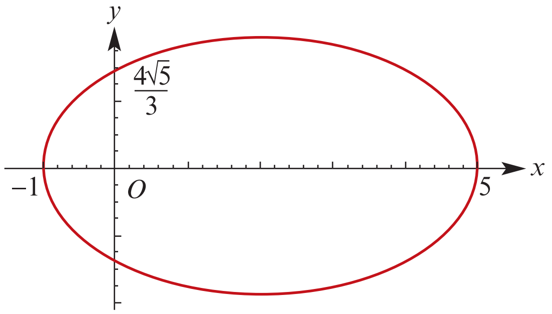
16 The ellipse shown has its centre on the x-axis. Its equation is



A $\frac{(x+2)^2}{9} + \frac{y^2}{16} = 1$

B $\frac{(x-2)^2}{9} + \frac{y^2}{16} = 1$

C $\frac{(x+2)^2}{3} + \frac{y^2}{4} = 1$

D $\frac{(x-2)^2}{3} + \frac{y^2}{4} = 1$

E $\frac{(x-2)^2}{9} - \frac{y^2}{16} = 1$

17 The ellipse with equation $\frac{x^2}{9} + \frac{y^2}{25} = 1$ has x-axis intercepts with coordinates

A $(-3, -5)$ and $(3, 5)$ **B** $(-5, -3)$ and $(5, 3)$ **C** $(0, -3)$ and $(0, 3)$
D $(-3, 0)$ and $(3, 0)$ **E** $(3, 0)$ and $(5, 0)$

18 The circle defined by the equation $x^2 + y^2 - 6x + 8y = 0$ has centre

A $(2, 4)$ **B** $(-5, 9)$ **C** $(4, -3)$ **D** $(3, -4)$ **E** $(6, -8)$

19 If the line $x = k$ is a tangent to the circle with equation $(x-1)^2 + (y+2)^2 = 1$, then k is equal to

A 1 or −2 **B** 1 or 3 **C** −1 or −3 **D** 0 or −2 **E** 0 or 2

20 The curve with equation $x^2 - 2x = y^2$ is

A an ellipse with centre $(1, 0)$ **B** a hyperbola with centre $(1, 0)$
C a circle with centre $(1, 0)$ **D** an ellipse with centre $(-1, 0)$
E a hyperbola with centre $(-1, 0)$

21 If $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{c} = -3\boldsymbol{j} + 4\boldsymbol{k}$, then $\boldsymbol{a} - 2\boldsymbol{b} - \boldsymbol{c}$ equals

A $3\boldsymbol{i} + 10\boldsymbol{j} - 12\boldsymbol{k}$ **B** $-3\boldsymbol{i} + 7\boldsymbol{j} - 12\boldsymbol{k}$ **C** $4\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$
D $-4\boldsymbol{j} + 4\boldsymbol{k}$ **E** $2\boldsymbol{j} - 4\boldsymbol{k}$

22 A vector of magnitude 6 and with direction opposite to $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ is

A $6\boldsymbol{i} - 12\boldsymbol{j} + 12\boldsymbol{k}$ **B** $-6\boldsymbol{i} + 12\boldsymbol{j} - 2\boldsymbol{k}$ **C** $-3\boldsymbol{i} + 6\boldsymbol{j} - 6\boldsymbol{k}$
D $-2\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$ **E** $\frac{2}{3}\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$

23 If $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$, then the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

A $7(-2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k})$ **B** $\frac{1}{7}(2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k})$ **C** $-\frac{1}{7}(2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k})$
D $-\frac{7}{11}(2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k})$ **E** $-\frac{19}{49}(-2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k})$

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24 If $\boldsymbol{a} = 3\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$, then a vector which is not perpendicular to $\boldsymbol{a}$ is

A $\frac{1}{35}(3\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k})$ **B** $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ **C** $\boldsymbol{i} - \boldsymbol{j} - 8\boldsymbol{k}$

D $-3\boldsymbol{i} + 5\boldsymbol{j} + 34\boldsymbol{k}$ **E** $\frac{1}{9}(-3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$

25 The magnitude of vector $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ is

A 3 **B** $\sqrt{17}$ **C** 35 **D** 17 **E** $\sqrt{35}$

26 If $\boldsymbol{u} = 2\boldsymbol{i} - \sqrt{2}\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} + \sqrt{2}\boldsymbol{j} - \boldsymbol{k}$, then the angle between the direction of $\boldsymbol{u}$ and $\boldsymbol{v}$, correct to two decimal places, is

A 92.05° **B** 87.95° **C** 79.11° **D** 100.89° **E** 180°

27 Let $\boldsymbol{u} = 2\boldsymbol{i} - a\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} - b\boldsymbol{k}$. Then $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular to each other when

A $a = 2$ and $b = -1$ **B** $a = -2$ and $b = 10$ **C** $a = \frac{1}{2}$ and $b = -5$

D $a = 0$ and $b = 0$ **E** $a = -1$ and $b = 5$

28 Let $\boldsymbol{u} = \boldsymbol{i} + a\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{v} = b\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$. Then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel to each other when

A $a = -2$ and $b = 1$ **B** $a = -\frac{8}{3}$ and $b = -\frac{3}{4}$ **C** $a = -\frac{3}{2}$ and $b = -\frac{3}{4}$

D $a = -\frac{8}{3}$ and $b = -\frac{4}{3}$ **E** none of these

29 Let $\boldsymbol{a} = \boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$. Then the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is

A $-\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k}$ **B** $\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ **C** $-5\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}$

D $5\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ **E** $\frac{5}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{5}{3}\boldsymbol{k}$

30 If points A, B and C are such that $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, which one of the following statements must be true?

A Either $\overrightarrow{AB}$ or $\overrightarrow{BC}$ is a zero vector.

B Vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same magnitude.

C The vector resolute of $\overrightarrow{AC}$ in the direction of $\overrightarrow{AB}$ is $\overrightarrow{AB}$.

D The vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$ is $\overrightarrow{AC}$.

E Points A, B and C are collinear.

31 If $\boldsymbol{u} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 4\boldsymbol{i} + 12\boldsymbol{j} - 3\boldsymbol{k}$, then $\boldsymbol{u} \cdot \boldsymbol{v}$ equals

A $4\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}$ **B** $5\boldsymbol{i} + 11\boldsymbol{j} - 4\boldsymbol{k}$ **C** -5

D 19 **E** $\frac{5}{13}$

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32 If $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$, then the scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

A $\frac{10}{49}(6\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$ **B** $\frac{10}{7}$ **C** $2\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} - 2\boldsymbol{k}$

D $\frac{10}{49}$ **E** $\frac{\sqrt{10}}{7}$

33 Let $\boldsymbol{a} = 3\boldsymbol{i} - 5\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} - 4\boldsymbol{k}$. The unit vector in the direction of $\boldsymbol{a} - \boldsymbol{b}$ is

A $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ **B** $\frac{1}{\sqrt{65}}(5\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k})$ **C** $\frac{1}{3}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

D $\frac{1}{9}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$ **E** $\frac{1}{3}(-\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$

34 If the points P, Q and R are collinear with $\overrightarrow{OP} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\overrightarrow{OQ} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{OR} = 2\boldsymbol{i} + p\boldsymbol{j} + q\boldsymbol{k}$, then

A $p = -3$ and $q = 2$ **B** $p = -\frac{7}{2}$ and $q = 2$ **C** $p = -\frac{1}{2}$ and $q = 0$

D $p = 3$ and $q = -2$ **E** $p = -\frac{1}{2}$ and $q = 2$

35 If $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{4}{3}$, where both α and β are acute, then $\sin(\alpha + \beta)$ equals

A $\frac{7}{5}$ **B** $\frac{24}{25}$ **C** $\frac{7}{25}$ **D** 0 **E** 1

36 If $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{x} = \boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{x} = s\boldsymbol{a} + t\boldsymbol{b}$, then the scalars s and t are given by

A $s = -1$ and $t = -1$ **B** $s = -1$ and $t = 1$ **C** $s = 1$ and $t = -1$

D $s = 1$ and $t = 1$ **E** $s = \sqrt{5}$ and $t = 5$

37 Given that $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and the points O, P and Q are not collinear, which one of the following points, whose position vectors are given, is not collinear with P and Q?

A $\frac{1}{2}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$ **B** $3\boldsymbol{p} - 2\boldsymbol{q}$ **C** $\boldsymbol{p} - \boldsymbol{q}$ **D** $\frac{1}{3}\boldsymbol{p} + \frac{2}{3}\boldsymbol{q}$ **E** $2\boldsymbol{p} - \boldsymbol{q}$

38 Assume that $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$, $t \in \mathbb{R}$, is a vector equation of a line that does not pass through the origin. Which one of the following is *not* the position vector of a point on the line?

A $\boldsymbol{a}$ **B** $\boldsymbol{b}$ **C** $\boldsymbol{a} + \boldsymbol{b}$ **D** $\boldsymbol{a} - \boldsymbol{b}$ **E** $\boldsymbol{a} - 7\boldsymbol{b}$

39 The two lines given by the vector equations $\boldsymbol{r} = 9\boldsymbol{i} - 2\boldsymbol{j} + \lambda(3\boldsymbol{i} - \boldsymbol{j})$, for $\lambda \in \mathbb{R}$, and $\boldsymbol{s} = 3\boldsymbol{i} - 2\boldsymbol{j} + \mu(3\boldsymbol{i} + \boldsymbol{j})$, for $\mu \in \mathbb{R}$, intersect at the point with coordinates

A $(12, -3)$ **B** $(6, -1)$ **C** $(0, -3)$ **D** $(3, 0)$ **E** $(0, 1)$

40 The plane with vector equation $\boldsymbol{r} \cdot (\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 2$ contains the point

A $(1, -1, 1)$ **B** $(-1, 1, 0)$ **C** $(0, 1, 1)$ **D** $(2, 0, 0)$ **E** $(0, 0, 0)$

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41 For the straight line ℓ given by the vector equation

$$\boldsymbol{r} = -\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}), \quad t \in \mathbb{R}$$

which one of the following is true?

A The line ℓ is parallel to the vector $-2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$.
B The line ℓ is perpendicular to the vector $\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$.
C The line ℓ passes through the point $(-2, -3, 6)$.
D The line ℓ passes through the origin.
E The line ℓ lies in the plane with equation $x + y - z = -1$.

42 Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be non-zero vectors in three dimensions. If $\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u} \times \boldsymbol{v}|$, then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is

A 0 **B** $\frac{\pi}{6}$ **C** $\frac{\pi}{4}$ **D** $\frac{\pi}{3}$ **E** $\frac{\pi}{2}$

43 $\cos^2\theta + 3\sin^2\theta$ equals

A $2 + \cos\theta$ **B** $3 - 2\cos(2\theta)$ **C** $2 - \cos\theta$
D $2\cos(2\theta) - 1$ **E** none of these

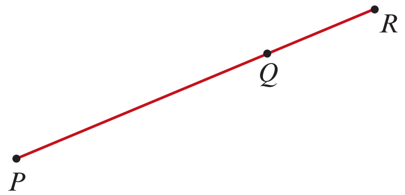
44 Assume that the two vector equations $\boldsymbol{r}_1 = \boldsymbol{a}_1 + t\boldsymbol{d}_1$, $t \in \mathbb{R}$, and $\boldsymbol{r}_2 = \boldsymbol{a}_2 + s\boldsymbol{d}_2$, $s \in \mathbb{R}$, represent the same line ℓ, where ℓ does not pass through the origin. Which one of the following is *not* true?

A $\boldsymbol{d}_1 = k\boldsymbol{d}_2$ for some $k \in \mathbb{R}$
B $\boldsymbol{a}_2 = \boldsymbol{a}_1 + t\boldsymbol{d}_1$ for some $t \in \mathbb{R}$
C $\boldsymbol{d}_2 = \boldsymbol{a}_1 + t\boldsymbol{d}_1$ for some $t \in \mathbb{R}$
D $\boldsymbol{a}_2 - \boldsymbol{a}_1 = k\boldsymbol{d}_2$ for some $k \in \mathbb{R}$
E $\boldsymbol{a}_2 + \boldsymbol{d}_1$ is the position vector of a point on the line ℓ

45 $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ equals

A $-\frac{5\pi}{6}$ **B** $-\frac{\pi}{2}$ **C** $-\frac{\pi}{6}$ **D** $\frac{\pi}{2}$ **E** $\frac{7\pi}{6}$

46 PQR is a straight line and $PQ = 2QR$.
If $\overrightarrow{OQ} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{OR} = \boldsymbol{i} + 3\boldsymbol{j}$, then $\overrightarrow{OP}$ is equal to



A $-\boldsymbol{i} + 8\boldsymbol{j}$ **B** $7\boldsymbol{i} - 12\boldsymbol{j}$ **C** $4\boldsymbol{i} - 10\boldsymbol{j}$
D $-4\boldsymbol{i} + 10\boldsymbol{j}$ **E** $-7\boldsymbol{i} + 12\boldsymbol{j}$

47 If $\overrightarrow{OP} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{PQ} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, then $|\overrightarrow{OQ}|$ equals

A $2\sqrt{5}$ **B** $3\sqrt{2}$ **C** 6 **D** 9 **E** 4

48 If $z_1 = 2 - i$ and $z_2 = 3 + 4i$, then $\left|\frac{z_2}{z_1}\right|^2$ equals

A $\sqrt{5}$ **B** 5 **C** $\frac{125}{9}$ **D** $\left(\frac{2 + 11i}{5}\right)^2$ **E** $\left(\frac{10 + 5i}{5}\right)^2$

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49 If $z = -1 - \sqrt{3}i$, then $\text{Arg}\, z$ equals

A $-\frac{2\pi}{3}$ **B** $-\frac{5\pi}{6}$ **C** $\frac{2\pi}{3}$ **D** $\frac{5\pi}{6}$ **E** $-\frac{\pi}{3}$

50 The vectors $p\boldsymbol{i} + 2\boldsymbol{j} - 3p\boldsymbol{k}$ and $p\boldsymbol{i} + \boldsymbol{k}$ are perpendicular when p is equal to

A 0 only **B** 3 only **C** 0 or 3 **D** 1 or 2 **E** 1 only

51 One solution of the equation $z^3 - 5z^2 + 17z - 13 = 0$ is $2 + 3i$. The other solutions are

A $-2 - 3i$ and 1 **B** $2 - 3i$ and 1 **C** $-2 + 3i$ and -1

D $2 - 3i$ and -1 **E** $-2 + 3i$ and 1

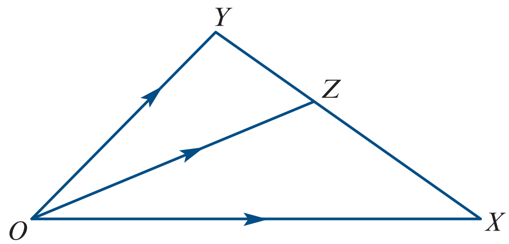
52 The value of $\dfrac{(\cos 60^\circ + i \sin 60^\circ)^4}{(\cos 30^\circ + i \sin 30^\circ)^2}$ is

A -1 **B** i **C** $-i$ **D** $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **E** $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

53 If $3\overrightarrow{OX} + 4\overrightarrow{OY} = 7\overrightarrow{OZ}$, then $\dfrac{XZ}{ZY}$ equals

A $\frac{3}{5}$ **B** $\frac{3}{4}$ **C** 1

D $\frac{4}{3}$ **E** $\frac{5}{3}$



54 $\cos\left(\tan^{-1}(1) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$ equals

A $\frac{\pi}{2}$ **B** 1 **C** 0 **D** $-\frac{1}{\sqrt{2}}$ **E** $-\frac{\sqrt{3}}{2}$

55 If $x + yi = \dfrac{1}{3 + 4i}$, where x and y are real, then

A $x = \frac{3}{25}$ and $y = -\frac{4}{25}$ **B** $x = \frac{3}{25}$ and $y = \frac{4}{25}$ **C** $x = -\frac{3}{7}$ and $y = \frac{4}{7}$

D $x = \frac{1}{3}$ and $y = \frac{1}{4}$ **E** $x = 3$ and $y = -4$

56 Let $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + p\boldsymbol{j} + \boldsymbol{k}$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then p equals

A $-\frac{7}{3}$ **B** -2 **C** $-\frac{5}{3}$ **D** 2 **E** $\frac{7}{3}$

57 Let $z = \dfrac{1}{1 - i}$. If $r = |z|$ and $\theta = \text{Arg}\, z$, then

A $r = 2$ and $\theta = \frac{\pi}{4}$ **B** $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ **C** $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$

D $r = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$ **E** $r = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$

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Revision

58 The maximal domain of $f(x) = \sin^{-1}(2x - 1)$ is

A $[-1, 1]$ **B** $(-1, 1)$ **C** $(0, 1)$ **D** $[0, 1]$ **E** $[-1, 0]$

59 If $u = 3\,\mathrm{cis}\left(\frac{\pi}{4}\right)$ and $v = 2\,\mathrm{cis}\left(\frac{\pi}{2}\right)$, then uv is equal to

A $\mathrm{cis}\left(\frac{7\pi}{4}\right)$ **B** $6\,\mathrm{cis}\left(\frac{\pi^2}{8}\right)$ **C** $6\,\mathrm{cis}^2\left(\frac{\pi^2}{8}\right)$ **D** $5\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$ **E** $6\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$

60 The exact value of $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ is

A $\frac{\sqrt{3}}{2}$ **B** $-\frac{1}{2}$ **C** 1 **D** $-\frac{\sqrt{3}}{2}$ **E** $\frac{1}{\sqrt{5}}$

61 If $-1 < x < 1$, then $\tan(\arcsin(x))$ is equal to

A $\frac{x}{\sqrt{1 - x^2}}$ **B** $-\frac{x}{\sqrt{1 - x^2}}$ **C** $\frac{\sqrt{1 - x^2}}{x}$ **D** $-\frac{\sqrt{1 - x^2}}{x}$ **E** $\sqrt{1 - x^2}$

62 The modulus of $12 - 5i$ is

A 119 **B** 7 **C** 13 **D** $\sqrt{119}$ **E** $\sqrt{7}$

63 When $\sqrt{3} - i$ is divided by $-1 - i$, the modulus and the principal argument of the quotient are

A $2\sqrt{2}$ and $\frac{7\pi}{12}$ **B** $\sqrt{2}$ and $-\frac{11\pi}{12}$ **C** $\sqrt{2}$ and $\frac{7\pi}{12}$

D $2\sqrt{2}$ and $-\frac{11\pi}{12}$ **E** $\sqrt{2}$ and $\frac{11\pi}{12}$

64 Let z be a complex number such that $|z + 4i| = 3$. Then the smallest and largest possible values of $|z + 3|$ are

A 2 and 8 **B** 3 and 4 **C** 4 and 8 **D** 5 and 8 **E** 4 and 10

65 Let $P(z)$ be a quadratic polynomial with real coefficients. Which one of the following is *not* possible?

A $P(z)$ has two real roots **B** $P(z)$ has two imaginary roots

C $P(z)$ has one real and one non-real root **D** $P(z)$ has two non-real roots

E $P(z)$ has a repeated real root

66 The product of the complex numbers $\frac{1 - i}{\sqrt{2}}$ and $\frac{\sqrt{3} + i}{2}$ has argument

A $-\frac{5\pi}{12}$ **B** $-\frac{\pi}{12}$ **C** $\frac{\pi}{12}$ **D** $\frac{5\pi}{12}$ **E** none of these

67 If $\tan\theta = \frac{1}{3}$, then $\tan(2\theta)$ equals

A $\frac{3}{5}$ **B** $\frac{2}{3}$ **C** $\frac{3}{4}$ **D** $\frac{4}{5}$ **E** $\frac{4}{3}$

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68 Which one of the following five expressions is not identical to any of the others?

A $\cos^4\theta - \sin^4\theta$ **B** $1 + \cos\theta$ **C** $\cos(2\theta)$ **D** $2\cos^2\left(\frac{\theta}{2}\right)$ **E** $1 - \cos\theta$

69 The modulus of $1 + \cos(2\theta) + i\sin(2\theta)$, where $0 < \theta < \frac{\pi}{2}$, is

A $4\cos^2\theta$ **B** $4\sin^2\theta$ **C** $2\cos\theta$ **D** $2\sin\theta$ **E** none of these

70 An expression for an argument of $1 + \cos\theta + i\sin\theta$ is

A $2\cos\left(\frac{\theta}{2}\right)$ **B** $2\sin\left(\frac{\theta}{2}\right)$ **C** θ **D** $\frac{\theta}{2}$ **E** $\frac{\pi}{2} - \frac{\theta}{2}$

71 A quadratic equation with solutions $2 + 3i$ and $2 - 3i$ is

A $x^2 + 4x + 13 = 0$ **B** $x^2 - 4x + 13 = 0$ **C** $x^2 + 4x - 13 = 0$
D $x^2 + 4x - 5 = 0$ **E** $x^2 - 4x - 5 = 0$

72 If $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1} x$, then x is

A 1 **B** $\frac{5}{6}$ **C** $\frac{5}{7}$ **D** $\frac{1}{5}$ **E** $\frac{1}{7}$

73 Which one of the following five expressions is not identical to any of the others?

A $\tan\theta + \cot\theta$ **B** $\operatorname{cosec}^2\theta - \cot^2\theta$ **C** 1
D $\operatorname{cosec}\theta\cot\theta$ **E** $2\operatorname{cosec}(2\theta)$

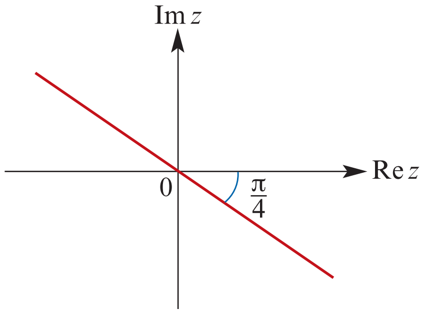
74 The subset of the complex plane defined by the equation $|z - 2| - |z + 2| = 0$ is

A a circle **B** an ellipse **C** a straight line
D the empty set **E** a hyperbola

75 The subset of the complex plane defined by the equation $|z - (2 - i)| = 6$ is

A a circle with centre at $-2 + i$ and radius 6
B a circle with centre at $2 - i$ and radius 6
C a circle with centre at $2 - i$ and radius 36
D a circle with centre at $-2 + i$ and radius 36
E a circle with centre at $-2 - i$ and radius 36

76 The line shown can be represented by the set



A $\left\{z : \operatorname{Arg} z = \frac{\pi}{4}\right\}$

B $\left\{z : \operatorname{Arg} z = -\frac{\pi}{4}\right\}$

C $\left\{z : \operatorname{Arg} z = \frac{7\pi}{4}\right\}$

D $\{z : \operatorname{Im} z + \operatorname{Re} z = 0\}$

E $\{z : \operatorname{Im} z - \operatorname{Re} z = 0\}$

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77 The subset of the complex plane defined by the equation $|z-2|-|z-2i|=0$ is

A a circle **B** an ellipse **C** a straight line
D the empty set **E** a hyperbola

78 Which one of the following subsets of the complex plane is *not* a circle?

A $\{z : |z| = 2\}$ **B** $\{z : |z-i| = 2\}$ **C** $\{z : z\bar{z} + 2\,\mathrm{Re}(iz) = 0\}$
D $\{z : |z-1| = 2\}$ **E** $\{z : |z| = 2i\}$

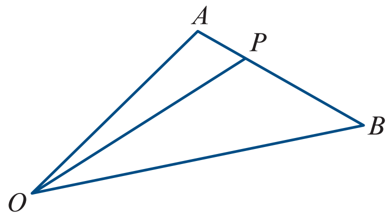
79 Which one of the following subsets of the complex plane is *not* a line?

A $\{z : \mathrm{Im}(z) = 0\}$ **B** $\{z : \mathrm{Im}(z) + \mathrm{Re}(z) = 1\}$ **C** $\{z : z + \bar{z} = 4\}$
D $\left\{z : \mathrm{Arg}(z) = \frac{\pi}{4}\right\}$ **E** $\{z : \mathrm{Re}(z) = \mathrm{Im}(z)\}$

80 Points P, Q, R and M are such that $\overrightarrow{PQ} = 5\boldsymbol{i}$, $\overrightarrow{PR} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\overrightarrow{RM}$ is parallel to $\overrightarrow{PQ}$ so that $\overrightarrow{RM} = \lambda\boldsymbol{i}$, where λ is a constant. The value of λ for which angle RQM is a right angle is

A 0 **B** $\frac{19}{4}$ **C** $\frac{21}{4}$ **D** 10 **E** 6

81 In this diagram, $\overrightarrow{OA} = 6\boldsymbol{i} - \boldsymbol{j} + 8\boldsymbol{k}$, $\overrightarrow{OB} = -3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $AP : PB = 1 : 2$.
The vector $\overrightarrow{OP}$ is equal to



A $\frac{7}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$ **B** $3\boldsymbol{i} + \frac{7}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$
C $3\boldsymbol{j} + 4\boldsymbol{k}$ **D** $3\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{14}{3}\boldsymbol{k}$
E none of these

82 In an Argand diagram, O is the origin, P is the point $(2, 1)$ and Q is the point $(1, 2)$. If P represents the complex number z and Q the complex number α, then α equals

A $\bar{z}$ **B** $i\bar{z}$ **C** $-\bar{z}$ **D** $-i\bar{z}$ **E** $z\bar{z}$

83 In an Argand diagram, the points that represent the complex numbers z, $-\bar{z}$, z^{-1} and $-\overline{(z^{-1})}$ necessarily lie at the vertices of a

A square **B** rectangle **C** parallelogram
D rhombus **E** trapezium

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7C Extended-response questions

1 **a** Points A, B and C are collinear with B between A and C. The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin O. If $\overrightarrow{AC} = \frac{3}{2}\overrightarrow{AB}$:

i express $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ **ii** express $\boldsymbol{c}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b The points A, B and C have position vectors $\boldsymbol{i}$, $2\boldsymbol{i} + 2\boldsymbol{j}$ and $4\boldsymbol{i} + \boldsymbol{j}$ respectively.

i Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

ii Show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have equal magnitudes.

iii Show that AB and BC are perpendicular.

iv Find the position vector of D such that $ABCD$ is a square.

c The triangle OAB is such that O is the origin, $\overrightarrow{OA} = 8\boldsymbol{i}$ and $\overrightarrow{OB} = 10\boldsymbol{j}$. The point P with position vector $\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is equidistant from O, A and B and is at a distance of 2 above the triangle. Find x, y and z.

2 **a** Solve the equation $2z^2 - 4z + 6 = 0$ for $z \in \mathbb{C}$.

b On an Argand diagram, let C be the circle with centre $1 + 0i$ that passes through the two points corresponding to the solutions from part **a**. Find the Cartesian equation of the circle C.

c Find the values of $d \in \mathbb{R}$ such the solutions of the equation $2z^2 - 4z + d = 0$ lie inside or on the circle C.

d Now let $a, b, c \in \mathbb{R}$ with $a \neq 0$, and assume that $b^2 \leq 4ac$.

i Solve the equation $az^2 - bz + c = 0$ for $z \in \mathbb{C}$.

ii State the condition under which there are two distinct solutions.

iii Under this condition, find the Cartesian equation of the circle with centre $\frac{b}{2a} + 0i$ that passes through the two points corresponding to the solutions.

3 **a** Let $S_1 = \{ z : |z| \leq 2 \}$ and $T_1 = \{ z : \text{Im}(z) + \text{Re}(z) \geq 4 \}$.

i On the same diagram, sketch S_1 and T_1, clearly indicating which boundary points are included.

ii Let $d = |z_1 - z_2|$, where $z_1 \in S_1$ and $z_2 \in T_1$. Find the minimum value of d.

b Let $S_2 = \{ z : |z - 1 - i| \leq 1 \}$ and $T_2 = \{ z : |z - 2 - i| \leq |z - i| \}$.

i On the same diagram, sketch S_2 and T_2, clearly indicating which boundaries are included.

ii If z belongs to $S_2 \cap T_2$, find the maximum and minimum values of $|z|$.

4 **a** Show the points $z = \sqrt{2}(1 + i)$, $w = \sqrt{3} - i$ and $z + w$ on an Argand diagram.

b Hence, use geometry to prove that $\tan\left(\frac{\pi}{24}\right) = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$.

Revision

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5 Suppose that $OACB$ is a trapezium with OB parallel to AC and $AC = 2OB$. Let D be the point of trisection of OC nearer to O.

a If $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$, find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{BD}$ **iii** $\overrightarrow{DA}$

b Hence prove that A, D and B are collinear.

6 **a** If $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = 12\boldsymbol{j} - 5\boldsymbol{k}$, find:

i the magnitude of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ to the nearest degree

ii the vector resolute of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$

iii real numbers x, y and z such that $x\boldsymbol{a} + y\boldsymbol{b} = 3\boldsymbol{i} - 30\boldsymbol{j} + z\boldsymbol{k}$.

b In triangle OAB, $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. Points P and Q are such that P is the point of trisection of AB nearer to B and $\overrightarrow{OQ} = 1.5\overrightarrow{OP}$.

i Find an expression for $\overrightarrow{AQ}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

ii Show that $\overrightarrow{OA}$ is parallel to $\overrightarrow{BQ}$.

7 **a** Show that if $2a + b - c = 0$ and $a - 4b - 2c = 0$, then $a : b : c = 2 : -1 : 3$.

b Assume that the vector $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is perpendicular to both $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$. Establish two equations in x, y and z, and find the ratio $x : y : z$.

c Hence, or otherwise, find any vector $\boldsymbol{v}$ which is perpendicular to both $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$.

d Show that the vector $4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}$ is also perpendicular to vector $\boldsymbol{v}$.

e Find the values of s and t such that $4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}$ can be expressed in the form $s(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + t(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$.

f Show that any vector $\boldsymbol{r} = s(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + t(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$ is perpendicular to vector $\boldsymbol{v}$ (where $s \in \mathbb{R}$ and $t \in \mathbb{R}$).

8 Consider a triangle with vertices O, A and B, where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Let θ be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

a Express $\cos\theta$ in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Hence express $\sin\theta$ in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

c Use the formula for the area of a triangle (area $= \frac{1}{2}ab\sin C$) to show that the area of triangle OAB is given by

$$\frac{1}{2}\sqrt{(\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}) - (\boldsymbol{a}\cdot\boldsymbol{b})^2}$$

9 In the quadrilateral $ABCD$, the points X and Y are the midpoints of the diagonals AC and BD respectively.

a Show that $\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BX}$.

b Show that $\overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{DC} = 4\overrightarrow{YX}$.

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10 Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be non-zero vectors in three dimensions such that

$$\boldsymbol{a} \times \boldsymbol{b} = 3\boldsymbol{a} \times \boldsymbol{c}$$

a Show that there exists $k \in \mathbb{R}$ such that $\boldsymbol{b} - 3\boldsymbol{c} = k\boldsymbol{a}$.

b Given that $|\boldsymbol{a}| = |\boldsymbol{c}| = 1$, $|\boldsymbol{b}| = 3$ and the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ is $\arccos\left(\frac{1}{3}\right)$, find:

i $\boldsymbol{b} \cdot \boldsymbol{c}$ **ii** $|\boldsymbol{b} - 3\boldsymbol{c}|$ **iii** the possible values of k.

c Hence find the two possible values for the cosine of the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$.

11 A vector equation of a plane Π is $\boldsymbol{r} \cdot \boldsymbol{n} = k$.

a Let ℓ be a line with vector equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$, $t \in \mathbb{R}$. Given that $\boldsymbol{b} \cdot \boldsymbol{n} \neq 0$, show that the plane Π meets the line ℓ at the point with position vector

$$\frac{(\boldsymbol{b} \cdot \boldsymbol{n})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{n})\boldsymbol{b} + k\boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{n}}$$

b Let P be a point, with position vector $\boldsymbol{p}$, such that P does not lie on the plane Π.

i Using part **a**, express the position vector of the point where the plane Π meets the line through P perpendicular to Π in terms of $\boldsymbol{p}$, $\boldsymbol{n}$ and k.

ii Express the distance from the point P to the plane Π in terms of $\boldsymbol{p}$, $\boldsymbol{n}$ and k.

12 The position vectors of the vertices of a triangle ABC, relative to a given origin O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Let P and Q be points on the line segments AB and AC respectively such that $AP : PB = 1 : 2$ and $AQ : QC = 2 : 1$. Let R be the point on the line segment PQ such that $PR : RQ = 2 : 1$.

a Prove that $\overrightarrow{OR} = \frac{4}{9}\boldsymbol{a} + \frac{1}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$.

b Let M be the midpoint of AC. Prove that R lies on the median BM.

c Find $BR : RM$.

13 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O. The point C lies on AB between A and B, and is such that $AC : CB = 2 : 1$, and D is the midpoint of OC. The line AD meets OB at E.

a Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OC}$ **ii** $\overrightarrow{AD}$

b Find the ratios:

i $OE : EB$ **ii** $AE : ED$

14 The position vectors of the vertices A, B and C of a triangle, relative to an origin O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively. The side BC is extended to D so that $BC = CD$. The point X divides side AB in the ratio $2 : 1$, and the point Y divides side AC in the ratio $4 : 1$. That is, $AX : XB = 2 : 1$ and $AY : YC = 4 : 1$.

a Express in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

i $\overrightarrow{OD}$ **ii** $\overrightarrow{OX}$ **iii** $\overrightarrow{OY}$

b Show that D, X and Y are collinear.

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15 Points A, B, C and D have position vectors $\boldsymbol{j} + 2\boldsymbol{k}$, $-\boldsymbol{i} - \boldsymbol{j}$, $4\boldsymbol{i} + \boldsymbol{k}$ and $3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ respectively.

a Prove that the triangle ABC is right-angled.

b Prove that the triangle ABD is isosceles.

c Show that BD passes through the midpoint, E, of AC and find the ratio $BE : ED$.

16 **a** For $\alpha = 1 - \sqrt{3}i$, write the product of $z - \alpha$ and $z - \overline{\alpha}$ as a quadratic expression in z with real coefficients, where $\overline{\alpha}$ denotes the complex conjugate of α.

b **i** Express α in polar form.

ii Find α^2 and α^3.

iii Show that α is a solution of $z^3 - z^2 + 2z + 4 = 0$, and find all three solutions of this equation.

c On an Argand diagram, plot the three points corresponding to the three solutions. Let A be the point in the first quadrant, let B be the point on the real axis and let C be the third point.

i Find the lengths AB and CB. **ii** Describe the triangle ABC.

17 Points A, B and C have position vectors $\overrightarrow{OA} = -\boldsymbol{j} + 2\boldsymbol{k}$, $\overrightarrow{OB} = \boldsymbol{i} + 2\boldsymbol{k}$ and $\overrightarrow{OC} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

a Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

c Using part **b**, find a Cartesian equation of the plane Π through points A, B and C.

Let D be the point with position vector $\overrightarrow{OD} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

d Find a vector equation of the line through D perpendicular to the plane Π.

e Find the position vector of the point of intersection of this line with the plane Π.

f Find the shortest distance from the point D to the plane Π.

18 **a** If $z = 1 + \sqrt{2}i$, express $p = z + \dfrac{1}{z}$ and $q = z - \dfrac{1}{z}$ in the form $a + bi$.

b On an Argand diagram, let P and Q be the points representing p and q respectively. Let O be the origin, let M be the midpoint of PQ and let G be the point on the line segment OM with $OG = \dfrac{2}{3}OM$. Denote vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Find each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{PQ}$ **ii** $\overrightarrow{OM}$ **iii** $\overrightarrow{OG}$ **iv** $\overrightarrow{GP}$ **v** $\overrightarrow{GQ}$

c Prove that angle PGQ is a right angle.

19 **a** Factorise $z^2 + 4$ into linear factors.

b Express $z^4 + 4$ as the product of two quadratic factors in $\mathbb{C}$.

c Show that:

i $(1 + i)^2 = 2i$ **ii** $(1 - i)^2 = -2i$

d Use the results of **c** to factorise $z^4 + 4$ into linear factors.

e Hence factorise $z^4 + 4$ into two quadratic factors with real coefficients.

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20 **a** Let $z_1 = 1 + 3i$ and $z_2 = 2 - i$. By evaluating $|z_1 - z_2|$, find the distance between the points z_1 and z_2 on an Argand diagram.

b Describe the locus of z on an Argand diagram such that $|z - (2 - i)| = \sqrt{5}$.

c Describe the locus of z such that $|z - (1 + 3i)| = |z - (2 - i)|$.

21 Let $z = 2 + i$.

a Express z^3 in the form $x + yi$, where x and y are integers.

b Let the polar form of $z = 2 + i$ be $r(\cos\alpha + i\sin\alpha)$. Using the polar form of z^3, but without evaluating α, find the value of:

i $\cos(3\alpha)$ **ii** $\sin(3\alpha)$

22 The cube roots of unity are often denoted by 1, ω and ω^2, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

a **i** Illustrate these three numbers on an Argand diagram.

ii Show that $(\omega^2)^2 = \omega$.

b By factorising the polynomial $z^3 - 1$, show that $\omega^2 + \omega + 1 = 0$.

c Evaluate:

i $(1 + \omega)(1 + \omega^2)$

ii $(1 + \omega^2)^3$

d Form the quadratic equation whose solutions are:

i $2 + \omega$ and $2 + \omega^2$

ii $3\omega - \omega^2$ and $3\omega^2 - \omega$

e Find the possible values of the expression $1 + \omega^n + \omega^{2n}$ for $n \in \mathbb{N}$.

23 **a** Find the fifth roots of unity in polar form.

b **i** Show that $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$.

ii Hence, explain why the solutions of $z^4 + z^3 + z^2 + z + 1 = 0$ are fifth roots of unity.

c Show that the solutions of $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$ are fifth roots of unity.

d Let $w = z + \frac{1}{z}$. Show that $w^2 + w - 1 = 0$ if and only if $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$.

e Solve the equation $w^2 + w - 1 = 0$ for w.

f Hence, write the solutions of the equation $z^4 + z^3 + z^2 + z + 1 = 0$ in Cartesian form using surd expressions.

g Hence, show that $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}$.

24 **a** Two complex variables w and z are related by $w = \frac{az + b}{z + c}$, where $a, b, c \in \mathbb{R}$.

Given that $w = 3i$ when $z = -3i$ and that $w = 1 - 4i$ when $z = 1 + 4i$, find the values of a, b and c.

b Let $z = x + yi$. Show that if $w = \bar{z}$, then z lies on a circle of centre $(4, 0)$, and state the radius of this circle.

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25 **a** Use De Moivre's theorem to show that $(1 + i\tan\theta)^5 = \dfrac{\text{cis}(5\theta)}{\cos^5(\theta)}$.

b Hence find expressions for $\cos(5\theta)$ and $\sin(5\theta)$ in terms of $\tan\theta$ and $\cos\theta$.

c Show that $\tan(5\theta) = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$.

d Use the result of **c** and an appropriate substitution to show that $\tan\left(\dfrac{\pi}{5}\right) = \sqrt{5 - 2\sqrt{5}}$.

26 **a** Express, in terms of θ, the solutions α and β of the equation $z + z^{-1} = 2\cos\theta$.

b If P and Q are points on the Argand diagram representing $\alpha^n + \beta^n$ and $\alpha^n - \beta^n$ respectively, show that PQ is of constant length for $n \in \mathbb{N}$.

27 **a** On the same set of axes, sketch the graphs of the following functions:

i $f(x) = \cos x, \ -\pi < x < \pi$ **ii** $g(x) = \tan^{-1} x, \ -\pi < x < \pi$

b Find correct to two decimal places:

i $\tan^{-1}\left(\dfrac{\pi}{4}\right)$ **ii** $\cos 1$

c Hence show that the graphs of $y = f(x)$ and $y = g(x)$ intersect in the interval $\left[\dfrac{\pi}{4}, 1\right]$.

d Using a CAS calculator, find the solution of $f(x) = g(x)$ correct to two decimal places.

e Show that $f(x) = g(x)$ has no other real solutions.

28 **a** On the same set of axes, sketch the graphs of the following functions:

i $f(x) = \sin x, \ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ **ii** $g(x) = \cos^{-1} x, \ -1 < x < 1$

b Find correct to two decimal places:

i $\sin\left(\dfrac{1}{2}\right)$ **ii** $\cos^{-1}\left(\dfrac{\pi}{4}\right)$

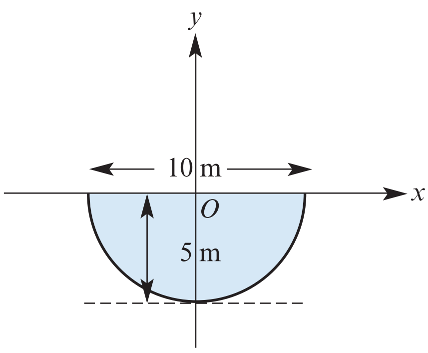
c Hence show that the graphs of $y = f(x)$ and $y = g(x)$ intersect in the interval $\left[\dfrac{1}{2}, \dfrac{\pi}{4}\right]$.

d Using a CAS calculator, find the coordinates of the point(s) of intersection of the graphs, correct to three decimal places.

29 The cross-section of a water channel is defined by the function

$$f(x) = a\sec\left(\frac{\pi}{15}x\right) + d$$

The top of the channel is level with the ground and is 10 m wide. At its deepest point, the channel is 5 m deep.



a Find a and d.

b Find, correct to two decimal places:

i the depth of the water when the width of the water surface is 7 m

ii the width of the water surface when the water is 2.5 m deep

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30 Let S and T be the subsets of the complex plane given by

$$S = \left\{ z : \sqrt{2} \le |z| \le 3 \text{ and } \frac{\pi}{2} < \operatorname{Arg} z \le \frac{3\pi}{4} \right\}$$

$$T = \{ z : z\bar{z} + 2\operatorname{Re}(iz) \le 0 \}$$

a Sketch S on an Argand diagram.

b Find $\{ z : z \in S \text{ and } z = x + yi \text{ where } x \text{ and } y \text{ are integers} \}$.

c On a separate diagram, sketch $S \cap T$.

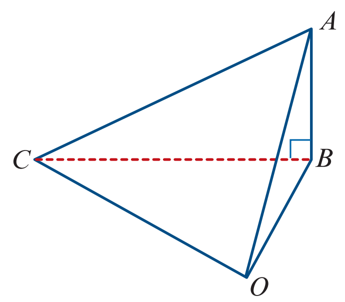
31 **a** Let $A = \left\{ z : \operatorname{Arg} z = \frac{\pi}{4} \right\}$ and $B = \left\{ z : \operatorname{Arg}(z - 4) = \frac{3\pi}{4} \right\}$.

Sketch A and B on the same Argand diagram, clearly labelling $A \cap B$.

b Let $C = \left\{ z : \left| \frac{z - \bar{z}}{z + \bar{z}} \right| \le 1 \right\}$ and $D = \{ z : z^2 + (\bar{z})^2 \le 2 \}$.

Sketch $C \cap D$ on an Argand diagram.

32 In the tetrahedron shown, $\overrightarrow{OB} = \boldsymbol{i}$, $\overrightarrow{OC} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{BA} = \sqrt{\lambda}\boldsymbol{k}$.



a Express $\overrightarrow{OA}$ and $\overrightarrow{CA}$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and $\sqrt{\lambda}$.

b Find the magnitude of $\angle CBO$ to the nearest degree.

c Find the value of λ, if the magnitude of $\angle OAC$ is $30°$.

33 Let $ABCD$ be a tetrahedron, where the vertices A, B, C and D have the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively.

a First assume that AB is perpendicular to CD and that AD is perpendicular to BC. Prove that AC is perpendicular to BD.

b Now assume that $ABCD$ is a regular tetrahedron.

The intersection point of the perpendicular bisectors of the edges of a triangle is called the circumcentre of the triangle. Let X, Y, Z and W be the circumcentres of faces ABC, ACD, ABD and BCD respectively.

i Find the position vectors of X, Y, Z and W.

ii Find the vectors $\overrightarrow{DX}$, $\overrightarrow{BY}$, $\overrightarrow{CZ}$ and $\overrightarrow{AW}$.

iii Let P be a point on DX such that $\overrightarrow{DP} = \frac{3}{4}\overrightarrow{DX}$. Find the position vector of P.

iv Hence find the position vectors of the points Q, R and S on BY, CZ and AW respectively such that $\overrightarrow{BQ} = \frac{3}{4}\overrightarrow{BY}$, $\overrightarrow{CR} = \frac{3}{4}\overrightarrow{CZ}$ and $\overrightarrow{AS} = \frac{3}{4}\overrightarrow{AW}$.

v Explain the geometric significance of results **iii** and **iv**.

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Revision

34 **a** Use mathematical induction to prove that, if $\operatorname{cis}(\theta) \neq 1$, then

$$1 + \operatorname{cis}(\theta) + \operatorname{cis}(2\theta) + \cdots + \operatorname{cis}(n\theta) = \frac{1 - \operatorname{cis}((n+1)\theta)}{1 - \operatorname{cis}(\theta)}$$

for each natural number n.

b Prove that $\dfrac{\operatorname{cis}(\theta) + 1}{\operatorname{cis}(\theta) - 1} = -i\cot\left(\dfrac{\theta}{2}\right)$.

35 **a** Let $x, y \geq 0$. Prove that if $x^2 \leq y^2$, then $x \leq y$. (Hint: Prove the contrapositive.)

b Prove that

$$\boldsymbol{a} \cdot \boldsymbol{b} \leq |\boldsymbol{a}|\,|\boldsymbol{b}|$$

for all vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. (Hint: Use the geometric description of the scalar product.)

c Triangle inequality Prove that

$$|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$$

for all vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. (Hint: Start by writing $|\boldsymbol{a} + \boldsymbol{b}|^2 = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$. You will need to use parts **a** and **b**.)

d Let ABC be a triangle and let X be a point on side AC. Use the triangle inequality to prove that $BX \leq \dfrac{p}{2}$, where p is the perimeter of the triangle.

36 Menelaus' theorem *Consider a triangle ABC. Let P, Q and R be points on the lines AB, BC and CA respectively, where* $\overrightarrow{AP} = p\overrightarrow{AB}$, $\overrightarrow{BQ} = q\overrightarrow{BC}$ *and* $\overrightarrow{CR} = r\overrightarrow{CA}$ *with* $p, q, r \in \mathbb{R} \setminus \{0, 1\}$. *Then the points P, Q and R are collinear if and only if*

$$\frac{p}{1-p} \times \frac{q}{1-q} \times \frac{r}{1-r} = -1$$

In this question, you will prove Menelaus' theorem. Let point C be the origin, and let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the position vectors of A and B respectively.

a Show that R has position vector $r\boldsymbol{a}$ and that Q has position vector $(1-q)\boldsymbol{b}$.

b Show that the line QR consists of all points with position vectors of the form $\lambda r\boldsymbol{a} + (1-\lambda)(1-q)\boldsymbol{b}$ for $\lambda \in \mathbb{R}$.

c Show that $\overrightarrow{AP} = p(\boldsymbol{b} - \boldsymbol{a})$ and hence that P has position vector $(1-p)\boldsymbol{a} + p\boldsymbol{b}$.

d Show that P, Q and R are collinear if and only if there is some $\lambda \in \mathbb{R}$ such that $\lambda r = 1 - p$ and $(1-\lambda)(1-q) = p$.

e Hence, show that P, Q and R are collinear if and only if

$$(1-p)(1-q) = r(1-p-q)$$

f Hence, show that P, Q and R are collinear if and only if

$$\frac{p}{1-p} \times \frac{q}{1-q} \times \frac{r}{1-r} = -1$$

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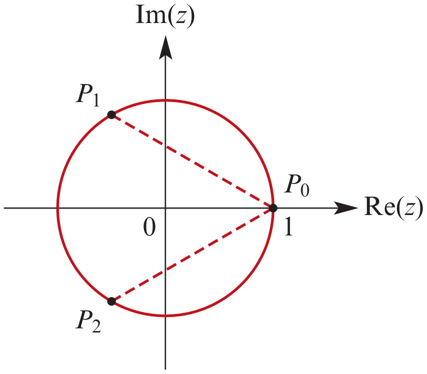
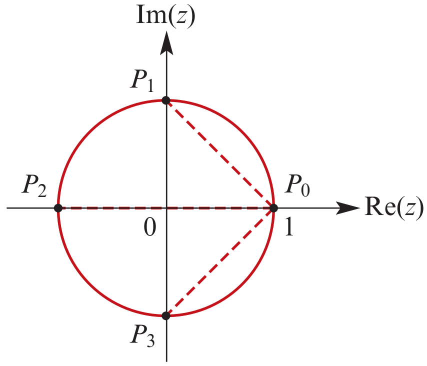
37 The first Argand diagram below shows the points representing the cube roots of unity; the second diagram shows the fourth roots of unity.



Cube roots of unity



Fourth roots of unity

In this question, we are using P_0P_i to denote the distance between points P_0 and P_i.

a For the cube roots of unity shown above, prove that $P_0P_1 \times P_0P_2 = 3$.

b For the fourth roots of unity shown above, prove that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$.

c Denote the nth roots of unity by $1, \omega_1, \omega_2, \dots, \omega_{n-1}$. Then we can factorise the polynomial $z^n - 1$ as follows:

$$z^n - 1 = (z-1)(1 + z + z^2 + \cdots + z^{n-1}) = (z-1)(z-\omega_1)(z-\omega_2)\cdots(z-\omega_{n-1})$$

Now let $P_0, P_1, \dots, P_{n-1}$ be the points representing the nth roots of unity on an Argand diagram, starting from 1 and moving anticlockwise around the unit circle. Prove that

$$P_0P_1 \times P_0P_2 \times \cdots \times P_0P_{n-1} = n$$

d State this result in terms of a regular polygon with n sides.

38 The vertices of a cube have the following position vectors:

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{c} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{d} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k},$$
$$\boldsymbol{e} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{f} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{g} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{h} = -\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

a Show that all the vertices are the same distance from the origin.

b Verify that for any vector $\boldsymbol{v}$ listed above, the vector $-\boldsymbol{v}$ is also listed. What is $\boldsymbol{v} \cdot (-\boldsymbol{v})$?

c Show that for any vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ listed above, we have $\boldsymbol{v} \cdot \boldsymbol{w} \in \{-3, -1, 1, 3\}$. Which pairs of vectors give the various values?

d Consider any vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ listed above such that $\boldsymbol{v} \neq \pm\boldsymbol{w}$. Show that the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ takes one of only two values α and β, where $\alpha + \beta = 180^\circ$.

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7D Algorithms and pseudocode

You may like to use a device to implement the algorithms in this section; see the coding appendices in the Interactive Textbook for instructions.

1 **Pythagorean triples** Recall that, for natural numbers a, b and c, the ordered triple (a, b, c) is called a Pythagorean triple if $a^2 + b^2 = c^2$.

a Here is a simple algorithm that finds all Pythagorean triples (up to the symmetry of swapping a and b) with natural numbers less than or equal to 100.

```
for a from 1 to 100
    for b from 1 to 100
        for c from 1 to 100
            if a^2 + b^2 = c^2 and a < b then
                print (a, b, c)
            end if
        end for
    end for
end for
```

i This algorithm uses a total of 100^3 steps in the nested loops. Try to improve the algorithm so that there are fewer steps in the nested loops.

ii This algorithm produces triples such as (3, 4, 5) and (6, 8, 10), where one is a multiple of the other. Write a pseudocode function *hcf* to find the highest common factor of two natural numbers, and use it to improve the algorithm to produce triples without common factors.

b Here is an algorithm that counts Pythagorean triples without common factors.

```
L ← [ ]
for c from 1 to 100
    for b from 1 to c
        for a from 1 to b
            if a^2 + b^2 = c^2 and hcf(a, b) = 1 then
                append (a, b, c) to L
            end if
        end for
    end for
end for
print L, length(L)
```

i Rewrite the code to verify that abc is divisible by $a + b + c$ for $c \le 100$.

ii Rewrite the code to verify that abc is divisible by 60 for $c \le 100$.

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c Up to symmetry, every Pythagorean triple (a, b, c) without common factors can be described by

$$a = m^2 - n^2, \quad b = 2mn \quad \text{and} \quad c = m^2 + n^2$$

for $m, n \in \mathbb{N}$ with $m > n$ such that $m + n$ is odd and $\text{HCF}(m, n) = 1$. Use this fact to prove that the two properties from part **b** hold for all Pythagorean triples.

d **i** Using pseudocode, write an algorithm with two nested loops to find the number of Pythagorean triples (a, b, c) such that $b = c - 1$, where $a < b < c \leq 1000$.

ii Prove that there are infinitely many Pythagorean triples (a, b, c) with $b = c - 1$.

iii Prove that, if a Pythagorean triple (a, b, c) satisfies $b = c - 1$, then $a^2 = b + c$.

iv Further investigate conditions of this type, such as $b = c - 9$.

2 **Equations over the natural numbers**

a **i** Using pseudocode, write an algorithm to solve the equation $a^3 + b^3 + c^3 = d^3$ over the natural numbers for $d \leq 100$.

ii Implement an algorithm to discover whether every solution (a, b, c, d) satisfies the property that $abcd$ is divisible by $a + b + c + d$.

b Using pseudocode, write an algorithm that will prove that the equation $x^n + y^n = z^n$ has no solutions over the natural numbers for $3 \leq n \leq 8$ and $x, y, z \leq 100$.

c **i** Implement an algorithm to find all solutions of the equation $a^2 + b^2 = q(ab + 1)$ over the natural numbers, for $a < b$ with $a \leq 150$ and $b \leq 1000$. Describe each solution as a triple (a, b, q).

ii Make a conjecture about the form of the solutions (a, b, q) in terms of powers of an integer. Verify that all triples of this form are solutions of the equation.

3 **Vectors** The pseudocode function shown here inputs the coefficients of two vectors $\boldsymbol{v}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{v}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$ and returns their scalar product.

```
define f(x1, y1, z1, x2, y2, z2):
  return x1x2 + y1y2 + z1z2
```

Write a pseudocode function that for given vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ will produce:

a the cross product of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

b the length of the projection of $\boldsymbol{v}_1$ onto $\boldsymbol{v}_2$.

4 **Circles in the complex plane**

a Use pseudocode to describe an algorithm that, given $a \in \mathbb{R}$ and $r_1, r_2 \in \mathbb{R}^+$, determines whether the two circles in the complex plane defined by

$$|z| = r_1 \quad \text{and} \quad |z - a| = r_2$$

coincide, intersect in two points, touch at one point or are disjoint. Adapt your algorithm so that, if the circles intersect, then it finds the points of intersection.

b Use pseudocode to describe an algorithm that, given $c_1, c_2 \in \mathbb{C}$ and $r_1, r_2 \in \mathbb{R}^+$, determines whether the two circles in the complex plane defined by

$$|z - c_1| = r_1 \quad \text{and} \quad |z - c_2| = r_2$$

coincide, intersect in two points, touch at one point or are disjoint.

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8

Differentiation and rational functions

Objectives

- ▶ To review differentiation.
- ▶ To use the rule $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ to obtain the derivative of a function of the form $x = f(y)$.
- ▶ To find the derivatives of the **inverse circular functions**.
- ▶ To find the derivative of the function $y = \log_e |x|$.
- ▶ To define the **second derivative** of a function.
- ▶ To define and investigate **points of inflection**.
- ▶ To apply the chain rule to problems involving **related rates**.
- ▶ To apply the chain rule to parametrically defined relations.
- ▶ To sketch the graphs of **rational functions**.
- ▶ To use **implicit differentiation**.

In this chapter we review the techniques of differentiation that you have met in Mathematical Methods Units 3 & 4. We also introduce important new techniques that will be used throughout the remainder of the book. Differentiation and integration are used in each of the following chapters, up to the chapters on statistical inference.

One of the new techniques is the use of the second derivative in sketching graphs. This will give you a greater ability both to sketch graphs and to understand a given sketch of a graph.

Another new technique is implicit differentiation, which is a valuable tool for determining the gradient at a point on a curve that is not the graph of a function.

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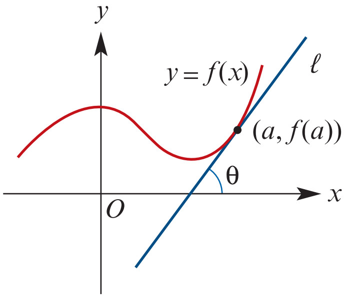
8A Differentiation

The derivative of a function f is denoted by f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative f' is also known as the gradient function.



If $(a, f(a))$ is a point on the graph of $y = f(x)$, then the gradient of the graph at that point is $f'(a)$.

If the line ℓ is the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ and ℓ makes an angle of θ with the positive direction of the x-axis, as shown, then

$$f'(a) = \text{gradient of } \ell = \tan\theta$$

Review of differentiation

Here we summarise basic derivatives and rules for differentiation covered in Mathematical Methods Units 3 & 4.

$f(x)$	$f'(x)$	
a	0	where $a \in \mathbb{R}$
x^n	nx^{n-1}	where $n \in \mathbb{R} \setminus \{0\}$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
e^x	e^x	
$\log_e x$	$\frac{1}{x}$	for $x > 0$

Product rule

- If $f(x) = g(x)\,h(x)$, then
  $$f'(x) = g'(x)\,h(x) + g(x)\,h'(x)$$
- If $y = uv$, then
  $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule

- If $f(x) = \frac{g(x)}{h(x)}$, then
  $$f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{(h(x))^2}$$
- If $y = \frac{u}{v}$, then
  $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain rule

- If $f(x) = h(g(x))$, then
  $$f'(x) = h'(g(x))\,g'(x)$$
- If $y = h(u)$ and $u = g(x)$, then
  $$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

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Example 1

Differentiate each of the following with respect to x:

a $\sqrt{x}\sin x$ **b** $\dfrac{x^2}{\sin x}$ **c** $\cos(x^2+1)$

Solution

a Let $f(x) = \sqrt{x}\sin x$.

Applying the product rule:

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}\sin x + x^{\frac{1}{2}}\cos x$$

$$= \frac{\sqrt{x}\sin x}{2x} + \sqrt{x}\cos x, \quad x > 0$$

b Let $f(x) = \dfrac{x^2}{\sin x}$.

Applying the quotient rule:

$$f'(x) = \frac{2x\sin x - x^2\cos x}{\sin^2 x}$$

c Let $y = \cos(x^2+1)$.

Let $u = x^2 + 1$. Then $y = \cos u$.

By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= -\sin u \cdot 2x$$

$$= -2x\sin(x^2+1)$$

The derivative of tan(*kx*)

Let $f(x) = \tan(kx)$. Then $f'(x) = k\sec^2(kx)$.

Proof Let $f(x) = \tan(kx) = \dfrac{\sin(kx)}{\cos(kx)}$.

The quotient rule yields

$$f'(x) = \frac{k\cos(kx)\cos(kx) + k\sin(kx)\sin(kx)}{\cos^2(kx)}$$

$$= \frac{k(\cos^2(kx) + \sin^2(kx))}{\cos^2(kx)}$$

$$= k\sec^2(kx)$$

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Example 2

Differentiate each of the following with respect to x:

a $\tan(5x^2 + 3)$ **b** $\tan^3 x$ **c** $\sec^2(3x)$

Solution

a Let $f(x) = \tan(5x^2 + 3)$.
By the chain rule with $g(x) = 5x^2 + 3$, we have

$$f'(x) = \sec^2(5x^2 + 3) \cdot 10x = 10x \sec^2(5x^2 + 3)$$

b Let $f(x) = \tan^3 x = (\tan x)^3$.
By the chain rule with $g(x) = \tan x$, we have

$$f'(x) = 3(\tan x)^2 \cdot \sec^2 x = 3 \tan^2 x \sec^2 x$$

c Let $y = \sec^2(3x)$

$= \tan^2(3x) + 1$ (using the Pythagorean identity)

$= (\tan(3x))^2 + 1$

Let $u = \tan(3x)$. Then $y = u^2 + 1$ and the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u \cdot 3\sec^2(3x) = 6\tan(3x)\sec^2(3x)$$

Operator notation

Sometimes it is appropriate to use notation which emphasises that differentiation is an operation on an expression. The derivative of $f(x)$ can be denoted by $\frac{d}{dx}(f(x))$.

Example 3

Find:

a $\frac{d}{dx}(x^2 + 2x + 3)$ **b** $\frac{d}{dx}(e^{x^2})$ **c** $\frac{d}{dz}(\sin^2(z))$

Solution

a $\frac{d}{dx}(x^2 + 2x + 3) = 2x + 2$

b Let $y = e^{x^2}$ and $u = x^2$. Then $y = e^u$.
The chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot 2x = 2xe^{x^2}$$

i.e. $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$

c Let $y = \sin^2(z)$ and $u = \sin z$. Then $y = u^2$.
The chain rule gives

$$\frac{dy}{dz} = \frac{dy}{du}\frac{du}{dz} = 2u\cos z = 2\sin z \cos z = \sin(2z)$$

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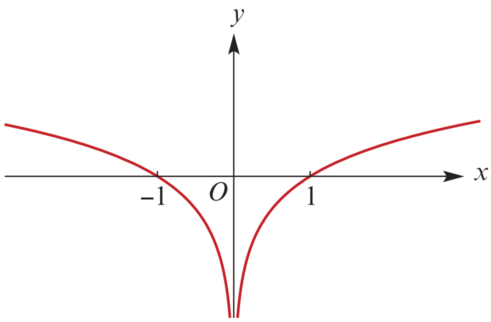
The derivative of $\log_e |x|$

The function

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \log_e |x|$$

is very important in this course.

The graph of the function is shown opposite. The derivative of this function is determined in the following example.



Example 4

a Find $\dfrac{d}{dx}(\log_e |x|)$ for $x \neq 0$.

b Find $\dfrac{d}{dx}(\log_e |\sec x|)$ for $x \notin \left\{ \dfrac{(2k+1)\pi}{2} : k \in \mathbb{Z} \right\}$.

Solution

a Let $y = \log_e |x|$.

If $x > 0$, then $y = \log_e x$, so

$$\frac{dy}{dx} = \frac{1}{x}$$

If $x < 0$, then $y = \log_e(-x)$, so the chain rule gives

$$\frac{dy}{dx} = \frac{1}{-x} \times (-1) = \frac{1}{x}$$

Hence

$$\frac{d}{dx}(\log_e |x|) = \frac{1}{x} \quad \text{for } x \neq 0$$

b Let $y = \log_e |\sec x|$

$$= \log_e \left| \frac{1}{\cos x} \right|$$

$$= \log_e \left(\frac{1}{|\cos x|} \right)$$

$$= -\log_e |\cos x|$$

Let $u = \cos x$. Then $y = -\log_e |u|$.

By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= -\frac{1}{u} \times (-\sin x)$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

Derivative of $\log_e |x|$

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \log_e |x|$. Then $f'(x) = \dfrac{1}{x}$.

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Skill-sheet

Exercise 8A

Example 1

1 Find the derivative of each of the following with respect to x:

a $x^5 \sin x$ **b** $\sqrt{x} \cos x$ **c** $e^x \cos x$ **d** $x^3 e^x$ **e** $\sin x \cos x$

Example 2

2 Find the derivative of each of the following with respect to x:

a $e^x \tan x$ **b** $x^4 \tan x$ **c** $\tan x \log_e x$ **d** $\sin x \tan x$ **e** $\sqrt{x} \tan x$

3 Find the derivative of each of the following using the quotient rule:

a $\dfrac{x}{\log_e x}$ **b** $\dfrac{\sqrt{x}}{\tan x}$ **c** $\dfrac{e^x}{\tan x}$ **d** $\dfrac{\tan x}{\log_e x}$

e $\dfrac{\sin x}{x^2}$ **f** $\dfrac{\tan x}{\cos x}$ **g** $\dfrac{\cos x}{e^x}$ **h** $\dfrac{\cos x}{\sin x}$ $(= \cot x)$

4 Find the derivative of each of the following using the chain rule:

a $\tan(x^2 + 1)$ **b** $\sin^2 x$ **c** $e^{\tan x}$ **d** $\tan^5 x$

e $\sin(\sqrt{x})$ **f** $\sqrt{\tan x}$ **g** $\cos\left(\dfrac{1}{x}\right)$ **h** $\sec^2 x$

i $\tan\left(\dfrac{x}{4}\right)$ **j** $\cot x$ Hint: Use $\cot x = \tan\left(\dfrac{\pi}{2} - x\right)$.

5 Use appropriate techniques to find the derivative of each of the following:

a $\tan(kx)$, $k \in \mathbb{R}$ **b** $e^{\tan(2x)}$ **c** $\tan^2(3x)$ **d** $\log_e(x)\, e^{\sin x}$

e $\sin^3(x^2)$ **f** $\dfrac{e^{3x+1}}{\cos x}$ **g** $e^{3x} \tan(2x)$ **h** $\sqrt{x} \tan(\sqrt{x})$

i $\dfrac{\tan^2 x}{(x + 1)^3}$ **j** $\sec^2(5x^2)$

6 Find $\dfrac{dy}{dx}$ for each of the following:

a $y = (x - 1)^5$ **b** $y = \log_e(4x)$ **c** $y = e^x \tan(3x)$ **d** $y = e^{\cos x}$

e $y = \cos^3(4x)$ **f** $y = (\sin x + 1)^4$ **g** $y = \sin(2x) \cos x$ **h** $y = \dfrac{x^2 + 1}{x}$

i $y = \dfrac{x^3}{\sin x}$ **j** $y = \dfrac{1}{x \log_e x}$

Example 3

7 For each of the following, determine the derivative:

a $\dfrac{d}{dx}(x^3)$ **b** $\dfrac{d}{dy}(2y^2 + 10y)$ **c** $\dfrac{d}{dz}(\cos^2 z)$

d $\dfrac{d}{dx}(e^{\sin^2 x})$ **e** $\dfrac{d}{dz}(1 - \tan^2 z)$ **f** $\dfrac{d}{dy}(\operatorname{cosec}^2 y)$

Example 4

8 For each of the following, find the derivative with respect to x:

a $\log_e |2x + 1|$ **b** $\log_e |-2x + 1|$ **c** $\log_e |\sin x|$

d $\log_e |\sec x + \tan x|$ **e** $\log_e |\operatorname{cosec} x + \tan x|$ **f** $\log_e |\tan(\frac{1}{2}x)|$

g $\log_e |\operatorname{cosec} x - \cot x|$ **h** $\log_e |x + \sqrt{x^2 - 4}|$ **i** $\log_e |x + \sqrt{x^2 + 4}|$

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9 Let $f(x) = \tan\left(\frac{x}{2}\right)$. Find the gradient of the graph of $y = f(x)$ at the point where:

a $x = 0$ **b** $x = \frac{\pi}{3}$ **c** $x = \frac{\pi}{2}$

10 Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \tan x$.

a Find the coordinates of the points on the graph where the gradient is 4.

b Find the equation of the tangent at each of these points.

11 Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \tan x - 8\sin x$.

a **i** Find the stationary points on the graph of $y = f(x)$.

ii State the nature of each of the stationary points.

b Sketch the graph of $y = f(x)$.

12 Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = e^x \sin x$.

a Find the gradient of $y = f(x)$ when $x = \frac{\pi}{4}$.

b Find the coordinates of the point where the gradient is zero.

13 Let $f: \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to \mathbb{R}$, $f(x) = \tan(2x)$. The tangent to the graph of $y = f(x)$ at $x = a$ makes an angle of 70° with the positive direction of the x-axis. Find the value(s) of a.

14 Let $f(x) = \sec\left(\frac{x}{4}\right)$.

a Find $f'(x)$. **b** Find $f'(\pi)$.

c Find the equation of the tangent to $y = f(x)$ at the point where $x = \pi$.

8B Derivatives of $x = f(y)$

From the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For the special case where $y = x$, this gives

$$\frac{dx}{dx} = \frac{dx}{du} \times \frac{du}{dx}$$

$$\therefore \quad 1 = \frac{dx}{du} \times \frac{du}{dx}$$

provided both derivatives exist.

This is restated in the standard form by replacing u with y in the formula:

$$\frac{dx}{dy} \times \frac{dy}{dx} = 1$$

We obtain the following useful result.

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$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \text{provided } \frac{dx}{dy} \neq 0$$

Note: We are assuming that $x = f(y)$ is a one-to-one function.

Example 5

Given $x = y^3$, find $\frac{dy}{dx}$.

Solution

We have

$$\frac{dx}{dy} = 3y^2$$

Hence

$$\frac{dy}{dx} = \frac{1}{3y^2}, \quad y \neq 0$$

Explanation

The power of this method can be appreciated by comparing it with an alternative approach as follows.

Let $x = y^3$. Then $y = \sqrt[3]{x} = x^{\frac{1}{3}}$.

Hence

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

i.e. $$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}, \quad x \neq 0$$

Note that $\frac{1}{3y^2} = \frac{1}{3\sqrt[3]{x^2}}$.

While the derivative expressed in terms of x is the familiar form, it is no less powerful when it is found in terms of y.

Note: Here x is a one-to-one function of y.

Example 6

Find the gradient of the curve $x = y^2 - 4y$ at the point where $y = 3$.

Solution

$$x = y^2 - 4y$$

$$\frac{dx}{dy} = 2y - 4$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2y - 4}, \quad y \neq 2$$

Hence the gradient at $y = 3$ is $\frac{1}{2}$.

Note: Here x is not a one-to-one function of y, but it is for $y \geq 2$, which is where we are interested in the curve for this example. In the next example, we can consider two one-to-one functions of y. One with domain $y \geq 2$ and the other with domain $y \leq 2$.

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Example 7

Find the gradient of the curve $x = y^2 - 4y$ at $x = 5$.

Solution

$$x = y^2 - 4y$$

$$\frac{dx}{dy} = 2y - 4$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2y-4}, \quad y \neq 2$$

Substituting $x = 5$ into $x = y^2 - 4y$ yields

$$y^2 - 4y = 5$$

$$y^2 - 4y - 5 = 0$$

$$(y-5)(y+1) = 0$$

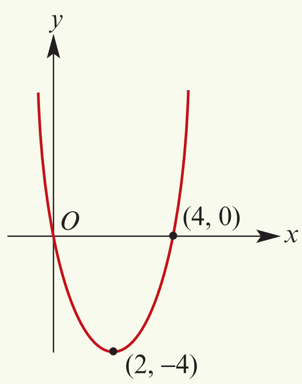
$$\therefore \quad y = 5 \quad \text{or} \quad y = -1$$

Substituting these two y-values into the derivative gives

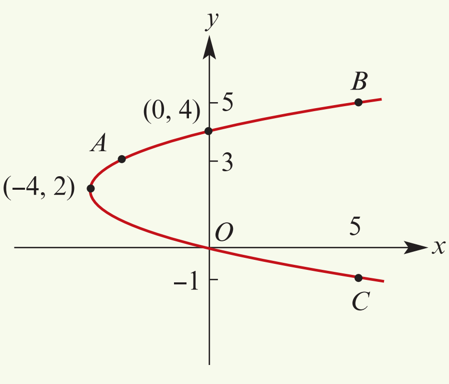
$$\frac{dy}{dx} = \frac{1}{6} \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{6}$$

Note: To explain the two answers here, we consider the graph of $x = y^2 - 4y$, which is the reflection of the graph of $y = x^2 - 4x$ in the line with equation $y = x$.

Graph of $y = x^2 - 4x$



Graph of $x = y^2 - 4y$



When $x = 5$, there are two points, B and C, on the graph of $x = y^2 - 4y$.

At B, $y = 5$ and $\frac{dy}{dx} = \frac{1}{6}$.

At C, $y = -1$ and $\frac{dy}{dx} = -\frac{1}{6}$.

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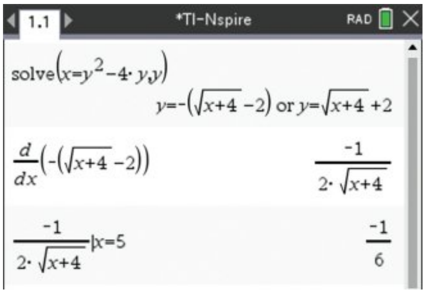
Using the TI-Nspire

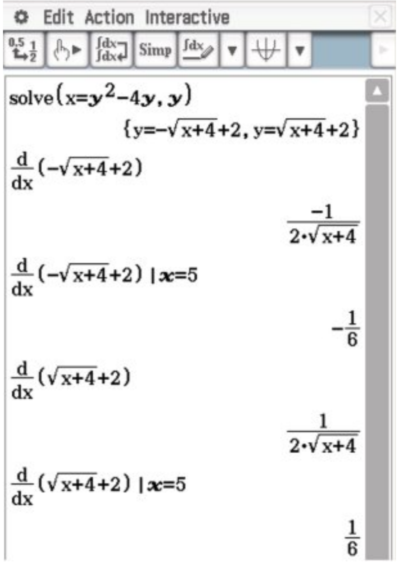
- First solve $x = y^2 - 4y$ for y.
- Differentiate each expression for y with respect to x and then substitute $x = 5$, as shown.

Note: Press [template key] to obtain the derivative template $\frac{d}{d\square}\square$.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the equation $x = y^2 - 4y$ and solve for y.
- Enter and highlight each expression for y as shown.
- Go to **Interactive** > **Calculation** > **diff**.
- Substitute $x = 5$.



Skill-sheet

Exercise 8B

Example 5

1 Using $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$, find $\dfrac{dy}{dx}$ for each of the following:

a $x = 2y + 6$ **b** $x = y^2$ **c** $x = (2y - 1)^2$ **d** $x = e^y$

e $x = \sin(5y)$ **f** $x = \log_e y$ **g** $x = \tan y$ **h** $x = y^3 + y - 2$

i $x = \dfrac{y - 1}{y}$ **j** $x = ye^y$

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Example 6
Example 7

2 For each of the following, find the gradient of the curve at the given value:

a $x = y^3$ at $y = \frac{1}{8}$ **b** $x = y^3$ at $x = \frac{1}{8}$

c $x = e^{4y}$ at $y = 0$ **d** $x = e^{4y}$ at $x = \frac{1}{4}$

e $x = (1 - 2y)^2$ at $y = 1$ **f** $x = (1 - 2y)^2$ at $x = 4$

g $x = \cos(2y)$ at $y = \frac{\pi}{6}$ **h** $x = \cos(2y)$ at $x = 0$

3 For each of the following, express $\frac{dy}{dx}$ in terms of y:

a $x = (2y - 1)^3$ **b** $x = e^{2y+1}$ **c** $x = \log_e(2y - 1)$ **d** $x = \log_e(2y) - 1$

4 For each relation in Question 3, by first making y the subject, express $\frac{dy}{dx}$ in terms of x.

5 Find the equations of the tangents to the curve with equation $x = 2 - 3y^2$ at the points where $x = -1$.

6 **a** Find the coordinates of the points of intersection of the graphs of the relations $x = y^2 - 4y$ and $y = x - 6$.

b Find the coordinates of the point at which the tangent to the graph of $x = y^2 - 4y$ is parallel to the line $y = x - 6$.

c Find the coordinates of the point at which the tangent to the graph of $x = y^2 - 4y$ is perpendicular to the line $y = x - 6$.

7 **a** Show that the graphs of $x = y^2 - y$ and $y = \frac{1}{2}x + 1$ intersect where $x = 2$ and find the coordinates of this point.

b Find, correct to two decimal places, the angle between the line $y = \frac{1}{2}x + 1$ and the tangent to the graph of $x = y^2 - y$ at the point of intersection found in part **a** (that is, at the point where $x = 2$).

8C Derivatives of inverse circular functions

The result established in the previous section

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

can be used to find the derivative of the inverse of a function, provided we know the derivative of the original function.

For example, for the function with rule $y = \log_e x$, the equivalent function is $x = e^y$. Given that we know $\frac{dx}{dy} = e^y$, we obtain $\frac{dy}{dx} = \frac{1}{e^y}$. But $x = e^y$, and therefore $\frac{dy}{dx} = \frac{1}{x}$.

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The derivative of $\sin^{-1}(x)$

If $f(x) = \sin^{-1}(x)$, then $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

Proof Let $y = \sin^{-1}(x)$, where $x \in [-1, 1]$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

The equivalent form is $x = \sin y$ and so $\dfrac{dx}{dy} = \cos y$.

Thus $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$ and $\cos y \neq 0$ for $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

The Pythagorean identity is used to express $\dfrac{dy}{dx}$ in terms of x:

$$\sin^2 y + \cos^2 y = 1$$
$$\cos^2 y = 1 - \sin^2 y$$
$$\cos y = \pm\sqrt{1 - \sin^2 y}$$

Therefore $\cos y = \sqrt{1 - \sin^2 y}$ since $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and so $\cos y > 0$

$= \sqrt{1 - x^2}$ since $x = \sin y$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$

The derivative of $\cos^{-1}(x)$

If $f(x) = \cos^{-1}(x)$, then $f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

Proof Let $y = \cos^{-1}(x)$, where $x \in [-1, 1]$ and $y \in [0, \pi]$.

The equivalent form is $x = \cos y$ and so $\dfrac{dx}{dy} = -\sin y$.

Thus $\dfrac{dy}{dx} = \dfrac{-1}{\sin y}$ and $\sin y \neq 0$ for $y \in (0, \pi)$.

Using the Pythagorean identity yields

$$\sin y = \pm\sqrt{1 - \cos^2 y}$$

Therefore $\sin y = \sqrt{1 - \cos^2 y}$ since $y \in (0, \pi)$ and so $\sin y > 0$

$= \sqrt{1 - x^2}$ since $x = \cos y$

Hence $\dfrac{dy}{dx} = \dfrac{-1}{\sin y} = \dfrac{-1}{\sqrt{1-x^2}}$

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The derivative of $\tan^{-1}(x)$

If $f(x) = \tan^{-1}(x)$, then $f'(x) = \dfrac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Proof Let $y = \tan^{-1}(x)$, where $x \in \mathbb{R}$ and $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Then $x = \tan y$. Therefore $\dfrac{dx}{dy} = \sec^2 y$, giving $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$.

Using the Pythagorean identity $\sec^2 y = 1 + \tan^2 y$, we have

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$$

$$= \frac{1}{1+x^2} \qquad \text{since } x = \tan y$$

For $a > 0$, the following results can be obtained using the chain rule.

Inverse circular functions

$f\colon (-a, a) \to \mathbb{R}$, $\quad f(x) = \sin^{-1}\left(\dfrac{x}{a}\right)$, $\quad f'(x) = \dfrac{1}{\sqrt{a^2 - x^2}}$

$f\colon (-a, a) \to \mathbb{R}$, $\quad f(x) = \cos^{-1}\left(\dfrac{x}{a}\right)$, $\quad f'(x) = \dfrac{-1}{\sqrt{a^2 - x^2}}$

$f\colon \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \tan^{-1}\left(\dfrac{x}{a}\right)$, $\quad f'(x) = \dfrac{a}{a^2 + x^2}$

Proof We show how to obtain the first result; the remaining two are left as an exercise.

Let $y = \sin^{-1}\left(\dfrac{x}{a}\right)$. Then by the chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \times \frac{1}{a} = \frac{1}{\sqrt{a^2\left(1 - \frac{x^2}{a^2}\right)}} = \frac{1}{\sqrt{a^2 - x^2}}$$

Example 8

Differentiate each of the following with respect to x:

a $\sin^{-1}\left(\dfrac{x}{3}\right)$ **b** $\cos^{-1}(4x)$ **c** $\tan^{-1}\left(\dfrac{2x}{3}\right)$ **d** $\sin^{-1}(x^2 - 1)$

Solution

a Let $y = \sin^{-1}\left(\dfrac{x}{3}\right)$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}$$

b Let $y = \cos^{-1}(4x)$ and $u = 4x$.

By the chain rule:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \times 4$$

$$= \frac{-4}{\sqrt{1 - 16x^2}}$$

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c Let $y = \tan^{-1}\left(\frac{2x}{3}\right)$ and $u = \frac{2x}{3}$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+u^2} \times \frac{2}{3} \\ &= \frac{1}{1+\left(\frac{2x}{3}\right)^2} \times \frac{2}{3} \\ &= \frac{9}{4x^2+9} \times \frac{2}{3} \\ &= \frac{6}{4x^2+9}\end{aligned}$$

d Let $y = \sin^{-1}(x^2 - 1)$ and $u = x^2 - 1$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-u^2}} \times 2x \\ &= \frac{2x}{\sqrt{1-(x^2-1)^2}} \\ &= \frac{2x}{\sqrt{1-(x^4-2x^2+1)}} \\ &= \frac{2x}{\sqrt{2x^2-x^4}} \\ &= \frac{2x}{\sqrt{x^2}\sqrt{2-x^2}} \\ &= \frac{2x}{|x|\sqrt{2-x^2}}\end{aligned}$$

Hence $\frac{dy}{dx} = \frac{2}{\sqrt{2-x^2}}$ for $0 < x < \sqrt{2}$

and $\frac{dy}{dx} = \frac{-2}{\sqrt{2-x^2}}$ for $-\sqrt{2} < x < 0$

Skill-sheet

Exercise 8C

Example 8

1 Find the derivative of each of the following with respect to x:

a $\sin^{-1}\left(\frac{x}{2}\right)$ **b** $\cos^{-1}\left(\frac{x}{4}\right)$ **c** $\tan^{-1}\left(\frac{x}{3}\right)$ **d** $\sin^{-1}(3x)$

e $\cos^{-1}(2x)$ **f** $\tan^{-1}(5x)$ **g** $\sin^{-1}\left(\frac{3x}{4}\right)$ **h** $\cos^{-1}\left(\frac{3x}{2}\right)$

i $\tan^{-1}\left(\frac{2x}{5}\right)$ **j** $\sin^{-1}(0.2x)$

2 Find the derivative of each of the following with respect to x:

a $\sin^{-1}(x+1)$ **b** $\cos^{-1}(2x+1)$ **c** $\tan^{-1}(x+2)$ **d** $\sin^{-1}(4-x)$

e $\cos^{-1}(1-3x)$ **f** $3\tan^{-1}(1-2x)$ **g** $2\sin^{-1}\left(\frac{3x+1}{2}\right)$ **h** $-4\cos^{-1}\left(\frac{5x-3}{2}\right)$

i $5\tan^{-1}\left(\frac{1-x}{2}\right)$ **j** $-\sin^{-1}(x^2)$ **k** $3\cos^{-1}(x^2-1)$

3 Find the derivative of each of the following with respect to x:

a $y = \cos^{-1}\left(\frac{3}{x}\right)$ where $x > 3$ **b** $y = \sin^{-1}\left(\frac{5}{x}\right)$ where $x > 5$

c $y = \cos^{-1}\left(\frac{3}{2x}\right)$ where $x > \frac{3}{2}$

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4 For a positive constant a, find the derivative of each of the following:

a $\sin^{-1}(ax)$ **b** $\cos^{-1}(ax)$ **c** $\tan^{-1}(ax)$

5 Let $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$.

a **i** Find the maximal domain of f. **ii** Find the range of f.

b Find the derivative of $f(x)$, and state the domain for which the derivative exists.

c Sketch the graph of $y = f'(x)$, labelling the turning points and the asymptotes.

6 Let $f(x) = 4\cos^{-1}(3x)$.

a **i** Find the maximal domain of f. **ii** Find the range of f.

b Find the derivative of $f(x)$, and state the domain for which the derivative exists.

c Sketch the graph of $y = f'(x)$, labelling the turning points and the asymptotes.

7 Let $f(x) = 2\tan^{-1}\left(\frac{x+1}{2}\right)$.

a **i** Find the maximal domain of f. **ii** Find the range of f.

b Find the derivative of $f(x)$.

c Sketch the graph of $y = f'(x)$, labelling the turning points and the asymptotes.

8 Differentiate each of the following with respect to x:

a $(\sin^{-1} x)^2$ **b** $\sin^{-1} x + \cos^{-1} x$ **c** $\sin(\cos^{-1} x)$

d $\cos(\sin^{-1} x)$ **e** $e^{\sin^{-1} x}$ **f** $\tan^{-1}(e^x)$

9 Find, correct to two decimal places where necessary, the gradient of the graph of each of the following functions at the value of x indicated:

a $f(x) = \sin^{-1}\left(\frac{x}{3}\right), \quad x = 1$ **b** $f(x) = 2\cos^{-1}(3x), \quad x = 0.1$

c $f(x) = 3\tan^{-1}(2x+1), \quad x = 1$

10 For each of the following, find the value(s) of a from the given information:

a $f(x) = 2\sin^{-1} x, \quad f'(a) = 4$ **b** $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right), \quad f'(a) = -10$

c $f(x) = \tan^{-1}(3x), \quad f'(a) = 0.5$ **d** $f(x) = \sin^{-1}\left(\frac{x+1}{2}\right), \quad f'(a) = 20$

e $f(x) = 2\cos^{-1}\left(\frac{2x}{3}\right), \quad f'(a) = -8$ **f** $f(x) = 4\tan^{-1}(2x-1), \quad f'(a) = 1$

11 Find, in the form $y = mx + c$, the equation of the tangent to the graph of:

a $y = \sin^{-1}(2x)$ at $x = \frac{1}{4}$ **b** $y = \tan^{-1}(2x)$ at $x = \frac{1}{2}$

c $y = \cos^{-1}(3x)$ at $x = \frac{1}{6}$ **d** $y = \cos^{-1}(3x)$ at $x = \frac{1}{2\sqrt{3}}$

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12 Let $f(x) = \cos^{-1}\left(\dfrac{6}{x}\right)$.

a Find the maximal domain of f.

b Find $f'(x)$ and show that $f'(x) > 0$ for $x > 6$.

c Sketch the graph of $y = f(x)$ and label endpoints and asymptotes.

8D Second derivatives

For the function f with rule $f(x)$, the derivative is denoted by f' and has rule $f'(x)$. This notation is extended to taking the derivative of the derivative: the new function is denoted by f'' and has rule $f''(x)$. This new function is known as the **second derivative**.

Consider the function g with rule $g(x) = 2x^3 - 4x^2$. The derivative has rule $g'(x) = 6x^2 - 8x$, and the second derivative has rule $g''(x) = 12x - 8$.

Note: The second derivative might not exist at a point even if the first derivative does.
For example, let $f(x) = x^{\frac{4}{3}}$. Then $f'(x) = \dfrac{4}{3}x^{\frac{1}{3}}$ and $f''(x) = \dfrac{4}{9}x^{-\frac{2}{3}}$.
We see that $f'(0) = 0$, but the second derivative $f''(x)$ is not defined at $x = 0$.

In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Example 9

Find the second derivative of each of the following with respect to x:

a $f(x) = 6x^4 - 4x^3 + 4x$

b $y = e^x \sin x$

Solution

a
$$f(x) = 6x^4 - 4x^3 + 4x$$
$$f'(x) = 24x^3 - 12x^2 + 4$$
$$f''(x) = 72x^2 - 24x$$

b
$$y = e^x \sin x$$
$$\frac{dy}{dx} = e^x \sin x + e^x \cos x \quad \text{(by the product rule)}$$
$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$
$$= 2e^x \cos x$$

A CAS calculator has the capacity to find the second derivative directly.

Using the TI-Nspire

- Press to obtain the 2nd derivative template $\frac{d^2}{d\square}\square$.
- Complete as shown.

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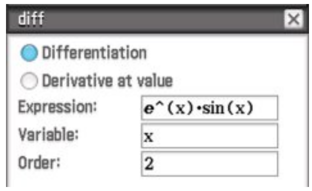
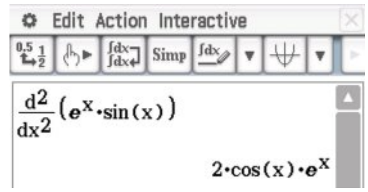
Using the Casio ClassPad

- Enter and highlight the expression $e^x \cdot \sin(x)$.
- Go to **Interactive** > **Calculation** > **diff** and change to order 2. Tap OK.





Example 10

If $f(x) = e^{2x}$, find $f''(0)$.

Solution

$$f(x) = e^{2x}$$
$$f'(x) = 2e^{2x}$$
$$f''(x) = 4e^{2x}$$

Therefore $f''(0) = 4e^0 = 4$.

Example 11

If $y = \cos(2x)$, find a simple expression for

$$\left(\frac{dy}{dx}\right)^2 + \frac{1}{4}\left(\frac{d^2y}{dx^2}\right)^2$$

Solution

$$y = \cos(2x)$$
$$\frac{dy}{dx} = -2\sin(2x)$$
$$\frac{d^2y}{dx^2} = -4\cos(2x)$$

Hence

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 + \frac{1}{4}\left(\frac{d^2y}{dx^2}\right)^2 &= (-2\sin(2x))^2 + \frac{1}{4}(-4\cos(2x))^2 \\ &= 4\sin^2(2x) + \frac{1}{4}(16\cos^2(2x)) \\ &= 4\sin^2(2x) + 4\cos^2(2x) \\ &= 4(\sin^2(2x) + \cos^2(2x)) \\ &= 4\end{aligned}$$

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Exercise 8D

Example 9

1 Find the second derivative of each of the following:

a $2x + 5$ **b** x^8 **c** $\sqrt{x}$ **d** $(2x + 1)^4$ **e** $\sin x$

f $\cos x$ **g** e^x **h** $\log_e x$ **i** $\frac{1}{x + 1}$ **j** $\tan x$

2 Find the second derivative of each of the following:

a $\sqrt{x^5}$ **b** $(x^2 + 3)^4$ **c** $\sin\left(\frac{x}{2}\right)$ **d** $3\cos(4x + 1)$ **e** $\frac{1}{2}e^{2x+1}$

f $\log_e(2x + 1)$ **g** $3\tan(x - 4)$ **h** $4\sin^{-1}(x)$ **i** $\tan^{-1}(x)$ **j** $2(1 - 3x)^5$

3 Find $f''(x)$ if $f(x)$ is equal to:

a $6e^{3-2x}$ **b** $-8e^{-0.5x^2}$ **c** $e^{\log_e x}$ **d** $\log_e(\sin x)$

e $3\sin^{-1}\left(\frac{x}{4}\right)$ **f** $\cos^{-1}(3x)$ **g** $2\tan^{-1}\left(\frac{2x}{3}\right)$ **h** $\frac{1}{\sqrt{1 - x}}$

i $5\sin(3 - x)$ **j** $\tan(1 - 3x)$ **k** $\sec\left(\frac{x}{3}\right)$ **l** $\operatorname{cosec}\left(\frac{x}{4}\right)$

Example 10

4 Find $f''(0)$ if $f(x)$ is equal to:

a $e^{\sin x}$ **b** $e^{-\frac{1}{2}x^2}$ **c** $\sqrt{1 - x^2}$ **d** $\tan^{-1}\left(\frac{1}{x - 1}\right)$

Example 11

5 If $y = e^{\sin^{-1} x}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$.

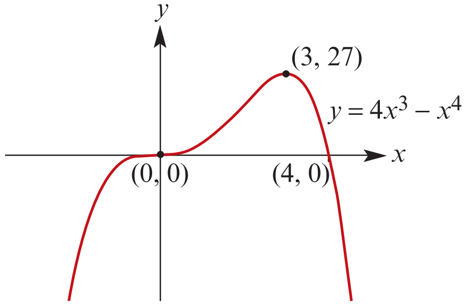
8E Points of inflection

In Mathematical Methods Units 3 & 4, you have sketched the graphs of polynomial functions and informally considered points of inflection. These are points on the graph where the gradient changes from decreasing to increasing, or vice versa.

In this section, we use the second derivative to study points of inflection more formally.

The graph of $y = 4x^3 - x^4$

As an example, we start by considering the graph of $y = 4x^3 - x^4$, which is shown below.



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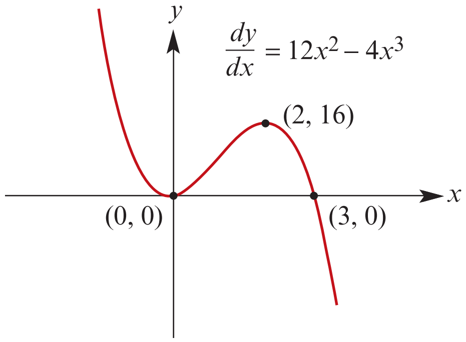
There is a local maximum at $(3, 27)$ and a stationary point of inflection at $(0, 0)$. These have been determined by considering the derivative function $\frac{dy}{dx} = 12x^2 - 4x^3$.

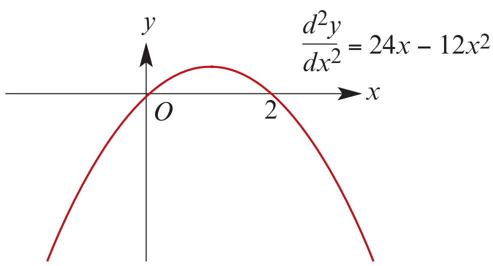
The graph of the derivative function



Note that the local maximum and the stationary point of inflection of the original graph correspond to the x-axis intercepts of the graph of the derivative. Also it can be seen that the gradient of the original graph is positive for $x < 0$ and $0 < x < 3$ and negative for $x > 3$.

The graph of the second derivative function

Further information can be obtained by considering the graph of the second derivative.



The graph of the second derivative reveals that, at the points on the original graph where $x = 0$ and $x = 2$, there are important changes in the gradient.

- At the point where $x = 0$, the gradient of $y = 4x^3 - x^4$ changes from decreasing (positive) to increasing (positive). This point is also a stationary point, but it is neither a local maximum nor a local minimum. It is known as a *stationary point of inflection*.
- At the point where $x = 2$, the gradient of $y = 4x^3 - x^4$ changes from increasing (positive) to decreasing (positive). This point is known as a *non-stationary point of inflection*. In this case, the point corresponds to a local maximum of the derivative graph.

  The gradient of $y = 4x^3 - x^4$ increases on the interval $(0, 2)$ and then decreases on the interval $(2, 3)$. The point $(2, 16)$ is the point of maximum gradient of $y = 4x^3 - x^4$ for the interval $(0, 3)$.

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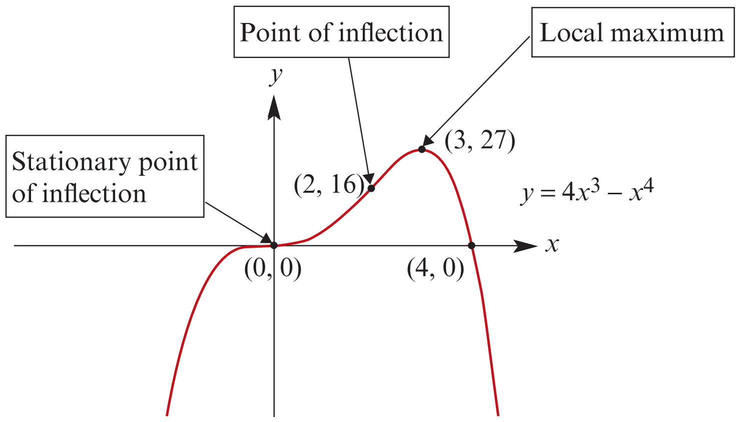


Behaviour of tangents

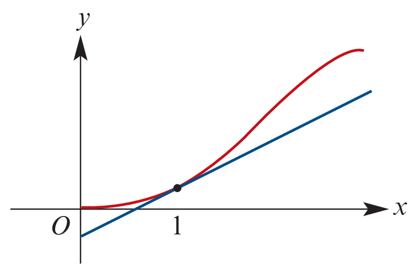
A closer look at the graph of $y = 4x^3 - x^4$ for the interval $(0, 3)$ and, in particular, the behaviour of the tangents to the graph in this interval will reveal more.

The tangents at $x = 1$, 2 and 2.5 have equations $y = 8x - 5$, $y = 16x - 16$ and $y = \frac{25}{2}x - \frac{125}{16}$ respectively. The following graphs illustrate the behaviour.

- The first diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 1$.

  The tangent lies *below* the graph in the immediate neighbourhood of $x = 1$.

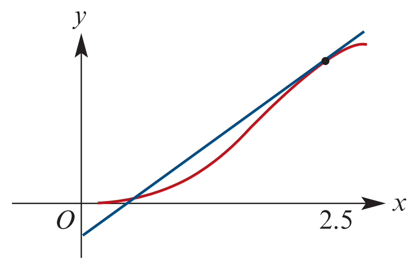
  For the interval $(0, 2)$, the gradient of the graph is increasing; the graph is said to be *concave up*.



- The second diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2.5$.

  The tangent lies *above* the graph in the immediate neighbourhood of $x = 2.5$.

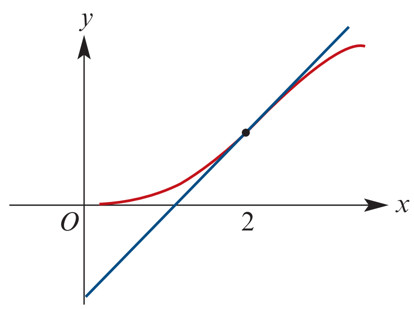
  For the interval $(2, 3)$, the gradient of the graph is decreasing; the graph is said to be *concave down*.



- The third diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2$.

  The tangent *crosses* the graph at the point $(2, 16)$.

  At $x = 2$, the gradient of the graph changes from increasing to decreasing; the point $(2, 16)$ is said to be a *point of inflection*.



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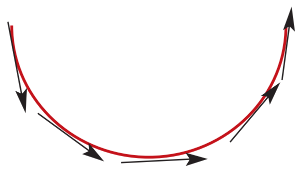
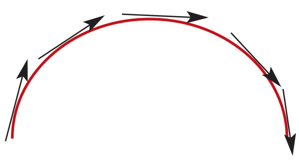
Concavity and points of inflection

We have met the ideas of concave up and concave down in the example at the beginning of this section. We now give the definitions of these ideas.

Concave up and concave down

For a curve $y = f(x)$:

- If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is strictly increasing over the interval (a, b). The curve is said to be **concave up**.
- If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is strictly decreasing over the interval (a, b). The curve is said to be **concave down**.

Concave up for an interval	Concave down for an interval
	
The tangent is below the curve at each point and the gradient is increasing i.e. $f''(x) > 0$	The tangent is above the curve at each point and the gradient is decreasing i.e. $f''(x) < 0$

Point of inflection

A point where a curve changes from concave up to concave down or from concave down to concave up is called a **point of inflection**. That is, a point of inflection occurs where the sign of the second derivative changes.

Note: At a point of inflection, the tangent will pass through the curve.

If there is a point of inflection on the graph of $y = f(x)$ at $x = a$, where both f' and f'' exist, then we must have $f''(a) = 0$. But the converse does not hold.

For example, consider $f(x) = x^4$. Then $f''(x) = 12x^2$ and so $f''(0) = 0$. But the graph of $y = x^4$ has a local minimum at $x = 0$.

From now on, we can use these new ideas in our graphing.

The graph of $y = \sin x$

Let $f\colon \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to \mathbb{R}$, $f(x) = \sin x$. Then $f'(x) = \cos x$ and $f''(x) = -\sin x$.

Hence $f'(x) = 0$ where $x = \dfrac{\pi}{2}$ and $x = -\dfrac{\pi}{2}$, and $f''(x) = 0$ where $x = 0$.

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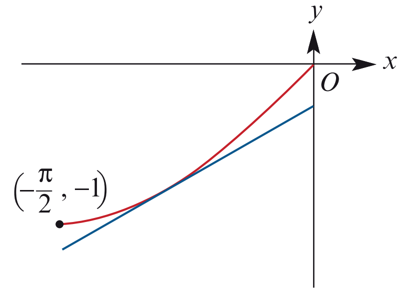
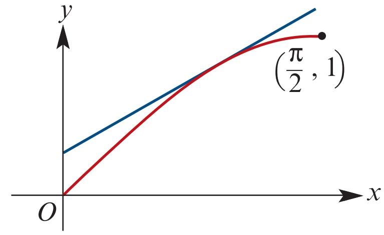
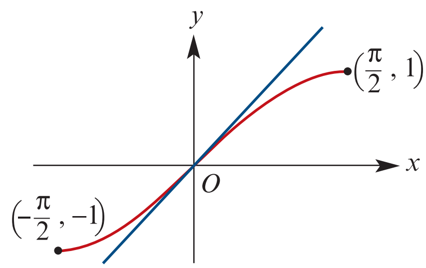
- **Concave up**
  In the interval $\left(-\frac{\pi}{2}, 0\right)$, $f'(x) > 0$ and $f''(x) > 0$.
  Note that the tangents to the curve lie below the curve and it is said to be concave up.



- **Concave down**
  In the interval $\left(0, \frac{\pi}{2}\right)$, $f'(x) > 0$ and $f''(x) < 0$.
  Note that the tangents to the curve lie above the curve and it is said to be concave down.



- **Point of inflection**
  Where $x = 0$, the tangent $y = x$ passes through the graph. There is a point of inflection at the origin. This is also the point of maximum gradient in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



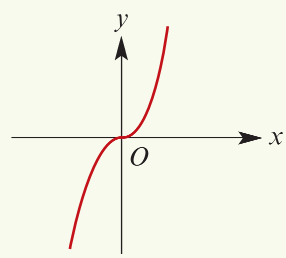
Example 12

For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up.

a $f(x) = x^3$ **b** $f(x) = -x^3$ **c** $f(x) = x^3 - 3x^2 + 1$ **d** $f(x) = \dfrac{1}{x^2 - 4}$

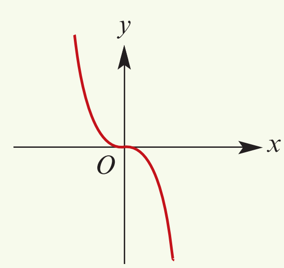
Solution

a
- There is a stationary point of inflection at $(0, 0)$.
  At $x = 0$, the gradient is zero and the curve changes from concave down to concave up.
- The curve is concave up on the interval $(0, \infty)$.
  The second derivative is positive on this interval.



Note: The tangent at $x = 0$ is the line $y = 0$.

b
- There is a stationary point of inflection at $(0, 0)$.
- The curve is concave up on the interval $(-\infty, 0)$.
  The second derivative is positive on this interval.



Note: The tangent at $x = 0$ is the line $y = 0$.

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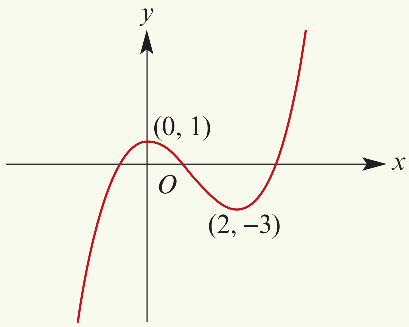
c $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

There is a local maximum at the point with coordinates (0, 1) and a local minimum at the point with coordinates (2, −3).



The second derivative is zero at $x = 1$, it is positive for $x > 1$, and it is negative for $x < 1$.

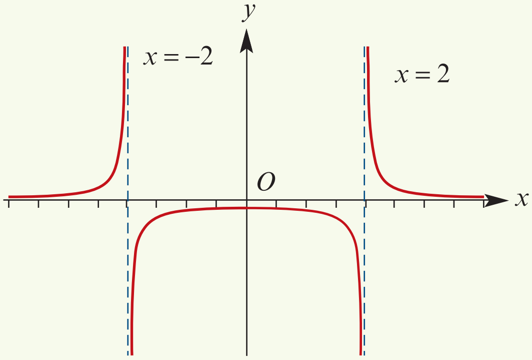
- There is a non-stationary point of inflection at $(1, -1)$.
- The curve is concave up on the interval $(1, \infty)$.

d $f(x) = \dfrac{1}{x^2 - 4}$

$f'(x) = \dfrac{-2x}{(x^2 - 4)^2}$

$f''(x) = \dfrac{2(3x^2 + 4)}{(x^2 - 4)^3}$

There is a local maximum at the point $(0, -\frac{1}{4})$.



- There is no point of inflection, as $f''(x) \neq 0$ for all x in the domain.
- $f''(x) > 0$ for $x^2 - 4 > 0$, i.e. for $x > 2$ or $x < -2$. The curve is concave up on $(2, \infty)$ and $(-\infty, -2)$.

Example 13

Sketch the graph of the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \dfrac{6}{x} - 6 + 3\log_e x$, showing all key features.

Solution

The derivative function has rule $f'(x) = \dfrac{3}{x} - \dfrac{6}{x^2} = \dfrac{3x - 6}{x^2}$.

The second derivative function has rule $f''(x) = \dfrac{12}{x^3} - \dfrac{3}{x^2} = \dfrac{12 - 3x}{x^3}$.

Stationary points

$f'(x) = 0$ implies $x = 2$. Also note that $f'(1) = -3 < 0$ and $f'(3) = \dfrac{1}{3} > 0$.

Hence there is a local minimum at the point with coordinates $(2, 3\log_e 2 - 3)$.

Points of inflection

$f''(x) = 0$ implies $x = 4$. Also note that $f''(3) = \dfrac{1}{9} > 0$ and $f''(5) = -\dfrac{3}{125} < 0$.

Hence there is a point of inflection at $\left(4, 6\log_e 2 - \dfrac{9}{2}\right)$.

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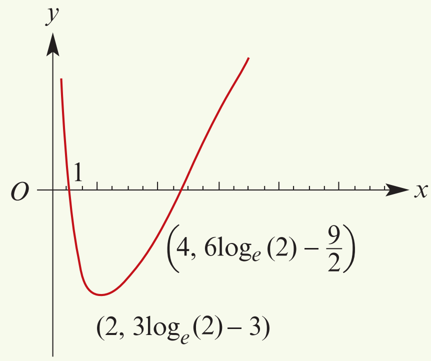
In the interval $(2, 4)$, $f''(x) > 0$, i.e. gradient is increasing. In the interval $(4, \infty)$, $f''(x) < 0$, i.e. gradient is decreasing.

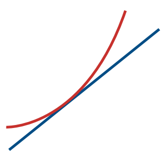
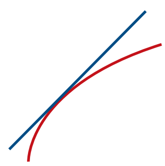
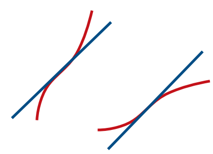
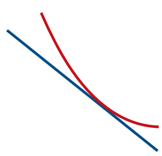
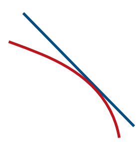
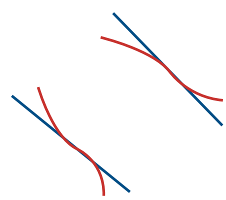
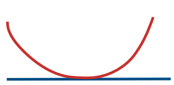
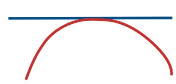
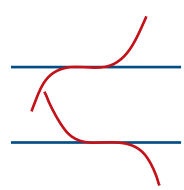
Notes:

- The point of inflection is the point of maximum gradient in the interval $(2, \infty)$.
- The x-axis intercepts of the graph occur at $x = 1$ and $x \approx 4.92$.



Use of the second derivative in graph sketching

The following table illustrates different situations for graphs of different functions $y = f(x)$.

	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} = 0$ and point of inflection
$\frac{dy}{dx} > 0$	 Curve rising and concave up	 Curve rising and concave down	 Point of inflection on rising curve
$\frac{dy}{dx} < 0$	 Curve falling and concave up	 Curve falling and concave down	 Point of inflection on falling curve
$\frac{dy}{dx} = 0$	 Local minimum	 Local maximum	 Stationary point of inflection

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The following test provides a useful method for identifying local maximums and minimums.

Second derivative test

For the graph of $y = f(x)$:

- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum, as the curve is concave up.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum, as the curve is concave down.
- If $f''(a) = 0$, then further investigation is necessary.

Example 14

Consider the function with rule $f(x) = e^{x^2}$.

a **i** Find $f'(x)$. **ii** Find $f''(x)$.

b On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [-1, 1]$. (Use a calculator to help.)

c Solve the equation $f'(x) = 0$.

d Show that $f''(x) > 0$ for all x.

e Show that the graph of $y = f(x)$ has a local minimum at the point $(0, 1)$.

f State the intervals for which:

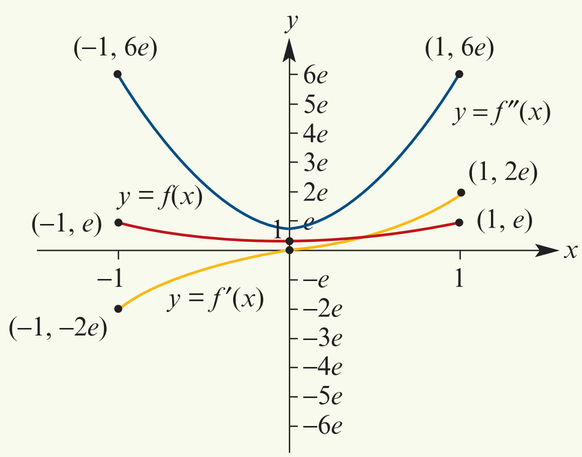
i $f'(x) > 0$ **ii** $f'(x) < 0$

Solution

a **i** For $f(x) = e^{x^2}$, the chain rule gives $f'(x) = 2xe^{x^2}$.

ii The product rule gives $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$.

b



c $f'(x) = 0$ implies $2xe^{x^2} = 0$. Thus $x = 0$.

d $f''(x) = e^{x^2}(2 + 4x^2) > 0$ for all x, as $e^{x^2} > 0$ and $2 + 4x^2 > 0$ for all x.

e Since $f'(0) = 0$ and $f''(0) = 2 > 0$, there is a local minimum at $(0, 1)$.

f **i** $f'(x) > 0$ for $x \in (0, \infty)$ **ii** $f'(x) < 0$ for $x \in (-\infty, 0)$

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Example 15

Consider the function with rule $g(x) = x^2 + 1$.

a On the one set of axes, sketch the graphs of $y = g(x)$, $y = g'(x)$ and $y = g''(x)$ for $x \in [-1, 1]$.

b Compare the graph of $y = g(x)$ with the graph of $y = f(x)$ in Example 14.

Solution

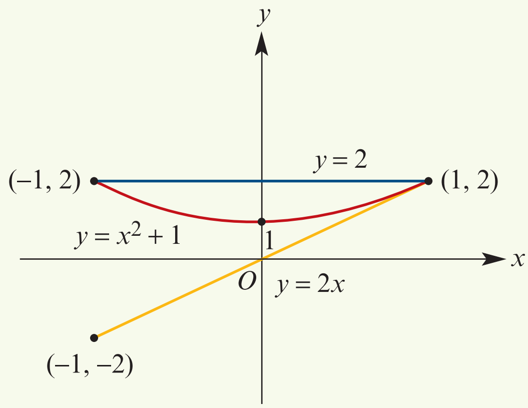
a $g(x) = x^2 + 1$

$g'(x) = 2x$

$g''(x) = 2$

The graphs of $y = g(x)$, $y = g'(x)$ and $y = g''(x)$ have been sketched using a similar scale to Example 14.

Since $g'(0) = 0$ and $g''(0) = 2 > 0$, there is a local minimum at $(0, 1)$.



b Similarities

- $g'(x) > 0$ for $x > 0$
- $g'(x) < 0$ for $x < 0$
- The graphs of $y = x^2 + 1$ and $y = e^{x^2}$ are symmetric about the y-axis.

Differences

The second derivatives reveal that the gradient of $y = e^{x^2}$ is increasing rapidly for $x > 0$, while the gradient of $y = x^2$ is increasing at a constant rate.

Example 16

Consider the graph of $y = f(x)$, where $f(x) = x^2(10 - x)$.

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 10]$.

Solution

We have $f(x) = x^2(10 - x) = 10x^2 - x^3$, $f'(x) = 20x - 3x^2$ and $f''(x) = 20 - 6x$.

a $f'(x) = 0$ implies $x(20 - 3x) = 0$, and therefore $x = 0$ or $x = \dfrac{20}{3}$.

Since $f''(0) = 20 > 0$, there is a local minimum at $(0, 0)$.

Since $f''\left(\dfrac{20}{3}\right) = -20 < 0$, there is a local maximum at $\left(\dfrac{20}{3}, \dfrac{4000}{27}\right)$.

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b $f''(x) = 0$ implies $x = \frac{10}{3}$.

We have $f''(x) < 0$ for $x > \frac{10}{3}$

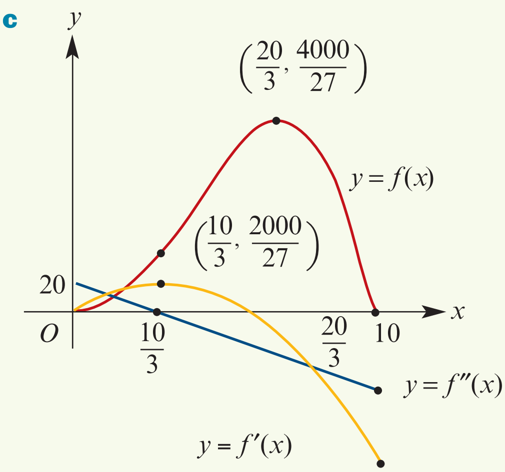
and $f''(x) > 0$ for $x < \frac{10}{3}$.

Hence there is a point of inflection at $\left(\frac{10}{3}, \frac{2000}{27}\right)$.

The gradient at this point is $\frac{100}{3}$.

Note: The maximum gradient of $y = f(x)$ is at the point of inflection.

c

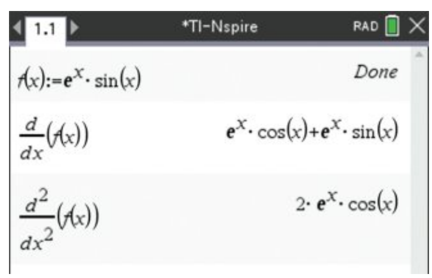


Example 17

Use a CAS calculator to find the stationary points and the points of inflection of the graph of $f(x) = e^x \sin x$ for $x \in [0, 2\pi]$.

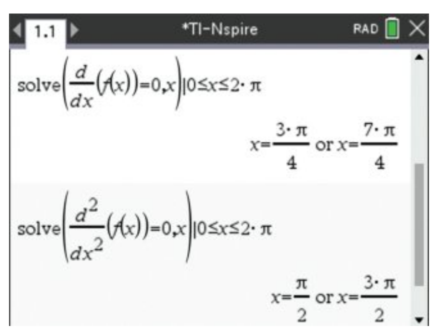
Using the TI-Nspire

- Define $f(x) = e^x \sin(x)$.
- To find the derivative, press [template key] to obtain the template $\frac{d}{d\square}\square$ and then complete as shown.
- To find the second derivative, press [template key] to obtain the template $\frac{d^2}{d\square^2}\square$ and then complete as shown.



Stationary points

- Solve the equation $\frac{d}{dx}(f(x)) = 0$ for x such that $0 \le x \le 2\pi$.
- Substitute to find the y-coordinates.
- The stationary points are $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}} e^{\frac{3\pi}{4}}\right)$ and $\left(\frac{7\pi}{4}, \frac{-1}{\sqrt{2}} e^{\frac{7\pi}{4}}\right)$.



Points of inflection

- Solve the equation $\frac{d^2}{dx^2}(f(x)) = 0$ for x such that $0 \le x \le 2\pi$.
- Note that the second derivative changes sign at each of these x-values.
- Substitute to find the y-coordinates.
- The points of inflection are $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right)$.

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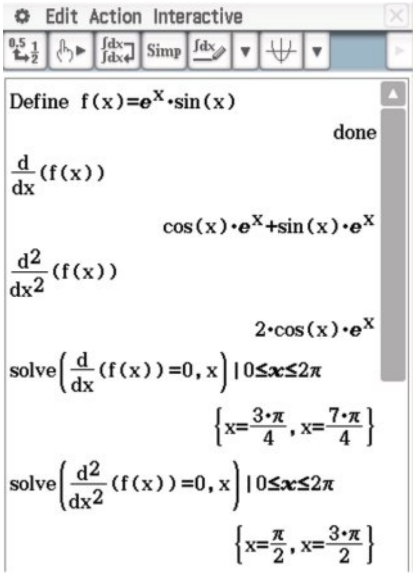
Using the Casio ClassPad

- Define $f(x) = e^x \sin(x)$.
- Find $\dfrac{d}{dx}(f(x))$ and $\dfrac{d^2}{dx^2}(f(x))$.



Stationary points

- Solve the equation $\dfrac{d}{dx}(f(x)) = 0$ for x such that $0 \le x \le 2\pi$.
- Substitute to find the y-coordinates.
- The stationary points are $\left(\dfrac{3\pi}{4}, \dfrac{1}{\sqrt{2}} e^{\frac{3\pi}{4}}\right)$ and $\left(\dfrac{7\pi}{4}, \dfrac{-1}{\sqrt{2}} e^{\frac{7\pi}{4}}\right)$.

Points of inflection

- Solve the equation $\dfrac{d^2}{dx^2}(f(x)) = 0$ for x such that $0 \le x \le 2\pi$.
- Note that the second derivative changes sign at each of these x-values.
- Substitute to find the y-coordinates.
- The points of inflection are $\left(\dfrac{\pi}{2}, e^{\frac{\pi}{2}}\right)$ and $\left(\dfrac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right)$.

Skill-sheet

Exercise 8E

1 Sketch a small portion of a continuous curve around a point $x = a$ having the property:

a $\dfrac{dy}{dx} > 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} > 0$ when $x = a$

b $\dfrac{dy}{dx} < 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} < 0$ when $x = a$

c $\dfrac{dy}{dx} > 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} < 0$ when $x = a$

d $\dfrac{dy}{dx} < 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} > 0$ when $x = a$

Example 12

2 For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

a $f(x) = x^3 - x$ **b** $f(x) = x^3 - x^2$ **c** $f(x) = x^2 - x^3$ **d** $f(x) = x^4 - x^3$

Example 13

3 Consider the graph of $y = \dfrac{1}{1 + x + x^2}$.

a Find the coordinates of the points of inflection.

b Find the coordinates of the point of intersection of the tangents at the points of inflection.

Example 14

4 Let $f(x) = xe^{x^2}$.

a **i** Find $f'(x)$. **ii** Find $f''(x)$.

b On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [-1, 1]$. (Use a calculator to help.)

c Show that $f'(x) > 0$ for all $x \in \mathbb{R}$.

d Show that $f''(0) = 0$ and that there is a point of inflection at $(0, 0)$.

e State the intervals for which:

i $f''(x) > 0$ **ii** $f''(x) < 0$

Example 16

5 Let $f\colon [0, 20] \to \mathbb{R}$, $f(x) = \dfrac{x^2}{10}(20 - x)$.

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 20]$.

6 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^3 + 6x^2 - 12$.

a **i** Find $f'(x)$. **ii** Find $f''(x)$.

b Find the coordinates of the stationary points and use the second derivative test to establish their nature.

c Use $f''(x)$ to find the coordinates of the point on the graph of $y = f(x)$ where the gradient is a minimum (the point of inflection).

7 Repeat Question 6 for each of the following functions:

a $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin x$

b $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = xe^x$

8 The graph of $y = f(x)$ has a local minimum at $x = a$ and no other stationary point 'close' to a.

a For a small value h, where $h > 0$, what can be said about the value of:

i $f'(a - h)$ **ii** $f'(a)$ **iii** $f'(a + h)$?

b What can be said about the gradient of $y = f'(x)$ for $x \in [a - h, a + h]$?

c What can be said about the value of $f''(a)$?

d Verify your observation by calculating the value of $f''(0)$ for each of the following functions:

i $f(x) = x^2$ **ii** $f(x) = -\cos x$ **iii** $f(x) = x^4$

e Can $f''(a)$ ever be less than zero if $y = f(x)$ has a local minimum at $x = a$?

9 Let $f(x) = \arctan(2x - 6) + 1$.

a Find the coordinates of the point of inflection of the graph of $y = f(x)$.

b Find the equations of the asymptotes of the graph of $y = f(x)$.

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10 Let $f: [0, 10] \to \mathbb{R}$, $f(x) = x(10 - x)e^x$.

a Find $f'(x)$ and $f''(x)$.

b Sketch the graphs of $y = f(x)$ and $y = f''(x)$ on the one set of axes for $x \in [0, 10]$.

c Find the value of x for which the gradient of the graph of $y = f(x)$ is a maximum and indicate this point on the graph of $y = f(x)$.

11 Find the coordinates of the points of inflection of $y = x - \sin x$ for $x \in [0, 4\pi]$.

12 For each of the following functions, find the values of x for which the graph of the function has a point of inflection:

a $y = \sin x$ **b** $y = \tan x$ **c** $y = \sin^{-1} x$

d $y = \sin(2x)$ **e** $y = (x + 1)\tan^{-1} x$ **f** $y = x^3 \log_e x$

13 Show that the parabola with equation $y = ax^2 + bx + c$ has no points of inflection.

14 For the curve with equation $y = 2x^3 - 9x^2 + 12x + 8$, find the values of x for which:

a $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$ **b** $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} < 0$

15 For each of the following functions, determine the coordinates of any points of inflection and the gradient of the graph at these points:

a $y = x^3 - 6x$ **b** $y = x^4 - 6x^2 + 4$ **c** $y = 3 - 10x^3 + 10x^4 - 3x^5$

d $y = (x^2 - 1)(x^2 + 1)$ **e** $y = \dfrac{x + 1}{x - 1}$ **f** $y = x\sqrt{x + 1}$

g $y = \dfrac{2x}{x^2 + 1}$ **h** $y = \sin^{-1} x$ **i** $y = \dfrac{x - 2}{(x + 2)^2}$

16 Determine the values of x for which the graph of $y = e^{-x} \sin x$ has:

a stationary points **b** points of inflection.

17 Given that $f(x) = x^3 + bx^2 + cx$ and $b^2 > 3c$, prove that:

a the graph of f has two stationary points

b the graph of f has one point of inflection

c the point of inflection is the midpoint of the line segment joining the stationary points.

18 Consider the function with rule $f(x) = 2x^2 \log_e(x)$.

a Find $f'(x)$. **b** Find $f''(x)$.

c Find the stationary points and the points of inflection of the graph of $y = f(x)$.

19 Let $f(x) = xe^{\frac{x}{3}}$. Find the stationary points and the points of inflection of the graph of f.

20 Let $f(x) = 2x\cos(5x) + (5x^2 - 6)\sin(5x)$. Show that if the graph of $y = f(x)$ has a point of inflection at $(a, f(a))$, then $\tan(5a) = \dfrac{10a}{25a^2 - 28}$.

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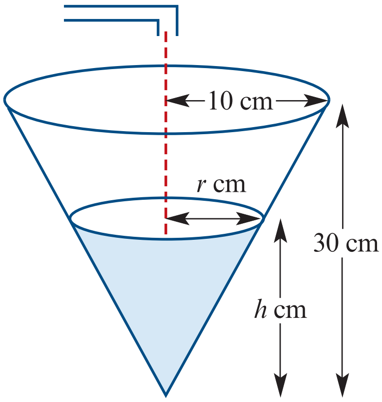
8F Related rates

Consider the situation of a right circular cone being filled from a tap.



At time t seconds:

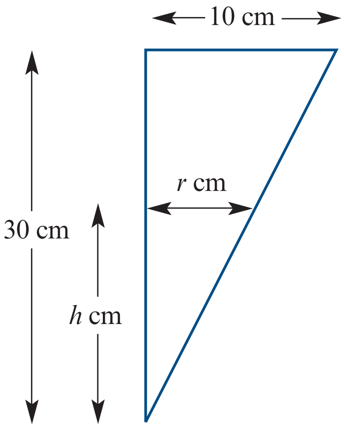
- the volume of water in the cone is V cm^3
- the height of the water in the cone is h cm
- the radius of the circular water surface is r cm.

As the water flows in, the values of V, h and r change:

- $\frac{dV}{dt}$ is the rate of change of volume with respect to time
- $\frac{dh}{dt}$ is the rate of change of height with respect to time
- $\frac{dr}{dt}$ is the rate of change of radius with respect to time.

It is clear that these rates are related to each other. The chain rule is used to establish these relationships.

For example, if the height of the cone is 30 cm and the radius of the cone is 10 cm, then similar triangles yield



$$\frac{r}{h} = \frac{10}{30}$$

$$\therefore \quad h = 3r$$

Then the chain rule is used:

$$\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dt}$$

$$= 3 \cdot \frac{dr}{dt}$$

The volume of a cone is given in general by $V = \frac{1}{3}\pi r^2 h$. Since $h = 3r$, we have

$$V = \pi r^3$$

Therefore by using the chain rule again:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$= 3\pi r^2 \cdot \frac{dr}{dt}$$

The relationships between the rates have been established.

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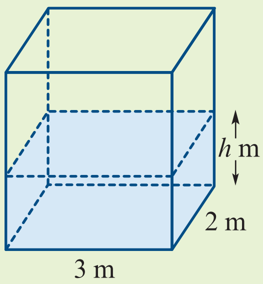
Example 18

A rectangular prism is being filled with water at a rate of 0.00042 m^3/s. Find the rate at which the height of the water is increasing.



Solution

Let t be the time in seconds after the prism begins to fill. Let V m^3 be the volume of water at time t, and let h m be the height of the water at time t.

We are given that $\frac{dV}{dt} = 0.00042$ and $V = 6h$.

Using the chain rule, the rate at which the height is increasing is

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$

Since $V = 6h$, we have $\frac{dV}{dh} = 6$ and so $\frac{dh}{dV} = \frac{1}{6}$.

Thus $\frac{dh}{dt} = \frac{1}{6} \times 0.00042$

$= 0.00007$ m/s

i.e. the height is increasing at a rate of 0.00007 m/s.

Example 19

As Steven's ice block melts, it forms a circular puddle on the floor. The radius of the puddle increases at a rate of 3 cm/min. When its radius is 2 cm, find the rate at which the area of the puddle is increasing.

Solution

The area, A, of a circle is given by $A = \pi r^2$, where r is the radius of the circle.

The rate of increase of the radius is $\frac{dr}{dt} = 3$ cm/min.

Using the chain rule, the rate of increase of the area is

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$

$$= 2\pi r \times 3$$

$$= 6\pi r$$

When $r = 2$, $\frac{dA}{dt} = 12\pi$.

Hence the area of the puddle is increasing at 12π cm^2/min.

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Example 20

A metal cube is being heated so that the side length is increasing at the rate of 0.02 cm per hour. Calculate the rate at which the volume is increasing when the side length is 5 cm.

Solution

Let x be the length of a side of the cube. Then the volume is $V = x^3$.

We are given that $\frac{dx}{dt} = 0.02$ cm/h.

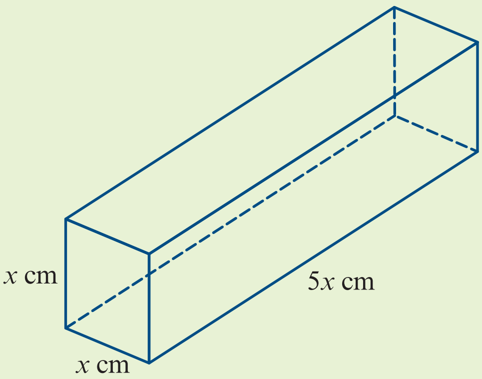
The rate of increase of volume is found using the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$$
$$= 3x^2 \times 0.02$$
$$= 0.06x^2$$

When $x = 5$, the volume of the cube is increasing at a rate of 1.5 cm^3/h.

Example 21

The diagram shows a rectangular block of ice that is x cm by x cm by $5x$ cm.



a Express the total surface area, A cm^2, in terms of x and then find $\frac{dA}{dx}$.

b If the ice is melting such that the total surface area is decreasing at a constant rate of 4 cm^2/s, calculate the rate of decrease of x when $x = 2$.

Solution

a
$$A = 4 \times 5x^2 + 2 \times x^2$$
$$= 22x^2$$
$$\frac{dA}{dx} = 44x$$

b The surface area is decreasing, so $\frac{dA}{dt} = -4$.

By the chain rule:

$$\frac{dx}{dt} = \frac{dx}{dA}\frac{dA}{dt}$$
$$= \frac{1}{44x} \times (-4)$$
$$= -\frac{1}{11x}$$

When $x = 2$, $\frac{dx}{dt} = -\frac{1}{22}$ cm/s.

Note: The rates of change of the lengths of the edges are $-\frac{1}{22}$ cm/s, $-\frac{1}{22}$ cm/s and $-\frac{5}{22}$ cm/s. The negative signs indicate that the lengths are decreasing.

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Parametric equations

Parametric equations were revised in Chapter 1. For example:

- The unit circle can be described by the parametric equations $x = \cos t$ and $y = \sin t$.
- The parabola $y^2 = 4ax$ can be described by the parametric equations $x = at^2$ and $y = 2at$.

In general, a parametric curve is specified by a pair of equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

For a point $(f(t), g(t))$ on the curve, we can consider the gradient of the tangent to the curve at this point. By the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

This gives the following result.

Gradient at a point on a parametric curve

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0$$

Note: A curve defined by parametric equations is not necessarily the graph of a function. However, each value of t determines a point on the curve, and we can use this technique to find the gradient of the curve at this point (given the tangent exists).

Example 22

A curve has parametric equations

$$x = 2t - \log_e(2t) \quad \text{and} \quad y = t^2 - \log_e(t^2)$$

Find:

a $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$

b $\dfrac{dy}{dx}$

Solution

a
$$x = 2t - \log_e(2t)$$
$$\therefore \frac{dx}{dt} = 2 - \frac{1}{t} = \frac{2t-1}{t}$$

$$y = t^2 - \log_e(t^2)$$
$$\therefore \frac{dy}{dt} = 2t - \frac{2}{t} = \frac{2t^2-2}{t}$$

b
$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{2t^2-2}{t} \times \frac{t}{2t-1} = \frac{2t^2-2}{2t-1}$$

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Example 23

For the curve defined by the given parametric equations, find the gradient of the tangent at a point $P(x, y)$ on the curve, in terms of the parameter t:

a $x = 16t^2$ and $y = 32t$

b $x = 2\sin(3t)$ and $y = -2\cos(3t)$

Solution

a $x = 16t^2$ and so $\dfrac{dx}{dt} = 32t$

$y = 32t$ and so $\dfrac{dy}{dt} = 32$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{32}{32t} = \frac{1}{t}$$

The gradient of the tangent at the point $P(16t^2, 32t)$ is $\dfrac{1}{t}$, for $t \neq 0$.

b $x = 2\sin(3t)$ and so $\dfrac{dx}{dt} = 6\cos(3t)$

$y = -2\cos(3t)$ and so $\dfrac{dy}{dt} = 6\sin(3t)$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6\sin(3t)}{6\cos(3t)} = \tan(3t)$$

The gradient of the tangent at the point $P(2\sin(3t), -2\cos(3t))$ is $\tan(3t)$.

The second derivative at a point on a parametric curve

If the parametric equations for a curve define a function for which the second derivative exists, then $\dfrac{d^2y}{dx^2}$ can be found as follows:

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} \quad \text{where } y' = \frac{dy}{dx}$$

Example 24

A curve is defined by the parametric equations $x = t - t^3$ and $y = t - t^2$. Find $\dfrac{d^2y}{dx^2}$.

Solution

Let $y' = \dfrac{dy}{dx}$. Then $y' = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$.

We have $x = t - t^3$ and $y = t - t^2$, giving $\dfrac{dx}{dt} = 1 - 3t^2$ and $\dfrac{dy}{dt} = 1 - 2t$.

Therefore

$$y' = \frac{1 - 2t}{1 - 3t^2}$$

Next differentiate y' with respect to t, using the quotient rule:

$$\frac{dy'}{dt} = \frac{-2(3t^2 - 3t + 1)}{(3t^2 - 1)^2}$$

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Hence

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{dy'}{dt} \div \frac{dx}{dt}\\ &= \frac{-2(3t^2-3t+1)}{(3t^2-1)^2} \times \frac{1}{1-3t^2}\\ &= \frac{-2(3t^2-3t+1)}{(1-3t^2)^3}\\ &= \frac{-6t^2+6t-2}{(1-3t^2)^3}\end{aligned}$$

Exercise 8F

Example 18
Example 19

1 The radius of a spherical balloon is 2.5 m and its volume is increasing at a rate of 0.1 m^3/min.

a At what rate is the radius increasing?

b At what rate is the surface area increasing?

Example 20

2 When a wine glass is filled to a depth of x cm, it contains V cm^3 of wine, where $V = 4x^{\frac{3}{2}}$. If the depth is 9 cm and wine is being poured into the glass at 10 cm^3/s, at what rate is the depth changing?

3 Variables x and y are connected by the equation $y = 2x^2 + 5x + 2$. Given that x is increasing at the rate of 3 units per second, find the rate of increase of y with respect to time when $x = 2$.

Example 21

4 If a hemispherical bowl of radius 6 cm contains water to a depth of x cm, the volume, V cm^3, of the water is given by

$$V = \frac{1}{3}\pi x^2(18 - x)$$

Water is poured into the bowl at a rate of 3 cm^3/s. Find the rate at which the water level is rising when the depth is 2 cm.

5 Variables p and v are linked by the equation $pv = 1500$. Given that p is increasing at the rate of 2 units per minute, find the rate of decrease of v at the instant when $p = 60$.

6 A circular metal disc is being heated so that the radius is increasing at the rate of 0.01 cm per hour. Find the rate at which the area is increasing when the radius is 4 cm.

7 The area of a circle is increasing at the rate of 4 cm^2 per second. At what rate is the circumference increasing at the instant when the radius is 8 cm?

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Example 22 **8** A curve has parametric equations $x = \frac{1}{1+t^2}$ and $y = \frac{t}{1+t^2}$.

a Find $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

b Find $\frac{dy}{dx}$.

9 A curve has parametric equations $x = 2t + \sin(2t)$ and $y = \cos(2t)$. Find $\frac{dy}{dx}$.

Example 23 **10** A curve has parametric equations $x = t - \cos t$ and $y = \sin t$. Find the equation of the tangent to the curve when $t = \frac{\pi}{6}$.

11 A point moves along the curve $y = x^2$ such that its velocity parallel to the x-axis is a constant 2 cm/s (i.e. $\frac{dx}{dt} = 2$). Find its velocity parallel to the y-axis (i.e. $\frac{dy}{dt}$) when:

a $x = 3$

b $y = 16$

12 Variables x and y are related by $y = \frac{2x-6}{x}$. They are given by $x = f(t)$ and $y = g(t)$, where f and g are functions of time. Find $f'(t)$ when $y = 1$, given that $g'(t) = 0.4$.

13 A particle moves along the curve

$$y = 10\cos^{-1}\left(\frac{x-5}{5}\right)$$

in such a way that its velocity parallel to the x-axis is a constant 3 cm/s. Find its velocity parallel to the y-axis when:

a $x = 6$

b $y = \frac{10\pi}{3}$

14 The radius, r cm, of a sphere is increasing at a constant rate of 2 cm/s. Find, in terms of π, the rate at which the volume is increasing at the instant when the volume is 36π cm^3.

15 Liquid is poured into a container at a rate of 12 cm^3/s. The volume of liquid in the container is V cm^3, where $V = \frac{1}{2}(h^2 + 4h)$ and h is the height of the liquid in the container. Find, when $V = 16$:

a the value of h

b the rate at which h is increasing

16 The area of an ink blot, which is always circular in shape, is increasing at a rate of 3.5 cm^2/s. Find the rate of increase of the radius when the radius is 3 cm.

17 A tank in the shape of a prism has constant cross-sectional area A cm^2. The amount of water in the tank at time t seconds is V cm^3 and the height of the water is h cm. Find the relationship between $\frac{dV}{dt}$ and $\frac{dh}{dt}$.

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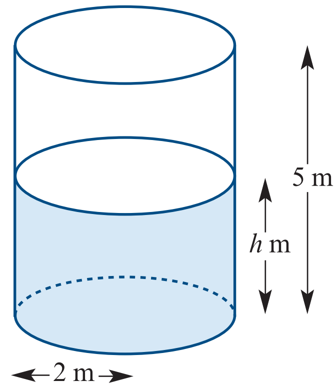
18 A cylindrical tank 5 m high with base radius 2 m is initially full of water. Water flows out through a hole at the bottom of the tank at the rate of $\sqrt{h}$ m^3/h, where h metres is the depth of the water remaining in the tank after t hours. Find:



a $\dfrac{dh}{dt}$

b **i** $\dfrac{dV}{dt}$ when $V = 10\pi$ m^3

ii $\dfrac{dh}{dt}$ when $V = 10\pi$ m^3

19 For the curve defined by the parametric equations $x = 2\cos t$ and $y = \sin t$, find the equation of the tangent to the curve at the point:

a $\left(\sqrt{2}, \dfrac{\sqrt{2}}{2}\right)$ **b** $(2\cos t, \sin t)$, where t is any real number.

20 For the curve defined by the parametric equations $x = 2\sec\theta$ and $y = \tan\theta$, find the equation of:

a the tangent at the point where $\theta = \dfrac{\pi}{4}$ **b** the normal at the point where $\theta = \dfrac{\pi}{4}$

c the tangent at the point $(2\sec\theta, \tan\theta)$.

21 For the curve with parametric equations $x = 2\sec t - 3$ and $y = 4\tan t + 2$, find:

a $\dfrac{dy}{dx}$ **b** the equation of the tangent to the curve when $t = \dfrac{\pi}{4}$.

22 A curve is defined by the parametric equations $x = \sec t$ and $y = \tan t$.

a Find the equation of the normal to the curve at the point $(\sec t, \tan t)$.

b Let A and B be the points of intersection of the normal to the curve with the x-axis and y-axis respectively, and let O be the origin. Find the area of $\triangle OAB$.

c Find the value of t for which the area of $\triangle OAB$ is $4\sqrt{3}$.

23 A curve is specified by the parametric equations $x = e^{2t} + 1$ and $y = 2e^t + 1$ for $t \in \mathbb{R}$.

a Find the gradient of the curve at the point $(e^{2t} + 1, 2e^t + 1)$.

b State the domain of the relation.

c Sketch the graph of the relation.

d Find the equation of the tangent at the point where $t = \log_e\left(\dfrac{1}{2}\right)$.

Example 24

24 For the parametric curve given by $x = t^2 + 1$ and $y = t(t-3)^2$, for $t \in \mathbb{R}$, find:

a $\dfrac{dy}{dx}$ **b** the coordinates of the stationary points

c $\dfrac{d^2y}{dx^2}$ **d** the coordinates of the points of inflection.

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8G Rational functions

A rational function has a rule of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. There is a huge variety of different types of curves in this particular family of functions. An example of a rational function is

$$f(x) = \frac{x^2 + 2x + 3}{x^2 + 4x - 1}$$

The following are also rational functions, but are not given in the form used in the definition of a rational function:

$$g(x) = 1 + \frac{1}{x} \qquad h(x) = x - \frac{1}{x^2 + 2}$$

Their rules can be rewritten as shown:

$$g(x) = \frac{x}{x} + \frac{1}{x} = \frac{x+1}{x} \qquad h(x) = \frac{x(x^2 + 2)}{x^2 + 2} - \frac{1}{x^2 + 2} = \frac{x^3 + 2x - 1}{x^2 + 2}$$

Graphing rational functions

For sketching graphs, it is also useful to write rational functions in the alternative form, that is, with a division performed if possible. For example:

$$f(x) = \frac{8x^2 - 3x + 2}{x} = \frac{8x^2}{x} - \frac{3x}{x} + \frac{2}{x} = 8x - 3 + \frac{2}{x}$$

For this example, we can see that $\frac{2}{x} \to 0$ as $x \to \pm\infty$, so the graph of $y = f(x)$ will approach the line $y = 8x - 3$ as $x \to \pm\infty$.

We say that the line $y = 8x - 3$ is a **non-vertical asymptote** of the graph. This is a line which the graph approaches as $x \to \pm\infty$.

Important features of a sketch graph are:

- asymptotes
- axis intercepts
- stationary points
- points of inflection.

Methods for sketching graphs of rational functions include:

- adding the y-coordinates (ordinates) of two simple graphs
- taking the reciprocals of the y-coordinates (ordinates) of a simple graph.

Addition of ordinates

Key points for addition of ordinates

- When the two graphs have the same ordinate, the y-coordinate of the resultant graph will be double this.
- When the two graphs have opposite ordinates, the y-coordinate of the resultant graph will be zero (an x-axis intercept).
- When one of the two ordinates is zero, the resulting ordinate is equal to the other ordinate.

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Example 25

Sketch the graph of $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = \dfrac{x^2+1}{x}$.

Solution

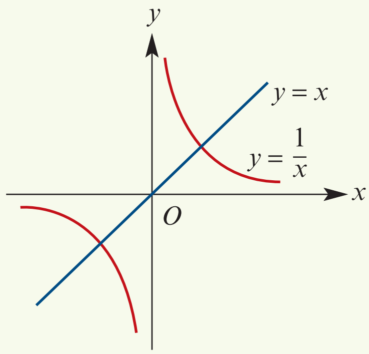
Asymptotes The vertical asymptote has equation $x = 0$, i.e. the y-axis.

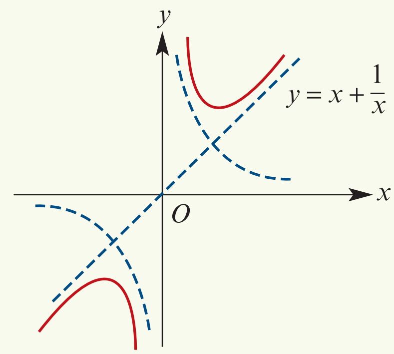
Dividing through gives

$$f(x) = \frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$$

Note that $\dfrac{1}{x} \to 0$ as $x \to \pm\infty$. Therefore the graph of $y = f(x)$ approaches the graph of $y = x$ as $x \to \pm\infty$. The non-vertical asymptote has equation $y = x$.

Addition of ordinates The graph of $y = f(x)$ can be obtained by adding the y-coordinates of the graphs of $y = x$ and $y = \dfrac{1}{x}$.





Intercepts There is no y-axis intercept, as the domain of f is $\mathbb{R}\setminus\{0\}$. There are no x-axis intercepts, as the equation $\dfrac{x^2+1}{x} = 0$ has no solutions.

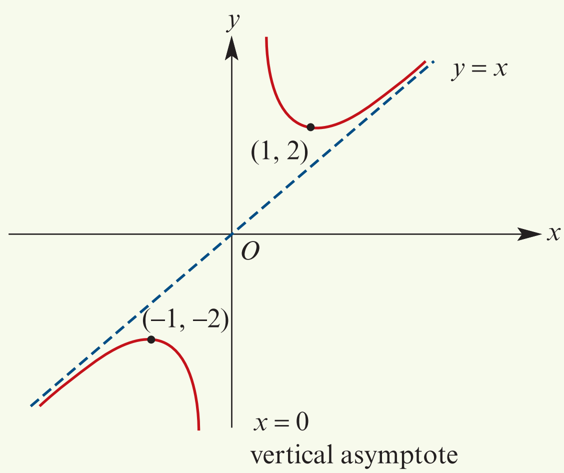
Stationary points

$$f(x) = x + \frac{1}{x}$$

$$\therefore \quad f'(x) = 1 - \frac{1}{x^2}$$

Thus $f'(x) = 0$ implies $x^2 = 1$, i.e. $x = \pm 1$.

As $f(1) = 2$ and $f(-1) = -2$, the stationary points are $(1, 2)$ and $(-1, -2)$.



Points of inflection

$$f''(x) = \frac{2}{x^3}$$

Therefore $f''(x) \neq 0$, for all x in the domain of f, and so there are no points of inflection.

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Example 26

Sketch the graph of $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{x^4 + 2}{x^2}$.

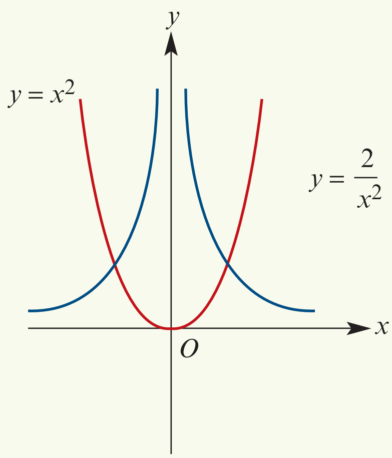
Solution

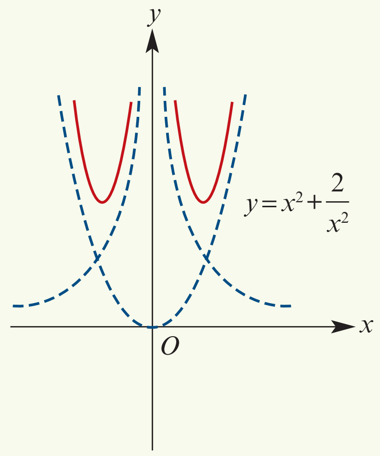
Asymptotes The vertical asymptote has equation $x = 0$.

Dividing through gives

$$f(x) = x^2 + \frac{2}{x^2}$$

Addition of ordinates





Intercepts There are no axis intercepts.

Stationary points

$$f(x) = x^2 + 2x^{-2}$$

$$\therefore \quad f'(x) = 2x - 4x^{-3}$$

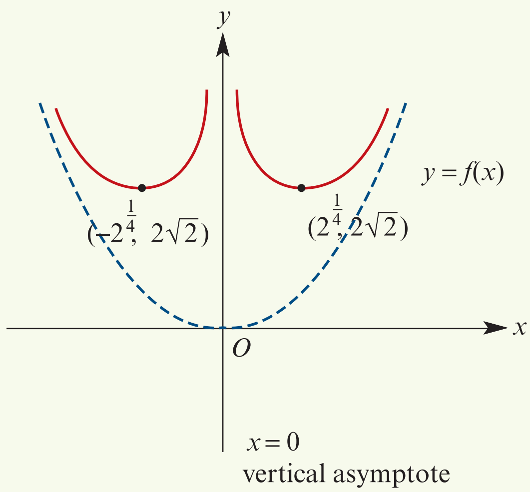
When $f'(x) = 0$,

$$2x - \frac{4}{x^3} = 0$$

$$2x^4 - 4 = 0$$

$$\therefore \quad x = \pm 2^{\frac{1}{4}}$$

The stationary points have coordinates $(2^{\frac{1}{4}}, 2\sqrt{2})$ and $(-2^{\frac{1}{4}}, 2\sqrt{2})$.



Points of inflection

Since $f''(x) = 2 + 12x^{-4} > 0$, there are no points of inflection.

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Example 27

Sketch the graph of $y = \dfrac{x^3 + 2}{x}$, $x \neq 0$.

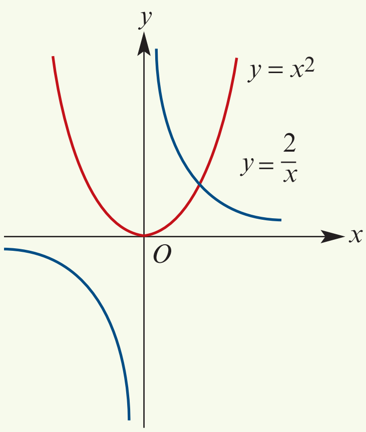
Solution

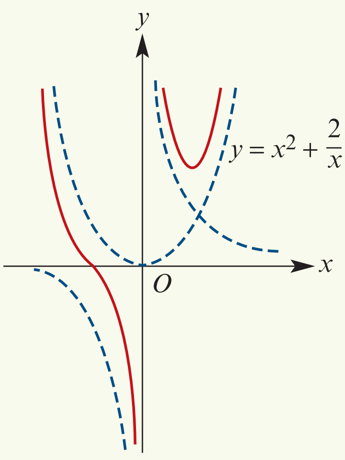
Asymptotes The vertical asymptote has equation $x = 0$.

Divide through to obtain

$$y = x^2 + \frac{2}{x}$$

Addition of ordinates





Intercepts Consider $y = 0$, which implies $x^3 + 2 = 0$, i.e. $x = -\sqrt[3]{2}$.

Stationary points

$$y = x^2 + 2x^{-1}$$

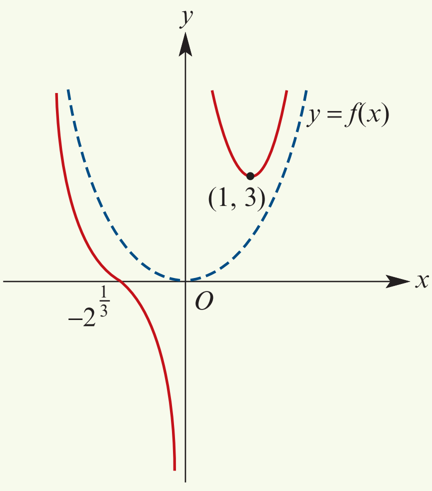
$$\therefore \quad \frac{dy}{dx} = 2x - 2x^{-2}$$

Thus $\dfrac{dy}{dx} = 0$ implies $x - \dfrac{1}{x^2} = 0$

$$x^3 = 1$$

$$\therefore \quad x = 1$$

The turning point has coordinates $(1, 3)$.



Points of inflection

$$\frac{d^2y}{dx^2} = 2 + 4x^{-3}$$

Thus $\dfrac{d^2y}{dx^2} = 0$ implies $x = -\sqrt[3]{2}$. There is a point of inflection at $(-\sqrt[3]{2}, 0)$.

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Reciprocal of ordinates

This is the second method for sketching graphs of rational functions. We will consider functions of the form $f(x) = \dfrac{1}{Q(x)}$, where $Q(x)$ is a quadratic function.

Example 28

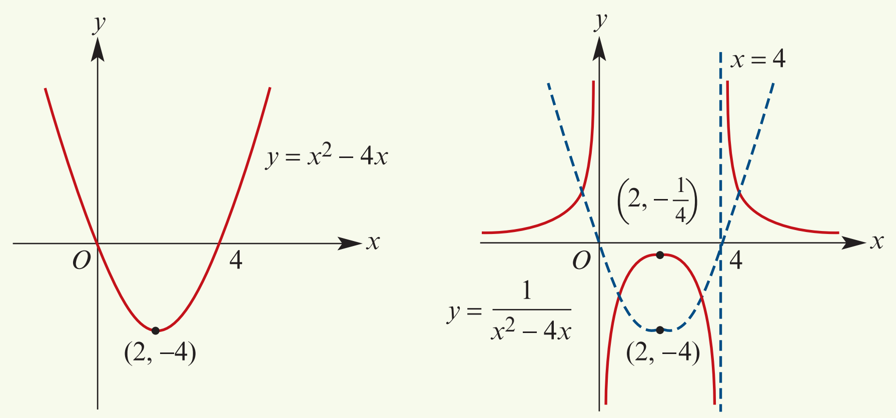
Sketch the graph of $f : \mathbb{R} \setminus \{0, 4\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2 - 4x}$.

Solution

$$f(x) = \frac{1}{x^2 - 4x} = \frac{1}{x(x-4)}$$

Asymptotes The vertical asymptotes have equations $x = 0$ and $x = 4$. The non-vertical asymptote has equation $y = 0$, since $f(x) \to 0$ as $x \to \pm\infty$.

Reciprocal of ordinates To sketch the graph of $y = f(x)$, first sketch the graph of $y = Q(x)$. In this case, we have $Q(x) = x^2 - 4x$.



Summary of properties of reciprocal functions

- The x-axis intercepts of the original function determine the equations of the vertical asymptotes for the reciprocal function.
- The reciprocal of a positive number is positive.
- The reciprocal of a negative number is negative.
- A graph and its reciprocal will intersect at a point if the y-coordinate is 1 or -1.
- Local maximums of the original function produce local minimums of the reciprocal.
- Local minimums of the original function produce local maximums of the reciprocal.
- If $g(x) = \dfrac{1}{f(x)}$, then $g'(x) = -\dfrac{f'(x)}{(f(x))^2}$. Therefore, at any given point, the gradient of the reciprocal function is opposite in sign to that of the original function.

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Further graphing

So far we have only started to consider the diversity of rational functions. Here we look at some further rational functions and employ a variety of techniques.

Example 29

Sketch the graph of $y = \dfrac{4x^2 + 2}{x^2 + 1}$.

Solution

Axis intercepts

When $x = 0$, $y = 2$.

Since $\dfrac{4x^2 + 2}{x^2 + 1} > 0$ for all x, there are no x-axis intercepts.

Stationary points

Using the quotient rule:

$$\frac{dy}{dx} = \frac{4x}{(x^2 + 1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{4(1 - 3x^2)}{(x^2 + 1)^3}$$

Thus $\dfrac{dy}{dx} = 0$ implies $x = 0$.

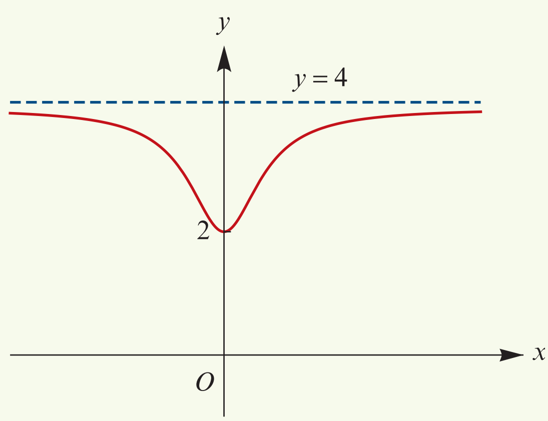
When $x = 0$, $\dfrac{d^2y}{dx^2} = 4 > 0$. Hence there is a local minimum at $(0, 2)$.

Points of inflection $\dfrac{d^2y}{dx^2} = 0$ implies $x = \pm\dfrac{\sqrt{3}}{3}$

Asymptotes

$$y = \frac{4x^2 + 2}{x^2 + 1} = 4 - \frac{2}{x^2 + 1}$$

The line $y = 4$ is a horizontal asymptote, since $\dfrac{2}{x^2 + 1} \to 0$ as $x \to \pm\infty$.



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Example 30

Sketch the graph of $y = \dfrac{4x^2 - 4x + 1}{x^2 - 1}$.

Solution

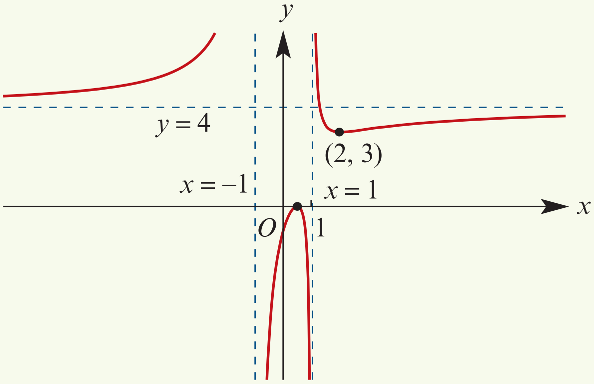
Axis intercepts

When $x = 0$, $y = -1$.

When $y = 0$, $4x^2 - 4x + 1 = 0$

$$(2x-1)^2 = 0$$

$$\therefore \quad x = \tfrac{1}{2}$$



Stationary points

Using the quotient rule:

$$\frac{dy}{dx} = \frac{2(2x^2 - 5x + 2)}{(x^2-1)^2}$$

Thus $\dfrac{dy}{dx} = 0$ implies $x = \tfrac{1}{2}$ or $x = 2$.

There is a local maximum at $(\tfrac{1}{2}, 0)$ and a local minimum at $(2, 3)$.

The nature of the stationary points can most easily be determined through using $\dfrac{dy}{dx} = \dfrac{2(2x-1)(x-2)}{(x^2-1)^2}$. (Observe that the denominator is always positive.)

Points of inflection

$$\frac{d^2y}{dx^2} = -\frac{2(4x^3 - 15x^2 + 12x - 5)}{(x^2-1)^3}$$

Thus $\dfrac{d^2y}{dx^2} = 0$ implies $4x^3 - 15x^2 + 12x - 5 = 0$, and so $x = \dfrac{1}{4}(5 + 3^{\frac{4}{3}} + 3^{\frac{2}{3}}) \approx 2.85171$

Asymptotes

By solving $x^2 - 1 = 0$, we find that the graph has vertical asymptotes $x = 1$ and $x = -1$.

Since $\dfrac{4x^2 - 4x + 1}{x^2 - 1} = 4 - \dfrac{4x - 5}{x^2 - 1}$, there is a horizontal asymptote $y = 4$.

The graph crosses this asymptote at the point $(\tfrac{5}{4}, 4)$.

While the next example is not a rational function, it can be graphed using similar techniques.

Example 31

Let $y = \dfrac{x+1}{\sqrt{x-1}}$.

a Find the maximal domain.

b Find the coordinates and the nature of any stationary points of the graph.

c Find the equation of the vertical asymptote.

d Sketch the graph.

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Solution

a For $\dfrac{x+1}{\sqrt{x-1}}$ to be defined, we require $\sqrt{x-1} > 0$, i.e. $x > 1$.

The maximal domain is $(1, \infty)$.

b Using the quotient and chain rules: $\dfrac{dy}{dx} = \dfrac{x-3}{2(x-1)^{\frac{3}{2}}}$ and $\dfrac{d^2y}{dx^2} = \dfrac{7-x}{4(x-1)^{\frac{5}{2}}}$

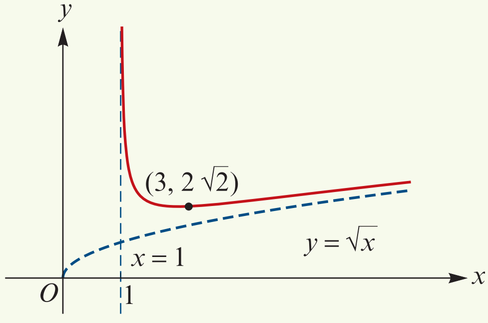
Thus $\dfrac{dy}{dx} = 0$ implies $x = 3$. When $x = 3$, $\dfrac{d^2y}{dx^2} > 0$.

There is a local minimum at $(3, 2\sqrt{2})$.

c As $x \to 1$, $y \to \infty$. Hence $x = 1$ is a vertical asymptote.

As $x \to \infty$, $y \to \dfrac{x}{\sqrt{x}} = \sqrt{x}$.

d



Skill-sheet

Exercise 8G

Example 25
Example 26
Example 27
Example 28

1 Sketch the graph of each of the following, labelling all axis intercepts, turning points and asymptotes:

a $y = \dfrac{1}{x^2 - 2x}$ **b** $y = \dfrac{x^4 + 1}{x^2}$ **c** $y = \dfrac{1}{(x-1)^2 + 1}$

d $y = \dfrac{x^2 - 1}{x}$ **e** $y = \dfrac{x^3 - 1}{x^2}$ **f** $y = \dfrac{x^2 + x + 1}{x}$

g $y = \dfrac{4x^3 - 8}{x}$ **h** $y = \dfrac{1}{x^2 + 1}$ **i** $y = \dfrac{1}{x^2 - 1}$

j $y = \dfrac{x^2}{x^2 + 1} = 1 - \dfrac{1}{x^2 + 1}$ **k** $y = \dfrac{1}{x^2 - x - 2}$ **l** $y = \dfrac{1}{4 + 3x - x^2}$

2 Sketch the graph of each of the following, labelling all axis intercepts, turning points and asymptotes:

a $f(x) = \dfrac{1}{9 - x^2}$ **b** $g(x) = \dfrac{1}{(x-2)(3-x)}$ **c** $h(x) = \dfrac{1}{x^2 + 2x + 4}$

d $f(x) = \dfrac{1}{x^2 + 2x + 1}$ **e** $g(x) = x^2 + \dfrac{1}{x^2} + 2$

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3 The equation of a curve is $y = 4x + \frac{1}{x}$. Find:

a the coordinates of the turning points

b the equation of the tangent to the curve at the point where $x = 2$.

4 Find the x-coordinates of the points on the curve $y = \frac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

5 Find the gradient of the curve $y = \frac{2x - 4}{x^2}$ at the point where it crosses the x-axis.

6 For the curve $y = x - 5 + \frac{4}{x}$, find:

a the coordinates of the points of intersection with the axes

b the equations of all asymptotes

c the coordinates of all turning points.

Use this information to sketch the curve.

7 If x is positive, find the least value of $x + \frac{4}{x^2}$.

8 For positive values of x, sketch the graph of $y = x + \frac{4}{x}$, and find the least value of y.

9 **a** Find the coordinates of the stationary points of the curve $y = \frac{(x - 3)^2}{x}$ and determine the nature of each stationary point.

b Sketch the graph of $y = \frac{(x - 3)^2}{x}$.

10 **a** Find the coordinates of the turning point(s) of the curve $y = 8x + \frac{1}{2x^2}$ and determine the nature of each point.

b Sketch the graph of $y = 8x + \frac{1}{2x^2}$.

11 Determine the asymptotes, intercepts and stationary points for the graph of the relation $y = \frac{x^3 + 3x^2 - 4}{x^2}$. Hence sketch the graph.

12 Consider the relation $y = \frac{4x^2 + 8}{2x + 1}$.

a State the maximal domain.

b Find $\frac{dy}{dx}$.

c Hence find the coordinates and nature of all stationary points.

d Find the equations of all asymptotes.

e State the range of this relation.

Example 29

13 Consider the function with rule $f(x) = \frac{x^2 + 4}{x^2 - 5x + 4}$.

a Find the equations of all asymptotes.

b Find the coordinates and nature of all stationary points.

c Sketch the graph of $y = f(x)$. Include the coordinates of the points of intersection of the graph with the horizontal asymptote.

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Example 30

14 Let $y = \dfrac{2x^2 + 2x + 3}{2x^2 - 2x + 5}$.

a Find the equations of all asymptotes.

b Find the coordinates and nature of all stationary points.

c Find the coordinates of all points of inflection.

d Sketch the graph of the relation, noting where the graph crosses any asymptotes.

15 Sketch the graph of each of the following, labelling all axis intercepts, turning points and asymptotes:

a $y = \dfrac{x^3 - 3x}{(x-1)^2}$ **b** $y = \dfrac{(x+1)(x-3)}{x^2 - 4}$ **c** $y = \dfrac{(x-2)(x+1)}{x(x-1)}$

d $y = \dfrac{x^2 - 2x - 8}{x^2 - 2x}$ **e** $y = \dfrac{8x^2 + 7}{4x^2 - 4x - 3}$

Example 31

16 Consider the function with rule $f(x) = \dfrac{x}{\sqrt{x-2}}$.

a Find the maximal domain. **b** Find $f'(x)$.

c Hence find the coordinates and nature of all stationary points.

d Find the equation of the vertical asymptote.

17 Consider the function with rule $f(x) = \dfrac{x^2 + x + 7}{\sqrt{2x+1}}$.

a Find the maximal domain. **b** Find $f(0)$. **c** Find $f'(x)$.

d Hence find the coordinates and nature of all stationary points.

e Find the equation of the vertical asymptote.

8H A summary of differentiation

It is appropriate at this stage to review the techniques of differentiation of Specialist Mathematics.

The derivatives of the standard functions also need to be reviewed in preparation for the chapters on antidifferentiation.

Differentiation techniques

Function	Derivative
$f(x)$	$f'(x)$
$af(x),\ a \in \mathbb{R}$	$af'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$
$f(x)\,g(x)$	$f'(x)\,g(x) + f(x)\,g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$
$f(g(x))$	$f'(g(x))\,g'(x)$

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Derivatives of standard functions

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\log_e \lvert ax \rvert$	$\frac{1}{x}$

$f(x)$	$f'(x)$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$

$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2 + x^2}$

- **First derivative** $f'(x)$ and $\frac{dy}{dx}$ are the first derivatives of $f(x)$ and y respectively.
- **Second derivative** $f''(x)$ and $\frac{d^2y}{dx^2}$ are the second derivatives of $f(x)$ and y respectively.
- **Chain rule** Using Leibniz notation, the chain rule is written as $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

  An important result from the chain rule is $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Exercise 8H

1 Find the second derivative of each of the following:

a x^{10} **b** $(2x+5)^8$ **c** $\sin(2x)$ **d** $\cos\left(\frac{x}{3}\right)$

e $\tan\left(\frac{3x}{2}\right)$ **f** e^{-4x} **g** $\log_e(6x)$ **h** $\sin^{-1}\left(\frac{x}{4}\right)$

i $\cos^{-1}(2x)$ **j** $\tan^{-1}\left(\frac{x}{2}\right)$ **k** $(x+2)\arctan(x-4)$

2 Find the first derivative of each of the following:

a $(1-4x^2)^3$ **b** $\frac{1}{\sqrt{2-x}}$ **c** $\sin(\cos x)$ **d** $\cos(\log_e x)$

e $\tan\left(\frac{1}{x}\right)$ **f** $e^{\cos x}$ **g** $\log_e(4-3x)$ **h** $\sin^{-1}(1-x)$

i $\cos^{-1}(2x+1)$ **j** $\tan^{-1}(x+1)$ **k** $\cos^{-1}\left(\frac{9}{x}\right)$

3 Find $\frac{dy}{dx}$ for each of the following:

a $y = \frac{\log_e x}{x}$ **b** $y = \frac{x^2+2}{x^2+1}$ **c** $y = 1 - \tan^{-1}(1-x)$

d $y = \log_e\left(\frac{e^x}{e^x+1}\right)$ **e** $x = \sqrt{\sin y + \cos y}$ **f** $y = \log_e(x + \sqrt{1+x^2})$

g $y = \sin^{-1}(e^x)$ **h** $y = \frac{\sin x}{e^x + 1}$

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4 **a** If $y = ax + \frac{b}{x}$, find:

i $\frac{dy}{dx}$ **ii** $\frac{d^2y}{dx^2}$

b Hence show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y$.

5 **a** If $y = \sin(2x) + 3\cos(2x)$, find:

i $\frac{dy}{dx}$ **ii** $\frac{d^2y}{dx^2}$

b Hence show that $\frac{d^2y}{dx^2} + 4y = 0$.

8I Implicit differentiation

The rules for circles, ellipses and many other curves are not expressible in the form $y = f(x)$ or $x = f(y)$. Equations such as

$$x^2 + y^2 = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{(y-3)^2}{4} = 1$$

are said to be implicit equations. In this section, we introduce a technique for finding $\frac{dy}{dx}$ for such relations. The technique is called **implicit differentiation**, and it involves differentiating both sides of an equation with respect to x.

If two algebraic expressions are always equal, then the value of each expression must change in an identical way as one of the variables changes.

That is, if p and q are expressions in x and y such that $p = q$, for all x and y, then

$$\frac{dp}{dx} = \frac{dq}{dx} \quad \text{and} \quad \frac{dp}{dy} = \frac{dq}{dy}$$

For example, consider the relation $x = y^3$. In Example 5, we found that $\frac{dy}{dx} = \frac{1}{3y^2}$.

We can also use implicit differentiation to obtain this result. Differentiate both sides of the equation $x = y^3$ with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(y^3) \qquad (1)$$

To simplify the right-hand side using the chain rule, we let $u = y^3$. Then

$$\frac{d}{dx}(y^3) = \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = 3y^2 \times \frac{dy}{dx}$$

Hence equation (1) becomes

$$1 = 3y^2 \times \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3y^2} \qquad \text{provided } y \neq 0$$

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Example 32

For each of the following, find $\frac{dy}{dx}$ by implicit differentiation:

a $x^3 = y^2$

b $xy = 2x + 1$

Solution

a Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(y^2)$$

$$3x^2 = 2y\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2}{2y}$$

b Differentiate both sides with respect to x:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(2x + 1)$$

$$\frac{d}{dx}(xy) = 2$$

Use the product rule on the left-hand side:

$$y + x\frac{dy}{dx} = 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{2 - y}{x}$$

Example 33

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$.

Solution

Note that $x^2 + y^2 = 1$ leads to

$$y = \pm\sqrt{1 - x^2} \quad \text{or} \quad x = \pm\sqrt{1 - y^2}$$

So y is not a function of x, and x is not a function of y. Implicit differentiation should be used. Since $x^2 + y^2 = 1$ is the unit circle, we can also find the derivative geometrically.

Method 1: Using geometry

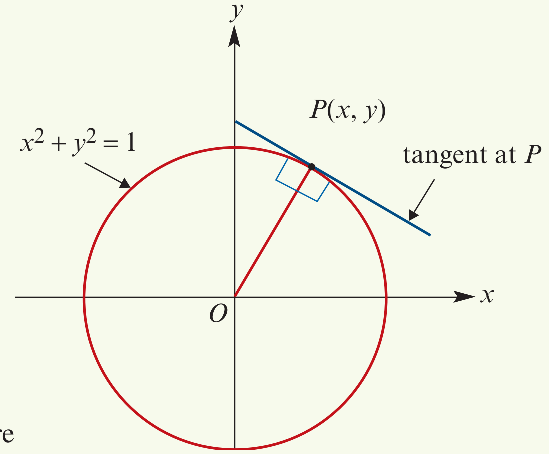
Let $P(x, y)$ be a point on the unit circle with $x \neq 0$.

The gradient of OP is $\frac{\text{rise}}{\text{run}} = \frac{y}{x}$.

Since the radius is perpendicular to the tangent for a circle, the gradient of the tangent is $-\frac{x}{y}$, provided $y \neq 0$.

That is, $\frac{dy}{dx} = -\frac{x}{y}$.

From the graph, when $y = 0$ the tangents are parallel to the y-axis, hence $\frac{dy}{dx}$ is not defined.



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Method 2: Using implicit differentiation

$$x^2 + y^2 = 1$$

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{(differentiate both sides with respect to } x\text{)}$$

$$\therefore \quad 2y\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y} \qquad \text{for } y \neq 0$$

Example 34

Given $xy - y - x^2 = 0$, find $\dfrac{dy}{dx}$.

Solution

Method 1: Expressing y as a function of x

$$xy - y - x^2 = 0$$

$$y(x-1) = x^2$$

$$y = \frac{x^2}{x-1}$$

Therefore $\quad y = x + 1 + \dfrac{1}{x-1} \qquad$ for $x \neq 1$

Hence $\quad \dfrac{dy}{dx} = 1 - \dfrac{1}{(x-1)^2}$

$$= \frac{(x-1)^2 - 1}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2} \qquad \text{for } x \neq 1$$

Method 2: Using implicit differentiation

$$xy - y - x^2 = 0$$

$$\therefore \quad \frac{d}{dx}(xy) - \frac{dy}{dx} - \frac{d}{dx}(x^2) = \frac{d}{dx}(0) \qquad \text{(differentiate both sides with respect to } x\text{)}$$

$$\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) - \frac{dy}{dx} - 2x = 0 \qquad \text{(product rule)}$$

$$x\frac{dy}{dx} - \frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx}(x-1) = 2x - y$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x-y}{x-1} \qquad \text{for } x \neq 1$$

This can be checked, by substitution of $y = \dfrac{x^2}{x-1}$, to confirm that the results are identical.

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Example 35

Consider the curve with equation $2x^2 - 2xy + y^2 = 5$.

a Find $\dfrac{dy}{dx}$.

b Find the gradient of the tangent to the curve at the point $(1, 3)$.

Solution

a Neither x nor y can be expressed as a function, so implicit differentiation must be used.

$$2x^2 - 2xy + y^2 = 5$$

$$\frac{d}{dx}(2x^2) - \frac{d}{dx}(2xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$

$$4x - \left(2x \cdot \frac{dy}{dx} + y \cdot 2\right) + 2y\frac{dy}{dx} = 0 \qquad \text{(by the product and chain rules)}$$

$$4x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 4x$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 4x$$

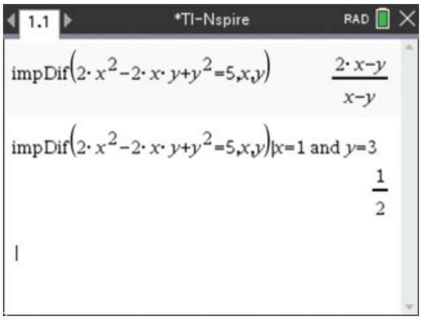
$$\therefore \quad \frac{dy}{dx} = \frac{2y - 4x}{2y - 2x}$$

$$= \frac{y - 2x}{y - x} \qquad \text{for } x \neq y$$

b When $x = 1$ and $y = 3$, the gradient is $\dfrac{3 - 2}{3 - 1} = \dfrac{1}{2}$.

Using the TI-Nspire

- For implicit differentiation, use (menu) > **Calculus** > **Implicit Differentiation** or just type impdif(.
- Complete as shown. This gives $\dfrac{dy}{dx}$ in terms of x and y.
- The gradient at the point $(1, 3)$ is found by substituting $x = 1$ and $y = 3$ as shown.



Note: If the positions of x and y are interchanged, then the result is $\dfrac{dx}{dy}$.

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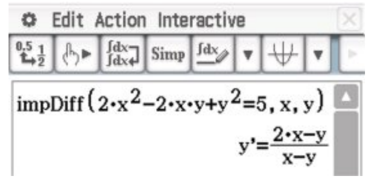
Using the Casio ClassPad

- Enter and highlight the equation $2x^2 - 2xy + y^2 = 5$.
- Go to **Interactive** > **Calculation** > **impDiff**.
- Complete with x as the independent variable and y as the dependent variable.



Skill-sheet

Exercise 8I

Example 32
Example 33

1 For each of the following, find $\frac{dy}{dx}$ using implicit differentiation:

a $x^2 - 2y = 3$ **b** $x^2y = 1$ **c** $x^3 + y^3 = 1$
d $y^3 = x^2$ **e** $x - \sqrt{y} = 2$ **f** $xy - 2x + 3y = 0$
g $y^2 = 4ax$ **h** $4x + y^2 - 2y - 2 = 0$

Example 34

2 Find $\frac{dy}{dx}$ for each of the following:

a $(x+2)^2 - y^2 = 4$ **b** $\frac{1}{x} + \frac{1}{y} = 1$ **c** $y = (x+y)^2$
d $x^2 - xy + y^2 = 1$ **e** $y = x^2e^y$ **f** $\sin y = \cos^2 x$
g $\sin(x-y) = \sin x - \sin y$ **h** $y^5 - x\sin y + 3y^2 = 1$

Example 35

3 For each of the following, find the equation of the tangent at the indicated point:

a $y^2 = 8x$ at $(2, -4)$ **b** $x^2 - 9y^2 = 9$ at $\left(5, \frac{4}{3}\right)$
c $xy - y^2 = 1$ at $\left(\frac{17}{4}, 4\right)$ **d** $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at $(0, -3)$

4 Find $\frac{dy}{dx}$ in terms of x and y, given that $\log_e(y) = \log_e(x) + 1$.

5 Find the gradient of the curve $x^3 + y^3 = 9$ at the point $(1, 2)$.

6 A curve is defined by the equation $x^3 + y^3 + 3xy - 1 = 0$. Find the gradient of the curve at the point $(2, -1)$.

7 Given that $\tan x + \tan y = 3$, find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$.

8 Find the gradient at the point $(1, -3)$ on the curve with equation $y^2 + xy - 2x^2 = 4$.

9 Consider the curve with equation $x^3 + y^3 = 28$.

a Obtain an expression for $\frac{dy}{dx}$. **b** Show that $\frac{dy}{dx}$ cannot be positive.
c Calculate the value of $\frac{dy}{dx}$ when $x = 1$.

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10 The equation of a curve is $2x^2 + 8xy + 5y^2 = -3$. Find the equation of the two tangents which are parallel to the x-axis.

11 The equation of a curve C is $x^3 + xy + 2y^3 = k$, where k is a constant.

a Find $\frac{dy}{dx}$ in terms of x and y.

b The curve C has a tangent parallel to the y-axis. Show that the y-coordinate at the point of contact satisfies $216y^6 + 4y^3 + k = 0$.

c Hence show that $k \leq \frac{1}{54}$.

d Find the possible value(s) of k in the case where $x = -6$ is a tangent to C.

12 The equation of a curve is $x^2 - 2xy + 2y^2 = 4$.

a Find an expression for $\frac{dy}{dx}$ in terms of x and y.

b Find the coordinates of each point on the curve at which the tangent is parallel to the x-axis.

13 Consider the curve with equation $y^2 + x^3 = 1$.

a Find $\frac{dy}{dx}$ in terms of x and y.

b Find the coordinates of the points where $\frac{dy}{dx} = 0$.

c Find the coordinates of the points where $\frac{dx}{dy} = 0$.

d Describe the behaviour as $x \to -\infty$.

e Express y in terms of x.

f Find the coordinates of the points of inflection of the curve.

g Use a calculator to help you sketch the graph of $y^2 + x^3 = 1$.

14 The equation of a circle is $(x - 2)^2 + (y - 2)^2 = 9$.

a Find $\frac{dy}{dx}$ in terms of x and y.

b Find the gradient of the tangent to the circle at the point in the first quadrant where $x = 1$.

15 Given that $\frac{dy}{dx} = (6 - y)^2$, find $\frac{d^2y}{dx^2}$ in terms of y.

16 For the curve given by $4x^2 + 3xy + y^2 = 14$, find the value of $\frac{d^2y}{dx^2}$ at the point $(1, 2)$.

17 Find the gradient of the curve with equation $2x^2\cos(y) + 2xy = \frac{\pi^2}{3}$ at the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

18 For the circle $x^2 + y^2 = 169$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(5, 12)$.

19 Find the equation of the tangent to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, 1)$.

20 For the curve with equation $4x^2 + y^2 = 1$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

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Chapter summary

Assign-ment

Nrich

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\log_e \lvert ax \rvert$	$\frac{1}{x}$

$f(x)$	$f'(x)$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$

$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2 + x^2}$

- If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ and $\frac{d^2y}{dx^2} = f''(x)$.

Rational functions

- A rational function has a rule of the form:

$$\begin{aligned} f(x) &= \frac{a(x)}{b(x)} && \text{where } a(x) \text{ and } b(x) \text{ are polynomials} \\ &= q(x) + \frac{r(x)}{b(x)} && \text{(quotient–remainder form)} \end{aligned}$$

- Vertical asymptotes occur where $b(x) = 0$.
- The non-vertical asymptote has equation $y = q(x)$.
- The x-axis intercepts occur where $a(x) = 0$.
- The y-axis intercept is $f(0) = \frac{a(0)}{b(0)}$, provided $b(0) \neq 0$.
- The stationary points occur where $f'(x) = 0$.
- If $f(x) = \frac{1}{b(x)}$, first sketch the graph of $y = b(x)$ and then use reciprocals of ordinates to sketch the graph of $y = f(x)$.
- If $f(x) = q(x) + \frac{r(x)}{b(x)}$, use addition of ordinates of $y = q(x)$ and $y = \frac{r(x)}{b(x)}$ to sketch the graph of $y = f(x)$.

Reciprocal functions

- The x-axis intercepts of the original function determine the equations of the vertical asymptotes for the reciprocal function.
- The reciprocal of a positive number is positive.
- The reciprocal of a negative number is negative.
- A graph and its reciprocal will intersect at a point if the y-coordinate is 1 or -1.
- Local maximums of the original function produce local minimums of the reciprocal.
- Local minimums of the original function produce local maximums of the reciprocal.
- If $g(x) = \frac{1}{f(x)}$, then $g'(x) = -\frac{f'(x)}{(f(x))^2}$. Therefore, at any given point, the gradient of the reciprocal function is opposite in sign to that of the original function.

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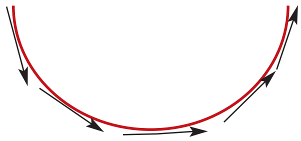
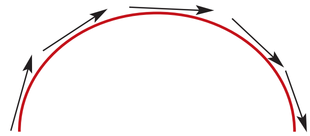
Use of the second derivative in graph sketching

- Concave up: $f''(x) > 0$



- Concave down: $f''(x) < 0$



- A **point of inflection** is where the curve changes from concave up to concave down or from concave down to concave up.
- At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.
- Second derivative test For the graph of $y = f(x)$:
  - If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
  - If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
  - If $f''(a) = 0$, then further investigation is necessary.

Implicit differentiation

- Many curves are not defined by a rule of the form $y = f(x)$ or $x = f(y)$; for example, the unit circle $x^2 + y^2 = 1$. Implicit differentiation is used to find the gradient at a point on such a curve. To do this, we differentiate both sides of the equation with respect to x.
- Using operator notation:

$$\frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx} \qquad \text{(use of chain rule)}$$

$$\frac{d}{dx}(x^2 y) = 2xy + x^2\frac{dy}{dx} \qquad \text{(use of product rule)}$$

Technology-free questions

1 Find $\frac{dy}{dx}$ if:

a $y = x\tan x$ **b** $y = \tan(\tan^{-1} x)$ **c** $y = \cos(\sin^{-1} x)$ **d** $y = \sin^{-1}(2x - 1)$

2 Find $f''(x)$ if:

a $f(x) = \tan x$ **b** $f(x) = \log_e(\tan x)$ **c** $f(x) = x\sin^{-1} x$ **d** $f(x) = \sin(e^x)$

3 For each of the following, state the coordinates of the point(s) of inflection:

a $y = x^3 - 8x^2$ **b** $y = \sin^{-1}(x - 2)$ **c** $y = \log_e(x) + \frac{1}{x}$ **d** $y = \frac{1}{x^2} - \frac{1}{x^3}$

4 Consider a function $f : \mathbb{R} \to \mathbb{R}$. Prove that if the graph of f is concave up, then the graph of $g(x) = e^{f(x)}$ is also concave up.

5 Let $g(x) = (x - a)^3 f(x)$, where f is twice differentiable. Show that $g''(a) = 0$.

6 Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that all the values of f are positive. Prove that if the graph of f is concave down, then the graph of $g(x) = \log_e(f(x))$ is also concave down.

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7 Let $f(x) = \dfrac{1}{\arccos(x)}$. Find the domain and rule of the derivative function f'.

8 Let $f : \left[\pi, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $f(x) = \sin x$.

a Sketch the graphs of f and f^{-1} on the same set of axes.

b Find the derivative of f^{-1}.

c Find the point on the graph of f^{-1} where the tangent has a gradient of -2.

9 A curve is defined parametrically by $x = \arcsin(t)$ and $y = \log_e(1 - t^2)$ for $t \in (-1, 1)$.

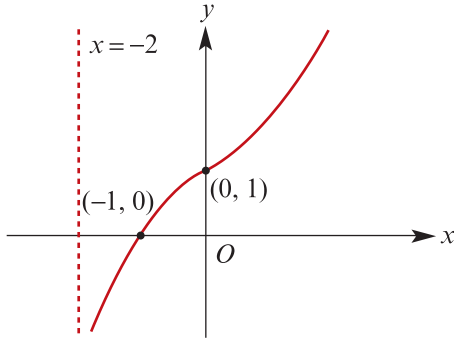
a Find the value of $\dfrac{dy}{dx}$ when $t = \dfrac{1}{2}$.

b Find the value of $\dfrac{d^2y}{dx^2}$ when $t = \dfrac{1}{2}$.

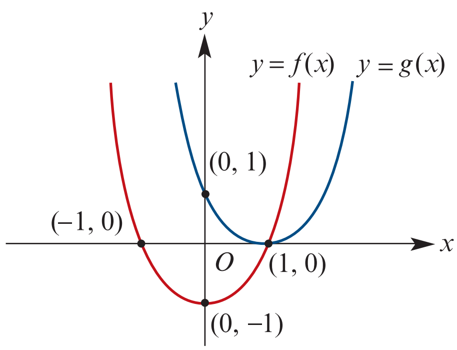
10 This is the graph of $y = f(x)$.
Sketch the graphs of:

a $y = \dfrac{1}{f(x)}$

b $y = f^{-1}(x)$



11 These are the graphs of $y = f(x)$ and $y = g(x)$, where f and g are quadratic functions.



a Sketch the graphs of:

i $y = f(x) + g(x)$

ii $y = \dfrac{1}{f(x) + g(x)}$

iii $y = \dfrac{1}{f(x)} + \dfrac{1}{g(x)}$

b Use the points given to determine the rules $y = f(x)$ and $y = g(x)$.

c Hence determine, in simplest form, the rules:

i $y = f(x) + g(x)$ **ii** $y = \dfrac{1}{f(x) + g(x)}$ **iii** $y = \dfrac{1}{f(x)} + \dfrac{1}{g(x)}$

12 Find $\dfrac{dy}{dx}$ by implicit differentiation:

a $x^2 + 2xy + y^2 = 1$ **b** $x^2 + 2x + y^2 + 6y = 10$ **c** $\dfrac{2}{x} + \dfrac{1}{y} = 4$

d $(x + 1)^2 + (y - 3)^2 = 1$ **e** $\cos(x) + \sin(y) = 1$ **f** $x \log_e(y) = 10$

13 A point moves along the curve $y = x^3$ in such a way that its velocity parallel to the x-axis is a constant 3 cm/s. Find its velocity parallel to the y-axis when:

a $x = 6$ **b** $y = 8$

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Review

14 Consider the function $f\colon \mathbb{R} \setminus \{-3, 3\} \to \mathbb{R}$, $f(x) = \dfrac{x+1}{x^2-9}$.

a Show that $f'(x) < 0$ for all $x \in \mathbb{R} \setminus \{-3, 3\}$.

b Find the coordinates of the axis intercepts of the graph of f.

c Find the equations of the asymptotes of the graph of f.

15 Given that the graph of $y = x^3 + ax^2 + bx - 5$ has a point of inflection at $(1, 5)$, find the values of a and b.

16 For the parametric curve defined by $x = t^2 + 1$ and $y = t^3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

17 Given that $\dfrac{dy}{dx} = e^{2x}\arctan(y)$, find $\dfrac{d^2y}{dx^2}$.

Multiple-choice questions

1 The equation of the tangent to $x^2 + y^2 = 1$ at the point with coordinates $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ is

A $y = -x$ **B** $y = -x + 2\sqrt{2}$ **C** $y = -x + 1$

D $y = -2\sqrt{x} + 2$ **E** $y = -x + \sqrt{2}$

2 If $f(x) = 2x^2 + 3x - 20$, then the graph of $y = \dfrac{1}{f(x)}$ has

A x-axis intercepts at $x = \dfrac{5}{2}$ and $x = -4$

B vertical asymptotes at $x = \dfrac{5}{2}$ and $x = 4$

C vertical asymptotes at $x = -\dfrac{5}{2}$ and $x = 4$

D a local minimum at $\left(-\dfrac{3}{4}, -\dfrac{169}{8}\right)$

E a local maximum at $\left(-\dfrac{3}{4}, -\dfrac{8}{169}\right)$

3 The coordinates of the points of inflection of $y = \sin x$ for $x \in [0, 2\pi]$ are

A $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(-\dfrac{\pi}{2}, -1\right)$ **B** $(\pi, 0)$ **C** $(0, 0)$, $(\pi, 0)$ and $(2\pi, 0)$

D $(1, 0)$ **E** $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$, $\left(\dfrac{3\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ and $\left(\dfrac{5\pi}{4}, -\dfrac{1}{\sqrt{2}}\right)$

4 Let $g(x) = e^{-x}f(x)$, where the function f is twice differentiable. There is a point of inflection on the graph of $y = g(x)$ at $(a, g(a))$. An expression for $f''(a)$ in terms of $f'(a)$ and $f(a)$ is

A $f''(a) = f(a) + f'(a)$ **B** $f''(a) = 2f(a)f'(a)$ **C** $f''(a) = 2f(a) + f'(a)$

D $f''(a) = \dfrac{f'(a)}{f(a)}$ **E** $f''(a) = 2f'(a) - f(a)$

5 If $x = t^2$ and $y = t^3$, then $\dfrac{dx}{dy}$ is equal to

A $\dfrac{1}{t}$ **B** $\dfrac{2}{3t}$ **C** $\dfrac{3t}{2}$ **D** $\dfrac{2t}{3}$ **E** $\dfrac{3}{2t}$

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6 If $y = \cos^{-1}\left(\frac{4}{x}\right)$ and $x > 4$, then $\frac{dy}{dx}$ is equal to

A $\frac{-1}{\sqrt{16 - x^2}}$ **B** $\frac{-4}{\sqrt{1 - 16x^2}}$ **C** $\frac{-4x}{\sqrt{x^2 - 16}}$ **D** $\frac{4}{x\sqrt{x^2 - 16}}$ **E** $\frac{4}{\sqrt{x^2 - 16}}$

7 The gradient of the line that is perpendicular to the curve $4y^2 - 6xy - 2x^2 = 2$ at the point $(1, 2)$ is

A $-\frac{2}{3}$ **B** $\frac{2}{5}$ **C** $-\frac{5}{8}$ **D** $\frac{7}{5}$ **E** $-\frac{3}{7}$

8 Let $y = \sin^{-1}\left(\frac{x}{2}\right)$ for $x \in [0, 1]$. Then $\frac{d^2y}{dx^2}$ is equal to

A $\cos^{-1}\left(\frac{x}{2}\right)$ **B** $x(4 - x^2)^{-\frac{3}{2}}$ **C** $\frac{-x}{\sqrt{4 - x^2}}$

D $\frac{-x}{\sqrt{4 - x^2}\,(4 - x^2)}$ **E** $\frac{-1}{\sqrt{4 - x^2}}$

9 If $y = \tan^{-1}\left(\frac{1}{3x}\right)$, then $\frac{dy}{dx}$ is equal to

A $\frac{1}{3(1 + x^2)}$ **B** $\frac{-1}{3(1 + x^2)}$ **C** $\frac{1}{3(1 + 9x^2)}$ **D** $\frac{-3}{9x^2 + 1}$ **E** $\frac{9x^2}{9x^2 + 1}$

10 Which of the following statements is false for the graph of $y = \cos^{-1}(x)$, for $x \in [-1, 1]$ and $y \in [0, \pi]$?

A The gradient of the graph is negative for $x \in (-1, 1)$.

B The graph has a point of inflection at $\left(0, \frac{\pi}{2}\right)$.

C The gradient of the graph has a minimum value of -1.

D The gradient of the graph is undefined at the point $(-1, \pi)$.

E At $x = \frac{1}{2}$, $y = \frac{\pi}{3}$.

11 Given that $\frac{dy}{dx} = e^x \cos^2(y)$, the value of $\frac{d^2y}{dx^2}$ at the point $\left(0, \frac{\pi}{6}\right)$ is

A $-\frac{\sqrt{3}}{2}$ **B** $\frac{3}{4}$ **C** $\frac{3}{8}(2 - \sqrt{3})$ **D** $\frac{3}{4}(2 - \sqrt{3})$ **E** $\frac{1}{2}$

12 If $\sin x = e^y$ for $0 < x < \pi$, then $\frac{dy}{dx}$ is equal to

A $\tan x$ **B** $\cot x$ **C** $\sec x$ **D** $\operatorname{cosec} x$ **E** $-\tan x$

13 Let $y = \tan t$, where $t = w - \frac{1}{w}$ and $w = \log_e x$. The value of $\frac{dy}{dx}$ at $x = e$ is

A 0 **B** $\frac{2}{e}$ **C** 12 **D** $\frac{12}{e}$ **E** $\tan e$

14 The second derivative of a function f is given by $f''(x) = x\sin x - 2$. How many points of inflection are there on the graph of $y = f(x)$ for $x \in (-10, 10)$?

A 0 **B** 2 **C** 4 **D** 6 **E** 8

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Review

Extended-response questions

1 **a** Consider the cubic polynomial $f(x) = 2x^3 - 5x^2 - 4x$.

i Find the coordinates of the two turning points A and B on the graph of f.

ii Find the coordinates of the point of inflection C on the graph of f.

iii Show that C is the midpoint of the line segment AB.

b Now consider a cubic polynomial $f(x) = ax^3 + bx^2 + cx$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

i Find the condition on a, b and c for which the graph of f has two turning points.

ii Prove that if the graph of f has two turning points A and B, then the point of inflection C is the midpoint of the line segment AB.

2 Assume that the graph of $f(x) = ax^3 + bx^2 + cx + d$ has a point of inflection at $P(p, q)$.

a Express p and q in terms of a, b, c and d.

b Define $g(x) = f(x + p) - q$. This translates the point of inflection to the origin. Show that g is an odd function. (That is, show that $g(-x) = -g(x)$ for all x.)

3 In this question, we consider higher derivatives. For example, the derivative of the second derivative of f is called the third derivative of f and is denoted by $f^{(3)}$. In general, the nth derivative of f is denoted by $f^{(n)}$.

a Find the first three derivatives of $f(x) = xe^x$.

b Conjecture a rule for the nth derivative of $f(x) = xe^x$.

c Prove that your rule is correct using induction.

d Conjecture a rule for the nth derivative of $f(x) = x^2e^x$, and then prove that it is correct using induction.

4 **a** Let $f(x) = x^n$, where $n \in \mathbb{N}$. Using mathematical induction, prove that $f'(x) = nx^{n-1}$. (Hint: You will need to use the product rule.)

b Let $f(x) = \dfrac{1}{g(x)}$. Using first principles, prove that

$$f'(x) = -\frac{g'(x)}{(g(x))^2}$$

c Now let $f(x) = x^{-n}$, where $n \in \mathbb{N}$. Using parts **a** and **b**, prove that $f'(x) = -nx^{-n-1}$.

5 Consider the curve defined by the parametric equations

$$x = \sin t \quad \text{and} \quad y = \sin\left(t + \frac{\pi}{3}\right) \qquad \text{for } 0 < t < \frac{\pi}{2}$$

a This curve can be described in the form $y = f(x)$ for a function f. Find the rule, domain and range of f.

b Find the equation of the tangent to the curve at $t = \dfrac{\pi}{6}$.

c Find the coordinates of the local maximum of the curve.

d Show that the curve is concave down.

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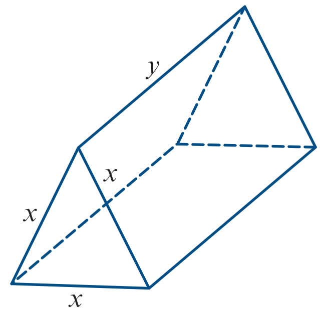
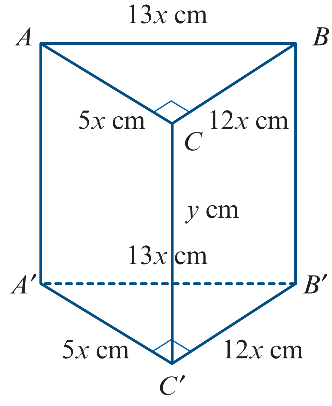
6 This diagram shows a solid triangular prism with edge lengths as shown. All measurements are in cm. The volume is 2000 cm^3. The surface area is A cm^2.



- **a** Express A in terms of x and y.
- **b** Establish a relationship between x and y.
- **c** Hence express A in terms of x.
- **d** Sketch the graph of A against x.
- **e** Hence determine the minimum surface area of the prism.

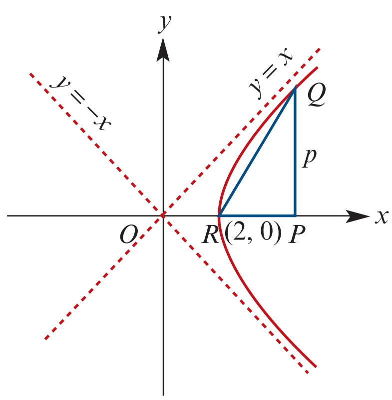
7 The triangular prism as shown in the diagram has a right-angled triangle as its cross-section. The right angle is at C and C' on the ends of the prism.

The volume of the prism is 3000 cm^3. The dimensions of the prism are shown on the diagram. Assume that the volume remains constant and x varies.



- **a** **i** Find y in terms of x.
  - **ii** Find the total surface area, S cm^2, in terms of x.
  - **iii** Sketch the graph of S against x for $x > 0$. Clearly label the asymptotes and the coordinates of the turning point.
- **b** Given that x is increasing at a constant rate of 0.5 cm/s, find the rate at which S is increasing when $x = 9$.
- **c** Find the values of x for which the surface area is 2000 cm^2, correct to two decimal places.

8 The diagram shows part of the curve $x^2 - y^2 = 4$. The line segment PQ is parallel to the y-axis, and R is the point $(2, 0)$. The length of PQ is p.



- **a** Find the area, A, of triangle PQR in terms of p.
- **b** **i** Find $\dfrac{dA}{dp}$.
  - **ii** Use your CAS calculator to help sketch the graph of A against p.
  - **iii** Find the value of p for which $A = 50$ (correct to two decimal places).
  - **iv** Prove that $\dfrac{dA}{dp} \geq 0$ for all p.
- **c** Point Q moves along the curve and point P along the x-axis so that PQ is always parallel to the y-axis and p is increasing at a rate of 0.2 units per second. Find the rate at which A is increasing, correct to three decimal places, when:

  i $p = 2.5$ **ii** $p = 4$ **iii** $p = 50$ **iv** $p = 80$

  (Use calculus to obtain the rate.)

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9 **a** Sketch the graph of $g\colon [0, 5] \to \mathbb{R}$, where $g(x) = 4 - \dfrac{8}{2 + x^2}$.

b **i** Find $g'(x)$. **ii** Find $g''(x)$.

c For what value of x is the gradient of the graph of $y = g(x)$ a maximum?

d Sketch the graph of $g\colon [-5, 5] \to \mathbb{R}$, where $g(x) = 4 - \dfrac{8}{2 + x^2}$.

10 Consider the family of cubic functions, i.e. $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = ax^3 + bx^2 + cx + d$.

a Find $f'(x)$.

b Find $f''(x)$.

c Under what conditions does the graph of f have no turning points?

d **i** Find the x-coordinate of the point where $y = f'(x)$ has a local minimum or maximum.

ii State the conditions for $y = f'(x)$ to have a local maximum.

e If $a = 1$, find the x-coordinate of the stationary point of $y = f'(x)$.

f For $y = x^3 + bx^2 + cx$, find the relationship between b and c if:

i there is only one x-axis intercept

ii there are two turning points but only one x-axis intercept.

11 A function is defined by the rule $f(x) = \dfrac{1 - x^2}{1 + x^2}$.

a **i** Show that $f'(x) = \dfrac{-4x}{(1 + x^2)^2}$. **ii** Find $f''(x)$.

b Sketch the graph of $y = f(x)$. Label the turning point and give the equation of the asymptote.

c With the aid of a CAS calculator, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [-2, 2]$.

d The graph of $y = f(x)$ crosses the x-axis at A and B and crosses the y-axis at C.

i Find the equations of the tangents at A and B.

ii Show that they intersect at C.

12 The volume, V litres, of water in a pool at time t minutes is given by the rule

$$V = -3000\pi\,(\log_e(1 - h) + h)$$

where h metres is the depth of water in the pool at time t minutes.

a **i** Find $\dfrac{dV}{dh}$ in terms of h.

ii Sketch the graph of $\dfrac{dV}{dh}$ against h for $0 \le h \le 0.9$.

b The maximum depth of the pool is 90 cm.

i Find the maximum volume of the pool to the nearest litre.

ii Sketch the graphs of $y = -3000\pi \log_e(1 - x)$ and $y = -3000\pi x$. Use addition of ordinates to sketch the graph of V against h for $0 \le h \le 0.9$.

c If water is being poured into the pool at 15 litres/min, find the rate at which the depth of the water is increasing when $h = 0.2$, correct to two significant figures.

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13 **a** Let $f(x) = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$, for $x \neq 0$.

i Find $f'(x)$. **ii** If $x > 0$, find $f(x)$. **iii** If $x < 0$, find $f(x)$.

b Let $y = \cot x$, where $x \in (0, \pi)$.

i Find $\frac{dy}{dx}$. **ii** Find $\frac{dy}{dx}$ in terms of y.

c Find the derivative with respect to x of the function $y = \cot^{-1} x$, where $y \in (0, \pi)$ and $x \in \mathbb{R}$.

d Find the derivative with respect to x of $\cot(x) + \tan(x)$, where $x \in \left(0, \frac{\pi}{2}\right)$.

14 Consider the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, where $f(x) = \frac{8}{x^2} - 32 + 16\log_e(2x)$.

a Find $f'(x)$. **b** Find $f''(x)$.

c Find the exact coordinates of any stationary points of the graph of $y = f(x)$.

d Find the exact value of x for which there is a point of inflection.

e State the interval for x for which $f'(x) > 0$.

f Find, correct to two decimal places, any x-axis intercepts other than $x = 0.5$.

g Sketch the graph of $y = f(x)$.

15 An ellipse is described by the parametric equations $x = 3\cos\theta$ and $y = 2\sin\theta$.

a Show that the tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ has equation $2x\cos\theta + 3y\sin\theta = 6$.

b The tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ meets the line with equation $x = 3$ at a point T.

i Find the coordinates of the point T.

ii Let A be the point with coordinates $(-3, 0)$ and let O be the origin. Prove that OT is parallel to AP.

c The tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ meets the x-axis at Q and the y-axis at R.

i Find the midpoint M of the line segment QR in terms of θ.

ii Find the locus of M as θ varies.

d $W(-3\sin\theta, 2\cos\theta)$ and $P(3\cos\theta, 2\sin\theta)$ are points on the ellipse.

i Find the equation of the tangent to the ellipse at W.

ii Find the coordinates of Z, the point of intersection of the tangents at P and W, in terms of θ.

iii Find the locus of Z as θ varies.

16 An ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at a point $P(a\cos\theta, b\sin\theta)$ intersects the axes at points M and N. The origin is O.

a Find the area of triangle OMN in terms of a, b and θ.

b Find the values of θ for which the area of triangle OMN is a minimum and state this minimum area in terms of a and b.

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17 A hyperbola is described by the parametric equations $x = a\sec\theta$ and $y = b\tan\theta$.

a Show that the equation of the tangent at the point $P(a\sec\theta, b\tan\theta)$ can be written as $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

b Find the coordinates of the points of intersection, Q and R, of the tangent with the asymptotes $y = \pm\frac{bx}{a}$ of the hyperbola.

c Find the coordinates of the midpoint of the line segment QR.

18 A section of an ellipse is described by the parametric equations

$$x = 2\cos\theta \quad \text{and} \quad y = \sin\theta \qquad \text{for } 0 < \theta < \frac{\pi}{2}$$

The normal to the ellipse at the point $P(2\cos\theta, \sin\theta)$ meets the x-axis at Q and the y-axis at R.

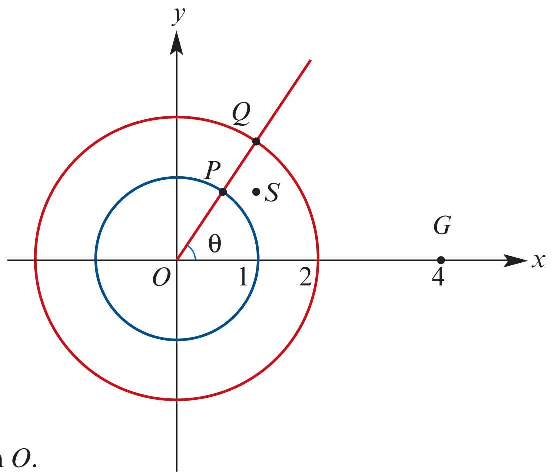
a Find the area of triangle OQR, where O is the origin, in terms of θ.

b Find the maximum value of this area and the value of θ for which this occurs.

c Find the midpoint, M, of the line segment QR in terms of θ.

d Find the locus of the point M as θ varies.

19 An electronic game appears on a flat screen, part of which is shown in the diagram. Concentric circles of radii one unit and two units appear on the screen.



Points P and Q move around the circles so that O, P and Q are collinear and OP makes an angle of θ with the x-axis.

A spaceship S moves around between the two circles and a gun is on the x-axis at G, which is 4 units from O.

The spaceship moves so that at any time it is at a point (x, y), where x is equal to the x-coordinate of Q and y is equal to the y-coordinate of P. The player turns the gun and tries to hit the spaceship.

a Find the Cartesian equation of the path C of S.

b Show that the equation of the tangent to C at the point (u, v) on C is $y = \frac{-u}{4v}x + \frac{1}{v}$.

c Show that in order to aim at the spaceship at any point on its path, the player needs to turn the gun through an angle of at most 2α, where $\tan\alpha = \frac{1}{6}\sqrt{3}$.

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9

Techniques of integration

Objectives

- To review **antidifferentiation by rule**.
- To investigate the relationship between the graph of a function and the graphs of its antiderivatives.
- To use the inverse circular functions to find antiderivatives of the form
  $$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx \quad \text{and} \quad \int \frac{a}{a^2 + x^2}\,dx$$
- To apply the technique of **substitution** to integration.
- To apply **trigonometric identities** to integration.
- To apply **partial fractions** to integration.
- To use **integration by parts**.

Integration is used in many areas of this course. In the next chapter, integration is used to find areas, volumes and lengths. In Chapter 11, it is used to help solve differential equations, which are of great importance in mathematical modelling.

We begin this chapter by reviewing the methods of integration developed in Mathematical Methods Units 3 & 4.

In the remainder of the chapter, we introduce techniques for integrating many more functions. We will use the inverse circular functions, trigonometric identities, partial fractions and two techniques which can be described as ‘reversing’ the chain rule and the product rule.

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9A Antidifferentiation

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

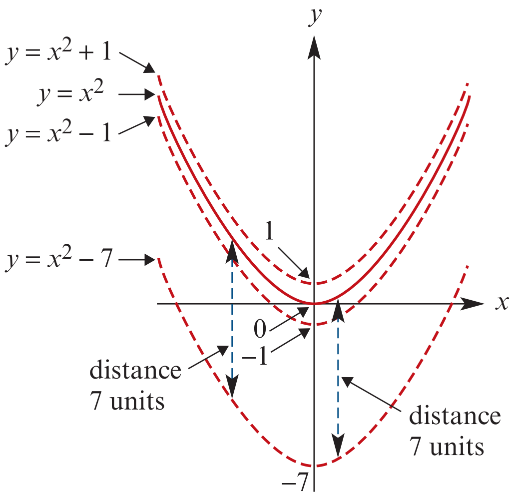
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$.

If two functions have the same derivative function, then they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several antiderivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



Notation for antiderivatives

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x \, dx = x^2 + c$$

This is read as 'the **general antiderivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we could write:

$$\begin{aligned}\int 2x \, dx &= \{ f(x) : f'(x) = 2x \} \\ &= \{ x^2 + c : c \in \mathbb{R} \}\end{aligned}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.

In general:

If $F'(x) = f(x)$, then $\int f(x) \, dx = F(x) + c$, where c is an arbitrary real number.

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Basic antiderivatives

The following antiderivatives are covered in Mathematical Methods Units 3 & 4.

$f(x)$	$\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\dfrac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
x^{-1}	$\log_e x + c$	for $x > 0$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\log_e(ax+b) + c$	for $ax + b > 0$
e^{ax+b}	$\dfrac{1}{a}e^{ax+b} + c$	
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b) + c$	
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b) + c$	

The definite integral

For a continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ denotes the signed area enclosed by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. By the fundamental theorem of calculus, we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

Note: In the expression $\int_a^b f(x)\,dx$, the number a is called the **lower limit** of integration and b the **upper limit** of integration. The function f is called the **integrand**.

We will review the fundamental theorem of calculus in Chapter 10. In this chapter, our focus is on developing techniques for calculating definite integrals using antidifferentiation.

Example 1

Find an antiderivative of each of the following:

a $\sin\left(3x - \dfrac{\pi}{4}\right)$ **b** e^{3x+4} **c** $6x^3 - \dfrac{2}{x^2}$

Solution

a $\sin\left(3x - \dfrac{\pi}{4}\right)$ is of the form $\sin(ax + b)$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\therefore \quad \int \sin\left(3x - \frac{\pi}{4}\right)dx = -\frac{1}{3}\cos\left(3x - \frac{\pi}{4}\right) + c$$

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b e^{3x+4} is of the form e^{ax+b}

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

$$\therefore \quad \int e^{3x+4}\,dx = \frac{1}{3}e^{3x+4} + c$$

c $\displaystyle\int 6x^3 - \frac{2}{x^2}\,dx = \int 6x^3 - 2x^{-2}\,dx$

$$= \frac{6x^4}{4} + 2x^{-1} + c$$

$$= \frac{3}{2}x^4 + \frac{2}{x} + c$$

Example 2

Evaluate each of the following integrals:

a $\displaystyle\int_0^{\frac{\pi}{2}} \cos(3x)\,dx$ **b** $\displaystyle\int_0^1 e^{2x} - e^x\,dx$ **c** $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$ **d** $\displaystyle\int_0^1 \sqrt{2x+1}\,dx$

Solution

a $\displaystyle\int_0^{\frac{\pi}{2}} \cos(3x)\,dx = \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{2}}$

$$= \frac{1}{3}\left(\sin\left(\frac{3\pi}{2}\right) - \sin 0\right)$$

$$= \frac{1}{3}(-1 - 0)$$

$$= -\frac{1}{3}$$

b $\displaystyle\int_0^1 e^{2x} - e^x\,dx = \left[\frac{1}{2}e^{2x} - e^x\right]_0^1$

$$= \frac{1}{2}e^2 - e^1 - \left(\frac{1}{2}e^0 - e^0\right)$$

$$= \frac{e^2}{2} - e - \left(\frac{1}{2} - 1\right)$$

$$= \frac{e^2}{2} - e + \frac{1}{2}$$

c From Chapter 8, we know that if $f(x) = \tan(ax+b)$, then $f'(x) = a\sec^2(ax+b)$. Hence

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

$$\therefore \quad \int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx = \left[\frac{1}{2}\tan(2x)\right]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \tan 0\right)$$

$$= \frac{1}{2}(1 - 0)$$

$$= \frac{1}{2}$$

d $\displaystyle\int_0^1 \sqrt{2x+1}\,dx = \int_0^1 (2x+1)^{\frac{1}{2}}\,dx$

$$= \left[\frac{1}{2 \times \frac{3}{2}}(2x+1)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{3}\left((2+1)^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(3^{\frac{3}{2}} - 1\right)$$

$$= \frac{1}{3}(3\sqrt{3} - 1)$$

In the previous chapter, we showed that the derivative of $\log_e |x|$ is $\dfrac{1}{x}$.

By the chain rule, the derivative of $\log_e |ax+b|$ is $\dfrac{a}{ax+b}$.

This gives the following antiderivative.

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e |ax+b| + c \qquad \text{for } ax+b \neq 0$$

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Example 3

a Find an antiderivative of $\dfrac{1}{4x+2}$.

b Evaluate $\int_0^1 \dfrac{1}{4x+2}\,dx$.

c Evaluate $\int_{-2}^{-1} \dfrac{1}{4x+2}\,dx$.

Solution

a $\dfrac{1}{4x+2}$ is of the form $\dfrac{1}{ax+b}$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c$$

$$\therefore \quad \int \frac{1}{4x+2}\,dx = \frac{1}{4}\log_e|4x+2| + c$$

b
$$\int_0^1 \frac{1}{4x+2}\,dx = \left[\frac{1}{4}\log_e|4x+2|\right]_0^1 = \frac{1}{4}(\log_e 6 - \log_e 2) = \frac{1}{4}\log_e 3$$

c
$$\int_{-2}^{-1} \frac{1}{4x+2}\,dx = \left[\frac{1}{4}\log_e|4x+2|\right]_{-2}^{-1} = \frac{1}{4}(\log_e|-2| - \log_e|-6|) = \frac{1}{4}\log_e\left(\frac{1}{3}\right) = -\frac{1}{4}\log_e 3$$

Graphs of functions and their antiderivatives

In each of the following examples in this section, the functions F and f are such that $F'(x) = f(x)$. That is, the function F is an antiderivative of f.

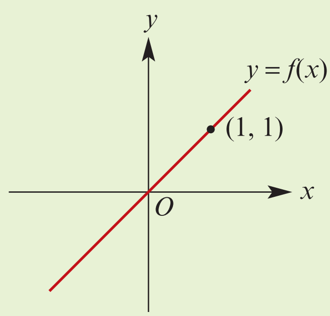
Example 4

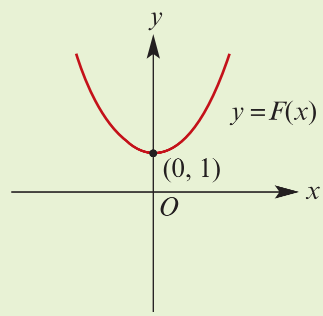
Consider the graphs of $y = f(x)$ and $y = F(x)$ shown.

Find:

a $f(x)$

b $F(x)$





Solution

a $f(x) = mx$

Since $f(1) = 1$, we have $m = 1$.

Hence $f(x) = x$.

b $F(x) = \dfrac{x^2}{2} + c$ (by antidifferentiation)

But $F(0) = 1$ and therefore $c = 1$.

Hence $F(x) = \dfrac{x^2}{2} + 1$.

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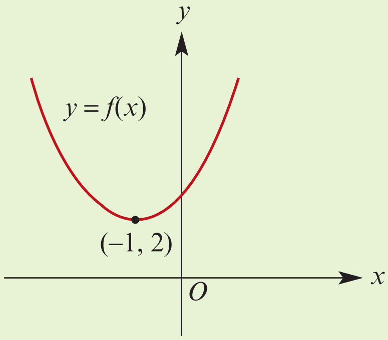
Note: The graph of $y = f(x)$ is the gradient graph for the graph of $y = F(x)$.
We have seen that there are infinitely many graphs defined by $\int f(x)\,dx$.

Example 5

The graph of $y = f(x)$ is as shown.

Sketch the graph of $y = F(x)$, given that $F(0) = 0$.

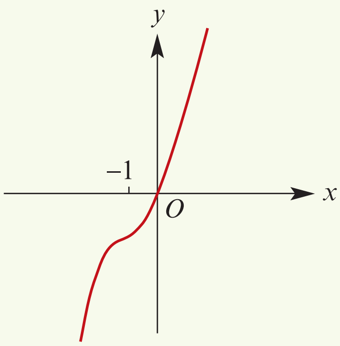


Solution

The given graph $y = f(x)$ is the gradient graph of $y = F(x)$.

Therefore the gradient of $y = F(x)$ is always positive.
The minimum gradient is 2 and this occurs when $x = -1$.
There is a line of symmetry $x = -1$, which indicates equal gradients for x-values equidistant from $x = -1$.
Also $F(0) = 0$.

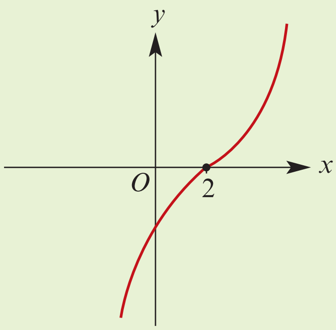
A possible graph is shown.



Example 6

The graph of $y = f(x)$ is as shown.

Sketch the graph of $y = F(x)$, given that $F(1) = 1$.

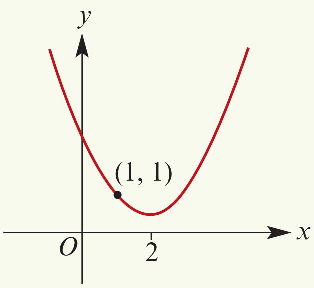


Solution

The given graph $y = f(x)$ is the gradient graph of $y = F(x)$.

Therefore the gradient of $y = F(x)$ is positive for $x > 2$, negative for $x < 2$ and zero for $x = 2$.

A possible graph is shown.



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Exercise 9A

Example 1

1 Find an antiderivative of each of the following:

a $\sin\left(2x + \frac{\pi}{4}\right)$ **b** $\cos(\pi x)$ **c** $\sin\left(\frac{2\pi x}{3}\right)$

d e^{3x+1} **e** $e^{5(x+4)}$ **f** $\frac{3}{2x^2}$

g $6x^3 - 2x^2 + 4x + 1$

Example 2

2 Evaluate each of the following integrals:

a $\int_{-1}^{1} e^x - e^{-x}\,dx$ **b** $\int_0^2 3x^2 + 2x + 4\,dx$ **c** $\int_0^{\frac{\pi}{2}} \sin(2x)\,dx$

d $\int_2^3 \frac{3}{x^3}\,dx$ **e** $\int_0^{\frac{\pi}{4}} \cos(x) + 2x\,dx$ **f** $\int_0^1 e^{3x} + x\,dx$

g $\int_0^{\frac{\pi}{2}} \cos(4x)\,dx$ **h** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(\frac{x}{2}\right)\,dx$ **i** $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

Example 3

3 **a** Find an antiderivative of $\frac{1}{2x-5}$.

b Evaluate $\int_0^1 \frac{1}{2x-5}\,dx$.

c Evaluate $\int_{-2}^{-1} \frac{1}{2x-5}\,dx$.

4 Evaluate each of the following integrals:

a $\int_0^1 \frac{1}{3x+2}\,dx$ **b** $\int_{-3}^{-1} \frac{1}{3x-2}\,dx$ **c** $\int_{-1}^{0} \frac{1}{4-3x}\,dx$

5 Find an antiderivative of each of the following:

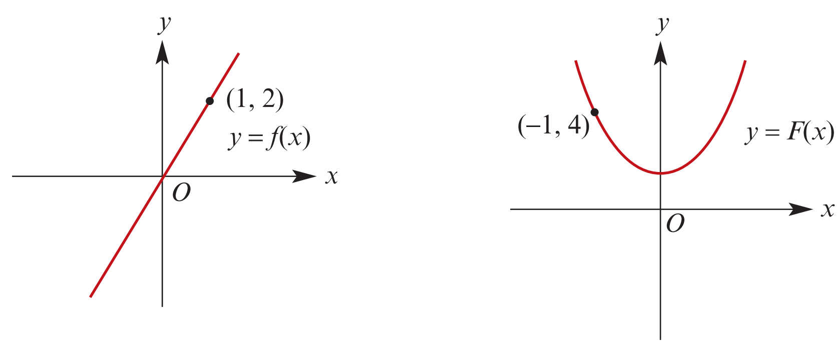
a $(3x+2)^5$ **b** $\frac{1}{3x-2}$ **c** $\sqrt{3x+2}$ **d** $\frac{1}{(3x+2)^2}$

e $\frac{3x+1}{x+1}$ **f** $\cos\left(\frac{3x}{2}\right)$ **g** $(5x-1)^{\frac{1}{3}}$ **h** $\frac{2x+1}{x+3}$

Example 4

6 For each of the following, find the rules for $f(x)$ and $F(x)$, where $F'(x) = f(x)$:

a



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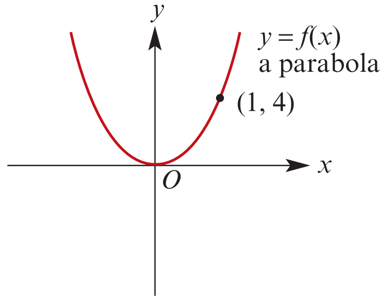
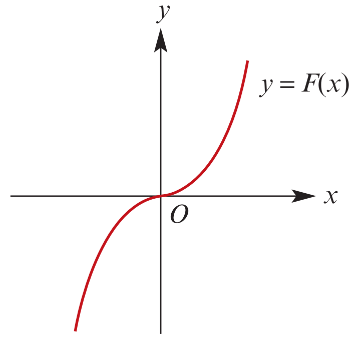
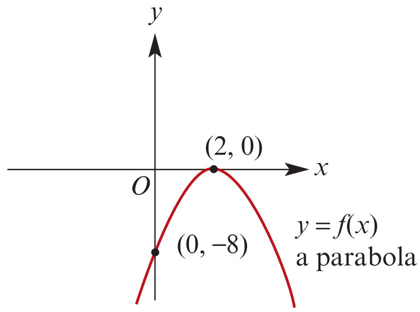
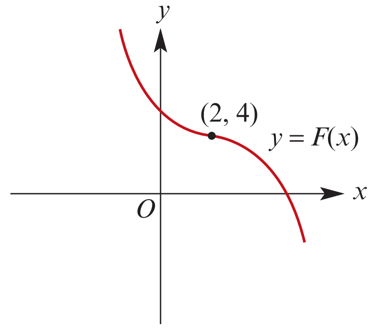
b





c





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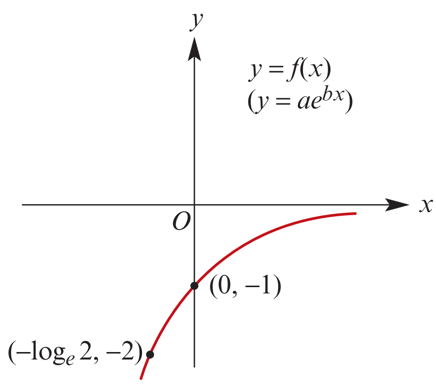
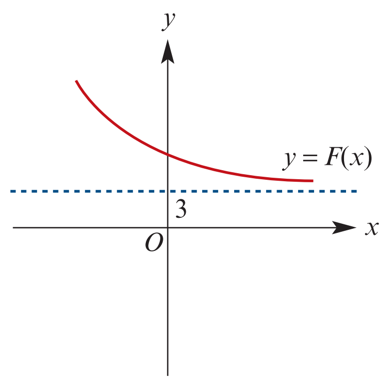
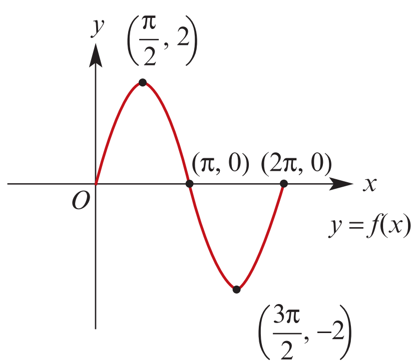
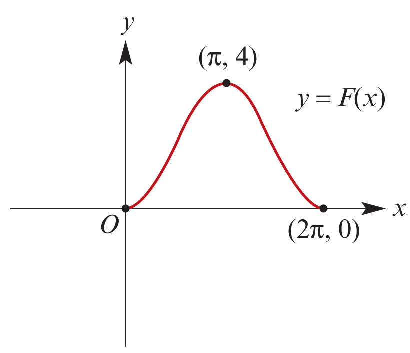




e





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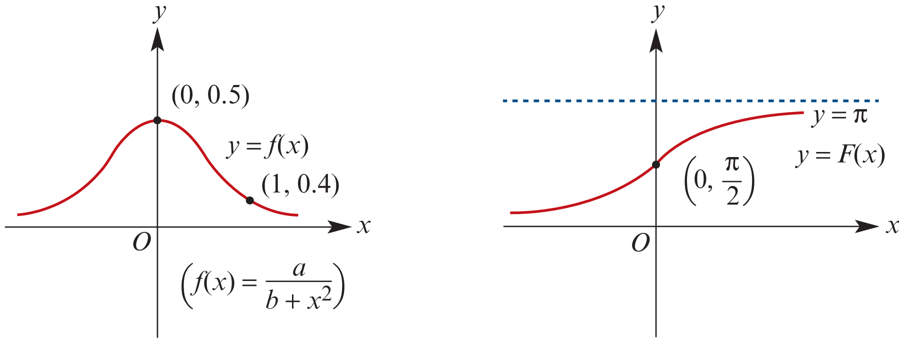
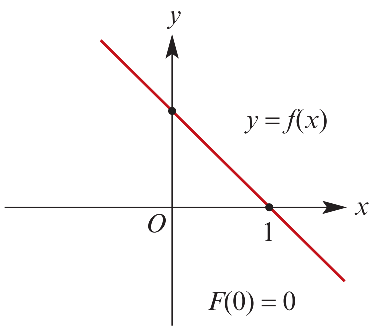
f



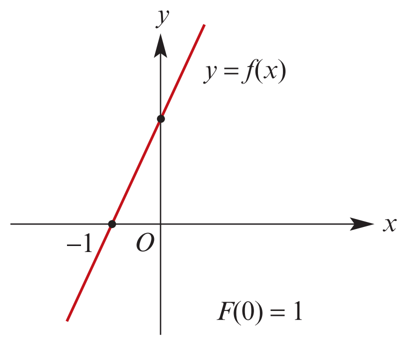
Example 5
Example 6

7 For each of the following, use the given graph of $y = f(x)$ and the given value of $F(0)$ to sketch a possible graph of $y = F(x)$, where $F'(x) = f(x)$:

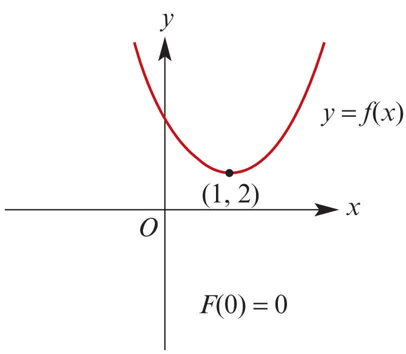
a



b



c



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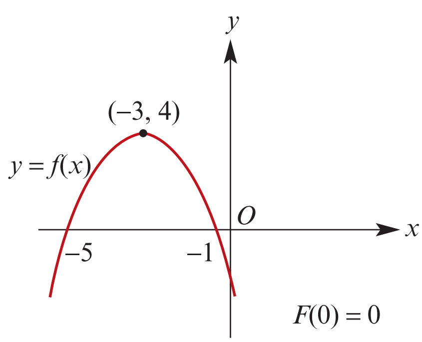

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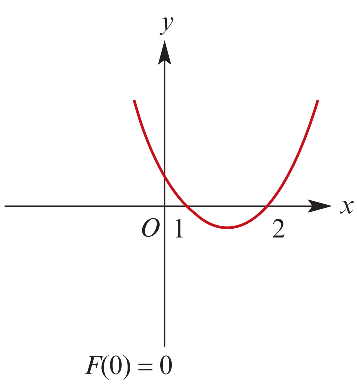

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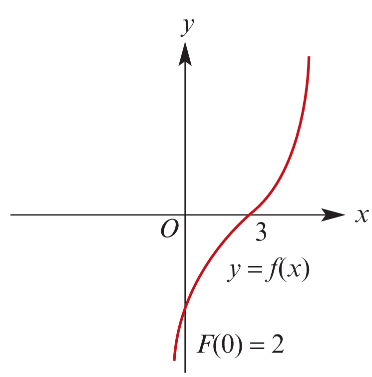

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9B Antiderivatives involving inverse circular functions

In Chapter 8, the following rules for differentiation of inverse circular functions were established:

$$f\colon (-a, a) \to \mathbb{R}, \qquad f(x) = \sin^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{1}{\sqrt{a^2 - x^2}}$$

$$f\colon (-a, a) \to \mathbb{R}, \qquad f(x) = \cos^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) = \tan^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{a}{a^2 + x^2}$$

From these results, the following can be stated:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in \mathbb{R}$$

Note: It follows that $\sin^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{x}{a}\right)$ must be constant for $x \in (-a, a)$.

By substituting $x = 0$, we can see that $\sin^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{x}{a}\right) = \frac{\pi}{2}$ for all $x \in (-a, a)$.

Example 7

Find an antiderivative of each of the following:

a $\dfrac{1}{\sqrt{9 - x^2}}$ **b** $\dfrac{1}{\sqrt{9 - 4x^2}}$ **c** $\dfrac{1}{9 + 4x^2}$

Solution

a $$\int \frac{1}{\sqrt{9 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + c$$

b $$\begin{aligned}\int \frac{1}{\sqrt{9 - 4x^2}}\,dx &= \int \frac{1}{2\sqrt{\frac{9}{4} - x^2}}\,dx \\ &= \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4} - x^2}}\,dx \\ &= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c\end{aligned}$$

c $$\begin{aligned}\int \frac{1}{9 + 4x^2}\,dx &= \int \frac{1}{4(\frac{9}{4} + x^2)}\,dx \\ &= \frac{2}{3}\int \frac{\frac{3}{2}}{4(\frac{9}{4} + x^2)}\,dx \\ &= \frac{1}{6}\int \frac{\frac{3}{2}}{\frac{9}{4} + x^2}\,dx \\ &= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c\end{aligned}$$

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Example 8

Evaluate each of the following definite integrals:

a $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ **b** $\int_0^2 \frac{1}{4+x^2}\,dx$ **c** $\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx$

Solution

a $$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0 = \frac{\pi}{6}$$

b $$\int_0^2 \frac{1}{4+x^2}\,dx = \frac{1}{2}\int_0^2 \frac{2}{4+x^2}\,dx = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0) = \frac{\pi}{8}$$

c $$\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx = \int_0^1 \frac{3}{2\sqrt{\frac{9}{4}-x^2}}\,dx = \frac{3}{2}\int_0^1 \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx = \frac{3}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^1 = \frac{3}{2}\sin^{-1}\left(\frac{2}{3}\right) \approx 1.095$$

Exercise 9B

Example 7

1 Find each of the following integrals:

a $\int \frac{1}{\sqrt{9-x^2}}\,dx$ **b** $\int \frac{1}{5+x^2}\,dx$ **c** $\int \frac{1}{1+t^2}\,dt$ **d** $\int \frac{5}{\sqrt{5-x^2}}\,dx$

e $\int \frac{3}{16+x^2}\,dx$ **f** $\int \frac{1}{\sqrt{16-4x^2}}\,dx$ **g** $\int \frac{10}{\sqrt{10-t^2}}\,dt$ **h** $\int \frac{1}{9+16t^2}\,dt$

i $\int \frac{1}{\sqrt{5-2x^2}}\,dx$ **j** $\int \frac{7}{3+y^2}\,dy$

Example 8

2 Evaluate each of the following:

a $\int_0^1 \frac{2}{1+x^2}\,dx$ **b** $\int_0^{\frac{1}{2}} \frac{3}{\sqrt{1-x^2}}\,dx$ **c** $\int_0^1 \frac{5}{\sqrt{4-x^2}}\,dx$ **d** $\int_0^5 \frac{6}{25+x^2}\,dx$

e $\int_0^{\frac{3}{2}} \frac{3}{9+4x^2}\,dx$ **f** $\int_0^2 \frac{1}{8+2x^2}\,dx$ **g** $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}\,dx$ **h** $\int_0^{\frac{3\sqrt{2}}{4}} \frac{1}{\sqrt{9-4x^2}}\,dx$

i $\int_0^{\frac{1}{3}} \frac{3}{\sqrt{1-9y^2}}\,dy$ **j** $\int_0^2 \frac{1}{1+3x^2}\,dx$

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9C Integration by substitution

In this section, we introduce the technique of substitution. The substitution will result in one of the forms for integrands covered in Sections 9A and 9B.

First consider the following example.

Example 9

Differentiate each of the following with respect to x:

a $(2x^2 + 1)^5$ **b** $\cos^3 x$ **c** e^{3x^2}

Solution

a Let $y = (2x^2 + 1)^5$ and $u = 2x^2 + 1$.

Then $y = u^5$, $\dfrac{dy}{du} = 5u^4$ and $\dfrac{du}{dx} = 4x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 5u^4 \cdot 4x\\ &= 20u^4 x\\ &= 20x(2x^2+1)^4\end{aligned}$$

b Let $y = \cos^3 x$ and $u = \cos x$.

Then $y = u^3$, $\dfrac{dy}{du} = 3u^2$ and $\dfrac{du}{dx} = -\sin x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 3u^2 \cdot (-\sin x)\\ &= 3\cos^2 x \cdot (-\sin x)\\ &= -3\cos^2 x \sin x\end{aligned}$$

c Let $y = e^{3x^2}$ and $u = 3x^2$.

Then $y = e^u$, $\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = 6x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= e^u \cdot 6x\\ &= 6xe^{3x^2}\end{aligned}$$

This example suggests that a ‘converse’ of the chain rule can be used to obtain a method for antidifferentiating functions of a particular form.

- From Example 9 **a**: $\int 20x(2x^2 + 1)^4 \, dx = (2x^2 + 1)^5 + c$

  This is of the form: $\int 5g'(x)\,(g(x))^4 \, dx = (g(x))^5 + c$ where $g(x) = 2x^2 + 1$
- From Example 9 **b**: $\int -3\cos^2 x \sin x \, dx = \cos^3 x + c$

  This is of the form: $\int 3g'(x)\,(g(x))^2 \, dx = (g(x))^3 + c$ where $g(x) = \cos x$
- From Example 9 **c**: $\int 6xe^{3x^2} \, dx = e^{3x^2} + c$

  This is of the form: $\int g'(x)\, e^{g(x)} \, dx = e^{g(x)} + c$ where $g(x) = 3x^2$

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This suggests a method that can be used for integration.

e.g. $\int 2x(x^2+1)^5\,dx = \dfrac{(x^2+1)^6}{6} + c \qquad [g(x) = x^2+1]$

$\int \cos x \sin x\,dx = \dfrac{\sin^2 x}{2} + c \qquad [g(x) = \sin x]$

A formalisation of this idea provides a method for integrating functions of this form.

Let $y = \int f(u)\,du$, where $u = g(x)$.

By the chain rule for differentiation:

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$
$$= f(u)\cdot\frac{du}{dx}$$
$$\therefore \quad y = \int f(u)\,\frac{du}{dx}\,dx$$

This gives the following technique for integration.

Integration by substitution

$$\int f(u)\,\frac{du}{dx}\,dx = \int f(u)\,du$$

This is also called the **change of variable rule**.

Example 10

Find an antiderivative of each of the following:

a $\sin x \cos^2 x$ **b** $5x^2(x^3-1)^{\frac{1}{2}}$ **c** $3xe^{x^2}$

Solution

a $\int \sin x \cos^2 x\,dx$

Let $u = \cos x$. Then $f(u) = u^2$ and $\dfrac{du}{dx} = -\sin x$.

$$\therefore \quad \int \sin x \cos^2 x\,dx = -\int \cos^2 x\cdot(-\sin x)\,dx$$
$$= -\int f(u)\,\frac{du}{dx}\,dx$$
$$= -\int f(u)\,du$$
$$= -\int u^2\,du$$
$$= -\frac{u^3}{3} + c$$
$$= -\frac{\cos^3 x}{3} + c$$

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b $\int 5x^2(x^3-1)^{\frac{1}{2}}\,dx$

Let $u = x^3 - 1$.

Then $f(u) = u^{\frac{1}{2}}$ and $\frac{du}{dx} = 3x^2$.

$$\begin{aligned}\therefore\quad \int 5x^2(x^3-1)^{\frac{1}{2}}\,dx &= \frac{5}{3}\int (x^3-1)^{\frac{1}{2}}\cdot 3x^2\,dx\\ &= \frac{5}{3}\int u^{\frac{1}{2}}\frac{du}{dx}\,dx\\ &= \frac{5}{3}\int u^{\frac{1}{2}}\,du\\ &= \frac{5}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right)+c\\ &= \frac{10}{9}u^{\frac{3}{2}}+c\\ &= \frac{10}{9}(x^3-1)^{\frac{3}{2}}+c\end{aligned}$$

c $\int 3xe^{x^2}\,dx$

Let $u = x^2$.

Then $f(u) = e^u$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned}\therefore\quad \int 3xe^{x^2}\,dx &= \frac{3}{2}\int e^u\cdot 2x\,dx\\ &= \frac{3}{2}\int e^u\frac{du}{dx}\,dx\\ &= \frac{3}{2}\int e^u\,du\\ &= \frac{3}{2}e^u+c\\ &= \frac{3}{2}e^{x^2}+c\end{aligned}$$

Example 11

Find an antiderivative of each of the following:

a $\dfrac{2}{x^2+2x+6}$

b $\dfrac{3}{\sqrt{9-4x-x^2}}$

Solution

a Completing the square gives

$$\begin{aligned}x^2+2x+6 &= x^2+2x+1+5\\ &= (x+1)^2+5\end{aligned}$$

Therefore

$$\int \frac{2}{x^2+2x+6}\,dx = \int \frac{2}{(x+1)^2+5}\,dx$$

Let $u = x + 1$. Then $\frac{du}{dx} = 1$ and hence

$$\begin{aligned}\int \frac{2}{(x+1)^2+5}\,dx &= \int \frac{2}{u^2+5}\,du\\ &= \frac{2}{\sqrt{5}}\int \frac{\sqrt{5}}{u^2+5}\,du\\ &= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right)+c\\ &= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+c\end{aligned}$$

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b $\displaystyle\int \frac{3}{\sqrt{9-4x-x^2}}\,dx$

Completing the square gives

$$\begin{aligned} 9-4x-x^2 &= -(x^2+4x-9) \\ &= -((x+2)^2-13) \\ &= 13-(x+2)^2 \end{aligned}$$

Therefore

$$\int \frac{3}{\sqrt{9-4x-x^2}}\,dx = \int \frac{3}{\sqrt{13-(x+2)^2}}\,dx$$

Let $u = x+2$. Then $\dfrac{du}{dx} = 1$ and hence

$$\begin{aligned} \int \frac{3}{\sqrt{13-(x+2)^2}}\,dx &= \int \frac{3}{\sqrt{13-u^2}}\,du \\ &= 3\sin^{-1}\left(\frac{u}{\sqrt{13}}\right) + c \\ &= 3\sin^{-1}\left(\frac{x+2}{\sqrt{13}}\right) + c \end{aligned}$$

Linear substitutions

Antiderivatives of expressions such as

$$(2x+3)\sqrt{3x-4}, \quad \frac{2x+5}{\sqrt{3x-4}}, \quad \frac{2x+5}{(x+2)^2}, \quad (2x+4)(x+3)^{20}, \quad x^2\sqrt{3x-1}$$

can be found using a linear substitution.

Example 12

Find an antiderivative of each of the following:

a $(2x+1)\sqrt{x+4}$ **b** $\dfrac{2x+1}{(1-2x)^2}$ **c** $x^2\sqrt{3x-1}$

Solution

a Let $u = x+4$. Then $\dfrac{du}{dx} = 1$ and $x = u-4$.

$$\begin{aligned} \therefore \quad \int (2x+1)\sqrt{x+4}\,dx &= \int (2(u-4)+1)\,u^{\frac{1}{2}}\,du \\ &= \int (2u-7)u^{\frac{1}{2}}\,du \\ &= \int 2u^{\frac{3}{2}} - 7u^{\frac{1}{2}}\,du \\ &= 2\left(\frac{2}{5}u^{\frac{5}{2}}\right) - 7\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c \\ &= \frac{4}{5}(x+4)^{\frac{5}{2}} - \frac{14}{3}(x+4)^{\frac{3}{2}} + c \end{aligned}$$

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b $\displaystyle\int \frac{2x+1}{(1-2x)^2}\,dx$

Let $u = 1 - 2x$. Then $\dfrac{du}{dx} = -2$ and $2x = 1 - u$.

Therefore

$$\begin{aligned}\int \frac{2x+1}{(1-2x)^2}\,dx &= -\frac{1}{2}\int \frac{2-u}{u^2}(-2)\,dx\\ &= -\frac{1}{2}\int \frac{2-u}{u^2}\frac{du}{dx}\,dx\\ &= -\frac{1}{2}\int 2u^{-2} - u^{-1}\,du\\ &= -\frac{1}{2}(-2u^{-1} - \log_e|u|) + c\\ &= u^{-1} + \frac{1}{2}\log_e|u| + c\\ &= \frac{1}{1-2x} + \frac{1}{2}\log_e|1-2x| + c\end{aligned}$$

c $\displaystyle\int x^2\sqrt{3x-1}\,dx$

Let $u = 3x - 1$. Then $\dfrac{du}{dx} = 3$.

We have $x = \dfrac{u+1}{3}$ and so $x^2 = \dfrac{(u+1)^2}{9}$.

Therefore

$$\begin{aligned}\int x^2\sqrt{3x-1}\,dx &= \int \frac{(u+1)^2}{9}\sqrt{u}\,dx\\ &= \frac{1}{27}\int (u+1)^2\,u^{\frac{1}{2}}\,(3)\,dx\\ &= \frac{1}{27}\int (u^2+2u+1)\,u^{\frac{1}{2}}\frac{du}{dx}\,dx\\ &= \frac{1}{27}\int u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du\\ &= \frac{1}{27}\left(\frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + c\\ &= \frac{2}{27}u^{\frac{3}{2}}\left(\frac{1}{7}u^2 + \frac{2}{5}u + \frac{1}{3}\right) + c\\ &= \frac{2}{2835}(3x-1)^{\frac{3}{2}}\left(15(3x-1)^2 + 42(3x-1) + 35\right) + c\\ &= \frac{2}{2835}(3x-1)^{\frac{3}{2}}(135x^2 + 36x + 8) + c\end{aligned}$$

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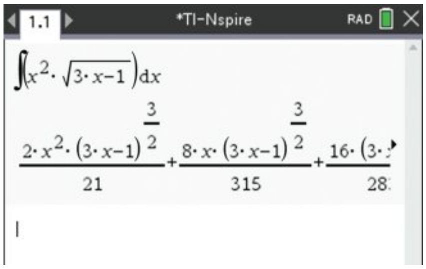
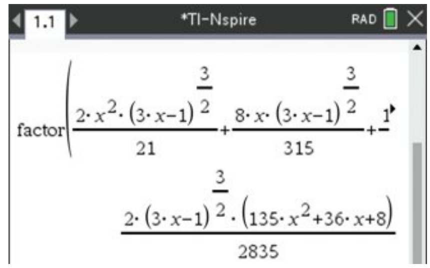
Using the TI-Nspire

- To find an antiderivative, use (menu) > **Calculus** > **Integral**.
- Use **factor** from the **Algebra** menu to obtain the required form.

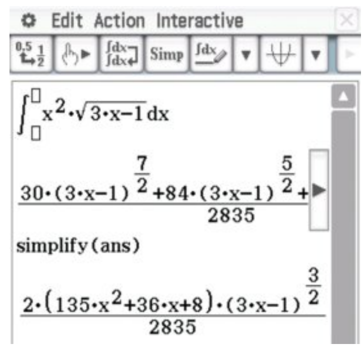




Note: The integral template can also be obtained directly from the 2D-template palette (template key) or by pressing (shift) (+).

Using the Casio ClassPad

- Enter and highlight the expression $x^2\sqrt{3x-1}$.
- Go to **Interactive** > **Calculation** > $\int$. Make sure that **Indefinite** is selected and that x is the variable.
- Tap on (Simp) to simplify the resulting expression.



Exercise 9C

Example 10

1 Find each of the following:

a $\int 2x(x^2+1)^3\,dx$ **b** $\int \frac{x}{(x^2+1)^2}\,dx$ **c** $\int \cos x \sin^3 x\,dx$ **d** $\int \frac{\cos x}{\sin^2 x}\,dx$

e $\int (2x+1)^5\,dx$ **f** $\int 5x\sqrt{9+x^2}\,dx$ **g** $\int x(x^2-3)^5\,dx$ **h** $\int \frac{x+1}{(x^2+2x)^3}\,dx$

i $\int \frac{2}{(3x+1)^3}\,dx$ **j** $\int \frac{1}{\sqrt{1+x}}\,dx$ **k** $\int (x^2-2x)(x^3-3x^2+1)^4\,dx$

l $\int \frac{3x}{x^2+1}\,dx$ **m** $\int \frac{3x}{2-x^2}\,dx$ **n** $\int \frac{\log_e x}{x}\,dx$ **o** $\int xe^{-4x^2}\,dx$

Example 11

2 Find an antiderivative of each of the following:

a $\frac{1}{x^2+2x+2}$ **b** $\frac{1}{x^2-x+1}$ **c** $\frac{1}{\sqrt{21-4x-x^2}}$

d $\frac{1}{\sqrt{10x-x^2-24}}$ **e** $\frac{1}{\sqrt{40-x^2-6x}}$ **f** $\frac{1}{3x^2+6x+7}$

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3 Consider the integral $\int \sin x \cos x \, dx$.

a Find the integral by using the substitution $u = \sin x$.

b Now find the same integral by using the substitution $u = \cos x$.

c By comparing the two answers, show that $\cos^2 x + \sin^2 x = c$, for some constant c.

d Find the value of c by letting $x = 0$. What does this give?

Example 12

4 Find an antiderivative of each of the following:

a $x\sqrt{2x+3}$ **b** $x\sqrt{1-x}$ **c** $6x(3x-7)^{-\frac{1}{2}}$ **d** $(2x+1)\sqrt{3x-1}$

e $\dfrac{2x-1}{(x-1)^2}$ **f** $(x+3)\sqrt{3x+1}$ **g** $(x+2)(x+3)^{\frac{1}{3}}$ **h** $\dfrac{5x-1}{(2x+1)^2}$

i $x^2\sqrt{x-1}$ **j** $\dfrac{x^2}{\sqrt{x-1}}$

9D Definite integrals by substitution

Example 13

Evaluate $\int_0^4 3x\sqrt{x^2+9}\, dx$.

Solution

Let $u = x^2 + 9$. Then $\dfrac{du}{dx} = 2x$ and so

$$\begin{aligned}\int 3x\sqrt{x^2+9}\, dx &= \frac{3}{2}\int \sqrt{x^2+9}\cdot 2x\, dx\\ &= \frac{3}{2}\int u^{\frac{1}{2}}\frac{du}{dx}\, dx\\ &= \frac{3}{2}\int u^{\frac{1}{2}}\, du\\ &= \frac{3}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c\\ &= u^{\frac{3}{2}} + c\\ &= (x^2+9)^{\frac{3}{2}} + c\end{aligned}$$

$$\begin{aligned}\therefore \quad \int_0^4 3x\sqrt{x^2+9}\, dx &= \left[(x^2+9)^{\frac{3}{2}}\right]_0^4\\ &= 25^{\frac{3}{2}} - 9^{\frac{3}{2}}\\ &= 125 - 27 = 98\end{aligned}$$

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In a definite integral which involves the change of variable rule, it is not necessary to return to an expression in x if the values of u corresponding to each of the limits of x are found.

For the previous example:

- $x = 0$ implies $u = 9$
- $x = 4$ implies $u = 25$

Therefore the integral can be evaluated as

$$\frac{3}{2}\int_9^{25} u^{\frac{1}{2}}\,du = \frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_9^{25} = 125 - 27 = 98$$

Example 14

Evaluate the following:

a $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

b $\int_0^1 2x^2 e^{x^3}\,dx$

Solution

a $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos x\,(\cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} \cos x\,(1 - \sin^2 x)\,dx$

Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$.

When $x = \frac{\pi}{2}$, $u = 1$ and when $x = 0$, $u = 0$.

Therefore the integral becomes

$$\int_0^1 (1 - u^2)\,du = \left[u - \frac{u^3}{3}\right]_0^1$$
$$= 1 - \frac{1}{3} = \frac{2}{3}$$

b $\int_0^1 2x^2 e^{x^3}\,dx$

Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$.

When $x = 1$, $u = 1$ and when $x = 0$, $u = 0$.

Therefore the integral becomes

$$\frac{2}{3}\int_0^1 e^{x^3}\cdot(3x^2)\,dx = \frac{2}{3}\int_0^1 e^u\,du$$
$$= \frac{2}{3}[e^u]_0^1$$
$$= \frac{2}{3}(e^1 - e^0)$$
$$= \frac{2}{3}(e - 1)$$

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Skill-sheet

Exercise 9D

Example 13
Example 14

1 Evaluate each of the following definite integrals:

a $\int_0^3 x\sqrt{x^2+16}\,dx$ **b** $\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\,dx$ **c** $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$

d $\int_3^4 x(x-3)^{17}\,dx$ **e** $\int_0^1 x\sqrt{1-x}\,dx$ **f** $\int_e^{e^2} \frac{1}{x\log_e x}\,dx$

g $\int_0^4 \frac{1}{\sqrt{3x+4}}\,dx$ **h** $\int_{-1}^1 \frac{e^x}{e^x+1}\,dx$ **i** $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx$

j $\int_0^1 \frac{2x+3}{x^2+3x+4}\,dx$ **k** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\,dx$ **l** $\int_{-4}^{-3} \frac{2x}{1-x^2}\,dx$

m $\int_{-2}^{-1} \frac{e^x}{1-e^x}\,dx$

2 By using $\tan x = \frac{\sin x}{\cos x}$, evaluate $\int_0^{\frac{\pi}{3}} \tan x\,dx$.

9E Use of trigonometric identities for integration

Products of sines and cosines

Integrals of the form $\int \sin^m x \cos^n x\,dx$, where m and n are non-negative integers, can be considered in the following three cases.

Case A: the power of sine is odd

If m is odd, write $m = 2k+1$. Then

$$\sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x$$

and the substitution $u = \cos x$ can now be made.

Case B: the power of cosine is odd

If m is even and n is odd, write $n = 2k+1$. Then

$$\cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x$$

and the substitution $u = \sin x$ can now be made.

Case C: both powers are even

If both m and n are even, then the identity $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, $\cos^2 x = \frac{1}{2}(1+\cos(2x))$ or $\sin(2x) = 2\sin x \cos x$ can be used.

Also note that $\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + c$. The identity $1 + \tan^2 x = \sec^2 x$ is used in the following example.

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Example 15

Find:

a $\int \cos^2 x \, dx$ **b** $\int \tan^2 x \, dx$ **c** $\int \sin(2x)\cos(2x) \, dx$

d $\int \cos^4 x \, dx$ **e** $\int \sin^3 x \cos^2 x \, dx$

Solution

a Use the identity $\cos(2x) = 2\cos^2 x - 1$. Rearranging gives

$$\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$$

$$\therefore \quad \int \cos^2 x \, dx = \frac{1}{2}\int \cos(2x) + 1 \, dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin(2x) + x\right) + c$$

$$= \frac{1}{4}\sin(2x) + \frac{x}{2} + c$$

b Use the identity $1 + \tan^2 x = \sec^2 x$. This gives $\tan^2 x = \sec^2 x - 1$ and so

$$\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx$$

$$= \tan x - x + c$$

c Use the identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

Let $\theta = 2x$. Then $\sin(4x) = 2\sin(2x)\cos(2x)$ and so $\sin(2x)\cos(2x) = \frac{1}{2}\sin(4x)$.

$$\therefore \quad \int \sin(2x)\cos(2x) \, dx = \frac{1}{2}\int \sin(4x) \, dx$$

$$= \frac{1}{2}\left(-\frac{1}{4}\cos(4x)\right) + c$$

$$= -\frac{1}{8}\cos(4x) + c$$

d $\cos^4 x = (\cos^2 x)^2 = \left(\dfrac{\cos(2x) + 1}{2}\right)^2 = \dfrac{1}{4}(\cos^2(2x) + 2\cos(2x) + 1)$

As $\cos(4x) = 2\cos^2(2x) - 1$, this gives

$$\cos^4 x = \frac{1}{4}\left(\frac{\cos(4x) + 1}{2} + 2\cos(2x) + 1\right)$$

$$= \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8}$$

$$\therefore \quad \int \cos^4 x \, dx = \int \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8} \, dx$$

$$= \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$$

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e $\int \sin^3 x \cos^2 x \, dx = \int \sin x \, (\sin^2 x) \cos^2 x \, dx$
$= \int \sin x \, (1 - \cos^2 x) \cos^2 x \, dx$

Now let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$. We obtain

$$\begin{aligned} \int \sin^3 x \cos^2 x \, dx &= -\int (-\sin x)(1 - u^2)(u^2) \, dx \\ &= -\int (1 - u^2) \, u^2 \frac{du}{dx} \, dx \\ &= -\int u^2 - u^4 \, du \\ &= -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + c \\ &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c \end{aligned}$$

Products to sums

We recall the following identities from Chapter 3.

Product-to-sum identities

$$\begin{aligned} 2\cos x \, \cos y &= \cos(x - y) + \cos(x + y) \\ 2\sin x \, \sin y &= \cos(x - y) - \cos(x + y) \\ 2\sin x \, \cos y &= \sin(x + y) + \sin(x - y) \end{aligned}$$

These identities enable us to determine further integrals involving the circular functions.

Example 16

Find:

a $\int \sin(3x) \sin(2x) \, dx$ **b** $\int \sin(5x) \cos(2x) \, dx$ **c** $\int_0^{\frac{\pi}{2}} \cos(3x) \cos(2x) \, dx$

Solution

a
$$\begin{aligned} \int \sin(3x) \sin(2x) \, dx &= \frac{1}{2} \int \cos(3x - 2x) - \cos(3x + 2x) \, dx \\ &= \frac{1}{2} \int \cos x - \cos(5x) \, dx \\ &= \frac{1}{2} \sin x - \frac{1}{10} \sin(5x) + c \end{aligned}$$

b
$$\begin{aligned} \int \sin(5x) \cos(2x) \, dx &= \frac{1}{2} \int \sin(5x + 2x) + \sin(5x - 2x) \, dx \\ &= \frac{1}{2} \int \sin(7x) + \sin(3x) \, dx \\ &= -\frac{1}{14} \cos(7x) - \frac{1}{6} \cos(3x) + c \end{aligned}$$

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c $\int_0^{\frac{\pi}{2}} \cos(3x)\cos(2x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(3x-2x)+\cos(3x+2x)\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x + \cos(5x)\,dx$

$= \left[\frac{1}{2}\sin x + \frac{1}{10}\sin(5x)\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2} + \frac{1}{10}$

$= \frac{3}{5}$

Skill-sheet

Exercise 9E

Example 15

1 Find an antiderivative of each of the following:

a $\sin^2 x$ **b** $\sin^4 x$ **c** $2\tan^2 x$

d $2\sin(3x)\cos(3x)$ **e** $\sin^2(2x)$ **f** $\tan^2(2x)$

g $\sin^2 x\cos^2 x$ **h** $\cos^2 x - \sin^2 x$ **i** $\cot^2 x$

j $\cos^3(2x)$

2 Find an antiderivative of each of the following:

a $\sec^2 x$ **b** $\sec^2(2x)$ **c** $\sec^2(\frac{1}{2}x)$

d $\sec^2(kx)$ **e** $\tan^2(3x)$ **f** $1-\tan^2 x$

g $\tan^2 x - \sec^2 x$ **h** $\operatorname{cosec}^2\left(x-\frac{\pi}{2}\right)$

3 Evaluate each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ **b** $\int_0^{\frac{\pi}{4}} \tan^3 x\,dx$ **c** $\int_0^{\frac{\pi}{2}} \sin^2 x\cos x\,dx$

d $\int_0^{\frac{\pi}{4}} \cos^4 x\,dx$ **e** $\int_0^{\pi} \sin^3 x\,dx$ **f** $\int_0^{\frac{\pi}{2}} \sin^2(2x)\,dx$

g $\int_0^{\frac{\pi}{3}} \sin^2 x\cos^2 x\,dx$ **h** $\int_0^{1} \sin^2 x + \cos^2 x\,dx$

4 Find an antiderivative of each of the following:

a $\cos^3 x$ **b** $\sin^3\left(\frac{x}{4}\right)$ **c** $\cos^2(4\pi x)$

d $7\cos^7 t$ **e** $\cos^3(5x)$ **f** $8\sin^4 x$

g $\sin^2 x\cos^4 x$ **h** $\cos^5 x$

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Example 16

5 Find an antiderivative of each of the following:

a $\sin(4x)\sin(2x)$ **b** $\cos(4x)\cos(2x)$ **c** $\sin(4x)\cos(2x)$

d $\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$ **e** $\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$ **f** $\sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)$

6 Evaluate each of the following definite integrals:

a $\int_0^{\pi} \cos(2x)\cos\left(\frac{x}{2}\right)\,dx$ **b** $\int_0^{\frac{\pi}{2}} \sin(2x)\cos(6x)\,dx$

c $\int_0^{\frac{\pi}{2}} \sin(8x)\cos(10x)\,dx$

9F Further substitution*

In Section 9C, we found the result

$$\int f(u)\,\frac{du}{dx}\,dx = \int f(u)\,du$$

If we interchange the variables x and u, then we can write this as follows.

$$\int f(x)\,dx = \int f(x)\,\frac{dx}{du}\,du$$

Note: For this substitution to work, the function that we substitute for x must be one-to-one. You will see this in the following examples.

Example 17

Find $\int \frac{1}{x^2+1}\,dx$ by using the substitution $x = \tan u$, where $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

Solution

Let $x = \tan u$. Then $\frac{dx}{du} = \sec^2 u$.

We substitute into $\int f(x)\,dx = \int f(x)\,\frac{dx}{du}\,du$.

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\tan^2 u + 1}\cdot \sec^2 u\,du \\ &= \int \frac{1}{\sec^2 u}\cdot \sec^2 u\,du \qquad \text{since } 1+\tan^2 u = \sec^2 u \\ &= \int 1\,du \\ &= u + c \\ &= \arctan x + c\end{aligned}$$

* This material is not required for examinations.

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Example 18

Find $\int \frac{1}{(x^2+1)^2}\,dx$ by using the substitution $x = \tan u$, where $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

Solution

Let $x = \tan u$. Then $\frac{dx}{du} = \sec^2 u$.

We substitute into $\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du$.

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1}{(\tan^2 u+1)^2}\cdot \sec^2 u\,du$$

$$= \int \frac{1}{(\sec^2 u)^2}\cdot \sec^2 u\,du \qquad \text{since } 1+\tan^2 u = \sec^2 u$$

$$= \int \cos^2 u\,du$$

$$= \frac{1}{2}\int \cos(2u) + 1\,du \qquad \text{since } \cos^2 u = \frac{1}{2}(1+\cos(2u))$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin(2u) + u\right) + c$$

Since $x = \tan u$, we have $\sin u = \frac{x}{\sqrt{x^2+1}}$ and $\cos u = \frac{1}{\sqrt{x^2+1}}$.

$$\therefore \quad \int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\sin u\cos u + \frac{u}{2} + c$$

$$= \frac{1}{2}\cdot\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} + \frac{1}{2}\arctan x + c$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x + c$$

Example 19

Find $\int_0^2 \sqrt{4-x^2}\,dx$ by using the substitution $x = 2\sin u$, where $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

Solution

Using the Pythagorean identity

Let $x = 2\sin u$. Then

$$4 - x^2 = 4 - 4\sin^2 u$$

$$= 4(1-\sin^2 u)$$

$$= 4\cos^2 u$$

$$\therefore \quad \sqrt{4-x^2} = \sqrt{4\cos^2 u}$$

$$= 2|\cos u|$$

$$= 2\cos u \qquad \text{since } -\frac{\pi}{2} \le u \le \frac{\pi}{2}$$

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We have shown that if $x = 2\sin u$, where $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$, then $\sqrt{4 - x^2} = 2\cos u$.

Limits of integration

- $x = 2$ implies $2 = 2\sin u$ and so $u = \frac{\pi}{2}$
- $x = 0$ implies $0 = 2\sin u$ and so $u = 0$

Changing variables

Since $\frac{dx}{du} = 2\cos u$, we obtain

$$\begin{aligned}\int_{x=0}^{x=2} \sqrt{4 - x^2}\, dx &= \int_{u=0}^{u=\frac{\pi}{2}} 2\cos u\, dx \\ &= \int_0^{\frac{\pi}{2}} 2\cos u \cdot 2\cos u\, du \\ &= \int_0^{\frac{\pi}{2}} 4\cos^2 u\, du\end{aligned}$$

We use the identity $\cos^2 u = \frac{1}{2}(1 + \cos(2u))$:

$$\begin{aligned}\int_{x=0}^{x=2} \sqrt{4 - x^2}\, dx &= \int_0^{\frac{\pi}{2}} 2 + 2\cos(2u)\, du \\ &= \Big[2u + \sin(2u)\Big]_0^{\frac{\pi}{2}} \\ &= \pi\end{aligned}$$

Exercise 9F

1 Find $\int \frac{1}{x^2 + 9}\, dx$ by substituting $x = 3\tan u$, where $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

2 Find $\int \frac{-1}{\sqrt{4 - x^2}}\, dx$ by substituting $x = 2\cos u$, where $0 \le u \le \pi$.

3 Find $\int \frac{1}{x + \sqrt{x}}\, dx$ by substituting $x = u^2$, where $u > 0$.

4 Find $\int \frac{1}{3\sqrt{x} + 4x}\, dx$ by substituting $x = u^2$, where $u > 0$.

5 Find $\int \frac{1}{\sqrt{9 - x^2}}\, dx$ by substituting $x = 3\sin u$, where $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

6 Find $\int \sqrt{9 - x^2}\, dx$ by substituting $x = 3\sin u$, where $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

7 Find $\int \frac{1}{x(1 + \sqrt[3]{x})}\, dx$ by substituting $x = u^3$, where $u > 0$.

8 Find $\int \frac{1}{(1 - x^2)^{\frac{3}{2}}}\, dx$ by substituting $x = \sin u$, where $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

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9G Partial fractions

We studied graphs of rational functions in Chapter 8.

If $g(x)$ and $h(x)$ are polynomials, then $f(x) = \dfrac{g(x)}{h(x)}$ is a rational function; e.g. $f(x) = \dfrac{4x+2}{x^2-1}$.

- If the degree of $g(x)$ is less than the degree of $h(x)$, then $f(x)$ is a **proper fraction**.
- If the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then $f(x)$ is an **improper fraction**.

A rational function may be expressed as a sum of simpler functions by resolving it into what are called **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

We will see that this is a useful technique for integration.

Proper fractions

For proper fractions, the method used for obtaining partial fractions depends on the type of factors in the denominator of the original algebraic fraction. We only consider examples where the denominators have factors that are either degree 1 (linear) or degree 2 (quadratic).

- For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax+b}$.
- For every repeated linear factor $(ax + b)^2$ in the denominator, there will be partial fractions of the form $\dfrac{A}{ax+b}$ and $\dfrac{B}{(ax+b)^2}$.
- For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$.
- For every repeated irreducible quadratic factor $(ax^2 + bx + c)^2$ in the denominator, there will be partial fractions of the form $\dfrac{Ax+B}{ax^2+bx+c}$ and $\dfrac{Cx+D}{(ax^2+bx+c)^2}$.

Note: A quadratic expression is **irreducible** if it cannot be factorised over $\mathbb{R}$, that is, if its discriminant is negative. For example, both $x^2 + 1$ and $x^2 + 4x + 10$ are irreducible.

To resolve an algebraic fraction into its partial fractions:

Step 1 Write a statement of identity between the original fraction and a sum of the appropriate number of partial fractions.

Step 2 Express the sum of the partial fractions as a single fraction, and note that the numerators of both sides are equivalent.

Step 3 Find the values of the introduced constants $A, B, C, \ldots$ by substituting appropriate values for x or by equating coefficients.

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Example 20

Resolve $\dfrac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution

Let

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \qquad (1)$$

for all $x \in \mathbb{R} \setminus \{1, -3\}$. Then

$$3x+5 = A(x+3) + B(x-1) \qquad (2)$$

Substitute $x = 1$ in equation (2):

$$8 = 4A$$
$$\therefore \quad A = 2$$

Substitute $x = -3$ in equation (2):

$$-4 = -4B$$
$$\therefore \quad B = 1$$

Hence $\dfrac{3x+5}{(x-1)(x+3)} = \dfrac{2}{x-1} + \dfrac{1}{x+3}$.

Explanation

We know that equation (2) is true for all $x \in \mathbb{R} \setminus \{1, -3\}$.

But if this is the case, then it also has to be true for $x = 1$ and $x = -3$.

Notes:

- You could substitute any values of x to find A and B in this way, but these values simplify the calculations.
- The method of equating coefficients could also be used here.

Example 21

Resolve $\dfrac{2x+10}{(x+1)(x-1)^2}$ into partial fractions.

Solution

Since the denominator has a repeated linear factor and a single linear factor, there are three partial fractions:

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

This gives the equation

$$2x+10 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

Let $x = 1$: $\quad 12 = 2C$

$\therefore \quad C = 6$

Let $x = -1$: $\quad 8 = 4A$

$\therefore \quad A = 2$

Let $x = 0$: $\quad 10 = A - B + C$

$\therefore \quad B = A + C - 10 = -2$

Hence $\dfrac{2x+10}{(x+1)(x-1)^2} = \dfrac{2}{x+1} - \dfrac{2}{x-1} + \dfrac{6}{(x-1)^2}$.

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Example 22

Resolve $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)}$ into partial fractions.

Solution

Since the denominator has a single linear factor and an irreducible quadratic factor (i.e. cannot be reduced to linear factors), there are two partial fractions:

$$\frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$

This gives the equation

$$x^2+6x+5 = A(x^2+x+1) + (Bx+C)(x-2) \qquad (1)$$

Subsituting $x = 2$:

$$\begin{aligned} 2^2+6(2)+5 &= A(2^2+2+1) \\ 21 &= 7A \\ \therefore \quad A &= 3 \end{aligned}$$

We can rewrite equation (1) as

$$\begin{aligned} x^2+6x+5 &= A(x^2+x+1) + (Bx+C)(x-2) \\ &= A(x^2+x+1) + Bx^2 - 2Bx + Cx - 2C \\ &= (A+B)x^2 + (A-2B+C)x + A - 2C \end{aligned}$$

Since $A = 3$, this gives

$$x^2+6x+5 = (3+B)x^2 + (3-2B+C)x + 3 - 2C$$

Equate coefficients:

$$\begin{aligned} 3+B &= 1 \qquad \text{and} \qquad & 3-2C &= 5 \\ \therefore \quad B &= -2 & \therefore \quad C &= -1 \end{aligned}$$

Check: $3-2B+C = 3-2(-2)+(-1) = 6$

Therefore

$$\begin{aligned} \frac{x^2+6x+5}{(x-2)(x^2+x+1)} &= \frac{3}{x-2} + \frac{-2x-1}{x^2+x+1} \\ &= \frac{3}{x-2} - \frac{2x+1}{x^2+x+1} \end{aligned}$$

Note: The values of B and C could also be found by substituting $x = 0$ and $x = 1$ in equation (1).

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Improper fractions

Improper algebraic fractions can be expressed as a sum of partial fractions by first dividing the denominator into the numerator to produce a quotient and a proper fraction. This proper fraction can then be resolved into its partial fractions using the techniques just introduced.

Example 23

Express $\dfrac{x^5 + 2}{x^2 - 1}$ as partial fractions.

Solution

Dividing through:

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2 - 1 & x^5 + 2 \\ & x^5 - x^3 \\ \hline & \quad x^3 + 2 \\ & \quad x^3 - x \\ \hline & \qquad x + 2 \end{array}$$

Therefore

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x + \frac{x + 2}{x^2 - 1}$$

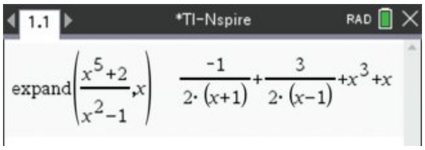
By expressing $\dfrac{x + 2}{x^2 - 1} = \dfrac{x + 2}{(x - 1)(x + 1)}$ as partial fractions, we obtain

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x - \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

Using the TI-Nspire

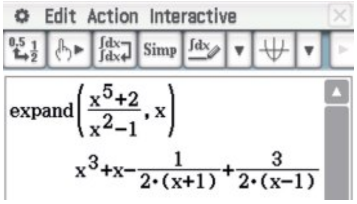
Use (menu) > **Algebra** > **Expand** as shown.

Note: The use of ‘, x’ is optional.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight $\dfrac{x^5 + 2}{x^2 - 1}$.
- Go to **Interactive** > **Transformation** > **expand** and choose the **Partial Fraction** option.
- Enter the variable and tap OK.



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Summary

- Examples of resolving a proper fraction into partial fractions:
  - Distinct linear factors
    $$\frac{3x-4}{(2x-3)(x+5)} = \frac{A}{2x-3} + \frac{B}{x+5}$$
  - Repeated linear factor
    $$\frac{3x-4}{(2x-3)(x+5)^2} = \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$
  - Irreducible quadratic factor
    $$\frac{3x-4}{(2x-3)(x^2+5)} = \frac{A}{2x-3} + \frac{Bx+C}{x^2+5}$$
  - Repeated irreducible quadratic factor
    $$\frac{3x-4}{(2x-3)(x^2+5)^2} = \frac{A}{2x-3} + \frac{Bx+C}{x^2+5} + \frac{Dx+E}{(x^2+5)^2}$$
- If $f(x) = \dfrac{g(x)}{h(x)}$ is an improper fraction, i.e. if the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then the division must be performed first.

These techniques work with more than two factors in the denominator.

- Distinct linear factors: $\dfrac{p(x)}{(x-a_1)(x-a_2)\dots(x-a_n)} = \dfrac{A_1}{x-a_1} + \dfrac{A_2}{x-a_2} + \dots + \dfrac{A_n}{x-a_n}$
- Repeated linear factor: $\dfrac{p(x)}{(x-a)^n} = \dfrac{A_1}{(x-a)} + \dfrac{A_2}{(x-a)^2} + \dots + \dfrac{A_n}{(x-a)^n}$

Using partial fractions for integration

We now use partial fractions to help perform integration.

Distinct linear factors

Example 24

Find $\displaystyle\int \frac{3x+5}{(x-1)(x+3)}\,dx$.

Solution

In Example 20, we found that

$$\frac{3x+5}{(x-1)(x+3)} = \frac{2}{x-1} + \frac{1}{x+3}$$

Therefore

$$\int \frac{3x+5}{(x-1)(x+3)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{1}{x+3}\,dx$$
$$= 2\log_e|x-1| + \log_e|x+3| + c$$
$$= \log_e\big((x-1)^2\,|x+3|\big) + c \qquad \text{(using logarithm rules)}$$

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Improper fractions

If the degree of the numerator is greater than or equal to the degree of the denominator, then division must take place first.

Example 25

Find $\int \frac{x^5 + 2}{x^2 - 1}\, dx$.

Solution

In Example 23, we divided through to find that

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x + \frac{x + 2}{x^2 - 1}$$

Expressing as partial fractions:

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x - \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

Hence

$$\begin{aligned}\int \frac{x^5 + 2}{x^2 - 1}\, dx &= \int x^3 + x - \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}\, dx \\ &= \frac{x^4}{4} + \frac{x^2}{2} - \frac{1}{2}\log_e |x + 1| + \frac{3}{2}\log_e |x - 1| + c \\ &= \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{2}\log_e\left(\frac{|x - 1|^3}{|x + 1|}\right) + c\end{aligned}$$

Repeated linear factor

Example 26

Express $\frac{3x + 1}{(x + 2)^2}$ in partial fractions and hence find $\int \frac{3x + 1}{(x + 2)^2}\, dx$.

Solution

Write $\quad \frac{3x + 1}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$

Then $\quad 3x + 1 = A(x + 2) + B$

Substituting $x = -2$ gives $-5 = B$.

Substituting $x = 0$ gives $1 = 2A + B$ and therefore $A = 3$.

Thus $\quad \frac{3x + 1}{(x + 2)^2} = \frac{3}{x + 2} - \frac{5}{(x + 2)^2}$

$$\begin{aligned}\therefore \quad \int \frac{3x + 1}{(x + 2)^2}\, dx &= \int \frac{3}{x + 2} - \frac{5}{(x + 2)^2}\, dx \\ &= 3\log_e |x + 2| + \frac{5}{x + 2} + c\end{aligned}$$

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Irreducible quadratic factor

Example 27

Find an antiderivative of $\dfrac{4}{(x+1)(x^2+1)}$ by first expressing it as partial fractions.

Solution

Write

$$\frac{4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Then

$$4 = A(x^2+1) + (Bx+C)(x+1)$$

Let $x = -1$: $\quad 4 = 2A$

$\therefore \quad A = 2$

Let $x = 0$: $\quad 4 = A + C$

$\therefore \quad C = 2$

Let $x = 1$: $\quad 4 = 2A + 2(B+C)$

$\therefore \quad B = -2$

Hence

$$\frac{4}{(x+1)(x^2+1)} = \frac{2}{x+1} + \frac{2-2x}{x^2+1}$$

We now turn to the integration:

$$\begin{aligned}\int \frac{4}{(x+1)(x^2+1)}\,dx &= \int \frac{2}{x+1} + \frac{2-2x}{x^2+1}\,dx \\ &= \int \frac{2}{x+1}\,dx + \int \frac{2}{x^2+1}\,dx - \int \frac{2x}{x^2+1}\,dx \\ &= 2\log_e|x+1| + 2\arctan x - \log_e(x^2+1) + c \\ &= \log_e\left(\frac{(x+1)^2}{x^2+1}\right) + 2\arctan x + c\end{aligned}$$

Skill-sheet

Exercise 9G

Example 20

1 Resolve the following rational expressions into partial fractions:

a $\dfrac{5x+1}{(x-1)(x+2)}$ **b** $\dfrac{-1}{(x+1)(2x+1)}$ **c** $\dfrac{3x-2}{x^2-4}$

d $\dfrac{4x+7}{x^2+x-6}$ **e** $\dfrac{7-x}{(x-4)(x+1)}$

Example 21

2 Resolve the following rational expressions into partial fractions:

a $\dfrac{2x+3}{(x-3)^2}$ **b** $\dfrac{9}{(1+2x)(1-x)^2}$ **c** $\dfrac{2x-2}{(x+1)(x-2)^2}$

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Example 22 **3** Resolve the following rational expressions into partial fractions:

a $\dfrac{3x+1}{(x+1)(x^2+x+1)}$ **b** $\dfrac{3x^2+2x+5}{(x^2+2)(x+1)}$ **c** $\dfrac{x^2+2x-13}{2x^3+6x^2+2x+6}$

Example 23 **4** Resolve $\dfrac{3x^2-4x-2}{(x-1)(x-2)}$ into partial fractions.

Example 24 **5** Decompose $\dfrac{9}{(x-10)(x-1)}$ into partial fractions and find its antiderivatives.

Example 25 **6** Decompose $\dfrac{x^4+1}{(x+2)^2}$ into partial fractions and find its antiderivatives.

Example 26 **7** Decompose $\dfrac{7x+1}{(x+2)^2}$ into partial fractions and find its antiderivatives.

Example 27 **8** Decompose $\dfrac{5}{(x^2+2)(x-4)}$ into partial fractions and find its antiderivatives.

9 Decompose each of the following into partial fractions and find their antiderivatives:

a $\dfrac{7}{(x-2)(x+5)}$ **b** $\dfrac{x+3}{x^2-3x+2}$ **c** $\dfrac{2x+1}{(x+1)(x-1)}$

d $\dfrac{2x^2}{x^2-1}$ **e** $\dfrac{2x+1}{x^2+4x+4}$ **f** $\dfrac{4x-2}{(x-2)(x+4)}$

10 Find an antiderivative of each of the following:

a $\dfrac{2x-3}{x^2-5x+6}$ **b** $\dfrac{5x+1}{(x-1)(x+2)}$ **c** $\dfrac{x^3-2x^2-3x+9}{x^2-4}$

d $\dfrac{4x+10}{x^2+5x+4}$ **e** $\dfrac{x^3+x^2-3x+3}{x+2}$ **f** $\dfrac{x^3+3}{x^2-x}$

11 Find an antiderivative of each of the following:

a $\dfrac{3x}{(x+1)(x^2+2)}$ **b** $\dfrac{2}{(x+1)^2(x^2+1)}$ **c** $\dfrac{5x^3}{(x-1)(x^2+4)}$

d $\dfrac{16(4x+1)}{(x-2)^2(x^2+4)}$ **e** $\dfrac{24(x+2)}{(x+2)^2(x^2+2)}$ **f** $\dfrac{8}{(x+1)^3(x^2-1)}$

12 Evaluate the following:

a $\displaystyle\int_1^2 \frac{1}{x(x+1)}\,dx$ **b** $\displaystyle\int_0^1 \frac{1}{(x+1)(x+2)}\,dx$ **c** $\displaystyle\int_2^3 \frac{x-2}{(x-1)(x+2)}\,dx$

d $\displaystyle\int_0^1 \frac{x^2}{x^2+3x+2}\,dx$ **e** $\displaystyle\int_2^3 \frac{x+7}{(x+3)(x-1)}\,dx$ **f** $\displaystyle\int_2^3 \frac{2x+6}{(x-1)^2}\,dx$

g $\displaystyle\int_2^3 \frac{x+2}{x(x+4)}\,dx$ **h** $\displaystyle\int_0^1 \frac{1-4x}{3+x-2x^2}\,dx$ **i** $\displaystyle\int_1^2 \frac{1}{x(x-4)}\,dx$

j $\displaystyle\int_{-3}^{-2} \frac{1-4x}{(x+6)(x+1)}\,dx$ **k** $\displaystyle\int_0^1 \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$

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13 Evaluate the following:

a $\displaystyle\int_0^1 \frac{10x}{(x+1)(x^2+1)}\,dx$

b $\displaystyle\int_0^{\sqrt{3}} \frac{x^3-8}{(x-2)(x^2+1)}\,dx$

c $\displaystyle\int_0^1 \frac{x^2-1}{x^2+1}\,dx$

d $\displaystyle\int_{-\frac{1}{2}}^0 \frac{6}{(x^2+x+1)(x-1)}\,dx$

14 Let $f(x) = \dfrac{x^2+6x+5}{(x-2)(x^2+x+1)}$.

a Express $f(x)$ as partial fractions.

b Hence find an antiderivative of $f(x)$.

c Hence evaluate $\int_{-2}^{-1} f(x)\,dx$.

15 In this question, we find an antiderivative of $\sec x$.

a Prove that $\sec x = \dfrac{\cos x}{(1-\sin x)(1+\sin x)}$.

b Hence, by using the substitution $u = \sin x$ and then partial fractions, show that

$$\int \sec x\,dx = \log_e\left|\sec x + \tan x\right| + c$$

16 We now find an antiderivative of $\sec x$ by using the substitution $x = 2\tan^{-1} t$.

a **i** Show that $\sin x = \dfrac{2t}{1+t^2}$.

ii Show that $\cos x = \dfrac{1-t^2}{1+t^2}$.

b Hence find $\displaystyle\int \frac{1}{\cos x}\,dx$ in terms of t.

c By using the same substitution, find $\displaystyle\int \frac{\sin x}{1+\sin x}\,dx$.

9H Integration by parts

The product rule is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrate both sides with respect to x:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

By rearranging this equation, we obtain the following technique for integration.

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Note: We can use integration by parts to find an integral $\int u\frac{dv}{dx}\,dx$ if the integral $\int v\frac{du}{dx}\,dx$ is easier to find.

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Example 28

Find an antiderivative of each of the following:

a $x \cos x$ **b** xe^x **c** $\arcsin x$

Solution

a Let $u = x$ and $\frac{dv}{dx} = \cos x$.

Then $\frac{du}{dx} = 1$ and $v = \sin x$. (Choose v to be the simplest antiderivative of $\frac{dv}{dx}$.)

So
$$\begin{aligned}\int x \cos x \, dx &= \int u \frac{dv}{dx} \, dx \\ &= uv - \int v \frac{du}{dx} \, dx \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + c\end{aligned}$$

b $\int xe^x \, dx$

Let $u = x$ and $\frac{dv}{dx} = e^x$. Then $\frac{du}{dx} = 1$ and $v = e^x$.

So
$$\begin{aligned}\int xe^x \, dx &= \int u \frac{dv}{dx} \, dx \\ &= uv - \int v \frac{du}{dx} \, dx \\ &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + c\end{aligned}$$

c $\int \arcsin x \, dx$

Let $u = \arcsin x$ and $\frac{dv}{dx} = 1$. Then $\frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}}$ and $v = x$.

So
$$\begin{aligned}\int \arcsin x \, dx &= \int u \frac{dv}{dx} \, dx \\ &= uv - \int v \frac{du}{dx} \, dx \\ &= x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} \, dx \\ &= x \arcsin x + \sqrt{1 - x^2} + c\end{aligned}$$

Note: We can find $\int \frac{x}{\sqrt{1 - x^2}} \, dx$ by using the substitution $w = 1 - x^2$.

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Using integration by parts more than once

In some cases, we need to use integration by parts more than once.

Example 29

Find $\int x^2 e^x \, dx$.

Solution

Let $u = x^2$ and $\frac{dv}{dx} = e^x$. Then $\frac{du}{dx} = 2x$ and $v = e^x$.

So
$$\begin{aligned}\int x^2 e^x \, dx &= \int u \frac{dv}{dx} \, dx \\ &= uv - \int v \frac{du}{dx} \, dx \\ &= x^2 e^x - \int 2xe^x \, dx \\ &= x^2 e^x - 2(xe^x - e^x) + c \qquad \text{(using Example 28 **b**)} \\ &= (x^2 - 2x + 2)e^x + c\end{aligned}$$

Using integration by parts by solving for the unknown integral

Integration by parts can be applied to expressions of the form $e^{ax} \sin(bx)$ and $e^{ax} \cos(bx)$ in a different way. Again, we use integration by parts twice. We form an equation which we can solve for the unknown integral.

Example 30

Find $\int e^x \cos x \, dx$.

Solution

Let $u = e^x$ and $\frac{dv}{dx} = \cos x$. Then $\frac{du}{dx} = e^x$ and $v = \sin x$.

So, using integration by parts, we obtain

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \qquad (1)$$

Similarly, we can use integration by parts to obtain

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx \qquad (2)$$

Substitute (2) in (1) and then rearrange:

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx\right)$$

$$\therefore \quad 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + c$$

Now dividing by 2 and renaming the constant gives

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c$$

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Using reduction formulas

In the next example, we find the integral $\int x^3 e^x \, dx$ by considering integrals of the general form $I_n = \int x^n e^x \, dx$. We will see that the problem of finding the integral I_n can be reduced to the simpler problem of finding the integral I_{n-1}.

Example 31

a Find a recursive formula for $\int x^n e^x \, dx$, where $n \in \mathbb{N}$.

b Use your formula to find $\int x^3 e^x \, dx$.

Solution

a For each $n \in \mathbb{N} \cup \{0\}$, define $I_n = \int x^n e^x \, dx$.

Now let $n \in \mathbb{N}$. Using integration by parts, we find

$$\begin{aligned} I_n &= \int x^n e^x \, dx \\ &= \int x^n \frac{d}{dx}(e^x) \, dx \\ &= x^n e^x - \int e^x \frac{d}{dx}(x^n) \, dx \\ &= x^n e^x - \int e^x n x^{n-1} \, dx \\ &= x^n e^x - n \int x^{n-1} e^x \, dx \\ &= x^n e^x - n I_{n-1} \end{aligned}$$

We have shown that

$$I_n = x^n e^x - n I_{n-1} \quad \text{for all } n \in \mathbb{N}$$

b Using the formula from part **a** gives

$$\begin{aligned} \int x^3 e^x \, dx &= I_3 && \text{(by the definition of } I_3\text{)} \\ &= x^3 e^x - 3I_2 && \text{(expressing } I_3 \text{ in terms of } I_2\text{)} \\ &= x^3 e^x - 3(x^2 e^x - 2I_1) && \text{(expressing } I_2 \text{ in terms of } I_1\text{)} \\ &= x^3 e^x - 3x^2 e^x + 6I_1 \\ &= x^3 e^x - 3x^2 e^x + 6(x e^x - I_0) && \text{(expressing } I_1 \text{ in terms of } I_0\text{)} \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6I_0 \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6 \int e^x \, dx && \text{(by the definition of } I_0\text{)} \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c \\ &= (x^3 - 3x^2 + 6x - 6)e^x + c \end{aligned}$$

A recursive formula like the one found in Example 31 **a**, which expresses an integral in terms of a simpler integral of the same form, is called a **reduction formula**.

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Using integration by parts for definite integrals

We can also use integration by parts to evaluate definite integrals.

$$\int_a^b u \frac{dv}{dx}\, dx = [uv]_a^b - \int_a^b v \frac{du}{dx}\, dx$$

Example 32

Evaluate $\int_1^2 \log_e x \, dx$.

Solution

Let $u = \log_e x$ and $\frac{dv}{dx} = 1$. Then $\frac{du}{dx} = \frac{1}{x}$ and $v = x$.

We have

$$\begin{aligned}\int_1^2 \log_e x\, dx &= [x \log_e x]_1^2 - \int_1^2 1\, dx \\ &= [x \log_e x]_1^2 - [x]_1^2 \\ &= 2\log_e 2 - (2 - 1) \\ &= 2\log_e 2 - 1\end{aligned}$$

Exercise 9H

Example 28

1 Find an antiderivative of each of the following:

a xe^{-x} **b** $\log_e x$ **c** $x \sin x$ **d** $\arccos x$
e $x\cos(3x)$ **f** $x\sec^2 x$ **g** $x\tan^2 x$ **h** $\arcsin(2x)$
i $\arctan x$ **j** $(x+1)e^{-x}$ **k** $x \arctan x$ **l** $x \log_e x$
m $x^2 \log_e x$ **n** $x^{-\frac{1}{2}} \log_e x$ **o** $(x+3)e^x$ **p** $x^5 \log_e x$
q xe^{2x+1} **r** $x\log_e(2x)$

Example 29

2 Find an antiderivative of each of the following:

a x^2e^{-x} **b** $x^2 \sin x$

Example 30

3 Find an antiderivative of each of the following:

a $e^x \sin x$ **b** $e^{2x}\cos(3x)$ **c** $e^{3x}\sin x$ **d** $e^x \sin\left(\frac{x}{2}\right)$

Example 31

4 Find a reduction formula for $\int x^n e^{2x}\, dx$ and then use it to find $\int x^3 e^{2x}\, dx$.

5 Find a reduction formula for $\int (\log_e x)^n\, dx$ and then use it to find $\int (\log_e x)^3\, dx$.
Hint: Write $(\log_e x)^n$ as $(\log_e x)^n \cdot 1$.

6 Let $I_n = \int \sin^n x\, dx$.

a Show that $I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$. **b** Hence find $\int \sin^5 x\, dx$.

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7 Let $I_n = \int \cos^n x \, dx$.

a Show that $I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$. **b** Hence find $\int \cos^5 x \, dx$.

8 For each $n \in \mathbb{N} \cup \{0\}$, define $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx$.

a Find the values of I_0 and I_1.

b Using integration by parts, show that $I_n = \frac{2n}{2n+1} I_{n-1}$ for all $n \in \mathbb{N}$.

c Hence find the values of I_2 and I_3.

d Prove that

$$I_n = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{4}{5} \times \frac{2}{3}$$

for all $n \in \mathbb{N}$.

Example 32

9 Evaluate each of the following:

a $\int_0^2 xe^{2x} \, dx$ **b** $\int_0^{2\pi} x \sin(4x) \, dx$ **c** $\int_0^{\frac{\pi}{4}} x \cos(4x) \, dx$

d $\int_0^1 2xe^{3x} \, dx$ **e** $\int_0^{\pi} (4x-3) \sin\left(\frac{x}{4}\right) dx$ **f** $\int_0^1 x^2 e^{3x-1} \, dx$

g $\int_1^2 \log_e(3x) \, dx$ **h** $\int_0^2 x^2 e^{2x} \, dx$ **i** $\int_1^3 x^2 \log_e x \, dx$

9I Further techniques and miscellaneous exercises

In this section, the different techniques are arranged so that a choice must be made of the most suitable one for a particular problem. Often there is more than one appropriate choice.

The relationship between a function and its derivative is also exploited. This is illustrated in the following example.

Example 33

a Find the derivative of $\sin^{-1}(x) + x\sqrt{1-x^2}$. **b** Hence evaluate $\int_0^{\frac{1}{2}} \sqrt{1-x^2} \, dx$.

Solution

a Let $y = \sin^{-1}(x) + x\sqrt{1-x^2}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \left(\sqrt{1-x^2} + \frac{(-x)x}{\sqrt{1-x^2}}\right) \qquad \text{(using the product rule for } x\sqrt{1-x^2}\text{)} \\ &= \frac{1}{\sqrt{1-x^2}} + \frac{1-x^2-x^2}{\sqrt{1-x^2}} \\ &= \frac{2(1-x^2)}{\sqrt{1-x^2}} \\ &= 2\sqrt{1-x^2}\end{aligned}$$

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b From part **a**, we have

$$\int 2\sqrt{1-x^2}\,dx = \sin^{-1}(x) + x\sqrt{1-x^2} + c$$

$$\therefore \quad \int_0^{\frac{1}{2}} 2\sqrt{1-x^2}\,dx = \left[\sin^{-1}(x) + x\sqrt{1-x^2}\right]_0^{\frac{1}{2}}$$

$$\therefore \quad \int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\sqrt{1-\left(\tfrac{1}{2}\right)^2} - \left(\sin^{-1}(0)+0\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

Skill-sheet

Exercise 9I

1 If $\int_0^1 \frac{1}{(x+1)(x+2)}\,dx = \log_e p$, find p.

2 Evaluate each of the following:

a $\int_0^{\frac{\pi}{6}} \sin^2 x \cos x\,dx$ **b** $\int_0^1 \frac{e^{2x}}{1+e^x}\,dx$

c $\int_0^{\frac{\pi}{3}} \sin^3 x \cos x\,dx$ **d** $\int_3^4 \frac{x}{(x-2)(x+1)}\,dx$

3 If $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx = \log_e c$, find c.

4 Find an antiderivative of $\sin(3x)\cos^5(3x)$.

5 If $\int_4^6 \frac{2}{x^2-4}\,dx = \log_e p$, find p.

6 If $\int_5^6 \frac{3}{x^2-5x+4}\,dx = \log_e p$, find p.

7 Find an antiderivative of each of the following:

a $\frac{\cos x}{\sin^3 x}$ **b** $x(4x^2+1)^{\frac{3}{2}}$ **c** $\sin^2 x \cos^3 x$ **d** $\frac{e^x}{e^{2x}-2e^x+1}$

8 Evaluate $\int_0^3 \frac{x}{\sqrt{25-x^2}}\,dx$.

9 Find an antiderivative of each of the following:

a $\frac{1}{(x+1)^2+4}$ **b** $\frac{1}{\sqrt{1-9x^2}}$ **c** $\frac{1}{\sqrt{1-4x^2}}$ **d** $\frac{1}{(2x+1)^2+9}$

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Example 33 **10** Let $f\colon (1, \infty) \to \mathbb{R}$, where $f(x) = \sin^{-1}\left(\frac{1}{\sqrt{x}}\right)$.

a Find $f'(x)$

b Using the result of **a**, find $\int_2^4 \frac{1}{x\sqrt{x-1}}\, dx$.

11 For each of the following, use an appropriate substitution to find an expression for the antiderivative in terms of $f(x)$:

a $\int f'(x)(f(x))^2\, dx$

b $\int \frac{f'(x)}{(f(x))^2}\, dx$

c $\int \frac{f'(x)}{f(x)}\, dx$, where $f(x) > 0$

d $\int f'(x)\sin(f(x))\, dx$

12 If $y = x\sqrt{4-x}$, find $\frac{dy}{dx}$ and simplify. Hence evaluate $\int_0^2 \frac{8-3x}{\sqrt{4-x}}\, dx$.

13 Find a, b and c such that $\frac{2x^3 - 11x^2 + 20x - 13}{(x-2)^2} = ax + b + \frac{c}{(x-2)^2}$ for all $x \neq 2$.

Hence find $\int \frac{2x^3 - 11x^2 + 20x - 13}{(x-2)^2}\, dx$.

14 Evaluate each of the following:

a $\int_0^{\frac{\pi}{4}} \sin^2(2x)\, dx$

b $\int_{-1}^{0} (14 - 2x)\sqrt{x^2 - 14x + 1}\, dx$

c $9\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\, dx$

d $\int_e^{e^2} \frac{1}{x\log_e x}\, dx$

e $\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$

f $\int_0^{\frac{\pi}{2}} \frac{\sin x}{2 + \cos x}\, dx$

15 Find $\int \sin x \cos x\, dx$ using:

a the substitution $u = \sin x$

b the identity $\sin(2x) = 2\sin x \cos x$

16 **a** If $y = \log_e(x + \sqrt{x^2+1})$, find $\frac{dy}{dx}$. Hence find $\int \frac{1}{\sqrt{x^2+1}}\, dx$.

b If $y = \log_e(x + \sqrt{x^2-1})$, find $\frac{dy}{dx}$. Hence show that $\int_2^7 \frac{1}{\sqrt{x^2-1}}\, dx = \log_e(2+\sqrt{3})$.

17 Find an antiderivative of each of the following:

a $\frac{1}{4+x^2}$

b $\frac{1}{4-x^2}$

c $\frac{4+x^2}{x}$

d $\frac{x}{4+x^2}$

e $\frac{x^2}{4+x^2}$

f $\frac{1}{1+4x^2}$

g $x\sqrt{4+x^2}$

h $x\sqrt{4+x}$

i $\frac{1}{\sqrt{4-x}}$

j $\frac{1}{\sqrt{4-x^2}}$

k $\frac{x}{\sqrt{4-x}}$

l $\frac{x}{\sqrt{4-x^2}}$

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18 Find constants c and d such that $\int_2^3 \frac{x^3 - x + 2}{x^2 - 1}\, dx = c + \log_e d$.

19 **a** Differentiate $f(x) = \sin x \cos^{n-1} x$.

b Hence verify that $n \int \cos^n x\, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx$.

c Hence evaluate:

i $\int_0^{\frac{\pi}{2}} \cos^4 x\, dx$ **ii** $\int_0^{\frac{\pi}{2}} \cos^6 x\, dx$

iii $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x\, dx$ **iv** $\int_0^{\frac{\pi}{4}} \sec^4 x\, dx$

20 Find:

a $\int \frac{x}{(x+1)^n}\, dx$ **b** $\int_1^2 x(x-1)^n\, dx$

21 **a** Evaluate $\int_0^1 (1 + ax)^2\, dx$.

b For what value of a is the value of this integral a minimum?

22 **a** Differentiate $\frac{a \sin x - b \cos x}{a \cos x + b \sin x}$ with respect to x.

b Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{(a \cos x + b \sin x)^2}\, dx$.

23 Let $U_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, where $n \in \mathbb{Z}$ with $n > 1$.

a Express $U_n + U_{n-2}$ in terms of n. **b** Hence show that $U_6 = \frac{13}{15} - \frac{\pi}{4}$.

24 **a** Simplify $\frac{1}{1 + \tan x} + \frac{1}{1 + \cot x}$.

b Let $\varphi = \frac{\pi}{2} - \theta$. Show that $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cot\varphi}\, d\varphi$.

c Use these results to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan\theta}\, d\theta$.

25 Consider the integral $\int e^{\sqrt{x}}\, dx$.

a By using the substitution $u = \sqrt{x}$, show that $\int e^{\sqrt{x}}\, dx = 2 \int ue^u\, du$.

b Hence, using integration by parts, show that $\int e^{\sqrt{x}}\, dx = e^{\sqrt{x}}(\sqrt{x} - 1) + c$.

26 **a** Find $\int \sin^4 x\, dx$ using a reduction formula.

b Find $\int \sin^4 x\, dx$ using the double angle formula $\cos(2x) = 1 - 2\sin^2 x$.

c Combine your two answers to obtain an impressive trigonometric identity.

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Review

Chapter summary

Assign-ment

Nrich

Antiderivatives involving inverse circular functions

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\, dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2 + x^2}\, dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

Integration by substitution

- The change of variable rule is

$$\int f(u)\frac{du}{dx}\, dx = \int f(u)\, du \qquad \text{where } u \text{ is a function of } x$$

- **Linear substitution**

  A linear substitution can be used to find antiderivatives of expressions such as

$$(2x+3)\sqrt{3x-4}, \quad \frac{2x+5}{\sqrt{3x-4}} \quad \text{and} \quad \frac{2x+5}{(x+2)^2}$$

  Consider $\int f(x)\, g(ax+b)\, dx$.

  Let $u = ax + b$. Then $x = \dfrac{u-b}{a}$ and so

$$\int f(x)\, g(ax+b)\, dx = \int f\left(\frac{u-b}{a}\right) g(u)\, dx$$
$$= \frac{1}{a}\int f\left(\frac{u-b}{a}\right) g(u)\, du$$

- Definite integration involving the change of variable rule:

  Let $u = g(x)$. Then

$$\int_a^b f(u)\frac{du}{dx}\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Useful trigonometric identities

$$\sin(2x) = 2\sin x\cos x$$

$$\cos(2x) = 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$= \cos^2 x - \sin^2 x$$

$$\sec^2 x = 1 + \tan^2 x$$

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Partial fractions

- A rational function may be expressed as a sum of simpler functions by resolving it into **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

- Examples of resolving a proper fraction into partial fractions:
  - Distinct linear factors

$$\frac{3x-4}{(2x-3)(x+5)} = \frac{A}{2x-3} + \frac{B}{x+5}$$

  - Repeated linear factor

$$\frac{3x-4}{(2x-3)(x+5)^2} = \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$

  - Irreducible quadratic factor

$$\frac{3x-4}{(2x-3)(x^2+5)} = \frac{A}{2x-3} + \frac{Bx+C}{x^2+5}$$

- A quadratic polynomial is **irreducible** if it cannot be factorised over $\mathbb{R}$.
- If $f(x) = \dfrac{g(x)}{h(x)}$ is an improper fraction, i.e. if the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then the division must be performed first. Write $f(x)$ in the form

$$\frac{g(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)}$$

where the degree of $r(x)$ is less than the degree of $h(x)$.

Integration by parts

$$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$$

$$\int_a^b u \frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v \frac{du}{dx}\,dx$$

Technology-free questions

1 Find an antiderivative of each of the following:

a $\cos^3(2x)$ **b** $\dfrac{2x+3}{4x^2+1}$ **c** $\dfrac{1}{1-4x^2}$ **d** $\dfrac{x}{\sqrt{1-4x^2}}$

e $\dfrac{x^2}{1-4x^2}$ **f** $x\sqrt{1-2x^2}$ **g** $\sin^2\left(x-\dfrac{\pi}{3}\right)$ **h** $\dfrac{x}{\sqrt{x^2-2}}$

i $\sin^2(3x)$ **j** $\sin^3(2x)$ **k** $x\sqrt{x+1}$ **l** $\dfrac{1}{1+\cos(2x)}$

m $\dfrac{e^{3x}+1}{e^{3x+1}}$ **n** $\dfrac{x}{x^2-1}$ **o** $\sin^2 x\cos^2 x$ **p** $\dfrac{x^2}{1+x}$

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Review

2 Evaluate each of the following integrals:

a $\int_0^{\frac{1}{2}} x(1-x^2)^{\frac{1}{2}}\,dx$ **b** $\int_0^{\frac{1}{2}} (1-x^2)^{-1}\,dx$ **c** $\int_0^{\frac{1}{2}} x(1+x^2)^{\frac{1}{2}}\,dx$

d $\int_1^2 \frac{1}{6x+x^2}\,dx$ **e** $\int_0^1 \frac{2x^2+3x+2}{x^2+3x+2}\,dx$ **f** $\int_0^1 \frac{1}{\sqrt{4-3x}}\,dx$

g $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ **h** $\int_0^{\frac{\pi}{2}} \sin^2(2x)\,dx$ **i** $\int_{-\pi}^{\pi} \sin^2 x \cos^2 x\,dx$

j $\int_0^{\frac{\pi}{2}} \sin^2(2x)\cos^2(2x)\,dx$ **k** $\int_0^{\frac{\pi}{4}} \frac{2\cos x - \sin x}{2\sin x + \cos x}\,dx$ **l** $\int_{-1}^2 x^2\sqrt{x^3+1}\,dx$

3 Show that $\frac{x}{x^2+2x+3} = \frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right) - \frac{1}{x^2+2x+3}$. Hence find $\int \frac{x}{x^2+2x+3}\,dx$.

4 **a** Differentiate $\sin^{-1}(\sqrt{x})$ and hence find $\int \frac{1}{\sqrt{x(1-x)}}\,dx$.

b Differentiate $\sin^{-1}(x^2)$ and hence find $\int \frac{2x}{\sqrt{1-x^4}}\,dx$.

5 Use integration by parts to find:

a $\int \sin^{-1} x\,dx$ **b** $\int \log_e x\,dx$ **c** $\int \tan^{-1} x\,dx$

6 Find an antiderivative of each of the following:

a $\sin(2x)\cos(2x)$ **b** $x^2(x^3+1)^2$ **c** $\frac{\cos\theta}{(3+2\sin\theta)^2}$

d xe^{1-x^2} **e** $\tan^2(x+3)$ **f** $\frac{2x}{\sqrt{6+2x^2}}$

g $\tan^2 x \sec^2 x$ **h** $\sec^3 x \tan x$ **i** $\tan^2(3x)$

7 Evaluate the following:

a $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$ **b** $\int_1^8 (13-5x)^{\frac{1}{3}}\,dx$ **c** $\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$

d $\int_1^2 (3-y)^{\frac{1}{2}}\,dy$ **e** $\int_0^{\pi} \sin^2 x\,dx$ **f** $\int_{-3}^{-1} \frac{x^2+1}{x^3+3x}\,dx$

8 Find the derivative of $2\left(x^2+\frac{1}{x}\right)^{\frac{1}{2}}$ and hence evaluate $\int_1^2 (2x-x^{-2})\left(x^2+\frac{1}{x}\right)^{-\frac{1}{2}}\,dx$.

9 Let $f(x) = \frac{4x^2+16x}{(x-2)^2(x^2+4)}$.

a Given that $f(x) = \frac{a}{x-2} + \frac{6}{(x-2)^2} - \frac{bx+4}{x^2+4}$, find a and b.

b Given that $\int_{-2}^0 f(x)\,dx = \frac{c-\pi-\log_e d}{2}$, find c and d.

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10 Find an antiderivative of each of the following:

a $e^{-2x}\cos(2x+3)$ **b** $x\sec^2 x$ **c** $e^{3x}\cos\left(\frac{x}{2}\right)$

11 Evaluate each of the following definite integrals:

a $\int_1^2 x^2 \log_e x\, dx$ **b** $\int_1^2 \frac{\log_e x}{x}\, dx$ **c** $\int_0^1 xe^{-2x}\, dx$

Multiple-choice questions

1 By using a suitable substitution, the integral $\int_1^3 x^2\sqrt{3-x}\, dx$ can be expressed as

A $\int_2^0 9u^{\frac{1}{2}} - 6u^{\frac{3}{2}} + u^{\frac{5}{2}}\, du$ **B** $-\int_2^0 9u^{\frac{1}{2}} + 6u^{\frac{3}{2}} + u^{\frac{5}{2}}\, du$ **C** $-\int_3^0 9u^{\frac{1}{2}} - 6u^{\frac{3}{2}} + u^{\frac{5}{2}}\, du$

D $\int_0^2 9u^{\frac{1}{2}} - 6u^{\frac{3}{2}} + u^{\frac{5}{2}}\, du$ **E** $\int_0^2 9u^{\frac{1}{2}} + 6u^{\frac{3}{2}} - u^{\frac{5}{2}}\, du$

2 If $\int_0^m \tan x \sec^2 x\, dx = \frac{3}{2}$, where $m \in \left(0, \frac{\pi}{2}\right)$, then the value of m is

A 0.5 **B** 1 **C** $\frac{\pi}{3}$ **D** $\frac{\pi}{6}$ **E** $\frac{\pi}{8}$

3 An antiderivative of $\tan(2x)$ is

A $\frac{1}{2}\sec^2(2x)$ **B** $\frac{1}{2}\log_e|\cos(2x)|$ **C** $\frac{1}{2}\log_e|\sec(2x)|$

D $\frac{1}{2}\log_e|\sin(2x)|$ **E** $\frac{1}{2}\tan^2(2x)$

4 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)}{2+\cos(2x)}\, dx$ is equal to

A $\frac{1}{\sqrt{2}}$ **B** $\log_e\left(\frac{1}{\sqrt{2}}\right)$ **C** $\log_e 2$ **D** $\frac{1}{2}\log_e 2$ **E** 1

5 $\int_0^{\frac{\pi}{3}} \sin x \cos^3 x\, dx$ written as an integral with respect to u, where $u = \cos x$, is

A $\int_{\frac{1}{2}}^1 u^3\, du$ **B** $\int_0^{\frac{\pi}{3}} u^3\, du$ **C** $\int_1^{\frac{1}{2}} u^3\sqrt{1-u^2}\, du$

D $\int_{\frac{1}{2}}^0 u^3\sqrt{1-u^2}\, du$ **E** $\int_1^{\frac{1}{2}} u^3\, du$

6 The value of $\int_0^2 \cos^2 x - \sin^2 x\, dx$, correct to four decimal places, is

A −0.0348 **B** 0.0349 **C** −0.3784 **D** 2.0000 **E** 0.3784

7 The substitution $u = \sin x$ is made to the integral $\int \cos^3 x \sin^n x\, dx$. If the resulting integral is $\int u^9 - u^{11}\, du$, then the value of n is

A 3 **B** 2 **C** 9 **D** 10 **E** 11

8 Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f(0) = 2$, $f(1) = 3$, $f'(0) = 6$ and $f'(1) = 10$. The value of $\int_0^1 5xf''(x)\, dx$ is

A −20 **B** −16 **C** 45 **D** 10 **E** 36

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Review

9 If $\frac{d}{dx}(xf(x)) = xf'(x) + f(x)$ and $xf'(x) = \frac{1}{1+x^2}$, then an antiderivative of $f(x)$ is

A $xf(x) - \tan^{-1}(x)$ **B** $\log_e(x^2+1)$ **C** $\frac{1}{2x}\log_e(x^2+1)$

D $f(x) - \tan^{-1}(x)$ **E** $\tan^{-1}(x)$

10 If $F'(x) = f(x)$, then an antiderivative of $3f(3-2x)$ is

A $\frac{3}{2}F(3-2x)$ **B** $-\frac{3}{4}(3-2x)^2$ **C** $\frac{3}{4}(3-2x)^2$ **D** $-\frac{3}{2}F(3-2x)$ **E** $-\frac{3}{2}f(3-2x)$

Extended-response questions

1 In this question, you will establish reduction formulas for $\int \tan^n x\,dx$ and $\int \sec^n x\,dx$.

a Prove that $\tan^n x = \tan^{n-2} x \sec^2 x - \tan^{n-2} x$, for all $n \in \mathbb{N}$ with $n \geq 2$.

b Define $I_n = \int \tan^n x\,dx$. Prove the reduction formula

$$I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$$

for all $n \in \mathbb{N}$ with $n \geq 2$.

c Use this reduction formula to find an antiderivative of each of the following:

i $\tan^2 x$ **ii** $\tan^3 x$ **iii** $\tan^4 x$ **iv** $\tan^5 x$

d Find $\int \sec x\,dx$ by using $\sec x = \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$.

e Use integration by parts to obtain a reduction formula for $\int \sec^n x\,dx$.

f Use this reduction formula to find an antiderivative of each of the following:

i $\sec^3 x$ **ii** $\sec^4 x$ **iii** $\sec^5 x$

2 Define $I_n = \int \frac{1}{(x^2+1)^n}\,dx$. Now let $n \in \mathbb{N}$ with $n \geq 2$.

a Using integration by parts, show that $I_n = \dfrac{x}{(x^2+1)^n} + 2n\int \dfrac{x^2}{(x^2+1)^{n+1}}\,dx$.

b Show that $\dfrac{x^2}{(x^2+1)^{n+1}} = \dfrac{1}{(x^2+1)^n} - \dfrac{1}{(x^2+1)^{n+1}}$.

c Prove the reduction formula $I_n = \dfrac{x}{2(n-1)(x^2+1)^{n-1}} + \dfrac{2n-3}{2(n-1)}I_{n-1}$.

d Use this reduction formula to evaluate each of the following:

i $\int_0^1 \frac{1}{(x^2+1)^2}\,dx$ **ii** $\int_0^1 \frac{1}{(x^2+1)^3}\,dx$ **iii** $\int_0^1 \frac{1}{(x^2+1)^4}\,dx$

3 **a** Let $k \in \mathbb{N}$. Find an expression in terms of k for $\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx$.

b The binomial theorem states that $(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$ for all $n \in \mathbb{N}$.

Use the binomial theorem to help show that

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx = \sum_{k=0}^{n} {}^nC_k \frac{(-1)^k}{2k+1}$$

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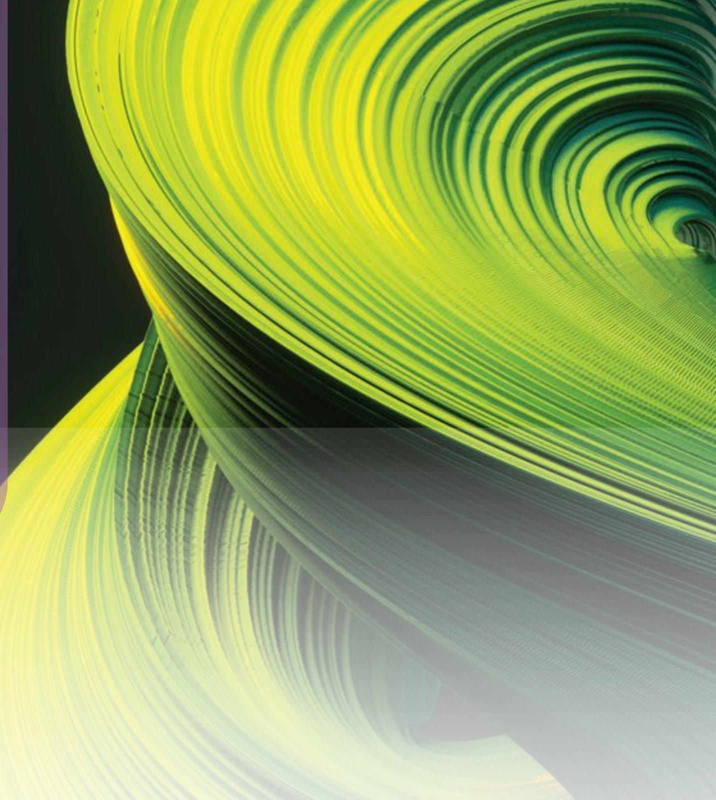
10

Applications of integration



Objectives

- To determine the **area** under a curve.
- To determine the **area** between two curves.
- To use a CAS calculator to evaluate definite integrals.
- To determine the **volume** of a solid of revolution.
- To determine the **length** of a section of a curve.
- To determine the **surface area** of a solid of revolution.

In this chapter we revisit the **fundamental theorem of calculus**. We will apply the theorem to the new functions introduced in this course, and use the integration techniques developed in the previous chapter. We then study three further applications of integration.

First, we will use integration to find the volume of a solid formed by revolving a bounded region defined by a curve around one of the axes.

If the region bounded by the curve with equation $y = f(x)$ and the lines $x = a$ and $x = b$ is rotated about the x-axis, then the volume V of the solid is given by

$$V = \pi \int_a^b (f(x))^2 \, dx$$

We will also use integration to find the surface area of such a solid.

These two applications can be used to derive the formulas for the volume and the surface area of a sphere, which you have used for several years.

As a third application, we will use integration to find the length of a section of a curve.

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10A The fundamental theorem of calculus

In this section we review integration from Mathematical Methods Units 3 & 4. We consider the graphs of some of the functions introduced in earlier chapters, and the areas of regions defined through these functions. It may be desirable to use a graphing package or a CAS calculator to help with the graphing in this section.

Signed area

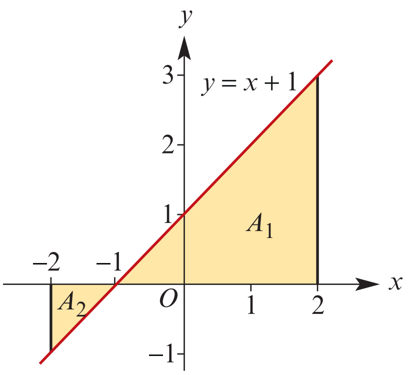
Consider the graph of $y = x + 1$ shown to the right.

$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$ (area of a triangle)

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

The total area is $A_1 + A_2 = 5$.

The **signed area** is $A_1 - A_2 = 4$.

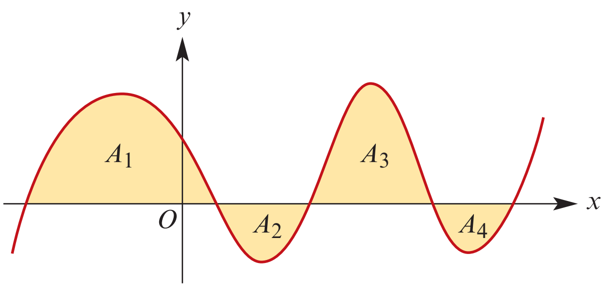


Regions above the x-axis have **positive signed area**.

Regions below the x-axis have **negative signed area**.

The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$.

The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.



The definite integral

Let f be a continuous function on a closed interval $[a, b]$. The signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$ is denoted by

$$\int_a^b f(x)\,dx$$

and is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$.

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

Notes:

- If $f(x) \geq 0$ for all $x \in [a, b]$, the area between $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
- If $f(x) \leq 0$ for all $x \in [a, b]$, the area between $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.

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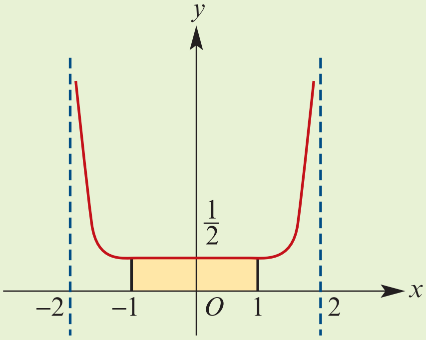
Example 1

The graph of $y = \dfrac{1}{\sqrt{4 - x^2}}$ is shown.

Find the area of the shaded region.



Solution

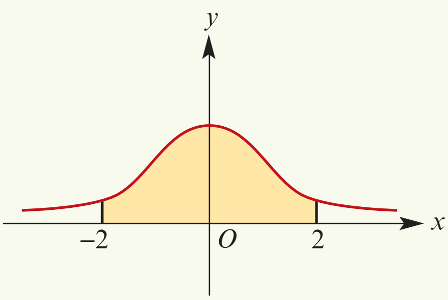
$$\begin{aligned}
\text{Area} &= \int_{-1}^{1} \frac{1}{\sqrt{4 - x^2}}\,dx \\
&= 2\int_{0}^{1} \frac{1}{\sqrt{4 - x^2}}\,dx \qquad \text{(by symmetry)} \\
&= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 \\
&= 2\sin^{-1}\left(\frac{1}{2}\right) \\
&= 2 \times \frac{\pi}{6} \\
&= \frac{\pi}{3}
\end{aligned}$$

Example 2

Find the area under the graph of $y = \dfrac{6}{4 + x^2}$ between $x = -2$ and $x = 2$.

Solution

$$\begin{aligned}
\text{Area} &= 6\int_{-2}^{2} \frac{1}{4 + x^2}\,dx \\
&= \frac{6}{2}\int_{-2}^{2} \frac{2}{4 + x^2}\,dx \\
&= 6\int_{0}^{2} \frac{2}{4 + x^2}\,dx \qquad \text{(by symmetry)} \\
&= 6\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 \\
&= 6\tan^{-1}(1) \\
&= 6 \times \frac{\pi}{4} \\
&= \frac{3\pi}{2}
\end{aligned}$$



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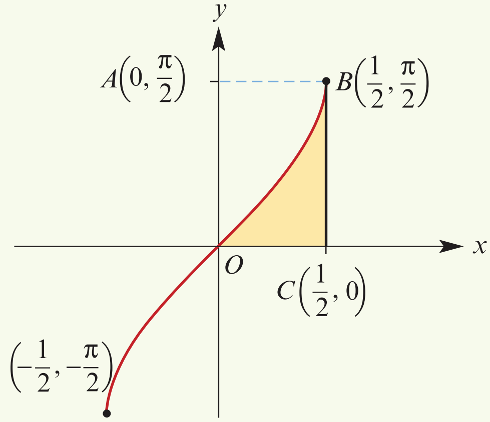
Example 3

Sketch the graph of $f\colon [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$, $f(x) = \sin^{-1}(2x)$. Shade the region defined by the inequalities $0 \le x \le \frac{1}{2}$ and $0 \le y \le f(x)$. Find the area of this region.

Solution

$$\text{Area} = \int_0^{\frac{1}{2}} \sin^{-1}(2x)\,dx$$

Note: This area could be calculated directly using integration by parts, but it is simpler to use the following alternative method.



$$\begin{aligned}\text{Area} &= \text{area rectangle } OABC - \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin y\,dy\\ &= \frac{\pi}{4} - \frac{1}{2}\Big[-\cos y\Big]_0^{\frac{\pi}{2}}\\ &= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

Example 4

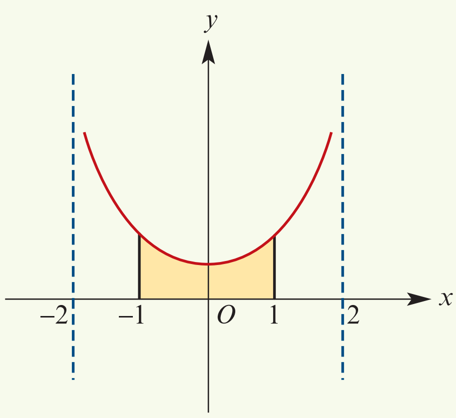
Sketch the graph of $y = \dfrac{1}{4 - x^2}$. Shade the region for the area determined by $\displaystyle\int_{-1}^{1} \frac{1}{4 - x^2}\,dx$ and find this area.

Solution

$$\begin{aligned}\text{Area} &= \int_{-1}^{1} \frac{1}{4 - x^2}\,dx\\ &= \frac{1}{4}\int_{-1}^{1} \frac{1}{2 - x} + \frac{1}{2 + x}\,dx\end{aligned}$$

By symmetry:

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^{1} \frac{1}{2 - x} + \frac{1}{2 + x}\,dx\\ &= \frac{1}{2}\left[\log_e\left(\frac{2 + x}{2 - x}\right)\right]_0^1\\ &= \frac{1}{2}(\log_e 3 - \log_e 1)\\ &= \frac{1}{2}\log_e 3\end{aligned}$$



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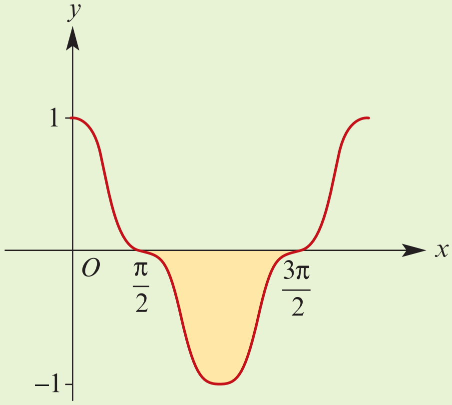
Example 5

The graph of $y = \cos^3 x$ is shown.

Find the area of the shaded region.



Solution

$$\text{Area} = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3 x \, dx$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \cos^2 x \, dx$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x (1 - \sin^2 x) \, dx$$

Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$.

When $x = \frac{\pi}{2}$, $u = 1$. When $x = \frac{3\pi}{2}$, $u = -1$.

$$\therefore \quad \text{Area} = -\int_{1}^{-1} 1 - u^2 \, du$$

$$= -\left[u - \frac{u^3}{3}\right]_1^{-1}$$

$$= -\left(-1 + \frac{1}{3} - \left(1 - \frac{1}{3}\right)\right)$$

$$= \frac{4}{3}$$

Properties of the definite integral

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
- $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

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Exercise 10A

Example 1

1 Sketch the graph of $f\colon \left(-\frac{3}{2}, \frac{3}{2}\right) \to \mathbb{R},\ f(x) = \frac{1}{\sqrt{9 - 4x^2}}$.

Find the area of the region defined by the inequalities $0 \le y \le f(x)$ and $-1 \le x \le 1$.

Example 2

2 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \frac{9}{4 + x^2}$.

Find the area of the region defined by the inequalities $0 \le y \le f(x)$ and $-2 \le x \le 2$.

3 Sketch the graph of $f(x) = x + \frac{2}{x}$. Shade the region for which the area is determined by the integral $\int_1^2 f(x)\,dx$ and evaluate this integral.

Example 3

4 For each of the following:

i sketch the appropriate graph and shade the required region

ii evaluate the integral.

a $\int_0^1 \tan^{-1} x\,dx$ **b** $\int_0^{\frac{1}{2}} \cos^{-1}(2x)\,dx$ **c** $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{-1}(2x)\,dx$

d $\int_0^1 2\sin^{-1} x\,dx$ **e** $\int_0^2 \sin^{-1}\left(\frac{x}{2}\right) dx$ **f** $\int_{-1}^2 \sin^{-1}\left(\frac{x}{2}\right) dx$

Example 4

5 Sketch the graph of $g\colon \mathbb{R} \setminus \{-3, 3\} \to \mathbb{R},\ g(x) = \frac{4}{9 - x^2}$ and find the area of the region with $-2 \le x \le 2$ and $0 \le y \le g(x)$.

6 For the curve with equation $y = -1 + \frac{2}{x^2 + 1}$, find:

a the coordinates of its turning point **b** the equation of its asymptote

c the area enclosed by the curve and the x-axis.

7 Consider the graph of $y = x - \frac{4}{x + 3}$.

a Find the coordinates of the intercepts with the axes.

b Find the equations of all asymptotes. **c** Sketch the graph.

d Find the area bounded by the curve, the x-axis and the line $x = 8$.

8 **a** State the implied domain of the function g with rule $g(x) = \frac{1}{(1 - x)(x - 2)}$.

b Sketch the graph of $y = g(x)$, indicating the equation of any asymptotes and the coordinates of the turning points.

c State the range of g.

d Find the area of the region bounded by the graph of $y = g(x)$, the x-axis and the lines $x = 4$ and $x = 3$.

9 Sketch the graph of $f\colon (-1, 1) \to \mathbb{R},\ f(x) = \frac{-3}{\sqrt{1 - x^2}}$. Evaluate $\int_0^{\frac{1}{2}} \frac{-3}{\sqrt{1 - x^2}}\,dx$.

10 Find the area of the region enclosed by the curve $y = \frac{1}{\sqrt{4 - x^2}}$, the x-axis and the lines $x = 1$ and $x = \sqrt{2}$.

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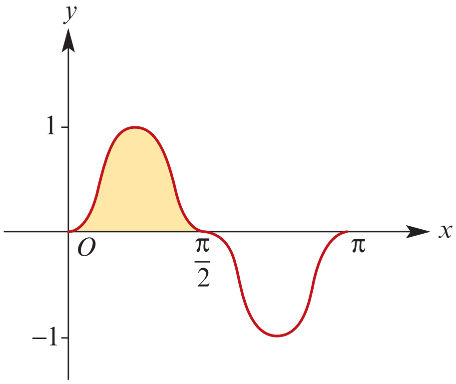
11 Sketch the curve with equation $y = \tan^{-1} x$. Find the area enclosed between this curve, the line $x = \sqrt{3}$ and the x-axis.

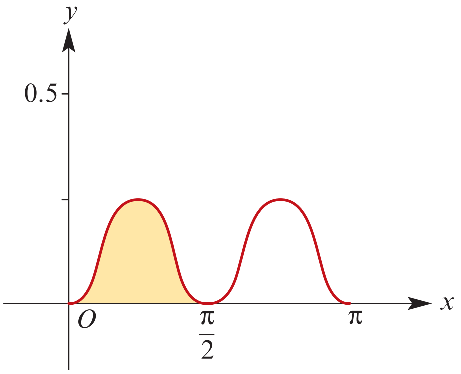
12 Find the area between the curve $y = \dfrac{2\log_e x}{x}$ and the x-axis from $x = 1$ to $x = e$.

Example 5

13 The graph of $y = \sin^3(2x)$ for $x \in [0, \pi]$ is as shown. Find the area of the shaded region.



14 The graph of $y = \sin x \cos^2 x$ for $x \in [0, \pi]$ is as shown. Find the area of the shaded region.



15 Sketch the curve with equation $y = \dfrac{2x}{x+3}$, showing clearly how the curve approaches its asymptotes. On your diagram, shade the finite region bounded by the curve and the lines $x = 0$, $x = 3$ and $y = 2$. Find the area of this region.

16 **a** Show that the curve $y = \dfrac{3}{(2x+1)(1-x)}$ has only one turning point.

b Find the coordinates of this point and determine its nature.

c Sketch the curve.

d Find the area of the region enclosed by the curve and the line $y = 3$.

17 Use integration by parts to find the area of the bounded region enclosed by:

a the graph of $y = xe^x$, the x-axis and the line $x = 1$

b the graph of $y = \arcsin x$, the x-axis and the line $x = \dfrac{\sqrt{3}}{2}$

c the graph of $y = x\log_e x$ and the x-axis

d the graph of $y = x\cos(4x)$ and the x-axis from $x = \dfrac{3\pi}{8}$ to $x = \dfrac{5\pi}{8}$

e the graph of $y = x^2 \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

18 Find an expression in terms of a for the area of the region enclosed by the curve $y = x\sqrt{a^2 - x^2}$, the x-axis and the line $x = \dfrac{a}{2}$, where $a > 0$.

19 Let $f(x) = \dfrac{2x^3 + 4}{x^2 + 1}$. Find the area of the region bounded by the graph of $y = f(x)$, the two coordinate axes and the line $x = 1$.

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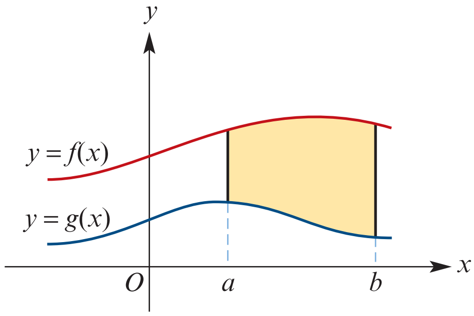
10B Area of a region between two curves

Let f and g be continuous functions on the interval $[a, b]$ such that

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b]$$

Then the area of the region bounded by the two curves and the lines $x = a$ and $x = b$ can be found by evaluating

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$



Example 6

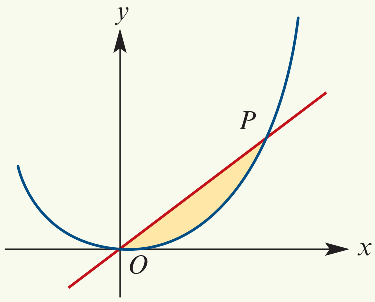
Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

We first find the coordinates of the point P:

$$x^2 = 2x$$
$$x(x-2) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 2$$

Therefore the coordinates of P are $(2, 4)$.



$$\begin{aligned}\text{Required area} &= \int_0^2 2x - x^2\,dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= 4 - \frac{8}{3} = \frac{4}{3}\end{aligned}$$

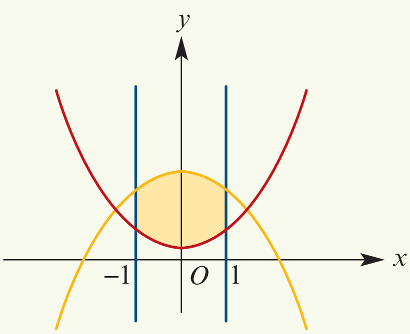
The area is $\frac{4}{3}$ square units.

Example 7

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.

Solution

$$\begin{aligned}\text{Required area} &= \int_{-1}^1 4 - x^2 - (x^2 + 1)\,dx \\ &= \int_{-1}^1 3 - 2x^2\,dx \\ &= \left[3x - \frac{2x^3}{3}\right]_{-1}^1 \\ &= 3 - \frac{2}{3} - \left(-3 + \frac{2}{3}\right) \\ &= \frac{14}{3}\end{aligned}$$



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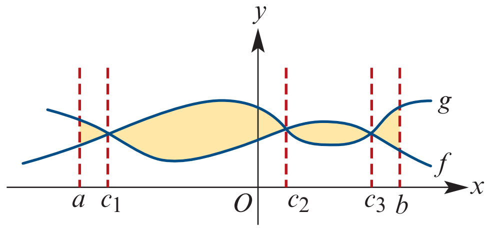
In the two examples considered so far in this section, the graph of one function is 'above' the graph of the other for all of the interval considered.

What happens when the graphs cross?



To find the area of the shaded region, we must consider the intervals $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$ separately. Thus, the shaded area is given by

$$\int_a^{c_1} f(x) - g(x)\, dx + \int_{c_1}^{c_2} g(x) - f(x)\, dx + \int_{c_2}^{c_3} f(x) - g(x)\, dx + \int_{c_3}^{b} g(x) - f(x)\, dx$$

The absolute value function could also be used here:

$$\left|\int_a^{c_1} f(x) - g(x)\, dx\right| + \left|\int_{c_1}^{c_2} f(x) - g(x)\, dx\right| + \left|\int_{c_2}^{c_3} f(x) - g(x)\, dx\right| + \left|\int_{c_3}^{b} f(x) - g(x)\, dx\right|$$

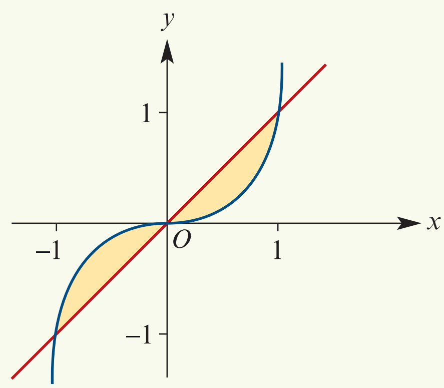
Example 8

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

Solution

The graphs intersect where $f(x) = g(x)$:

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$\therefore \quad x = 0 \text{ or } x = \pm 1$$



We see that:

- $f(x) \geq g(x)$ for $-1 \leq x \leq 0$
- $f(x) \leq g(x)$ for $0 \leq x \leq 1$

Thus the area is given by

$$\int_{-1}^{0} f(x) - g(x)\, dx + \int_0^1 g(x) - f(x)\, dx = \int_{-1}^{0} x^3 - x\, dx + \int_0^1 x - x^3\, dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= -\left(-\frac{1}{4}\right) + \frac{1}{4}$$

$$= \frac{1}{2}$$

Note: The result could also be obtained by observing the symmetry of the graphs, finding the area of the region where both x and y are non-negative, and then multiplying by 2.

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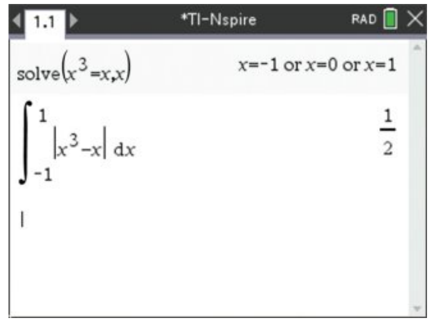
Using the TI-Nspire

Method 1

In a **Calculator** page:

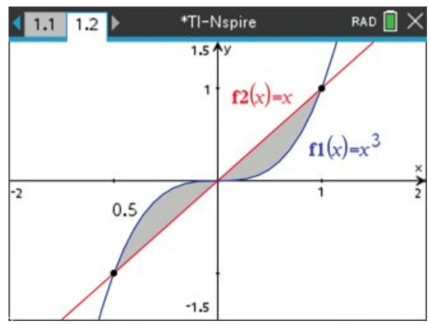
- Enter the integral as shown.
  (Use the 2D-template palette [template key] for the definite integral and the absolute value.)



Method 2

In a **Graphs** page:

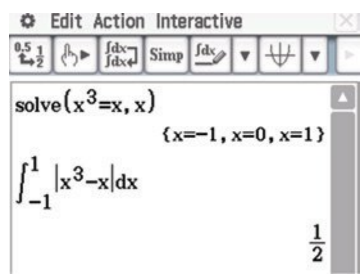
- Enter the functions $f1(x) = x^3$ and $f2(x) = x$ as shown.
- To find the area of the bounded region, use (menu) > **Analyze Graph** > **Bounded Area** and click on the lower and upper intersections of the graphs.



Using the Casio ClassPad

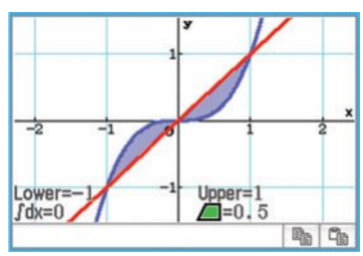
Method 1

- In Main $\sqrt{\alpha}$, solve the equation $x^3 = x$ to find the limits for the integral.
- Enter and highlight $|x^3 - x|$.
- Go to **Interactive** > **Calculation** > $\int$.
- Select **Definite**. Enter -1 for the lower limit and 1 for the upper limit. Then tap OK.



Method 2

- Graph the functions $y1 = x^3$ and $y2 = x$.
- Go to **Analysis** > **G-Solve** > **Integral** > $\int$**dx intersection**.
- Press execute at $x = -1$. Use the cursor key to go to $x = 1$ and press execute again.



Note: Here the absolute value function is used to simplify the process of finding areas with a CAS calculator. This technique is not helpful when doing these problems by hand.

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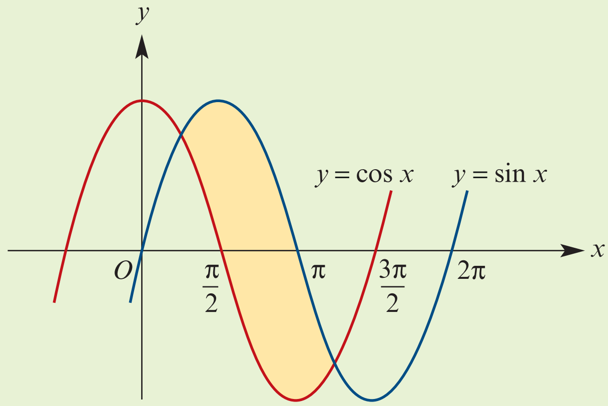
Example 9

Find the area of the shaded region.



Solution

First find the x-coordinates of the two points of intersection:

$$\sin x = \cos x$$

$$\therefore \quad \tan x = 1$$

Therefore $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$ for $x \in [0, 2\pi]$.

From the graph, we see that $\sin x \geq \cos x$ for all $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Hence

$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx \\
&= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= \frac{4}{\sqrt{2}} \\
&= 2\sqrt{2}
\end{aligned}$$

The area is $2\sqrt{2}$ square units.

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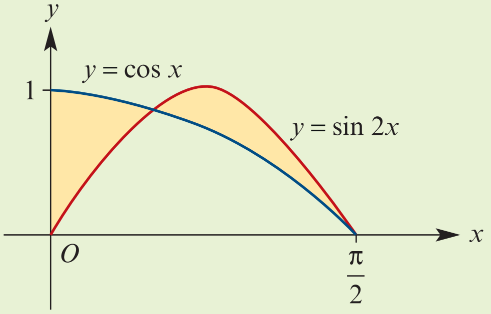
Example 10

Find the area of the shaded region.



Solution

First find the x-coordinates of the points of intersection:

$$\cos x = \sin(2x)$$
$$\cos x = 2\sin x \cos x$$
$$0 = \cos x\,(2\sin x - 1)$$
$$\therefore \quad \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

Therefore $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$ for $x \in \left[0, \frac{\pi}{2}\right]$.

$$\text{Area} = \int_0^{\frac{\pi}{6}} \cos x - \sin(2x)\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin(2x) - \cos x\,dx$$
$$= \left[\sin x + \frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{6}} + \left[-\frac{1}{2}\cos(2x) - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= \left(\frac{1}{2} + \frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - 1 - \left(-\frac{1}{4} - \frac{1}{2}\right)\right)$$
$$= \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{1}{2}$$

Skill-sheet

Exercise 10B

Example 6

1 Find the points of intersection of the two curves with equations $y = x^2 - 2x$ and $y = -x^2 + 8x - 12$. Find the area of the region enclosed between the two curves.

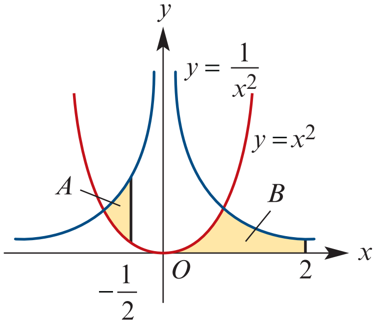
Example 7

2 Find the area of the region enclosed by the graphs of $y = -x^2$ and $y = x^2 - 2x$.

3 Find the area of:

a region A

b region B



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4 Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 4$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{16}{f(x)}$ on the same set of axes. Find the area of the region bounded by the two graphs and the lines $x = 1$ and $x = -1$.

5 The area of the region bounded by $y = \dfrac{12}{x}$, $x = 1$ and $x = a$ is 24. Find the value of a.

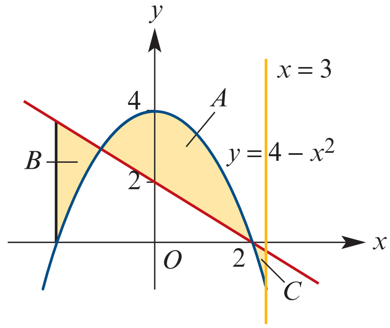
Example 8

6 Find the area of:

a region A

b region B

c region C



Example 9

Example 10

7 For each of the following, find the area of the region enclosed by the lines and curves. Draw a sketch graph and shade the appropriate region for each example.

a $y = 2\sin x$ and $y = \sin(2x)$, for $0 \le x \le \pi$

b $y = \sin(2x)$ and $y = \cos x$, for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

c $y = \sqrt{x}$, $y = 6 - x$ and $y = 1$

d $y = \dfrac{2}{1 + x^2}$ and $y = 1$

e $y = \sin^{-1} x$, $x = \dfrac{1}{2}$ and $y = 0$

f $y = \cos(2x)$ and $y = 1 - \sin x$, for $0 \le x \le \pi$

g $y = \dfrac{1}{3}(x^2 + 1)$ and $y = \dfrac{3}{x^2 + 1}$

8 Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = xe^x$.

a Find the derivative of f.

b Find $\{ x : f'(x) = 0 \}$.

c Sketch the curve $y = f(x)$.

d Find the equation of the tangent to this curve at $x = -1$.

e Find the area of the region bounded by this tangent, the curve and the y-axis.

9 Let P be the point with coordinates $(1, 1)$ on the curve with equation $y = 1 + \log_e x$.

a Find the equation of the normal to the curve at P.

b Find the area of the region enclosed by the normal, the curve and the x-axis.

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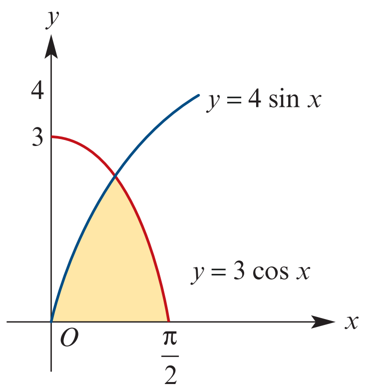
10 **a** Find the coordinates of the points of intersection of the curves with equations $y = (x-1)(x-2)$ and $y = \dfrac{3(x-1)}{x}$.

b Sketch the two curves on the one set of axes.

c Find the area of the region bounded by the two curves for $1 \leq x \leq 3$.

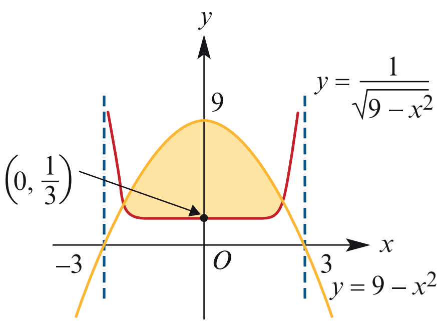
11 Show that the area of the shaded region is 2.



12 The graphs of $y = 9 - x^2$ and $y = \dfrac{1}{\sqrt{9 - x^2}}$ are as shown.

a Find the coordinates of the points of intersection of the two graphs.

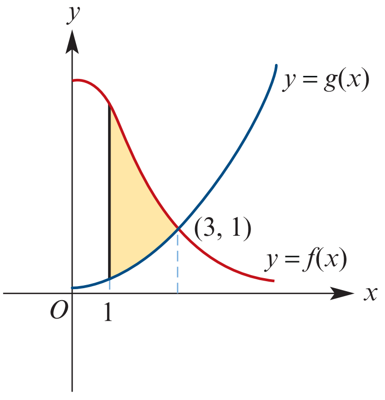
b Find the area of the shaded region.



13 Find the area enclosed by the graphs of $y = x^2$ and $y = x + 2$.

14 Consider the functions $f(x) = \dfrac{10}{1 + x^2}$ for $x \geq 0$ and $g(x) = e^{x-3}$ for $x \geq 0$.

The graphs of $y = f(x)$ and $y = g(x)$ intersect at the point $(3, 1)$. Find, correct to three decimal places, the area of the region enclosed by the two graphs and the line with equation $x = 1$.



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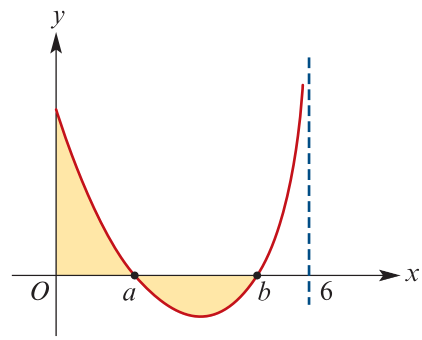
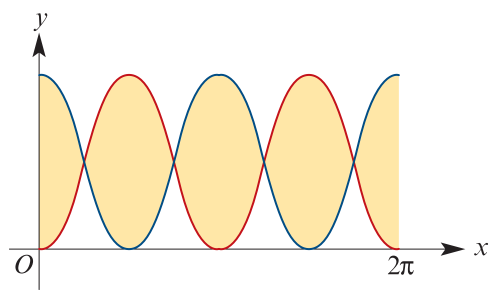
15 The graph of the function $f\colon [0,6) \to \mathbb{R}$, where $f(x) = \dfrac{8\sqrt{5}}{\sqrt{36 - x^2}} - x$, is shown.

a Find the values of a and b.

b Find the total area of the shaded regions.



16 The graphs of $y = \cos^2 x$ and $y = \sin^2 x$ are shown for $0 \le x \le 2\pi$. Find the total area of the shaded regions.



10C Integration using a CAS calculator

In Chapter 9, we discussed methods of integration by rule. In this section, we consider the use of a CAS calculator in evaluating definite integrals. It is often not possible to determine the antiderivative of a given function by rule, and so we will also look at numerical evaluation of definite integrals.

Using a calculator to find exact values of definite integrals

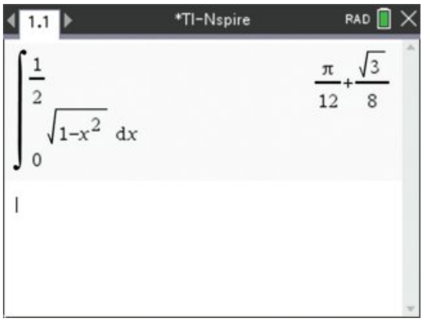
Example 11

Use a CAS calculator to evaluate $\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx$.

Using the TI-Nspire

To find a definite integral, use (menu) > **Calculus** > **Integral**.

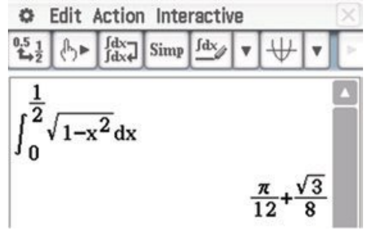
Note: The integral template can also be obtained directly from the 2D-template palette (template key) or by pressing (shift) (+).



Using the Casio ClassPad

- Enter and highlight the expression $\sqrt{1 - x^2}$.
- Go to **Interactive** > **Calculation** > $\int$.
- Select **Definite**. Enter 0 for the lower limit and $\frac{1}{2}$ for the upper limit. Then tap OK.

Note: The integral template can also be found in the Math2 keyboard.



Using the inverse function to find a definite integral

Example 12

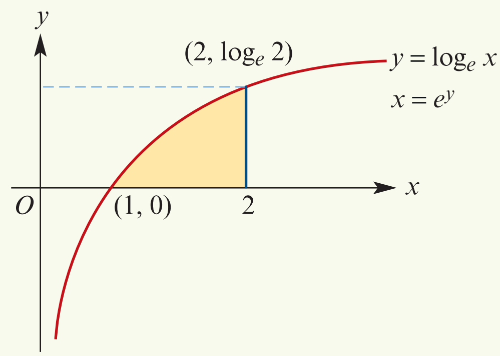
Find the area of the region bounded by the graph of $y = \log_e x$, the line $x = 2$ and the x-axis by using the inverse function.

Solution

From the graph, we see that

$$\int_1^2 \log_e x \, dx = 2\log_e 2 - \int_0^{\log_e 2} e^y \, dy$$
$$= 2\log_e 2 - (e^{\log_e 2} - e^0)$$
$$= 2\log_e 2 - (2 - 1)$$
$$= 2\log_e 2 - 1$$

The area is $2\log_e 2 - 1$ square units.



Note: Alternatively, you could use integration by parts to evaluate the integral.

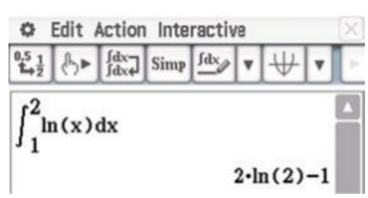
This area can also be found by using a CAS calculator to evaluate $\int_1^2 \log_e x \, dx$.

Using the TI-Nspire

To find a definite integral, use menu > **Calculus** > **Integral** or select the integral template from the 2D-template palette.

Using the Casio ClassPad

- Enter and highlight the expression ln(x).
- Go to **Interactive** > **Calculation** > $\int$.
- Select **Definite**, enter the lower and upper limits and tap OK.



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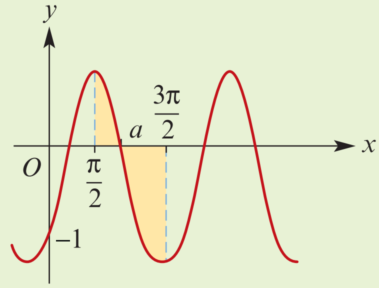
Using a calculator to find approximate values of definite integrals

Example 13

The graph of $y = e^{\sin x} - 2$ is as shown. Using a CAS calculator, find the area of the shaded regions.



Solution

Using a CAS calculator, first find the value of a, which is approximately 2.37575.

$$\begin{aligned}\text{Required area} &= \int_{\frac{\pi}{2}}^{a} (e^{\sin x} - 2)\, dx - \int_{a}^{\frac{3\pi}{2}} (e^{\sin x} - 2)\, dx \\ &= 0.369\,213\ldots + 2.674\,936\ldots \\ &= 3.044\,149\ldots\end{aligned}$$

The area is approximately 3.044 square units.

Using the fundamental theorem of calculus

We have used the fundamental theorem of calculus to find areas using antiderivatives. We can also use the theorem to define antiderivatives using area functions.

If F is an antiderivative of a continuous function f, then $F(b) - F(a) = \int_a^b f(x)\, dx$. Using a dummy variable t, we can write $F(x) - F(a) = \int_a^x f(t)\, dt$, giving $F(x) = F(a) + \int_a^x f(t)\, dt$.

If we define a function by $G(x) = \int_a^x f(t)\, dt$, then F and G differ by a constant, and so G is also an antiderivative of f.

Example 14

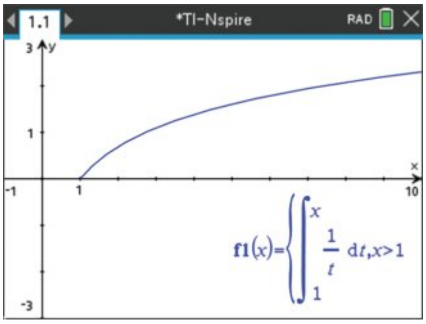
Plot the graph of $F(x) = \int_1^x \frac{1}{t}\, dt$ for $x > 1$.

Using the TI-Nspire

In a **Graphs** page, enter the function

$$f1(x) = \int_1^x \frac{1}{t}\, dt$$

Note: The integral template can be obtained from the 2D-template palette.



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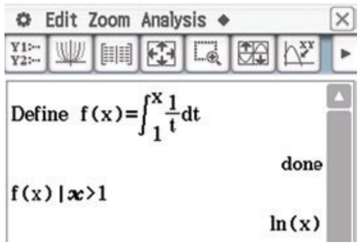
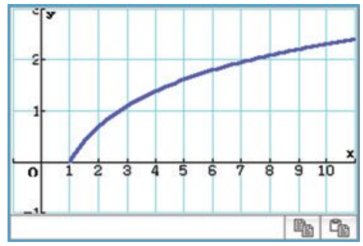
Using the Casio ClassPad

- Enter and define the function as shown.
- Graph the function with the restricted domain.





Note: The natural logarithm function can be defined by $\ln(x) = \int_1^x \frac{1}{t}\,dt$.
The number e can then be defined to be the unique real number a such that $\ln(a) = 1$.

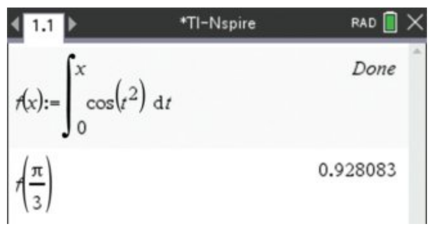
Example 15

Use a CAS calculator to find an approximate value of $\int_0^{\frac{\pi}{3}} \cos(x^2)\,dx$ and to plot the graph of $f(x) = \int_0^x \cos(t^2)\,dt$ for $-\frac{\pi}{4} \le x \le \pi$.

Using the TI-Nspire

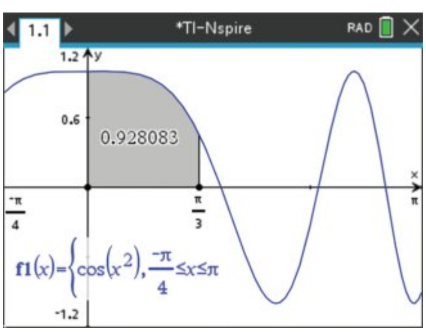
Method 1

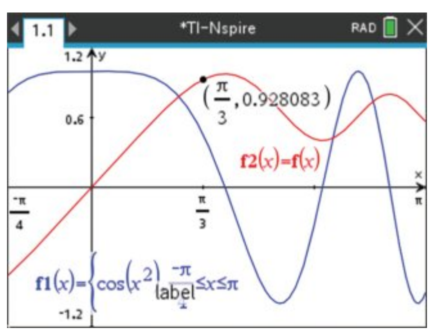
- Define the function f as shown and evaluate at $x = \frac{\pi}{3}$.



Method 2

- Plot the graph of $f1(x) = \cos(x^2)$ for $-\frac{\pi}{4} \le x \le \pi$.
- To find the required area, use the integral measurement tool from (menu) > **Analyze Graph** > **Integral**. Type in the lower limit 0 and press (enter). Move to the right, type in the upper limit $\pi/3$ and press (enter).





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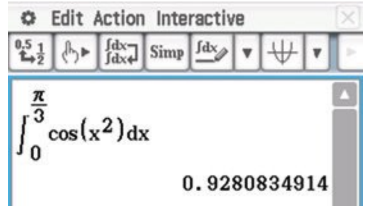
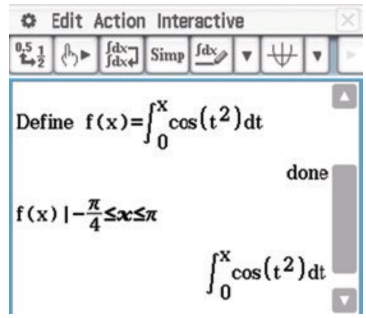
Using the Casio ClassPad

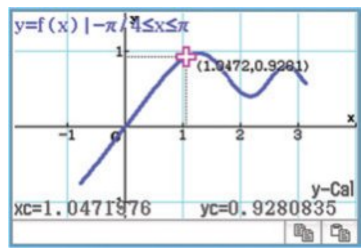
- Enter and highlight the expression $\cos(x^2)$.
- Go to **Interactive** > **Calculation** > $\int$.
- Select **Definite** and enter the lower and upper limits as shown.



- Define the function $f(x) = \int_0^x \cos(t^2)\, dt$.
- Graph the function with the restricted domain.



- The approximate value of $f\left(\frac{\pi}{3}\right)$ can now be found graphically using **Analysis** > **G-Solve** > **y-Cal**.



Exercise 10C

Example 11

1 For each of the following, evaluate the integral using a CAS calculator to obtain an exact value:

a $\int_0^3 \sqrt{9 - x^2}\, dx$ **b** $\int_0^3 \sqrt{9x^2 - x^3}\, dx$ **c** $\int_0^3 \log_e(x^2 + 1)\, dx$

Example 12

2 For each of the following, determine the exact value both by using the inverse function and by using your CAS calculator:

a $\int_0^{\frac{1}{2}} \arcsin(2x)\, dx$ **b** $\int_3^4 \log_e(x - 2)\, dx$ **c** $\int_0^{\frac{1}{2}} \arctan(2x)\, dx$

Example 13

3 Using a CAS calculator, evaluate each of the following correct to two decimal places:

a $\int_0^2 e^{\sin x}\, dx$ **b** $\int_0^{\pi} x \sin x\, dx$ **c** $\int_1^3 (\log_e x)^2\, dx$ **d** $\int_{-1}^1 \cos(e^x)\, dx$

e $\int_{-1}^2 \frac{e^x}{e^x + e^{-x}}\, dx$ **f** $\int_0^2 \frac{x}{x^4 + 1}\, dx$ **g** $\int_1^2 x \log_e x\, dx$ **h** $\int_{-1}^1 x^2 e^x\, dx$

i $\int_0^1 \sqrt{1 + x^4}\, dx$ **j** $\int_0^{\frac{\pi}{2}} \sin(x^2)\, dx$

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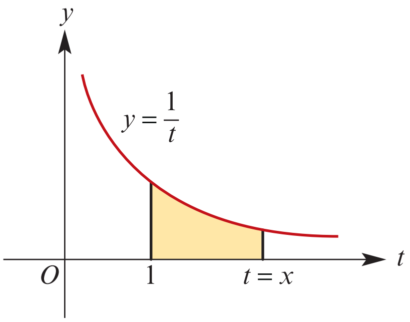
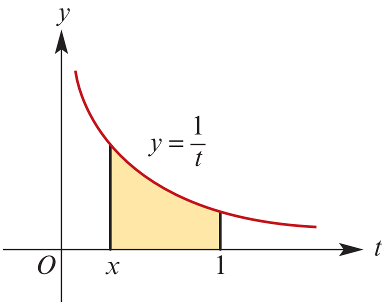
4 In each of the following, the rule of the function is defined as an area function. Find $f(x)$ in each case.

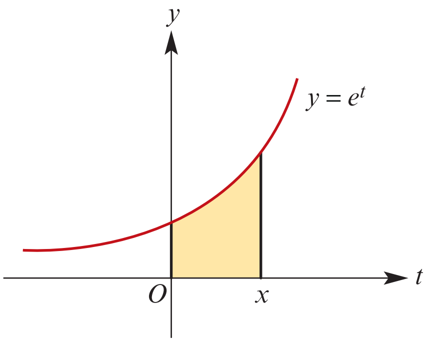
a $f(x) = \int_1^x \frac{1}{t}\,dt$, for $x > 1$



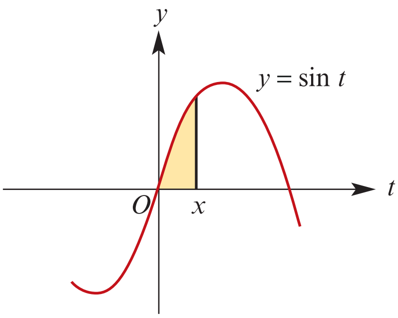
b $f(x) = \int_x^1 \frac{1}{t}\,dt$, for $0 < x < 1$



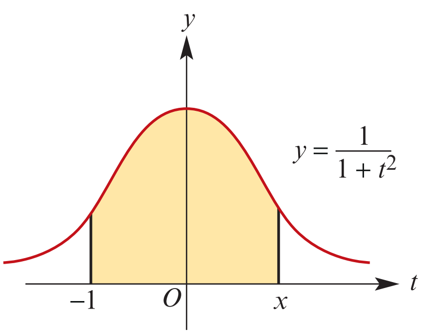
c $f(x) = \int_0^x e^t\,dt$, for $x \in \mathbb{R}$



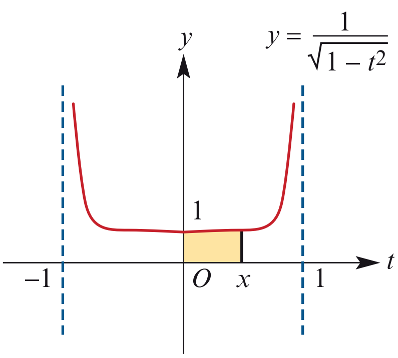
d $f(x) = \int_0^x \sin t\,dt$, for $x \in \mathbb{R}$



e $f(x) = \int_{-1}^x \frac{1}{1+t^2}\,dt$, for $x \in \mathbb{R}$



f $f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$, for $-1 < x < 1$



Example 14

5 Use a CAS calculator to plot the graph of each of the following:

a $f(x) = \int_0^x \tan^{-1} t\,dt$

b $f(x) = \int_0^x e^{t^2}\,dt$

c $f(x) = \int_0^x \sin^{-1} t\,dt$

d $f(x) = \int_0^x \sin(t^2)\,dt$

e $f(x) = \int_1^x \frac{\sin t}{t}\,dt$, $x > 1$

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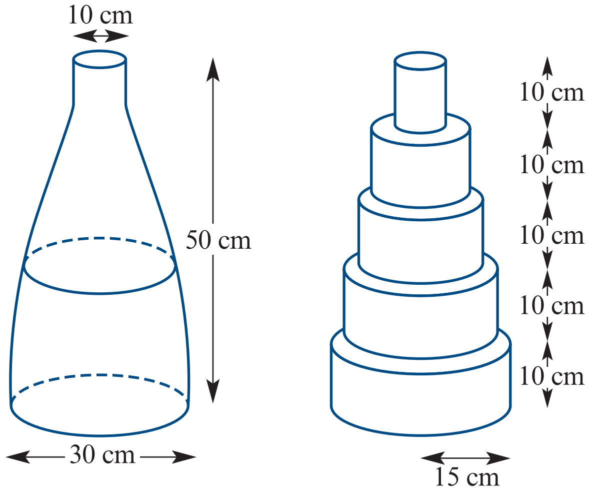
10D Volumes of solids of revolution

A large glass flask has a shape as illustrated in the figure below. In order to find its approximate volume, consider the flask as a series of cylinders.



Radius of cylinder
5 cm
9 cm
11 cm
13 cm
15 cm

$\therefore$ Volume of flask $\approx \pi(15^2 + 13^2 + 11^2 + 9^2 + 5^2) \times 10$

$\approx 19\,509.29 \text{ cm}^3$

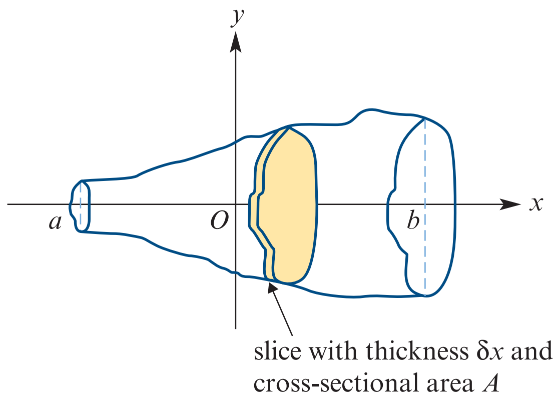
≈ 19 litres

This estimate can be improved by taking more cylinders to obtain a better approximation.

In Mathematical Methods Units 3 & 4, it was shown that areas defined by well-behaved functions can be determined as the limit of a sum.

This can also be done for volumes. The volume of a typical thin slice is $A\delta x$, and the approximate total volume is

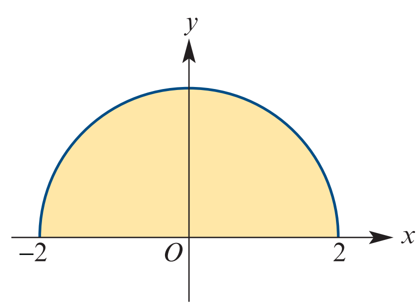
$$\sum_{x=a}^{x=b} A\delta x$$



Volume of a sphere

Consider the graph of $f(x) = \sqrt{4 - x^2}$.

If the shaded region is rotated around the x-axis, it will form a sphere of radius 2.



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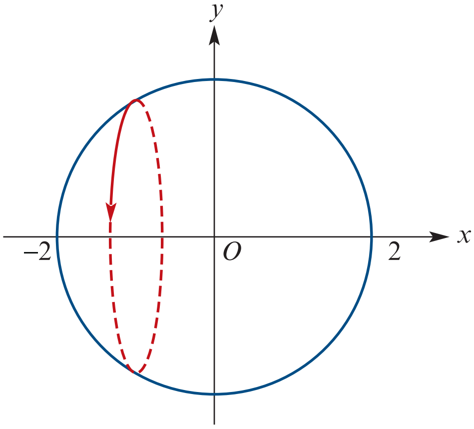
Divide the interval $[-2, 2]$ into n subintervals $[x_{i-1}, x_i]$ with $x_0 = -2$ and $x_n = 2$.



The volume of a typical slice (a cylinder) is approximately $\pi(f(c_i))^2(x_i - x_{i-1})$, where $c_i \in [x_{i-1}, x_i]$.

The total volume will be approximated by the sum of the volumes of these slices. As the number of slices n gets larger and larger:

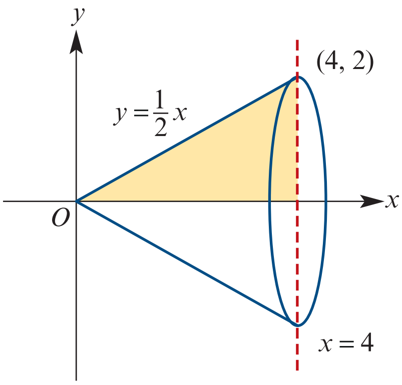
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(f(c_i))^2(x_i - x_{i-1})$$

It has been seen that the limit of such a sum is an integral and therefore:

$$\begin{aligned} V &= \int_{-2}^{2} \pi(f(x))^2 \, dx \\ &= \int_{-2}^{2} \pi(4 - x^2) \, dx \\ &= \pi\left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \pi\left(8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)\right) \\ &= \pi\left(16 - \frac{16}{3}\right) \\ &= \frac{32\pi}{3} \end{aligned}$$

Volume of a cone

If the region between the line $y = \frac{1}{2}x$, the line $x = 4$ and the x-axis is rotated around the x-axis, then a solid in the shape of a cone is produced.



The volume of the cone is given by:

$$\begin{aligned} V &= \int_{0}^{4} \pi y^2 \, dx \\ &= \int_{0}^{4} \pi\left(\frac{1}{2}x\right)^2 \, dx \\ &= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_{0}^{4} \\ &= \frac{\pi}{4} \times \frac{64}{3} \\ &= \frac{16\pi}{3} \end{aligned}$$

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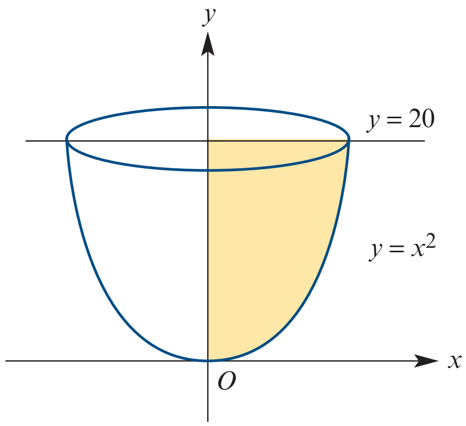
Solids of revolution

In general, the solid formed by rotating a region about a line is called a **solid of revolution**.

For example, if the region between the graph of $y = x^2$, the line $y = 20$ and the y-axis is rotated about the y-axis, then a solid in the shape of the top of a wine glass is produced.



Volume of a solid of revolution

- **Rotation about the x-axis**
  If the region to be rotated is bounded by the curve with equation $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis, then

$$V = \int_{x=a}^{x=b} \pi y^2 \, dx$$
$$= \pi \int_a^b (f(x))^2 \, dx$$

- **Rotation about the y-axis**
  If the region to be rotated is bounded by the curve with equation $x = f(y)$, the lines $y = a$ and $y = b$ and the y-axis, then

$$V = \int_{y=a}^{y=b} \pi x^2 \, dy$$
$$= \pi \int_a^b (f(y))^2 \, dy$$

Example 16

Find the volume of the solid of revolution formed by rotating the curve $y = x^3$ about:

a the x-axis for $0 \le x \le 1$

b the y-axis for $0 \le y \le 1$

Solution

a
$$V = \pi \int_0^1 y^2 \, dx$$
$$= \pi \int_0^1 x^6 \, dx$$
$$= \pi \left[\frac{x^7}{7} \right]_0^1$$
$$= \frac{\pi}{7}$$

b
$$V = \pi \int_0^1 x^2 \, dy$$
$$= \pi \int_0^1 y^{\frac{2}{3}} \, dy$$
$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}} \right]_0^1$$
$$= \frac{3\pi}{5}$$

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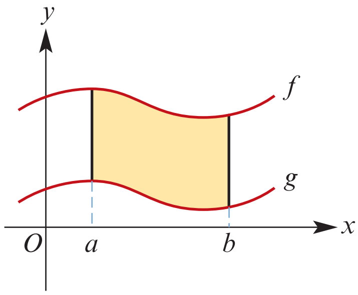
Regions not bounded by the x-axis

If the shaded region is rotated about the x-axis, then the volume V is given by

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx$$



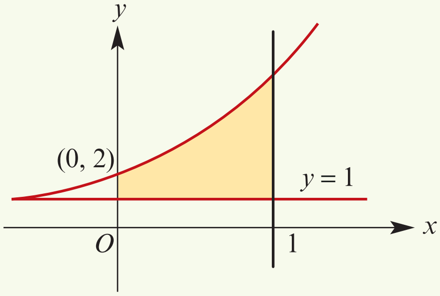
Example 17

Find the volume of the solid of revolution when the region bounded by the graphs of $y = 2e^{2x}$, $y = 1$, $x = 0$ and $x = 1$ is rotated around the x-axis.

Solution

The volume is given by

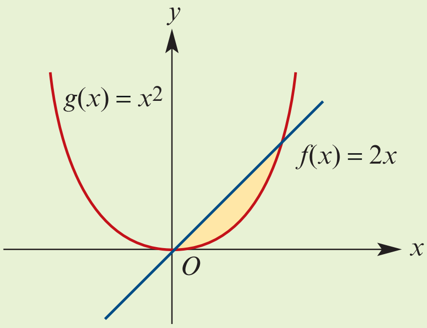
$$\begin{aligned} V &= \pi \int_0^1 4e^{4x} - 1 \, dx \\ &= \pi \Big[e^{4x} - x \Big]_0^1 \\ &= \pi(e^4 - 1 - (1)) \\ &= \pi(e^4 - 2) \end{aligned}$$



Note: Here $f(x) = 2e^{2x}$ and $g(x) = 1$.

Example 18

The shaded region is rotated around the x-axis.
Find the volume of the resulting solid.



Solution

The graphs meet where $2x = x^2$, i.e. at the points with coordinates $(0, 0)$ and $(2, 4)$.

$$\begin{aligned} \text{Volume} &= \pi \int_0^2 (f(x))^2 - (g(x))^2 \, dx \\ &= \pi \int_0^2 4x^2 - x^4 \, dx \\ &= \pi \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 \\ &= \pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{64\pi}{15} \end{aligned}$$

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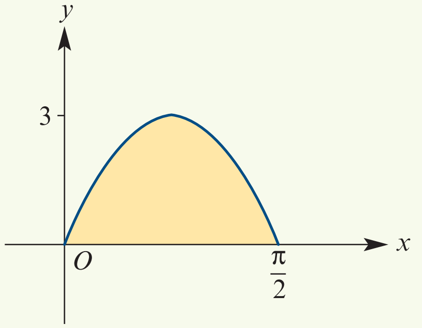
Example 19

A solid is formed when the region bounded by the x-axis and the graph of $y = 3\sin(2x)$, $0 \le x \le \frac{\pi}{2}$, is rotated around the x-axis. Find the volume of this solid.

Solution

$$
\begin{aligned}
V &= \pi \int_0^{\frac{\pi}{2}} (3\sin(2x))^2 \, dx \\
&= \pi \int_0^{\frac{\pi}{2}} 9\sin^2(2x) \, dx \\
&= 9\pi \int_0^{\frac{\pi}{2}} \sin^2(2x) \, dx \\
&= 9\pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(4x)) \, dx \\
&= \frac{9\pi}{2} \int_0^{\frac{\pi}{2}} 1 - \cos(4x) \, dx \\
&= \frac{9\pi}{2} \left[x - \frac{1}{4}\sin(4x) \right]_0^{\frac{\pi}{2}} \\
&= \frac{9\pi}{2} \left(\frac{\pi}{2} \right) \\
&= \frac{9\pi^2}{4}
\end{aligned}
$$

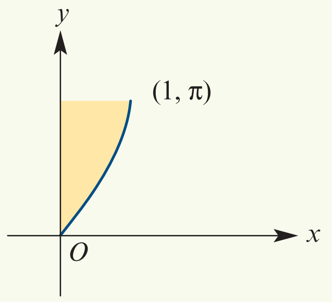


Example 20

The curve $y = 2\sin^{-1} x$, $0 \le x \le 1$, is rotated around the y-axis to form a solid of revolution. Find the volume of this solid.

Solution

$$
\begin{aligned}
V &= \pi \int_0^{\pi} \sin^2\left(\frac{y}{2}\right) dy \\
&= \frac{\pi}{2} \int_0^{\pi} 1 - \cos y \, dy \\
&= \frac{\pi}{2} \left[y - \sin y \right]_0^{\pi} \\
&= \frac{\pi^2}{2}
\end{aligned}
$$



Skill-sheet

Exercise 10D

Example 16

1 Find the area of the region bounded by the x-axis and the curve whose equation is $y = 4 - x^2$. Also find the volume of the solid formed when this region is rotated about the y-axis.

2 Find the volume of the solid of revolution when the region bounded by the given curve, the x-axis and the given lines is rotated about the x-axis:

a $f(x) = \sqrt{x},\ x = 4$

b $f(x) = 2x + 1,\ x = 0,\ x = 4$

c $f(x) = 2x - 1,\ x = 4$

d $f(x) = \sin x,\ 0 \leq x \leq \dfrac{\pi}{2}$

e $f(x) = e^x,\ x = 0,\ x = 2$

f $f(x) = \sqrt{9 - x^2},\ -3 \leq x \leq 3$

3 The hyperbola $x^2 - y^2 = 1$ is rotated around the x-axis to form a surface of revolution. Find the volume of the solid enclosed by this surface between $x = 1$ and $x = \sqrt{3}$.

4 Find the volumes of the solids generated by rotating about the x-axis each of the regions bounded by the following curves and lines:

a $y = \dfrac{1}{x},\ y = 0,\ x = 1,\ x = 4$

b $y = x^2 + 1,\ y = 0,\ x = 0,\ x = 1$

c $y = \sqrt{x},\ y = 0,\ x = 2$

d $y = \sqrt{a^2 - x^2},\ y = 0$

e $y = \sqrt{9 - x^2},\ y = 0$

f $y = \sqrt{9 - x^2},\ y = 0,\ x = 0,$ given $x \geq 0$

Example 17
Example 18

5 The region bounded by the line $y = 5$ and the curve $y = x^2 + 1$ is rotated about the x-axis. Find the volume generated.

Example 19

6 The region, for which $x \geq 0$, bounded by the curves $y = \cos x$ and $y = \sin x$ and the y-axis is rotated around the x-axis, forming a solid of revolution. By using the identity $\cos(2x) = \cos^2 x - \sin^2 x$, obtain a volume for this solid.

7 The region enclosed by $y = \dfrac{4}{x^2}$, $x = 4$, $x = 1$ and the x-axis is rotated about the x-axis. Find the volume generated.

8 The region enclosed by $y = x^2$ and $y^2 = x$ is rotated about the x-axis. Find the volume generated.

9 A region is bounded by the curve $y = \sqrt{6 - x}$, the straight line $y = x$ and the positive x-axis. Find the volume of the solid of revolution formed by rotating this figure about the x-axis.

10 The region bounded by the x-axis, the line $x = \dfrac{\pi}{2}$ and the curve $y = \tan\left(\dfrac{x}{2}\right)$ is rotated about the x-axis. Prove that the volume of the solid of revolution is $\dfrac{\pi}{2}(4 - \pi)$.

Hint: Use the result that $\tan^2\left(\dfrac{x}{2}\right) = \sec^2\left(\dfrac{x}{2}\right) - 1$.

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11 Sketch the graphs of $y = \sin x$ and $y = \sin(2x)$ for $0 \le x \le \frac{\pi}{2}$. Show that the area of the region bounded by these graphs is $\frac{1}{4}$ square unit, and the volume formed by rotating this region about the x-axis is $\frac{3}{16}\pi\sqrt{3}$ cubic units.

12 Let V be the volume of the solid formed when the region enclosed by $y = \frac{1}{x}$, $y = 0$, $x = 4$ and $x = b$, where $0 < b < 4$, is rotated about the x-axis. Find the value of b for which $V = 3\pi$.

13 Find the volume of the solid generated when the region enclosed by $y = \sqrt{3x+1}$, $y = \sqrt{3x}$, $y = 0$ and $x = 1$ is rotated about the x-axis.

Example 20

14 Find the volumes of the solids formed when the following regions are rotated around the y-axis:

a $x^2 = 4y^2 + 4$ for $0 \le y \le 1$

b $y = \log_e(2 - x)$ for $0 \le y \le 2$

15 **a** Find the area of the region bounded by the curve $y = e^x$, the tangent at the point $(1, e)$ and the y-axis.

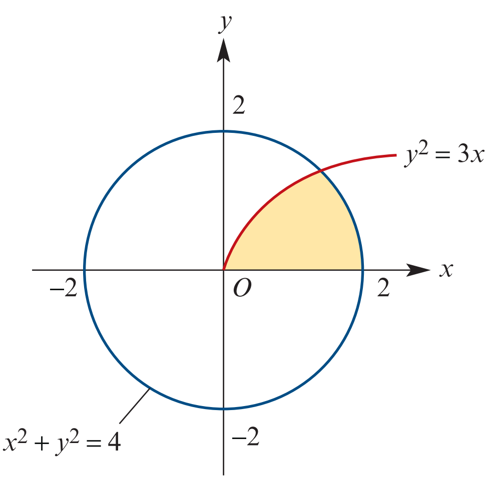
b Find the volume of the solid formed by rotating this region through a complete revolution about the x-axis.

16 The region defined by the inequalities $y \ge x^2 - 2x + 4$ and $y \le 4$ is rotated about the line $y = 4$. Find the volume generated.

17 The region enclosed by $y = \cos\left(\frac{x}{2}\right)$ and the x-axis, for $0 \le x \le \pi$, is rotated about the x-axis. Find the volume generated.

18 Find the volume generated by revolving the region enclosed between the parabola $y = 3x - x^2$ and the line $y = 2$ about the x-axis.

19 The shaded region is rotated around the x-axis to form a solid of revolution. Find the volume of this solid.



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20 The region enclosed between the curve $y = e^x - 1$, the x-axis and the line $x = \log_e 2$ is rotated around the x-axis to form a solid of revolution. Find the volume of this solid.

21 Show that the volume of the solid of revolution formed by rotating about the x-axis the region bounded by the curve $y = e^{-2x}$ and the lines $x = 0$, $y = 0$ and $x = \log_e 2$ is $\dfrac{15\pi}{64}$.

22 Find the volume of the solid generated by revolving about the x-axis the region bounded by the graph of $y = 2\tan x$ and the lines $x = -\dfrac{\pi}{4}$, $x = \dfrac{\pi}{4}$ and $y = 0$.

23 The region bounded by the parabola $y^2 = 4(1 - x)$ and the y-axis is rotated about:

a the x-axis

b the y-axis.

Prove that the volumes of the solids formed are in the ratio 15 : 16.

24 The region bounded by the graph of $y = \dfrac{1}{\sqrt{x^2 + 9}}$, the x-axis, the y-axis and the line $x = 4$ is rotated about:

a the x-axis

b the y-axis.

Find the volume of the solid formed in each case.

25 A bucket is defined by rotating the curve with equation

$$y = 40\log_e\left(\frac{x - 20}{10}\right), \quad 0 \le y \le 40$$

about the y-axis. If x and y are measured in centimetres, find the maximum volume of liquid that the bucket could hold. Give the answer to the nearest cm^3.

26 An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the volume of the solid generated when the region bounded by the ellipse is rotated about:

a the x-axis

b the y-axis.

27 The diagram shows part of the curve $y = \dfrac{12}{x}$.

Points $P(2, 6)$ and $Q(6, 2)$ lie on the curve.

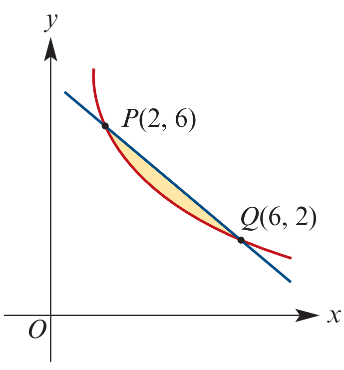
Find:

a the equation of the line PQ

b the volume obtained when the shaded region is rotated about:

i the x-axis

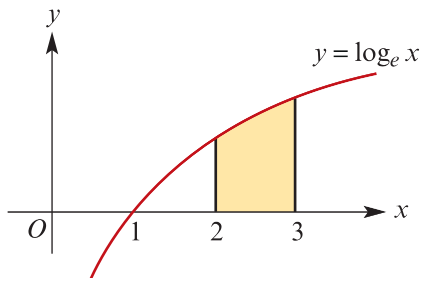
ii the y-axis.



28 **a** Sketch the graph of $y = 2x + \frac{9}{x}$.

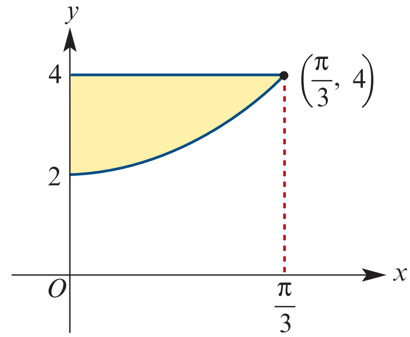
b Find the volume generated when the region bounded by the curve $y = 2x + \frac{9}{x}$ and the lines $x = 1$ and $x = 3$ is rotated about the x-axis.

29 The region shown is rotated about the x-axis to form a solid of revolution. Find the volume of the solid, correct to three decimal places.



30 The graphs of $y = 2\sec x$ and $y = 4$ are shown for $0 \leq x \leq \frac{\pi}{3}$.

The shaded region is rotated about the x-axis to form a solid of revolution. Calculate the exact volume of this solid.



31 The graph of $y = \frac{\sqrt{4x^2 - 1}}{2}$, where $0 \leq y \leq \frac{\sqrt{3}}{2}$, is rotated about the y-axis. Find the volume of the solid of revolution.

32 The region bounded by the curve $y = \frac{2x}{x + 2}$, the x-axis and the line $x = 2$ is rotated about the x-axis to form a solid of revolution. Find the volume of this solid.
Hint: Use the substitution $u = x + 2$.

33 The region bounded by the graph of $y = \sin^{-1}(2x^2 - 1)$ and the line $y = \frac{\pi}{2}$ is rotated about the y-axis to form a solid of revolution. Find the volume of this solid.

34 The region bounded by the curve $y = \sqrt{2 - \cos^2 x}$, the coordinate axes and the line $x = \pi$ is rotated about the x-axis to form a solid of revolution. Find the volume of this solid.

35 The region bounded by the graph of $y = \sqrt{\frac{6 - 4x}{4 + x^2}}$ and the coordinate axes is rotated about the x-axis to form a solid of revolution. Find the volume of this solid.

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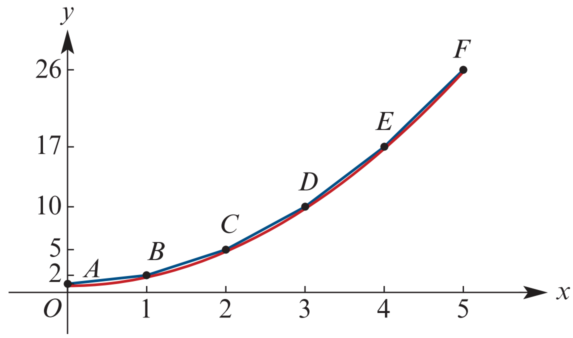
10E Lengths of curves in the plane

We have seen how the area under a curve may be found as the limit of a sum of areas of rectangles, and how the volume of a solid of revolution may be found as the limit of a sum of volumes of cylinders. We can do something very similar to find the length of a curve. We can define the length as the limit of a sum of lengths of line segments. This is discussed here.

Note: In this course, you are expected to find lengths of parametric curves only. However, we start by considering a curve of the form $y = f(x)$ to illustrate the ideas.

The graph of $f(x) = x^2 + 1$, $0 \le x \le 5$, is shown.



The points $A(0, f(0))$, $B(1, f(1))$, $\dots$, $F(5, f(5))$ on the curve are shown, as well as the line segments AB, BC, CD, DE and EF. The length of the curve is approximated by the sum of the lengths of these line segments.

We can use this idea to find the length of the curve by integral calculus. The following is not a rigorous proof, but will help you to understand how integral calculus can be applied.

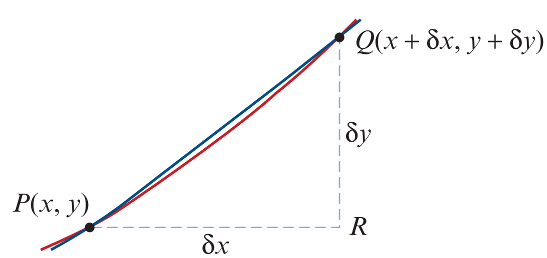
A portion of a curve is shown below. Let δs be the length of the curve from P to Q, let $PR = \delta x$ and let $QR = \delta y$.

By Pythagoras' theorem applied to the right-angled triangle PQR, we have

$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$$

$$\therefore \quad \left(\frac{\delta s}{\delta x}\right)^2 \approx 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

$$\therefore \quad \delta s \approx \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\ \delta x$$



We can think of the length of the curve as the limit as $\delta x \to 0$ of the sum of these lengths. Formally, we can state the result as follows.

Length of a curve

The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_a^b \sqrt{1 + (f'(x))^2}\ dx$$

Note: We are assuming that f is differentiable on $[a, b]$ and that f' is continuous.

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The length of a parametric curve

Now consider a curve defined by parametric equations $x = f(t)$ and $y = g(t)$. We can give another very informal argument to motivate the formula for the length of the curve using the derivatives of x and y with respect to t:

$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$$

$$\therefore \quad \left(\frac{\delta s}{\delta t}\right)^2 \approx \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

$$\therefore \quad \delta s \approx \sqrt{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2}\, \delta t$$

This leads to the following result, if you consider $\delta t \to 0$.

Length of a parametric curve

Consider a curve defined by the parametric equations $x = f(t)$ and $y = g(t)$ for $a \le t \le b$. If the point $P(f(t), g(t))$ traces the curve exactly once from $t = a$ to $t = b$, then the length of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note: We are assuming that f and g are differentiable on $[a, b]$, with f' and g' continuous.

Example 21

Find the length of the curve defined by the parametric equations

$$x = t \quad \text{and} \quad y = t^{\frac{3}{2}} \quad \text{for } 1 \le t \le 4$$

Solution

We obtain $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = \dfrac{3}{2}t^{\frac{1}{2}}$.

Therefore the length is

$$\begin{aligned}\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_1^4 \sqrt{1 + \frac{9t}{4}}\, dt \\ &= \frac{1}{2}\int_1^4 \sqrt{4 + 9t}\, dt \\ &= \frac{1}{2}\left[\frac{2(4+9t)^{\frac{3}{2}}}{27}\right]_1^4 \\ &= \left(\frac{40^{\frac{3}{2}}}{27}\right) - \left(\frac{13^{\frac{3}{2}}}{27}\right) \\ &= \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})\end{aligned}$$

Alternative

The Cartesian equation of this curve is $y = x^{\frac{3}{2}}$.

Alternatively, you could use the formula for the length of a curve $y = f(x)$.

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Example 22

Find the length of the curve defined by the parametric equations

$$x = \cos t \quad \text{and} \quad y = \sin t \quad \text{for } 0 \le t \le 2\pi$$

Solution

We obtain $\dfrac{dx}{dt} = -\sin t$ and $\dfrac{dy}{dt} = \cos t$.

Thus the length is

$$\begin{aligned}\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt \\ &= \int_0^{2\pi} \sqrt{1}\, dt \\ &= [t]_0^{2\pi} \\ &= 2\pi\end{aligned}$$

Many apparently easy curve-length questions will produce integrals that you cannot evaluate. Sometimes it will be possible to evaluate these integrals exactly using a CAS calculator, but sometimes it will only be possible to obtain an approximate answer.

Skill-sheet

Exercise 10E

Example 21

1 Find the length of each of the following curves:

a $x = t$ and $y = 2t^{\frac{3}{2}}$, for $0 \le t \le 1$

b $x = t$ and $y = 2t + 1$, for $0 \le t \le 3$

c $x = t$ and $y = \dfrac{1}{3}(t^2 + 2)^{\frac{3}{2}}$, for $0 \le t \le 6$

2 Find the length of each of the following curves:

a $x = t - 1$ and $y = t^{\frac{3}{2}}$, for $0 \le t \le 1$

b $x = t^3 + 3t^2$ and $y = t^3 - 3t^2$, for $0 \le t \le 3$

c $x = e^t$ and $y = \dfrac{2}{3}e^{\frac{3t}{2}}$, for $\log_e 2 \le t \le \log_e 3$

d $x = \dfrac{1}{2}t^2$ and $y = \dfrac{1}{3}t^3$, for $0 \le t \le \sqrt{3}$

Example 22

3 Consider the curve defined by the parametric equations

$$x = 3\sin(2t) \quad \text{and} \quad y = 3\cos(2t) \quad \text{for } 0 \le t \le \frac{\pi}{6}$$

Find the length of this curve.

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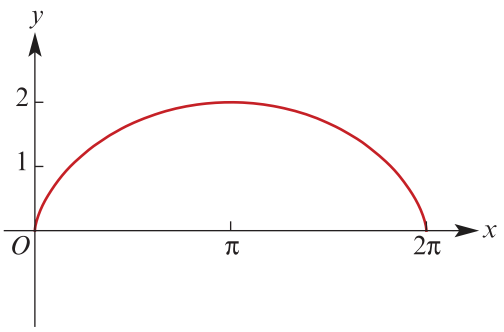
4 A curve is specified parametrically by the equations

$$x = t - \sin t, \quad y = 1 - \cos t$$

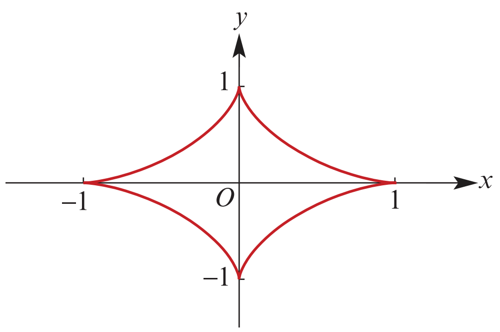
Find the length of the curve from $t = 0$ to $t = 2\pi$.



5 A curve is specified parametrically by the equations

$$x = \cos^3 t, \quad y = \sin^3 t$$

The graph of the curve is shown. Find the length of the curve.



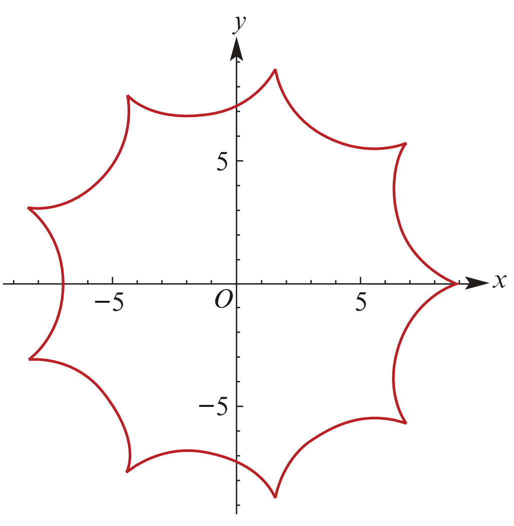
6 Find the length of the curve defined by $x = e^t \sin(2t)$ and $y = e^t \cos(2t)$ for $0 \le t \le \pi$.

7 The curve shown on the right is defined by the parametric equations

$$x = 8\cos(t) + \cos(8t)$$

$$y = 8\sin(t) - \sin(8t)$$

Find the length of this curve.



8 A parametric curve is defined by

$$x = \frac{t^3}{3} \quad \text{and} \quad y = \sin^{-1}(t) + t\sqrt{1 - t^2} \quad \text{for } 0 \le t \le \frac{1}{2}$$

Find the length of the curve.

9 A parametric curve is defined by

$$x = 4\cos(t) + \cos(2t) \quad \text{and} \quad y = \sin(2t) + 4\sin(t) + 2t \quad \text{for } 0 \le t \le \frac{\pi}{4}$$

Find the length of the curve.

10F Areas of surfaces of revolution

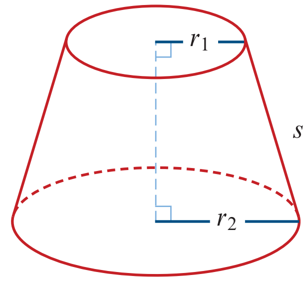
In this section, we use integration to find the areas of surfaces of revolution. The approach is a little different from that used to find volumes in Section 10D. Here we use truncated cones instead of cylinders for our approximations.

Surface area of a frustum

A truncated cone is called a **frustum**. The area of the curved surface of a frustum is given by

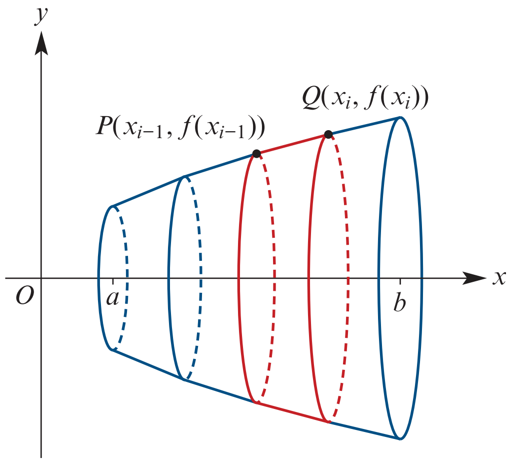
$$A = \pi(r_1 + r_2)s$$

where r_1 and r_2 are the radii of the two circular faces and s is the length of the slope.



Approximating surface area using frustums

Assume that the curve $y = f(x)$ from $x = a$ to $x = b$ is above the x-axis. We can rotate this curve about the x-axis to form a curved surface, called a **surface of revolution**.



To find the area, A, of this curved surface, we approximate the surface using frustums, as shown.

Say that we use n frustums, determined by the x-values $a = x_0, x_1, x_2, \dots, x_n = b$.

Now consider the frustum between points $P(x_{i-1}, f(x_{i-1}))$ and $Q(x_i, f(x_i))$. The radii of the two circles are $r_1 = f(x_{i-1})$ and $r_2 = f(x_i)$. The length of the slope is

$$s = \sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2}$$

Therefore the area of the curved surface of this frustum is

$$A_i = \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2}$$

The total area is approximated by adding the surface areas of all n frustums:

$$\begin{aligned} A &\approx \sum_{i=1}^{n} \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2} \\ &= \sum_{i=1}^{n} \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\,(x_i - x_{i-1}) \end{aligned}$$

As $n \to \infty$, this sum converges to the integral

$$A = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

A more rigorous version of this argument can be used to prove the following result.

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Area of a surface of revolution

- **Rotation about the x-axis** If the curve $y = f(x)$ from $x = a$ to $x = b$ is rotated about the x-axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- **Rotation about the y-axis** If the curve $x = f(y)$ from $y = a$ to $y = b$ is rotated about the y-axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Note: We are assuming that f is non-negative and differentiable on $[a, b]$, with f' continuous.

Example 23

Find the area of the surface generated by revolving the part of the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$ about the x-axis.

Solution

We have $y = x^3$ and $\frac{dy}{dx} = 3x^2$.

The surface area is

$$A = 2\pi \int_0^2 x^3\sqrt{1 + (3x^2)^2}\, dx$$

$$= 2\pi \int_0^2 x^3\sqrt{1 + 9x^4}\, dx$$

Let $u = 1 + 9x^4$. Then $\frac{du}{dx} = 36x^3$. Therefore

$$A = \frac{\pi}{18}\int_1^{145} \sqrt{u}\, du$$

$$= \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{145}$$

$$= \frac{\pi}{27}(145\sqrt{145} - 1)$$

Surface area using inverse functions

We can reframe this result in the special case that the function $f\colon [a, b] \to \mathbb{R}$ is one-to-one with inverse function $g\colon [c, d] \to \mathbb{R}$, where g is differentiable on $[c, d]$.

- **Rotation about the x-axis** If the curve $x = g(y)$ from $y = c$ to $y = d$ is rotated about the x-axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_c^d y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

- **Rotation about the y-axis** If the curve $y = g(x)$ from $x = c$ to $x = d$ is rotated about the y-axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

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Surface area using parameters

Area of a surface of revolution formed by a parametric curve

Consider a curve defined by the parametric equations $x = f(t)$ and $y = g(t)$ for $a \le t \le b$. If the point $P(f(t), g(t))$ traces the curve exactly once from $t = a$ to $t = b$, then the area of the surface of revolution formed by rotating the curve about the x-axis is given by

$$A = 2\pi \int_a^b |g(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notes:

- We are assuming that f and g are differentiable on $[a, b]$, with f' and g' continuous.
- For rotation about the x-axis, we are assuming that the curve is the graph of a function.
- The area of a surface formed by rotating a parametric curve about the y-axis can be found in a similar way by replacing $g(t)$ with $f(t)$ in this formula.

Example 24

A semicircle with radius r is defined by the parametric equations

$$x = r\cos t \quad \text{and} \quad y = r\sin t \quad \text{for } 0 \le t \le \pi$$

Find the area of the surface generated by revolving this semicircle about the x-axis.

Solution

We have $\frac{dx}{dt} = -r\sin t$ and $\frac{dy}{dt} = r\cos t$.

The surface area is

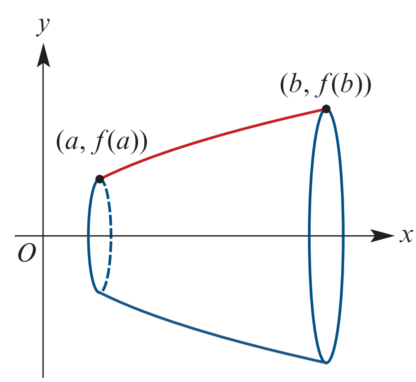
$$\begin{aligned} A &= 2\pi \int_0^{\pi} r\sin t \cdot \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= 2\pi r^2 \int_0^{\pi} \sin t\, dt \\ &= 2\pi r^2 \Big[-\cos t\Big]_0^{\pi} \\ &= 4\pi r^2 \end{aligned}$$

Note: This is the surface area of a sphere with radius r.

Total surface area of a solid of revolution

Suppose that the region under the curve $y = f(x)$ from $x = a$ to $x = b$ is rotated about the x-axis to form a solid of revolution. To find the total surface area of this solid, we first find the area, A, of the curved surface of revolution and then add the areas of the circular discs at each end:

$$\text{total surface area} = A + \pi(f(a))^2 + \pi(f(b))^2$$



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Exercise 10F

Example 23

1 For each of the following, find the area of the surface generated when the given curve is rotated about the x-axis:

a $y = \frac{3}{4}x$ for $0 \le x \le 8$

b $y = \frac{1}{3}x + 4$ for $0 \le x \le 3$

c $y = \frac{1}{4}x^3$ for $0 \le x \le 1$

d $y = \sqrt{4 - x^2}$ for $0 \le x \le 1$

e $y = \sqrt{4 - x^2}$ for $-1 \le x \le 1$

f $y = \frac{x^3}{6} + \frac{1}{2x}$ for $\frac{1}{2} \le x \le 1$

2 For each of the following, find the area of the surface generated when the given curve is rotated about the y-axis:

a $y = \frac{3}{4}x$ for $0 \le y \le 10$

b $y = -\frac{3}{4}x + 3$ for $0 \le y \le 4$

c $y = x^2$ for $4 \le y \le 9$

d $x = \sqrt{2y - y^2}$ for $0 \le y \le 1$

e $y = \sqrt[3]{3x}$ for $0 \le y \le 2$

f $y = x^2$ for $1 \le x \le 3$

3 Find the area of the curved surface formed by rotating the part of the graph of $y = x^2$ from $(1, 1)$ to $(2, 4)$ about the y-axis by using:

a $y = x^2$ and $\frac{dy}{dx} = 2x$

b $x = \sqrt{y}$ and $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

4 Find the area of the surface formed by rotating the parabola $y = 1 - x^2$, for $x \in [0, 1]$, about the y-axis.

Example 24

5 A curve is defined parametrically by

$$x = 4\cos(2t) \quad \text{and} \quad y = 4\sin(2t) \quad \text{for } 0 \le t \le \frac{\pi}{2}$$

Find the area of the surface obtained by revolving this curve about the x-axis.

6 For each of the following, find the area of the surface obtained by revolving the given parametric curve about the given axis:

a $x = 6 + 2t^2$ and $y = 4t$, for $0 \le t \le 4$, about the x-axis

b $x = 1 - t^2$ and $y = 2t$, for $0 \le t \le 1$, about the x-axis

c $x = 3t - t^3$ and $y = 3t^2$, for $0 \le t \le 1$, about the x-axis

d $x = t$ and $y = t^2 - 2$, for $0 \le t \le 3$, about the y-axis

e $x = t + \sqrt{3}$ and $y = \frac{1}{2}t^2 + \sqrt{3}t$, for $-\sqrt{3} \le t \le \sqrt{3}$, about the y-axis

f $x = 3 + 2\cos t$ and $y = 4 + 2\sin t$, for $0 \le t \le \frac{\pi}{2}$, about the y-axis

g $x = 4t$ and $y = t^2 - 2\log_e t$, for $1 \le t \le 3$, about the x-axis

7 A curve is defined parametrically by $x = \cos t$ and $y = 4 + \sin t$, for $0 \le t \le \pi$. Find the area of the surface obtained by revolving this curve about the x-axis.

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8 A curve is defined parametrically by $x = \cos^3 t$ and $y = \sin^3 t$, for $0 \le t \le \frac{\pi}{2}$. Find the area of the surface obtained by revolving this curve about the x-axis.

9 Let $a > 0$. A parametric curve is defined by $x = a\cos^2 t$ and $y = a\sin^2 t$, for $0 \le t \le \frac{\pi}{2}$. Find the area of the surface obtained by revolving this curve about the x-axis.

10 Let r and h be positive constants. Find the area of the surface that is formed by rotating the graph of $y = \frac{r}{h}x$ over the interval $0 \le x \le h$ about the x-axis.

11 A sphere of radius r is cut by two parallel planes at a distance h apart. Prove that the surface area of the part of the sphere cut off by the two planes is $2\pi rh$.

12 Use your CAS calculator to find the surface area of the solid of revolution formed by rotating the half ellipse defined by the parametric equations $x = 3\cos t$ and $y = 2\sin t$, for $0 \le t \le \pi$, about the x-axis.

13 Let $R > r > 0$. The circle with centre $(R, 0)$ and radius r is rotated about the y-axis. What is the surface area of the resulting solid?

Hint: The resulting solid is called a **torus** and looks like a doughnut.
Work towards obtaining the integral $2\pi r \int_{R-r}^{R+r} \frac{x}{\sqrt{r^2 - (x-R)^2}}\, dx$.

14 Let $M > 1$.

a Sketch the curve $y = \frac{1}{x}$ over the interval $[1, M]$.

b Sketch the surface obtained when this curve is rotated about the x-axis.

c Show that the volume of the solid of revolution is $\pi\left(1 - \frac{1}{M}\right)$.

d Show that the curved surface area of the solid of revolution is greater than $2\pi \log_e M$.

e Determine the volume and surface area as $M \to \infty$.

Note: The solid of revolution obtained in the limit is called **Gabriel's horn**.
Remarkably, it has a finite volume and an infinite surface area.

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Chapter summary

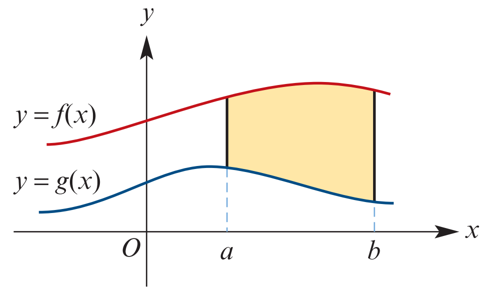
Fundamental theorem of calculus

- If f is a continuous function on an interval $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.
- If f is a continuous function and the function G is defined by $G(x) = \int_a^x f(t)\,dt$, then G is an antiderivative of f.

Areas of regions between curves

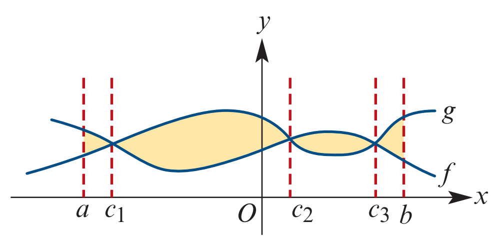
- If f and g are continuous functions such that $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region bounded by the curves and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x) - g(x)\,dx$$



- For graphs that cross, consider intervals. For example, the area of the shaded region is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx$$
$$+ \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$



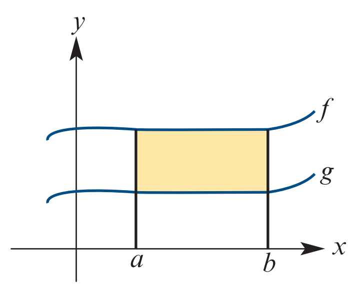
Volumes of solids of revolution

- Region bounded by the x-axis If the region to be rotated about the x-axis is bounded by the curve with equation $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis, then the volume V is given by

$$V = \int_a^b \pi y^2\,dx = \pi \int_a^b (f(x))^2\,dx$$

- Region not bounded by the x-axis If the shaded region is rotated about the x-axis, then the volume V is given by

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2\,dx$$



Lengths of curves

- The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

- For a parametric curve defined by $x = f(t)$ and $y = g(t)$, if the point $P(f(t), g(t))$ traces the curve exactly once from $t = a$ to $t = b$, then the length of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

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Areas of surfaces of revolution

- **Rotation about the x-axis** If the curve $y = f(x)$ from $x = a$ to $x = b$ is rotated about the x-axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- **Rotation about the y-axis** If the curve $x = f(y)$ from $y = a$ to $y = b$ is rotated about the y-axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

- **Parametric curve** Consider a curve defined by $x = f(t)$ and $y = g(t)$ for $a \le t \le b$. If the point $P(f(t), g(t))$ traces the curve exactly once from $t = a$ to $t = b$, then the area of the surface of revolution formed by rotating the curve about the x-axis is given by

$$A = 2\pi \int_a^b |g(t)|\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Technology-free questions

1 Calculate the area of the region enclosed by the graph of $y = \dfrac{x}{\sqrt{x-2}}$ and the line $y = 3$.

2 **a** If $y = 1 - \cos x$, find the value of $\int_0^{\frac{\pi}{2}} y\, dx$. On a sketch graph, indicate the region for which the area is represented by this integral.

b Hence find $\int_0^1 x\, dy$.

3 Find the volume of revolution of each of the following. (Rotation is about the x-axis.)

a $y = \sec x$ between $x = 0$ and $x = \dfrac{\pi}{4}$

b $y = \sin x$ between $x = 0$ and $x = \dfrac{\pi}{4}$

c $y = \cos x$ between $x = 0$ and $x = \dfrac{\pi}{4}$

d the region between $y = x^2$ and $y = 4x$

e $y = \sqrt{1 + x}$ between $x = 0$ and $x = 8$

4 Find the volume generated when the region bounded by the curve $y = 1 + \sqrt{x}$, the x-axis and the lines $x = 1$ and $x = 4$ is rotated about the x-axis.

5 The region S in the first quadrant of the Cartesian plane is bounded by the axes, the line $x = 3$ and the curve $y = \sqrt{1 + x^2}$. Find the volume of the solid formed when S is rotated:

a about the x-axis

b about the y-axis.

6 Sketch the graph of $y = \sec x$ for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Find the volume of the solid of revolution obtained by rotating this curve about the x-axis for $x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$.

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7 **a** Find the coordinates of the points of intersection of the graphs of $y^2 = 8x$ and $y = 2x$.

b Find the volume of the solid formed when the area enclosed by these graphs is rotated about the x-axis.

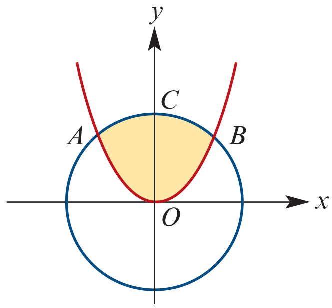
8 **a** On the one set of axes, sketch the graphs of $y = 1 - x^2$ and $y = x - x^3 = x(1 - x^2)$. (Turning points of the second graph do not have to be determined.)

b Find the area of the region enclosed between the two graphs.

9 The curves $y = x^2$ and $x^2 + y^2 = 2$ meet at the points A and B.

a Find the coordinates of A, B and C.

b Find the volume of the solid of revolution formed by rotating the shaded region about the x-axis.



10 **a** Sketch the graph of $y = 2x - x^2$ for $y \geq 0$.

b Find the area of the region enclosed between this curve and the x-axis.

c Find the volume of the solid of revolution formed by rotating this region about the x-axis.

11 **a** Let the curve $f\colon [0, b] \to \mathbb{R}, f(x) = x^2$ be rotated:

i around the x-axis to define a solid of revolution, and find the volume of this solid in terms of b (where the region rotated is between the curve and the x-axis)

ii around the y-axis to define a solid of revolution, and find the volume of this solid in terms of b (where the region rotated is between the curve and the y-axis).

b For what value of b are the two volumes equal?

12 **a** Sketch the graph of $\left\{ (x, y) : y = \dfrac{1}{4x^2 + 1} \right\}$.

b Find $\dfrac{dy}{dx}$ and hence find the equation of the tangent to this curve at $x = \frac{1}{2}$.

c Find the area of the region bounded by the curve and the tangent to the curve at $x = \frac{1}{2}$.

13 Let $f\colon \mathbb{R} \to \mathbb{R}, f(x) = x$ and $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, g(x) = \dfrac{9}{x}$.

a Sketch, on the same set of axes, the graphs of $f + g$ and $f - g$.

b Find the area of the region bounded by the two graphs sketched in part **a** and the lines $x = 1$ and $x = 3$.

14 Sketch the graph of $\left\{ (x, y) : y = x - 5 + \dfrac{4}{x} \right\}$. Find the area of the region bounded by this curve and the x-axis.

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15 Sketch the graph of $\left\{(x, y) : y = \dfrac{1}{2 + x - x^2}\right\}$. Find the area of the region bounded by this graph and the line $y = \frac{1}{2}$.

16 The region bounded by the graph of $y = x^{\frac{1}{2}} \sin(2x)$, the x-axis and the line $x = \dfrac{\pi}{4}$ is rotated about the x-axis to form a solid of revolution. Find its volume.

17 Find the length of the curve defined by $x = 2t - 2\sin t$ and $y = 2 - 2\cos t$ for $\dfrac{\pi}{3} \leq t \leq 2\pi$.

18 Find the length of the curve defined by $x = \cos^3 t$ and $y = \sin^3 t$ for $0 \leq t \leq \dfrac{\pi}{4}$.

19 Consider the graph of $y = xe^x$ over the interval $0 \leq x \leq 1$. Using integration by parts twice, find the volume obtained when this curve is rotated about the x-axis.

20 Let $a > b > 0$. The curve defined parametrically by $x = a\cos\theta$ and $y = b\sin\theta$ is an ellipse. Show that the circumference of this ellipse is given by

$$C = a\int_0^{2\pi} \sqrt{1 - e^2\cos^2\theta}\, d\theta \quad \text{where } e = \sqrt{1 - \frac{b^2}{a^2}}$$

(The quantity e is known as the eccentricity of the ellipse.)

Multiple-choice questions

1 The volume of the solid of revolution formed when the region bounded by the axes, the line $x = 1$ and the curve with equation $y = \dfrac{1}{\sqrt{4 - x^2}}$ is rotated about the x-axis is

A $\dfrac{\pi^2}{6}$ **B** $\dfrac{\pi^2}{3}$ **C** $\dfrac{\pi}{4}\log_e(3)$ **D** $\pi\sqrt{3}\log_e(3)$ **E** $\dfrac{2\pi^2}{3}$

2 The shaded region shown below is enclosed by the curve $y = \dfrac{6}{\sqrt{5 + x^2}}$, the straight line $y = 2$ and the y-axis. The region is rotated about the x-axis to form a solid of revolution. The volume of this solid, in cubic units, is given by

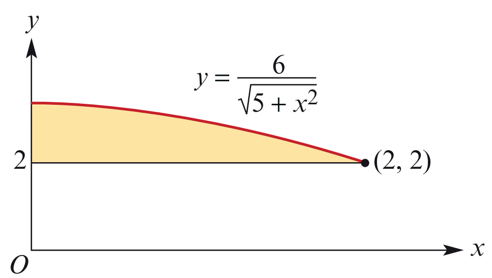
A $\pi\displaystyle\int_0^2 \left(\frac{6}{\sqrt{5 + x^2}} - 2\right)^2 dx$

B $6\pi\tan^{-1}\left(\dfrac{2}{5}\right)$

C $\dfrac{36\pi}{\sqrt{5}}\tan^{-1}\left(\dfrac{2}{\sqrt{5}}\right)$

D $\pi\displaystyle\int_0^2 \left(\frac{6}{\sqrt{5 + x^2}}\right)^2 - 4\, dx$

E 36π



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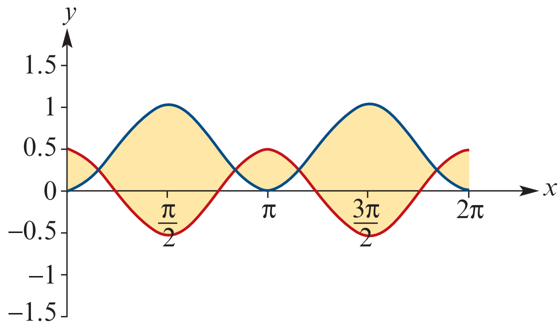
3 The graphs of $y = \sin^2 x$ and $y = \frac{1}{2}\cos(2x)$ are shown in the diagram. The total area of the shaded regions is equal to



A $\int_0^{2\pi} \sin^2 x - \frac{1}{2}\cos(2x)\,dx$

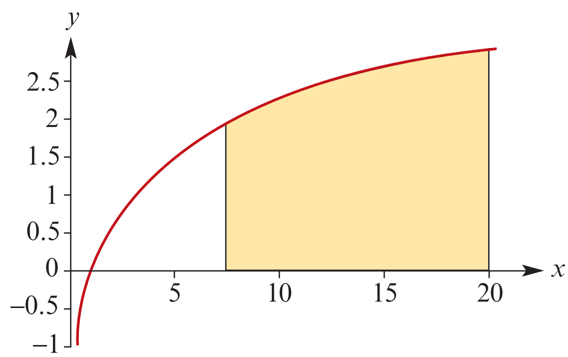
B $4\int_0^{\frac{\pi}{6}} \frac{1}{2}\cos(2x) - \sin^2 x\,dx + 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x - \frac{1}{2}\cos(2x)\,dx$

C 3.14

D π

E $\frac{\sqrt{3}}{2} + \frac{\pi}{3}$

4 The shaded region in the diagram is bounded by the lines $x = e^2$ and $x = e^3$, the x-axis and the graph of $y = \log_e x$. The volume of the solid of revolution formed by rotating this region about the x-axis is equal to



A $\pi\int_2^3 e^{2x}\,dx$

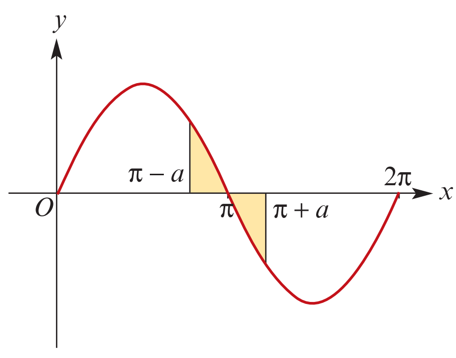
B $\pi\int_7^{20} (\log_e x)^2\,dx$

C $\pi\int_{e^2}^{e^3} (\log_e x)^2\,dx$

D $\pi(e^3 - e^2)$

E $\pi^2\int_{e^2}^{e^3} (\log_e x)^2\,dx$

5 The graph represents the function $y = \sin x$ where $0 \le x \le 2\pi$. The total area of the shaded regions is



A $1 - \cos a$

B $-2\sin a$

C $2(1 - \cos a)$

D 0

E $-2(1 - \cos a)$

6 The area of the region enclosed between the curve with equation $y = \sin^3 x$, $x \in [0, a]$, the x-axis and the line with equation $x = a$, where $0 < a < \frac{\pi}{2}$, is

A $3\cos^2 a$

B $\frac{2}{3} - \frac{1}{3}\sin^3 a$

C $\left(\frac{2}{3} - \frac{1}{3}\sin^2 a\right)\cos a + \frac{2}{3}$

D $\frac{1}{3}\cos^3 a \sin a$

E $\frac{2}{3} - \cos a + \frac{1}{3}\cos^3 a$

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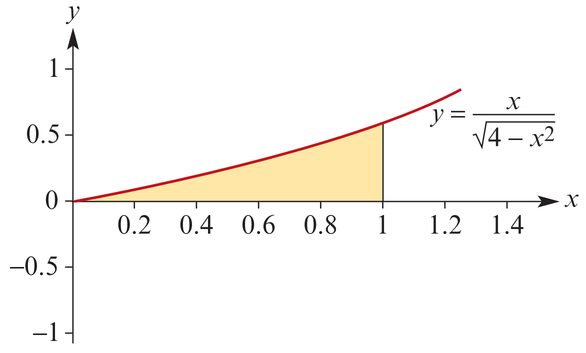
Review

7 The shaded region shown is rotated around the x-axis to form a solid of revolution. The volume of the solid of revolution is



A $1 - \log_e(\frac{1}{3})$

B $\pi(\log_e 3 - 1)$

C 0.099

D $\pi(-1 + \log_e(\frac{1}{3}))$

E 0.1

8 The shaded region shown in the diagram is rotated around the x-axis to form a solid of revolution, where $f'(x) > 0$ and $f''(x) > 0$ for all $x \in [a, b]$ and the volume of the solid of revolution is V cubic units. Which of the following statements is false?

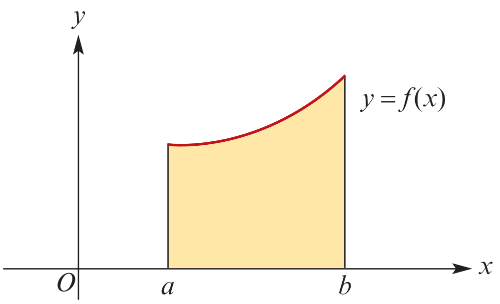
A $V < \pi(f(b))^2(b - a)$

B $V > \pi(f(a))^2(b - a)$

C $V = \pi \int_a^b (f(x))^2\, dx$

D $V = \pi\left((F(b))^2 - (F(a))^2\right)$, where $F'(x) = f(x)$

E $V < \pi\left((f(b))^2 b - (f(a))^2 a\right)$



9 The length of the curve defined by the parametric equations $x = 4\sin t$ and $y = 3\cos t$, for $0 \le t \le \pi$, is given by

A $\int_0^\pi \sqrt{16\cos^2 t - 9\sin^2 t}\, dt$

B $\int_0^\pi \sqrt{7 + 9\sin^2 t}\, dt$

C $\int_0^\pi \sqrt{9\cos^2 t + 16\sin^2 t}\, dt$

D $\int_0^\pi 4\cos^2 t + 9\sin^2 t\, dt$

E $\int_0^\pi \sqrt{9 + 7\cos^2 t}\, dt$

10 The region bounded by the coordinate axes and the graph of $y = \cos x$, for $0 \le x \le \frac{\pi}{2}$, is rotated about the y-axis to form a solid of revolution. The volume of the solid is given by

A $\pi \int_0^1 \cos^2 x\, dx$

B $\pi \int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

C $\pi \int_0^1 \cos^{-1} y\, dx$

D $\pi \int_0^{\frac{\pi}{2}} (\cos^{-1} y)^2\, dy$

E $\pi \int_0^1 (\cos^{-1} y)^2\, dy$

11 The area of the surface formed by rotating the curve $y = \frac{1}{x}$, for $1 \le x \le a$, about the x-axis is given by

A $2\pi \int_1^a \frac{\sqrt{1 + x^2}}{x}\, dx$

B $2\pi \int_1^a 1 + \frac{1}{x^2}\, dx$

C $2\pi \int_1^a \frac{\sqrt{1 + x^4}}{x^3}\, dx$

D $2\pi \int_1^a \sqrt{1 + \frac{1}{x^4}}\, dx$

E $2\pi \int_1^a \sqrt{x^2 + \frac{1}{x^4}}\, dx$

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12 The area of the region enclosed by the graph of $y = 2x^2 \sec^2 x \tan x$, the x-axis and the line $x = \frac{\pi}{4}$ is given by

A $\frac{\pi^2}{8} - \int_0^{\frac{\pi}{4}} 2x \sec^2 x \, dx$

B $\frac{3\pi}{4} - \int_0^{\frac{\pi}{4}} x^2 \tan^2 x \, dx$

C $\frac{\pi^2}{16} - \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$

D $\frac{\pi^2}{8} - \int_0^{\frac{\pi}{4}} 2x^2 \sec^2 x \tan^2 x \, dx$

E $-\frac{\pi}{2} + \frac{\pi^2}{8} - \int_0^{\frac{\pi}{4}} 2x \sec^2 x \, dx$

Extended-response questions

1 **a** Sketch the curve with equation $y = 1 - \frac{1}{x+2}$.

b Find the area of the region bounded by the x-axis, the curve and the lines $x = 0$ and $x = 2$.

c Find the volume of the solid of revolution formed when this region is rotated around the x-axis.

2 **a** **i** Using integration by parts twice, find $\int (\log_e x)^2 \, dx$.

ii Hence find the volume of the solid of revolution formed by rotating the curve $y = \log_e x$ from $x = 1$ to $x = e$ about the x-axis.

b Sketch the graph of $f\colon [-2, 2] \to \mathbb{R}$,

$$f(x) = \begin{cases} e^x & x \in [0, 2] \\ e^{-x} & x \in [-2, 0) \end{cases}$$

c The interior of a wine glass is formed by rotating the curve $y = e^x$ from $x = 0$ to $x = 2$ about the y-axis. If the units are in centimetres find, correct to two significant figures, the volume of liquid that the glass contains when full.

3 A bowl is modelled by rotating the curve $y = x^2$ for $0 \le x \le 1$ around the y-axis.

a Find the volume of the bowl.

b If liquid is poured into the bowl at a rate of R units of volume per second, find the rate of increase of the depth of liquid in the bowl when the depth is $\frac{1}{4}$.

Hint: Use the chain rule: $\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt}$.

c **i** Find the volume of liquid in the bowl when the depth of liquid is $\frac{1}{2}$.

ii Find the depth of liquid in the bowl when it is half full.

4 **a** On the same set of axes, sketch the graphs of $y = 3\sec^2 x$ and $y = 16\sin^2 x$ for $0 \le x \le \frac{\pi}{4}$.

b Find the coordinates of the point of intersection of these two curves.

c Find the area of the region bounded by the two curves and the y-axis.

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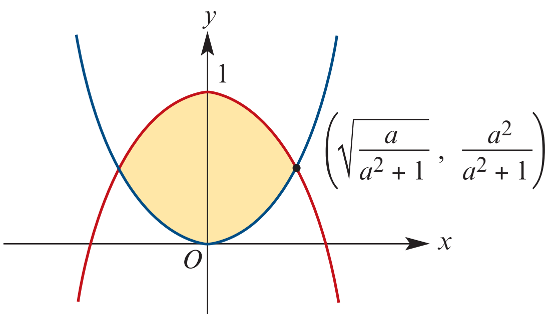
Review

5 The curves $y = ax^2$ and $y = 1 - \dfrac{x^2}{a}$ are shown, where $a > 0$.



a Show that the area enclosed by the two curves is $\dfrac{4}{3}\sqrt{\dfrac{a}{a^2+1}}$.

b **i** Find the value of a which gives the maximum area.

ii Find the maximum area.

c Find the volume of the solid formed when the region bounded by these curves is rotated about the y-axis.

6 Consider the curve defined by the parametric equations

$$x = 3 + 2\cos(2t) \quad \text{and} \quad y = 2\sin(2t) \quad \text{for } 0 \leq t \leq \frac{\pi}{4}$$

a This curve can be described in the form $y = f(x)$ for a function f. Find the rule, domain and range of f.

b Find the equation of the tangent to the curve at $t = \dfrac{\pi}{8}$.

c Find the length of the curve.

d Show that the area of the region bounded by the curve, the line $x = 3$ and the x-axis is equal to π.

Hint: Use the observation that $\int y\, dx = \int y \dfrac{dx}{dt}\, dt$.

e Find the total surface area of the solid of revolution formed when the region bounded by the curve, the line $x = 3$ and the x-axis is rotated about the x-axis.

7 Let $f\colon (1, \infty) \to \mathbb{R}$ be such that:

- $f'(x) = \dfrac{1}{x - a}$, where a is a positive constant
- $f(2) = 1$
- $f(1 + e^{-1}) = 0$

a Find a and use it to determine $f(x)$.

b Sketch the graph of f.

c If f^{-1} is the inverse of f, show that $f^{-1}(x) = 1 + e^{x-1}$. Give the domain and range of f^{-1}.

d Find the area of the region enclosed by $y = f^{-1}(x)$, the x-axis, the y-axis and the line $x = 1$.

e Find $\int_{1+e^{-1}}^{2} f(x)\, dx$.

8 The curves $cy^2 = x^3$ and $y^2 = ax$ (where $a > 0$ and $c > 0$) intersect at the origin, O, and at a point P in the first quadrant. The areas of the regions enclosed by the curves OP, the x-axis and the vertical line through P are A_1 and A_2 respectively for the two curves. The volumes of the two solids formed by rotating these regions about the x-axis are V_1 and V_2 respectively. Show that $A_1 : A_2 = 3 : 5$ and $V_1 : V_2 = 1 : 2$.

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9 Let $f\colon [0, a] \to \mathbb{R}$, where $f(x) = 3\cos(\frac{1}{2}x)$.

a Find the largest value of a for which f has an inverse function, f^{-1}.

b **i** State the domain and range of f^{-1}.

ii Find $f^{-1}(x)$.

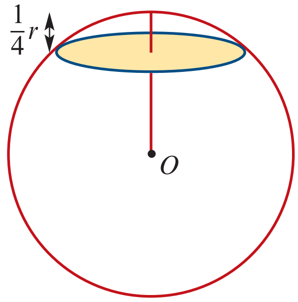
iii Sketch the graph of f^{-1}.

c Find the gradient of the curve $y = f^{-1}(x)$ at the point where the curve crosses the y-axis.

d Let V_1 be the volume of the solid of revolution formed by rotating the curve $y = f(x)$ between $x = 0$ and $x = \pi$ about the x-axis. Let V_2 be the volume of the solid of revolution formed by rotating the curve $y = f^{-1}(x)$ between $y = 0$ and $y = \pi$ about the y-axis. Find V_1 and hence find V_2.

10 **a** Find the area of the circle formed when a sphere is cut by a plane at a distance y from the centre, where $y < r$.

b By integration, prove that the volume of a 'cap' of height $\frac{1}{4}r$ cut from the top of the sphere, as shown in the diagram, is $\dfrac{11\pi r^3}{192}$.



11 Consider the section of a hyperbola with $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $a \le x \le 2a$ (where $a > 0$). Find the volume of the solid formed when region bounded by the hyperbola and the line with equation $x = 2a$ is rotated about:

a the x-axis

b the y-axis.

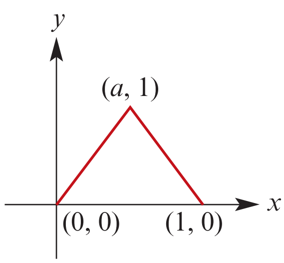
12 **a** Show that the line $y = \dfrac{3x}{2}$ does not meet the curve $y = \dfrac{1}{\sqrt{1 - x^2}}$.

b Find the area of the region bounded by the curve with equation $y = \dfrac{1}{\sqrt{1 - x^2}}$ and the lines $y = \dfrac{3x}{2}$, $x = 0$ and $x = \dfrac{1}{2}$.

c Find the volume of the solid of revolution formed by rotating the region defined in **b** about the x-axis. Express your answer in the form $\pi(a + \log_e b)$.

13 **a** For $0 \le a \le 1$, let T_a be the triangle whose vertices are $(0, 0)$, $(1, 0)$ and $(a, 1)$. Find the volume of the solid of revolution when T_a is rotated about the x-axis.

b For $0 \le k \le 1$, let T_k be the triangle whose vertices are $(0, 0)$, $(k, 0)$ and $(0, \sqrt{1 - k^2})$. The triangle T_k is rotated about the x-axis. What value of k gives the maximum volume? What is the maximum volume?



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14 A model for a bowl is formed by rotating a section of the graph of a cubic function $f(x) = ax^3 + bx^2 + cx + d$ around the x-axis to form a solid of revolution. The cubic is chosen to pass through the points with coordinates $(0, 0)$, $(5, 1)$, $(10, 2.5)$ and $(30, 10)$.

a **i** Write down the four simultaneous equations that can be used to determine the coefficients a, b, c and d.

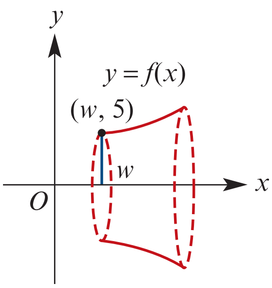
ii Using a CAS calculator, or otherwise, find the values of a, b, c and d. (Exact values should be stated.)

b Find the area of the region enclosed by the curve and the line $x = 30$.

c **i** Write the expression that can be used to determine the volume of the solid of revolution when the section of the curve $0 \leq x \leq 30$ is rotated around the x-axis.

ii Use a CAS calculator to determine this volume.

d Using the initial design, the bowl is unstable. The designer is very fond of the cubic $y = f(x)$, and modifies the design so that the base of the bowl has radius 5 units. Using a CAS calculator:

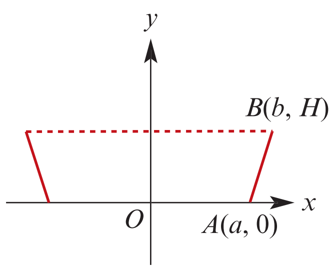


i find the value of w such that $f(w) = 5$

ii find the new volume, correct to four significant figures.

e A mathematician looks at the design and suggests that it may be more pleasing to the eye if the base is chosen to occur at a point where $x = p$ and $f''(p) = 0$. Find the values of coordinates of the point $(p, f(p))$.

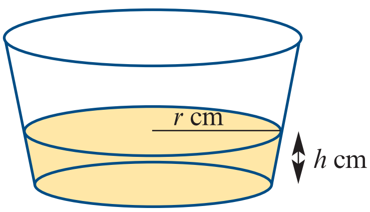
15 A model of a bowl is formed by rotating the line segment AB about the y-axis to form a solid of revolution.



a Find the volume, V cm^3, of the bowl in terms of a, b and H. (Units are centimetres.)

b If the bowl is filled with water to a height $\frac{H}{2}$, find the volume of water.

c Find an expression for the volume of water in the bowl when the radius of the water surface is r cm. (The constants a, b and H are to be used.)



d **i** Find $\frac{dV}{dr}$.

ii Find an expression for the depth of the water, h cm, in terms of r.

e Now assume that $a = 10$, $b = 20$ and $H = 20$.

i Find $\frac{dV}{dr}$ in terms of r.

ii If water is being poured into the bowl at 3 cm^3/s, find $\frac{dr}{dt}$ and $\frac{dh}{dr}$ when $r = 12$.

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11 Differential equations

Objectives

- ► To **verify** a solution of a **differential equation**.
- ► To apply techniques to **solve** differential equations of the form $\frac{dy}{dx} = f(x)$ and $\frac{d^2y}{dx^2} = f(x)$.
- ► To apply techniques to **solve** differential equations of the form $\frac{dy}{dx} = g(y)$.
- ► To **construct** differential equations from a given situation.
- ► To **apply** differential equations to solve problems.
- ► To solve differential equations which can be written in the form $\frac{dy}{dx} = f(x)\,g(y)$ using **separation of variables**.
- ► To solve differential equations using a CAS calculator.
- ► To use **Euler's method** to obtain approximate solutions to a given differential equation.
- ► To construct a **slope field** for a given differential equation.

Differential equations arise when we have information about the rate of change of a quantity, rather than the quantity itself. For example, we know that the rate of decay of a radioactive substance is proportional to the mass m of substance remaining at time t. We can write this as a differential equation:

$$\frac{dm}{dt} = -km$$

where k is a constant. What we would really like is an expression for the mass m at time t. Using techniques developed in this chapter, we will find that the general solution to this differential equation is $m = Ae^{-kt}$.

Differential equations have many applications in science, engineering and economics, and their study is a major branch of mathematics. For Specialist Mathematics, we consider only a limited variety of differential equations.

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11A An introduction to differential equations

A differential equation contains derivatives of a particular function or variable. The following are examples of differential equations:

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = \frac{y}{y+1}$$

The solution of a differential equation is a clear definition of the function or relation, without any derivatives involved.

For example, if $\frac{dy}{dx} = \cos x$, then $y = \int \cos x \, dx$ and so $y = \sin x + c$.

Here $y = \sin x + c$ is the **general solution** of the differential equation $\frac{dy}{dx} = \cos x$.

This example displays the main features of such solutions. The general solution of a differential equation is a family of functions or relations.

To obtain a **particular solution**, we require further information, which is usually given as an ordered pair belonging to the function or relation. For equations with second derivatives, we need two items of information.

Verifying a solution of a differential equation

We can verify that a particular expression is a solution of a differential equation by substitution. This is demonstrated in the following examples.

We will use the following notation to denote the y-value for a given x-value:

$y(0) = 3$ will mean that when $x = 0$, $y = 3$.

We consider y as a function of x. This notation is useful in differential equations.

Example 1

a Verify that $y = Ae^x - x - 1$ is a solution of the differential equation $\frac{dy}{dx} = x + y$.

b Hence find the particular solution of the differential equation given that $y(0) = 3$.

Solution

a Let $y = Ae^x - x - 1$. We need to check that $\frac{dy}{dx} = x + y$.

$$\begin{aligned} \text{LHS} &= \frac{dy}{dx} \\ &= Ae^x - 1 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= x + y \\ &= x + Ae^x - x - 1 \\ &= Ae^x - 1 \end{aligned}$$

Hence LHS = RHS and so $y = Ae^x - x - 1$ is a solution of $\frac{dy}{dx} = x + y$.

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b $y(0) = 3$ means that when $x = 0$, $y = 3$.

Substituting in the solution $y = Ae^x - x - 1$ verified in **a**:

$$3 = Ae^0 - 0 - 1$$
$$3 = A - 1$$
$$\therefore \quad A = 4$$

The particular solution is $y = 4e^x - x - 1$.

Example 2

Verify that $y = e^{2x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$.

Solution

Let $\quad y = e^{2x}$

Then $\quad \dfrac{dy}{dx} = 2e^{2x}$

and $\quad \dfrac{d^2y}{dx^2} = 4e^{2x}$

Now consider the differential equation:

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y \\ &= 4e^{2x} + 2e^{2x} - 6e^{2x} \qquad \text{(from above)} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Example 3

Verify that $y = ae^{2x} + be^{-3x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$.

Solution

Let $\quad y = ae^{2x} + be^{-3x}$

Then $\quad \dfrac{dy}{dx} = 2ae^{2x} - 3be^{-3x}$

and $\quad \dfrac{d^2y}{dx^2} = 4ae^{2x} + 9be^{-3x}$

So

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y \\ &= (4ae^{2x} + 9be^{-3x}) + (2ae^{2x} - 3be^{-3x}) - 6(ae^{2x} + be^{-3x}) \\ &= 4ae^{2x} + 9be^{-3x} + 2ae^{2x} - 3be^{-3x} - 6ae^{2x} - 6be^{-3x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

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Example 4

Find the constants a and b if $y = e^{4x}(2x + 1)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0$$

Solution

Let $y = e^{4x}(2x + 1)$

Then
$$\frac{dy}{dx} = 4e^{4x}(2x + 1) + 2e^{4x}$$
$$= 2e^{4x}(4x + 2 + 1)$$
$$= 2e^{4x}(4x + 3)$$

and
$$\frac{d^2y}{dx^2} = 8e^{4x}(4x + 3) + 4 \times 2e^{4x}$$
$$= 8e^{4x}(4x + 3 + 1)$$
$$= 8e^{4x}(4x + 4)$$
$$= 32e^{4x}(x + 1)$$

If $y = e^{4x}(2x + 1)$ is a solution of the differential equation, then

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0$$

i.e. $32e^{4x}(x + 1) - 2ae^{4x}(4x + 3) + be^{4x}(2x + 1) = 0$

We can divide through by e^{4x} (since $e^{4x} > 0$):

$$32x + 32 - 8ax - 6a + 2bx + b = 0$$

i.e. $(32 - 8a + 2b)x + (32 - 6a + b) = 0$

Thus

$$32 - 8a + 2b = 0 \quad (1)$$
$$32 - 6a + b = 0 \quad (2)$$

Multiply (2) by 2 and subtract from (1):

$$-32 + 4a = 0$$

Hence $a = 8$ and $b = 16$.

Exercise 11A

Example 1

1 For each of the following, verify that the given function or relation is a solution of the differential equation. Hence find the particular solution from the given information.

	Differential equation	Function or relation	Added information
a	$\frac{dy}{dt} = 2y + 4$	$y = Ae^{2t} - 2$	$y(0) = 2$

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	Differential equation	Function or relation	Added information
b	$\frac{dy}{dx} = \log_e \lvert x \rvert$	$y = x\log_e \lvert x \rvert - x + c$	$y(1) = 3$
c	$\frac{dy}{dx} = \frac{1}{y}$	$y = \sqrt{2x + c}$	$y(1) = 9$
d	$\frac{dy}{dx} = \frac{y+1}{y}$	$y - \log_e \lvert y + 1 \rvert = x + c$	$y(3) = 0$
e	$\frac{d^2y}{dx^2} = 6x^2$	$y = \frac{x^4}{2} + Ax + B$	$y(0) = 2,\ y(1) = 2$
f	$\frac{d^2y}{dx^2} = 4y$	$y = Ae^{2x} + Be^{-2x}$	$y(0) = 3,\ y(\log_e 2) = 9$
g	$\frac{d^2x}{dt^2} + 9x = 18$	$x = A\sin(3t) + B\cos(3t) + 2$	$x(0) = 4,\ x\left(\frac{\pi}{2}\right) = -1$

Example 2
Example 3

2 For each of the following, verify that the given function is a solution of the differential equation:

a $\frac{dy}{dx} = 2y, \quad y = 4e^{2x}$

b $\frac{dy}{dx} = -4xy^2, \quad y = \frac{1}{2x^2}$

c $\frac{dy}{dx} = 1 + \frac{y}{x}, \quad y = x\log_e \lvert x \rvert + x$

d $\frac{dy}{dx} = \frac{2x}{y^2}, \quad y = \sqrt[3]{3x^2 + 27}$

e $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0, \quad y = e^{-2x} + e^{3x}$

f $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0, \quad y = e^{4x}(x + 1)$

g $\frac{d^2y}{dx^2} = -n^2y, \quad y = a\sin(nx)$

h $\frac{d^2y}{dx^2} = n^2y, \quad y = e^{nx} + e^{-nx}$

i $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}, \quad y = \frac{x + 1}{1 - x}$

j $y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2, \quad y = \frac{4}{x + 1}$

Example 4

3 If the differential equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 10y = 0$ has a solution $y = ax^n$, find the possible values of n.

4 Find the constants a, b and c if $y = a + bx + cx^2$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 4x^2$.

5 Find the constants a and b if $x = t(a\cos(2t) + b\sin(2t))$ is a solution of the differential equation $\frac{d^2x}{dt^2} + 4x = 2\cos(2t)$.

6 Find the constants a, b, c and d if $y = ax^3 + bx^2 + cx + d$ is a solution to the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^3$.

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11B Differential equations involving a function of the independent variable

In this section we solve differential equations of the following two forms:

$$\frac{dy}{dx} = f(x) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = f(x)$$

Solving differential equations of the form $\frac{dy}{dx} = f(x)$

The simplest differential equations are those of the form

$$\frac{dy}{dx} = f(x)$$

Such a differential equation can be solved provided an antiderivative of $f(x)$ can be found.

> If $\frac{dy}{dx} = f(x)$, then $y = \int f(x)\, dx$.

Example 5

Find the general solution of each of the following:

a $\frac{dy}{dx} = x^4 - 3x^2 + 2$

b $\frac{dy}{dt} = \sin(2t)$

c $\frac{dx}{dt} = e^{-3t} + \frac{1}{t}$

d $\frac{dx}{dy} = \frac{1}{1 + y^2}$

Solution

a $\frac{dy}{dx} = x^4 - 3x^2 + 2$

$\therefore\ y = \int x^4 - 3x^2 + 2\, dx$

$\therefore\ y = \frac{x^5}{5} - x^3 + 2x + c$

b $\frac{dy}{dt} = \sin(2t)$

$\therefore\ y = \int \sin(2t)\, dt$

$\therefore\ y = -\frac{1}{2}\cos(2t) + c$

c $\frac{dx}{dt} = e^{-3t} + \frac{1}{t}$

$\therefore\ x = \int e^{-3t} + \frac{1}{t}\, dt$

$\therefore\ x = -\frac{1}{3}e^{-3t} + \log_e|t| + c$

d $\frac{dx}{dy} = \frac{1}{1 + y^2}$

$\therefore\ x = \int \frac{1}{1 + y^2}\, dy$

$\therefore\ x = \tan^{-1}(y) + c$

This can also be written as $y = \tan(x - c)$.

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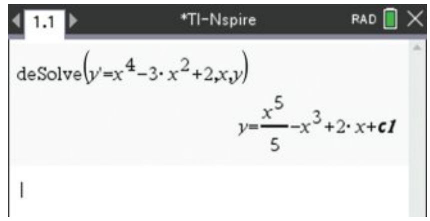
Using the TI-Nspire

a Use (menu) > **Calculus** > **Differential Equation Solver** and complete as shown.

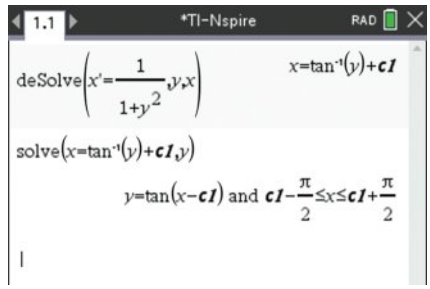
Note: Access the derivative symbol (′) using (ctrl)(⧉) or (π▸).



d ■ Use (menu) > **Calculus** > **Differential Equation Solver** and complete as shown.

Note: This differential equation is of the form $\frac{dx}{dy} = f(y)$, so the roles of the variables x and y are reversed.

■ Solve for y in terms of x.



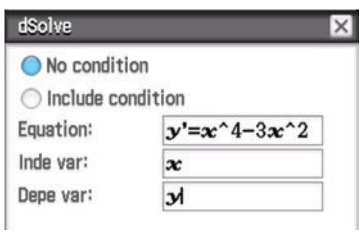
Using the Casio ClassPad

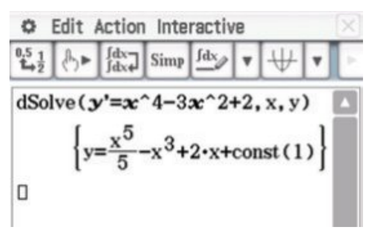
a ■ In Main $\sqrt{\alpha}$, enter and highlight the differential equation $y' = x^4 - 3x^2 + 2$.

Note: The derivative symbol (′) is found in the (Math3) keyboard.

■ Select **Interactive** > **Advanced** > **dSolve**.

■ Enter x for the *Independent variable* and y for the *Dependent variable*. Tap OK.



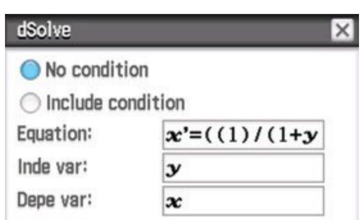


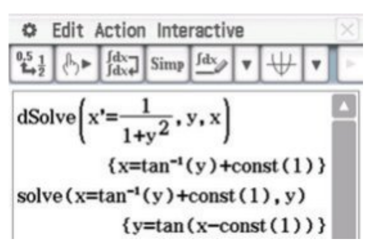
d ■ In Main $\sqrt{\alpha}$, enter and highlight the differential equation $x' = \frac{1}{1+y^2}$.

■ Select **Interactive** > **Advanced** > **dSolve**.

■ Enter y for the *Independent variable* and x for the *Dependent variable*. Tap OK.

■ Solve for y in terms of x.





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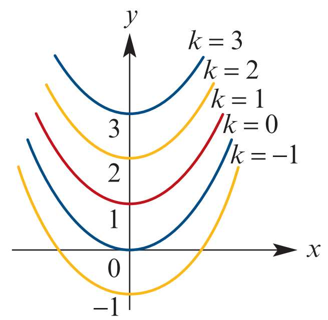
Families of solution curves

Solving a differential equation requires finding an equation that connects the variables, but does not contain a derivative. There are no specific values for the variables. By solving differential equations, it is possible to determine what function or functions might model a particular situation or physical law.

For example, if $\frac{dy}{dx} = x$, then $y = \frac{1}{2}x^2 + k$, where k is a constant.



The **general solution** of the differential equation $\frac{dy}{dx} = x$ is $y = \frac{1}{2}x^2 + k$.

If different values of the constant k are taken, then a family of curves is obtained. This differential equation represents the family of curves $y = \frac{1}{2}x^2 + k$, where $k \in \mathbb{R}$.

A **particular solution** of the differential equation corresponds to a particular curve from the family. This can be found if we know a specific point in the plane through which the curve passes.

For instance, the particular solution of $\frac{dy}{dx} = x$ for which $y = 2$ when $x = 4$ is the solution curve of the differential equation that passes through the point $(4, 2)$.

From above:

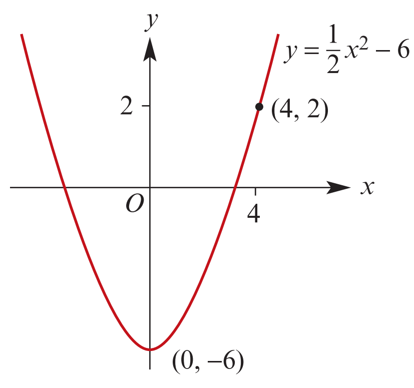
$$y = \tfrac{1}{2}x^2 + k$$

$$\therefore \quad 2 = \tfrac{1}{2} \times 16 + k$$

$$2 = 8 + k$$

$$\therefore \quad k = -6$$

Thus the solution is $y = \frac{1}{2}x^2 - 6$.



Example 6

a Find the family of solution curves for the differential equation $\frac{dy}{dx} = e^{2x}$.

b Find the particular solution if the solution curve passes through the point $(0, 3)$.

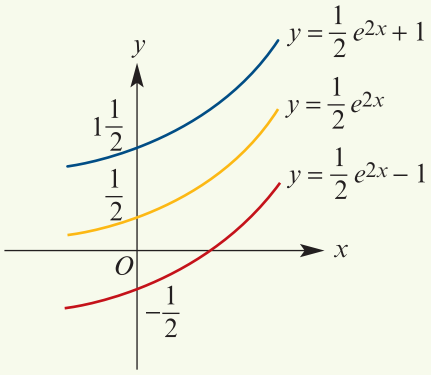
Solution

a $\frac{dy}{dx} = e^{2x}$

$\therefore \; y = \int e^{2x}\, dx$

$= \frac{1}{2}e^{2x} + c$

The general solution $y = \frac{1}{2}e^{2x} + c$ represents a family of curves, since c can take any real number value. The diagram shows some of these curves.



b Substituting $x = 0$ and $y = 3$ into the general equation $y = \frac{1}{2}e^{2x} + c$, we have

$$3 = \frac{1}{2}e^{0} + c$$

$$\therefore \quad c = \frac{5}{2}$$

The equation is $y = \frac{1}{2}e^{2x} + \frac{5}{2}$.

Solving differential equations of the form $\frac{d^2y}{dx^2} = f(x)$

These differential equations are similar to those discussed above, with antidifferentiation being applied twice.

Let $p = \frac{dy}{dx}$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = f(x)$.

The technique involves first finding p as the solution of the differential equation $\frac{dp}{dx} = f(x)$, and then substituting p into $\frac{dy}{dx} = p$ and solving this differential equation.

Example 7

Find the general solution of each of the following:

a $\frac{d^2y}{dx^2} = 10x^3 - 3x + 4$

b $\frac{d^2y}{dx^2} = \cos(3x)$

c $\frac{d^2y}{dx^2} = e^{-x}$

d $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x+1}}$

Solution

a Let $p = \frac{dy}{dx}$.

Then $\frac{dp}{dx} = 10x^3 - 3x + 4$

$$\therefore \quad p = \frac{5x^4}{2} - \frac{3x^2}{2} + 4x + c$$

$$\therefore \quad \frac{dy}{dx} = \frac{5x^4}{2} - \frac{3x^2}{2} + 4x + c$$

$$\therefore \quad y = \frac{x^5}{2} - \frac{x^3}{2} + 2x^2 + cx + d, \quad \text{where } c, d \in \mathbb{R}$$

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b $\frac{d^2y}{dx^2} = \cos(3x)$

Let $p = \frac{dy}{dx}$. Then $\frac{dp}{dx} = \cos(3x)$.

$$\text{Thus} \quad p = \int \cos(3x)\, dx$$
$$= \frac{1}{3}\sin(3x) + c$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{3}\sin(3x) + c$$
$$\therefore \quad y = \int \frac{1}{3}\sin(3x) + c\, dx$$
$$= -\frac{1}{9}\cos(3x) + cx + d, \quad \text{where } c, d \in \mathbb{R}$$

The p substitution can be omitted:

c $\frac{d^2y}{dx^2} = e^{-x}$

$$\therefore \quad \frac{dy}{dx} = \int e^{-x}\, dx$$
$$= -e^{-x} + c$$
$$\therefore \quad y = \int -e^{-x} + c\, dx$$
$$= e^{-x} + cx + d \qquad (c, d \in \mathbb{R})$$

d $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x+1}}$

$$\therefore \quad \frac{dy}{dx} = \int (x+1)^{-\frac{1}{2}}\, dx$$
$$= 2(x+1)^{\frac{1}{2}} + c$$
$$\therefore \quad y = \int 2(x+1)^{\frac{1}{2}} + c\, dx$$
$$= \frac{4}{3}(x+1)^{\frac{3}{2}} + cx + d \qquad (c, d \in \mathbb{R})$$

Example 8

Consider the differential equation $\frac{d^2y}{dx^2} = \cos^2 x$.

a Find the general solution.

b Find the solution given that when $x = 0$, $y = -\frac{1}{8}$ and $\frac{dy}{dx} = 0$.

Solution

a Now $\frac{d^2y}{dx^2} = \cos^2 x$

$$\therefore \quad \frac{dy}{dx} = \int \cos^2 x\, dx$$

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Use the trigonometric identity $\cos(2x) = 2\cos^2 x - 1$:

$$\frac{dy}{dx} = \int \cos^2 x \, dx$$

$$= \int \frac{1}{2}(\cos(2x) + 1)\, dx$$

$$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$$

$$\therefore \quad y = \int \frac{1}{4}\sin(2x) + \frac{1}{2}x + c \, dx$$

Hence $y = -\frac{1}{8}\cos(2x) + \frac{1}{4}x^2 + cx + d$ is the general solution.

b First use that $\frac{dy}{dx} = 0$ when $x = 0$. We have

$$\frac{dy}{dx} = \frac{1}{4}\sin(2x) + \frac{1}{2}x + c \qquad \text{(from } \mathbf{a}\text{)}$$

$$0 = \frac{1}{4}\sin 0 + 0 + c \qquad \text{(substituting given condition)}$$

$$\therefore \quad c = 0$$

$$\therefore \quad y = -\frac{1}{8}\cos(2x) + \frac{1}{4}x^2 + d$$

Now use that $y = -\frac{1}{8}$ when $x = 0$. We substitute and find

$$-\frac{1}{8} = -\frac{1}{8}\cos 0 + 0 + d$$

$$\therefore \quad d = 0$$

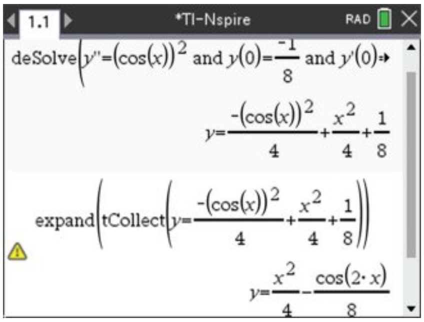
Hence $y = -\frac{1}{8}\cos(2x) + \frac{1}{4}x^2$ is the solution.

Using the TI-Nspire

- Use (menu) > **Calculus** > **Differential Equation Solver** and complete as follows:

  $\text{deSolve}\left(y'' = (\cos(x))^2 \text{ and } y(0) = \frac{-1}{8} \text{ and } y'(0) = 0, x, y\right)$

- The answer can be simplified using **expand** and **tCollect** ((menu) > **Algebra** > **Trigonometry** > **Collect**).



Note: Access the derivative symbol (′) using (ctrl)(📖) or (π▸). To enter the second derivative y'', use the derivative symbol (′) twice.

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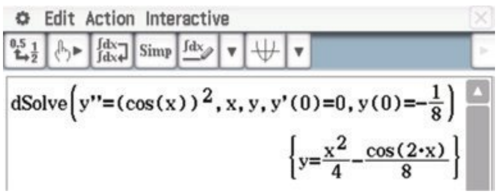
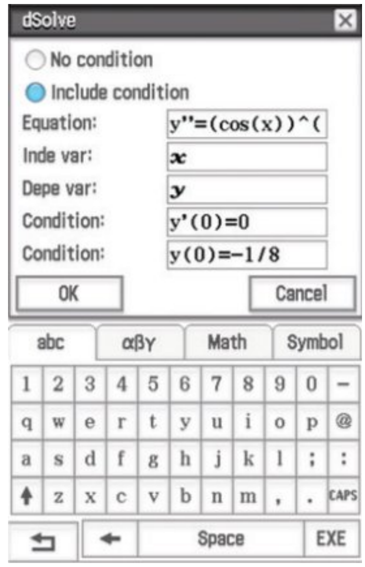
Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, enter and highlight the differential equation $y'' = (\cos(x))^2$.
- Select **Interactive** > **Advanced** > **dSolve**.
- Tap *Include condition*.
- Enter x for *Inde var* and y for *Depe var*.
- Enter the conditions $y'(0) = 0$ and $y(0) = -1/8$.

Note: You must enter y using the abc keyboard.

- Tap OK to obtain the solution.





Skill-sheet Exercise 11B

Example 5

1 Find the general solution of each of the following differential equations:

a $\dfrac{dy}{dx} = x^2 - 3x + 2$ **b** $\dfrac{dy}{dx} = \dfrac{x^2 + 3x - 1}{x}$ **c** $\dfrac{dy}{dx} = (2x + 1)^3$

d $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$ **e** $\dfrac{dy}{dt} = \dfrac{1}{2t - 1}$ **f** $\dfrac{dy}{dt} = \sin(3t - 2)$

g $\dfrac{dy}{dt} = \tan(2t)$ **h** $\dfrac{dx}{dy} = e^{-3y}$ **i** $\dfrac{dx}{dy} = \dfrac{1}{\sqrt{4 - y^2}}$

j $\dfrac{dx}{dy} = -\dfrac{1}{(1 - y)^2}$

Example 7

2 Find the general solution of each of the following differential equations:

a $\dfrac{d^2y}{dx^2} = 5x^3$ **b** $\dfrac{d^2y}{dx^2} = \sqrt{1 - x}$ **c** $\dfrac{d^2y}{dx^2} = \sin\left(2x + \dfrac{\pi}{4}\right)$

d $\dfrac{d^2y}{dx^2} = e^{\frac{x}{2}}$ **e** $\dfrac{d^2y}{dx^2} = \dfrac{1}{\cos^2 x}$ **f** $\dfrac{d^2y}{dx^2} = \dfrac{1}{(x + 1)^2}$

Example 6

3 Find the solution for each of the following differential equations:

a $\dfrac{dy}{dx} = \dfrac{1}{x^2}$, given that $y = \dfrac{3}{4}$ when $x = 4$

b $\dfrac{dy}{dx} = e^{-x}$, given that $y(0) = 0$

c $\dfrac{dy}{dx} = \dfrac{x^2 - 4}{x}$, given that $y = \dfrac{3}{2}$ when $x = 1$

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d $\frac{dy}{dx} = \frac{x}{x^2 - 4}$, given that $y(2\sqrt{2}) = \log_e 2$

e $\frac{dy}{dx} = x\sqrt{x^2 - 4}$, given that $y = \frac{1}{4\sqrt{3}}$ when $x = 4$

f $\frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}$, given that $y(1) = \frac{\pi}{3}$

g $\frac{dy}{dx} = \frac{1}{4 - x^2}$, given that $y = 2$ when $x = 0$

h $\frac{dy}{dx} = \frac{1}{4 + x^2}$, given that $y(2) = \frac{3\pi}{8}$

i $\frac{dy}{dx} = x\sqrt{4 - x}$, given that $y = -\frac{8}{15}$ when $x = 0$

j $\frac{dy}{dx} = \frac{e^x}{e^x + 1}$, given that $y(0) = 0$

Example 8

4 Find the solution for each of the following differential equations:

a $\frac{d^2y}{dx^2} = e^{-x} - e^x$, given that $y(0) = 0$ and that $\frac{dy}{dx} = 0$ when $x = 0$

b $\frac{d^2y}{dx^2} = 2 - 12x$, given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

c $\frac{d^2y}{dx^2} = 2 - \sin(2x)$, given that when $x = 0$, $y = -1$ and $\frac{dy}{dx} = \frac{1}{2}$

d $\frac{d^2y}{dx^2} = 1 - \frac{1}{x^2}$, given that $y(1) = \frac{3}{2}$ and that $\frac{dy}{dx} = 0$ when $x = 1$

e $\frac{d^2y}{dx^2} = \frac{2x}{(1 + x^2)^2}$, given that when $x = 0$, $\frac{dy}{dx} = 0$ and that when $x = 1$, $y = 1$

f $\frac{d^2y}{dx^2} = 24(2x + 1)$, given that $y(-1) = -2$ and that $\frac{dy}{dx} = 6$ when $x = -1$

g $\frac{d^2y}{dx^2} = \frac{x}{(4 - x^2)^{\frac{3}{2}}}$, given that when $x = 0$, $\frac{dy}{dx} = \frac{1}{2}$ and when $x = -2$, $y = -\frac{\pi}{2}$

5 Find the family of curves defined by each of the following differential equations:

a $\frac{dy}{dx} = 3x + 4$ **b** $\frac{d^2y}{dx^2} = -2x$ **c** $\frac{dy}{dx} = \frac{1}{x - 3}$

6 Find the equation of the curve defined by each of the following:

a $\frac{dy}{dx} = 2 - e^{-x}$, $y(0) = 1$ **b** $\frac{dy}{dx} = x + \sin(2x)$, $y(0) = 4$

c $\frac{dy}{dx} = \frac{1}{2 - x}$, $y(3) = 2$

7 Assume that $\frac{dx}{dy}$ is inversely proportional to y. That is, assume that $\frac{dx}{dy} = \frac{k}{y}$, for $k \in \mathbb{R}^+$. Given that when $x = 0$, $y = 2$ and when $x = 2$, $y = 4$, find y when $x = 3$.

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11C Differential equations involving a function of the dependent variable

In this section we solve differential equations of the form

$$\frac{dy}{dx} = g(y)$$

Using the identity $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$, this becomes $\dfrac{dx}{dy} = \dfrac{1}{g(y)}$.

If $\dfrac{dy}{dx} = g(y)$, then $x = \displaystyle\int \frac{1}{g(y)}\,dy$.

Note: This is a special case of separation of variables, which is covered in Section 11F.

Example 9

Find the general solution of each of the following differential equations:

a $\dfrac{dy}{dx} = 2y + 1$, for $y > -\frac{1}{2}$

b $\dfrac{dy}{dx} = e^{2y}$

c $\dfrac{dy}{dx} = \sqrt{1 - y^2}$, for $y \in (-1, 1)$

d $\dfrac{dy}{dx} = 1 - y^2$, for $-1 < y < 1$

Solution

a $\dfrac{dy}{dx} = 2y + 1$ gives $\dfrac{dx}{dy} = \dfrac{1}{2y+1}$

Therefore $x = \displaystyle\int \frac{1}{2y+1}\,dy$

$= \frac{1}{2}\log_e |2y + 1| + k$ where $k \in \mathbb{R}$

$= \frac{1}{2}\log_e(2y + 1) + k$ as $y > -\frac{1}{2}$

So $2(x - k) = \log_e(2y + 1)$

$2y + 1 = e^{2(x-k)}$

i.e. $y = \frac{1}{2}(e^{2(x-k)} - 1)$

This can also be written as $y = \frac{1}{2}(Ae^{2x} - 1)$, where $A = e^{-2k}$.

Note: For $y < -\frac{1}{2}$, the general solution is $y = -\frac{1}{2}(Ae^{2x} + 1)$, where $A = e^{-2k}$.

b $\dfrac{dy}{dx} = e^{2y}$ gives $\dfrac{dx}{dy} = e^{-2y}$

Thus $x = \displaystyle\int e^{-2y}\,dy$

$x = -\frac{1}{2}e^{-2y} + c$

$e^{-2y} = -2(x - c)$

$-2y = \log_e(-2(x - c))$

$\therefore$ $y = -\frac{1}{2}\log_e(-2(x - c))$

$= -\frac{1}{2}\log_e(2c - 2x)$, $x < c$

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c $\dfrac{dy}{dx} = \sqrt{1 - y^2}$ gives $\dfrac{dx}{dy} = \dfrac{1}{\sqrt{1 - y^2}}$

So $\quad x = \displaystyle\int \frac{1}{\sqrt{1 - y^2}}\, dy$

$x = \sin^{-1}(y) + c$

$\therefore \quad y = \sin(x - c)$

d $\dfrac{dy}{dx} = 1 - y^2$ gives $\dfrac{dx}{dy} = \dfrac{1}{1 - y^2}$

Thus $\quad x = \displaystyle\int \frac{1}{1 - y^2}\, dy$

$$= \int \frac{1}{2(1 - y)} + \frac{1}{2(1 + y)}\, dy$$

$$= -\tfrac{1}{2}\log_e(1 - y) + \tfrac{1}{2}\log_e(1 + y) + c \qquad (\text{since } -1 < y < 1)$$

So $\quad x - c = \tfrac{1}{2}\log_e\left(\dfrac{1 + y}{1 - y}\right)$

$$e^{2(x-c)} = \frac{1 + y}{1 - y}$$

Let $A = e^{-2c}$. Then

$$Ae^{2x} = \frac{1 + y}{1 - y}$$

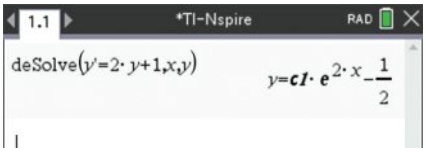
$$Ae^{2x}(1 - y) = 1 + y$$

$$Ae^{2x} - 1 = y(1 + Ae^{2x})$$

$$\therefore \quad y = \frac{Ae^{2x} - 1}{Ae^{2x} + 1}$$

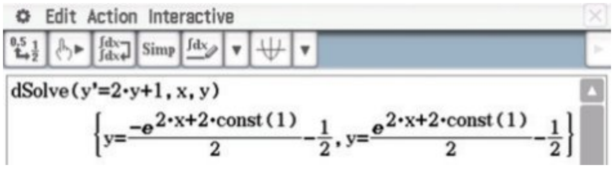
Using the TI-Nspire

Use (menu) > **Calculus** > **Differential Equation Solver** and complete as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the differential equation.
- Go to **Interactive** > **Advanced** > **dSolve**.
- Enter x for *Inde var* and y for *Depe var*.
- Tap OK.



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Constant solutions

When we convert $\frac{dy}{dx} = g(y)$ into $\frac{dx}{dy} = \frac{1}{g(y)}$, we are assuming that $g(y) \neq 0$.

We need to take care when dividing that we do not lose constant solutions. This is demonstrated in the next example.

Example 10

Find the general solution of the differential equation $\frac{dy}{dx} = y - 2$.

Solution

Constant solution

First observe that the constant function $y = 2$ is a solution.

Non-constant solutions

The solution curves do not intersect. Since the line $y = 2$ is a solution curve, every other solution curve must satisfy either $y > 2$ or $y < 2$. Therefore

$$\frac{dy}{dx} = y - 2 \quad \text{gives} \quad \frac{dx}{dy} = \frac{1}{y-2} \qquad \text{since } y \neq 2$$

$$x = \int \frac{1}{y-2}\, dy$$

$$= \log_e |y - 2| + k \qquad \text{where } k \in \mathbb{R}$$

$$x - k = \log_e |y - 2|$$

$$|y - 2| = e^{x-k}$$

This can be written as $|y - 2| = Ae^x$, where $A = e^{-k} \in \mathbb{R}^+$.

There are two cases:

- If $y > 2$, then $y - 2 = Ae^x$ and so $y = Ae^x + 2$.
- If $y < 2$, then $-(y - 2) = Ae^x$ and so $y = -Ae^x + 2$.

In both cases, we can write the solution as $y = Be^x + 2$, for some constant $B \in \mathbb{R} \setminus \{0\}$.

General solution

Note that the constant solution $y = 2$ is covered by the case $B = 0$. Therefore we can write the general solution as $y = Be^x + 2$, for $B \in \mathbb{R}$.

Skill-sheet

Exercise 11C

Example 9

1 Find the general solution of each of the following differential equations:

a $\frac{dy}{dx} = 3y - 5, \quad y > \frac{5}{3}$ **b** $\frac{dy}{dx} = 1 - 2y, \quad y > \frac{1}{2}$ **c** $\frac{dy}{dx} = e^{2y-1}$

d $\frac{dy}{dx} = \cos^2 y, \quad |y| < \frac{\pi}{2}$ **e** $\frac{dy}{dx} = \cot y, \quad y \in \left(0, \frac{\pi}{2}\right)$ **f** $\frac{dy}{dx} = y^2 - 1, \quad |y| < 1$

g $\frac{dy}{dx} = 1 + y^2$ **h** $\frac{dy}{dx} = \frac{1}{5y^2 + 2y}$ **i** $\frac{dy}{dx} = \sqrt{y}, \quad y > 0$

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2 Find the solution for each of the following differential equations:

a $\frac{dy}{dx} = y$, given that $y = e$ when $x = 0$ **b** $\frac{dy}{dx} = y + 1$, given that $y(4) = 0$

c $\frac{dy}{dx} = 2y$, given that $y = 1$ when $x = 1$ **d** $\frac{dy}{dx} = 2y + 1$, given that $y(0) = -1$

e $\frac{dy}{dx} = \frac{e^y}{e^y + 1}$, if $y = 0$ when $x = 0$ **f** $\frac{dy}{dx} = \sqrt{9 - y^2}$, given that $y(0) = 3$

g $\frac{dy}{dx} = 9 - y^2$, if $y = 0$ when $x = \frac{7}{6}$ **h** $\frac{dy}{dx} = 1 + 9y^2$, given that $y\left(-\frac{\pi}{12}\right) = -\frac{1}{3}$

i $\frac{dy}{dx} = \frac{y^2 + 2y}{2}$, given that $y = -4$ when $x = 0$

Example 10

3 Find the general solution of each of the following differential equations. Take care to consider constant solutions separately.

a $\frac{dy}{dx} = y + 3$ **b** $\frac{dy}{dx} = 2y - 1$ **c** $\frac{dy}{dx} = y(y + 1)$ **d** $\frac{dy}{dx} = (y - 3)(y - 4)$

11D Applications of differential equations

Many differential equations arise from scientific or business situations and are constructed from observations and data obtained from experiment.

For example, the following two results from science are described by differential equations:

- **Newton's law of cooling** The rate at which a body cools is proportional to the difference between its temperature and the temperature of its immediate surroundings.
- **Radioactive decay** The rate at which a radioactive substance decays is proportional to the mass of the substance remaining.

These two results will be investigated further in worked examples in this section.

Example 11

The table gives the observed rate of change of a variable x with respect to time t.

t	0	1	2	3	4
$\frac{dx}{dt}$	0	2	8	18	32

a Construct the differential equation which applies to this situation.

b Solve the differential equation to find x in terms of t, given that $x = 2$ when $t = 0$.

Solution

a From the table, it can be established that $\frac{dx}{dt} = 2t^2$.

b Therefore $x = \int 2t^2 \, dt = \frac{2t^3}{3} + c$.

When $t = 0$, $x = 2$. This gives $2 = 0 + c$ and so $c = 2$. Hence $x = \frac{2t^3}{3} + 2$.

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Differential equations can also be constructed from statements, as shown in the following.

Example 12

The population of a city is P at time t years from a certain date. The population increases at a rate that is proportional to the square root of the population at that time. Construct and solve the appropriate differential equation and sketch the population–time graph.

Solution

Remembering that the derivative is a rate, we have $\frac{dP}{dt} \propto \sqrt{P}$. Therefore $\frac{dP}{dt} = k\sqrt{P}$, where k is a constant. Since the population is increasing, we have $k > 0$.

The differential equation is

$$\frac{dP}{dt} = k\sqrt{P}, \quad k > 0$$

Since there are no initial conditions given here, only a general solution for this differential equation can be found. Note that it is of the form $\frac{dy}{dx} = g(y)$.

Now $\frac{dt}{dP} = \frac{1}{k\sqrt{P}}$

$$\therefore \quad t = \frac{1}{k}\int P^{-\frac{1}{2}}\,dP$$

$$= \frac{1}{k}\cdot 2P^{\frac{1}{2}} + c$$

The general solution is

$$t = \frac{2}{k}\sqrt{P} + c \qquad \text{where } c \in \mathbb{R}$$

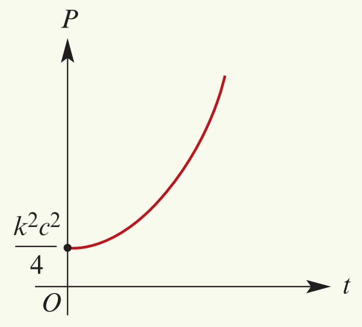
Rearranging to make P the subject:

$$t = \frac{2}{k}\sqrt{P} + c$$

$$\sqrt{P} = \frac{k}{2}(t - c)$$

$$\therefore \quad P = \frac{k^2}{4}(t - c)^2$$

The graph is a section of the parabola $P = \frac{k^2}{4}(t - c)^2$ with vertex at $(c, 0)$.



Example 13

In another city, with population P at time t years after a certain date, the population increases at a rate proportional to the population at that time. Construct and solve the appropriate differential equation and sketch the population–time graph.

Solution

Here $\frac{dP}{dt} \propto P$.

The differential equation is

$$\frac{dP}{dt} = kP, \qquad k > 0$$

$$\therefore \quad \frac{dt}{dP} = \frac{1}{kP}$$

$$\therefore \quad t = \frac{1}{k}\int \frac{1}{P}\, dP$$

$$\therefore \quad t = \frac{1}{k}\log_e P + c \quad \text{since } P > 0$$

This is the general solution.

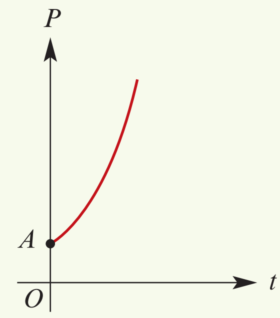
Rearranging to make P the subject:

$$k(t - c) = \log_e P$$

$$e^{k(t-c)} = P$$

$$\therefore \quad P = Ae^{kt}, \quad \text{where } A = e^{-kc}$$

The graph is a section of the exponential curve $P = Ae^{kt}$.



Example 14

Suppose that a tank containing liquid has a vent at the top and an outlet at the bottom through which the liquid drains.

Torricelli's law states that if, at time t seconds after opening the outlet, the depth of the liquid is h m and the surface area of the liquid is A m^2, then

$$\frac{dh}{dt} = \frac{-k\sqrt{h}}{A} \quad \text{where } k > 0$$

(The constant k depends on factors such as the viscosity of the liquid and the cross-sectional area of the outlet.)

Apply Torricelli's law to a cylindrical tank that is initially full, with a height of 1.6 m and a radius length of 0.4 m. Use $k = 0.025$. Construct the appropriate differential equation, solve it and find how many seconds it will take the tank to empty.

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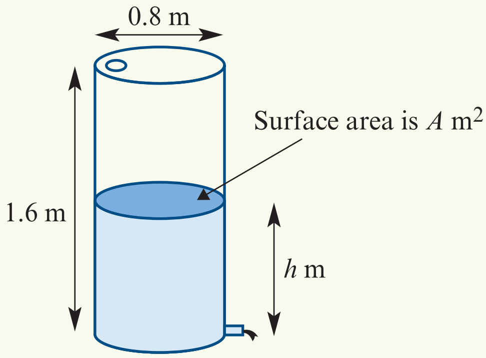
Solution

We start by drawing a diagram.



Since the surface area is a circle with constant area $A = \pi \times 0.4^2$, we have

$$\frac{dh}{dt} = \frac{-0.025\sqrt{h}}{\pi \times 0.4^2} = \frac{-0.025\sqrt{h}}{0.16\pi} = \frac{-5\sqrt{h}}{32\pi}$$

The appropriate differential equation is

$$\frac{dh}{dt} = \frac{-5\sqrt{h}}{32\pi}$$

$$\therefore \quad \frac{dt}{dh} = \frac{-32\pi}{5} \cdot h^{-\frac{1}{2}}$$

$$\therefore \quad t = \frac{-32\pi}{5} \int h^{-\frac{1}{2}}\, dh$$

$$\therefore \quad t = \frac{-32\pi}{5} \cdot 2h^{\frac{1}{2}} + c$$

$$\therefore \quad t = \frac{-64\pi}{5}\sqrt{h} + c$$

Now use the given condition that the tank is initially full: when $t = 0$, $h = 1.6$.

By substitution:

$$0 = \frac{-64\pi}{5}\sqrt{1.6} + c$$

$$\therefore \quad c = \frac{64\pi}{5}\sqrt{1.6}$$

So the particular solution for this differential equation is

$$t = \frac{-64\pi}{5}\sqrt{h} + \frac{64\pi}{5}\sqrt{1.6}$$

$$\therefore \quad t = \frac{-64\pi}{5}(\sqrt{h} - \sqrt{1.6})$$

Now we find the time when the tank is empty. That is, we find t when $h = 0$.

By substitution:

$$t = \frac{64\pi}{5}(\sqrt{1.6})$$

$$\therefore \quad t \approx 50.9$$

It will take approximately 51 seconds to empty this tank.

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The following example uses Newton's law of cooling.

Example 15

An iron bar is placed in a room which has a temperature of 20°C. The iron bar initially has a temperature of 80°C. It cools to 70°C in 5 minutes. Let T be the temperature of the bar at time t minutes.

a Construct a differential equation.

b Solve this differential equation.

c Sketch the graph of T against t.

d How long does it take the bar to cool to 40°C?

Solution

a Newton's law of cooling yields

$$\frac{dT}{dt} = -k(T - 20) \quad \text{where } k \in \mathbb{R}^+$$

(Note the use of the negative sign as the temperature is decreasing.)

b $$\frac{dt}{dT} = \frac{-1}{k(T - 20)}$$

$$\therefore \; t = -\frac{1}{k}\log_e(T - 20) + c, \qquad T > 20$$

When $t = 0$, $T = 80$. This gives

$$0 = -\frac{1}{k}\log_e(80 - 20) + c$$

$$c = \frac{1}{k}\log_e 60$$

$$\therefore \quad t = \frac{1}{k}\log_e\left(\frac{60}{T - 20}\right)$$

When $t = 5$, $T = 70$. This gives

$$\frac{1}{k} = \frac{5}{\log_e(\frac{6}{5})}$$

$$\therefore \quad t = \frac{5}{\log_e(\frac{6}{5})}\log_e\left(\frac{60}{T - 20}\right)$$

This equation can be rearranged to make T the subject:

$$\frac{t}{5} \cdot \log_e\left(\frac{6}{5}\right) = \log_e\left(\frac{60}{T - 20}\right)$$

$$\log_e\left(\left(\frac{6}{5}\right)^{\frac{t}{5}}\right) = \log_e\left(\frac{60}{T - 20}\right)$$

$$\left(\frac{6}{5}\right)^{\frac{t}{5}} = \frac{60}{T - 20}$$

Hence $T = 20 + 60\left(\frac{5}{6}\right)^{\frac{t}{5}}$.

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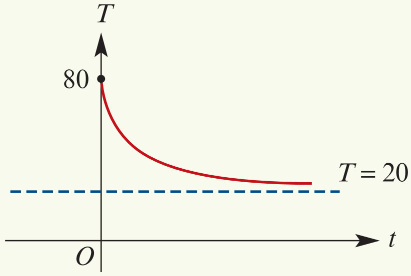
c



d When $T = 40$, we have

$$t = \frac{5}{\log_e\left(\frac{6}{5}\right)} \log_e\left(\frac{60}{40 - 20}\right)$$
$$= 30.1284\ldots$$

The bar reaches a temperature of 40°C after 30.1 minutes.

Difference of rates

Consider the following two situations:

- A population is increasing due to births, but at the same time is diminishing due to deaths.
- A liquid is being poured into a container, while at the same time the liquid is flowing out.

In both of these situations:

rate of change = rate of increase − rate of decrease

For example, if water is flowing into a container at 8 litres per minute and at the same time water is flowing out of the container at 6 litres per minute, then the overall rate of change is $\frac{dV}{dt} = 8 - 6 = 2$, where the volume of water in the container is V litres at time t minutes.

Example 16

A certain radioactive isotope decays at a rate that is proportional to the mass, m kg, present at any time t years. The rate of decay is $2m$ kg per year. The isotope is formed as a byproduct from a nuclear reactor at a constant rate of 0.5 kg per year. None of the isotope was present initially.

a Construct a differential equation.
b Solve the differential equation.
c Sketch the graph of m against t.
d How much isotope is there after two years?

Solution

a $\frac{dm}{dt}$ = rate of increase − rate of decrease $= 0.5 - 2m$

i.e. $\frac{dm}{dt} = \frac{1 - 4m}{2}$

b $\frac{dt}{dm} = \frac{2}{1 - 4m}$

Thus $t = -\frac{2}{4}\log_e|1 - 4m| + c$

$= -\frac{1}{2}\log_e(1 - 4m) + c$ (since $0.5 - 2m > 0$)

When $t = 0$, $m = 0$ and therefore $c = 0$.

So $-2t = \log_e(1 - 4m)$

$e^{-2t} = 1 - 4m$

$\therefore m = \frac{1}{4}(1 - e^{-2t})$

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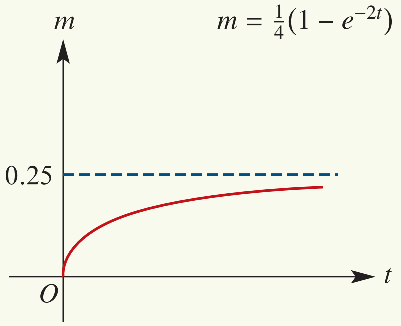
c



d When $t = 2$,

$$m = \tfrac{1}{4}(1 - e^{-4})$$
$$= 0.245\ldots$$

After two years, the mass of the isotope is 0.245 kg.

Example 17

Pure oxygen is pumped into a 50-litre tank of air at 5 litres per minute. The oxygen is well mixed with the air in the tank. The mixture is removed at the same rate.

a Construct a differential equation, given that plain air contains 23% oxygen.

b After how many minutes does the mixture contain 50% oxygen?

Solution

a Let Q litres be the volume of oxygen in the tank at time t minutes.
When $t = 0$, $Q = 50 \times 0.23 = 11.5$.

$$\frac{dQ}{dt} = \text{rate of inflow} - \text{rate of outflow}$$
$$= 5 - \frac{Q}{50} \times 5$$

i.e. $$\frac{dQ}{dt} = \frac{50 - Q}{10}$$

b $$\frac{dt}{dQ} = \frac{10}{50 - Q}$$

$$\therefore \quad t = -10\log_e|50 - Q| + c$$
$$= -10\log_e(50 - Q) + c \qquad (\text{as } Q < 50)$$

When $t = 0$, $Q = 11.5$. Therefore

$$c = 10\log_e(38.5)$$

$$\therefore \quad t = 10\log_e\left(\frac{77}{2(50 - Q)}\right)$$

When the mixture is 50% oxygen, we have $Q = 25$ and so

$$t = 10\log_e\left(\frac{77}{2 \times 25}\right)$$
$$= 10\log_e\left(\frac{77}{50}\right)$$
$$= 4.317\ldots$$

The tank contains 50% oxygen after 4 minutes and 19.07 seconds.

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Exercise 11D

Example 11

1 Each of the following tables gives the results of an experiment where a rate of change was found to be a linear function of time, i.e. $\dfrac{dx}{dt} = at + b$. For each table, set up a differential equation and solve it using the additional information.

a

t	0	1	2	3
$\dfrac{dx}{dt}$	1	3	5	7

and $x(0) = 3$

b

t	0	1	2	3
$\dfrac{dx}{dt}$	−1	2	5	8

and $x(1) = 1$

c

t	0	1	2	3
$\dfrac{dx}{dt}$	8	6	4	2

and $x(2) = -3$

2 For each of the following, construct (but do not attempt to solve) a differential equation:

a A family of curves is such that the gradient at any point (x, y) is the reciprocal of the y-coordinate (for $y \neq 0$).

b A family of curves is such that the gradient at any point (x, y) is the square of the reciprocal of the y-coordinate (for $y \neq 0$).

c The rate of increase of a population of size N at time t years is inversely proportional to the square of the population.

d A particle moving in a straight line is x m from a fixed point O after t seconds. The rate at which the particle is moving is inversely proportional to the distance from O.

e The rate of decay of a radioactive substance is proportional to the mass of substance remaining. Let m kg be the mass of the substance at time t minutes.

f The gradient of the normal to a curve at any point (x, y) is three times the gradient of the line joining the same point to the origin.

Example 12
Example 13

3 A city, with population P at time t years after a certain date, has a population which increases at a rate proportional to the population at that time.

a **i** Set up a differential equation to describe this situation.

ii Solve to obtain a general solution.

b If the initial population was 1000 and after two years the population had risen to 1100:

i find the population after five years

ii sketch a graph of P against t.

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Example 14

4 An island has a population of rabbits of size P at time t years after 1 January 2010. Due to a virus, the population is decreasing at a rate proportional to the square root of the population at that time.

a **i** Set up a differential equation to describe this situation.

ii Solve to obtain a general solution.

b If the population was initially 15 000 and decreased to 13 500 after five years:

i find the population after 10 years

ii sketch a graph of P against t.

5 A city has population P at time t years from a certain date. The population increases at a rate inversely proportional to the population at that time.

a **i** Set up a differential equation to describe this situation.

ii Solve to obtain a general solution.

b Initially the population was 1 000 000, but after four years it had risen to 1 100 000.

i Find an expression for the population in terms of t.

ii Sketch the graph of P against t.

6 A curve has the property that its gradient at any point is one-tenth of the y-coordinate at that point. It passes through the point (0, 10). Find the equation of the curve.

Example 15

7 A body at a temperature of 80°C is placed in a room which is kept at a constant temperature of 20°C. After 20 minutes, the temperature of the body is 60°C. Assuming Newton's law of cooling, find the temperature after a further 20 minutes.

8 If the thermostat in an electric heater fails, the rate of increase in its temperature, $\frac{d\theta}{dt}$, is 0.01θ K per minute, where the temperature θ is measured in kelvins (K) and the time t in minutes. If the heater is switched on at a room temperature of 300 K and the thermostat does not function, what is the temperature of the heater after 10 minutes?

9 The rate of decay of a radioactive substance is proportional to the amount Q of matter present at any time t. The differential equation for this situation is $\frac{dQ}{dt} = -kQ$, where k is a constant. Given that $Q = 50$ when $t = 0$ and that $Q = 25$ when $t = 10$, find the time t at which $Q = 10$.

10 The rate of decay of a substance is km, where k is a positive constant and m is the mass of the substance remaining. Show that the half-life (i.e. the time in which the amount of the original substance remaining is halved) is given by $\frac{1}{k}\log_e 2$.

11 The concentration, x grams per litre, of salt in a solution at time t minutes is given by $\frac{dx}{dt} = \frac{20 - 3x}{30}$.

a If the initial concentration was 2 grams per litre, solve the differential equation, giving x in terms of t.

b Find the time taken, to the nearest minute, for the salt concentration to rise to 6 grams per litre.

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12 If $\frac{dy}{dx} = 10 - \frac{y}{10}$ and $y = 10$ when $x = 0$, find y in terms of x. Sketch the graph of the equation for $x \geq 0$.

13 The number n of bacteria in a colony grows according to the law $\frac{dn}{dt} = kn$, where k is a positive constant. If the number increases from 4000 to 8000 in four days, find, to the nearest hundred, the number of bacteria after three days more.

14 A town had a population of 10 000 in 2010 and 12 000 in 2020. If the population is N at a time t years after 2010, find the predicted population in the year 2030 assuming:

a $\frac{dN}{dt} \propto N$ **b** $\frac{dN}{dt} \propto \frac{1}{N}$ **c** $\frac{dN}{dt} \propto \sqrt{N}$

15 For each of the following, construct a differential equation, but do not solve it:

a Water is flowing into a tank at a rate of 0.3 m^3 per hour. At the same time, water is flowing out through a hole in the bottom of the tank at a rate of $0.2\sqrt{V}$ m^3 per hour, where V m^3 is the volume of the water in the tank at time t hours. (Find an expression for $\frac{dV}{dt}$.)

b A tank initially contains 200 litres of pure water. A salt solution containing 5 kg of salt per litre is added at the rate of 10 litres per minute, and the mixed solution is drained simultaneously at the rate of 12 litres per minute. There is m kg of salt in the tank after t minutes. (Find an expression for $\frac{dm}{dt}$.)

c A partly filled tank contains 200 litres of water in which 1500 grams of salt have been dissolved. Water is poured into the tank at a rate of 6 L/min. The mixture, which is kept uniform by stirring, leaves the tank through a hole at a rate of 5 L/min. There is x grams of salt in the tank after t minutes. (Find an expression for $\frac{dx}{dt}$.)

Example 16

16 A certain radioactive isotope decays at a rate that is proportional to the mass, m kg, present at any time t years. The rate of decay is m kg per year. The isotope is formed as a byproduct from a nuclear reactor at a constant rate of 0.25 kg per year. None of the isotope was present initially.

a Construct a differential equation.

b Solve the differential equation.

c Sketch the graph of m against t.

d How much isotope is there after two years?

Example 17

17 A tank holds 100 litres of water in which 20 kg of sugar was dissolved. Water runs into the tank at the rate of 1 litre per minute. The solution is continually stirred and, at the same time, the solution is being pumped out at 1 litre per minute. At time t minutes, there is m kg of sugar in the solution.

a At what rate is the sugar being removed at time t minutes?

b Set up a differential equation to represent this situation.

c Solve the differential equation.

d Sketch the graph of m against t.

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18 A tank holds 100 litres of pure water. A sugar solution containing 0.25 kg per litre is being run into the tank at the rate of 1 litre per minute. The liquid in the tank is continuously stirred and, at the same time, liquid from the tank is being pumped out at the rate of 1 litre per minute. After t minutes, there is m kg of sugar dissolved in the solution.

- **a** At what rate is the sugar being added to the solution at time t?
- **b** At what rate is the sugar being removed from the tank at time t?
- **c** Construct a differential equation to represent this situation.
- **d** Solve this differential equation.
- **e** Find the time taken for the concentration in the tank to reach 0.1 kg per litre.
- **f** Sketch the graph of m against t.

19 A laboratory tank contains 100 litres of a 20% serum solution (i.e. 20% of the contents is pure serum and 80% is distilled water). A 10% serum solution is then pumped in at the rate of 2 litres per minute, and an amount of the solution currently in the tank is drawn off at the same rate.

- **a** Set up a differential equation to show the relation between x and t, where x litres is the amount of pure serum in the tank at time t minutes.
- **b** How long will it take for there to be an 18% solution in the tank? (Assume that at all times the contents of the tank form a uniform solution.)

20 A tank initially contains 400 litres of water in which is dissolved 10 kg of salt. A salt solution of concentration 0.2 kg/L is poured into the tank at the rate of 2 L/min. The mixture, which is kept uniform by stirring, flows out at the rate of 2 L/min.

- **a** If the mass of salt in the tank is x kg after t minutes, set up and solve the differential equation for x in terms of t.
- **b** If instead the mixture flows out at 1 L/min, set up (but do not solve) the differential equation for the mass of salt in the tank.

21 A tank contains 20 litres of water in which 10 kg of salt is dissolved. Pure water is poured in at a rate of 2 litres per minute, mixing occurs uniformly (owing to stirring) and the water is released at 2 litres per minute. The mass of salt in the tank is x kg at time t minutes.

- **a** Construct a differential equation representing this information, expressing $\frac{dx}{dt}$ as a function of x.
- **b** Solve the differential equation.
- **c** Sketch the mass–time graph.
- **d** How long will it take the original mass of salt to be halved?

22 A country's population N at time t years after 1 January 2020 changes according to the differential equation $\frac{dN}{dt} = 0.1N - 5000$. (There is a 10% growth rate and 5000 people leave the country every year.)

- **a** Given that the population was 5 000 000 at the start of 2020, find N in terms of t.
- **b** In which year will the country have a population of 10 million?

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11E The logistic differential equation

In the previous section, we modelled the growth of a population, P, over time, t, using a differential equation of the form

$$\frac{dP}{dt} = kP$$

The solution is $P = P_0 e^{kt}$, where P_0 is the initial population.

This exponential growth model can be appropriate for a short time. However, it is not realistic over a long period of time. This model implies that the population will grow without limit. But the population will be limited by the available resources, such as food and space.

We need a model which acknowledges that there is an upper limit to growth.

Example 18

A population grows according to the differential equation

$$\frac{dP}{dt} = 0.025P\left(1 - \frac{P}{1000}\right), \quad 0 < P < 1000$$

where P is the population at time t. When $t = 0$, $P = 20$.

a Find the population P at time t.

b Sketch the graph of P against t.

c Find the population P when the rate of growth is at a maximum.

Solution

a Write $\dfrac{dP}{dt} = \dfrac{P(1000 - P)}{40\,000}$

Then

$$\begin{aligned} t &= \int \frac{40\,000}{P(1000 - P)}\, dP \\ &= 40 \int \frac{1}{P} + \frac{1}{1000 - P}\, dP \\ &= 40\left(\log_e |P| - \log_e |1000 - P|\right) + c \\ &= 40 \log_e\left(\frac{P}{1000 - P}\right) + c \qquad \text{since } 0 < P < 1000 \end{aligned}$$

$$\therefore \quad e^{\frac{t-c}{40}} = \frac{P}{1000 - P}$$

Let $A = e^{-\frac{c}{40}}$. Then we have

$$Ae^{\frac{t}{40}} = \frac{P}{1000 - P}$$

When $t = 0$, $P = 20$. This implies that $A = \dfrac{1}{49}$, and so

$$\begin{aligned} (1000 - P)e^{\frac{t}{40}} &= 49P \\ 1000e^{\frac{t}{40}} &= 49P + Pe^{\frac{t}{40}} \end{aligned}$$

Hence $P = \dfrac{1000e^{\frac{t}{40}}}{49 + e^{\frac{t}{40}}}$.

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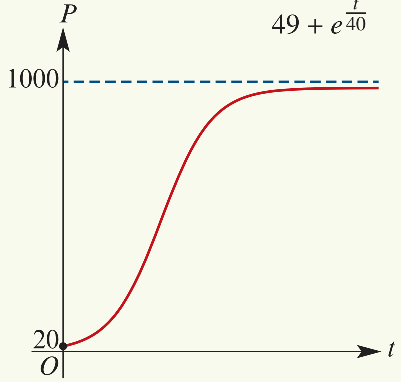
b $P = \dfrac{1000e^{\frac{t}{40}}}{49 + e^{\frac{t}{40}}}$



c The maximum rate of increase occurs at the point of inflection on the graph.
We have

$$\frac{dP}{dt} = \frac{1000P - P^2}{40\,000}$$

The chain rule gives

$$\frac{d^2P}{dt^2} = \frac{1000 - 2P}{40\,000} \cdot \frac{dP}{dt}$$

Since $0 < P < 1000$, we have $\dfrac{dP}{dt} \neq 0$.

Therefore $\dfrac{d^2P}{dt^2} = 0$ implies $P = 500$.

Note: Since $\dfrac{dP}{dt}$ is a quadratic in P, the maximum rate of increase occurs at the vertex of the parabola, which is midway between its intercepts at $P = 0$ and $P = 1000$.

Logistic differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad 0 < P < K$$

This differential equation can be used to model a population P at time t, where:

- the constant r is called the **growth parameter**
- the constant K is called the **carrying capacity**.

Notes:

- As in the example, we can show that the solution of this differential equation is

$$P(t) = \frac{P_0K}{P_0 + (K - P_0)e^{-rt}} = \frac{P_0Ke^{rt}}{P_0e^{rt} + (K - P_0)} \quad \text{where } P_0 = P(0)$$

- The carrying capacity K is the upper limit on the population: the rate of increase approaches 0 as P approaches K; the population P approaches K as $t \to \infty$.
- The maximum rate of increase occurs when $P = \dfrac{K}{2}$.

Exercise 11E

1 Solve the differential equation $\dfrac{dP}{dt} = P(1 - P)$, where $P(0) = 2$.

Example 18

2 A population grows according to the differential equation

$$\frac{dP}{dt} = 0.02P\left(1 - \frac{P}{500}\right), \quad 0 < P < 500$$

where P is the population at time t. When $t = 0$, $P = 100$.

a Find the population P at time t.
b Sketch the graph of P against t.
c Find the population P when the rate of growth is at a maximum.

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3 Let $P(t)$ be the population of a species of fish in a lake after t years. Suppose that $P(t)$ is modelled by a logistic differential equation with a growth parameter of $r = 0.3$ and a carrying capacity of $K = 10\,000$.

- **a** Write down the logistic differential equation for this situation.
- **b** If $P(0) = 2500$, solve the differential equation for $P(t)$.
- **c** Sketch the graph of $P(t)$ against t.
- **d** Find the number of fish in the lake after 5 years.
- **e** Find the time that it will take for there to be 5000 fish in the lake.

4 A population of wasps is growing according to the logistic differential equation, where P is the number of wasps after t months. If the carrying capacity is 500 and the growth parameter is 0.1, what is the maximum possible growth rate for the population?

5 A population of bacteria grows according to the differential equation

$$\frac{dP}{dt} = 0.05P(1 - 0.001P), \quad P_0 = 300, \quad 0 < P < 1000$$

Find the population P at time t.

6 Suppose that t weeks after the start of an epidemic in a certain community, the number of people who have caught the disease, $P(t)$, is given by the logistic function

$$P(t) = \frac{2000}{5 + 395e^{-\frac{4t}{5}}}$$

- **a** How many people had the disease when the epidemic began?
- **b** Approximately how many people in total will get the disease?
- **c** When was the disease spreading most rapidly?
- **d** How fast was the disease spreading at the peak of the epidemic?
- **e** At what rate was the disease spreading when 300 people had caught the disease?

7 Consider the differential equation $\dfrac{dP}{dt} = 0.01P\left(1 - \dfrac{P}{1000}\right)$. For each of the following cases, solve the differential equation and sketch the graph of P against t:

a $P_0 = 1500$ and $P > 1000$ **b** $P_0 = 200$ and $0 < P < 1000$ **c** $P_0 = 1000$

8 A population of rabbits grows in a way described by the logistic differential equation

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{25\,000}\right)$$

where P is the number of rabbits after t months, and the initial population is $P_0 = 2000$.

- **a** Solve the differential equation for P.
- **b** How many rabbits are there after:
  - **i** 6 months **ii** 5 years?
- **c** After how many months is the population increasing most rapidly?
- **d** How long does it take for the population to reach 20 000?
- **e** Sketch the graph of P against t.

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9 Consider the differential equation

$$\frac{dy}{dx} = -\left(1 - \frac{y}{K_1}\right)\left(1 - \frac{y}{K_2}\right)$$

where K_1 and K_2 are positive constants. Taking $K_1 = 5$ and $K_2 = 10$, solve the differential equation for each of the following cases:

a $y(0) = 20, \ y > 10$ **b** $y(0) = 8, \ 5 < y < 10$ **c** $y(0) = 3, \ 0 < y < 5$

11F Separation of variables

A first-order differential equation is **separable** if it can be written in the form

$$\frac{dy}{dx} = f(x)\,g(y)$$

Divide both sides by $g(y)$ (for $g(y) \neq 0$):

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrate both sides with respect to x:

$$\begin{aligned}\int f(x)\,dx &= \int \frac{1}{g(y)}\frac{dy}{dx}\,dx \\ &= \int \frac{1}{g(y)}\,dy\end{aligned}$$

If $\dfrac{dy}{dx} = f(x)\,g(y)$, then $\displaystyle\int f(x)\,dx = \int \frac{1}{g(y)}\,dy$.

Example 19

Solve the differential equation $\dfrac{dy}{dx} = e^{2x}(1 + y^2)$.

Solution

First we write the equation in the form

$$\int f(x)\,dx = \int \frac{1}{g(y)}\,dy$$

i.e. $$\int e^{2x}\,dx = \int \frac{1}{1 + y^2}\,dy$$

Integrating gives

$$\frac{1}{2}e^{2x} + c_1 = \tan^{-1}(y) + c_2$$

Solve for y:

$$\tan^{-1}(y) = \frac{1}{2}e^{2x} + c \qquad \text{(where } c = c_1 - c_2\text{)}$$

$$\therefore \quad y = \tan\left(\frac{1}{2}e^{2x} + c\right)$$

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Example 20

Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{\sin^2 x}{y^2}$$

that also satisfies $y(0) = 1$.

Solution

First we write the equation in the form

$$\int f(x)\, dx = \int \frac{1}{g(y)}\, dy$$

i.e. $\displaystyle \int \sin^2 x\, dx = \int y^2\, dy$

Left-hand side

We use the trigonometric identity $\cos(2x) = 1 - 2\sin^2 x$, which transforms to

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$\therefore \quad \int \sin^2 x\, dx = \frac{1}{2}\int 1 - \cos(2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + c_1$$

Right-hand side

$$\int y^2\, dy = \frac{y^3}{3} + c_2$$

General solution

We now obtain

$$\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + c_1 = \frac{y^3}{3} + c_2$$

$$\therefore \quad \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) = \frac{y^3}{3} + c \qquad (\text{where } c = c_2 - c_1)$$

Particular solution

By substituting $y(0) = 1$, we find that $c = -\dfrac{1}{3}$. Hence

$$\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) = \frac{y^3}{3} - \frac{1}{3}$$

Making y the subject:

$$y^3 = 3\left(\tfrac{1}{2}(x - \tfrac{1}{2}\sin(2x)) + \tfrac{1}{3}\right)$$

$$\therefore \quad y = \sqrt[3]{\tfrac{3}{2}(x - \tfrac{1}{2}\sin(2x)) + 1}$$

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Example 21

A tank contains 30 litres of a solution of a chemical in water. The concentration of the chemical is reduced by running pure water into the tank at a rate of 1 litre per minute and allowing the solution to run out of the tank at a rate of 2 litres per minute. The tank contains x litres of the chemical at time t minutes after the dilution starts.

a Show that $\dfrac{dx}{dt} = \dfrac{-2x}{30-t}$.

b Find the general solution of this differential equation.

c Find the fraction of the original chemical still in the tank after 20 minutes.

Solution

a At time t minutes, the volume of solution in the tank is $30 - t$ litres, since solution is flowing out at 2 litres per minute and water is flowing in at 1 litre per minute.

At time t minutes, the fraction of the solution which is the chemical is $\dfrac{x}{30-t}$.

Hence the rate of flow of the chemical out of the tank is $2 \cdot \dfrac{x}{30-t}$.

Therefore $\dfrac{dx}{dt} = \dfrac{-2x}{30-t}$.

b Using separation of variables, we have

$$\int \frac{1}{30-t}\,dt = \int \frac{-1}{2x}\,dx$$

$$\therefore \quad -\log_e(30-t) + c_1 = -\tfrac{1}{2}\log_e x + c_2$$

$$\therefore \quad \log_e x = 2\log_e(30-t) + c \qquad (\text{where } c = 2c_2 - 2c_1)$$

Let A_0 be the initial amount of chemical in the solution.
Thus $x = A_0$ when $t = 0$, and therefore

$$c = \log_e(A_0) - 2\log_e(30) = \log_e\left(\frac{A_0}{900}\right)$$

Hence

$$\log_e x = 2\log_e(30-t) + \log_e\left(\frac{A_0}{900}\right)$$

$$\log_e x = \log_e\left(\frac{A_0}{900}(30-t)^2\right)$$

$$\therefore \quad x = \frac{A_0}{900}(30-t)^2$$

c When $t = 20$, $x = \dfrac{1}{9}A_0$. The amount of chemical is one-ninth of the original amount.

Notes on separation of variables

- We observe that differential equations of the form $\dfrac{dy}{dx} = g(y)$ can also be solved by separation of variables if $g(y) \neq 0$. The solution will be given by $\displaystyle\int \frac{1}{g(y)}\,dy = \int 1\,dx$.
- When undertaking separation of variables, be careful that you do not lose solutions when dividing. For example, the differential equation $\dfrac{dy}{dx} = y - 2$ has a constant solution $y = 2$.

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Skill-sheet

Exercise 11F

Example 19

1 Find the general solution of each of the following:

a $\frac{dy}{dx} = yx, \quad y > 0$ **b** $\frac{dy}{dx} = \frac{x}{y}$ **c** $\frac{4}{x^2}\frac{dy}{dx} = y, \quad y > 0$ **d** $\frac{dy}{dx} = \frac{1}{xy}$

2 Determine the general solution of the differential equation

$$\frac{dy}{dx} = xy^2$$

using separation of variables. First consider what happens if $y = 0$ as a separate case.

3 Determine the general solution of the differential equation

$$\frac{dy}{dx} = y \sin x - \sin x$$

using separation of variables. First consider what happens if $y = 1$ as a separate case.

4 Determine the general solution of the differential equation

$$\frac{dy}{dx} = 2x(1 - y)^2$$

using separation of variables. First consider what happens if $y = 1$ as a separate case.

Example 20

5 **a** Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, given that $y(1) = 1$.

b Solve the differential equation $\frac{dy}{dx} = \frac{y}{x}$, given that $y(1) = 1$.

c Sketch the graphs of both solutions on the one set of axes.

6 Solve $(1 + x^2)\frac{dy}{dx} = 4xy$ if $y = 2$ when $x = 1$.

7 Find the equation of the curve which satisfies the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and passes through the point $(2, 3)$.

8 Solve the differential equation $\frac{dy}{dx} = \frac{x + 1}{3 - y}$ and describe the solution curves.

9 Find the general solution of the differential equation $y^2\frac{dy}{dx} = \frac{1}{x^3}$.

10 Find the general solution of the differential equation $x^3\frac{dy}{dx} = y^2(x - 3), \ y \neq 0$.

11 Find the general solution of each of the following:

a $\frac{dy}{dx} = y(1 + e^x)$ **b** $\frac{dy}{dx} = 9x^2 y$ **c** $\frac{4}{y^3}\frac{dy}{dx} = \frac{1}{x}$

d $\frac{dy}{dx} = \frac{\log_e x}{yx}$ **e** $\frac{dy}{dx} = yxe^{x^2}$ **f** $\frac{dy}{dx} = 2y^2 x\sqrt{1 - x^2}$

12 Solve each of the following differential equations:

a $y\frac{dy}{dx} = 1 + x^2, \quad y(0) = 1$ **b** $x^2\frac{dy}{dx} = \cos^2 y, \quad y(1) = \frac{\pi}{4}$

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13 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x^2 - x}{y^2 - y}$.

Example 21 **14** A tank contains 50 litres of a solution of a chemical in water. The concentration of the chemical is reduced by running pure water into the tank at a rate of 2 litres per minute and allowing the solution to run out of the tank at a rate of 4 litres per minute. The tank contains x litres of the chemical at time t minutes after the dilution starts.

a Show that $\frac{dx}{dt} = \frac{-4x}{50 - 2t}$.

b Find the general solution of this differential equation.

c Find the fraction of the original chemical still in the tank after 10 minutes.

15 Bacteria in a tank of water increase at a rate proportional to the number present. Water is drained out of the tank, initially containing 100 litres, at a steady rate of 2 litres per hour. Let N be the number of bacteria present at time t hours after the draining starts.

a Show that $\frac{dN}{dt} = kN - \frac{2N}{100 - 2t}$.

b If $k = 0.6$ and at $t = 0$, $N = N_0$, find in terms of N_0 the number of bacteria after 24 hours.

16 Solve the differential equation $x\frac{dy}{dx} = y + x^2y$, given that $y = 2\sqrt{e}$ when $x = 1$.

17 Find y in terms of x if $\frac{dy}{dx} = (1 + y)^2 \sin^2 x \cos x$ and $y = 2$ when $x = 0$.

11G Differential equations with related rates

In Chapter 8, the concept of related rates was introduced. This is a useful technique for constructing and solving differential equations in a variety of situations.

Example 22

For the variables x, y and t, it is known that $\frac{dx}{dt} = \tan t$ and $y = 3x$.

a Find $\frac{dy}{dt}$ as a function of t.

b Find the solution of the resulting differential equation.

Solution

a We are given that $y = 3x$ and $\frac{dx}{dt} = \tan t$.

Using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = 3\tan t$$

b $\frac{dy}{dt} = \frac{3\sin t}{\cos t}$

Let $u = \cos t$. Then $\frac{du}{dt} = -\sin t$.

$$\therefore \quad y = -3\int \frac{1}{u}\,du$$

$$= -3\log_e |u| + c$$

$$\therefore \quad y = -3\log_e |\cos t| + c$$

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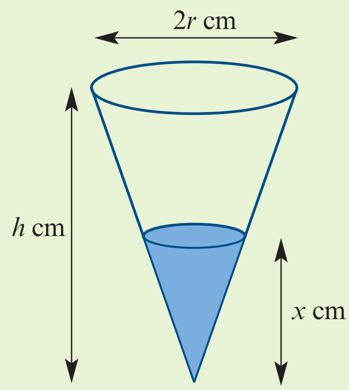
Example 23

An inverted cone has height h cm and radius length r cm. It is being filled with water, which is flowing from a tap at k litres per minute. The depth of water in the cone is x cm at time t minutes.

Construct an appropriate differential equation for $\dfrac{dx}{dt}$ and solve it, given that initially the cone was empty.



Solution

Let V cm^3 be the volume of water at time t minutes.

Since k litres is equal to $1000k$ cm^3, the given rate of change is $\dfrac{dV}{dt} = 1000k$, where $k > 0$.

To find an expression for $\dfrac{dx}{dt}$, we can use the chain rule:

$$\frac{dx}{dt} = \frac{dx}{dV}\,\frac{dV}{dt} \qquad (1)$$

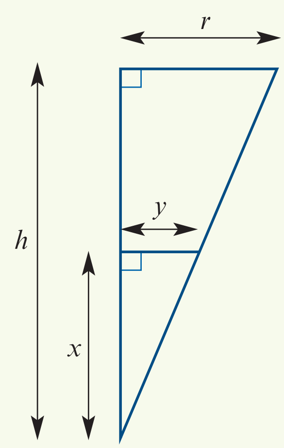
To find $\dfrac{dx}{dV}$, we first need to establish the relationship between x and V.

The formula for the volume of a cone gives

$$V = \frac{1}{3}\pi y^2 x \qquad (2)$$

where y cm is the radius length of the surface when the depth is x cm.

By similar triangles:



$$\frac{y}{r} = \frac{x}{h}$$

$$\therefore \quad y = \frac{rx}{h}$$

So $\quad V = \dfrac{1}{3}\pi \cdot \dfrac{r^2x^2}{h^2} \cdot x \qquad$ (substitution into (2))

$$\therefore \quad V = \frac{\pi r^2}{3h^2} \cdot x^3$$

$\therefore \quad \dfrac{dV}{dx} = \dfrac{\pi r^2}{h^2} \cdot x^2 \qquad$ (by differentiation)

$$\therefore \quad \frac{dx}{dV} = \frac{h^2}{\pi r^2} \cdot \frac{1}{x^2}$$

So $\quad \dfrac{dx}{dt} = \dfrac{h^2}{\pi r^2} \cdot \dfrac{1}{x^2} \cdot 1000k \qquad$ (substitution into (1))

$\therefore \quad \dfrac{dx}{dt} = \dfrac{1000kh^2}{\pi r^2} \cdot \dfrac{1}{x^2} \qquad$ where $k > 0$

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To solve this differential equation:

$$\frac{dt}{dx} = \frac{\pi r^2}{1000kh^2} \cdot x^2$$

$$\therefore \quad t = \frac{\pi r^2}{1000kh^2} \int x^2 \, dx$$

$$= \frac{\pi r^2}{1000kh^2} \cdot \frac{x^3}{3} + c$$

$$\therefore \quad t = \frac{\pi r^2 x^3}{3000kh^2} + c$$

The cone was initially empty, so $x = 0$ when $t = 0$, and therefore $c = 0$.

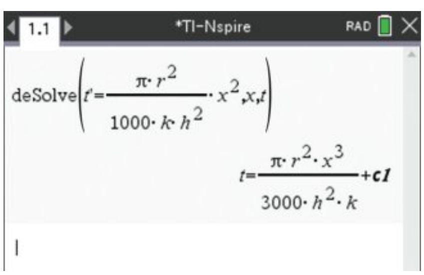
$$\therefore \quad t = \frac{\pi r^2 x^3}{3000kh^2}$$

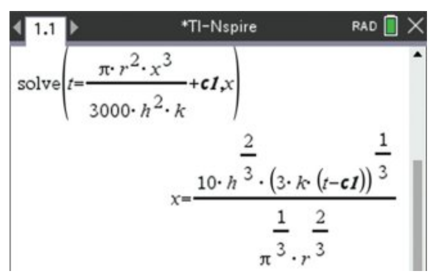
$$\therefore \quad x^3 = \frac{3000kh^2 t}{\pi r^2}$$

Hence $x = \sqrt[3]{\dfrac{3000kh^2 t}{\pi r^2}}$ is the solution of the differential equation.

Using the TI-Nspire

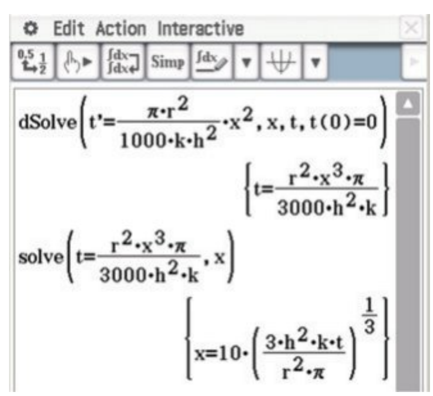
- Use (menu) > **Calculus** > **Differential Equation Solver** and complete as shown.
- Solve for x in terms of t.





Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the differential equation $t' = \dfrac{\pi r^2}{1000kh^2} \times x^2$.
- Select **Interactive** > **Advanced** > **dSolve**.
- Tap *Include condition*.
- Enter x for *Inde var* and t for *Depe var*.
- Enter the condition $t(0) = 0$. (You must select t from the (abc) keyboard.) Tap OK.
- Copy the answer to the next entry line and solve for x.



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Skill-sheet

Exercise 11G

1 Construct, but do not solve, a differential equation for each of the following:

a An inverted cone with depth 50 cm and radius 25 cm is initially full of water, which drains out at 0.5 litres per minute. The depth of water in the cone is h cm at time t minutes. (Find an expression for $\frac{dh}{dt}$.)

b A tank with a flat bottom and vertical sides has a constant horizontal cross-section of A m^2. The tank has a tap in the bottom through which water is leaving at a rate of $c\sqrt{h}$ m^3 per minute, where h m is the height of the water in the tank and c is a constant. Water is being poured into the tank at a rate of Q m^3 per minute. (Find an expression for $\frac{dh}{dt}$.)

c Water is flowing at a constant rate of 0.3 m^3 per hour into a tank. At the same time, water is flowing out through a hole in the bottom of the tank at the rate of $0.2\sqrt{V}$ m^3 per hour, where V m^3 is the volume of the water in the tank at time t hours. It is known that $V = 6\pi h$, where h m is the height of the water at time t. (Find an expression for $\frac{dh}{dt}$.)

d A cylindrical tank 4 m high with base radius 1.5 m is initially full of water. The water starts flowing out through a hole at the bottom of the tank at the rate of $\sqrt{h}$ m^3 per hour, where h m is the depth of water remaining in the tank after t hours. (Find an expression for $\frac{dh}{dt}$.)

Example 22

2 For the variables x, y and t, it is known that $\frac{dx}{dt} = \sin t$ and $y = 5x$.

a Find $\frac{dy}{dt}$ as a function of t.

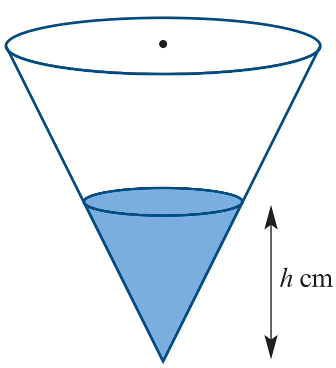
b Find the solution of the resulting differential equation.

Example 23

3 A conical tank has a radius length at the top equal to its height. Water, initially with a depth of 25 cm, leaks out through a hole in the bottom of the tank at the rate of $5\sqrt{h}$ cm^3 per minute, where the depth is h cm at time t minutes.

a Construct a differential equation expressing $\frac{dh}{dt}$ as a function of h, and solve it.

b Hence find how long it will take for the tank to empty.



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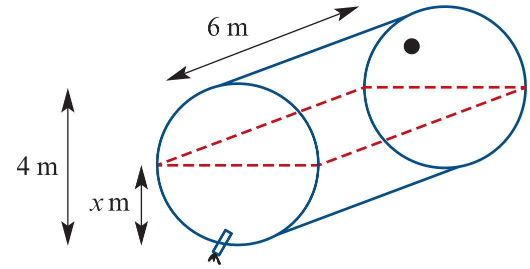
4 A cylindrical tank is lying on its side. The tank has a hole in the top, and another in the bottom so that the water in the tank leaks out. The depth of water is x m at time t minutes and

$$\frac{dx}{dt} = \frac{-0.025\sqrt{x}}{A}$$

where A m^2 is the surface area of the water at time t minutes.



a Construct the differential equation expressing $\frac{dx}{dt}$ as a function of x only.
b Solve the differential equation given that initially the tank was full.
c Find how long it will take to empty the tank.

5 A spherical drop of water evaporates so that the volume remaining is V mm^3 and the surface area is A mm^2 when the radius is r mm at time t seconds.
Given that $\frac{dV}{dt} = -2A^2$:
a Construct the differential equation expressing $\frac{dr}{dt}$ as a function of r.
b Solve the differential equation given that the initial radius was 2 mm.
c Sketch the graph of A against t and the graph of r against t.

6 A water tank of uniform cross-sectional area A cm^2 is being filled by a pipe which supplies Q litres of water every minute. The tank has a small hole in its base through which water leaks at a rate of kh litres every minute, where h cm is the depth of water in the tank at time t minutes. Initially the depth of the water is h_0 cm.
a Construct the differential equation expressing $\frac{dh}{dt}$ as a function of h.
b Solve the differential equation if $Q > kh_0$.
c Find the time taken for the depth to reach $\frac{Q + kh_0}{2k}$.

7 A tank has the shape of an inverted cone with a height of 1 m and a top radius of 1 m.
a First suppose that the tank is initially empty and that water flows into the tank at a rate of 0.1 m^3 per minute. Let h m be the depth of water in the tank after t minutes.
  i Find a formula for the rate of change of the depth of water in the tank (in metres per minute) when the depth of water is h m.
  ii Find a formula for the depth of water in the tank (in metres) after t minutes.
b Now suppose that the tank is initially full and that water begins to flow out of the tank at a rate of $0.1\sqrt{t}$ m^3 per minute, where t is the time in minutes since the tank began to empty. Find a formula for the depth of water in the tank (in metres) after t minutes.

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11H Using a definite integral to solve a differential equation

There are many situations in which an exact solution to a differential equation $\frac{dy}{dx} = f(x)$ is not required. Indeed, in some cases it may not even be possible to obtain an exact solution. However, we can find an approximate solution by numerically evaluating a definite integral.

Using antidifferentiation

For a differential equation of the form $\frac{dy}{dx} = f(x)$, consider the problem of finding the value of y when $x = b$, given that we know the value of y when $x = a$.

Example 24

For the differential equation $\frac{dy}{dx} = x^2 + 2$, given that $y = 7$ when $x = 1$, find y when $x = 3$.

Solution

$$\frac{dy}{dx} = x^2 + 2$$

$$\therefore \quad y = \frac{x^3}{3} + 2x + c$$

Since $y = 7$ when $x = 1$, we have

$$7 = \frac{1}{3} + 2 + c$$

$$\therefore \quad c = \frac{14}{3}$$

$$\therefore \quad y = \frac{x^3}{3} + 2x + \frac{14}{3}$$

When $x = 3$:

$$y = \frac{1}{3} \times 3^3 + 2 \times 3 + \frac{14}{3} = \frac{59}{3}$$

Using a definite integral

Again, consider a differential equation of the form $\frac{dy}{dx} = f(x)$, and suppose that we know the value of y when $x = a$. We have

$$\frac{dy}{dx} = f(x)$$

$$y = F(x) + c \qquad \text{where } F \text{ is an antiderivative of } f$$

$$y(a) = F(a) + c \qquad \text{substituting } y = y(a) \text{ when } x = a$$

$$c = y(a) - F(a)$$

$$\therefore \quad y = F(x) - F(a) + y(a)$$

Using the fundamental theorem of calculus, we obtain

$$y = \int_a^x f(t)\,dt + y(a)$$

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For a differential equation of the form $\frac{dy}{dx} = f(x)$, with $y = y(a)$ when $x = a$, the values of y can be found using

$$y = \int_a^x f(t)\, dt + y(a)$$

Example 25

Use a definite integral for each of the following:

a Solve the differential equation $\frac{dy}{dx} = x^2 + 2$ at $x = 3$, given that $y = 7$ at $x = 1$.

b Solve the differential equation $\frac{dy}{dx} = \cos x$ at $x = \frac{\pi}{4}$, given that $y = 1$ at $x = 0$.

Solution

a When $x = 3$:

$$\begin{aligned} y &= \int_1^3 t^2 + 2\, dt + 7 \\ &= \left[\frac{t^3}{3} + 2t\right]_1^3 + 7 \\ &= \frac{1}{3} \times 3^3 + 2 \times 3 - \left(\frac{1}{3} + 2\right) + 7 \\ &= \frac{59}{3} \end{aligned}$$

b When $x = \frac{\pi}{4}$:

$$\begin{aligned} y &= \int_0^{\frac{\pi}{4}} \cos t\, dt + 1 \\ &= [\sin t]_0^{\frac{\pi}{4}} + 1 \\ &= \sin\left(\frac{\pi}{4}\right) + 1 \\ &= \frac{1}{\sqrt{2}} + 1 = \frac{2 + \sqrt{2}}{2} \end{aligned}$$

This idea is very useful for solving a differential equation that cannot be antidifferentiated, because we can use numerical methods to approximate the definite integral.

Example 26

Solve the differential equation $f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ at $x = 1$, given that $f(0) = 0.5$.

Give your answer correct to four decimal places.

Solution

Calculus methods are not available for this differential equation and, since an approximate answer is acceptable, the use of a CAS calculator is appropriate.

The fundamental theorem of calculus gives

$$f(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, dt + 0.5$$

So $\quad f(1) = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, dt + 0.5$

The required answer is 0.8413, correct to four decimal places.

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Exercise 11H

Example 25

1 For each of the following, use a definite integral to find the required value:

a $\frac{dy}{dx} = \sin(2x)$ and $y = 2$ when $x = \frac{\pi}{2}$. Find y when $x = \frac{3\pi}{4}$.

b $\frac{dy}{dx} = e^{2x}$ and $y = 3$ when $x = 1$. Find y when $x = 2$.

c $\frac{dy}{dx} = \frac{2}{4 - x^2}$ and $y = 2$ when $x = 1$. Find y when $x = \frac{3}{2}$.

Example 26

2 For each of the following, use a calculator to find values correct to four decimal places:

a $\frac{dy}{dx} = \sqrt{\cos x}$ and $y = 1$ when $x = 0$. Find y when $x = \frac{\pi}{4}$.

b $\frac{dy}{dx} = \frac{1}{\sqrt{\cos x}}$ and $y = 1$ when $x = 0$. Find y when $x = \frac{\pi}{4}$.

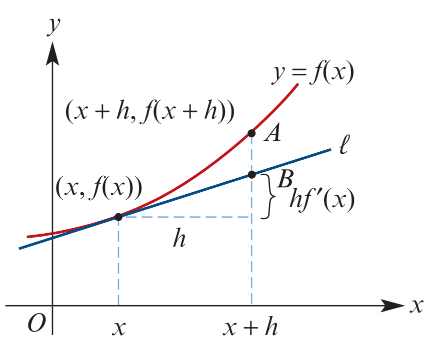
c $\frac{dy}{dx} = \log_e(x^2)$ and $y = 2$ when $x = 1$. Find y when $x = e$.

d $\frac{dy}{dx} = \sqrt{\log_e x}$ and $y = 2$ when $x = 1$. Find y when $x = e$.

11I Using Euler's method to solve a differential equation

In this section we discuss a method of finding an approximate solution to a differential equation. This is done by finding a sequence of points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ that lie on a curve that approximates the solution curve of the given differential equation.

Linear approximation to a curve



From the diagram, we have

$$\frac{f(x+h) - f(x)}{h} \approx f'(x) \quad \text{for small } h$$

Rearranging this equation gives

$$f(x+h) \approx f(x) + hf'(x)$$

This is shown on the diagram. The line ℓ is a tangent to $y = f(x)$ at the point with coordinates $(x, f(x))$.

This gives an approximation to the curve $y = f(x)$ in that the y-coordinate of B is an approximation to the y-coordinate of A on the graph of $y = f(x)$.

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The start of the process

For example, consider the differential equation

$$f'(x) = x^2 - 2x \quad \text{with} \quad f(3) = 0$$

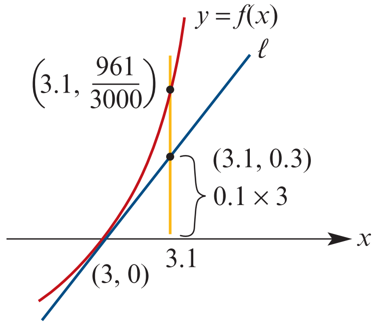
We start with the point $(x_0, y_0) = (3, 0)$.

The graph shown is a section of the solution curve for the differential equation. In this case, we are taking $h = 0.1$, and so $f(x + h) \approx f(x) + hf'(x)$ gives

$$f(3.1) \approx 0 + 0.1 \times 3 = 0.3$$

So the next point in the sequence is $(x_1, y_1) = (3.1, 0.3)$.

Note that the actual value of $f(3.1)$ is $\dfrac{961}{3000} \approx 0.32$.



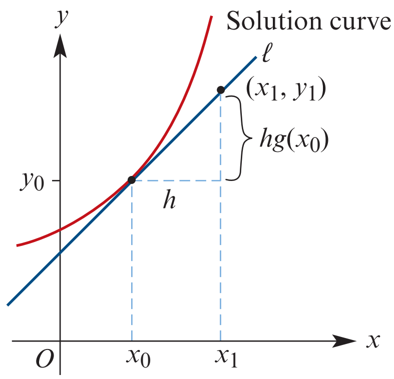
The general process

This process can be repeated to generate a longer sequence of points.

We start again at the beginning. Consider the differential equation

$$\frac{dy}{dx} = g(x) \quad \text{with} \quad y(x_0) = y_0$$

Then $x_1 = x_0 + h$ and $y_1 = y_0 + hg(x_0)$.



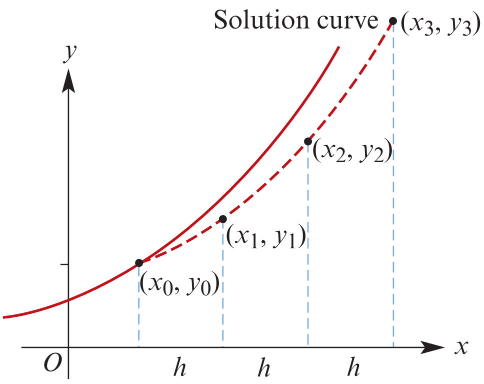
The process is now applied repeatedly to approximate the value of the function at $x_2, x_3, \dots$.

The result is:

$$x_2 = x_1 + h \quad \text{and} \quad y_2 = y_1 + hg(x_1)$$
$$x_3 = x_2 + h \quad \text{and} \quad y_3 = y_2 + hg(x_2)$$

and so on.

The point (x_n, y_n) is found in the nth step of the iterative process.



This iterative process can be summarised as follows.

Formula for Euler's method

If $\dfrac{dy}{dx} = g(x)$ with $y = y_0$ when $x = x_0$, then

$$x_{n+1} = x_n + h \quad \text{and} \quad y_{n+1} = y_n + hg(x_n)$$

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The accuracy of this formula, and the associated process, can be checked against the values obtained through the solution of the differential equation, where the result is known.

Euler's method for $f'(x) = x^2 - 2x$

The table gives the sequence of points (x_i, y_i), $0 \le i \le 10$, when Euler's method is applied to the differential equation

$$f'(x) = x^2 - 2x \quad \text{with} \quad f(3) = 0$$

using a step size of $h = 0.1$.

The solution to this differential equation is

$$f(x) = \frac{x^3}{3} - x^2$$

The values $f(x_i)$ of the solution are given in the last column of the table.

As can be seen, the y-values obtained using Euler's method are reasonably close to the actual values of the solution.

i	x_i	y_i	$g(x_i)$	$f(x_i)$
0	3	0	3	0
1	3.1	0.3	3.41	0.320
2	3.2	0.641	3.84	0.683
3	3.3	1.025	4.29	1.089
4	3.4	1.454	4.76	1.541
5	3.5	1.93	5.25	2.042
6	3.6	2.455	5.76	2.592
7	3.7	3.031	6.29	3.194
8	3.8	3.66	6.84	3.851
9	3.9	4.344	7.41	4.563
10	4.0	5.085		5.333

A smaller step size h would yield a better approximation. For example, using $h = 0.01$, the approximation to $f(4)$ is 5.3085. The percentage error for $x = 4$ using $h = 0.1$ is 4.65%, but using $h = 0.01$ the error is 0.46%.

Example 27

Let $\frac{dy}{dx} = 2x$ with $y(0) = 3$. Apply Euler's method to find y_4 using steps of 0.1.

Solution

Here $g(x) = 2x$ and $h = 0.1$.

Step 0 $x_0 = 0$ and $y_0 = 3$

Step 1 $x_1 = 0 + 0.1 = 0.1$ and $y_1 = 3 + 0.1 \times 2 \times 0 = 3$

Step 2 $x_2 = 0.1 + 0.1 = 0.2$ and $y_2 = 3 + 0.1 \times 2 \times 0.1 = 3.02$

Step 3 $x_3 = 0.2 + 0.1 = 0.3$ and $y_3 = 3.02 + 0.1 \times 2 \times 0.2 = 3.06$

Step 4 $x_4 = 0.3 + 0.1 = 0.4$ and $y_4 = 3.06 + 0.1 \times 2 \times 0.3 = 3.12$

General version of Euler's method

We can apply Euler's method to a more general type of differential equation.

General formula for Euler's method

If $\frac{dy}{dx} = g(x, y)$ with $y = y_0$ when $x = x_0$, then

$$x_{n+1} = x_n + h \quad \text{and} \quad y_{n+1} = y_n + hg(x_n, y_n)$$

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Example 28

Let $\dfrac{dy}{dx} = x^2y$ with $y(1) = 4$. Apply Euler's method to find y_3 using steps of 0.2.

Solution

Here $g(x, y) = x^2y$ and $h = 0.2$.

Step 0	$x_0 = 1$	and	$y_0 = 4$
Step 1	$x_1 = 1 + 0.2 = 1.2$	and	$y_1 = 4 + 0.2 \times 1^2 \times 4 = 4.8$
Step 2	$x_2 = 1.2 + 0.2 = 1.4$	and	$y_2 = 4.8 + 0.2 \times (1.2)^2 \times 4.8 = 6.1824$
Step 3	$x_3 = 1.4 + 0.2 = 1.6$	and	$y_3 = 6.1824 + 0.2 \times (1.4)^2 \times 6.1824 = 8.6059\ldots$

Using a calculator for Euler's method

Example 29

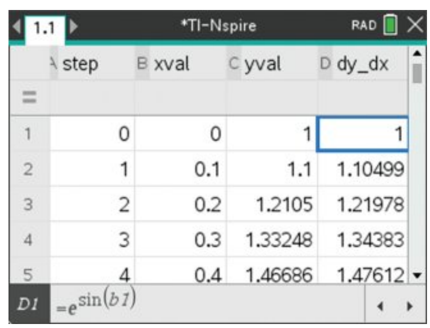
Use a CAS calculator to approximate the solution of the differential equation $\dfrac{dy}{dx} = e^{\sin x}$ with $y(0) = 1$:

a using step size 0.1

b using step size 0.01.

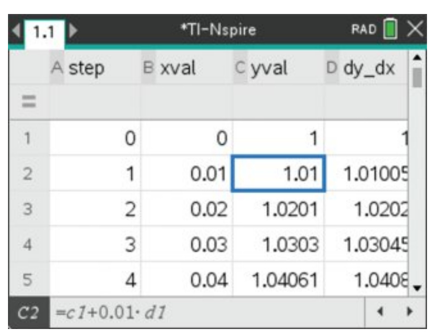
Using the TI-Nspire

- Choose a **Lists & Spreadsheet** application.
- Add the column headings as shown.
- Enter 0 in A1, 0 in B1, 1 in C1, and $= e^{\sin(b1)}$ in D1.
- Fill down in Column D. To do this, select cell D1 and then (menu) > **Data** > **Fill**. Use the arrow keys to go down to cell D10 and press (enter).



a Now in A2, enter $= \text{a}1 + 1$.
In B2, enter $= \text{b}1 + 0.1$.
In C2, enter $= \text{c}1 + 0.1 \times \text{d}1$.
Select A2, B2 and C2 and fill down to row 10.

b In B2, enter $= \text{b}1 + 0.01$.
In C2, enter $= \text{c}1 + 0.01 \times \text{d}1$.
Select B2 and C2 and fill down to row 10.



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Using the Casio ClassPad

In Menu, select **Sequence**. Tap the **Recursive** tab and choose the setting $\frac{n+1}{a0}$.

a To generate the x-values with step size 0.1:

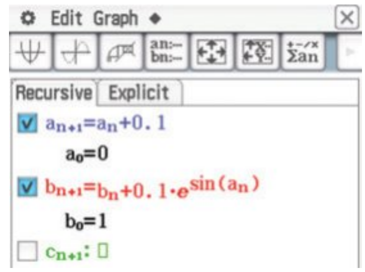
- Tap on a_{n+1} and enter $a_n + 0.1$. (Note that a_n can be selected from the menu bar.)
- Tap on a_0 and enter the initial value 0.

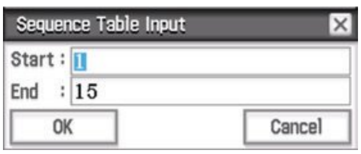
To generate the y-values:

- Tap on b_{n+1} and enter $b_n + 0.1e^{\sin(a_n)}$.
- Tap on b_0 and enter the initial value 1.

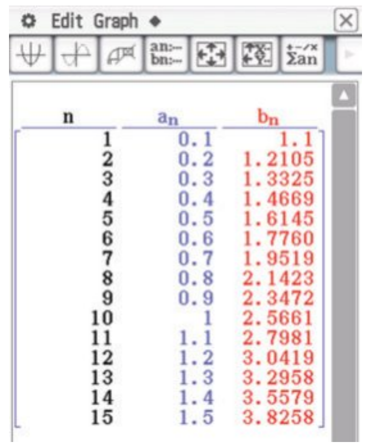
To view the table of values:

- First tap to set the table to 15 rows.





- Tick all boxes and tap the table icon.
- Resize to view all 15 rows.



n	a_n	b_n
1	0.1	1.1
2	0.2	1.2105
3	0.3	1.3325
4	0.4	1.4669
5	0.5	1.6145
6	0.6	1.7760
7	0.7	1.9519
8	0.8	2.1423
9	0.9	2.3472
10	1	2.5661
11	1.1	2.7981
12	1.2	3.0419
13	1.3	3.2958
14	1.4	3.5579
15	1.5	3.8258

b To generate the x-values with step size 0.01:

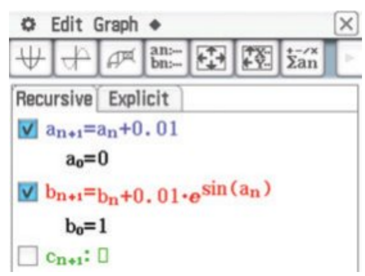
- Tap on a_{n+1} and enter $a_n + 0.01$.
- Tap on a_0 and enter the initial value 0.

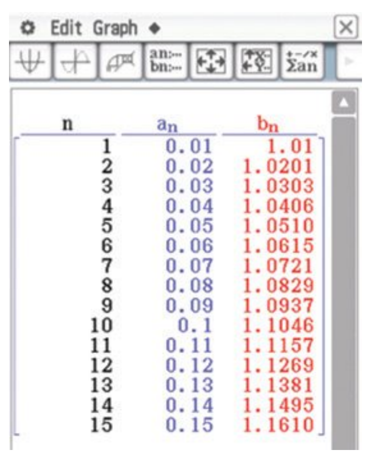
To generate the y-values:

- Tap on b_{n+1} and enter $b_n + 0.01e^{\sin(a_n)}$.
- Tap on b_0 and enter the initial value 1.

To view the table of values:

- Tick all boxes and tap the table icon.
- Resize to view all 15 rows.





n	a_n	b_n
1	0.01	1.01
2	0.02	1.0201
3	0.03	1.0303
4	0.04	1.0406
5	0.05	1.0510
6	0.06	1.0615
7	0.07	1.0721
8	0.08	1.0829
9	0.09	1.0937
10	0.1	1.1046
11	0.11	1.1157
12	0.12	1.1269
13	0.13	1.1381
14	0.14	1.1495
15	0.15	1.1610

Note: By moving the cursor to a particular cell in the table, you can see a fuller expression of the value in that cell at the bottom of the window.

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Using pseudocode for Euler's method

We now describe Euler's method as an algorithm using pseudocode. As an example, we will apply Euler's method to solve the differential equation

$$\frac{dy}{dx} = \frac{x}{2y} \quad \text{with} \quad y(x_0) = y_0$$

First we define the function for the right-hand side of the differential equation.

```
define g(x, y):
  return x/(2y)
```

Now we define a function $euler(x_0, y_0, h, n)$, where h is the step size and n is the number of iterations to perform.

```
define euler(x0, y0, h, n):
  x ← x0
  y ← y0
  for i from 1 to n
      y ← y + h × g(x, y)
      x ← x + h
      print i, (x, y)
  end for
  return
```

This function will print the values of $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

Note: For instructions on how to implement Euler's method as a program, see the coding appendices in the Interactive Textbook.

Exercise 11I

Example 27

1 For each of the following, apply Euler's method to find the indicated y_n-value using the given step size h. Give each answer correct to four decimal places.

a $\frac{dy}{dx} = \cos x$, given $y_0 = y(0) = 1$, find y_3 using $h = 0.1$

b $\frac{dy}{dx} = \frac{1}{x^2}$, given $y_0 = y(1) = 0$, find y_4 using $h = 0.01$

c $\frac{dy}{dx} = \sqrt{x}$, given $y_0 = y(1) = 1$, find y_3 using $h = 0.1$

d $\frac{dy}{dx} = \frac{1}{x^2 + 3x + 2}$, given $y_0 = y(0) = 0$, find y_3 using $h = 0.01$

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Example 29

2 Solve each of the following differential equations using:

i a calculus method

ii a spreadsheet or a program with a step size of 0.01.

a $\frac{dy}{dx} = \cos x$, given $y(0) = 1$, find $y(1)$
b $\frac{dy}{dx} = \frac{1}{x^2}$, given $y(1) = 0$, find $y(2)$

c $\frac{dy}{dx} = \sqrt{x}$, given $y(1) = 1$, find $y(2)$
d $\frac{dy}{dx} = \frac{1}{x^2 + 3x + 2}$, given $y(0) = 0$, find $y(2)$

3 Solve the differential equation $\frac{dy}{dx} = \sec^2 x$ at $x = 1$, given that $y = 2$ when $x = 0$, using:

a a calculus method

b a spreadsheet or a program with a step size of:

i 0.1 **ii** 0.05 **iii** 0.01

Example 28

4 Use Euler's method with steps of size 0.1 to find an approximate value of y at $x = 0.5$ if $\frac{dy}{dx} = y^3$ and $y = 1$ when $x = 0$.

5 Use Euler's method with steps of size 0.1 to find an approximate value of y at $x = 1$ if $\frac{dy}{dx} = y^2 + 1$ and $y = 1$ when $x = 0$.

6 Use Euler's method with steps of size 0.1 to find an approximate value of y at $x = 1$ if $\frac{dy}{dx} = xy$ and $y = 1$ when $x = 0$.

7 Use Euler's method with steps of size 0.1 to find an approximate value of y at $x = 1$ if $\frac{dy}{dx} = y - x$ and $y = \frac{1}{2}$ when $x = 0$.

8 The graph for the standard normal distribution is given by the rule

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Probabilities can be found using

$$\Pr(Z \le z) = \int_{-\infty}^{z} f(x)\,dx = \frac{1}{2} + \int_{0}^{z} f(x)\,dx$$

Let $y = \Pr(Z \le z)$. Then $\frac{dy}{dz} = f(z)$ with $y(0) = \frac{1}{2}$.

a Use Euler's method with a step size of 0.1 to find an approximation for $\Pr(Z \le z)$, where $z = 0, 0.1, 0.2, \dots, 0.9, 1$.

b Compare the values found in **a** with the probabilities found using a CAS calculator.

c Use a step size of 0.01 to obtain an approximation for:

i $\Pr(Z \le 0.5)$ **ii** $\Pr(Z \le 1)$

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11J Slope field for a differential equation

Consider a differential equation of the form $\frac{dy}{dx} = f(x)$.

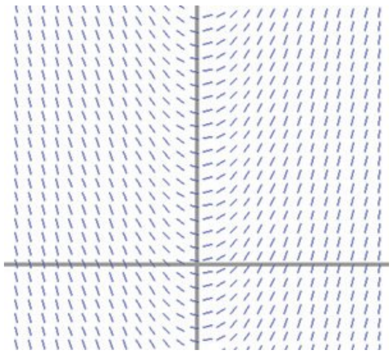
The **slope field** of this differential equation assigns to each point $P(x, y)$ in the plane (for which x is in the domain of f) the number $f(x)$, which is the gradient of the solution curve through P.

For the differential equation $\frac{dy}{dx} = 2x$, a gradient value is assigned for each point $P(x, y)$.

- For $(1, 3)$ and $(1, 5)$, the gradient value is 2.
- For $(-2, 5)$ and $(-2, -2)$, the gradient value is -4.

A slope field can, of course, be represented in a graph.

The slope field for $\frac{dy}{dx} = 2x$ is shown opposite.

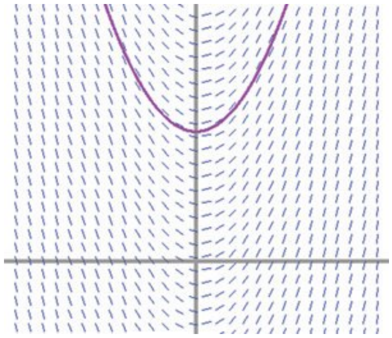


When initial conditions are given, a particular solution curve can be drawn.

Here the solution curve with $y = 2$ when $x = 0$ has been superimposed on the slope field for $\frac{dy}{dx} = 2x$.

Changing the initial conditions changes the particular solution.

A slope field is defined similarly for any differential equation of the form $\frac{dy}{dx} = f(x, y)$.



Example 30

a Use a CAS calculator to plot the slope field for the differential equation $\frac{dy}{dx} = y$.

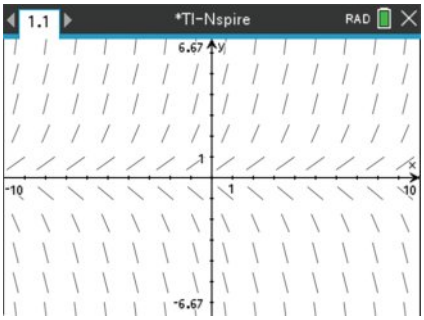
b On the plot of the slope field, plot the graphs of the particular solutions for:

i $y = 2$ when $x = 0$ **ii** $y = -3$ when $x = 1$.

Using the TI-Nspire

a
- In a **Graphs** application, select (menu) > **Graph Entry/Edit** > **Diff Eq**.
- Enter the differential equation as $y1' = y1$.
- Press (enter) to plot the slope field.

Note: The notation must match when entering the differential equation. (Here $y1$ is used for y.)



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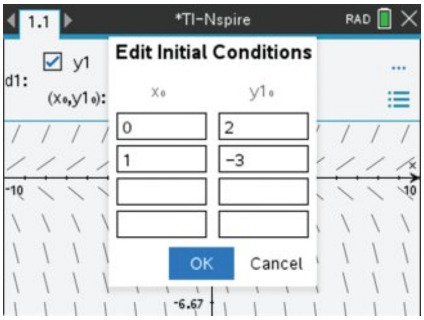
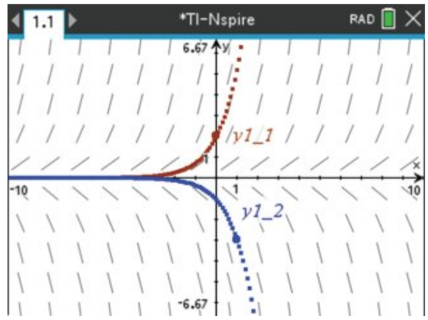
b In the graph entry line, you have the option of adding several initial conditions.

- To show the graph entry line, press (tab) or double click in an open area.
- Arrow up to $y1'$ and add the first set of initial conditions: $x = 0$ and $y1 = 2$.
- Click on the 'plus' icon to add more initial conditions: $x = 1$ and $y1 = -3$.
- Select OK to plot the solution curves for the given initial conditions.



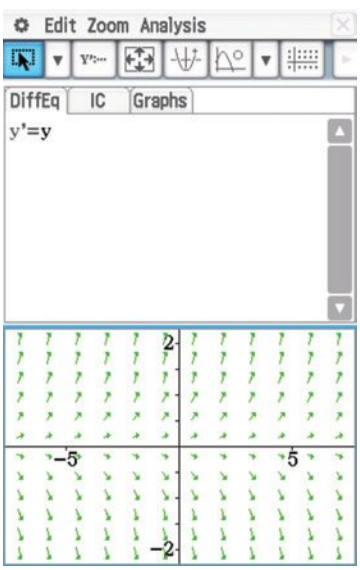


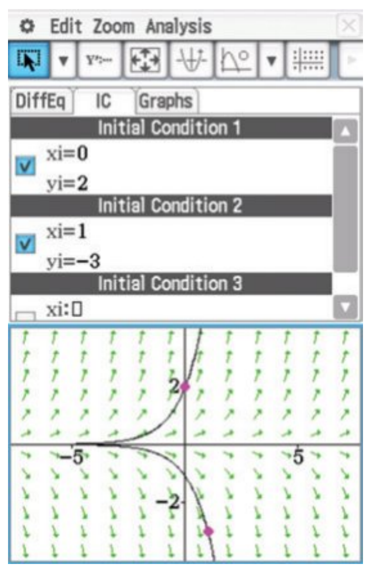
Note: You can grab the initial point and drag to show differing initial conditions.

Using the Casio ClassPad

a
- Open the menu .
- Select **DiffEqGraph** .
- Tap on y' and type y.
- Tap the slope field icon .

b
- Tap the **IC** window.
- Enter the initial conditions as shown.
- Tap the slope field icon .
- Tap to adjust the window.





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The differential equation $\frac{dy}{dx} = y$ from Example 30 can be solved as follows.

Note that $y = 0$ is a constant solution. For $y \neq 0$, we can write $\frac{dx}{dy} = \frac{1}{y}$.

Then $x = \log_e |y| + c$, which implies $|y| = e^{x-c} = Ae^x$.

- If $y = 2$ when $x = 0$, then $A = 2$ and therefore $y = 2e^x$, as $y > 0$.
- If $y = -3$ when $x = 1$, then $A = 3e^{-1}$ and therefore $y = -3e^{x-1}$, as $y < 0$.

Exercise 11J

1 For each of the following differential equations, sketch a slope field graph and the solution curve for the given initial conditions, using $-3 \le x \le 3$ and $-3 \le y \le 3$. Use calculus to solve the differential equation in each case.

a $\frac{dy}{dx} = 3x^2$, given $y = 0$ when $x = 1$

b $\frac{dy}{dx} = \sin x$, given $y = 0$ when $x = 0$ (use radian mode)

c $\frac{dy}{dx} = e^{-2x}$, given $y = 1$ when $x = 0$

d $\frac{dy}{dx} = y^2$, given $y = 1$ when $x = 1$

e $\frac{dy}{dx} = y^2$, given $y = -1$ when $x = 1$

f $\frac{dy}{dx} = y(y - 1)$, given $y = -1$ when $x = 0$

g $\frac{dy}{dx} = y(y - 1)$, given $y = 2$ when $x = 0$

h $\frac{dy}{dx} = \tan x$, given $y = 0$ when $x = 0$

2 For each of the following differential equations, sketch a slope field graph and the solution curve for the given initial conditions, using $-3 \le x \le 3$ and $-3 \le y \le 3$.

a $\frac{dy}{dx} = -\frac{x}{y}$, given that at $x = 0$, $y = \pm 1$

b $\frac{dy}{dx} = -\frac{x}{y}$, given that at $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$

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Review

Assign-ment

Nrich

Chapter summary

- A differential equation is an equation that contains at least one derivative.
- A solution of a differential equation is a function that satisfies the differential equation when it and its derivatives are substituted. The general solution is the family of functions that satisfy the differential equation.

Differential equation	Method of solution
$\frac{dy}{dx} = f(x)$	$\frac{dy}{dx} = f(x)$ $\therefore\ y = \int f(x)\,dx$ $= F(x) + c, \quad \text{where } F'(x) = f(x)$
$\frac{d^2y}{dx^2} = f(x)$	$\frac{d^2y}{dx^2} = f(x)$ $\frac{dy}{dx} = \int f(x)\,dx$ $= F(x) + c, \quad \text{where } F'(x) = f(x)$ $\therefore\ y = \int F(x) + c\,dx$ $= G(x) + cx + d, \quad \text{where } G'(x) = F(x)$
$\frac{dy}{dx} = g(y)$	$\frac{dy}{dx} = g(y)$ $\frac{dx}{dy} = \frac{1}{g(y)}$ $\therefore\ x = \int \frac{1}{g(y)}\,dy$ $= F(y) + c, \quad \text{where } F'(y) = \frac{1}{g(y)}$
$\frac{dy}{dx} = f(x)\,g(y)$	$\frac{dy}{dx} = f(x)\,g(y)$ $f(x) = \frac{1}{g(y)}\frac{dy}{dx}$ $\int f(x)\,dx = \int \frac{1}{g(y)}\,dy$

- Slope field

  The slope field of a differential equation

$$\frac{dy}{dx} = f(x, y)$$

  assigns to each point $P(x, y)$ in the plane (for which $f(x, y)$ is defined) the number $f(x, y)$, which is the gradient of the solution curve through P.

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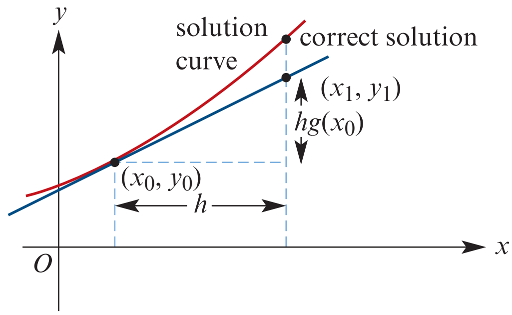
- Euler's method

For $\dfrac{dy}{dx} = g(x)$ with $y = y_0$ when $x = x_0$:

$$\begin{aligned} x_1 &= x_0 + h \quad \text{and} \quad y_1 = y_0 + hg(x_0) \\ x_2 &= x_1 + h \quad \text{and} \quad y_2 = y_1 + hg(x_1) \\ x_3 &= x_2 + h \quad \text{and} \quad y_3 = y_2 + hg(x_2) \\ &\vdots \qquad\qquad\qquad\quad \vdots \\ x_{n+1} &= x_n + h \quad \text{and} \quad y_{n+1} = y_n + hg(x_n) \end{aligned}$$



The sequence of points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ approximate a solution curve for the differential equation.

Technology-free questions

1 Find the general solution of each of the following differential equations:

a $\dfrac{dy}{dx} = \dfrac{x^2+1}{x^2}, \quad x > 0$

b $\dfrac{1}{y} \cdot \dfrac{dy}{dx} = 10, \quad y > 0$

c $\dfrac{d^2y}{dt^2} = \dfrac{1}{2}(\sin(3t) + \cos(2t)), \quad t \geq 0$

d $\dfrac{d^2y}{dx^2} = \dfrac{e^{-x} + e^x}{e^{2x}}$

e $\dfrac{dy}{dx} = \dfrac{3-y}{2}, \quad y < 3$

f $\dfrac{dy}{dx} = \dfrac{3-x}{2}$

2 Find the solution of the following differential equations under the stated conditions:

a $\dfrac{dy}{dx} = \pi\cos(2\pi x)$, if $y = -1$ when $x = \dfrac{5}{2}$

b $\dfrac{dy}{dx} = \cot(2x)$, if $y = 0$ when $x = \dfrac{\pi}{4}$

c $\dfrac{dy}{dx} = \dfrac{1+x^2}{x}$, if $y = 0$ when $x = 1$

d $\dfrac{dy}{dx} = \dfrac{x}{1+x^2}$, if $y(0) = 1$

e $6\dfrac{dy}{dx} = -3y$, if $y = e^{-1}$ when $x = 2$

f $\dfrac{d^2x}{dt^2} = -10$, given that $\dfrac{dx}{dt} = 4$ when $x = 0$ and that $x = 0$ when $t = 4$

3 **a** If $y = x\sin x$ is a solution of the differential equation $x^2\dfrac{d^2y}{dx^2} - kx\dfrac{dy}{dx} + (x^2 - m)y = 0$, find k and m.

b Show that $y = xe^{2x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 3e^{2x} = 2xe^{2x}$.

4 The curve with equation $y = f(x)$ passes through the point $P\left(\dfrac{\pi}{4}, 3\right)$, with a gradient of 1 at this point, and $f''(x) = 2\sec^2(x)$.

a Find the gradient of the curve at $x = \dfrac{\pi}{6}$.

b Find $f\left(\dfrac{\pi}{6}\right)$.

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5 Find all real values of n such that $y = e^{nx}$ is a solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 0$.

6 Let $\frac{dy}{dx} = (y+4)^2 + 9$ and $y_0 = y(0) = 0$.

a Solve this differential equation, giving y as a function of x.

b Using Euler's method with a step size of 0.2, find y_1.

7 **a** Use Euler's method to find y_2 if $\frac{dy}{dx} = \frac{1}{x^2}$, given that $y_0 = y(1) = \frac{1}{2}$ and $h = 0.1$.

b Solve the differential equation.

c Find the value of y approximated by y_2.

8 Consider the differential equation $\frac{dy}{dx} = 4 + y^2$.

a Sketch the slope field of this differential equation for $y = -2, -1, 0, 1, 2$ at $x = -2, -1, 0, 1, 2$.

b If $y = -1$ when $x = 2$, solve the differential equation, giving your answer with y in terms of x.

9 A container of water is heated to boiling point (100°C) and then placed in a room with a constant temperature of 25°C. After 10 minutes, the temperature of the water is 85°C. Newton's law of cooling gives $\frac{dT}{dt} = -k(T - 25)$, where T°C is the temperature of the water at time t minutes after being placed in the room.

a Find the value of k.

b Find the temperature of the water 15 minutes after it was placed in the room.

10 Solve the differential equation $\frac{dy}{dx} = 2x\sqrt{25 - x^2}$, for $-5 \le x \le 5$, given that $y = 25$ when $x = 4$.

11 If $y = e^x \sin(x)$ is a solution to the differential equation $\frac{d^2y}{dx^2} + k\frac{dy}{dx} + y = e^x \cos(x)$, find the value of k.

12 If a hemispherical bowl of radius 6 cm contains water to a depth of x cm, the volume, V cm^3, is given by

$$V = \frac{\pi}{3}x^2(18 - x)$$

If water is poured into the bowl at the rate of 3 cm^3/s, construct the differential equation expressing $\frac{dx}{dt}$ as a function of x.

13 A circle has area A cm^2 and circumference C cm at time t seconds. If the area is increasing at a rate of 4 cm^2/s, construct the differential equation expressing $\frac{dC}{dt}$ as a function of C.

14 A population of size x is decreasing according to the law $\frac{dx}{dt} = -\frac{x}{100}$, where t denotes the time in days. If initially the population is of size x_0, find to the nearest day how long it takes for the size of the population to be halved.

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15 Some students put 3 kilograms of soap powder into a water fountain. The soap powder totally dissolved in the 1000 litres of water, thus forming a solution in the fountain. When the soap solution was discovered, clean water was run into the fountain at the rate of 40 litres per minute. The clean water and the solution in the fountain mixed instantaneously and the excess mixture was removed immediately at a rate of 40 litres per minute. If S kilograms was the amount of soap powder in the fountain t minutes after the soap solution was discovered, construct and solve the differential equation to fit this situation.

16 A metal rod that is initially at a temperature of 10°C is placed in a warm room. After t minutes, the temperature, θ°C, of the rod is such that $\dfrac{d\theta}{dt} = \dfrac{30 - \theta}{20}$.

- **a** Solve this differential equation, expressing θ in terms of t.
- **b** Calculate the temperature of the rod after one hour has elapsed, giving the answer correct to the nearest degree.
- **c** Find the time taken for the temperature of the rod to rise to 20°C, giving the answer correct to the nearest minute.

17 A fire broke out in a forest and, at the moment of detection, covered an area of 0.5 hectares. From an aerial surveillance, it was estimated that the fire was spreading at a rate of increase in area of 2% per hour. If the area of the fire at time t hours is denoted by A hectares:

- **a** Write down the differential equation that relates $\dfrac{dA}{dt}$ and A.
- **b** What would be the area of the fire 10 hours after it is first detected?
- **c** When would the fire cover an area of 3 hectares (to the nearest quarter-hour)?

18 A flexible beam is supported at its ends, which are at the same horizontal level and at a distance L apart. The deflection, y, of the beam, measured downwards from the horizontal through the supports, satisfies the differential equation

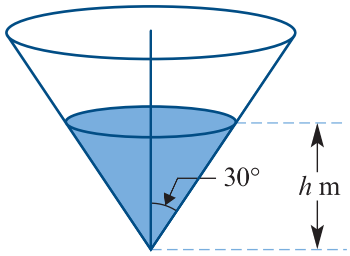
$$16\frac{d^2y}{dx^2} = L - 3x, \quad 0 \le x \le L$$

where x is the horizontal distance from one end. Find where the deflection has its greatest magnitude, and also the value of this magnitude.

19 A vessel in the shape of a right circular cone has a vertical axis and a semi-vertex angle of 30°.

There is a small hole at the vertex so that liquid leaks out at the rate of $0.05\sqrt{h}$ m^3 per hour, where h m is the depth of liquid in the vessel at time t hours.

Given that the liquid is poured into this vessel at a constant rate of 2 m^3 per hour, set up (but do not attempt to solve) a differential equation for h.



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Multiple-choice questions

1 The acceleration, a m/s^2, of an object moving in a straight line at time t seconds is given by $a = \sin(2t)$. If the object has an initial velocity of 4 m/s, then v is equal to

A $2\cos(2t) + 4$ **B** $2\cos(2t) + 2$ **C** $\int_0^t \sin(2x)\,dx + 4$

D $-\frac{1}{2}\cos(2t) + 4$ **E** $\int_0^t \sin(2x)\,dx - 4$

2 If $f'(x) = x^2 - 1$ and $f(1) = 3$, an approximate value of $f(1.4)$ using Euler's method with a step size of 0.2 is

A 3.88 **B** 3.688 **C** $3.\dot{6}$ **D** 3.088 **E** 3

3 Euler's method with a step size of 0.1 is used to approximate the solution of the differential equation $\frac{dy}{dx} = x\log_e x$ with $y(2) = 2$. When $x = 2.2$, the value obtained for y is closest to

A 2.314 **B** 2.294 **C** 2.291 **D** 2.287 **E** 2.277

4 Assume that $\frac{dy}{dx} = \frac{2-y}{4}$ and that $x = 3$ when $y = 1$. The value of x when $y = \frac{1}{2}$ is given by

A $x = \int_1^{\frac{1}{2}} \frac{4}{2-t}\,dt + 3$ **B** $x = \int_3^{\frac{1}{2}} \frac{4}{2-t}\,dt + 1$ **C** $x = \int_1^{\frac{1}{2}} \frac{2-t}{4}\,dt + 3$

D $x = \int_3^{\frac{1}{2}} \frac{2-t}{4}\,dt + 1$ **E** $x = \int_1^{\frac{1}{2}} \frac{2-y}{4}\,dy + 3$

5 If $\frac{dy}{dx} = \frac{2x+1}{4}$ and $y = 0$ when $x = 2$, then y is equal to

A $\frac{1}{4}(x^2 + x) + \frac{1}{2}$ **B** $\frac{x(x+1)}{4}$ **C** $\frac{1}{4}(x^2 + x) + 2$

D $\frac{1}{4}(x^2 + x - 1)$ **E** $\frac{1}{4}(x^2 + x - 6)$

6 If $\frac{dy}{dx} = \frac{1}{5}(y-1)^2$ and $y = 0$ when $x = 0$, then y is equal to

A $\frac{5}{1-x} - 5$ **B** $1 + \frac{5}{x+5}$ **C** $\frac{x}{x+5}$ **D** $\frac{5}{x+5} - 1$ **E** $1 - \frac{5}{x}$

7 The solution of the differential equation $\frac{dy}{dx} = e^{-x^2}$, where $y = 4$ when $x = 1$, is

A $y = \int_1^4 e^{-x^2}\,dx$ **B** $y = \int_1^4 e^{-x^2}\,dx + 4$ **C** $y = \int_1^x e^{-u^2}\,du - 4$

D $y = \int_1^x e^{-u^2}\,du + 4$ **E** $y = \int_4^x e^{-u^2}\,du + 1$

8 For which one of the following differential equations is $y = 2xe^{2x}$ a solution?

A $\frac{dy}{dx} - 2y = 0$ **B** $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ **C** $\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

D $\frac{d^2y}{dx^2} - 4y = e^{2x}$ **E** $\frac{d^2y}{dx^2} - 4y = 8e^{2x}$

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9 Water is leaking from an initially full container with a depth of 40 cm. The volume, V cm^3, of water in the container is given by $V = \pi(5h^2 + 225h)$, where h cm is the depth of the water at time t minutes.

If water leaks out at the rate of $\dfrac{5\sqrt{h}}{2h+45}$ cm^3/min, then the rate of change of the depth is

A $\dfrac{-\sqrt{h}}{\pi(2h+45)^2}$ cm/min **B** $5\pi(2h+45)$ cm/min **C** $\dfrac{\sqrt{h}}{\pi(2h+45)^2}$ cm/min

D $\dfrac{1}{5\pi(2h+45)}$ cm/min **E** $\dfrac{-1}{5\pi(2h+45)}$ cm/min

10 The solution of the differential equation $\dfrac{dy}{dx} = y$, where $y = 2$ when $x = 0$, is

A $y = e^{2x}$ **B** $y = e^{\frac{x}{2}}$ **C** $y = 2e^x$ **D** $y = \frac{1}{2}e^x$ **E** $y = \log_e\left(\frac{x}{2}\right)$

11 The rate at which a particular disease spreads through a population of 2000 cattle is proportional to the product of the number of infected cows and the number of non-infected cows. Initially four cows are infected. If N denotes the number of infected cows at time t days, then a differential equation to describe this is

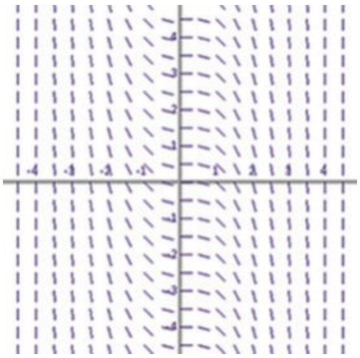
A $\dfrac{dN}{dt} = kN(2000 - N)$ **B** $\dfrac{dN}{dt} = k(4 - N)(200 - N)$ **C** $\dfrac{dN}{dt} = kN(200 - N)$

D $\dfrac{dN}{dt} = kN^2(2000 - N^2)$ **E** $\dfrac{dN}{dt} = \dfrac{k(2000 - N)}{2000}$

12 Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x^2 + 2x + 2}$ with $y_0 = 2$ and $x_0 = 0$. Using Euler's method with a step size of 0.1, the value of y_2, correct to three decimal places, is

A 2.123 **B** 2.675 **C** 2.567 **D** 1.987 **E** 2.095

13 The differential equation that best matches the slope field shown is



A $\dfrac{dy}{dx} = x$ **B** $\dfrac{dy}{dx} = -x$ **C** $\dfrac{dy}{dx} = x^2$

D $\dfrac{dy}{dx} = -x^2$ **E** $\dfrac{dy}{dx} = \dfrac{x}{y}$

14 The amount of a salt Q in a tank at time t is given by the differential equation

$$\frac{dQ}{dt} = 3 - \frac{5}{5 - t} \quad \text{with} \quad Q_0 = Q(0) = 10$$

Using Euler's method with a step size of 0.5 in the values of t, the value of Q correct to three decimal places when $t = 1$ is

A 12.123 **B** 9.675 **C** 8.967 **D** 10.587 **E** 11.944

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Review

15 Water containing 3 grams of salt per litre flows at the rate of 20 litres per minute into a tank that initially contained 100 litres of pure water. The concentration of salt in the tank is kept uniform by stirring, and the mixture flows out of the tank at the rate of 10 litres per minute. If M grams is the amount of salt in the tank t minutes after the water begins to flow, the differential equation relating M to t is

A $\frac{dM}{dt} = 60 - \frac{10M}{100 - 10t}$ **B** $\frac{dM}{dt} = 3 - \frac{10M}{100 - 10t}$ **C** $\frac{dM}{dt} = 60 - \frac{10M}{100 + 10t}$

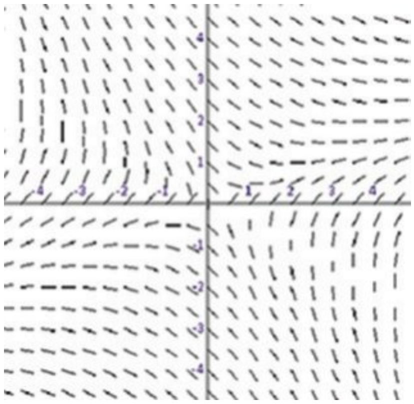
D $\frac{dM}{dt} = 20 - 10t$ **E** $\frac{dM}{dt} = -\frac{10M}{100 + 10t}$

16 The differential equation that best matches the slope field shown is

A $\frac{dy}{dx} = \frac{y}{x}$ **B** $\frac{dy}{dx} = -\frac{x^2}{y}$

C $\frac{dy}{dx} = \frac{x - 2y}{2y + x}$ **D** $\frac{dy}{dx} = -\frac{y}{x}$

E $\frac{dy}{dx} = \frac{x}{y}$



Extended-response questions

1 The percentage of radioactive carbon-14 in living matter decays, from the time of death, at a rate proportional to the percentage present.

a If x% is present t years after death:

i Construct an appropriate differential equation.

ii Solve the differential equation, given that carbon-14 has a half-life of 5760 years, i.e. 50% of the original amount will remain after 5760 years.

b A sample was taken from a tree buried by volcanic ash and was found to contain 45.1% of the amount of carbon-14 present in living timber. How long ago did the eruption occur?

c Sketch the graph of x against t.

2 Two chemicals, A and B, are put together in a solution, where they react to form a compound, X. The rate of increase of the mass, x kg, of X is proportional to the product of the masses of *unreacted* A and B present at time t minutes. It takes 1 kg of A and 3 kg of B to form 4 kg of X. Initially, 2 kg of A and 3 kg of B are put together in solution, and 1 kg of X forms in 1 minute.

a Set up the appropriate differential equation expressing $\frac{dx}{dt}$ as a function of x.

b Solve the differential equation. **c** Find the time taken to form 2 kg of X.

d Find the mass of X formed after 2 minutes.

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3 Newton's law of cooling states that the rate of cooling of a body is proportional to the excess of its temperature above that of its surroundings. The body has a temperature of T°C at time t minutes, while the temperature of the surroundings is a constant T_S°C.

a Construct a differential equation expressing $\frac{dT}{dt}$ as a function of T.

b A teacher pours a cup of coffee at lunchtime. The lunchroom is at a constant temperature of 22°C, while the coffee is initially 72°C. The coffee becomes undrinkable (too cold) when its temperature drops below 50°C. After 5 minutes, the temperature of the coffee has fallen to 65°C. Find correct to one decimal place:

i the length of time, after it was poured, that the coffee remains drinkable

ii the temperature of the coffee at the end of 30 minutes.

4 On a cattle station there were p head of cattle at time t years after 1 January 2015. The population naturally increases at a rate proportional to p. Every year 1000 head of cattle are withdrawn from the herd.

a Show that $\frac{dp}{dt} = kp - 1000$, where k is a constant.

b If the herd initially had 5000 head of cattle, find an expression for t in terms of k and p.

c The population increased to 6000 head of cattle after 5 years.

i Show that $5k = \log_e\left(\frac{6k - 1}{5k - 1}\right)$.

ii Use a CAS calculator to find an approximation for the value of k.

d Sketch a graph of p against t.

5 In the main lake of a trout farm, the trout population is N at time t days after 1 January 2020. The number of trout harvested on a particular day is proportional to the number of trout in the lake at that time. Every day 100 trout are added to the lake.

a Construct a differential equation with $\frac{dN}{dt}$ in terms of N and k, where k is a constant.

b Initially the trout population was 1000. Find an expression for t in terms of k and N.

c The trout population decreases to 700 after 10 days. Use a CAS calculator to find an approximation for the value of k.

d Sketch a graph of N against t.

e If the procedure at the farm remains unchanged, find the eventual trout population in the lake.

6 A thin horizontal beam, AB, of length L cm, is bent under a load so that the deflection, y cm at a point x cm from the end A, satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{9}{40L^2}(3x - L), \quad 0 \le x \le L$$

Given that the deflection of the beam and its inclination to the horizontal are both zero at A, find:

a where the maximum deflection occurs

b the magnitude of the maximum deflection.

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7 The water in a hot-water tank cools at a rate which is proportional to $T - T_0$, where T°C is the temperature of the water at time t minutes and T_0°C is the temperature of the surrounding air. When $T = 60$, the water is cooling at 1°C per minute. When switched on, the heater supplies sufficient heat to raise the water temperature by 2°C each minute (neglecting heat loss by cooling). If $T = 20$ when the heater is switched on and $T_0 = 20$:

a Construct a differential equation for $\frac{dT}{dt}$ as a function of T (where both heating and cooling are taking place).

b Solve the differential equation.

c Find the temperature of the water 30 minutes after turning on the heater.

d Sketch the graph of T against t.

8 **a** The rate of growth of a population of iguanas on an island is $\frac{dW}{dt} = 0.04W$, where W is the number of iguanas alive after t years. Initially there were 350 iguanas.

i Solve the differential equation.

ii Sketch the graph of W against t.

iii Give the value of W to the nearest integer when $t = 50$.

b If $\frac{dW}{dt} = kW$ and there are initially 350 iguanas, find the value of k for which the population remains constant.

c A more realistic population model for the iguanas is determined by the logistic differential equation $\frac{dW}{dt} = (0.04 - 0.00005W)W$. Initially there were 350 iguanas.

i Solve the differential equation.

ii Sketch the graph of W against t.

iii Find the population after 50 years.

9 A hospital patient is receiving a drug at a constant rate of R mg per hour through a drip. At time t hours, the amount of the drug in the patient is x mg. The rate of loss of the drug from the patient is proportional to x.

a When $t = 0$, $x = 0$:

i Show that $\frac{dx}{dt} = R - kx$, where k is a positive constant.

ii Find an expression for x in terms of t, k and R.

b If $R = 50$ and $k = 0.05$:

i Sketch the graph of x against t.

ii Find the time taken for there to be 200 mg in the patient, correct to two decimal places.

c When the patient contains 200 mg of the drug, the drip is disconnected.

i Assuming that the rate of loss remains the same, find the time taken for the amount of the drug in the patient to fall to 100 mg, correct to two decimal places.

ii Sketch the graph of x against t, showing the rise to 200 mg and fall to 100 mg.

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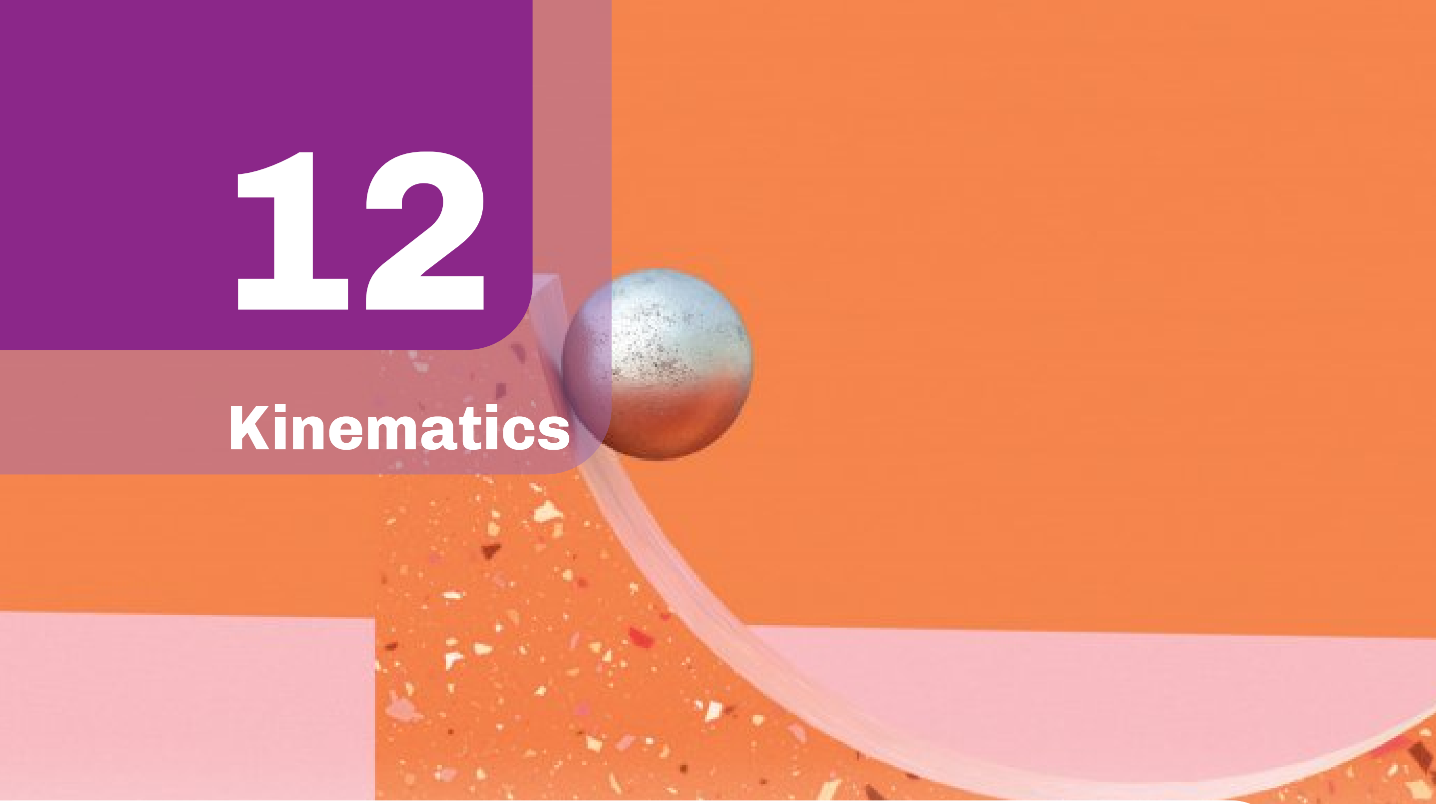


12 Kinematics

Objectives

- To model **motion in a straight line**.
- To use **calculus** to solve problems involving motion in a straight line with constant or variable acceleration.
- To use **graphical methods** to solve problems involving motion in a straight line.
- To use techniques for solving **differential equations** to solve problems of the form

$$v = f(x), \quad a = f(v) \quad \text{and} \quad a = f(x)$$

where x, v and a represent position, velocity and acceleration respectively.

Kinematics is the study of motion without reference to the cause of the motion.

In this chapter, we will consider the motion of a particle in a straight line only. Such motion is called **rectilinear motion**. When referring to the motion of a particle, we may in fact be referring to an object of any size. However, for the purposes of studying its motion, we can assume that all forces acting on the object, causing it to move, are acting through a single point. Hence we can consider the motion of a car or a train in the same way as we would consider the motion of a dimensionless particle.

When studying motion, it is important to make a distinction between vector quantities and scalar quantities:

Vector quantities Position, displacement, velocity and acceleration must be specified by both magnitude and direction.

Scalar quantities Distance, time and speed are specified by their magnitude only.

Since we are considering movement in a straight line, the *direction* of each vector quantity is simply specified by the *sign* of the numerical value.

12A Position, velocity and acceleration

Position

The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.

Consider a particle which starts at O and begins to move. The position of the particle at any instant can be specified by a real number x. For example, if the unit is metres and if $x = -3$, the position is 3 m to the left of O; while if $x = 3$, the position is 3 m to the right of O.

Sometimes there is a rule that enables the position at any instant to be calculated. In this case, we can view x as being a function of t. Hence $x(t)$ is the position at time t.

For example, imagine that a stone is dropped from the top of a vertical cliff 45 metres high. Assume that the stone is a particle travelling in a straight line. Let $x(t)$ metres be the downwards position of the particle from O, the top of the cliff, t seconds after the particle is dropped. If air resistance is neglected, then an approximate model for the position is

$$x(t) = 5t^2 \quad \text{for } 0 \le t \le 3$$

Example 1

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \ge 0$.

a Find its initial position. **b** Find its position at $t = 4$.

Solution

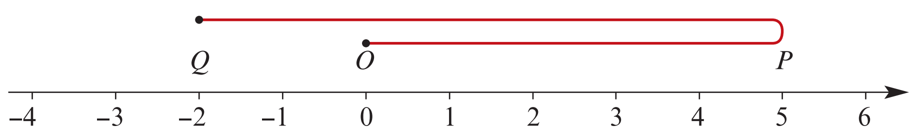
a At $t = 0$, $x = +6$, i.e. the particle is 6 cm to the right of O.

b At $t = 4$, $x = (4)^2 - 7(4) + 6 = -6$, i.e. the particle is 6 cm to the left of O.

Displacement and distance

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction). For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is -2. So the displacement of the particle is -2 units. However, the distance it has travelled is 12 units.

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Velocity and speed

You are already familiar with rates of change through your studies in Mathematical Methods.

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_1 is the position at time t_1 and x_2 is the position at time t_2.

Instantaneous velocity

The instantaneous rate of change of position with respect to time is **instantaneous velocity**. We will refer to the instantaneous velocity as simply the **velocity**.

If a particle's position, x, at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

If x is the position of a particle at time t, then

$$\text{velocity } v = \frac{dx}{dt}$$

Note: Velocity is also denoted by $\dot{x}$ or $\dot{x}(t)$.

Velocity is a vector quantity. For motion in a straight line, the direction is specified by the sign of the numerical value.

If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

Speed and average speed

Speed is a scalar quantity; its value is always non-negative.

- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is given by $\dfrac{\text{distance travelled}}{t_2 - t_1}$

Units of measurement

Common units for velocity (and speed) are:

1 metre per second = 1 m/s = 1 m s^{-1}

1 centimetre per second = 1 cm/s = 1 cm s^{-1}

1 kilometre per hour = 1 km/h = 1 km h^{-1}

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The first and third units are connected in the following way:

$$1 \text{ km/h} = 1000 \text{ m/h} = \frac{1000}{60 \times 60} \text{ m/s} = \frac{5}{18} \text{ m/s}$$

$$\therefore \quad 1 \text{ m/s} = \frac{18}{5} \text{ km/h}$$

Example 2

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = 3t - t^3$, for $t \geq 0$. Find:

a its initial position

b its position when $t = 2$

c its initial velocity

d its velocity when $t = 2$

e its speed when $t = 2$

f when and where the velocity is zero.

Solution

a When $t = 0$, $x = 0$. The particle is initially at O.

b When $t = 2$, $x = 3 \times 2 - 8 = -2$. The particle is 2 cm to the left of O.

c Given $x = 3t - t^3$, the velocity is

$$v = \frac{dx}{dt} = 3 - 3t^2$$

When $t = 0$, $v = 3 - 3 \times 0 = 3$.

The velocity is 3 cm/s. The particle is initially moving to the right.

d When $t = 2$, $v = 3 - 3 \times 4 = -9$.

The velocity is -9 cm/s. The particle is moving to the left.

e When $t = 2$, the speed is 9 cm/s. (The speed is the magnitude of the velocity.)

f $v = 0$ implies

$$3 - 3t^2 = 0$$

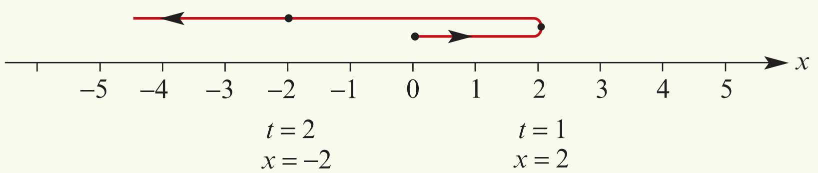
$$3(1 - t^2) = 0$$

$$\therefore \quad t = 1 \text{ or } t = -1$$

But $t \geq 0$ and so $t = 1$. When $t = 1$, $x = 3 \times 1 - 1 = 2$.

At time $t = 1$ second, the particle is at rest 2 cm to the right of O.

Note: The motion of the particle can now be shown on a number line.



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Example 3

The motion of a particle moving along a straight line is defined by $x(t) = t^2 - t$, where x m is the position of the particle relative to O at time t seconds ($t \geq 0$). Find:

a the average velocity of the particle in the first 3 seconds

b the distance travelled by the particle in the first 3 seconds

c the average speed of the particle in the first 3 seconds.

Solution

a
$$\text{Average velocity} = \frac{x(3) - x(0)}{3} = \frac{6-0}{3} = 2 \text{ m/s}$$

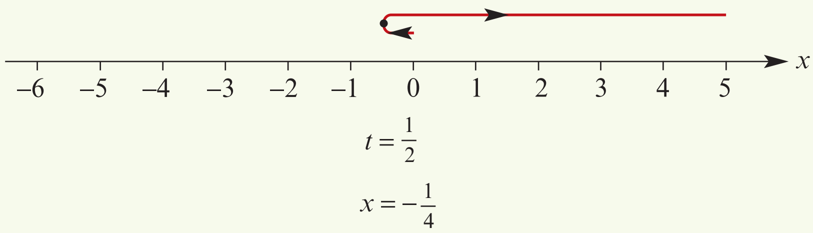
b To find the distance travelled in the first 3 seconds, it is useful to show the motion of the particle on a number line. The critical points are where it starts and where and when it changes direction.

The particle starts at the origin. The turning points occur when the velocity is zero.

We have $v = \dfrac{dx}{dt} = 2t - 1$. Therefore $v = 0$ when $t = \frac{1}{2}$.

The particle changes direction when $t = \frac{1}{2}$ and $x = (\frac{1}{2})^2 - \frac{1}{2} = -\frac{1}{4}$.

When $0 \leq t < \frac{1}{2}$, v is negative and when $t > \frac{1}{2}$, v is positive.



From the number line, the particle travels a distance of $\frac{1}{4}$ m in the first $\frac{1}{2}$ second. It then changes direction. When $t = 3$, the particle's position is $x(3) = 6$ m to the right of O, so the particle has travelled a distance of $6 + \frac{1}{4} = 6\frac{1}{4}$ m from when it changed direction.

The total distance travelled by the particle in the first 3 seconds is $\frac{1}{4} + 6\frac{1}{4} = 6\frac{1}{2}$ m.

c
$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}} = 6\tfrac{1}{2} \div 3 = \frac{13}{2} \div 3 = \frac{13}{6} \text{ m/s}$$

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Acceleration

The acceleration of a particle is the rate of change of its velocity with respect to time.

- **Average acceleration** for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Note: The second derivative $\dfrac{d^2x}{dt^2}$ is also denoted by $\ddot{x}$ or $\ddot{x}(t)$.

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

Example 4

An object travelling in a horizontal line has position x metres, relative to an origin O, at time t seconds, where $x = -4t^2 + 8t + 12$, $t \geq 0$.

a Sketch the position–time graph, showing key features.

b Find the velocity at time t seconds and sketch the velocity–time graph.

c Find the acceleration at time t seconds and sketch the acceleration–time graph.

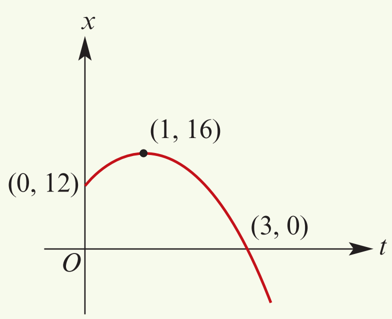
d Represent the motion of the object on a number line.

e Find the displacement of the object in the third second.

f Find the distance travelled in the first 3 seconds.

Solution

a $x = -4t^2 + 8t + 12$, for $t \geq 0$



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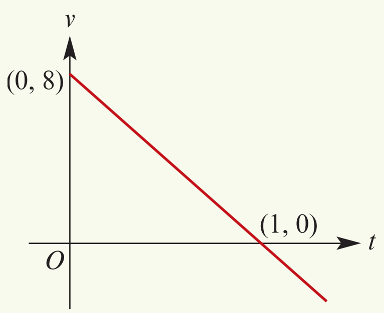
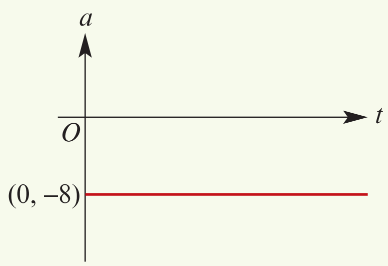
b $v = \dfrac{dx}{dt} = -8t + 8$, for $t \geq 0$



When $t \in [0, 1)$, the velocity is positive.
When $t > 1$, the velocity is negative.

c $a = \dfrac{dv}{dt} = -8$, for $t \geq 0$

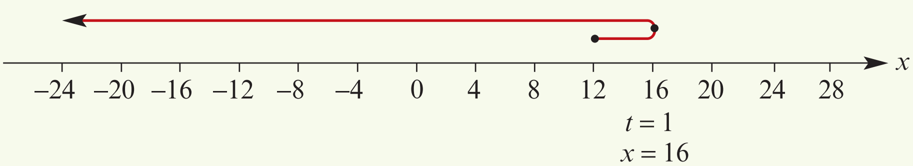


The acceleration is -8 m/s^2.
The direction of the acceleration is always to the left.

d Starting point: When $t = 0$, $x = 12$.

Turning point: When $v = -8t + 8 = 0$, $t = 1$ and $x = 16$.

When $0 \leq t < 1$, $v > 0$ and when $t > 1$, $v < 0$. That is, when $0 \leq t < 1$, the object is moving to the right, and when $t > 1$, the object is moving to the left.



e The displacement of the object in the third second is given by

$$\begin{aligned} x(3) - x(2) &= 0 - 12 \\ &= -12 \end{aligned}$$

The displacement is 12 metres to the left.

f From the position–time graph in **a**, the distance travelled in the first 3 seconds is $4 + 16 = 20$ m.

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Example 5

An object moves in a horizontal line such that its position, x m, relative to a fixed point at time t seconds is given by $x = -t^3 + 3t + 2$, $t \geq 0$. Find:

a when the position is zero, and the velocity and acceleration at that time

b when the velocity is zero, and the position and acceleration at that time

c when the acceleration is zero, and the position and velocity at that time

d the distance travelled in the first 3 seconds.

Solution

Now $x = -t^3 + 3t + 2$

$$v = \dot{x} = -3t^2 + 3$$

$$a = \ddot{x} = -6t$$

(The acceleration is variable in this case.)

a $x = 0$ when $\quad -t^3 + 3t + 2 = 0$

$$t^3 - 3t - 2 = 0$$

$$(t - 2)(t + 1)^2 = 0$$

Therefore $t = 2$, since $t \geq 0$.

At $t = 2$, $v = -3 \times 2^2 + 3 = -9$.

At $t = 2$, $a = -6 \times 2 = -12$.

When the position is zero, the velocity is -9 m/s and the acceleration is -12 m/s^2.

b $v = 0$ when $-3t^2 + 3 = 0$

$$t^2 = 1$$

Therefore $t = 1$, since $t \geq 0$.

At $t = 1$, $x = -1^3 + 3 \times 1 + 2 = 4$.

At $t = 1$, $a = -6 \times 1 = -6$.

When the object is at rest, the position is 4 m and the acceleration is -6 m/s^2.

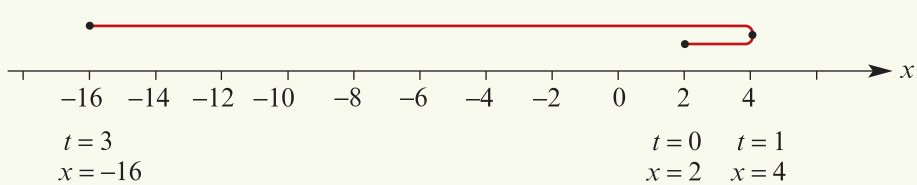
c $a = 0$ when $-6t = 0$

$$\therefore \quad t = 0$$

At $t = 0$, $x = 2$ and $v = 3$.

When the object has zero acceleration, the position is 2 m and the velocity is 3 m/s.

d



The distance travelled is $2 + 4 + 16 = 22$ metres.

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Using antidifferentiation

In the previous examples, we were given a rule for the position of a particle in terms of time, and from it we derived rules for the velocity and the acceleration by differentiation.

We may be given a rule for the acceleration of a particle in terms of time, and by using antidifferentiation and some additional information, we can deduce rules for both velocity and position.

Example 6

The acceleration of a particle moving in a straight line, in m/s^2, is given by

$$\frac{d^2y}{dt^2} = \cos(\pi t)$$

at time t seconds. The particle's initial velocity is 3 m/s and its initial position is $y = 6$. Find its position, y m, at time t seconds.

Solution

Find the velocity by antidifferentiating the acceleration:

$$\begin{aligned}\frac{dy}{dt} &= \int \frac{d^2y}{dt^2}\, dt \\ &= \int \cos(\pi t)\, dt \\ &= \frac{1}{\pi}\sin(\pi t) + c\end{aligned}$$

When $t = 0$, $\frac{dy}{dt} = 3$, so $c = 3$.

$$\therefore \quad \frac{dy}{dt} = \frac{1}{\pi}\sin(\pi t) + 3$$

Antidifferentiating again:

$$\begin{aligned}y &= \int \frac{dy}{dt}\, dt \\ &= \int \frac{1}{\pi}\sin(\pi t) + 3\, dt \\ &= -\frac{1}{\pi^2}\cos(\pi t) + 3t + d\end{aligned}$$

When $t = 0$, $y = 6$:

$$6 = -\frac{1}{\pi^2} + d$$

$$\therefore \quad d = \frac{1}{\pi^2} + 6$$

Hence $y = -\frac{1}{\pi^2}\cos(\pi t) + 3t + \frac{1}{\pi^2} + 6$

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Example 7

A cricket ball projected vertically upwards from ground level experiences a gravitational acceleration of 9.8 m/s^2. If the initial speed of the cricket ball is 25 m/s, find:

a its speed after 2 seconds

b its height after 2 seconds

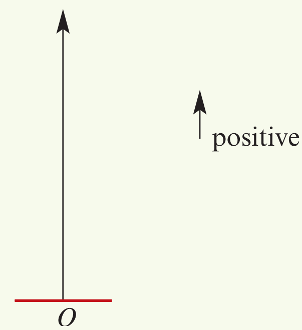
c the greatest height

d the time it takes to return to ground level.

Solution

A frame of reference is required. The path of the cricket ball is considered as a vertical straight line with origin O at ground level. Vertically up is taken as the positive direction.

We are given $a = -9.8$, $v(0) = 25$ and $x(0) = 0$.



a $a = \dfrac{dv}{dt} = -9.8$

$$v = \int \frac{dv}{dt}\, dt = \int -9.8\, dt = -9.8t + c$$

Since $v(0) = 25$, we have $c = 25$ and therefore

$$v = -9.8t + 25$$

When $t = 2$, $v = -9.8 \times 2 + 25 = 5.4$.
The speed of the cricket ball is 5.4 m/s after 2 seconds.

b $v = \dfrac{dx}{dt} = -9.8t + 25$

$$x = \int -9.8t + 25\, dt = -4.9t^2 + 25t + d$$

Since $x(0) = 0$, we have $d = 0$ and therefore

$$x = -4.9t^2 + 25t$$

When $t = 2$, $x = -19.6 + 50 = 30.4$.
The ball is 30.4 m above the ground after 2 seconds.

c The greatest height is reached when the ball is instantaneously at rest, i.e. when $v = -9.8t + 25 = 0$, which implies $t = \dfrac{25}{9.8}$.

When $t = \dfrac{25}{9.8}$, $x = -4.9 \times \left(\dfrac{25}{9.8}\right)^2 + 25 \times \dfrac{25}{9.8} \approx 31.89$.

The greatest height reached is 31.89 m.

d The cricket ball reaches the ground again when $x = 0$.

$$x = 0 \text{ implies } \quad 25t - 4.9t^2 = 0$$
$$t(25 - 4.9t) = 0$$
$$\therefore \quad t = 0 \text{ or } t = \frac{25}{4.9}$$

The ball returns to ground level after $\dfrac{25}{4.9} \approx 5.1$ seconds.

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Example 8

A particle travels in a line such that its velocity, v m/s, at time t seconds is given by

$$v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right), \quad t \geq 0$$

The initial position of the particle is $-2\sqrt{2}$ m, relative to O.

a **i** Find the particle's initial velocity.

ii Find the particle's maximum and minimum velocities.

iii For $0 \leq t \leq 4\pi$, find the times when the particle is instantaneously at rest.

iv Determine the period of the motion.

Use this information to sketch the graph of velocity against time.

b **i** Find the particle's position at time t.

ii Find the particle's maximum and minimum position.

iii Find when the particle first passes through the origin.

iv Find the relation between the particle's velocity and position.

c **i** Find the particle's acceleration at time t.

ii Find the particle's maximum and minimum acceleration.

iii Find the relation between the particle's acceleration and position.

iv Find the relation between the particle's acceleration and velocity.

d Use the information obtained in **a–c** to describe the motion of the particle.

Solution

a **i** $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

At $t = 0$, $\quad v = 2\cos\left(-\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ m/s.

ii By inspection, $v_{max} = 2$ m/s and $v_{min} = -2$ m/s.

iii $v = 0$ implies

$$\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right) = 0$$

$$\frac{1}{2}t - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\frac{1}{2}t = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$$

$$t = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$$

For $0 \leq t \leq 4\pi$, the velocity is zero at $t = \frac{3\pi}{2}$ and $t = \frac{7\pi}{2}$.

iv The period of $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ is $2\pi \div \frac{1}{2} = 4\pi$ seconds.

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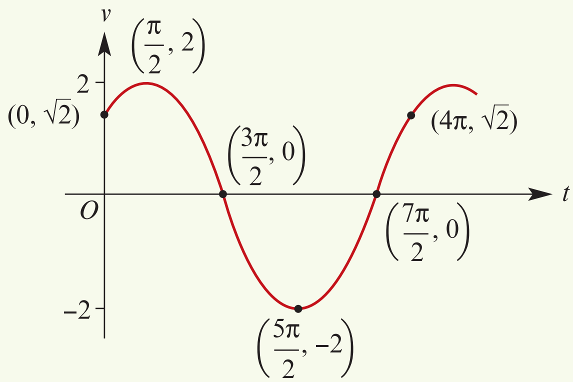
$v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$



b **i** $x = \int v\,dt = \int 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)dt$

Let $u = \frac{1}{2}t - \frac{\pi}{4}$. Then $\frac{du}{dt} = \frac{1}{2}$ and so

$$x = 2\int 2\cos u \frac{du}{dt}\,dt$$
$$= 4\int \cos u\,du$$
$$= 4\sin u + c$$

$\therefore \quad x = 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right) + c$

Substituting $x = -2\sqrt{2}$ at $t = 0$:

$$-2\sqrt{2} = 4\sin\left(-\frac{\pi}{4}\right) + c$$
$$-2\sqrt{2} = 4 \times \left(-\frac{1}{\sqrt{2}}\right) + c$$

$\therefore \quad c = 0$

Hence $\quad x = 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

ii By inspection, $x_{max} = 4$ m and $x_{min} = -4$ m.

iii The particle passes through the origin when $x = 0$, which implies

$$\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right) = 0$$
$$\frac{1}{2}t - \frac{\pi}{4} = 0,\ \pi,\ 2\pi,\ \dots$$
$$\frac{1}{2}t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \dots$$

$\therefore \quad t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots$

Thus the particle first passes through the origin at $t = \frac{\pi}{2}$ seconds.

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iv We have $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ and $x = 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$.

Using the Pythagorean identity:

$$\cos^2\left(\frac{1}{2}t - \frac{\pi}{4}\right) + \sin^2\left(\frac{1}{2}t - \frac{\pi}{4}\right) = 1$$

This gives

$$\left(\frac{v}{2}\right)^2 + \left(\frac{x}{4}\right)^2 = 1$$

$$\frac{v}{2} = \pm\sqrt{1 - \frac{x^2}{16}}$$

$$\frac{v}{2} = \pm\frac{1}{4}\sqrt{16 - x^2}$$

$$\therefore \quad v = \pm\frac{1}{2}\sqrt{16 - x^2}$$

c **i** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)\right)$

$\therefore \quad a = -\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ (using the chain rule)

ii By inspection, $a_{max} = 1 \text{ m/s}^2$ and $a_{min} = -1 \text{ m/s}^2$.

iii We have $a = -\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ and $x = 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$.

Therefore $a = -\dfrac{x}{4}$.

iv We have $a = -\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ and $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$.

Using the Pythagorean identity again:

$$a^2 + \left(\frac{v}{2}\right)^2 = 1$$

$$a = \pm\sqrt{1 - \frac{v^2}{4}}$$

$$\therefore \quad a = \pm\frac{1}{2}\sqrt{4 - v^2}$$

d The particle oscillates between positions ±4 m, relative to O, taking 4π seconds for each cycle. The particle's velocity oscillates between ±2 m/s, and its acceleration oscillates between $\pm 1 \text{ m/s}^2$.

Maximum and minimum acceleration occurs when the particle is at the maximum distance from the origin; this is where the particle is instantaneously at rest.

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Skill-sheet

Exercise 12A

Example 2
Example 3

1 The position of a particle travelling in a horizontal line, relative to a point O on the line, is x metres at time t seconds. The position is described by $x = 3t - t^2$, $t \geq 0$.

- **a** Find the position of the particle at times $t = 0, 1, 2, 3, 4$ and illustrate the motion of the particle on a number line.
- **b** Find the displacement of the particle in the fifth second.
- **c** Find the average velocity in the first 4 seconds.
- **d** Find the relation between velocity, v m/s, and time, t s.
- **e** Find the velocity of the particle when $t = 2.5$.
- **f** Find when and where the particle changes direction.
- **g** Find the distance travelled in the first 4 seconds.
- **h** Find the particle's average speed for the first 4 seconds.

Example 4

2 An object travelling in a horizontal line has position x metres, relative to an origin O, at time t seconds, where $x = -3t^2 + 10t + 8$, $t \geq 0$.

- **a** Sketch the position–time graph, showing key features.
- **b** Find the velocity at time t seconds and sketch the velocity–time graph.
- **c** Find the acceleration at time t seconds and sketch the acceleration–time graph.
- **d** Represent the motion of the object on a number line for $0 \leq t \leq 6$.
- **e** Find the displacement of the object in the third second.
- **f** Find the distance travelled in the first 3 seconds.

Example 5

3 A particle travels in a straight line through a fixed point O. Its position, x metres, relative to O is given by $x = t^3 - 9t^2 + 24t$, $t \geq 0$, where t is the time in seconds after passing O. Find:

- **a** the values of t for which the velocity is instantaneously zero
- **b** the acceleration when $t = 5$
- **c** the average velocity of the particle during the first 2 seconds
- **d** the average speed of the particle during the first 4 seconds.

4 A particle moves in a straight line. Relative to a fixed point O on the line, the particle's position, x m, at time t seconds is given by $x = t(t - 3)^2$. Find:

- **a** the velocity of the particle after 2 seconds
- **b** the values of t for which the particle is instantaneously at rest
- **c** the acceleration of the particle after 4 seconds.

5 A particle moving in a straight line has position given by $x = 2t^3 - 4t^2 - 100$. Find the time(s) when the particle has zero velocity.

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6 A particle moving in a straight line passes through a fixed point O. Its velocity, v m/s, at time t seconds after passing O is given by $v = 4 + 3t - t^2$. Find:

a the maximum value of v **b** the distance of the particle from O when $t = 4$.

7 A particle moves in a straight line such that, at time t seconds after passing through a fixed point O, its velocity, v m/s, is given by $v = 3t^2 - 30t + 72$. Find:

a the initial acceleration of the particle

b the two values of t for which the particle is instantaneously at rest

c the distance moved by the particle during the interval between these two values

d the total distance moved by the particle between $t = 0$ and $t = 7$.

Example 6

8 A particle moving in a straight line passes through a fixed point O with velocity 8 m/s. Its acceleration, a m/s^2, at time t seconds after passing O is given by $a = 12 - 6t$. Find:

a the velocity of the particle when $t = 2$

b the displacement of the particle from O when $t = 2$.

9 A particle moving in a straight line passes through a fixed point O on the line with a velocity of 30 m/s. The acceleration, a m/s^2, of the particle at time t seconds after passing O is given by $a = 13 - 6t$. Find:

a the velocity of the particle 3 seconds after passing O

b the time taken to reach the maximum distance from O in the initial direction of motion

c the value of this maximum distance.

Example 7

10 An object is dropped down a well. It takes 2 seconds to reach the bottom. During its fall, the object travels under a gravitational acceleration of 9.8 m/s^2.

a Find an expression in terms of t for:

i the velocity, v m/s **ii** the position, x m, measured from the top of the well.

b Find the depth of the well.

c At what speed does the object hit the bottom of the well?

Example 8

11 An object travels in a line such that its velocity, v m/s, at time t seconds is given by $v = \cos\left(\frac{t}{2}\right)$, $t \in [0, 4\pi]$. The initial position of the object is 0.5 m, relative to O.

a Find an expression for the position, x m, of the object in terms of t.

b Sketch the position–time graph for the motion, indicating clearly the values of t at which the object is instantaneously at rest.

c Find an expression for the acceleration, a m/s^2, of the object in terms of t.

d Find a relation (not involving t) between:

i position and acceleration **ii** position and velocity

iii velocity and acceleration.

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12 A particle moves horizontally in a line such that its position, x m, relative to O at time t seconds is given by $x = t^3 - \frac{15}{2}t^2 + 12t + 10$. Find:

a when and where the particle has zero velocity

b the average velocity during the third second

c the velocity at $t = 2$

d the distance travelled in the first 2 seconds

e the closest the particle comes to O.

13 An object moves in a line such that at time t seconds the acceleration, $\ddot{x}$ m/s^2, is given by $\ddot{x} = 2\sin\left(\frac{1}{2}t\right)$. The initial velocity is 1 m/s.

a Find the maximum velocity.

b Find the time taken for the object to first reach the maximum velocity.

14 From a balloon ascending with a velocity of 10 m/s, a stone was dropped and reached the ground in 12 seconds. Given that the gravitational acceleration is 9.8 m/s^2, find:

a the height of the balloon when the stone was dropped

b the greatest height reached by the stone.

15 An object moves in a line with acceleration, $\ddot{x}$ m/s^2, given by $\ddot{x} = \frac{1}{(2t+3)^2}$. If the object starts from rest at the origin, find the position–time relationship.

16 A particle moves in a line with acceleration, $\ddot{x}$ m/s^2, given by $\ddot{x} = \frac{2t}{(1+t^2)^2}$. If the initial velocity is 0.5 m/s, find the distance travelled in the first $\sqrt{3}$ seconds.

17 An object moves in a line with velocity, $\dot{x}$ m/s, given by $\dot{x} = \frac{t}{1+t^2}$. The object starts from the origin. Find:

a the initial velocity

b the maximum velocity

c the distance travelled in the third second

d the position–time relationship

e the acceleration–time relationship

f the average acceleration over the third second

g the minimum acceleration.

18 An object moves in a horizontal line such that its position, x m, at time t seconds is given by $x = 2 + \sqrt{t+1}$. Find when the acceleration is -0.016 m/s^2.

19 A particle moves in a straight line such that the position, x metres, of the particle relative to a fixed origin at time t seconds is given by $x = 2\sin t + \cos t$, for $t \geq 0$. Find the first value of t for which the particle is instantaneously at rest.

20 The acceleration of a particle moving in a straight line, in m/s^2, at time t seconds is given by $\frac{d^2x}{dt^2} = 8 - e^{-t}$. If the initial velocity is 3 m/s, find the velocity when $t = 2$.

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12B Constant acceleration

If an object is moving due to a constant force (for example, gravity), then its acceleration is constant. There are several useful formulas that apply in this situation.

Formulas for constant acceleration

For a particle moving in a straight line with constant acceleration a, we can use the following formulas, where u is the initial velocity, v is the final velocity, s is the displacement and t is the time taken:

1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$

Proof 1 We can write

$$\frac{dv}{dt} = a$$

where a is a constant and v is the velocity at time t. By antidifferentiating with respect to t, we obtain

$$v = at + c$$

where the constant c is the initial velocity. We denote the initial velocity by u, and therefore $v = u + at$.

2 We now write

$$\frac{dx}{dt} = v = u + at$$

where x is the position at time t. By antidifferentiating again, we have

$$x = ut + \frac{1}{2}at^2 + d$$

where the constant d is the initial position. The particle's displacement (change in position) is given by $s = x - d$, and so we obtain the second equation.

3 Transform the first equation $v = u + at$ to make t the subject:

$$t = \frac{v - u}{a}$$

Now substitute this into the second equation:

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{u(v - u)}{a} + \frac{a(v - u)^2}{2a^2}$$

$$\begin{aligned} 2as &= 2u(v - u) + (v - u)^2 \\ &= 2uv - 2u^2 + v^2 - 2uv + u^2 \\ &= v^2 - u^2 \end{aligned}$$

4 Similarly, the fourth equation can be derived from the first and second equations.

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These four formulas are very useful, but it must be remembered that they only apply when the acceleration is constant.

When approaching problems involving constant acceleration, it is a good idea to list the quantities you are given, establish which quantity or quantities you require, and then use the appropriate formula. Ensure that all quantities are converted to compatible units.

Example 9

An object is moving in a straight line with uniform acceleration. Its initial velocity is 12 m/s and after 5 seconds its velocity is 20 m/s. Find:

a the acceleration

b the distance travelled during the first 5 seconds

c the time taken to travel a distance of 200 m.

Solution

We are given $u = 12$, $v = 20$ and $t = 5$.

a Find a using

$$v = u + at$$
$$20 = 12 + 5a$$
$$a = 1.6$$

The acceleration is 1.6 m/s^2.

b Find s using

$$s = ut + \frac{1}{2}at^2$$
$$= 12(5) + \frac{1}{2}(1.6)5^2 = 80$$

The distance travelled is 80 m.

Note: Since the object is moving in one direction, the distance travelled is equal to the displacement.

c We are now given $a = 1.6$, $u = 12$ and $s = 200$.

Find t using $s = ut + \frac{1}{2}at^2$

$$200 = 12t + \frac{1}{2} \times 1.6 \times t^2$$
$$200 = 12t + \frac{4}{5}t^2$$
$$1000 = 60t + 4t^2$$
$$250 = 15t + t^2$$
$$t^2 + 15t - 250 = 0$$
$$(t - 10)(t + 25) = 0$$
$$\therefore \quad t = 10 \text{ or } t = -25$$

As $t \geq 0$, the only allowable solution is $t = 10$.

The object takes 10 s to travel a distance of 200 m.

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Example 10

A body is moving in a straight line with uniform acceleration and an initial velocity of 12 m/s. If the body stops after 20 metres, find the acceleration of the body.

Solution

We are given $u = 12$, $v = 0$ and $s = 20$.

Find a using

$$v^2 = u^2 + 2as$$
$$0 = 144 + 2 \times a \times 20$$
$$0 = 144 + 40a$$
$$\therefore \quad a = -\frac{144}{40}$$

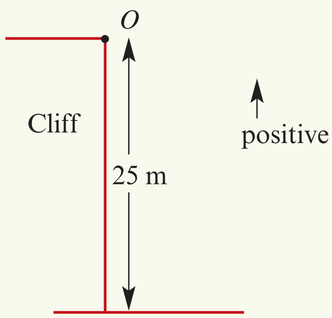
The acceleration is $-\frac{18}{5}$ m/s^2.

Example 11

A stone is thrown vertically upwards from the top of a cliff which is 25 m high. The velocity of projection of the stone is 22 m/s. Find the time it takes to reach the base of the cliff. (Give answer correct to two decimal places.)

Solution

Take the origin at the top of the cliff and vertically upwards as the positive direction.



We are given $s = -25$, $u = 22$ and $a = -9.8$.

Find t using

$$s = ut + \tfrac{1}{2}at^2$$
$$-25 = 22t + \tfrac{1}{2} \times (-9.8) \times t^2$$
$$-25 = 22t - 4.9t^2$$

Therefore

$$4.9t^2 - 22t - 25 = 0$$

By the quadratic formula:

$$t = \frac{22 \pm \sqrt{22^2 - 4 \times 4.9 \times (-25)}}{2 \times 4.9}$$

$$\therefore \quad t = 5.429\ldots \text{ or } t = -0.9396\ldots$$

But $t \geq 0$, so the only allowable solution is $t = 5.429\ldots$.

It takes 5.43 seconds for the stone to reach the base of the cliff.

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Skill-sheet

Exercise 12B

1 An object with constant acceleration starts with a velocity of 15 m/s. At the end of the eleventh second, its velocity is 48 m/s. What is its acceleration?

2 A car accelerates uniformly from 5 km/h to 41 km/h in 10 seconds. Express this acceleration in:

a km/h^2 **b** m/s^2

Example 9

3 An object is moving in a straight line with uniform acceleration. Its initial velocity is 10 m/s and after 5 seconds its velocity is 25 m/s. Find:

a the acceleration

b the distance travelled during the first 5 seconds

c the time taken to travel a distance of 100 m.

Example 10

4 A body moving in a straight line has uniform acceleration and an initial velocity of 20 m/s. If the body stops after 40 metres, find the acceleration of the body.

5 A particle starts from a fixed point O with an initial velocity of -10 m/s and a uniform acceleration of 4 m/s^2. Find:

a the displacement of the particle from O after 6 seconds

b the velocity of the particle after 6 seconds

c the time when the velocity is zero

d the distance travelled in the first 6 seconds.

Example 11

6 **a** A stone is thrown vertically upwards from ground level at 21 m/s. The acceleration due to gravity is 9.8 m/s^2.

i What is its height above the ground after 2 seconds?

ii What is the maximum height reached by the stone?

b If the stone is thrown vertically upwards from a cliff 17.5 m high at 21 m/s:

i How long will it take to reach the ground at the base of the cliff?

ii What is the velocity of the stone when it hits the ground?

7 A basketball is thrown vertically upwards with a velocity of 14 m/s. The acceleration due to gravity is 9.8 m/s^2. Find:

a the time taken by the ball to reach its maximum height

b the greatest height reached by the ball

c the time taken for the ball to return to the point from which it is thrown.

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8 A car sliding on ice is decelerating at the rate of 0.1 m/s^2. Initially the car is travelling at 20 m/s. Find:

a the time taken before it comes to rest

b the distance travelled before it comes to rest.

9 An object is dropped from a point 100 m above the ground. The acceleration due to gravity is 9.8 m/s^2. Find:

a the time taken by the object to reach the ground

b the velocity at which the object hits the ground.

10 An object is projected vertically upwards from a point 50 m above ground level. (Acceleration due to gravity is 9.8 m/s^2.) If the initial velocity is 10 m/s, find:

a the time the object takes to reach the ground (correct to two decimal places)

b the object's velocity when it reaches the ground.

11 A book is pushed across a table and is subjected to a retardation of 0.8 m/s^2 due to friction. (Retardation is acceleration in the opposite direction to motion.) If the initial speed of the book is 1 m/s, find:

a the time taken for the book to stop

b the distance over which the book slides.

12 A box is pushed across a bench and is subjected to a constant retardation, a m/s^2, due to friction. The initial speed of the box is 1.2 m/s and the box travels 3.2 m before stopping. Find:

a the value of a

b the time taken for the box to come to rest.

13 A particle travels in a straight line with a constant velocity of 4 m/s for 12 seconds. It is then subjected to a constant acceleration in the opposite direction for 20 seconds, which returns the particle to its original position. Find the acceleration of the particle.

14 A child slides from rest down a slide 4 m long. The child undergoes constant acceleration and reaches the end of the slide travelling at 2 m/s. Find:

a the time taken to go down the slide

b the acceleration which the child experiences.

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12C Velocity–time graphs

Velocity–time graphs are valuable when considering motion in a straight line.

Information from a velocity–time graph

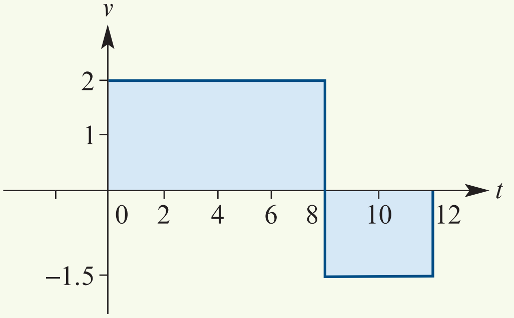
- **Acceleration** is given by the gradient.
- **Displacement** is given by the signed area bounded by the graph and the t-axis.
- **Distance travelled** is given by the total area bounded by the graph and the t-axis.

Example 12

A person walks east for 8 seconds at 2 m/s and then west for 4 seconds at 1.5 m/s. Sketch the velocity–time graph for this journey and find the displacement from the start of the walk and the total distance travelled.

Solution

The velocity–time graph is as shown.



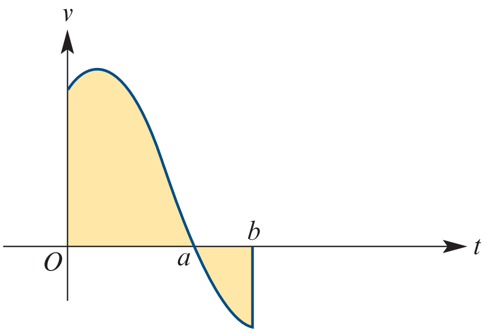
Distance travelled to the east
$= 8 \times 2 = 16$ m

Distance travelled to the west
$= 4 \times 1.5 = 6$ m

Displacement (signed area)
$= 8 \times 2 + 4 \times (-1.5) = 10$ m

Distance travelled (total area)
$= 8 \times 2 + 4 \times 1.5 = 22$ m

Consider a particle moving in a straight line with its motion described by the velocity–time graph shown opposite.



The shaded area represents the total distance travelled by the particle from $t = 0$ to $t = b$.

The signed area represents the displacement (change in position) of the particle for this time interval.

Using integral notation to describe the areas yields the following:

- Distance travelled over the time interval $[0, a]$ $= \int_0^a v(t)\,dt$
- Distance travelled over the time interval $[a, b]$ $= -\int_a^b v(t)\,dt$
- Total distance travelled over the time interval $[0, b]$ $= \int_0^a v(t)\,dt - \int_a^b v(t)\,dt$
- Displacement over the time interval $[0, b]$ $= \int_0^b v(t)\,dt$

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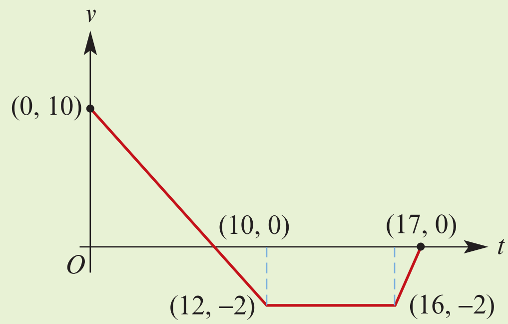
Example 13

The graph shows the motion of a particle.

a Describe the motion.

b Find the distance travelled.

Velocity is measured in m/s and time in seconds.



Solution

a The particle decelerates uniformly from an initial velocity of 10 m/s. After 10 seconds, it is instantaneously at rest before it accelerates uniformly in the opposite direction for 2 seconds, until its velocity reaches −2 m/s. It continues to travel in this direction with a constant velocity of −2 m/s for a further 4 seconds. Finally, it decelerates uniformly until it comes to rest after 17 seconds.

b $$\begin{aligned}\text{Distance travelled} &= (\tfrac{1}{2} \times 10 \times 10) + (\tfrac{1}{2} \times 2 \times 2) + (4 \times 2) + (\tfrac{1}{2} \times 1 \times 2)\\ &= 61 \text{ m}\end{aligned}$$

Example 14

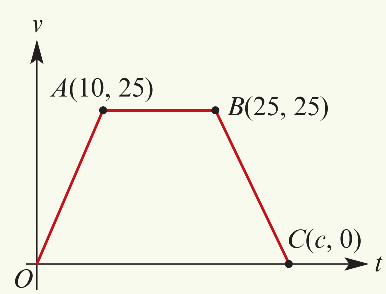
A car travels from rest for 10 seconds, with uniform acceleration, until it reaches a speed of 90 km/h. It then travels with this constant speed for 15 seconds and finally decelerates at a uniform 5 m/s^2 until it stops. Calculate the distance travelled from start to finish.

Solution

First convert the given speed to standard units:

$$90 \text{ km/h} = 90\,000 \text{ m/h} = \frac{90\,000}{3600} \text{ m/s} = 25 \text{ m/s}$$

Now sketch a velocity–time graph showing the given information.



The gradient of BC is −5 (deceleration):

$$\begin{aligned}\text{gradient} = \frac{25}{25 - c} &= -5\\ -5(25 - c) &= 25\\ -125 + 5c &= 25\\ \therefore \qquad c &= 30\end{aligned}$$

Now calculate the distance travelled using the area of trapezium $OABC$:

$$\text{area} = \tfrac{1}{2}(15 + 30) \times 25 = 562.5$$

The total distance travelled in 562.5 metres.

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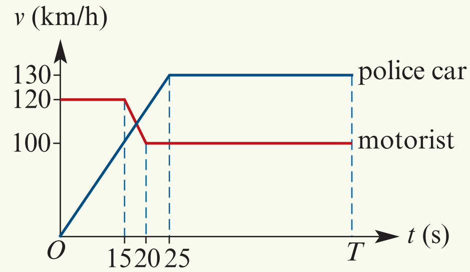
Example 15

A motorist is travelling at a constant speed of 120 km/h when he passes a stationary police car. He continues at that speed for another 15 s before uniformly decelerating to 100 km/h in 5 s. The police car takes off after the motorist the instant that he passes. It accelerates uniformly for 25 s, by which time it has reached 130 km/h. It continues at that speed until it catches up to the motorist. After how long does the police car catch up to the motorist and how far has he travelled in that time?

Solution

We start by representing the information on a velocity–time graph.



The distances travelled by the motorist and the police car will be the same, so the areas under the two velocity–time graphs will be equal. This fact can be used to find T, the time taken for the police car to catch up to the motorist.

Note: The factor $\frac{5}{18}$ changes velocities from km/h to m/s.

The distances travelled (in metres) after T seconds are given by

$$\begin{aligned}\text{Distance for motorist} &= \frac{5}{18}\left(120 \times 15 + \frac{1}{2}(120 + 100) \times 5 + 100(T - 20)\right)\\ &= \frac{5}{18}(1800 + 550 + 100T - 2000)\\ &= \frac{5}{18}(100T + 350)\end{aligned}$$

$$\begin{aligned}\text{Distance for police car} &= \frac{5}{18}\left(\frac{1}{2} \times 25 \times 130 + 130(T - 25)\right)\\ &= \frac{5}{18}(130T - 1625)\end{aligned}$$

When the police car catches up to the motorist:

$$\begin{aligned}100T + 350 &= 130T - 1625\\ 30T &= 1975\\ T &= \frac{395}{6}\end{aligned}$$

The police car catches up to the motorist after 65.83 s.

$$\begin{aligned}\therefore \quad \text{Distance for motorist} &= \frac{5}{18}(100T + 350) \qquad \text{where } T = \frac{395}{6}\\ &= \frac{52\,000}{27}\text{ m}\\ &= 1.926\text{ km}\end{aligned}$$

The motorist has travelled 1.926 km when the police car catches up.

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Example 16

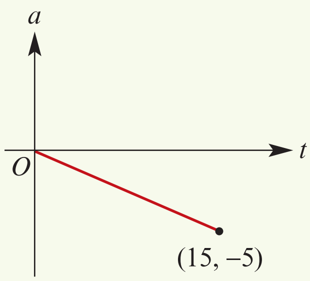
An object travels in a line. Its acceleration decreases uniformly from 0 m/s^2 to −5 m/s^2 in 15 seconds. If the initial velocity was 24 m/s, find:

a the velocity at the end of the 15 seconds

b the distance travelled in the 15 seconds.

Solution

a The acceleration–time graph shows the uniform change in acceleration from 0 m/s^2 to −5 m/s^2 in 15 seconds.



From the graph, we can write $a = mt + c$.

But $m = \frac{-5}{15} = -\frac{1}{3}$ and $c = 0$, giving

$$a = -\frac{1}{3}t$$

$$\therefore \quad v = -\frac{1}{6}t^2 + d$$

At $t = 0$, $v = 24$, so $d = 24$.

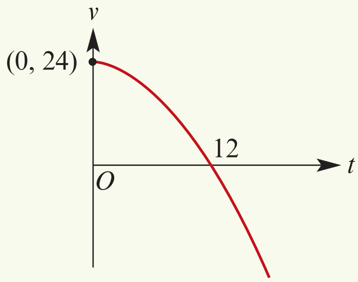
$$\therefore \quad v = -\frac{1}{6}t^2 + 24$$

Now, at $t = 15$,

$$v = -\frac{1}{6} \times 15^2 + 24$$
$$= -13.5$$

The velocity at 15 seconds is −13.5 m/s.

b To sketch the velocity–time graph, first find the t-axis intercepts:

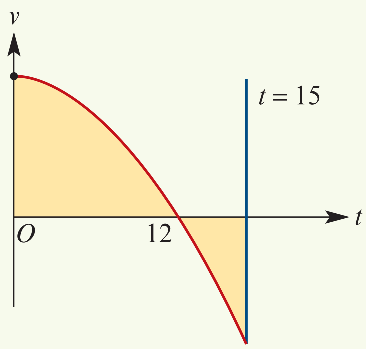


$$-\frac{1}{6}t^2 + 24 = 0$$

$$\therefore \quad t^2 = 144$$

$$\therefore \quad t = 12 \quad (\text{since } t \geq 0)$$

The distance travelled is given by the area of the shaded region.



$$\text{Area} = \int_0^{12} \left(-\frac{1}{6}t^2 + 24\right) dt + \left|\int_{12}^{15} \left(-\frac{1}{6}t^2 + 24\right) dt\right|$$

$$= 192 + |-19.5|$$

$$= 211.5$$

The distance travelled in 15 seconds is 211.5 metres.

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Skill-sheet

Exercise 12C

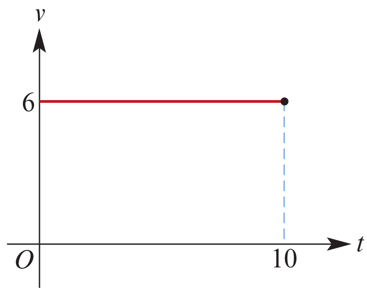
Example 13

1 Each of the following graphs shows the motion of a particle. For each graph:

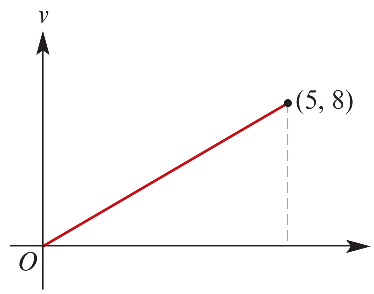
i describe the motion **ii** find the distance travelled.

Velocity is measured in m/s and time in seconds.

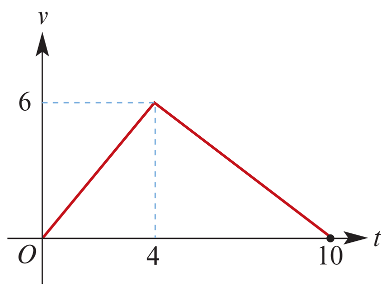
a



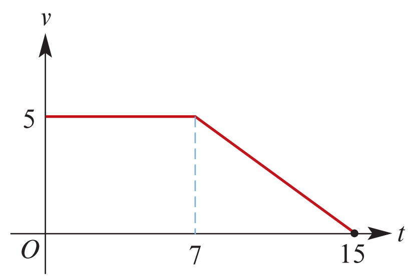
b



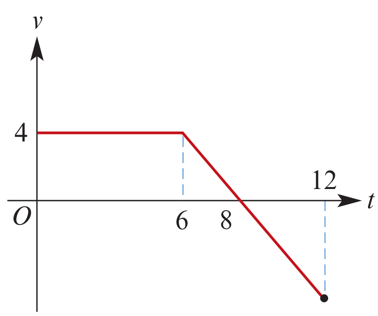
c



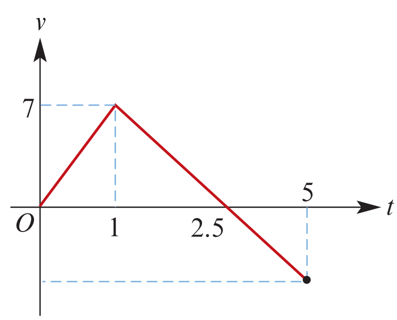
d



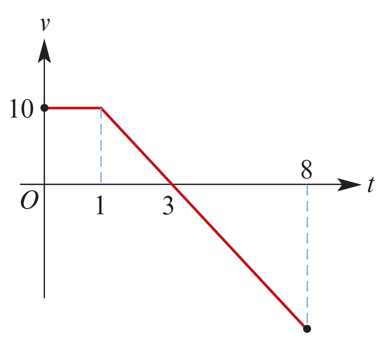
e



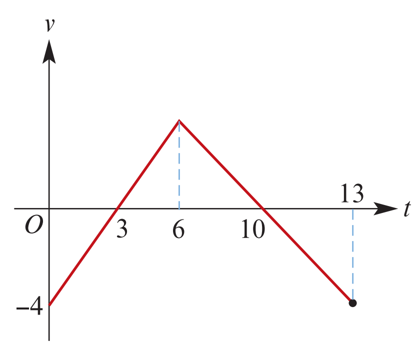
f



g



h



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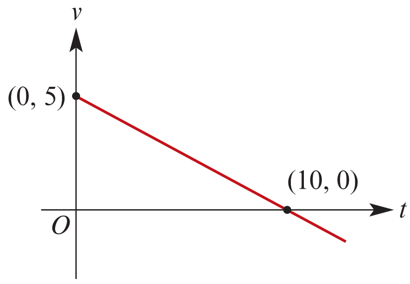
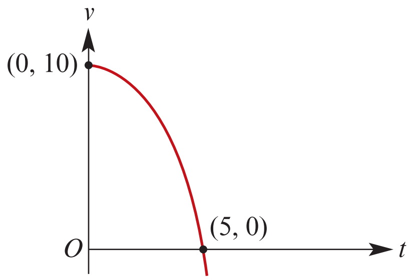
2 For each of the following velocity–time graphs, the object starts from the origin and moves in a line. In each case, find the relationship between time and:

i velocity **ii** acceleration **iii** position.

a

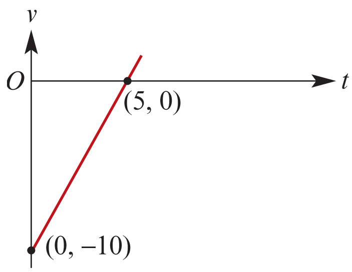


b

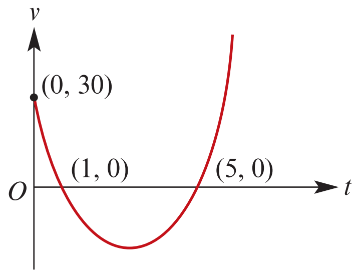


This is a curve of the form $v = at^2 + b$

c

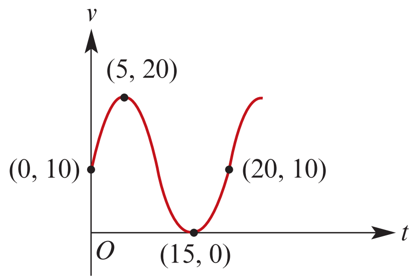


d



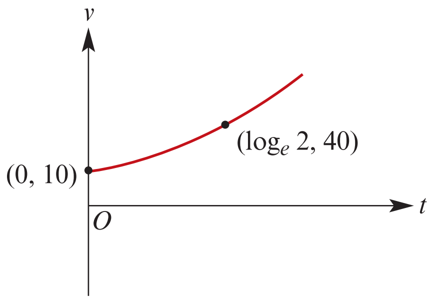
This is a curve of the form $v = at^2 + bt + c$

e



This is a curve of the form $v = a\sin(bt) + c$

f



This is a curve of the form $v = ae^{bt}$

Example 14

3 A car travels from rest for 15 seconds, with uniform acceleration, until it reaches a speed of 100 km/h. It then travels with this constant speed for 120 seconds and finally decelerates at a uniform 8 m/s^2 until it stops. Calculate the total distance travelled.

4 A particle moves in a straight line with a constant velocity of 20 m/s for 10 seconds. It is then subjected to a constant acceleration of 5 m/s^2 in the opposite direction for T seconds, at which time the particle is back to its original position.

a Sketch the velocity–time graph representing the motion.

b Find how long it takes the particle to return to its original position.

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5 An object travels in a line starting from rest. It accelerates uniformly for 3 seconds until it reaches a speed of 14 m/s. It then travels at this speed for 10 seconds. Finally, it decelerates uniformly to rest in 4 seconds. Sketch a velocity–time graph and find the total distance travelled.

6 Two tram stops, A and B, are 500 metres apart. A tram starts from A and travels with acceleration a m/s^2 to a certain point. It then decelerates at $4a$ m/s^2 until it stops at B. The total time taken is 2 minutes. Sketch a velocity–time graph. Find the value of a and the maximum speed reached by the tram.

7 The maximum rate at which a bus can accelerate or decelerate is 2 m/s^2. It has a maximum speed of 60 km/h. Find the shortest time the bus can take to travel between two bus stops 1 km apart on a straight stretch of road.

8 A car being tested on a straight level road starts from rest and accelerates uniformly to 90 km/h. It travels at this speed for a time, then comes to rest with a uniform retardation of 1.25 m/s^2. The total distance travelled is 525 metres and the total time is 36 seconds. Find the time taken in the acceleration phase and how far the car travels at 90 km/h.

Example 15

9 Cars A and B are stationary on a straight road, side by side. Car A moves off with acceleration 1 m/s^2, which it maintains for 20 seconds, after which it moves at constant speed. Car B starts 20 seconds after car A; it sets off with acceleration 2 m/s^2, until it draws level with A. Find the time taken and the distance travelled by B to catch A.

Example 16

10 An object is travelling in a line with an initial velocity of 6 m/s. The deceleration changes uniformly from 1 m/s^2 to 3 m/s^2 over 1 second. If this deceleration continues until the object comes to rest, find:

a the time taken

b the distance travelled.

11 A stationary police motorcycle is passed by a car travelling at 72 km/h. The motorcycle starts in pursuit 3 seconds later. Moving with constant acceleration for a distance of 300 metres, it reaches a speed of 108 km/h, which it maintains. Find the time it takes the motorcyclist to catch the car (from when the motorcycle starts pursuit).

12 Two cars A and B, each moving with constant acceleration, are travelling in the same direction along the parallel lanes of a divided road. When A passes B, the speeds are 64 km/h and 48 km/h respectively. Three minutes later, B passes A, travelling at 96 km/h. Find:

a the distance travelled by A and B at this instant (since they first passed) and the speed of A

b the instant at which both are moving with the same speed, and the distance between them at this time.

13 A particle, starting from rest, falls vertically with acceleration, $\ddot{y}$ m/s^2, at time t seconds given by $\ddot{y} = ke^{-t}$, where $k < 0$.

a Find the velocity–time relationship and sketch the velocity–time graph.

b Briefly describe the motion.

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12D Differential equations of the form $v = f(x)$ and $a = f(v)$

When we are given information about the motion of an object in one of the forms

$$v = f(x) \quad \text{or} \quad a = f(v)$$

we can apply techniques for solving differential equations to obtain other information about the motion.

Example 17

The velocity of a particle moving along a straight line is inversely proportional to its position. The particle is initially 1 m from point O and is 2 m from point O after 1 second.

a Find an expression for the particle's position, x m, at time t seconds.

b Find an expression for the particle's velocity, v m/s, at time t seconds.

Solution

a The information can be written as

$$v = \frac{k}{x} \quad \text{for } k \in \mathbb{R}^+, \quad x(0) = 1 \quad \text{and} \quad x(1) = 2$$

This gives

$$\frac{dx}{dt} = \frac{k}{x}$$

$$\therefore \quad \frac{dt}{dx} = \frac{x}{k}$$

$$\therefore \quad t = \int \frac{x}{k}\,dx$$

$$= \frac{x^2}{2k} + c$$

Since $x(0) = 1$: $\quad 0 = \dfrac{1}{2k} + c \quad (1)$

Since $x(1) = 2$: $\quad 1 = \dfrac{4}{2k} + c \quad (2)$

Subtracting (1) from (2) yields $1 = \dfrac{3}{2k}$ and therefore $k = \dfrac{3}{2}$.

Substituting in (1) yields $c = -\dfrac{1}{2k} = -\dfrac{1}{3}$.

Now $\quad t = \dfrac{x^2}{3} - \dfrac{1}{3}$

$$x^2 = 3t + 1$$

$$\therefore \quad x = \pm\sqrt{3t + 1}$$

But when $t = 0$, $x = 1$ and therefore

$$x = \sqrt{3t + 1}$$

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b $x = \sqrt{3t+1}$ implies

$$v = \frac{dx}{dt} = 3 \times \frac{1}{2} \times \frac{1}{\sqrt{3t+1}}$$
$$= \frac{3}{2\sqrt{3t+1}}$$

Example 18

A body moving in a straight line has an initial velocity of 25 m/s and its acceleration, a m/s^2, is given by $a = -k(50 - v)$, where k is a positive constant and v m/s is its velocity. Find v in terms of t and sketch the velocity–time graph for the motion.

(The motion stops when the body is instantaneously at rest for the first time.)

Solution

$$a = -k(50 - v)$$
$$\frac{dv}{dt} = -k(50 - v)$$
$$\frac{dt}{dv} = \frac{1}{-k(50 - v)}$$
$$t = -\frac{1}{k}\int \frac{1}{50 - v}\, dv$$
$$= -\frac{1}{k}\Big(-\log_e|50 - v|\Big) + c$$

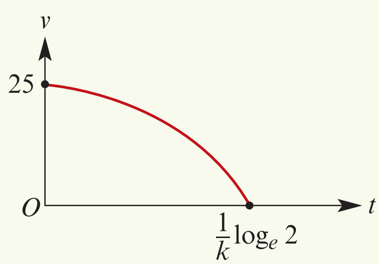
$\therefore \quad t = \frac{1}{k}\log_e(50 - v) + c \qquad$ (Note that $v \le 25$ since $a < 0$.)

When $t = 0$, $v = 25$, and so $c = -\frac{1}{k}\log_e 25$.

Thus $\quad t = \frac{1}{k}\log_e\left(\frac{50 - v}{25}\right)$

$$e^{kt} = \frac{50 - v}{25}$$

$\therefore \quad v = 50 - 25e^{kt}$



Example 19

The acceleration, a, of an object moving along a line is given by $a = -(v + 1)^2$, where v is the velocity of the object at time t. Also $v(0) = 10$ and $x(0) = 0$, where x is the position of the object at time t. Find:

a an expression for the velocity of the object in terms of t

b an expression for the position of the object in terms of t.

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Solution

a $a = -(v+1)^2$ gives

$$\frac{dv}{dt} = -(v+1)^2$$

$$\frac{dt}{dv} = \frac{-1}{(v+1)^2}$$

$$t = -\int \frac{1}{(v+1)^2}\,dv$$

$$\therefore \quad t = \frac{1}{v+1} + c$$

Since $v(0) = 10$, we obtain $c = -\dfrac{1}{11}$ and so

$$t = \frac{1}{v+1} - \frac{1}{11}$$

This can be rearranged as

$$v = \frac{11}{11t+1} - 1$$

b $\dfrac{dx}{dt} = v = \dfrac{11}{11t+1} - 1$

$$\therefore \; x = \int \frac{11}{11t+1} - 1\,dt$$

$$= \log_e |11t+1| - t + c$$

Since $x(0) = 0$, $c = 0$ and therefore $x = \log_e |11t+1| - t$.

Exercise 12D

Example 17

1 A particle moves in a line such that the velocity, $\dot{x}$ m/s, is given by $\dot{x} = \dfrac{1}{2x-4}$, $x > 2$. If $x = 3$ when $t = 0$, find:

- **a** the position at 24 seconds
- **b** the distance travelled in the first 24 seconds.

2 A particle moves in a straight line such that its velocity, v m/s, and position, x m, are related by $v = 1 + e^{-2x}$.

- **a** Find x in terms of time t seconds ($t \geq 0$), given that $x = 0$ when $t = 0$.
- **b** Hence find the acceleration when $t = \log_e 5$.

Example 18

3 An object moves in a straight line such that its acceleration, a m/s^2, and velocity, v m/s, are related by $a = 3 + v$. If the object is initially at rest at the origin, find:

- **a** v in term of t
- **b** a in terms of t
- **c** x in terms of t

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4 An object falls from rest with acceleration, a m/s^2, given by $a = g - kv$, $k > 0$. Find:

a an expression for the velocity, v m/s, at time t seconds

b the terminal velocity, i.e. the limiting velocity as $t \to \infty$.

Example 19

5 A body is projected along a horizontal surface. Its deceleration is $0.3(v^2 + 1)$, where v m/s is the velocity of the body at time t seconds. If the initial velocity is $\sqrt{3}$ m/s, find:

a an expression for v in terms of t

b an expression for x m, the displacement of the body from its original position, in terms of t.

6 The velocity, v m/s, and acceleration, a m/s^2, of an object t seconds after it is dropped from rest are related by $a = \dfrac{450 - v}{50}$ for $v < 450$. Express v in terms of t.

7 The brakes are applied in a car travelling in a straight line. The acceleration, a m/s^2, of the car is given by $a = -0.4\sqrt{225 - v^2}$. If the initial velocity of the car was 12 m/s, find an expression for v, the velocity of the car, in terms of t, the time after the brakes were first applied.

8 An object moves in a straight line such that its velocity is directly proportional to x m, its position relative to a fixed point O on the line. The object starts 5 m to the right of O with a velocity of 2 m/s.

a Express x in terms of t, where t is the time after the motion starts.

b Find the position of the object after 10 seconds.

9 The velocity, v m/s, and the acceleration, a m/s^2, of an object t seconds after it is dropped from rest are related by the equation $a = \dfrac{1}{50}(500 - v)$, $0 \le v < 500$.

a Express t in terms of v.

b Express v in terms of t.

10 A particle is travelling in a horizontal straight line. The initial velocity of the particle is u and the acceleration is given by $-k(2u - v)$, where v is the velocity of the particle at any instant and k is a positive constant. Find the time taken for the particle to come to rest.

11 A boat is moving at 8 m/s. When the boat's engine stops, its acceleration is given by $\dfrac{dv}{dt} = -\dfrac{1}{5}v$. Express v in terms of t and find the velocity when $t = 4$.

12 A particle, initially at a point O, slows down under the influence of an acceleration, a m/s^2, such that $a = -kv^2$, where v m/s is the velocity of the particle at any instant. Its initial velocity is 30 m/s and its initial acceleration is -20 m/s^2. Find:

a its velocity at time t seconds

b its position relative to the point O when $t = 10$.

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12E Other expressions for acceleration

In the earlier sections of this chapter, we have written acceleration as $\dfrac{dv}{dt}$ and $\dfrac{d^2x}{dt^2}$. In this section, we use two further expressions for acceleration.

Expressions for acceleration

$$a = v\,\frac{dv}{dx} \quad \text{and} \quad a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Proof Using the chain rule:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\,\frac{dx}{dt} = \frac{dv}{dx}\,v$$

Using the chain rule again:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} = v\,\frac{dv}{dx} = a$$

The different expressions for acceleration are useful in different situations:

Given	Initial conditions	Useful form
$a = f(t)$	in terms of t and v	$a = \dfrac{dv}{dt}$
$a = f(v)$	in terms of t and v	$a = \dfrac{dv}{dt}$
$a = f(v)$	in terms of x and v	$a = v\,\dfrac{dv}{dx}$
$a = f(x)$	in terms of x and v	$a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

Note: In the last case, it is also possible to use $a = v\,\dfrac{dv}{dx}$ and separation of variables.

Example 20

An object travels in a line such that the velocity, v m/s, is given by $v^2 = 4 - x^2$. Find the acceleration at $x = 1$.

Solution

Given $v^2 = 4 - x^2$, we can use implicit differentiation to obtain:

$$\frac{d}{dx}(v^2) = \frac{d}{dx}(4 - x^2)$$

$$2v\,\frac{dv}{dx} = -2x$$

$$\therefore \quad a = -x$$

So, at $x = 1$, $a = -1$. The acceleration at $x = 1$ is -1 m/s^2.

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Example 21

An object moves in a line so that the acceleration, $\ddot{x}$ m/s^2, is given by $\ddot{x} = 1 + v$. Its velocity at the origin is 1 m/s. Find the position of the object when its velocity is 2 m/s.

Solution

Since we are given a as a function of v and initial conditions involving x and v, it is appropriate to use the form $a = v\dfrac{dv}{dx}$.

Now $\quad \ddot{x} = 1 + v$

$$v\frac{dv}{dx} = 1 + v$$

$$\frac{dv}{dx} = \frac{1+v}{v}$$

$$\frac{dx}{dv} = \frac{v}{1+v}$$

$$\therefore \quad x = \int \frac{v}{1+v}\,dv$$

$$= \int 1 - \frac{1}{1+v}\,dv$$

$$\therefore \quad x = v - \log_e|1+v| + c$$

Since $v = 1$ when $x = 0$, we have

$$0 = 1 - \log_e 2 + c$$

$$\therefore \quad c = \log_e 2 - 1$$

Hence $\quad x = v - \log_e|1+v| + \log_e 2 - 1$

$$= v + \log_e\left(\frac{2}{1+v}\right) - 1 \qquad \text{(as } v > 0\text{)}$$

Now, when $v = 2$,

$$x = 2 + \log_e\left(\tfrac{2}{3}\right) - 1$$

$$= 1 + \log_e\left(\tfrac{2}{3}\right)$$

$$\approx 0.59$$

So, when the velocity is 2 m/s, the position is 0.59 m.

Example 22

A particle is moving in a straight line. Its acceleration, a m/s^2, is described by $a = -\sqrt{x}$, where x m is its position with respect to an origin O. Find a relation between v and x which describes the motion, given that $v = 2$ m/s when the particle is at the origin.

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Solution

Given $a = -\sqrt{x}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -x^{\frac{1}{2}}$$

$$\frac{1}{2}v^2 = -\frac{2}{3}x^{\frac{3}{2}} + c$$

When $x = 0$, $v = 2$, and therefore $c = 2$.

Thus $\frac{1}{2}v^2 = 2 - \frac{2}{3}x^{\frac{3}{2}}$

$\therefore$ $v^2 = \frac{4}{3}\left(3 - x^{\frac{3}{2}}\right)$

Note: This problem can also be solved using $a = v\frac{dv}{dx}$ and separation of variables.

Example 23

An object falls from a hovering helicopter over the ocean 1000 m above sea level. Find the velocity of the object when it hits the water:

a neglecting air resistance **b** assuming air resistance is $0.2v^2$.

Solution

a An appropriate starting point is $\ddot{y} = -9.8$.

Since the initial conditions involve y and v, use $\ddot{y} = \frac{d}{dy}\left(\frac{1}{2}v^2\right)$.

Now $\frac{d}{dy}\left(\frac{1}{2}v^2\right) = -9.8$

$$\frac{1}{2}v^2 = -9.8y + c$$

Using $v = 0$ at $y = 1000$ gives

$$0 = -9.8 \times 1000 + c$$

$\therefore$ $c = 9800$

Hence $\frac{1}{2}v^2 = -9.8y + 9800$

$\therefore$ $v^2 = -19.6y + 19\,600$

The object is falling, so $v < 0$.

$$v = -\sqrt{19\,600 - 19.6y}$$

At sea level, $y = 0$ and therefore

$$v = -\sqrt{19\,600} = -140$$

The object has a velocity of -140 m/s at sea level (504 km/h).

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b In this case, we have

$$\ddot{y} = -9.8 + 0.2v^2$$
$$= \frac{v^2 - 49}{5}$$

Because of the initial conditions given, use $\ddot{y} = v\dfrac{dv}{dy}$:

$$v\frac{dv}{dy} = \frac{v^2 - 49}{5}$$
$$\frac{dv}{dy} = \frac{v^2 - 49}{5v}$$
$$y = \int \frac{5v}{v^2 - 49}\,dv$$
$$= \frac{5}{2}\int \frac{2v}{v^2 - 49}\,dv$$
$$\therefore \quad y = \frac{5}{2}\log_e |v^2 - 49| + c$$

Now, when $v = 0$, $y = 1000$, and so $c = 1000 - \frac{5}{2}\log_e 49$.

$$\therefore \quad y = \frac{5}{2}\log_e |49 - v^2| + 1000 - \frac{5}{2}\log_e 49$$
$$= \frac{5}{2}\left(\log_e |49 - v^2| - \log_e 49\right) + 1000$$
$$= \frac{5}{2}\log_e \left|\frac{49 - v^2}{49}\right| + 1000$$

Assume that $-7 < v < 7$. Then

$$y - 1000 = \frac{5}{2}\log_e\left(1 - \frac{v^2}{49}\right)$$
$$\frac{2}{5}(y - 1000) = \log_e\left(1 - \frac{v^2}{49}\right)$$
$$e^{\frac{2}{5}(y-1000)} = 1 - \frac{v^2}{49}$$
$$\therefore \quad v^2 = 49\left(1 - e^{\frac{2}{5}(y-1000)}\right)$$

But the object is falling and thus $v < 0$. Therefore

$$v = -7\sqrt{1 - e^{\frac{2}{5}(y-1000)}}$$

At sea level, $y = 0$ and therefore

$$v = -7\sqrt{1 - e^{-400}}$$

The object has a velocity of approximately −7 m/s at sea level (25.2 km/h).

Note: If $v < -7$, then $v^2 = 49\left(1 + e^{\frac{2}{5}(y-1000)}\right)$ and the initial conditions are not satisfied.

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Skill-sheet

Exercise 12E

Example 20

1 An object travels in a line such that the velocity, v m/s, is given by $v^2 = 9 - x^2$. Find the acceleration at $x = 2$.

Example 21
Example 22

2 For each of the following, a particle moves in a horizontal line such that, at time t seconds, the position is x m, the velocity is v m/s and the acceleration is a m/s^2.

a If $a = -x$ and $v = 0$ at $x = 4$, find v at $x = 0$.

b If $a = 2 - v$ and $v = 0$ when $t = 0$, find t when $v = -2$.

c If $a = 2 - v$ and $v = 0$ when $x = 0$, find x when $v = -2$.

3 The motion of a particle is in a horizontal line such that, at time t seconds, the position is x m, the velocity is v m/s and the acceleration is a m/s^2.

a If $a = -v^3$ and $v = 1$ when $x = 0$, find v in terms of x.

b If $v = x + 1$ and $x = 0$ when $t = 0$, find:

i x in terms of t **ii** a in terms of t **iii** a in terms of v.

4 An object is projected vertically upwards from the ground with an initial velocity of 100 m/s. Assuming that the acceleration, a m/s^2, is given by $a = -g - 0.2v^2$, find x in terms of v. Hence find the maximum height reached.

5 The velocity, v m/s, of a particle moving along a line is given by $v = 2\sqrt{1 - x^2}$. Find:

a the position, x m, in terms of time t seconds, given that when $t = 0$, $x = 1$

b the acceleration, a m/s^2, in terms of x.

6 Each of the following gives the acceleration, a m/s^2, of an object travelling in a line. Given that $v = 0$ and $x = 0$ when $t = 0$, solve for v in each case.

a $a = \dfrac{1}{1 + t}$ **b** $a = \dfrac{1}{1 + x}$, $x > -1$ **c** $a = \dfrac{1}{1 + v}$

7 A particle moves in a straight line from a position of rest at a fixed origin O. Its velocity is v when its displacement from O is x. If its acceleration is $(2 + x)^{-2}$, find v in terms of x.

8 A particle moves in a straight line and, at time t, its position relative to a fixed origin is x and its velocity is v.

a If its acceleration is $1 + 2x$ and $v = 2$ when $x = 0$, find v when $x = 2$.

b If its acceleration is $2 - v$ and $v = 0$ when $x = 0$, find the position at which $v = 1$.

Example 23

9 A particle is projected vertically upwards. The speed of projection is 50 m/s. The acceleration of the particle, a m/s^2, is given by $a = -\frac{1}{5}(v^2 + 50)$, where v m/s is the velocity of the particle when it is x m above the point of projection. Find:

a the height reached by the particle

b the time taken to reach this highest point.

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Review

Chapter summary

Assign-ment

Nrich

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the origin, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.
- **Displacement** is the change in position (i.e. final position minus initial position).
- **Average velocity** $= \dfrac{\text{change in position}}{\text{change in time}}$
- For a particle moving in a straight line with position x at time t:
  - **velocity** (v) is the rate of change of position with respect to time
  - **acceleration** (a) is the rate of change of velocity with respect to time

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

  - velocity at time t is also denoted by $\dot{x}(t)$
  - acceleration at time t is also denoted by $\ddot{x}(t)$
- Scalar quantities
  - **Distance travelled** means the total distance travelled.
  - **Speed** is the magnitude of the velocity.
  - **Average speed** $= \dfrac{\text{distance travelled}}{\text{change in time}}$
- Constant acceleration

  If acceleration is constant, then the following formulas can be used (for acceleration a, initial velocity u, final velocity v, displacement s and time taken t):

  1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$
- Velocity–time graphs
  - Acceleration is given by the gradient.
  - Displacement is given by the signed area bounded by the graph and the t-axis.
  - Distance travelled is given by the total area bounded by the graph and the t-axis.
- Acceleration $\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

Technology-free questions

1 A particle is moving in a straight line with position, x metres, at time t seconds ($t \geq 0$) given by $x = t^2 - 7t + 10$. Find:

a when its velocity equals zero

b its acceleration at this time

c the distance travelled in the first 5 seconds

d when and where its velocity is -2 m/s.

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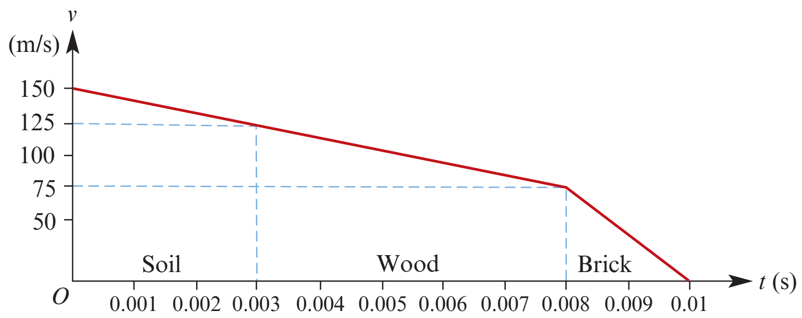
2 An object moves in a straight line so that its acceleration, a m/s^2, at time t seconds ($t \geq 0$) is given by $a = 2t - 3$. Initially, the position of the object is 2 m to the right of a point O and its velocity is 3 m/s. Find the position and velocity after 10 seconds.

3 Two tram stops are 800 m apart. A tram starts at rest from the first stop and accelerates at a constant rate of a m/s^2 for a certain time and then decelerates at a constant rate of $2a$ m/s^2, before coming to rest at the second stop. The time taken to travel between the stops is 1 minute 40 seconds. Find:

a the maximum speed reached by the tram in km/h
b the time at which the brakes are applied
c the value of a.

4 The velocity–time graph shows the journey of a bullet fired into the wall of a practice range made up of three successive layers of soil, wood and brick.



Calculate:

a the deceleration of the bullet as it passes through the soil
b the thickness of the layer of soil
c the deceleration of the bullet as it passes through the wood
d the thickness of the layer of wood
e the deceleration of the bullet passing through the brick
f the depth penetrated by the bullet into the layer of brick.

5 A helicopter climbs vertically from the top of a 110-metre tall building, so that its height in metres above the ground after t seconds is given by $h = 110 + 55t - 5.5t^2$. Calculate:

a the average velocity of the helicopter from $t = 0$ to $t = 2$
b its instantaneous velocity at time t
c its instantaneous velocity at time $t = 1$
d the time at which the helicopter's velocity is zero
e the maximum height reached above the ground.

6 A golf ball is putted across a level putting green with an initial velocity of 8 m/s. Owing to friction, the velocity decreases at the rate of 2 m/s^2. How far will the golf ball roll?

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7 A particle moves in a straight line such that after t seconds its position, x metres, relative to a point O on the line is given by $x = \sqrt{9 - t^2}$, $0 \le t < 3$.

a When is the position $\sqrt{5}$?

b Find expressions for the velocity and acceleration of the particle at time t.

c Find the particle's maximum distance from O.

d When is the velocity zero?

8 A particle moving in a straight line passes through a fixed point O with velocity 8 m/s. Its acceleration, a m/s^2, at time t seconds after passing O is given by $a = 12 - 6t$. Find:

a the velocity of the particle when $t = 2$

b the displacement of the particle from O when $t = 2$.

9 A particle travels at 12 m/s for 5 seconds. It then accelerates uniformly for the next 8 seconds to a velocity of x m/s, and then decelerates uniformly to rest during the next 3 seconds. Sketch a velocity–time graph. Given that the total distance travelled is 218 m, calculate:

a the value of x **b** the average velocity.

10 A ball is thrown vertically upwards from ground level with an initial velocity of 35 m/s. Let g m/s^2 be the acceleration due to gravity. Find:

a the velocity, in terms of g, and the direction of motion of the ball after:

i 3 seconds **ii** 5 seconds

b the total distance travelled by the ball, in terms of g, when it reaches the ground again

c the velocity with which the ball strikes the ground.

11 A car is uniformly accelerated from rest at a set of traffic lights until it reaches a speed of 10 m/s in 5 seconds. It then continues to move at the same constant speed of 10 m/s for 6 seconds before the car's brakes uniformly retard it at 5 m/s^2 until it comes to rest at a second set of traffic lights. Draw a velocity–time graph of the car's journey and calculate the distance between the two sets of traffic lights.

12 A particle moves in a straight line so that its position, x, relative to a fixed point O on the line at any time $t \ge 2$ is given by $x = 4\log_e(t - 1)$. Find expressions for the velocity and acceleration at time t.

13 A missile is fired vertically upwards from a point on the ground, level with the base of a tower 64 m high. The missile is level with the top of the tower 0.8 seconds after being fired. Let g m/s^2 be the acceleration due to gravity. Find in terms of g:

a the initial velocity of the missile

b the time taken to reach its greatest height

c the greatest height

d the length of time for which the missile is higher than the top of the tower.

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Multiple-choice questions

1 A particle moves in a straight line so that its position, x cm, relative to a point O at time t seconds ($t \geq 0$) is given by $x = t^3 - 9t^2 + 24t - 1$. The position (in cm) of the particle at $t = 3$ is

A 17 **B** 16 **C** 24 **D** −17 **E** 8

2 A particle moves in a straight line so that its position, x cm, relative to a fixed point O at time t seconds ($t \geq 0$) is given by $x = t^3 - 9t^2 + 24t - 1$. The average speed (in cm/s) of the particle in the first 2 seconds is

A 0 **B** −12 **C** 10 **D** −10 **E** 9.5

3 A body is projected up from the ground with a velocity of 30 m/s. Its acceleration due to gravity is -10 m/s^2. The body's velocity (in m/s) at time $t = 2$ seconds is

A 10 **B** −10 **C** 0 **D** 20 **E** −20

4 A car accelerating uniformly from rest reaches a speed of 50 km/h in 5 seconds. The car's acceleration during the 5 seconds is

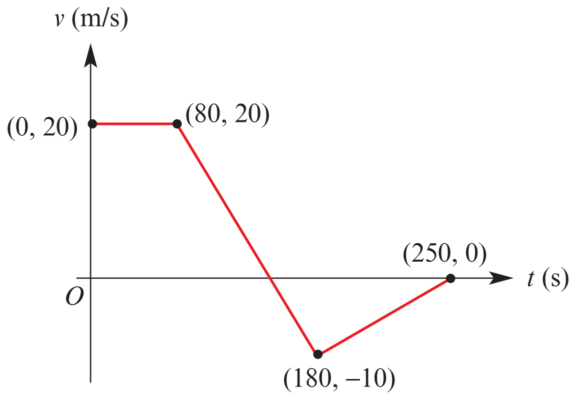
A 10 km/s^2 **B** 10 m/s^2 **C** 2.78 m/s^2 **D** $\frac{25}{9}$ m/s^2 **E** $\frac{25}{3}$ m/s^2

5 A particle moves in a straight line such that, at time t ($t \geq 0$), its velocity v is given by $v = 5 - \frac{2}{t+2}$. The initial acceleration of the particle is

A 0 **B** $\frac{1}{2}$ **C** 1 **D** 2 **E** 4

6 The velocity–time graph shown describes the motion of a particle. The time (in seconds) when the velocity of the particle is first zero is closest to

A 0 **B** 125
C 147 **D** 150
E 250



7 A particle is travelling in a straight line. Its position, x metres, relative to the origin is given by $x = 2t^3 - 10t^2 - 44t + 112$. In the interval $0 \leq t \leq 10$, the number of times that the particle passes through the origin is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

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8 An object is moving in a straight line. Its acceleration, a m/s^2, and its position relative to the origin, x m, are related by $a = -x$, where $-\sqrt{3} \le x \le \sqrt{3}$. If the object starts from the origin with a velocity of $\sqrt{3}$ m/s, then its velocity, v m/s, is given by

A $-\sqrt{3 - x^2}$ **B** $\sqrt{3 - x^2}$ **C** $\pm\sqrt{3 - x^2}$ **D** $-\sqrt{x^2 - 3}$ **E** $\sqrt{x^2 - 3}$

9 The position, x metres, with respect to an origin of a particle travelling in a straight line is given by $x = 2 - 2\cos\left(\frac{3\pi}{2}t - \frac{\pi}{2}\right)$. The velocity (in m/s) at time $t = \frac{8}{3}$ seconds is

A -3π **B** 3π **C** 0 **D** $-\frac{3\pi}{2}$ **E** $\frac{3\pi}{2}$

10 An object starting at the origin has a velocity given by $v = 10\sin(\pi t)$. The distance that the object travels from $t = 0$ to $t = 1.6$, correct to two decimal places, is

A 1.60 **B** 2.20 **C** 4.17 **D** 6.37 **E** 10.53

Extended-response questions

1 A stone initially at rest is released and falls vertically. Its velocity, v m/s, at time t seconds satisfies $5\frac{dv}{dt} + v = 50$.

a Find the acceleration of the stone when $t = 0$.

b Find v in terms of t.

c **i** Sketch the graph of v against t.

ii Find the value of t for which $v = 47.5$. (Give your answer correct to two decimal places.)

d Let x m be the distance fallen after t seconds.

i Find x in terms of t.

ii Sketch the graph of x against t ($t \ge 0$).

iii After how many seconds has the stone fallen 8 metres? (Give your answer correct to two decimal places.)

2 A particle is moving along a straight line. At time t seconds after it passes a point O on the line, its velocity is v m/s, where $v = A - \log_e(t + B)$ for positive constants A and B.

a If $A = 1$ and $B = 0.5$:

i Sketch the graph of v against t.

ii Find the position of the particle when $t = 3$ (correct to two decimal places).

iii Find the distance travelled by the particle in the 3 seconds after passing O (correct to two decimal places).

b If the acceleration of the particle is $-\frac{1}{20}$ m/s^2 when $t = 10$ and the particle comes to rest when $t = 100$, find the exact value of B and the value of A correct to two decimal places.

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3 The velocity, v km/h, of a train which moves along a straight track from station A, where it starts at rest, to station B, where it next stops, is given by

$$v = kt(1 - \sin(\pi t))$$

where t hours is the time measured from when the train left station A and k is a positive constant.

a Find the time that the train takes to travel from A to B.

b **i** Find an expression for the acceleration at time t.

ii Find the interval of time for which the velocity is increasing. (Give your answer correct to two decimal places.)

c Given that the distance from A to B is 20 km, find the value of k. (Give your answer correct to three significant figures.)

4 A particle A moves along a horizontal line so that its position, x m, relative to a point O is given by $x = 28 + 4t - 5t^2 - t^3$, where t is the time in seconds after the motion starts.

a Find:

i the velocity of A in terms of t

ii the acceleration of A in terms of t

iii the value of t for which the velocity is zero (to two decimal places)

iv the times when the particle is 28 m to the right of O (to two decimal places)

v the time when the particle is 28 m to the left of O (to two decimal places).

b A second particle B moves along the same line as A. It starts from O at the same time that A begins to move. The initial velocity of B is 2 m/s and its acceleration at time t is $(2 - 6t)$ m/s^2.

i Find the position of B at time t.

ii Find the time at which A and B collide.

iii At the time of collision are they going in the same direction?

5 A particle moves in a straight line. At time t seconds its position, x cm, with respect to a fixed point O on the line is given by $x = 5\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$.

a Find:

i the velocity in terms of t **ii** the acceleration in terms of t.

b Find:

i the velocity in terms of x **ii** the acceleration in terms of x.

c Find the speed of the particle when $x = -2.5$, correct to one decimal place.

d Find the acceleration when $t = 0$, correct to two decimal places.

e Find:

i the maximum distance of the particle from O

ii the maximum speed of the particle

iii the maximum magnitude of acceleration of the particle.

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6 In a tall building, two lifts simultaneously pass the 40th floor, each travelling downwards at 24 m/s. One lift immediately slows down with a constant retardation of $\frac{6}{7}$ m/s^2. The other continues for 6 seconds at 24 m/s and then slows down with a retardation of $\frac{1}{3}(t-6)$ m/s^2, where t seconds is the time that has elapsed since passing the 40th floor. Find the difference between the heights of the lifts when both have come to rest.

7 The motion of a bullet through a special shield is modelled by the equation $a = -30(v+110)^2$, $v \geq 0$, where a m/s^2 is its acceleration and v m/s its velocity t seconds after impact. When $t = 0$, $v = 300$.

a Find v in terms of t.

b Sketch the graph of v against t.

c Let x m be the penetration into the shield at time t seconds.

- **i** Find x in terms of t
- **ii** Find x in terms of v.
- **iii** Find how far the bullet penetrates the shield before coming to rest.

d Another model for the bullet's motion is $a = -30(v^2 + 11\,000)$, $v \geq 0$. Given that when $t = 0$, $v = 300$:

- **i** Find t in terms of v.
- **ii** Find v in terms of t.
- **iii** Sketch the graph of v against t.
- **iv** Find the distance travelled by the bullet in the first 0.0001 seconds after impact.

8 A motorist is travelling at 25 m/s along a straight road and passes a stationary police officer on a motorcycle. Four seconds after the motorist passes, the police officer starts in pursuit. The police officer's motion for the first 6 seconds is described by

$$v(t) = \frac{-3}{10}\left(t^3 - 21t^2 + \frac{364}{3}t - \frac{1281}{6}\right), \quad 4 \leq t \leq 10$$

where $v(t)$ m/s is his speed t seconds after the motorist has passed. After 6 seconds, he reaches a speed of v_1 m/s, which he maintains until he overtakes the motorist.

a Find the value of v_1.

b **i** Find $\frac{dv}{dt}$ for $4 \leq t \leq 10$.

- **ii** Find the time when the police officer's acceleration is a maximum.

c On the same set of axes, sketch the velocity–time graphs for the motorist and the police officer.

d **i** How far has the police officer travelled when he reaches his maximum speed at $t = 10$?

- **ii** Write down an expression for the distance travelled by the police officer for $t \in [4, 10]$.

e For what value of t does the police officer draw level with the motorist? (Give your answer correct to two decimal places.)

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9 Two cyclists, A and B, pass a starting post together (but at different velocities) and race along a straight road. They are able to pass each other. At time t hours after they pass the post, their velocities (in km/h) are given by

$$V_A = \begin{cases} 9 - t^2 & \text{for } 0 \le t \le 3 \\ 2t - 6 & \text{for } t > 3 \end{cases} \qquad \text{and} \qquad V_B = 8, \quad \text{for } t \ge 0$$

a On the one set of axes, draw the velocity–time graphs for the two cyclists.

b Find the times at which the two cyclists have the same velocity.

c Find the time in hours, correct to one decimal place, when:

i A passes B **ii** B passes A.

10 Two particles, P and Q, move along the same straight path and can overtake each other. Their velocities are $V_P = 2 - t + \frac{1}{4}t^2$ and $V_Q = \frac{3}{4} + \frac{1}{2}t$ respectively at time t, for $t \ge 0$.

a **i** Find the times when the velocities of P and Q are the same.

ii On the same diagram, sketch velocity–time graphs to represent the motion of P and the motion of Q.

b If the particles start from the same point at time $t = 0$:

i Find the time when P and Q next meet again (correct to one decimal place).

ii State the times during which P is further than Q from the starting point (correct to one decimal place).

11 Annabelle and Cuthbert are ants on a picnic table. Annabelle falls off the edge of the table at point X. She falls 1.2 m to the ground. (Assume $g = 9.8$ for this question.)

a Assuming that Annabelle's acceleration down is g m/s^2, find:

i Annabelle's velocity when she hits the ground, correct to two decimal places

ii the time it takes for Annabelle to hit the ground, correct to two decimal places.

b Assume now that Annabelle's acceleration is slowed by air resistance and is given by $(g - t)$ m/s^2, where t is the time in seconds after leaving the table.

i Find Annabelle's velocity, v m/s, at time t.

ii Find Annabelle's position, x m, relative to X at time t.

iii Find the time in seconds, correct to two decimal places, when Annabelle hits the ground.

c When Cuthbert reaches the edge of the table, he observes Annabelle groaning on the ground below. He decides that action must be taken and fashions a parachute from a small piece of potato chip. He jumps from the table and his acceleration is $\frac{g}{2}$ m/s^2 down.

i Find an expression for x, the distance in metres that Cuthbert is from the ground at time t seconds.

ii Unfortunately, Annabelle is very dizzy and on seeing Cuthbert coming down jumps vertically with joy. Her initial velocity is 1.4 m/s up and her acceleration is g m/s^2 down. She jumps 0.45 seconds after Cuthbert leaves the top of the table. How far above the ground (to the nearest cm) do the two ants collide?

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12 On a straight road, a car starts from rest with an acceleration of 2 m/s^2 and travels until it reaches a velocity of 6 m/s. The car then travels with constant velocity for 10 seconds before the brakes cause a deceleration of $(v + 2)$ m/s^2 until it comes to rest, where v m/s is the velocity of the car.

a For how long is the car accelerating?

b Find an expression for v, the velocity of the car, in terms of t, the time in seconds after it starts.

c Find the total time taken for the motion of the car, to the nearest tenth of a second.

d Draw a velocity–time graph of the motion.

e Find the total distance travelled by the car to the nearest tenth of a metre.

13 A particle is first observed at time $t = 0$ and its position at this point is taken as its initial position. The particle moves in a straight line such that its velocity, v, at time t is given by

$$v = \begin{cases} 3 - (t-1)^2 & \text{for } 0 \le t \le 2 \\ 6 - 2t & \text{for } t > 2 \end{cases}$$

a Draw the velocity–time graph for $t \ge 0$.

b Find the distance travelled by the particle from its initial position until it first comes to rest.

c If the particle returns to its original position at $t = T$, calculate T correct to two decimal places.

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13

Vector functions and vector calculus

Objectives

- ▶ To sketch the graphs of curves in the plane specified by **vector functions**.
- ▶ To understand the concept of **position vectors** as a function of time.
- ▶ To represent the path of a particle moving in two dimensions as a **vector function**.
- ▶ To differentiate and antidifferentiate vector functions.
- ▶ To use **vector calculus** to analyse the motion of a particle along a curve, by finding the velocity, acceleration and speed.
- ▶ To find the distance travelled by a particle moving along a curve.

In Chapter 4, we introduced vectors and applied them to physical and geometric situations.

In Chapter 12, we studied motion in a straight line and used the vector quantities of position, displacement, velocity and acceleration to describe this motion. In this chapter, we consider motion in two dimensions and we again employ vectors.

The motion of a particle in space can be described by giving its position vector with respect to an origin in terms of a variable t. The variable in this situation is referred to as a **parameter**. This idea has been used in Section 1G, where parametric equations were introduced to describe circles, ellipses and hyperbolas, and also in Chapter 5, where vector equations were introduced to describe straight lines. Differentiation involving parametric equations was used in Chapter 8.

In two dimensions, the position vector can be described through the use of two functions. The position vector at time t is given by

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

We say that $\boldsymbol{r}(t)$ is a **vector function**.

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13A Vector functions

Describing a particle's path using a vector function

Consider the vector $\boldsymbol{r} = (3 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j}$, where $t \in \mathbb{R}$.

Then $\boldsymbol{r}$ represents a family of vectors defined by different values of t.

If the variable t represents time, then $\boldsymbol{r}$ is a vector function of time. We write

$$\boldsymbol{r}(t) = (3 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j}, \quad t \in \mathbb{R}$$

Further, if $\boldsymbol{r}(t)$ represents the position of a particle with respect to time, then the graph of the endpoints of $\boldsymbol{r}(t)$ will represent the path of the particle in the Cartesian plane.

A table of values for a range of values of t is given below. These position vectors can be represented in the Cartesian plane as shown in Figure A.

t	-3	-2	-1	0	1	2	3
$\boldsymbol{r}(t)$	$7\boldsymbol{j}$	$\boldsymbol{i} + 5\boldsymbol{j}$	$2\boldsymbol{i} + 3\boldsymbol{j}$	$3\boldsymbol{i} + \boldsymbol{j}$	$4\boldsymbol{i} - \boldsymbol{j}$	$5\boldsymbol{i} - 3\boldsymbol{j}$	$6\boldsymbol{i} - 5\boldsymbol{j}$

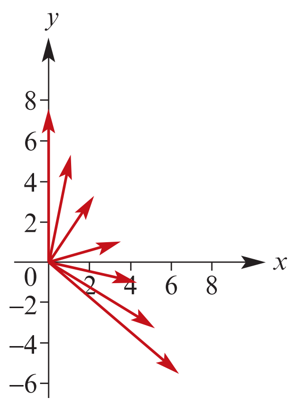

Figure A

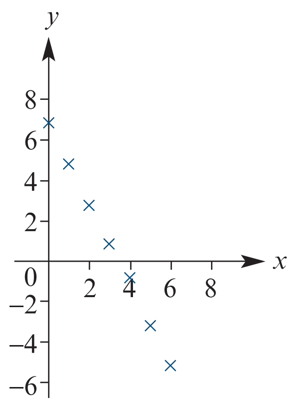

Figure B

The graph of the position vectors (Figure A) is not helpful. But when only the endpoints are plotted (Figure B), the pattern of the path is more obvious. We can find the Cartesian equation for the path as follows.

Let (x, y) be the point on the path at time t.

Then $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$ and therefore

$$x\boldsymbol{i} + y\boldsymbol{j} = (3 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j}$$

This implies that

$$x = 3 + t \quad (1) \qquad \text{and} \qquad y = 1 - 2t \quad (2)$$

Now we eliminate the parameter t from the equations.

From (1), we have $t = x - 3$. Substituting in (2) gives $y = 1 - 2(x - 3) = 7 - 2x$.

The particle's path is the straight line with equation $y = 7 - 2x$.

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Describing curves in the plane using vector functions

Now consider the Cartesian equation $y = x^2$. The graph can also be described by a vector function using a parameter t, which does not necessarily represent time.

Define the vector function $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\boldsymbol{j},\ t \in \mathbb{R}$.

Using similar reasoning as before, if $x\boldsymbol{i} + y\boldsymbol{j} = t\,\boldsymbol{i} + t^2\boldsymbol{j}$, then $x = t$ and $y = t^2$, so eliminating t yields $y = x^2$.

This representation is not unique. It is clear that $\boldsymbol{r}(t) = t^3\boldsymbol{i} + t^6\boldsymbol{j},\ t \in \mathbb{R}$, also represents the graph with Cartesian equation $y = x^2$. Note that if these two vector functions are used to describe the motion of particles, then the paths are the same, but the particles are at different locations at a given time (with the exception of $t = 0$ and $t = 1$).

Also note that $\boldsymbol{r}(t) = t^2\boldsymbol{i} + t^4\boldsymbol{j},\ t \in \mathbb{R}$, only represents the equation $y = x^2$ for $x \geq 0$.

In the rest of this section, we consider graphs defined by vector functions, but without relating them to the motion of a particle. We view a vector function as a mapping from a subset of the real numbers into the set of all two-dimensional vectors.

Example 1

Find the Cartesian equation for the graph represented by each vector function:

a $\boldsymbol{r}(t) = (2 - t)\boldsymbol{i} + (3 + t^2)\boldsymbol{j},\quad t \in \mathbb{R}$

b $\boldsymbol{r}(t) = (1 - \cos t)\,\boldsymbol{i} + \sin t\,\boldsymbol{j},\quad t \in \mathbb{R}$

Solution

a Let (x, y) be any point on the curve.

Then $x = 2 - t$ (1)

and $y = 3 + t^2$ (2)

Equation (1) gives $t = 2 - x$.

Substitute in (2):

$$y = 3 + (2 - x)^2$$

$$\therefore\quad y = x^2 - 4x + 7,\quad x \in \mathbb{R}$$

b Let (x, y) be any point on the curve.

Then $x = 1 - \cos t$ (3)

and $y = \sin t$ (4)

From (3): $\cos t = 1 - x$

From (4):

$$\begin{aligned} y^2 = \sin^2 t &= 1 - \cos^2 t \\ &= 1 - (1 - x)^2 \\ &= -x^2 + 2x \end{aligned}$$

The Cartesian equation is $y^2 = -x^2 + 2x$.

For a vector function $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$:

- The **domain** of the Cartesian relation is given by the range of the function $x(t)$.
- The **range** of the Cartesian relation is given by the range of the function $y(t)$.

In Example 1 **b**, the domain of the corresponding Cartesian relation is the range of the function $x(t) = 1 - \cos t$, which is $[0, 2]$. The range of the Cartesian relation is the range of the function $y(t) = \sin t$, which is $[-1, 1]$.

Note that the Cartesian equation $y^2 = -x^2 + 2x$ can be written as $(x - 1)^2 + y^2 = 1$; it is the circle with centre $(1, 0)$ and radius 1.

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Example 2

Find the Cartesian equation of each of the following. State the domain and range and sketch the graph of each of the relations.

a $\boldsymbol{r}(t) = \cos^2(t)\,\boldsymbol{i} + \sin^2(t)\,\boldsymbol{j}, \quad t \in \mathbb{R}$ **b** $\boldsymbol{r}(t) = t\,\boldsymbol{i} + (1 - t)\boldsymbol{j}, \quad t \in \mathbb{R}$

Solution

a Let (x, y) be any point on the curve defined by $\boldsymbol{r}(t) = \cos^2(t)\,\boldsymbol{i} + \sin^2(t)\,\boldsymbol{j}$, $t \in \mathbb{R}$. Then

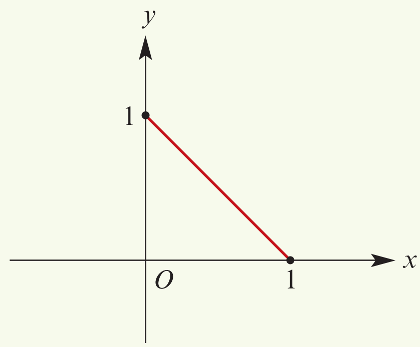
$$x = \cos^2(t) \quad \text{and} \quad y = \sin^2(t)$$

Therefore

$$y = \sin^2(t) = 1 - \cos^2(t) = 1 - x$$

Hence $y = 1 - x$.

Note that $0 \le \cos^2(t) \le 1$ and $0 \le \sin^2(t) \le 1$, for all $t \in \mathbb{R}$. The domain of the relation is $[0, 1]$ and the range is $[0, 1]$.

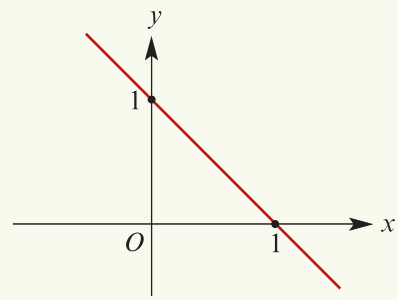


b Let (x, y) be any point on the curve defined by $\boldsymbol{r}(t) = t\,\boldsymbol{i} + (1 - t)\boldsymbol{j}$, $t \in \mathbb{R}$. Then

$$x = t \quad \text{and} \quad y = 1 - t$$

Hence $y = 1 - x$.

The domain is $\mathbb{R}$ and the range is $\mathbb{R}$.



Example 3

For each of the following, state the Cartesian equation, the domain and range of the corresponding Cartesian relation and sketch the graph:

a $\boldsymbol{r}(\lambda) = (1 - 2\cos(\lambda))\,\boldsymbol{i} + 3\sin(\lambda)\,\boldsymbol{j}$ **b** $\boldsymbol{r}(\lambda) = 2\sec(\lambda)\,\boldsymbol{i} + \tan(\lambda)\,\boldsymbol{j}$

Solution

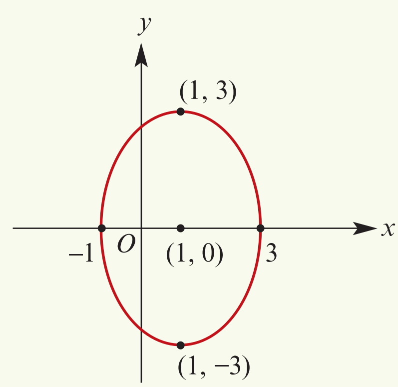
a Let $x = 1 - 2\cos(\lambda)$ and $y = 3\sin(\lambda)$. Then

$$\frac{x-1}{-2} = \cos(\lambda) \quad \text{and} \quad \frac{y}{3} = \sin(\lambda)$$

Squaring each and adding yields

$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = \cos^2(\lambda) + \sin^2(\lambda) = 1$$

The graph is an ellipse with centre $(1, 0)$. The domain of the relation is $[-1, 3]$ and the range is $[-3, 3]$.



Note: The entire ellipse is obtained by taking $\lambda \in [0, 2\pi]$.

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b $\boldsymbol{r}(\lambda) = 2\sec(\lambda)\,\boldsymbol{i} + \tan(\lambda)\,\boldsymbol{j}$, for $\lambda \in \mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$

Let (x, y) be any point on the curve. Then

$$x = 2\sec(\lambda) \quad \text{and} \quad y = \tan(\lambda)$$
$$\therefore \quad x^2 = 4\sec^2(\lambda) \quad \text{and} \quad y^2 = \tan^2(\lambda)$$
$$\therefore \quad \frac{x^2}{4} = \sec^2(\lambda) \quad \text{and} \quad y^2 = \tan^2(\lambda)$$

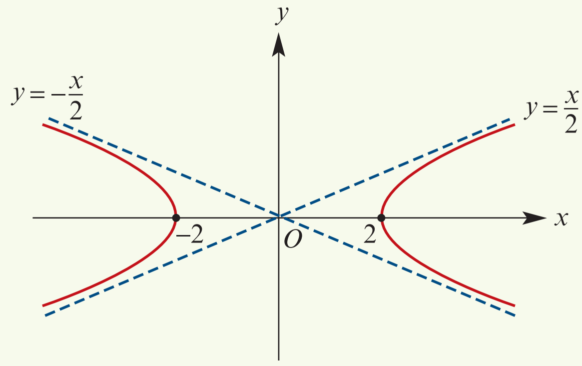
But $\sec^2(\lambda) - \tan^2(\lambda) = 1$ and therefore

$$\frac{x^2}{4} - y^2 = 1$$

The domain of the relation is the range of $x(\lambda) = 2\sec(\lambda)$, which is $(-\infty, -2] \cup [2, \infty)$.
The range of the relation is the range of $y(\lambda) = \tan(\lambda)$, which is $\mathbb{R}$.

The graph is a hyperbola centred at the origin with asymptotes $y = \pm\frac{x}{2}$.

Note: The graph is produced for $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.



Exercise 13A

Example 1
Example 2

1 For each of the following vector functions, find the corresponding Cartesian equation, and state the domain and range of the Cartesian relation:

a $\boldsymbol{r}(t) = t\,\boldsymbol{i} + 2t\boldsymbol{j}, \quad t \in \mathbb{R}$

b $\boldsymbol{r}(t) = 2\boldsymbol{i} + 5t\boldsymbol{j}, \quad t \in \mathbb{R}$

c $\boldsymbol{r}(t) = -t\,\boldsymbol{i} + 7\boldsymbol{j}, \quad t \in \mathbb{R}$

d $\boldsymbol{r}(t) = (2 - t)\boldsymbol{i} + (t + 7)\boldsymbol{j}, \quad t \in \mathbb{R}$

e $\boldsymbol{r}(t) = t^2\boldsymbol{i} + (2 - 3t)\boldsymbol{j}, \quad t \in \mathbb{R}$

f $\boldsymbol{r}(t) = (t - 3)\boldsymbol{i} + (t^3 + 1)\boldsymbol{j}, \quad t \in \mathbb{R}$

g $\boldsymbol{r}(t) = (2t + 1)\boldsymbol{i} + 3^t\boldsymbol{j}, \quad t \in \mathbb{R}$

h $\boldsymbol{r}(t) = \left(t - \frac{\pi}{2}\right)\boldsymbol{i} + \cos(2t)\,\boldsymbol{j}, \quad t \in \mathbb{R}$

i $\boldsymbol{r}(t) = \frac{1}{t+4}\boldsymbol{i} + (t^2 + 1)\boldsymbol{j}, \quad t \neq -4$

j $\boldsymbol{r}(t) = \frac{1}{t}\boldsymbol{i} + \frac{1}{t+1}\boldsymbol{j}, \quad t \neq 0, -1$

Example 3

2 For each of the following vector functions, find the corresponding Cartesian relation, state the domain and range of the relation and sketch the graph:

a $\boldsymbol{r}(t) = 2\cos(t)\,\boldsymbol{i} + 3\sin(t)\,\boldsymbol{j}, \quad t \in \mathbb{R}$

b $\boldsymbol{r}(t) = 2\cos^2(t)\,\boldsymbol{i} + 3\sin^2(t)\,\boldsymbol{j}, \quad t \in \mathbb{R}$

c $\boldsymbol{r}(t) = t\,\boldsymbol{i} + 3t^2\boldsymbol{j}, \quad t \geq 0$

d $\boldsymbol{r}(t) = t^3\boldsymbol{i} + 3t^2\boldsymbol{j}, \quad t \geq 0$

e $\boldsymbol{r}(\lambda) = \cos(\lambda)\,\boldsymbol{i} + \sin(\lambda)\,\boldsymbol{j}, \quad \lambda \in \left[0, \frac{\pi}{2}\right]$

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f $\boldsymbol{r}(\lambda) = 3\sec(\lambda)\,\boldsymbol{i} + 2\tan(\lambda)\,\boldsymbol{j}, \quad \lambda \in \left(0, \frac{\pi}{2}\right)$

g $\boldsymbol{r}(t) = 4\cos(2t)\,\boldsymbol{i} + 4\sin(2t)\,\boldsymbol{j}, \quad t \in \left[0, \frac{\pi}{2}\right]$

h $\boldsymbol{r}(\lambda) = 3\sec^2(\lambda)\,\boldsymbol{i} + 2\tan^2(\lambda)\,\boldsymbol{j}, \quad \lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

i $\boldsymbol{r}(t) = (3 - t)\boldsymbol{i} + (5t^2 + 6t)\boldsymbol{j}, \quad t \in \mathbb{R}$

3 Find a vector function which corresponds to each of the following. Note that the answers given are the 'natural choice', but your answers could be different.

a $y = 3 - 2x$ **b** $x^2 + y^2 = 4$ **c** $(x - 1)^2 + y^2 = 4$

d $x^2 - y^2 = 4$ **e** $y = (x - 3)^2 + 2(x - 3)$ **f** $2x^2 + 3y^2 = 12$

4 A circle of radius 5 has its centre at the point C with position vector $2\boldsymbol{i} + 6\boldsymbol{j}$ relative to the origin O. A general point P on the circle has position $\boldsymbol{r}$ relative to O. The angle between $\boldsymbol{i}$ and $\overrightarrow{CP}$, measured anticlockwise from $\boldsymbol{i}$ to $\overrightarrow{CP}$, is denoted by θ.

a Give the vector function for P. **b** Give the Cartesian equation for P.

13B Position vectors as a function of time

Consider a particle travelling at a constant speed along a circular path with radius length 1 unit and centre O. The path is represented in **Cartesian form** as

$$\{(x, y) : x^2 + y^2 = 1\}$$

If the particle starts at the point $(1, 0)$ and travels anticlockwise, taking 2π units of time to complete one circle, then its path is represented in **parametric form** as

$$\{(x, y) : x = \cos t \text{ and } y = \sin t, \text{ for } t \geq 0\}$$

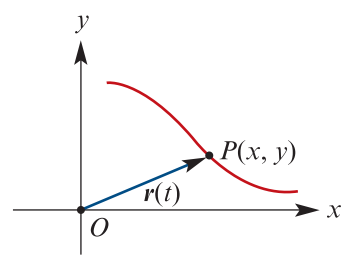
This is expressed in vector form as

$$\boldsymbol{r}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}$$

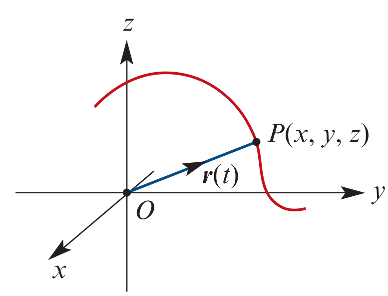
where $\boldsymbol{r}(t)$ is the position vector of the particle at time t.

The graph of a vector function is the set of points determined by the function $\boldsymbol{r}(t)$ as t varies.

In two dimensions, the x- and y-axes are used.



In three dimensions, three mutually perpendicular axes are used. It is best to consider the x- and y-axes as in the horizontal plane and the z-axis as vertical and through the point of intersection of the x- and y-axes.



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Information from the vector function

The vector function gives much more information about the motion of the particle than the Cartesian equation of its path.

For example, the vector function $\boldsymbol{r}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}$, $t \geq 0$, indicates that:

- At time $t = 0$, the particle has position vector $\boldsymbol{r}(0) = \boldsymbol{i}$. That is, the particle starts at $(1, 0)$.
- The particle moves with constant speed on the curve with equation $x^2 + y^2 = 1$.
- The particle moves in an anticlockwise direction.
- The particle moves around the circle with a period of 2π, i.e. it takes 2π units of time to complete one circle.

The vector function $\boldsymbol{r}(t) = \cos(2\pi t)\,\boldsymbol{i} + \sin(2\pi t)\,\boldsymbol{j}$ describes a particle moving anticlockwise around the circle with equation $x^2 + y^2 = 1$, but this time the period is 1 unit of time.

The vector function $\boldsymbol{r}(t) = -\cos(2\pi t)\,\boldsymbol{i} + \sin(2\pi t)\,\boldsymbol{j}$ again describes a particle moving around the unit circle, but the particle starts at $(-1, 0)$ and moves clockwise.

Example 4

Sketch the path of a particle where the position at time t is given by

$$\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + t^2\boldsymbol{j}, \quad t \geq 0$$

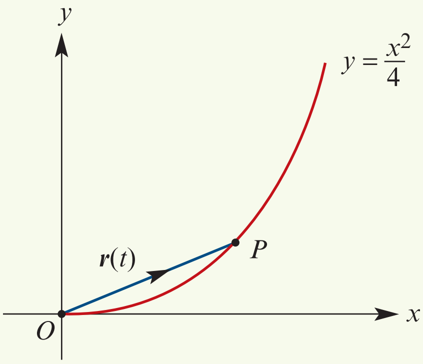
Solution

Now $x = 2t$ and $y = t^2$.

This implies $t = \dfrac{x}{2}$ and so $y = \left(\dfrac{x}{2}\right)^2$.

The Cartesian form is $y = \dfrac{x^2}{4}$, for $x \geq 0$.

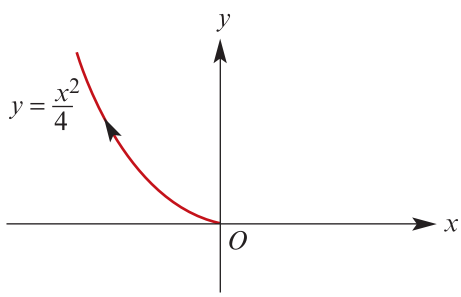
Since $\boldsymbol{r}(0) = \boldsymbol{0}$ and $\boldsymbol{r}(1) = 2\boldsymbol{i} + \boldsymbol{j}$, it can be seen that the particle starts at the origin and moves along the parabola $y = \dfrac{x^2}{4}$ with $x \geq 0$.



Notes:

- The equation $\boldsymbol{r}(t) = t\,\boldsymbol{i} + \frac{1}{4}t^2\boldsymbol{j}$, $t \geq 0$, gives the same Cartesian path, but the rate at which the particle moves along the path is different.
- If $\boldsymbol{r}(t) = -t\,\boldsymbol{i} + \frac{1}{4}t^2\boldsymbol{j}$, $t \geq 0$, then again the Cartesian equation is $y = \dfrac{x^2}{4}$, but $x \leq 0$.

  Hence the motion is along the curve shown and in the direction indicated.



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- **Motion in two dimensions**
  When a particle moves along a curve in a plane, its position is specified by a vector function of the form

  $$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

- **Motion in three dimensions**
  When a particle moves along a curve in three-dimensional space, its position is specified by a vector function of the form

  $$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

Example 5

An object moves along a path where the position vector is given by

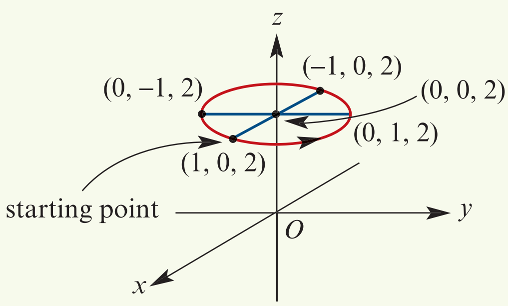
$$\boldsymbol{r}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j} + 2\boldsymbol{k}, \quad t \geq 0$$

Describe the motion of the object.

Solution

Being unfamiliar with the graphs of relations in three dimensions, it is probably best to determine a number of position vectors (points) and try to visualise joining the dots.

t	$\boldsymbol{r}(t)$	Point
0	$\boldsymbol{i} + 2\boldsymbol{k}$	$(1, 0, 2)$
$\frac{\pi}{2}$	$\boldsymbol{j} + 2\boldsymbol{k}$	$(0, 1, 2)$
π	$-\boldsymbol{i} + 2\boldsymbol{k}$	$(-1, 0, 2)$
$\frac{3\pi}{2}$	$-\boldsymbol{j} + 2\boldsymbol{k}$	$(0, -1, 2)$
2π	$\boldsymbol{i} + 2\boldsymbol{k}$	$(1, 0, 2)$



The object is moving along a circular path, with centre $(0, 0, 2)$ and radius length 1, starting at $(1, 0, 2)$ and moving anticlockwise when viewed from above, always at a distance of 2 above the x–y plane (horizontal plane).

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Example 6

The motion of two particles is given by the vector functions $\boldsymbol{r}_1(t) = (2t - 3)\boldsymbol{i} + (t^2 + 10)\boldsymbol{j}$ and $\boldsymbol{r}_2(t) = (t + 2)\boldsymbol{i} + 7t\boldsymbol{j}$, where $t \geq 0$. Find:

a the point at which the particles collide

b the points at which the two paths cross

c the distance between the particles when $t = 1$.

Solution

a The two particles collide when they share the same position at the same time:

$$\boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$$

$$(2t - 3)\boldsymbol{i} + (t^2 + 10)\boldsymbol{j} = (t + 2)\boldsymbol{i} + 7t\boldsymbol{j}$$

Therefore

$$2t - 3 = t + 2 \quad (1) \qquad \text{and} \qquad t^2 + 10 = 7t \quad (2)$$

From (1), we have $t = 5$.

Check in (2): $t^2 + 10 = 35 = 7t$.

The particles are at the same point when $t = 5$, i.e. they collide at the point $(7, 35)$.

b At the points where the paths cross, the two paths share common points which may occur at different times for each particle. Therefore we need to distinguish between the two time variables:

$$\boldsymbol{r}_1(t) = (2t - 3)\boldsymbol{i} + (t^2 + 10)\boldsymbol{j}$$

$$\boldsymbol{r}_2(s) = (s + 2)\boldsymbol{i} + 7s\boldsymbol{j}$$

When the paths cross:

$$2t - 3 = s + 2 \quad (3)$$

$$t^2 + 10 = 7s \quad (4)$$

We now solve these equations simultaneously.

Equation (3) becomes $s = 2t - 5$.

Substitute in (4):

$$t^2 + 10 = 7(2t - 5)$$

$$t^2 - 14t + 45 = 0$$

$$(t - 9)(t - 5) = 0$$

$$\therefore \quad t = 5 \text{ or } t = 9$$

The corresponding values for s are 5 and 13.

These values can be substituted back into the vector equations to obtain the points at which the paths cross, i.e. $(7, 35)$ and $(15, 91)$.

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c When $t = 1$: $\boldsymbol{r}_1(1) = -\boldsymbol{i} + 11\boldsymbol{j}$

$\boldsymbol{r}_2(1) = 3\boldsymbol{i} + 7\boldsymbol{j}$

The vector representing the displacement between the two particles after 1 second is

$$\boldsymbol{r}_1(1) - \boldsymbol{r}_2(1) = -4\boldsymbol{i} + 4\boldsymbol{j}$$

The distance between the two particles is $\sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ units.

Exercise 13B

Example 4

1 The path of a particle with respect to an origin is described as a function of time, t, by the vector equation $\boldsymbol{r}(t) = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}$, $t \geq 0$.

- **a** Find the Cartesian equation of the path.
- **b** Sketch the path of the particle.
- **c** Find the times at which the particle crosses the y-axis.

2 Repeat Question 1 for the paths described by the following vector functions:

- **a** $\boldsymbol{r}(t) = (t^2 - 9)\boldsymbol{i} + 8t\boldsymbol{j}, \quad t \geq 0$
- **b** $\boldsymbol{r}(t) = (t + 1)\boldsymbol{i} + \dfrac{1}{t + 2}\boldsymbol{j}, \quad t > -2$
- **c** $\boldsymbol{r}(t) = \dfrac{t - 1}{t + 1}\boldsymbol{i} + \dfrac{2}{t + 1}\boldsymbol{j}, \quad t > -1$

Example 6

3 The paths of two particles with respect to time t are described by the vector equations $\boldsymbol{r}_1(t) = (3t - 5)\boldsymbol{i} + (8 - t^2)\boldsymbol{j}$ and $\boldsymbol{r}_2(t) = (3 - t)\boldsymbol{i} + 2t\boldsymbol{j}$, where $t \geq 0$. Find:

- **a** the point at which the two particles collide
- **b** the points at which the two paths cross
- **c** the distance between the two particles when $t = 3$.

4 Repeat Question 3 for the paths described by the vector equations $\boldsymbol{r}_1(t) = (2t^2 + 4)\boldsymbol{i} + (t - 2)\boldsymbol{j}$ and $\boldsymbol{r}_2(t) = 9t\,\boldsymbol{i} + 3(t - 1)\boldsymbol{j}$, where $t \geq 0$.

5 The path of a particle defined as a function of time t is given by the vector equation $\boldsymbol{r}(t) = (1 + t)\boldsymbol{i} + (3t + 2)\boldsymbol{j}$. Find:

- **a** the distance of the particle from the origin when $t = 3$
- **b** the times at which the distance of the particle from the origin is 1 unit.

6 Let $\boldsymbol{r}(t) = t\,\boldsymbol{i} + 2t\boldsymbol{j} - 3\boldsymbol{k}$ be the vector equation representing the motion of a particle with respect to time t, where $t \geq 0$. Find:

- **a** the position, A, of the particle when $t = 3$
- **b** the distance of the particle from the origin when $t = 3$
- **c** the position, B, of the particle when $t = 4$
- **d** the displacement of the particle in the fourth second in vector form.

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7 Let $\boldsymbol{r}(t) = (t+1)\boldsymbol{i} + (3-t)\boldsymbol{j} + 2t\boldsymbol{k}$ be the vector equation representing the motion of a particle with respect to time t, where $t \geq 0$. Find:

a the position of the particle when $t = 2$

b the distance of the particle from the point $(4, -1, 1)$ when $t = 2$.

8 Let $\boldsymbol{r}(t) = at^2\boldsymbol{i} + (b-t)\boldsymbol{j}$ be the vector equation representing the motion of a particle with respect to time t. When $t = 3$, the position of the particle is $(6, 4)$. Find a and b.

9 A particle travels in a path such that the position vector, $\boldsymbol{r}(t)$, at time t is given by $\boldsymbol{r}(t) = 3\cos(t)\,\boldsymbol{i} + 2\sin(t)\,\boldsymbol{j}$, $t \geq 0$.

a Express this vector function as a Cartesian relation.

b Find the initial position of the particle.

c The positive y-axis points north and the positive x-axis points east. Find, correct to two decimal places, the bearing of the point P, the position of the particle at $t = \dfrac{3\pi}{4}$, from:

i the origin **ii** the initial position.

10 An object moves so that the position vector at time t is given by $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + e^{-t}\boldsymbol{j}$, $t \geq 0$.

a Express this vector function as a Cartesian relation.

b Find the initial position of the object.

c Sketch the graph of the path travelled by the object, indicating the direction of motion.

11 An object is moving so that its position, $\boldsymbol{r}$, at time t is given by $\boldsymbol{r}(t) = (e^t + e^{-t})\boldsymbol{i} + (e^t - e^{-t})\boldsymbol{j}$, $t \geq 0$.

a Find the initial position of the object. **b** Find the position at $t = \log_e 2$.

c Find the Cartesian equation of the path.

12 An object is projected so that its position, $\boldsymbol{r}$, at time t is given by $\boldsymbol{r}(t) = 100t\,\boldsymbol{i} + (100\sqrt{3}t - 5t^2)\boldsymbol{j}$, for $0 \leq t \leq 20\sqrt{3}$.

a Find the initial and final positions of the object.

b Find the Cartesian form of the path.

c Sketch the graph of the path, indicating the direction of motion.

13 Two particles A and B have position vectors $\boldsymbol{r}_A(t)$ and $\boldsymbol{r}_B(t)$ respectively at time t, given by $\boldsymbol{r}_A(t) = 6t^2\boldsymbol{i} + (2t^3 - 18t)\boldsymbol{j}$ and $\boldsymbol{r}_B(t) = (13t - 6)\boldsymbol{i} + (3t^2 - 27)\boldsymbol{j}$, where $t \geq 0$. Find where and when the particles collide.

Example 5 **14** The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = 3\cos t\,\boldsymbol{i} + 3\sin t\,\boldsymbol{j} + \boldsymbol{k}$, $t \geq 0$. Describe the motion of the particle.

15 The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = t\,\boldsymbol{i} + 3t\boldsymbol{j} + t\boldsymbol{k}$, $t \geq 0$. Describe the motion of the particle.

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16 The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = (1 - 2\cos(2t))\,\boldsymbol{i} + (3 - 5\sin(2t))\,\boldsymbol{j}$, for $t \geq 0$. Find:

a the Cartesian equation of the path

b the position at:

i $t = 0$ **ii** $t = \frac{\pi}{4}$ **iii** $t = \frac{\pi}{2}$

c the time taken by the particle to return to its initial position

d the direction of motion along the curve.

17 For each of the following vector equations:

i find the Cartesian equation of the particle's path

ii sketch the path

iii describe the motion of the particle.

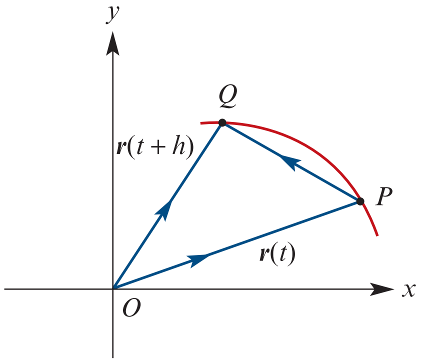
a $\boldsymbol{r}(t) = \cos^2(3\pi t)\,\boldsymbol{i} + 2\cos^2(3\pi t)\,\boldsymbol{j},\ t \geq 0$

b $\boldsymbol{r}(t) = \cos(2\pi t)\,\boldsymbol{i} + \cos(4\pi t)\,\boldsymbol{j},\ t \geq 0$

c $\boldsymbol{r}(t) = e^{t}\,\boldsymbol{i} + e^{-2t}\boldsymbol{j},\ t \geq 0$

13C Vector calculus

Consider the curve defined by $\boldsymbol{r}(t)$.



Let P and Q be points on the curve with position vectors $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+h)$ respectively.

Then $\overrightarrow{PQ} = \boldsymbol{r}(t+h) - \boldsymbol{r}(t)$.

It follows that

$$\frac{1}{h}(\boldsymbol{r}(t+h) - \boldsymbol{r}(t))$$

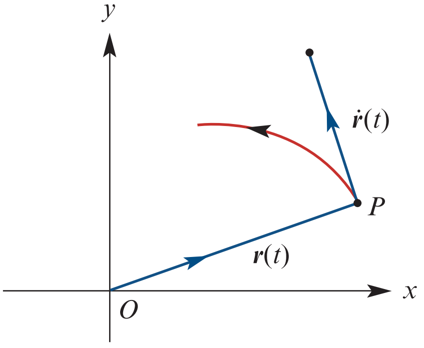
is a vector parallel to $\overrightarrow{PQ}$.

As $h \to 0$, the point Q approaches P along the curve.

The derivative of $\boldsymbol{r}$ with respect to t is denoted by $\dot{\boldsymbol{r}}$ and is defined by

$$\dot{\boldsymbol{r}}(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$

provided that this limit exists.



The vector $\dot{\boldsymbol{r}}(t)$ points along the tangent to the curve at P, in the direction of increasing t.

Note: The derivative of a vector function $\boldsymbol{r}(t)$ is also denoted by $\frac{d\boldsymbol{r}}{dt}$ or $\boldsymbol{r}'(t)$.

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Derivative of a vector function

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. If both $x(t)$ and $y(t)$ are differentiable, then

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j}$$

Proof By the definition, we have

$$\begin{aligned}
\dot{\mathbf{r}}(t) &= \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\
&= \lim_{h\to 0} \frac{(x(t+h)\mathbf{i} + y(t+h)\mathbf{j}) - (x(t)\mathbf{i} + y(t)\mathbf{j})}{h} \\
&= \lim_{h\to 0} \frac{x(t+h)\mathbf{i} - x(t)\mathbf{i}}{h} + \lim_{h\to 0} \frac{y(t+h)\mathbf{j} - y(t)\mathbf{j}}{h} \\
&= \left(\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\right)\mathbf{i} + \left(\lim_{h\to 0} \frac{y(t+h) - y(t)}{h}\right)\mathbf{j}
\end{aligned}$$

$$\therefore \quad \dot{\mathbf{r}}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

The second derivative of $\mathbf{r}(t)$ is

$$\ddot{\mathbf{r}}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j}$$

This can be extended to three-dimensional vector functions:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$\dot{\mathbf{r}}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\ddot{\mathbf{r}}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

Example 7

Find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ if $\mathbf{r}(t) = 20t\,\mathbf{i} + (15t - 5t^2)\mathbf{j}$.

Solution

$$\dot{\mathbf{r}}(t) = 20\mathbf{i} + (15 - 10t)\mathbf{j}$$

$$\ddot{\mathbf{r}}(t) = -10\mathbf{j}$$

Example 8

Find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ if $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + 5t\mathbf{k}$.

Solution

$$\dot{\mathbf{r}}(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 5\mathbf{k}$$

$$\ddot{\mathbf{r}}(t) = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$$

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Example 9

If $\boldsymbol{r}(t) = t\,\boldsymbol{i} + ((t-1)^3 + 1)\boldsymbol{j}$, find $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$, where $\boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j}$.

Solution

$$\boldsymbol{r}(t) = t\,\boldsymbol{i} + ((t-1)^3 + 1)\boldsymbol{j}$$
$$\dot{\boldsymbol{r}}(t) = \boldsymbol{i} + 3(t-1)^2\boldsymbol{j}$$
$$\ddot{\boldsymbol{r}}(t) = 6(t-1)\boldsymbol{j}$$

We have

$$\boldsymbol{r}(\alpha) = \alpha\boldsymbol{i} + ((\alpha-1)^3 + 1)\boldsymbol{j} = \boldsymbol{i} + \boldsymbol{j}$$

Therefore $\alpha = 1$, and $\dot{\boldsymbol{r}}(1) = \boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(1) = \boldsymbol{0}$.

Example 10

If $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + ((e^t-1)^3 + 1)\boldsymbol{j}$, find $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$, where $\boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j}$.

Solution

$$\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + ((e^t-1)^3 + 1)\boldsymbol{j}$$
$$\dot{\boldsymbol{r}}(t) = e^t\,\boldsymbol{i} + 3e^t(e^t-1)^2\boldsymbol{j}$$
$$\ddot{\boldsymbol{r}}(t) = e^t\,\boldsymbol{i} + (6e^{2t}(e^t-1) + 3e^t(e^t-1)^2)\boldsymbol{j}$$

We have

$$\boldsymbol{r}(\alpha) = e^\alpha\,\boldsymbol{i} + ((e^\alpha-1)^3 + 1)\boldsymbol{j} = \boldsymbol{i} + \boldsymbol{j}$$

Therefore $\alpha = 0$, and $\dot{\boldsymbol{r}}(0) = \boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(0) = \boldsymbol{i}$.

Example 11

A curve is described by the vector equation $\boldsymbol{r}(t) = 2\cos t\,\boldsymbol{i} + 3\sin t\,\boldsymbol{j}$.

a Find:

i $\dot{\boldsymbol{r}}(t)$ **ii** $\ddot{\boldsymbol{r}}(t)$

b Find the gradient of the curve at the point (x, y), where $x = 2\cos t$ and $y = 3\sin t$.

Solution

a **i** $\dot{\boldsymbol{r}}(t) = -2\sin t\,\boldsymbol{i} + 3\cos t\,\boldsymbol{j}$

ii $\ddot{\boldsymbol{r}}(t) = -2\cos t\,\boldsymbol{i} - 3\sin t\,\boldsymbol{j}$

b We can find $\dfrac{dy}{dx}$ using related rates:

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}, \qquad \frac{dx}{dt} = -2\sin t, \qquad \frac{dy}{dt} = 3\cos t$$

$$\therefore \quad \frac{dy}{dx} = 3\cos t \cdot \frac{1}{-2\sin t} = -\frac{3}{2}\cot t$$

Note that the gradient is undefined when $\sin t = 0$.

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Example 12

A curve is described by the vector equation $\boldsymbol{r}(t) = \sec(t)\,\boldsymbol{i} + \tan(t)\,\boldsymbol{j}$, with $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

a Find the gradient of the curve at the point (x, y), where $x = \sec(t)$ and $y = \tan(t)$.

b Find the gradient of the curve where $t = \frac{\pi}{4}$.

Solution

a $x = \sec(t) = \dfrac{1}{\cos(t)} = (\cos t)^{-1}$ and $y = \tan(t)$

$$\frac{dx}{dt} = -(\cos t)^{-2}(-\sin t) = \frac{\sin(t)}{\cos^2(t)} = \tan(t)\sec(t)$$

$$\frac{dy}{dt} = \sec^2(t)$$

Hence

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \sec^2(t)\cdot\frac{1}{\tan(t)\sec(t)} = \sec(t)\cot(t) = \frac{1}{\sin(t)}$$

b When $t = \frac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$

We have the following results for differentiating vector functions.

Properties of the derivative of a vector function

- $\dfrac{d}{dt}(\boldsymbol{c}) = \boldsymbol{0}$, where $\boldsymbol{c}$ is a constant vector
- $\dfrac{d}{dt}(k\boldsymbol{r}(t)) = k\,\dfrac{d}{dt}(\boldsymbol{r}(t))$, where k is a real number
- $\dfrac{d}{dt}(\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)) = \dfrac{d}{dt}(\boldsymbol{r}_1(t)) + \dfrac{d}{dt}(\boldsymbol{r}_2(t))$
- $\dfrac{d}{dt}(f(t)\,\boldsymbol{r}(t)) = f(t)\,\dfrac{d}{dt}(\boldsymbol{r}(t)) + \dfrac{d}{dt}(f(t))\,\boldsymbol{r}(t)$, where f is a real-valued function

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Antidifferentiation

Consider $\int \boldsymbol{r}(t)\,dt = \int x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}\,dt$

$$= \left(\int x(t)\,dt\right)\boldsymbol{i} + \left(\int y(t)\,dt\right)\boldsymbol{j} + \left(\int z(t)\,dt\right)\boldsymbol{k}$$

$$= X(t)\boldsymbol{i} + Y(t)\boldsymbol{j} + Z(t)\boldsymbol{k} + \boldsymbol{c}$$

where $\dfrac{dX}{dt} = x(t)$, $\dfrac{dY}{dt} = y(t)$, $\dfrac{dZ}{dt} = z(t)$ and $\boldsymbol{c}$ is a constant vector. Note that $\dfrac{d\boldsymbol{c}}{dt} = \boldsymbol{0}$.

Example 13

Given that $\ddot{\boldsymbol{r}}(t) = 10\boldsymbol{i} - 12\boldsymbol{k}$, find:

a $\dot{\boldsymbol{r}}(t)$ if $\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$

b $\boldsymbol{r}(t)$ if also $\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 2\boldsymbol{k}$

Solution

a $\dot{\boldsymbol{r}}(t) = 10t\,\boldsymbol{i} - 12t\boldsymbol{k} + \boldsymbol{c}_1$, where $\boldsymbol{c}_1$ is a constant vector

$\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$

Thus $\boldsymbol{c}_1 = 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$

and $\dot{\boldsymbol{r}}(t) = 10t\,\boldsymbol{i} - 12t\boldsymbol{k} + 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$

$= (10t + 30)\boldsymbol{i} - 20\boldsymbol{j} + (10 - 12t)\boldsymbol{k}$

b $\boldsymbol{r}(t) = (5t^2 + 30t)\boldsymbol{i} - 20t\boldsymbol{j} + (10t - 6t^2)\boldsymbol{k} + \boldsymbol{c}_2$, where $\boldsymbol{c}_2$ is a constant vector

$\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 2\boldsymbol{k}$

Thus $\boldsymbol{c}_2 = 2\boldsymbol{k}$

and $\boldsymbol{r}(t) = (5t^2 + 30t)\boldsymbol{i} - 20t\boldsymbol{j} + (10t - 6t^2 + 2)\boldsymbol{k}$

Example 14

Given $\ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$ with $\boldsymbol{r}(0) = \boldsymbol{0}$ and $\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} + 40\boldsymbol{j}$, find $\boldsymbol{r}(t)$.

Solution

$\ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$

$\therefore \quad \dot{\boldsymbol{r}}(t) = \left(\int 0\,dt\right)\boldsymbol{i} + \left(\int -9.8\,dt\right)\boldsymbol{j}$

$= -9.8t\boldsymbol{j} + \boldsymbol{c}_1$

But $\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} + 40\boldsymbol{j}$, giving $\boldsymbol{c}_1 = 30\boldsymbol{i} + 40\boldsymbol{j}$.

$\therefore \quad \dot{\boldsymbol{r}}(t) = 30\boldsymbol{i} + (40 - 9.8t)\boldsymbol{j}$

Thus $\boldsymbol{r}(t) = \left(\int 30\,dt\right)\boldsymbol{i} + \left(\int 40 - 9.8t\,dt\right)\boldsymbol{j}$

$= 30t\,\boldsymbol{i} + (40t - 4.9t^2)\boldsymbol{j} + \boldsymbol{c}_2$

Now $\boldsymbol{r}(0) = \boldsymbol{0}$ and therefore $\boldsymbol{c}_2 = \boldsymbol{0}$.

Hence $\boldsymbol{r}(t) = 30t\,\boldsymbol{i} + (40t - 4.9t^2)\boldsymbol{j}$.

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Skill-sheet

Exercise 13C

Example 7
Example 8

1 Find $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ for each of the following:

a $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + e^{-t}\,\boldsymbol{j}$

b $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\boldsymbol{j}$

c $\boldsymbol{r}(t) = \frac{1}{2}t\,\boldsymbol{i} + t^2\boldsymbol{j}$

d $\boldsymbol{r}(t) = 16t\,\boldsymbol{i} - 4(4t-1)^2\boldsymbol{j}$

e $\boldsymbol{r}(t) = \sin(t)\,\boldsymbol{i} + \cos(t)\,\boldsymbol{j}$

f $\boldsymbol{r}(t) = (3+2t)\boldsymbol{i} + 5t\boldsymbol{j}$

g $\boldsymbol{r}(t) = 100t\,\boldsymbol{i} + (100\sqrt{3}t - 4.9t^2)\boldsymbol{j}$

h $\boldsymbol{r}(t) = \tan(t)\,\boldsymbol{i} + \cos^2(t)\,\boldsymbol{j}$

Example 9
Example 10

2 Sketch graphs for each of the following, for $t \geq 0$, and find $\boldsymbol{r}(t_0)$, $\dot{\boldsymbol{r}}(t_0)$ and $\ddot{\boldsymbol{r}}(t_0)$ for the given t_0:

a $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + e^{-t}\boldsymbol{j}, \quad t_0 = 0$

b $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\boldsymbol{j}, \quad t_0 = 1$

c $\boldsymbol{r}(t) = \sin(t)\,\boldsymbol{i} + \cos(t)\,\boldsymbol{j}, \quad t_0 = \dfrac{\pi}{6}$

d $\boldsymbol{r}(t) = 16t\,\boldsymbol{i} - 4(4t-1)^2\boldsymbol{j}, \quad t_0 = 1$

e $\boldsymbol{r}(t) = \dfrac{1}{t+1}\boldsymbol{i} + (t+1)^2\boldsymbol{j}, \quad t_0 = 1$

Example 11
Example 12

3 Find the gradient at the point on the curve determined by the given value of t for each of the following:

a $\boldsymbol{r}(t) = \cos(t)\,\boldsymbol{i} + \sin(t)\,\boldsymbol{j}, \quad t = \dfrac{\pi}{4}$

b $\boldsymbol{r}(t) = \sin(t)\,\boldsymbol{i} + \cos(t)\,\boldsymbol{j}, \quad t = \dfrac{\pi}{2}$

c $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + e^{-2t}\boldsymbol{j}, \quad t = 1$

d $\boldsymbol{r}(t) = 2t^2\boldsymbol{i} + 4t\boldsymbol{j}, \quad t = 2$

e $\boldsymbol{r}(t) = (t+2)\boldsymbol{i} + (t^2 - 2t)\boldsymbol{j}, \quad t = 3$

f $\boldsymbol{r}(t) = \cos(\pi t)\,\boldsymbol{i} + \cos(2\pi t)\,\boldsymbol{j}, \quad t = \dfrac{1}{4}$

Example 13
Example 14

4 Find $\boldsymbol{r}(t)$ for each of the following:

a $\dot{\boldsymbol{r}}(t) = 4\boldsymbol{i} + 3\boldsymbol{j}$, where $\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

b $\dot{\boldsymbol{r}}(t) = 2t\,\boldsymbol{i} + 2\boldsymbol{j} - 3t^2\boldsymbol{k}$, where $\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

c $\dot{\boldsymbol{r}}(t) = e^{2t}\boldsymbol{i} + 2e^{0.5t}\boldsymbol{j}$, where $\boldsymbol{r}(0) = \frac{1}{2}\boldsymbol{i}$

d $\ddot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2t\boldsymbol{j}$, where $\dot{\boldsymbol{r}}(0) = \boldsymbol{i}$ and $\boldsymbol{r}(0) = \boldsymbol{0}$

e $\ddot{\boldsymbol{r}}(t) = \sin(2t)\,\boldsymbol{i} - \cos(\frac{1}{2}t)\,\boldsymbol{j}$, where $\dot{\boldsymbol{r}}(0) = -\frac{1}{2}\boldsymbol{i}$ and $\boldsymbol{r}(0) = 4\boldsymbol{j}$

5 The position of a particle at time t is given by $\boldsymbol{r}(t) = \sin(t)\,\boldsymbol{i} + t\boldsymbol{j} + \cos(t)\,\boldsymbol{k}$, where $t \geq 0$. Prove that $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are always perpendicular.

6 The position of a particle at time t is given by $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + 16t^2(3-t)\boldsymbol{j}$, where $t \geq 0$. Find:

a when $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are perpendicular

b the pairs of perpendicular vectors $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$.

7 A particle has position $\boldsymbol{r}(t)$ at time t determined by $\boldsymbol{r}(t) = at\,\boldsymbol{i} + \dfrac{a^2t^2}{4}\boldsymbol{j}$, $a > 0$ and $t \geq 0$.

a Sketch the graph of the path of the particle.

b Find when the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ is 45°.

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8 A particle has position $\boldsymbol{r}(t)$ at time t specified by $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + (t^2 - 4)\boldsymbol{j}$, where $t \geq 0$.

a Sketch the graph of the path of the particle.

b Find the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ at $t = 1$.

c Find when the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ is 30°.

9 Given $\boldsymbol{r} = 3t\,\boldsymbol{i} + \frac{1}{3}t^3\boldsymbol{j} + t^3\boldsymbol{k}$, find:

a $\dot{\boldsymbol{r}}$

b $|\dot{\boldsymbol{r}}|$

c $\ddot{\boldsymbol{r}}$

d $|\ddot{\boldsymbol{r}}|$

e t when $|\ddot{\boldsymbol{r}}| = 16$

10 Given that $\boldsymbol{r} = (V\cos\alpha)t\,\boldsymbol{i} + \left((V\sin\alpha)t - \frac{1}{2}gt^2\right)\boldsymbol{j}$ specifies the position of an object at time $t \geq 0$, find:

a $\dot{\boldsymbol{r}}$

b $\ddot{\boldsymbol{r}}$

c when $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are perpendicular

d the position of the object when $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are perpendicular.

13D Velocity and acceleration for motion along a curve

Consider a particle moving along a curve in the plane, with position vector at time t given by

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

We can find the particle's velocity and acceleration at time t as follows.

Velocity

Velocity is the rate of change of position.

Therefore $\boldsymbol{v}(t)$, the velocity at time t, is given by

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \dot{x}(t)\boldsymbol{i} + \dot{y}(t)\boldsymbol{j}$$

The velocity vector gives the direction of motion at time t.

Acceleration

Acceleration is the rate of change of velocity.

Therefore $\boldsymbol{a}(t)$, the acceleration at time t, is given by

$$\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) = \ddot{x}(t)\boldsymbol{i} + \ddot{y}(t)\boldsymbol{j}$$

Speed

Speed is the magnitude of velocity. At time t, the speed is $|\dot{\boldsymbol{r}}(t)|$.

Distance between two points on the curve

The (shortest) distance between two points on the curve is found using $|\boldsymbol{r}(t_1) - \boldsymbol{r}(t_0)|$.

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Example 15

The position of an object is $\boldsymbol{r}(t)$ metres at time t seconds, where $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + \frac{2}{9}e^{2t}\boldsymbol{j},\ t \geq 0$.

Find at time t:

a the velocity vector **b** the acceleration vector **c** the speed.

Solution

a $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = e^t\,\boldsymbol{i} + \frac{4}{9}e^{2t}\boldsymbol{j}$

b $\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t) = e^t\,\boldsymbol{i} + \frac{8}{9}e^{2t}\boldsymbol{j}$

c Speed $= |\boldsymbol{v}(t)| = \sqrt{(e^t)^2 + (\frac{4}{9}e^{2t})^2} = \sqrt{e^{2t} + \frac{16}{81}e^{4t}}$ m/s

Example 16

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = (2t - t^2)\boldsymbol{i} + (t^2 - 3t)\boldsymbol{j} + 2t\boldsymbol{k}$, where $t \geq 0$. Find:

a the velocity of the particle at time t

b the speed of the particle at time t

c the minimum speed of the particle.

Solution

a $\dot{\boldsymbol{r}}(t) = (2 - 2t)\boldsymbol{i} + (2t - 3)\boldsymbol{j} + 2\boldsymbol{k}$

b
$$\begin{aligned}\text{Speed} = |\dot{\boldsymbol{r}}(t)| &= \sqrt{4 - 8t + 4t^2 + 4t^2 - 12t + 9 + 4} \\ &= \sqrt{8t^2 - 20t + 17}\end{aligned}$$

c Minimum speed occurs when $8t^2 - 20t + 17$ is a minimum.

$$\begin{aligned}8t^2 - 20t + 17 &= 8\left(t^2 - \frac{5t}{2} + \frac{17}{8}\right) \\ &= 8\left(t^2 - \frac{5t}{2} + \frac{25}{16} + \frac{17}{8} - \frac{25}{16}\right) \\ &= 8\left(\left(t - \frac{5}{4}\right)^2 + \frac{9}{16}\right) \\ &= 8\left(t - \frac{5}{4}\right)^2 + \frac{9}{2}\end{aligned}$$

Hence the minimum speed is $\sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$.

(This occurs when $t = \frac{5}{4}$.)

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Example 17

The position of a projectile at time t is given by $\boldsymbol{r}(t) = 400t\,\boldsymbol{i} + (500t - 5t^2)\boldsymbol{j}$, for $t \geq 0$, where $\boldsymbol{i}$ is a unit vector in a horizontal direction and $\boldsymbol{j}$ is a unit vector vertically up. The projectile is fired from a point on the ground. Find:

a the time taken to reach the ground again

b the speed at which the projectile hits the ground

c the maximum height of the projectile

d the initial speed of the projectile.

Solution

a The projectile is at ground level when the $\boldsymbol{j}$-component of $\boldsymbol{r}$ is zero:

$$500t - 5t^2 = 0$$
$$5t(100 - t) = 0$$
$$\therefore \quad t = 0 \text{ or } t = 100$$

The projectile reaches the ground again at $t = 100$.

b $\dot{\boldsymbol{r}}(t) = 400\boldsymbol{i} + (500 - 10t)\boldsymbol{j}$

The velocity of the projectile when it hits the ground is

$$\dot{\boldsymbol{r}}(100) = 400\boldsymbol{i} - 500\boldsymbol{j}$$

Therefore the speed is

$$|\dot{\boldsymbol{r}}(100)| = \sqrt{400^2 + 500^2} = 100\sqrt{41}$$

The projectile hits the ground with speed $100\sqrt{41}$.

c The projectile reaches its maximum height when the $\boldsymbol{j}$-component of $\dot{\boldsymbol{r}}$ is zero:

$$500 - 10t = 0$$
$$\therefore \quad t = 50$$

The maximum height is $500 \times 50 - 5 \times 50^2 = 12\,500$.

d The initial velocity is

$$\dot{\boldsymbol{r}}(0) = 400\boldsymbol{i} + 500\boldsymbol{j}$$

So the initial speed is

$$|\dot{\boldsymbol{r}}(0)| = \sqrt{400^2 + 500^2} = 100\sqrt{41}$$

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Example 18

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = 2\sin(2t)\,\boldsymbol{i} + \cos(2t)\,\boldsymbol{j} + 2t\boldsymbol{k}$, where $t \geq 0$. Find:

a the velocity at time t

b the speed of the particle at time t

c the maximum speed

d the minimum speed.

Solution

a $\dot{\boldsymbol{r}}(t) = 4\cos(2t)\,\boldsymbol{i} - 2\sin(2t)\,\boldsymbol{j} + 2\boldsymbol{k}$

b Speed $= |\dot{\boldsymbol{r}}(t)| = \sqrt{16\cos^2(2t) + 4\sin^2(2t) + 4}$

$$= \sqrt{12\cos^2(2t) + 8}$$

c Maximum speed $= \sqrt{20} = 2\sqrt{5}$, when $\cos(2t) = 1$

d Minimum speed $= \sqrt{8} = 2\sqrt{2}$, when $\cos(2t) = 0$

Example 19

The position vectors, at time $t \geq 0$, of particles A and B are given by

$$\boldsymbol{r}_A(t) = (t^3 - 9t + 8)\boldsymbol{i} + t^2\boldsymbol{j}$$
$$\boldsymbol{r}_B(t) = (2 - t^2)\boldsymbol{i} + (3t - 2)\boldsymbol{j}$$

Prove that A and B collide while travelling at the same speed but at right angles to each other.

Solution

When the particles collide, they must be at the same position at the same time:

$$(t^3 - 9t + 8)\boldsymbol{i} + t^2\boldsymbol{j} = (2 - t^2)\boldsymbol{i} + (3t - 2)\boldsymbol{j}$$

Thus $\quad t^3 - 9t + 8 = 2 - t^2 \quad (1)$

and $\quad t^2 = 3t - 2 \quad (2)$

From (1): $\quad t^3 + t^2 - 9t + 6 = 0 \quad (3)$

From (2): $\quad t^2 - 3t + 2 = 0 \quad (4)$

Equation (4) is simpler to solve:

$$(t - 2)(t - 1) = 0$$

$\therefore \quad t = 2$ or $t = 1$

Now check in (3):

$t = 1 \quad$ LHS $= 1 + 1 - 9 + 6 = -1 \neq$ RHS

$t = 2 \quad$ LHS $= 8 + 4 - 18 + 6 = 0 =$ RHS

The particles collide when $t = 2$.

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Now consider the speeds when $t = 2$.

$$\dot{\boldsymbol{r}}_A(t) = (3t^2 - 9)\boldsymbol{i} + 2t\boldsymbol{j} \qquad \dot{\boldsymbol{r}}_B(t) = -2t\,\boldsymbol{i} + 3\boldsymbol{j}$$

$$\therefore \quad \dot{\boldsymbol{r}}_A(2) = 3\boldsymbol{i} + 4\boldsymbol{j} \qquad \ddot{\boldsymbol{r}}_B(2) = -4\boldsymbol{i} + 3\boldsymbol{j}$$

The speed of particle A is $\sqrt{3^2 + 4^2} = 5$.

The speed of particle B is $\sqrt{(-4)^2 + 3^2} = 5$.

The speeds of the particles are equal at the time of collision.

Consider the scalar product of the velocity vectors for A and B at time $t = 2$.

$$\begin{aligned} \dot{\boldsymbol{r}}_A(2) \cdot \dot{\boldsymbol{r}}_B(2) &= (3\boldsymbol{i} + 4\boldsymbol{j}) \cdot (-4\boldsymbol{i} + 3\boldsymbol{j}) \\ &= -12 + 12 \\ &= 0 \end{aligned}$$

Hence the velocities are perpendicular at $t = 2$.

The particles are travelling at right angles at the time of collision.

Distance travelled along a curve

In Section 10E, we considered the length of a curve defined by parametric equations. We can use the same result to find the distance travelled by a particle along a curve.

If $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$ describes the path of a particle, then the distance travelled along the path in the time interval from $t = a$ to $t = b$ is given by

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 20

A particle moves along a line such that its position at time t is given by the vector function

$$\boldsymbol{r}(t) = (3t - 2)\boldsymbol{i} + (4t + 3)\boldsymbol{j}, \qquad t \geq 0$$

How far along the line does the particle travel from $t = 1$ to $t = 3$?

Solution

We have $x(t) = 3t - 2$ and $y(t) = 4t + 3$.

Hence the distance travelled is

$$\begin{aligned} \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_1^3 \sqrt{3^2 + 4^2}\, dt \\ &= \int_1^3 5\, dt \\ &= [5t]_1^3 \\ &= 10 \end{aligned}$$

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Example 21

A particle moves along a curve such that its position vector at time t is given by

$$\boldsymbol{r}(t) = \sin(t)\,\boldsymbol{i} + \tfrac{1}{2}\sin(2t)\,\boldsymbol{j}, \quad t \geq 0$$

a How far along the curve does the particle travel from $t = 0$ to $t = \dfrac{\pi}{3}$? (Give your answer correct to three decimal places.)

b Find the shortest distance between these two points.

Solution

a $x = \sin(t)$ and $y = \tfrac{1}{2}\sin(2t)$

$\dfrac{dx}{dt} = \cos(t)$ and $\dfrac{dy}{dt} = \cos(2t)$

Hence the distance travelled is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\frac{\pi}{3}} \sqrt{\cos^2(t) + \cos^2(2t)}\,dt \approx 1.061$$

using a calculator.

b We have $\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j}$ and $\boldsymbol{r}\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}\boldsymbol{i} + \dfrac{\sqrt{3}}{4}\boldsymbol{j}$.

Thus $\boldsymbol{r}\left(\dfrac{\pi}{3}\right) - \boldsymbol{r}(0) = \dfrac{\sqrt{3}}{2}\boldsymbol{i} + \dfrac{\sqrt{3}}{4}\boldsymbol{j}$.

Hence the shortest distance between the two points is $\left|\dfrac{\sqrt{3}}{2}\boldsymbol{i} + \dfrac{\sqrt{3}}{4}\boldsymbol{j}\right| = \dfrac{\sqrt{15}}{4} \approx 0.968$.

Skill-sheet

Exercise 13D

All distances are measured in metres and time in seconds.

Example 15

1 The position of a particle at time t is given by $\boldsymbol{r}(t) = t^2\boldsymbol{i} - (1 + 2t)\boldsymbol{j}$, for $t \geq 0$. Find:

a the velocity at time t

b the acceleration at time t

c the average velocity for the first 2 seconds, i.e. $\dfrac{\boldsymbol{r}(2) - \boldsymbol{r}(0)}{2}$.

2 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = -g\boldsymbol{j}$, where $g = 9.8$. Find:

a the velocity at time t if $\dot{\boldsymbol{r}}(0) = 2\boldsymbol{i} + 6\boldsymbol{j}$

b the position at time t if also $\boldsymbol{r}(0) = 0\boldsymbol{i} + 6\boldsymbol{j}$.

Example 16

3 The velocity of a particle at time t is given by $\dot{\boldsymbol{r}}(t) = 3\boldsymbol{i} + 2t\boldsymbol{j} + (1 - 4t)\boldsymbol{k}$, for $t \geq 0$.

a Find the acceleration of the particle at time t.

b Find the position of the particle at time t if initially the particle is at $\boldsymbol{j} + \boldsymbol{k}$.

c Find an expression for the speed at time t.

d **i** Find the time at which the minimum speed occurs.

ii Find this minimum speed.

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4 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = 10\boldsymbol{i} - g\boldsymbol{k}$, where $g = 9.8$. Find:

a the velocity of the particle at time t, given that $\dot{\boldsymbol{r}}(0) = 20\boldsymbol{i} - 20\boldsymbol{j} + 40\boldsymbol{k}$

b the position of the particle at time t, given also that $\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$.

5 The position of an object at time t is given by $\boldsymbol{r}(t) = 5\cos(1+t^2)\,\boldsymbol{i} + 5\sin(1+t^2)\,\boldsymbol{j}$. Find the speed of the object at time t.

6 The position of a particle, $\boldsymbol{r}(t)$, at time t seconds is given by $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + (t^2 - 4)\boldsymbol{j}$. Find the magnitude of the angle between the velocity and acceleration vectors at $t = 1$.

7 The position vector of a particle is given by $\boldsymbol{r}(t) = 12\sqrt{t}\,\boldsymbol{i} + t^{\frac{3}{2}}\boldsymbol{j}$, for $t \geq 0$. Find the minimum speed of the particle and its position when it has this speed.

Example 17

8 The position, $\boldsymbol{r}(t)$, of a projectile at time t is given by $\boldsymbol{r}(t) = 400t\,\boldsymbol{i} + (300t - 4.9t^2)\boldsymbol{j}$, for $t \geq 0$. If the projectile is initially at ground level, find:

a the time taken to return to the ground

b the speed at which the object hits the ground

c the maximum height reached

d the initial speed of the object

e the initial angle of projection from the horizontal.

9 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = -3(\sin(3t)\,\boldsymbol{i} + \cos(3t)\,\boldsymbol{j})$.

a Find the position vector $\boldsymbol{r}(t)$, given that $\dot{\boldsymbol{r}}(0) = \boldsymbol{i}$ and $\boldsymbol{r}(0) = -3\boldsymbol{i} + 3\boldsymbol{j}$.

b Show that the path of the particle is circular and state the position of its centre.

c Show that the acceleration is always perpendicular to the velocity.

Example 18

10 The position vector of a particle at time t is $\boldsymbol{r}(t) = 2\cos(t)\,\boldsymbol{i} + 4\sin(t)\,\boldsymbol{j} + 2t\boldsymbol{k}$. Find the maximum and minimum speeds of the particle.

11 The velocity vector of a particle at time t seconds is given by

$$\boldsymbol{v}(t) = (2t+1)^2\boldsymbol{i} + \frac{1}{\sqrt{2t+1}}\,\boldsymbol{j}$$

a Find the magnitude and direction of the acceleration after 1 second.

b Find the position vector at time t seconds if the particle is initially at O.

12 The acceleration of a particle moving in the x–y plane is $-g\boldsymbol{j}$. The particle is initially at O with velocity given by $V\cos(\alpha)\,\boldsymbol{i} + V\sin(\alpha)\,\boldsymbol{j}$, for some positive real number α.

a Find $\boldsymbol{r}(t)$, the position vector at time t.

b Prove that the particle's path has Cartesian equation $y = x\tan\alpha - \dfrac{gx^2}{2V^2}\sec^2\alpha$.

Example 19

13 Particles A and B move in the x–y plane with constant velocities.

- $\dot{\boldsymbol{r}}_A(t) = \boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{r}_A(2) = 3\boldsymbol{i} + 4\boldsymbol{j}$
- $\dot{\boldsymbol{r}}_B(t) = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{r}_B(3) = \boldsymbol{i} + 3\boldsymbol{j}$

Prove that the particles collide, finding:

a the time of collision **b** the position vector of the point of collision.

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14 A body moves horizontally with a constant speed of 20 m/s on a bearing of $(360 - \alpha)°$, where $0 < \alpha < 90$ and $\tan(\alpha°) = \frac{4}{3}$. If $\boldsymbol{i}$ is a horizontal unit vector due east and $\boldsymbol{j}$ is a horizontal unit vector due north, find:

a the velocity of the body at time t

b the position of the body after 5 seconds.

15 The position vector of a particle at time t is $\boldsymbol{r}(t) = 4\sin(2t)\,\boldsymbol{i} + 4\cos(2t)\,\boldsymbol{j}$, $t \geq 0$. Find:

a the velocity at time t

b the speed at time t

c the acceleration in terms of $\boldsymbol{r}$.

16 The velocity of a particle is given by $\dot{\boldsymbol{r}}(t) = (2t - 5)\boldsymbol{i}$, $t \geq 0$. Initially, the position of the particle relative to an origin O is $-2\boldsymbol{i} + 2\boldsymbol{j}$.

a Find the position of the particle at time t.

b Find the position of the particle when it is instantaneously at rest.

c Find the Cartesian equation of the path followed by the particle.

17 A particle has path defined by $\boldsymbol{r}(t) = 6\sec(t)\,\boldsymbol{i} + 4\tan(t)\,\boldsymbol{j}$, $t \geq 0$. Find:

a the Cartesian equation of the path

b the particle's velocity at time t.

18 A particle moves such that its position vector, $\boldsymbol{r}(t)$, at time t is given by $\boldsymbol{r}(t) = 4\cos(t)\,\boldsymbol{i} + 3\sin(t)\,\boldsymbol{j}$, $0 \leq t \leq 2\pi$.

a Find the Cartesian equation of the path of the particle and sketch the path.

b **i** Find when the velocity of the particle is perpendicular to its position vector.

ii Find the position vector of the particle at each of these times.

c **i** Find the speed of the particle at time t.

ii Write the speed in terms of $\cos^2 t$.

iii State the maximum and minimum speeds of the particle.

Example 20

19 A particle moves along a line such that its position at time t is given by the vector function $\boldsymbol{r}(t) = (t + 2)\boldsymbol{i} + (6t + 1)\boldsymbol{j}$, $t \geq 0$. How far along the line does the particle travel from $t = 1$ to $t = 3$?

20 A particle moves around a circle such that its position at time t is given by the vector function $\boldsymbol{r}(t) = \cos(2t)\,\boldsymbol{i} + \sin(2t)\,\boldsymbol{j}$, $t \geq 0$. How far along the circle does the particle travel from $t = 0$ to $t = \dfrac{\pi}{4}$?

Example 21

21 A particle moves along a curve such that its position at time t is given by the vector function $\boldsymbol{r}(t) = \sqrt{t}\,\boldsymbol{i} + (2t + 4)\boldsymbol{j}$, $t \geq 0$.

a How far along the curve does the particle travel from $t = 1$ to $t = 4$? (Give your answer correct to three decimal places.)

b Find the shortest distance between these two points.

22 A particle moves around an ellipse such that its position vector at time t is given by $\boldsymbol{r}(t) = 4\cos(t)\,\boldsymbol{i} + 3\sin(t)\,\boldsymbol{j}$, $0 \leq t \leq 2\pi$.

a How far along the ellipse does the particle travel from $t = 0$ to $t = \dfrac{\pi}{4}$? (Give your answer correct to three decimal places.)

b Find the shortest distance between these two points.

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Review

Assign-ment

Nrich

Chapter summary

- We state the following results for motion in three dimensions. The statements for motion in two dimensions are analogous.
  - The position of a particle at time t can be described by a vector function:
  $$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$
  - The velocity of the particle at time t is
  $$\dot{\boldsymbol{r}}(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$$
  - The acceleration of the particle at time t is
  $$\ddot{\boldsymbol{r}}(t) = f''(t)\boldsymbol{i} + g''(t)\boldsymbol{j} + h''(t)\boldsymbol{k}$$
- The velocity vector $\dot{\boldsymbol{r}}(t)$ has the direction of the motion of the particle at time t.
- Speed is the magnitude of velocity. At time t, the speed is $|\dot{\boldsymbol{r}}(t)|$.
- The (shortest) distance between the points on the path corresponding to $t = t_0$ and $t = t_1$ is given by $|\boldsymbol{r}(t_1) - \boldsymbol{r}(t_0)|$.
- If $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$ describes the path of a particle, then the distance travelled along the path in the time interval from $t = a$ to $t = b$ is given by
$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Technology-free questions

1 The position, $\boldsymbol{r}(t)$ metres, of a particle moving in a plane is given by $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + (t^2 - 4)\boldsymbol{j}$ at time t seconds.

a Find the velocity and acceleration when $t = 2$.

b Find the Cartesian equation of the path.

2 Find the velocity and acceleration vectors of the position vectors:

a $\boldsymbol{r} = 2t^2\boldsymbol{i} + 4t\boldsymbol{j} + 8\boldsymbol{k}$ **b** $\boldsymbol{r} = 4\sin t\,\boldsymbol{i} + 4\cos t\,\boldsymbol{j} + t^2\boldsymbol{k}$

3 At time t, a particle has coordinates $(6t, t^2 + 4)$. Find the unit vector along the tangent to the path when $t = 4$.

4 The position vector of a particle is given by $\boldsymbol{r}(t) = 10\sin(2t)\,\boldsymbol{i} + 5\cos(2t)\,\boldsymbol{j}$.

a Find its position vector when $t = \dfrac{\pi}{6}$.

b Find the cosine of the angle between its directions of motion at $t = 0$ and $t = \dfrac{\pi}{6}$.

5 Find the unit tangent vector of the curve $\boldsymbol{r} = (\cos t + t\sin t)\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j}$, $t > 0$.

6 A particle moves on a curve with equation $\boldsymbol{r} = 5(\cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j})$. Find:

a the velocity at time t **b** the speed at time t

c the acceleration at time t **d** $\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}$, and comment.

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7 Particles A and B move with velocities $\boldsymbol{V}_A = \cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}$ and $\boldsymbol{V}_B = \sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j}$ respectively. At time $t = 0$, the position vectors of A and B are $\boldsymbol{r}_A = \boldsymbol{i}$ and $\boldsymbol{r}_B = \boldsymbol{j}$. Prove that the particles collide, finding the time of collision.

8 The position vector of a particle at any time t is given by $\boldsymbol{r} = (1 + \sin t)\boldsymbol{i} + (1 - \cos t)\boldsymbol{j}$.

- **a** Show that the magnitudes of the velocity and acceleration are constants.
- **b** Find the Cartesian equation of the path described by the particle.
- **c** Find the first instant that the position is perpendicular to the velocity.

9 The velocities of two particles A and B are given by $\boldsymbol{V}_A = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{V}_B = 3\boldsymbol{i} - 4\boldsymbol{j}$. The initial position vector of particle A is $\boldsymbol{r}_A = \boldsymbol{i} - \boldsymbol{j}$. If the particles collide after 3 seconds, find the initial position vector of particle B.

10 A particle starts from point $\boldsymbol{i} - 2\boldsymbol{j}$ and travels with a velocity given by $t\,\boldsymbol{i} + \boldsymbol{j}$, at time t seconds from the start. A second particle travels in the same plane and its position vector is given by $\boldsymbol{r} = (s - 4)\boldsymbol{i} + 3\boldsymbol{j}$, at time s seconds after it started.

- **a** Find an expression for the position of the first particle.
- **b** Find the point at which their paths cross.
- **c** If the particles actually collide, find the time between the two starting times.

11 A particle travels with constant acceleration, given by $\ddot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2\boldsymbol{j}$. Two seconds after starting, the particle passes through the point $\boldsymbol{i}$, travelling at a velocity of $2\boldsymbol{i} - \boldsymbol{j}$. Find:

- **a** an expression for the velocity of the particle at time t
- **b** an expression for its position
- **c** the initial position and velocity of the particle.

12 Two particles travel with constant acceleration given by $\ddot{\boldsymbol{r}}_1(t) = \boldsymbol{i} - \boldsymbol{j}$ and $\ddot{\boldsymbol{r}}_2(t) = 2\boldsymbol{i} + \boldsymbol{j}$. The initial velocity of the second particle is $-4\boldsymbol{i}$ and that of the first particle is $k\boldsymbol{j}$.

- **a** Find an expression for:
  - **i** the velocity of the second particle
  - **ii** the velocity of the first particle.
- **b** At one instant both particles have the same velocity. Find:
  - **i** the time elapsed before that instant
  - **ii** the value of k
  - **iii** the common velocity.

13 The position of an object is given by $\boldsymbol{r}(t) = e^t\,\boldsymbol{i} + 4e^{2t}\boldsymbol{j}$, $t \geq 0$.

- **a** Show that the path of the particle is the graph of $f\colon [1, \infty) \to \mathbb{R}$, $f(x) = 4x^2$.
- **b** Find:
  - **i** the velocity vector at time t
  - **ii** the initial velocity
  - **iii** the time at which the velocity is parallel to the vector $\boldsymbol{i} + 12\boldsymbol{j}$.

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Review

14 The velocity of a particle is given by $\dot{\boldsymbol{r}}(t) = (t-3)\boldsymbol{j}$, $t \geq 0$.

a Show that the path of this particle is linear.

b Initially, the position of the particle is $2\boldsymbol{i} + \boldsymbol{j}$.

i Find the Cartesian equation of the path followed by the particle.

ii Find the point at which the particle is momentarily at rest.

Multiple-choice questions

1 A particle moves in a plane such that, at time t, its position is $\boldsymbol{r}(t) = 2t^2\boldsymbol{i} + (3t-1)\boldsymbol{j}$. Its acceleration at time t is given by

A $4t\,\boldsymbol{i} + 3\boldsymbol{j}$ **B** $\frac{2}{3}t^3\boldsymbol{i} + (\frac{3}{2}t^2 - t)\,\boldsymbol{j}$ **C** $4\boldsymbol{i} + 3\boldsymbol{j}$ **D** $0\boldsymbol{i} + 0\boldsymbol{j}$ **E** $4\boldsymbol{i} + 0\boldsymbol{j}$

2 The position vector of a particle at time t, $t \geq 0$, is given by $\boldsymbol{r} = \sin(3t)\,\boldsymbol{i} - 2\cos(t)\,\boldsymbol{j}$. The speed of the particle when $t = \pi$ is

A 2 **B** $2\sqrt{2}$ **C** $\sqrt{5}$ **D** 0 **E** 3

3 A particle moves with constant velocity $5\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$. Its initial position is $3\boldsymbol{i} - 6\boldsymbol{k}$. Its position vector at time t is given by

A $(3t+5)\boldsymbol{i} - 4\boldsymbol{j} + (2-6t)\boldsymbol{k}$ **B** $(5t+3)\boldsymbol{i} - 4t\boldsymbol{j} + (2t-6)\boldsymbol{k}$

C $5t\,\boldsymbol{i} - 4t\boldsymbol{j} + 2t\boldsymbol{k}$ **D** $-5t\,\boldsymbol{i} - 4t\boldsymbol{j} + 2t\boldsymbol{k}$ **E** $(5t-3)\boldsymbol{i} + (2t-6)\boldsymbol{k}$

4 A particle moves with its position vector defined with respect to time t by the vector function $\boldsymbol{r}(t) = (2t^3-1)\boldsymbol{i} + (2t^2+3)\,j + 6t\boldsymbol{k}$. The acceleration when $t = 1$ is given by

A $6\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$ **B** $12\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$ **C** $12\boldsymbol{i}$ **D** $2\sqrt{10}$ **E** $12\boldsymbol{i} + 4\boldsymbol{j}$

5 The position vector of a particle at time t seconds is $\boldsymbol{r}(t) = (t^2 - 4t)(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$, measured in metres from a fixed point. The distance in metres travelled in the first 4 seconds is

A 0 **B** $4\sqrt{3}$ **C** $8\sqrt{3}$ **D** 4 **E** $\sqrt{3}$

6 The initial position, velocity and constant acceleration of a particle are given by $3\boldsymbol{i}$, $2\boldsymbol{j}$ and $2\boldsymbol{i} - \boldsymbol{j}$ respectively. The position vector of the particle at time t is given by

A $(2\boldsymbol{i} - \boldsymbol{j})t + 3\boldsymbol{i}$ **B** $t^2\boldsymbol{i} - \frac{1}{2}t^2\boldsymbol{j}$ **C** $(t^2+3)\boldsymbol{i} + (2t - \frac{1}{2}t^2)\,\boldsymbol{j}$

D $3\boldsymbol{i} + 2t\boldsymbol{j}$ **E** $\frac{1}{2}t^2(2\boldsymbol{i} - \boldsymbol{j})$

7 The position of a particle at time $t = 0$ is $\boldsymbol{r}(0) = \boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k}$. The position of the particle at time $t = 3$ is $\boldsymbol{r}(3) = 7\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}$. The average velocity for the interval $[0, 3]$ is

A $\frac{1}{3}(8\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$ **B** $\frac{1}{3}(21\boldsymbol{i} + 21\boldsymbol{j} - 12\boldsymbol{k})$ **C** $2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$

D $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ **E** $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

8 A particle is moving so its velocity vector at time t is $\dot{\boldsymbol{r}}(t) = 2t\,\boldsymbol{i} + 3\boldsymbol{j}$, where $\boldsymbol{r}(t)$ is the position vector at time t. If $\boldsymbol{r}(0) = 3\boldsymbol{i} + \boldsymbol{j}$, then $\boldsymbol{r}(t)$ is equal to

A $2\boldsymbol{i}$ **B** $(3t+1)\boldsymbol{i} + (3t^2+1)\boldsymbol{j}$ **C** $2t^2\boldsymbol{i} + 3t\boldsymbol{j} + 3\boldsymbol{i} + \boldsymbol{j}$

D $5\boldsymbol{i} + 3\boldsymbol{j}$ **E** $(t^2+3)\boldsymbol{i} + (3t+1)\boldsymbol{j}$

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9 The velocity of a particle is given by the vector $\dot{\boldsymbol{r}}(t) = t\,\boldsymbol{i} + e^t\boldsymbol{j}$. At time $t = 0$, the position of the particle is given by $\boldsymbol{r}(0) = 3\boldsymbol{i}$. The position of the particle at time t is given by

A $\boldsymbol{r}(t) = \frac{1}{2}t^2\boldsymbol{i} + e^t\boldsymbol{j}$ **B** $\boldsymbol{r}(t) = \frac{1}{2}(t^2+3)\boldsymbol{i} + e^t\boldsymbol{j}$ **C** $\boldsymbol{r}(t) = (\frac{1}{2}t^2+3)\boldsymbol{i} + (e^t - 1)\boldsymbol{j}$

D $\boldsymbol{r}(t) = (\frac{1}{2}t^2+3)\boldsymbol{i} + e^t\boldsymbol{j}$ **E** $\boldsymbol{r}(t) = \frac{1}{2}(t^2+3)\boldsymbol{i} + (e^t - 1)\boldsymbol{j}$

10 A curve is described by the vector equation $\boldsymbol{r}(t) = 2\cos(\pi t)\,\boldsymbol{i} + 3\sin(\pi t)\,\boldsymbol{j}$. With respect to a set of Cartesian axes, the gradient of the curve at the point $(\sqrt{3}, 1.5)$ is

A $-\frac{\sqrt{3}}{2}$ **B** $-(\pi\boldsymbol{i} + 3\sqrt{3}\pi\boldsymbol{j})$ **C** $\pi\boldsymbol{i} + 3\sqrt{3}\pi\boldsymbol{j}$ **D** $-\frac{3\sqrt{3}}{2}\pi$ **E** $-\frac{3\sqrt{3}}{2}$

Extended-response questions

1 Two particles P and Q are moving in a horizontal plane. The particles are moving with velocities $9\boldsymbol{i} + 6\boldsymbol{j}$ m/s and $5\boldsymbol{i} + 4\boldsymbol{j}$ m/s respectively.

a Determine the speeds of the particles.

b At time $t = 4$, particles P and Q have position vectors $\boldsymbol{r}_P(4) = 96\boldsymbol{i} + 44\boldsymbol{j}$ and $\boldsymbol{r}_Q(4) = 100\boldsymbol{i} + 96\boldsymbol{j}$. (Distances are measured in metres.)

i Find the position vectors of P and Q at time $t = 0$.

ii Find the vector $\overrightarrow{PQ}$ at time t.

c Find the time at which P and Q are nearest to each other and the magnitude of $\overrightarrow{PQ}$ at this instant.

2 Two particles A and B move in the plane. The velocity of A is $(-3\boldsymbol{i} + 29\boldsymbol{j})$ m/s while that of B is $v(\boldsymbol{i} + 7\boldsymbol{j})$ m/s, where v is a constant. (All distances are measured in metres.)

a Find the vector $\overrightarrow{AB}$ at time t seconds, given that when $t = 0$, $\overrightarrow{AB} = -56\boldsymbol{i} + 8\boldsymbol{j}$.

b Find the value of v such that the particles collide.

c If $v = 3$:

i Find $\overrightarrow{AB}$. **ii** Find the time when the particles are closest.

3 A child is sitting still in some long grass watching a bee. The bee flies at a constant speed in a straight line from its beehive to a flower and reaches the flower 3 seconds later. The position vector of the beehive relative to the child is $10\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$ and the position vector of the flower relative to the child is $7\boldsymbol{i} + 8\boldsymbol{j}$, where all the distances are measured in metres.

a If B is the position of the beehive and F the position of the flower, find $\overrightarrow{BF}$.

b Find the distance BF.

c Find the speed of the bee.

d Find the velocity of the bee.

e Find the time when the bee is closest to the child and its distance from the child at this time.

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Review

4 Initially, a motorboat is at a point J at the end of a jetty and a police boat is at a point P. The position vector of P relative to J is $400\boldsymbol{i} - 600\boldsymbol{j}$. The motorboat leaves the point J and travels with constant velocity $6\boldsymbol{i}$. At the same time, the police boat leaves its position at P and travels with constant velocity $u(8\boldsymbol{i} + 6\boldsymbol{j})$, where u is a real number. All distances are measured in metres and all times are measured in seconds.

a If the police boat meets the motorboat after t seconds, find:

i the value of t

ii the value of u

iii the speed of the police boat

iv the position of the point where they meet.

b Find the time at which the police boat was closest to J and its distance from J at this time.

5 A particle A is at rest on a smooth horizontal table at a point with position vector $-\boldsymbol{i} + 2\boldsymbol{j}$, relative to an origin O. Point B is on the table such that $\overrightarrow{OB} = 2\boldsymbol{i} + \boldsymbol{j}$. (All distances are measured in metres and time in seconds.) At time $t = 0$, the particle is projected along the table with velocity $(6\boldsymbol{i} + 3\boldsymbol{j})$ m/s.

a Determine:

i $\overrightarrow{OA}$ at time t **ii** $\overrightarrow{BA}$ at time t.

b Find the time when $|\overrightarrow{BA}| = 5$.

c Using the time found in **b**:

i Find a unit vector $\boldsymbol{c}$ along $\overrightarrow{BA}$.

ii Find a unit vector $\boldsymbol{d}$ perpendicular to $\overrightarrow{BA}$.

Hint: The vector $y\boldsymbol{i} - x\boldsymbol{j}$ is perpendicular to $x\boldsymbol{i} + y\boldsymbol{j}$.

iii Express $6\boldsymbol{i} + 3\boldsymbol{j}$ in the form $p\boldsymbol{c} + q\boldsymbol{d}$.

6 **a** Sketch the graph of the Cartesian relation corresponding to the vector equation

$$\boldsymbol{r}(\theta) = \cos(\theta)\,\boldsymbol{i} - \sin(\theta)\,\boldsymbol{j}, \quad 0 < \theta < \frac{\pi}{2}$$

b A particle P describes a circle of radius 16 cm about the origin. It completes the circle every π seconds. At $t = 0$, P is at the point $(16, 0)$ and is moving in a clockwise direction. It can be shown that $\overrightarrow{OP} = a\cos(nt)\,\boldsymbol{i} + b\sin(nt)\,\boldsymbol{j}$.
Find the values of:

i a **ii** b **iii** n **iv** State the velocity and acceleration of P at time t.

c A second particle Q has position vector given by $\overrightarrow{OQ} = 8\sin(t)\,\boldsymbol{i} + 8\cos(t)\,\boldsymbol{j}$, where measurements are in centimetres. Obtain an expression for:

i $\overrightarrow{PQ}$ **ii** $|\overrightarrow{PQ}|^2$

d Find the minimum distance between P and Q.

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7 At time t, a particle has velocity $\boldsymbol{v} = (2\cos t)\boldsymbol{i} - (4\sin t\ \cos t)\boldsymbol{j}$, $t \geq 0$. At time $t = 0$, it is at the point with position vector $3\boldsymbol{j}$.

- **a** Find the position of the particle at time t.
- **b** Find the position of the particle when it first comes to rest.
- **c** **i** Find the Cartesian equation of the path of the particle.
  - **ii** Sketch the path of the particle.
- **d** Express $|\boldsymbol{v}|^2$ in terms of $\cos t$ and, without using calculus, find the maximum speed of the particle.
- **e** Give the time at which the particle is at rest for the second time.
- **f** **i** Show that the distance, d, of the particle from the origin at time t is given by $d^2 = \cos^2(2t) + 2\cos(2t) + 6$.
  - **ii** Find the time(s) at which the particle is closest to the origin.

8 A golfer hits a ball from a point referred to as the origin with a velocity of $a\boldsymbol{i} + b\boldsymbol{j} + 20\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors horizontally forwards, horizontally to the right and vertically upwards respectively. After being hit, the ball is subject to an acceleration $2\boldsymbol{j} - 10\boldsymbol{k}$. (All distances are measured in metres and all times in seconds.) Find:

- **a** the velocity of the ball at time t
- **b** the position vector of the ball at time t
- **c** the time of flight of the ball
- **d** the values of a and b if the golfer wishes to hit a *direct* hole-in-one, where the position vector of the hole is $100\boldsymbol{i}$
- **e** the angle of projection of the ball if a hole-in-one is achieved.

9 Particles P and Q have variable position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, given by $\boldsymbol{p}(t) = \cos(t)\,\boldsymbol{i} + \sin(t)\,\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{q}(t) = \cos(2t)\,\boldsymbol{i} - \sin(2t)\,\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$, where $0 \leq t \leq 2\pi$.

- **a** **i** For $\boldsymbol{p}(t)$, describe the path.
  - **ii** Find the distance of particle P from the origin at time t.
  - **iii** Find the velocity of particle P at time t.
  - **iv** Show that the vector $\cos(t)\,\boldsymbol{i} + \sin(t)\,\boldsymbol{j}$ is perpendicular to the velocity vector of P for any value of t.
  - **v** Find the acceleration, $\ddot{\boldsymbol{p}}(t)$, at time t.
- **b** **i** Find the vector $\overrightarrow{PQ}$ at time t.
  - **ii** Show that the distance between P and Q at time t is $\sqrt{\frac{17}{4} - 2\cos(3t)}$.
  - **iii** Find the maximum distance between the particles.
  - **iv** Find the times at which this maximum occurs.
  - **v** Find the minimum distance between the particles.
  - **vi** Find the times at which this minimum occurs.
- **c** **i** Show that $\boldsymbol{p}(t) \cdot \boldsymbol{q}(t) = \cos(3t) - \frac{1}{2}$.
  - **ii** Find an expression for $\cos(\angle POQ)$.
  - **iii** Find the greatest magnitude of angle POQ.

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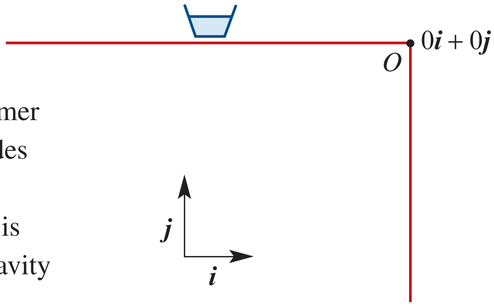
10 Particles A and B move such that, at any time $t \geq 0$, their position vectors are $\boldsymbol{r}_A = 2t\,\boldsymbol{i} + t\boldsymbol{j}$ and $\boldsymbol{r}_B = (4 - 4\sin(\alpha t))\,\boldsymbol{i} + 4\cos(\alpha t)\,\boldsymbol{j}$, where α is a positive constant.

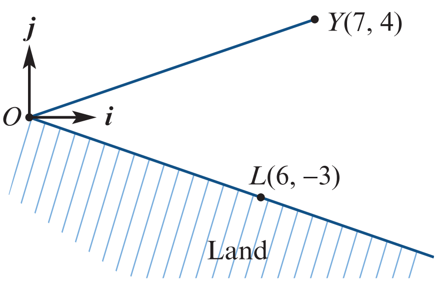
- **a** Find the speed of B in terms of α.
- **b** Find the Cartesian equations of the paths of A and B.
- **c** On the same set of axes, sketch the paths of A and B, showing directions of travel.
- **d** Find the coordinates of the points where the paths of A and B cross.
- **e** Find the least value of α, correct to two decimal places, for which particles A and B will collide.

11 A bartender slides a glass along a bar for a customer to collect. Unfortunately, the customer has turned to speak to a friend. The glass slides over the edge of the bar with a horizontal velocity of 2 m/s. Assume that air resistance is negligible and that the acceleration due to gravity is 9.8 m/s^2 in a downwards direction.



- **a** **i** Give the acceleration of the glass as a vector expression.
  - **ii** Give the vector expression for the velocity of the glass at time t seconds, where t is measured from when the glass leaves the bar.
  - **iii** Give the position of the glass with respect to the edge of the bar, O, at time t seconds.
- **b** It is 0.8 m from O to the floor directly below. Find:
  - **i** the time it takes for the glass to hit the floor
  - **ii** the horizontal distance from the bar where the glass hits the floor.

12 A yacht is returning to its marina at O. At midday, the yacht is at Y. The yacht takes a straight-line course to O. Point L is the position of a navigation sign on the shore. Coordinates represent distances east and north of the marina, measured in kilometres.



- **a** **i** Write down the position vector of the navigation sign L.
  - **ii** Find the unit vector in the direction of $\overrightarrow{OL}$.
- **b** Find the vector resolute of $\overrightarrow{OY}$ in the direction of $\overrightarrow{OL}$ and hence find the coordinates of the point on shore closest to the yacht at midday.
- **c** The yacht sails towards O. The position vector at time t hours after 12 p.m. is given by $\boldsymbol{r}(t) = (7 - \frac{7}{2}t)\,\boldsymbol{i} + (4 - 2t)\boldsymbol{j}$.
  - **i** Find an expression for $\overrightarrow{LP}$, where P is the position of the yacht at time t.
  - **ii** Find the time when the yacht is closest to the navigation sign.
  - **iii** Find the closest distance between the sign and the yacht.

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14 Revision of Chapters 8–13

14A Technology-free questions

1 Find the derivative of each of the following with respect to x:

a $\dfrac{1}{\arcsin x}$ **b** $\dfrac{1}{\arctan x}$ **c** $\dfrac{1}{(\arcsin x)^2}$

2 A tank initially contains 20 L of water in which 1 kg of salt has been dissolved. Pure water flows into the tank at a rate of 4 L per minute. The mixture flows out of the tank at a rate of 2 L per minute. Let Q kg be the amount of salt in the tank at time t minutes ($t \geq 0$).

a Construct a differential equation to model this situation.

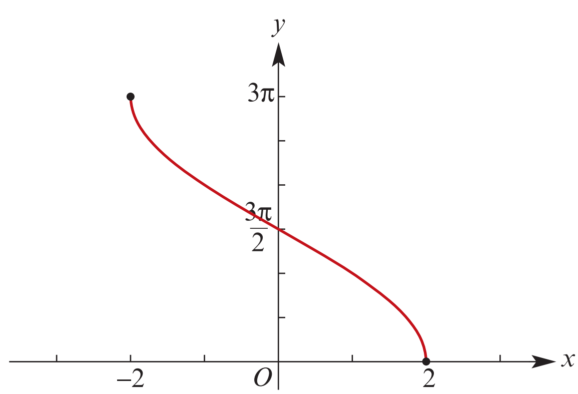
b Solve the differential equation to find Q in terms of t.

3 Solve the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{4 + x^2}$ given that $y = 2$ when $x = 1$.

4 The graph of $y = 3\arccos\left(\dfrac{x}{2}\right)$ is shown opposite.

a Find the area bounded by the graph, the x-axis and the line $x = -2$.

b Find the volume of the solid of revolution formed when the graph is rotated about the y-axis.



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Revision

5 Consider the relation $5x^2 + 2xy + y^2 = 13$.

a Find the gradient of each of the tangents to the graph at the points where $x = 1$.

b Find the equation of the normal to the graph at the point in the first quadrant where $x = 1$.

6 The motion of a particle in a straight line is modelled by

$$\frac{dx}{dt} = x^2 \sin(2t)$$

where x cm is the position of the particle relative to an origin O at time t seconds ($t \geq 0$). The initial position of the particle is $x = \frac{1}{2}$.

a Find an expression for x in terms of t.

b Find the maximum distance of the particle from the origin, and the times at which this occurs.

7 Sketch the graph of $y = \dfrac{4 - x^3}{3x^2}$. Give the coordinates of any turning points and axis intercepts and state the equations of all asymptotes.

8 Let $f(x) = \dfrac{1 + x^2}{4 - x^2}$.

a Express $f(x)$ as partial fractions.

b Find the area enclosed by the graph of $y = f(x)$ and the lines $x = 1$ and $x = -1$.

9 Consider the curve defined by the parametric equations

$$x = 2\cos^2(\theta) \quad \text{and} \quad y = \sin(2\theta) \quad \text{for } 0 \leq \theta \leq \frac{\pi}{4}$$

a Compute the length of the curve.

b This curve can be described in the form $y = f(x)$ for a function f. Find the rule, domain and range of f.

c Find the area of the surface of revolution formed by rotating the curve $y = f(x)$ about the x-axis.

10 Find each of the following antiderivatives:

a $\int x \sec^2(2x)\, dx$ **b** $\int \log_e(x + 5)\, dx$ **c** $\int e^{2x} \sin x\, dx$

11 Find y as a function of x given that $\dfrac{dy}{dx} = e^{2y}\sin(2x)$ and that $y = 0$ when $x = 0$.

12 Find the solution of the differential equation $(1 + x^2)\dfrac{dy}{dx} = 2xy$, given $y = 2$ when $x = 0$.

13 Let $f(x) = \arcsin(4x^2 - 3)$. Find the maximal domain of f.

14 Sketch the graph of $f(x) = \dfrac{4x^2 + 5}{x^2 + 1}$.

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15 For the curve defined by the parametric equations

$$x = 2\sin t + 1 \quad \text{and} \quad y = 2\cos t - 3$$

find $\frac{dy}{dx}$ and its value at $t = \frac{\pi}{4}$.

16 Find the length of the curve defined by $x = e^t - t$ and $y = 4e^{\frac{t}{2}}$ for $0 \le t \le 1$.

17 Let a be a positive constant. The curve defined by the parametric equations

$$x = at^2 \quad \text{and} \quad y = 2at \quad \text{for } 0 \le t \le \sqrt{3}$$

is rotated about the x-axis to form a surface of revolution.

a Find the volume of the corresponding solid of revolution. (Hint: First find the Cartesian equation of the curve.)

b Find the area of the surface of revolution.

18 Evaluate:

a $\int_0^1 e^{2x}\cos(e^{2x})\,dx$ **b** $\int_1^2 (x-1)\sqrt{2-x}\,dx$ **c** $\int_0^1 \frac{x-2}{x^2-7x+12}\,dx$

d $\int_3^5 \frac{6}{x^2-6x+4}\,dx$ **e** $\int_2^7 \frac{2+x}{\sqrt{2+x}}\,dx$ **f** $\int_0^{\frac{\pi}{4}} \sec^3 x \tan x\,dx$

19 For the differential equation $\frac{dy}{dx} = -2x^2$ with $y = 2$ when $x = 1$, find y_3 using Euler's method with step size 0.1.

20 Find the volume of the solid formed when the region bounded by the x-axis and the curve with equation $y = a - \frac{x^2}{16a^3}$, where $a > 0$, is rotated about the y-axis.

21 A particle is moving in a straight line and is subject to a retardation of $1 + v^2$ m/s^2, where v m/s is the speed of the particle at time t seconds. The initial speed is u m/s. Find an expression for the distance travelled, in metres, for the particle to come to rest.

22 A particle falls vertically from rest such that the acceleration, a m/s^2, is given by $a = g - 0.4v$, where v m/s is the speed at time t seconds. Find an expression for v in terms of t in the form $v = A(1 - e^{-Bt})$, where A and B are positive constants. Hence state the values of A and B.

23 A train, when braking, has an acceleration, a m/s^2, given by

$$a = -\left(1 + \frac{v}{100}\right)$$

where v m/s is the velocity. The brakes are applied when the train is moving at 20 m/s and it travels x metres after the brakes are applied. Find the distance that the train travels to come to rest in the form $x = A\log_e(B) + C$, where A, B and C are positive constants.

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24 Consider the graph of $f(x) = \dfrac{2x}{x^2 + 1}$.

a Show that $\dfrac{dy}{dx} = \dfrac{-2(x^2 - 1)}{(x^2 + 1)^2}$.

b Find the coordinates of any points of inflection.

25 The position of a particle at time t seconds, relative to an origin O, is given by

$$\boldsymbol{r}(t) = \sin(t)\,\boldsymbol{i} + \frac{1}{2}\sin(2t)\,\boldsymbol{j}, \quad t \geq 0$$

a Find the velocity of the particle at time t.

b Find the acceleration at time t.

c Find an expression for the distance of the particle from the origin at time t in terms of $\sin(t)$.

d Find an expression for the speed of the particle at time t in terms of $\sin(t)$.

e Find the Cartesian equation of the path of the particle.

26 The position vector of a particle moving relative to the origin at time t seconds is given by $\boldsymbol{r}(t) = 2\sec(t)\,\boldsymbol{i} + \dfrac{1}{2}\tan(t)\,\boldsymbol{j}$, for $t \in \left[0, \dfrac{\pi}{2}\right)$.

a Find the Cartesian equation of the path.

b Find the velocity of the particle at time t.

c Find the speed of the particle when $t = \dfrac{\pi}{3}$.

27 A particle moves such that, at time t seconds, the velocity, $\boldsymbol{v}$ m/s, is given by $\boldsymbol{v} = e^{2t}\boldsymbol{i} - e^{-2t}\boldsymbol{k}$. Given that, at $t = 0$, the position of the particle is $\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, find the position at $t = \log_e 2$.

28 A particle has acceleration, $\boldsymbol{a}$ m/s^2, given by $\boldsymbol{a} = -g\boldsymbol{j}$, where $\boldsymbol{j}$ is a unit vector vertically upwards. Let $\boldsymbol{i}$ be a horizontal unit vector in the plane of the particle's motion. The particle is projected from the origin with an initial speed of 20 m/s at an angle of 60° to the horizontal.

a Prove that the velocity, in m/s, at t seconds is given by $\boldsymbol{v} = 10\boldsymbol{i} + (10\sqrt{3} - gt)\boldsymbol{j}$.

b Hence find the Cartesian equation of the path of the particle.

29 The velocity, $\boldsymbol{v}$, of a particle at time t seconds is given by

$$\boldsymbol{v}(t) = -2\sin(2t)\,\boldsymbol{i} + 2\cos(2t)\,\boldsymbol{j}, \quad 0 \leq t \leq 2\pi$$

The particle moves in the horizontal plane. Let $\boldsymbol{i}$ be the unit vector in the easterly direction and $\boldsymbol{j}$ be the unit vector in the northerly direction. Find:

a the position vector, $\boldsymbol{r}(t)$, given that $\boldsymbol{r}(0) = 2\boldsymbol{i} - \boldsymbol{j}$

b the Cartesian equation of the path of the particle

c the time(s) when the particle is moving in the westerly direction.

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30 A particle is projected from the origin such that its position vector, $\boldsymbol{r}(t)$ metres, after t seconds is given by

$$\boldsymbol{r}(t) = 14\sqrt{3}t\,\boldsymbol{i} + \left(14t - \frac{g}{2}t^2\right)\boldsymbol{j}$$

where $\boldsymbol{i}$ is the unit vector in the direction of the x-axis, horizontally, and $\boldsymbol{j}$ is the unit vector in the direction of the y-axis, vertically. The x-axis represents ground level. Find:

a the time (in seconds) taken for the particle to reach the ground, in terms of g

b the Cartesian equation of the parabolic path

c the maximum height reached by the particle (in metres), in terms of g.

31 **a** Solve the differential equation

$$\frac{dy}{dx} = y(1+y)(1-x)$$

given that $y = 1$ when $x = 1$. Write your answer in the form $y = f(x)$.

b Find the local maximum of the curve $y = f(x)$.

32 The curve defined parametrically by $x = t$ and $y = \frac{1}{2}(e^t + e^{-t})$ is called a **catenary**; it has the shape of a string suspended from both ends. Find the length of this curve from $t = -1$ to $t = 1$.

33 On the moon, the acceleration due to gravity is 1.6 m/s^2. An astronaut is standing on the moon and then jumps upwards with an initial velocity of 3.2 m/s.

a How high does she get off the surface of the moon?

b For how long is she off the surface of the moon?

34 **a** Let $f\colon [0, a] \to \mathbb{R}$ be continuous. Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

b Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$.

14B Multiple-choice questions

1 The graph of the function $f(x) = \dfrac{x^2 + x + 2}{x}$ has asymptotes with equations

A $y = x$ and $y = x^2 + x + 2$
B $y = x$ and $y = x + 1$
C $x = 0$ and $y = x^2 + x + 2$
D $x = 0$ and $y = x + 1$
E $y = \dfrac{2}{x}$ and $y = x + 1$

2 One solution of the differential equation $\dfrac{d^2y}{dx^2} = 2\cos x + 1$ is

A $y = -4 + \cos x + x$
B $y = 2\sin x + x + 1$
C $y = -\dfrac{1}{4}\cos(2x) + \dfrac{x^2}{2} + x$
D $y = -2\cos x + \dfrac{x^2}{2} + x$
E $y = 2\cos x + \dfrac{x^2}{2} + x$

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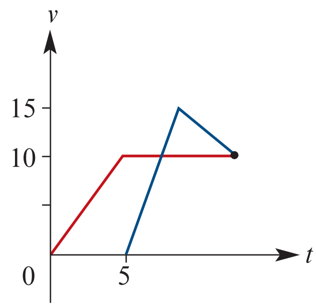
3 A curve passes through the point $(2, 3)$ and is such that the tangent to the curve at each point (a, b) is perpendicular to the tangent to $y = 2x^3$ at $(a, 2a^3)$. The equation of the curve can be found by using the differential equation

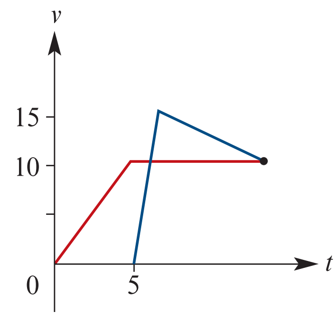
A $\dfrac{dy}{dx} = 2x^3$ **B** $\dfrac{dy}{dx} = -\dfrac{1}{6x^2}$ **C** $\dfrac{dy}{dx} = -6x^2$ **D** $\dfrac{dy}{dx} = \dfrac{2}{x} + c$ **E** $\dfrac{dy}{dx} = -\dfrac{1}{2x^3}$

4 Car P leaves a garage, accelerates at a constant rate to a speed of 10 m/s and continues at that speed. Car Q leaves the garage 5 seconds later, accelerates at the same rate as car P to a speed of 15 m/s and continues at that speed until it hits the back of car P. Which one of the following pairs of graphs represents the motion of these cars?

A



B



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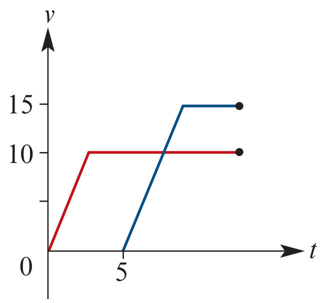
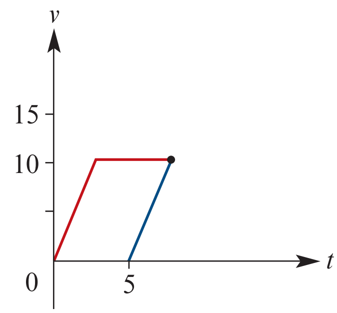
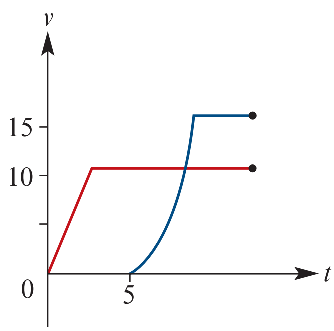


D



E



5 A curve passes through the point $(1, 1)$ and is such that the gradient at any point is twice the reciprocal of the x-coordinate. The equation of this curve can be found by solving the differential equation with the given boundary condition

A $x\dfrac{dy}{dx} = 2, \quad y(1) = 1$ **B** $\dfrac{d^2y}{dx^2} = \dfrac{x}{2}, \quad y(1) = 1$ **C** $y\dfrac{dy}{dx} = 2, \quad y(1) = 1$

D $\dfrac{dy}{dx} = x, \quad y(1) = 1$ **E** $\dfrac{1}{2}\dfrac{dy}{dx} = x, \quad y(1) = 1$

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6 A container initially holds 20 L of pure water. A salt solution of concentration 3 g/L is poured into the container at a rate of 2 L/min. The mixture is kept uniform by stirring and flows out at a rate of 2 L/min. If Q g is the amount of salt in the container t minutes after pouring begins, then Q satisfies the equation

A $\frac{dQ}{dt} = \frac{Q}{10}$ **B** $\frac{dQ}{dt} = Q$ **C** $\frac{dQ}{dt} = 6 - \frac{Q}{10}$

D $\frac{dQ}{dt} = 6 - \frac{Q}{10 + t}$ **E** $\frac{dQ}{dt} = 6 - \frac{Q}{20}$

7 The equation of the particular member of the family of curves defined by $\frac{dy}{dx} = 3x^2 + 1$ that passes through the point $(1, 3)$ is

A $y = 6x$ **B** $y = x^3 + x^2 + 1$ **C** $y = x^3 + x + 1$

D $y = x^3 + x + 3$ **E** $y = \frac{x^3}{3} + x$

8 For which one of the following derivative functions does the graph of $y = g(x)$ have no points of inflection?

A $g'(x) = 5(x - 4)^3 + 5$ **B** $g'(x) = 6x^3 - 4x$ **C** $g'(x) = (x - 2)^2 - 3$

D $g'(x) = (x - 2)^2 - 3x$ **E** $g'(x) = 4x - 5(x - 2)^3$

9 One solution of the differential equation $\frac{d^2y}{dx^2} = e^{3x}$ is

A $y = 3e^{3x}$ **B** $y = \frac{1}{3}e^{3x}$ **C** $y = \frac{1}{3}e^{3x} + x$

D $y = 9e^{3x} + x$ **E** $y = \frac{1}{9}e^{3x} + x$

10 A body initially travelling at 12 m/s is subject to a constant deceleration of 4 m/s^2. The time taken to come to rest (t seconds) and the distance travelled before it comes to rest (s metres) are

A $t = 3,\ s = 24$ **B** $t = 3,\ s = 18$ **C** $t = 3,\ s = 8$

D $t = 4,\ s = 18$ **E** $t = 4,\ s = 8$

11 A tank initially contains 50 litres of water in which 0.5 kg of salt has been dissolved. Fresh water flows into the tank at a rate of 10 litres per minute, and the solution (kept uniform by stirring) flows out at a rate of 8 litres per minute. Let x kg be the mass of salt in solution in the tank after t minutes. Which one of the following describes the relationship between x and t?

A $\int \frac{1}{4x}\,dx = -\int \frac{1}{25 + t}\,dt$ **B** $\int \frac{1}{4x}\,dx = \int \frac{1}{25 + t}\,dt$

C $\int \frac{1}{6x}\,dx = -\int \frac{1}{25 - t}\,dt$ **D** $\int \frac{1}{6x}\,dx = \int \frac{1}{25 - t}\,dt$

E $\int \frac{1}{10x}\,dx = -\int \frac{6}{25 - t}\,dt$

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Revision

12 If $x = 2\sin^2(y)$, then $\dfrac{dy}{dx}$ equals

A $4\sin(y)$ **B** $\dfrac{1}{2}\operatorname{cosec}(2y)$ **C** $4\sqrt{\dfrac{x}{2}}$ **D** $2\sqrt{2x}$ **E** $\dfrac{1}{2}\sin^{-1}(2y)$

13 The rate of decay of a radioactive substance is proportional to the amount, x, of the substance present. This is described by the differential equation $\dfrac{dx}{dt} = -kx$, where k is a positive constant. Given that initially $x = 20$ and that $x = 5$ when $t = 20$, the time at which $x = 2$ is closest to

A 22.33 **B** 10.98 **C** 50 **D** 30.22 **E** 33.22

14 $\int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x\, dx$ is given by

A $\int_0^{\sqrt{3}} u^2\, du$ **B** $\int_0^{\frac{2}{\sqrt{3}}} 1 - u^2\, du$ **C** $\int_0^{\frac{\pi}{3}} u^2(1 - u^2)\, du$

D $\int_0^{\frac{\pi}{3}} u^2(1 + u^2)\, du$ **E** $\int_0^{\frac{\pi}{3}} 1 - u^2\, du$

15 Assume that $\ddot{y} = e^x + e^{-2x}$. If $y = 0$ and $\dot{y} = \dfrac{1}{2}$ when $x = 0$, then

A $y = e^x + \dfrac{1}{4}e^{-2x} - \dfrac{5}{4}$ **B** $y = e^x + e^{-2x} - \dfrac{1}{2}$ **C** $y = e^x + e^{-2x}$

D $y = e^x + e^{-2x} + \dfrac{1}{2}$ **E** $y = e^x + e^{-2x} + \dfrac{5}{4}x - \dfrac{5}{4}$

16 If $\dfrac{dy}{dx} = 2y + 1$ and $y = 3$ when $x = 0$, then

A $y = \dfrac{7e^{2x} - 1}{2}$ **B** $y = \dfrac{1}{2}\log_e(2x + 1)$ **C** $y = y^2 + y + 1$

D $y = e^{2x}$ **E** $y = \dfrac{2e^{2x} + 1}{7}$

17 If $y = x\tan^{-1}(x)$, then $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2} + \tan^{-1}(x)$. It follows that an antiderivative of $\tan^{-1}(x)$ is

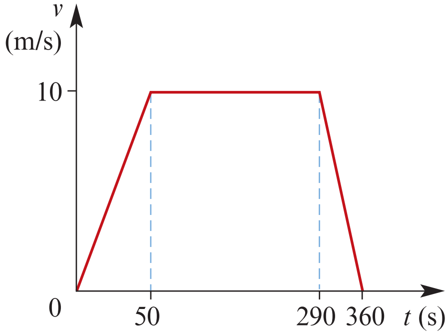
A $x\tan^{-1}(x)$ **B** $x\tan^{-1}(x) - \dfrac{x}{1 + x^2}$ **C** $x\tan^{-1}(x) - \log_e(\sqrt{1 + x^2})$

D $\dfrac{1}{1 + x^2} + \dfrac{1}{x}\tan^{-1}(x)$ **E** $\dfrac{x}{1 + x^2}$

18 The velocity–time graph shows the motion of a train between two stations. The distance between the stations, in metres, is

A 2500 **B** 2900 **C** 3000

D 3400 **E** 5800



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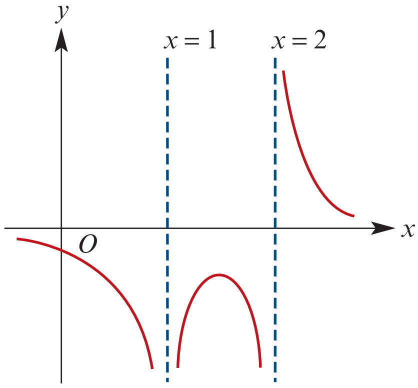
19 The equation of the particular member of the family of curves defined by $\frac{dy}{dx} = 1 - e^{-x}$ that passes through the point (0, 6) is

A $y = x - e^{-x} + 5$ **B** $y = x + e^{-x} + 5$ **C** $y = x + e^{-x} + 7$
D $y = x + e^{-x} + 6$ **E** $y = x - e^{-x} + 6$

20 This could be the graph of



A $y = \frac{1}{(x-1)(x-2)}$

B $y = \frac{x}{(x-1)(x-2)}$

C $y = \frac{(x-1)(x-2)}{x}$

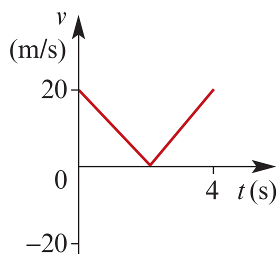
D $y = \frac{1}{(x-2)(x-1)^2}$

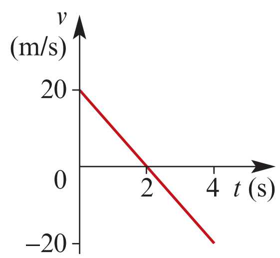
E $y = \frac{1}{(x-1)(x-2)^2}$

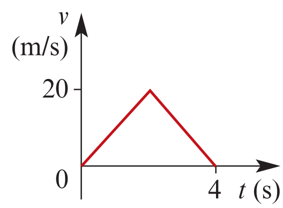
21 The values of m for which $y = e^{mx}$ satisfies the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$ are

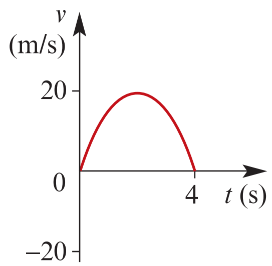
A $m = 1,\ m = 2$ **B** $m = 3,\ m = -1$ **C** $m = -2,\ m = 3$
D $m = \pm 1$ **E** $m = \pm 3$

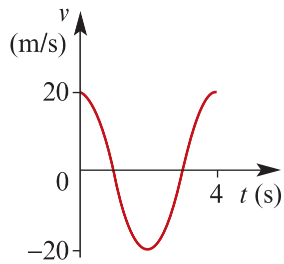
22 A particle is projected vertically upwards from ground level with a velocity of 20 m/s and returns to the point of projection. The velocity–time graph illustrating this could be

A


B


C


D


E


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23 Which one of the following differential equations is satisfied by $y = e^{3x}$ for all values of x?

A $\dfrac{d^2y}{dx^2} + 9y = 0$ **B** $\dfrac{d^2y}{dx^2} - 9y = 0$ **C** $\dfrac{d^2y}{dx^2} + \dfrac{y}{9} = 0$

D $\dfrac{d^2y}{dx^2} - 27y = 0$ **E** $\dfrac{d^2y}{dx^2} - 8y = 0$

24 A particle has initial velocity 3 m/s and its acceleration t seconds later is given by $(6t^2 + 5t - 3)$ m/s^2. After 2 seconds, its velocity in m/s is

A 15 **B** 18 **C** 21 **D** 27 **E** 23

25 A particle starts from rest at a point O and moves in a straight line so that after t seconds its velocity, v, is given by $v = 4\sin(2t)$. Its displacement from O is given by

A $s = 8\cos(2t)$ **B** $s = 2\cos(2t)$ **C** $s = -2\cos(2t)$

D $s = 8\cos(2t) - 8$ **E** $s = 2 - 2\cos(2t)$

26 The volume of the solid of revolution when the shaded region of the diagram is rotated about the y-axis is given by

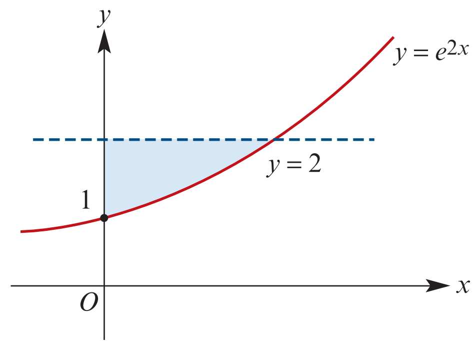
A $\pi \int_0^{\frac{1}{2}\log_e 2} e^{2x}\,dx$

B $\pi \int_0^2 \frac{1}{2}\log_e y\,dy$

C $\pi\left(\log_e 2 - \int_0^{\frac{1}{2}\log_e 2} e^{2x}\,dx\right)$

D $\pi \int_0^2 \frac{1}{4}(\log_e y)^2\,dy - \frac{\pi}{2}$

E $\pi \int_1^2 \frac{1}{4}(\log_e y)^2\,dy$



27 The area of the shaded region in the graph is

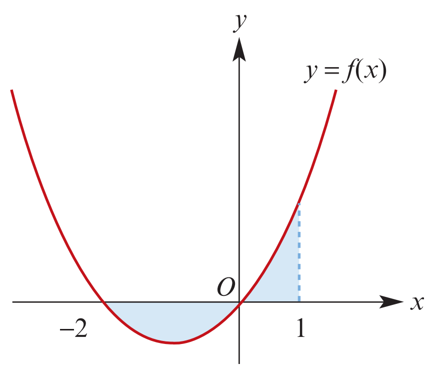
A $\int_0^1 f(x)\,dx + \int_0^{-2} f(x)\,dx$

B $\int_{-2}^1 f(x)\,dx$

C $\int_{-2}^0 f(x)\,dx + \int_0^1 f(x)\,dx$

D $-\int_1^0 f(x)\,dx + \int_0^{-2} f(x)\,dx$

E $-\int_0^{-2} f(x)\,dx + \int_0^1 f(x)\,dx$



28 An arrangement of the integrals

$$P = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx, \quad Q = \int_0^{\frac{\pi}{4}} \cos^2 x\,dx, \quad R = \int_0^{\frac{\pi}{4}} \sin^2 x\,dx$$

in ascending order of magnitude is

A $P,\ R,\ Q$ **B** $Q,\ P,\ R$ **C** $R,\ Q,\ P$ **D** $R,\ P,\ Q$ **E** $Q,\ R,\ P$

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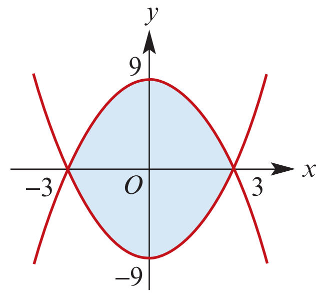
29 In the diagram on the right, the area of the region enclosed between the graphs with equations $y = x^2 - 9$ and $y = 9 - x^2$ is given by

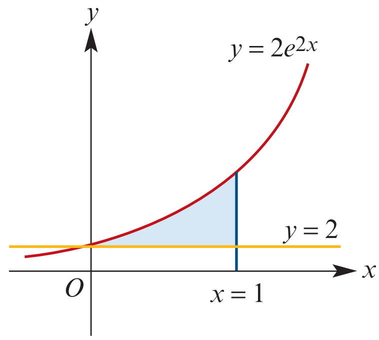


A $\int_{-3}^{3} 2x^2 - 18\,dx$ **B** $\int_{-3}^{3} 18 - 2x^2\,dx$

C 0 **D** $\int_{-9}^{9} 2x^2 - 18\,dx$

E $\int_{-9}^{9} 18 - 2x^2\,dx$

30 The volume of the solid of revolution when the shaded region of this graph is rotated about the x-axis is given by



A $\pi \int_0^1 4e^{4x} - 4\,dx$

B $\pi \int_0^1 e^{2x} - 4\,dx$

C $\pi \int_0^1 (2e^{2x} - 2)^2\,dx$

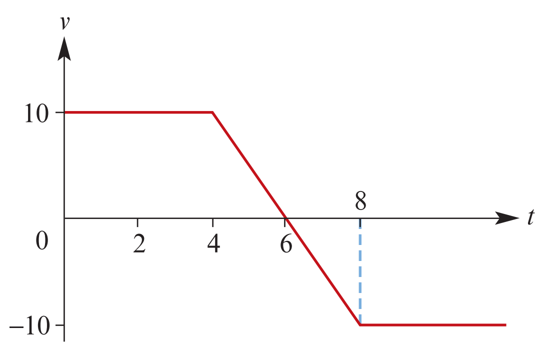
D $\pi \int_2^{2e} 1\,dy$

E $\pi \int_0^1 4 - 4e^{2x}\,dx$

31 A body moves in a straight line so that its acceleration (in m/s^2) at time t seconds is given by $\dfrac{d^2x}{dt^2} = 4 - e^{-t}$. If the body's initial velocity is 3 m/s, then when $t = 2$ its velocity (in m/s) is

A e^{-2} **B** $2 + e^{-2}$ **C** $8 + e^{-2}$ **D** $10 + e^{-2}$ **E** $12 + e^{-2}$

32 A particle moves with velocity v m/s. The distance travelled, in metres, by the particle in the first 8 seconds is



A 40 **B** 50

C 60 **D** 70

E 80

33 A particle moves in a straight line such that its velocity, v m/s, at time t seconds is given by $v = \dfrac{\log_e t}{t}$, for $t \geq 1$. When does the particle have its maximum velocity?

A $t = 1$ **B** $t = e^{\frac{1}{2}}$ **C** $t = e$ **D** $t = e^{\frac{3}{2}}$ **E** $t = e^2$

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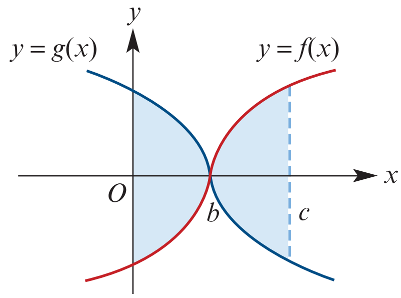
Revision

34 If $\frac{dy}{dx} = -\frac{x}{ye^{x^2}}$ and $y = 2$ when $x = 0$, then

A $y = 3 - e^{-x^2}$ **B** $y^2 = 3 + e^{-x^2}$ **C** $y = \frac{5}{2} - e^{-x^2}$ **D** $y^2 = \frac{5}{2} - e^{-x^2}$ **E** $y^2 = 3 + e^{x^2}$

35 The area of the region shaded in the graph is equal to



A $\int_0^c f(x) - g(x)\,dx$

B $\int_b^c f(x) - g(x)\,dx + \int_0^b f(x) - g(x)\,dx$

C $\int_b^c f(x) - g(x)\,dx + \int_b^0 f(x) - g(x)\,dx$

D $\int_b^c f(x)\,dx + \int_0^b g(x)\,dx$

E $\int_0^c f(x) + g(x)\,dx$

36 A partial fraction expansion of $\frac{1}{(2x+6)(x-4)}$ shows that it has an antiderivative $\frac{a}{2}\log_e|2x+6| + b\log_e|x-4|$, where

A $a = -\frac{1}{7},\ b = \frac{1}{14}$ **B** $a = 1,\ b = 1$ **C** $a = \frac{1}{2},\ b = \frac{1}{2}$

D $a = -1,\ b = -1$ **E** $a = \frac{1}{11},\ b = \frac{1}{7}$

37 $\int_0^1 x\sqrt{2x+1}\,dx$ is equal to

A $\frac{1}{2}\int_0^1 (u-1)\sqrt{u}\,du$ **B** $\int_0^1 u\sqrt{u}\,du$ **C** $\frac{1}{4}\int_1^3 \sqrt{u}\,du$

D $2\int_1^3 \sqrt{u}\,du$ **E** $\frac{1}{4}\int_1^3 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

38 $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$ is equal to

A $\int_1^e \frac{u}{(1+u)^2}\,du$ **B** $\int_0^1 \frac{u}{(1+u)^2}\,du$ **C** $\int_0^1 \frac{1}{(1+u)^2}\,du$

D $\int_1^e \frac{1}{(1+u)^2}\,du$ **E** $\int_1^e \frac{u}{1+u^2}\,du$

39 If $\int_0^{\frac{\pi}{6}} \sin^n x \cos x\,dx = \frac{1}{64}$, then n equals

A 6 **B** 5 **C** 4 **D** 3 **E** 7

40 Given that $\int x^2 \sin x\,dx = g(x) + \int 2x\cos x\,dx$, the rule for g can be written in the form

A $g(x) = -x^2\cos x + c$ **B** $g(x) = 2\cos x + 2x\sin x + c$

C $g(x) = (2 - x^2)\cos x + 2x\sin x + c$ **D** $g(x) = 4\sin x - 2x\cos x + c$

E $g(x) = (2x - x^2)\sin x + c$

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41 Of the integrals

$$\int_0^{\pi} \sin^3\theta \cos^3\theta \, d\theta, \quad \int_0^2 t^3(4-t^2)^2 \, dt, \quad \int_0^{\pi} x^2 \cos x \, dx$$

one is negative, one is positive and one is zero. Without evaluating them, determine which is the correct order of signs.

A $- \ 0 \ +$ **B** $+ \ - \ 0$ **C** $+ \ 0 \ -$ **D** $0 \ - \ +$ **E** $0 \ + \ -$

42 $\int_0^{\frac{\pi}{4}} \cos(2x) \, dx$ is equal to

A $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2x) \, dx$ **B** $\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2x) \, dx$ **C** $\int_{-\frac{\pi}{4}}^{0} \sin(2x) \, dx$

D $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(4x) \, dx$ **E** $\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(4x) \, dx$

43 $\int_{-a}^{a} \tan x \, dx$ can be evaluated if a equals

A $\frac{\pi}{2}$ **B** $\frac{3\pi}{2}$ **C** $\frac{\pi}{4}$ **D** π **E** $-\frac{3\pi}{2}$

44 If $e^{f(x)} = x^2 + 9$, then $f'(x)$ is equal to

A $\frac{1}{x^2+9}$ **B** $\frac{2x}{x^2+9}$ **C** $2x(x^2+9)$ **D** $2xe^{2x}$ **E** $2x\log_e(x^2+9)$

45 Which one of the following expressions is equivalent to $\int x^3 \sin(3x) \, dx$?

A $\frac{x^3}{3}\cos(3x) - \frac{1}{3}\int x^3 \cos(3x) \, dx$ **B** $-\frac{x^3}{3}\cos(3x) + \int x^2 \cos(3x) \, dx$

C $\frac{x^3}{3}\cos(3x) + c$ **D** $x^2\cos(3x) + c$ **E** $-\frac{1}{3}\cos(3x) + c$

46 Let $f(x) = \log_e(x^2 + 5)$. The graph of $y = f(x)$ is concave up for

A $-\sqrt{5} < x < \sqrt{5}$ **B** $-\sqrt{6} < x < \sqrt{6}$ **C** $x > \sqrt{5}$ or $x < -\sqrt{5}$

D $x > 2$ or $x < -2$ **E** $-2\sqrt{5} < x < 2\sqrt{5}$

47 The volume of the solid of revolution formed by rotating the region bounded by the curve $y = 2\sin x - 1$ and the lines with equations $x = 0$, $x = \frac{\pi}{4}$ and $y = 0$ about the x-axis is given by

A $\int_0^{\frac{\pi}{2}} \pi^2(2\sin x - 1)^2 \, dx$ **B** $\int_0^{\frac{\pi}{4}} \pi(4\sin^2 x - 1) \, dx$ **C** $\int_0^{\frac{\pi}{4}} \pi(1 - 2\sin x)^2 \, dx$

D $\int_0^{\frac{\pi}{4}} (2\sin x - 1)^2 \, dx$ **E** $\int_0^{\frac{\pi}{4}} \pi(2\sin x - 1) \, dx$

48 The area of the region bounded by the graphs of $f\colon \left[0, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \sin x$ and $g\colon \left[0, \frac{\pi}{2}\right) \to \mathbb{R}$, $g(x) = \sin(2x)$ is

A $\int_0^{\frac{\pi}{2}} \sin x - x\sin(2x) \, dx$ **B** $\int_0^{\frac{\pi}{3}} \sin(2x) - \sin x \, dx$ **C** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x) - \sin x \, dx$

D $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \sin(2x) \, dx$ **E** $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sin(2x) - \sin x \, dx$

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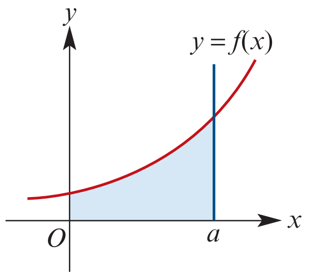
49 The number of bacteria in a culture at time t is growing at a rate of $4000e^{0.4t}$ bacteria per unit of time. When $t = 0$, the number of bacteria was 10 000. When $t = 10$, the number of bacteria will be closest to

A $5000e^4$ **B** $10\,000e^2$ **C** $7000e^3$ **D** $10\,000e^4$ **E** $7500e^4$

50 The shaded region is bounded by the curve $y = f(x)$, the coordinate axes and the line $x = a$. Which one of the following statements is false?



A The area of the shaded region is $\int_0^a f(x)\,dx$.

B The volume of the solid of revolution formed by rotating the region about the x-axis is $\int_0^a \pi(f(x))^2\,dx$.

C The volume of the solid of revolution formed by rotating the region about the y-axis is $\int_{f(0)}^{f(a)} \pi x^2\,dy$.

D The area of the shaded region is greater than $af(0)$.

E The area of the shaded region is less than $af(a)$.

51 The general solution of the differential equation $\frac{dy}{dx} + y = 1$ (with P being an arbitrary constant) is

A $2x + (1 - y)^2 = P$ **B** $2x - (1 - y)^2 = P$ **C** $y = 1 + Pe^x$

D $y = 1 + Pe^{-x}$ **E** $y = Pe^{-x} - 1$

52 Air leaks from a spherical balloon at a constant rate of 2 m^3/s. When the radius of the balloon is 5 m, the rate (in m^2/s) at which the surface area is decreasing is

A $\frac{4}{5}$ **B** $\frac{8}{5}$ **C** $\frac{1}{50}\pi$ **D** $\frac{1}{100}\pi$ **E** none of these

53 A curve is defined parametrically by $x = \sin^2(2t)$ and $y = \cos(3t)$ for $0 \le t \le \frac{\pi}{2}$. The length of this curve is given by

A $\int_0^{\frac{\pi}{2}} \sqrt{2\sin(4t) - 3\sin(3t)}\,dt$ **B** $\int_0^{\frac{\pi}{2}} \sqrt{\sin^4(2t) + \cos^2(3t)}\,dt$

C $\int_0^{\frac{\pi}{2}} \sqrt{2\sin(4t) + 3\sin(3t)}\,dt$ **D** $\int_0^{\frac{\pi}{2}} \sqrt{4\cos^4(2t) + 9\sin^2(3t)}\,dt$

E $\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2(4t) + 9\sin^2(3t)}\,dt$

54 At a certain instant, a sphere is of radius 10 cm and the radius is increasing at a rate of 2 cm/s. The rate of increase (in cm^3/s) of the volume of the sphere is

A 80π **B** $\frac{800\pi}{3}$ **C** 400π **D** 800π **E** $\frac{8000\pi}{3}$

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55 $\frac{d}{d\theta}\left(\log_e(\sec\theta + \tan\theta)\right)$ equals

A $\sec\theta$ **B** $\sec^2\theta$ **C** $\sec\theta\tan\theta$ **D** $\cot\theta - \tan\theta$ **E** $\tan\theta$

56 A particle is moving along the x-axis such that $x = 3\cos(2t)$ at time t. When $t = \frac{\pi}{2}$, the acceleration of the particle in the positive x-direction is

A -12 **B** -6 **C** 0 **D** 6 **E** 12

57 A particle moves in the Cartesian plane such that its position vector, $\boldsymbol{r}$ metres, at time t seconds is given by $\boldsymbol{r} = 2t^2\boldsymbol{i} + t^3\boldsymbol{j}$. When $t = 1$, the speed of the particle (in m/s) is

A $\frac{3}{4}$ **B** $\sqrt{5}$ **C** 5 **D** 7 **E** 25

58 The position of a particle at time $t = 0$ is $\boldsymbol{r}(0) = 2\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$, and its position at time $t = 2$ is $\boldsymbol{r}(2) = 4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$. The average velocity for the time interval $[0, 2]$ is

A $\frac{1}{2}(6\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k})$ **B** $\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ **C** $24\boldsymbol{i} + \boldsymbol{k}$
D $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ **E** $\frac{1}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$

59 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{x}}(t) = 2\boldsymbol{i} + t\boldsymbol{j}$. If the velocity of the particle at time $t = 0$ is described by the vector $2\boldsymbol{i}$, then the velocity at time t is

A $\dot{\boldsymbol{x}}(t) = 2t\,\boldsymbol{i} + \frac{1}{2}t^2\boldsymbol{j}$ **B** $\dot{\boldsymbol{x}}(t) = (2t + 2)\boldsymbol{i} + \frac{1}{2}t^2\boldsymbol{j}$ **C** $\dot{\boldsymbol{x}}(t) = 2\boldsymbol{i} + (2\boldsymbol{i} + t\boldsymbol{j})t$
D $\dot{\boldsymbol{x}}(t) = 2(2\boldsymbol{i} + t\boldsymbol{j})$ **E** $\dot{\boldsymbol{x}}(t) = 2 + 2t\,\boldsymbol{i} + \frac{1}{2}t^2\boldsymbol{j}$

60 A particle is moving so its velocity vector at time t is $\dot{\boldsymbol{r}}(t) = 2t\,\boldsymbol{i} + 3\boldsymbol{j}$, where $\boldsymbol{r}(t)$ is the position vector of the particle at time t. If $\boldsymbol{r}(0) = 3\boldsymbol{i} + \boldsymbol{j}$, then $\boldsymbol{r}(t)$ is equal to

A $2\boldsymbol{i}$ **B** $5\boldsymbol{i} + 3\boldsymbol{j}$ **C** $(3t + 1)\boldsymbol{i} + (3t^2 + 1)\boldsymbol{j}$
D $(t^2 + 3)\boldsymbol{i} + (3t + 1)\boldsymbol{j}$ **E** $2t^2\boldsymbol{i} + 3t\boldsymbol{j} + 3\boldsymbol{i} + \boldsymbol{j}$

61 A particle has its position in metres from a given point at time t seconds defined by the vector $\boldsymbol{r}(t) = 4t\,\boldsymbol{i} - \frac{1}{3}t^2\boldsymbol{j}$. The magnitude of the displacement in the third second is

A 4 m **B** $3\frac{2}{3}$ m **C** $4\frac{1}{3}$ m **D** $6\frac{2}{3}$ m **E** 9 m

62 The position of a particle at time t seconds is given by $\boldsymbol{r}(t) = (t^2 - 2t)(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$, measured in metres from a fixed point. The distance travelled by the particle in the first 2 seconds is

A 0 m **B** 2 m **C** −2 m **D** 6 m **E** 10 m

63 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that:

- $f(0) = 3$
- $f(3) = 6$
- $f'(0) = 8$
- $f'(3) = 11$

Then the value of $\int_0^3 4xf''(x)\,dx$ is

A 98 **B** 112 **C** 120 **D** 132 **E** 142

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Revision

64 The position of a particle at time t seconds is given by the vector

$$\boldsymbol{r}(t) = \left(\frac{1}{3}t^3 - 4t^2 + 15t\right)\boldsymbol{i} + \left(t^3 - \frac{15}{2}t^2\right)\boldsymbol{j}$$

When the particle is instantaneously at rest, its acceleration vector is given by

A $15\boldsymbol{i}$ **B** $-18\boldsymbol{j}$ **C** $2\boldsymbol{i} + 15\boldsymbol{j}$ **D** $-8\boldsymbol{i} - 15\boldsymbol{j}$ **E** $-2\boldsymbol{i} + 3\boldsymbol{j}$

65 A particle moves with its position defined with respect to time t by the vector function

$$\boldsymbol{r}(t) = (3t^3 - t)\boldsymbol{i} + (2t^2 + 1)\boldsymbol{j} + 5t\boldsymbol{k}$$

When $t = \frac{1}{2}$, the magnitude of the acceleration is

A 12 **B** 17 **C** $4\sqrt{3}$ **D** $4\sqrt{5}$ **E** none of these

66 The velocity of a particle is given by the vector $\dot{\boldsymbol{r}}(t) = \sin(t)\,\boldsymbol{i} + \cos(2t)\,\boldsymbol{j}$. At time $t = 0$, the position of the particle is given by the vector $6\boldsymbol{i} - 4\boldsymbol{j}$. The position of the particle at time t is given by

A $(6 - \cos t)\boldsymbol{i} + (\frac{1}{2}\sin(2t) + 4)\boldsymbol{j}$ **B** $(5 - \cos t)\boldsymbol{i} + (\frac{1}{2}\sin(2t) - 3)\boldsymbol{j}$

C $(5 + \cos t)\boldsymbol{i} + (2\sin(2t) - 4)\boldsymbol{j}$ **D** $(6 + \cos t)\boldsymbol{i} + (2\sin(2t) - 4)\boldsymbol{j}$

E $(7 - \cos t)\boldsymbol{i} + (\frac{1}{2}\sin(2t) - 4)\boldsymbol{j}$

67 The initial position, velocity and constant acceleration of a particle are given by $2\boldsymbol{i}$, $3\boldsymbol{j}$ and $\boldsymbol{i} - \boldsymbol{j}$ respectively. The position of the particle at time t is given by

A $(4 + t)\boldsymbol{i} + (3 - \frac{1}{2}t^2)\boldsymbol{j}$ **B** $2\boldsymbol{i} + 3t\boldsymbol{j}$ **C** $2t\,\boldsymbol{i} + 3t\boldsymbol{j}$

D $(2 + \frac{1}{2}t^2)\boldsymbol{i} + (3t - \frac{1}{2}t^2)\boldsymbol{j}$ **E** $(2 + t)\boldsymbol{i} + (3 - t)\boldsymbol{j}$

14C Extended-response questions

1 A bowl can be described as the solid of revolution formed by rotating the graph of $y = \dfrac{1}{4}x^2$ around the y-axis for $0 \le y \le 25$.

a Find the volume of the bowl.

b The bowl is filled with water and then, at time $t = 0$, the water begins to run out of a small hole in the base. The rate at which the water runs out is proportional to the depth, h, of the water at time t. Let V denote the volume of water at time t.

i Show that $\dfrac{dh}{dt} = \dfrac{-k}{4\pi}$, where $k > 0$.

ii Given that the bowl is empty after 30 seconds, find the value of k.

iii Find h in terms of t.

iv Find V in terms of t.

c Sketch the graph of:

i V against h

ii V against t

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2 Let $I_n = \int_0^1 x^n \tan^{-1}(x)\, dx$, where $n \in \mathbb{N} \cup \{0\}$.

a For each $n \in \mathbb{N} \cup \{0\}$, show that $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\, dx$.

b Evaluate I_0 and I_1.

c For each $n \in \mathbb{N} \cup \{0\}$, show that $(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \frac{1}{n+2}$.

d Hence evaluate:

i I_2 **ii** I_3 **iii** I_4 **iv** I_5

3 Let $I_n = \int_0^1 x^n \log_e(x+1)$, where $n \in \mathbb{N} \cup \{0\}$.

a Using integration by parts, find the value of I_0.

b For each $n \in \mathbb{N}$, show that $(n+1)I_n = 2\log_e 2 - \frac{1}{n+1} - nI_{n-1}$.

c Hence prove that, if n is odd, then $(n+1)I_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{n+1}$.

4 Consider the graph of the relation $2(x+y) = (x-y)^2$.

a Find the coordinates of the axis intercepts of the graph.

b Find the coordinates of the point of intersection of the graph with the line $y = x$.

c Explain why the image of the graph reflected in the line $y = x$ is itself.

d **i** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y. **ii** If $\frac{dy}{dx} = 0$, find the values of x and y.

iii If $\frac{dx}{dy} = 0$, find the values of x and y. **iv** If $\frac{dy}{dx} = 0$, find the value of $\frac{d^2y}{dx^2}$.

e Let F be the point with coordinates $(\frac{1}{4}, \frac{1}{4})$ and let $P(x, y)$ be any point on the graph of the relation $2(x+y) = (x-y)^2$. Show that $PF = PM$, where M is the point on the line $y = -x - \frac{1}{2}$ such that the line PM has gradient 1.

f A linear transformation maps the graph of $y^2 = x$ to the graph of $2(x+y) = (x-y)^2$. Determine the 2×2 matrix that represents this transformation.

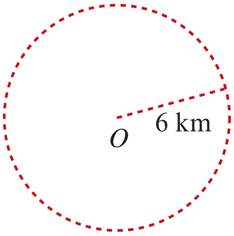
5 **a** Sketch the curve with equation $y + 3 = \frac{6}{x-1}$.

b Find the coordinates of the points where the line $y + 3x = 9$ intersects the curve.

c Find the area of the region enclosed between the curve and the line.

d Find the equations of two tangents to the curve that are parallel to the line.

6 Point O is the centre of a city with a population of 600 000. All of the population lives within 6 km of the city centre. The number of people who live within r km $(0 \le r \le 6)$ of the city centre is given by $\int_0^r 2\pi k(6-x)^{\frac{1}{2}} x^2\, dx$.



a Find the value of k, correct to three significant figures.

b Find the number of people who live within 3 km of the city centre, correct to three significant figures.

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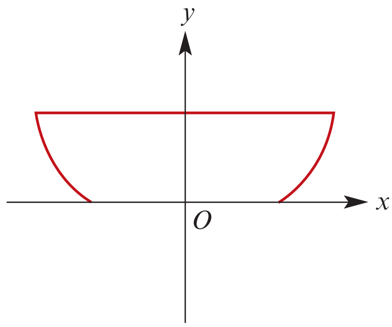
Revision

7 The vertical cross-section of a bucket is shown in this diagram. The sides are arcs of a parabola with the y-axis as the central axis and the horizontal cross-sections are circular. The depth is 36 cm, the radius length of the base is 10 cm and the radius length of the top is 20 cm.



a Prove that the parabolic sides are arcs of the parabola $y = 0.12x^2 - 12$.

b Prove that the bucket holds 9π litres when full.

Water starts leaking from the bucket, initially full, at the rate given by $\frac{dv}{dt} = \frac{-\sqrt{h}}{A}$, where at time t seconds the depth is h cm, the surface area is A cm^2 and the volume is v cm^3.

c Prove that $\frac{dv}{dt} = \frac{-3\sqrt{h}}{25\pi(h + 12)}$.

d Show that $v = \pi \int_0^h \left(\frac{25y}{3} + 100\right) dy$.

e Hence construct a differential equation expressing:

i $\frac{dv}{dh}$ as a function of h **ii** $\frac{dh}{dt}$ as a function of h

f Hence find the time taken for the bucket to empty.

8 A hemispherical bowl can be described as the solid of revolution generated by rotating $x^2 + y^2 = a^2$ about the y-axis for $-a \le y \le 0$. The bowl is filled with water. At time $t = 0$, water starts running out of a small hole in the bottom of the bowl, so that the depth of water in the bowl at time t is h cm. The rate at which the volume is decreasing is proportional to h. (All length units are centimetres.)

a **i** Show that, when the depth of water is h cm, the volume, V cm^3, of water remaining is $V = \pi(ah^2 - \frac{1}{3}h^3)$, where $0 < h \le a$.

ii If $a = 10$, find the depth of water in the bowl if the volume is 1 litre.

b Show that $\pi(2ah - h^2)\frac{dh}{dt} = -kh$, for a positive constant k.

c Given that the bowl is empty after time T, show that $k = \frac{3\pi a^2}{2T}$.

d If $a = 10$ and $T = 30$, find k (correct to three significant figures).

e Sketch the graph of:

i $\frac{dV}{dt}$ against h for $0 \le h \le a$ **ii** $\frac{dh}{dt}$ against h for $0 \le h \le a$

f Find the rate of change of the depth with respect to time when:

i $h = \frac{a}{2}$ **ii** $h = \frac{a}{4}$

g If $a = 10$ and $T = 30$, find the rate of change of depth with respect to time when there is 1 litre of water in the bowl.

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9 Let N be the number of insects in a colony at time t weeks, where $N(0) = 1000$.

a First consider the model $\dfrac{dN}{dt} = \dfrac{k}{N}$, where $k \in \mathbb{R}$. If $N(1) = 1500$, find N in terms of t.

b Now consider the model $\dfrac{dN}{dt} = \dfrac{N^2}{4000e^{0.2t}}$.

i Find N in terms of t.

ii According to this model, when will there be 1500 insects in the colony?

c Compare the long-term behaviour of the two models.

10 Consider the function with rule $f(x) = \dfrac{1}{ax^2 + bx + c}$, where $a \neq 0$.

a Find $f'(x)$.

b State the coordinates of the turning point and state the nature of this turning point if:

i $a > 0$ **ii** $a < 0$

c If $b^2 - 4ac < 0$, sketch the graph of $y = f(x)$ for:

i $a > 0$ **ii** $a < 0$

State the equations of all asymptotes.

d If $b^2 - 4ac = 0$, sketch the graph of $y = f(x)$ for:

i $a > 0$ **ii** $a < 0$

e If $b^2 - 4ac > 0$ and $a > 0$, sketch the graph of $y = f(x)$, stating the equations of all asymptotes.

11 Consider the family of curves with equations of the form $y = ax^2 + \dfrac{b}{x^2}$, where $a, b \in \mathbb{R}^+$.

a Find $\dfrac{dy}{dx}$.

b State the coordinates of the turning points of a member of this family in terms of a and b, and state the nature of each.

c Consider the family $y = ax^2 + \dfrac{1}{x^2}$. Show that the coordinates of the turning points are $\left(\dfrac{1}{\sqrt[4]{a}}, 2\sqrt{a}\right)$ and $\left(\dfrac{-1}{\sqrt[4]{a}}, 2\sqrt{a}\right)$.

12 Let $f\colon [0, 4\pi] \to \mathbb{R}$, $f(x) = e^{-x}\sin x$.

a Find $\{\, x : f'(x) = 0 \,\}$.

b Determine the ratio $f(a + 2\pi) : f(a)$.

c Find the coordinates of all stationary points for $x \in [0, 4\pi]$, and state their nature.

d Evaluate $\int_0^{\pi} e^{-x}\sin x \, dx$.

e Use the results of **b** and **d** to determine $\int_{2\pi}^{3\pi} f(x)\, dx$.

13 **a** Evaluate $\int_0^{\frac{\pi}{4}} \tan^4\theta \sec^2\theta \, d\theta$.

b Hence show that $\int_0^{\frac{\pi}{4}} \tan^6\theta \, d\theta = \dfrac{1}{5} - \int_0^{\frac{\pi}{4}} \tan^4\theta \, d\theta$.

c Deduce that $\int_0^{\frac{\pi}{4}} \tan^6\theta \, d\theta = \dfrac{13}{15} - \dfrac{\pi}{4}$.

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14 A disease spreads through a population. Let p denote the proportion of the population who have the disease at time t. The rate of change of p is proportional to the product of p and the proportion $1 - p$ who do not have the disease.

When $t = 0$, $p = \frac{1}{10}$ and when $t = 2$, $p = \frac{1}{5}$.

a **i** Show that $t = \frac{1}{k}\log_e\left(\frac{9p}{1-p}\right)$, where $k = \log_e\left(\frac{3}{2}\right)$.

ii Hence show that $\frac{9p}{1-p} = \left(\frac{3}{2}\right)^t$.

b Find p when $t = 4$.

c Find p in terms of t.

d Find the values of t for which $p > \frac{1}{2}$.

e Sketch the graph of p against t.

15 A car moves along a straight level road. Its speed, v, is related to its displacement, x, by the differential equation

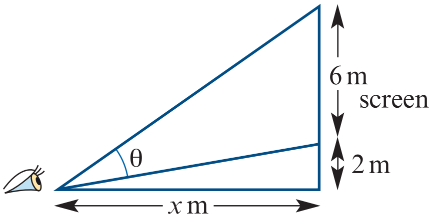
$$v\frac{dv}{dx} = \frac{p}{v} - kv^2$$

where p and k are constants.

a Given that $v = 0$ when $x = 0$, show that $v^3 = \frac{1}{k}(p - pe^{-3x})$.

b Find $\lim_{x\to\infty} v$.

16 A projection screen is 6 metres in height and has its lower edge 2 metres above the eye level of an observer. The angle between the lines of sight of the upper and lower edges of the screen is θ. Let x m be the horizontal distance from the observer to the screen.



a Find θ in terms of x.

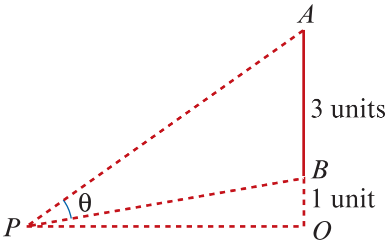
b Find $\frac{d\theta}{dx}$.

c What values can θ take?

d Sketch the graph of θ against x.

e If $1 \le x \le 25$, find the minimum value of θ.

17 A vertical rod AB of length 3 units is held with its lower end, B, at a distance 1 unit vertically above a point O. The angle subtended by AB at a variable point P on the horizontal plane through O is θ.



a Show that $\theta = \tan^{-1}(x) - \tan^{-1}\left(\frac{x}{4}\right)$, where $x = OP$.

b Prove that:

i θ is a maximum when $x = 2$

ii the maximum value of θ is $\tan^{-1}\left(\frac{3}{4}\right)$.

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18 An open rectangular tank is to have a square base. The capacity of the tank is to be 4000 m^3. Let x m be the length of an edge of the square base and A m^2 be the amount of sheet metal used to construct the tank.

a Show that $A = x^2 + \dfrac{16\,000}{x}$.

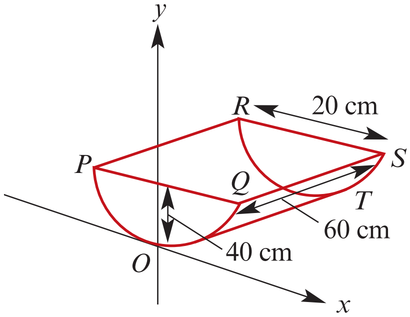
b Sketch the graph of A against x.

c Find, correct to two decimal places, the value(s) of x for which 2500 m^2 of sheet metal is used.

d Find the value of x for which A is a minimum.

19 A closed rectangular box is made of very thin sheet metal and its length is three times its width. If the volume of the box is 288 cm^3, show that its surface area, $A(x)$ cm^2, is given by $A(x) = \dfrac{768}{x} + 6x^2$, where x cm is the width of the box. Find the minimum surface area of the box.

20 This container has an open rectangular horizontal top, $PQSR$, and parallel vertical ends, PQO and RST. The ends are parabolic in shape. The x-axis and y-axis intersect at O, with the x-axis horizontal and the y-axis the line of symmetry of the end PQO. The dimensions are shown on the diagram.



a Find the equation of the parabolic arc QOP.

b If water is poured into the container to a depth of y cm, with a volume of V cm^3, find the relationship between V and y.

c Calculate the depth, to the nearest mm, when the container is half full.

d Water is poured into the empty container so that the depth is y cm at time t seconds. If the water is poured in at the rate of 60 cm^3/s, construct a differential equation expressing $\dfrac{dy}{dt}$ as a function of y and solve it.

e Calculate, to the nearest second:

i how long it will take the water to reach a depth of 20 cm

ii how much longer it will take for the container to be completely full.

21 Moving in the same direction along parallel tracks, objects A and B pass the point O simultaneously with speeds of 20 m/s and 10 m/s respectively.

From then on, the deceleration of A is $\dfrac{v^3}{400}$ m/s^2 and the deceleration of B is $\dfrac{v^2}{100}$ m/s^2, when the speeds are v m/s.

a Find the speeds of A and B at time t seconds after passing O.

b Find the positions of A and B at time t seconds after passing O.

c Use a CAS calculator to plot the graphs of the positions of objects A and B.

d Use a CAS calculator to find, to the nearest second, when the objects pass.

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22 A stone, initially at rest, is released and falls vertically. Its velocity, v m/s, at time t s after release is determined by the differential equation $5\dfrac{dv}{dt} + v = 50$.

a Find an expression for v in terms of t.

b Find v when $t = 47.5$.

c Sketch the graph of v against t.

d **i** Let x be the displacement from the point of release at time t. Find an expression for x in terms of t.

ii Find x when $t = 6$.

23 The rate of change of a population, y, is given by $\dfrac{dy}{dt} = \dfrac{2y(N - y)}{N}$, where N is a positive constant. When $t = 0$, $y = \dfrac{N}{4}$.

a Find y in terms of t and find $\dfrac{dy}{dt}$ in terms of t.

b What limiting value does the population size approach for large values of t?

c Explain why the population is always increasing.

d What is the population when the population is increasing most rapidly?

e For $N = 10^6$:

i Sketch the graph of $\dfrac{dy}{dt}$ against y.

ii At what time is the population increasing most rapidly?

24 An object projected vertically upwards from the surface of the Earth experiences an acceleration of a m/s^2 at a point x m from the centre of the Earth (neglecting air resistance). This acceleration is given by $a = \dfrac{-gR^2}{x^2}$, where g m/s^2 is the acceleration due to gravity and R m is the radius length of the Earth.

a Given that $g = 9.8$, $R = 6.4 \times 10^6$ and the object has an upwards velocity of u m/s at the Earth's surface:

i Express v^2 in terms of x, using $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$.

ii Use the result of part **i** to find the position of the object when it has zero velocity.

iii For what values of u does the result in part **ii** not exist?

b The minimum value of u for which the object does not fall back to Earth is called the escape velocity. Determine the escape velocity in km/h.

25 Define $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

a Find $f(0)$.

b Find $\lim\limits_{x\to\infty} f(x)$.

c Find $\lim\limits_{x\to-\infty} f(x)$.

d Find $f'(x)$.

e Sketch the graph of f.

f Find $f^{-1}(x)$.

g If $g(x) = f^{-1}(x)$, find $g'(x)$.

h Sketch the graph of g' and prove that the area measure of the region bounded by the graph of $y = g'(x)$, the x-axis, the y-axis and the line $x = \frac{1}{2}$ is $\log_e(\sqrt{3})$.

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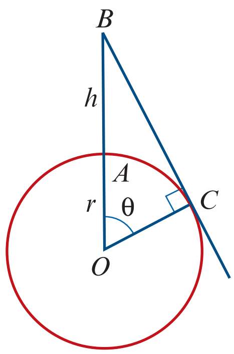
26 The diagram shows a plane circular section through O, the centre of the Earth (which is assumed to be stationary for the purpose of this problem).



From the point A on the surface, a rocket is launched vertically upwards. After t hours, the rocket is at B, which is h km above A. Point C is on the horizon as seen from B, and the length of the chord AC is y km. The angle AOC is θ radians. The radius of the Earth is r km.

a **i** Express y in terms of r and θ.

ii Express $\cos\theta$ in terms of r and h.

b Suppose that after t hours the vertical velocity of the rocket is $\dfrac{dh}{dt} = r\sin t$, $t \in [0, \pi)$. Assume that $r = 6000$.

i Find $\dfrac{dy}{d\theta}$ and $\dfrac{dy}{dt}$.

ii How high is the rocket when $t = \dfrac{\pi}{2}$?

iii Find $\dfrac{dy}{dt}$ when $t = \dfrac{\pi}{2}$.

27 **a** Differentiate $f(x) = e^{-x}x^n$ and hence prove that

$$\int e^{-x}x^n\,dx = n\int e^{-x}x^{n-1}\,dx - e^{-x}x^n$$

b Let $g\colon \mathbb{R}^+ \to \mathbb{R}$, $g(n) = \int_0^\infty e^{-x}x^n\,dx$.

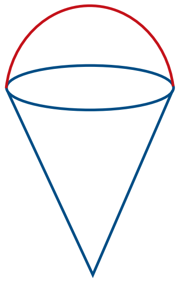
Note: $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$

i Show that $g(0) = 1$.

ii Using the answer to **a**, show that $g(n) = ng(n-1)$.

iii Using your answers to **b i** and **b ii**, show that $g(n) = n!$, for $n = 0, 1, 2, 3, \dots$.

28 A large weather balloon is in the shape of a hemisphere on a cone, as shown in this diagram. When inflated, the height of the cone is twice the radius length of the hemisphere. The shapes and conditions are true as long as the radius of the hemisphere is at least 2 metres.



At time t minutes, the radius length of the hemisphere is r metres and the volume of the balloon is V m^3, for $r \geq 2$.

The balloon has been inflated so that the radius length is 10 m and it is ready to be released, when a leak develops. The gas leaks out at the rate of t^2 m^3 per minute.

a Find the relationship between V and r.

b Construct a differential equation of the form $f(r)\dfrac{dr}{dt} = g(t)$.

c Solve the differential equation with respect to t, given that the initial radius length is 10 m.

d Find how long it will take for the radius length to reduce to 2 metres.

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29 The position vector of a particle at time t seconds is given by $\boldsymbol{r}_1(t) = 2t\,\boldsymbol{i} - (t^2 + 2)\boldsymbol{j}$, where distances are measured in metres.

a What is the average velocity of the particle for the interval [0, 10]?

b By differentiation, find the velocity at time t.

c In what direction is the particle moving when $t = 3$?

d When is the particle moving with minimum speed?

e At what time is the particle moving at the average velocity for the first 10 seconds?

f A second particle has its position at time t seconds given by $\boldsymbol{r}_2(t) = (t^3 - 4)\boldsymbol{i} - 3t\boldsymbol{j}$. Are the two particles coincident at any time t?

30 The acceleration vector, $\ddot{\boldsymbol{r}}(t)$ m/s^2, of a particle at time t seconds is given by $\ddot{\boldsymbol{r}}(t) = -16(\cos(4t)\,\boldsymbol{i} + \sin(4t)\,\boldsymbol{j})$.

a Find the position vector, $\boldsymbol{r}(t)$ m, given that $\dot{\boldsymbol{r}}(0) = 4\boldsymbol{j}$ and $\boldsymbol{r}(0) = \boldsymbol{j}$.

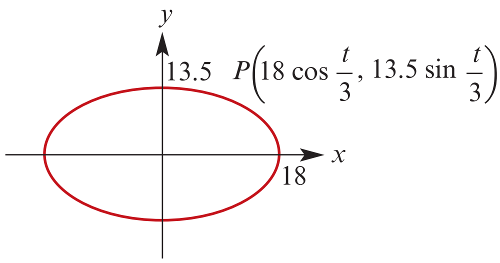
b Show that the path of the particle is a circle and state the position vector of its centre.

c Show that the acceleration is always perpendicular to the velocity.

31 An ice-skater describes an elliptic path. His position at time t seconds is given by

$$\boldsymbol{r} = 18\cos\left(\frac{t}{3}\right)\boldsymbol{i} + 13.5\sin\left(\frac{t}{3}\right)\boldsymbol{j}$$

When $t = 0$, $\boldsymbol{r} = 18\boldsymbol{i}$.



a How long does the skater take to go around the path once?

b **i** Find the velocity of the ice-skater at $t = 2\pi$.

ii Find the acceleration of the ice-skater at $t = 2\pi$.

c **i** Find an expression for the speed of the ice-skater at time t.

ii At what time is his speed greatest?

d Prove that the acceleration satisfies $\ddot{\boldsymbol{r}} = k\boldsymbol{r}$, and hence find when the acceleration has a maximum magnitude.

32 **a** The velocity vector of a particle P at time t is

$$\dot{\boldsymbol{r}}_1(t) = 3\cos(2t)\,\boldsymbol{i} + 4\sin(2t)\,\boldsymbol{j}$$

where $\boldsymbol{r}_1(t)$ is the position relative to O at time t. Find:

i $\boldsymbol{r}_1(t)$, given that $\boldsymbol{r}_1(0) = -2\boldsymbol{j}$

ii the acceleration vector at time t

iii the times when the position and velocity vectors are perpendicular

iv the Cartesian equation of the path.

b At time t, a second particle Q has a position vector (relative to O) given by

$$\boldsymbol{r}_2(t) = \tfrac{3}{2}\sin(2t)\,\boldsymbol{i} + 2\cos(2t)\,\boldsymbol{j} + (a - t)\boldsymbol{k}$$

Find the possible values of a in order for the particles to collide.

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33 An aircraft takes off from the end of a runway in a southerly direction and climbs at an angle of $\tan^{-1}(\frac{1}{2})$ to the horizontal at a speed of $225\sqrt{5}$ km/h.

a Show that, t seconds after take-off, the position vector $\boldsymbol{r}$ of the aircraft with respect to the end of the runway is given by $\boldsymbol{r}_1 = \frac{t}{16}(2\boldsymbol{i} + \boldsymbol{k})$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are vectors of magnitude 1 km in the directions south, east and vertically upwards respectively.

b At time $t = 0$, a second aircraft, flying horizontally south-west at $720\sqrt{2}$ km/h, has position vector $-1.2\boldsymbol{i} + 3.2\boldsymbol{j} + \boldsymbol{k}$.

i Find its position vector $\boldsymbol{r}_2$ at time t in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

ii Show that there will be a collision and state the time at which it will occur.

34 A particle is fired from the top of a cliff h m above sea level with an initial velocity of V m/s inclined at an angle α above the horizontal. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ define the horizontal and vertically upwards vectors in the plane of the particle's path.

a Define:

i the initial position vector of the particle

ii the particle's initial velocity.

b The acceleration vector of the particle under gravity is given by $\boldsymbol{a} = -g\boldsymbol{j}$. Find:

i the velocity vector of the particle t seconds after it is projected

ii the corresponding position vector.

c Use the velocity vector to find the time at which the particle reaches its highest point.

d Show that the time at which the particle hits the sea is given by

$$t = \frac{V\sin\alpha + \sqrt{(V\sin\alpha)^2 + 2gh}}{g}$$

35 A particle travels on a path given by the Cartesian equation $y = x^2 + 2x$.

a Show that one possible vector representing the position of the particle is

$$\boldsymbol{r}(t) = (t-1)\boldsymbol{i} + (t^2-1)\boldsymbol{j}$$

b Show that another possible vector representing the position of the particle is

$$\boldsymbol{r}(t) = (e^{-t}-1)\boldsymbol{i} + (e^{-2t}-1)\boldsymbol{j}$$

c Two particles travel simultaneously. At time $t \geq 0$, the positions of the two particles are given by

$$\boldsymbol{r}_1(t) = (t-1)\boldsymbol{i} + (t^2-1)\boldsymbol{j}$$
$$\boldsymbol{r}_2(t) = (e^{-t}-1)\boldsymbol{i} + (e^{-2t}-1)\boldsymbol{j}$$

i Find the initial positions of the two particles.

ii Show that the particles travel in opposite directions along the path $y = x^2 + 2x$.

iii Find, correct to two decimal places, the point at which the two particles collide.

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36 A ball is projected against a wall that rebounds the ball in its plane of flight. If the ball has velocity $a\boldsymbol{i} + b\boldsymbol{j}$ just before hitting the wall, its velocity of rebound is given by $-0.8a\boldsymbol{i} + b\boldsymbol{j}$. The ball is projected from ground level, and its position vector before hitting the wall is defined by $\boldsymbol{r}(t) = 10t\,\boldsymbol{i} + t(10\sqrt{3} - 4.9t)\boldsymbol{j}$, $t \geq 0$.

a Find:

i the initial position vector of the ball

ii the initial velocity vector of the ball, and hence the magnitude of the velocity and direction (to be stated as an angle of elevation)

iii an expression for the acceleration of the ball.

b The wall is at a horizontal distance x from the point of projection. Find in terms of x:

i the time taken by the ball to reach the wall

ii the position vector of the ball at impact

iii the velocity of the ball immediately before impact with the wall

iv the velocity of the ball immediately after impact.

c Let the second part of the flight of the ball be defined in terms of t_1, a time variable, where $t_1 = 0$ at impact. Assuming that the ball is under the same acceleration vector, find in terms of x and t_1:

i a new velocity vector of the rebound

ii a new position vector of the rebound.

d Find the time taken for the ball to hit the ground after the rebound.

e Find the value of x (correct to two decimal places) for which the ball will return to its initial position.

37 An aeroplane takes off from an airport and, with respect to a given frame of reference, its path with respect to time t is described by the vector $\boldsymbol{r}(t) = (5 - 3t)\boldsymbol{i} + 2t\boldsymbol{j} + t\boldsymbol{k}$, for $t \geq 0$, where $t = 0$ seconds at the time of take-off.

a Find the position vector that represents the position of the plane at take-off.

b Find:

i the position of the plane at times t_1 and t_2

ii the vector which defines the displacement between these two positions in terms of t_1 and t_2 ($t_2 > t_1$).

c Hence show that the plane is travelling along a straight line and state a position vector parallel to the flight.

d A road on the ground is defined by the vector $\boldsymbol{r}_1(s) = s\boldsymbol{i}$, $s \leq 0$.

i Find the magnitude of the acute angle between the path of the plane and the road, correct to two decimal places.

ii Hence, or otherwise, find the shortest distance from the plane to the road 6 seconds after take-off, correct to two decimal places.

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38 Two trains, T_1 and T_2, are moving on perpendicular tracks that cross at the point O. Relative to O, the position vectors of T_1 and T_2 at time t are given by $\boldsymbol{r}_1 = Vt\,\boldsymbol{i}$ and $\boldsymbol{r}_2 = 2V(t - t_0)\boldsymbol{j}$ respectively, where V and t_0 are positive constants.

a **i** Which train goes through O first?

ii How much later does the other train go through O?

b **i** Show that the trains are closest together when $t = \dfrac{4t_0}{5}$.

ii Calculate their distance apart at this time.

iii Draw a diagram to show the positions of the trains at this time. Also show the directions in which they are moving.

39 The vector function $\boldsymbol{r}_1(t) = (2 - t)\boldsymbol{i} + (2t + 1)\boldsymbol{j}$ represents the path of a particle with respect to time t, measured in seconds.

a Find the Cartesian equation that describes the path of the particle. (Assume $t \geq 0$.)

b **i** Rearrange the rule for the vector function in the form $\boldsymbol{r}_1(t) = \boldsymbol{a} + t\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors.

ii Describe the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ geometrically with respect to the path of the particle.

c A second particle which started at the same time as the first particle travels along a path that is represented by $\boldsymbol{r}_2(t) = \boldsymbol{c} + t(2\boldsymbol{i} + \boldsymbol{j})$, $t \geq 0$. The particles collide after 5 seconds.

i Find $\boldsymbol{c}$.

ii Find the distance between the two starting points.

40 The paths of two aeroplanes in an aerial display are simultaneously defined by the vectors

$$\boldsymbol{r}_1(t) = (16 - 3t)\boldsymbol{i} + t\boldsymbol{j} + (3 + 2t)\boldsymbol{k}$$
$$\boldsymbol{r}_2(t) = (3 + 2t)\boldsymbol{i} + (1 + t)\boldsymbol{j} + (11 - t)\boldsymbol{k}$$

where t represents the time in minutes. Find:

a the position of the first plane after 1 minute

b the unit vector in the direction of motion for each of the two planes

c the acute angle between their lines of flight, correct to two decimal places

d the point at which their two paths cross

e the vector which represents the displacement between the two planes after t seconds

f the shortest distance between the two planes during their flight.

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Revision

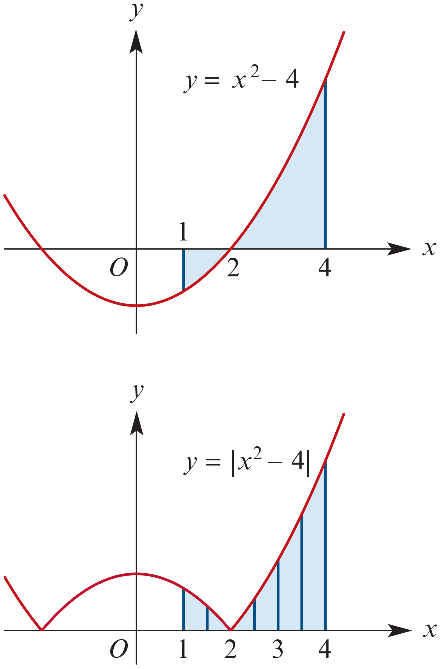
14D Algorithms and pseudocode

You may like to use a device to implement the algorithms in this section; see the coding appendices in the Interactive Textbook for instructions.

1 **Riemann sums** In Section 1H, we used Riemann sums to approximate the area under a curve. In this question, we also approximate volume, arc length and surface area.

a Area The following algorithm finds the trapezoidal estimate for the area between the curve $y = x^2 - 4$ and the x-axis from $x = 1$ to $x = 4$. We use the curve $y = |x^2 - 4|$.

```
define f(x):
  return abs(x^2 − 4)
a ← 1
b ← 4
n ← 6
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    area ← 1/2(f(x) + f(x + h)) × h
    sum ← sum + area
    x ← x + h
end for
print sum
```



i Perform a desk check for this algorithm.

ii Modify the algorithm to estimate the area between the curves $y = \sin(2x)$ and $y = \cos x$ for $0 \le x \le 2\pi$ with 12 strips. (Hint: Use $y = |\cos x - \sin(2x)|$.)

b Volume In Section 10D, we used cylinders to approximate the volume of the solid of revolution formed by rotating a curve $y = f(x)$ about the x-axis for $x \in [a, b]$. Here we will divide the interval $[a, b]$ into n equal subintervals $[x_{i-1}, x_i]$, each of length h, and so we can write

$$V \approx \sum_{i=1}^{n} \pi(f(x_{i-1}))^2 h$$

i Write an algorithm in pseudocode that finds an estimate for the volume of the solid of revolution formed by rotating the curve $y = \sin^{-1}(x)$ about the x-axis for $x \in [0, 0.5]$. Use five cylinders.

ii Perform a desk check for your algorithm.

iii Modify the algorithm to estimate the volume of the solid formed when the region enclosed by the graphs of $y = x^2$ and $y = x$ is rotated about the x-axis.

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c Arc length In Section 10E, we saw that we can approximate the length of a curve by forming line segments along the curve and finding the sum of their lengths.

i Consider the algorithm on the right. What does this algorithm find an estimate for?

ii Perform a desk check for this algorithm.

iii Modify the algorithm to find an estimate for the length of the curve $y = x^3$ from $x = 0$ to $x = 2$. Use 10 line segments.

```
define f(x):
  return x^2

a ← 0
b ← 1
n ← 5
h ← (b − a)/n
x ← a
sum ← 0
for i from 1 to n
    length ← √(h^2 + (f(x + h) − f(x))^2)
    sum ← sum + length
    x ← x + h
end for
print sum
```

d Surface area In Section 10F, we used frustums to approximate the area of a surface of revolution. The algorithm for arc length can be adapted for this task using:

$$area \leftarrow \pi \times (f(x) + f(x+h)) \times \sqrt{h^2 + (f(x+h) - f(x))^2}$$

i Write an algorithm in pseudocode to estimate the area of the surface formed by rotating the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$ about the x-axis. Use five frustums.

ii Write an algorithm in pseudocode to estimate the area of the surface formed by rotating the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$ about the y-axis. Use five frustums.

2 Numerical solution of differential equations

a Using Euler's method We introduced Euler's method in Section 11I. Consider the differential equation

$$\frac{dy}{dx} = x^2 y \quad \text{where } y = 1 \text{ when } x = 0$$

Using pseudocode, write an algorithm that applies Euler's method to estimate the value of y when $x = 3$.

b Using a definite integral In Section 11H, we saw that for a differential equation of the form $\frac{dy}{dx} = f(x)$, the values of y can be found using $y = \int_a^x f(t)\,dt + y(a)$.

Consider the differential equation

$$\frac{dy}{dx} = \sqrt{\sin x} \quad \text{where } y = 0 \text{ when } x = 0$$

Using pseudocode, write an algorithm to estimate the value of y when $x = \frac{\pi}{4}$. Use the trapezoidal estimate for the definite integral.

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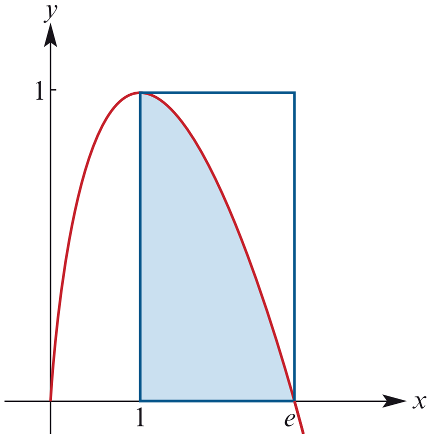
3 Monte Carlo integration

The graph of $y = x - x\log_e x$ is shown below. The given algorithm finds an estimate for the area of the shaded region, which is

$$\int_1^e x - x\log_e x \, dx$$

The enclosing rectangle has area $e - 1$. By randomly choosing points in this rectangle, we estimate what proportion of the rectangle lies below the curve.

```
define f(x):
  return x − x log_e x
count ← 0
for i from 1 to 10^6
    x ← (e − 1) × random( ) + 1
    y ← random( )
    if y < f(x) then
        count ← count + 1
    end if
end for
area ← count/10^6 × (e − 1)
print area
```



Note: Here we are using the pseudocode function *random*( ) to generate a random number in the interval (0, 1). So the instruction $b \times random() + a$ produces a random number in the interval $(a, a + b)$.

a Write an algorithm in pseudocode that uses a Monte Carlo method to estimate the value of $\int_0^\pi \sin^3 x \, dx$.

b Write an algorithm in pseudocode that uses a Monte Carlo method to estimate the value of $\int_0^\pi \sqrt{\sin x} \, dx$.

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15

Linear combinations of random variables and the sample mean

Objectives

- ▶ To investigate the distribution of a **linear function** of a random variable.
- ▶ To determine the mean and variance of a linear function of a random variable.
- ▶ To determine the mean and variance of a **linear combination** of independent random variables.
- ▶ To investigate the behaviour of a linear combination of independent normal random variables.
- ▶ To understand the **sample mean** $\bar{X}$ as a random variable.
- ▶ To use simulation to understand the distribution of the sample mean $\bar{X}$.
- ▶ To introduce the **central limit theorem**.
- ▶ To use the central limit theorem to understand the normal approximation to the binomial distribution.

Some of the most interesting and useful applications of probability are concerned not with a single random variable, but with combinations of random variables.

For example, the time that it takes to build a house (which is a random variable) is the sum of the times taken for each of the component parts of the build, such as digging the foundations, constructing the frame, installing the plumbing, and so on. Each component is a random variable in its own right, and so has a distribution which can be examined and understood.

In this chapter, we consider linear combinations of independent random variables, and their application to the sample mean.

Note: The statistics material in Specialist Mathematics Units 3 & 4 requires a knowledge of probability and statistics from Mathematical Methods Units 3 & 4.

15A Linear functions of a random variable

In this section, we consider a random variable Y which is a linear function of another random variable X. That is,

$$Y = aX + b$$

where a and b are constants. We can consider b as a location parameter and a as a scale parameter.

Discrete random variables

If X is a discrete random variable, then $Y = aX + b$ is also a discrete random variable. We can determine probabilities associated with Y by using the original probability distribution of X, as illustrated in the following example.

Example 1

The probability distribution of X, the number of cars that Matt sells in a week, is given in the following table.

Number of cars sold, x	0	1	2	3	4
$\Pr(X = x)$	0.45	0.25	0.20	0.08	0.02

Suppose that Matt is paid \$750 each week, plus \$1000 commission on each car sold.

a Express S, Matt's weekly salary, as a linear function of X.

b What is the probability distribution of S?

c What is the probability that Matt earns more than \$2000 in any given week?

Solution

a $S = 1000X + 750$

b We can use the rule from part **a** to determine the possible values of S.

Weekly salary, s	750	1750	2750	3750	4750
$\Pr(S = s)$	0.45	0.25	0.20	0.08	0.02

c From the table, we have

$$\Pr(S > 2000) = 0.20 + 0.08 + 0.02 = 0.30$$

Continuous random variables

A continuous random variable X has a probability density function f such that:

1 $f(x) \geq 0$ for all x

2 $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Moreover, we have

$$\Pr(X \leq c) = \int_{-\infty}^{c} f(x)\,dx$$

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If X is a continuous random variable and $a \neq 0$, then $Y = aX + b$ is also a continuous random variable. If $a > 0$, then

$$\Pr(Y \leq y) = \Pr(aX + b \leq y) = \Pr\left(X \leq \frac{y-b}{a}\right)$$

giving

$$\Pr(Y \leq y) = \int_{-\infty}^{\frac{y-b}{a}} f(x)\,dx$$

Example 2

Assume that the random variable X has density function f given by

$$f(x) = \begin{cases} 1.5(1-x^2) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

a Find $\Pr(X \leq 0.5)$.

b Let $Y = 2X + 3$. Find $\Pr(Y \leq 3.5)$.

Solution

a
$$\begin{aligned} \Pr(X \leq 0.5) &= \int_0^{0.5} f(x)\,dx \\ &= \int_0^{0.5} 1.5(1-x^2)\,dx \\ &= 1.5\left[x - \frac{x^3}{3}\right]_0^{0.5} \\ &= 1.5\left(0.5 - \frac{0.5^3}{3}\right) \\ &= 0.6875 \end{aligned}$$

b
$$\begin{aligned} \Pr(Y \leq 3.5) &= \int_0^{\frac{3.5-3}{2}} f(x)\,dx \\ &= \int_0^{0.25} 1.5(1-x^2)\,dx \\ &= 1.5\left[x - \frac{x^3}{3}\right]_0^{0.25} \\ &= 1.5\left(0.25 - \frac{0.25^3}{3}\right) \\ &= 0.3672 \end{aligned}$$

The mean of a linear function of a random variable

Now we consider the mean of Y, where $Y = aX + b$.

Discrete random variables

For a discrete random variable X, by definition we have

$$\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x)$$

Thus
$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(aX + b) \\ &= \sum_x (ax + b) \cdot \Pr(X = x) \\ &= \sum_x ax \cdot \Pr(X = x) + \sum_x b \cdot \Pr(X = x) \\ &= a\sum_x x \cdot \Pr(X = x) + b\sum_x \Pr(X = x) \\ &= a\,\mathrm{E}(X) + b \qquad \text{since } \sum_x \Pr(X = x) = 1 \end{aligned}$$

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Continuous random variables

Similarly, for a continuous random variable X, we have

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Thus

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(aX + b) \\ &= \int_{-\infty}^{\infty} (ax + b) \cdot f(x)\, dx \\ &= \int_{-\infty}^{\infty} ax \cdot f(x)\, dx + \int_{-\infty}^{\infty} b \cdot f(x)\, dx \\ &= a\int_{-\infty}^{\infty} x \cdot f(x)\, dx + b\int_{-\infty}^{\infty} f(x)\, dx \\ &= a\,\mathrm{E}(X) + b \qquad \text{since } \int_{-\infty}^{\infty} f(x)\, dx = 1\end{aligned}$$

Mean of a linear function of a random variable

Let X be a random variable with mean μ. If $Y = aX + b$, where a and b are constants, then

$$\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b = a\mu + b$$

The variance of a linear function of a random variable

What can we say about the variance of Y, where $Y = aX + b$? Whether the random variable X is discrete or continuous, we have

$$\mathrm{Var}(aX + b) = \mathrm{E}[(aX + b)^2] - [\mathrm{E}(aX + b)]^2$$

Now

$$\begin{aligned}[\mathrm{E}(aX + b)]^2 &= [a\,\mathrm{E}(X) + b]^2 \\ &= (a\mu + b)^2 \\ &= a^2\mu^2 + 2ab\mu + b^2\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}[(aX + b)^2] &= \mathrm{E}(a^2X^2 + 2abX + b^2) \\ &= a^2\,\mathrm{E}(X^2) + 2ab\mu + b^2\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Var}(aX + b) &= a^2\,\mathrm{E}(X^2) + 2ab\mu + b^2 - a^2\mu^2 - 2ab\mu - b^2 \\ &= a^2\,\mathrm{E}(X^2) - a^2\mu^2 \\ &= a^2\,\mathrm{Var}(X)\end{aligned}$$

Note: This calculation uses sums of random variables, which we discuss in the next section.

Variance of a linear function of a random variable

Let X be a random variable with variance σ^2. If $Y = aX + b$, where a and b are constants, then

$$\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X) = a^2\sigma^2$$

$$\mathrm{sd}(Y) = \mathrm{sd}(aX + b) = \sqrt{a^2\sigma^2} = |a|\sigma$$

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Although initially the absence of b in the variance may seem surprising, on reflection it makes sense that adding a constant merely changes the location of the distribution, and has no effect on its spread. Similarly, multiplying by a is in effect a scale change, and this is consistent with the result obtained.

Example 3

Suppose that X is a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 2$.

a Find $\mathrm{E}(2X + 1)$.

b Find $\mathrm{Var}(1 - 3X)$.

Solution

a $\mathrm{E}(2X + 1) = 2\,\mathrm{E}(X) + 1$
$= 2 \times 10 + 1 = 21$

b $\mathrm{Var}(1 - 3X) = (-3)^2\,\mathrm{Var}(X)$
$= 9 \times 2 = 18$

Skill-sheet

Exercise 15A

Example 1

1 The number of chocolate bars produced by a manufacturer in any week has the following distribution.

x	1000	1500	2000	2500	3000	4000
$\Pr(X = x)$	0.05	0.15	0.35	0.25	0.15	0.05

It costs the manufacturer \$450 per week, plus an additional 50 cents per chocolate bar, to produce the bars.

a Express C, the manufacturer's weekly cost of production, as a linear function of X.

b What is the probability distribution of C?

c What is the probability that the cost is more than \$2000 in any given week?

2 Sam plays a game with his sister Annabelle. He tosses a coin three times, and counts the number of times that the coin comes up heads. Annabelle charges him \$5 to play, and gives him \$2.50 for each head that he tosses.

a Express W, the net amount he wins, in terms of X, the number of heads observed in the three tosses.

b What is the probability distribution of W?

c What is the probability that the net amount he wins in a game is more than \$2?

Example 2

3 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X < 0.3)$.

b Let $Y = X + 1$. Find $\Pr(Y \le 1.5)$.

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4 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{4}\right) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X < 0.5)$.

b Let $Y = 3X - 1$. Find $\Pr(Y > 2)$.

5 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x+2}{16} & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X < 2.5)$.

b Let $Y = 4X + 2$. Find $\Pr(Y > 2)$.

Example 3

6 Suppose that X is a random variable with mean $\mu = 25$ and variance $\sigma^2 = 9$.

a Let $Y = 3X + 2$. Find $E(Y)$ and $Var(Y)$.

b Let $U = 5 - 2X$. Find $E(U)$ and $sd(U)$.

c Let $V = 4 - 0.5X$. Find $E(V)$ and $Var(V)$.

7 Suppose that X is a random variable with mean $\mu = 25$ and variance $\sigma^2 = 16$. Let $Y = mX + n$, where $m \in \mathbb{R}^+$ and $n \in \mathbb{R}$.

a Given that $E(Y) = 45$ and $Var(Y) = 64$, find the values of m and n.

b Hence find the value of Y when the value of X is 20.

8 A random variable X has density function f given by

$$f(x) = \begin{cases} 0.2 & \text{if } -1 \le x \le 0 \\ 0.2 + 1.2x & \text{if } 0 < x \le 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$$

a Find $E(X)$.

b Find $Var(X)$.

c Hence find $E(4X + 2)$ and $sd(4X + 2)$.

9 A machine dispenses soft drink into cylindrical cans of diameter 6 cm. The machine has an automatic switch that stops the liquid flowing when it reaches a depth of X cm in the can. The random variable X has a mean of 15 cm and a standard deviation of 2 mm. Let V mL be the volume of soft drink in a can. (Note that 1 mL = 1 cm^3.)

a Find the expected value of V. Give your answer correct to one decimal place.

b Find the variance of V. Give your answer correct to one decimal place.

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10 An investment account pays simple interest, with the interest rate being fixed over the period of the investment. The interest rate, $X\%$ p.a., is a random variable with a mean of 2% and a standard deviation of 0.2%. Suppose that an initial amount of \$100 000 is invested for five years.

a Find the expected value of the investment at the end of five years.

b Find the standard deviation of the value of the investment at the end of five years.

11 The probability distribution of X, the number of houses that Madeline sells in a week, is given in the following table.

x	0	1	2	3	4
$\Pr(X = x)$	0.40	0.33	0.22	0.04	0.01

Suppose that Madeline is paid \$1000 each week, plus \$5000 commission on each house sold.

a Find Madeline's expected total weekly income, correct to the nearest dollar.

b Find the standard deviation of Madeline's total weekly income, correct to the nearest dollar.

c How much extra commission on a sale should Madeline request so that her expected weekly income increases by \$500?

15B Linear combinations of random variables

From Mathematical Methods, you are familiar with the idea of independent events, that is, events A and B such that

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

The term independent can also be applied to random variables. While a formal definition of independent random variables is beyond the scope of this course, we say that two random variables are **independent** if their joint probability function is a product of their individual probability functions.

The sum of two independent identically distributed random variables

We start by investigating sums of random variables that are not only independent, but also identically distributed. This means that they will have the same values for their means and standard deviations.

Consider, for example, the numbers observed when two six-sided dice are rolled. Let X_1 be the number observed when the first die is rolled, and X_2 be the number observed when the second die is rolled. The two random variables X_1 and X_2 are independent and have identical distributions.

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What can we say about the distribution of $X_1 + X_2$?

Since the rolling of these two dice can be considered as independent events, we can find probabilities associated with the sum by multiplying probabilities associated with each individual random variable. For example:

$$\begin{aligned}\Pr(X_1 + X_2 = 2) &= \Pr(X_1 = 1,\ X_2 = 1) \\ &= \Pr(X_1 = 1) \times \Pr(X_2 = 1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}\end{aligned}$$

Example 4

Suppose that X_1 is the number observed when one fair die is rolled, and X_2 is the number observed when another fair die is rolled. Find the probability distribution of $Y = X_1 + X_2$.

Solution

We can construct the following table to determine the possible values of $Y = X_1 + X_2$.

X_1 \ X_2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

For example, consider $Y = 4$. The possible outcomes for this value are:

- $X_1 = 1,\ X_2 = 3$
- $X_1 = 2,\ X_2 = 2$
- $X_1 = 3,\ X_2 = 1$

Therefore

$$\begin{aligned}&\Pr(Y = 4) \\ &= \Pr(X_1 = 1,\ X_2 = 3) + \Pr(X_1 = 2,\ X_2 = 2) + \Pr(X_1 = 3,\ X_2 = 1) \\ &= \Pr(X_1 = 1) \times \Pr(X_2 = 3) + \Pr(X_1 = 2) \times \Pr(X_2 = 2) + \Pr(X_1 = 3) \times \Pr(X_2 = 1) \\ &= \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{3}{36}\end{aligned}$$

Continuing in this way, we can obtain the probability distribution of $Y = X_1 + X_2$.

y	2	3	4	5	6	7	8	9	10	11	12
$\Pr(Y = y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

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The mean and variance of the sum of two independent identically distributed random variables

We can consider the mean and variance of the sum of two independent identically distributed random variables using the previous simple example.

Example 5

Consider again the random variable $Y = X_1 + X_2$ from Example 4. Find:

a $E(Y)$

b $\text{Var}(Y)$

Solution

Using the probability distribution of $Y = X_1 + X_2$ from Example 4:

a
$$E(Y) = \sum_y y \cdot \Pr(Y = y) = \frac{2 + 6 + 12 + \cdots + 12}{36} = \frac{252}{36} = 7$$

b
$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$$
$$E(Y^2) = \sum_y y^2 \cdot \Pr(Y = y) = \frac{4 + 18 + 48 + \cdots + 144}{36} = \frac{1974}{36}$$
$$\therefore \quad \text{Var}(Y) = \frac{1974}{36} - 49 = \frac{35}{6}$$

How do these values compare to the mean and variance of X_1 and X_2?

We can easily determine that $E(X_1) = E(X_2) = 3.5$, and we know that $E(X_1 + X_2) = 7$. Thus we have

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

Similarly, we can calculate $\text{Var}(X_1) = \text{Var}(X_2) = \frac{35}{12}$, and we know that $\text{Var}(X_1 + X_2) = \frac{35}{6}$. Thus we have

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$$

These results hold for any two *independent* identically distributed random variables X_1 and X_2, and we can generalise our findings as follows.

Let X be a random variable with mean μ and variance σ^2. Then if X_1 and X_2 are independent random variables with identical distributions to X, we have

$$E(X_1 + X_2) = E(X_1) + E(X_2) = 2\mu$$
$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 2\sigma^2$$
$$\text{sd}(X_1 + X_2) = \sqrt{\text{Var}(X_1 + X_2)} = \sqrt{2}\sigma$$

Note: Since $\text{sd}(X_1) + \text{sd}(X_2) = 2\sigma$, we see that $\text{sd}(X_1 + X_2) \neq \text{sd}(X_1) + \text{sd}(X_2)$ for $\sigma \neq 0$.

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The sum of *n* independent identically distributed random variables

So far we have looked at the sum of two independent identically distributed random variables. In the next example, we consider a sum of three random variables.

Example 6

Consider a random variable X which has a probability distribution as follows:

x	0	1	2
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Let X_1, X_2 and X_3 be independent random variables with identical distributions to X.

a Find the probability distribution of $X_1 + X_2 + X_3$.

b Hence find the mean, variance and standard deviation of $X_1 + X_2 + X_3$.

Solution

a Using a tree diagram or a similar strategy, we can list all the possible combinations of values of X_1, X_2 and X_3 as follows:

$$(0,0,0),\ (0,0,1),\ (0,0,2),\ (0,1,0),\ (0,1,1),\ (0,1,2),\ (0,2,0),\ (0,2,1),\ (0,2,2)$$
$$(1,0,0),\ (1,0,1),\ (1,0,2),\ (1,1,0),\ (1,1,1),\ (1,1,2),\ (1,2,0),\ (1,2,1),\ (1,2,2)$$
$$(2,0,0),\ (2,0,1),\ (2,0,2),\ (2,1,0),\ (2,1,1),\ (2,1,2),\ (2,2,0),\ (2,2,1),\ (2,2,2)$$

The value of $X_1 + X_2 + X_3$ can be determined for each of the 27 outcomes. Since the three random variables are independent, we can determine the probability of each outcome by multiplying the probabilities of the individual outcomes.

For example:

$$\begin{aligned}\Pr(X_1 + X_2 + X_3 = 0) &= \Pr(X_1 = 0,\ X_2 = 0,\ X_3 = 0)\\ &= \Pr(X_1 = 0) \times \Pr(X_2 = 0) \times \Pr(X_3 = 0)\\ &= \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}\\ &= \frac{1}{64}\end{aligned}$$

Continuing in this way, we can obtain the probability distribution of $X_1 + X_2 + X_3$.

y	0	1	2	3	4	5	6
$\Pr(X_1 + X_2 + X_3 = y)$	$\frac{1}{64}$	$\frac{3}{32}$	$\frac{15}{64}$	$\frac{5}{16}$	$\frac{15}{64}$	$\frac{3}{32}$	$\frac{1}{64}$

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b Using the probability distribution from part **a**, we have

$$\mathrm{E}(X_1 + X_2 + X_3) = 0 \times \frac{1}{64} + 1 \times \frac{3}{32} + 2 \times \frac{15}{64} + \cdots + 6 \times \frac{1}{64} = 3$$

$$\mathrm{E}[(X_1 + X_2 + X_3)^2] = 0^2 \times \frac{1}{64} + 1^2 \times \frac{3}{32} + 2^2 \times \frac{15}{64} + \cdots + 6^2 \times \frac{1}{64} = \frac{21}{2}$$

Thus $\mathrm{Var}(X_1 + X_2 + X_3) = \frac{21}{2} - 3^2 = \frac{3}{2}$

and $\mathrm{sd}(X_1 + X_2 + X_3) = \sqrt{\frac{3}{2}} = 1.225$

It is easy to verify in the previous example that

$$\mathrm{E}(X_1 + X_2 + X_3) = 3\,\mathrm{E}(X)$$

$$\mathrm{Var}(X_1 + X_2 + X_3) = 3\,\mathrm{Var}(X)$$

We can extend our findings in this section to the sum of n independent identically distributed random variables.

The sum of n independent identically distributed random variables

Let X be a random variable with mean μ and variance σ^2. Then if $X_1, X_2, \ldots, X_n$ are independent random variables with identical distributions to X, we have

$$\mathrm{E}(X_1 + X_2 + \cdots + X_n) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \cdots + \mathrm{E}(X_n) = n\mu$$

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

$$\mathrm{sd}(X_1 + X_2 + \cdots + X_n) = \sqrt{\mathrm{Var}(X_1 + X_2 + \cdots + X_n)} = \sqrt{n}\sigma$$

Note: The result for the expected value holds even if the random variables $X_1, X_2, \ldots, X_n$ are not independent.

Example 7

Let X be a random variable with mean $\mu = 10$ and variance $\sigma^2 = 9$. If X_1, X_2, X_3, X_4 are independent random variables with identical distributions to X, find:

a $\mathrm{E}(X_1 + X_2 + X_3 + X_4)$ **b** $\mathrm{Var}(X_1 + X_2 + X_3 + X_4)$ **c** $\mathrm{sd}(X_1 + X_2 + X_3 + X_4)$

Solution

a $\mathrm{E}(X_1 + X_2 + X_3 + X_4)$
$= 4\mu = 40$

b $\mathrm{Var}(X_1 + X_2 + X_3 + X_4)$
$= 4\sigma^2 = 36$

c $\mathrm{sd}(X_1 + X_2 + X_3 + X_4)$
$= \sqrt{4}\sigma = 2\sigma = 6$

We can see from Example 6 that determining the probability distribution of a sum of random variables from first principles is very tedious by hand, even for a simple example. However, such probability distributions can be determined efficiently using a computer program.

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Linear combinations of two independent random variables

We now consider a random variable Y which is a linear combination of two independent random variables X_1 and X_2. That is,

$$Y = a_1X_1 + a_2X_2$$

where a_1 and a_2 are constants.

We will consider linear combinations of independent random variables that do not necessarily have the same distribution (and so they most likely have different values for their means and standard deviations). We begin with a simple example to illustrate the distribution of a linear combination of two independent random variables.

Example 8

Let X_1 and X_2 be independent random variables with the probability distributions given in the following tables. Find the probability distribution of $Y = 2X_1 + 3X_2$.

x_1	0	1	2
$\Pr(X_1 = x_1)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

x_2	0	1
$\Pr(X_2 = x_2)$	$\frac{1}{2}$	$\frac{1}{2}$

Solution

We can construct the following table to determine the possible values of $Y = 2X_1 + 3X_2$.

		X_2 = 0	X_2 = 1
X_1	0	$0 + 0 = 0$	$0 + 3 = 3$
	1	$2 + 0 = 2$	$2 + 3 = 5$
	2	$4 + 0 = 4$	$4 + 3 = 7$

Since X_1 and X_2 are independent, we can determine the probability of each outcome by multiplying the probabilities of the individual outcomes. For example:

$$\begin{aligned}\Pr(Y = 7) &= \Pr(X_1 = 2,\ X_2 = 1)\\ &= \Pr(X_1 = 2) \times \Pr(X_2 = 1)\\ &= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}\end{aligned}$$

Continuing in this way, we can obtain the probability distribution of $Y = 2X_1 + 3X_2$.

y	0	2	3	4	5	7
$\Pr(Y = y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

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The mean and variance of a linear combination of two independent random variables

We can consider the mean and variance of a linear combination of two independent random variables using the previous simple example.

Example 9

Consider again the random variable $Y = 2X_1 + 3X_2$ from Example 8. Find:

a $\mathrm{E}(Y)$

b $\mathrm{Var}(Y)$

Solution

Using the probability distribution of $Y = 2X_1 + 3X_2$ from Example 8:

a
$$\mathrm{E}(Y) = \sum_y y \cdot \Pr(Y = y) = \frac{0+2+3+4+5+7}{6} = \frac{21}{6} = \frac{7}{2}$$

b $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2$

$$\mathrm{E}(Y^2) = \sum_y y^2 \cdot \Pr(Y = y) = \frac{0+4+9+16+25+49}{6} = \frac{103}{6}$$

$$\therefore \quad \mathrm{Var}(Y) = \frac{103}{6} - \frac{49}{4} = \frac{59}{12}$$

Again we should compare these values to the means and variances of X_1 and X_2.

We can easily determine that $\mathrm{E}(X_1) = 1$ and $\mathrm{E}(X_2) = \frac{1}{2}$, and we know that $\mathrm{E}(2X_1 + 3X_2) = \frac{7}{2}$. Thus we have

$$\mathrm{E}(2X_1 + 3X_2) = 2\,\mathrm{E}(X_1) + 3\,\mathrm{E}(X_2)$$

We can calculate $\mathrm{Var}(X_1) = \frac{2}{3}$ and $\mathrm{Var}(X_2) = \frac{1}{4}$, and we know that $\mathrm{Var}(2X_1 + 3X_2) = \frac{59}{12}$. Thus we have

$$\mathrm{Var}(2X_1 + 3X_2) = 2^2\,\mathrm{Var}(X_1) + 3^2\,\mathrm{Var}(X_2)$$

These results hold for any two *independent* random variables X_1 and X_2, and we can generalise our findings as follows.

Let X_1 and X_2 be independent random variables, where X_1 has mean μ_1 and variance σ_1^2, and X_2 has mean μ_2 and variance σ_2^2. Then if a_1 and a_2 are constants, we have

$$\mathrm{E}(a_1X_1 + a_2X_2) = a_1\,\mathrm{E}(X_1) + a_2\,\mathrm{E}(X_2) = a_1\mu_1 + a_2\mu_2$$

$$\mathrm{Var}(a_1X_1 + a_2X_2) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$$

$$\mathrm{sd}(a_1X_1 + a_2X_2) = \sqrt{\mathrm{Var}(a_1X_1 + a_2X_2)} = \sqrt{a_1^2\sigma_1^2 + a_2^2\sigma_2^2}$$

Note: For the sum of two independent random variables X_1 and X_2, take $a_1 = 1$ and $a_2 = 1$.

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Example 10

A manufacturing process involves two stages:

- The time taken to complete the first stage, X_1 hours, is a continuous random variable with mean $\mu_1 = 4$ and standard deviation $\sigma_1 = 1.5$.
- The time taken to complete the second stage, X_2 hours, is a continuous random variable with mean $\mu_2 = 7$ and standard deviation $\sigma_2 = 1$.

Assume that the second stage is able to commence immediately after the first stage ends.

a Find the mean and standard deviation of the total processing time, if the times taken at each stage are independent.

b If the cost of processing is \$200 per hour for the first stage and \$300 per hour for the second stage, find the mean and standard deviation of the total processing cost.

Solution

a The total processing time is $X_1 + X_2$ hours.

The mean of the total processing time is

$$\begin{aligned} E(X_1 + X_2) &= E(X_1) + E(X_2) \\ &= 4 + 7 = 11 \text{ hours} \end{aligned}$$

The variance of the total processing time is

$$\begin{aligned} \text{Var}(X_1 + X_2) &= \text{Var}(X_1) + \text{Var}(X_2) \\ &= 1.5^2 + 1^2 = 3.25 \end{aligned}$$

Hence the standard deviation of the total processing time is

$$\text{sd}(X_1 + X_2) = \sqrt{3.25} = 1.803 \text{ hours}$$

b Let \$$C$ be the total processing cost. Then $C = 200X_1 + 300X_2$.

The mean of the total processing cost is

$$\begin{aligned} E(C) &= E(200X_1 + 300X_2) \\ &= 200\,E(X_1) + 300\,E(X_2) \\ &= 200 \times 4 + 300 \times 7 \\ &= \$2900 \end{aligned}$$

The variance of the total processing cost is

$$\begin{aligned} \text{Var}(C) &= \text{Var}(200X_1 + 300X_2) \\ &= 200^2\,\text{Var}(X_1) + 300^2\,\text{Var}(X_2) \\ &= 200^2 \times 1.5^2 + 300^2 \times 1^2 \\ &= 180\,000 \end{aligned}$$

Hence the standard deviation of the total processing cost is

$$\text{sd}(C) = \sqrt{180\,000} = \$424.26$$

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Linear combinations of n independent random variables

The results of this section can be generalised to linear combinations of more than two independent random variables, as follows.

A linear combination of n independent random variables

Let $X_1, X_2, \dots, X_n$ be independent random variables with means $\mu_1, \mu_2, \dots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$ respectively. Then if $a_1, a_2, \dots, a_n$ are constants, we have

$$\begin{aligned} \mathrm{E}(a_1X_1 + a_2X_2 + \dots + a_nX_n) &= a_1\,\mathrm{E}(X_1) + a_2\,\mathrm{E}(X_2) + \dots + a_n\,\mathrm{E}(X_n) \\ &= a_1\mu_1 + a_2\mu_2 + \dots + a_n\mu_n \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(a_1X_1 + a_2X_2 + \dots + a_nX_n) &= a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2) + \dots + a_n^2\,\mathrm{Var}(X_n) \\ &= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \dots + a_n^2\sigma_n^2 \end{aligned}$$

$$\mathrm{sd}(a_1X_1 + a_2X_2 + \dots + a_nX_n) = \sqrt{a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \dots + a_n^2\sigma_n^2}$$

Note: The result for the expected value holds even if the random variables $X_1, X_2, \dots, X_n$ are not independent.

Skill-sheet

Exercise 15B

Example 4

1 Pippi is going on a school family picnic. Family groups will sit together on tables in the park. The number of children in a family, X, follows the distribution shown.

x	1	2	3	4
$\Pr(X = x)$	0.5	0.3	0.15	0.05

The number of children in a family is independent of the number of children in any other family. Find the probability that, if two families sit at the one table, there will be more than three children in the combined group.

Example 5

2 Suppose that X_1 is the number observed when a five-sided die is rolled, and X_2 is the number observed when another five-sided die is rolled. Find from first principles:

a $\mathrm{E}(X_1)$ **b** $\mathrm{Var}(X_1)$
c $\mathrm{E}(X_1 + X_2)$ **d** $\mathrm{Var}(X_1 + X_2)$

Example 7

3 The random variables X_1 and X_2 are independent and identically distributed, with mean $\mu = 10$ and variance $\sigma^2 = 9$. Find:

a $\mathrm{E}(X_1 + X_2)$ **b** $\mathrm{Var}(X_1 + X_2)$ **c** $\mathrm{sd}(X_1 + X_2)$

4 The random variables X_1, X_2, X_3, X_4 and X_5 are independent and identically distributed, with mean $\mu = 7$ and standard deviation $\sigma = 2$. Find:

a $\mathrm{E}(X_1 + X_2 + X_3 + X_4 + X_5)$ **b** $\mathrm{Var}(X_1 + X_2 + X_3 + X_4 + X_5)$
c $\mathrm{sd}(X_1 + X_2 + X_3 + X_4 + X_5)$

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5 Let X_1, X_2 and X_3 be independent identically distributed random variables, each with the probability density function given by

$$f(x) = \begin{cases} 2x^3 - x + 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find:

a $E(X_1 + X_2 + X_3)$ **b** $\text{Var}(X_1 + X_2 + X_3)$ **c** $\text{sd}(X_1 + X_2 + X_3)$

Example 8

6 The independent random variables X and Y have probability distributions as shown.

x	1	2	3
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

y	2	4
$\Pr(Y = y)$	$\frac{1}{3}$	$\frac{2}{3}$

Let $S = X + Y$.

a Complete a table to show the probability distribution of S.

b Find $\Pr(S \leq 5)$.

7 Suppose that X_1 is the number observed when a six-sided die is rolled, and X_2 is the number observed when another six-sided die is rolled.

a Find $\Pr(X_1 - X_2 = 0)$. **b** Find $\Pr(X_1 + 3X_2 = 6)$.

Example 9

8 Use the probability distribution of S from Question 6 to find:

a $E(S)$ **b** $\text{sd}(S)$

Example 10

9 To get to school, Jasmine first rides her bike to her friend's house and then travels by car the rest of the way.

- The time taken for the bike ride, X_1 minutes, is a continuous random variable with mean $\mu_1 = 17$ and standard deviation $\sigma_1 = 4.9$.
- The time taken for the car journey, X_2 minutes, is a continuous random variable with mean $\mu_2 = 32$ and standard deviation $\sigma_2 = 7$.

If they leave by car immediately after Jasmine arrives at her friend's house, find the mean and standard deviation of the total time taken for her to get to school (assuming that the times taken for each part of the journey are independent).

10 A coffee machine automatically dispenses coffee into a cup, followed by hot milk. The volume of coffee dispensed has a mean of 50 mL and a standard deviation of 5 mL. The volume of hot milk dispensed has a mean of 145 mL and a standard deviation of 10 mL.

a What are the mean and standard deviation of the total amount of liquid dispensed by the machine?

b If the cost of the coffee is \$10 per litre and the cost of the milk is \$4 per litre, what are the mean and standard deviation of the total cost of a cup of coffee?

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11 The random variables X and Y are independent. The mean and variance of X are 2 and 3 respectively, while the mean and variance of Y are 3 and 4 respectively. Find the values of $a, b \in \mathbb{N}$ if the mean and variance of $aX + bY$ are 19 and 111 respectively.

12 At a greengrocer, the bags of apples have a mean weight of 1000 g, with a variance of 50 g^2, and the bags of bananas have a mean weight of 750 g, with a variance of 25 g^2.

a If the cost of apples is \$4.50 per kg and the cost of bananas is \$3 per kg, find the mean and standard deviation of the total cost of buying one bag of each.

b Suppose that Danni buys two bags of apples and three bags of bananas. What are the mean and standard deviation of the total weight of her purchases?

15C Linear combinations of normal random variables

In the previous section, we looked at the mean and variance of a linear combination of independent random variables. However, we were not able to say much about the form of the distribution or to calculate probabilities, except in very simple examples. In this section, we investigate the special case when the random variables are normally distributed.

It can be proved, but is beyond the scope of this course, that a linear combination of independent normal random variables is also normally distributed.

A linear combination of n independent normal random variables

Let $X_1, X_2, \dots, X_n$ be independent normal random variables and let $a_1, a_2, \dots, a_n$ be constants. Then the random variable $a_1X_1 + a_2X_2 + \cdots + a_nX_n$ is also normally distributed.

Example 11

The time taken to prepare a house for painting is known to be normally distributed with a mean of 10 hours and a standard deviation of 4 hours. The time taken to paint the house is independent of the preparation time, and is normally distributed with a mean of 20 hours and a standard deviation of 3 hours. What is the probability that the total time taken to prepare and paint the house is more than 35 hours?

Solution

Let X represent the time taken to prepare the house, and Y the time taken to paint the house. Since X and Y are independent normal random variables, the distribution of $X + Y$ is also normal, with

$$\begin{aligned} \mathrm{E}(X+Y) &= \mathrm{E}(X) + \mathrm{E}(Y) = 10 + 20 = 30 \\ \mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) = 4^2 + 3^2 = 25 \\ \mathrm{sd}(X+Y) &= \sqrt{25} = 5 \end{aligned}$$

Therefore

$$\Pr(X + Y > 35) = \Pr\left(Z > \frac{35 - 30}{5}\right) = \Pr(Z > 1) = 0.1587$$

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Skill-sheet

Exercise 15C

Note that many of the real-world variables used in this exercise can only be approximately normally distributed, as there are practical limitations to the values they can take.

Example 11

1 The time taken to prepare a meal at a certain restaurant is normally distributed, with a mean of 12 minutes and a standard deviation of 3 minutes. The time taken to cook the meal is independent of the preparation time, and is normally distributed with a mean of 14 minutes and a standard deviation of 3 minutes. What is the probability that a diner will have to wait more than 30 minutes for their meal to be served?

2 Batteries of type A have a mean voltage of 5.0 volts, with variance 0.0225. Type B batteries have a mean voltage of 8.0 volts, with variance 0.04. If we form a series connection containing one battery of each type, what is the probability that the combined voltage exceeds 13.4 volts?

3 Scores on the mathematics component of a standardised test are normally distributed with a mean of 63 and a standard deviation of 10. Scores on the English component of the test are normally distributed with a mean of 68 and a standard deviation of 7. Assuming that the two components of the test are independent of each other, find the probability that a student's mathematics score is higher than their English score.

4 The clearance between two components of a device is important, as component A must fit inside component B. The outer diameter of component A is normally distributed with mean $\mu_A = 0.425$ cm and variance $\sigma_A^2 = 0.0001$, and the inner diameter of component B is normally distributed with mean $\mu_B = 0.428$ cm and variance $\sigma_B^2 = 0.0004$. What is the probability that component A will not fit inside component B?

5 Two students are known to have equal ability in playing an electronic game, so that each of their scores are normally distributed with mean 25 000 and standard deviation 3000. The two scores are independent. What is the probability that, in a particular game, the students' scores will differ by more than 7500 points?

6 The weight of bananas is normally distributed with a mean of 180 g and a standard deviation of 20 g. The bananas are packed in bags of six. Find the probability that the weight of a randomly chosen bag of six bananas is less than 1 kg.

7 Suppose that the weights of people are normally distributed with a mean of 82 kg and a standard deviation of 9 kg. What is the maximum number of people who can get into an elevator which has a weight limit of 680 kg, if we want to be at least 99% sure that the elevator does not exceed capacity?

8 An alarm system has 20 batteries that are connected so that, when one battery fails, the next one takes over. (Only one battery is working at any one time.) The batteries operate independently, and each has a mean life of 7 hours and a standard deviation of 0.5 hours. What is the probability that the alarm system is still working after 145 hours?

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9 Certain machine components have lifetimes, in hours, which are independent and normally distributed with mean 300 and variance 100. Find the probability that:

a the total life of three components is more than 950 hours

b the total life of four components is more than 1250 hours.

10 The independent random variables X and Y each have a normal distribution. The means of X and Y are 10 and 12 respectively, and the standard deviations are 3 and 4 respectively. Find $\Pr(X < Y)$.

15D The sample mean of a normal random variable

In Section 15C, we saw that a linear combination of independent normal random variables is also normally distributed. An important application of this result is to the sample mean, $\bar{X}$.

Populations and samples

You met the following concepts in Specialist Mathematics Units 1 & 2:

- A **population** is the set of all eligible members of a group which we intend to study. A population does not have to be a group of people. For example, it could consist of all apples produced in a particular area, or all components produced by a factory.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- The simplest way to obtain a valid sample is to choose a **random sample**, where every member of the population has an equal chance of being included in the sample.
- The **population mean** μ is the mean of all values of a measure in the entire population; the **sample mean** $\bar{x}$ is the mean of these values in a particular sample.
- Since $\bar{x}$ varies according to the contents of the random samples, we consider the sample means $\bar{x}$ as being the values of a random variable, which we denote by $\bar{X}$.

The sample mean

Let X be a normal random variable which represents a particular measure on a population (for example, IQ scores or rope lengths). The mean of X is μ and the standard deviation is σ.

Samples of size n selected from this population can be described by independent random variables $X_1, X_2, \dots, X_n$ with identical distributions to X.

The sample mean is defined as

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

Since $\bar{X}$ is a linear combination of independent normal random variables, the random variable $\bar{X}$ is also normally distributed.

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The expected value of $\bar{X}$ can be found using our general result for linear combinations:

$$\begin{aligned}
E(\bar{X}) &= E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= E\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n\right) \\
&= \frac{1}{n}E(X_1) + \frac{1}{n}E(X_2) + \cdots + \frac{1}{n}E(X_n) && \text{where } a_1 = a_2 = \cdots = a_n = \frac{1}{n} \\
&= n \times \frac{1}{n} \times \mu && \text{since } E(X_i) = E(X) = \mu \\
&= \mu
\end{aligned}$$

Similarly, we can find the variance of $\bar{X}$:

$$\begin{aligned}
\mathrm{Var}(\bar{X}) &= \mathrm{Var}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= \frac{1}{n^2}\mathrm{Var}(X_1) + \frac{1}{n^2}\mathrm{Var}(X_2) + \cdots + \frac{1}{n^2}\mathrm{Var}(X_n) \\
&= n \times \frac{1}{n^2} \times \sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

We can summarise our results as follows.

The sample mean of a normal random variable

Let X be a normally distributed random variable with mean μ and standard deviation σ. Let $X_1, X_2, \ldots, X_n$ represent a sample of size n selected from this population. The sample mean is defined as

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

The sample mean $\bar{X}$ is normally distributed with $E(\bar{X}) = \mu$ and $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

If we know that a random variable has a normal distribution and we know its mean and standard deviation, then we know exactly the distribution of the sample mean.

Example 12

Experience has shown that the heights of a certain population of women can be assumed to be normally distributed with mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. What can be said about the distribution of the sample mean for a sample of size 16?

Solution

Let X be the height of a woman chosen at random from this population.

The distribution of the sample mean $\bar{X}$ is normal with mean $\mu_{\bar{X}} = \mu = 160$ and standard deviation $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{8}{\sqrt{16}} = 2$.

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Example 13

Consider the population described in Example 12. What is the probability that:

a a woman chosen at random has a height greater than 168 cm

b a sample of four women chosen at random has an average height greater than 168 cm?

Solution

a $\Pr(X > 168) = \Pr\left(Z > \frac{168 - 160}{8}\right) = \Pr(Z > 1) = 0.1587$

b The distribution of the sample mean $\bar{X}$ is normal with mean $\mu_{\bar{X}} = \mu = 160$ and standard deviation $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{8}{\sqrt{4}} = 4$.

Thus $\Pr(\bar{X} > 168) = \Pr\left(Z > \frac{168 - 160}{4}\right) = \Pr(Z > 2) = 0.0228$

Skill-sheet

Exercise 15D

Example 12

1 The distribution of final marks in an examination is normal with a mean of 74 and a standard deviation of 8. A random sample of three students is selected and their mean mark calculated. What are the mean and standard deviation of this sample mean?

2 A machine produces nails which have an intended diameter of $\mu = 25.025$ mm, with a standard deviation of $\sigma = 0.003$ mm. A sample of five nails is selected for inspection each hour and their average diameter calculated. What are the mean and standard deviation of this average diameter?

Example 13

3 The distribution of final marks in a statistics course is normal with a mean of 70 and a standard deviation of 6.

a Find the probability that a randomly selected student has a final mark above 80.

b Find the probability that the mean final mark for two randomly selected students is above 80.

c Compare the answers to parts **a** and **b**.

4 Suppose that IQ in a certain population is a normally distributed random variable, X, with mean $\mu = 100$ and standard deviation $\sigma = 15$.

a Find the probability that a randomly selected individual has an IQ greater than 120.

b Find the probability that the mean IQ of three randomly selected individuals is greater than 120.

c Compare the answers to parts **a** and **b**.

5 Gestation time for pregnancies without problems in humans is approximately normally distributed, with a mean of $\mu = 266$ days and a standard deviation of $\sigma = 16$ days. In the maternity ward of a large hospital, a random sample of seven women who had just given birth after pregnancies without problems was selected. What is the probability that the average gestation period for these seven pregnancies exceeded 280 days?

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6 The annual income for those in the 18–25 age group living in a certain town is normally distributed with mean $\mu = \$42\,500$ and standard deviation $\sigma = \$6000$. What is the probability that 10 randomly chosen individuals in this age group have an average income of less than \$38 000?

7 The IQ scores of adults are known to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. Find the probability that a randomly chosen group of 25 adults will have an average IQ of more than 105.

8 The actual weight of sugar in a 1 kg package produced by a food-processing company is normally distributed with mean $\mu = 1.00$ kg and standard deviation $\sigma = 0.03$ kg. What is the probability that the average weight for a randomly chosen sample of 20 packages is less than 0.98 kg?

9 The adult length of a certain species of fish is known to be normally distributed with mean $\mu = 10$ cm and standard deviation $\sigma = 0.5$ cm. A random sample of 50 fish is chosen and the average length determined. Find the probability that this average is more than 10.1 cm.

10 The time for a customer to be served at a fast-food outlet is normally distributed with a mean of 3.5 minutes and a standard deviation of 1.0 minutes. What is the probability that 20 customers can be served in less than one hour?

15E Investigating the distribution of the sample mean using simulation

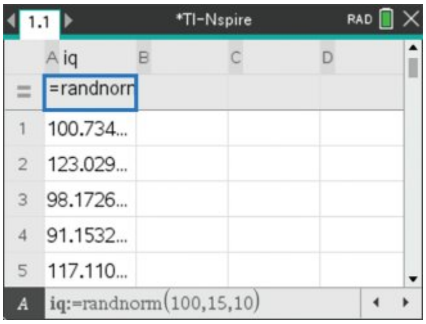
In the previous section, we made assertions about the distribution of the sample mean $\bar{X}$, when X is a normally distributed random variable. In this section, we use simulation to validate these assertions empirically.

Consider the random variable IQ, which we assume is normally distributed with a mean of $\mu = 100$ and a standard deviation of $\sigma = 15$ in a given population. We will begin by simulating the drawing of a random sample of size 10 from this population.

Using the TI-Nspire

To generate a random sample of size 10 from a normal population with $\mu = 100$ and $\sigma = 15$:

- Start from a **Lists & Spreadsheet** page.
- Name the list 'iq' in Column A.
- In the formula cell of Column A, enter the formula using (menu) > **Data** > **Random** > **Normal** and complete as: = randnorm(100, 15, 10)



Note: The syntax is: randnorm(*mean*, *standard deviation*, *sample size*)

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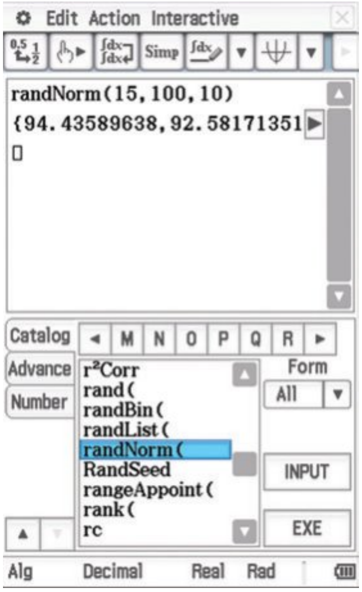
Using the Casio ClassPad

To generate a random sample of size 10 from a normal population with $\mu = 100$ and $\sigma = 15$:

- In $\sqrt[\text{Main}]{\alpha}$, press the Keyboard button.
- Find and then select Catalog by first tapping ▼ at the bottom of the left sidebar.
- Scroll across the alphabet to the letter R.
- Select **randNorm(** and type: 15, 100, 10)
- Tap ► to view all the values.



Notes:

- The syntax is: randNorm(*standard deviation*, *mean*, *sample size*)
- Alternatively, the random sample can be generated in the **Statistics** application.

One random sample of 10 scores, obtained by simulation, is

105, 109, 104, 86, 118, 100, 81, 94, 70, 88

Recall that the sample mean is denoted by $\bar{x}$ and that

$$\bar{x} = \frac{\sum x}{n}$$

where $\sum$ means 'sum' and n is the size of the sample.

Here the sample mean is

$$\bar{x} = \frac{105 + 109 + 104 + 86 + 118 + 100 + 81 + 94 + 70 + 88}{10} = 95.5$$

A second sample, also obtained by simulation, is

114, 124, 128, 133, 95, 107, 117, 91, 115, 104

with sample mean

$$\bar{x} = \frac{114 + 124 + 128 + 133 + 95 + 107 + 117 + 91 + 115 + 104}{10} = 112.8$$

Since $\bar{x}$ varies according to the contents of the random samples, we consider the sample means $\bar{x}$ as being the values of a random variable, which we denote by $\bar{X}$.

Since $\bar{x}$ is a statistic which is calculated from a sample, the probability distribution of the random variable $\bar{X}$ is called a **sampling distribution**.

The sampling distribution of the sample mean

Generating random samples and then calculating the mean from the sample is quite a tedious process if we wish to investigate the sampling distribution of $\bar{X}$ empirically. Luckily, we can also use technology to simulate values of the sample mean.

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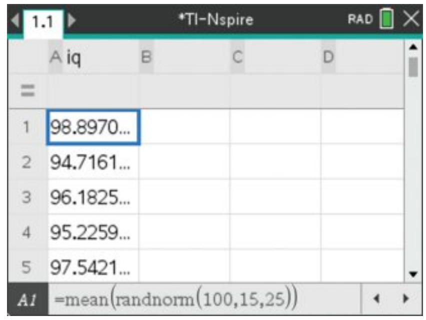
Using the TI-Nspire

To generate the sample means for 10 random samples of size 25 from a normal population with $\mu = 100$ and $\sigma = 15$:



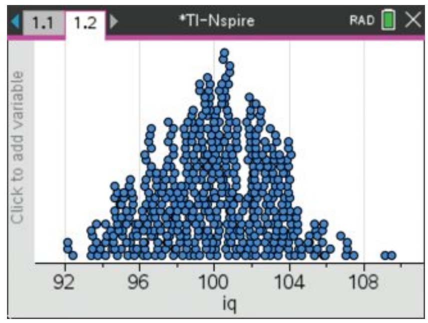
- Start from a **Lists & Spreadsheet** page.
- Name the list ‘iq’ in Column A.
- In cell A1, enter the formula using (menu) > **Data** > **Random** > **Normal** and complete as:
  = mean(randnorm(100, 15, 25))
- Fill down to obtain the sample means for 10 random samples.

For a large number of simulations, an alternative method is easier.

To generate the sample means for 500 random samples of size 25, enter the following formula in the formula cell of Column A:

$$= \text{seq}(\text{mean}(\text{randnorm}(100, 15, 25)), k, 1, 500)$$

The dotplot on the right was created this way.



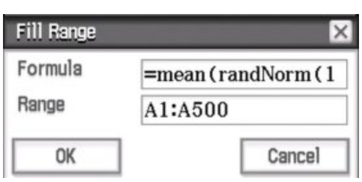
Using the Casio ClassPad

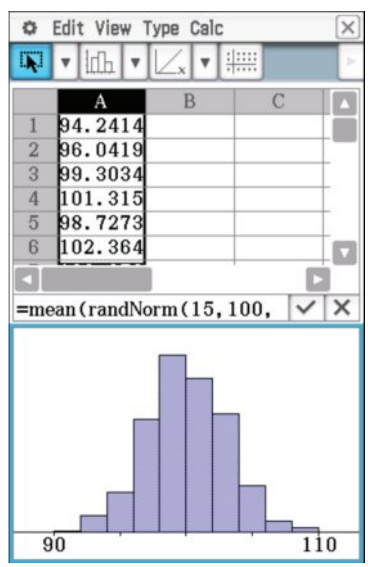
To generate the sample means for 500 random samples of size 25 from a normal population with $\mu = 100$ and $\sigma = 15$:

- Open the **Spreadsheet** application .
- Tap in cell A1.
- Type: = mean(randNorm(15, 100, 25))

Note: The commands **mean(** and **randNorm(** can be selected from (Catalog).

- Tap the tick icon and then tap again in cell A1.
- Go to **Edit** > **Fill** > **Fill Range**.
- Enter A1:A500 for the range, using the symbols A and : from the toolbar. Tap OK.





To sketch a histogram of these sample means:

- Select Column A by tapping on the column header ‘A’ above cell A1.
- Select **Graph** and tap **Histogram**.

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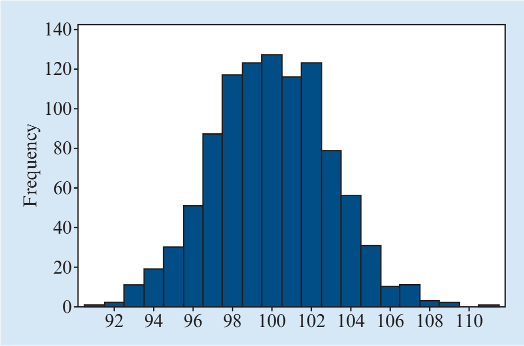
This histogram shows the distribution of the sample mean when 1000 samples (each of size 25) were selected from a population with mean 100 and standard deviation 15.

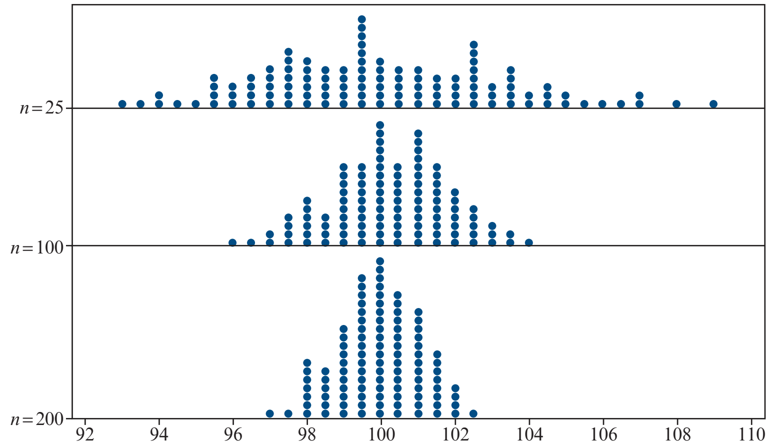


We see from this plot that the distribution of sample means is symmetric and bell-shaped, confirming that the sampling distribution of the sample mean may also be described by the normal distribution.

The mean and standard deviation of the sample mean

We can also use simulation to explore the mean and standard deviation of the sample mean. We know from Section 15D that $E(\bar{X}) = \mu$ and $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}}$, where n is the sample size.

The following dotplots show the sample means $\bar{x}$ obtained when 200 samples of size 25, then size 100 and then size 200 were chosen from a population.



Each symbol represents up to 2 observations.

We can see from the dotplots that all three sampling distributions appear to be centred at 100, the value of the population mean μ. Furthermore, as the sample size increases, the values of the sample mean $\bar{x}$ are more tightly clustered around that value.

These observations are confirmed in the following table, which gives the mean and standard deviation for each of the three simulated sampling distributions shown in the dotplots. The theoretical values of the mean and standard deviation of $\bar{X}$ are included for comparison.

Sample size	25	100	200
Theoretical mean of $\bar{X}$	100	100	100
Mean of the values of $\bar{x}$	99.24	100.24	100.03
Theoretical standard deviation of $\bar{X}$	3	1.5	1.06
Standard deviation of the values of $\bar{x}$	3.05	1.59	1.06

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Example 14

The sizes of kindergarten classes in a certain city are normally distributed, with a mean size of $\mu = 24$ children and a standard deviation of $\sigma = 2$.

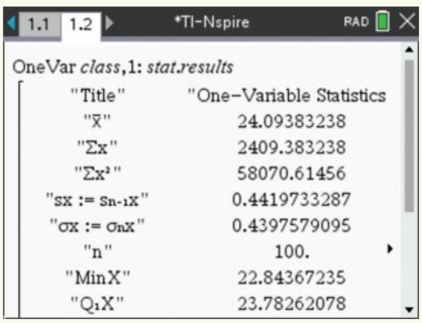
a Use your calculator to generate the sample means for 100 samples, each of size 20. Find the mean and standard deviation of these values of the sample mean.

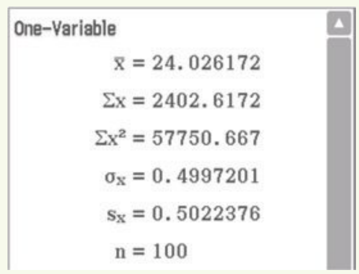
b Use your calculator to generate the sample means for 100 samples, each of size 50. Find the mean and standard deviation of these values of the sample mean.

c Compare the values of the mean and standard deviation calculated in **a** and **b**.

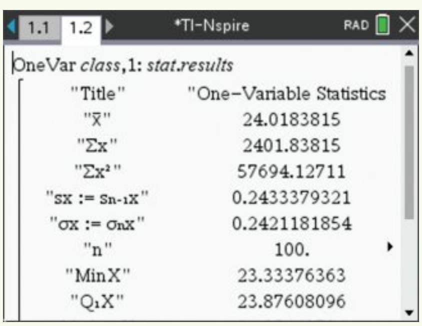
Solution

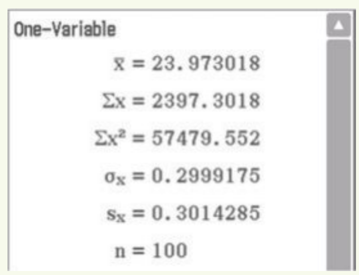
a





b





c The means determined from the simulations are very similar, and close to the population mean of 24, as expected. The standard deviation for the samples of size 50 is much smaller than the standard deviation for the samples of size 20.

Exercise 15E

Example 14

1 The lengths of a species of fish are normally distributed with mean length $\mu = 40$ cm and standard deviation $\sigma = 4$ cm.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 50 drawn from this population of fish.

b Summarise the values obtained in part **a** in a dotplot.

c Find the mean and standard deviation of these values of the sample mean.

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2 The marks in a statistics examination in a certain university are normally distributed with a mean of $\mu = 48$ marks and a standard deviation of $\sigma = 15$ marks.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 20 drawn from the students at this university.

b Summarise the values obtained in part **a** in a dotplot.

c Find the mean and standard deviation of these values of the sample mean.

3 At the Fizzy Drinks Company, the volume of soft drink in a 1 litre bottle is normally distributed with mean $\mu = 1$ litre and standard deviation $\sigma = 0.01$ litres.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of 25 bottles from this company. Determine the mean and standard deviation of these values of the sample mean.

b Determine the theoretical mean and standard deviation of the sample mean, and compare them with your answers from part **a**.

15F The distribution of the sample mean

We know that the sample mean $\bar{X}$ is normally distributed if the random variable X is normally distributed. What can we say about the distribution of $\bar{X}$ if X is not normally distributed?

First consider a discrete random variable X with a uniform distribution given by

$$\Pr(X = x) = \frac{1}{10} \quad \text{for } x = 1, 2, \dots, 10$$

We can use a calculator to generate sample means for 500 random samples of size 50 and to summarise the results in an appropriate plot.

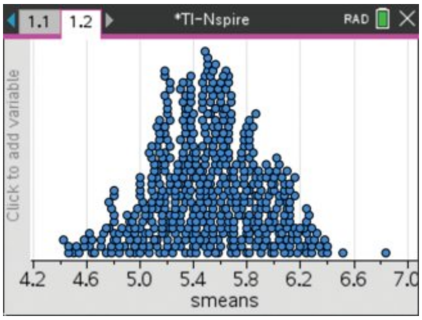
Using the TI-Nspire

To generate the sample means for 500 random samples of size 50 from the uniform distribution with values $1, 2, \dots, 10$:

- Start from a **Lists & Spreadsheet** page, and give Column A the name 'smeans'.
- In the formula cell of Column A, enter:
  = approx(seq(mean(randInt(1, 10, 50)), k, 1, 500))

Note: The approximate command ensures that the sample means are in decimal form.

- Open a **Data & Statistics** page, and generate a dotplot of the sample means.



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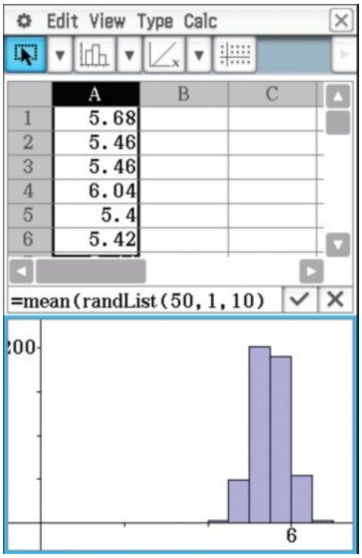
Using the Casio ClassPad

To generate the sample means for 500 random samples of size 50 from the uniform distribution with values $1, 2, \ldots, 10$:

- Open the **Spreadsheet** application.
- Tap in cell A1.
- Type: = mean(randList(50, 1, 10))
- Tap the tick icon and then tap again in cell A1.
- Go to **Edit** > **Fill** > **Fill Range**.
- Enter A1:A500 for the range. Tap OK.
- Tap on the column header A.
- Select **Graph** and tap **Histogram**.



We can see in this example that, even though the distribution of X is clearly not normal, the sampling distribution of $\bar{X}$ is quite well approximated by a normal distribution.

Let's look at another example. This time we will consider a binomial random variable X with parameters $n = 20$ and $p = 0.3$. Here we have

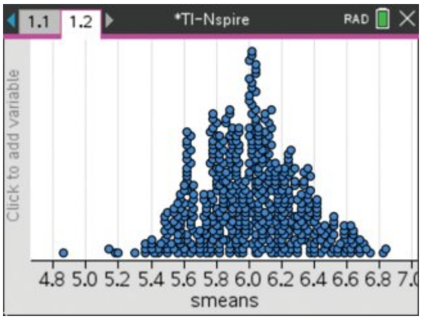
$$\Pr(X = x) = \binom{20}{x} 0.3^x \, 0.7^{20-x} \quad \text{for } x = 0, 1, \ldots, 20$$

Again, we can use a calculator to generate sample means for 500 random samples of size 50 and to summarise the results in an appropriate plot.

Using the TI-Nspire

To generate the sample means for 500 random samples of size 50 from a binomial distribution with $n = 20$ and $p = 0.3$:

- Start from a **Lists & Spreadsheet** page, and give Column A the name 'smeans'.
- In the formula cell of Column A, enter:
  = approx(seq(mean(randBin(20, 0.3, 50)), k, 1, 500))
- Open a **Data & Statistics** page, and generate a dotplot of the sample means.



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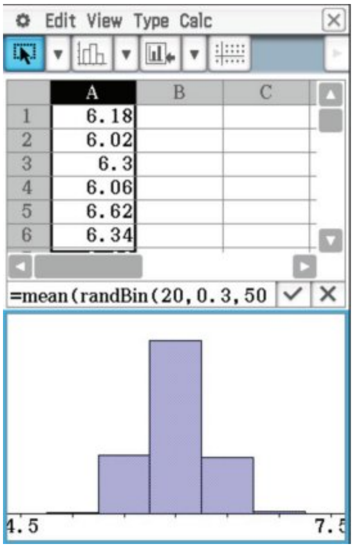
Using the Casio ClassPad

To generate the sample means for 500 random samples of size 50 from a binomial distribution with $n = 20$ and $p = 0.3$:

- Open the **Spreadsheet** application.
- Tap in cell A1.
- Type: = mean(randBin(20, 0.3, 50))
- Tap the tick icon and then tap again in cell A1.
- Go to **Edit** > **Fill** > **Fill Range**.
- Enter A1:A500 for the range. Tap OK.
- Tap on the column header A.
- Select **Graph** and tap **Histogram**.



Once again we can see in this example that, even though the distribution of X is not normal, the sampling distribution of $\bar{X}$ is quite well approximated by a normal distribution.

The central limit theorem

From these two examples we have found that, for different underlying distributions, the sampling distribution of the sample mean is approximately normal, provided the sample size n is large enough. Furthermore, the approximation to the normal distribution improves as the sample size increases. This fact is known as the **central limit theorem**.

Central limit theorem

Let X be any random variable, with mean μ and standard deviation σ. Then, provided that the sample size n is large enough, the distribution of the sample mean $\bar{X}$ is approximately normal with mean $E(\bar{X}) = \mu$ and standard deviation $\text{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

Note: The required sample size n depends on the symmetry of the distribution of X. Unless the distribution is very skewed, a sample size of 25 to 30 is sufficient. We will assume for the remainder of this chapter that the central limit theorem can be applied.

The central limit theorem can be used to solve problems associated with sample means, as illustrated in the following example.

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Example 15

The amount of coffee, X mL, dispensed by a machine has a distribution with probability density function f defined by

$$f(x) = \begin{cases} \frac{1}{20} & \text{if } 160 \le x \le 180 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that the average amount of coffee contained in 25 randomly chosen cups will be more than 173 mL.

Solution

The central limit theorem tells us that the sample mean $\bar{X}$ has an approximately normal distribution. To find the mean and standard deviation of $\bar{X}$, we first find the mean and standard deviation of X:

$$\mathrm{E}(X) = \int_{160}^{180} \frac{x}{20}\,dx = \left[\frac{x^2}{40}\right]_{160}^{180} = 170$$

and $$\mathrm{E}(X^2) = \int_{160}^{180} \frac{x^2}{20}\,dx = \left[\frac{x^3}{60}\right]_{160}^{180} = 28\,933.33$$

So $$\mathrm{sd}(X) = \sqrt{28\,933.33 - 170^2} = 5.77$$

By the central limit theorem, the sample mean $\bar{X}$ is (approximately) normally distributed with

$$\mathrm{E}(\bar{X}) = \mathrm{E}(X) = 170 \qquad \text{and} \qquad \mathrm{sd}(\bar{X}) = \frac{\mathrm{sd}(X)}{\sqrt{n}} = \frac{5.77}{5} = 1.15$$

Therefore

$$\Pr(\bar{X} > 173) = \Pr\left(Z > \frac{173 - 170}{1.15}\right) = \Pr(Z > 2.61) = 1 - 0.9955 = 0.0045$$

The normal approximation to the binomial distribution

The fact that the binomial distribution can be well approximated by the normal distribution was discussed in Mathematical Methods Units 3 & 4.

If X is a binomial random variable with parameters n and p, then the distribution of X is approximately normal, with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$, provided $np > 5$ and $n(1-p) > 5$.

This approximation can now be justified using the central limit theorem.

We know that a binomial random variable, X, is the number of successes in n independent trials, each with probability of success p. We can express X as the sum of n independent random variables $Y_1, Y_2, \dots, Y_n$, called **Bernoulli random variables**.

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Each Y_i takes values 0 and 1, with $\Pr(Y_i = 1) = p$ and $\Pr(Y_i = 0) = 1 - p$, where the value 1 corresponds to success and the value 0 corresponds to failure. We can write

$$X = Y_1 + Y_2 + \cdots + Y_n$$

and therefore

$$\frac{X}{n} = \frac{Y_1 + Y_2 + \cdots + Y_n}{n} = \bar{Y}$$

By the central limit theorem, the sample mean $\bar{Y}$ has an approximately normal distribution, for large n. Since $X = n\bar{Y}$, we see that X also has an approximately normal distribution.

Note: For a binomial random variable X, we can consider the sample mean $\frac{X}{n}$, with

$$\mathrm{E}\left(\frac{X}{n}\right) = \frac{\mathrm{E}(X)}{n} = \frac{np}{n} = p$$

$$\mathrm{Var}\left(\frac{X}{n}\right) = \frac{\mathrm{Var}(X)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

This random variable is denoted by $\hat{P}$ in Mathematical Methods Units 3 & 4.

Example 16

The population in a particular state is known to be 50% female. What is the probability that a random sample of 100 people will contain less than 45% females?

Solution

Let X denote the number of females in the sample. Then X has a binomial distribution with $n = 100$ and $p = 0.5$.

By the central limit theorem, the distribution of the sample mean $\frac{X}{n}$ is approximately normal, with

$$\mathrm{E}\left(\frac{X}{n}\right) = p = 0.5 \quad \text{and} \quad \mathrm{Var}\left(\frac{X}{n}\right) = \frac{p(1-p)}{n} = \frac{0.5 \times 0.5}{100} = 0.0025$$

Thus

$$\Pr\left(\frac{X}{n} < 0.45\right) = \Pr\left(Z < \frac{0.45 - 0.5}{0.05}\right) = \Pr(Z < -1) = 0.1587$$

Skill-sheet

Exercise 15F

Example 15

1 The lengths of blocks of cheese, X cm, produced by a machine have a distribution with probability density function

$$f(x) = \begin{cases} 5 & \text{if } 10.0 \le x \le 10.2 \\ 0 & \text{otherwise} \end{cases}$$

a Find the probability that a randomly selected block is more than 10.1 cm long.

b Find the probability that the average length of 30 randomly selected blocks is more than 10.12 cm.

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2 The number of accidents per week at an intersection has a mean of 0.25 and a standard deviation of 0.10. What is the probability that the average number of accidents per week at the intersection over a year is less than 0.22?

3 The working life of a particular brand of electric light bulb has a mean of 1200 hours and a standard deviation of 200 hours. What is the probability that the mean life of a sample of 64 bulbs is less than 1150 hours?

4 The amount of pollutant emitted from a smokestack in a day, X kg, has probability density function f defined by

$$f(x) = \begin{cases} \frac{4}{9}x(5 - x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

a Find the probability that the amount of pollutant emitted on any one day is more than 0.5 kg.

b Find the probability that the average amount of pollutant emitted on a random sample of 30 days is more than 0.5 kg.

5 The incubation period for a certain disease is between 5 and 11 days after contact. The probability of showing the first symptoms at various times during the incubation period is described by the probability density function

$$f(x) = \begin{cases} \frac{1}{36}(t - 5)(11 - t) & \text{if } 5 \le x \le 11 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that the average time for the appearance of symptoms for a random sample of 40 people with the disease was less than 7.5 days.

Example 16

6 The manager of a car-hire company knows from experience that 55% of their customers prefer automatic cars. If there are 50 automatic cars available on a particular day, use the normal approximation to the binomial distribution to estimate the probability that the company will not be able to meet the demand of the next 100 customers.

7 If 15% of people are left-handed, use the normal approximation to the binomial distribution to find the probability that at least 200 people in a randomly selected group of 1000 people are left-handed.

8 The thickness of silicon wafers is normally distributed with mean 1 mm and standard deviation 0.1 mm. A wafer is acceptable if it has a thickness between 0.85 and 1.1.

a What is the probability that a wafer is acceptable?

b If 200 wafers are selected, estimate the probability that between 140 and 160 wafers are acceptable.

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Chapter summary

Assign-ment

Nrich

Linear functions of a random variable

- If $Y = aX + b$ is a linear function of a random variable X, where a and b are constants with $a > 0$, then $\Pr(Y \le y) = \Pr\left(X \le \dfrac{y-b}{a}\right)$.
- For a random variable X and constants a and b:
  - $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$
  - $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$
  - $\mathrm{sd}(aX + b) = |a|\,\mathrm{sd}(X)$

Linear combinations of independent random variables

- Let X be a random variable with mean μ and variance σ^2. Then if $X_1, X_2, \dots, X_n$ are independent random variables with identical distributions to X:
  - $\mathrm{E}(X_1 + X_2 + \cdots + X_n) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \cdots + \mathrm{E}(X_n) = n\mu$
  - $\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$
  - $\mathrm{sd}(X_1 + X_2 + \cdots + X_n) = \sqrt{\mathrm{Var}(X_1 + X_2 + \cdots + X_n)} = \sqrt{n}\sigma$
- Let $X_1, X_2, \dots, X_n$ be independent random variables with means $\mu_1, \mu_2, \dots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$ respectively. Then if $a_1, a_2, \dots, a_n$ are constants:
  - $\mathrm{E}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1\,\mathrm{E}(X_1) + a_2\,\mathrm{E}(X_2) + \cdots + a_n\,\mathrm{E}(X_n)$
    $= a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$
  - $\mathrm{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2) + \cdots + a_n^2\,\mathrm{Var}(X_n)$
    $= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2$
  - $\mathrm{sd}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = \sqrt{a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2}$
- Let $X_1, X_2, \dots, X_n$ be independent normal random variables and let $a_1, a_2, \dots, a_n$ be constants. Then the random variable $a_1X_1 + a_2X_2 + \cdots + a_nX_n$ is also normally distributed.

Distribution of the sample mean

- The **population mean** μ is the mean of all values of a measure in a population. The **sample mean** $\bar{x}$ is the mean of these values in a particular sample.
- The sample mean $\bar{X}$ can be viewed as a random variable, and its distribution is called a **sampling distribution**.
- If X is a normally distributed random variable with mean μ and standard deviation σ, then the distribution of the sample mean $\bar{X}$ will also be normal, with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$, where n is the sample size.
- Central limit theorem

  Let X be any random variable, with mean μ and standard deviation σ. Then, provided that the sample size n is large enough, the distribution of the sample mean $\bar{X}$ is approximately normal with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.
- If X is a binomial random variable with parameters n and p, then the distribution of X is approximately normal, with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$, provided $np > 5$ and $n(1-p) > 5$.

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Review

Technology-free questions

1 Suppose that X is a random variable with mean $\mu = 15$ and variance $\sigma^2 = 25$.

a Let $Y = 2X + 1$. Find $E(Y)$ and $Var(Y)$.

b Let $U = 10 - 3X$. Find $E(U)$ and $sd(U)$.

c Let $V = Y + 2U$. Find $E(V)$ and $Var(V)$.

2 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} 2\left(1 - \dfrac{1}{x^2}\right) & \text{if } 1 \le x \le 2 \\ 0 & \text{if } x < 1 \text{ or } x > 2 \end{cases}$$

a Find $\Pr(X \le 1.6)$.

b Let $Y = 2X - 1$. Find $\Pr(Y \le 2.5)$.

3 A machine that produces plastic blocks dispenses the melted plastic into a mould in the shape of a rectangular prism. The block produced has a square base 3 cm by 3 cm and a height of X cm, where X is a random variable with a mean of 3 cm and a standard deviation of 0.01 cm.

a Find the expected volume of the blocks (in cm^3).

b Find the variance of the volume of the blocks (in cm^6).

c Find the expected surface area of the blocks (in cm^2).

4 A coffee machine automatically dispenses coffee into a cup, followed by hot milk. The volume of coffee dispensed has a mean of 60 mL and a standard deviation of 5 mL. The volume of hot milk dispensed has a mean of 140 mL and a standard deviation of 12 mL.

a What are the mean and standard deviation of the total amount of liquid dispensed by the machine?

b If the cost of the coffee is \$12 per litre and the cost of the milk is \$3 per litre, what is the mean total cost of a cup of coffee?

5 The random variables X and Y are independent. The mean and variance of X are 3 and 5 respectively, while the mean and variance of Y are 2 and 4 respectively. Find the values of $a, b \in \mathbb{N}$ if the mean and variance of $aX - bY$ are 7 and 49 respectively.

6 At a supermarket, the bags of oranges have a mean weight of 500 g, with a variance of 25 g^2, and the bags of lemons have a mean weight of 585 g, with a variance of 36 g^2.

a If there are four oranges in each bag, what are the mean and standard deviation of the weight of an individual orange?

b If there are nine lemons in each bag, what are the mean and standard deviation of the weight of an individual lemon?

c If you purchase three bags of oranges and three bags of lemons, what are the mean and variance of the total weight of the six bags?

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7 A random variable X has probability density function given by

$$f(x) = \begin{cases} 0.2 & \text{if } -1 \le x \le 0 \\ 0.2 + 1.2x & \text{if } 0 < x \le 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$$

Let X_1, X_2, X_3 and X_4 be independent random variables, each with the same probability density function as X. If $V = X_1 + X_2 + X_3 + X_4$, find the mean of V.

8 The final marks in a mathematics examination are normally distributed with mean 65 and standard deviation 7. A random sample of 25 students are selected and their mean mark calculated. What are the mean and standard deviation of this sample mean?

9 A machine produces nails which have an intended diameter of $\mu = 25.025$ mm, with a standard deviation of $\sigma = 0.003$ mm. A sample of nails is selected for inspection each hour, and the average diameter of the nails in the sample, $\bar{X}$, is determined. If the manufacturer requires the standard deviation of the average diameter to be less than 0.00075 mm, how many nails should be included in the sample?

Multiple-choice questions

1 Let P and Q be independent normally distributed random variables, where P has a mean of 10 and a variance of 4, and Q has a mean of 5 and a variance of 1. Define the random variable $R = 2P - 2Q + 2$. In terms of the standard normal random variable Z, the probability $\Pr(R > 6)$ is equal to

A $\Pr\left(Z > \frac{-2}{\sqrt{5}}\right)$ **B** $\Pr\left(Z > \frac{-3}{\sqrt{6}}\right)$ **C** $\Pr\left(Z < \frac{-3}{\sqrt{5}}\right)$ **D** $\Pr\left(Z > \frac{-3}{\sqrt{5}}\right)$ **E** $\Pr\left(Z > \frac{3}{\sqrt{5}}\right)$

2 An aeroplane is only allowed a total passenger weight of 10 000 kg. If the weights of people are normally distributed with a mean of 80 kg and a standard deviation of 10 kg, the probability that the combined weight of 100 passengers will exceed 10 000 kg is

A 0.0228 **B** 0.0022 **C** 0 **D** 0.9772 **E** 0.0013

3 Mangoes from a certain supplier have a mean weight of 150 g, with variance 4 g^2. Pineapples from the same supplier have a mean weight of 1000 g, with variance 36 g^2. The mean and standard deviation of the weight (in grams) of a bag containing four mangoes and two pineapples are

A $2600, 2\sqrt{22}$ **B** $2600, 4\sqrt{13}$ **C** $4300, 2\sqrt{38}$ **D** 2600, 20 **E** 2600, 400

4 The time required to assemble an electronic component is normally distributed, with a mean of 10 minutes and a standard deviation of 1.5 minutes. The probability that the time required to assemble a box of 12 components is greater than 130 minutes is

A 0.2892 **B** 0.7108 **C** 0.0092 **D** 0.9910 **E** 0.0271

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Review

5 Suppose that X is a random variable with mean $\mu = 3.6$ and variance $\sigma^2 = 1.44$. If $Y = 3 - 4X$, then $\mathrm{E}(Y)$ and $\mathrm{sd}(Y)$ are

A $\mathrm{E}(Y) = -11.4,\ \mathrm{sd}(Y) = 4.8$ **B** $\mathrm{E}(Y) = -11.4,\ \mathrm{sd}(Y) = 5.76$
C $\mathrm{E}(Y) = -11.4,\ \mathrm{sd}(Y) = 23.04$ **D** $\mathrm{E}(Y) = -3.6,\ \mathrm{sd}(Y) = 4.8$
E $\mathrm{E}(Y) = -3.6,\ \mathrm{sd}(Y) = 5.76$

6 Let X be a random variable with $\mathrm{E}(X) = 2.0$ and $\mathrm{Var}(X) = 0.5$. Let $Y = mX + n$, where $m \in \mathbb{R}^+$ and $n \in \mathbb{R}$. If $\mathrm{E}(Y) = 2.1$ and $\mathrm{Var}(Y) = 0.32$, then when the value of X is 2.2, the value of Y is closest to

A 2.2 **B** 2.3 **C** 2.5 **D** 2.9 **E** 3.0

7 The monthly mortgage payments for recent home buyers are normally distributed with mean \$2489 and standard deviation \$554. A random sample of 100 recent home buyers is selected. The distribution of the mean of this sample is normal with

A mean \$24.89, sd \$5.54 **B** mean \$2489, sd \$55.40 **C** mean \$2489, sd \$5.54
D mean \$2489, sd \$554 **E** mean \$248.90, sd \$55.40

8 The lengths of a species of fish are normally distributed with a mean of 40 cm and a standard deviation of 4 cm. The probability that the mean length of a sample of 25 of these fish is greater than 42 cm is

A 0 **B** 0.0062 **C** 0.3085 **D** 0.6915 **E** 0.9938

9 Jo and Ann regularly play golf together. The distance that Jo hits her driver is normally distributed, with mean 150 m and standard deviation 20 m. The distance that Ann hits her driver is normally distributed, with mean 140 m and standard deviation 25 m. The percentage of times that Ann hits her driver further than Jo is closest to

A 41% **B** 62% **C** 59% **D** 38% **E** 51%

Extended-response questions

1 Jan uses the lift in her multi-storey office building each day. She has noted that, when she goes to her office each morning, the time she waits for the lift is normally distributed with a mean of 60 seconds and a standard deviation of 20 seconds.

a What is the probability that Jan will wait less than 54 seconds on a particular day?

b Find a and b such that the probability that Jan waits between a seconds and b seconds is 0.95.

c During a five-day working week, find the probability that:

i Jan's average waiting time is less than 54 seconds

ii Jan's total waiting time is less than 270 seconds

iii she waits for less than 54 seconds on more than two days in the week.

d Find c and d such that there is a probability of 0.95 that her average waiting time over a five-day period is between c seconds and d seconds.

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2 The daily rainfall in Briswin is normally distributed with mean μ mm and standard deviation σ mm. The rainfall on one day is independent of the rainfall on any other day. On a randomly selected day, there is a 5% chance that the rainfall is more than 10.2 mm. In a randomly selected seven-day week, there is a probability of 0.025 that the mean daily rainfall is less than 6.1 mm. Find the values of μ and σ.

3 An aeroplane is licensed to carry 100 passengers.

a If the weights of passengers are normally distributed with a mean of 80 kg and a standard deviation of 20 kg, find the probability that the combined weight of 100 passengers will exceed 8500 kg.

b The weight of the luggage that passengers check in before they travel is normally distributed, with a mean of 27 kg and a standard deviation of 4 kg. Find the probability that the combined weight of the checked luggage of 100 passengers is more than 2850 kg.

c Passengers are also allowed to take hand luggage on the plane. The weight of the hand luggage that they carry is normally distributed, with a mean of 8 kg and a standard deviation of 2.5 kg. Find the probability that the combined weight of the hand luggage for 100 passengers is more than 900 kg.

d What is the probability that the combined weight of the 100 passengers, their checked luggage and their hand luggage is more than 12 000 kg?

4 A probability density function for the lifetime, T hours, of a brand of battery is

$$f(t) = \frac{1}{250}e^{-\frac{t}{250}}, \quad t > 0$$

a If batteries are sold in boxes of 100, find the probability that the average lifetime of batteries in a randomly chosen box is less than 220 hours.

b The manufacturer would like the probability that the average lifetime of batteries in a box is greater than 220 hours to be 0.80. What is the minimum number of batteries that should be packed in a box?

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16

Confidence intervals and hypothesis testing for the mean

Objectives

- To introduce the concept of a **confidence interval** when estimating the mean.
- To construct approximate confidence intervals for the mean.
- To introduce the logic of hypothesis testing, including the formulation of a **null hypothesis** and an **alternative hypothesis**.
- To introduce the concept of a ***p*-value**.
- To determine the *p*-value for the sample mean of a sample drawn from a normal distribution with known variance, or for the sample mean of a large sample.
- To understand the implications of **one-tail** and **two-tail tests** on the *p*-value.
- To introduce **Type I** and **Type II errors** in hypothesis testing.

Statistical inference involves making a decision about a population (an inference) based on the information which has been collected from a sample. There are two key components of statistical inference which will be addressed in this chapter:

- **Estimation** This involves using the sample mean to determine an interval estimate (confidence interval) for the value of the population mean, which is unknown.
- **Hypothesis testing** Here we ask the question: 'Has the population mean changed?' For example, suppose that medical researchers know that the mean time for recovery from a certain virus using the drug currently being prescribed is five days. They have developed a new drug for the treatment of this virus, which they hope will result in a speedier recovery. Thus their question is: 'Is the mean time for recovery using the new drug less than five days?' Such a question is addressed using the discipline of hypothesis testing.

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16A Confidence intervals for the population mean

In practice, the reason we analyse samples is to further our understanding of the population from which they are drawn. That is, we know what is in the sample, and from that knowledge we would like to infer something about the population.

Point estimates

Suppose, for example, we are interested in the mean IQ score of all Year 12 mathematics students in Australia. The value of the population mean μ is unknown. Collecting information about the whole population is not feasible, and so a random sample must suffice.

What information can be obtained from a single sample? Certainly, the sample mean $\bar{x}$ gives some indication of the value of the population mean μ, and can be used when we have no other information.

> The value of the sample mean $\bar{x}$ can be used to estimate the population mean μ. Since this is a single-valued estimate, it is called a **point estimate** of μ.

Thus, if we select a random sample of 100 Year 12 mathematics students and find that their mean IQ is 108.6, then the value $\bar{x} = 108.6$ serves as an estimate of the population mean μ.

Interval estimates

The value of the sample mean $\bar{x}$ obtained from a single sample is going to change from sample to sample, and while sometimes the value will be close to the population mean μ, at other times it will not. To use a single value to estimate μ can be rather risky. What is required is an interval that we are reasonably sure contains the parameter value μ.

> An **interval estimate** for the population mean μ is called a **confidence interval** for μ.

Approximate 95% confidence intervals

The central limit theorem was introduced in Section 15F. This theorem tells us that, whatever the underlying distribution of the random variable X, if the sample size n is large, then the sampling distribution of $\bar{X}$ is approximately normal with

$$\mathrm{E}(\bar{X}) = \mu \quad \text{and} \quad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

For the standard normal random variable Z, we have

$$\Pr(-1.9600 < Z < 1.9600) = 0.95$$

So we can state that, for large n:

$$\Pr\left(-1.9600 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.9600\right) \approx 0.95$$

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Multiplying through gives

$$\Pr\left(-1.9600\frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < 1.9600\frac{\sigma}{\sqrt{n}}\right) \approx 0.95$$

Further simplifying, we obtain

$$\Pr\left(\bar{X} - 1.9600\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.9600\frac{\sigma}{\sqrt{n}}\right) \approx 0.95$$

This final expression gives us an interval which, with 95% probability, will contain the value of the population mean μ (which we do not know).

95% confidence interval

An approximate **95% confidence interval** for μ is given by

$$\left(\bar{x} - 1.9600\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.9600\frac{\sigma}{\sqrt{n}}\right)$$

where:

- μ is the population mean (unknown)
- $\bar{x}$ is a value of the sample mean
- σ is the value of the population standard deviation
- n is the size of the sample from which $\bar{x}$ was calculated.

Note: Often when determining a confidence interval for the population mean, the population standard deviation σ is unknown. If the sample size is large (say $n \geq 30$), then we can use the sample standard deviation s in this formula as an approximation to the population standard deviation σ.

Example 1

Find an approximate 95% confidence interval for the mean IQ of Year 12 mathematics students in Australia, if we select a random sample of 100 students and find the sample mean $\bar{x}$ to be 108.6. Assume that the standard deviation for this population is 15.

Solution

The interval is found by substituting $\bar{x} = 108.6$, $n = 100$ and $\sigma = 15$ into the expression for an approximate 95% confidence interval:

$$\left(\bar{x} - 1.9600\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.9600\frac{\sigma}{\sqrt{n}}\right)$$

$$= \left(108.6 - 1.9600 \times \frac{15}{\sqrt{100}},\ 108.6 + 1.9600 \times \frac{15}{\sqrt{100}}\right)$$

$$= (105.66, 111.54)$$

Thus, based on a sample of size 100 and a sample estimate of 108.6, an approximate 95% confidence interval for the population mean μ is (105.66, 111.54).

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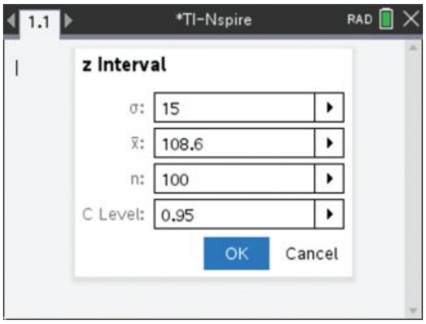
Using the TI-Nspire

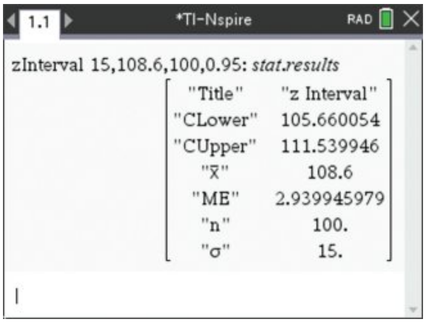
In a **Calculator** page:

- Use (menu) > **Statistics** > **Confidence Intervals** > **z Interval**.
- If necessary, change the **Data Input Method** to **Stats**.
- Enter the given values and the confidence level as shown.



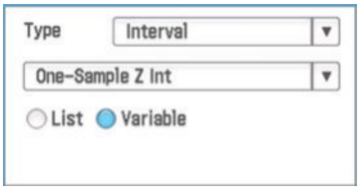
- The 'CLower' and 'CUpper' values give the 95% confidence interval (105.66, 111.54).

Note: 'ME' stands for margin of error, which is covered later in this section.

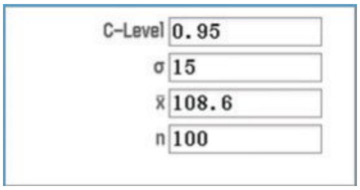


Using the Casio ClassPad

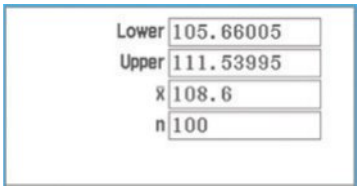
- In Statistics, go to **Calc** > **Interval**.
- Select **One–Sample Z Int** and **Variable**. Tap Next.



- Enter the confidence level and the given values as shown. Tap Next.



- The 'Lower' and 'Upper' values give the 95% confidence interval (105.66, 111.54).



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Changing the level of confidence

We can find an approximate confidence interval with a level of confidence other than 95% by using the same principles. For example, since we know that

$$\Pr(-1.6449 < Z < 1.6449) = 0.90$$

an approximate 90% confidence interval for μ is given by

$$\left(\bar{x} - 1.6449\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.6449\frac{\sigma}{\sqrt{n}}\right)$$

We can generalise these two examples as follows.

C% confidence interval

An approximate ***C*% confidence interval** for μ is given by

$$\left(\bar{x} - z\frac{\sigma}{\sqrt{n}},\ \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$$

where:

- z is such that $\Pr(-z < Z < z) = C\%$
- μ is the population mean (unknown)
- $\bar{x}$ is a value of the sample mean
- σ is the value of the population standard deviation
- n is the size of the sample from which $\bar{x}$ was calculated.

Note: The values of z (to four decimal places) for commonly used confidence intervals are:

- 90% $z = 1.6449$
- 95% $z = 1.9600$
- 99% $z = 2.5758$

Example 2

Calculate and compare 90%, 95% and 99% confidence intervals for the mean IQ of Year 12 mathematics students in Australia, if we select a random sample of 100 students and find the sample mean $\bar{x}$ to be 108.6. (Assume that $\sigma = 15$.)

Solution

From Example 1, we know that the 95% confidence interval is (105.66, 111.54).

The 90% confidence interval is

$$\left(108.6 - \frac{1.6449 \times 15}{10},\ 108.6 + \frac{1.6449 \times 15}{10}\right) = (106.13, 111.07)$$

The 99% confidence interval is

$$\left(108.6 - \frac{2.5758 \times 15}{10},\ 108.6 + \frac{2.5758 \times 15}{10}\right) = (104.74, 112.46)$$

We see that increasing the level of confidence increases the width of the confidence interval.

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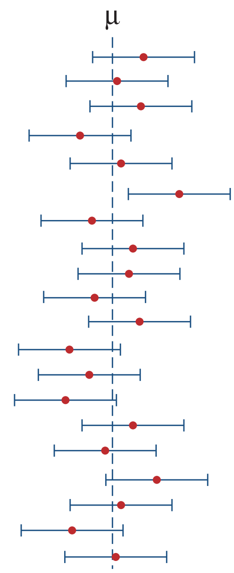
Interpretation of confidence intervals

The 95% confidence interval found in Example 1 should not be interpreted as meaning that $\Pr(105.66 < \mu < 111.54) = 0.95$. Since μ is a constant, the value either does or does not lie in the stated interval.

The particular confidence interval found is just one of any number of confidence intervals which could be found for the population mean μ, each one depending on the particular value of the sample mean $\bar{x}$.

The correct interpretation of a 95% confidence interval is that we expect approximately 95% of such intervals to contain the population mean μ. Whether or not the particular confidence interval obtained contains the population mean μ is generally not known.

If we were to repeat the process of taking a sample and calculating a confidence interval many times, the result would be something like that indicated in the diagram.



The diagram shows the confidence intervals obtained when 20 different samples were drawn from the same population. The round dot indicates the value of the sample estimate in each case. The intervals vary, because the samples themselves vary. The value of the population mean μ is indicated by the vertical line, and it is of course constant.

It is quite easy to see from the diagram that none of the values of the sample estimate is exactly the same as the population mean, but that all the intervals except one (19 out of 20, or 95%) have captured the value of the population mean, as would be expected in the case of a 95% confidence interval.

Example 3

Suppose that the process of taking a sample and determining a confidence interval based on the sample mean was repeated 200 times. How many of these intervals would be expected to contain the value of the population mean μ if the level of confidence is:

a 90% **b** 95% **c** 99%?

Solution

a We expect $0.90 \times 200 = 180$ of the 90% confidence intervals to contain μ.

b We expect $0.95 \times 200 = 190$ of the 95% confidence intervals to contain μ.

c We expect $0.99 \times 200 = 198$ of the 99% confidence intervals to contain μ.

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Effect of sample size on the width of confidence intervals

We saw in Example 2 that increasing the level of confidence increases the width of the confidence interval. The width of a confidence interval is important, as for a confidence interval to be useful it should not be too wide. The distance between the sample mean and the endpoints of a confidence interval is called the **margin of error**. The smaller the margin of error, the better the estimate of the population mean.

Since the width of the confidence interval is inversely proportional to the square root of the sample size, it makes sense that a better way to decrease the width of the confidence interval is to increase the sample size.

Example 4

Calculate and compare 95% confidence intervals for the mean IQ of Year 12 mathematics students in Australia, if:

- we select a random sample of 100 students and find the sample mean $\bar{x}$ to be 108.6
- we select a random sample of 400 students and find the sample mean $\bar{x}$ to be 108.6.

(Assume that $\sigma = 15$.)

Solution

From Example 1, the first 95% confidence interval is (105.66, 111.54).

The second 95% confidence interval is

$$\left(108.6 - 1.9600 \times \frac{15}{\sqrt{400}},\ 108.6 + 1.9600 \times \frac{15}{\sqrt{400}}\right) = (107.13, 110.07)$$

Thus the confidence interval based on a sample of size 400 is narrower than the confidence interval based on a sample of size 100.

In this example, by increasing the sample size, we obtained a narrower 95% confidence interval and therefore a better estimate for the population mean μ. In fact, since we have increased the sample size by a factor of 4 (from 100 to 400), we can readily verify that we have decreased the width of the confidence interval by a factor of 2.

Example 5

A confidence interval is used to estimate the population mean μ based on a sample mean $\bar{x}$. By what factor must the sample size be increased in order to decrease the width of the confidence interval by 80%?

Solution

Let n_1 be the current sample size, and let n_2 be the new sample size.

Let W_1 be the width of the current confidence interval, and let W_2 be the width of the new confidence interval. Then

$$W_1 = 2z\frac{\sigma}{\sqrt{n_1}} \qquad \text{and} \qquad W_2 = 2z\frac{\sigma}{\sqrt{n_2}}$$

where the value of z is determined by the level of confidence.

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For the width to decrease by 80%, we require

$$W_2 = 0.2 \times W_1$$

$$2z\frac{\sigma}{\sqrt{n_2}} = 0.2 \times 2z\frac{\sigma}{\sqrt{n_1}}$$

$$\sqrt{n_2} = 5\sqrt{n_1}$$

$$\therefore \quad n_2 = 25n_1$$

The sample size should be increased by a factor of 25.

Example 6

Consider again the problem of estimating the mean IQ of Year 12 mathematics students in Australia. What size sample is required in order to ensure that the difference between the sample mean and the population mean is 1.5 points or less at the 95% confidence level? (Assume that $\sigma = 15$.)

Solution

The distance between the sample mean $\bar{x}$ and the endpoints of the 95% confidence interval should be less than or equal to 1.5. Therefore we require

$$1.9600 \times \frac{15}{\sqrt{n}} \le 1.5$$

Hence

$$n \ge \left(\frac{1.9600 \times 15}{1.5}\right)^2 = 384.16$$

We require a sample of at least 385 students.

Skill-sheet

Exercise 16A

Example 1

1 A university lecturer selects a sample of 40 of her first-year students to determine how many hours per week they spend on study outside class time. She finds that their average study time is 7.4 hours. If the standard deviation of study time, σ, is 1.8 hours, find a 95% confidence interval for the mean study time for the population of first-year students.

Example 2

2 Calculate and compare approximate 90%, 95% and 99% confidence intervals for the mean battery life for a certain brand of batteries, if the mean life of 64 batteries was found to be 35.7 hours. Assume that $\sigma = 15$ hours.

3 In an investigation of physical fitness of students, resting heart rates were recorded for a sample of 15 female students. The sample had a mean of 71.1 beats per minute. The investigator knows from experience that resting heart rates are normally distributed and have a standard deviation of 6.4 beats per minute. Find a 99% confidence interval for the mean resting heart rate of the relevant population of female students.

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4 A random sample of 49 of a certain brand of batteries was found to last an average of 14.6 hours. If the standard deviation of battery life is known to be 90 minutes, find a 90% confidence interval for the mean time that the batteries will last.

5 The lengths of time (in seconds) for which each of a randomly selected sample of 12-year-old girls could hold their breath are as follows.

14	43	16	25	25	35	14	42	23	33	20	60
39	68	18	20	25	30	20	32	54	35	45	48

If breath-holding time is known to be normally distributed, with a standard deviation of 15 seconds, find a 98% confidence interval for the mean time for which a 12-year-old girl can hold her breath.

6 Twenty-two air samples taken at the same place over a period of six months showed the following amounts of suspended matter (in micrograms per cubic metre of air).

68	22	36	32	42	24	28	38	39	26	21
79	45	57	59	34	43	57	30	31	28	30

Assuming these measurements to be a random sample from a normally distributed population with standard deviation 10, construct an approximate 95% confidence interval for the mean amount of suspended matter during that time period.

7 The birth weights, in kilograms, of a random sample of 30 full-term babies with no complications born at a hospital are as follows.

2.9	2.7	3.5	3.6	2.8	3.6	3.7	3.6	3.6	2.9	3.7	3.6	3.2	2.9	3.2
2.5	2.6	3.8	3.0	4.2	2.8	3.5	3.3	3.1	3.0	4.2	3.2	2.4	4.3	3.2

Find an approximate 99% confidence interval for the mean weight of full-term babies with no complications, if the birth weights of full-term babies are normally distributed with a standard deviation of 400 g.

8 Fifty plots are planted with a new variety of corn. The average yield for these plots is 130 bushels per acre. Assume that the standard deviation is equal to 10.

a Find an approximate 95% confidence interval for the mean yield of this variety.

b Find an approximate 99% confidence interval for the mean yield of this variety.

c Compare your answers to parts **a** and **b**.

9 The average amount of time (in hours per week) spent in physical exercise by a random sample of 24 male Year 12 students is as follows.

4.0	3.3	4.5	0.0	8.0	2.0	3.3	2.5	7.0	2.0	12.0	4.0
8.0	3.0	6.0	2.5	1.0	0.5	5.0	6.0	4.0	1.0	0.0	7.0

Assume that the time spent per week in physical exercise by Year 12 males is normally distributed with a mean of μ hours and a standard deviation of 3 hours.

a Find an approximate 90% confidence interval for μ.

b Find an approximate 95% confidence interval for μ.

c Compare your answers to parts **a** and **b**.

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10 A random sample of 100 married males were asked to give the age at which they married. The average age given by this sample was 29.5 years. Assume that the standard deviation of age at marriage is known to be 8 years.

a Find an approximate 90% confidence interval for the mean age at marriage for males.

b Find an approximate 98% confidence interval for the mean age at marriage for males.

c Compare your answers to parts **a** and **b**.

Example 3

11 Suppose that the process of taking a sample and determining a confidence interval based on the sample mean was repeated 100 times. How many of these intervals would be expected to contain the value of the population mean μ if the level of confidence is:

a 80% **b** 85% **c** 90%?

12 A researcher determines an approximate 95% confidence interval for the mean weight of a certain species of frog. Her colleague, using a different sample of data, determines a 95% confidence interval for the mean weight of the same species of frog.

a What is the probability that both of these confidence intervals contain the population mean weight for this species of frog?

b What is the probability that at least one of these confidence intervals contains the population mean weight for this species of frog?

Example 4

13 A quality-control engineer in a factory needs to estimate the mean weight, μ grams, of bags of potato chips that are packed by a machine. The engineer knows by experience that $\sigma = 2.0$ grams for this machine.

a Find a 95% confidence interval for μ, if the engineer takes a random sample of 36 bags and finds the sample mean to be 25.4 grams.

b Find a 95% confidence interval for μ, if the engineer takes a random sample of 100 bags and finds the sample mean to be 25.4 grams.

c Compare your confidence intervals in parts **a** and **b**.

14 A confidence interval is used to estimate the population mean μ based on a sample mean $\bar{x}$.

Example 5

a By what factor must the sample size be increased in order to decrease the width of the confidence interval by 50%?

b What percentage increase in the sample size is required in order to decrease the width of the confidence interval by 20%?

c If the sample size is increased by a factor of 9, what is the change in the width of the confidence interval?

d If the sample size is decreased by a factor of 16, what is the change in the width of the confidence interval?

Example 6

15 For a population with a standard deviation of 100, how large a random sample is needed in order to be 95% confident that the sample mean is within 20 of the population mean?

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16 A quality-control engineer in a factory needs to estimate the mean weight, μ grams, of bags of potato chips that are packed by a machine. The engineer knows by experience that $\sigma = 2.0$ grams for this machine. What size sample is required to ensure that we can be 95% confident that the estimate will be within 0.5 grams of μ?

17 The number of customers per day at a fast-food outlet is known to have a standard deviation of 50. What size sample is required so that the owner can be 99% confident that the difference between the sample mean and the true mean is not more than 10?

18 A manufacturer knows that the standard deviation of the lifetimes of their light bulbs is 150 hours. What size sample is required so that the manufacturer can be 90% confident that the sample mean, $\bar{x}$, will be within 20 hours of the population mean?

19 Consider once again the problem of estimating the mean IQ score, μ, of Year 12 mathematics students. (Assume that $\sigma = 15$.)

a What size sample is required to ensure with 95% confidence that the sample mean IQ will be within 2 points of μ?

b What size sample is required to ensure with 99% confidence that the sample mean IQ will be within 2 points of μ?

16B Hypothesis testing for the mean

The mean and standard deviation for IQ scores in the general population are $\mu = 100$ and $\sigma = 15$. Suppose we believe that, in general, Year 12 mathematics students score higher on IQ tests than members of the general population. To investigate, we select a random sample of 100 Year 12 mathematics students and determine their mean IQ to be 103.6. This is 3.6 points higher than the mean IQ of people in general.

Is it reasonable to conclude that Year 12 mathematics students score higher on IQ tests than the general public? We already know that sample means will vary from sample to sample, and we would not expect the mean of an individual sample to have exactly the same value as the mean of the population from which it is drawn.

- One explanation is that Year 12 mathematics students perform no better on IQ tests than members of the general public, and the difference between the mean score of the sample, $\bar{x} = 103.6$, and that of the general population, $\mu = 100$, is due to sampling variability.
- Another explanation is that Year 12 mathematics students actually do better than average on IQ tests, and a sample mean of $\bar{x} = 103.6$ is consistent with this explanation.

Hypothesis testing is concerned with deciding which of the two explanations is more likely, which we do on the basis of probability.

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The logic of a hypothesis test

A hypothesis test can be likened to a trial in a court of law. We begin with a hypothesis that we wish to find evidence to support. In a court, as a prosecutor, your intention is to show that the person is guilty. However, the starting point in the trial is that the person is innocent. It is up to the prosecutor to provide enough evidence to show that this assumption is untenable.

The assumption of innocence in hypothesis-testing terms is called the **null hypothesis**, denoted by H_0. If we can collect evidence to show that the null hypothesis is untenable, we can conclude that there is support for an **alternative hypothesis**, denoted by H_1.

Setting up the hypotheses

In this IQ example, our hypothesis is that Year 12 mathematics students perform better than the general population on IQ tests. To test this with a hypothesis test, we start by *assuming the opposite*: we assume that Year 12 mathematics students perform no better on IQ tests than members of the general public. In statistical terms, we are saying that the distribution of IQ scores for these students is the same as for the general public.

For the general public, we know that IQ is normally distributed with a mean of $\mu = 100$ and a standard deviation of $\sigma = 15$. The null hypothesis is that the students are drawn from a population in which the mean is $\mu = 100$. We express this null hypothesis symbolically as

$$H_0: \ \mu = 100$$

The null hypothesis

The null hypothesis, H_0, says that the sample is drawn from a population which has the same mean as before (i.e. the population mean has not changed). Under the null hypothesis, any difference between the values of a sample statistic and the population parameter is explained by sample-to-sample variation.

In this case, we are hypothesising that the mean IQ of Year 12 mathematics students is higher than that of the general population – that the sample comes from a population with mean $\mu > 100$. We express this alternative hypothesis symbolically as

$$H_1: \ \mu > 100$$

The alternative hypothesis

The alternative hypothesis, H_1, says that the population mean has changed. That is, while there will always be some sampling variability, the amount of variation is so much that it is more likely that the sample has been drawn from a population with a different mean.

Note: Hypotheses are always expressed in terms of population parameters.

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Example 7

The average fuel consumption for a particular model of car is 13.7 litres per 100 km. The manufacturer is claiming that the new model will use less petrol. A sample of 25 of the new model cars had an average fuel consumption of 12.5 litres per 100 km. Write down the null and alternative hypotheses that the manufacturer will use in testing this claim.

Solution

We start by assuming that the new model of the car is no better than the previous model, and that the difference between the population mean $\mu = 13.7$ and the sample mean $\bar{x} = 12.5$ is due only to sampling variability. Thus:

$$H_0: \ \mu = 13.7$$

The alternative hypothesis asserts that the sample mean is lower than the previous population mean because the sample has been drawn from a population with a mean that is lower than that of the previous model. That is:

$$H_1: \ \mu < 13.7$$

The test statistic

How do we decide between the two hypotheses? Both in a court of law and in statistical hypothesis testing, evidence is collected. This evidence is then weighed up (considered) so that a decision can be made. In the court room, the jury functions as the decision maker, weighing the evidence to make a decision of guilty (the alternative hypothesis) or not guilty (the null hypothesis). In hypothesis testing, the evidence is contained in the sample data.

To help us make our decision, we generally summarise the data into a single statistic, called the **test statistic**. There are many test statistics that can be used. If we are testing a hypothesis about a population mean μ, then the obvious test statistic is the sample mean $\bar{x}$.

If we find that the sample mean observed is very unlikely to have been obtained from a sample drawn from the hypothesised population, this will cause us to doubt the credibility of that hypothesised population mean. The statistical tool we use to determine the likelihood of this value of a test statistic is the distribution of sample means.

The *p*-value

Hypothesis testing requires us to make a decision between the null and alternative hypotheses. To do this, we determine the probability of obtaining a value of the sample statistic as extreme as or more extreme than the one found from the sample, assuming that the null hypothesis is true. This probability is known as the *p*-value of the test.

The *p*-value

The ***p*-value** is the probability of observing a value of the sample statistic as extreme as or more extreme than the one observed, assuming that the null hypothesis is true.

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Consider again the hypothesis that the mean IQ of Year 12 mathematics students is higher than that of the general population.

We have hypotheses

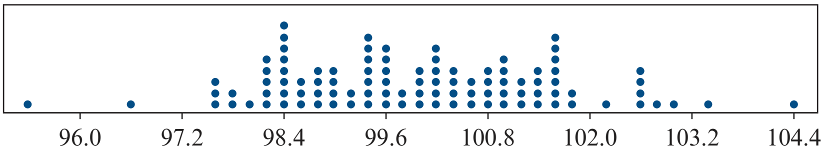
$$H_0: \ \mu = 100$$
$$H_1: \ \mu > 100$$

and the mean of a sample of size 100 is $\bar{x} = 103.6$.

Thus we can write:

$$p\text{-value} = \Pr(\bar{X} \geq 103.6 \mid \mu = 100)$$

To get a picture as to how much we could reasonably expect the sample mean to vary from sample to sample, we can use simulation. The following dotplot shows the values of $\bar{x}$ obtained from 100 samples (each of size 100) taken from a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$.



We can clearly see from the dotplot that a sample mean of $\bar{x} = 103.6$ is very unlikely. In fact, we obtained a sample mean as big as or bigger than this only once in 100 samples.

To determine the p-value exactly, we can use a result from Chapter 15:

Distribution of the sample mean

If X is a normally distributed random variable with mean μ and standard deviation σ, then the distribution of the sample mean $\bar{X}$ will also be normal, with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$, where n is the sample size.

Thus, if the null hypothesis is true, then $\bar{X}$ is normally distributed with

$$\mathrm{E}(\bar{X}) = \mu = 100 \quad \text{and} \quad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{100}} = 1.5$$

Therefore

$$\begin{aligned} p\text{-value} &= \Pr(\bar{X} \geq 103.6 \mid \mu = 100) \\ &= \Pr\left(Z \geq \frac{103.6 - 100}{1.5}\right) \\ &= \Pr(Z \geq 2.4) \\ &= 0.0082 \end{aligned}$$

Thus, the p-value tells us that, if the mean IQ of Year 12 mathematics students is 100, then the likelihood of observing a sample mean as high as or higher than 103.6 is extremely small, only 0.0082.

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Example 8

Consider again Example 7, where we are testing the hypotheses:

$H_0\colon\ \mu = 13.7$

$H_1\colon\ \mu < 13.7$

Assume that fuel consumption is normally distributed with a standard deviation of $\sigma = 2.8$ litres per 100 km. If the average fuel consumption for a sample of 25 cars is $\bar{x} = 12.5$ litres per 100 km, determine the p-value for this test.

Solution

If the null hypothesis is true, then $\bar{X}$ is normally distributed with

$$\mathrm{E}(\bar{X}) = \mu = 13.7 \quad \text{and} \quad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{2.8}{\sqrt{25}} = 0.56$$

Thus

$$\begin{aligned} p\text{-value} &= \Pr(\bar{X} \le 12.5 \mid \mu = 13.7) \\ &= \Pr\left(Z \le \frac{12.5 - 13.7}{0.56}\right) \\ &= \Pr(Z \le -2.143) \\ &= 0.0161 \end{aligned}$$

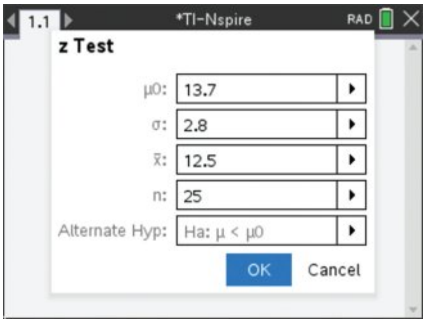
Using the TI-Nspire

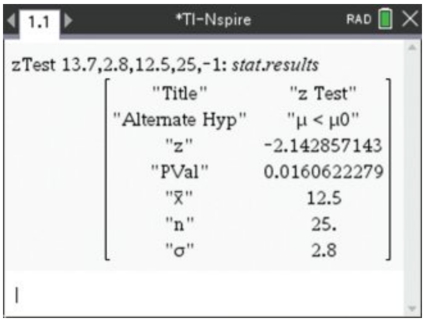
In a **Calculator** page:

- Use (menu) > **Statistics** > **Stat Tests** > **z Test**.
- If necessary, change the **Data Input Method** to **Stats**.
- Enter the given values as shown and select the form of the alternative hypothesis.

Note: Check carefully the sign required for the **Alternate Hyp**. If necessary, use the right arrow to access the dropdown menu. In this case, use $H_a\colon\ \mu < \mu_0$.

- The result 'PVal' gives the p-value 0.0161.





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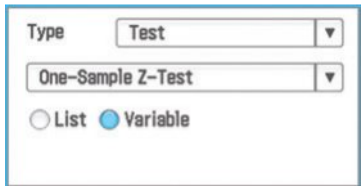
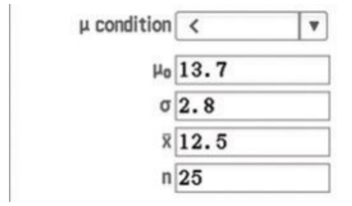
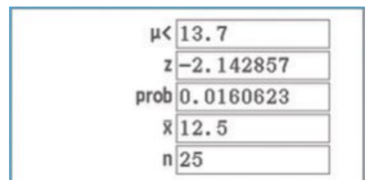
Using the Casio ClassPad

- In Statistics, go to **Calc** > **Test**.
- Select **One–Sample Z–Test** and **Variable**. Tap `Next`.
- Set the μ condition to < and enter the given values as shown below. Tap `Next`.





- The result 'prob' gives the p-value 0.0161.



Strength of evidence

Consider again our IQ example. The more unlikely it is that the sample we observed could be drawn from a population with a mean IQ of 100, the more convinced we are that the sample must come from a population with a higher IQ.

In general, the *smaller the p-value*, the smaller the probability that the sample is from a population with the mean under the null hypothesis, and thus the *stronger the evidence* against the null hypothesis.

How small does the p-value have to be to provide convincing evidence against the null hypothesis? The following table gives some conventions.

p-value		Conclusion
p-value > 0.05		insufficient evidence against H_0
p-value < 0.05	(5%)	good evidence against H_0
p-value < 0.01	(1%)	strong evidence against H_0
p-value < 0.001	(0.1%)	very strong evidence against H_0

For our IQ example, we interpret the p-value of 0.0082 as *strong* evidence against the null hypothesis and in support of our hypothesis that Year 12 mathematics students perform better than the general population on IQ tests.

Example 9

In Example 8, we obtained a p-value of 0.0161. How do we interpret this p-value?

Solution

We interpret this p-value of 0.0161 as *good* evidence against the null hypothesis and in support of the hypothesis that fuel consumption is lower for the new model.

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Statistical significance

Our goal in hypothesis testing is generally to choose between the two hypotheses under consideration. Therefore, we need to decide just how unlikely a sample result must be in order to throw sufficient doubt on the null hypothesis that we should reject it and choose the alternative hypothesis. We need an agreed value against which we can compare the p-value of the test. This value is called the significance level of the test, and is generally denoted by the Greek letter α.

Statistical significance

The **significance level of a test**, α, is the condition for rejecting the null hypothesis:

- If the p-value is less than α, then we reject the null hypothesis in favour of the alternative hypothesis.
- If the p-value is greater than α, then we do not reject the null hypothesis.

The most commonly used value for the significance level is 0.05 (5%), although 0.01 (1%) and 0.001 (0.1%) are sometimes used.

- If the p-value is less than the significance level, say 0.05, then we say that the result is statistically significant at the 5% level.
- If the p-value is greater than the significance level, then we say that the result is not statistically significant at the 5% level.

This approach to hypothesis testing is commonly used.

Example 10

The lifetimes of a certain brand of 'long-life' batteries are normally distributed, with a mean of 240 hours and a standard deviation of 40 hours. After introducing a new manufacturing process, the company has had a number of customer complaints that have led them to believe that the batteries may have a shorter life than before. In order to check the length of battery life, a random sample of 25 batteries was selected and the mean battery life found to be 230 hours.

a Write down the null and alternative hypotheses for this test.

b Determine the p-value for this test.

c Has the lifetime of the batteries decreased? Test at the 5% level of significance.

Solution

a We are using the sample data to decide whether the mean battery life is still 240 hours or has decreased. That is:

$$H_0: \ \mu = 240$$

$$H_1: \ \mu < 240$$

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b If the null hypothesis is true, then $\bar{X}$ is normally distributed with

$$E(\bar{X}) = \mu = 240 \quad \text{and} \quad sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{40}{\sqrt{25}} = 8$$

Therefore

$$\begin{aligned} p\text{-value} &= \Pr(\bar{X} \leq 230 \mid \mu = 240) \\ &= \Pr\left(Z \leq \frac{230 - 240}{8}\right) \\ &= \Pr(Z \leq -1.25) \\ &= 0.1056 \end{aligned}$$

c Since the p-value (0.1056) is greater than the significance level (0.05), we fail to reject the null hypothesis. We do not have enough evidence to conclude that the mean battery life has decreased.

Since we are able to use the normal distribution to determine the p-value for a hypothesis test for the mean of a normal distribution, this hypothesis test is named appropriately:

z-test

The hypothesis test for a mean of a sample drawn from a normally distributed population with known standard deviation is called a ***z*-test**.

Large samples

The central limit theorem tells us that, if the sample size is large enough, then the distribution of the sample mean of any random variable is approximately normal. Thus, a z-test can be used even when the distribution of the random variable is not known, provided the sample size is large enough. (For most distributions, a sample size of 30 is sufficient.)

Skill-sheet

Exercise 16B

Example 7

1 In a certain country, the average number of children per family in the 1990s was 2.4. Researchers believe that the average number of children has decreased over the last 30 years. To test this hypothesis, they select a random sample of 20 families and find the average number of children to be 2.2. Write down the null and alternative hypotheses for this test.

2 A local school reports that its students' GPA scores are normally distributed with a mean of 2.66. After the introduction of a new program designed to improve GPA at the school, they find that the mean GPA for a group of 25 randomly selected students is 2.78. Write down the null and alternative hypotheses which could be used to test the effectiveness of the new program.

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Example 8

3 The weight of a certain species of fish living in the Yerra River is known to be normally distributed with a mean of 60 g and a standard deviation of 4.5 g. Researchers believe that the same species of fish living in the Merry River grow larger than this. To test the hypotheses

$$H_0: \mu = 60$$
$$H_1: \mu > 60$$

the researchers select a random sample of 10 fish from the Merry River, and find that their average weight is 65.8 g. Assuming that the standard deviation of the weight of fish from the Merry River is 4.5 g, determine the p-value for this test.

4 The concentration of a certain chemical pollutant in Rapid River is monitored at a testing station every hour. The concentration is normally distributed with mean $\mu = 34$ ppm (parts per million) and standard deviation $\sigma = 8$ ppm. A representative of a company that discharges liquids into the river is now claiming that they have lowered the mean concentration of the pollutant by using improved filtration devices. A scientist selects 50 random samples of water from various locations along the river and finds a mean concentration of the chemical of 32.5 ppm.

$$H_0: \mu = 34$$
$$H_1: \mu < 34$$

What is the p-value for this test?

Example 9

5 Write a statement interpreting each of the following p-values in terms of the strength of evidence it provides against the null hypothesis:

a p-value = 0.033 **b** p-value = 0.245 **c** p-value = 0.003
d p-value = 0.0049 **e** p-value = 0.0008

6 Suppose that, when testing the following hypotheses, we find a p-value of 0.0355.

$$H_0: \mu = 50$$
$$H_1: \mu < 50$$

What would you conclude based on this p-value?

7 Suppose that, when testing the following hypotheses, we find a p-value of 0.099.

$$H_0: \mu = 10$$
$$H_1: \mu > 10$$

What would you conclude based on this p-value?

8 Suppose that, when testing the following hypotheses, we find a p-value of 0.013.

$$H_0: \mu = 40$$
$$H_1: \mu < 40$$

What would you conclude based on this p-value?

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Example 10

9 The monthly weight gain of a certain breed of cattle (when aged between one year and two years) is normally distributed, with a mean of 2.9 kg and a standard deviation of 1 kg. A researcher believes that a special high-protein feed will result in higher monthly weight gain. To test this hypothesis, she feeds a random sample of 30 cattle with the special feed for a month, and notes that their average weight gain is 3.4 kg.

a Write down the null and alternative hypotheses for this test.

b Determine the *p*-value for this test.

c Can the researcher conclude that the special high-protein feed will increase weight gain in this breed of cattle? Test at the 5% level of significance.

10 According to a census held in 1992, the mean number of residents per household in an inner suburb, Richthorn, was 3.6, with a standard deviation of 1.2. An urban planner believes that the mean has reduced over the last 30 years, due to the increasing number of apartments and townhouses in the suburb. In 2022, a random sample of 11 households was drawn from the suburb and the mean number of residents per household was found to be 2.6.

a Write down the null and alternative hypotheses for this test.

b Determine the *p*-value for this test.

c Can we conclude that the mean size of households has reduced? Use $\alpha = 0.05$.

11 The yearly income for families living in a certain state is normally distributed with a mean of $\mu = \$42\,150$ and a standard deviation of $\sigma = \$10\,000$. A social researcher believes that the residents living in a particular country town have lower incomes than this. She takes a random sample of 20 families from this town and finds that they have an average yearly income of $39 500.

a Write down the null and alternative hypotheses for this test.

b Determine the *p*-value for this test.

c Can the social researcher conclude that average income for families in this town is lower than that for the rest of the state? Test at the 5% level of significance.

12 The length of time taken for a customer to be served at a fast-food outlet has a mean of 3.5 minutes and a standard deviation of 1.5 minutes. After the introduction of a new range of products, the manager feels that the mean time for serving a customer has increased. To test this, he records the service time for a random sample of 50 customers and finds the average service time to be 4.0 minutes.

a Write down the null and alternative hypotheses for this test.

b Determine the *p*-value for this test.

c Can we conclude that the average service time has increased? Use $\alpha = 0.05$.

13 A researcher predicts that sleeping for at least 8 hours before taking a test will improve test scores. The scores for a certain test are known to be normally distributed with a mean of 20 and a standard deviation of 3. She obtains a sample mean of 23 for the test scores of 12 randomly chosen students who had at least 8 hours of sleep. Is this evidence that students who sleep for at least 8 hours before taking the test have better test scores? Test at the 1% level of significance.

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16C One-tail and two-tail tests

In the previous section, we considered only situations where we had a pretty good idea as to the direction in which the mean might have changed. That is, we considered only that the mean IQ of Year 12 mathematics students might be higher than the general population, or that the fuel consumption of the new model car might be lower than the previous model. These are examples of directional hypotheses. When we translate these hypotheses into testable alternative hypotheses, we say that our sample has come from a population with mean more than 100 (for the IQ example) or less than 13.7 (for the fuel-consumption example).

The presence of a ‘less than’ sign ($<$) or a ‘greater than’ sign ($>$) in the alternative hypothesis indicates that we are dealing with a directional hypothesis. Only values of the sample mean more than 100 (for the IQ example) or less than 13.7 (for the fuel-consumption example) will lend support to the alternative hypothesis.

Now suppose that we do not know whether the fuel consumption of our new model car has increased or decreased. In this case, we would hypothesise that the fuel consumption is different for the new model (a non-directional hypothesis). We have to allow for the possibility of the sample mean being less than or greater than 13.7 litres per 100 km.

We express this symbolically by using a ‘not equal to’ sign ($\neq$) in the alternative hypothesis:

$$H_1: \ \mu \neq 13.7$$

The presence of the ‘not equal to’ sign ($\neq$) in the alternative hypothesis indicates that we are dealing with a non-directional hypothesis. A sample mean either greater than 13.7 or less than 13.7 could provide evidence to support this hypothesis.

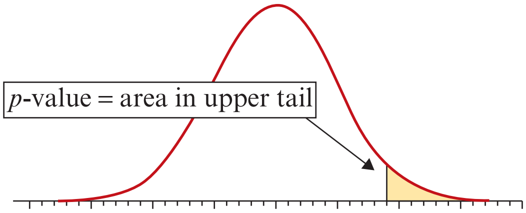
One-tail tests

The directionality of the alternative hypothesis H_1 determines how the p-value is calculated.

For the directional hypothesis

$$H_1: \ \mu > 13.7$$

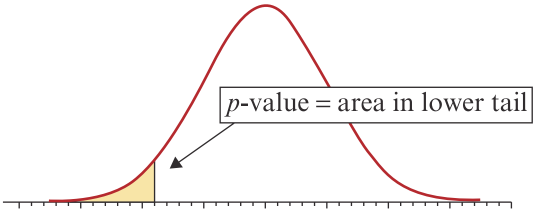
only a sample mean considerably greater than 13.7 will lend support to this hypothesis. Thus, in calculating the p-value, we only consider values in the upper tail of the normal curve.



For the directional hypothesis

$$H_1: \ \mu < 13.7$$

only a sample mean considerably less than 13.7 will lend support to this hypothesis. Thus, in calculating the p-value, we only consider values in the lower tail of the normal curve.



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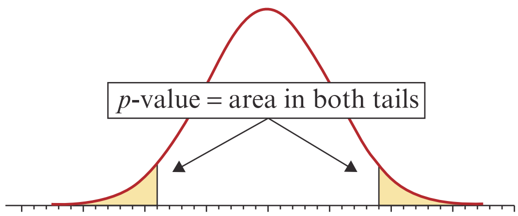
Because the p-values for directional tests are given by an area in just one tail of the curve, these tests are commonly called **one-tail tests**.

Two-tail tests

For the non-directional hypothesis

$$H_1: \ \mu \neq 13.7$$

a sample mean that is either considerably less than 13.7 or considerably greater than 13.7 will lend support to this hypothesis. Thus, in calculating the p-value, we need to consider values in both tails of the normal curve.



Because the p-values for non-directional tests are given by an area in both tails of the curve, these tests are commonly called **two-tail tests**.

As can be seen from the diagram for a two-tail test, the areas in the two tails of the distribution are equal, so the p-value for a two-tail test is twice the p-value for a one-tail test.

One-tail and two-tail tests

- When the alternative hypothesis is directional (< or >), we carry out a one-tail test.
- When the alternative hypothesis is non-directional (≠), we carry out a two-tail test.

p-value (two-tail test) = 2 × p-value (one-tail test)

Example 11

The volume of coffee dispensed by a coffee machine is known to be normally distributed, with a mean of 200 mL and a standard deviation of 5 mL. After a routine service, a test was carried out on the machine to check that it is still functioning properly. A random sample of 15 cups yielded a mean volume of 197.7 mL.

a Write down the null and alternative hypotheses for this test.

b Use the given data to test whether the mean volume of coffee dispensed by the machine is still 200 mL. Test at the 5% level of significance.

Solution

a Since we do not know before we collect the data whether the mean volume is more or less than 200 mL, we should carry out a two-tail test.

$$H_0: \ \mu = 200$$
$$H_1: \ \mu \neq 200$$

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b If the null hypothesis is true, then $\bar{X}$ is normally distributed with

$$\mathrm{E}(\bar{X}) = \mu = 200 \quad \text{and} \quad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{15}} = 1.291$$

Therefore

$$\begin{aligned} p\text{-value} &= 2 \times \Pr(\bar{X} \le 197.7 \mid \mu = 200) \\ &= 2 \times \Pr\left(Z \le \frac{197.7 - 200}{1.291}\right) \\ &= 2 \times \Pr(Z \le -1.782) \\ &= 2 \times 0.0374 \\ &= 0.0748 \end{aligned}$$

Since the p-value (0.0748) is greater than the significance level (0.05), we fail to reject the null hypothesis. We do not have enough evidence to conclude that the mean volume of coffee dispensed has changed.

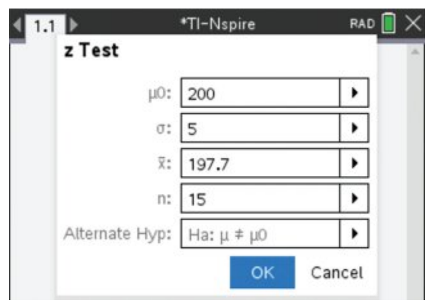
Using the TI-Nspire

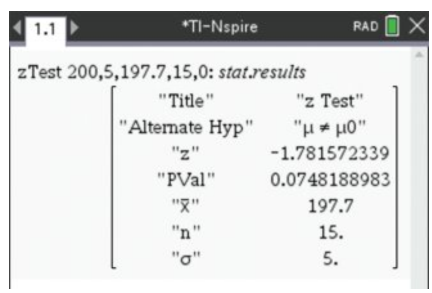
In a **Calculator** page:

- Use menu > **Statistics** > **Stat Tests** > **z Test**.
- If necessary, change the **Data Input Method** to **Stats**.
- Enter the given values as shown and select the form of the alternative hypothesis.

Note: Check carefully the sign required for the **Alternate Hyp**. If necessary, use the right arrow to access the dropdown menu. In this case, use $H_a: \ \mu \ne \mu_0$.

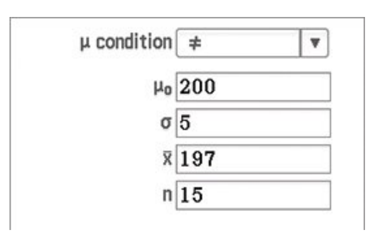
- The result 'PVal' gives the p-value 0.0748.





Using the Casio ClassPad

- In Statistics, go to **Calc** > **Test**.
- Select **One-Sample Z-Test** and **Variable**. Tap Next.
- Set the μ condition to ≠ and enter the given values as shown. Tap Next.



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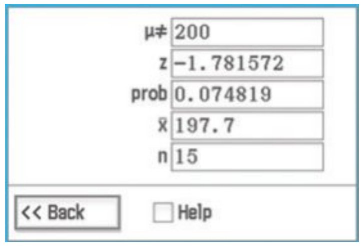
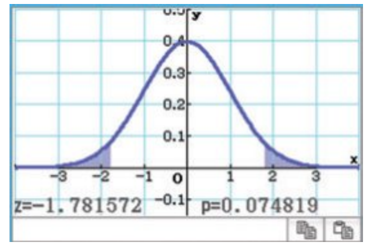
- Tap [⊎] to view the graph.





When do we use a two-tail test?

The decision of whether to use a one-tail test or a two-tail test is important, as it may mean the difference between rejecting or not rejecting the null hypothesis.

In Example 11, we carried out a two-tail test, and thus calculated a p-value of 0.0748. This was greater than the significance level, and thus we had insufficient evidence to conclude that the coffee machine was malfunctioning. If we had carried out a one-tail test, we would have calculated a p-value of 0.0374. This is less than the significance level, and thus we would have had sufficient evidence to conclude that the coffee machine was malfunctioning.

A two-tail test is more conservative than a one-tail test, requiring the sample mean to be more different from the population mean in order to reject the null hypothesis.

In practice, you should only use a one-tail test when you have a very good theoretical reason to expect that the difference will be in a particular direction. In practice, the hypotheses are established before the data is collected, so we cannot use the direction of the difference seen in the data to establish the hypotheses.

Relating a two-tail test to a confidence interval

We established in Section 16A that a 95% confidence interval for the population mean μ is given by

$$\left(\bar{x} - 1.9600\,\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.9600\,\frac{\sigma}{\sqrt{n}}\right)$$

There is a close relationship between confidence intervals and two-tail hypothesis tests. To explain this, we will use the following basic fact about intervals of the real number line:

$$a \in (b - c, b + c) \quad \Leftrightarrow \quad |a - b| < c \quad \Leftrightarrow \quad b \in (a - c, a + c)$$

Now suppose that we are testing the hypotheses

$$H_0\colon\ \mu = \mu_0$$
$$H_1\colon\ \mu \neq \mu_0$$

Then we have

$$\mu_0 \notin \left(\bar{x} - 1.9600\,\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.9600\,\frac{\sigma}{\sqrt{n}}\right) \quad \Leftrightarrow \quad \bar{x} \notin \left(\mu_0 - 1.9600\,\frac{\sigma}{\sqrt{n}},\ \mu_0 + 1.9600\,\frac{\sigma}{\sqrt{n}}\right)$$

Hence, the 95% confidence interval does not contain μ_0 if and only if we should reject the null hypothesis at the 5% level of significance.

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Example 12

a Use the information from Example 11 to determine a 95% confidence interval for the mean volume of coffee dispensed by the coffee machine after the service.

b Use this confidence interval to test the following hypotheses at the 5% level of significance. How does this compare with your answer for Example 11 **b**?

$H_0: \mu = 200$

$H_1: \mu \neq 200$

c Use this confidence interval to test the following hypotheses at the 5% level of significance.

$H_0: \mu = 198$

$H_1: \mu \neq 198$

Solution

a Based on the sample mean $\bar{x} = 197.7$, a 95% confidence interval is $(195.17, 200.23)$.

b Since the interval contains the hypothesised mean value of 200, we would not reject the null hypothesis. This is consistent with the conclusion reached in Example 11 **b**, where we also did not reject the null hypothesis.

c Since the interval contains the hypothesised mean value of 198, we would again not reject the null hypothesis.

In Example 12, we failed to reject both that the population mean is 200 mL, and that the population mean is 198 mL. Remember that in hypothesis testing, just like in a court of law, we cannot conclude that the null hypothesis is true (innocent), only that we do not have sufficient evidence to say that it is false (guilty).

Skill-sheet

Exercise 16C

Example 11

1 A manufacturing process produces ball bearings with diameters that are normally distributed with a mean of 0.50 cm and a standard deviation of 0.04 cm. Ball bearings with diameters that are too small or too large are unacceptable. In order to test whether or not the machine is still producing acceptable ball bearings, a sample of 25 ball bearings was selected at random, and the mean diameter found to be 0.52 cm.

a Write down the null and alternative hypotheses for this test.

b Determine the p-value for this test.

c Can we conclude that the average diameter has changed? Use $\alpha = 0.05$.

2 The weight of sugar in a 2 kg package produced by a food-processing company is normally distributed, with a mean of $\mu = 2.00$ kg and a standard deviation of $\sigma = 0.02$ kg. A new packing machine has been introduced, and a random sample of 20 packages was found to have an average weight of sugar of 1.99 kg. Can the company conclude that the average weight of sugar in a 2 kg package has changed? Use $\alpha = 0.05$.

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3 The mean length of stay in hospital among patients with different diagnoses is of interest to health planners. The number of days that patients suffering from disease A remain in hospital is known to be approximately normally distributed with a mean of 40 days and a standard deviation of 10 days. A random sample of 56 patients with disease A, admitted to a particular hospital, remained in that hospital an average of 43 days. Test, at the 5% level of significance, the hypothesis that the mean length of stay in this hospital is different from the other hospitals.

4 Over the past three years, the number of visitors per day to a city museum had a mean of 484 people and a standard deviation of 42. In order to test whether this has recently changed, the manager collected data on the number of visitors on each of 30 randomly chosen days, and found the mean to be 456. Is this evidence that the average number of daily visitors to the museum has changed? Use $\alpha = 0.01$.

5 In the 1990s, the number of hours of television watched each day by school children in a certain town was known to be approximately normally distributed with a mean of 2 and a standard deviation of 1.2. To see if this has changed, a researcher collected the number of hours of television watched in a day by a randomly selected group of school children. Use these data to test the hypothesis that the average number of hours of television watched by school children is no longer 2. Test at the 5% level of significance.

4	1	1	1	2	4	6	4	1	1
2	2	3	4	6	2	8	2	1	2

6 The lifetime of a certain brand of batteries has a mean of 60 hours and a standard deviation of 10 hours. After implementing a new process, the manufacturer finds that the mean life of a random sample of 30 batteries is 65 hours. Is this evidence that the mean battery life has changed? Use $\alpha = 0.05$.

Example 12

7 Suppose that the number of hours that children sleep per night in a certain community is approximately normally distributed with a mean of 9 hours and a standard deviation of 2 hours. A study was conducted to see if this average has changed. The study was based on a sample of 20 children, and their sample mean number of hours slept was 8.5 hours.

a Does this data provide evidence that the mean number of hours slept per night by children in this community has changed? Use a significance level of 0.05.

b Determine a 95% confidence interval for the mean number of hours slept by children in this community.

c Use this confidence interval to test the hypotheses from part **a** at the 5% level of significance. How does your conclusion compare with your answer for part **a**?

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8 According to the records, the average starting salary for a university graduate in a certain state is \$55 000, with a standard deviation of \$5000. The vice-chancellor of a large university wishes to determine whether their graduates earn more or less than this. A group of 50 randomly selected graduates are surveyed, and their average salary is found to be \$53 445.

a Does this data provide evidence that the average starting salary for a graduate from this university is different from the rest of the state? Use a significance level of 0.05.

b Determine a 95% confidence interval for the average starting salary for a graduate from this university.

c Use this confidence interval to test the hypotheses from part **a** at the 5% level of significance. How does your conclusion compare with your answer for part **a**?

16D Two-tail tests revisited

Before we revisit two-tail tests, it is useful to consider probability statements which include the absolute value function, as illustrated in the following two examples.

Example 13

Suppose that Z is a standard normal random variable. Find $\Pr(|Z| \geq 2)$.

Solution

$$\begin{aligned}\Pr(|Z| \geq 2) &= \Pr(Z \leq -2) + \Pr(Z \geq 2)\\ &= 2 \times \Pr(Z \leq -2)\\ &= 2 \times 0.02275\\ &= 0.0455\end{aligned}$$

Explanation

Since the standard normal distribution is symmetric about 0, we have $\Pr(Z \geq 2) = \Pr(Z \leq -2)$.

In order to apply the symmetry of the normal distribution to determine such probabilities, the random variable must first be standardised.

Example 14

Suppose that X is a normally distributed random variable with mean $\mu = 10$ and standard deviation $\sigma = 5$. Find the probability that a single value of X is at least 2 units from the mean.

Solution

$$\begin{aligned}\Pr(|X - \mu| \geq 2) &= \Pr\left(\left|\frac{X-\mu}{\sigma}\right| \geq \frac{2}{5}\right)\\ &= \Pr(|Z| \geq 0.4)\\ &= 2 \times \Pr(Z \leq -0.4)\\ &= 2 \times 0.3446\\ &= 0.6892\end{aligned}$$

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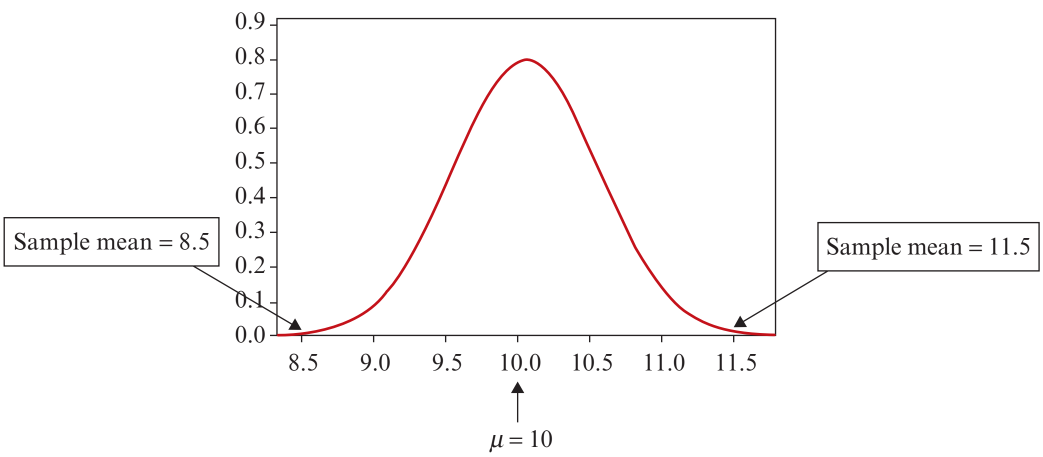
We can use the concept of absolute value to reconsider the definition of the *p*-value for a two-tail test. We perform a two-tail test when addressing the question: 'Has the population mean changed or is it still the same?' That is, we don't know whether the population mean may have increased or decreased from the previously accepted value. Thus an observed value of the sample mean which is either much larger or much smaller than the hypothesised mean can be taken as evidence against the null hypothesis.

Suppose that the fuel consumption for another model of car is known to be normally distributed with a mean of 10 litres per 100 km, and we are again testing the claim that the fuel consumption is different for the newer model of this car. Here we have the hypotheses:

H_0: $\mu = 10$

H_1: $\mu \neq 10$

To test these hypotheses, we compare the observed value of the sample mean with the value of the population mean under the null hypothesis. If the null hypothesis is true, and the standard deviation for fuel consumption is $\sigma = 2.5$ litres per 100 km, then the sample mean $\bar{X}$ for samples of size 25 has the following distribution.



- If we observe a value of the sample mean as large as 11.5, for example, then we are likely to think that the mean fuel consumption for the new model of the car is more than 10 litres per 100 km.
- If we observe a value of the sample mean as small as 8.5, for example, then we are likely to think that the mean fuel consumption for the new model of the car is less than 10 litres per 100 km.

In general terms, we will be persuaded to reject the null hypothesis if the distance between the observed sample mean and the hypothesised population mean is more than would be explained by normal sampling variability. If $\bar{X}$ is the random variable representing the mean of a sample of size *n*, we can write this distance symbolically as $|\bar{X} - \mu|$.

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The p-value for a two-tail test

For a two-tail test, we can define

$$\begin{aligned} p\text{-value} &= \Pr(|\bar{X} - \mu| \geq |\bar{x} - \mu|) \\ &= \Pr\left(|Z| \geq \left|\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right|\right) \end{aligned}$$

where:

- μ is the population mean under the null hypothesis
- $\bar{x}$ is the observed value of the sample mean
- σ is the value of the population standard deviation
- n is the sample size.

Example 15

Suppose that the weight, X kg, of sand in a bag is a normally distributed random variable with a mean of 50 kg and a standard deviation of 1.5 kg. A random sample of 10 bags is taken.

a Find the probability that the mean weight of the 10 bags in the sample differs by 1 kg or more from the population mean of 50 kg.

b Suppose that the mean weight of the 10 bags is 49.1 kg.

i Determine the p-value appropriate to test the hypotheses:

$$H_0\colon\ \mu = 50$$
$$H_1\colon\ \mu \neq 50$$

ii Based on this p-value, what is your conclusion? (Use $\alpha = 0.05$.)

Solution

a
$$\begin{aligned} \Pr(|\bar{X} - \mu| \geq 1) &= \Pr\left(\left|\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right| \geq \frac{\sqrt{10}}{1.5}\right) \\ &= \Pr(|Z| \geq 2.108) \\ &= 2 \times \Pr(Z \leq -2.108) \\ &= 2 \times 0.0175 \\ &= 0.035 \end{aligned}$$

b **i**
$$\begin{aligned} p\text{-value} &= \Pr(|\bar{X} - \mu| \geq |\bar{x} - \mu|) \\ &= \Pr\left(|Z| \geq \left|\frac{49.1 - 50}{1.5/\sqrt{10}}\right|\right) \qquad \text{(standardising)} \\ &= \Pr(|Z| \geq |-1.897|) \\ &= \Pr(|Z| \geq 1.897) \\ &= 2 \times 0.0289 \\ &= 0.0578 \end{aligned}$$

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ii Since the p-value is greater than 0.05, there is insufficient evidence to conclude that the mean weight of the bags of sand is not 50 kg.

Exercise 16D

Example 13

1 Suppose that Z is a standard normal random variable. Find:

a $\Pr(|Z| \geq 1)$ **b** $\Pr(|Z| \leq 0.5)$ **c** $\Pr(|Z| \geq 1.75)$

d $\Pr(|Z| \leq 2.1)$ **e** $\Pr(|Z| \geq 0.995)$

Example 14

2 Suppose that X is a normally distributed random variable with mean $\mu = 5$ and standard deviation $\sigma = 5$. Find $\Pr(|X - \mu| \geq 5)$.

3 Suppose that X is a normally distributed random variable with mean $\mu = 47.5$ and standard deviation $\sigma = 6.4$. Find $\Pr(|X - \mu| \geq 8.5)$.

4 Suppose that X is a normally distributed random variable with mean $\mu = 620$ and variance $\sigma^2 = 100$. Find $\Pr(|X - \mu| \geq 23)$.

Example 15a

5 Suppose that X is a normally distributed random variable with mean $\mu = 10$ and standard deviation $\sigma = 5$. Let $\bar{X}$ represent the mean of a random sample of size 20 drawn from this population. Find the probability that the sample mean differs from the population mean by at least 1 unit.

Example 15b

6 Suppose that X is a normally distributed random variable with mean $\mu = 2.56$ and standard deviation $\sigma = 0.09$. If $\bar{X}$ represents the mean of a random sample of size 30 drawn from this population and $\bar{x}$ represents an observed value of the sample mean, find $\Pr(|\bar{X} - \mu| \geq |\bar{x} - \mu|)$ when:

a $\bar{x} = 2.52$ **b** $\bar{x} = 2.57$

7 Suppose that X is a normally distributed random variable with mean $\mu = 27\,583$ and standard deviation $\sigma = 13\,525$. If $\bar{X}$ represents the mean of a random sample of size 100 drawn from this population and $\bar{x}$ represents an observed value of the sample mean, find $\Pr(|\bar{X} - \mu| \geq |\bar{x} - \mu|)$ when:

a $\bar{x} = 25\,450$ **b** $\bar{x} = 30\,000$

8 Scores for a certain aptitude test are known to be normally distributed with a mean of 30 and a standard deviation of 7. A group of 25 students are randomly selected to take the test. Find the probability that the mean test score of this group differs by 3 points or more from the population mean of 30.

9 The weights of a certain species of fish are normally distributed with a mean of 2 kg and a standard deviation of 0.5 kg. A researcher collects a random sample of 10 fish from a particular lake. Find the probability that the mean weight of this group of fish differs from the population mean by 0.25 kg or more.

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10 To plan its work schedule, a manufacturing company uses the knowledge that the time taken to assemble a certain component is normally distributed with a mean of 15 minutes and a standard deviation of 5 minutes.

a If the actual mean assembly time for 20 randomly selected components is recorded, what is the probability that this will differ by at least 2 minutes from the accepted mean of 15 minutes?

b If a difference of at least 2 minutes was observed, would this cause you to question whether the mean assembly time was actually 15 minutes? Explain your answer in terms of an appropriate hypothesis test. (Use $\alpha = 0.05$.)

c What size difference between a sample mean determined from a sample of size 20 and the hypothesised population mean would lead you to reject the hypothesis that the population mean is 15 minutes?

16E Errors in hypothesis testing

As discussed in Section 16B, the logic of a hypothesis test parallels that of a criminal trial. In a trial, it is always possible that the verdict will be wrong. This can happen in two ways:

- The first is to convict an innocent person. In the language of hypothesis testing, this is called a **Type I error**.
- The second is to let a guilty person go free. In the language of hypothesis testing, this is called a **Type II error**.

The following table shows how Type I and Type II errors can arise in a court of law.

Situation: A person is to be tried for a crime by a jury

H_0: The person is not guilty
H_1: The person is guilty

	Actual situation	
Jury's decision	Did not commit crime (H_0 true)	Did commit crime (H_0 not true)
Guilty (reject H_0)	Type I error	Correct decision
Not guilty (do not reject H_0)	Correct decision	Type II error

Type I and Type II errors are always potentially present in hypothesis testing and are formally defined as follows.

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Type I and Type II errors

- A **Type I error** occurs if we reject the null hypothesis H_0 when it is true.
- A **Type II error** occurs if we do not reject the null hypothesis H_0 when it is false.

Example 16

Suppose that we are testing a new drug for controlling migraine, with hypotheses:

H_0: The drug is ineffective in controlling migraine

H_1: The drug is effective in controlling migraine

Describe the Type I and Type II errors in this situation.

Solution

- A Type I error would be committed if we decided that the drug is effective when it is not.
- A Type II error would be committed if we decided that the drug is not effective when it really does work.

Probability of Type I and Type II errors

We consider the probability of making a Type I or Type II error after we have set up our test (i.e. after deciding on our hypotheses, sample size and significance level), but before we have selected our random sample and determined the sample mean.

Probability of a Type I error

If the null hypothesis is false, then we cannot make a Type I error. If the null hypothesis is true, then

$$\Pr(\text{Type I error}) = \Pr(H_0 \text{ is rejected} \mid H_0 \text{ is true})$$

This is equal to the probability of selecting a random sample that produces a p-value less than α, given that the null hypothesis is true. But the definition of p-value means that this probability is equal to α.

Thus the probability of committing a Type I error is directly related to the significance level of the test, α. If the null hypothesis is true and we decide to reject the null hypothesis for a p-value less than 0.05, then the chance of committing a Type I error is 0.05, or 5%. We can reduce this chance by testing at a lower significance level, say 1%.

Probability of a Type II error

If the alternative hypothesis is true, then

$$\Pr(\text{Type II error}) = \Pr(H_0 \text{ is not rejected} \mid H_1 \text{ is true})$$

This probability cannot be calculated without making some further assumptions about the true value of the population mean μ, as illustrated in the next example.

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Example 17

Student scores on a spelling test are known to be normally distributed with a mean of 20.0 and a standard deviation of 6.0. A new method of teaching spelling has been introduced, which teachers believe will improve the students' results. A random sample of 16 students were taught using the new method, and their average score $\bar{x}$ determined.

a At the 5% level of significance, find the values of the sample mean that would support the conclusion that the mean score achieved using the new teaching method is greater than 20.0.

b Suppose that the true mean score achieved using the new teaching method is 24.0. Find the probability that the teachers conclude that the new teaching method is no better than the original teaching method (that is, a Type II error). Assume $\sigma = 6.0$.

Solution

a We wish to find the values of c such that $\Pr(\bar{X} \geq c \mid \mu = 20.0) < 0.05$.

If the null hypothesis is true, then $\bar{X}$ is normally distributed with

$$\mathrm{E}(\bar{X}) = \mu = 20.0 \quad \text{and} \quad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{6.0}{\sqrt{16}} = 1.5$$

Therefore we require

$$\Pr(\bar{X} \geq c \mid \mu = 20.0) = \Pr\left(Z \geq \frac{c - 20.0}{1.5}\right) < 0.05$$

$$\frac{c - 20.0}{1.5} > 1.6449$$

$$c > 22.467$$

For any value of the sample mean greater than 22.467 (from a sample of size 16), the teachers will conclude that the population mean for the new teaching method is greater than 20.0.

b If the null hypothesis is not true and $\mu = 24.0$, then $\bar{X}$ is normally distributed with

$$\mathrm{E}(\bar{X}) = \mu = 24.0 \quad \text{and} \quad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{6.0}{\sqrt{16}} = 1.5$$

The teachers will conclude that the new teaching method is no better than the original teaching method if $\bar{x} < 22.467$. We have

$$\Pr(\bar{X} < 22.467 \mid \mu = 24.0) = \Pr\left(Z < \frac{22.467 - 24.0}{1.5}\right) = 0.153$$

Thus there is a probability of 0.153 that the teachers will make a Type II error.

Part of the researcher's job is to reduce the probability of committing these errors, as the nature of hypothesis testing (where decisions are made on the basis of probabilities) means that the potential for such errors to occur is always there.

However, this is a balancing act, as decreasing the chance of a Type I error will increase the chance of a Type II error. One way of decreasing the chance of a Type II error (without increasing the chance of a Type I error) is to increase the size of the sample used.

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Exercise 16E

Example 16

1 Researchers test the hypothesis that cattle given a special high-protein feed for a month will have a higher average weight gain than those given regular feed.

a Describe a Type I error in this scenario.

b Describe a Type II error in this scenario.

2 In testing for tuberculosis (TB), there are always a certain proportion of patients who show up as having TB but do not actually have the disease. In medical testing, this is called a 'false positive'.

a In hypothesis testing, does this correspond to a Type I or a Type II error?

b In testing for TB, what would be a 'false negative'? Would this be a Type I or a Type II error?

Example 17

3 Gavin knows that the time it takes him to drive to work is normally distributed with a mean of 28.3 minutes and a standard deviation of 5.1 minutes. Road works have recently been undertaken to remove a level crossing on his route, and he is hoping that his mean travel time has been reduced. He measured his travel time on a random sample of 20 days after the road works, and determined his average travel time, $\bar{x}$ minutes.

a At the 1% level of significance, find the largest value of the sample mean that would support the conclusion that Gavin's mean travel time is now less than 28.3 minutes. That is, find the value of c such that $\Pr(\bar{X} \leq c \mid \mu = 28.3) = 0.01$. Give your answer correct to three decimal places.

b Suppose that the true mean travel time after the road works is 24.0 minutes. Find the probability that Gavin concludes that his travel time has not been reduced. That is, find $\Pr(\bar{X} > c \mid \mu = 24.0)$. Assume the standard deviation is 5.1 minutes. Give your answer correct to three decimal places.

4 Scores on a statewide examination are normally distributed with a mean of 60 and a standard deviation of 14. A new syllabus has been introduced, which teachers feel students might find more difficult, resulting in a lower mean score. A random sample of 100 students were given the examination, and their mean score $\bar{x}$ determined.

a At the 5% level of significance, find the largest value of the sample mean that would support the conclusion that the mean examination score is now less than 60. That is, find the value of c such that $\Pr(\bar{X} \leq c \mid \mu = 60) = 0.05$. Give your answer correct to three decimal places.

b Suppose that the true mean examination score is 58 under the new syllabus. Find the probability that teachers conclude that the new syllabus has not resulted in a lower mean score. That is, find $\Pr(\bar{X} > c \mid \mu = 58)$. Assume the standard deviation is 14. Give your answer correct to three decimal places.

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5 According to the records of a university, the average age of students graduating from a Bachelor degree is 23.6 years, with a standard deviation of 2.8 years. A lecturer thinks that the average age has increased over the last few years. She determines the average age at graduation, $\bar{x}$ years, for a random sample of 100 recent graduates.

a At the 1% level of significance, find the values of the sample mean that would support the conclusion that the average age at graduation is more than 23.6 years. Give your answer correct to three decimal places.

b Suppose that the average age at graduation is now 24.5 years. Find the probability that the lecturer concludes that the average age has not increased (a Type II error). Assume the standard deviation is 2.8 years. Give your answer correct to three decimal places.

6 The daily sales in a boutique are known to be normally distributed with a mean of \$2000 and a standard deviation of \$500. The owner has engaged a marketing firm to promote her boutique, which she hopes will increase sales. To see if sales have increased, a random sample of 10 days was selected and the mean daily sales determined.

a At the 5% level of significance, find the values of the sample mean that would support the conclusion that sales have increased after the marketing campaign. Give your answer correct to the nearest dollar.

b Suppose that the true mean daily sales after the marketing campaign is \$2400. Find the probability that the boutique owner concludes that sales have not increased (a Type II error). Assume the standard deviation is \$500. Give your answer correct to three decimal places.

7 The time taken to complete a task is normally distributed with a mean of 27.5 seconds and standard deviation of 3.2 seconds. In an experiment to investigate the effect of alcohol on manual dexterity, a researcher asked a random sample of 25 adults to complete this task with a blood alcohol content of 0.05%. The researcher has already decided that if the sample mean is more than 29.3 seconds, then he will conclude that the population mean time to complete this task with a blood alcohol content of 0.05% is more than 27.5 seconds.

a What is probability that the researcher rejects the null hypothesis when it is true (a Type I error)? Give your answer correct to four decimal places.

b If the population mean time to complete this task with a blood alcohol content of 0.05% is actually 29.0 seconds, what is the probability that the researcher fails to reject the null hypothesis (a Type II error)? Assume the standard deviation is 3.2 seconds. Give your answer correct to four decimal places.

c Do you think the researcher has set up an effective study? Explain your answer.

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Review

Chapter summary

Assign-ment

Nrich

Confidence intervals for the mean

- The value of the sample mean $\bar{x}$ can be used to estimate the population mean μ. Since this is a single-valued estimate, it is called a **point estimate** of μ.
- An **interval estimate** for the population mean μ is called a **confidence interval** for μ.
- An approximate ***C*% confidence interval** for the population mean μ is given by

$$\left(\bar{x} - z\frac{\sigma}{\sqrt{n}},\ \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$$

  where:
  - z is such that $\Pr(-z < Z < z) = C\%$
  - $\bar{x}$ is a value of the sample mean
  - σ is the value of the population standard deviation
  - n is the size of the sample from which $\bar{x}$ was calculated.
- The values of z (to four decimal places) for commonly used confidence intervals are:
  - 90% $z = 1.6449$
  - 95% $z = 1.9600$
  - 99% $z = 2.5758$

Hypothesis testing for the mean

- When carrying out a hypothesis test for the mean, we are choosing between two scenarios:
  - The **null hypothesis**, H_0, asserts that the sample is drawn from a population with the same mean as before.
  - The **alternative hypothesis**, H_1, asserts that the sample is drawn from a population with a mean which differs from that of the original population.
- Symbolically, we can express the null and alternative hypotheses in one of the following three forms:

$$H_0: \mu = \mu_0 \qquad H_0: \mu = \mu_0 \qquad H_0: \mu = \mu_0$$
$$H_1: \mu > \mu_0 \qquad H_1: \mu < \mu_0 \qquad H_1: \mu \neq \mu_0$$

- The ***p*-value** is the probability of observing a value of the sample statistic as extreme as or more extreme than the one observed, assuming that the null hypothesis is true.
- The **significance level of a test**, α, is the condition for rejecting the null hypothesis:
  - If the p-value is less than α, then we reject the null hypothesis in favour of the alternative hypothesis.
  - If the p-value is greater than α, then we do not reject the null hypothesis.
- The hypothesis test for a mean of a sample drawn from a normally distributed population with known standard deviation is called a ***z*-test**.
- When the alternative hypothesis is directional (< or >), we carry out a **one-tail test**.
- When the alternative hypothesis is non-directional (≠), we carry out a **two-tail test**.
- p-value (two-tail test) = 2 × p-value (one-tail test)
- A **Type I error** occurs if we reject the null hypothesis H_0 when it is true.
- A **Type II error** occurs if we do not reject the null hypothesis H_0 when it is false.

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Technology-free questions

Note: *For Questions 1–3, use the approximation* $\Pr(-2 < Z < 2) \approx 0.95$ *to simplify your calculations.*

1 The number of customers per day at a fast-food outlet is known to be normally distributed with a standard deviation of 50. In a sample of 25 randomly chosen days, a total of 4000 customers were served.

a Give a point estimate for μ, the mean number of customers served per day.

b Calculate an approximate 95% confidence interval for μ.

2 A manufacturer knows that the lifetimes of their light bulbs are normally distributed with a standard deviation of 150 hours.

a What size sample is required in order to ensure that the distance between the sample mean and the population mean is less than 20 hours at the 95% confidence level?

b If the number of light bulbs in the sample were doubled, what would be the effect on the width of the confidence interval?

3 The heights of trees of a particular species are normally distributed with a mean of μ cm and a standard deviation of 200 cm.

a From a random sample of n trees of this species, a 95% confidence interval for μ was calculated to be $(2430, 2530)$. Find the value of the sample mean, $\bar{x}$ cm, and the value of the sample size n.

b What size sample is required to ensure that a 95% confidence interval for μ has a width of 80 cm?

4 Suppose that 60 independent random samples are taken from a large population and a 95% confidence interval for the population mean is computed from each of them.

a How many of the 95% confidence intervals would you expect to contain the population mean μ?

b Write down an expression for the probability that all 60 confidence intervals contain the population mean μ.

5 For each of the following p-values:

i What is the decision if $\alpha = 0.05$?

ii What is the decision if $\alpha = 0.01$?

a $p = 0.1000$

b $p = 0.0250$

c $p = 0.0050$

d $p = 0.0001$

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6 In order to see if the level of background noise reduces concentration, an experiment is carried out as follows. A randomly selected group of students are given puzzles to complete under noisy conditions, and their mean completion time is compared with the mean found when there is no background noise. The p-value is 0.02.

a Write down (in words) the null and alternative hypotheses for this experiment.

b What conclusion can you draw about the statistical significance of the effect of noise level on concentration and why?

c How often are you likely to see a p-value less than 0.02 if the noise level has not reduced concentration?

7 A psychologist studies the effects of praise on happiness. She believes that children who receive praise are happier overall than children who do not receive praise. She measures happiness by counting the number of times that a child smiles in a one-hour period. She knows that children who do not receive praise smile an average of 4 times per hour, with a standard deviation of 0.5, and that these data are normally distributed. She selects a sample of 25 children who she knows receive praise, and finds that they smile an average of 4.3 times per hour. Given that $\Pr(-3 < Z < 3) = 0.9973$:

a Write down appropriate null and alternative hypotheses for this research.

b Determine the p-value for this test, correct to three decimal places.

c At the 1% level of significance, what would be your conclusion?

8 A training college has established that the time taken to learn a specific technology system is normally distributed, with a mean of 50 hours and a standard deviation of 10 hours. A new version of the technology system has been released, and the company believes that the time to learn it has been reduced. A random sample of 49 new employees were trained using the new version, and their mean time to learn the technology was 46 hours. Given that $\Pr(-2.8 < Z < 2.8) = 0.9949$:

a Write down appropriate null and alternative hypotheses for this research.

b Determine the p-value for this test, correct to three decimal places.

c At the 5% level of significance, what would be your conclusion?

9 Will each of the following increase, decrease or have no effect on the p-value of a z-test (if everything else stays the same)?

a The sample size is increased.

b The population variance is decreased.

c The sample variance is doubled.

d The difference between the sample mean and the population mean is decreased.

10 Let Z represent the standard normal random variable. Given that $\Pr(Z > 1.5) = 0.0668$ and $\Pr(Z < -1.7) = 0.0446$, find:

a $\Pr(|Z| > 1.5)$

b $\Pr(|Z| < 1.7)$

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11 To investigate the hypotheses

$H_0: \mu = 20$

$H_1: \mu \neq 20$

a researcher collected a random sample, determined the sample mean $\bar{x}$ and used her results to determine $\Pr(|\bar{X} - \mu| \geq 2) = 0.044$.

a What value of the sample mean did the researcher observe?

b What is the p-value for the hypothesis test, based on her results?

c What conclusion should she reach? (Use $\alpha = 0.05$.)

Multiple-choice questions

1 The amount of money that customers spend at the supermarket each week in a certain town is known to be normally distributed with a standard deviation of \$84. If the average amount spent by a random sample of 50 customers is \$162, then a 95% confidence interval for the population mean is

A (\$39.10, \$128.90) **B** (−\$233.50, \$401.51)
C (\$151.31, \$172.69) **D** (\$138.72, \$185.28)
E (\$15.36, \$84.64)

2 A random sample of 100 observations is taken from a population that is known to be normally distributed with a standard deviation of 25. The sample mean is 45. At the 95% confidence level, the distance between the sample mean and the population mean is at most

A 4.9 **B** 0.49 **C** 0.98 **D** 40.1 **E** 9.8

3 In order to be 95% confident that the sample mean is within 1.4 of the population mean when a random sample is drawn from a population with a standard deviation of 6.7, the minimum size of the sample should be

A 10 **B** 14 **C** 56 **D** 67 **E** 88

4 If 50 random samples are chosen from a population and a 90% confidence interval for the population mean μ is computed from each sample, then on average we would expect the number of intervals which contain μ to be

A 50 **B** 48 **C** 45 **D** 40 **E** none of these

5 If the sample mean remains unchanged, then an increase in the level of confidence will lead to a confidence interval which is

A narrower **B** wider **C** unchanged **D** asymmetric
E cannot be determined from the information given

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6 A confidence interval is to be used to estimate the population mean μ based on a sample mean $\bar{x}$. To decrease the width of the confidence interval by $\frac{2}{3}$, the sample size must be increased by a factor of

A $\frac{2}{3}$ **B** $\frac{9}{4}$ **C** 3 **D** 9 **E** 16

7 Which of the following statements is true?

I The centre of a confidence interval is a population parameter.
II The higher the level of confidence, the smaller the confidence interval.
III The confidence interval is a type of point estimate.
IV The true value of a population mean is an example of a point estimate.

A I only **B** II only **C** III only **D** IV only **E** none of these

8 If a researcher increases her sample size by a factor of 4, then the width of a 95% confidence interval would

A increase by a factor of 2
B increase by a factor of 4
C decrease by a factor of 2
D decrease by a factor of 4
E none of these

9 A 95% confidence interval for the mean attention span of an audience in a lecture, based on a sample of 16 participants, was found to be from 6.7 minutes to 10.5 minutes. The standard deviation of the attention span of the audience is closest to

A 8.6 minutes **B** 3.8 minutes **C** 1.0 minutes
D 1.9 minutes **E** 3.9 minutes

10 A significance level of 0.05 means that

A if H_0 is true, then there is a 5% chance that it will be wrongly rejected
B there is more than a 95% chance that H_0 is not true
C if you retain the null hypothesis, then you have at least a 5% chance of making the wrong decision
D if you make a Type II error, there is a 95% chance of making a Type I error as well
E the probability of making a Type I error is less than the probability of making a Type II error

11 Suppose the null hypothesis is that you are not guilty of murder. If you are found 'not guilty', then

A a Type I error is possible
B a Type II error is possible
C there is no error
D both A and B
E none of these

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12 If the p-value for a test is less than 0.01, then

A you have strong evidence that the null hypothesis is true

B if the null hypothesis is true, then fewer than 1% of samples would give a result as extreme as or more extreme than the observed result

C there is a 1% chance that both hypotheses are true

D you have failed to reject H_0

E there is more than a 99% chance that H_0 is not true

13 A local gymnasium instructor found that, during a recent power blackout, the intensity levels of the aerobics participants seemed higher than usual. The average intensity levels (measured in heartbeats per minute) in a well-lit room has been established as normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 10$. In a follow-up study, an aerobics class was run in the dark, and the mean intensity level for the 25 participants was 76.5. The p-value for the two-tail test is closest to

A 0.0017 **B** 0.9991 **C** 0.0012 **D** 3.25 **E** 2.0

14 Suppose that you are a medical researcher who is trying to establish that the new drug you have developed is more effective than the existing drug. Which outcome would you most prefer?

A $p < 0.01$ **B** $p < 0.05$ **C** $p > 0.05$ **D** $\alpha = 0.05$ **E** $\alpha = 0.01$

15 A fast-food franchiser is considering building a restaurant at a certain location. Based on financial analysis, a site is acceptable only if the number of pedestrians passing the location averages at least 100 per hour. A random sample of 50 hours produced a sample mean of $\bar{x} = 96$ pedestrians. If the standard deviation is $\sigma = 21$, what is the probability that a sample mean as small as or smaller than 96 would be observed if the average number of pedestrians is 100 per hour?

A 0.05 **B** 0.9109 **C** 0.1780 **D** 0.4245 **E** 0.0890

16 In a one-sided statistical test at the 1% level of significance, it would be concluded that

A H_0 should not be rejected if $p = 0.006$

B H_0 should be rejected if $p = 0.008$

C H_0 should be rejected if $p = 0.06$

D H_0 should not be rejected if $p \neq 0.01$

E H_0 should be rejected if $p > 0.05$

17 Which of the following is a two-tail test?

A a test to see whether women smoke cigarettes more than men

B a test to see whether exercise promotes weight loss

C a test to see whether the mean age of Year 12 students is 18 years old

D a test to see whether test scores of students who have tutors are higher on average than those of high-income students

E a test to see whether people who are stressed tend to eat more

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18 The number of hours that people sleep at night in a certain community is normally distributed with a mean of 8 hours and a standard deviation of 2 hours. A study was conducted to see whether Year 12 students sleep less than 8 hours on average. The study was based on a sample of 25 students, and the sample mean was 7.5 hours. What is the p-value for this test?

A 0.932 **B** 0.2113 **C** 0.4013 **D** 0.8944 **E** 0.1056

19 When carrying out a z-test, increasing the sample size (and keeping everything else constant) has the effect of

A increasing the chance of a Type I error **B** increasing the chance of a Type II error
C increasing the p-value **D** decreasing the p-value
E increasing the level of significance

20 Suppose that X is a normally distributed random variable with mean $\mu = 34$ and variance $\sigma^2 = 10$. If $\bar{X}$ represents the mean of a random sample of size 12 drawn from this population and $\bar{x} = 31.5$ is an observed value of the sample mean, then $\Pr(|\bar{X} - \mu| \geq |\bar{x} - \mu|)$ is equal to

A 0.0031 **B** 0.0062 **C** 0.1932 **D** 0.2145 **E** 0.3865

21 Let Z be a standard normal random variable and let $a > 0$. If $\Pr(Z < a) = k$, then $\Pr(|Z| > a) =$

A k **B** $2k$ **C** $1 - k$ **D** $2(1 - k)$ **E** $2(k - 1)$

Extended-response questions

1 **a** Researchers have established that the time it takes for a certain drug to cure a headache is normally distributed, with a mean of 14.5 minutes and a standard deviation of 2.4 minutes. Find the probability that in a random sample of 20 patients, the mean time for the headache to be cured is between 12 and 15 minutes.

b The researchers modify the formula for the drug, and carry out some trials to determine the new mean time for a headache to be cured.

i Determine a 95% confidence interval for the mean time for a headache to be cured, if the average time it took for the headache to be cured in a random sample of 20 subjects was 12.5 minutes. (Assume that $\sigma = 2.4$.)

ii Determine a 95% confidence interval for the mean time for a headache to be cured, if the average time it took for the headache to be cured in a random sample of 50 subjects was 13.5 minutes. (Assume that $\sigma = 2.4$.)

iii Determine a 95% confidence interval for the mean time for a headache to be cured based on the combined data from the two studies in **i** and **ii**.

iv The researchers want to determine a 95% confidence interval for the mean time for a headache to be cured that has a width of at most 1 minute. What size sample should the researchers use?

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2 A sociologist asked randomly selected workers in two different industries to fill out a questionnaire on job satisfaction. The answers were scored from 1 to 20, with higher scores indicating greater job satisfaction.

a The scores on the questionnaire for industry A are known to be normally distributed with a standard deviation of 2.2. The mean score on the questionnaire from a random sample of 30 people from industry A was 15.3. Find a 95% confidence interval for μ_A, the mean satisfaction score in industry A.

b The scores on the questionnaire for industry B were known to be normally distributed with a mean of 11.3 and a standard deviation of 3.1. The management team in industry B has now introduced programs to increase job satisfaction. To test the effectiveness of the new programs, they selected a random sample of 35 people from industry B, and found their mean score on the questionnaire to be 12.1.

i Write down suitable null and alternative hypotheses for this test.

ii Determine the p-value for this test, correct to three decimal places.

iii At the 5% level of significance, what would be your conclusion?

c Another random sample of 25 people from industry B is given the questionnaire.

i At the 5% level of significance, find the range of values for the sample mean that would support the hypothesis that the mean job satisfaction score for industry B is more than 11.3. Give your answer correct to three decimal places.

ii Suppose that the new mean for industry B is in fact 13.0. Find the probability that the management team conclude that their new programs have not been effective. Assume the standard deviation is 3.1. Give your answer correct to three decimal places.

3 The time that it takes to assemble a bookcase is normally distributed, with a mean of 42 minutes and a standard deviation of 5 minutes. The manufacturers have developed a new model of the bookcase, which they claim is assembled more quickly. A random sample of 20 of the new bookcases are assembled, and the sample mean is found to be 40 minutes.

a Find a 99% confidence interval for the mean time to assemble the new bookcase.

b The manufacturer decides to carry out a hypothesis test to determine whether the mean time has decreased.

i Write down the null and alternative hypotheses for this test.

ii Find the p-value for this test, correct to three decimal places.

iii If the significance level is 0.05, what is your conclusion based on this p-value?

c At the 5% level of significance, find the largest value of the mean assembly time for 20 bookcases that would support the null hypothesis being rejected. Give your answer correct to three decimal places.

d Suppose that the mean time to assemble the new model bookcase is actually 37 minutes. What is the probability that the null hypothesis will not be rejected? Give your answer correct to three decimal places.

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4 For a certain model of phone, the length of time between battery charges is normally distributed with mean 70 hours and standard deviation 10 hours. The manufacturer brings out a new model of the phone with an enhanced battery, which they claim lasts longer than the previous battery. To test this claim, a random sample of 25 phones was selected, and the average time between charges was found to be 75 hours.

a Write down the null and alternative hypotheses for this test.

b Find the p-value for this test, correct to three decimal places.

c If the significance level is 5%, what is your conclusion based on this p-value?

d What is the largest value of the mean time between charges for 25 phones for which the null hypothesis would not be rejected at the 5% level of significance? Give your answer correct to three decimal places.

e Suppose that the actual mean time between charges for the new model is k hours, where $k > 70$. If the probability that the null hypothesis will not be rejected at the 5% level of significance is equal to 0.20, what is the value of k? Give your answer correct to one decimal place.

5 When using the recommended fertiliser, the monthly growth in a certain species of plant is normally distributed with a mean of 8.2 cm and a standard deviation of 1.2 cm. A gardener decides to start using a new brand of fertiliser on his plants, but he is unsure what effect this will have on their growth. To investigate, he uses the new fertiliser for a month on a sample of 36 plants, and finds their mean growth to be 8.7 cm.

a **i** Write down suitable null and alternative hypotheses for this test.

ii Determine the p-value for this test, correct to four decimal places.

iii At the 5% level of significance, what would be your conclusion?

b Another gardener wants to conduct the same test, and she also uses the new fertiliser for a month on a sample of 36 plants.

i Find the value of c such that $\Pr(\bar{X} \leq c \mid \mu = 8.2) = 0.025$. Give your answer correct to three decimal places.

ii Find the value of d such that $\Pr(\bar{X} \geq d \mid \mu = 8.2) = 0.025$. Give your answer correct to three decimal places.

iii Suppose that the actual mean monthly growth using the new fertiliser is 8.6 cm. Find the probability that the second gardener concludes that the new fertiliser changes the mean monthly growth. Assume the standard deviation is 1.2. Give your answer correct to three decimal places.

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17

Revision of Chapters 15–16

17A Technology-free questions

1 Suppose that X is a random variable with mean $\mu = 5$ and variance $\sigma^2 = 16$.

a Let $Y = 3X - 1$. Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

b Let $U = 3 - 2X$. Find $\mathrm{E}(U)$ and $\mathrm{sd}(U)$.

c Let $V = 2Y - 2U$. Find $\mathrm{E}(V)$ and $\mathrm{Var}(V)$.

2 A machine produces components in the shape of a cone. The base of the cone is a circle of area 1.5 cm^2 and the height of the cone is X cm, where X is a random variable with a mean of 2 cm and a standard deviation of 0.02 cm. The volume of a cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

where r is the radius of the base and h is the height.

a Find the expected volume of the components (in cm^3).

b Find the variance of the volume of the components (in cm^6).

3 A factory produces nuts and bolts. The mass of each nut is normally distributed with mean 5 g and standard deviation 0.2 g. The mass of each bolt is normally distributed with mean 20 g and standard deviation 0.1 g. For distribution, two nuts are screwed onto each bolt. What are the mean and standard deviation of the resulting total mass?

4 The random variables X and Y are independent. The mean and variance of X are 2 and 3 respectively, while the mean and variance of Y are -2 and 3 respectively. Find the values of $a, b \in \mathbb{N}$ if the mean and variance of $aX + bY$ are -2 and 75 respectively.

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5 At a farmers' market, the bags of mandarins have a mean weight of 500 g, with a variance of 25 g^2, and the bags of passionfruit have a mean weight of 160 g, with a variance of 10 g^2.

a If there are five mandarins in each bag, what are the mean and variance of the weight of an individual mandarin?

b If there are eight passionfruit in each bag, what are the mean and variance of the weight of an individual passionfruit?

c If you buy two bags of mandarins and three bags of passionfruit from the market, what are the mean and variance of the total weight of the five bags?

6 A random variable X has probability density function f given by

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a Let $Y = 3X$. Find the mean and variance of Y.

b Let X_1, X_2 and X_3 be independent random variables, each with the same probability density function as X. If $V = X_1 + X_2 + X_3$, find the mean and variance of V.

7 The final marks in a chemistry examination are normally distributed with mean 68 and standard deviation 8. A random sample of 16 students are selected and their mean mark calculated. What are the mean and standard deviation of this sample mean?

8 The cost of a certain brand of phone has a mean of $\mu = \$1600$ and a standard deviation of $\sigma = \$200$. A sample of phones is selected and the average price of the phones in the sample, $\$\bar{X}$, determined. In order for the standard deviation of the average price to be less than \$25, how many phones should be included in the sample?

9 A random sample of 36 fish was removed from a large nursery tank. The average weight of these fish was 84.0 grams, and the population weight is known to have a standard deviation of 12.0 grams. Using $\Pr(-1.96 < Z < 1.96) = 0.95$, find an approximate 95% confidence interval for the mean weight of the fish in the tank.

10 For this question, use the approximation $\Pr(-2 < Z < 2) \approx 0.95$ in your calculations. The adult weights of a particular species of fish are normally distributed with a mean of μ g and a standard deviation of 25 g.

a From a random sample of n fish of this species, a 95% confidence interval for μ was calculated to be $(430, 450)$. Find the value of the sample mean, $\bar{x}$ g, and the value of the sample size n.

b What size sample is required to ensure that a 95% confidence interval for μ has a width of 4 g?

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11 Consider the following hypotheses:

H_0: $\mu = 20$

H_1: $\mu < 20$

a If the p-value is 0.045, what is your conclusion (at the 5% level of significance)?

b Suppose that the same data are used to carry out a two-tail test.

i What is the p-value for the two-tail test?

ii What is your conclusion for the two-tail test (at the 5% level of significance)?

12 The time that students take to complete a puzzle is normally distributed, with a mean of 95 seconds and a standard deviation of 15 seconds. Researchers believe that students who meditate for 20 minutes before they do the puzzle will complete it more quickly. A random sample of 25 students, who first meditated, completed the puzzle in an average time of 89 seconds. Given that $\Pr(-2 < Z < 2) = 0.9545$:

a Write down appropriate null and alternative hypotheses for this research.

b Determine the p-value for this test, correct to three decimal places.

c At the 5% level of significance, what would be your conclusion?

17B Multiple-choice questions

1 Let V and W be independent normally distributed random variables, where V has a mean of 4 and a variance of 2, and W has a mean of 3 and a variance of 5. Define the random variable $X = 2V - 2W + 3$. In terms of the standard normal random variable Z, the probability $\Pr(X > 4)$ is equal to

A $\Pr\left(Z > \frac{-1}{\sqrt{28}}\right)$ **B** $\Pr\left(Z > \frac{1}{\sqrt{28}}\right)$ **C** $\Pr\left(Z < \frac{-1}{\sqrt{28}}\right)$ **D** $\Pr\left(Z > \frac{-1}{\sqrt{14}}\right)$ **E** $\Pr\left(Z > \frac{-1}{\sqrt{31}}\right)$

2 The random variable X is normally distributed with mean 58 and standard deviation 8. The random variable Y is normally distributed with mean 52 and standard deviation 6. If X and Y are independent, then $\Pr(X < Y)$ is equal to

A 0.3341 **B** 0.2743 **C** 0.0013 **D** 0.7257 **E** 0.6659

3 The weight of a certain type of large dog is normally distributed with mean 42 kg and standard deviation 4.5 kg. The probability that the average weight of 20 of these dogs, randomly selected, is between 38 kg and 43 kg is closest to

A 0.8398 **B** 0.4009 **C** 0.7564 **D** 0.6862 **E** 0.9332

4 The weight of a large loaf of bread is normally distributed with mean 420 g and standard deviation 30 g. The weight of a small loaf of bread is normally distributed with mean 220 g and standard deviation 10 g. The mean, μ g, and standard deviation, σ g, of the total weight of 5 large loaves and 10 small loaves are

A $\mu = 4300,\ \sigma = 10\sqrt{55}$ **B** $\mu = 4300,\ \sigma = 250$ **C** $\mu = 4300,\ \sigma = 50\sqrt{13}$
D $\mu = 5300,\ \sigma = 250$ **E** $\mu = 5300,\ \sigma = 10\sqrt{55}$

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5 The mean cost of a 500 g loaf of bread is \$2.84, with a standard deviation of \$0.88. A random sample of 16 loaves of bread is selected. The mean and standard deviation of the mean cost of the loaves in this sample are

A mean \$2.84, sd \$0.88 **B** mean \$2.84, sd \$0.22 **C** mean \$0.71, sd \$0.22
D mean \$45.44, sd \$14.08 **E** mean \$2.84, sd \$0.06

6 The mean cost of a 1 kg bag of bananas is \$3.68, with a standard deviation of \$1.05. The probability that the mean cost of a random sample of 25 bags of bananas is less than \$3.60 is

A 0.3516 **B** 0.6484 **C** 0.4696 **D** 0.5304 **E** 0.0939

7 For a statistician to be 99% confident that the sample mean will differ by less than 0.3 units from the population mean, given that the population standard deviation is 1.365, the minimum sample size should be

A 56 **B** 80 **C** 113 **D** 138 **E** 145

8 The time taken to complete task A is normally distributed with a mean of 5 hours and a standard deviation of 1 hour. The time taken to complete task B is independent of the time taken to complete task A, and has a mean of 8 hours and a standard deviation of 1.5 hours. A tradesperson wishes to quote a total completion time for both tasks that he will be 99% certain to achieve. This quote, in hours, would be closest to

A 14.5 **B** 15.2 **C** 16.5 **D** 17.2 **E** 18.5

9 A 98% confidence interval for the mean amount spent per person in a restaurant, based on a sample of 21 patrons, was found to be from \$32.00 to \$45.00. The standard deviation of the amount spent per person is closest to

A \$14.89 **B** \$50.67 **C** \$6.40 **D** \$38.50 **E** \$12.80

10 A production line is designed to produce bicycle wheels with mean diameter 42 cm. It is known that the diameters are normally distributed with standard deviation 1.5 cm. In order to test the hypothesis that the mean diameter is indeed 42 cm, a random sample of 25 wheels is selected. The sample mean is found to be 41.5 cm. The p-value for a two-tail test is closest to

A 0.9522 **B** 0.0956 **C** 0.0372 **D** 0.0477 **E** 0.0556

11 The VCAA scores in all studies (the population) have a mean of 30 and a standard deviation of 7. A Specialist Mathematics teacher takes her class of 15 students to be a random sample. Her class mean score was 36.2. If these data are used to test the hypotheses $H_0: \mu = 30$ and $H_1: \mu > 30$, with $\alpha = 0.05$, then the p-value for her class and the conclusion are

A $p = 0.0030$ and reject H_0 **B** $p = 0.0030$ and do not reject H_0
C $p = 0.3000$ and reject H_0 **D** $p = 0.0003$ and reject H_0
E $p = 0.0003$ and do not reject H_0

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Revision

12 Which of the following statements is true about hypothesis testing for μ with known σ?

A The hypothesis test can be conducted even if α is unknown.

B The p-value is independent of H_0.

C The p-value is a statistic calculated as $(\bar{x} - \mu)\sqrt{n}/\sigma$.

D If the p-value is greater than α, where α is the significance level, this is insufficient evidence to reject H_0.

E The hypothesis test is only valid if the population from which the sample is selected is normally distributed.

13 A confidence interval is to be used to estimate the population mean μ based on a sample mean $\bar{x}$. To decrease the width of the confidence interval by 75%, the sample size must be multiplied by a factor of

A $\frac{3}{4}$ **B** $\frac{1}{4}$ **C** 4 **D** 16 **E** 64

14 An engineer is checking the quality of a shipment of electronic components. If the rating (on a scale of 1 to 10) is less than 7, then the shipment will be rejected. A random sample of 12 components is selected, and their quality level is rated by the engineer. He determines that the average rating for this sample is 6.8. He must now decide whether to reject the shipment. In this situation, the alternative hypothesis H_1 is

A $\mu < 6.8$ **B** $\mu \neq 7$ **C** $\mu \neq 6.8$ **D** $\mu > 6.8$ **E** $\mu < 7$

15 In a one-sided statistical test at the 5% level of significance, it would be concluded that

A H_0 should not be rejected if $p = 0.026$

B H_0 should be rejected if $p = 0.048$

C H_0 should be rejected if $p = 0.06$

D H_0 should not be rejected if $p \neq 0.05$

E H_0 should not be rejected if $p < 0.05$

17C Extended-response questions

1 Let $X_1, X_2, \dots, X_{30}$ be independent random variables, each having a probability distribution given by

$$\Pr(X = x) = 0.4^{x-1} \times 0.6 \quad \text{for } x = 1, 2, 3, \dots$$

with $\mathrm{E}(X) = \frac{5}{3}$ and $\mathrm{Var}(X) = \frac{10}{9}$. Find:

a $\Pr(X = 4)$ **b** $\Pr(X > 4)$

Given that $Y = X_1 + X_2 + \cdots + X_{30}$, and using the central limit theorem, find:

c $\mathrm{E}(Y)$ **d** $\mathrm{Var}(Y)$ **e** $\Pr(Y > 60)$, correct to two decimal places.

2 The volume of liquid in a 1 litre bottle is normally distributed with a mean of μ mL and a standard deviation of σ mL. In a randomly selected bottle, there is a probability of 0.057 that there is more than 1.02 litres. In a randomly selected six-pack of bottles, there is a probability of 0.033 that the mean volume of liquid is more than 1.01 litres. Find the values of μ and σ.

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3 Suppose that people's weights, X kg, are normally distributed with a mean of 80 kg and a standard deviation of 20 kg.

a Find k_1 and k_2 such that, for a person chosen at random, $\Pr(k_1 < X < k_2) = 0.95$.

b Suppose that we plan to take a random sample of 20 people and determine their mean weight, $\bar{X}$ kg. Find c_1 and c_2 such that $\Pr(c_1 < \bar{X} < c_2) = 0.95$.

c Suppose that researchers are no longer sure that the mean weight of people is 80 kg. They believe that it might have changed, due to changes in diet. To investigate this possibility, they take a random sample of 20 people and determine a sample mean of 85 kg. Based on this value (and a standard deviation of 20 kg), determine a 95% confidence interval for the mean.

4 A machine packs sugar into 1 kg bags. A random sample of 10 bags was taken and their weights, in grams, were as follows:

1000, 998, 1005, 999, 1002, 1001, 999, 1000, 1003, 1001

It is suspected that the machine overfills the bags and needs adjustment. It is known that the weights of the bags are normally distributed with a standard deviation of 1.75 g.

a The manufacturer decides to carry out a hypothesis test to determine whether the mean weight of the bags is more than 1 kg.

i Write down the null and alternative hypotheses for this test.

ii Find the p-value for this test, correct to three decimal places.

iii If the significance level is 0.05, what is your conclusion based on this p-value?

b At the 5% level of significance, find the largest value of the mean weight for a random sample of 10 bags that would support the null hypothesis being rejected. Give your answer correct to two decimal places.

c Suppose that the actual mean weight is 1.002 kg. What is the probability that the null hypothesis will not be rejected? Give your answer correct to three decimal places.

5 Linh rides her bike to work each day. She knows that the time it takes is normally distributed with a mean of 55 minutes and a standard deviation of 5 minutes.

a A new bike path has recently been opened, and Linh thinks that her average riding time may have decreased. In the week following the opening of the new bike path, she determines that her average riding time for the 10 trips is 50 minutes.

i Write down the null and alternative hypotheses for a test to determine if Linh's average riding time has decreased.

ii Find the p-value for this test, correct to four decimal places.

iii If the significance level is 0.05, what is your conclusion based on this p-value?

b At the 5% level of significance, find the largest value of the average riding time for 10 trips that would support the null hypothesis being rejected. Give your answer correct to three decimal places.

c Suppose that Linh's average riding time is now actually 52 minutes. What is the probability that the null hypothesis will not be rejected? Give your answer correct to three decimal places.

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17D Algorithms and pseudocode

You may like to use a device to implement the algorithms in this section; see the coding appendices in the Interactive Textbook for instructions.

1 Three dice

Let X_1, X_2 and X_3 represent the numbers observed when three fair dice are rolled. In this question, we consider the probability distribution of the sum $X_1 + X_2 + X_3$.

The pseudocode function given on the right outputs a sorted list of the values of $X_1 + X_2 + X_3$ for all possible combinations of values of X_1, X_2 and X_3.

```
define sums():
  S ← []
  for i from 1 to 6
      for j from 1 to 6
          for k from 1 to 6
              append i + j + k to S
          end for
      end for
  end for
  sort S into ascending order
  return S
```

The list produced by this function is

$$sums() = [3, 4, 4, 4, 5, 5, 5, 5, 5, 5, \dots, 17, 17, 17, 18]$$

The pseudocode function given on the right converts a sorted list of values into a frequency table.

```
define freq(A):
  T ← []
  n ← length(A)
  for i from 1 to n
      if i = 1 or A[i] ≠ A[i − 1] then
          count ← 1
          for j from i + 1 to n
              if A[j] = A[i] then
                  count ← count + 1
              end if
          end for
          append [A[i], count] to T
      end if
  end for
  return T
```

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If we apply the second function to the sorted list produced by the first function, then we obtain

$$freq(sums(\,)) = [[3, 1], [4, 3], [5, 6], [6, 10], [7, 15], [8, 21], \dots, [17, 3], [18, 1]]$$

This corresponds to the following frequency table for the values of $X_1 + X_2 + X_3$.

Sum	3	4	5	6	7	8	...	17	18
Frequency	1	3	6	10	15	21	...	3	1

a Perform a desk check for the function *freq*(A) using $A = [1, 1, 2, 3, 3]$.

b By modifying the function *freq*(A), write a new pseudocode function *relfreq*(A) that converts a sorted list of values into a relative frequency table.

c Using the two pseudocode functions *sums*( ) and *relfreq*(A), write an algorithm in pseudocode to find:

i $\Pr(X_1 + X_2 + X_3 = 5)$ **ii** $\Pr(7 \le X_1 + X_2 + X_3 \le 12)$ **iii** $E(X_1 + X_2 + X_3)$

2 Sum of two random variables

In this question, we consider the sum of two independent discrete random variables X_1 and X_2 that do not have uniform distributions.

x_1	0	1	2	3
$\Pr(X_1 = x_1)$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

x_2	0	1	2
$\Pr(X_2 = x_2)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

We can represent these two distributions with the following four lists:

$X1 = [0, 1, 2, 3]$ $X2 = [0, 1, 2]$

$P1 = [3/8, 1/4, 1/4, 1/8]$ $P2 = [1/2, 1/3, 1/6]$

a Write a pseudocode function *sums*(A, B) that inputs two lists of values A and B, and outputs a sorted list of all the possible sums of one value from A and one value from B. (Hint: Look at the function *sums*( ) from Question 1.)

b Write a pseudocode function *unique*(A) that inputs a sorted list of values A, and outputs the list A with repetitions removed. (Hint: Look at the function *freq*(A) from Question 1.)

We can apply these two functions to find the list of unique values of $X_1 + X_2$:

$$Y = unique(sums(X1, X2)) = [0, 1, 2, 3, 4, 5]$$

We now aim to calculate the probability distribution of $X_1 + X_2$:

$$D = [[0, 0.1875], [1, 0.25], [2, 0.2708\dot{3}], [3, 0.1875], [4, 0.08\dot{3}], [5, 0.0208\dot{3}]]$$

c Using pseudocode, write an algorithm that produces the probability distribution D of $X_1 + X_2$. (Hint: You can do this using three nested `for` loops, where the outer loop goes over the list Y and the inner two loops go over the lists $X1$ and $X2$.)

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Revision

18 Revision of Chapters 1–17

18A Technology-free questions

1 Prove by induction that for all $n \in \mathbb{N}$:

a $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

b $\begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}^n = \begin{bmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{bmatrix}$

2 Consider the points $A(3, 2, 4)$, $B(-2, 3, 5)$ and $C(1, 0, 2)$ in three-dimensional space.

a Express the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ in component form.

b Find $\overrightarrow{CA} \times \overrightarrow{CB}$.

c Find the area of the triangle ABC.

d Find the area of the parallelogram spanned by the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

3 Let ℓ be the line with vector equation $\boldsymbol{r} = -8\boldsymbol{i} + 4\boldsymbol{j} + 10\boldsymbol{k} + t(\boldsymbol{i} + 7\boldsymbol{j} - 2\boldsymbol{k})$, $t \in \mathbb{R}$, and let Π be the plane with Cartesian equation $12x - 2y - z = 17$. Show that the line ℓ and the plane Π do not intersect.

4 The line given by $\boldsymbol{r} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} + t(-3\boldsymbol{i} + 9\boldsymbol{j} + \boldsymbol{k})$, $t \in \mathbb{R}$, crosses the x–z plane and the y–z plane at the points A and B respectively. What is the length of the line segment AB?

5 Find a vector equation that represents the line of intersection of the planes defined by the equations $3x - y + 2z = 100$ and $x + 3y = 45$.

6 Determine the distance between the two parallel lines $5x + 5y - 11 = 0$ and $x + y - 1 = 0$ in the Cartesian plane.

7 Let n be a natural number, and consider the statement:

- If n has an odd number of factors, then n is a perfect square.

For this statement, write:

a the converse **b** the contrapositive **c** the negation.

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8 Prove by induction that $2^{n+2} + 3^{2n+1}$ is divisible by 7, for each positive integer n.

9 **a** Find the gradient of the curve $2y^2 - xy^3 = 8$ at the point where $y = -1$.

b Find the length of the parametric curve defined by $x = 3\sin(2t)$ and $y = -3\cos(2t)$, for $\frac{\pi}{6} \le t \le \frac{2\pi}{3}$.

10 Let $f(x) = 4\arccos(2x - 1)$. Find:

a the maximal domain

b the range

c $f(\frac{1}{2})$

d a, if $f(a) = 3\pi$

e the equation of the tangent to the graph at the point where $x = \frac{1}{2}$.

11 A tank originally holds 40 litres of water, in which 10 grams of a chemical is dissolved. Pure water is poured into the tank at 4 litres per minute. The mixture is well stirred and flows out at 6 litres per minute until the tank is empty.

a State how long it takes the tank to empty.

b Set up a differential equation for the mass, m grams, of chemical in the tank at time t minutes, including the initial condition.

c Express m in terms of t.

d Hence determine how long it takes for the concentration of the solution to reach 0.2 grams per litre.

12 For the graph of $f(x) = \dfrac{x+3}{x^2+3}$, find:

a the equations of any asymptotes

b the coordinates of any stationary points

c the area bounded by the x-axis, the y-axis, the line $x = 3$ and the graph of $y = f(x)$.

13 A curve is defined by the parametric equations $x = t$ and $y = 3t^{\frac{3}{2}} - 1$, for $0 \le t \le 1$. Let P and Q be the points $(0, -1)$ and $(1, 2)$ respectively.

a Find the length of the arc PQ.

b Find the length of the line segment PQ.

14 **a** Find each of the following complex numbers in Cartesian form:

i $(5 + i)(4 + i)$ **ii** $(\sqrt{3} + i)(-2\sqrt{3} + i)$ **iii** $\left(\frac{1}{2} + i\right)\left(-\frac{3}{4} + i\right)$ **iv** $(1.2 - i)(0.4 + i)$

b Let $z = a + i$ and $w = b + i$, where both a and b are integers.

i Find zw, in terms of a and b.

ii If $\text{Re}(zw) = \text{Im}(zw)$, express b in terms of a.

iii Hence sketch the graph of b against a.

15 The random variable X takes values $-1, 0, 1$ with probabilities $\frac{1}{6}, \frac{1}{2}, \frac{1}{3}$ respectively. Let X_1 and X_2 be independent random variables with this same distribution and let $Y = X_1 + X_2$. Find:

a $\Pr(Y = 2)$

b $\Pr(Y = 0)$

c $\Pr(Y = 1)$

d $\text{E}(Y)$

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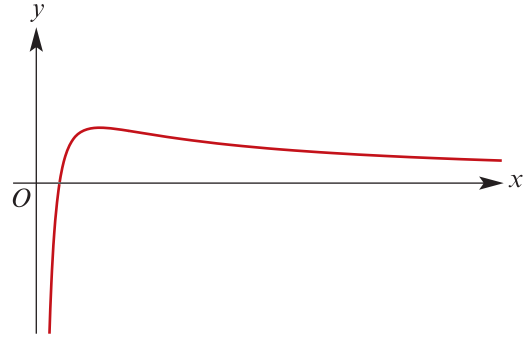
16 Suppose that the mean weight of males is 80 kg with a standard deviation of 12 kg, and the mean weight of females is 70 kg with a standard deviation of 10 kg. If five men and five women get into a lift, what are the mean and standard deviation of the total weight in the lift?

17 The graph of $y = \dfrac{\log_e x}{x}$ is shown.

Point P is the stationary point, and Q is the point of intersection of the graph with the x-axis.



a Find the coordinates of P and Q.

b Find the area of the region bounded by the x-axis, the curve and the line $x = e$.

18 A machine dispenses liquid into bottles. The volume of liquid in a bottle, V mL, has a mean of 502 mL and a standard deviation of 1 mL. A sample of bottles is selected for inspection each hour, and the average volume of liquid in the sample bottles, $\bar{V}$ mL, determined. If the manufacturer requires the standard deviation of the average volume to be less than 0.2 mL, how many bottles should be included in each sample?

19 **a** Solve the differential equation $\dfrac{dy}{dx} = e^{x+y}$, $y(1) = 1$, expressing y as a function of x.

b State the maximal domain of this function.

c Find the equation of the tangent to the curve at $x = 0$.

20 **a** Solve the differential equation $\dfrac{dy}{dx} = x(4 + y^2)$, $y(0) = 2$, expressing y as a function of x.

b State the maximal domain of this function.

c Find the equation of the normal to the curve at $x = \dfrac{1}{2}\sqrt{\dfrac{\pi}{3}}$.

21 **a** Express $\dfrac{x}{(1-x)^2}$ as partial fractions.

b Hence find the area of the region defined by the graphs of $y = \dfrac{x}{(1-x)^2}$, $x = 2$, $x = 4$ and the x-axis.

22 **a** Show that

$$\frac{x}{\sqrt{x-1}} = \sqrt{x-1} + \frac{1}{\sqrt{x-1}}$$

b The graph of $f(x) = \dfrac{x}{\sqrt{x-1}}$, for $x \in [2, a]$, is rotated about the x-axis to form a solid of revolution. Find the volume of this solid in terms of a.

23 Determine the asymptotes, intercepts and stationary points for the graph of the relation $y = \dfrac{x^3 + 3x^2 - 4}{x^2}$. Hence sketch the graph.

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24 Let P be a point on the line $x + y = 1$ and write $\overrightarrow{OP} = m\boldsymbol{i} + n\boldsymbol{j}$, where O is the origin and $m, n \in \mathbb{R}$.

a Find the unit vectors parallel to the line $x + y = 1$.

b Find a relation between m and n, and hence express $\overrightarrow{OP}$ in terms of m only.

c Find the two values of m such that $\overrightarrow{OP}$ makes an angle of 60° with the line $x + y = 1$.

25 Points A, B and C are represented by position vectors $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $2\boldsymbol{i} + m\boldsymbol{j} + \boldsymbol{k}$ and $3\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ respectively.

a The position vector $\boldsymbol{r} = \overrightarrow{OA} + t\overrightarrow{AC}$, $t \in \mathbb{R}$ can be used to represent any point on the line AC. Find the value of t for which $\boldsymbol{r}$ is perpendicular to $\overrightarrow{AC}$.

b Find the value of m such that $\angle BAC$ is a right angle.

26 Let $f(x) = \dfrac{4x^2 + 16x}{(x-2)^2(x^2+4)}$.

a Given that $f(x) = \dfrac{a}{x-2} + \dfrac{6}{(x-2)^2} - \dfrac{bx+4}{x^2+4}$, find a and b.

b Given that $\int_{-2}^{0} f(x)\,dx = \dfrac{c - \pi - \log_e d}{2}$, find c and d.

27 Consider the polynomial $P(z) = z^2 - (m + 2i)z + n(1 + i)$, where m and n are real numbers such that $P(1 + 3i) = 0$.

a Determine the values of m and n.

b Solve the equation $P(z) = 0$ for z.

28 Find an antiderivative of each of the following:

a $(2x - 6)e^x$

b $x \log_e(2x)$

c $x \sec^2(3x)$

d $x \tan^2(x)$

29 Prove by induction that

$$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n} \quad \text{for all } n \in \mathbb{N}$$

30 **a** Let $\mathbf{M}$ be a 2×2 matrix and assume that

$$\mathbf{M} = \mathbf{A}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\mathbf{A}^{-1}$$

where $\mathbf{A}$ is an invertible 2×2 matrix and $a, b \in \mathbb{R}$. Prove by induction that

$$\mathbf{M}^n = \mathbf{A}\begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}\mathbf{A}^{-1} \quad \text{for all } n \in \mathbb{N}$$

b Show that $\begin{bmatrix} -4 & 2 \\ -21 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1}$. Hence find a formula for $\begin{bmatrix} -4 & 2 \\ -21 & 9 \end{bmatrix}^n$.

31 The cost of a 1 L carton of milk in Australia is known to be normally distributed with a standard deviation of \$0.50. A random sample of 25 cartons was selected from different shops across the country, and the mean cost for this sample was \$1.70.

a Give a point estimate for μ, the mean cost of a 1 L carton of milk in Australia.

b Calculate an approximate 95% confidence interval for μ.
Note: Use the approximation $\Pr(-2 < Z < 2) \approx 0.95$ in your calculation.

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32 On average, a certain type of battery lasts for 8.3 hours before it requires recharging, with a standard deviation of 2.4 hours. The manufacturers are introducing a new model of the battery, which they hope will last longer. To test this, they select a random sample of 36 new-model batteries and find that they last on average 8.9 hours.

a Write down appropriate null and alternative hypotheses for this test.

b Given that $\Pr(Z < 1.5) = 0.9332$, determine the p-value for this test, correct to three decimal places.

c At the 5% level of significance, what would be your conclusion?

33 Let $I_n = \int_0^1 (1 - x^2)^n \, dx$, where $n \in \mathbb{N} \cup \{0\}$.

a Use integration by parts to show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$ for all $n \in \mathbb{N}$.

Hint: After applying integration by parts, you may need to use $x^2 = 1 - (1 - x^2)$.

b Hence prove by induction that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$ for all $n \in \mathbb{N}$.

18B Multiple-choice questions

1 A normal vector to the plane with Cartesian equation $8x + 6y - 3z = -12$ is

A $8\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$ **B** $-8\boldsymbol{i} - 6\boldsymbol{j} - 3\boldsymbol{k}$ **C** $6\boldsymbol{i} + 3\boldsymbol{j} - 8\boldsymbol{k}$

D $-12\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$ **E** $6\boldsymbol{i} - 3\boldsymbol{j} - 12\boldsymbol{k}$

2 Consider the statement:

- For all integers n, if n is a multiple of 9, then n is a multiple of 3.

Which one of the following is the contrapositive of this statement?

A For all integers n, if n is a multiple of 3, then n is a multiple of 9.

B For all integers n, if n is not a multiple of 9, then n is not a multiple of 3.

C For all integers n, if n is not a multiple of 3, then n is not a multiple of 9.

D There exists an integer n such that n is a multiple of 9 and not a multiple of 3.

E There exists an integer n such that n is a multiple of 3 and not a multiple of 9.

3 Consider the statement:

- For all integers n, if n is a multiple of 9, then n is a multiple of 3.

Which one of the following is the negation of this statement?

A For all integers n, if n is a multiple of 3, then n is a multiple of 9.

B For all integers n, if n is not a multiple of 9, then n is not a multiple of 3.

C For all integers n, if n is not a multiple of 3, then n is not a multiple of 9.

D There exists an integer n such that n is a multiple of 9 and not a multiple of 3.

E There exists an integer n such that n is a multiple of 3 and not a multiple of 9.

4 Let $z = a + i$, where $a \in \mathbb{R}$. If $\operatorname{Arg}(z^9) = \operatorname{Arg}(z)$, then a possible value of a is

A $-\sqrt{2}$ **B** $-\dfrac{1}{\sqrt{2}}$ **C** 0 **D** $\dfrac{1}{\sqrt{2}}$ **E** $\sqrt{2}$

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5 The stationary points of the function $f(x) = \dfrac{2x^2 - x + 1}{x - 1}$ occur when x equals

A 1 **B** 0 or 2 **C** 0 only **D** $\frac{1}{4}$ **E** -1

6 The point of inflection of the graph of $y = \dfrac{x^2 - 3x + 2}{x^2}$ has x-coordinate

A 0 **B** -1 **C** 1 **D** 2 **E** -2

7 The gradient of the curve with equation $x^3 + y^3 + 3xy = 1$ at the point $(2, -1)$ is

A 0 **B** -1 **C** 1 **D** -2 **E** 2

8 The graph of the function $f(x) = e^x \sin x$, $0 \le x \le \pi$, has a maximum gradient of

A 1 **B** $\dfrac{\pi}{2}$ **C** $e^{-\frac{\pi}{2}}$ **D** e^{π} **E** $e^{\frac{\pi}{2}}$

9 If $\int_0^k xe^{-x}\,dx = 0.5$ and $k > 0$, then k is closest to

A 0.7 **B** 1.7 **C** 2.7 **D** 3.7 **E** 4.7

10 If $\dfrac{dy}{dx} = x \log_e x$ with $y(2) = 2$, then $y(3)$ is closest to

A 4.31 **B** 2.3 **C** -1.7 **D** 0 **E** 1.3

11 Consider the surface of revolution formed by revolving the curve $y = \dfrac{4}{x}$, for $1 \le x \le b$, about the x-axis. The surface area is given by

A $8\pi \int_1^b \dfrac{1}{x}\,dx$ **B** $8\pi \int_1^b \dfrac{1}{x^2}\,dx$ **C** $8\pi \int_1^b \dfrac{\sqrt{16 + x^4}}{x^3}\,dx$

D $8\pi \int_1^b \sqrt{1 + \dfrac{16}{x^2}}\,dx$ **E** $8\pi \int_1^b \sqrt{1 + \dfrac{16}{x^4}}\,dx$

12 The solution of the inequality $\cot\left(\dfrac{\theta}{2}\right) \ge \sqrt{3}$, for $-\pi \le \theta \le \pi$, is

A $\left(-\pi, \dfrac{\pi}{3}\right)$ **B** $\left[-\pi, \dfrac{\pi}{3}\right)$ **C** $\left[0, \dfrac{\pi}{3}\right]$ **D** $\left(0, \dfrac{\pi}{3}\right]$ **E** $\left[\dfrac{\pi}{3}, \pi\right]$

13 The velocity, v m/s, of a particle at time t seconds is given by $v = \dfrac{4t}{1 + t^2}$, $t \ge 0$. The distance, in metres, travelled by the particle in the first 10 seconds is closest to

A 9.23 **B** 533.33 **C** 1 **D** 2 **E** 1.73

14 A small rocket is fired vertically upwards. The initial speed of the rocket is 200 m/s. The acceleration of the rocket, a m/s^2, is given by

$$a = -\frac{20 + v^2}{50}$$

where v m/s is the velocity of the rocket at time t seconds. The time that the rocket takes to reach the highest point, in seconds, is closest to

A 5 **B** 8 **C** 12 **D** 17 **E** 25

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Revision

15 The graph of $y = -\sec(ax + b)$ is identical to the graph of $y = \operatorname{cosec}\left(x + \frac{\pi}{3}\right)$. The values of a and b could be

A $a = 1$ and $b = \frac{\pi}{6}$ **B** $a = -1$ and $b = \frac{\pi}{6}$ **C** $a = 1$ and $b = \frac{2\pi}{3}$

D $a = -1$ and $b = \frac{7\pi}{6}$ **E** none of these

16 $\frac{d}{dx}(x \log_e y) - \frac{x}{y}\frac{dy}{dx} =$

A 0 **B** $\log_e y$ **C** $x \log_e y$

D $\log_e y - \frac{x}{y}\frac{dy}{dx}$ **E** $\frac{1-x}{y}\frac{dy}{dx}$

17 The graph with parametric equations $x = 2 + 3\sec(t)$ and $y = 1 + 2\tan(t)$, where $t \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, has

A two asymptotes, $y = \frac{2x}{3} - \frac{1}{3}$ and $y = -\frac{2x}{3} + \frac{7}{3}$

B two asymptotes, $y = \frac{2}{3}(x-1)$ and $y = -\frac{2}{3}(x-1)$

C two asymptotes, $y - 1 = \frac{3}{2}(x-2)$ and $y - 1 = -\frac{3}{2}(x-2)$

D one asymptote, $y = \frac{2x}{3} - \frac{1}{3}$ **E** one asymptote, $3y = 7 - 2x$

18 Consider the vectors $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$. Solving the equation $3\boldsymbol{i} = m\boldsymbol{a} + n\boldsymbol{b} + p\boldsymbol{c}$ produces

A $m = 1, n = -1, p = 1$ and $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent vectors

B $m = 1, n = \frac{3}{8}, p = \frac{1}{8}$ and $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent vectors

C $m = 1, n = -1, p = 1$ and $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent vectors

D $m = 1, n = \frac{3}{8}, p = \frac{1}{8}$ and $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent vectors

E no values of m, n and p satisfy this equation

19 $\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \cos^2(2x)\, dx$ is not equal to

A $\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \cos^2(x)\, dx$ **B** $\frac{\pi}{2} - \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin^2(2x)\, dx$ **C** $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} 1 + \cos(4x)\, dx$

D $\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin^2(\frac{1}{2}(\pi - 4x))\, dx$ **E** $\left[\frac{1}{6}\cos^3(2x)\right]_{\frac{\pi}{6}}^{\frac{2\pi}{3}}$

20 Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z^2(1 + i) = 2 - 2i$, then the Cartesian form of one value of z could be

A $\sqrt{2}i$ **B** $-\sqrt{2}i$ **C** $-1 - i$ **D** $-1 + i$ **E** $\sqrt{-2}$

21 The gradient of the tangent to the graph of $y = e^{xy}$ at the point where $x = 0$ is

A 0 **B** 1 **C** 2 **D** $\log_e 2$ **E** undefined

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22 Let $\boldsymbol{a} = p\boldsymbol{i} + q\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$. If the scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\frac{2}{3}$ and the scalar resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is 2, then the values of p and q are

A $p = 0$ and $q = 0$

B $p = 2 - \sqrt{7}$ and $q = \sqrt{7}$

C $p = \dfrac{8 + \sqrt{10}}{5}$ and $q = \dfrac{4\sqrt{10}}{5}$

D $p = 1$ and $q = 0.5$

E $p = -2$ and $q = -1$

23 Let $f(x) = a\cos(x + c)$ for $x \in \left[\pi - c, \dfrac{3\pi}{2} - c\right]$, where $a > 0$. Then $f^{-1}(x) =$

A $a\cos^{-1}(x - c)$

B $\cos^{-1}\left(\dfrac{x}{a} - c\right)$

C $\pi - c - \cos^{-1}\left(\dfrac{x}{a}\right)$

D $\pi + c - \cos^{-1}\left(\dfrac{x}{a}\right)$

E $2\pi - c - \cos^{-1}\left(\dfrac{x}{a}\right)$

24 The position of a particle at time t seconds is defined by $\boldsymbol{r} = \dfrac{a}{t+1}\boldsymbol{i} + (1 + t^2)\boldsymbol{j}$, $t \geq 0$, where $a > 0$. The Cartesian equation which represents the path of the particle is

A $y = \dfrac{a^2}{x^2}$, for $x \in [0, \infty)$

B $y = \dfrac{a^2 - 2ax + 2x^2}{x^2}$, for $x \in [a, \infty)$

C $y = \left(\dfrac{a}{x} - 1\right)^2 + 1$, for $x \in (0, a]$

D $y = \dfrac{x^2 - 2ax + 2a^2}{a^2}$, for $x \in \mathbb{R} \setminus \{-1\}$

E $y = \dfrac{a^2}{(x-1)^2} + 1$, for $x \in [0, \infty)$

25 Using an appropriate substitution, the integral $\int_1^2 x(2 - x)(x^3 - 3x^2 + 4)\,dx$ can be expressed as

A $3\int_1^2 u\,du$

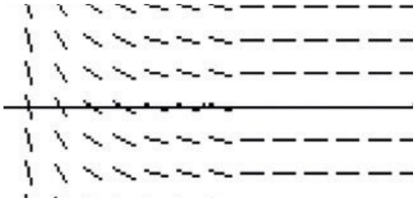
B $\dfrac{1}{3}\int_2^1 u\,du$

C $\dfrac{1}{6}\int_2^0 u^2\,du$

D $\int_2^0 3u\,du$

E $-\dfrac{1}{3}\int_2^0 u\,du$

26 This is the slope field for a differential equation, produced by a calculator, with $0 \leq x \leq 2$ and $-3 \leq y \leq 3$.



A solution for the differential equation could be

A $y = -\dfrac{1}{x^2}$

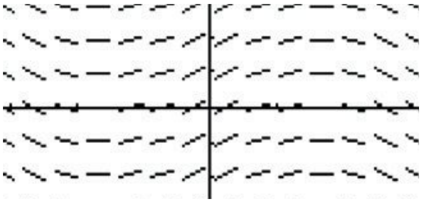
B $y = -\dfrac{1}{x^3}$

C $y = \dfrac{1}{x}$

D $y = e^x$

E $y = -\dfrac{1}{\sqrt{x}}$

27 This is the slope field for a differential equation, produced by a calculator, with $-\pi \leq x \leq \pi$ and $-3 \leq y \leq 3$.



The differential equation could be

A $\dfrac{dy}{dx} = \sin x$

B $\dfrac{dy}{dx} = -\cos x$

C $\dfrac{dy}{dx} = \tan x$

D $\dfrac{dy}{dx} = \sin(2x)$

E $\dfrac{dy}{dx} = \cos x$

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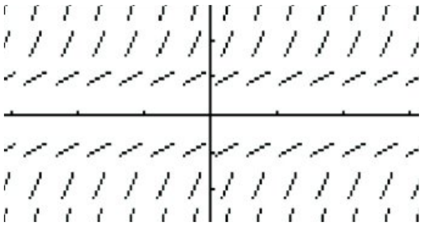
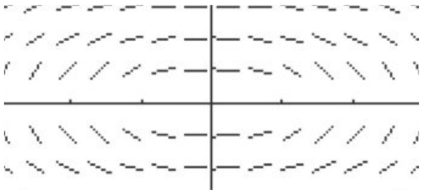
28 This is the slope field for a differential equation, produced by a calculator, with $-3 \le x \le 3$ and $-3 \le y \le 3$.



A solution for the differential equation could be

A $y = \frac{1}{x}$ **B** $x = y^3$ **C** $y = \frac{1}{x^2}$ **D** $x = -\frac{1}{y}$ **E** $x = \log_e y$

29 This is the slope field for a differential equation, produced by a calculator, with $-3 \le x \le 3$ and $-3 \le y \le 3$.



The differential equation could be

A $\frac{dy}{dx} = x^2$ **B** $\frac{dy}{dx} = -\frac{y}{x}$ **C** $\frac{dy}{dx} = \frac{y}{x}$ **D** $\frac{dy}{dx} = \frac{x}{y}$ **E** $\frac{dy}{dx} = -\frac{x}{y}$

30 A particle is projected at an angle of $\arctan\left(\frac{3}{4}\right)$ to the horizontal with a speed of 40 m/s. After 2 seconds, the particle is moving in a direction at an angle of θ to the horizontal. If the acceleration due to gravity is taken as $g = 10$ m/s^2, then $\tan\theta$ is equal to

A $\frac{1}{8}$ **B** $\frac{1}{2}$ **C** 2 **D** 4 **E** 8

31 Which one of the following points is equidistant from the two planes given by the equations $-2x + y - 2z + 3 = 0$ and $-2x + y - 2z + 21 = 0$?

A $(4, 3, 4)$ **B** $(2, 4, 6)$ **C** $(3, -5, 5)$ **D** $(1, 3, 5)$ **E** $(-1, 4, -6)$

32 A random sample is drawn from a population with a standard deviation of 15.6. For the difference between the sample mean and the population mean to be less than 2.0 at the 99% confidence level, the minimum size of the sample should be

A 41 **B** 404 **C** 403 **D** 1614 **E** 1615

33 A manufacturer claims that the average lifetime of their tyres is $\mu = 40\,000$ km. Laboratory testing of 30 tyres produced a sample mean lifetime of $\bar{x} = 38\,500$ km. Given that the standard deviation is $\sigma = 3000$ km, what is the probability that a sample mean as small as or smaller than 38 500 km would be observed if the average lifetime of the tyres is 40 000 km?

A 0.05 **B** 0.0031 **C** 0.0062 **D** 0.6915 **E** 0.9969

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18C Extended-response questions

1 **a** Let a and b be integers.

i Show that if both a and b are even, then there exist integers x and y such that $x + y = a$ and $x - y = b$.

ii Show that if both a and b are odd, then there exist integers x and y such that $x + y = a$ and $x - y = b$.

iii However, show that if a is even and b is odd, then there do not exist integers x and y such that $x + y = a$ and $x - y = b$.

b Using part **a**, prove each of the following:

i If an integer n is the product of two even integers a and b, then n is a difference of perfect squares.

ii If an integer n is the product of two odd integers a and b, then n is a difference of perfect squares.

c Using part **b**, prove each of the following:

i Every odd integer is a difference of perfect squares.

ii Every multiple of 4 is a difference of perfect squares.

2 A curve has equation $x^2 + y^2 + 2x + 4y = 24$.

a Find the gradient of this curve at the point $(1, 3)$.

b Find the equation of the tangent to this curve at the point $(1, 3)$.

c The curve can be described by parametric equations of the form

$$x = -1 + b\cos t \quad \text{and} \quad y = -2 + d\sin t$$

where b and d are positive constants. Find the values of b and d.

d Using the parametric representation, find the area of the surface of revolution formed by rotating the curve:

i about the x-axis for $\frac{\pi}{6} \le t \le \frac{\pi}{3}$ **ii** about the y-axis for $\frac{\pi}{6} \le t \le \frac{\pi}{3}$.

e A particle moves along the curve. When the particle is at the point $(1, 3)$, its y-coordinate is increasing by 2 units per second. Find the corresponding rate of change in its x-coordinate.

3 **a** Use mathematical induction to prove that

$$\sum_{r=1}^{n} r(r^2 + 1) = \frac{1}{4}n(n + 1)(n^2 + n + 2)$$

for every natural number n.

b Using pseudocode, describe an algorithm to find the smallest value of n such that

$$\sum_{r=1}^{n} r^3(r^2 + 1) \ge 10\,000$$

Hint: Use a `while` loop in your algorithm.

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4 The points A, B and C have position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ respectively, with respect to an origin O.

a Find a vector equation of the line BC.

b Find a vector equation of the plane Π that contains the point A and is perpendicular to the line OA.

c Show that the line BC is parallel to the plane Π.

d A circle with centre O passes through A and B. Find the length of the minor arc AB.

e Verify that $2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ is perpendicular to the plane OAB. Write down a vector perpendicular to the plane OAC. Hence, find the acute angle between the planes OAB and OAC.

5 The plane Π_1 is given by the Cartesian equation $3x + 2y - z = -1$, and the line ℓ_1 is given by the vector equation $\boldsymbol{r} = (4 - t)\boldsymbol{i} + (2t - 3)\boldsymbol{j} + (t + 7)\boldsymbol{k}$, $t \in \mathbb{R}$.

a Show that the line ℓ_1 lies in the plane Π_1.

The line ℓ_2 is given by the vector equation $\boldsymbol{r} = 10\boldsymbol{j} + 7\boldsymbol{k} + t(\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k})$, $t \in \mathbb{R}$, and the line ℓ_2 intersects the plane Π_1 at the point A.

b Find the coordinates of the point A.

c Find a Cartesian equation of the plane Π_2 that passes through the point A and is perpendicular to the line ℓ_1.

d Find the coordinates of the point where the line ℓ_1 intersects the plane Π_2.

e Find a vector equation of the line ℓ_3 that lies in the plane Π_1 and is perpendicular to the line ℓ_1.

6 A plane Π_1 in three-dimensional space has vector equation $\boldsymbol{r} \cdot (2\boldsymbol{i} + 3\boldsymbol{j}) = -6$.

a Find a vector equation of the line that is normal to Π_1 and passes through $P(2, 1, 4)$.

b Find the coordinates of Q, the foot of the perpendicular on Π_1 from the point P.

c Find the angle between OQ and Π_1.

d Planes Π_2 and Π_3 have vector equations $\boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 5$ and $\boldsymbol{r} \cdot \boldsymbol{i} = 0$ respectively. Find the point of intersection of the three planes Π_1, Π_2 and Π_3.

7 The independent random variables R and S each have a normal distribution. The means of R and S are 10 and 12 respectively, and the variances are 9 and 16 respectively. Find the following probabilities, giving your answers correct to three decimal places:

a $\Pr(R < S)$

b $\Pr(2R > S_1 + S_2)$, where S_1 and S_2 are independent random variables, each with the same distribution as S.

8 A machine produces sheets of paper, the thickness of which are normally distributed with mean 0.1 mm and standard deviation 0.005 mm.

a Find the mean and standard deviation of the normal random variable of the total thickness of eight randomly selected sheets of paper.

b Find the mean and standard deviation of the normal random variable of the total thickness if a single sheet of paper is folded three times to give eight 'thicknesses'.

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9 The length of a rectangular tile is a normal random variable with mean 20 cm and standard deviation 0.1 cm. The width is an independent normal random variable with mean 10 cm and standard deviation 0.1 cm.

a Find the probability that the sum of the lengths of four randomly chosen tiles exceeds 80 cm.

b Find the probability that the width of a randomly chosen tile is less than half its length.

c Let S be the random variable formed from the sum of the lengths of 50 randomly chosen tiles, and let T be the random variable formed from the sum of the widths of 80 randomly chosen tiles. Find the mean and variance of $S - T$.

10 Consider the function

$$f(x) = \begin{cases} x \log_e x - 3x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

a Find the derivative for $x > 0$.

b One x-axis intercept is at $(0, 0)$. Find the coordinates of the other x-axis intercept, A.

c Find the equation of the tangent at A.

d Find the ratio of the area of the region bounded by the tangent and the coordinate axes to the area of the region bounded by the graph of $y = f(x)$ and the x-axis.

11 **a** Consider $y = \dfrac{a + b\sin x}{b + a\sin x}$, where $0 < a < b$.

i Find $\dfrac{dy}{dx}$.

ii Find the maximum and minimum values of y.

b For the graph of $y = \dfrac{1 + 2\sin x}{2 + \sin x}$, $-\pi \le x \le 2\pi$:

i State the coordinates of the y-axis intercept.

ii Determine the coordinates of the x-axis intercepts.

iii Determine the coordinates of the stationary points.

iv Sketch the graph of $y = f(x)$.

v Find the area measure of the region bounded by the graph and the line with equation $y = -1$.

12 Consider the function

$$f(x) = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$$

Given that $f(x)$ can be expressed in the form $r\cos(x - a)$, where $r > 0$ and $0 < a < \dfrac{\pi}{2}$:

a Find the values of r and a.

b Find the range of the function.

c Find the y-axis intercept.

d Find the x-axis intercepts.

e Find x, if $f(x) = \sqrt{2}$.

f If $g(x) = \dfrac{1}{f(x)}$, evaluate $\int_0^{\frac{\pi}{2}} g(x)\,dx$.

g Find the volume measure of the solid formed when the region bounded by the graph of $y = f(x)$, the x-axis and the y-axis is rotated about the x-axis.

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Revision

13 A particle moves in a line such that the velocity, v m/s, at time t seconds ($t \geq 0$) satisfies the differential equation

$$\frac{dv}{dt} = \frac{-v}{50}(1 + v^2)$$

The particle starts from O with an initial velocity of 10 m/s.

a **i** Express as an integral the time taken for the particle's velocity to decrease from 10 m/s to 5 m/s.

ii Hence calculate the time taken for this to occur.

b **i** Show that, for $v \geq 0$, the motion of this particle is described by the differential equation

$$\frac{dv}{dx} = \frac{-(1 + v^2)}{50}$$

where x metres is the displacement from O.

ii Given that $v = 10$ when $x = 0$, solve this differential equation, expressing x in terms of v.

iii Hence show that

$$v = \frac{10 - \tan\left(\frac{x}{50}\right)}{1 + 10\tan\left(\frac{x}{50}\right)}$$

iv Hence find the displacement of the particle from O, to the nearest metre, when it first comes to rest.

14 Let $f(x) = \sin(\pi x) + px$, $x \in [0, 1]$.

a **i** Find the value of p for which $f'(1) = 0$.

ii Hence show that $f'(x) \geq 0$ for $x \in [0, 1]$.

b Sketch the graph of $y = f(x)$, $x \in [0, 1]$.

c Find the exact value for the volume of revolution formed when the graph of $y = f(x)$, $x \in [0, 1]$, is rotated around the x-axis.

d For $g(x) = k\arcsin(x)$, $x \in [0, 1]$, find the value of k such that $f(1) = g(1)$.

e Find the area of the region enclosed by the graphs of $y = f(x)$ and $y = g(x)$, correct to three decimal places.

f If $f(x) - g(x)$ has a maximum at $x = a$, find a, correct to three decimal places.

15 **a** Let $z^5 - 1 = (z - 1)P(z)$, where $P(z)$ is a polynomial. Find $P(z)$ by division.

b Show that $z = \text{cis}\left(\frac{2\pi}{5}\right)$ is a solution of the equation $z^5 - 1 = 0$.

c Hence find another complex solution of the equation $z^5 - 1 = 0$.

d Find all the complex solutions of $z^5 - 1 = 0$.

e Hence factorise $P(z)$ as a product of two quadratic polynomials with real coefficients.

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16 Points A and B are represented by position vectors $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = m(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$ respectively, relative to a point O, where $m > 0$.

a Find the value of m for which A and B are equidistant from O.

Points A and B lie on a circle with centre O. Point C is represented by the position vector $-\boldsymbol{a}$.

b **i** Give reasons why C also lies on the circle.

ii By using the scalar product, show that $\angle ABC = 90°$.

Now assume that all points on this circle can be represented by the general position vector $\boldsymbol{d} = k\boldsymbol{a} + \ell\boldsymbol{b}$, for different values of k and ℓ.

c **i** Show that the relation between k and ℓ is given by $9k^2 - 2\sqrt{3}k\ell + 9\ell^2 = 9$.

ii When $k = 1$, find the two position vectors that represent points on the circle.

d Let P be a point on the circle such that OP bisects AB. Find the position vectors which represent P. Do not attempt to simplify your answer.

A particle is travelling such that its position at time t seconds is given by

$$\boldsymbol{r} = (5 - t)\boldsymbol{i} + (2 + t)\boldsymbol{j} + (t - 3)\boldsymbol{k}$$

e Find the value of t when $\boldsymbol{r}$ can be expressed in the form $k\boldsymbol{a} + \ell\boldsymbol{b}$, and find the corresponding values of k and ℓ.

f Hence determine whether the particle lies inside, outside or on this circle at this time.

17 A curve is defined by the parametric equations $x = 3\sin(t)$ and $y = 6\cos(t) - a$, where $0 \le a < 6$.

a **i** Find the Cartesian equation of the curve.

ii Find the intercepts of the curve with the x-axis.

b Define the function which represents the part of the curve above the x-axis.

c Differentiate $x\sqrt{9 - x^2}$.

d **i** Show that $\dfrac{x^2}{\sqrt{9 - x^2}}$ can be expressed in the form $\dfrac{A}{\sqrt{9 - x^2}} - \sqrt{9 - x^2}$ by finding the appropriate value for A.

ii Hence show that the result in **c** can be written as $2\sqrt{9 - x^2} - \dfrac{9}{\sqrt{9 - x^2}}$.

e Use this result and calculus to find an antiderivative of $\sqrt{9 - x^2}$.

f Hence find the area of the region enclosed by the curve above the x-axis.

g For $a = 0$, find the area of the region enclosed by the curve.

h For $a = 0$, find the volume of the solid of revolution formed when the curve is rotated about its horizontal axis.

18 A curve is defined by the parametric equations $x = t^2$ and $y = \frac{1}{3}t^3 - t$.

a The curve can be described by a Cartesian equation of the form $y^2 = g(x)$. Find $g(x)$.

b Find the coordinates of the stationary points of the curve.

c Find the area of a region enclosed by the curve.

d Find the volume of the solid formed by rotating this region around the x-axis.

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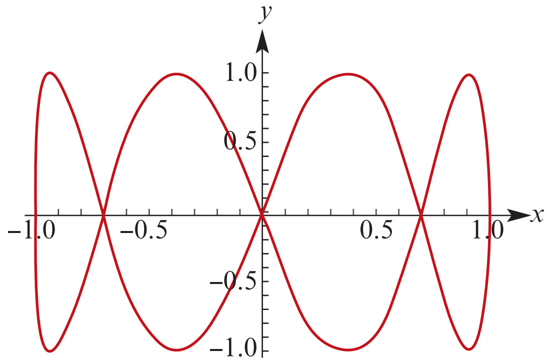
19 A curve is defined by the parametric equations

$$x = \sin(t), \quad y = \sin(4t)$$

for $0 \le t \le 2\pi$. The graph is shown on the right.



a Find the Cartesian equation of the curve with y in terms of x.

b Find $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ in terms of t.

c **i** Find the values of t for which $\frac{dy}{dx} = 0$.

ii Find the values of x for which $\frac{dy}{dx} = 0$.

iii Find the coordinates of the stationary points of the graph.

iv Find the gradients of the graph at $x = \frac{1}{\sqrt{2}}$, at $x = \frac{-1}{\sqrt{2}}$ and at the origin.

v Show that the gradient is undefined when $x = -1$ or $x = 1$.

d Find the total area of the regions enclosed by the curve.

e Find the volume of the solid of revolution formed by rotating the curve around the x-axis.

20 Let $f(x) = \frac{x^3}{x^2 + a}$, where a is a positive real constant.

a Find $f'(x)$ and $f''(x)$.

b Find the coordinates of the stationary point and state its nature.

c Find the coordinates of the points of inflection (non-stationary).

d Find the equation of the asymptote of the graph of f.

e Sketch the graph of f.

f Find the value of a such that the area between the curve, the line $y = x$ and the line $x = a$ is equal to $\frac{1}{2}\log_e 2$.

21 Let $f(x) = \frac{x^3}{x^2 - a}$, where a is a positive real constant.

a Find $f'(x)$ and $f''(x)$.

b Find the coordinates of the stationary points of f in terms of a and state their nature.

c Find the coordinates of the point of inflection of f.

d Find the equation of the asymptotes of the graph of f.

e Sketch the graph of f.

f Find the value of a if a stationary point of f occurs where $x = 4\sqrt{3}$.

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22 Let $f: [-1, 1] \to \mathbb{R}$, $f(x) = x \arcsin(x)$ and $g: [-1, 1] \to \mathbb{R}$, $g(x) = \arcsin(x)$.

a Find $f'(x)$ and the coordinates of any turning points for $x \in (-1, 1)$.

b Find $f''(x)$ and show that there are no points of inflection for $x \in (-1, 1)$.

c Prove that $f(x) \geq 0$ for all $x \in [-1, 1]$.

d Find the values of x for which $f(x) = g(x)$.

e Sketch the graphs of f and g on the one set of axes.

f Find the area of the region enclosed by the graphs of f and g.

23 The coordinates, $P(x, y)$, of points on a curve satisfy the differential equations

$$\frac{dx}{dt} = -3y \quad \text{and} \quad \frac{dy}{dt} = \sin(2t)$$

and when $t = 0$, $y = -\frac{1}{2}$ and $x = 0$.

a Find x and y in terms of t.

b Find the Cartesian equation of the curve.

c Find the gradient of the tangent to the curve at a point $P(x, y)$ in terms of t.

d Find the axis intercepts of the tangent in terms of t.

e Let the x- and y-axis intercepts of the tangent be points A and B respectively, and let O be the origin. Find an expression for the area of triangle AOB in terms of t, and hence find the minimum area of this triangle and the values of t for which this occurs.

f Give a pair of parametric equations in terms of t which describe the circle with centre the origin and the same x-axis intercepts as the curve.

g Find the volume of the solid formed by rotating the region between the circle and the curve about the x-axis.

24 Linh rides her bike to work each day. She knows that the time it takes is normally distributed with a mean of 55 minutes and a standard deviation of 5 minutes.

a What is the probability that Linh will ride to work in less than 48 minutes on a particular day?

b Find k_1 and k_2 such that the probability that Linh takes between k_1 and k_2 minutes to ride to work is 0.95.

c During a five-day working week, Linh makes the ride 10 times. Find the probability that, in a randomly chosen week:

i Linh's average riding time is less than 50 minutes

ii Linh's total riding time is more than 580 minutes

iii the ride takes less than 50 minutes more than three times during the week.

d Find c_1 and c_2 such that there is a probability of 0.95 that her average riding time over a five-day period is between c_1 and c_2 minutes.

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25 A continuous random variable X has the probability density function

$$f(x) = \begin{cases} \dfrac{1}{b} & \text{if } 0 \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where b is a positive constant.

a Find the mean and standard deviation of X in terms of b.

b Find the mean and standard deviation of $\bar{X}$ in terms of b and n, where $\bar{X}$ is the mean of a random sample of size n.

c For a particular random sample of size 50, the sample mean was 2.4. Give an expression in terms of b for a 90% confidence interval for the mean of X.

d What does this confidence interval tell us about the value of b?

26 Let z be a non-zero complex number such that

$$z + \frac{4}{z} = k \quad \text{for some } k \in \mathbb{R}$$

and let z be written in Cartesian form as $z = x + yi$, where $x, y \in \mathbb{R}$.

a Prove that $y = 0$ or $x^2 + y^2 = 4$.

b Prove that if $y = 0$, then $|k| \ge 4$.

c Prove that if $x^2 + y^2 = 4$, then $|k| \le 4$.

27 Riley's factory has two machines, A and B, for making nails. The nails produced by machine A have a mean diameter of 3 mm, with a standard deviation of 0.03 mm. The nails produced by machine B have a mean diameter of 3.01 mm, with a standard deviation of 0.02 mm.

a A random sample of 30 nails is collected from machine A. Find the approximate probability that the mean diameter of the nails in this sample is less than 2.99 mm or greater than 3.01 mm.

b A random sample of 30 nails is collected from machine B. Find the approximate probability that the mean diameter of the nails in this sample is less than 2.99 mm or greater than 3.01 mm.

c Morgan brings Riley another random sample of 30 nails, but cannot remember which of the two machines they were collected from. Riley finds that this sample of nails has a mean diameter of 3.0 mm, with a standard deviation of 0.025 mm.

i Use this sample to determine a 95% confidence interval for the mean diameter of the population of nails.

ii Which machine do you think this sample came from, and why?

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28 The population, P, of goats on an island grows at a rate proportional to $P - 0.5P_0$, where P_0 is the initial population and time t is measured in years.

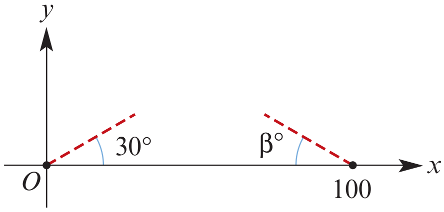
a Write down a differential equation that represents this situation.

b Solve the differential equation, given that the initial population was 1000 and the population increased to 1100 after 1 year.

c Find the increase in population in the third year.

d Find the time taken for the population to reach 2000. (Answer in years, correct to two decimal places.)

29 In the diagram, the x-axis represents horizontal ground, the y-axis is vertical and the unit of distance is metres.



Particle A is projected with speed 60 m/s from the origin at an angle of 30° to the positive x-direction. At the same time, particle B is projected with speed 50 m/s from the point $(100, 0)$ at an angle of $\beta°$ to the negative x-direction.

a Give expressions for $\boldsymbol{r}_A(t)$ and $\boldsymbol{r}_B(t)$, the position vectors of particles A and B after t seconds. (Let g m/s^2 be the acceleration due to gravity.)

b Given that the two particles collide, find the value of β.

c Find the time of collision (in seconds, correct to two decimal places).

d Determine the coordinates of the point of collision (correct to two decimal places).

30 A light-bulb manufacturer advertises that 90% of the light bulbs they produce will last longer than 100 hours.

a Assume that the lifetimes of the light bulbs are normally distributed with a standard deviation of 10 hours. If the manufacturer's claim is true, what is the mean lifetime, μ hours, of their light bulbs? Give your answer correct to one decimal place.

b A consumer-protection association suspects that the actual mean lifetime of the light bulbs is less than the value determined in part **a**. They select a random sample of 40 light bulbs and determine that their average lifetime is $\bar{x} = 110.5$ hours.

i Write down the null and alternative hypotheses for the one-tail test to investigate the manufacturer's claim.

ii Determine the p-value for this test. (Again assume that the lifetimes of the light bulbs are normally distributed with a standard deviation of 10 hours.)

iii At the 5% level of significance, what can be concluded about the manufacturer's advertising?

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Glossary

A

Absolute value function [p. 29] The absolute value of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *modulus function*

Acceleration [pp. 542, 600] the rate of change of velocity with respect to time

Acceleration, average [p. 542]

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

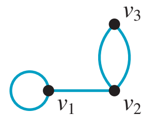
Acceleration, instantaneous [pp. 542, 569]

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Addition of complex numbers [p. 240] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.

Addition of vectors [p. 145] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$.

Adjacency matrix [SM1&2] a matrix that represents a graph. The entries of the matrix give the number of edges joining each pair of vertices. For example:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

Adjacent vertices [SM1&2] Two vertices of a graph are adjacent if they are joined by an edge.

Algorithm [p. 51] a finite, unambiguous sequence of instructions for performing a specific task

Alternative hypothesis, H_1 [p. 693] asserts that the sample is drawn from a population with a mean which differs from that of the original population

Amplitude of circular functions [p. 4] The distance between the mean position and the maximum position is called the amplitude. The graph of $y = a \sin x$ has an amplitude of $|a|$.

Angle between a vector and an axis [p. 159] If the vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ makes angles α, β and γ with the positive directions of the x-, y- and z-axes respectively, then

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

Angle between two lines [p. 210] Let θ be the angle between two vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ that are parallel to the two lines. The angle between the lines is θ or $180^\circ - \theta$, whichever is in $[0^\circ, 90^\circ]$.

Angle between two planes [p. 225] Let θ be the angle between two vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ that are normal to the two planes. The angle between the planes is θ or $180^\circ - \theta$, whichever is in $[0^\circ, 90^\circ]$.

Angle between two vectors [p. 169] can be found using the scalar product:

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

Antiderivative [p. 382] To find the general antiderivative of $f(x)$: If $F'(x) = f(x)$, then

$\int f(x)\,dx = F(x) + c$

where c is an arbitrary real number.

Antiderivative of a vector function [p. 598] If $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, then

$$\int \boldsymbol{r}(t)\,dt = X(t)\boldsymbol{i} + Y(t)\boldsymbol{j} + Z(t)\boldsymbol{k} + \boldsymbol{c}$$

where $\dfrac{dX}{dt} = x(t)$, $\dfrac{dY}{dt} = y(t)$, $\dfrac{dZ}{dt} = z(t)$ and $\boldsymbol{c}$ is a constant vector.

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Arccos [p. 116] *see* inverse cosine function

Arcsin [p. 115] *see* inverse sine function

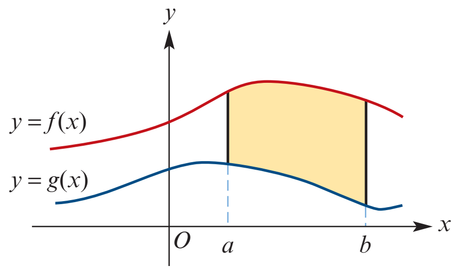
Arctan [p. 116] *see* inverse tangent function

Area of a parallelogram [p. 212] The area of the parallelogram spanned by two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $|\boldsymbol{a} \times \boldsymbol{b}|$.

Area of a region between two curves [p. 436]

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

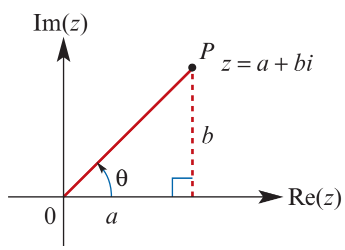
where $f(x) \geq g(x)$ for all $x \in [a, b]$



Area of a surface of revolution [p. 462] For a differentiable function $f\colon [a,b] \to \mathbb{R}$ with non-negative values:

- **Rotation about the x-axis**
  If the curve $y = f(x)$ from $x = a$ to $x = b$ is rotated about the x-axis, then the area of the surface of revolution is given by
  $$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
- **Rotation about the y-axis**
  If the curve $x = f(y)$ from $y = a$ to $y = b$ is rotated about the y-axis, then the area of the surface of revolution is given by
  $$A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Argand diagram [p. 242] a geometric representation of the set of complex numbers



Argument of a complex number [p. 251]

- An argument of a non-zero complex number z is an angle θ from the positive direction of the real axis to the line segment joining the origin to z.
- The *principal value* of the argument, denoted by Arg z, is the angle in the interval $(-\pi, \pi]$.

Argument, properties [pp. 256, 257]

- $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1
- $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1
- $\operatorname{Arg}\left(\frac{1}{z}\right) = -\operatorname{Arg}(z)$, provided z is not a negative real number

Arithmetic sequence [p. 20] a sequence in which each successive term is found by adding a fixed amount to the previous term; e.g. $2, 5, 8, 11, \dots$. An arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is the common difference. The nth term can be found using $t_n = a + (n-1)d$, where $a = t_1$.

Arithmetic series [p. 21] the sum of the terms in an arithmetic sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

where $a = t_1$ and d is the common difference.

C

$\mathbb{C}$ [p. 238] the set of complex numbers: $\mathbb{C} = \{\, a + bi : a, b \in \mathbb{R} \,\}$

Cartesian equation An equation in variables x and y can describe a curve in the plane by giving the relationship between the x- and y-coordinates of the points on the curve; e.g. $y = x^2 + 1$.
An equation in x, y and z can describe a surface in three-dimensional space; e.g. $x + 2y + z = 3$.

Cartesian form of a complex number [p. 242] A complex number is expressed in Cartesian form as $z = a + bi$, where a is the real part of z and b is the imaginary part of z.

Central limit theorem [p. 673] Let X be any random variable, with mean μ and standard deviation σ. Then, provided that the sample size n is large enough, the distribution of the sample mean $\bar{X}$ is approximately normal with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

Chain rule [p. 317]

- If $f(x) = h(g(x))$, then $f'(x) = h'(g(x))\,g'(x)$.
- If $y = h(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

Change of variable rule [p. 393] *see* integration by substitution

Circle, general Cartesian equation [p. 34] The circle with radius r and centre (h, k) has equation $(x-h)^2 + (y-k)^2 = r^2$.

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Circular functions [p. 2] the sine, cosine and tangent functions

cis θ [p. 251] $\cos\theta + i\sin\theta$

Collinear points [p. 176] Three or more points are collinear if they all lie on a single line.
Three distinct points A, B, C (with position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$) are collinear if and only if there exists a non-zero number m such that $\boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b}$.

Common difference, *d* [p. 20] the difference between two consecutive terms of an arithmetic sequence, i.e. $d = t_n - t_{n-1}$

Common ratio, *r* [p. 21] the quotient of two consecutive terms of a geometric sequence, i.e. $r = \dfrac{t_n}{t_{n-1}}$

Complex conjugate, $\bar{z}$ [pp. 246, 252]
- If $z = a + bi$, then $\bar{z} = a - bi$.
- If $z = r\operatorname{cis}\theta$, then $\bar{z} = r\operatorname{cis}(-\theta)$.

Complex conjugate, properties [p. 247]
- $z + \bar{z} = 2\operatorname{Re}(z)$
- $z\bar{z} = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Complex number [p. 238] an expression of the form $a + bi$, where a and b are real numbers

Complex plane [p. 242] *see* Argand diagram

Compound angle formulas [p. 109]
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Concavity [p. 336]
- If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval; the curve is said to be *concave up*.
- If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval; the curve is said to be *concave down*.

Concurrent lines [p. 180] Three or more lines are concurrent if they all pass through a single point.

Conditional statement [p. 68] a statement of the form 'If P is true, then Q is true', which can be abbreviated to $P \Rightarrow Q$

Confidence interval [p. 683] an interval estimate for the population mean μ based on the value of the sample mean $\bar{x}$

Conjugate root theorem [p. 267] If a polynomial has real coefficients, then the complex roots occur in conjugate pairs.

Constant acceleration formulas [p. 553]
- $v = u + at$
- $s = ut + \frac{1}{2}at^2$
- $v^2 = u^2 + 2as$
- $s = \frac{1}{2}(u + v)t$

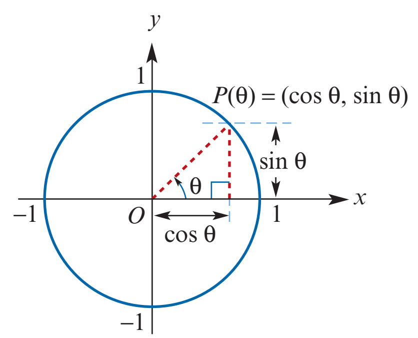
Contrapositive [p. 69] The contrapositive of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P). The contrapositive is equivalent to the original statement.

Convergent series [p. 23] An infinite series $t_1 + t_2 + t_3 + \cdots$ is convergent if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. An infinite geometric series is convergent if $|r| < 1$, where r is the common ratio.

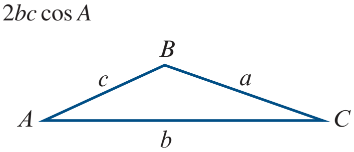
Converse [p. 70] The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Cosecant function [p. 102] $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ for $\sin\theta \neq 0$

Cosine function [p. 2] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Cosine rule [p. 16] For triangle ABC:
$a^2 = b^2 + c^2 - 2bc\cos A$



Cotangent function [p. 103] $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ for $\sin\theta \neq 0$

Counterexample [p. 76] an example that shows that a universal statement is false. For example, the number 2 is a counterexample to the claim 'Every prime number is odd.'

Cross product [p. 212] *see* vector product

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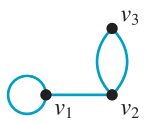
D

De Moivre's theorem [p. 258]
$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta)$, where $n \in \mathbb{Z}$

De Morgan's laws [p. 68]
- 'not (P and Q)' is '(not P) or (not Q)'
- 'not (P or Q)' is '(not P) and (not Q)'

Definite integral [pp. 383, 430] $\int_a^b f(x)\,dx$ denotes the signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

Degree of a vertex of a graph [SM1&2] the number of edges that end at the vertex, with each edge that is a loop counted twice. For example:



Vertex	Degree
v_1	3
v_2	3
v_3	2

Derivative function [p. 317] also called the gradient function. The derivative f' of a function f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative of a vector function [p. 594]

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

$$\dot{\boldsymbol{r}}(t) = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

$$\ddot{\boldsymbol{r}}(t) = \frac{d^2x}{dt^2}\boldsymbol{i} + \frac{d^2y}{dt^2}\boldsymbol{j} + \frac{d^2z}{dt^2}\boldsymbol{k}$$

Derivatives, basic [pp. 317, 318, 320]

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\log_e \lvert ax \rvert$	$\frac{1}{x}$

$f(x)$	$f'(x)$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$

Derivatives, inverse circular [pp. 327–328]

$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2 + x^2}$

Desk check [p. 52] To carry out a desk check of an algorithm, you carefully follow the algorithm step by step, and construct a table of the values of all the variables after each step.

Determinant of a matrix [SM1&2] Associated with each square matrix **A**, there is a real number called the determinant of **A**, which is denoted by det(**A**). A square matrix **A** has an inverse if and only if det(**A**) ≠ 0.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

Differential equation [p. 478] an equation involving derivatives of a particular function or variable; e.g.

$$\frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = \frac{y}{y+1}$$

Differential equation, general solution [p. 478] $y = \sin x + c$ is the general solution of the differential equation $\frac{dy}{dx} = \cos x$.

Differential equation, particular solution [p. 478] $y = \sin x$ is the particular solution of the differential equation $\frac{dy}{dx} = \cos x$, given $y(0) = 0$.

Direct proof [p. 68] To give a direct proof of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Displacement [p. 538] the change in position. If a particle moves from point A to point B, then its displacement is described by the vector $\overrightarrow{AB}$.

Distance from a point *P* to a line [p. 202] given by $|\overrightarrow{PQ}|$, where Q is the point on the line such that PQ is perpendicular to the line

Distance from a point *P* to a plane [p. 223] given by $|\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}|$, where $\hat{\boldsymbol{n}}$ is a unit vector normal to the plane and Q is any point on the plane

Distance in the complex plane [p. 277] The distance between complex numbers z_1 and z_2 is equal to $|z_2 - z_1|$.

Divisible [p. 66] For two integers a and b, we say that a is divisible by b if there exists an integer k such that $a = bk$.

Division of complex numbers [pp. 248, 257]

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Divisor [p. 66] For two integers a and b, we say that b is a divisor of a if there exists an integer k such that $a = bk$.

Dot product [p. 167] *see* scalar product

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Double angle formulas [p. 112]

- $\cos(2x) = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$
- $\sin(2x) = 2 \sin x \cos x$
- $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$

E

Ellipse [p. 36] The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).

Equality of complex numbers [p. 240]

$a + bi = c + di$ if and only if $a = c$ and $b = d$

Equivalence of vectors [p. 155]
Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$.
If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Equivalent statements [p. 71]
Statements P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$; this is abbreviated to $P \Leftrightarrow Q$.
For equivalent statements P and Q, we also say 'P is true *if and only if* Q is true'.

Euler's method [p. 518] a numerical method for solving a differential equation.
If $\dfrac{dy}{dx} = g(x)$ with $y = y_0$ when $x = x_0$, then

$x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hg(x_n)$

The sequence of points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots$ approximates a solution curve.

Existence statement [p. 75] a statement claiming that a property holds for some member of a given set. Such a statement can be written using the quantifier 'there exists'.

Expected value of a random variable, E(X) [p. 647] also called the mean, μ.

- For a discrete random variable X:
  $\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x)$
- For a continuous random variable X:
  $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

F

Factor theorem [p. 266] A polynomial $P(z)$ has $z - \alpha$ as a factor if and only if $P(\alpha) = 0$.

Factorise [p. 270] In the complex number system, every non-constant polynomial can be expressed as a product of linear factors.

Fundamental theorem of algebra [p. 270]
Every non-constant polynomial with complex coefficients has at least one linear factor in the complex number system.

Fundamental theorem of calculus [p. 430]
If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any antiderivative of f and $\int_a^b f(x)\, dx$ is the definite integral from a to b.

G

Geometric sequence [p. 21] a sequence in which each successive term is found by multiplying the previous term by a fixed amount; e.g. $2, 6, 18, 54, \dots$. A geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is the common ratio. The nth term can be found using $t_n = ar^{n-1}$, where $a = t_1$.

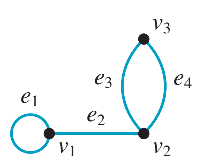
Geometric series [p. 22] the sum of the terms in a geometric sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Gradient function *see* derivative function

Graph [SM1&2] A graph consists of a finite non-empty set of *vertices*, a finite set of *edges* and an *edge-endpoint function* that maps each edge to a set of either one or two vertices. A graph can be represented by a diagram, where the vertices are shown as points and the edges as lines connecting the vertices. For example:



Edge	Endpoints
e_1	$\{v_1\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2, v_3\}$

H

Hyperbola [p. 39] The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k); the asymptotes are given by

$$y - k = \pm\frac{b}{a}(x - h)$$

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I

Imaginary number i [p. 238] $i^2 = -1$

Imaginary part of a complex number [p. 238] If $z = a + bi$, then $\text{Im}(z) = b$.

Implication [p. 68] *see* conditional statement

Implicit differentiation [p. 365] used to find the gradient at a point on a curve such as $x^2 + y^2 = 1$, which is not defined by a rule of the form $y = f(x)$ or $x = f(y)$

Infinite geometric series [p. 22] If $|r| < 1$, then the sum to infinity is given by

$$S_\infty = \frac{a}{1-r}$$

where $a = t_1$ and r is the common ratio.

Integers [p. 66] $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Integrals, standard [pp. 383, 390]

$f(x)$	$\int f(x)\,dx$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e\lvert ax+b\rvert + c$
e^{ax+b}	$\frac{1}{a}e^{ax+b} + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1}\left(\frac{x}{a}\right) + c$
$\frac{a}{a^2+x^2}$	$\tan^{-1}\left(\frac{x}{a}\right) + c$

Integration by parts [p. 415]

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Integration by substitution [p. 393]

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

Inverse cosine function (arccos) [p. 116]

$\cos^{-1} x = y$ if $\cos y = x$,

for $x \in [-1, 1]$ and $y \in [0, \pi]$

Inverse sine function (arcsin) [p. 115]

$\sin^{-1} x = y$ if $\sin y = x$,

for $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Inverse tangent function (arctan) [p. 116]

$\tan^{-1} x = y$ if $\tan y = x$,

for $x \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Iteration [p. 51] In an algorithm, we can use looping constructs to repeat steps in a controlled way; e.g. `for` loops and `while` loops.

Iterative rule [p. 19] *see* recurrence relation

L

Length of a parametric curve [p. 459] If the point $P(f(t), g(t))$ traces the curve exactly once from $t = a$ to $t = b$, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Limits of integration [p. 383] In the expression $\int_a^b f(x)\,dx$, the number a is called the *lower limit* of integration and b the *upper limit* of integration.

Line in three dimensions [pp. 198, 200] can be described as follows, where $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is the position vector of a point A on the line, and $\boldsymbol{d} = d_1\boldsymbol{i} + d_2\boldsymbol{j} + d_3\boldsymbol{k}$ is parallel to the line:

Vector equation	$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d},\ t \in \mathbb{R}$
Parametric equations	$x = a_1 + d_1t$ $y = a_2 + d_2t$ $z = a_3 + d_3t$
Cartesian form	$\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$

Linear approximation formula [p. 518]

$f(x+h) \approx f(x) + hf'(x)$

Linear combination of independent normal random variables [p. 661] If X and Y are independent normal random variables and $a, b \in \mathbb{R}$, then $aX + bY$ is also a normal random variable (provided a and b are not both zero).

Linear combination of independent random variables [p. 656] If X and Y are independent random variables and $a, b \in \mathbb{R}$, then:

- $\text{E}(aX + bY) = a\,\text{E}(X) + b\,\text{E}(Y)$
- $\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$

Linear combination of vectors [p. 150] A vector $\boldsymbol{w}$ is a linear combination of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ if it can be expressed in the form

$\boldsymbol{w} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \dots + k_n\boldsymbol{v}_n$

where $k_1, k_2, \dots, k_n$ are real numbers.

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Linear dependence [p. 150]
- A set of vectors is linearly dependent if at least one of its members can be expressed as a linear combination of other vectors in the set.
- Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent if there exist real numbers k, ℓ and m, not all zero, such that $k\boldsymbol{a} + \ell\boldsymbol{b} + m\boldsymbol{c} = \mathbf{0}$.

Linear function of a random variable [p. 646]
- $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$
- $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

Linear independence [p. 150]
- A set of vectors is linearly independent if no vector in the set is expressible as a linear combination of other vectors in the set.
- Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent if $k\boldsymbol{a} + \ell\boldsymbol{b} + m\boldsymbol{c} = \mathbf{0}$ implies $k = \ell = m = 0$.

Local maximum stationary point [p. 340] If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum, as the curve is concave down.

Local minimum stationary point [p. 340] If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum, as the curve is concave up.

Locus [p. 277] a set of points described by a geometric condition; e.g. the locus of the equation $|z - 1 - i| = 2$ is the circle with centre $1 + i$ and radius 2

Logistic differential equation [p. 505]

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad 0 < P < K$$

This differential equation can be used to model a population P at time t, where:
- the constant r is called the *growth parameter*
- the constant K is called the *carrying capacity*.

Loop in a graph [SM1&2] an edge that joins a vertex to itself

Loop in an algorithm [p. 51] a sequence of instructions that is to be repeated. Each repeat is a *pass* of the loop.

M

Magnitude of a vector [p. 144] the length of a directed line segment corresponding to the vector.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

Mathematical induction [p. 84] a proof technique for showing that a statement is true for all natural numbers; uses the *principle of mathematical induction*

Matrices, multiplication [SM1&2] If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

For 2×2 matrices:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

Matrix [SM1&2] a rectangular array of numbers. A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Matrix, identity [SM1&2] For square matrices of a given size (e.g. 2×2), there is a multiplicative identity matrix $\mathbf{I}$.

For 2×2 matrices, the identity is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

and $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ for each 2×2 matrix $\mathbf{A}$.

Matrix, inverse [SM1&2] If $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

provided $ad - bc \neq 0$.

Matrix, square [SM1&2] A matrix with the same number of rows and columns is called a square matrix; e.g. a 2×2 matrix.

Mean of a random variable, μ [p. 647] *see* expected value of a random variable, $\mathrm{E}(X)$

Modulus–argument form [p. 251] *see* polar form of a complex number

Modulus function [p. 29] The modulus of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *absolute value function*

Modulus of a complex number, $|z|$ [pp. 246, 251] the distance of the complex number from the origin. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

Modulus, properties [p. 246] For complex numbers z_1 and z_2:
- $|z_1z_2| = |z_1|\,|z_2|$ (the modulus of a product is the product of the moduli)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)

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Multiplication of a complex number by a real number [pp. 241, 256]

- If $z = a + bi$ and $k \in \mathbb{R}$, then $kz = ka + kbi$.
- If $z = r \operatorname{cis} \theta$ and $k > 0$, then $kz = kr \operatorname{cis} \theta$.
- If $z = r \operatorname{cis} \theta$ and $k < 0$, then $kz = |k| r \operatorname{cis}(\theta + \pi)$.

Multiplication of a vector by a scalar [p. 145] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $m \in \mathbb{R}$, then $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j} + ma_3\boldsymbol{k}$.

Multiplication of complex numbers [pp. 244, 256] If $z_1 = a + bi$ and $z_2 = c + di$, then

$z_1 z_2 = (ac - bd) + (ad + bc)i$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

N

Natural numbers [p. 66] $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Negation [pp. 68, 75–76] The negation of a statement P is the opposite statement, called 'not P'.

Normal vector to a plane [p. 217] a vector that is perpendicular to the plane

Null hypothesis, H_0 [p. 693] asserts that the sample is drawn from a population with the same mean as before

O

One-tail test [p. 703] used when the alternative hypothesis is directional (< or >)

Operator notation for differentiation [p. 319] emphasises that differentiation is an operation on an expression; e.g.

$$\frac{d}{dx}(x^2 + 5x + 3) = 2x + 5$$

P

***p*-value** [p. 694] the probability of observing a value of the sample statistic as extreme as or more extreme than the one observed, assuming that the null hypothesis is true

Parametric equations [p. 43] A pair of equations of the form $x = f(t)$ and $y = g(t)$ describes a curve in the plane, where t is called the *parameter* of the curve. For example:

- Circle $x = a\cos t$ and $y = a\sin t$
- Ellipse $x = a\cos t$ and $y = b\sin t$
- Hyperbola $x = a\sec t$ and $y = b\tan t$

Similarly, equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ describe a curve in three-dimensional space.

Partial fractions [p. 407] Some rational functions may be expressed as a sum of partial fractions; e.g.

$$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{ex^2+fx+g}$$

Particle model [p. 537] an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

Period of a function [p. 4] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f.

- Sine and cosine have period 2π.
- Tangent has period π.
- A function of the form $y = a\cos(nx + \varepsilon) + b$ or $y = a\sin(nx + \varepsilon) + b$ has period $\frac{2\pi}{n}$.

Pi notation [p. 89] *see* product notation

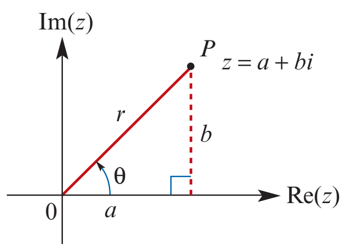
Plane in three dimensions [p. 217] can be described as follows, where $\boldsymbol{a}$ is the position vector of a point A on the plane, $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$ is normal to the plane, and $k = \boldsymbol{a} \cdot \boldsymbol{n}$:

Vector equation	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation	$n_1 x + n_2 y + n_3 z = k$

Point estimate [p. 683] If the value of the sample mean $\bar{x}$ is used as an estimate of the population mean μ, then it is called a point estimate of μ.

Point of inflection [p. 336] a point where a curve changes from concave up to concave down or from concave down to concave up. That is, a point of inflection occurs where the sign of the second derivative changes.

Polar form of a complex number [p. 251] A complex number is expressed in polar form as $z = r\operatorname{cis}\theta$, where r is the modulus of z and θ is an argument of z.



Population [p. 663] the set of all eligible members of a group which we intend to study

Population mean, μ [p. 663] the mean of all values of a measure in the entire population

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Population parameter a statistical measure that is based on the whole population; the value is constant for a given population

Position [p. 538] For a particle moving in a straight line, the position of the particle relative to a point O on the line is determined by its distance from O and whether it is to the right or left of O. The direction to the right of O is positive.

Position vector [p. 147] A position vector, $\overrightarrow{OP}$, indicates the position in space of the point P relative to the origin O.

Principle of mathematical induction [p. 84] To prove that a statement $P(n)$ is true for every natural number n:

- Show that $P(1)$ is true.
- Show that, for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

Product notation [p. 89] used for writing products concisely. For example:

$$\prod_{k=1}^{n}(2k-1) = 1 \times 3 \times 5 \times \cdots \times (2n-1)$$

Product rule [p. 317]

- If $f(x) = g(x)\,h(x)$, then $f'(x) = g'(x)\,h(x) + g(x)\,h'(x)$.
- If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Product-to-sum identities [p. 129]

- $2\cos x\cos y = \cos(x-y) + \cos(x+y)$
- $2\sin x\sin y = \cos(x-y) - \cos(x+y)$
- $2\sin x\cos y = \sin(x+y) + \sin(x-y)$

Proof by contradiction [p. 69] a proof that begins by assuming the negation of what is to be proved

Proof by induction [p. 84] a proof that uses the *principle of mathematical induction*

Pseudocode [p. 51] a notation for describing algorithms that is less formal than a programming language

Pythagorean identity [pp. 6, 105]

- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

Q

Quadratic formula [p. 264] An equation of the form $az^2 + bz + c = 0$, with $a \neq 0$, may be solved using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quantifier [p. 75] 'for all', 'there exists'

Quotient rule [p. 317]

- If $f(x) = \dfrac{g(x)}{h(x)}$, then $f'(x) = \dfrac{g'(x)\,h(x) - g(x)\,h'(x)}{(h(x))^2}$.
- If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

R

Radian [p. 3] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Random sample [p. 663] a sample chosen using a random process so that each member of the population has an equal chance of being included

Rational function [p. 354] a function of the form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials

Real part of a complex number [p. 238] If $z = a + bi$, then $\operatorname{Re}(z) = a$.

Reciprocal circular functions [p. 102] the cosecant, secant and cotangent functions

Reciprocal function [p. 358] The reciprocal of the function $y = f(x)$ is defined by $y = \dfrac{1}{f(x)}$.

Recurrence relation [p. 19] a rule which enables each subsequent term of a sequence to be found from previous terms; e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$

Reduction formula [p. 418] a recursive formula that expresses an integral in terms of a simpler integral of the same form

Remainder theorem [p. 266] Let $\alpha \in \mathbb{C}$. When a polynomial $P(z)$ is divided by $z - \alpha$, the remainder is $P(\alpha)$.

Restricted cosine function [p. 116] $f\colon [0, \pi] \to \mathbb{R},\ f(x) = \cos x$

Restricted sine function [p. 115] $f\colon \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to \mathbb{R},\ f(x) = \sin x$

Restricted tangent function [p. 116] $f\colon \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R},\ f(x) = \tan x$

Roots of a complex number [p. 274] The nth roots of a complex number a are the solutions of the equation $z^n = a$. If $a = 1$, then the solutions are called the *nth roots of unity*.

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S

Sample [p. 663] a subset of the population which we select in order to make inferences about the whole population

Sample mean, $\bar{x}$ [p. 663] the mean of all values of a measure in a particular sample. The values $\bar{x}$ are the values of a random variable $\bar{X}$.

Sample statistic a statistical measure that is based on a sample from the population; the value varies from sample to sample

Scalar [p. 145] a real number; name used when working with vectors or matrices

Scalar product [p. 167] The scalar product of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is given by
$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$

Scalar product, properties [p. 168]
- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Scalar quantity [p. 537] a quantity determined by its magnitude; e.g. distance, time, speed

Scalar resolute [p. 172] The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

Secant function [p. 102] $\sec\theta = \dfrac{1}{\cos\theta}$ for $\cos\theta \neq 0$

Second derivative [p. 331]
- The second derivative of a function f with rule $f(x)$ is denoted by f'' and has rule $f''(x)$.
- The second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Second derivative test [p. 340]
- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
- If $f''(a) = 0$, then further investigation is necessary.

Selection [p. 51] In an algorithm, we can use decision-making constructs to specify whether certain steps should be followed based on some condition; e.g. `if-then` blocks.

Separation of variables [p. 507]
If $\dfrac{dy}{dx} = f(x)\,g(y)$, then $\int f(x)\,dx = \int \dfrac{1}{g(y)}\,dy$.

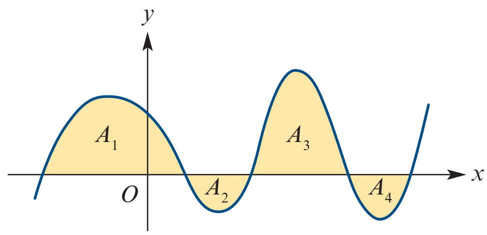
Sequence [p. 19] a list of numbers, with the order being important; e.g. $1, 1, 2, 3, 5, 8, 13, \ldots$ The numbers of a sequence are called its *terms*, and the nth term is often denoted by t_n.

Series [p. 21] the sum of the terms in a sequence

Sigma notation [p. 87] *see* summation notation

Signed area [p. 430]
- Regions *above* the x-axis are defined to have *positive* signed area.
- Regions *below* the x-axis are defined to have *negative* signed area.

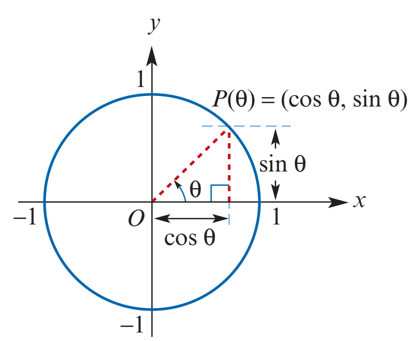
For example, the signed area of the shaded region in the following graph is $A_1 - A_2 + A_3 - A_4$.



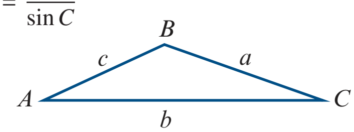
Significance level, α [p. 698] the condition for rejecting the null hypothesis:
- If the p-value is less than α, then we reject the null hypothesis in favour of the alternative hypothesis.
- If the p-value is greater than α, then we do not reject the null hypothesis.

Simulation [p. 666] using technology (calculators or computers) to repeat a random process many times; e.g. random sampling

Sine function [p. 2] sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Sine rule [p. 14] For triangle ABC:
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$



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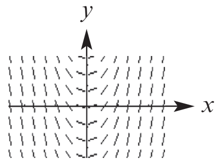
Skew lines [p. 207] In three-dimensional space, two lines are skew if they do not intersect and are not parallel.

Slope field [p. 525] The slope field of a differential equation

$$\frac{dy}{dx} = f(x, y)$$



assigns to each point $P(x, y)$ in the plane the number $f(x, y)$, which is the gradient of the solution curve through P.

Solid of revolution [p. 451] the solid formed by rotating a region about a line

Speed [pp. 539, 600] the magnitude of velocity

Speed, average [p. 539]

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Standard deviation of a random variable, σ a measure of the spread or variability, given by $\text{sd}(X) = \sqrt{\text{Var}(X)}$

Subtraction of complex numbers [p. 240] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 - z_2 = (a - c) + (b - d)i$.

Subtraction of vectors [p. 146] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$.

Sum of independent random variables [pp. 651, 656] If X and Y are independent random variables, then:

- $\text{E}(X + Y) = \text{E}(X) + \text{E}(Y)$
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

Sum to infinity [p. 23] The sum to infinity of an infinite geometric series exists provided $|r| < 1$ and is given by

$$S_\infty = \frac{a}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Sum-to-product identities [p. 130]

- $\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$
- $\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$
- $\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$
- $\sin x - \sin y = 2\sin\left(\frac{x - y}{2}\right)\cos\left(\frac{x + y}{2}\right)$

Summation notation [p. 87] used for writing sums concisely. For example:

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

Surface area of a solid of revolution [p. 464] equals the area of the curved surface of revolution plus the areas of the circular discs at each end

Surface of revolution [p. 462] the curved surface formed by rotating a section of a curve about a line

T

Tangent function [p. 2] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$

Telescopic cancelling [p. 82] can be used to find the partial sums of some sequences

Two-tail test [p. 703] used when the alternative hypothesis is non-directional (≠)

Type I error [p. 713] occurs if we reject the null hypothesis H_0 when it is true

Type II error [p. 713] occurs if we do not reject the null hypothesis H_0 when it is false

U

Unit vector [p. 155] a vector of magnitude 1. The unit vectors in the positive directions of the x-, y- and z-axes are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. The unit vector in the direction of $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Universal statement [p. 75] a statement claiming that a property holds for all members of a given set. Such a statement can be written using the quantifier 'for all'.

V

Variance of a random variable, σ² [p. 648] a measure of the spread or variability, defined by $\text{Var}(X) = \text{E}[(X - \mu)^2]$

An alternative (computational) formula is $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

Vector [p. 144] a set of equivalent directed line segments

Vector equation [pp. 198, 584] An equation of the form $\boldsymbol{r} = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$ describes a curve in the plane. This curve can also be described by the parametric equations $x = f(t)$ and $y = g(t)$. Similarly, an equation $\boldsymbol{r} = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ describes a curve in three-dimensional space.

Vector function [p. 584] If $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$, then we say that $\boldsymbol{r}$ is a vector function of t.

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Vector product, formula [p. 214]
For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is given by
$(a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$

Vector product, geometric properties [p. 212]
- Magnitude $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, where θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
- Direction $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ (if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel non-zero vectors)

Vector product, properties [pp. 212–214]
- $k(\boldsymbol{a} \times \boldsymbol{b}) = (k\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (k\boldsymbol{b})$
- $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$
- $\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b})$
- $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{0} = \boldsymbol{0}$

Vector quantity [p. 537] a quantity determined by its magnitude and direction; e.g. position, displacement, velocity, acceleration

Vector resolute [p. 172] The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by
$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Vectors, parallel [p. 147] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = k\boldsymbol{b}$ for some $k \in \mathbb{R} \setminus \{0\}$.

Vectors, perpendicular [p. 168] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Vectors, properties [p. 148]

$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$	commutative law
$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$	associative law
$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$	zero vector
$\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$	additive inverse
$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$	distributive law

Vectors, resolution [p. 172] A vector $\boldsymbol{a}$ is resolved into rectangular components by writing it as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

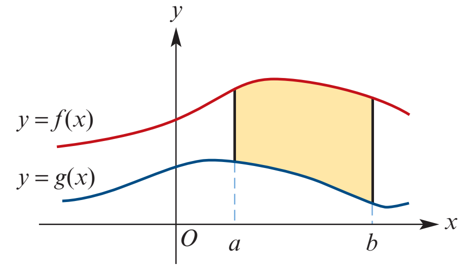
Velocity [pp. 539, 600] the rate of change of position with respect to time

Velocity, average [p. 539]
$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Velocity, instantaneous [p. 539] $v = \dfrac{dx}{dt}$

Velocity–time graph [p. 558]
- Acceleration is given by the gradient.
- Displacement is given by the signed area bounded by the graph and the t-axis.
- Distance travelled is given by the total area bounded by the graph and the t-axis.

Volume of a solid of revolution [p. 451]
- Rotation about the x-axis
  If the region is bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis, then
  $V = \int_a^b \pi y^2 \, dx = \pi \int_a^b (f(x))^2 \, dx$
- Rotation about the y-axis
  If the region is bounded by the curve $x = f(y)$, the lines $y = a$ and $y = b$ and the y-axis, then
  $V = \int_a^b \pi x^2 \, dy = \pi \int_a^b (f(y))^2 \, dy$
- Region not bounded by the x-axis
  If the shaded region is rotated about the x-axis, then the volume V is given by
  $V = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx$



Z

***z*-test** [p. 699] the hypothesis test for a mean of a sample drawn from a normally distributed population with known standard deviation

Zero vector, 0 [p. 146] a line segment of zero length with no direction

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Answers

Chapter 1

Exercise 1A

1 a i 4π **ii** 3π **iii** $-\frac{5\pi}{2}$

iv $\frac{\pi}{12}$ **v** $-\frac{\pi}{18}$ **vi** $-\frac{7\pi}{4}$

b i $225°$ **ii** $-120°$ **iii** $105°$

iv $-330°$ **v** $260°$ **vi** $-165°$

2 a i 0.12^c **ii** -1.75^c **iii** -0.44^c

iv 0.89^c **v** 3.60^c **vi** -7.16^c

b i $97.40°$ **ii** $-49.85°$ **iii** $160.43°$

iv $5.73°$ **v** $-171.89°$ **vi** $-509.93°$

3 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $\frac{\sqrt{3}}{2}$

d $-\frac{1}{2}$ **e** $\frac{1}{\sqrt{2}}$ **f** $\frac{\sqrt{3}}{2}$

4 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{2}$

d $-\frac{1}{\sqrt{2}}$ **e** $\frac{1}{\sqrt{2}}$ **f** $-\frac{\sqrt{3}}{2}$

g $-\frac{\sqrt{3}}{2}$ **h** $-\frac{\sqrt{3}}{2}$ **i** $\frac{1}{2}$

5 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{3}}$

6 a $-\frac{\sqrt{51}}{10}$ **b** $\frac{\sqrt{51}}{7}$

7 a $-\frac{\sqrt{3}}{2}$ **b** $\frac{1}{\sqrt{3}}$

8 a $\frac{\sqrt{91}}{10}$ **b** $-\frac{3\sqrt{91}}{91}$

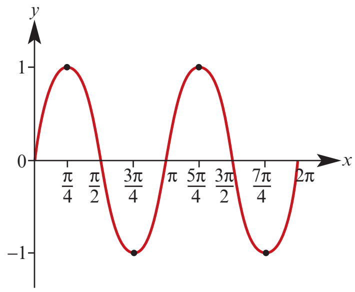
9 $2\pi - a,\ 2\pi - b,\ 2\pi - c,\ 2\pi - d$

10 a $\frac{4\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

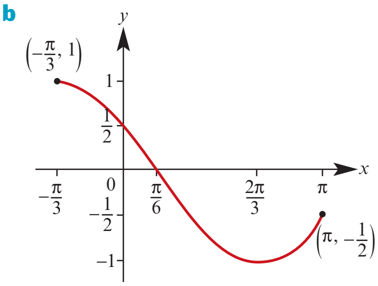
c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **d** $\frac{5\pi}{6}, \frac{3\pi}{2}$

e $0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ **f** $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

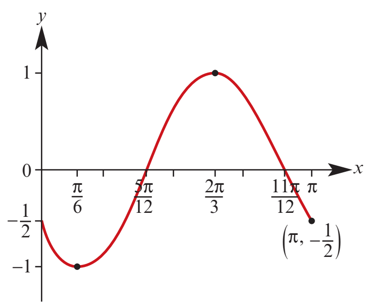
11 a



b



c



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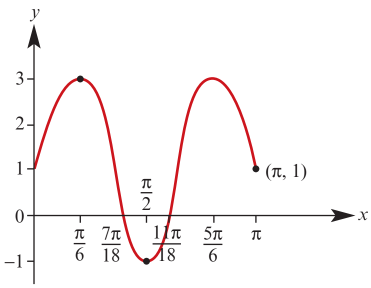
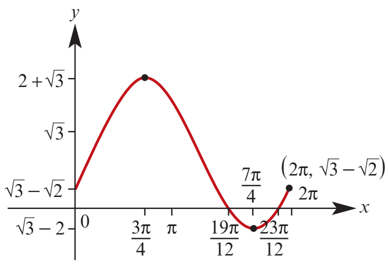
d



e



12 a 1 **b** $\sqrt{3}$ **c** $\frac{1}{\sqrt{3}}$ **d** $\sqrt{3}$

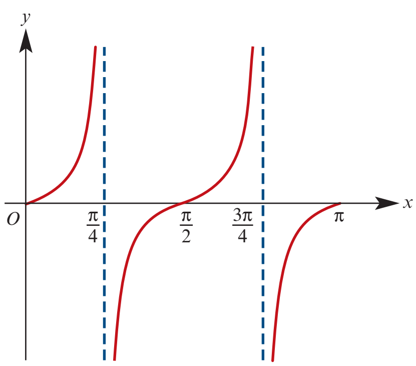
13 a $-\frac{\sqrt{17}}{17}$ **b** $-\frac{4\sqrt{17}}{17}$ **c** $-\frac{1}{4}$ **d** $-\frac{1}{4}$

14 a $\frac{\sqrt{21}}{7}$ **b** $-\frac{2\sqrt{7}}{7}$ **c** $\frac{\sqrt{3}}{2}$ **d** $-\frac{\sqrt{3}}{2}$

15 a $\frac{2\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

c $\frac{3\pi}{2}$ **d** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

16 a



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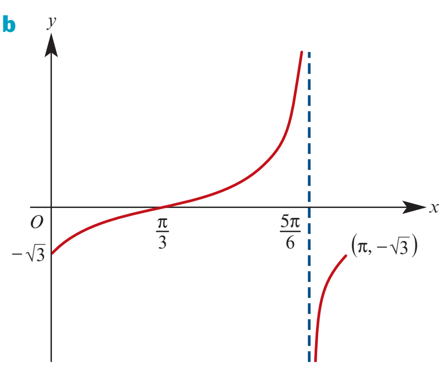

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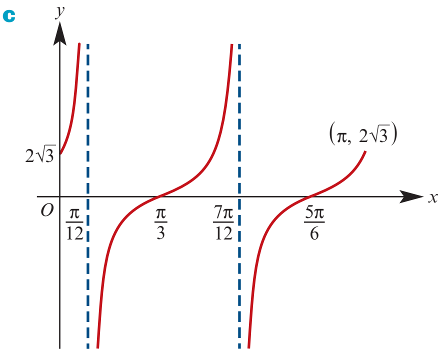
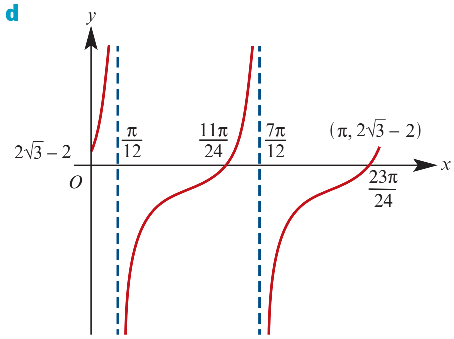


d



Exercise 1B

1 a 11.67 cm **b** 9.62 cm

2 a 58.08°, 121.92° **b** 10.01 cm, 4.09 cm

3 a 7.15 cm **b** 50.43°

4 a 54.90° **b** 100.95°

5 16.71 cm

6 a 6.71 cm
b 121.33° (acute angle is inconsistent)

7 $6\sqrt{6}$ cm

8 $\sqrt{7}$ cm

9 30.10 cm

10 $5\sqrt{3} \pm \sqrt{39}$ cm

Exercise 1C

1 3, −1, −5, −9 **2** $t_1 = -2,\ t_n = -3t_{n-1}$

3 −1, 1, 3, 5

4 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

5 −31 **6** −39 366

7 210 **8** −9840

9 $\frac{3}{4}$

10 a 20 **b** $\frac{4}{5}$ **c** $\frac{4^{10}}{5^7}$

11 $a(2+\sqrt{2})$

12 a $4\left(1-\left(\frac{3}{4}\right)^{10}\right)$

b i $-2 < x < 2$ **ii** $\pm 2^{\frac{9}{10}}$

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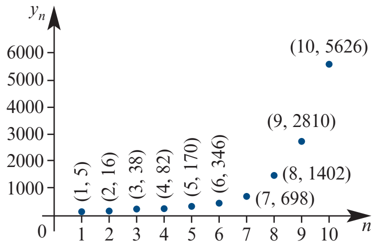
13 a $\dfrac{1}{1-\sin\theta}$

b $\dfrac{\pi}{6}+2k\pi,\ \dfrac{5\pi}{6}+2k\pi$, for $k \in \mathbb{Z}$

14 $t_2 = 17,\ t_3 = 50,\ t_8 = 12\,029$

15 $y_2 = 16,\ y_3 = 38,\ y_{10} = 5626$



16 a $t_n = 8$

b $t_n = 8 \times 2^{n-1} + 2$

c $t_n = \dfrac{28}{2^{n-1}} + 12$

17 $t_n = 2^{n-1} + 5$

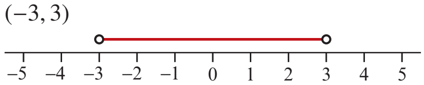
18 $t_n = \dfrac{1}{4}(15 \times 5^{n-1} + 17)$

Exercise 1D

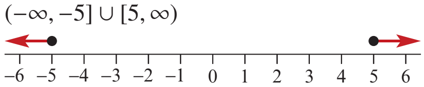
1 a 8 **b** 8 **c** 2 **d** −2 **e** −2 **f** 4

2 a 3, −1 **b** $\frac{7}{2}, -\frac{1}{2}$ **c** $\frac{12}{5}, -\frac{6}{5}$ **d** 12, −6 **e** −1, 7 **f** $\frac{4}{3}, -4$ **g** $-\frac{2}{5}, -4$

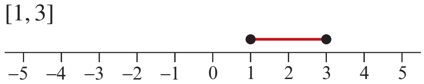
3 a $(-3, 3)$



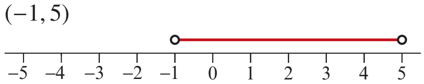
b $(-\infty, -5] \cup [5, \infty)$



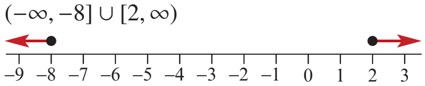
c $[1, 3]$



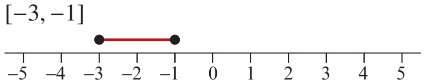
d $(-1, 5)$



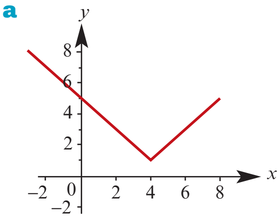
e $(-\infty, -8] \cup [2, \infty)$



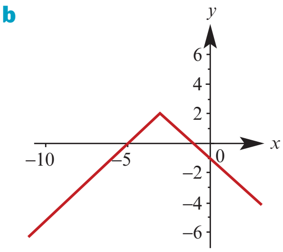
f $[-3, -1]$



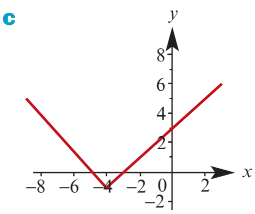
4 a Range $[1, \infty)$



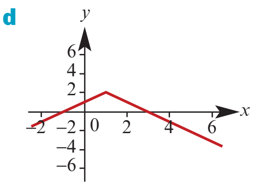
b Range $(-\infty, 2]$



c Range $[-1, \infty)$



d Range $(-\infty, 2]$



5 a $\{x : -5 \le x \le 5\}$

b $\{x : x \le -2\} \cup \{x : x \ge 2\}$

c $\{x : 1 \le x \le 2\}$ **d** $\{x : -\frac{1}{5} < x < 1\}$

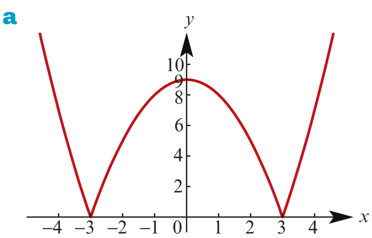
e $\{x : x \le -4\} \cup \{x : x \ge 10\}$

f $\{x : 1 \le x \le 3\}$

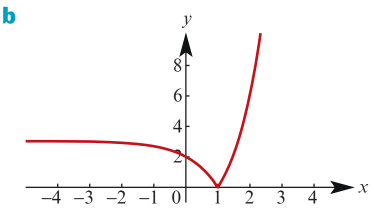
6 a $x \le -2$ **b** $x = -9$ or $x = 11$

c $x = -\dfrac{5}{4}$ or $x = \dfrac{15}{4}$

7 a



b



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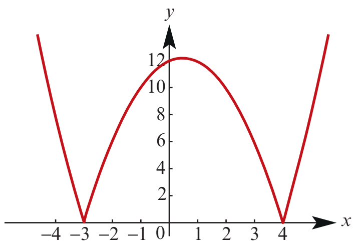
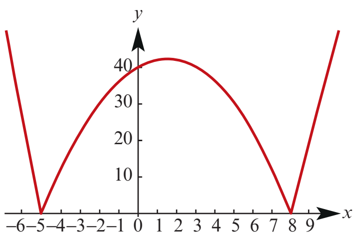
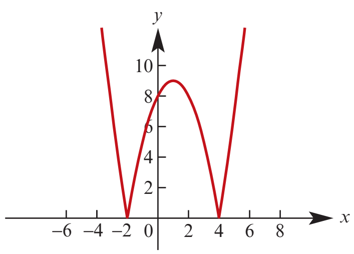
c



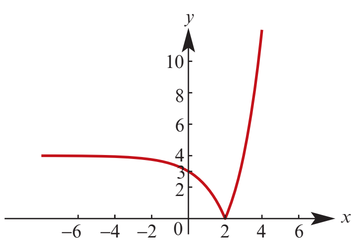
d



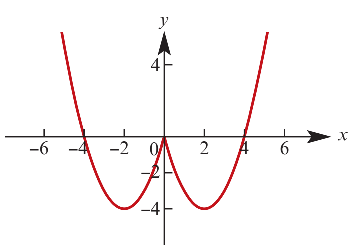
e



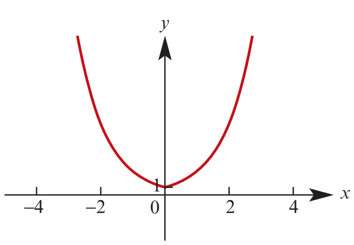
f



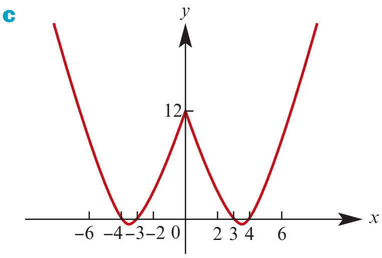
8 a



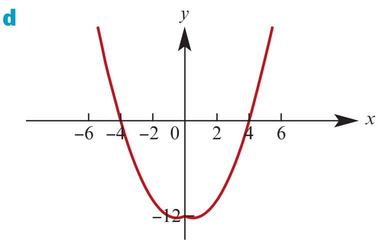
b



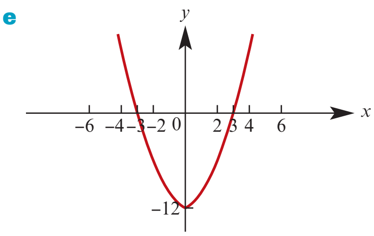
c



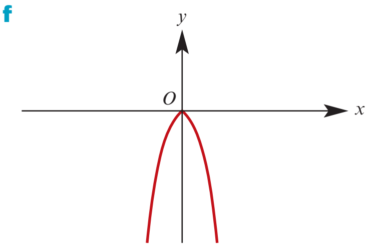
d



e



f



9 $a = 1, \; b = 1$

Exercise 1E

1 a $(x-2)^2 + (y-3)^2 = 1$

b $(x+3)^2 + (y-4)^2 = 25$

c $x^2 + (y+5)^2 = 25$

d $(x-3)^2 + y^2 = 2$

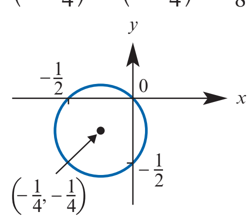
2 a Centre $(-2, 3)$; radius 1

b Centre $(1, 2)$; radius 2

c Centre $\left(\frac{3}{2}, 0\right)$; radius $\frac{3}{2}$

d Centre $(-2, 5)$; radius 2

3 a $\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{1}{8}$



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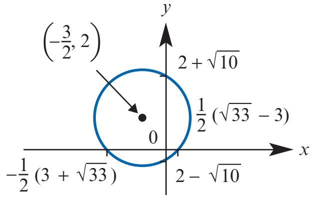
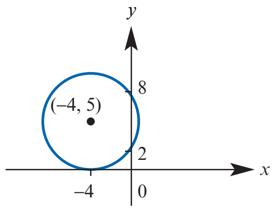
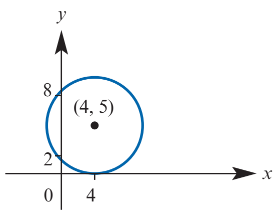
b $\left(x + \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{49}{4}$



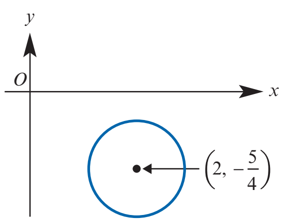
c $(x + 4)^2 + (y - 5)^2 = 25$



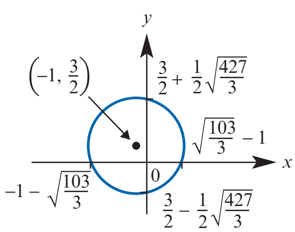
d $(x - 4)^2 + (y - 5)^2 = 25$



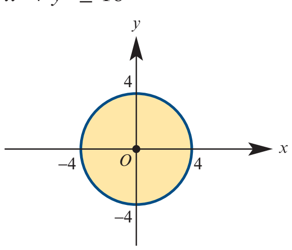
e $(x - 2)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{9}{16}$



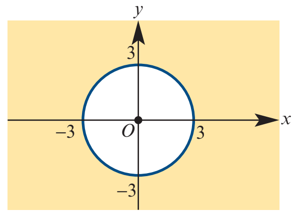
f $(x + 1)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{439}{12}$



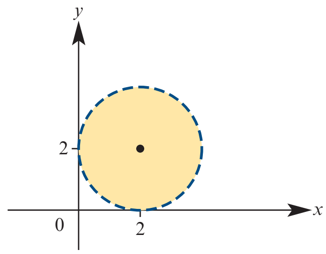
4 a $x^2 + y^2 \le 16$



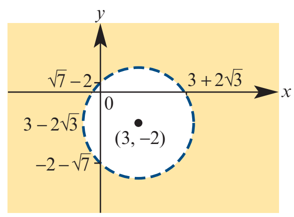
b $x^2 + y^2 \ge 9$



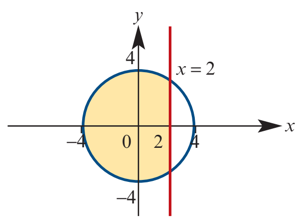
c $(x - 2)^2 + (y - 2)^2 < 4$



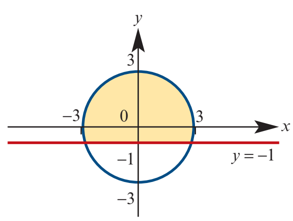
d $(x - 3)^2 + (y + 2)^2 > 16$



e $x^2 + y^2 \le 16$ and $x \le 2$



f $x^2 + y^2 \le 9$ and $y \ge -1$



5 Centre $(5, 3)$; radius $\sqrt{10}$

6 $(x - 2)^2 + (y + 3)^2 = 9$

7 $(x - 5)^2 + (y - 4)^2 = 13$

8 a First circle: centre $\left(\frac{15}{2}, \frac{19}{2}\right)$; radius $\frac{5\sqrt{2}}{2}$

Second circle: centre $(5, 7)$; radius 5

b $(5, 12), (10, 7)$

9 a $\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right), \left(-\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$

b $(\sqrt{5}, 2\sqrt{5}), (-\sqrt{5}, -2\sqrt{5})$

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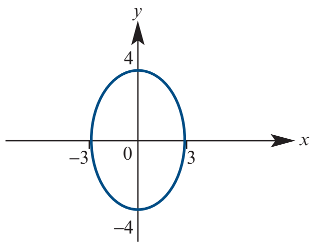
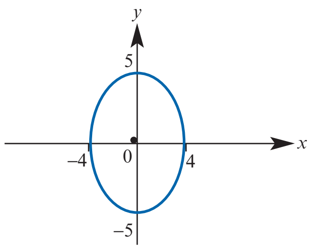
Exercise 1F

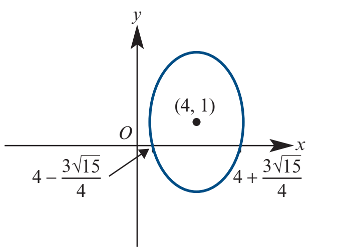
1 a $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, centre $(0, 0)$



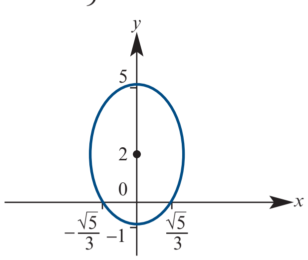
b $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$, centre $(0, 0)$



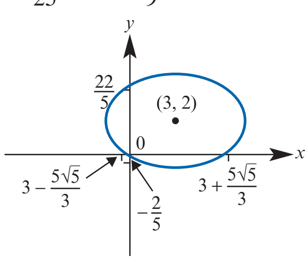
c $\dfrac{(x-4)^2}{9} + \dfrac{(y-1)^2}{16} = 1$, centre $(4, 1)$



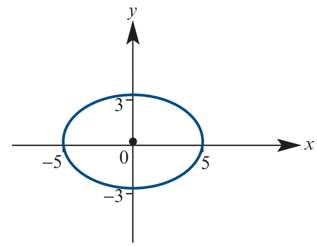
d $x^2 + \dfrac{(y-2)^2}{9} = 1$, centre $(0, 2)$



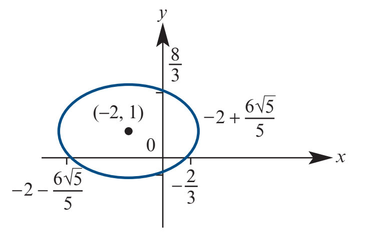
e $\dfrac{(x-3)^2}{25} + \dfrac{(y-2)^2}{9} = 1$, centre $(3, 2)$



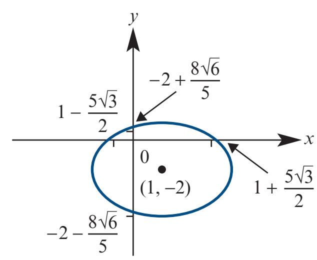
f $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, centre $(0, 0)$



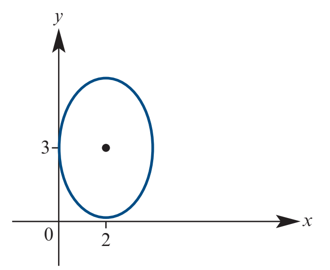
g $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{5} = 1$, centre $(-2, 1)$



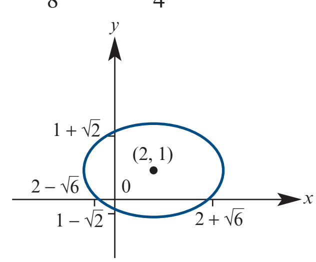
h $\dfrac{(x-1)^2}{25} + \dfrac{(y+2)^2}{16} = 1$, centre $(1, -2)$



i $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{9} = 1$, centre $(2, 3)$



j $\dfrac{(x-2)^2}{8} + \dfrac{(y-1)^2}{4} = 1$, centre $(2, 1)$



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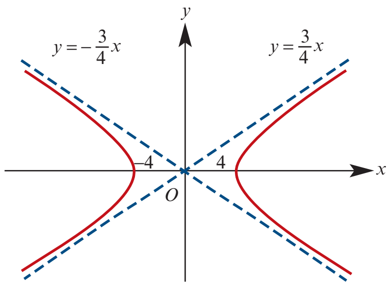
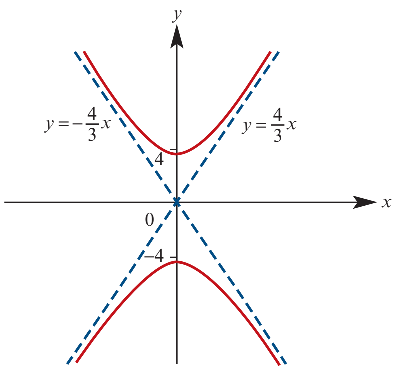
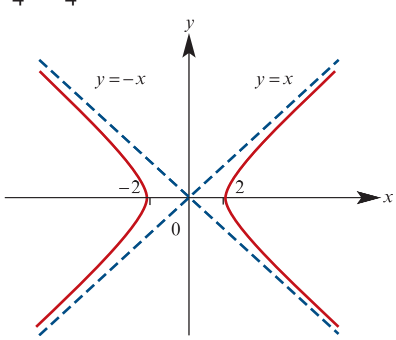
2 a $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, asymptotes $y = \pm\dfrac{3}{4}x$



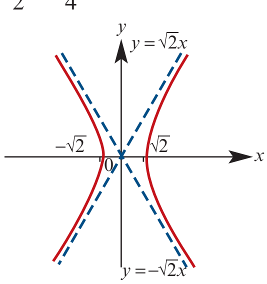
b $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$, asymptotes $y = \pm\dfrac{4}{3}x$



c $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$, asymptotes $y = \pm x$

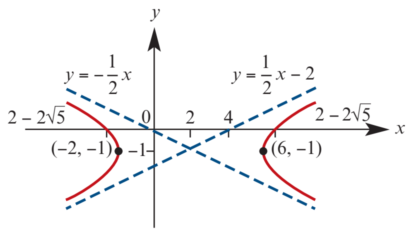


d $\dfrac{x^2}{2} - \dfrac{y^2}{4} = 1$, asymptotes $y = \pm\sqrt{2}x$



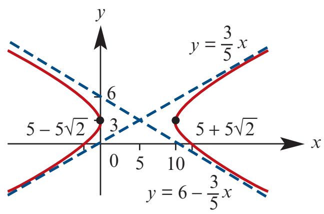
e $\dfrac{(x-2)^2}{16} - \dfrac{(y+1)^2}{4} = 1$,

asymptotes $y = \dfrac{1}{2}x - 2,\ y = -\dfrac{1}{2}x$



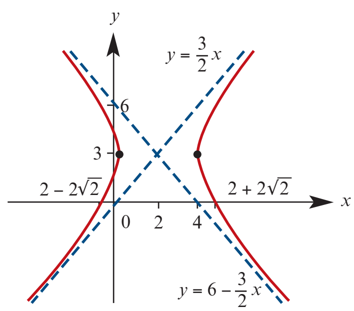
f $\dfrac{(x-5)^2}{25} - \dfrac{(y-3)^2}{9} = 1$,

asymptotes $y = \dfrac{3}{5}x,\ y = 6 - \dfrac{3}{5}x$



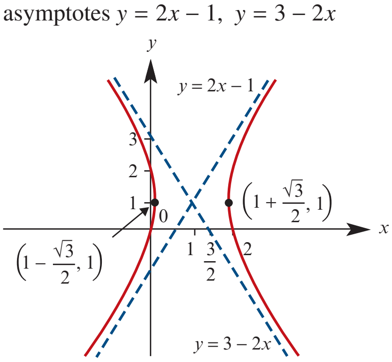
g $\dfrac{(x-2)^2}{4} - \dfrac{(y-3)^2}{9} = 1$,

asymptotes $y = \dfrac{3}{2}x,\ y = 6 - \dfrac{3}{2}x$



h $\dfrac{4(x-1)^2}{3} - \dfrac{(y-1)^2}{3} = 1$,

asymptotes $y = 2x - 1,\ y = 3 - 2x$



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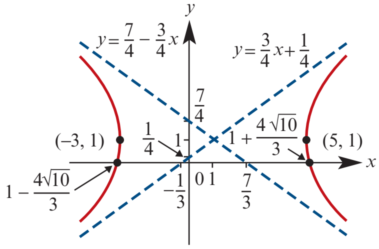
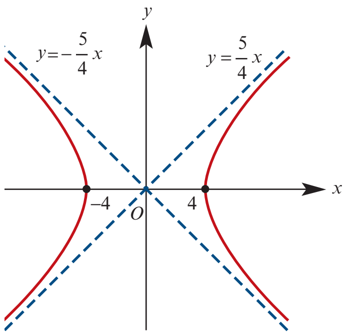
i $\dfrac{(x-1)^2}{16} - \dfrac{(y-1)^2}{9} = 1$,

asymptotes $y = \dfrac{3}{4}x + \dfrac{1}{4},\ y = \dfrac{7}{4} - \dfrac{3}{4}x$



j $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$, asymptotes $y = \pm\dfrac{5}{4}x$

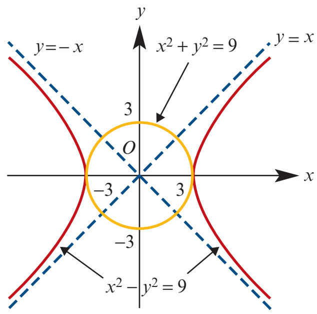


3 a $\left(\dfrac{2\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right), \left(-\dfrac{2\sqrt{3}}{3}, -\dfrac{\sqrt{3}}{3}\right)$

b $\left(\sqrt{2}, \dfrac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\dfrac{\sqrt{2}}{2}\right)$

6 $\left(-2\sqrt{2}, -\dfrac{5\sqrt{2}}{2}\right), \left(2\sqrt{2}, \dfrac{5\sqrt{2}}{2}\right)$

7



Exercise 1G

1 $x^2 + y^2 = 4$, dom = $[-2, 2]$, ran = $[-2, 2]$

2 a $y^2 = 16x$ **b** $x = 4$ **c** $32\sqrt{2}$

3 $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{4} = 1$;

ellipse with centre $(2, 3)$

4 $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1,\ x \le -2$;

left branch of hyperbola with centre $(0, 0)$ and x-axis intercept $(-2, 0)$; asymptotes $y = \pm\dfrac{3x}{2}$

5 a $x^2 + y^2 = 16$ **b** $x^2 + y^2 = 4$

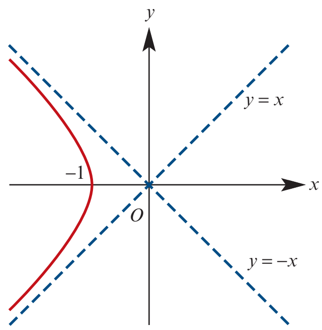
c $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ **d** $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

e $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$ **f** $y = x^2 - 2x - 3$

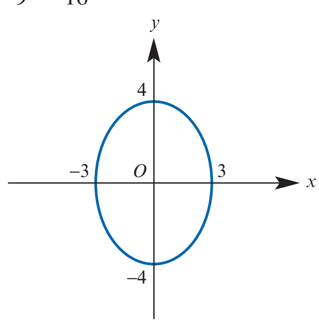
g $y = \dfrac{1}{x-2}$ **h** $y = x + 2$

i $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$

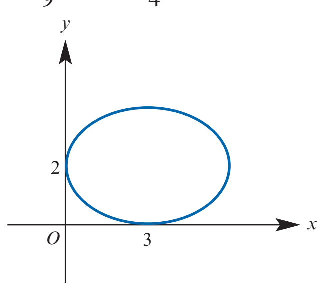
6 a $x^2 - y^2 = 1,\ x \in (-\infty, -1]$



b $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$



c $\dfrac{(x-3)^2}{9} + \dfrac{(y-2)^2}{4} = 1$



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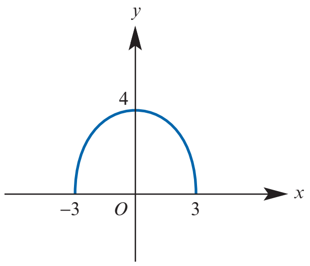
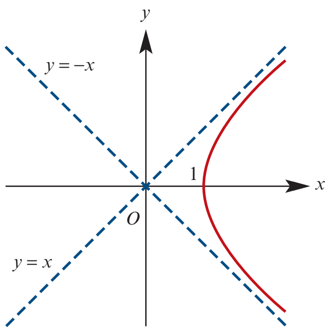
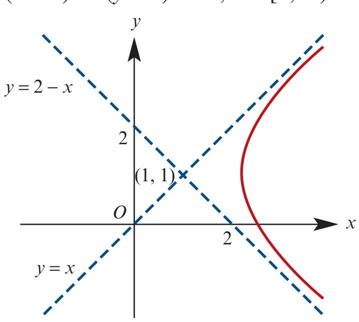
d $\frac{x^2}{9} + \frac{y^2}{16} = 1, x \in [-3, 3], y \in [0, 4]$



e $x^2 - y^2 = 1, x \in [1, \infty)$



f $(x-1)^2 - (y-1)^2 = 1, x \in [2, \infty)$



7 a $P = (-1, -\sqrt{3})$

b $\sqrt{3}x + 3y = -4\sqrt{3}$

8 a $x = 4\cos t$ $\quad y = 4\sin t$

b $x = 3\sec t$ $\quad y = 2\tan t$

c $x = 3\cos t + 1$ $\quad y = 3\sin t - 2$

d $x = 9\cos t + 1$ $\quad y = 6\sin t - 3$

9 $a = 1,\ b = 2,\ c = 3,\ d = 2$

10 $x = 4\cos t,\ y = 3\sin t$

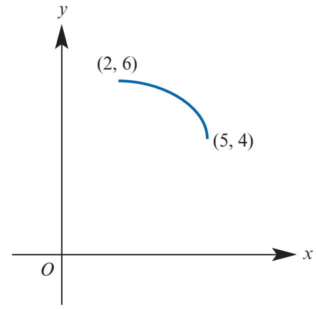
11 a $x = 2\cos t,\ y = 6\sin t$

b $\frac{x^2}{4} + \frac{y^2}{36} = 1$

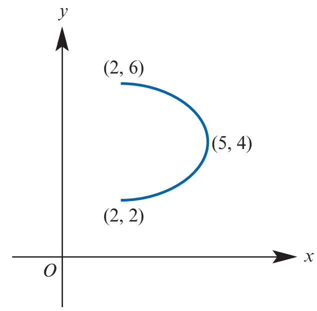
12 a $x = -2\cos\left(\frac{t}{2}\right),\ y = 2 + 3\sin\left(\frac{t}{2}\right)$

b $\frac{x^2}{4} + \frac{(y-2)^2}{9} = 1$

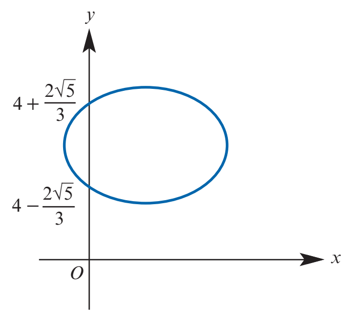
13 a dom = $[2, 5]$, ran = $[4, 6]$



b dom = $[2, 5]$, ran = $[2, 6]$



c dom = $[-1, 5]$, ran = $[2, 6]$



Exercise 1H

1

```
n ← 1
x ← 3
while x ≤ 100
  n ← n + 1
  x ← 2x + 3
end while
print n
```

n	x
1	3
2	9
3	21
4	45
5	93
6	189

2

```
define evenprod(n):
  product ← 1
  for i from 1 to n
    product ← product × 2i
  end for
  return product
```

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3
```
define powers(n):
  A ← []
  for i from 1 to n
    append 2^(i−1) to A
  end for
  return A
```

4 a
```
for x from 1 to 22
  for y from 1 to 22
    for z from 1 to 22
      if x^2 + y^2 + z^2 = 500 then
        print (x, y, z)
      end if
    end for
  end for
end for
```

b
```
for x from 1 to 99
  for y from 1 to 99
    for z from 1 to 99
      if x^3 + y^3 + z^3 = 1 000 000 then
        print (x, y, z)
      end if
    end for
  end for
end for
```

5 a i 0.099 833 **ii** 0.841 468 **iii** 0.907 937

6 a
```
define cossum(x, n):
  sum ← 0
  for k from 1 to n
    sum ← sum + ((−1)^(k+1) × x^(2k−2)) / factorial(2k − 2)
  end for
  return sum
```

b $-\frac{19}{45} \approx -0.422$, $\cos 2 \approx -0.416$

7 a

i	*strip*	*sum*	*left*
		0	0
1	1.5	1.5	0.5
2	1.8125	3.3125	1
3	3	6.3125	1.5
4	5.4375	11.75	2
5	9.5	21.25	2.5
6	15.5625	36.8125	3
7	24	60.8125	3.5
8	35.1875	96	4
9	49.5	145.5	4.5
10	67.3125	212.8125	5

b
```
define f(x):
  return x^3 + 2x^2 + 3

a ← 0
b ← 5
n ← 50
h ← (b − a)/n
left ← a
right ← a + h
sum ← 0
for i from 1 to n
  strip ← 0.5 × (f(left) + f(right)) × h
  sum ← sum + strip
  left ← left + h
  right ← right + h
end for
print sum
```

8 a 1.259 921 **b** 3.141 593

Chapter 1 review

Technology-free questions

1 $f_n = 5^n$

2 $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{16} = 1$

3 $\frac{7}{\sqrt{113}}$

4 $\frac{9}{2}$

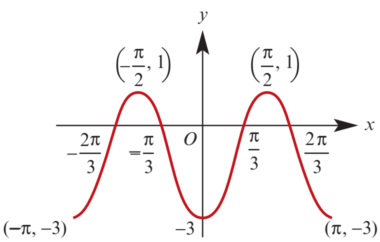
5 a $\frac{1}{\sqrt{2}}$ **b** $-\frac{4}{5}$

c 210° is a possible answer

6 $\tan^{-1}(3\sqrt{2})$

7 a $\left\{-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}\right\}$

b



c $\left[-\pi, -\frac{2\pi}{3}\right) \cup \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right]$

8 a 90° **b** 45°, $\tan^{-1}\left(\frac{3}{4}\right)$, $\tan^{-1}\left(\frac{5}{4}\right)$

9 a $3\sqrt{97}$ nautical miles

b $5\sqrt{97}$ nautical miles

10 $9\sqrt{2}$

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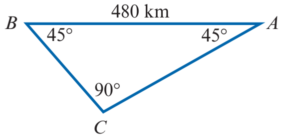
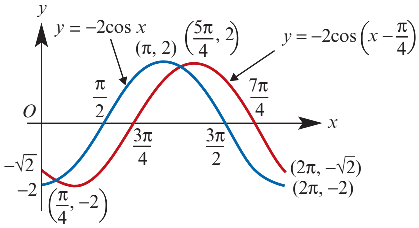
11 a 

b $240\sqrt{2}$ km **c** $480\sqrt{2}$ km

12 $y = 3x + 2,\ y = -3x + 2$

13 $\frac{(x-2)^2}{9} + (y+6)^2 = 1$

14 $x^2 + (y-4)^2 = 4$

15 a 

b $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ **c** $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$

16 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{6}, \frac{11\pi}{6}$ **c** $\frac{\pi}{4}, \frac{5\pi}{4}$

17 $a = 1,\ c = 2,\ b = d = 3$

18 Centre $(-4, 6)$; radius 7

19 $(\pm 9, 0), (0, \pm 3)$

20 a i $n = 7p + 7$

ii $S_n = 70p^2 + 147p + 77$

21 a $t_n = 3^{n-1}$ **b** 3^{190}

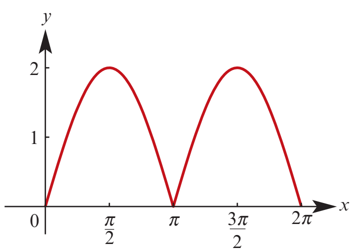
22 a 9 **b** $\frac{1}{400}$ **c** 4

d 4 **e** $\pi - 3$ **f** $4 - \pi$

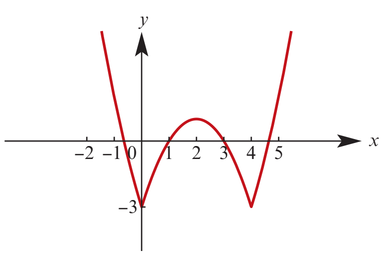
23 a $\left(0, \frac{1}{10^4}\right)$ **b** $(100, \infty)$

24 $x = 0$ or $x = 2$ or $x = 4$

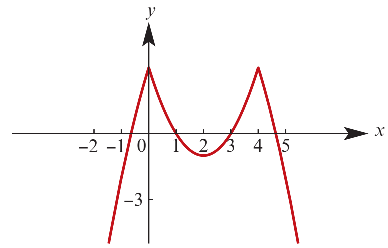
25 a Range $[0, 2]$



b Range $[-3, \infty)$



c Range $(-\infty, 3]$



Multiple-choice questions

1 B	**2** D	**3** C	**4** A	**5** D
6 B	**7** C	**8** D	**9** D	

Extended-response questions

1 a $a = \sqrt{2},\ w = \frac{3-\sqrt{3}}{2},\ x = \frac{1+\sqrt{3}}{2},$
$y = \frac{\sqrt{3}-1}{2},\ z = 15$

b $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}, \cos 15° = \frac{\sqrt{2}+\sqrt{6}}{4},$
$\tan 15° = 2 - \sqrt{3}$

c $\sin 75° = \frac{\sqrt{2}+\sqrt{6}}{4}, \cos 75° = \frac{\sqrt{6}-\sqrt{2}}{4},$
$\tan 75° = \frac{1}{2-\sqrt{3}} = 2 + \sqrt{3}$

2 a 10.2 km

b 049°

c i 11.08 km **ii** 031°

d 11.93 km

3 a i $[-\sqrt{2}, \sqrt{2}]$ **ii** $[-3-\sqrt{5}, -3+\sqrt{5}]$

iii $(0, -3)$

b 2, 3, 1, 2 **c** $\left(\frac{37}{13}, \frac{11}{13}\right)$ **d** $\left(0, \frac{48}{13}\right)$

e $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{35}{26}\right)^2 = \frac{3890}{676}$

4 d Centre $(2, 2)$ and radius 2;
centre $(10, 10)$ and radius 10

e Gradient undefined; gradient $\frac{3}{4}$

f $y = 4;\ y = -\frac{4}{3}x + \frac{20}{3}$

5 a $y = (\tan t)x$ **b** $(-a\cos t, -a\sin t)$

c $y - a\sin t = -\frac{\cos t}{\sin t}(x - a\cos t)$

d $A\left(\frac{a}{\cos t}, 0\right), B\left(0, \frac{a}{\sin t}\right)$

e Area $= \frac{a^2}{2\sin t\,\cos t} = \frac{a^2}{\sin(2t)}$;
minimum when $t = \frac{\pi}{4}$

6 a 100°, 15°, 65° **b** 2.63 km, 4.56 km

c 346° **d** 14.18 km

7 a $(0, 0), (a, 0)$ **b** $(0, 0)$

c $\frac{a^2}{4}$ **d** 3, −5

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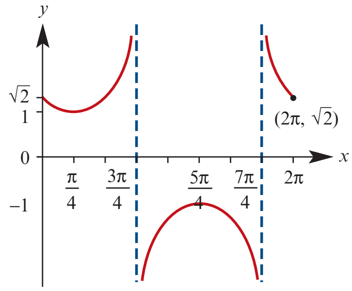
Chapter 2

See solutions supplement

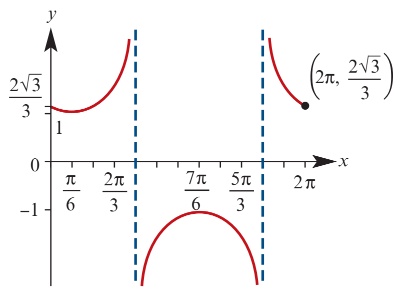
Chapter 3

Exercise 3A

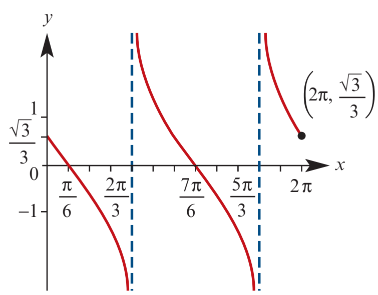
1 a



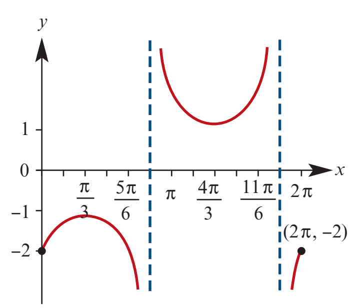
b



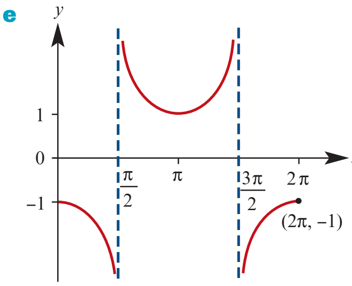
c



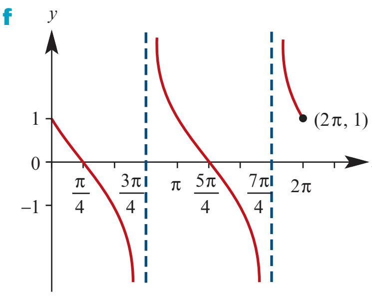
d



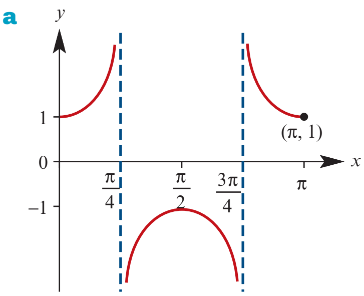
e



f



2 a



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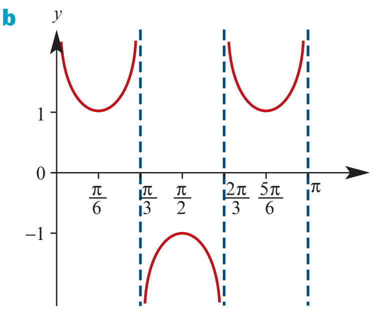

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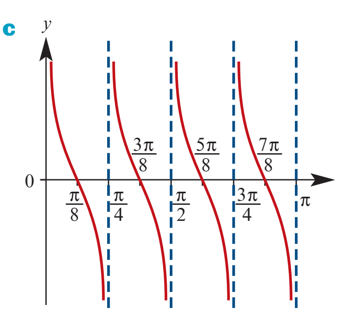

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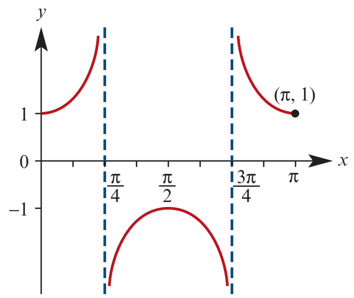
d



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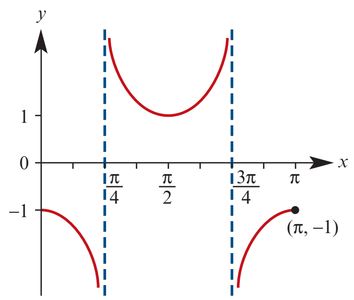

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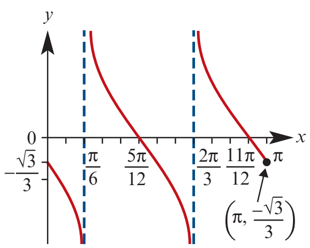
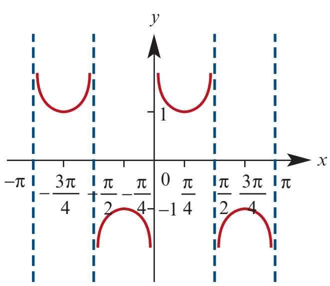
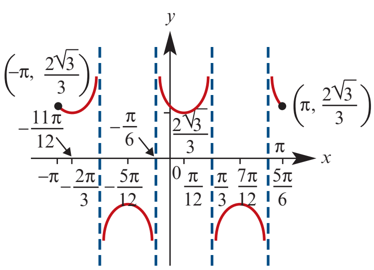


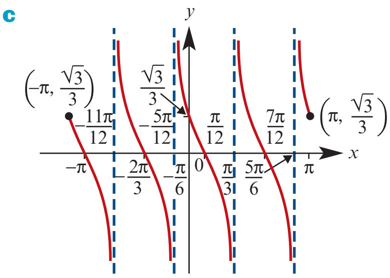
3 a



b



c



4 a $\cot x = \frac{5}{8}$, $\sec x = \frac{\sqrt{89}}{5}$, $\operatorname{cosec} x = \frac{\sqrt{89}}{8}$

b $\cot x = \frac{2\sqrt{6}}{5}$, $\sec x = \frac{7\sqrt{6}}{12}$, $\operatorname{cosec} x = \frac{7}{5}$

c $\cot x = \frac{7\sqrt{2}}{8}$, $\sec x = \frac{9}{7}$, $\operatorname{cosec} x = \frac{9\sqrt{2}}{8}$

5 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{2}}{2}$ **c** -1 **d** 2

e $\sqrt{2}$ **f** $-\sqrt{3}$ **g** $-\frac{\sqrt{2}}{2}$ **h** $-\frac{\sqrt{3}}{3}$

i 2 **j** $\sqrt{2}$ **k** 1 **l** $\frac{1}{2}$

6 a 1 **b** -1 **c** $\operatorname{cosec}^2 x$ **d** $\sec x$

e $\sin^2 x - \cos^2 x = -\cos(2x)$

f $\tan x \sec^2 x$

7 a $\sqrt{17}$ **b** $\frac{\sqrt{17}}{17}$ **c** $-\frac{\sqrt{17}}{4}$

8 a $-\sqrt{10}$ **b** $-\frac{\sqrt{10}}{10}$ **c** $-\frac{\sqrt{10}}{3}$

9 a $-3\sqrt{11}$ **b** $-\frac{3\sqrt{11}}{10}$

10 a $-\sqrt{35}$ **b** $\frac{\sqrt{35}}{6}$

11 a $-\frac{\sqrt{3}}{2}$ **b** $-\sqrt{3}$ **c** 2

12 a $-\frac{1}{3}$ **b** $-\frac{2\sqrt{2}}{3}$ **c** $-\frac{3\sqrt{2}}{4}$

13 a $\frac{\sqrt{51}}{10}$ **b** $-\frac{\sqrt{51}}{7}$ **c** $-\frac{7\sqrt{51}}{51}$

14 a 0.2 **b** $-\frac{2\sqrt{6}}{5}$ **c** $-\frac{\sqrt{6}}{12}$

15 a 0 **b** $\frac{1}{2}\sin(2\theta)$ **c** 1 **d** 1

16 $x - \frac{1}{x} = -2\tan\theta$

Exercise 3B

1 a $\frac{\sqrt{2}}{4}(\sqrt{3} - 1)$ **b** $2 + \sqrt{3}$

c $\frac{\sqrt{2}}{4}(1 - \sqrt{3})$ **d** $2 - \sqrt{3}$

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2 a $\sin(2x)\cos(5y) - \cos(2x)\sin(5y)$

b $\cos(x^2)\cos(y) - \sin(x^2)\sin(y)$

c $\dfrac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}$

3 a $\sin(x - 2y)$ **b** $\cos x$ **c** $\tan B$ **d** $\sin(2A)$ **e** $\cos y$

4 a $\sin x \cos(2x) + \cos x \sin(2x)$

b $3\sin x - 4\sin^3 x$

5 a $\cos x \cos(2x) - \sin x \sin(2x)$

b $4\cos^3 x - 3\cos x$

6 a -0.8 **b** 2.6 **c** $\frac{5}{13}$ **d** $\frac{12}{13}$ **e** -0.75 **f** $\frac{16}{65}$ **g** $\frac{63}{65}$ **h** $\frac{33}{56}$ **i** $-\frac{837}{116}$

7 a $-\frac{\sqrt{51}}{10}$ **b** $\frac{\sqrt{21}}{5}$ **c** 0.40 **d** -0.36

8 a $\frac{1}{4}\sin(2x)$ **b** $-\cos(2x)$ **c** $\frac{1}{2}\tan(2x)$ **d** -1 **e** $-2\tan x$ **f** $\sin(2x)$

9 a 0.96 **b** -0.28 **c** $-\frac{24}{7}$

10 a $-\frac{3}{4}$ **b** $\frac{9}{13}$

11 a -0.66 **b** 0.91

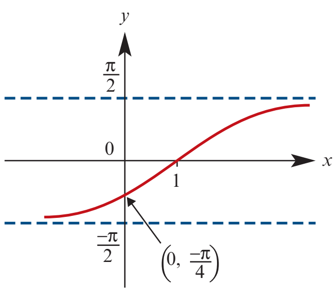
12 $\sqrt{2} - 1$

13 0.97

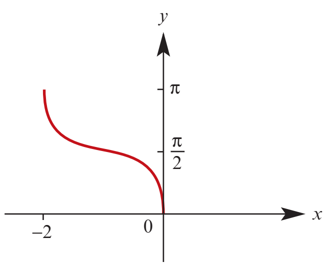
14 a $\frac{12}{5}$ **b** $\frac{2}{3}$ **c** $3\frac{1}{3}$ m

Exercise 3C

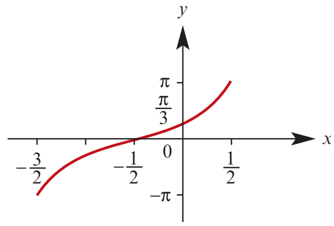
1 a dom $= \mathbb{R}$, ran $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



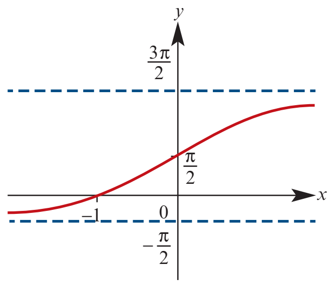
b dom $= [-2, 0]$, ran $= [0, \pi]$



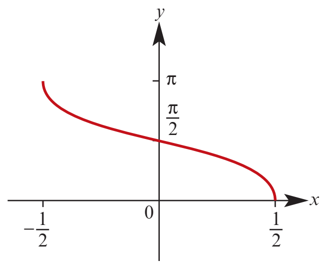
c dom $= \left[-\frac{3}{2}, \frac{1}{2}\right]$, ran $= [-\pi, \pi]$



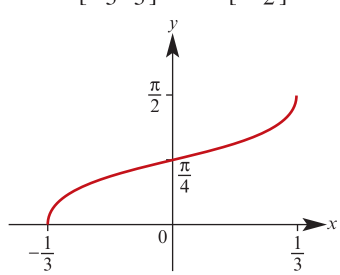
d dom $= \mathbb{R}$, ran $= \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$



e dom $= \left[-\frac{1}{2}, \frac{1}{2}\right]$, ran $= [0, \pi]$



f dom $= \left[-\frac{1}{3}, \frac{1}{3}\right]$, ran $= \left[0, \frac{\pi}{2}\right]$



2 a $\frac{\pi}{2}$ **b** $-\frac{\pi}{4}$ **c** $\frac{\pi}{6}$ **d** $\frac{5\pi}{6}$ **e** $\frac{\pi}{3}$ **f** $\frac{\pi}{4}$ **g** $-\frac{\pi}{3}$ **h** $\frac{\pi}{6}$ **i** π

3 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\pi}{3}$ **c** -1 **d** $\frac{\sqrt{2}}{2}$ **e** $\frac{\pi}{4}$ **f** $\sqrt{3}$ **g** $\frac{\pi}{3}$ **h** $-\frac{\pi}{3}$ **i** $-\frac{\pi}{4}$ **j** $\frac{5\pi}{6}$ **k** π **l** $-\frac{\pi}{4}$

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4 a $f^{-1}: [-1, 1] \to \mathbb{R}, f^{-1}(x) = y,$
where $\sin y = x$ and $y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$(f^{-1}(x) = \pi - \sin^{-1}(x))$

b i 1 **ii** $\frac{1}{\sqrt{2}}$ **iii** $-\frac{1}{2}$ **iv** $\frac{3\pi}{2}$ **v** π **vi** $\frac{5\pi}{6}$

5 a $[1, 3], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **b** $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right], [-1, 1]$

c $\left[-\frac{5}{2}, -\frac{3}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **d** $\left[-\frac{\pi}{18}, \frac{5\pi}{18}\right], [-1, 1]$

e $\left[\frac{\pi}{6}, \frac{7\pi}{6}\right], [-1, 1]$ **f** $[-2, 0], [0, \pi]$

g $[-1, 1], \left[0, \frac{\pi}{2}\right]$ **h** $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right], [-1, 1]$

i $\mathbb{R}, \left[0, \frac{\pi}{2}\right)$ **j** $\left(0, \frac{\pi}{2}\right), \mathbb{R}$ **k** $\mathbb{R}, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

l $\left(-\frac{\sqrt{2}\pi}{2}, \frac{\sqrt{2}\pi}{2}\right), \mathbb{R}^+ \cup \{0\}$

6 a $\frac{3}{5}$ **b** $\frac{12}{5}$ **c** $\frac{24}{25}$

d $\frac{40}{9}$ **e** $\sqrt{3}$ **f** $\frac{\sqrt{5}}{3}$

g $-\frac{2\sqrt{5}}{5}$ **h** $\frac{2\sqrt{10}}{7}$ **i** $\frac{7\sqrt{149}}{149}$

7 a i $\frac{4}{5}$ **ii** $\frac{12}{13}$

8 a $[0, \pi], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **b** $[0, 1], [0, 1]$

c $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right], [0, \pi]$ **d** $[0, 1], [-1, 0]$

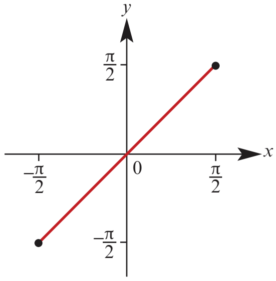
e $[0, 1], [-1, 1]$ **f** $[0, \pi], \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

g $\mathbb{R}^+ \cup \{0\}, (0, 1]$ **h** $\mathbb{R}, (-1, 1)$

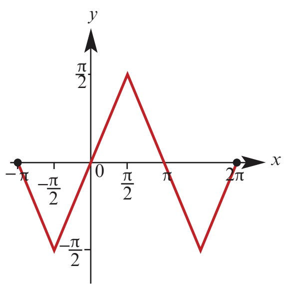
11 a $[-1, 1]$

12 a dom $= \mathbb{R}$, ran $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b



c



Exercise 3D

1 a $\frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{\pi}{12}, \frac{17\pi}{12}$ **c** $\frac{\pi}{6}, \frac{11\pi}{6}$

d $\frac{\pi}{4}, \frac{5\pi}{4}$ **e** $\frac{5\pi}{6}, \frac{11\pi}{6}$

f $\frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$

2 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{5\pi}{6}, \frac{7\pi}{6}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$

d $\frac{3\pi}{4}, \frac{7\pi}{4}$ **e** $\frac{2\pi}{3}, \frac{4\pi}{3}$ **f** $\frac{5\pi}{4}, \frac{7\pi}{4}$

3 a $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{3\pi}{4} + 2n\pi, n \in \mathbb{Z}$

b $x = 2n\pi, n \in \mathbb{Z}$ **c** $x = \frac{\pi}{6} + n\pi, n \in \mathbb{Z}$

d $x = \frac{(12n - 5)\pi}{12}$ or $x = \frac{(4n + 1)\pi}{4}, n \in \mathbb{Z}$

e $x = \frac{(2n - 1)\pi}{3}$ or $x = \frac{2(3n + 1)\pi}{9}, n \in \mathbb{Z}$

f $x = \frac{2n\pi}{3}$ or $x = \frac{(6n + 1)\pi}{9}, n \in \mathbb{Z}$

g $x = \frac{(3n - 2)\pi}{6}, n \in \mathbb{Z}$

h $x = \frac{n\pi}{2}, n \in \mathbb{Z}$

i $x = \frac{(8n - 5)\pi}{8}, n \in \mathbb{Z}$

4 a ± 1.16 **b** $-0.20, -2.94$ **c** $1.03, -2.11$

5 a $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$ **b** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

c $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

d $\frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{5\pi}{8}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24},$
$\frac{9\pi}{8}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{13\pi}{8}, \frac{41\pi}{24}, \frac{15\pi}{8}$

e $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ **f** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

g $0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$ **h** $\frac{3\pi}{4}, \frac{7\pi}{4}$

i $\frac{\pi}{3}, \frac{5\pi}{3}$ **j** $0, \frac{\pi}{2}, 2\pi$

6 a max = 3, min = 1 **b** max = 1, min = $\frac{1}{3}$

c max = 5, min = 4 **d** max = $\frac{1}{4}$, min = $\frac{1}{5}$

e max = 3, min = −1 **f** max = 9, min = 5

7 a $(-1.14, -2.28), (0, 0), (1.14, 2.28)$
b $(-1.24, -1.24), (0, 0), (1.24, 1.24)$
c $(3.79, -0.79)$ **d** $(0, 0), (4.49, 4.49)$

8 $2\pi - q$

9 a $\pi + \alpha, 2\pi - \alpha$ **b** $\frac{\pi}{2} - \alpha, \frac{3\pi}{2} + \alpha$

10 a $\pi - \beta, \beta - \pi$ **b** $\frac{\pi}{2} - \beta, \beta - \frac{3\pi}{2}$

11 a $2\pi - \gamma, 3\pi - \gamma$ **b** $\frac{3\pi}{2} - \gamma, \frac{5\pi}{2} - \gamma$

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12 0, 0.33, 2.16 **13** 1.50
14 b 45.07
15 0.86 **16** 1.94
17 b 1.113
18 When $t = 0$, $x_A = x_B = 0$; when $t = 1.29$, $x_A = x_B = 0.48$
19 b 0.94

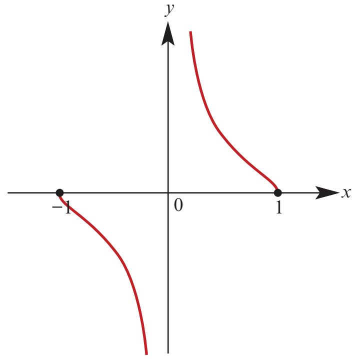
Exercise 3E

1 a $\sin(11\pi t) - \sin(3\pi t)$ **b** $\frac{1}{2}(\sin 60° + \sin 40°)$
c $\frac{3}{2}\left(\sin(\pi x) + \sin\left(\frac{\pi x}{3}\right)\right)$
d $\sin(A) + \sin(B + C)$ **e** $\cos\left(\frac{x}{2}\right) - \cos\left(\frac{5x}{2}\right)$
f $\cos\left(\frac{\pi x}{2}\right) + \cos(\pi x)$
2 $\cos(3\theta) - \cos(5\theta)$
3 $\sin x - \sin y$
5 a $2 \sin 50° \cos 16°$ **b** $2 \cos 50° \cos 16°$
c $2 \sin 16° \cos 50°$ **d** $-2 \sin 50° \sin 16°$
6 a $2 \sin(5A) \cos(3A)$ **b** $2 \cos\left(\frac{5x}{2}\right)\cos\left(\frac{3x}{2}\right)$
c $2 \sin(x) \cos(5x)$ **d** $-2 \sin(4A) \sin(A)$
11 a $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$
b $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$
c $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$
d $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$
12 $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$
13 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$
c $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$
d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$
14 a 0 **b** -1
15 b $\frac{\pi}{2}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$

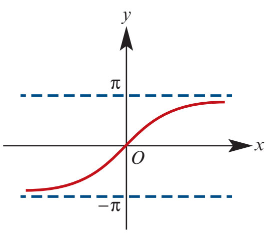
Chapter 3 review

Technology-free questions

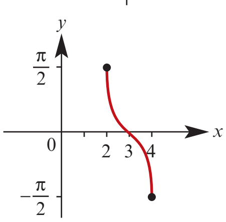
1 a $\frac{7}{25}$ **b** $\frac{24}{25}$ **c** $\frac{24}{7}$ **d** $\frac{5}{3}$ **e** $\frac{4}{3}$
2 a $0, \pi, 2\pi, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$
b $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$
c $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ **d** $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$
e $\frac{\pi}{2}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, -\frac{5\pi}{6}$
f $0, 2\pi, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$
3 a $\frac{7\pi}{6}, \frac{11\pi}{6}, \sin^{-1}\left(\frac{1}{3}\right), \pi - \sin^{-1}\left(\frac{1}{3}\right)$
b $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $\frac{\pi}{4}, \frac{5\pi}{4}$
d $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
4 a $\frac{2\sqrt{3}}{3}$ **b** 2 **c** 2 **d** 1 **e** $-\sqrt{3}$
5 a $-p$ **b** $-p$ **c** $\frac{1}{p}$ **d** $-\frac{1}{p}$ **e** $-p$
6 a dom $= [-1, 1] \setminus \{0\}$, ran $= \mathbb{R}$
c



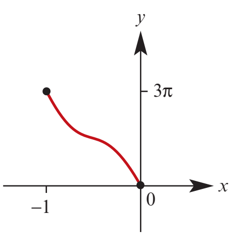
7 a $\frac{\pi}{3}$ **b** $\frac{1}{2}$ **c** $\frac{2\pi}{3}$ **d** $\frac{2\pi}{3}$ **e** $\frac{\sqrt{3}}{2}$ **f** $\frac{\sqrt{2}}{2}$
8 a $\left[-\frac{b}{a}, \frac{1-b}{a}\right]$ **b** $\left(-\infty, -\frac{2}{a}\right] \cup \left[\frac{2}{a}, \infty\right)$
c $\left[\frac{2}{a}, \frac{6}{a}\right]$ **d** $\left[\frac{1}{a}, \frac{2}{a}\right]$
9 a



b



c



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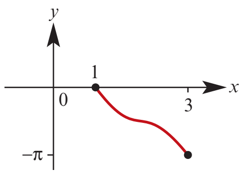
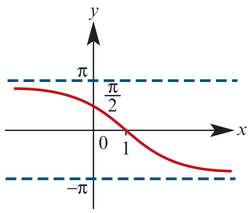
d



e



10 $0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi$

Multiple-choice questions

1 C **2** C **3** E **4** D **5** A
6 A **7** E **8** D **9** E **10** E

Extended-response questions

1 a i x **ii** $\sqrt{1-x^2}$ **iii** $\frac{x}{\sqrt{1-x^2}}$

iv $2x$ **v** $\sqrt{1-4x^2}$ **vi** $\frac{2x}{\sqrt{1-4x^2}}$

b i $2x\sqrt{(1-x^2)} - x\sqrt{1-4x^2}$

ii $\sqrt{(1-4x^2)(1-x^2)} + 2x^2$

iii $\frac{2x\sqrt{1-x^2} - x\sqrt{1-4x^2}}{\sqrt{(1-4x^2)(1-x^2)} + 2x^2}$

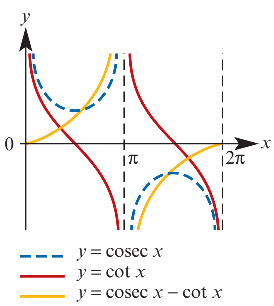
iv $\frac{2x\sqrt{1-x^2}}{1-2x^2}$

v $2x\sqrt{1-x^2}$

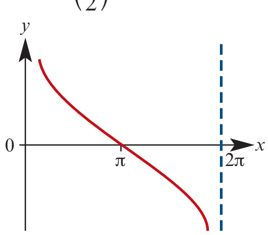
vi $1-2x^2$

c $\angle B_2AB_1 = 0.34$, $2\alpha = 0.61$

2 a



c $y = \cot\left(\frac{x}{2}\right)$, $y = \operatorname{cosec}(x) + \cot(x)$



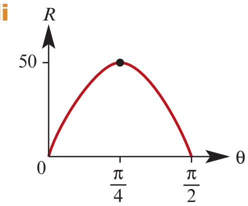
d ii $\cot\left(\frac{\pi}{8}\right) = 1+\sqrt{2}$, $\cot\left(\frac{\pi}{12}\right) = 2+\sqrt{3}$

iii $\frac{1}{\sqrt{4+2\sqrt{2}}}$

e $\cot\left(\frac{\theta}{2}\right) - \cot(4\theta)$

3 a i $100\sin\theta\cos\theta$

ii



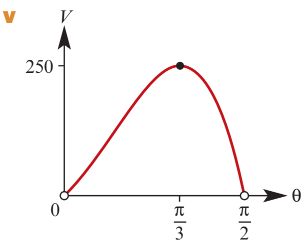
iii 50 **iv** $\frac{\pi}{4}$

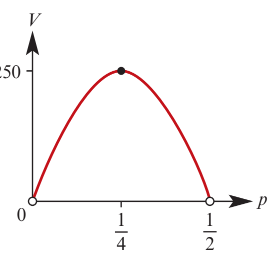
b ii $a = 2000$, $b = -4000$

iii $V = 2000p - 4000p^2$

iv $0 < p < \frac{1}{2}$

v

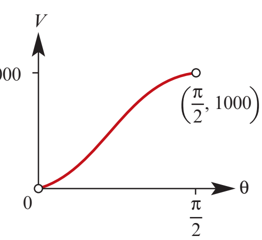




vi $V_{max} = 250$ when $p = \frac{1}{4}, \theta = \frac{\pi}{3}$

c i $V = 1000\sin^2\theta$, for $0 < \theta < \frac{\pi}{2}$

ii



iii V is an increasing function: as the angle θ gets larger, so does the volume of the cuboid

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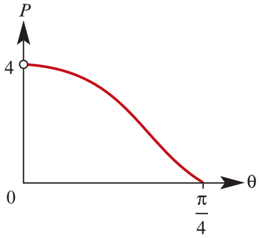
4 b $p = 8\cos^3\theta - 4\cos\theta$

c iii $\dfrac{\pi}{6}$ **iv** 1

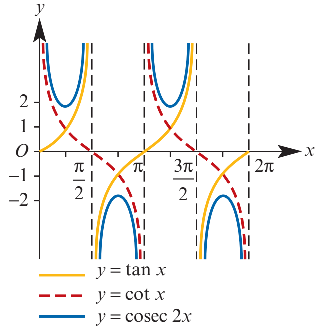
d



e $\dfrac{\pi}{4}$

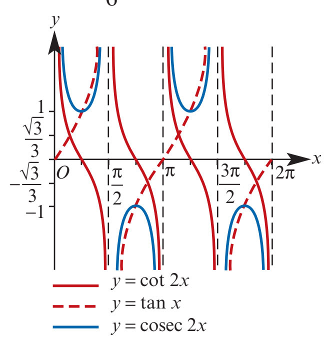
5 a ii $x = \pm\dfrac{\pi}{4} + n\pi, n \in \mathbb{Z}$

iii



b ii $x = n\pi \pm \dfrac{\pi}{6}, n \in \mathbb{Z}$

iii

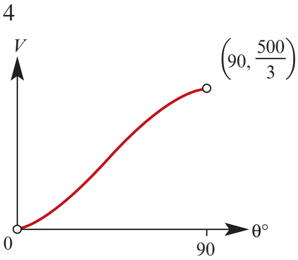


6 a i $\angle BAE = 72°$, $\angle AEC = 72°$, $\angle ACE = 72°$

ii 36°

e $\dfrac{\sqrt{5} - 1}{4}$

7 a ii



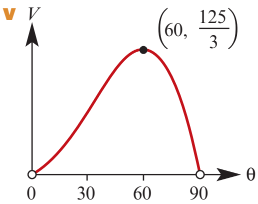
iii V is an increasing function: as the angle θ gets larger, so does the volume of the pyramid

b ii $\theta \in (0, 90)$

iii $V = -\dfrac{2000}{3}a^2 + \dfrac{1000}{3}a$

iv $V_{max} = \dfrac{125}{3}$ when $\theta = 60$

v



8 a i $V = \dfrac{500}{3}\cos(\theta°)\sin^2(\theta°)$

ii $V_{max} = 64.15$ when $\theta = 54.74$

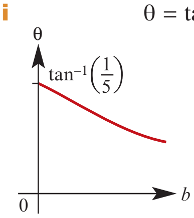
b ii $\theta \in (0, 90)$

c $V_{max} = 24.69$ when $a = 0.67$, $\theta = 48.19$

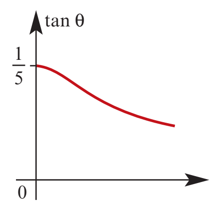
9 c i $x = \dfrac{a \pm \sqrt{a^2 - 4b(a+b)}}{2}$ **ii** $1 + \sqrt{2}$

d 0.62

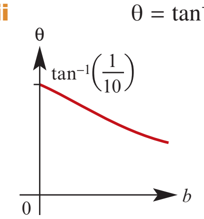
e i $\theta = \tan^{-1}\left(\dfrac{b+1}{5}\right) - \tan^{-1}\left(\dfrac{b}{5}\right)$



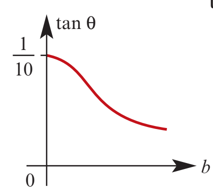
$\tan\theta = \dfrac{5}{25 + b + b^2}$



ii $\theta = \tan^{-1}\left(\dfrac{b+1}{10}\right) - \tan^{-1}\left(\dfrac{b}{10}\right)$



$\tan\theta = \dfrac{10}{100 + b + b^2}$



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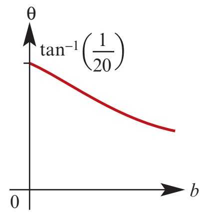
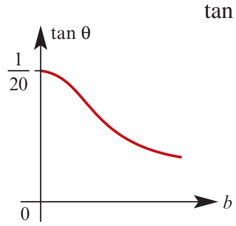
iii $\theta = \tan^{-1}\left(\frac{b+1}{20}\right) - \tan^{-1}\left(\frac{b}{20}\right)$

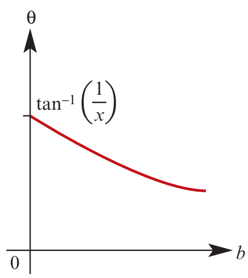


$\tan\theta = \dfrac{20}{400 + b + b^2}$



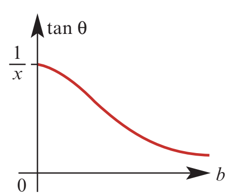
f Graph of $\theta = \tan^{-1}\left(\frac{b+1}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$:

- the b-axis is a horizontal asymptote
- domain is $[0, \infty)$; range is $\left(0, \tan^{-1}\left(\frac{1}{x}\right)\right)$
- the θ-axis intercept is $\tan^{-1}\left(\frac{1}{x}\right)$
- θ decreases as b increases



Graph of $\tan\theta = \dfrac{x}{x^2 + b + b^2}$:

- the b-axis as a horizontal asymptote
- the $\tan\theta$-axis intercept is $\frac{1}{x}$
- domain is $[0, \infty)$; range is $\left(0, \frac{1}{x}\right]$
- $\tan\theta$ decreases as b increases



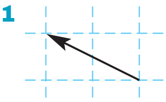
10 a Each triangle has a right angle, and angle CAD is common to both triangles
b $(\cos(2\theta), \sin(2\theta))$
c i $2\cos\theta$ **ii** $2\sin\theta$

12 a $t_2 = \sin^2(2\theta)$, $t_3 = \sin^2(4\theta)$
b $t_n = \sin^2(2^{n-1}\theta)$

15 a $\dfrac{\sqrt{2+\sqrt{2}}}{2}$ **b** $\dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

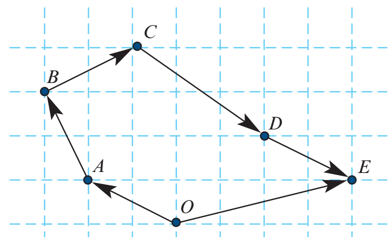
Chapter 4

Exercise 4A

1  Magnitude $= \sqrt{5}$

2 $a = 3,\ b = 2$

3



4 a i $2\mathbf{b}$ **ii** $4\mathbf{a}$ **iii** $2\mathbf{a} + \frac{3}{2}\mathbf{b}$
iv $\frac{1}{2}\mathbf{b} - 2\mathbf{a}$ **v** $2\mathbf{a} - \frac{3}{2}\mathbf{b}$
b i 4 **ii** 4 **iii** $\sqrt{13}$

5 a 6 **b** $\frac{9}{2}$ **c** $\frac{3}{2}$

6 a $\frac{21}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} - 21\mathbf{c}$ **b** $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$

7 a i $\frac{1}{4}\mathbf{a}$ **ii** $\frac{1}{4}\mathbf{b}$ **iii** $\frac{1}{4}(\mathbf{b} - \mathbf{a})$
iv $\mathbf{b} - \mathbf{a}$
b i $\frac{1}{2}\mathbf{a}$ **ii** $\frac{1}{2}\mathbf{b}$ **iii** $\frac{1}{2}(\mathbf{b} - \mathbf{a})$

8 a $\mathbf{b} - \mathbf{a}$ **b** $\frac{1}{2}(\mathbf{b} - \mathbf{a})$ **c** $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

9 a $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

10 a $\mathbf{a} + \mathbf{c} - \mathbf{b}$ **b** $\mathbf{a} + \mathbf{c} - 2\mathbf{b}$

11 a $-\mathbf{c}$ **b** $\mathbf{c}$ **c** $-\frac{1}{2}\mathbf{a}$
d $\mathbf{c} + \mathbf{g} + \frac{1}{2}\mathbf{a}$ **e** $\mathbf{c} + \mathbf{g} + \frac{1}{2}\mathbf{a}$

12 a i $\mathbf{b} - \mathbf{a}$ **ii** $\mathbf{c} - \mathbf{d}$ **iii** $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$
b i $\mathbf{c} - \mathbf{b}$ **ii** $-\frac{1}{2}\mathbf{a} + \mathbf{b} - \mathbf{c}$

13 a Linearly independent
b Linearly independent
c Linearly dependent
d Linearly independent

14 a $k = 3,\ \ell = \frac{1}{2}$ **b** $k = \frac{55}{2},\ \ell = -10$

15 a i $k(2\mathbf{a} - \mathbf{b})$ **ii** $(2m + 1)\mathbf{a} + (4 - 3m)\mathbf{b}$
b $k = \frac{11}{4},\ m = \frac{9}{4}$
c $\frac{11}{4}(2\mathbf{a} - \mathbf{b})$

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16 a i $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$ **ii** $\frac{4}{5}(\boldsymbol{a}+\boldsymbol{b})$

iii $\frac{1}{5}(4\boldsymbol{b}-\boldsymbol{a})$ **iv** $\frac{4}{5}(4\boldsymbol{b}-\boldsymbol{a})$

b $\overrightarrow{RP}=4\overrightarrow{AR}$, $1:4$

c $\lambda=4$

17 a $x=0,\ y=1$ **b** $x=-1,\ y=\frac{7}{3}$

c $x=-\frac{5}{2},\ y=0$

18 a $\frac{AX}{AB}=k$ **c** $\frac{AX}{XB}=\frac{k}{1-k}$ **d** $k=\frac{m}{1+m}$

Exercise 4B

1 a i $3\boldsymbol{i}+\boldsymbol{j}$ **ii** $-2\boldsymbol{i}+3\boldsymbol{j}$ **iii** $-3\boldsymbol{i}-2\boldsymbol{j}$

iv $4\boldsymbol{i}-3\boldsymbol{j}$

b i $-5\boldsymbol{i}+2\boldsymbol{j}$ **ii** $7\boldsymbol{i}-\boldsymbol{j}$ **iii** $-\boldsymbol{i}+4\boldsymbol{j}$

c i $\sqrt{10}$ **ii** $\sqrt{29}$ **iii** $\sqrt{17}$

2 a $\boldsymbol{i}+4\boldsymbol{j}$ **b** $4\boldsymbol{i}+4\boldsymbol{j}+2\boldsymbol{k}$ **c** $6\boldsymbol{j}-3\boldsymbol{k}$

d $-8\boldsymbol{i}-8\boldsymbol{j}+8\boldsymbol{k}$ **e** $\sqrt{6}$ **f** 4

3 a i $-5\boldsymbol{i}$ **ii** $3\boldsymbol{k}$ **iii** $2\boldsymbol{j}$ **iv** $5\boldsymbol{i}+3\boldsymbol{k}$

v $5\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ **vi** $5\boldsymbol{i}+2\boldsymbol{j}$

vii $-5\boldsymbol{i}-3\boldsymbol{k}$ **viii** $2\boldsymbol{j}-3\boldsymbol{k}$

ix $-5\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$ **x** $-5\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$

xi $5\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$ **xii** $5\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$

b i $\sqrt{34}$ **ii** $\sqrt{38}$ **iii** $\sqrt{29}$

c i $\frac{5}{2}\boldsymbol{i}$ **ii** $\frac{5}{2}\boldsymbol{i}+2\boldsymbol{j}$ **iii** $-\frac{5}{2}\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$

d i $-\frac{4}{3}\boldsymbol{j}$ **ii** $\frac{2}{3}\boldsymbol{j}$ **iii** $\frac{2}{3}\boldsymbol{j}+3\boldsymbol{k}$

iv $5\boldsymbol{i}-\frac{2}{3}\boldsymbol{j}-3\boldsymbol{k}$ **v** $\frac{5}{2}\boldsymbol{i}+\frac{4}{3}\boldsymbol{j}-3\boldsymbol{k}$

e i $\frac{\sqrt{613}}{6}$ **ii** $\frac{\sqrt{77}}{2}$ **iii** $\frac{\sqrt{310}}{3}$

4 a $x=3,\ y=-\frac{1}{3}$ **b** $x=4,\ y=\frac{2}{5}$

c $x=-\frac{3}{2},\ y=7$

5 a i $-2\boldsymbol{i}+4\boldsymbol{j}$ **ii** $3\boldsymbol{i}+2\boldsymbol{j}$ **iii** $-2\boldsymbol{i}-12\boldsymbol{j}$

b $(-1,2)$

c $(-8,-32)$

6 $\left(3,-\frac{7}{2},8\right)$

7 a i $4\boldsymbol{i}-2\boldsymbol{j}-4\boldsymbol{k}$ **ii** $-5\boldsymbol{i}+4\boldsymbol{j}+9\boldsymbol{k}$

iii $2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$ **iv** $-\boldsymbol{i}-\boldsymbol{j}-3\boldsymbol{k}$

b i $\sqrt{30}$ **ii** $\sqrt{67}$

c $\overrightarrow{AB}$, $\overrightarrow{CD}$

8 a i $2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k}$ **ii** $\frac{4}{5}(2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k})$

iii $\frac{1}{5}(13\boldsymbol{i}-7\boldsymbol{j}-9\boldsymbol{k})$

b $\left(\frac{13}{5},-\frac{7}{5},-\frac{9}{5}\right)$

10 $x=\frac{13}{9}$

11 a i $\overrightarrow{OA}=2\boldsymbol{i}+\boldsymbol{j}$ **ii** $\overrightarrow{AB}=-\boldsymbol{i}-4\boldsymbol{j}$

iii $\overrightarrow{BC}=-6\boldsymbol{i}+5\boldsymbol{j}$ **iv** $\overrightarrow{BD}=2\boldsymbol{i}+8\boldsymbol{j}$

b $\overrightarrow{BD}=-2\overrightarrow{AB}$

c Points A, B and D are collinear

12 a i $\overrightarrow{OB}=2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$

ii $\overrightarrow{AC}=-\boldsymbol{i}-5\boldsymbol{j}+8\boldsymbol{k}$

iii $\overrightarrow{BD}=2\boldsymbol{i}+2\boldsymbol{j}+5\boldsymbol{k}$

iv $\overrightarrow{CD}=4\boldsymbol{i}+6\boldsymbol{j}+2\boldsymbol{k}$

b $\overrightarrow{CD}=2(2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})=2\overrightarrow{OB}$

13 a i $\overrightarrow{AB}=2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$

ii $\overrightarrow{BC}=-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$

iii $\overrightarrow{CD}=-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$

iv $\overrightarrow{DA}=\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$

b Parallelogram

14 a $(-6,3)$ **b** $(6,5)$ **c** $\left(\frac{3}{2},-\frac{3}{2}\right)$

15 a i $\overrightarrow{BC}=6\boldsymbol{i}+3\boldsymbol{j}$

ii $\overrightarrow{AD}=(x-2)\boldsymbol{i}+(y-1)\boldsymbol{j}$

b $(8,4)$

16 a $(1.5,1.5,4)$

b $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$

17 $\left(\frac{17}{5},\frac{8}{5},-3\right)$ **18** $\left(\frac{17}{2},3\right)$

19 $\left(-11,-\frac{11}{3}\right)$

21 a i $\boldsymbol{i}+\boldsymbol{j}$ **ii** $-\boldsymbol{i}-6\boldsymbol{j}$ **iii** $-\boldsymbol{i}-15\boldsymbol{j}$

b $k=\frac{19}{8},\ \ell=-\frac{1}{4}$

22 a i $2\boldsymbol{i}+4\boldsymbol{j}-9\boldsymbol{k}$ **ii** $14\boldsymbol{i}-8\boldsymbol{j}+3\boldsymbol{k}$

iii $5.7\boldsymbol{i}-0.3\boldsymbol{j}-1.6\boldsymbol{k}$

b There are no values for k and ℓ such that $k\boldsymbol{a}+\ell\boldsymbol{b}=\boldsymbol{c}$

23 a i $\sqrt{29}$ **ii** $\sqrt{13}$ **iii** $\sqrt{97}$ **iv** $\sqrt{19}$

b i 21.80° anticlockwise

ii 23.96° clockwise **iii** 46.51°

24 a $-3.42\boldsymbol{i}+9.40\boldsymbol{j}$ **b** $-2.91\boldsymbol{i}-7.99\boldsymbol{j}$

c $4.60\boldsymbol{i}+3.86\boldsymbol{j}$ **d** $2.50\boldsymbol{i}-4.33\boldsymbol{j}$

25 a $-6.43\boldsymbol{i}+1.74\boldsymbol{j}+7.46\boldsymbol{k}$

b $5.14\boldsymbol{i}+4.64\boldsymbol{j}-4\boldsymbol{k}$

c $6.13\boldsymbol{i}-2.39\boldsymbol{j}-2.39\boldsymbol{k}$

d $-6.26\boldsymbol{i}+9.77\boldsymbol{j}+3.07\boldsymbol{k}$

26 c $\frac{1}{2}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}+\frac{1}{\sqrt{2}}\boldsymbol{k}$

27 a $|\overrightarrow{AB}|=|\overrightarrow{AC}|=3$ **b** $\overrightarrow{OM}=-\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}$

c $\overrightarrow{AM}=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ **d** $3\sqrt{2}$

28 a $5\boldsymbol{i}+5\boldsymbol{j}$ **b** $\frac{1}{2}(5\boldsymbol{i}+5\boldsymbol{j})$

c $\frac{5}{2}\boldsymbol{i}+\frac{5}{2}\boldsymbol{j}+3\boldsymbol{k}$ **d** $-\frac{5}{2}\boldsymbol{i}-\frac{5}{2}\boldsymbol{j}+3\boldsymbol{k}$

e $\frac{\sqrt{86}}{2}$

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29 a $\overrightarrow{MN} = \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

b $\overrightarrow{MN} \parallel \overrightarrow{AB}$, $MN = \frac{1}{2}AB$

30 a $\frac{\sqrt{3}}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$ **b** $\frac{3\sqrt{3}}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$

c $\frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{7}{2}\boldsymbol{j}$ **d** $\sqrt{19}$ km

31 a $\overrightarrow{OA} = 50\boldsymbol{k}$

b i $-80\boldsymbol{i} + 20\boldsymbol{j} - 10\boldsymbol{k}$ **ii** $10\sqrt{69}$ m

c $-80\boldsymbol{i} + 620\boldsymbol{j} + 100\boldsymbol{k}$

32 a 2.66 km

b i $-0.5\boldsymbol{i} - \boldsymbol{j} + 0.1\boldsymbol{k}$ **ii** 1.12 km

c $-0.6\boldsymbol{i} - 0.8\boldsymbol{j}$

33 a $-100\sqrt{2}\boldsymbol{i} + 100\sqrt{2}\boldsymbol{j}$ **b** $50\boldsymbol{j}$

c $-100\sqrt{2}\boldsymbol{i} + (50 + 100\sqrt{2})\boldsymbol{j}$ **d** $30\boldsymbol{k}$

e $-100\sqrt{2}\boldsymbol{i} + (50 + 100\sqrt{2})\boldsymbol{j} + 30\boldsymbol{k}$

34 a $\overrightarrow{OP} = 50\sqrt{2}\boldsymbol{i} + 50\sqrt{2}\boldsymbol{j}$

b i $(50\sqrt{2} - 100)\boldsymbol{i} + 50\sqrt{2}\boldsymbol{j}$ **ii** 337.5°

35 $m = \frac{2n - 9}{n + 3}$

36 a $-\boldsymbol{i} - 8\boldsymbol{j} + 16\boldsymbol{k}$ **b** $m = \frac{3}{4}$

37 a $\boldsymbol{c} = (3m + 1)\boldsymbol{i} - \boldsymbol{j} + (1 - 3m)\boldsymbol{k}$ **b** $p = -5$

Exercise 4C

1 a 66 **b** 22 **c** 6 **d** 11 **e** 25

f 86 **g** −43

2 a 14 **b** 13 **c** 0 **d** −8 **e** 14

3 a 21 **b** −21

4 a $\boldsymbol{a} \cdot \boldsymbol{a} + 4(\boldsymbol{a} \cdot \boldsymbol{b}) + 4(\boldsymbol{b} \cdot \boldsymbol{b})$ **b** $4(\boldsymbol{a} \cdot \boldsymbol{b})$

c $\boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{b}$ **d** $|\boldsymbol{a}|$

5 a −4 **b** 5 **c** 5 **d** −6 or 1

6 a $\overrightarrow{AB} = -2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ **b** $|\overrightarrow{AB}| = 3$

c 105.8°

7 $\sqrt{66}$

8 a i $\boldsymbol{c}$ **ii** $\boldsymbol{a} + \boldsymbol{c}$ **iii** $\boldsymbol{c} - \boldsymbol{a}$

b 0

9 $\boldsymbol{d}$ and $\boldsymbol{f}$; $\boldsymbol{a}$ and $\boldsymbol{e}$; $\boldsymbol{b}$ and $\boldsymbol{c}$

10 b 109.47°

11 a $\overrightarrow{AP} = -\boldsymbol{a} + q\boldsymbol{b}$ **b** $q = \frac{13}{15}$

c $\left(\frac{26}{15}, \frac{13}{3}, -\frac{13}{15}\right)$

12 $x = 1$, $y = -3$

13 a 2.45 **b** 1.11 **c** 0.580 **d** 2.01

15 a $\overrightarrow{OM} = \frac{3}{2}\boldsymbol{i} + \boldsymbol{j}$ **b** 36.81° **c** 111.85°

16 a i $-\boldsymbol{i} + 3\boldsymbol{j}$ **ii** $3\boldsymbol{j} - 2\boldsymbol{k}$

b 37.87° **c** 31.00°

17 a i $\frac{1}{2}(4\boldsymbol{i} + 5\boldsymbol{j})$ **ii** $\frac{1}{2}(2\boldsymbol{i} + 7\boldsymbol{k})$

b 80.12° **c** 99.88°

18 69.71°

Exercise 4D

1 a $\frac{\sqrt{11}}{11}(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$ **b** $\frac{1}{3}(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$

c $\frac{\sqrt{10}}{10}(-\boldsymbol{j} + 3\boldsymbol{k})$

2 a i $\frac{\sqrt{26}}{26}(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$ **ii** $\frac{\sqrt{3}}{3}(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

b $\frac{\sqrt{78}}{26}(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$

4 a i $\frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$ **ii** $\frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{k})$

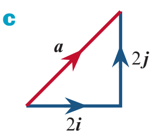
b $\frac{\sqrt{510}}{510}(19\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k})$

5 a $-\frac{11}{18}(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$ **b** $-\frac{1}{9}(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$

c $\frac{13}{17}(4\boldsymbol{i} - \boldsymbol{k})$

6 a 2 **b** $\frac{\sqrt{5}}{5}$ **c** $\frac{2\sqrt{21}}{7}$ **d** $-\frac{(1 + 4\sqrt{5})\sqrt{17}}{17}$

7 a $2\boldsymbol{i}$ **b** 2 **c**



8 a $\frac{9}{26}(5\boldsymbol{i} - \boldsymbol{k})$, $\frac{1}{26}(7\boldsymbol{i} + 26\boldsymbol{j} + 35\boldsymbol{k})$

b $\frac{3}{2}(\boldsymbol{i} + \boldsymbol{k})$, $\frac{3}{2}\boldsymbol{i} + \boldsymbol{j} - \frac{3}{2}\boldsymbol{k}$

c $-\frac{1}{9}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$, $-\frac{7}{9}\boldsymbol{i} + \frac{11}{9}\boldsymbol{j} + \frac{8}{9}\boldsymbol{k}$

9 a $\boldsymbol{j} + \boldsymbol{k}$ **b** $\frac{1}{3}(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$

10 a $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ **b** $3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ **c** $\sqrt{14}$

11 a i $\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ **ii** $\boldsymbol{i} - 5\boldsymbol{j}$

b $\frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$ **c** $\frac{2}{13}\sqrt{195}$ **d** $\sqrt{30}$

12 b i $\frac{2}{7}(\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$ **ii** $\frac{1}{3}(5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

c $\frac{1}{21}(\boldsymbol{i} + 11\boldsymbol{j} - 16\boldsymbol{k})$

13 a $\boldsymbol{d} = \frac{\boldsymbol{c} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a}$ **b** $\boldsymbol{e} = \frac{\boldsymbol{c} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$

c $\boldsymbol{f} = \boldsymbol{c} - \frac{\boldsymbol{c} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} - \frac{\boldsymbol{c} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$

Exercise 4E

1 a $\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$ **b** $\frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{b}$

2 a $\frac{5}{2}\boldsymbol{i} - \boldsymbol{j} + \frac{5}{2}\boldsymbol{k}$ **b** $\frac{5}{3}\boldsymbol{i} - \frac{8}{3}\boldsymbol{j}$

c $\frac{10}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + 5\boldsymbol{k}$

3 b 2 : 1

4 a $\frac{a + x}{2}\boldsymbol{i} + \frac{y}{2}\boldsymbol{j}$ **b** $x^2 + y^2 = a^2$

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5 b $1:5$

6 a $\overrightarrow{OB} = -\boldsymbol{i} + 7\boldsymbol{j}$ b $\overrightarrow{OD} = -2\boldsymbol{i} + \frac{17}{3}\boldsymbol{j}$

c $\lambda = \frac{2}{5}$

7 b i $\overrightarrow{OP} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

ii $\overrightarrow{OP} = \frac{18}{11}\boldsymbol{i} + \frac{15}{11}\boldsymbol{j} - \frac{1}{11}\boldsymbol{k}$

iii $\overrightarrow{OP} = \frac{7}{4}\boldsymbol{i} + \frac{5}{4}\boldsymbol{j} + \frac{1}{4}\boldsymbol{k}$

Exercise 4F

12 a i $\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$ ii $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

b $\frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b})$

13 c $3:1$

14 a i $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$ ii $\boldsymbol{a} + 2\boldsymbol{b}$ iii $2\boldsymbol{b}$

15 a $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{t}$

b $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{r} + \boldsymbol{s}),\ \boldsymbol{v} = \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

16 b $\overrightarrow{AB} = \boldsymbol{i} - 3\boldsymbol{j},\ \overrightarrow{DC} = \boldsymbol{i} - \boldsymbol{j}$ c $4\boldsymbol{i} + 2\boldsymbol{j}$

e $4\boldsymbol{j}$

18 $\frac{2}{3}\boldsymbol{b} - \frac{5}{12}\boldsymbol{a}$

20 b $\lambda = \frac{k+2}{2},\ \mu = \frac{k+2}{2}$

c $\lambda = \frac{3}{2},\ \mu = \frac{3}{2}$

21 b $12r^2$

23 a $\overrightarrow{OX} = \frac{c^2}{a^2 + c^2 + d^2}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})$

b $\overrightarrow{OY} = \frac{a^2 + c^2}{a^2 + c^2 + d^2}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})$

c i $\overrightarrow{OX} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d}),\ \overrightarrow{OY} = \frac{2}{3}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})$

ii $120°$ iii $120°$

26 a $\overrightarrow{OG} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

30 a $\lambda = \frac{1}{2}$ c $\cos^{-1}\left(\frac{1}{3}\right)$

31 a $\overrightarrow{OG} = \boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e},\ \overrightarrow{DF} = \boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e},$
$\overrightarrow{BH} = -\boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e},\ \overrightarrow{CE} = -\boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e}$

b $|\overrightarrow{OG}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(\boldsymbol{b} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{e} + \boldsymbol{d} \cdot \boldsymbol{e})$

$|\overrightarrow{DF}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(-\boldsymbol{b} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{e} - \boldsymbol{d} \cdot \boldsymbol{e})$

$|\overrightarrow{BH}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(-\boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{b} \cdot \boldsymbol{e} + \boldsymbol{d} \cdot \boldsymbol{e})$

$|\overrightarrow{CE}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(\boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{b} \cdot \boldsymbol{e} - \boldsymbol{d} \cdot \boldsymbol{e})$

Chapter 4 review

Technology-free questions

1 a $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ b $\frac{\sqrt{2}}{3}$

2 a i $\frac{3}{7}(-3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$ ii $\frac{1}{7}(6\boldsymbol{i} - 11\boldsymbol{j} - 12\boldsymbol{k})$

3 a $x = 5$ b $y = 2.8,\ z = -4.4$

4 a $\cos\theta = \frac{1}{3}$ b 6

5 a $\frac{1}{9}(43\boldsymbol{i} - 46\boldsymbol{j} + 20\boldsymbol{k})$ b $-\frac{61}{549}(3\boldsymbol{i} - 6\boldsymbol{j} + 4\boldsymbol{k})$

6 a i $(2 - 3t)\boldsymbol{j} + (-3 - 2t)\boldsymbol{k}$

ii $(-2 - 3t)\boldsymbol{j} + (3 - 2t)\boldsymbol{k}$

b ± 1

7 a i $2\sqrt{17}$ ii $4\sqrt{3}$ iii -40

b $\cos^{-1}\left(\frac{5\sqrt{51}}{51}\right)$

8 a $3\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$ b $\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + 4\boldsymbol{k}$ c $\frac{8\sqrt{5}}{21}$

9 a $34 - 4p$ b 8.5 c $\frac{5}{13}$

10 -6.5

11 $\lambda = \frac{3}{2},\ \mu = -\frac{3}{2}$

12 $AB \parallel CD,\ AB : CD = 1:2$

13 $\frac{\sqrt{19}}{5}$

14 a $(-1, 10)$ b $h = 3,\ k = -2$

15 a $2\boldsymbol{c},\ 2\boldsymbol{c} - \boldsymbol{a}$ b $\frac{1}{2}\boldsymbol{a} + \boldsymbol{c}$ c 1.5

16 $h = \frac{2}{3},\ k = \frac{3}{4}$

17 $3(\boldsymbol{i} + \boldsymbol{j})$

18 a $\boldsymbol{c} - \boldsymbol{a}$

19 a i $\frac{1}{3}\boldsymbol{c}$ ii $\frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$ iii $\frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{c}$

20 a $\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$

b i $\frac{\lambda}{4}\boldsymbol{a} + \left(\frac{3\lambda}{4} - 1\right)\boldsymbol{b}$ ii $\frac{4}{3}$

21 $m = \frac{3(n-6)}{n+2}$

22 a $\boldsymbol{v} = \frac{6}{5}\boldsymbol{i} + \boldsymbol{j} - \frac{2}{5}\boldsymbol{k}$

Multiple-choice questions

1 C	2 D	3 B	4 B	5 C
6 C	7 E	8 E	9 D	10 B
11 C	12 B	13 D		

Extended-response questions

1 a i $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ ii $\sqrt{3}$

b i $(\lambda - 0.5)\boldsymbol{i} + (\lambda - 1)\boldsymbol{j} + (\lambda - 0.5)\boldsymbol{k}$

ii $\lambda = \frac{2}{3},\ \overrightarrow{OQ} = \frac{1}{3}(8\boldsymbol{i} + 11\boldsymbol{j} + 5\boldsymbol{k})$

c $5\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$

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2 a i $|\overrightarrow{OA}| = \sqrt{14}$, $|\overrightarrow{OB}| = \sqrt{14}$ **ii** $\boldsymbol{i} - 5\boldsymbol{j}$

b i $\frac{1}{2}(5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

c $5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

e i $5\boldsymbol{i} + \boldsymbol{j} - 13\boldsymbol{k}$ or $-5\boldsymbol{i} - \boldsymbol{j} + 13\boldsymbol{k}$

iii The vector is perpendicular to the plane containing $OACB$

3 a $\overrightarrow{OX} = 7\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$, $\overrightarrow{OY} = 2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$, $\overrightarrow{OZ} = 6\boldsymbol{i} + 4\boldsymbol{j}$, $\overrightarrow{OD} = 6\boldsymbol{i} + 3\boldsymbol{k}$, $|\overrightarrow{OD}| = 3\sqrt{5}$, $|\overrightarrow{OY}| = \sqrt{29}$

b 48.27°

c i $\left(\frac{5\lambda}{\lambda+1} + 1\right)\boldsymbol{i} + 4\boldsymbol{j}$ **ii** $-\frac{1}{6}$

4 a i $\boldsymbol{b} - \boldsymbol{a}$ **ii** $\boldsymbol{c} - \boldsymbol{b}$ **iii** $\boldsymbol{a} - \boldsymbol{c}$

iv $\frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$ **v** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$ **vi** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

5 a $\frac{1}{3}\boldsymbol{b} + \frac{2}{3}\boldsymbol{c}$ **c ii** 5 : 1 **d** 1 : 3

6 a i $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ **ii** $-\frac{1}{2}\boldsymbol{a} + \left(\lambda - \frac{1}{2}\right)\boldsymbol{b}$

7 a i $12(1 - a)$ **ii** 1

b i $x - 4y + 2 = 0$ **ii** $x = -2,\ y = 0$

c i $\boldsymbol{j} + 4\boldsymbol{k}$ **ii** $\boldsymbol{i} - 12\boldsymbol{j} + 5\boldsymbol{k}$

iii $3\boldsymbol{i} - 11\boldsymbol{j} + 7\boldsymbol{k}$

d X has height 5 units; Y has height 7 units

8 a i $\frac{3}{4}\boldsymbol{c}$ **ii** $\frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$ **iii** $-\boldsymbol{a} + \frac{3}{4}\boldsymbol{c}$

b $\mu = \frac{5}{6}$, $\lambda = \frac{2}{3}$

9 a $\boldsymbol{b} = q\boldsymbol{i} - p\boldsymbol{j}$, $\boldsymbol{c} = -q\boldsymbol{i} + p\boldsymbol{j}$

b i $\overrightarrow{AB} = -(x + 1)\boldsymbol{i} - y\boldsymbol{j}$, $\overrightarrow{AC} = (1 - x)\boldsymbol{i} - y\boldsymbol{j}$

ii $\overrightarrow{AE} = y\boldsymbol{i} + (1 - x)\boldsymbol{j}$, $\overrightarrow{AF} = -y\boldsymbol{i} + (x + 1)\boldsymbol{j}$

10 a i $\overrightarrow{BC} = m\boldsymbol{v}$, $\overrightarrow{BE} = n\boldsymbol{v}$, $\overrightarrow{CA} = m\boldsymbol{w}$, $\overrightarrow{CF} = n\boldsymbol{w}$

ii $|\overrightarrow{AE}| = \sqrt{m^2 - mn + n^2}$, $|\overrightarrow{FB}| = \sqrt{m^2 - mn + n^2}$

11 a $\overrightarrow{CF} = \frac{1}{2}\boldsymbol{a} - c$, $\overrightarrow{OE} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$

b ii 60°

c ii HX is parallel to EX; KX is parallel to FX; HK is parallel to EF

12 a $\overrightarrow{OA} = -2(\boldsymbol{i} + \boldsymbol{j})$, $\overrightarrow{OB} = 2(\boldsymbol{i} - \boldsymbol{j})$, $\overrightarrow{OC} = 2(\boldsymbol{i} + \boldsymbol{j})$, $\overrightarrow{OD} = -2(\boldsymbol{i} - \boldsymbol{j})$

b $\overrightarrow{PM} = \boldsymbol{i} + 3\boldsymbol{j} + h\boldsymbol{k}$, $\overrightarrow{QN} = -3\boldsymbol{i} - \boldsymbol{j} + h\boldsymbol{k}$

c $\overrightarrow{OX} = \frac{1}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{h}{2}\boldsymbol{k}$

d i $\sqrt{2}$ **ii** 71°

e ii $\sqrt{6}$

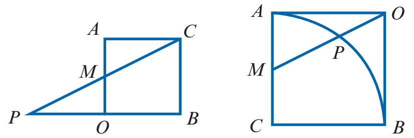
13 a i $\overrightarrow{OM} = \frac{a}{2}\boldsymbol{j}$ **ii** $\overrightarrow{MC} = a\boldsymbol{i} + \frac{a}{2}\boldsymbol{j}$

b $\overrightarrow{MP} = a\lambda\boldsymbol{i} + \frac{a\lambda}{2}\boldsymbol{j}$, $\overrightarrow{BP} = a(\lambda - 1)\boldsymbol{i} + \frac{a}{2}(\lambda + 1)\boldsymbol{j}$, $\overrightarrow{OP} = a\lambda\boldsymbol{i} + \frac{a}{2}(\lambda + 1)\boldsymbol{j}$

c i $\lambda = \frac{3}{5}$, $|\overrightarrow{BP}| = \frac{2\sqrt{5}a}{5}$, $|\overrightarrow{OP}| = a$, $|\overrightarrow{OB}| = a$

ii $\frac{\sqrt{5}}{5}$

d $\lambda = -1$ or $\lambda = \frac{3}{5}$



e $\overrightarrow{OY} = \frac{14}{15}a\boldsymbol{i} + \frac{29}{30}a\boldsymbol{j} + \frac{1}{6}a\boldsymbol{k}$

Chapter 5

Exercise 5A

1 a Yes **b** Yes **c** No

2 a $\boldsymbol{r} = \boldsymbol{i} + (1 + 2t)\boldsymbol{j}$

b $\boldsymbol{r} = \boldsymbol{i} - 3\boldsymbol{k} + t(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

c $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} + t(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

d $\boldsymbol{r} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} + t(-4\boldsymbol{i} + 3\boldsymbol{j})$

3 a $\boldsymbol{r} = 3\boldsymbol{i} + \boldsymbol{j} + t(-5\boldsymbol{i} + \boldsymbol{j})$

b $\boldsymbol{r} = -\boldsymbol{i} + 5\boldsymbol{j} + t(3\boldsymbol{i} - 6\boldsymbol{j})$

c $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k})$

d $\boldsymbol{r} = \boldsymbol{i} - 4\boldsymbol{j} + t(\boldsymbol{i} + 7\boldsymbol{j} + \boldsymbol{k})$

4 a i $x = 3 - 5t,\ y = 1 + t$

ii $y = \frac{1}{5}(8 - x)$

b i $x = -1 + 3t,\ y = 5 - 6t$

ii $y = -2x + 3$

c i $x = 1 + t,\ y = 2 - 2t,\ z = 3 - 4t$

ii $x - 1 = \frac{2 - y}{2} = \frac{3 - z}{4}$

d i $x = 1 + t,\ y = -4 + 7t,\ z = t$

ii $x - 1 = \frac{y + 4}{7} = z$

5 a $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + t(3\boldsymbol{i} - 2\boldsymbol{j})$

b $\boldsymbol{r} = 4\boldsymbol{j} + t(-9\boldsymbol{i} + 6\boldsymbol{j})$

c $\boldsymbol{r} = 6\boldsymbol{i} + t(-6\boldsymbol{i} + 4\boldsymbol{j})$

6 a $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(-3\boldsymbol{i} + \boldsymbol{j})$

b $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(\boldsymbol{i} + 3\boldsymbol{j})$

7 a $\boldsymbol{r} = t(2\boldsymbol{j} - \boldsymbol{k})$ **b** $\boldsymbol{r} = t(\boldsymbol{j} + 2\boldsymbol{k})$

8 a $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + t(-3\boldsymbol{i} + 2\boldsymbol{j})$

b i No **ii** No **iii** Yes

9 a $\boldsymbol{r} = \boldsymbol{j} + \boldsymbol{k} + t(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

c $m = -\frac{5}{3}$, $n = -\frac{4}{3}$

10 a $4\boldsymbol{i} + 3\boldsymbol{j}$ **b** $\boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{j} + t(4\boldsymbol{i} + 3\boldsymbol{j})$

c $\left(-\frac{7}{3}, 0\right)$, $\left(0, \frac{7}{4}\right)$

11 a $x = 2 - 3t,\ y = 5 + t,\ z = 4 - 2t$; $\frac{2 - x}{3} = y - 5 = \frac{4 - z}{2}$

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b $x = 2t,\ y = 2 + t,\ z = -1 + 4t;$
$\frac{x}{2} = y - 2 = \frac{z+1}{4}$

12 a $\frac{11\sqrt{114}}{38}$ **b** $\frac{\sqrt{23\ 123}}{19}$ **c** $\sqrt{17}$ **d** 3

13 c $t \in [-3, 2]$

14 $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

15 $\boldsymbol{r} = -\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k});\ \sqrt{5}$

16 $\left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$ **17** $\boldsymbol{r} = (t-2)\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

18 $\frac{\sqrt{165}}{3}$

19 a $(-1, -1, 3), (-5, 1, 7)$
b $(2, 6, -4), (5, 0, 2)$

20 a $(1, 1, 2)$ **c** $t = 2$

21 a $(1+t)\boldsymbol{i} + (-4+2t)\boldsymbol{j} + (1-t)\boldsymbol{k}$
b $\sqrt{18 - 16t + 6t^2}$ **c** $\frac{\sqrt{66}}{3}$ **d** $\frac{\sqrt{786}}{6}$

Exercise 5B

1 $\frac{17}{2}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}$

2 $(-1, 2, 3)$

4 a **i** No **ii** No **iii** No **iv** $(-7, -6)$
b **i** No **ii** Yes **iii** No **iv** $(-1, 1)$
c **i** Yes **ii** No **iii** Yes
d **i** Yes **ii** No **iii** No **iv** None
e **i** No **ii** Yes **iii** No **iv** $(3, 1, -2)$
f **i** No **ii** No **iii** No **iv** None
g **i** No **ii** No **iii** No **iv** $(3, 0, -1)$
h **i** Yes **ii** No **iii** Yes
i **i** No **ii** No **iii** No **iv** $(0, 1, -2)$
j **i** Yes **ii** No **iii** Yes

5 a $(1, 2, -1)$ **b** None **c** None **d** None

6 a $25.21°$ **b** $0°$

7 a $30°$

8 a $(3, 3, 1)$ **b** $\frac{1}{\sqrt{15}}$

9
- Lines ℓ_1 and ℓ_2 do not intersect
- Lines ℓ_1 and ℓ_3 intersect at $(2, 3, -1)$
- Lines ℓ_2 and ℓ_3 intersect at $(4, -5, -1)$

Exercise 5C

1 a $-3\boldsymbol{i} + 4\boldsymbol{j} + 19\boldsymbol{k}$ **b** $\boldsymbol{i} - 7\boldsymbol{j} - 4\boldsymbol{k}$
c $\boldsymbol{i} - \boldsymbol{j}$ **d** $\boldsymbol{i} + 2\boldsymbol{k}$
e $-9\boldsymbol{i} - 26\boldsymbol{j} - 12\boldsymbol{k}$ **f** $2\boldsymbol{j} + \boldsymbol{k}$
g $2\boldsymbol{j} + \boldsymbol{k}$ **h** $\boldsymbol{i} - 2\boldsymbol{k}$

2 a $\boldsymbol{a} \times \boldsymbol{b}$ **b** $\boldsymbol{0}$ **c** $2(\boldsymbol{a} \times \boldsymbol{b})$
d $(\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$ **e** 0 **f** 0

3 $\frac{\sqrt{10}}{6}(4\boldsymbol{i} - 5\boldsymbol{j} - 7\boldsymbol{k})$

4 $\boldsymbol{i} + \boldsymbol{j}$ is a possible answer

5 1

6 $\frac{\sqrt{374}}{2}$

Exercise 5D

1 a $\boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 3,\ x + y + z = 3$
b $\boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{k}) = 3,\ x - 2z = 3$
c $\boldsymbol{r} \cdot (2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = 0,\ 2x + 3y - z = 0$
d $\boldsymbol{r} \cdot (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = -8,\ x + 3y - z = -8$

2 $\frac{1}{\sqrt{170}}(12\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}),\ \boldsymbol{r} \cdot (12\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}) = 28$

3 $\boldsymbol{r} \cdot (\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}) = -1,\ x - y - 3z = -1$

4 a $5\boldsymbol{i} + 4\boldsymbol{j} + 13\boldsymbol{k}$
b $\boldsymbol{r} = -3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + t(5\boldsymbol{i} + 4\boldsymbol{j} + 13\boldsymbol{k})$

5 $\frac{1}{\sqrt{77}}(-6\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}),\ \boldsymbol{r} \cdot (-6\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}) = 11$

8 a $x = 0$ **b** $x = 6$ **c** $x = 3$ **d** $x = 4$

9 $6x + 2y + z = 10$ **10** $x - 2y + 8z = 7$

11 $5x - 3y + 2z = 27$ **12** $13x + 7y + 9z = 61$

13 $-3x + 8y + 7z = 41$ **14** $x + 2y = 5$

Exercise 5E

1 a 2 **b** $\frac{22}{9}$

2 $\frac{8}{3}$

3 a $80.41°$ **b** $\boldsymbol{r} = 22\boldsymbol{j} + 14\boldsymbol{k} + t(\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k})$

4 a $(-1, -9, 7)$ **b** $7.82°$

5 a $7\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$ **b** $\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$
c $72.98°$

6 a $(2, -2, -1),\ 29.50°$
b $\left(\frac{7}{2}, -\frac{3}{2}, -\frac{5}{2}\right),\ 32.98°$
c $\left(\frac{1}{2}, \frac{3}{2}, -\frac{7}{2}\right),\ 79.98°$ **d** $(-7, 4, -3),\ 7.45°$

7 a $\boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}) = -9$ **b** $\frac{9}{\sqrt{41}}$

8 a $\frac{7}{3}$ **b** $\frac{1}{3}(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$ **c** 1 **d** $\frac{4}{3}$

9 $\frac{5}{3}$

10 a $(5, -1, -1)$ **b** $25.7°$

11 a $x + y + z = 4$ **b** $2\sqrt{3}$
c $\frac{4}{3}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

12 a $88.18°$
b $\boldsymbol{r} = -\frac{5}{2}\boldsymbol{j} - 9\boldsymbol{k} + t\left(\boldsymbol{i} + \frac{19}{2}\boldsymbol{j} + 30\boldsymbol{k}\right)$

13 a $\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k}$ **b** $2\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k}$ **c** $43.12°$

14 a $-6\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$
b $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t(-6\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$

15 a 2 **b** $\frac{3}{\sqrt{2}}$ **c** $\frac{4}{\sqrt{5}}$

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Chapter 5 review

Technology-free questions

1 $4\boldsymbol{i} - \boldsymbol{k}$

3 $4x + 5y + 6z = 32$

4 $\boldsymbol{r} = (t-2)\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

5 $\left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$

6 $\frac{\sqrt{165}}{3}$

7 $(-1, -1, 4)$

8 $2\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}$

9 $\boldsymbol{r} = 5\boldsymbol{i} + 6\boldsymbol{k} + t(6\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k})$; $(1, -2, 0)$

10 $\frac{\sqrt{3}}{2}$

11 $2x - 8y + 5z = 18$

12 $12x + 8y + 20z = 16$

13 a $\boldsymbol{r} \cdot (\boldsymbol{i} + 10\boldsymbol{j} + 6\boldsymbol{k}) = 19$
b $x + 10y + 6z = 19$

14 a $x - 2y + z = 0$ **b** $\frac{\sqrt{6}}{2}$ **c** $\left(0, \frac{4}{3}, \frac{8}{3}\right)$

15 $(1, -2, 1)$ or $(1, 1, -2)$

Multiple-choice questions

1 C	**2** D	**3** D	**4** D	**5** D
6 C	**7** A	**8** D	**9** E	**10** C

Extended-response questions

1 a $(3, -1, 2)$ **b** $\frac{\sqrt{30}}{3}$

2 c $\sqrt{30}$ **d** $47.73°$ **e** $k = 2$ or $k = 80$

3 c $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(5\boldsymbol{i} - 7\boldsymbol{j} + \boldsymbol{k})$
d $(13, -12, 3)$, $10\sqrt{3}$

5 a $\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$ **b** $\sqrt{19}$
c $8x - 11y - 9z = 0$ **d** $29.9°$

6 d $(x - 2y + 6)^2 + (x - 8z - 10)^2 = 0$
e $(x + y - 5)^2 + (5x - z - 10)^2 = 0$

12 a $\sqrt{2}$ **b** $\frac{\sqrt{3}}{2}$
c $x + y + z = 1$, $x - y - z = -1$, $\cos^{-1}\left(\frac{1}{3}\right)$

13 a $\sqrt{2}$ **b** $\frac{\sqrt{3}}{2}$
d $x + y + z = 1$, $x - y - z = 1$, $\cos^{-1}\left(-\frac{1}{3}\right)$

Chapter 6

Exercise 6A

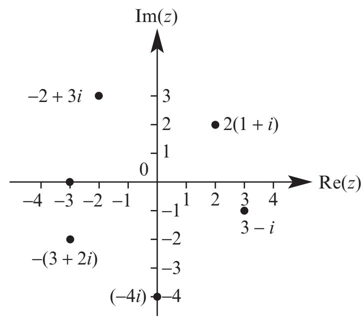
1 a 6 **b** −7 **c** 13

2 a $5i$ **b** $3\sqrt{3}i$ **c** $-5i$ **d** $13i$ **e** $5\sqrt{2}i$
f $-2\sqrt{3}$ **g** $-1 + 2i$ **h** 4 **i** 0

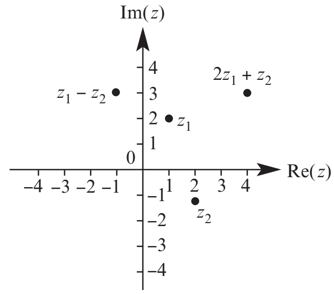
3 a $x = 5$, $y = 0$ **b** $x = 0$, $y = 2$
c $x = 0$, $y = 0$ **d** $x = 9$, $y = -4$
e $x = -2$, $y = -2$ **f** $x = 13$, $y = 6$

4 a $5 + i$ **b** $4 + 4i$ **c** $5 - 5i$
d $4 - 3i$ **e** $-1 + i$ **f** 2
g 2 **h** 1 **i** $3 - 2i$

5



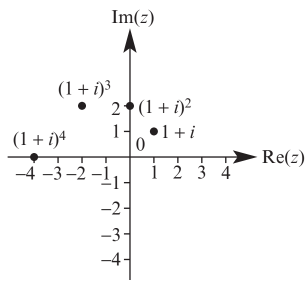
6



7 a $11 + 3i$ **b** $-23 + 41i$ **c** 13
d $-8 + 6i$ **e** $3 - 4i$ **f** $-2 + 2i$
g 1 **h** $5 - 6i$ **i** −1

8 a $x = 4$, $y = -3$ **b** $x = -2$, $y = 5$
c $x = -3$ **d** $x = 3$, $y = -3$ or $x = -3$, $y = 3$
e $x = 3$, $y = 2$

9 a



b Anticlockwise turn by $\frac{\pi}{4}$ about the origin; distance from origin increases by factor $\sqrt{2}$

10 a $\overrightarrow{PQ} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \overrightarrow{OR}$ **b** $|\overrightarrow{PQ}| = \sqrt{10}$

11 1

Exercise 6B

1 a $\sqrt{3}$ **b** $-8i$ **c** $4 + 3i$
d $-1 + 2i$ **e** $4 - 2i$ **f** $-3 + 2i$

2 a i **b** $\frac{3}{10} - \frac{1}{10}i$ **c** $-3 + 4i$
d $\frac{17}{5} + \frac{1}{5}i$ **e** $\frac{-1 - \sqrt{3}}{2} + \frac{\sqrt{3} - 1}{2}i$
f $4 + i$

4 a $5 - 5i$ **b** $6 + i$ **c** $2 + 3i$
d $\frac{2 - i}{5}$ **e** $-8i$ **f** $8 + 6i$

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5 a $a^2 + b^2$ **b** $\frac{a}{a^2+b^2} + \frac{b}{a^2+b^2}i$
c $2a$ **d** $2bi$
e $\frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i$
f $\frac{a^2-b^2}{a^2+b^2} - \frac{2ab}{a^2+b^2}i$

8 b $1^2 + 8^2$ or $4^2 + 7^2$
c $(n^2+2)^2 + n^2$ or $(n^2-2)^2 + (3n)^2$

Exercise 6C

1 a $3; \pi$ **b** $5; \frac{\pi}{2}$ **c** $\sqrt{2}; \frac{3\pi}{4}$
d $2; \frac{\pi}{6}$ **e** $4; -\frac{\pi}{3}$ **f** $16; -\frac{2\pi}{3}$

2 a 1.18 **b** 2.06 **c** −2.50
d −0.96 **e** 0.89 **f** −1.98

3 a $\frac{5\pi}{3}$ **b** $\frac{3\pi}{2}$ **c** $\frac{5\pi}{6}$
d $\frac{\pi}{4}$ **e** $-\frac{11\pi}{6}$ **f** $-\frac{3\pi}{2}$

4 a $-\frac{3\pi}{4}$ **b** $\frac{5\pi}{6}$ **c** $\frac{\pi}{8}$ **d** $-\frac{\pi}{2}$

5 a $\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ **b** $\operatorname{cis}\left(-\frac{\pi}{3}\right)$
c $\sqrt{6}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **d** $\frac{2}{3}\operatorname{cis}\left(\frac{\pi}{6}\right)$
e $2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}\right)$ **f** $4\operatorname{cis}\left(\frac{5\pi}{6}\right)$

6 a $-\sqrt{2} + \sqrt{2}i$ **b** $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$ **c** $2 + 2i$
d $-\frac{3\sqrt{3}}{2} - \frac{3}{2}i$ **e** $6i$ **f** -4

8 a $2\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ **b** $7\operatorname{cis}\left(\frac{2\pi}{3}\right)$
c $3\operatorname{cis}\left(\frac{\pi}{3}\right)$ **d** $5\operatorname{cis}\left(\frac{\pi}{4}\right)$

Exercise 6D

1 $(2\sqrt{3} - 3) + (3\sqrt{3} + 2)i$

2 a $12\operatorname{cis}\left(-\frac{7\pi}{12}\right)$ **b** $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$
c $\frac{7}{6}\operatorname{cis}\left(-\frac{\pi}{15}\right)$ **d** $8\operatorname{cis}\left(-\frac{19\pi}{20}\right)$ **e** $-\frac{1}{8}$

3 a $8\operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $\frac{8}{27}\operatorname{cis}\left(\frac{\pi}{8}\right)$
c $27\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **d** $-32i$ **e** -216
f $1024\operatorname{cis}\left(-\frac{\pi}{12}\right)$ **g** $\frac{27}{4}\operatorname{cis}\left(-\frac{\pi}{20}\right)$

4 a $\operatorname{Arg}(z_1z_2) = \frac{7\pi}{12}$; $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{7\pi}{12}$;
$\operatorname{Arg}(z_1z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$
b $\operatorname{Arg}(z_1z_2) = \frac{7\pi}{12}$; $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = -\frac{17\pi}{12}$;
$\operatorname{Arg}(z_1z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2\pi$
c $\operatorname{Arg}(z_1z_2) = -\frac{5\pi}{6}$; $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{7\pi}{6}$;
$\operatorname{Arg}(z_1z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) - 2\pi$

6 a $\frac{\pi}{4}$ **b** $-\frac{3\pi}{4}$ **c** $-\frac{\pi}{4}$

7 $\left(\frac{2\sqrt{3}+3}{2}, \frac{3\sqrt{3}-2}{2}\right)$

8 b i $\operatorname{cis}\left(\frac{3\pi}{2} - 7\theta\right)$ **ii** i
iii $\operatorname{cis}(4\theta)$ **iv** $\operatorname{cis}(\pi - \theta - \varphi)$

9 b i $\operatorname{cis}(-5\theta)$ **ii** $\operatorname{cis}(3\theta)$
iii 1 **iv** $\operatorname{cis}\left(\frac{\pi}{2} - 2\theta\right)$

10 b i $\operatorname{cis}(6\theta - 3\pi)$ **ii** $\operatorname{cis}(\pi - 2\theta)$
iii $\operatorname{cis}(\theta - \pi)$ **iv** $-i$

11 a i $\sec\theta \operatorname{cis}\theta$
ii $\operatorname{cosec}\theta \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$
iii $\frac{1}{\sin\theta\cos\theta}\operatorname{cis}\theta = \operatorname{cosec}\theta \sec\theta \operatorname{cis}\theta$
b i $\sec^2\theta \operatorname{cis}(2\theta)$
ii $\sin^3\theta \operatorname{cis}\left(3\theta - \frac{3\pi}{2}\right)$
iii $\operatorname{cosec}\theta \sec\theta \operatorname{cis}(-\theta)$

12 a $64\operatorname{cis}0 = 64$ **b** $\frac{\sqrt{2}}{8}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$
c $128\operatorname{cis}\left(-\frac{2\pi}{3}\right)$
d $\frac{\sqrt{3}}{72}\operatorname{cis}\left(-\frac{\pi}{2}\right) = -\frac{\sqrt{3}}{72}i$
e $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **f** $\frac{64\sqrt{3}}{3}\operatorname{cis}\left(\frac{3\pi}{4}\right)$
g $\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}i$ **h** $\frac{1}{4}\operatorname{cis}\left(-\frac{2\pi}{15}\right)$
i $8\sqrt{2}\operatorname{cis}\left(\frac{11\pi}{12}\right)$

13 $\frac{1}{2}(1+i)$

14 a $-32 + 32i$ **c** $32 + 33i$

Exercise 6E

1 a $(z + 4i)(z - 4i)$
b $(z + \sqrt{5}i)(z - \sqrt{5}i)$
c $(z + 1 + 2i)(z + 1 - 2i)$
d $\left(z - \frac{3}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{3}{2} - \frac{\sqrt{7}}{2}i\right)$
e $2\left(z - 2 + \frac{\sqrt{2}}{2}i\right)\left(z - 2 - \frac{\sqrt{2}}{2}i\right)$
f $3\left(z + 1 + \frac{\sqrt{3}}{3}i\right)\left(z + 1 - \frac{\sqrt{3}}{3}i\right)$
g $3\left(z + \frac{1}{3} + \frac{\sqrt{5}}{3}i\right)\left(z + \frac{1}{3} - \frac{\sqrt{5}}{3}i\right)$
h $2\left(z - \frac{1}{4} + \frac{\sqrt{23}}{4}i\right)\left(z - \frac{1}{4} - \frac{\sqrt{23}}{4}i\right)$

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2 a $5i,\ -5i$ **b** $2\sqrt{2}i,\ -2\sqrt{2}i$
c $2+i,\ 2-i$
d $-\frac{7}{6}+\frac{\sqrt{11}}{6}i,\ -\frac{7}{6}-\frac{\sqrt{11}}{6}i$
e $1-\sqrt{2}i,\ 1+\sqrt{2}i$
f $\frac{3}{10}+\frac{\sqrt{11}}{10}i,\ \frac{3}{10}-\frac{\sqrt{11}}{10}i$
g $-i,\ -1-i$ **h** $i,\ -1-i$

Exercise 6F

1 a $(z-5)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$
b $(z+2)\left(z-\frac{3}{2}+\frac{\sqrt{11}}{2}i\right)\left(z-\frac{3}{2}-\frac{\sqrt{11}}{2}i\right)$
c $3(z-4)\left(z-\frac{1}{6}+\frac{\sqrt{11}}{6}i\right)\left(z-\frac{1}{6}-\frac{\sqrt{11}}{6}i\right)$
d $2(z+3)\left(z-\frac{3}{4}+\frac{\sqrt{31}}{4}i\right)\left(z-\frac{3}{4}-\frac{\sqrt{31}}{4}i\right)$
e $(z+i)(z-i)(z-2+i)$

2 b $z-1+i$
c $(z+6)(z-1+i)(z-1-i)$

3 b $z+2+i$
c $(2z+1)(z+2+i)(z+2-i)$

4 b $z-1-3i$
c $(z-1+3i)(z-1-3i)(z+1+i)(z+1-i)$

5 a $(z+3)(z-3)(z+3i)(z-3i)$
b $(z+2)(z-2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)$
$(z+1+\sqrt{3}i)(z+1-\sqrt{3}i)$

6 a $(z-i)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$
b $(z+i)(z-1+\sqrt{2})(z-1-\sqrt{2})$
c $(z-2i)(z-3)(z+1)$
d $2(z-i)\left(z+\frac{1}{4}+\frac{\sqrt{41}}{4}\right)\left(z+\frac{1}{4}-\frac{\sqrt{41}}{4}\right)$

7 a 8 **b** -4 **c** -6

8 a $3,\ -2\pm\sqrt{2}i$ **b** $5,\ \frac{1\pm\sqrt{23}i}{2}$
c $-1,\ \frac{5\pm\sqrt{7}i}{2}$ **d** $-2,\ 3,\ \frac{1\pm\sqrt{23}i}{2}$

9 a $a=0,\ b=4$ **b** $a=-6,\ b=13$
c $a=2,\ b=10$

10 a $1-3i,\ \frac{1}{3}$ **b** $-2+i,\ 2\pm\sqrt{2}i$

11 $P(x)=-2x^3+10x^2-18x+10$;
$x=1$ or $x=2\pm i$

12 $a=6,\ b=-8$

13 a $z^2-4z+5,\ a=-7, b=6$
b $z=2\pm i$ or $z=-\frac{1}{2}$

14 a $P(1+i)=(-4a+d-2)+2(a-1)i$
b $a=1,\ d=6$
c $z=1\pm i$ or $z=-1\pm\sqrt{2}i$

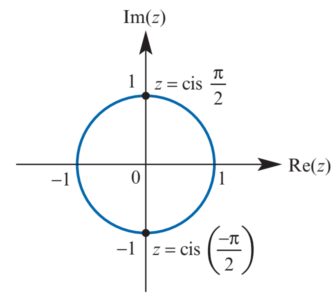
15 $p=-(5+4i),\ q=1+7i$

16 $z=1+i$ or $z=2$

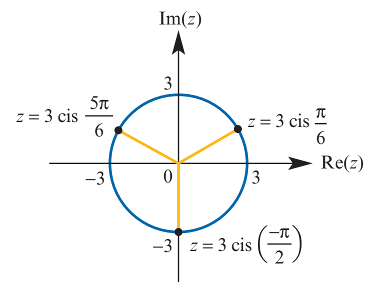
17 a $3+i$ **b** $2i,\ \pm\sqrt{6}$
c $1,\ \pm\sqrt{6}i$ **d** $2,\ \frac{-1\pm\sqrt{15}i}{2}$
e $\frac{\sqrt{2}\pm\sqrt{14}i}{4}$ **f** $0,\ -1\pm 2\sqrt{2}i$

Exercise 6G

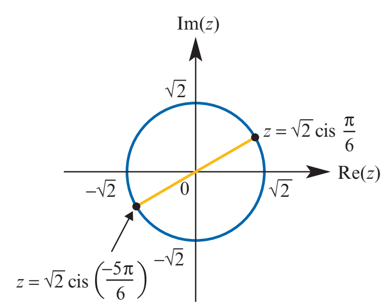
1 a $z=i$ or $z=-i$



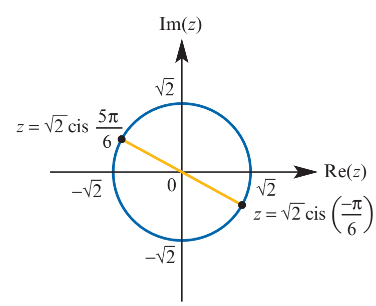
b $z=3\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right), z=3\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ or
$z=-3i$



c $z=\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ or $z=\sqrt{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$



d $z=\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$ or $z=\sqrt{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$



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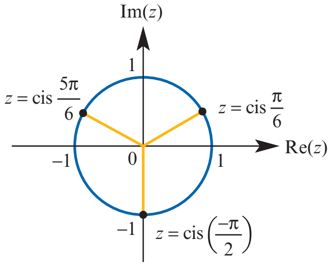
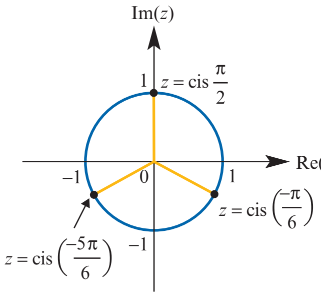
e $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ or $z = -i$



f $z = \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ z = i$ or $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$



2 a $2\operatorname{cis}\left(-\frac{\pi}{12}\right),\ 2\operatorname{cis}\left(\frac{7\pi}{12}\right),\ 2\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

b $2\operatorname{cis}\left(\frac{\pi}{4}\right),\ 2\operatorname{cis}\left(\frac{11\pi}{12}\right),\ 2\operatorname{cis}\left(-\frac{5\pi}{12}\right)$

c $2\operatorname{cis}\left(-\frac{5\pi}{18}\right),\ 2\operatorname{cis}\left(\frac{7\pi}{18}\right),\ 2\operatorname{cis}\left(-\frac{17\pi}{18}\right)$

d $2\operatorname{cis}\left(-\frac{\pi}{18}\right),\ 2\operatorname{cis}\left(\frac{11\pi}{18}\right),\ 2\operatorname{cis}\left(-\frac{13\pi}{18}\right)$

e $5\operatorname{cis}\left(-\frac{\pi}{6}\right),\ 5\operatorname{cis}\left(\frac{\pi}{2}\right),\ 5\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

f $2^{\frac{1}{6}}\operatorname{cis}\left(\frac{\pi}{4}\right),\ 2^{\frac{1}{6}}\operatorname{cis}\left(\frac{11\pi}{12}\right),\ 2^{\frac{1}{6}}\operatorname{cis}\left(-\frac{5\pi}{12}\right)$

3 a $a^2 - b^2 = 3,\ 2ab = 4$

b $a = \pm 2,\ b = \pm 1$;
square roots of $3 + 4i$ are $\pm(2 + i)$

4 a $\pm(1 - 4i)$ **b** $\pm\frac{\sqrt{2}}{2}(7 + i)$

c $\pm(1 + 2i)$ **d** $\pm(3 + 4i)$

5 $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{6}\right),\ \sqrt{2}\operatorname{cis}\left(-\frac{5\pi}{6}\right),\ \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}\right),$
$\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{6}\right)$

6 a $\frac{1}{2}(1 \pm \sqrt{5})$ **b** ± 1

8 $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ or $z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$;
$\left(z - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\left(z + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

9 $z = \operatorname{cis}\left(\frac{\pi}{8}\right),\ \operatorname{cis}\left(\frac{3\pi}{8}\right),\ \operatorname{cis}\left(\frac{5\pi}{8}\right),\ \operatorname{cis}\left(\frac{7\pi}{8}\right),$
$\operatorname{cis}\left(\frac{9\pi}{8}\right),\ \operatorname{cis}\left(\frac{11\pi}{8}\right),\ \operatorname{cis}\left(\frac{13\pi}{8}\right)$ or $\operatorname{cis}\left(\frac{15\pi}{8}\right)$;
$\left(z - \operatorname{cis}\left(\frac{\pi}{8}\right)\right)\left(z - \operatorname{cis}\left(\frac{3\pi}{8}\right)\right)\left(z - \operatorname{cis}\left(\frac{5\pi}{8}\right)\right)$
$\left(z - \operatorname{cis}\left(\frac{7\pi}{8}\right)\right)\left(z - \operatorname{cis}\left(\frac{9\pi}{8}\right)\right)\left(z - \operatorname{cis}\left(\frac{11\pi}{8}\right)\right)$
$\left(z - \operatorname{cis}\left(\frac{13\pi}{8}\right)\right)\left(z - \operatorname{cis}\left(\frac{15\pi}{8}\right)\right)$

10 a $a + bi = \pm\frac{\sqrt{2}}{2}\left((1 + \sqrt{2})^{\frac{1}{2}} + (\sqrt{2} - 1)^{\frac{1}{2}}\, i\right)$

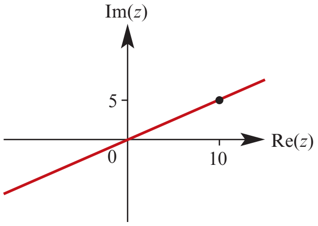
b $2^{\frac{1}{4}}\operatorname{cis}\left(\frac{\pi}{8}\right),\ 2^{\frac{1}{4}}\operatorname{cis}\left(-\frac{7\pi}{8}\right)$

c $\cos\left(\frac{\pi}{8}\right) = \frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2},\ \sin\left(\frac{\pi}{8}\right) = \frac{(2 - \sqrt{2})^{\frac{1}{2}}}{2}$

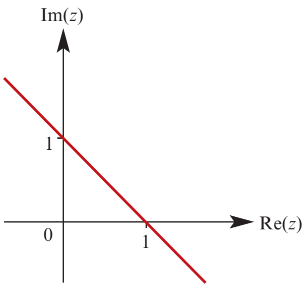
d $z = \frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2} + \frac{(2 - \sqrt{2})^{\frac{1}{2}}}{2}i,$
$z = -\frac{(2 - \sqrt{2})^{\frac{1}{2}}}{2} + \frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2}i,$
$z = -\frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2} - \frac{(2 - \sqrt{2})^{\frac{1}{2}}}{2}i$
or $z = \frac{(2 - \sqrt{2})^{\frac{1}{2}}}{2} - \frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2}i$

Exercise 6H

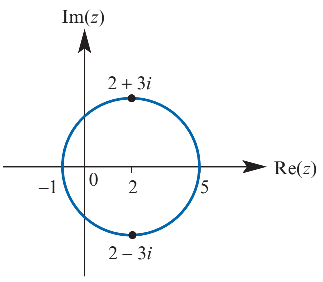
1 a



b



c



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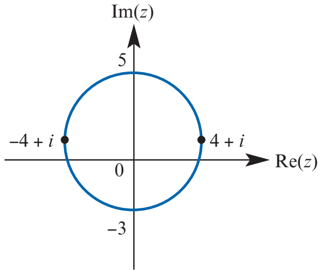
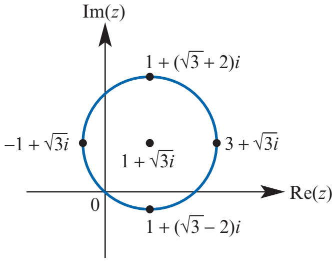
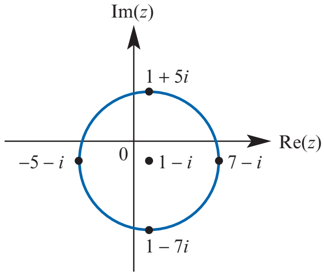
d



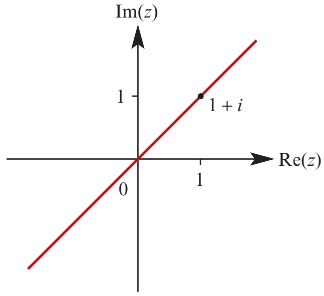
e



f

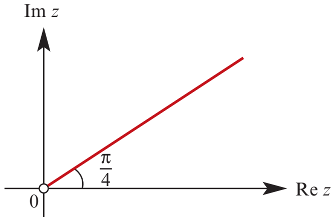


2



3 The imaginary axis, i.e. $\{z : \text{Re}(z) = 0\}$

4 a



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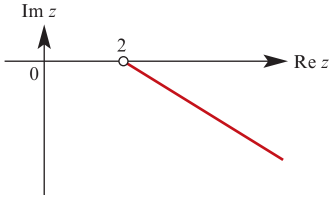

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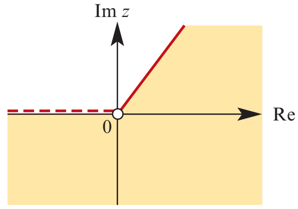
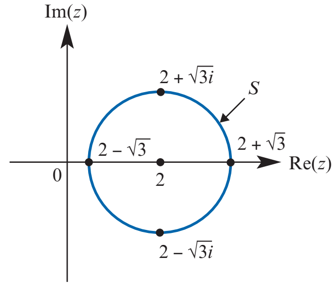
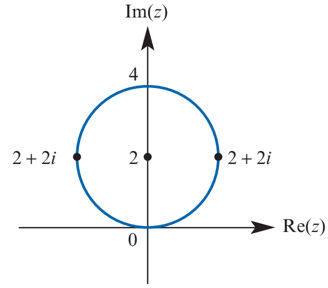


5



6 a



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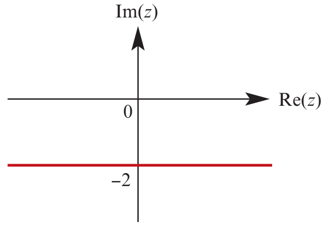

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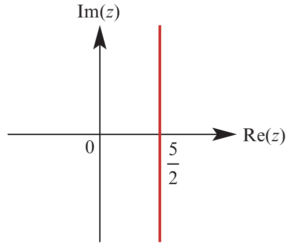

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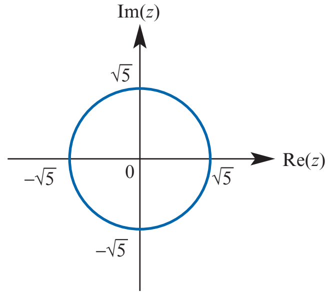

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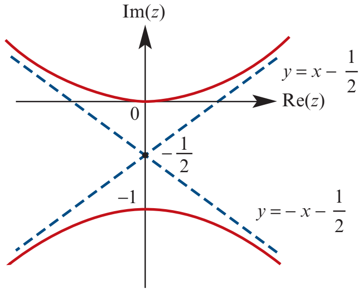
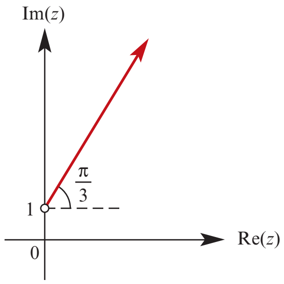
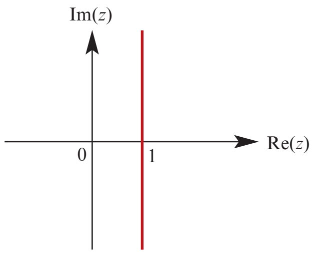
e



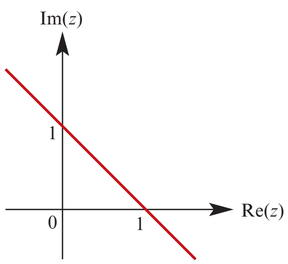
f



7 a



b



8 $x^2 + y^2 = 1$

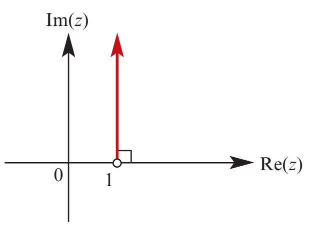
9 Centre $\left(\frac{8}{3}, -2\right)$; radius $\frac{4\sqrt{10}}{3}$

10 $|z|^2 : 1$

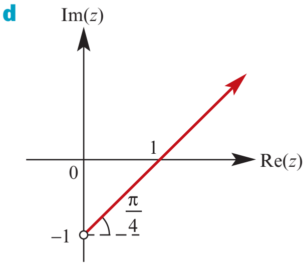
11 a Circle with centre (1, 1) and radius 1

b $y = -x$

c



d



12 Circle with centre $2 + 4i$ and radius 6

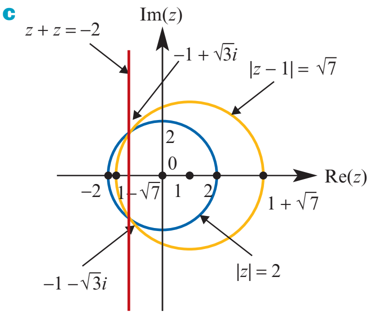
13 a $z = -1 \pm \sqrt{3}i$

b **i** $|z| = 2$

ii $|z - 1| = \sqrt{7}$

iii $z + \bar{z} = -2$

c



Chapter 6 review

Technology-free questions

1 a $8 - 5i$ **b** $-i$ **c** $29 + 11i$

d 13 **e** $\frac{6}{13} + \frac{4}{13}i$ **f** $\frac{9}{5} - \frac{7}{5}i$

g $\frac{3}{5} + \frac{6}{5}i$ **h** $-8 - 6i$ **i** $\frac{43}{10} + \frac{81}{10}i$

2 a $2 \pm 3i$ **b** $-6 + 2i$ **c** $-3 \pm \sqrt{3}i$

d $\frac{3}{\sqrt{2}}(1 \pm i), \frac{3}{\sqrt{2}}(-1 \pm i)$

e $3, \frac{3}{2}(-1 \pm \sqrt{3}i)$ or $3\operatorname{cis}\left(\pm\frac{2\pi}{3}\right)$

f $-\frac{3}{2}, \frac{3}{4}(1 \pm \sqrt{3}i)$ or $\frac{3}{2}\operatorname{cis}\left(\pm\frac{\pi}{3}\right)$

3 a $2 - i, 2 + i, -2$ **b** $3 - 2i, 3 + 2i, -1$

c $1 + i, 1 - i, 2$

4 a $2\left(x + \frac{3}{4} + \frac{\sqrt{7}}{4}i\right)\left(x + \frac{3}{4} - \frac{\sqrt{7}}{4}i\right)$

b $(x - 1)(x + i)(x - i)$

c $(x + 2)^2(x - 2)$

5 2 and −1; −2 and 1

6 a iv **b** ii **c** i **d** iii

7 −1 and 5; 1 and −5

8 $a = 2, b = 5$

9 $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$

10 $a = \frac{3}{2} - \frac{\sqrt{3}}{2},\ b = \frac{1}{2} + \frac{3\sqrt{3}}{2}$

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11 a $2 + 2i$ **b** $\frac{1}{2}(1 + i)$ **c** $8\sqrt{2}$ **d** $\frac{\pi}{4}$

12 $a = 2$ or $a = \frac{9}{2}$

13 a i $\sqrt{2}$ **ii** 2 **iii** $\frac{\pi}{4}$ **iv** $-\frac{\pi}{3}$

b $\frac{\sqrt{2}}{2}, -\frac{\pi}{12}$

14 $2\operatorname{cis}\left(\frac{\pi}{6}\right), -64\sqrt{3} - 64i$

15 $\pm 3, \pm 3i, 1 \pm i$

16 $16 - 16i$

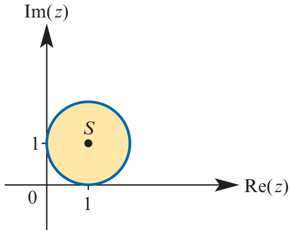
17 $-2i, i, -2, k = -2$ or 1

18 a $(z + 2)(z - 1 + i)(z - 1 - i)$ **b** 25

19 $-1 + 2i, \ -1 - \frac{1}{2}i$

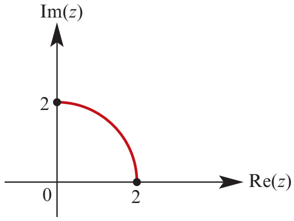
20 a $(x - 1)^2 + (y - 1)^2 \le 1$

b

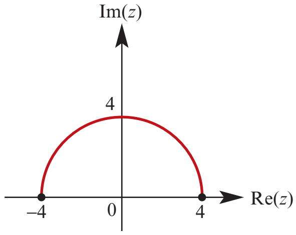


21 The real axis, i.e. $\{z : \operatorname{Im}(z) = 0\}$

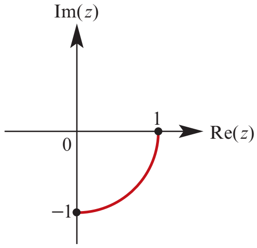
22 a



b



c



23 $\left(\frac{5}{6}, -\frac{7}{6}\right)$

24 a $4 - 3i$

b $c = 12 + 3i, \ d = 9 - i$ or
$c = 4 + 9i, \ d = 1 + 5i$

25 a $2\operatorname{cis}\left(\frac{\pi}{3}\right), \ 2\operatorname{cis}\pi, \ 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$

b $2\operatorname{cis}\left(\frac{\pi}{6}\right), \ 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

26 a $x^6 - 1 = (x + 1)(x - 1)(x^2 - x + 1)(x^2 + x + 1)$

b $x^6 - 1 = (x + 1)(x - 1)$
$\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

c $-1, \ 1, \ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

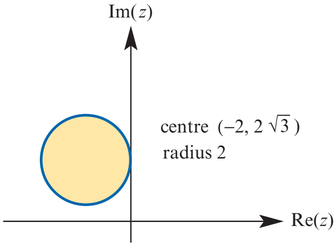
27 a 1 **b** 1 **c** 0

28 $\frac{3}{4}$

29 $-\frac{\pi}{4}$

30 a $-2 + 2\sqrt{3}i$ **b** $-3 - 6i$

31 a



b i 2 **ii** $\frac{5\pi}{6}$

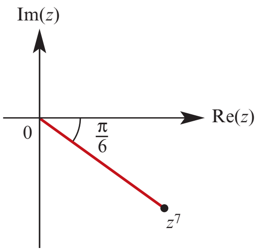
Multiple-choice questions

1 E	**2** C	**3** D	**4** E	**5** D
6 B	**7** B	**8** C	**9** B	**10** A

Extended-response questions

1 a $|z^7| = 16\,384; \quad \operatorname{Arg}(z^7) = -\frac{\pi}{6}$

b



c $2\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$

d $z = -2\sqrt{3} + 2i, \ w = 1 + i,$
$\frac{z}{w} = (1 - \sqrt{3}) + (1 + \sqrt{3})i$

e $-2 - \sqrt{3}$

f $\frac{1}{\sqrt{3}}$

2 b $3, \ 2 - i$

d $z^5 - 9z^4 + 36z^3 - 84z^2 + 115z - 75$

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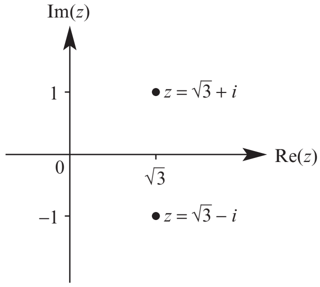
3 a $z = \sqrt{3} \pm i$

b i

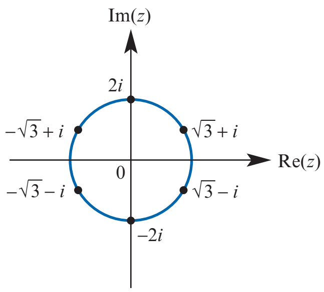


ii $x^2 + y^2 = 4$

iii $a = 2$

iv $P(z) = z^2 + 2\sqrt{3}z + 4$

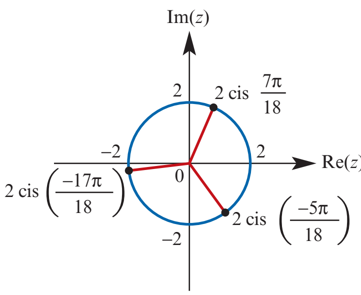
The solutions of the equation $z^6 + 64 = 0$ are equally spaced around the circle $x^2 + y^2 = 4$, and represent the sixth roots of -64. Three of the solutions are the conjugates of the other three solutions.



4 a $8\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

b $2\operatorname{cis}\left(-\frac{5\pi}{18}\right)$, $2\operatorname{cis}\left(\frac{7\pi}{18}\right)$, $2\operatorname{cis}\left(-\frac{17\pi}{18}\right)$

c



d i $(z - \sqrt{3}i)^3 = -4\sqrt{3} - 4i$

ii $2\cos\left(-\frac{5\pi}{18}\right) + \left(2\sin\left(-\frac{5\pi}{18}\right) + \sqrt{3}\right)i$,

$2\cos\left(\frac{7\pi}{18}\right) + \left(2\sin\left(\frac{7\pi}{18}\right) + \sqrt{3}\right)i$,

$2\cos\left(-\frac{17\pi}{18}\right) + \left(2\sin\left(-\frac{17\pi}{18}\right) + \sqrt{3}\right)i$

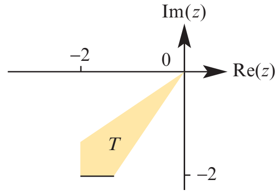
5 a $\overrightarrow{XY} = \sqrt{3}\boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{XZ} = 2\sqrt{3}\boldsymbol{i} - 2\boldsymbol{j}$

b $z_3 = 1 + \sqrt{3}i$

c $z_3 = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$; W corresponds to $6\sqrt{3}$

d $(4\sqrt{3}, 0)$

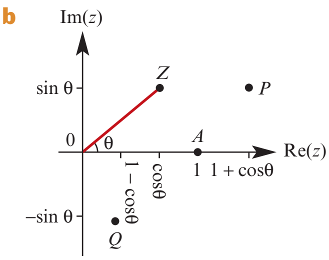
6 a



b $T = \{z : \text{Re}(z) > -2\} \cap \{z : \text{Im}(z) \geq -2\} \cap \left\{z : -\frac{5\pi}{6} < \text{Arg}(z) < -\frac{2\pi}{3}\right\}$

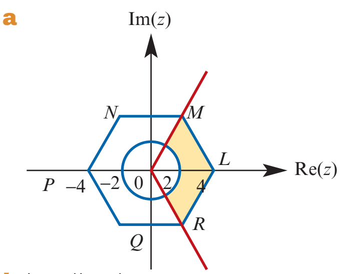
7 a $k > -\frac{5}{4}$ **b** $k = -\frac{5}{4}$ **c** $-2 < k < -\frac{5}{4}$

8 b



c $\operatorname{cosec}\theta + \cot\theta = \cot\left(\frac{\theta}{2}\right)$

9 a



b $|z - 4| = 4$

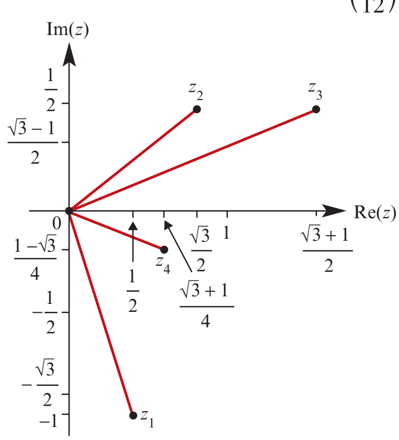
c N is $4\operatorname{cis}\left(\frac{2\pi}{3}\right)$; Q is $4\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

d New position of N is $4\operatorname{cis}\left(\frac{5\pi}{12}\right)$;

new position of Q is $4\operatorname{cis}\left(-\frac{11\pi}{12}\right)$

10 b $z_3 = \sqrt{2}\operatorname{cis}(\tan^{-1}(2 - \sqrt{3})) = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)$

c



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11 a ii $q = 2k^3$

b $b = -1 - i,\ c = 2 + 2i$

12 a i $6\sqrt{2}$ **ii** 6

b ii Isosceles

13 a i 13 **ii** $157.38° = 2.75^c$

b i $\cos\alpha = -\dfrac{12}{13}, \sin\alpha = \dfrac{5}{13}$

ii $r = \sqrt{13}, \cos(2\theta) = -\dfrac{12}{13}, \sin(2\theta) = \dfrac{5}{13}$

iii $\sin\theta = \pm\dfrac{5\sqrt{26}}{26}, \cos\theta = \pm\dfrac{\sqrt{26}}{26}$

iv $w = \pm\dfrac{\sqrt{2}}{2}(1 + 5i)$

d $\pm\dfrac{\sqrt{2}}{2}(5 + i)$; a reflection of the square roots of $-12 + 5i$ in the line $\text{Re}(z) = \text{Im}(z)$

14 a $\left(x + \dfrac{3}{2}\right)^2 + y^2 = \dfrac{29}{4}$

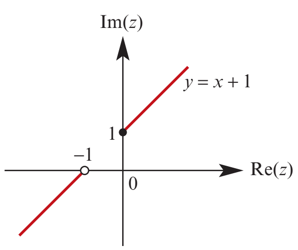
b $\left(x + \dfrac{3}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{15}{2}$

c $\left(x + \dfrac{\beta}{\alpha}\right)^2 + y^2 = \dfrac{\beta^2 - \alpha\gamma}{\alpha^2}$

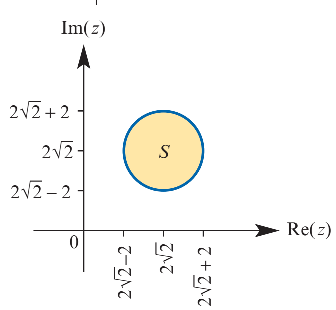
d $\left(x + \dfrac{a}{\alpha}\right)^2 + \left(y - \dfrac{b}{\alpha}\right)^2 = \dfrac{a^2 + b^2 - \alpha\gamma}{\alpha^2}$, where $\beta = a + bi$

15 a $(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + (5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)i$

16 a



b i



ii max = 6, min = 2

iii max = $75° = \dfrac{5\pi}{12}$, min = $15° = \dfrac{\pi}{12}$

17 a $2\,\text{cis}\left(\pm\dfrac{2\pi}{3}\right)$

c $z^2 + (2 - 2\sqrt{3}i)z - 4\sqrt{3}i = 0$ or $z^2 + (2 + 2\sqrt{3}i)z + 4\sqrt{3}i = 0$

d −4

18 a i $z = 2\,\text{cis}\,\theta + \frac{1}{2}\,\text{cis}(-\theta)$

b i $z = 2i\,\text{cis}\,\theta - \frac{1}{2}i\,\text{cis}(-\theta)$

Chapter 7

Technology-free questions

1 a If n is even, then $n^2 - 6n + 5$ is odd.

c If n is odd, then $n^2 - 6n + 5$ is even.

2 b $m = 1$ and $n = 1$

8 Many possible answers. For example:

a $p = 3$ and $q = 3$ **b** $a = 3$

14 a $\dfrac{2 + \sqrt{3}}{4}$

b i $\sqrt{5} - 1$ **ii** $5 - 2\sqrt{5}$

15 $[-1, 0], \left[\dfrac{8 - 3\pi}{2}, \dfrac{8 + 3\pi}{2}\right]$

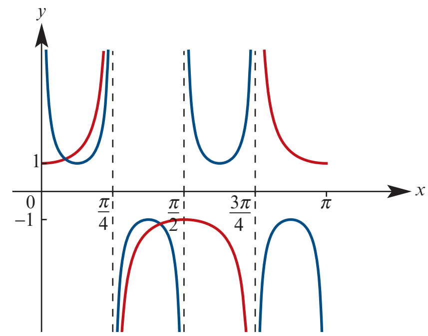
16 $\left(\dfrac{3}{4}, 2\right), \left(\dfrac{9}{4}, 2\right), \left(\dfrac{15}{4}, 2\right), \left(\dfrac{21}{4}, 2\right)$

17 $x = \dfrac{(2n + 1)\pi}{2}, \dfrac{\pi}{6} + 2n\pi, \dfrac{5\pi}{6} + 2n\pi$

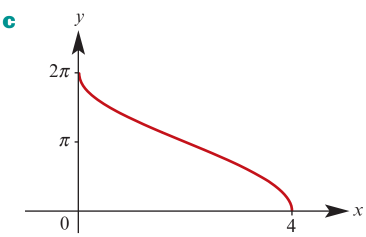
18 a $\dfrac{\pi}{12}, \dfrac{\pi}{4}, \dfrac{5\pi}{12}, \dfrac{3\pi}{4}$

b i $\left(\dfrac{\pi}{12}, \dfrac{2\sqrt{3}}{3}\right), \left(\dfrac{5\pi}{12}, -\dfrac{2\sqrt{3}}{3}\right)$

ii

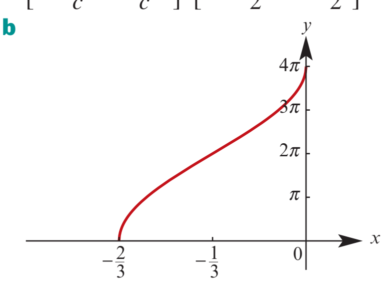


c



19 a $\left[-\dfrac{1 + d}{c}, \dfrac{1 - d}{c}\right], \left[a - \dfrac{\pi b}{2}, a + \dfrac{\pi b}{2}\right]$

b



20 −1

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21 $(9, 4, -5), (-1, 6, 1), (-5, -2, 9)$

22 $\frac{2}{3}(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}),\quad \frac{1}{3}(5\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k})$

23 a $-i$ **b** $\frac{7\sqrt{6}}{18}$ **c** $\frac{\sqrt{5}}{2}$

24 a $m = \pm 5$ **b** $m = -\frac{5}{4}$ **c** $4\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k}$

d $m = \frac{7}{2}$

25 b $\lambda = \frac{2}{7}$

26 a $\frac{\sqrt{13}}{13}(3\boldsymbol{i} + 2\boldsymbol{j})$

b i $\frac{10}{13}(3\boldsymbol{i} + 2\boldsymbol{j})$ **ii** $\frac{10\sqrt{13}}{13}$

27 a $m = -3,\ n = 2$ **b** $\lambda = -\frac{1}{5}$

28 a $m = \frac{1}{7}$

b i $\frac{19}{7}$ **ii** $\frac{4}{7}(2\boldsymbol{j} - 2\boldsymbol{k})$

29 3

30 a $\sqrt{3}$ **b** $\sqrt{2}$

31 $a = \pm\sqrt{2}$

32 a $\boldsymbol{r} = \lambda(3\boldsymbol{i} + 4\boldsymbol{k}),\ \lambda \in \mathbb{R}$

b $\boldsymbol{r} = 2\boldsymbol{j} + \boldsymbol{k} + \lambda(-\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}),\ \lambda \in \mathbb{R}$

c $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k} + \lambda(-3\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}),\ \lambda \in \mathbb{R}$

33 a $\boldsymbol{r} \cdot (\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 0$ **b** $\boldsymbol{r} \cdot (-2\boldsymbol{i} + 2\boldsymbol{k}) = 6$

c $\boldsymbol{r} \cdot (4\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k}) = -6$

34 a $\frac{4}{3}$ **b** $\frac{\sqrt{14}}{7}$ **c** 0 **d** $\frac{2\sqrt{3}}{3}$

35 $(3, -1, -3)$

36 a $(0, 1, 0)$ **b** $(2, -1, 2)$ **c** $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

37 3

38 a $a = -3$

b $\boldsymbol{r} = \frac{1}{11}(25 - 14\lambda)\boldsymbol{i} + \frac{27}{11}(\lambda - 1)\boldsymbol{j} + \lambda\boldsymbol{k}$

39 a $\frac{\sqrt{21}}{14}$ **b** $\boldsymbol{r} = \boldsymbol{j} + t(-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

40 c $\boldsymbol{r} = 8\boldsymbol{j} + 5\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

41 a $\frac{1}{\sqrt{51}}$ **b** $\frac{2}{29}$

42 a $\boldsymbol{r} = 3\boldsymbol{i} + 5\boldsymbol{j} + 9\boldsymbol{k} + t(-2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$

b $\frac{x - 3}{-2} = \frac{y - 5}{4} = z - 9$

c $x = 3 - 2t,\ y = 5 + 4t,\ z = 9 + t$

43 $z = \pm 2,\ z = \pm\sqrt{3}i$

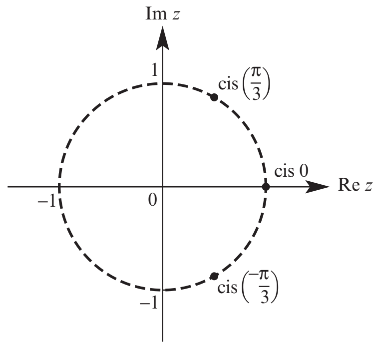
44 $\frac{\pi}{12}$

45 $z = 1, 2, -2 + i, -2 - i$

46 a $z = 1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

b $\text{cis}(0), \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(-\frac{\pi}{3}\right)$

c

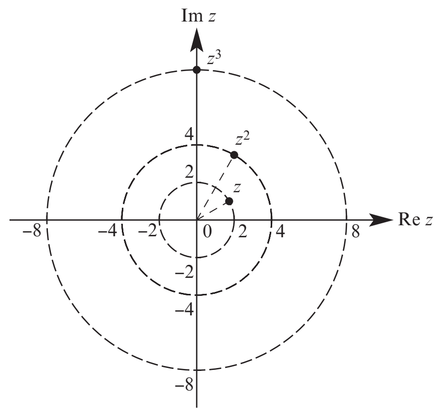


47 $(-2\sqrt{2}, \sqrt{2})$

48 $\cos\theta - (\sin\theta)i$

49 $z = 2\,\text{cis}\left(\frac{\pi}{6}\right),\ z^2 = 4\,\text{cis}\left(\frac{\pi}{3}\right),$

$z^3 = 8\,\text{cis}\left(\frac{\pi}{2}\right) = 8i$



50 b $(z - 1 - i)(z - 2 + 3i)(z - 2 - 3i)$

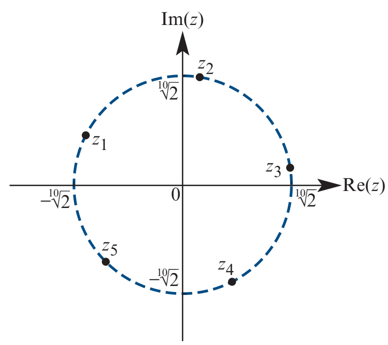
51 b i $(-1 \pm \sqrt{2})i$ **ii** i **iii** $\pm 1 - i$

52 a $a = 3,\ b = 4,\ c = 2$ **b** $-\sqrt{3} + i$

53 $z_1 = \sqrt[10]{2}\,\text{cis}\left(\frac{17\pi}{20}\right),\quad z_2 = \sqrt[10]{2}\,\text{cis}\left(\frac{9\pi}{20}\right),$

$z_3 = \sqrt[10]{2}\,\text{cis}\left(\frac{\pi}{20}\right),\quad z_4 = \sqrt[10]{2}\,\text{cis}\left(-\frac{7\pi}{20}\right),$

$z_5 = \sqrt[10]{2}\,\text{cis}\left(-\frac{15\pi}{20}\right)$



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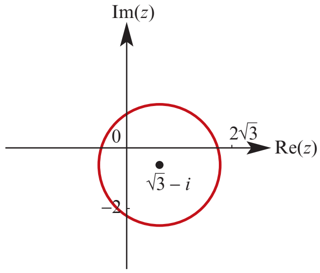
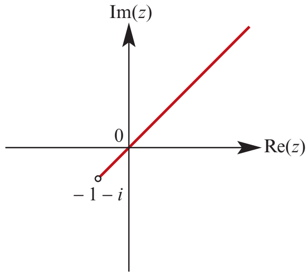
54 a $z_1 = 2\operatorname{cis}\left(-\frac{\pi}{6}\right),\ z_2 = \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

b $\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$ **c** $\sqrt{2}\operatorname{cis}\left(-\frac{7\pi}{12}\right)$

d i



ii



55 a i $2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **ii** $2^n\operatorname{cis}\left(\frac{n\pi}{3}\right)$

iii Multiples of 3 **iv** None

b $z_1^2 = -2 + 2\sqrt{3}i,\ z_1^3 = -8$

c $a = 1,\ b = -2$

d $z = -\frac{5}{2}, 1 + \sqrt{3}i, 1 - \sqrt{3}i$

56 a $C = 1 - 2i,\ D = 3 + 2i$ **b** Centre i

59 $c = 1,\ r = \sqrt{2}$

60 $a = 3,\ k = -30$

Multiple-choice questions

1 C	**2** B	**3** B	**4** D	**5** C
6 A	**7** A	**8** E	**9** B	**10** E
11 D	**12** E	**13** E	**14** C	**15** D
16 B	**17** D	**18** D	**19** E	**20** B
21 C	**22** D	**23** B	**24** A	**25** E
26 D	**27** C	**28** E	**29** A	**30** C
31 C	**32** B	**33** C	**34** C	**35** E
36 C	**37** C	**38** B	**39** B	**40** D
41 E	**42** C	**43** E	**44** C	**45** E
46 B	**47** E	**48** B	**49** A	**50** C
51 B	**52** A	**53** D	**54** C	**55** A
56 B	**57** E	**58** D	**59** E	**60** A
61 A	**62** C	**63** C	**64** A	**65** C
66 B	**67** C	**68** E	**69** C	**70** D
71 B	**72** A	**73** D	**74** C	**75** B
76 D	**77** C	**78** E	**79** D	**80** C
81 D	**82** B	**83** E		

Extended-response questions

1 a i $\frac{3}{2}(\boldsymbol{b} - \boldsymbol{a})$ **ii** $\frac{1}{2}(3\boldsymbol{b} - \boldsymbol{a})$

b i $\overrightarrow{AB} = \boldsymbol{i} + 2\boldsymbol{j},\ \overrightarrow{BC} = 2\boldsymbol{i} - \boldsymbol{j}$ **iv** $3\boldsymbol{i} - \boldsymbol{j}$

c $x = 4,\ y = 5,\ z = 2$

2 a $z = 1 \pm \sqrt{2}i$

b $(x - 1)^2 + y^2 = 2$

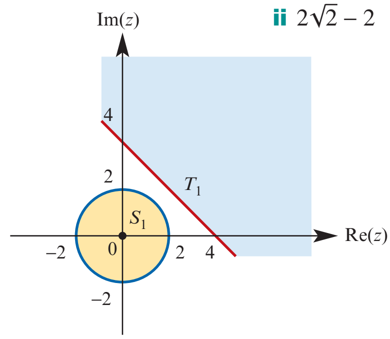
c $0 \le d \le 6$

d i $z = \frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}i$

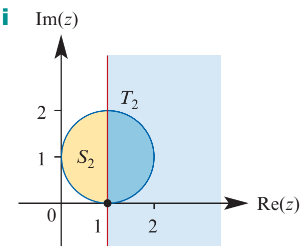
ii $b^2 < 4ac$

iii $\left(x - \frac{b}{2a}\right)^2 + y^2 = \frac{4ac - b^2}{4a^2}$

3 a i **ii** $2\sqrt{2} - 2$

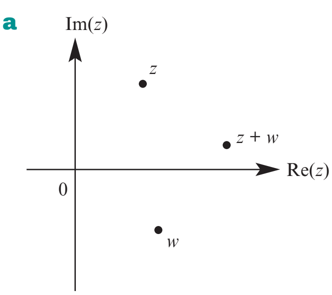


b i



ii Maximum $\sqrt{2} + 1$; minimum 1

4 a



5 a i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ **iii** $\frac{2}{3}(\boldsymbol{a} - \boldsymbol{b})$

b $\overrightarrow{DA} = 2\overrightarrow{BD}$

6 a i 151° **ii** $\frac{1}{9}(34\boldsymbol{i} + 40\boldsymbol{j} + 23\boldsymbol{k})$

iii $x = 3,\ y = -2,\ z = 16$

b i $\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ **ii** $\overrightarrow{OA} = 2\overrightarrow{BQ}$

7 b 4 : 1 : 3 **c** $4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

e $s = 3,\ t = -2$

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8 a $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$ **b** $\dfrac{\sqrt{(\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b})-(\boldsymbol{a}\cdot\boldsymbol{b})^2}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$

10 a Since $\boldsymbol{a}\times(\boldsymbol{b}-3\boldsymbol{c})=\boldsymbol{0}$ and $\boldsymbol{a}\neq\boldsymbol{0}$, we must have $\boldsymbol{b}-3\boldsymbol{c}=k\boldsymbol{a}$ for some $k\in\mathbb{R}$.

b i 1 **ii** $2\sqrt{3}$ **iii** $\pm 2\sqrt{3}$

c $\pm\dfrac{1}{\sqrt{3}}$

11 b i $\boldsymbol{p}+\dfrac{(k-\boldsymbol{p}\cdot\boldsymbol{n})\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}$

ii $\left|\dfrac{(k-\boldsymbol{p}\cdot\boldsymbol{n})\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}\right|=\dfrac{|k-\boldsymbol{p}\cdot\boldsymbol{n}|}{|\boldsymbol{n}|}$

12 c $8:1$

13 a i $\dfrac{1}{3}(\boldsymbol{a}+2\boldsymbol{b})$ **ii** $\dfrac{1}{6}(2\boldsymbol{b}-5\boldsymbol{a})$

b i $2:3$ **ii** $6:1$

14 a i $2\boldsymbol{c}-\boldsymbol{b}$ **ii** $\dfrac{1}{3}(\boldsymbol{a}+2\boldsymbol{b})$ **iii** $\dfrac{1}{5}(\boldsymbol{a}+4\boldsymbol{c})$

15 c $3:1$

16 a z^2-2z+4

b i $2\operatorname{cis}\left(-\dfrac{\pi}{3}\right)$ **ii** $4\operatorname{cis}\left(-\dfrac{2\pi}{3}\right)$, -8

iii $1\pm\sqrt{3}i$, -1

c i $\sqrt{7}$, $\sqrt{7}$ **ii** Isosceles

17 a $\overrightarrow{AB}=\boldsymbol{i}+\boldsymbol{j}$, $\overrightarrow{AC}=2\boldsymbol{i}-\boldsymbol{k}$

b $-\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$ **c** $x-y+2z=5$

d $\boldsymbol{r}=\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}+t(\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})$

e $\dfrac{5}{3}\boldsymbol{i}+\dfrac{4}{3}\boldsymbol{j}+\dfrac{7}{3}\boldsymbol{k}$ **f** $\dfrac{2\sqrt{6}}{3}$

18 a $p=\dfrac{1}{3}(4+2\sqrt{2}i)$, $q=\dfrac{1}{3}(2+4\sqrt{2}i)$

b i $\boldsymbol{b}-\boldsymbol{a}$ **ii** $\dfrac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$ **iii** $\dfrac{1}{3}(\boldsymbol{a}+\boldsymbol{b})$

iv $\dfrac{1}{3}(2\boldsymbol{a}-\boldsymbol{b})$ **v** $\dfrac{1}{3}(2\boldsymbol{b}-\boldsymbol{a})$

19 a $(z+2i)(z-2i)$ **b** $(z^2+2i)(z^2-2i)$

d $(z-1-i)(z+1+i)(z-1+i)(z+1-i)$

e $(z^2-2z+2)(z^2+2z+2)$

20 a $\sqrt{17}$

b Circle centre $2-i$ and radius $\sqrt{5}$

c Perpendicular bisector of line joining $1+3i$ and $2-i$

21 a $2+11i$

b i $\dfrac{2\sqrt{5}}{25}$ **ii** $\dfrac{11\sqrt{5}}{25}$

22 c i 1 **ii** -1

d i $z^2-3z+3=0$ **ii** $z^2+2z+13=0$

e 0, 3

23 a $\operatorname{cis}(0), \operatorname{cis}\left(\dfrac{2\pi}{5}\right), \operatorname{cis}\left(\dfrac{4\pi}{5}\right), \operatorname{cis}\left(-\dfrac{4\pi}{5}\right), \operatorname{cis}\left(-\dfrac{2\pi}{5}\right)$

e $w=\dfrac{-1\pm\sqrt{5}}{2}$

f $z=-\dfrac{1+\sqrt{5}}{4}\pm\dfrac{\sqrt{10-2\sqrt{5}}}{4}i$,

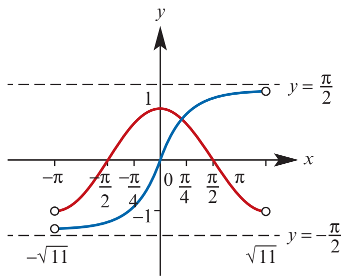
$z=\dfrac{\sqrt{5}-1}{4}\pm\dfrac{\sqrt{10+2\sqrt{5}}}{4}i$

24 a 4, 9, -4 **b** 5

25 b $\cos(5\theta)=\cos^5\theta\,(1-10\tan^2\theta+5\tan^4\theta)$, $\sin(5\theta)=\cos^5\theta\,(5\tan\theta-10\tan^3\theta+\tan^5\theta)$

26 a $\operatorname{cis}(\pm\theta)$

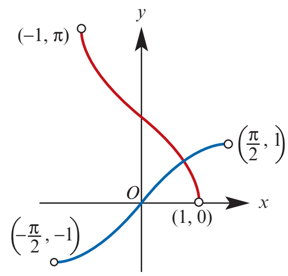
27 a



b i 0.67 **ii** 0.54

d 0.82

28 a



b i 0.48 **ii** 0.67

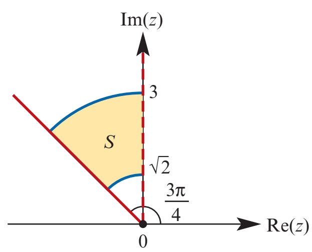
d (0.768, 0.695)

29 a $a=5$, $d=-10$

b i 1.73 metres

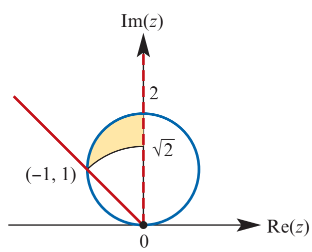
ii 8.03 metres

30 a



b $\{-1+i,\ -1+2i,\ -2+2i\}$

c



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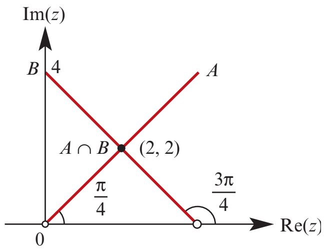
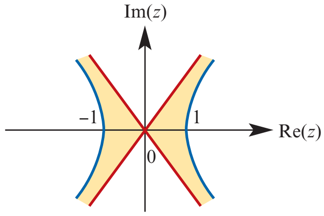
31 a



b



32 a $\overrightarrow{OA} = \boldsymbol{i} + \sqrt{\lambda}\boldsymbol{k}, \quad \overrightarrow{CA} = 2\boldsymbol{i} - 3\boldsymbol{j} + \sqrt{\lambda}\boldsymbol{k}$

b $56°$ **c** $13 + 8\sqrt{3}$ since $\lambda > 0$

33 b i $\overrightarrow{OX} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}), \overrightarrow{OY} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}),$
$\overrightarrow{OZ} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{d}), \overrightarrow{OW} = \frac{1}{3}(\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$

ii $\overrightarrow{DX} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) - \boldsymbol{d},$
$\overrightarrow{BY} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}) - \boldsymbol{b},$
$\overrightarrow{CZ} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{d}) - \boldsymbol{c},$
$\overrightarrow{AW} = \frac{1}{3}(\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) - \boldsymbol{a}$

iii $\overrightarrow{OP} = \frac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$

iv $\overrightarrow{OQ} = \overrightarrow{OR} = \overrightarrow{OS} = \frac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$

v $Q = R = S = P$, which is the centre of the sphere that circumscribes the tetrahedron

38 a Distance $\sqrt{3}$

b -3

c
- $\boldsymbol{v}\cdot\boldsymbol{w} = 3$ if no components differ (i.e. if $\boldsymbol{v} = \boldsymbol{w}$)
- $\boldsymbol{v}\cdot\boldsymbol{w} = -3$ if all components differ (i.e. if $\boldsymbol{v} = -\boldsymbol{w}$)
- $\boldsymbol{v}\cdot\boldsymbol{w} = 1$ if one component differs
- $\boldsymbol{v}\cdot\boldsymbol{w} = -1$ if two components differ

d $\cos^{-1}(\frac{1}{3}) \approx 70.53°, \ \cos^{-1}(-\frac{1}{3}) \approx 109.47°$

Algorithms and pseudocode

See solutions supplement

Chapter 8

Exercise 8A

1 a $x^4(5\sin x + x\cos x)$ **b** $\sqrt{x}\left(\dfrac{\cos x}{2x} - \sin x\right)$

c $e^x(\cos x - \sin x)$ **d** $x^2e^x(3 + x)$

e $\cos^2 x - \sin^2 x = \cos(2x)$

2 a $e^x(\tan x + \sec^2 x)$ **b** $x^3(4\tan x + x\sec^2 x)$

c $\sec^2 x \log_e x + \dfrac{\tan x}{x}$

d $\sin x(1 + \sec^2 x)$ **e** $\sqrt{x}\left(\dfrac{\tan x}{2x} + \sec^2 x\right)$

3 a $\dfrac{\log_e x - 1}{(\log_e x)^2}$

b $\sqrt{x}\left(\dfrac{\cot x}{2x} - \operatorname{cosec}^2 x\right)$

c $e^x(\cot x - \operatorname{cosec}^2 x)$

d $\dfrac{\sec^2 x}{\log_e x} - \dfrac{\tan x}{x(\log_e x)^2}$

e $\dfrac{\cos x}{x^2} - \dfrac{2\sin x}{x^3}$

f $\sec x(\sec^2 x + \tan^2 x)$

g $\dfrac{-(\sin x + \cos x)}{e^x}$

h $-\operatorname{cosec}^2 x$

4 a $2x\sec^2(x^2 + 1)$ **b** $\sin(2x)$

c $e^{\tan x}\sec^2 x$ **d** $5\tan^4 x \sec^2 x$

e $\dfrac{\sqrt{x}\cos(\sqrt{x})}{2x}$ **f** $\dfrac{1}{2}\sec^2 x\sqrt{\cot x}$

g $x^{-2}\sin\left(\dfrac{1}{x}\right)$ **h** $2\tan x \sec^2 x$

i $\dfrac{1}{4}\sec^2\left(\dfrac{x}{4}\right)$ **j** $-\operatorname{cosec}^2 x$

5 a $k\sec^2(kx)$

b $2\sec^2(2x)\,e^{\tan(2x)}$

c $6\tan(3x)\sec^2(3x)$

d $e^{\sin x}\left(\dfrac{1}{x} + \log_e x\cos x\right)$

e $6x\sin^2(x^2)\cos(x^2)$

f $e^{3x+1}\sec^2 x(3\cos x + \sin x)$

g $e^{3x}(3\tan(2x) + 2\sec^2(2x))$

h $\dfrac{\sqrt{x}\tan(\sqrt{x})}{2x} + \dfrac{\sec^2(\sqrt{x})}{2}$

i $\dfrac{2(x+1)\tan x\sec^2 x - 3\tan^2 x}{(x+1)^4}$

j $20x\sec^3(5x^2)\sin(5x^2)$

6 a $5(x-1)^4$ **b** $\dfrac{1}{x}$

c $e^x(3\sec^2(3x) + \tan(3x))$

d $-\sin x\, e^{\cos x}$ **e** $-12\cos^2(4x)\sin(4x)$

f $4\cos x(\sin x + 1)^3$

g $-\sin x\sin(2x) + 2\cos(2x)\cos x$ **h** $1 - \dfrac{1}{x^2}$

i $\dfrac{x^2(3\sin x - x\cos x)}{\sin^2 x}$ **j** $\dfrac{-(1 + \log_e x)}{(x\log_e x)^2}$

7 a $3x^2$ **b** $4y + 10$

c $-\sin(2z)$ **d** $\sin(2x)\,e^{\sin^2 x}$

e $-2\tan z\sec^2 z$ **f** $-2\cos y\operatorname{cosec}^3 y$

8 a $\dfrac{2}{2x+1}$ **b** $\dfrac{2}{2x-1}$ **c** $\cot x$ **d** $\sec x$

e $\dfrac{\sin^2 x - \cos^3 x}{\sin x\cos x(\cos x + \sin^2 x)}$ **f** $\operatorname{cosec} x$

g $\operatorname{cosec} x$ **h** $\dfrac{1}{\sqrt{x^2 - 4}}, x \neq \pm 2$ **i** $\dfrac{1}{\sqrt{x^2 + 4}}$

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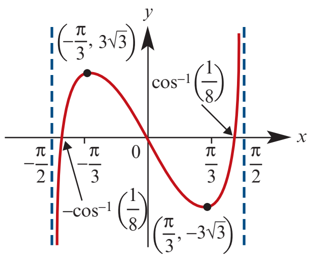
9 a $\frac{1}{2}$ **b** $\frac{2}{3}$ **c** 1

10 a $\left(-\frac{\pi}{3}, -\sqrt{3}\right), \left(\frac{\pi}{3}, \sqrt{3}\right)$

b $y = 4x - \frac{4\pi}{3} + \sqrt{3}, \quad y = 4x + \frac{4\pi}{3} - \sqrt{3}$

11 a $\left(-\frac{\pi}{3}, 3\sqrt{3}\right)$ is a local maximum; $\left(\frac{\pi}{3}, -3\sqrt{3}\right)$ is a local minimum

b



12 a $\sqrt{2}e^{\frac{\pi}{4}}$ **b** $\left(-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}\right)$

13 $\pm\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{2\tan\left(\frac{7\pi}{18}\right)}}{\tan\left(\frac{7\pi}{18}\right)}\right)$

14 a $\frac{1}{4}\sin\left(\frac{x}{4}\right)\sec^2\left(\frac{x}{4}\right)$ **b** $\frac{\sqrt{2}}{4}$

c $y = \frac{\sqrt{2}}{4}(x - \pi + 4)$

Exercise 8B

1 a $\frac{1}{2}$ **b** $\frac{1}{2y}$ **c** $\frac{1}{4(2y-1)}$

d e^{-y} **e** $\frac{1}{5\cos(5y)}$ **f** y

g $\cos^2 y$ **h** $\frac{1}{3y^2+1}$ **i** y^2

j $\frac{1}{e^y(y+1)}$

2 a $\frac{64}{3}$ **b** $\frac{4}{3}$ **c** $\frac{1}{4}$ **d** 1

e $\frac{1}{4}$ **f** $\pm\frac{1}{8}$ **g** $-\frac{\sqrt{3}}{3}$ **h** $\pm\frac{1}{2}$

3 a $\frac{1}{6(2y-1)^2}$ **b** $\frac{1}{2e^{2y+1}}$

c $\frac{1}{2}(2y-1)$ **d** y

4 a $\frac{1}{6\sqrt[3]{x^2}}$ **b** $\frac{1}{2x}$ **c** $\frac{1}{2}e^x$ **d** $\frac{1}{2}e^{x+1}$

5 $y = \frac{1}{6}x - \frac{5}{6}, \quad y = -\frac{1}{6}x + \frac{5}{6}$

6 a $(5, -1), (12, 6)$ **b** $\left(-\frac{15}{4}, \frac{5}{2}\right)$

c $\left(-\frac{15}{4}, \frac{3}{2}\right)$

7 a $(2, 2)$ **b** $8.13°$

Exercise 8C

1 a $\frac{1}{\sqrt{4-x^2}}, x \in (-2, 2)$

b $\frac{-1}{\sqrt{16-x^2}}, x \in (-4, 4)$ **c** $\frac{3}{9+x^2}$

d $\frac{3}{\sqrt{1-9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

e $\frac{-2}{\sqrt{1-4x^2}}, x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ **f** $\frac{5}{1+25x^2}$

g $\frac{3}{\sqrt{16-9x^2}}, x \in \left(-\frac{4}{3}, \frac{4}{3}\right)$

h $\frac{-3}{\sqrt{4-9x^2}}, x \in \left(-\frac{2}{3}, \frac{2}{3}\right)$ **i** $\frac{10}{25+4x^2}$

j $\frac{1}{\sqrt{25-x^2}}, x \in (-5, 5)$

2 a $\frac{1}{\sqrt{-x(x+2)}}, x \in (-2, 0)$

b $\frac{-1}{\sqrt{-x(x+1)}}, x \in (-1, 0)$ **c** $\frac{1}{x^2+4x+5}$

d $\frac{-1}{\sqrt{-x^2+8x-15}}, x \in (3, 5)$

e $\frac{3}{\sqrt{6x-9x^2}}, x \in \left(0, \frac{2}{3}\right)$

f $\frac{-3}{2x^2-2x+1}$

g $\frac{6}{\sqrt{-3(3x^2+2x-1)}}, x \in \left(-1, \frac{1}{3}\right)$

h $\frac{20}{\sqrt{-5(5x^2-6x+1)}}, x \in \left(\frac{1}{5}, 1\right)$

i $\frac{-10}{x^2-2x+5}$ **j** $\frac{-2x}{\sqrt{1-x^4}}, x \in (-1, 1)$

k $\frac{-6x}{|x|\sqrt{2-x^2}}$

3 a $\frac{3}{x\sqrt{x^2-9}}$ **b** $\frac{-5}{x\sqrt{x^2-25}}$ **c** $\frac{3}{x\sqrt{4x^2-9}}$

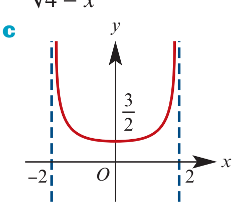
4 a $\frac{a}{\sqrt{1-a^2x^2}}, x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$

b $\frac{-a}{\sqrt{1-a^2x^2}}, x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$ **c** $\frac{a}{1+a^2x^2}$

5 a i $[-2, 2]$ **ii** $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

b $\frac{3}{\sqrt{4-x^2}}, x \in (-2, 2)$

c



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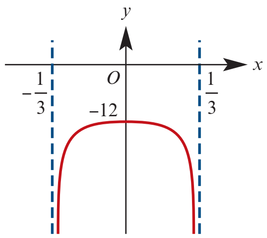
6 a i $\left[-\frac{1}{3}, \frac{1}{3}\right]$ **ii** $[0, 4\pi]$

b $f'(x) = \frac{-12}{\sqrt{1-9x^2}}$, domain $= \left(-\frac{1}{3}, \frac{1}{3}\right)$

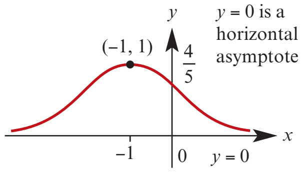
c



7 a i $\mathbb{R}$ **ii** $(-\pi, \pi)$

b $f'(x) = \frac{4}{x^2 + 2x + 5}$

c



8 a $f'(x) = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}, x \in (-1, 1)$

b $f'(x) = 0, x \in (-1, 1)$

c $f'(x) = \frac{-x}{\sqrt{1-x^2}}, x \in (-1, 1)$

d $f'(x) = \frac{-x}{\sqrt{1-x^2}}, x \in (-1, 1)$

e $f'(x) = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1, 1)$

f $f'(x) = \frac{e^x}{1 + e^{2x}}$

9 a 0.35 **b** −6.29 **c** $\frac{3}{5}$

10 a $\pm\frac{\sqrt{3}}{2}$ **b** $\pm\frac{\sqrt{391}}{10}$ **c** $\pm\frac{\sqrt{5}}{3}$

d $-1 \pm \frac{\sqrt{1599}}{20}$ **e** $\pm\frac{\sqrt{35}}{4}$

f $\frac{1}{2}(1 \pm \sqrt{7})$

11 a $y = \frac{4\sqrt{3}}{3}x - \frac{\sqrt{3}}{3} + \frac{\pi}{6}$

b $y = x - \frac{1}{2} + \frac{\pi}{4}$

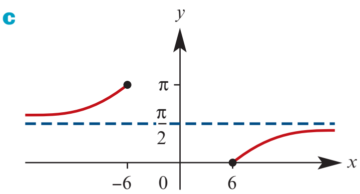
c $y = -2\sqrt{3}x + \frac{\sqrt{3} + \pi}{3}$

d $y = -6x + \sqrt{3} + \frac{\pi}{6}$

12 a $(-\infty, -6] \cup [6, \infty)$

b $f'(x) = \frac{6}{|x|\sqrt{x^2 - 36}}, x < -6 \text{ or } x > 6$

c



Exercise 8D

1 a $f''(x) = 0$ **b** $f''(x) = 56x^6$

c $f''(x) = -\frac{1}{4\sqrt{x^3}}$ **d** $f''(x) = 48(2x+1)^2$

e $f''(x) = -\sin x$ **f** $f''(x) = -\cos x$

g $f''(x) = e^x$ **h** $f''(x) = -\frac{1}{x^2}$

i $f''(x) = \frac{2}{(x+1)^3}$

j $f''(x) = 2\sin x \sec^3 x$

2 a $\frac{d^2y}{dx^2} = \frac{15\sqrt{x}}{4}$

b $\frac{d^2y}{dx^2} = 8(x^2+3)^2(7x^2+3)$

c $\frac{d^2y}{dx^2} = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$

d $\frac{d^2y}{dx^2} = -48\cos(4x+1)$

e $\frac{d^2y}{dx^2} = 2e^{2x+1}$ **f** $\frac{d^2y}{dx^2} = \frac{-4}{(2x+1)^2}$

g $\frac{d^2y}{dx^2} = 6\sin(x-4)\sec^3(x-4)$

h $\frac{d^2y}{dx^2} = \frac{4x}{\sqrt{(1-x^2)^3}}$ **i** $\frac{d^2y}{dx^2} = \frac{-2x}{(1+x^2)^2}$

j $\frac{d^2y}{dx^2} = 360(1-3x)^3$

3 a $f''(x) = 24e^{3-2x}$

b $f''(x) = 8e^{-0.5x^2}(1-x^2)$

c $f''(x) = 0$ **d** $f''(x) = -\operatorname{cosec}^2 x$

e $f''(x) = \frac{3x}{\sqrt{(16-x^2)^3}}$

f $f''(x) = \frac{-27x}{\sqrt{(1-9x^2)^3}}$

g $f''(x) = \frac{-96x}{(9+4x^2)^2}$ **h** $f''(x) = \frac{3}{4\sqrt{(1-x)^5}}$

i $f''(x) = -5\sin(3-x)$

j $f''(x) = 18\sin(1-3x)\sec^3(1-3x)$

k $f''(x) = \frac{1}{9}\sec\left(\frac{x}{3}\right)\left(2\tan^2\left(\frac{x}{3}\right) + 1\right)$

l $f''(x) = \frac{1 + \cos^2\left(\frac{x}{4}\right)}{16\sin^3\left(\frac{x}{4}\right)}$

4 a 1 **b** −1 **c** −1 **d** $-\frac{1}{2}$

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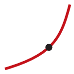
Exercise 8E

1 a  **b**  **c**  **d** 

2 a Point of inflection $(0, 0)$; concave up on $(0, \infty)$

b Point of inflection $\left(\frac{1}{3}, -\frac{2}{27}\right)$; concave up on $\left(\frac{1}{3}, \infty\right)$

c Point of inflection $\left(\frac{1}{3}, \frac{2}{27}\right)$; concave up on $\left(-\infty, \frac{1}{3}\right)$

d Points of inflection $(0, 0)$, $\left(\frac{1}{2}, -\frac{1}{16}\right)$; concave up on $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$

3 a $(-1, 1)$, $(0, 1)$ **b** $\left(-\frac{1}{2}, \frac{3}{2}\right)$

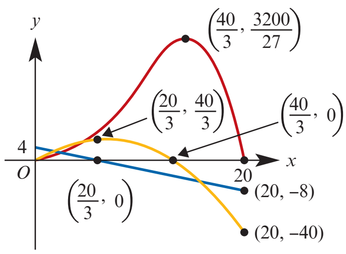
4 a i $(2x^2 + 1)e^{x^2}$ **ii** $2x(2x^2 + 3)e^{x^2}$

e i $(0, \infty)$ **ii** $(-\infty, 0)$

5 a Local min $(0, 0)$; local max $\left(\frac{40}{3}, \frac{3200}{27}\right)$

b $\left(\frac{20}{3}, \frac{1600}{27}\right)$; gradient $= \frac{40}{3}$

c



6 a i $6x^2 + 12x$ **ii** $12x + 12$

b Local min $(0, -12)$; local max $(-2, 4)$

c $(-1, -8)$

7 a $f'(x) = \cos x$; $f''(x) = -\sin x$; $\left(\frac{\pi}{2}, 1\right)$, $\left(\frac{3\pi}{2}, -1\right)$, $(\pi, 0)$

b $f'(x) = e^x(x + 1)$; $f''(x) = e^x(x + 2)$; $(-2, -2e^{-2})$, $(-1, -e^{-1})$

8 a i $f'(a - h) < 0$ **ii** $f'(a) = 0$

iii $f'(a + h) > 0$

b Non-negative

c $f''(a) \geq 0$

d i $f''(0) = 2$ **ii** $f''(0) = 1$

iii $f''(0) = 0$

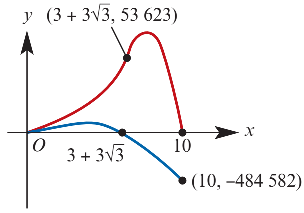
e No

9 a $(3, 1)$

b $y = -\frac{\pi}{2} + 1$, $y = \frac{\pi}{2} + 1$

10 a $f'(x) = e^x(10 + 8x - x^2)$, $f''(x) = e^x(18 + 6x - x^2)$

b



c $3 + 3\sqrt{3}$, $(3 + 3\sqrt{3}, 53\,623)$

11 $(0, 0)$, (π, π), $(2\pi, 2\pi)$, $(3\pi, 3\pi)$, $(4\pi, 4\pi)$

12 a $x = k\pi, k \in \mathbb{Z}$ **b** $x = k\pi, k \in \mathbb{Z}$

c $x = 0$ **d** $x = \frac{1}{2}k\pi, k \in \mathbb{Z}$

e $x = 1$ **f** $x = e^{-\frac{5}{6}}$

14 a $\left(\frac{3}{2}, 2\right)$ **b** $\left(1, \frac{3}{2}\right)$

15 a $(0, 0)$, -6 **b** $(-1, -1)$, 8; $(1, -1)$, -8

c $(0, 3)$, 0 **d** No points of inflection

e No points of inflection

f No points of inflection

g $\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$, $-\frac{1}{4}$; $(0, 0)$, 2; $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$, $-\frac{1}{4}$

h $(0, 0)$, 1 **i** $\left(10, \frac{1}{18}\right)$, $-\frac{1}{432}$

16 a $x = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}$ **b** $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

18 a $f'(x) = 2x(1 + 2\log_e x)$

b $f''(x) = 2(3 + 2\log_e x)$

c Stationary point at $(e^{-\frac{1}{2}}, -e^{-1})$; point of inflection at $(e^{-\frac{3}{2}}, -3e^{-3})$

19 Local minimum at $(-3, -3e^{-1})$; point of inflection at $(-6, -6e^{-2})$

Exercise 8F

1 a $\frac{dr}{dt} \approx 0.00127$ m/min

b $\frac{dA}{dt} = 0.08$ m^2/min

2 $\frac{dx}{dt} \approx 0.56$ cm/s

3 $\frac{dy}{dt} = 39$ units/s

4 $\frac{dx}{dt} = \frac{3}{20\pi} \approx 0.048$ cm/s

5 $\frac{dv}{dt} = -\frac{5}{6}$ units/min

6 $\frac{dA}{dt} = 0.08\pi \approx 0.25$ cm^2/h

7 $\frac{dc}{dt} = \frac{1}{2}$ cm/s

8 a $\frac{dy}{dt} = \frac{1 - t^2}{(1 + t^2)^2}$, $\frac{dx}{dt} = \frac{-2t}{(1 + t^2)^2}$

b $\frac{dy}{dx} = \frac{t^2 - 1}{2t}$

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9 $\dfrac{dy}{dx} = \dfrac{-\sin(2t)}{1+\cos(2t)} = -\tan t$

10 $y = \dfrac{\sqrt{3}}{3}x - \dfrac{\pi\sqrt{3}}{18} + 1$

11 a $\dfrac{dy}{dt} = 12$ cm/s **b** $\dfrac{dy}{dt} = \pm 16$ cm/s

12 2.4

13 a $-\dfrac{5\sqrt{6}}{2}$ cm/s **b** $-4\sqrt{3}$ cm/s

14 72π cm^3/s

15 a 4 **b** 2 cm/s

16 $\dfrac{7}{12\pi}$ cm/s

17 $\dfrac{dV}{dt} = A\dfrac{dh}{dt}$

18 a $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{4\pi}$

b i $\dfrac{dV}{dt} = -\dfrac{\sqrt{10}}{2}$ m^3/h **ii** $\dfrac{dh}{dt} = -\dfrac{\sqrt{10}}{8\pi}$ m/h

19 a $y = -\dfrac{1}{2}x + \sqrt{2}$ **b** $y = \dfrac{-\cos t}{2\sin t}x + \dfrac{1}{\sin t}$

20 a $y = \dfrac{\sqrt{2}}{2}x - 1$ **b** $y = -\sqrt{2}x + 5$

c $y = \dfrac{1}{2\sin\theta}x - \dfrac{\cos\theta}{\sin\theta}$

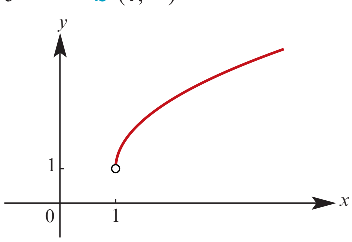
21 a $2\operatorname{cosec} t$ **b** $y = 2\sqrt{2}x + 6\sqrt{2} - 2$

22 a $y = -\sin(t)x + 2\tan(t)$

b $\dfrac{2\sin t}{\cos^2 t}$ **c** $\dfrac{\pi}{3}$

23 a e^{-t} **b** $(1, \infty)$

c



d $4x - 2y = 1$

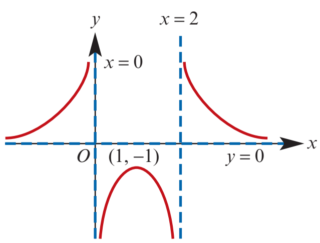
24 a $\dfrac{3(t-3)(t-1)}{2t}, t \neq 0$

b $(2, 4), (10, 0)$ **c** $\dfrac{3(t^2-3)}{4t^3}, t \neq 0$

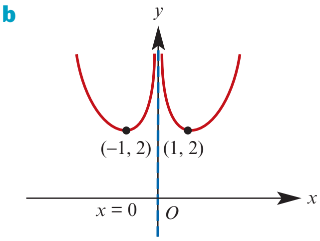
d $(4, 12\sqrt{3} - 18), (4, -12\sqrt{3} - 18)$

Exercise 8G

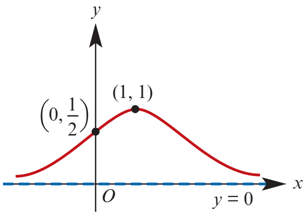
1 a



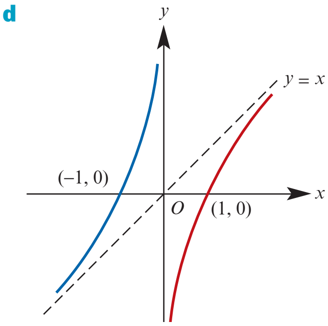
b



c



d



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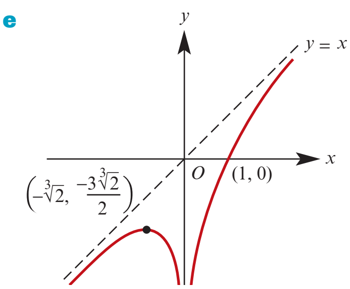

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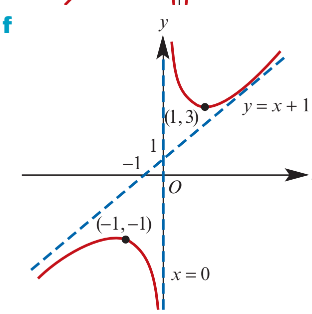

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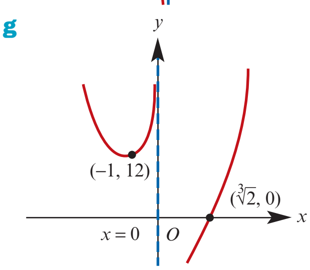

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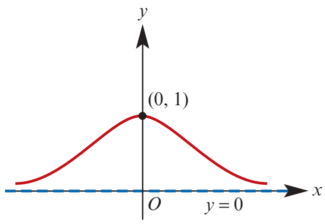
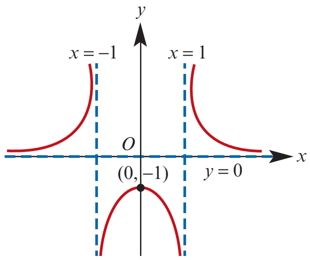
h



i



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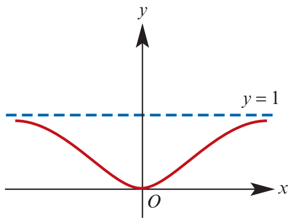

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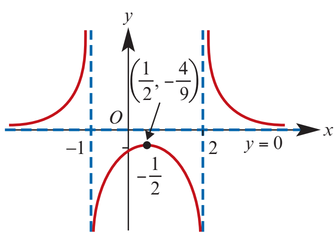
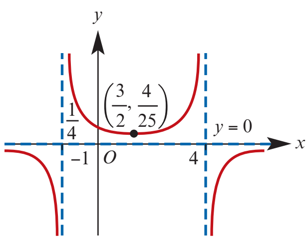
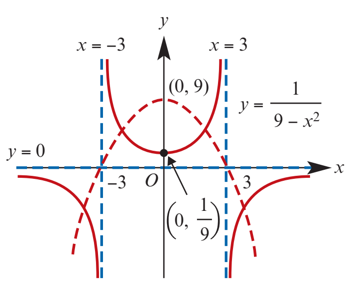


l



2 a



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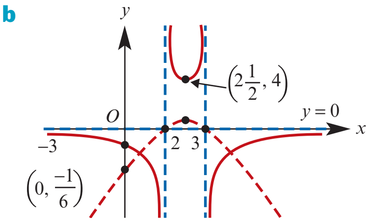

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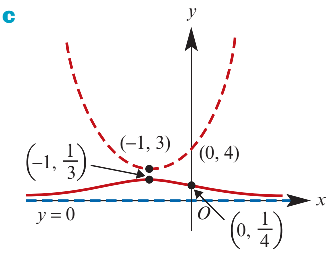

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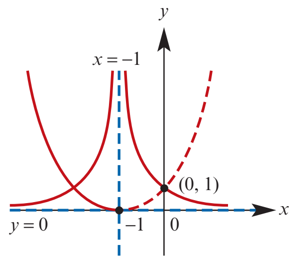

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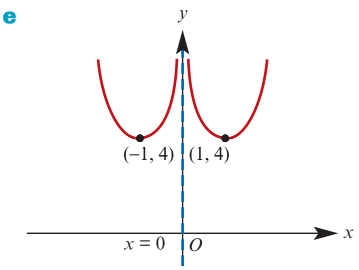


3 a min $\left(\frac{1}{2}, 4\right)$; max $\left(-\frac{1}{2}, -4\right)$

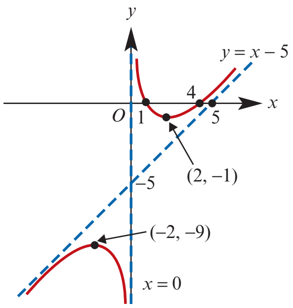
b $y = \frac{15}{4}x + 1$

4 $x = \pm\frac{1}{2}$

5 Gradient $= \frac{1}{2}$

6 a (1, 0); (4, 0) **b** $x = 0, y = x - 5$

c min (2, −1); max (−2, −9)



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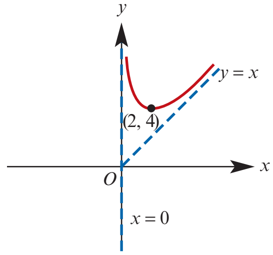
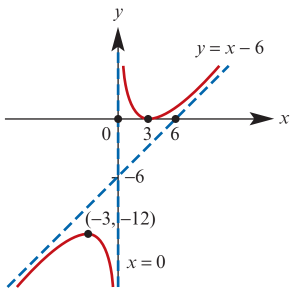
7 Least value = 3

8 Least value = 4



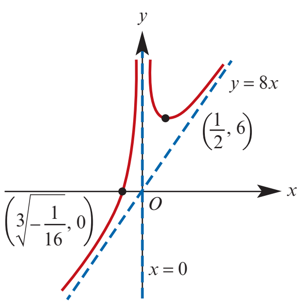
9 a min $(3, 0)$; max $(-3, -12)$

b

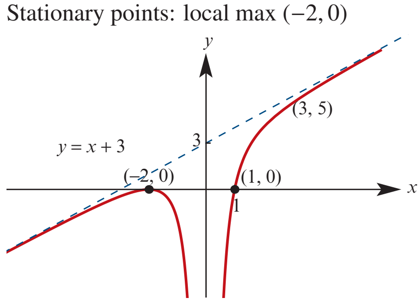


10 a min $\left(\frac{1}{2}, 6\right)$

b



11 Asymptotes: $y = x + 3$, $x = 0$;
Intercepts: $(-2, 0)$, $(1, 0)$;
Stationary points: local max $(-2, 0)$



12 a $\mathbb{R} \setminus \{-\frac{1}{2}\}$ **b** $\dfrac{8(x^2 + x - 2)}{(2x + 1)^2}$

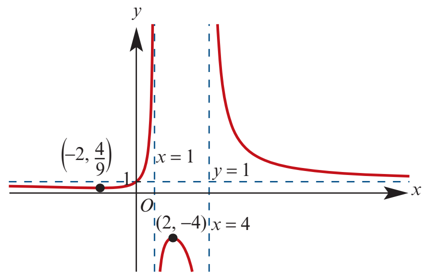
c Local min $(1, 4)$; local max $(-2, -8)$

d $x = -\dfrac{1}{2}$, $y = 2x - 1$ **e** $\mathbb{R} \setminus (-8, 4)$

13 a $x = 4$, $x = 1$, $y = 1$

b Local max $(2, -4)$; local min $\left(-2, \dfrac{4}{9}\right)$

c

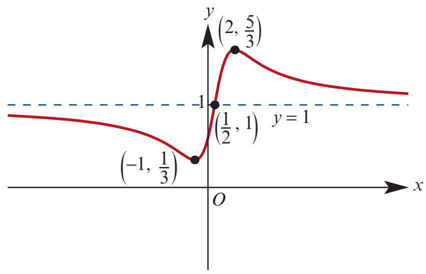


14 a $y = 1$

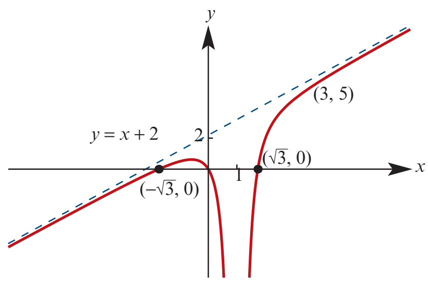
b Local min $\left(-1, \dfrac{1}{3}\right)$; local max $\left(2, \dfrac{5}{3}\right)$

c Points of inflection $\left(\dfrac{1}{2}, 1\right)$,
$\left(\dfrac{1 - 3\sqrt{3}}{2}, \dfrac{3 - \sqrt{3}}{3}\right)$, $\left(\dfrac{1 + 3\sqrt{3}}{2}, \dfrac{3 + \sqrt{3}}{3}\right)$

d



15 a



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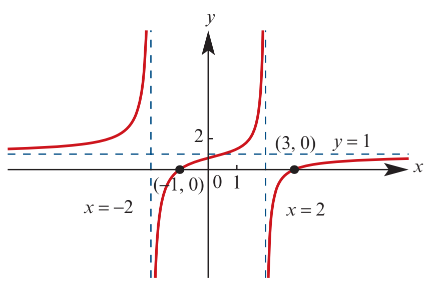

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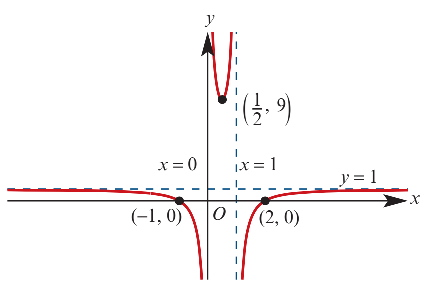

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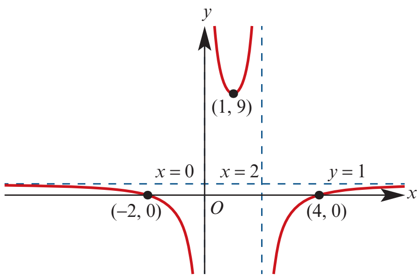
d



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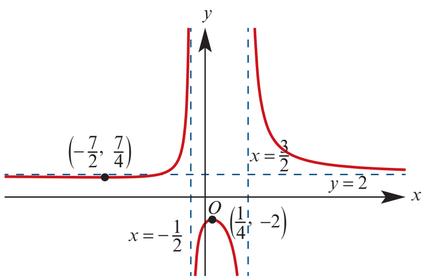

16 a $x > 2$ **b** $\dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$

c $(4, 2\sqrt{2})$, local minimum **d** $x = 2$

17 a $x > -\dfrac{1}{2}$ **b** 7

c $\dfrac{3x^2+3x-6}{(2x+1)^{\frac{3}{2}}}$

d $(1, 3\sqrt{3})$, local minimum **e** $x = -\dfrac{1}{2}$

Exercise 8H

1 a $f''(x) = 90x^8$

b $f''(x) = 224(2x+5)^6$

c $f''(x) = -4\sin(2x)$ **d** $f''(x) = -\dfrac{1}{9}\cos\left(\dfrac{x}{3}\right)$

e $f''(x) = \dfrac{9}{2}\sin\left(\dfrac{3x}{2}\right)\sec^3\left(\dfrac{3x}{2}\right)$

f $f''(x) = 16e^{-4x}$ **g** $f''(x) = \dfrac{-1}{x^2}$

h $f''(x) = \dfrac{x}{\sqrt{(16-x^2)^3}}$

i $f''(x) = \dfrac{-8x}{\sqrt{(1-4x^2)^3}}$

j $f''(x) = \dfrac{-4x}{(4+x^2)^2}$

k $f''(x) = \dfrac{50-12x}{(x^2-8x+17)^2}$

2 a $\dfrac{dy}{dx} = -24x(1-4x^2)^2$

b $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{(2-x)^3}}$

c $\dfrac{dy}{dx} = -\sin x\cos(\cos x)$

d $\dfrac{dy}{dx} = -\dfrac{\sin(\log_e x)}{x}$ **e** $\dfrac{dy}{dx} = -\dfrac{\sec^2\left(\frac{1}{x}\right)}{x^2}$

f $\dfrac{dy}{dx} = -\sin x\, e^{\cos x}$ **g** $\dfrac{dy}{dx} = \dfrac{3}{3x-4}$

h $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{x(2-x)}}$ **i** $\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{-4x(x+1)}}$

j $\dfrac{dy}{dx} = \dfrac{1}{x^2+2x+2}$ **k** $\dfrac{dy}{dx} = \dfrac{9}{|x|\sqrt{x^2-81}}$

3 a $\dfrac{1-\log_e x}{x^2}$ **b** $\dfrac{-2x}{(x^2+1)^2}$

c $\dfrac{1}{x^2-2x+2}$ **d** $\dfrac{1}{e^x+1}$

e $\dfrac{2\sqrt{\sin y+\cos y}}{\cos y-\sin y}$ **f** $\dfrac{1}{\sqrt{1+x^2}}$

g $\dfrac{e^x}{\sqrt{1-e^{2x}}}$

h $\dfrac{e^x(\cos x-\sin x)+\cos x}{(e^x+1)^2}$

4 a i $a - \dfrac{b}{x^2}$ **ii** $\dfrac{2b}{x^3}$

5 a i $2\cos(2x) - 6\sin(2x)$

ii $-4\sin(2x) - 12\cos(2x)$

Exercise 8I

1 a x **b** $-\dfrac{2y}{x}$ **c** $-\dfrac{x^2}{y^2}$ **d** $\dfrac{2x}{3y^2}$

e $2\sqrt{y}$ **f** $\dfrac{2-y}{x+3}$ **g** $\dfrac{2a}{y}$ **h** $\dfrac{2}{1-y}$

2 a $\dfrac{x+2}{y}$ **b** $-\dfrac{y^2}{x^2}$

c $\dfrac{2(x+y)}{1-2(x+y)}$ **d** $\dfrac{y-2x}{2y-x}$

e $\dfrac{2xe^y}{1-x^2e^y}$ **f** $-\dfrac{\sin(2x)}{\cos y}$

g $\dfrac{\cos x-\cos(x-y)}{\cos y-\cos(x-y)}$ **h** $\dfrac{\sin y}{5y^4-x\cos y+6y}$

3 a $x+y=-2$ **b** $5x-12y=9$

c $16x-15y=8$ **d** $y=-3$

4 $\dfrac{dy}{dx} = \dfrac{y}{x}$ **5** $-\dfrac{1}{4}$

6 -1 **7** $-\dfrac{2}{5}$

8 $-\dfrac{7}{5}$

9 a $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$ **c** $-\dfrac{1}{9}$

10 $y=-1,\ y=1$

11 a $\dfrac{dy}{dx} = \dfrac{-(3x^2+y)}{x+6y^2}$ **d** -220 or -212

12 a $\dfrac{dy}{dx} = \dfrac{y-x}{2y-x}$ **b** $(-2,-2), (2,2)$

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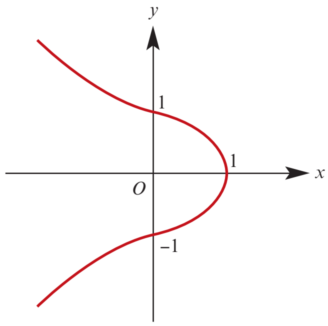
13 a $\frac{dy}{dx} = -\frac{3x^2}{2y}$ **b** $(0, -1), (0, 1)$

c $(1, 0)$ **e** $y = \pm\sqrt{1 - x^3}$

f $(0, -1), (0, 1)$

g



14 a $\frac{dy}{dx} = \frac{2 - x}{y - 2}$ **b** $\frac{\sqrt{2}}{4}$

15 $\frac{d^2y}{dx^2} = -2(6 - y)^3$

16 $-\frac{4}{7}$

17 $\frac{12}{\sqrt{3}\pi - 6}$

18 $\frac{dy}{dx} = -\frac{5}{12}, \frac{d^2y}{dx^2} = -\frac{169}{1728}$

19 $9x + 13y = 40$

20 $\frac{dy}{dx} = -\frac{4x}{y}, \frac{d^2y}{dx^2} = -\frac{16x^2 + 4y^2}{y^3}$

Chapter 8 review

Technology-free questions

1 a $\tan x + x\sec^2 x$

b $\frac{1}{1 + x^2}\sec^2(\tan^{-1} x) = 1$

c $\frac{-x}{\sqrt{1 - x^2}}$ **d** $\frac{1}{\sqrt{x - x^2}}$

2 a $2\sec^2 x \tan x$

b $\frac{\sec^2 x - 2}{\sin^2 x} = -4\cot(2x)\operatorname{cosec}^2 x = -\operatorname{cosec}^2 x + \sec^2 x$

c $\frac{2 - x^2}{(1 - x^2)^{\frac{3}{2}}}$

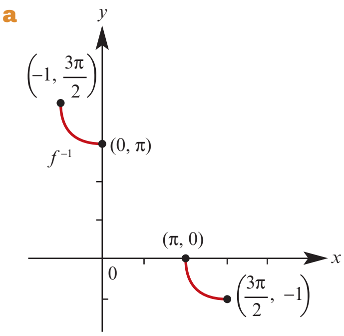
d $e^x(\cos e^x - e^x \sin e^x)$

3 a $\left(\frac{8}{3}, -\frac{1024}{27}\right)$ **b** $(2, 0)$

c $\left(2, \log_e 2 + \frac{1}{2}\right)$ **d** $\left(2, \frac{1}{8}\right)$

7 $f' : (-1, 1) \to \mathbb{R}, f'(x) = \frac{1}{\arccos^2(x)\sqrt{1 - x^2}}$

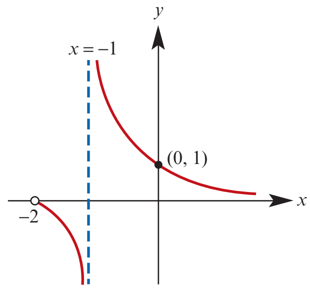
8 a



b $-\frac{1}{\sqrt{1 - x^2}}$ **c** $\left(-\frac{\sqrt{3}}{2}, \frac{4\pi}{3}\right)$

9 a $-\frac{2\sqrt{3}}{3}$ **b** $-\frac{8}{3}$

10 a



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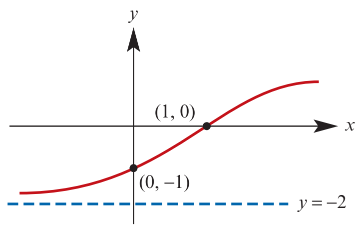
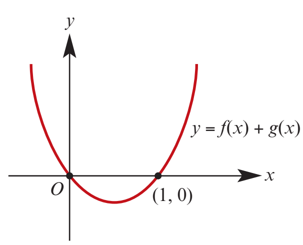
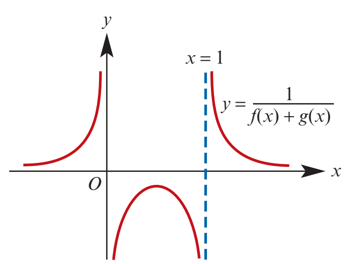


11 a i



ii



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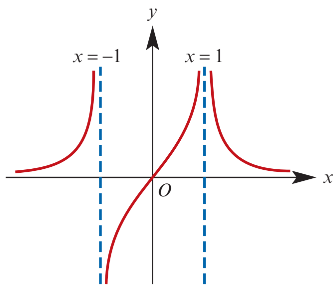
iii



b $f(x) = x^2 - 1, \quad g(x) = (x-1)^2$

c **i** $f(x) + g(x) = 2x^2 - 2x$

ii $\dfrac{1}{f(x)+g(x)} = \dfrac{1}{2x^2 - 2x}$

iii $\dfrac{1}{f(x)} + \dfrac{1}{g(x)} = \dfrac{2x}{(x-1)^2(x+1)}$

12 **a** -1 **b** $\dfrac{-(x+1)}{y+3}$ **c** $-\dfrac{2y^2}{x^2}$

d $\dfrac{-(x+1)}{y-3}$ **e** $\dfrac{\sin(x)}{\cos(y)}$ **f** $-\dfrac{y\log_e(y)}{x}$

13 **a** 324 cm/s **b** 36 cm/s

14 **b** $(-1, 0), \left(0, -\dfrac{1}{9}\right)$ **c** $x = \pm 3, \ y = 0$

15 $a = -3, \ b = 12$

16 $\dfrac{dy}{dx} = \dfrac{3t}{2}, \ \dfrac{d^2y}{dx^2} = \dfrac{3}{4t}$

17 $e^{2x}\arctan(y)\left(2 + \dfrac{e^{2x}}{1+y^2}\right)$

Multiple-choice questions

1 E	**2** E	**3** B	**4** E	**5** B
6 D	**7** C	**8** B	**9** D	**10** C
11 C	**12** B	**13** B	**14** C	

Extended-response questions

1 **a** **i** $\left(-\dfrac{1}{3}, \dfrac{19}{27}\right), (2, -12)$ **ii** $\left(\dfrac{5}{6}, -\dfrac{305}{54}\right)$

b $b^2 - 4ac > 0$

2 **a** $p = -\dfrac{b}{3a}, \ q = f\left(-\dfrac{b}{3a}\right)$

3 **a** $f'(x) = (x+1)e^x, \ f''(x) = (x+2)e^x,$
$f^{(3)}(x) = (x+3)e^x$

b $f^{(n)}(x) = (x+n)e^x$

d $f^{(n)}(x) = (x^2 + 2nx + n^2 - n)e^x$

5 **a** $f(x) = \dfrac{x}{2} + \dfrac{\sqrt{3}}{2}\sqrt{1-x^2},$

dom $= (0, 1)$, ran $= \left(\dfrac{1}{2}, 1\right]$

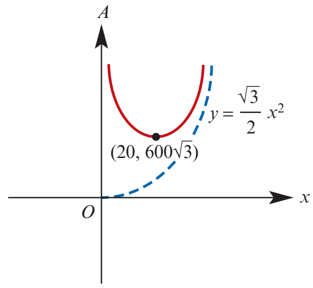
b $y = 1$

c $\left(\dfrac{1}{2}, 1\right)$

6 **a** $A = \dfrac{\sqrt{3}}{2}x^2 + 3xy$ **b** $y = \dfrac{8000\sqrt{3}}{3x^2}$

c $A = \dfrac{\sqrt{3}}{2}x^2 + \dfrac{8000\sqrt{3}}{x}$

d

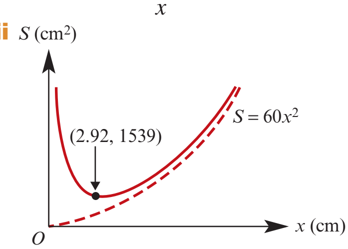


e Minimum surface area $= 600\sqrt{3}$ cm^2

7 **a** **i** $y = \dfrac{100}{x^2}$

ii $S = 60x^2 + \dfrac{3000}{x}$

iii



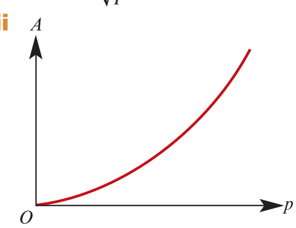
b $521\frac{13}{27}$ cm^2/s

c 1.63 cm or 4.78 cm

8 **a** $A = \dfrac{p\sqrt{p^2+4}}{2} - p$

b **i** $\dfrac{dA}{dp} = \dfrac{p^2}{2\sqrt{p^2+4}} + \dfrac{\sqrt{p^2+4}}{2} - 1$

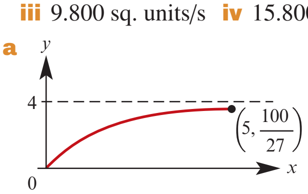
ii



iii 10.95

c **i** 0.315 sq. units/s **ii** 0.605 sq. units/s
iii 9.800 sq. units/s **iv** 15.800 sq. units/s

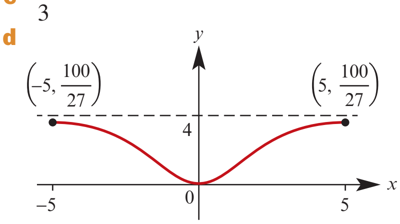
9 **a**



b **i** $\dfrac{16x}{(2+x^2)^2}$ **ii** $\dfrac{16}{(2+x^2)^2}\left(1 - \dfrac{4x^2}{2+x^2}\right)$

c $\dfrac{\sqrt{6}}{3}$

d



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10 a $3ax^2 + 2bx + c$

b $6ax + 2b$

c $b^2 \le 3ac$

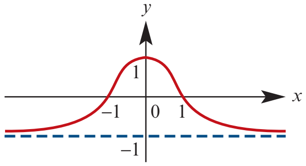
d i $x = -\dfrac{b}{3a}$ **ii** $\max a < 0,\ \min a > 0$

e $-\dfrac{b}{3}$

f i $b^2 < 4c$ **ii** $3c < b^2 < 4c$

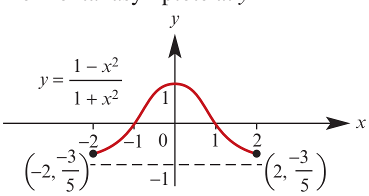
11 a ii $\dfrac{4(3x^2 - 1)}{(1 + x^2)^3}$

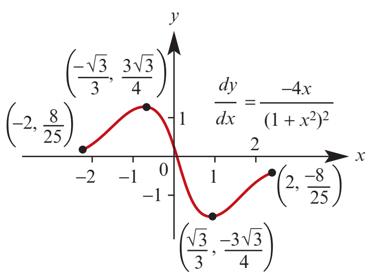
b

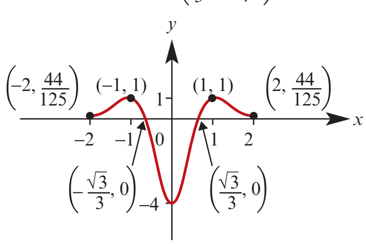


Horizontal asymptote at $y = -1$

c





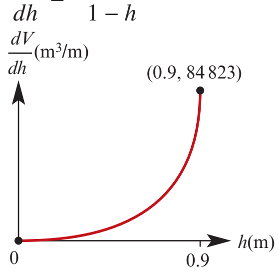


$\dfrac{d^2y}{dx^2} = \dfrac{4(3x^2 - 1)}{(1 + x^2)^3}$

d i $y = x + 1,\ y = -x + 1$

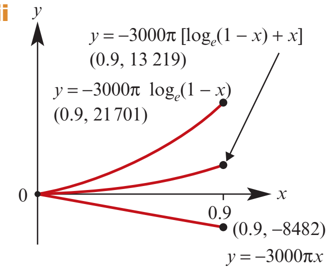
12 a i $\dfrac{dV}{dh} = \dfrac{3000\pi h}{1 - h}$

ii



b i 13 219 litres

ii



c 0.0064 m/min

13 a i $f'(x) = 0$ **ii** $f(x) = \dfrac{\pi}{2}$ **iii** $f(x) = -\dfrac{\pi}{2}$

b i $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$ **ii** $\dfrac{dy}{dx} = -(1 + y^2)$

c $\dfrac{-1}{1 + x^2}$

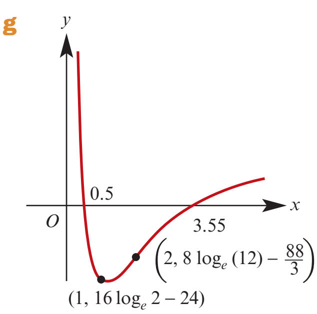
d $-\operatorname{cosec}^2 x + \sec^2 x$

14 a $f'(x) = -\dfrac{16}{x^3} + \dfrac{16}{x}$ **b** $f''(x) = \dfrac{48}{x^4} - \dfrac{16}{x^2}$

c $(1, 16\log_e 2 - 24)$ **d** $x = \sqrt{3}$

e $(1, \infty)$ **f** $x = 3.55$

g



15 b i $\left(3, \dfrac{2 - 2\cos\theta}{\sin\theta}\right)$

c i $M = \left(\dfrac{3}{2\cos\theta}, \dfrac{1}{\sin\theta}\right)$

ii $\dfrac{9}{4x^2} + \dfrac{1}{y^2} = 1$

d i $y = \dfrac{2\sin\theta}{3\cos\theta}x + \dfrac{6}{3\cos\theta}$

ii $Z = (3(\cos\theta - \sin\theta), 2(\cos\theta + \sin\theta))$

iii $(2x + 3y)^2 + (3y - 2x)^2 = 144$

16 a $\left|\dfrac{ab}{\sin(2\theta)}\right|$

b $\theta = (2n + 1)\dfrac{\pi}{4},\ n \in \mathbb{Z}$; minimum area $= ab$

17 b $Q = \left(\dfrac{a}{\sec\theta - \tan\theta}, \dfrac{b}{\sec\theta - \tan\theta}\right)$;

$R = \left(\dfrac{a}{\sec\theta + \tan\theta}, \dfrac{-b}{\sec\theta + \tan\theta}\right)$

c Midpoint $= (a\sec\theta, b\tan\theta)$

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18 a $\dfrac{9\sin\theta\cos\theta}{4}$

b Maximum area $= \dfrac{9}{8}$ when $\theta = \dfrac{\pi}{4}$

c $M = \left(\dfrac{3\cos\theta}{4}, \dfrac{3\sin\theta}{2}\right)$

d $\dfrac{16x^2}{9} + \dfrac{4y^2}{9} = 1$

19 a $\dfrac{x^2}{4} + y^2 = 1$

Chapter 9

Exercise 9A

1 a $-\dfrac{1}{2}\cos\left(2x + \dfrac{\pi}{4}\right)$ **b** $\dfrac{1}{\pi}\sin(\pi x)$

c $-\dfrac{3}{2\pi}\cos\left(\dfrac{2\pi x}{3}\right)$ **d** $\dfrac{1}{3}e^{3x+1}$ **e** $\dfrac{1}{5}e^{5(x+4)}$

f $-\dfrac{3}{2x}$ **g** $\dfrac{3}{2}x^4 - \dfrac{2}{3}x^3 + 2x^2 + x$

2 a 0 **b** 20 **c** 1

d $\dfrac{5}{24}$ **e** $\dfrac{1}{\sqrt{2}} + \dfrac{\pi^2}{16}$ **f** $\dfrac{e^3}{3} + \dfrac{1}{6}$

g 0 **h** 0 **i** 1

3 a $\dfrac{1}{2}\log_e|2x - 5|$ **b** $\dfrac{1}{2}\log_e\left(\dfrac{3}{5}\right)$

c $\dfrac{1}{2}\log_e\left(\dfrac{7}{9}\right)$

4 a $\dfrac{1}{3}\log_e\left(\dfrac{5}{2}\right)$ **b** $\dfrac{1}{3}\log_e\left(\dfrac{5}{11}\right)$ **c** $\dfrac{1}{3}\log_e\left(\dfrac{7}{4}\right)$

5 a $\dfrac{(3x+2)^6}{18}$ **b** $\dfrac{1}{3}\log_e|3x - 2|$

c $\dfrac{2}{9}(3x+2)^{\frac{3}{2}}$ **d** $-\dfrac{1}{3(3x+2)}$

e $3x - 2\log_e|x + 1|$ **f** $\dfrac{2}{3}\sin\left(\dfrac{3x}{2}\right)$

g $\dfrac{3}{20}(5x-1)^{\frac{4}{3}}$ **h** $2x - 5\log_e|x + 3|$

6 a $f(x) = 2x$, $F(x) = x^2 + 3$

b $f(x) = 4x^2$, $F(x) = \dfrac{4}{3}x^3$

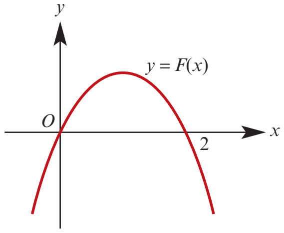
c $f(x) = -2x^2 + 8x - 8$,
$F(x) = -\dfrac{2}{3}x^3 + 4x^2 - 8x + \dfrac{28}{3}$

d $f(x) - e^{-x}$, $F(x) = e^{-x} + 3$

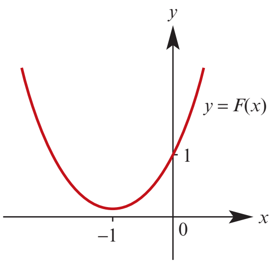
e $f(x) = 2\sin x$, $F(x) = 2 - 2\cos x$

f $f(x) = \dfrac{2}{4 + x^2}$, $F(x) = \tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{\pi}{2}$

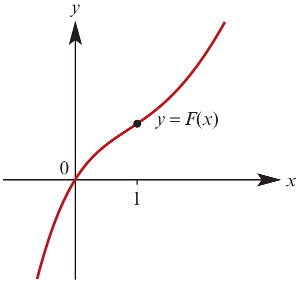
7 a



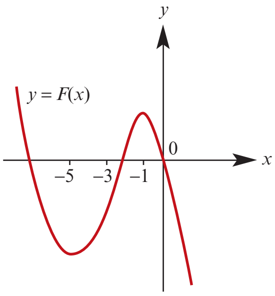
b



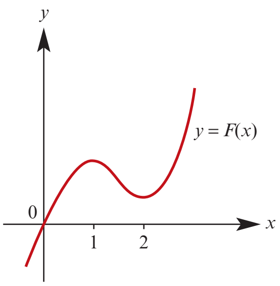
c



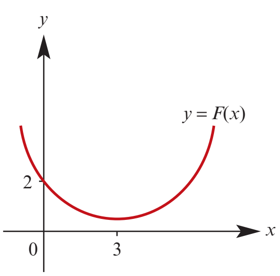
d



e



f



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Exercise 9B

1 a $\sin^{-1}\left(\frac{x}{3}\right)+c$ **b** $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$

c $\tan^{-1}(t)+c$ **d** $5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$

e $\frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$ **f** $\frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right)+c$

g $10\sin^{-1}\left(\frac{t}{\sqrt{10}}\right)+c$ **h** $\frac{1}{12}\tan^{-1}\left(\frac{4t}{3}\right)+c$

i $\frac{\sqrt{2}}{2}\sin^{-1}\left(\frac{x\sqrt{10}}{5}\right)+c$

j $\frac{7}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right)+c$

2 a $\frac{\pi}{2}$ **b** $\frac{\pi}{2}$ **c** $\frac{5\pi}{6}$ **d** $\frac{3\pi}{10}$

e $\frac{\pi}{8}$ **f** $\frac{\pi}{16}$ **g** $\frac{\pi}{6}$ **h** $\frac{\pi}{8}$

i $\frac{\pi}{2}$ **j** $\frac{1}{\sqrt{3}}\tan^{-1}(2\sqrt{3})$

Exercise 9C

1 a $\frac{(x^2+1)^4}{4}+c$ **b** $-\frac{1}{2(x^2+1)}+c$

c $\frac{1}{4}\sin^4 x+c$ **d** $-\frac{1}{\sin x}+c$

e $\frac{1}{12}(2x+1)^6+c$ **f** $\frac{5}{3}(9+x^2)^{\frac{3}{2}}+c$

g $\frac{1}{12}(x^2-3)^6+c$ **h** $-\frac{1}{4(x^2+2x)^2}+c$

i $-\frac{1}{3(3x+1)^2}+c$ **j** $2\sqrt{1+x}+c$

k $\frac{1}{15}(x^3-3x^2+1)^5+c$

l $\frac{3}{2}\log_e(x^2+1)+c$ **m** $-\frac{3}{2}\log_e|2-x^2|+c$

n $\frac{1}{2}(\log_e x)^2+c$ **o** $-\frac{1}{8}e^{-4x^2}+c$

2 a $\tan^{-1}(x+1)+c$

b $\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(2x-1)}{3}\right)+c$

c $\sin^{-1}\left(\frac{x+2}{5}\right)+c$ **d** $\sin^{-1}(x-5)+c$

e $\sin^{-1}\left(\frac{x+3}{7}\right)+c$

f $\frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{\sqrt{3}(x+1)}{2}\right)+c$

3 a $\frac{1}{2}\sin^2 x+c$ **b** $-\frac{1}{2}\cos^2 x+c$

d $c=1$; Pythagorean identity

4 a $-\frac{1}{2}(2x+3)^{\frac{3}{2}}+\frac{1}{10}(2x+3)^{\frac{5}{2}}+c$

b $\frac{2(1-x)^{\frac{5}{2}}}{5}-\frac{2(1-x)^{\frac{3}{2}}}{3}+c$

c $\frac{4}{9}(3x-7)^{\frac{3}{2}}+\frac{28}{3}(3x-7)^{\frac{1}{2}}+c$

d $\frac{4}{45}(3x-1)^{\frac{5}{2}}+\frac{10}{27}(3x-1)^{\frac{3}{2}}+c$

e $2\log_e|x-1|-\frac{1}{x-1}+c$

f $\frac{2}{45}(3x+1)^{\frac{5}{2}}+\frac{16}{27}(3x+1)^{\frac{3}{2}}+c$

g $\frac{3}{7}(x+3)^{\frac{7}{3}}-\frac{3}{4}(x+3)^{\frac{4}{3}}+c$

h $\frac{5}{4}\log_e|2x+1|+\frac{7}{4(2x+1)}+c$

i $\frac{2}{105}(x-1)^{\frac{3}{2}}(15x^2+12x+8)+c$

j $\frac{2\sqrt{x-1}}{15}(3x^2+4x+8)+c$

Exercise 9D

1 a $\frac{61}{3}$ **b** $\frac{1}{16}$ **c** $\frac{1}{3}$ **d** $\frac{25}{114}$ **e** $\frac{4}{15}$

f $\log_e 2$ **g** $\frac{4}{3}$ **h** 1 **i** $\frac{1}{2}$

j $\log_e 2$ **k** $\log_e\left(\frac{\sqrt{6}}{2}\right)$ **l** $\log_e\left(\frac{15}{8}\right)$

m $\log_e\left(\frac{e+1}{e}\right)=\log_e(e+1)-1$

2 $\log_e 2$

Exercise 9E

1 a $\frac{1}{2}x-\frac{1}{4}\sin(2x)+c$

b $\frac{1}{32}\sin(4x)-\frac{1}{4}\sin(2x)+\frac{3}{8}x+c$

c $2\tan x-2x+c$ **d** $-\frac{1}{6}\cos(6x)+c$

e $\frac{1}{2}x-\frac{1}{8}\sin(4x)+c$ **f** $\frac{1}{2}\tan(2x)-x+c$

g $\frac{1}{8}x-\frac{1}{32}\sin(4x)+c$

h $\frac{1}{2}\sin(2x)+c$ **i** $-\cot x-x+c$

j $\frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)+c$

2 a $\tan x \quad (c=0)$ **b** $\frac{1}{2}\tan(2x) \quad (c=0)$

c $2\tan\left(\frac{1}{2}x\right) \quad (c=0)$ **d** $\frac{1}{k}\tan(kx) \quad (c=0)$

e $\frac{1}{3}\tan(3x)-x \quad (c=0)$

f $2x-\tan x \quad (c=0)$ **g** $-x \quad (c=0)$

h $\tan x \quad (c=0)$

3 a $\frac{\pi}{4}$ **b** $\frac{1}{2}+\log_e\left(\frac{\sqrt{2}}{2}\right)=\frac{1}{2}-\frac{1}{2}\log_e 2$

c $\frac{1}{3}$ **d** $\frac{1}{4}+\frac{3\pi}{32}$ **e** $\frac{4}{3}$ **f** $\frac{\pi}{4}$

g $\frac{\pi}{24}+\frac{\sqrt{3}}{64}$ **h** 1

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4 a $\sin x - \dfrac{\sin^3 x}{3} + c$

b $\dfrac{4}{3}\cos^3\left(\dfrac{x}{4}\right) - 4\cos\left(\dfrac{x}{4}\right) + c$

c $\dfrac{1}{2}x + \dfrac{1}{16\pi}\sin(8\pi x) + c$

d $7\sin t\left(\cos^2 t + \dfrac{3}{5}\sin^4 t - \dfrac{1}{7}\sin^6 t\right) + c$

e $\dfrac{1}{5}\sin(5x) - \dfrac{1}{15}\sin^3(5x) + c$

f $3x - 2\sin(2x) + \dfrac{1}{4}\sin(4x) + c$

g $\dfrac{1}{48}\sin^3(2x) - \dfrac{1}{64}\sin(4x) + \dfrac{x}{16} + c$

h $\sin x - \dfrac{2\sin^3 x}{3} + \dfrac{\sin^5 x}{5} + c$

5 a $-\dfrac{1}{12}(\sin(6x) - 3\sin(2x))$

b $\dfrac{1}{12}(\sin(6x) + 3\sin(2x))$

c $-\dfrac{1}{12}(\cos(6x) + 3\cos(2x))$

d $-\dfrac{1}{4}(\cos(2x) + 2\cos x)$

e $\dfrac{1}{4}(\sin(2x) + 2\sin x)$

f $-\dfrac{1}{4}(\sin(2x) - 2\sin x)$

6 a $-\dfrac{2}{15}$ **b** 0 **c** $-\dfrac{4}{9}$

Exercise 9F

1 $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x}{3}\right) + c$ **2** $\cos^{-1}\left(\dfrac{x}{2}\right) + c$

3 $2\log_e(\sqrt{x} + 1) + c$ **4** $\dfrac{1}{2}\log_e(4\sqrt{x} + 3) + c$

5 $\sin^{-1}\left(\dfrac{x}{3}\right) + c$

6 $\dfrac{9}{2}\sin^{-1}\left(\dfrac{x}{3}\right) + \dfrac{x\sqrt{9 - x^2}}{2} + c$

7 $\log_e\left|\dfrac{x}{(\sqrt[3]{x} + 1)^3}\right| + c$ **8** $\dfrac{x}{\sqrt{1 - x^2}} + c$

Exercise 9G

1 a $\dfrac{2}{x - 1} + \dfrac{3}{x + 2}$ **b** $\dfrac{1}{x + 1} - \dfrac{2}{2x + 1}$

c $\dfrac{2}{x + 2} + \dfrac{1}{x - 2}$ **d** $\dfrac{1}{x + 3} + \dfrac{3}{x - 2}$

e $\dfrac{3}{5(x - 4)} - \dfrac{8}{5(x + 1)}$

2 a $\dfrac{2}{x - 3} + \dfrac{9}{(x - 3)^2}$

b $\dfrac{4}{1 + 2x} + \dfrac{2}{1 - x} + \dfrac{3}{(1 - x)^2}$

c $\dfrac{-4}{9(x + 1)} + \dfrac{4}{9(x - 2)} + \dfrac{2}{3(x - 2)^2}$

3 a $\dfrac{-2}{x + 1} + \dfrac{2x + 3}{x^2 + x + 1}$ **b** $\dfrac{x + 1}{x^2 + 2} + \dfrac{2}{x + 1}$

c $\dfrac{x - 2}{x^2 + 1} - \dfrac{1}{2(x + 3)}$

4 $3 + \dfrac{3}{x - 1} + \dfrac{2}{x - 2}$

5 $\dfrac{1}{x - 10} - \dfrac{1}{x - 1}$; $\log_e\left|\dfrac{x - 10}{x - 1}\right| + c$

6 $x^2 - 4x + 12 - \dfrac{32}{x + 2} + \dfrac{17}{(x + 2)^2}$;

$\dfrac{x^3}{3} - 2x^2 + 12x - \dfrac{17}{x + 2} - 32\log_e|x + 2| + c$

7 $\dfrac{7}{x + 2} - \dfrac{13}{(x + 2)^2}$; $7\log_e|x + 2| + \dfrac{13}{x + 2} + c$

8 $\dfrac{5}{18(x - 4)} - \dfrac{5x}{18(x^2 + 2)} - \dfrac{10}{9(x^2 + 2)}$;

$\dfrac{1}{36}\left(5\log_e\left(\dfrac{(x - 4)^2}{x^2 + 2}\right) - 20\sqrt{2}\arctan\left(\dfrac{\sqrt{2}x}{2}\right)\right) + c$

9 a $\log_e\left|\dfrac{x - 2}{x + 5}\right| + c$ **b** $\log_e\left|\dfrac{(x - 2)^5}{(x - 1)^4}\right| + c$

c $\dfrac{1}{2}\log_e|(x + 1)(x - 1)^3| + c$

d $2x + \log_e\left|\dfrac{x - 1}{x + 1}\right| + c$

e $2\log_e|x + 2| + \dfrac{3}{x + 2} + c$

f $\log_e|(x - 2)(x + 4)^3| + c$

10 a $\log_e\left|\dfrac{(x - 3)^3}{x - 2}\right| + c$

b $\log_e|(x - 1)^2(x + 2)^3| + c$

c $\dfrac{x^2}{2} - 2x + \log_e\left|(x + 2)^{\frac{1}{4}}(x - 2)^{\frac{3}{4}}\right| + c$

d $\log_e((x + 1)^2(x + 4)^2) + c$

e $\dfrac{x^3}{3} - \dfrac{x^2}{2} - x + 5\log_e|x + 2| + c$

f $\dfrac{x^2}{2} + x + \log_e\left|\dfrac{(x - 1)^4}{x^3}\right| + c$

11 a $\dfrac{1}{2}\left(\log_e\left(\dfrac{x^2 + 2}{(x + 1)^2}\right) + 2\sqrt{2}\arctan\left(\dfrac{\sqrt{2}x}{2}\right)\right)$

b $\dfrac{1}{2}\log_e\left(\dfrac{(x + 1)^2}{x^2 + 1}\right) - \dfrac{1}{x + 1}$

c $\dfrac{1}{5}\left(\log_e((x^2 + 4)^2|x - 1|) - 8\arctan\left(\dfrac{x}{2}\right)\right) + 5x$

d $\dfrac{1}{2}\log_e\left(\dfrac{x^2 + 4}{(x - 2)^2}\right) - 8\arctan\left(\dfrac{x}{2}\right) - \dfrac{18}{x - 2}$

e $2\log_e\left(\dfrac{(x + 2)^2}{x^2 + 2}\right) + 4\sqrt{2}\arctan\left(\dfrac{\sqrt{2}x}{2}\right)$

f $\dfrac{1}{2}\log_e\left|\dfrac{x - 1}{x + 1}\right| + \dfrac{3x^2 + 9x + 10}{3(x + 1)^3}$

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12 a $\log_e\left(\frac{4}{3}\right)$ **b** $\log_e\left(\frac{4}{3}\right)$ **c** $\frac{1}{3}\log_e\left(\frac{625}{512}\right)$

d $1+\log_e\left(\frac{32}{81}\right)$ **e** $\log_e\left(\frac{10}{3}\right)$

f $\log_e 4+4$ **g** $\frac{1}{2}\log_e\left(\frac{7}{4}\right)$

h $\log_e\left(\frac{2}{3}\right)$ **i** $\frac{1}{4}\log_e\left(\frac{1}{3}\right)$

j $5\log_e\left(\frac{3}{4}\right)-\log_e 2$ **k** $\log_e 2+\frac{1}{6}$

13 a $-\frac{5}{4}(2\log_e 2-\pi)$ **b** $2\log_e 2+\pi+\sqrt{3}$

c $1-\frac{\pi}{2}$ **d** $-\frac{1}{3}(3\log_e 3+\pi\sqrt{3})$

14 a $\frac{3}{x-2}-\frac{1+2x}{x^2+x+1}$

b $\log_e\left(\frac{|x-2|^3}{x^2+x+1}\right)+c$ **c** $2\log_e\left(\frac{9}{8}\right)$

16 b $\log_e\left|\frac{1+t}{1-t}\right|+c=\log_e\left|\frac{1+\tan(\frac{x}{2})}{1-\tan(\frac{x}{2})}\right|+c$

c $2\tan^{-1}t+\frac{2}{1+t}+c=x+\frac{2}{1+\tan(\frac{x}{2})}+c$

Exercise 9H

1 a $(-x-1)e^{-x}$ **b** $x\log_e x-x$

c $\sin x-x\cos x$ **d** $x\arccos(x)-\sqrt{1-x^2}$

e $\frac{1}{9}\cos(3x)+\frac{1}{3}x\sin(3x)$

f $\log_e|\cos x|+x\tan x$

g $-\frac{1}{2}x^2+x\tan x+\log_e|\cos x|$

h $x\arcsin(2x)+\frac{1}{2}\sqrt{1-4x^2}$

i $x\arctan x-\frac{1}{2}\log_e(1+x^2)$ **j** $(-x-2)e^{-x}$

k $\frac{1}{2}(-x+\arctan x+x^2\arctan x)$

l $\frac{1}{4}x^2(2\log_e x-1)$ **m** $\frac{1}{9}x^3(3\log_e x-1)$

n $2\sqrt{x}(\log_e x-2)$ **o** $(x+2)e^x$

p $\frac{1}{36}x^6(6\log_e x-1)$ **q** $\frac{1}{4}(2x-1)e^{2x+1}$

r $\frac{1}{4}x^2(2\log_e(2x)-1)$

2 a $-(x^2+2x+2)e^{-x}$

b $(2-x^2)\cos x+2x\sin x$

3 a $\frac{1}{2}e^x(\sin x-\cos x)$

b $\frac{1}{13}e^{2x}(3\sin(3x)+2\cos(3x))$

c $-\frac{1}{10}e^{3x}(\cos x-3\sin x)$

d $-\frac{2}{5}e^x\left(\cos\left(\frac{x}{2}\right)-2\sin\left(\frac{x}{2}\right)\right)$

4 $I_n=\frac{1}{2}(x^ne^{2x}-nI_{n-1}),\ n\in\mathbb{N};$

$I_3=\frac{1}{8}(4x^3-6x^2+6x-3)e^{2x}+c$

5 $I_n=x(\log_e x)^n-nI_{n-1},\ n\in\mathbb{N};$

$I_3=x((\log_e x)^3-3(\log_e x)^2+6\log_e x-6)+c$

6 b $-\frac{1}{15}\cos x\left(3\sin^4 x+4\sin^2 x+8\right)+c$

7 b $\frac{1}{15}\sin x\left(3\cos^4 x+4\cos^2 x+8\right)+c$

8 a $I_0=1,\ I_1=\frac{2}{3}$ **c** $I_2=\frac{8}{15},\ I_3=\frac{16}{35}$

9 a $\frac{1}{4}(1+3e^4)$ **b** $-\frac{\pi}{2}$ **c** $-\frac{1}{8}$ **d** $\frac{2}{9}(1+2e^3)$

e $-12+38\sqrt{2}-8\sqrt{2}\pi$ **f** $\frac{-2+5e^3}{27e}$

g $\log_e(12)-1$ **h** $\frac{1}{4}(5e^4-1)$

i $3\log_e(27)-\frac{26}{9}$

Exercise 9I

1 $p=\frac{4}{3}$

2 a $\frac{1}{24}$ **b** $e-1-\log_e\left(\frac{1+e}{2}\right)$

c $\frac{9}{64}$ **d** $\frac{1}{3}\log_e 5$

3 $c=\frac{3}{2}$ **4** $-\frac{1}{18}\cos^6(3x)+c$

5 $p=\left(\frac{3}{2}\right)^{\frac{1}{2}}$ **6** $p=\frac{8}{5}$

7 a $-\frac{1}{2\sin^2 x}+c$ **b** $\frac{1}{20}(4x^2+1)^{\frac{5}{2}}+c$

c $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+c$

d $\frac{1}{1-e^x}+c$

8 1

9 a $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$ **b** $\frac{1}{3}\sin^{-1}(3x)+c$

c $\frac{1}{2}\sin^{-1}(2x)+c$ **d** $\frac{1}{6}\tan^{-1}\left(\frac{2x+1}{3}\right)+c$

10 a $-\frac{1}{2x\sqrt{x-1}}$ **b** $\frac{\pi}{6}$

11 a $\frac{1}{3}(f(x))^3+c$ **b** $-\frac{1}{f(x)}+c$

c $\log_e(f(x))+c$ **d** $-\cos(f(x))+c$

12 $\frac{dy}{dx}=\frac{8-3x}{2\sqrt{4-x}};\quad 4\sqrt{2}$

13 $a=2,\ b=-3,\ c=-1;\quad x^2-3x+\frac{1}{x-2}+c$

14 a $\frac{\pi}{8}$ **b** 42 **c** 0 **d** $\log_e 2$

e $1-\frac{\pi}{4}$ **f** $\log_e\left(\frac{3}{2}\right)$

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15 a $\frac{1}{2}\sin^2 x + c$ **b** $-\frac{1}{4}\cos(2x) + c$

16 a $\frac{1}{\sqrt{x^2+1}}$; $\log_e|x + \sqrt{x^2+1}| + c$

b $\frac{1}{\sqrt{x^2-1}}$

17 a $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$ **b** $\frac{1}{4}\log_e\left|\frac{x+2}{2-x}\right|$

c $4\log_e|x| + \frac{1}{2}x^2$ **d** $\frac{1}{2}\log_e(4 + x^2)$

e $x - 2\tan^{-1}\left(\frac{x}{2}\right)$ **f** $\frac{1}{2}\tan^{-1}(2x)$

g $\frac{1}{3}(4 + x^2)^{\frac{3}{2}}$

h $\frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}}$

i $-2\sqrt{4-x}$ **j** $\sin^{-1}\left(\frac{x}{2}\right)$

k $-8\sqrt{4-x} + \frac{2}{3}(4-x)^{\frac{3}{2}}$

l $-\sqrt{4-x^2}$

18 $c = \frac{5}{2},\ d = \frac{3}{2}$

19 a $f'(x) = -(n-1)\sin^2 x\cos^{n-2} x + \cos^n x$

c i $\frac{3\pi}{16}$ **ii** $\frac{5\pi}{32}$ **iii** $\frac{\pi}{32}$ **iv** $\frac{4}{3}$

20 a $\frac{1}{2-n}(x+1)^{2-n} - \frac{1}{1-n}(x+1)^{1-n} + c$

b $\frac{1}{n+2} + \frac{1}{n+1}$

21 a $\frac{1}{3}a^2 + a + 1$ **b** $-\frac{3}{2}$

22 a $\frac{a^2+b^2}{(a\cos x + b\sin x)^2}$ **b** $\frac{1}{ab}$

23 a $U_n + U_{n-2} = \frac{1}{n-1}$

24 a 1 **c** $\frac{\pi}{4}$

26 a $\frac{3x}{8} - \frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + c$

b $\frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$

c $\sin(4x) = 4\sin x\cos x - 8\sin^3 x\cos x$

Chapter 9 review

Technology-free questions

1 a $\frac{1}{6}\sin(2x)\,(3 - \sin^2(2x))$

b $\frac{1}{4}(\log_e(4x^2+1) + 6\tan^{-1}(2x))$

c $\frac{1}{4}\log_e\left|\frac{1+2x}{1-2x}\right|$ **d** $-\frac{1}{4}\sqrt{1-4x^2}$

e $-\frac{1}{4}x + \frac{1}{16}\log_e\left|\frac{1+2x}{1-2x}\right|$ **f** $-\frac{1}{6}(1-2x^2)^{\frac{3}{2}}$

g $\frac{1}{2}x - \frac{1}{4}\sin\left(2x - \frac{2\pi}{3}\right)$ **h** $(x^2-2)^{\frac{1}{2}}$

i $\frac{1}{2}x - \frac{1}{12}\sin(6x)$

j $\frac{1}{6}\cos(2x)\,(\cos^2(2x) - 3)$

k $2(x+1)^{\frac{3}{2}}\left(\frac{1}{5}(x+1) - \frac{1}{3}\right)$ **l** $\frac{1}{2}\tan x$

m $\frac{x}{e} - \frac{1}{3e^{3x+1}}$ **n** $\frac{1}{2}\log_e|x^2-1|$

o $\frac{x}{8} - \frac{\sin 4x}{32}$ **p** $\frac{1}{2}x^2 - x + \log_e|1+x|$

2 a $\frac{1}{3} - \frac{\sqrt{3}}{8}$ **b** $\frac{1}{2}\log_e 3$ **c** $\frac{1}{3}\left(\frac{5\sqrt{5}}{8} - 1\right)$

d $\frac{1}{6}\log_e\left(\frac{7}{4}\right)$ **e** $2 + \log_e\left(\frac{32}{81}\right)$

f $\frac{2}{3}$ **g** $\frac{\pi}{6}$ **h** $\frac{\pi}{4}$

i $\frac{\pi}{4}$ **j** $\frac{\pi}{16}$ **k** $\log_e\left(\frac{3\sqrt{2}}{2}\right)$

l 6

3 $\frac{1}{2}\log_e|x^2+2x+3| - \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}(x+1)}{2}\right) + c$

4 a $\frac{1}{2\sqrt{x(1-x)}}$; $2\sin^{-1}(\sqrt{x}) + c$

b $\frac{2x}{\sqrt{1-x^4}}$; $\sin^{-1}(x^2) + c$

5 a $x\sin^{-1}x + \sqrt{1-x^2} + c$

b $x\log_e|x| - x + c$

c $x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2) + c$

6 a $-\frac{1}{8}\cos(4x)$ **b** $\frac{1}{9}(x^3+1)^3$

c $\frac{-1}{2(3+2\sin\theta)}$ **d** $-\frac{1}{2}e^{1-x^2}$

e $\tan(x+3) - x$ **f** $\sqrt{6+2x^2}$

g $\frac{1}{3}\tan^3 x$ **h** $\frac{1}{3\cos^3 x}$ **i** $\frac{1}{3}\tan(3x) - x$

7 a $\frac{8}{15}$ **b** $-\frac{39}{4}$ **c** $\frac{1}{2}$

d $\frac{2}{3}(2\sqrt{2} - 1)$ **e** $\frac{\pi}{2}$ **f** $\frac{1}{3}\log_e\left(\frac{1}{9}\right)$

8 $\left(x^2 + \frac{1}{x}\right)^{-\frac{1}{2}}(2x - x^{-2})$; $\sqrt{2}$

9 a $1, 1$ **b** $3, 2$

10 a $\frac{1}{4}e^{-2x}\left(\sin(2x+3) - \cos(2x+3)\right)$

b $\log_e|\cos x| + x\tan x$

c $\frac{1}{37}e^{3x}\left(12\cos\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\right)$

11 a $\frac{8}{3}\log_e 2 - \frac{7}{9}$ **b** $\frac{1}{2}(\log_e 2)^2$

c $\frac{1}{4} - \frac{3}{4e^2}$

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Multiple-choice questions

1 D **2** C **3** C **4** D **5** A
6 C **7** C **8** C **9** A **10** D

Extended-response questions

1 c i $I_2 = \tan x - x + c$

ii $I_3 = \frac{1}{2}\sec^2 x + \log_e|\cos x| + c$

iii $I_4 = x + \frac{1}{3}\tan x(\sec^2 x - 4) + c$

iv $I_5 = \frac{1}{4}\sec^4 x - \sec^2 x - \log_e|\cos x| + c$

d $\int \sec x\, dx = \log_e|\sec x + \tan x| + c$

e $I_n = \frac{1}{n-1}\tan x \sec^{n-2} x + \frac{n-2}{n-1} I_{n-2}, \ n \geq 3$

f i $I_3 = \frac{1}{2}(\tan x \sec x + \log_e|\sec x + \tan x|) + c$

ii $I_4 = \frac{1}{3}(\tan x \sec^2 x + 2\tan x) + c$

iii $I_5 = \frac{1}{8}\Big(2\tan x \sec^2 x + 3\tan x \sec x + 3\log_e|\sec x + \tan x|\Big) + c$

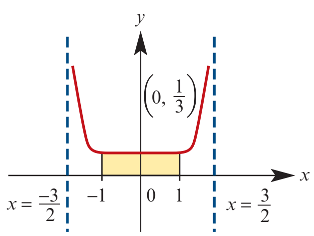
2 d i $\frac{\pi}{8} + \frac{1}{4}$ **ii** $\frac{3\pi}{32} + \frac{1}{4}$ **iii** $\frac{5\pi}{64} + \frac{11}{48}$

3 a $\frac{(2k-1)\cdot(2k-3)\cdot(2k-5)\cdot \cdots \cdot 3\cdot 1}{2k\cdot(2k-2)\cdot(2k-4)\cdot \cdots \cdot 4\cdot 2}\cdot\frac{\pi}{2}$

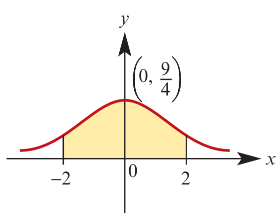
Chapter 10

Exercise 10A

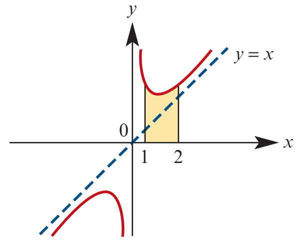
1 Area = $\sin^{-1}\left(\frac{2}{3}\right)$ square units



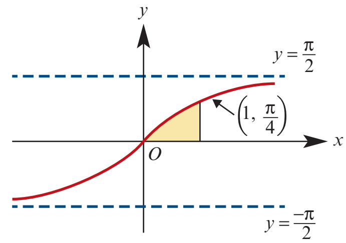
2 Area = $\frac{9\pi}{4}$ square units



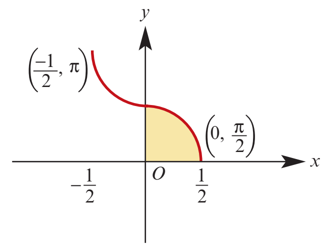
3 Area = $\frac{3}{2} + 2\log_e 2$ square units



4 a Area = $\frac{\pi}{4} - \log_e \sqrt{2}$ square units

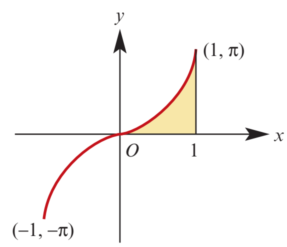


b Area = $\frac{1}{2}$ square units

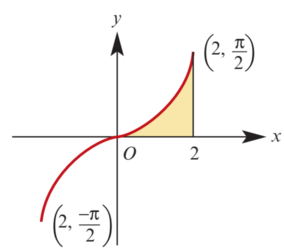


c Area = $\frac{\pi}{2}$ square units

d Area = $\pi - 2$ square units



e Area = $\pi - 2$ square units



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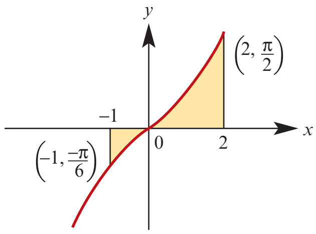
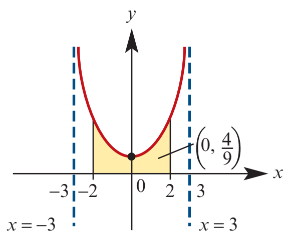
f Integral $= \frac{5\pi}{6} - \sqrt{3}$



5 Area $= \frac{4}{3}\log_e 5$ square units

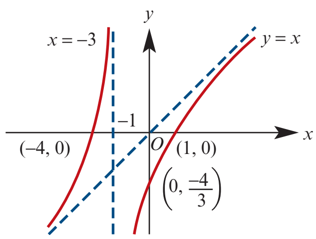


6 a $(0, 1)$ **b** $y = -1$ **c** $\pi - 2$ square units

7 a $\left(0, -\frac{4}{3}\right), (-4, 0), (1, 0)$

b $y = x,\ \ x = -3$

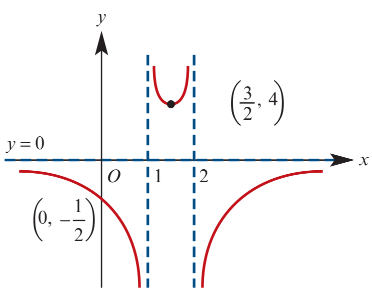
c



d Area $= 31\frac{1}{2} + 4\log_e\left(\frac{4}{11}\right)$ square units

8 a $\mathbb{R} \setminus \{1, 2\}$

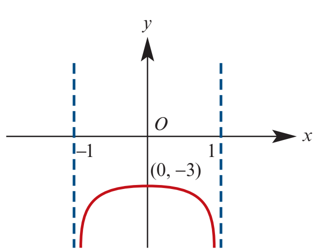
b



c $\mathbb{R}^- \cup [4, \infty)$

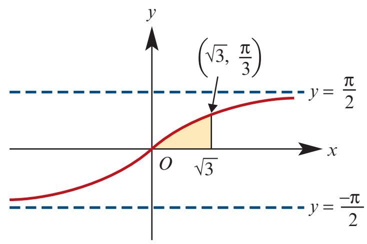
d Area $= -\log_e\left(\frac{3}{4}\right) = \log_e\left(\frac{4}{3}\right)$ square units

9 Integral $= -\frac{\pi}{2}$



10 $\frac{\pi}{12}$ square units

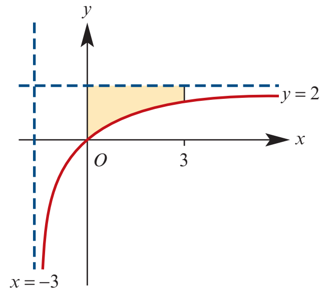
11 Area $= \frac{\pi\sqrt{3}}{3} - \log_e 2$ square units



12 1 square unit **13** $\frac{2}{3}$ square units

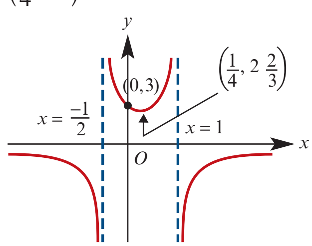
14 $\frac{1}{3}$ square units

15 Area $= 6\log_e 2$ square units



16 b $\left(\frac{1}{4}, 2\frac{2}{3}\right)$ local minimum

c



d $\frac{3}{2} - \log_e 4$ square units

17 a 1 **b** $\frac{\sqrt{3}\pi}{6} - \frac{1}{2}$ **c** $\frac{1}{4}$

d $\frac{\pi}{4}$ **e** $\pi^2 - 4$

18 $a^3\left(\frac{1}{3} - \frac{\sqrt{3}}{8}\right)$

19 $\pi + 1 - \log_e 2$

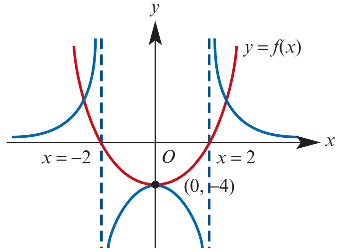
Exercise 10B

1 $(3, 3), (2, 0)$; $\frac{1}{3}$ square units

2 $\frac{1}{3}$ square units

3 a $\frac{17}{24}$ square units **b** $\frac{5}{6}$ square units

4 Area $= 8\log_e 3 - \frac{22}{3}$ square units

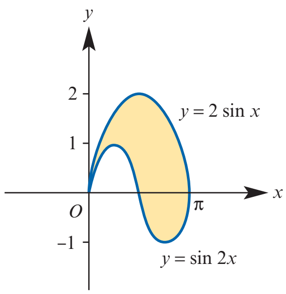


5 $a = e^2$

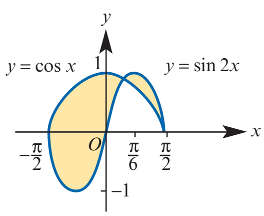
6 a $4\frac{1}{2}$ square units **b** $\frac{11}{6}$ square units

c $\frac{11}{6}$ square units

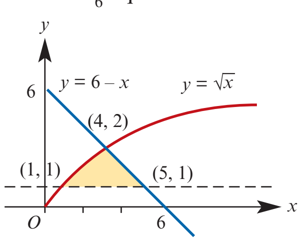
7 a Area = 4 square units



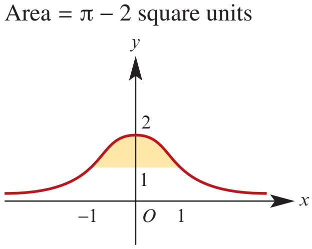
b Area $= 2\frac{1}{2}$ square units



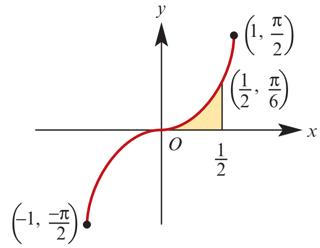
c Area $= 2\frac{1}{6}$ square units



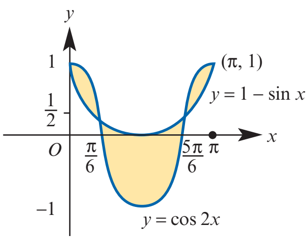
d Area $= \pi - 2$ square units



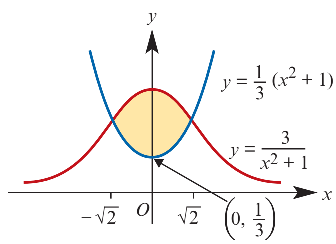
e Area $= \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$ square units



f Area $= 2 + \frac{\pi}{3} - \sqrt{3}$ square units

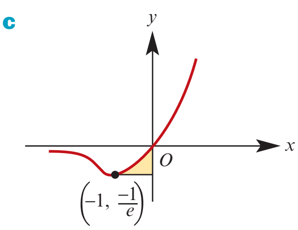


g Area ≈ 4.161 square units



8 a $f'(x) = e^x + xe^x$ **b** $x = -1$

c

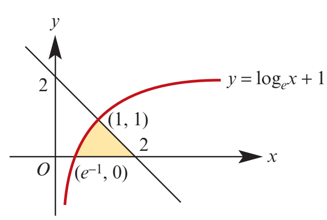


d $y = -\frac{1}{e}$

e Area $= \frac{3}{e} - 1$ square units

Note: As $f'(x) = e^x + xe^x$,
$\int xe^x \, dx = xe^x - e^x + c$

9



a $y = 2 - x$

b Area $= \frac{1}{2} + \frac{1}{e}$ square units

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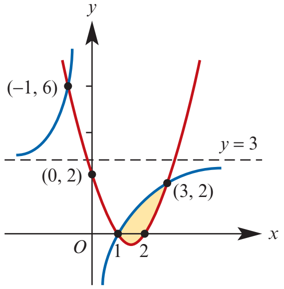
10 a $(-1, 6), (1, 0), (3, 2)$

b



c Area $= \frac{16}{3} - 3\log_e 3$ square units

12 a $(-2\sqrt{2}, 1), (2\sqrt{2}, 1)$ **b** 33.36

13 $\frac{9}{2}$

14 3.772

15 a $a = 4, b = 2\sqrt{5}$ **b** 5.06

16 4

Exercise 10C

1 a $\frac{9\pi}{4}$ **b** $\frac{324 - 108\sqrt{6}}{5}$

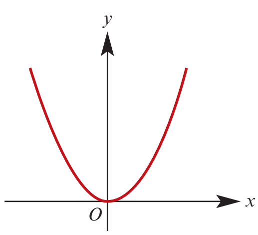
c $3\log_e(10) - 2\arctan\left(\frac{1}{3}\right) + \pi - 6$

2 a $\frac{\pi}{4} - \frac{1}{2}$ **b** $2\log_e 2 - 1$ **c** $\frac{\pi}{8} - \frac{\log_e 2}{4}$

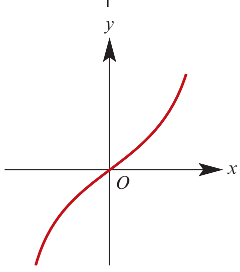
3 a 4.24 **b** 3.14 **c** 1.03 **d** 0.67
e 1.95 **f** 0.66 **g** 0.64 **h** 0.88
i 1.09 **j** 0.83

4 a $\log_e x$ **b** $-\log_e x$ **c** $e^x - 1$
d $1 - \cos x$ **e** $\tan^{-1}(x) + \frac{\pi}{4}$
f $\sin^{-1}(x)$

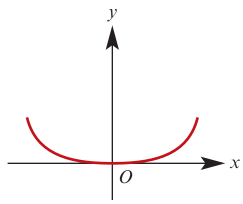
5 a



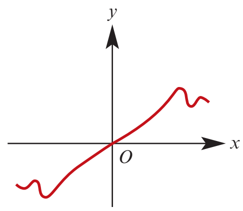
b



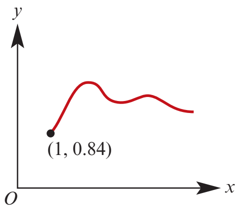
c



d



e



Exercise 10D

1 Area $= \frac{32}{3}$ square units;
Volume $= 8\pi$ cubic units

2 a 8π cubic units **b** $\frac{364\pi}{3}$ cubic units

c $\frac{343\pi}{6}$ cubic units **d** $\frac{\pi^2}{4}$ cubic units

e $\frac{\pi}{2}(e^4 - 1)$ cubic units

f 36π cubic units

3 $\frac{2\pi}{3}$ cubic units

4 a $\frac{3\pi}{4}$ cubic units **b** $\frac{28\pi}{15}$ cubic units

c 2π cubic units **d** $\frac{4\pi a^3}{3}$ cubic units

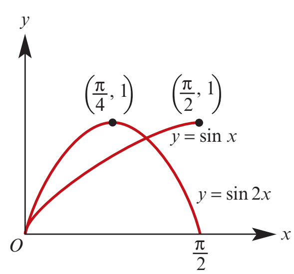
e 36π cubic units **f** 18π cubic units

5 $\frac{1088\pi}{15}$ cubic units **6** $\frac{\pi}{2}$ cubic units

7 $\frac{21\pi}{4}$ cubic units **8** $\frac{3\pi}{10}$ cubic units

9 $\frac{32\pi}{3}$ cubic units

11



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12 $b = \frac{4}{13}$

13 $\frac{7\pi}{6}$

14 a $\frac{16\pi}{3}$ **b** $\pi\left(\frac{e^4}{2} - 4e^2 + \frac{23}{2}\right)$

15 a $\frac{e}{2} - 1$ **b** $\frac{\pi}{6}(e^2 - 3)$

16 $\frac{16\pi}{15}$ cubic units

17 $\frac{\pi^2}{2}$ cubic units

18 $\frac{7\pi}{10}$ cubic units

19 $\frac{19\pi}{6}$ cubic units

20 $\pi\left(\log_e 2 - \frac{1}{2}\right)$ cubic units

22 $8\pi - 2\pi^2$ cubic units

24 a $\frac{\pi}{3}\tan^{-1}\left(\frac{4}{3}\right)$ **b** 4π

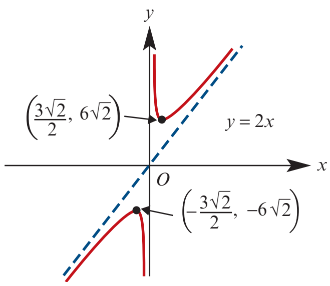
25 176 779 cm^3

26 a $\frac{4\pi ab^2}{3}$ **b** $\frac{4\pi a^2 b}{3}$

27 a $x + y = 8$

b i $\frac{64\pi}{3}$ **ii** $\frac{64\pi}{3}$

28 a



b $\frac{482\pi}{3}$

29 2.642 cubic units

30 $4\pi\left(\frac{4\pi}{3} - \sqrt{3}\right)$

31 $\frac{\sqrt{3}\pi}{4}$

32 $4\pi(3 - 4\log_e 2)$

33 $\frac{\pi^2 + 2\pi}{4}$

34 $\frac{3\pi^2}{2}$

35 $\pi\left(4\log_e\left(\frac{4}{5}\right) + 3\tan^{-1}\left(\frac{3}{4}\right)\right)$

Exercise 10E

1 a $\frac{1}{27}(20\sqrt{10} - 2)$ **b** $3\sqrt{5}$ **c** 78

2 a $\frac{1}{27}(13\sqrt{13} - 8)$ **b** $13\sqrt{26} - 8\sqrt{2}$

c $\frac{16}{3} - 2\sqrt{3}$ **d** $\frac{7}{3}$

3 π **4** 8 **5** 6

6 $\sqrt{5}(e^{\pi} - 1)$ **7** $\frac{64}{9}$ **8** $\frac{23}{24}$

9 $2\sqrt{2} + \pi$

Exercise 10F

1 a 60π **b** $9\pi\sqrt{10}$ **c** $\frac{61\pi}{432}$ **d** 4π

e 8π **f** $\frac{263\pi}{256}$

2 a $\frac{2000\pi}{9}$ **b** $\frac{160\pi}{9}$ **c** $\frac{\pi}{6}(37\sqrt{37} - 17\sqrt{17})$

d 2π **e** $\frac{\pi}{9}(17\sqrt{17} - 1)$

f $\frac{\pi}{6}(37\sqrt{37} - 5\sqrt{5})$

3 $\frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$

4 $\frac{\pi}{6}(5\sqrt{5} - 1)$

5 64π

6 a $\frac{32\pi}{3}(17\sqrt{17} - 1)$ **b** $\frac{8\pi}{3}(2\sqrt{2} - 1)$

c $\frac{48\pi}{5}$ **d** $\frac{\pi}{6}(17\sqrt{17} - 1)$

e $\frac{2\pi}{3}(13\sqrt{13} - 1)$ **f** $2\pi(3\pi + 4)$

g $4\pi(28 - 9\log_e 3 - (\log_e 3)^2)$

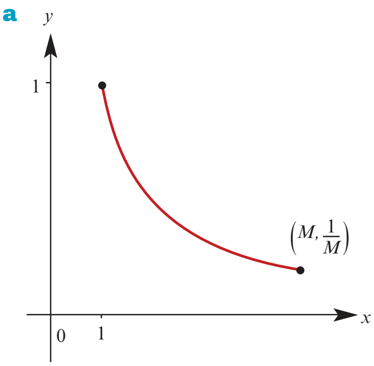
7 $4\pi(2\pi + 1)$ **8** $\frac{6\pi}{5}$

9 $2a^2\pi$ **10** $\pi r\sqrt{h^2 + r^2}$

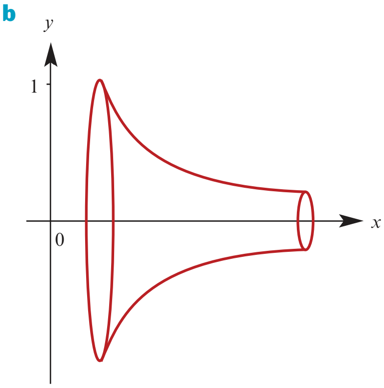
12 $\frac{4\pi}{5}\left(9\sqrt{5}\cos^{-1}\left(\frac{2}{3}\right) + 10\right)$

13 $2\pi^2 rR$

14 a



b



e Volume → π; surface area → ∞

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Chapter 10 review

Technology-free questions

1 $\frac{1}{3}$

2 **a** $\frac{\pi}{2} - 1$ **b** 1

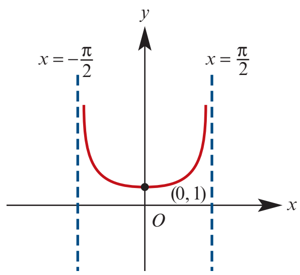
3 **a** π **b** $\frac{\pi}{8}(\pi - 2)$ **c** $\frac{\pi}{8}(\pi + 2)$

d $\frac{2048\pi}{15}$ **e** 40π

4 $\frac{119\pi}{6}$

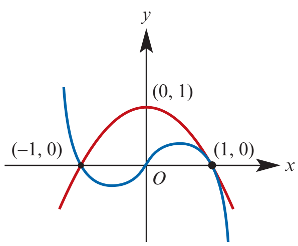
5 **a** 12π **b** $\frac{20\sqrt{10}\pi}{3} - \frac{2\pi}{3}$

6 Volume $= 2\pi$



7 **a** $(0, 0), (2, 4)$ **b** $\frac{16\pi}{3}$

8 **a**

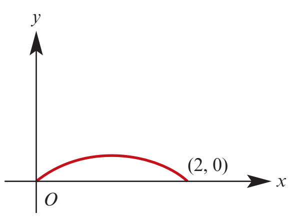


b $\frac{4}{3}$

9 **a** $A = (-1, 1),\ B = (1, 1),\ C = (0, \sqrt{2})$

b $\frac{44\pi}{15}$

10 **a**

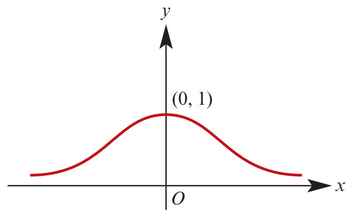


b $\frac{4}{3}$ **c** $\frac{16\pi}{15}$

11 **a** **i** $\frac{\pi b^5}{5}$ **ii** $\frac{\pi b^4}{2}$

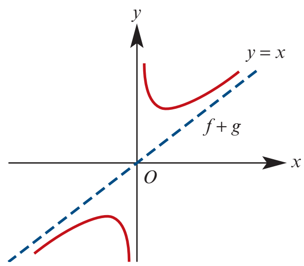
b $b = 2.5$

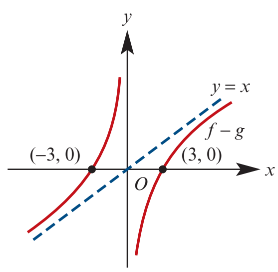
12 **a**



b $\frac{dy}{dx} = \frac{-8x}{(4x^2 + 1)^2},\ x + y = 1$ **c** $\frac{\pi - 3}{8}$

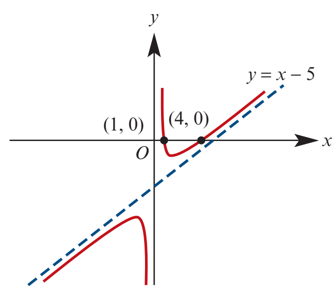
13 **a**



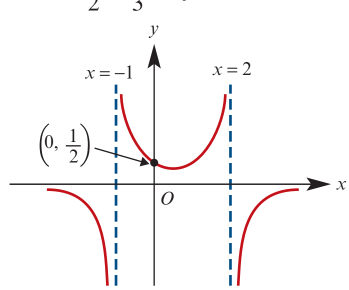


b $18 \log_e 3$

14 Area $= 7.5 - 4\ \log_e 4$



15 Area $= \frac{1}{2} - \frac{1}{3}\log_e 4$



16 $\frac{\pi(\pi^2 + 4)}{64}$

17 $4(2 + \sqrt{3})$

18 $\frac{3}{4}$

19 $\frac{(e^2 - 1)\pi}{4}$

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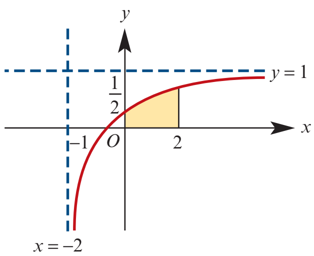
Multiple-choice questions

1 C **2** D **3** B **4** C **5** C **6** E
7 B **8** D **9** E **10** E **11** C **12** A

Extended-response questions

1 a



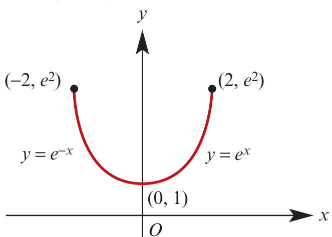
b $2 - \log_e 2$ square units

c $2\pi\left(\frac{9}{8} - \log_e 2\right)$ cubic units

2 a i $x(\log_e x)^2 - 2x\log_e x + 2x + c$

ii $\pi(e - 2)$ cubic units

b



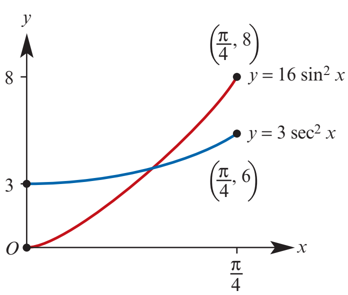
c $V = 2\pi(e^2 - 1)\text{ cm}^3 \approx 40\text{ cm}^3$

3 a $\frac{\pi}{2}$ cubic units

b $\frac{4R}{\pi}$ units per second

c i $\frac{\pi}{8}$ cubic units **ii** $\frac{\sqrt{2}}{2}$ units

4 a



b $\left(\frac{\pi}{6}, 4\right)$ **c** $3\sqrt{3} - \frac{4\pi}{3}$

5 b i $a = 1$ **ii** $\frac{2\sqrt{2}}{3}$

c $\frac{\pi a}{2(a^2 + 1)}$

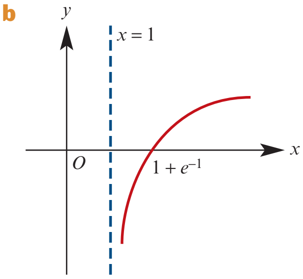
6 a $f(x) = \sqrt{4 - (x - 3)^2}$;
domain $= [3, 5]$; range $= [0, 2]$

b $y = -x + 3 + 2\sqrt{2}$ **c** π

e $8\pi + 4\pi = 12\pi$

7 a $a = 1$; $f(x) = \log_e(x - 1) + 1$

b



c Domain $= \mathbb{R}$; range $= (1, \infty)$ **d** $2 - e^{-1}$

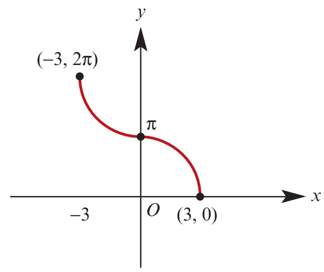
e e^{-1}

9 a $a = 2\pi$

b i Domain $= [-3, 3]$; range $= [0, 2\pi]$

ii $f^{-1}(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

iii



c $-\frac{2}{3}$ **d** $V_1 = V_2 = \frac{9\pi^2}{2}$ cubic units

10 a Area $= \pi(r^2 - y^2)$

11 a $\frac{4\pi ab^2}{3}$ **b** $4\sqrt{3}\pi a^2 b$

12 b $\frac{\pi}{6} - \frac{3}{16}$

c $\frac{\pi}{2}\left(\frac{-3}{16} + \log_e 3\right) = \pi\left(\frac{-3}{32} + \log_e(\sqrt{3})\right)$

13 a $\frac{2a\pi}{3}$ **b** $k = \frac{\sqrt{3}}{3}$; $\frac{2\pi\sqrt{3}}{27}$ cubic units

14 a i

$$d = 0$$
$$125a + 25b + 5c = 1$$
$$1000a + 100b + 10c = 2.5$$
$$27\,000a + 900b + 30c = 10$$

ii $a = \frac{-7}{30000}$, $b = \frac{27}{2000}$, $c = \frac{83}{600}$, $d = 0$

b $\frac{273}{2}$

c i $V = \frac{\pi}{900\,000\,000}$
$\times \int_0^{30} (-7x^3 + 405x^2 + 4150x)^2\, dx$

ii $\frac{362\,083\pi}{400}$

d i $w = 16.729335$

ii $\frac{1978\,810\,99\pi}{2\,500\,00} \approx 2487$

e $\left(\frac{135}{7}, \frac{1179}{196}\right)$

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15 a $\frac{\pi H}{3}(a^2 + ab + b^2)\ \text{cm}^3$

b $\frac{\pi H}{24}(7a^2 + 4ab + b^2)\ \text{cm}^3$

c $V = \frac{\pi H(r^3 - a^3)}{3(b - a)}$

d i $\frac{dV}{dr} = \frac{\pi H r^2}{b - a}$ **ii** $h = \frac{H(r - a)}{b - a}$

e i $\frac{dV}{dr} = 2\pi r^2$

ii $\frac{dr}{dt} = \frac{1}{96\pi}$; $\frac{dh}{dt} = \frac{1}{48\pi}$

Chapter 11

Exercise 11A

1 a $y = 4e^{2t} - 2$ **b** $y = x\log_e|x| - x + 4$

c $y = \sqrt{2x + 79}$

d $y - \log_e|y + 1| = x - 3$

e $y = \frac{1}{2}x^4 - \frac{1}{2}x + 2$ **f** $y = \frac{11}{5}e^{2x} + \frac{4}{5}e^{-2x}$

g $x = 3\sin(3t) + 2\cos(3t) + 2$

3 $-2, 5$

4 $a = 0,\ b = -1,\ c = 1$

5 $a = 0,\ b = \frac{1}{2}$

6 $a = 1,\ b = -6,\ c = 18,\ d = -24$

Exercise 11B

1 a $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c$

b $y = \frac{1}{2}x^2 + 3x - \log_e|x| + c$

c $y = 2x^4 + 4x^3 + 3x^2 + x + c$

d $y = 2\sqrt{x} + c$

e $y = \frac{1}{2}\log_e|2t - 1| + c$

f $y = -\frac{1}{3}\cos(3t - 2) + c$

g $y = -\frac{1}{2}\log_e|\cos(2t)| + c$

h $x = -\frac{1}{3}e^{-3y} + c$ **i** $x = \sin^{-1}\left(\frac{y}{2}\right) + c$

j $x = \frac{1}{y - 1} + c$

2 a $y = \frac{1}{4}x^5 + cx + d$

b $y = \frac{4}{15}(1 - x)^{\frac{5}{2}} + cx + d$

c $y = -\frac{1}{4}\sin\left(2x + \frac{\pi}{4}\right) + cx + d$

d $y = 4e^{\frac{x}{2}} + cx + d$

e $y = -\log_e|\cos x| + cx + d$

f $y = -\log_e|x + 1| + cx + d$

3 a $y = \frac{x - 1}{x}$ **b** $y = 1 - e^{-x}$

c $y = \frac{1}{2}x^2 - 4\log_e x + 1$

d $y = \frac{1}{2}\log_e|x^2 - 4|$

e $y = \frac{1}{3}(x^2 - 4)^{\frac{3}{2}} - \frac{95\sqrt{3}}{12}$

f $y = \sin^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{6}$

g $y = \frac{1}{4}\log_e\left|\frac{2 + x}{2 - x}\right| + 2$

h $y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{4}$

i $y = \frac{2}{5}(4 - x)^{\frac{5}{2}} - \frac{8}{3}(4 - x)^{\frac{3}{2}} + 8$

j $y = \log_e\left(\frac{e^x + 1}{2}\right)$

4 a $y = e^{-x} - e^x + 2x$ **b** $y = x^2 - 2x^3$

c $y = x^2 + \frac{1}{4}\sin(2x) - 1$

d $y = \frac{1}{2}x^2 - 2x + \log_e|x| + 3$

e $y = x - \tan^{-1}x + \frac{\pi}{4}$ **f** $y = 8x^3 + 12x^2 + 6x$

g $y = \sin^{-1}\left(\frac{x}{2}\right)$

5 a $y = \frac{3}{2}x^2 + 4x + c$ **b** $y = -\frac{1}{3}x^3 + cx + d$

c $y = \log_e|x - 3| + c$

6 a $y = 2x + e^{-x}$

b $y = \frac{1}{2}x^2 - \frac{1}{2}\cos(2x) + \frac{9}{2}$

c $y = 2 - \log_e|2 - x|$

7 $4\sqrt{2}$

Exercise 11C

1 a $y = \frac{1}{3}(Ae^{3x} + 5)$ **b** $y = \frac{1}{2}(Ae^{-2x} + 1)$

c $y = \frac{1}{2} - \frac{1}{2}\log_e|2c - 2x|$

d $y = \tan^{-1}(x - c)$ **e** $y = \cos^{-1}(e^{c-x})$

f $y = \frac{1 - Ae^{2x}}{1 + Ae^{2x}}$ **g** $y = \tan(x - c)$

h $x = \frac{5}{3}y^3 + y^2 + c$ **i** $y = \frac{1}{4}(x - c)^2$

2 a $y = e^{x+1}$ **b** $y = e^{x-4} - 1$

c $y = e^{2x-2}$ **d** $y = -\frac{1}{2}(e^{2x} + 1)$

e $x = y - e^{-y} + 1$ **f** $y = 3$

g $y = \frac{3(e^{6x-7} - 1)}{e^{6x-7} + 1}$

h $y = \frac{1}{3}\tan(3x),\ -\frac{\pi}{6} < x < \frac{\pi}{6}$

i $y = \frac{4}{e^{-x} - 2}$

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3 a $y = Ae^x - 3$ for $A \in \mathbb{R}$

b $y = Ae^{2x} + \dfrac{1}{2}$ for $A \in \mathbb{R}$

c $y = -1$ or $y = \dfrac{Ae^x}{1 - Ae^x}$ for $A \in \mathbb{R}$

d $y = 3$ or $y = \dfrac{3Ae^x - 4}{Ae^x - 1}$ for $A \in \mathbb{R}$

Exercise 11D

1 a $\dfrac{dx}{dt} = 2t + 1, \quad x = t^2 + t + 3$

b $\dfrac{dx}{dt} = 3t - 1, \quad x = \dfrac{3}{2}t^2 - t + \dfrac{1}{2}$

c $\dfrac{dx}{dt} = -2t + 8, \quad x = -t^2 + 8t - 15$

2 a $\dfrac{dy}{dx} = \dfrac{1}{y},\ y \neq 0$ **b** $\dfrac{dy}{dx} = \dfrac{1}{y^2},\ y \neq 0$

c $\dfrac{dN}{dt} = \dfrac{k}{N^2},\ N \neq 0,\ k > 0$

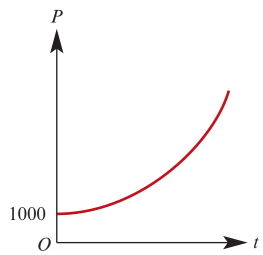
d $\dfrac{dx}{dt} = \dfrac{k}{x},\ x \neq 0,\ k > 0$

e $\dfrac{dm}{dt} = km,\ k < 0$ **f** $\dfrac{dy}{dx} = -\dfrac{x}{3y},\ y \neq 0$

3 a i $\dfrac{dP}{dt} = kP$

ii $t = \dfrac{1}{k}\log_e P + c,\ P > 0$

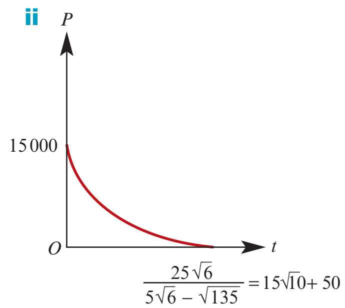
b i 1269 **ii** $P = 1000(1.1)^{\frac{t}{2}},\ t \geq 0$



4 a i $\dfrac{dP}{dt} = k\sqrt{P},\ k < 0,\ P > 0$

ii $t = \dfrac{2\sqrt{P}}{k} + c,\ k < 0$

b i 12 079 **ii**

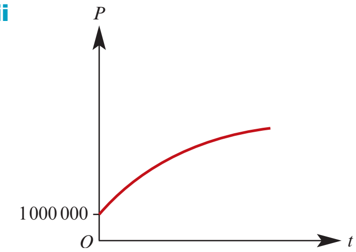


5 a i $\dfrac{dP}{dt} = \dfrac{k}{P},\ k > 0,\ P > 0$

ii $t = \dfrac{1}{2k}P^2 + c$

b i $P = 50\,000\sqrt{21t + 400},\ t \geq 0$

ii

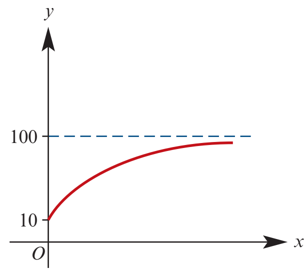


6 $y = 10e^{\frac{x}{10}}$ **7** $\dfrac{420}{9}$ °C

8 $\theta = 331.55$ K **9** 23.22

11 a $x = \dfrac{1}{3}\left(20 - 14e^{\frac{-t}{10}}\right)$ **b** 19 minutes

12 $y = 100 - 90e^{\frac{-x}{10}}$



13 13 500

14 a 14 400 **b** 13 711 **c** 14 182

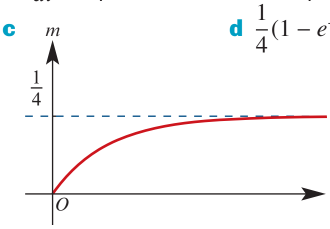
15 a $\dfrac{dV}{dt} = 0.3 - 0.2\sqrt{V},\ V > 0$

b $\dfrac{dm}{dt} = 50 - \dfrac{6m}{100 - t},\ 0 \leq t < 100$

c $\dfrac{dx}{dt} = \dfrac{-5x}{200 + t},\ t \geq 0$

16 a $\dfrac{dm}{dt} = \dfrac{1}{4}(1 - 4m)$ **b** $m = \dfrac{1}{4}(1 - e^{-t})$

c

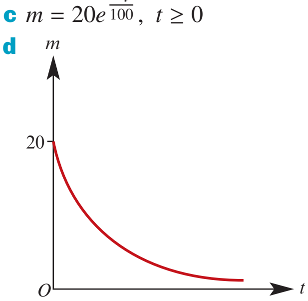


d $\dfrac{1}{4}(1 - e^{-2})$ kg

17 a $\dfrac{m}{100}$ kg/min **b** $\dfrac{dm}{dt} = -\dfrac{m}{100}$

c $m = 20e^{\frac{-t}{100}},\ t \geq 0$

d



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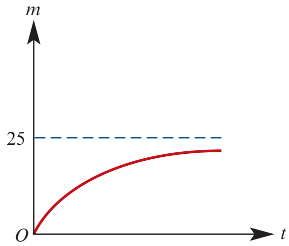
18 a 0.25 kg/min **b** $\frac{m}{100}$ kg/min

c $\frac{dm}{dt} = 0.25 - \frac{m}{100}$

d $m = 25\left(1 - e^{\frac{-t}{100}}\right), \ t \geq 0$

e 51 minutes

f



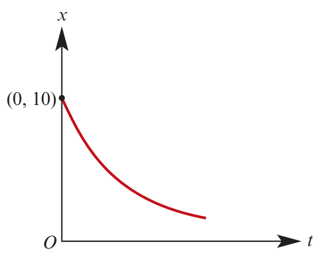
19 a $\frac{dx}{dt} = \frac{10 - x}{50}$ **b** 11.16 minutes

20 a $\frac{dx}{dt} = \frac{80 - x}{200}, \quad x = 80 - 70e^{\frac{-t}{200}}$

b $\frac{dx}{dt} = 0.4 - \frac{x}{400 + t}$

21 a $\frac{dx}{dt} = -\frac{x}{10}$ **b** $x = 10e^{\frac{-t}{10}}$

c



d $10 \log_e 2 \approx 6.93$ minutes

22 a $N = 50\,000\left(99e^{\frac{t}{10}} + 1\right), \ t \geq 0$

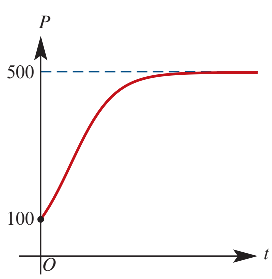
b At the end of 2026

Exercise 11E

1 $P = \frac{2e^t}{2e^t - 1}$

2 a $P = \frac{500e^{0.02t}}{4 + e^{0.02t}}$

b

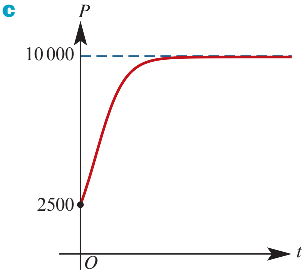


c 250

3 a $P'(t) = 0.3P\left(1 - \frac{P}{10\,000}\right)$

b $P(t) = \frac{10\,000e^{0.3t}}{3 + e^{0.3t}}$

c



d 5990

e $\frac{10}{3} \log_e 3 \approx 3.66$ years

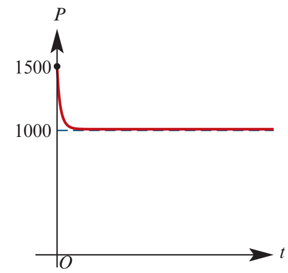
4 12.5 wasps per month

5 $P = \frac{3000e^{0.05t}}{7 + 3e^{0.05t}}$

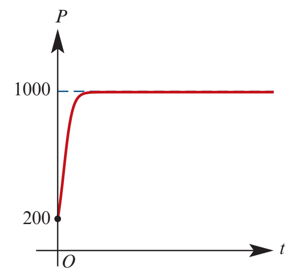
6 a 5 **b** 400 **c** $t = \frac{5}{4} \log_e 79$

d 80 cases per week **e** 60 cases per week

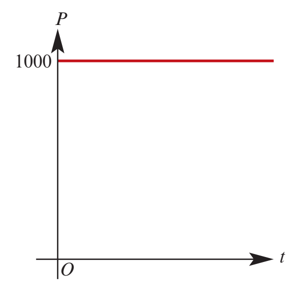
7 a $P = \frac{3000e^{0.1t}}{3e^{0.1t} - 1}$



b $P = \frac{1000e^{0.1t}}{e^{0.1t} + 4}$



c $P = 1000$



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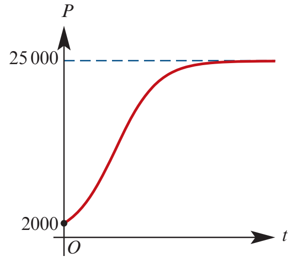
8 a $P = \dfrac{50\,000e^{0.1t}}{23 + 2e^{0.1t}}$

b i 3419 **ii** 24 307

c 24 months **d** 38 months

e



9 a $y = \dfrac{30 - 10e^{-0.1x}}{3 - 2e^{-0.1x}}, \ x \geq 0$

b $y = \dfrac{30 + 10e^{-0.1x}}{3 + 2e^{-0.1x}}, \ x \geq 0$

c $y = \dfrac{20 - 35e^{-0.1x}}{2 - 7e^{-0.1x}}, \ 0 \leq x \leq 10\log_e\left(\dfrac{7}{4}\right)$

Exercise 11F

1 a $y = Ae^{\frac{x^2}{2}}$ **b** $y^2 = x^2 + c$

c $y = Ae^{\frac{x^3}{12}}$ **d** $y^2 = 2\log_e|x| + c$

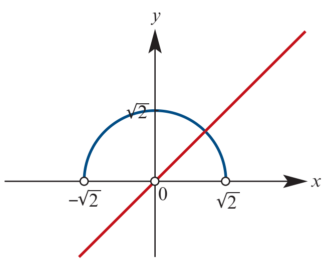
2 $y = 0$ or $y = -\dfrac{2}{x^2 + c}$

3 $y = Ae^{-\cos x} + 1$

4 $y = 1$ or $y = 1 - \dfrac{1}{x^2 + c}$

5 a $y = \sqrt{2 - x^2}$ **b** $y = x$

c



6 $y = \dfrac{1}{2}(x^2 + 1)^2$ **7** $y^2 - x^2 = 5$

8 Circles centre $(-1, 3)$ **9** $y^3 = c - \dfrac{3}{2x^2}$

10 $y = \dfrac{-2x^2}{2Ax^2 - 2x + 3}$

11 a $y = Ae^{e^x + x}$ **b** $y = Ae^{3x^3}$

c $y^2 = -\dfrac{2}{\log_e x} + c$ **d** $y^2 = (\log_e x)^2 + c$

e $y = 0$ or $\log_e|y| = \dfrac{1}{2}e^{x^2} + c$

f $y = 0$ or $y = \dfrac{3}{2(1 - x^2)^{\frac{3}{2}} + c}$

12 a $y = \sqrt{\dfrac{2x^3}{3} + 2x + 1}$ **b** $\tan y = 2 - \dfrac{1}{x}$

13 $\dfrac{y^3}{3} - \dfrac{y^2}{2} = \dfrac{x^3}{3} - \dfrac{x^2}{2} + c$

14 b $x = A(t - 25)^2$ **c** $\dfrac{9}{25}$

15 b $\dfrac{13}{25}e^{\frac{72}{5}}N_0$

16 $y = 2xe^{\frac{x^2}{2}}$

17 $\dfrac{-3}{\sin^3 x - 1} - 1$

Exercise 11G

1 a $\dfrac{dh}{dt} = -\dfrac{2000}{\pi h^2}, \ h > 0$

b $\dfrac{dh}{dt} = \dfrac{1}{A}(Q - c\sqrt{h}), \ h > 0$

c $\dfrac{dh}{dt} = \dfrac{3 - 2\sqrt{V}}{60\pi}, \ V > 0$

d $\dfrac{dh}{dt} = -\dfrac{4\sqrt{h}}{9\pi}, \ h > 0$

2 a $\dfrac{dy}{dt} = 5\sin t$ **b** $y = -5\cos t + c$

3 a $t = -\dfrac{2\pi}{25}h^{\frac{5}{2}} + 250\pi$ **b** 13 hrs 5 mins

4 a $\dfrac{dx}{dt} = -\dfrac{1}{480\sqrt{4 - x}}$ **b** $t = 320(4 - x)^{\frac{3}{2}}$

c 42 hrs 40 mins

5 a $\dfrac{dr}{dt} = -8\pi r^2$ **b** $r = \dfrac{2}{16\pi t + 1}$

6 a $\dfrac{dh}{dt} = \dfrac{1000}{A}(Q - kh), \ h > 0$

b $t = \dfrac{A}{1000k}\log_e\left(\dfrac{Q - kh_0}{Q - kh}\right), \ Q > kh_0$

c $\dfrac{A}{1000k}\log_e 2$ minutes

7 a i $\dfrac{dh}{dt} = \dfrac{1}{10\pi h^2}$ **ii** $h = \left(\dfrac{3t}{10\pi}\right)^{\frac{1}{3}}$

b $h = \left(1 - \dfrac{1}{5\pi}t^{\frac{3}{2}}\right)^{\frac{1}{3}}$

Exercise 11H

1 a $\dfrac{3}{2}$ **b** $\dfrac{e^4}{2} - \dfrac{e^2}{2} + 3$ **c** $\dfrac{1}{2}\log_e\left(\dfrac{7}{3}\right) + 2$

2 a 1.7443 **b** 1.8309 **c** 4 **d** 3.2556

Exercise 11I

1 a $y_3 \approx 1.2975$ **b** $y_4 \approx 0.0388$

c $y_3 \approx 1.3144$ **d** $y_3 \approx 0.0148$

2 a i 1.8415 **ii** Euler 1.8438

b i 0.5 **ii** Euler 0.5038

c i 2.2190 **ii** Euler 2.2169

d i 0.4055 **ii** Euler 0.4076

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3 a $\tan(1) + 2 \approx 3.5574$

b **i** 3.444969502 **ii** 3.498989223

iii 3.545369041

4 2.205 **5** 30.69

6 1.547 **7** 0.7031

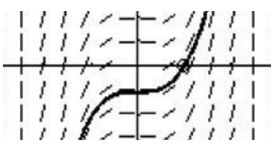
8 a b

	Pr($Z \le z$)	
z	Euler	CAS
0	0.5	0.5
0.1	0.53989423	0.53983
0.2	0.57958948	0.57926
0.3	0.61869375	0.61791
0.4	0.65683253	0.65542
0.5	0.69365955	0.69146
0.6	0.72886608	0.72575
0.7	0.76218854	0.75804
0.8	0.79341393	0.78814
0.9	0.82238309	0.81594
1	0.84899161	0.84134

c **i** 0.69169538 **ii** 0.84212759

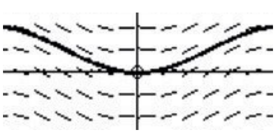
Exercise 11J

1 a



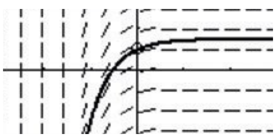
$y = x^3 - 1$

b



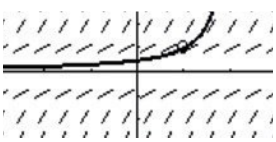
$y = 1 - \cos x$

c



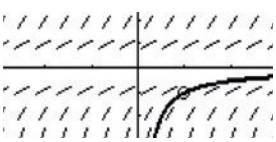
$y = \frac{1}{2}(3 - e^{-2x})$

d



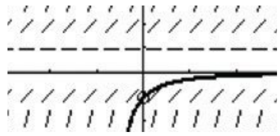
$y = \frac{1}{2 - x}, \ x < 2$

e



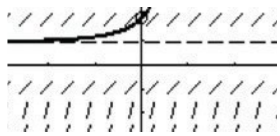
$y = -\frac{1}{x}, \ x > 0$

f



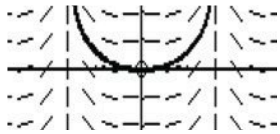
$y = \frac{1}{1 - 2e^x}, \ x > -\log_e 2$

g



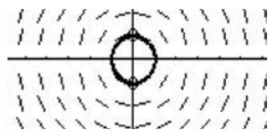
$y = \frac{2}{2 - e^x}, \ x < \log_e 2$

h

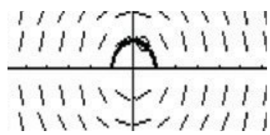


$y = -\log_e(\cos x), \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

2 a



b



Chapter 11 review

Technology-free questions

1 a $y = x - \frac{1}{x} + c$

b $y = e^{10x+c}$

c $y = -\frac{1}{2}\left(\frac{\sin(3t)}{9} + \frac{\cos(2t)}{4}\right) + at + b$

d $y = \frac{e^{-3x}}{9} + e^{-x} + ax + b$

e $y = 3 - e^{-\frac{x}{2}+c}$

f $y = \frac{3x}{2} - \frac{1}{4}x^2 + c$

2 a $y = \frac{1}{2}\sin(2\pi x) - 1$

b $y = \frac{1}{2}\log_e|\sin(2x)|$

c $y = \log_e|x| + \frac{1}{2}x^2 - \frac{1}{2}$

d $y = \frac{1}{2}\log_e(1 + x^2) + 1$

e $y = e^{-\frac{x}{2}}$

f $x = 64 + 4t - 5t^2$

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3 a $k = 2, \; m = -2$

4 a $\frac{2}{\sqrt{3}} - 1$ **b** $\frac{8}{3}$

5 $n = -3, \; n = 5$

6 a $y = 3\tan\left(3x + \arctan\left(\frac{4}{3}\right)\right) - 4$ **b** 5

7 a 0.6826 **b** $y = \frac{3}{2} - \frac{1}{x}$ **c** $\frac{2}{3}$

8 b $y = 2\tan\left(2x - \arctan\left(\frac{1}{2}\right) - 4\right)$

9 a $k = \frac{1}{10}\log_e\left(\frac{5}{4}\right)$ **b** 78.67°C

10 $y = 43 - \frac{2(25 - x^2)^{\frac{3}{2}}}{3}$

11 $k = -1$

12 $\frac{dx}{dt} = \frac{3}{\pi x(12 - x)}$

13 $\frac{dC}{dt} = \frac{8\pi}{C}$

14 $100\log_e 2 \approx 69$ days

15 $\frac{dS}{dt} = -\frac{S}{25}, \quad S = 3e^{\frac{-t}{25}}$

16 a $\theta = 30 - 20e^{\frac{-t}{20}}$ **b** 29°C **c** 14 mins

17 a $\frac{dA}{dt} = 0.02A$ **b** $0.5e^{0.2}$ ha

c $89\frac{1}{2}$ h

18 $x = \frac{2L}{3}$; maximum deflection $= \frac{L^3}{216}$

19 $\frac{dh}{dt} = \frac{6 - 0.15\sqrt{h}}{\pi h^2}$

Multiple-choice questions

1 C **2** D **3** B **4** A **5** E **6** C
7 D **8** E **9** A **10** C **11** A **12** E
13 D **14** E **15** C **16** C

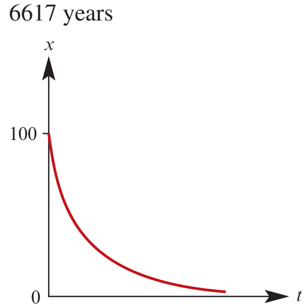
Extended-response questions

1 a i $\frac{dx}{dt} = -kx, \; k > 0$

ii $x = 100e^{\frac{-t\log_e 2}{5760}} = 100 \cdot 2^{\frac{-t}{5760}}, \; t \geq 0$

b 6617 years

c



2 a $\frac{dx}{dt} = \frac{3k}{16}(8 - x)(4 - x)$

b $t = \frac{1}{\log_e(\frac{7}{6})}\log_e\left(\frac{8 - x}{8 - 2x}\right)$

c 2 mins 38 secs **d** $\frac{52}{31}$ kg

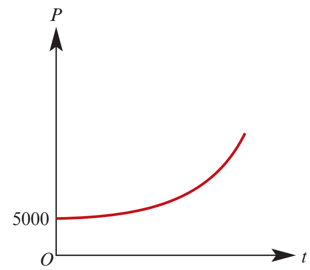
3 a $\frac{dT}{dt} = k(T - T_s), \; k < 0$

b i 19.2 mins **ii** 42.2°C

4 b $t = \frac{1}{k}\log_e\left(\frac{kp - 1000}{5000k - 1000}\right), \; kp > 1000$

c ii 0.22

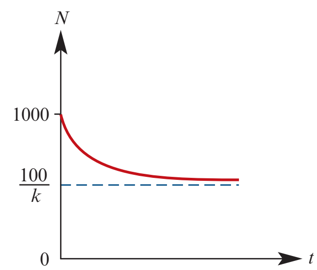
d $p = \frac{1}{k}(e^{kt}(5000k - 1000) + 1000)$



5 a $\frac{dN}{dt} = 100 - kN, \; k > 0$

b $t = \frac{1}{k}\log_e\left(\frac{100 - 1000k}{100 - kN}\right)$ **c** 0.16

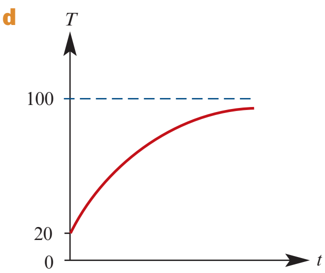
d $N = \frac{1}{k}(100 - e^{kt}(100 - 1000k))$ **e** $\frac{100}{k}$



6 a $\frac{2L}{3}$ **b** $\frac{L}{60}$

7 a $\frac{dT}{dt} = \frac{100 - T}{40}$ **b** $T = 100 - 80e^{\frac{-t}{40}}$

c 62.2°C **d**



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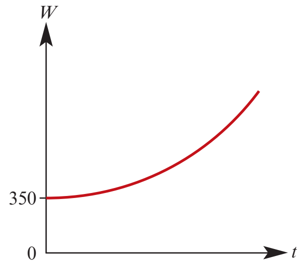
8 a i $t = 25\log_e\left(\dfrac{W}{350}\right),\ W > 0$

ii $W = 350e^{\frac{t}{25}}$

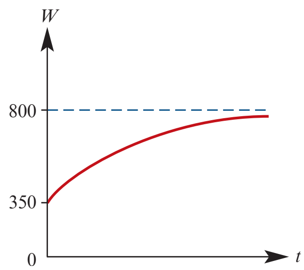


iii 2586

b 0

c i $t = 25\log_e\left(\dfrac{9W}{7(800 - W)}\right),\ 0 < W < 800$

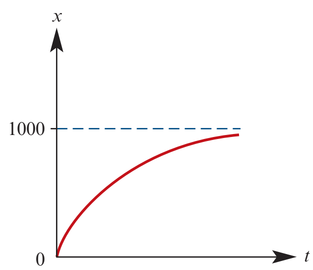
ii $W = \dfrac{5600e^{\frac{t}{25}}}{9 + 7e^{\frac{t}{25}}}$



iii 681

9 a ii $x = \dfrac{R}{k}(1 - e^{-kt})$

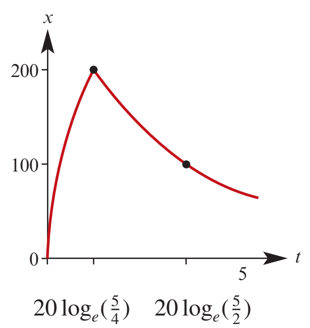
b i



ii 4.46 hours

c i 13.86 hours after drip is disconnected

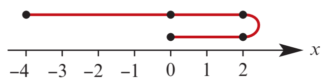
ii $x = \begin{cases} 1000\left(1 - e^{\frac{-t}{20}}\right) & 0 \le t \le 20\log_e\left(\frac{5}{4}\right) \\ 250e^{-\frac{t}{20}} & t > 20\log_e\left(\frac{5}{4}\right) \end{cases}$



Chapter 12

Exercise 12A

1 a $t = 0, x = 0;\ t = 1, x = 2;\ t = 2, x = 2;$
$t = 3, x = 0;\ t = 4, x = -4$

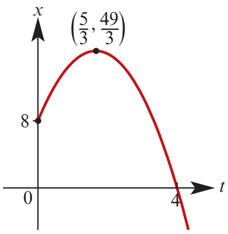


b −6 m **c** −1 m/s **d** $v = 3 - 2t$

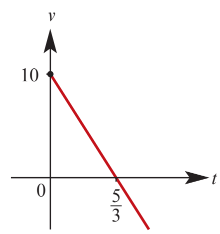
e −2 m/s **f** $x = \dfrac{9}{4},\ t = \dfrac{3}{2}$

g $\dfrac{17}{2}$ m **h** $\dfrac{17}{8}$ m/s

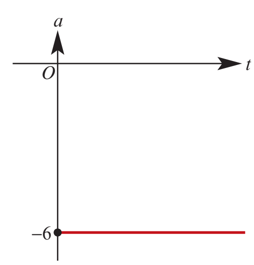
2 a



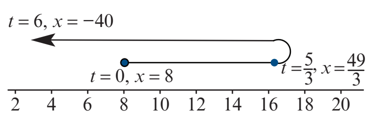
b $v = -6t + 10$



c $a = -6$



d



e −5 m

f $\dfrac{41}{3}$ m

3 a 2, 4 **b** 12 m/s^2 **c** 10 m/s **d** 6 m/s

4 a −3 m/s **b** 1, 3 **c** 12 m/s^2

5 $0, \dfrac{4}{3}$

6 a $\dfrac{25}{4}$ m/s **b** $\dfrac{56}{3}$ m

7 a −30 m/s^2 **b** 4, 6 **c** 4 m

d 120 m

Answers 12A

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8 a 20 m/s **b** 32 m

9 a 42 m/s **b** 6 s **c** 198 m

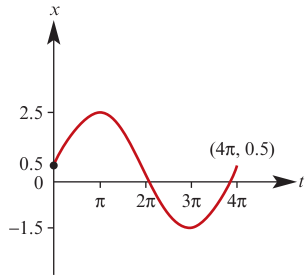
10 a i $v = 9.8t$ **ii** $x = 4.9t^2$

b 19.6 m

c 19.6 m/s

11 a $x = 2\sin\left(\frac{t}{2}\right) + 0.5$

b



Object is stationary at $t = \pi, 3\pi$

c $a = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$

d i $x = -4a + 0.5$

ii $(x - 0.5)^2 = 4 - 4v^2$

iii $v^2 = 1 - 4a^2$

12 a 1 s and 15.5 m; 4 s and 2 m **b** −6.5 m/s

c −6 m/s **d** 9 m **e** 2 m

13 a 9 m/s **b** 2π s

14 a 585.6 m **b** 590.70 m

15 $x = \frac{1}{6}t - \frac{1}{4}\log_e\left(\frac{2t+3}{3}\right)$

16 $\left(\frac{3\sqrt{3}}{2} - \frac{\pi}{3}\right)$ m

17 a 0 m/s **b** $\frac{1}{2}$ m/s **c** $\frac{1}{2}\log_e 2$ m

d $x = \frac{1}{2}\log_e(1 + t^2)$ **e** $\ddot{x} = \frac{1 - t^2}{(1 + t^2)^2}$

f -0.1 m/s^2 **g** $-\frac{1}{8}$ m/s^2

18 5.25 s

19 1.1 s

20 18.14 m/s

Exercise 12B

1 3 m/s^2

2 a 12 960 km/h^2 **b** 1 m/s^2

3 a 3 m/s^2 **b** $\frac{175}{2}$ m **c** $\frac{10(\sqrt{7} - 1)}{3}$ s

4 −5 m/s^2

5 a 12 m **b** 14 m/s **c** 2.5 s **d** 37 m

6 a i 22.4 m **ii** 22.5 m

b i 5 s **ii** −28 m/s

7 a $\frac{10}{7}$ s **b** 10 m **c** $\frac{20}{7}$ s

8 a 200 s **b** 2 km

9 a $\frac{10\sqrt{10}}{7}$ s **b** $14\sqrt{10}$ m/s

10 a 4.37 s **b** $-6\sqrt{30}$ m/s

11 a 1.25 s **b** 62.5 cm

12 a $a = 0.23$ **b** $5\frac{1}{3}$ s

13 −0.64 m/s^2

14 a 4 s **b** $\frac{1}{2}$ m/s^2

Exercise 12C

1 a 60 m **b** 20 m **c** 30 m **d** 55 m

e 44 m **f** $\frac{70}{3}$ m **g** $\frac{165}{2}$ m **h** $\frac{49}{2}$ m

2 a $v = -\frac{1}{2}t + 5;\ a = -\frac{1}{2};\ x = -\frac{t^2}{4} + 5t$

b $v = -\frac{2}{5}t^2 + 10;\ a = -\frac{4}{5}t;\ x = -\frac{2}{15}t^3 + 10t$

c $v = 2t - 10;\ a = 2;\ x = t^2 - 10t$

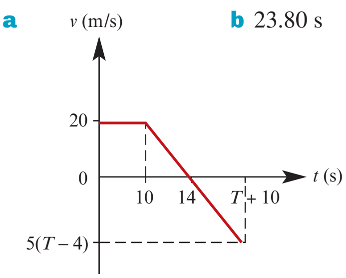
d $v = 6(t-1)(t-5);\ a = 12(t-3);$
$x = 2(t^3 - 9t^2 + 15t)$

e $v = 10\sin\left(\frac{\pi}{10}t\right) + 10;\ a = \pi\cos\left(\frac{\pi}{10}t\right);$
$x = 10\left(t + \frac{10}{\pi} - \frac{10}{\pi}\cos\left(\frac{\pi}{10}t\right)\right)$

f $v = 10e^{2t};\ a = 20e^{2t};\ x = 5e^{2t} - 5$

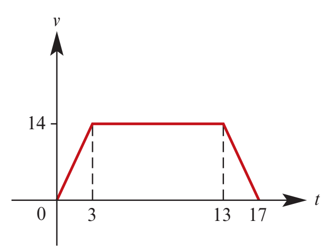
3 3589.89 m

4 a

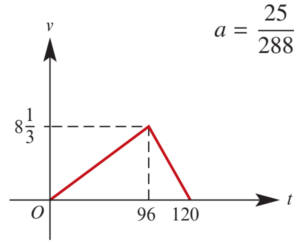


b 23.80 s

5 189 m



6 $a = \frac{25}{288}$, $\dot{x}_{max} = 8\frac{1}{3}$ m/s



7 $68\frac{1}{3}$ s

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8 10 s, 150 m

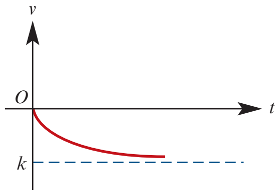
9 $10(3+\sqrt{3})$ s, $200(2+\sqrt{3})$ m

10 a 2 s **b** $7\frac{1}{3}$ m

11 36 s

12 a 3600 m, 80 km/h
b 90 s after A passed B, 200 m

13 a $\dot{y} = k(1 - e^{-t})$



b Limiting velocity of k m/s

Exercise 12D

1 a 7 m **b** 4 m

2 a $x = \frac{1}{2}\log_e(2e^{2t} - 1)$ **b** $-\frac{100}{2401}$ m/s^2

3 a $v = 3(e^t - 1)$ **b** $a = 3e^t$
c $x = 3(e^t - t - 1)$

4 a $v = \frac{g}{k}(1 - e^{-kt})$ **b** $\frac{g}{k}$

5 a $v = \tan\left(\frac{\pi}{3} - \frac{3t}{10}\right)$
b $x = \frac{10}{3}\log_e\left(2\cos\left(\frac{\pi}{3} - \frac{3t}{10}\right)\right)$

6 $v = 450\left(1 - e^{\frac{-t}{50}}\right)$

7 $v = 15\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \frac{2t}{5}\right)$

8 a $x = 5e^{\frac{2t}{5}}$ **b** 273 m

9 a $t = 50\log_e\left(\frac{500}{500 - v}\right)$
b $v = 500\left(1 - e^{\frac{-t}{50}}\right)$

10 $\frac{1}{k}\log_e 2$

11 $v = 8e^{\frac{-t}{5}}$; 3.59 m/s

12 a $v = \frac{90}{2t + 3}$ **b** 91.66 m

Exercise 12E

1 -2 m/s^2

2 a $v = \pm 4$ **b** $t = -\log_e 2$
c $x = 2(1 - \log_e 2)$

3 a $v = \frac{1}{x + 1}$
b i $x = e^t - 1$ **ii** $a = e^t$ **iii** $a = v$

4 $x = -\frac{5}{2}\log_e\left(\frac{g + 0.2v^2}{g + 2000}\right)$;
$x_{\max} = \frac{5}{2}\log_e\left(\frac{g + 2000}{g}\right)$

5 a $x = \cos(2t)$ **b** $a = -4x$

6 a $v = \log_e(1 + t)$ **b** $v^2 = 2\log_e(1 + x)$
c $v = \sqrt{2t + 1} - 1$

7 $v^2 = \frac{x}{2 + x}$

8 a 4 **b** $2\log_e 2 - 1$

9 a 9.83 m **b** 1.01 s

Chapter 12 review

Technology-free questions

1 a After 3.5 seconds
b 2 m/s^2
c 14.5 m
d When $t = 2.5$ s and the particle is 1.25 m to the left of O

2 $x = 215\frac{1}{3}$, $v = 73$

3 a 57.6 km/h
b After 1 minute $6\frac{2}{3}$ seconds **c** $a = 0.24$

4 a $\frac{25\,000}{3}$ m/s^2 **b** 0.4125 m
c 10 000 m/s^2 **d** 0.5 m
e 37 500 m/s^2 **f** 0.075 m

5 a 44 m/s **b** $v = 55 - 11t$ m/s **c** 44 m/s
d 5 s **e** 247.5 m

6 16 m

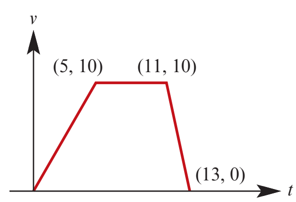
7 a 2 s **b** $v = \frac{-t}{\sqrt{9 - t^2}}$, $a = \frac{-9}{(9 - t^2)^{\frac{3}{2}}}$
c 3 m **d** $t = 0$

8 a 20 m/s **b** 32 m

9 a $x = 20$ **b** $\frac{109}{8}$ m/s

10 a i $v = 35 - 3g$ up **ii** $v = 5g - 35$ down
b $\frac{35^2}{g}$ m
c -35 m/s

11 Distance = 95 m



12 $v = \frac{4}{t - 1}$, $a = -\frac{4}{(t - 1)^2}$

13 a $80 + 0.4g$ m/s **b** $\frac{80 + 0.4g}{g}$ s
c $\frac{(80 + 0.4g)^2}{2g}$ m **d** $\frac{2(80 - 0.4g)}{g}$ s

Multiple-choice questions

1 A **2** C **3** A **4** D **5** B
6 C **7** C **8** C **9** A **10** E

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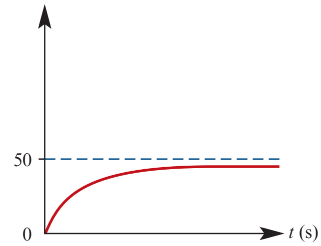
Extended-response questions

1 a $10\ \text{m/s}^2$

b $v = 50\left(1 - e^{\frac{-t}{5}}\right)$

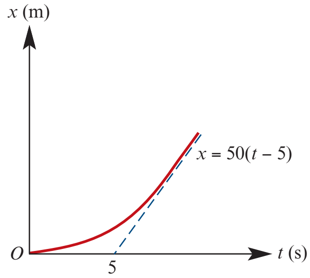
c i



ii 14.98

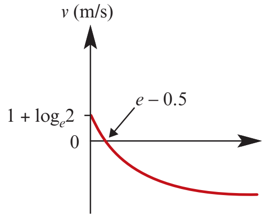
d i $x = 50\left(t + 5e^{\frac{-t}{5}} - 5\right)$

ii



iii 1.32 s

2 a i



ii 1.27 m **iii** 1.47 m

b $B = 10,\ A = 4.70$

3 a 30 minutes

b i $a = -k(\sin(\pi t) + \pi t\cos(\pi t) - 1)$

ii From 0 h to 0.18 h

c $k = 845$

4 a i $v = 4 - 10t - 3t^2$ **ii** $a = -10 - 6t$

iii $t = 0.36$ **iv** $t = 0$ or $t = 0.70$

v $t = 2.92$

b i $x = t^2 - t^3 + 2t$ **ii** $\frac{7}{3}$ s **iii** Yes

5 a i $v = -\frac{5\pi}{4}\sin\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

ii $a = -\frac{5\pi^2}{16}\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

b i $v = \pm\frac{\pi}{4}\sqrt{25 - x^2}$ **ii** $a = -\frac{\pi^2 x}{16}$

c 3.4 cm/s

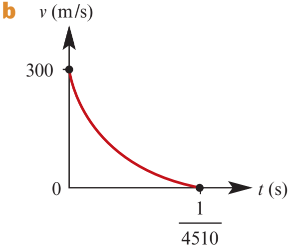
d $-1.54\ \text{cm/s}^2$

e i 5 cm **ii** $\frac{5\pi}{4}$ cm/s **iii** $\frac{5\pi^2}{16}\ \text{cm/s}^2$

6 0 m

7 a $v = \frac{300(1 - 4510t)}{12\,300t + 1},\ 0 \le t \le \frac{1}{4510}$

b



c i $x = -110t + \frac{1}{30}\log_e(12\,300t + 1)$

ii $x = \frac{1}{30}\left(\log_e\left(\frac{410}{v + 110}\right) - \frac{110}{v + 110} + \frac{11}{41}\right)$

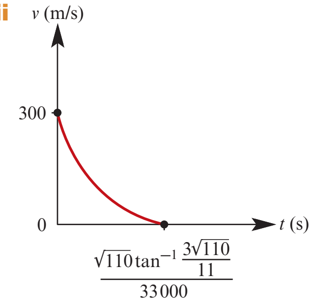
iii 19 mm

d i $t = \frac{\sqrt{110}}{33\,000} \times \left(\tan^{-1}\left(\frac{3\sqrt{110}}{11}\right) - \tan^{-1}\left(\frac{v\sqrt{110}}{1100}\right)\right)$

ii $v = 10\sqrt{110} \times \tan\left(\tan^{-1}\left(\frac{3\sqrt{110}}{11}\right) - 300\sqrt{110}t\right),$

for $0 \le t \le \dfrac{\sqrt{110}\tan^{-1}\left(\frac{3\sqrt{110}}{11}\right)}{33\,000}$

iii



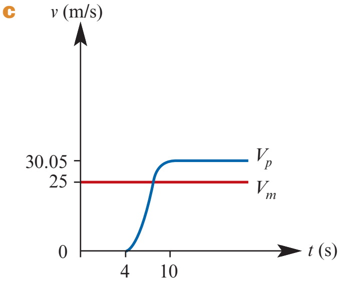
iv 20 mm

8 a $v_1 = 30.05$

b i $\frac{dv}{dt} = \frac{-3}{10}\left(3t^2 - 42t + \frac{364}{3}\right),\ 4 \le t \le 10$

ii $t = 7$ (Chasing for 3 s)

c



d i 90.3 m

ii $x_p = -\frac{3}{40}t^4 + \frac{21}{10}t^3 - \frac{91}{5}t^2 - \frac{1281}{20}t - \frac{401}{5}$, for $t \in [4, 10]$

e 41.62 s

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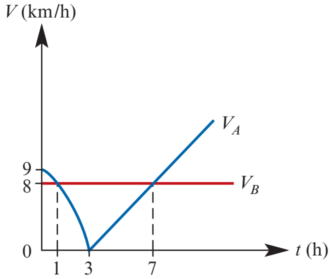
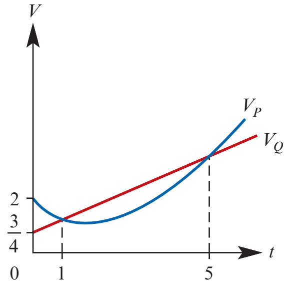
9 a 

b $t = 1$ or $t = 7$

c i 11.7 h **ii** 1.7 h

10 a i $t = 1$ or $t = 5$

ii 

b i 2.2 **ii** $0 < t < 2.2,\ t > 6.8$

11 a i 4.85 m/s **ii** 0.49 s

b i $v = 9.8t - \frac{1}{2}t^2$ **ii** $x = 4.9t^2 - \frac{1}{6}t^3$

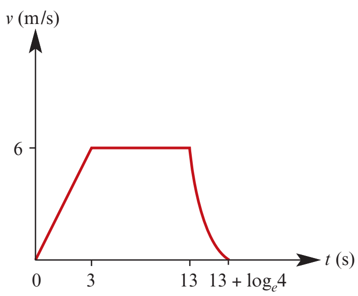
iii 0.50 s

c i $x = 1.2 - 2.45t^2$ **ii** 6 cm

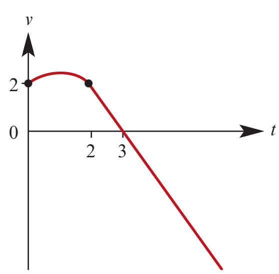
12 a 3 s

b $v = \begin{cases} 2t, & 0 \le t \le 3 \\ 6, & 3 < t \le 13 \\ 8e^{13-t} - 2, & 13 < t \le 13 + \log_e 4 \end{cases}$

c 14.4 s

d 

e 72.2 m

13 a 

b $\frac{19}{3}$

c $T = 5.52$

Chapter 13

Exercise 13A

1 a $y = 2x$; dom $= \mathbb{R}$; ran $= \mathbb{R}$

b $x = 2$; dom $= \{2\}$; ran $= \mathbb{R}$

c $y = 7$; dom $= \mathbb{R}$; ran $= \{7\}$

d $y = 9 - x$; dom $= \mathbb{R}$; ran $= \mathbb{R}$

e $x = \frac{1}{9}(2 - y)^2$; dom $= [0, \infty)$; ran $= \mathbb{R}$

f $y = (x + 3)^3 + 1$; dom $= \mathbb{R}$; ran $= \mathbb{R}$

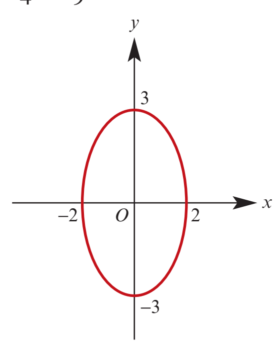
g $y = 3^{\frac{x-1}{2}}$; dom $= \mathbb{R}$; ran $= (0, \infty)$

h $y = \cos(2x + \pi) = -\cos(2x)$;
dom $= \mathbb{R}$; ran $= [-1, 1]$

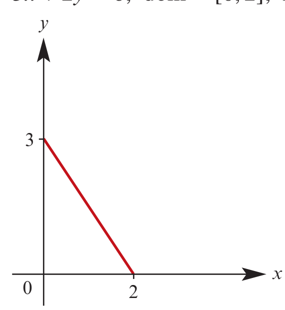
i $y = \left(\frac{1}{x} - 4\right)^2 + 1$;
dom $= \mathbb{R} \setminus \{0\}$; ran $= [1, \infty)$

j $y = \frac{x}{1 + x}$;
dom $= \mathbb{R} \setminus \{-1, 0\}$; ran $= \mathbb{R} \setminus \{0, 1\}$

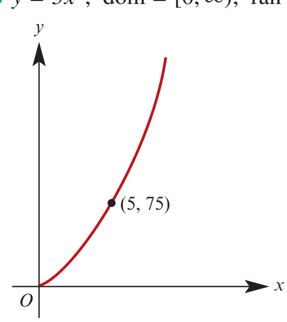
2 a $\frac{x^2}{4} + \frac{y^2}{9} = 1$; dom $= [-2, 2]$; ran $= [-3, 3]$



b $3x + 2y = 6$; dom $= [0, 2]$; ran $= [0, 3]$



c $y = 3x^2$; dom $= [0, \infty)$; ran $= [0, \infty)$



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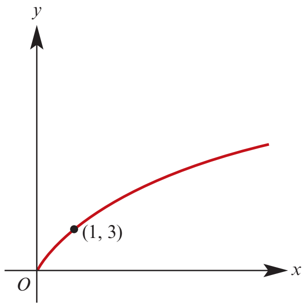
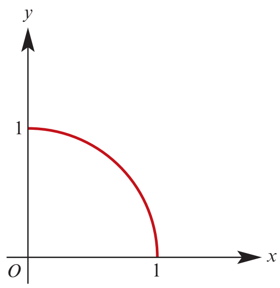
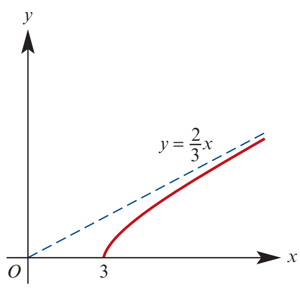
d $y = 3x^{\frac{2}{3}}$; dom = $[0, \infty)$; ran = $[0, \infty)$



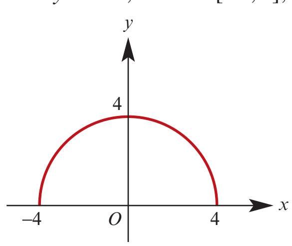
e $x^2 + y^2 = 1$; dom = $[0, 1]$; ran = $[0, 1]$



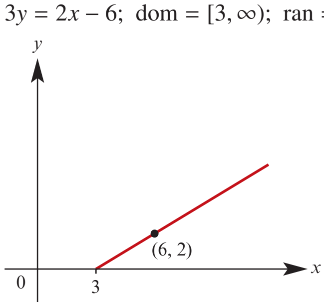
f $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$; dom = $(3, \infty)$; ran = $(0, \infty)$



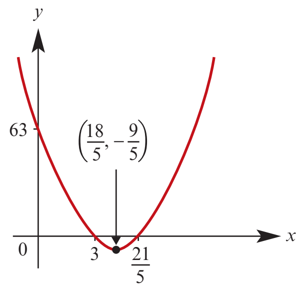
g $x^2 + y^2 = 16$; dom = $[-4, 4]$; ran = $[0, 4]$



h $3y = 2x - 6$; dom = $[3, \infty)$; ran = $[0, \infty)$



i $y = 5x^2 - 36x + 63$;
dom = $\mathbb{R}$; ran = $[-\frac{9}{5}, \infty)$

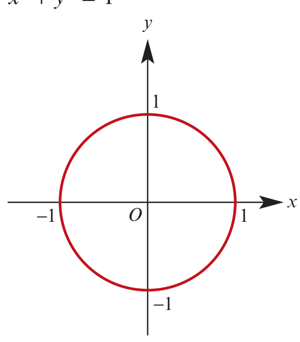


3 **a** $\boldsymbol{r}(t) = t\,\boldsymbol{i} + (3 - 2t)\boldsymbol{j},\ t \in \mathbb{R}$
b $\boldsymbol{r}(t) = 2\cos t\,\boldsymbol{i} + 2\sin t\,\boldsymbol{j},\ t \in \mathbb{R}$
c $\boldsymbol{r}(t) = (2\cos t + 1)\boldsymbol{i} + 2\sin t\,\boldsymbol{j},\ t \in \mathbb{R}$
d $\boldsymbol{r}(t) = 2\sec t\,\boldsymbol{i} + 2\tan t\,\boldsymbol{j},\ t \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
e $\boldsymbol{r}(t) = t\,\boldsymbol{i} + ((t-3)^2 + 2(t-3))\boldsymbol{j},\ t \in \mathbb{R}$
f $\boldsymbol{r}(t) = \sqrt{6}\cos t\,\boldsymbol{i} + 2\sin t\,\boldsymbol{j},\ t \in \mathbb{R}$

4 **a** $\boldsymbol{r}(\theta) = (2 + 5\cos\theta)\boldsymbol{i} + (6 + 5\sin\theta)\boldsymbol{j}$
b $(x-2)^2 + (y-6)^2 = 25$

Exercise 13B

1 **a** $x^2 + y^2 = 1$
b

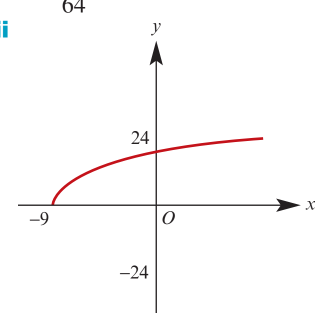


c $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

i.e. $t = \dfrac{(2n-1)\pi}{2},\ n \in \mathbb{N}$

2 **a** **i** $x = \dfrac{y^2}{64} - 9$

ii



iii $t = 3$

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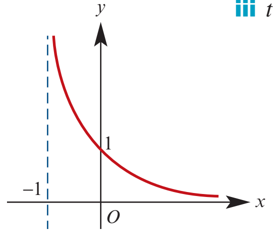
b **i** $y = \dfrac{1}{1+x}, \quad x > -1$

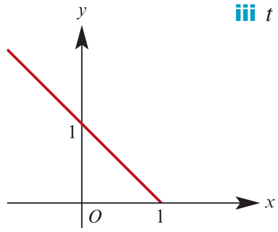
ii



iii $t = -1$

c **i** $y = 1 - x, \quad x < 1$

ii



iii $t = 1$

3 **a** Position vector $\boldsymbol{i} + 4\boldsymbol{j}$; coordinates $(1, 4)$
b $(1, 4), (7, -8)$ **c** $\sqrt{65}$

4 **a** $\dfrac{9}{2}\boldsymbol{i} - \dfrac{3}{2}\boldsymbol{j}, \left(\dfrac{9}{2}, -\dfrac{3}{2}\right)$ **b** $(6, -1), \left(\dfrac{9}{2}, -\dfrac{3}{2}\right)$
c $5\sqrt{2}$

5 **a** $\sqrt{137}$ **b** $t = -\dfrac{2}{5}, -1$

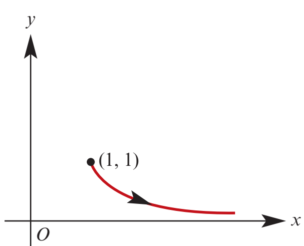
6 **a** $3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$ **b** $3\sqrt{6}$
c $4\boldsymbol{i} + 8\boldsymbol{j} - 3\boldsymbol{k}$ **d** $\boldsymbol{i} + 2\boldsymbol{j}$

7 **a** $3\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$ **b** $\sqrt{14}$

8 $a = \dfrac{2}{3}, \quad b = 7$

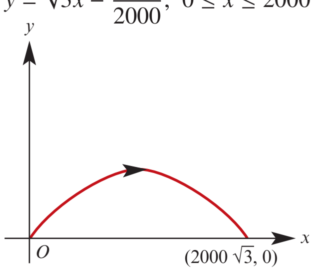
9 **a** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **b** $3\boldsymbol{i}$
c **i** 303.69° **ii** 285.44°

10 **a** $y = \dfrac{1}{x}$, for $x \geq 1$ **b** $\boldsymbol{i} + \boldsymbol{j}$
c



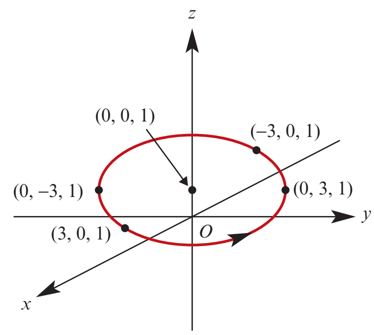
11 **a** $\boldsymbol{r}(0) = 2\boldsymbol{i}$ **b** $\dfrac{5}{2}\boldsymbol{i} + \dfrac{3}{2}\boldsymbol{j}$ **c** $x^2 - y^2 = 4$

12 **a** $\boldsymbol{r}(0) = \boldsymbol{0}, \quad \boldsymbol{r}(20\sqrt{3}) = 2000\sqrt{3}\boldsymbol{i}$
b $y = \sqrt{3}x - \dfrac{x^2}{2000}, \quad 0 \leq x \leq 2000\sqrt{3}$
c

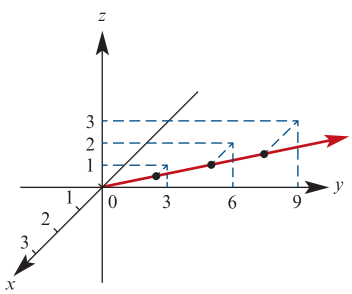


13 Collide when $t = \dfrac{3}{2}$; $\boldsymbol{r}\left(\dfrac{3}{2}\right) = \dfrac{27}{2}\boldsymbol{i} - \dfrac{81}{4}\boldsymbol{j}$

14 Particle is moving along a circular path, with centre $(0, 0, 1)$ and radius 3, starting at $(3, 0, 1)$ and moving anticlockwise; always a distance of 1 above the x–y plane. It takes 2π units of time to complete one circle.

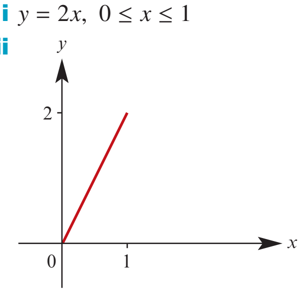


15 Particle is moving along a straight line, starting at $(0, 0, 0)$, and moving 'forward 1', 'across 3' and 'up 1' at each step.



16 **a** $\dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{25} = 1$
b **i** $(-1, 3)$ **ii** $(1, -2)$ **iii** $(3, 3)$
c π units of time
d Anticlockwise

17 **a** **i** $y = 2x, \quad 0 \leq x \leq 1$
ii



iii Particle starts at $(1, 2)$ and moves along a linear path towards the origin. When it reaches $(0, 0)$, it reverses direction and heads towards $(1, 2)$. It continues in this pattern, taking $\frac{1}{3}$ units of time to complete each cycle.

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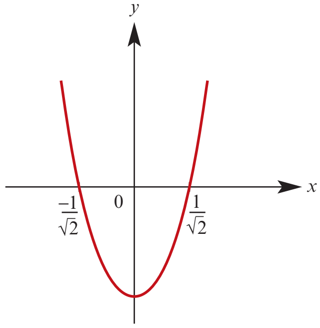
b **i** $y = 2x^2 - 1, \ -1 \le x \le 1$

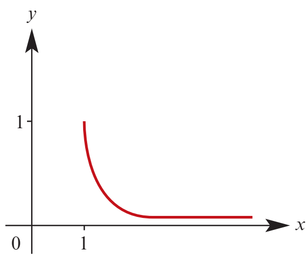
ii



iii Particle is moving along a parabolic path, starting at (1, 1) and reversing direction at (−1, 1). It takes 1 unit of time for each cycle.

c **i** $y = \dfrac{1}{x^2}, \ x \ge 1$

ii



iii Particle is moving along a 'truncus' path, starting at (1, 1) and moving to the 'right' indefinitely.

Exercise 13C

1 **a** $\dot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} - e^{-t}\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$

b $\dot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2t\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = 2\boldsymbol{j}$

c $\dot{\boldsymbol{r}}(t) = \frac{1}{2}\boldsymbol{i} + 2t\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = 2\boldsymbol{j}$

d $\dot{\boldsymbol{r}}(t) = 16\boldsymbol{i} - 32(4t - 1)\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = -128\boldsymbol{j}$

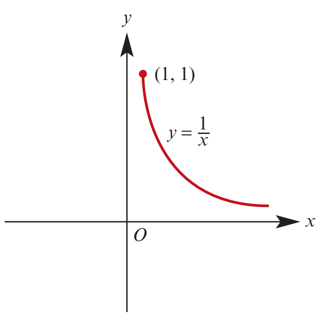
e $\dot{\boldsymbol{r}}(t) = \cos t\,\boldsymbol{i} - \sin t\,\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = -\sin t\,\boldsymbol{i} - \cos t\,\boldsymbol{j}$

f $\dot{\boldsymbol{r}}(t) = 2\boldsymbol{i} + 5\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = \boldsymbol{0}$

g $\dot{\boldsymbol{r}}(t) = 100\boldsymbol{i} + (100\sqrt{3} - 9.8t)\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$

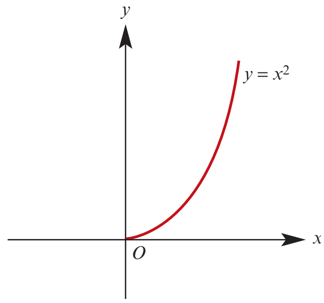
h $\dot{\boldsymbol{r}}(t) = \sec^2 t\,\boldsymbol{i} - \sin(2t)\,\boldsymbol{j},$
$\ddot{\boldsymbol{r}}(t) = (2\sec^2 t\,\tan t)\boldsymbol{i} - 2\cos(2t)\boldsymbol{j}$

2 **a** $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$



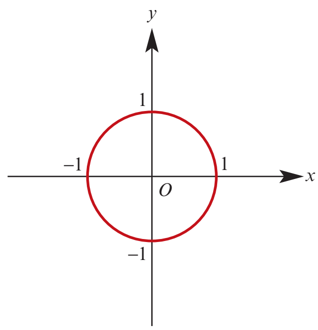
$\boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{j}, \quad \dot{\boldsymbol{r}}(0) = \boldsymbol{i} - \boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(0) = \boldsymbol{i} + \boldsymbol{j}$

b $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\boldsymbol{j}$



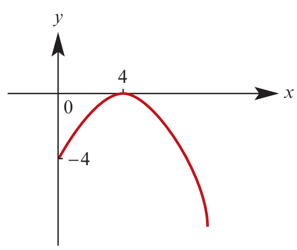
$\boldsymbol{r}(1) = \boldsymbol{i} + \boldsymbol{j}, \quad \dot{\boldsymbol{r}}(1) = \boldsymbol{i} + 2\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(1) = 2\boldsymbol{j}$

c $\boldsymbol{r}(t) = \sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j}$



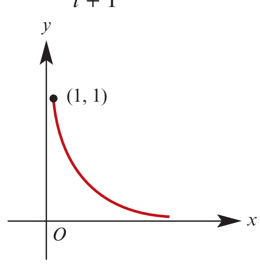
$\boldsymbol{r}\left(\frac{\pi}{6}\right) = \frac{1}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}, \quad \dot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j},$
$\ddot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = -\frac{1}{2}\boldsymbol{i} - \frac{\sqrt{3}}{2}\boldsymbol{j}$

d $\boldsymbol{r}(t) = 16t\,\boldsymbol{i} - 4(4t - 1)^2\boldsymbol{j}$



$\boldsymbol{r}(1) = 16\boldsymbol{i} - 36\boldsymbol{j}, \quad \dot{\boldsymbol{r}}(1) = 16\boldsymbol{i} - 96\boldsymbol{j},$
$\ddot{\boldsymbol{r}}(1) = -128\boldsymbol{j}$

e $\boldsymbol{r}(t) = \dfrac{1}{t+1}\boldsymbol{i} + (t + 1)^2\boldsymbol{j}$



$\boldsymbol{r}(1) = \frac{1}{2}\boldsymbol{i} + 4\boldsymbol{j}, \quad \dot{\boldsymbol{r}}(1) = -\frac{1}{4}\boldsymbol{i} + 4\boldsymbol{j},$

$\ddot{\boldsymbol{r}}(1) = \frac{1}{4}\boldsymbol{i} + 2\boldsymbol{j}$

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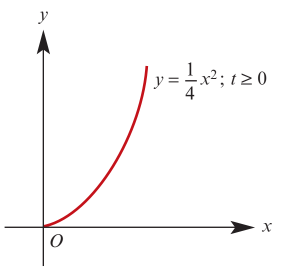
3 a -1 **b** Undefined **c** $-2e^{-3}$
d $\frac{1}{2}$ **e** 4 **f** $2\sqrt{2}$

4 a $\boldsymbol{r}(t) = (4t+1)\boldsymbol{i} + (3t-1)\boldsymbol{j}$
b $\boldsymbol{r}(t) = (t^2+1)\boldsymbol{i} + (2t-1)\boldsymbol{j} - t^3\boldsymbol{k}$
c $\boldsymbol{r}(t) = \frac{1}{2}e^{2t}\boldsymbol{i} + 4(e^{0.5t}-1)\boldsymbol{j}$
d $\boldsymbol{r}(t) = \left(\frac{t^2+2t}{2}\right)\boldsymbol{i} + \frac{1}{3}t^3\boldsymbol{j}$
e $\boldsymbol{r}(t) = -\frac{1}{4}\sin(2t)\,\boldsymbol{i} + 4\cos\left(\frac{1}{2}t\right)\boldsymbol{j}$

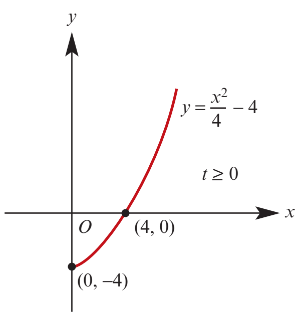
6 a $t = 0, 2$
b $\dot{\boldsymbol{r}}(0) = 2\boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(0) = 96\boldsymbol{j}$; $\dot{\boldsymbol{r}}(2) = 2\boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(2) = -96\boldsymbol{j}$

7 a



b $t = \frac{2}{a}$

8 a



b $45°$ **c** $t = \sqrt{3}$

9 a $\dot{\boldsymbol{r}} = 3\boldsymbol{i} + t^2\boldsymbol{j} + 3t^2\boldsymbol{k}$ **b** $|\dot{\boldsymbol{r}}| = \sqrt{9+10t^4}$
c $\ddot{\boldsymbol{r}} = 2t\,\boldsymbol{j} + 6t\boldsymbol{k}$ **d** $|\ddot{\boldsymbol{r}}| = 2t\sqrt{10}$
e $t = \frac{4\sqrt{10}}{5}$

10 a $\dot{\boldsymbol{r}} = V\cos\alpha\,\boldsymbol{i} + (V\sin\alpha - gt)\boldsymbol{j}$ **b** $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$
c $t = \frac{V\sin\alpha}{g}$
d $\boldsymbol{r} = \frac{V^2\sin(2\alpha)}{2g}\boldsymbol{i} + \frac{V^2\sin^2\alpha}{2g}\boldsymbol{j}$

Exercise 13D

1 a $2t\,\boldsymbol{i} - 2\boldsymbol{j}$ **b** $2\boldsymbol{i}$ **c** $2\boldsymbol{i} - 2\boldsymbol{j}$

2 a $2\boldsymbol{i} + (6-9.8t)\boldsymbol{j}$
b $2t\,\boldsymbol{i} + (6t - 4.9t^2 + 6)\boldsymbol{j}$

3 a $2\boldsymbol{j} - 4\boldsymbol{k}$
b $3t\,\boldsymbol{i} + (t^2+1)\boldsymbol{j} + (t - 2t^2 + 1)\boldsymbol{k}$
c $\sqrt{20t^2 - 8t + 10}$
d i $t = \frac{1}{5}$ **ii** $\frac{1}{5}\sqrt{230}$ m/s

4 a $(10t+20)\boldsymbol{i} - 20\boldsymbol{j} + (40 - 9.8t)\boldsymbol{k}$
b $(5t^2+20t)\boldsymbol{i} - 20t\boldsymbol{j} + (40t - 4.9t^2)\boldsymbol{k}$

5 Speed $= 10t$

6 $45°$

7 Minimum speed $= 3\sqrt{2}$; position $= 24\boldsymbol{i} + 8\boldsymbol{j}$

8 a $t = 61\frac{11}{49}$ s **b** 500 m/s **c** $\frac{225\,000}{49}$ m
d 500 m/s **e** $\theta = 36.87°$

9 a $\boldsymbol{r}(t) = (\frac{1}{3}\sin(3t) - 3)\boldsymbol{i} + (\frac{1}{3}\cos(3t) + \frac{8}{3})\boldsymbol{j}$
b $(x+3)^2 + (y - \frac{8}{3})^2 = \frac{1}{9}$; centre $(-3, \frac{8}{3})$

10 Max speed $= 2\sqrt{5}$; min speed $= 2\sqrt{2}$

11 a Magnitude $\frac{\sqrt{11\,667}}{9}$ m/s^2;
direction $\frac{1}{\sqrt{11\,667}}(108\boldsymbol{i} - \sqrt{3}\boldsymbol{j})$
b $\boldsymbol{r}(t) = (\frac{4}{3}t^3 + 2t^2 + t)\boldsymbol{i} + (\sqrt{2t+1} - 1)\boldsymbol{j}$

12 a $\boldsymbol{r}(t) = V\cos(\alpha)\,t\,\boldsymbol{i} + \left(V\sin(\alpha)\,t - \frac{gt^2}{2}\right)\boldsymbol{j}$

13 a $t = 6$ **b** $7\boldsymbol{i} + 12\boldsymbol{j}$

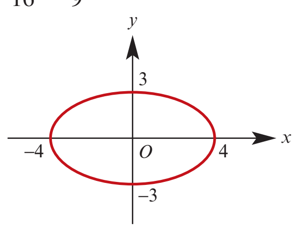
14 a $-16\boldsymbol{i} + 12\boldsymbol{j}$ **b** $-80\boldsymbol{i} + 60\boldsymbol{j}$

15 a $8\cos(2t)\,\boldsymbol{i} - 8\sin(2t)\,\boldsymbol{j},\ t \geq 0$
b 8 **c** $-4\boldsymbol{r}$

16 a $(t^2 - 5t - 2)\boldsymbol{i} + 2\boldsymbol{j}$ **b** $-\frac{33}{4}\boldsymbol{i} + 2\boldsymbol{j}$
c $y = 2$ for $x \geq -8.25$

17 a $\frac{x^2}{36} - \frac{y^2}{16} = 1$
b $6\tan(t)\sec(t)\,\boldsymbol{i} + 4\sec^2(t)\,\boldsymbol{j},\ t \geq 0$

18 a $\frac{x^2}{16} + \frac{y^2}{9} = 1$



b i $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$
ii $\boldsymbol{r}(0) = 4\boldsymbol{i}$, $\boldsymbol{r}\left(\frac{\pi}{2}\right) = 3\boldsymbol{j}$, $\boldsymbol{r}(\pi) = -4\boldsymbol{i}$, $\boldsymbol{r}\left(\frac{3\pi}{2}\right) = -3\boldsymbol{j}$, $\boldsymbol{r}(2\pi) = 4\boldsymbol{i}$
c i $\sqrt{9 + 7\sin^2 t}$ **ii** $\sqrt{16 - 7\cos^2 t}$
iii Max speed $= 4$; min speed $= 3$

19 $2\sqrt{37}$

20 $\frac{\pi}{2}$

21 a 6.086 **b** $\sqrt{37} \approx 6.083$

22 a 2.514 **b** 2.423

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Chapter 13 review

Technology-free questions

1 a $2\boldsymbol{i} + 4\boldsymbol{j},\ 2\boldsymbol{j}$
b $4y = x^2 - 16$

2 a $\dot{\boldsymbol{r}}(t) = 4t\,\boldsymbol{i} + 4\boldsymbol{j}, \quad \ddot{\boldsymbol{r}}(t) = 4\boldsymbol{i}$
b $\dot{\boldsymbol{r}}(t) = 4\cos t\,\boldsymbol{i} - 4\sin t\,\boldsymbol{j} + 2t\boldsymbol{k}$,
$\ddot{\boldsymbol{r}}(t) = -4\sin t\,\boldsymbol{i} - 4\cos t\,\boldsymbol{j} + 2\boldsymbol{k}$

3 $0.6\boldsymbol{i} + 0.8\boldsymbol{j}$

4 a $5\sqrt{3}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$ **b** $\frac{2\sqrt{7}}{7}$

5 $\cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j}$

6 a $5(-\sin t\,\boldsymbol{i} + \cos t\,\boldsymbol{j})$ **b** 5
c $-5(\cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j})$
d 0, acceleration perpendicular to velocity

7 $\frac{3\pi}{4}$ s

8 a $|\dot{\boldsymbol{r}}| = 1,\ |\ddot{\boldsymbol{r}}| = 1$
b $(x-1)^2 + (y-1)^2 = 1$ **c** $\frac{3\pi}{4}$

9 $-2\boldsymbol{i} + 20\boldsymbol{j}$

10 a $\boldsymbol{r} = \left(\frac{t^2}{2} + 1\right)\boldsymbol{i} + (t-2)\boldsymbol{j}$ **b** $(13.5, 3)$
c 12.5 s

11 a $\dot{\boldsymbol{r}} = t\,\boldsymbol{i} + (2t-5)\boldsymbol{j}$
b $\boldsymbol{r} = \left(\frac{t^2}{2} - 1\right)\boldsymbol{i} + (t^2 - 5t + 6)\boldsymbol{j}$
c $-\boldsymbol{i} + 6\boldsymbol{j},\quad -5\boldsymbol{j}$

12 a i $\dot{\boldsymbol{r}}_2(t) = (2t-4)\boldsymbol{i} + t\boldsymbol{j}$
ii $\dot{\boldsymbol{r}}_1(t) = t\,\boldsymbol{i} + (k-t)\boldsymbol{j}$
b i 4 **ii** 8 **iii** $4(\boldsymbol{i} + \boldsymbol{j})$

13 b i $\dot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + 8e^{2t}\boldsymbol{j}$ **ii** $\boldsymbol{i} + 8\boldsymbol{j}$
iii $\log_e 1.5$

14 b i $x = 2$ for $y \geq -3.5$ **ii** $(2, -3.5)$

Multiple-choice questions

1 E	**2** E	**3** B	**4** E	**5** C
6 C	**7** C	**8** E	**9** C	**10** E

Extended-response questions

1 a Speed of P is $3\sqrt{13}$ m/s;
speed of Q is $\sqrt{41}$ m/s
b i Position of P is $60\boldsymbol{i} + 20\boldsymbol{j}$;
position of Q is $80\boldsymbol{i} + 80\boldsymbol{j}$
ii $\overrightarrow{PQ} = (20 - 4t)\boldsymbol{i} + (60 - 2t)\boldsymbol{j}$
c 10 seconds, $20\sqrt{5}$ metres

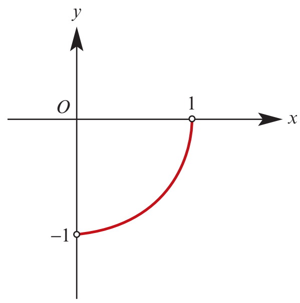
2 a $\overrightarrow{AB} = ((v+3)t - 56)\boldsymbol{i} + ((7v - 29)t + 8)\boldsymbol{j}$
b 4
c i $\overrightarrow{AB} = (6t - 56)\boldsymbol{i} + (8 - 8t)\boldsymbol{j}$
ii 4 seconds

3 a $\overrightarrow{BF} = -3\boldsymbol{i} + 6\boldsymbol{j} - 6\boldsymbol{k}$
b 9 m **c** 3 m/s
d $(-\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$ m/s
e 2 seconds, $2\sqrt{26}$ metres

4 a i 200 s **ii** $\frac{1}{2}$ **iii** 5 m/s **iv** $(1200, 0)$
b 8 seconds, 720 metres

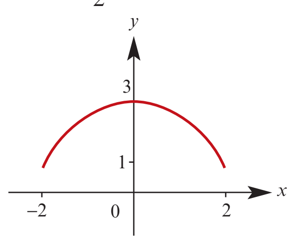
5 a i $\overrightarrow{OA} = (6t - 1)\boldsymbol{i} + (3t + 2)\boldsymbol{j}$
ii $\overrightarrow{BA} = (6t - 3)\boldsymbol{i} + (3t + 1)\boldsymbol{j}$
b 1 second
c i $\boldsymbol{c} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ **ii** $\boldsymbol{d} = \frac{1}{5}(4\boldsymbol{i} - 3\boldsymbol{j})$
iii $6\boldsymbol{c} + 3\boldsymbol{d}$

6 a



b i $a = 16$ **ii** $b = -16$ **iii** $n = 2$
iv $\boldsymbol{v}(t) = -32\sin(2t)\,\boldsymbol{i} - 32\cos(2t)\,\boldsymbol{j}$
$\boldsymbol{a}(t) = -4(16\cos(2t)\,\boldsymbol{i} - 16\sin(2t)\,\boldsymbol{j})$
c i $\overrightarrow{PQ} = 8((\sin t - 2\cos(2t))\boldsymbol{i}$
$+ (\cos t + 2\sin(2t))\boldsymbol{j})$
ii $|\overrightarrow{PQ}|^2 = 64(5 + 4\sin t)$
d 8 cm

7 a $(2\sin t)\,\boldsymbol{i} + (\cos(2t) + 2)\boldsymbol{j},\ t \geq 0$
b $2\boldsymbol{i} + \boldsymbol{j}$
c i $y = 3 - \frac{x^2}{2},\ -2 \leq x \leq 2$
ii



d $|\boldsymbol{v}|^2 = -16\cos^4 t + 20\cos^2 t$,
max speed $= \frac{5}{2}$
e $\frac{3\pi}{2}$
f ii $t = \frac{(2k-1)\pi}{2},\ k \in \mathbb{N}$

8 a $a\,\boldsymbol{i} + (b + 2t)\boldsymbol{j} + (20 - 10t)\boldsymbol{k}$
b $at\,\boldsymbol{i} + (bt + t^2)\boldsymbol{j} + (20t - 5t^2)\boldsymbol{k}$ **c** 4 s
d $a = 25,\ b = -4$ **e** 38.3°

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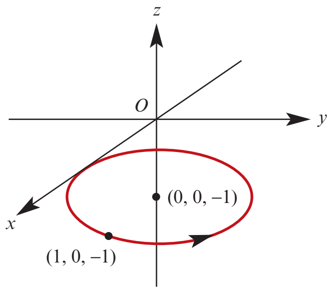
9 a i Particle P is moving on a circular path, with centre $(0, 0, -1)$ and radius 1, starting at $(1, 0, -1)$ and moving 'anticlockwise' a distance of 1 'below' the x–y plane. The particle finishes at $(1, 0, -1)$ after one revolution.



ii $\sqrt{2}$

iii $-\sin(t)\,\boldsymbol{i} + \cos(t)\,\boldsymbol{j}, \ 0 \le t \le 2\pi$

v $\ddot{\boldsymbol{p}}(t) = -\cos(t)\,\boldsymbol{i} - \sin(t)\,\boldsymbol{j}, \ 0 \le t \le 2\pi$

b i $\overrightarrow{PQ} = (\cos(2t) - \cos t)\boldsymbol{i} + (-\sin t - \sin(2t))\boldsymbol{j} + \frac{3}{2}\boldsymbol{k}$

iii $\dfrac{5}{2}$ **iv** $\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$ **v** $\dfrac{3}{2}$

vi $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 2\pi$

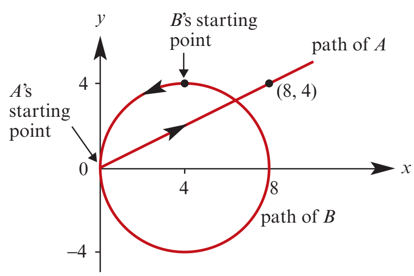
c ii $\dfrac{\sqrt{10}}{5}\left(\cos(3t) - \dfrac{1}{2}\right)$ **iii** 162°

10 a 4α

b A: $y = \dfrac{x}{2}$ for $x \ge 0$;

B: $(x-4)^2 + y^2 = 16$

c



d $(0, 0), \left(\dfrac{32}{5}, \dfrac{16}{5}\right)$

e 1.76

11 a i $-9.8\boldsymbol{j}$ **ii** $2\boldsymbol{i} - 9.8t\boldsymbol{j}$ **iii** $2t\,\boldsymbol{i} - 4.9t^2\,\boldsymbol{j}$

b i $\dfrac{2\sqrt{2}}{7}$ seconds **ii** $\dfrac{4\sqrt{2}}{7}$ metres

12 a i $6\boldsymbol{i} - 3\boldsymbol{j}$ **ii** $\dfrac{\sqrt{5}}{5}(2\boldsymbol{i} - \boldsymbol{j})$

b $4\boldsymbol{i} - 2\boldsymbol{j}, \ (4, -2)$

c i $\overrightarrow{LP} = (1 - \frac{7}{2}t)\boldsymbol{i} + (7 - 2t)\boldsymbol{j}$ **ii** 1:05 p.m.

iii $\dfrac{9\sqrt{65}}{13}$ km

Chapter 14

Technology-free questions

1 a $\dfrac{-1}{\sqrt{1-x^2}\,(\arcsin x)^2}$ **b** $\dfrac{-1}{(x^2+1)(\arctan x)^2}$

c $\dfrac{-2}{\sqrt{1-x^2}\,(\arcsin x)^3}$

2 a $\dfrac{dQ}{dt} = -\dfrac{Q}{10+t}$ **b** $Q = \dfrac{10}{t+10}$

3 $y = \dfrac{1}{2}\log_e\left(\dfrac{5}{x^2+4}\right) + 2$

4 a 6π **b** $6\pi^2$

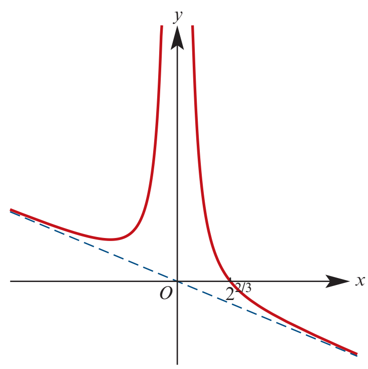
5 a $\dfrac{1}{3}, -\dfrac{7}{3}$ **b** $3x - 7y = -11$

6 a $x = \dfrac{2}{\cos(2t) + 3}$

b 1 cm, $t = \dfrac{(2n-1)\pi}{2}, n \in \mathbb{N}$

7 Asymptotes $y = -\dfrac{x}{3}, \ x = 0$;

axis intercept $(\sqrt[3]{4}, 0)$; stat point $(-2, 1)$



8 a $-1 + \dfrac{5}{4(x+2)} - \dfrac{5}{4(x-2)}$

b $\dfrac{1}{2}(5\log_e 3 - 4)$

9 a $\dfrac{\pi}{2}$

b $f(x) = \sqrt{x(2-x)}$, dom $= [1, 2]$, ran $= [0, 1]$

c 2π

10 a $\dfrac{1}{4}\log_e|\cos(2x)| + \dfrac{1}{2}x\tan(2x) + c$

b $(x+5)\log_e(x+5) - x + c$

c $\dfrac{1}{5}e^{2x}(2\sin x - \cos x) + c$

11 $y = -\dfrac{1}{2}\log_e(\cos(2x))$

12 $y = 2(1 + x^2)$

13 $\left[-1, -\dfrac{\sqrt{2}}{2}\right] \cup \left[\dfrac{\sqrt{2}}{2}, 1\right]$

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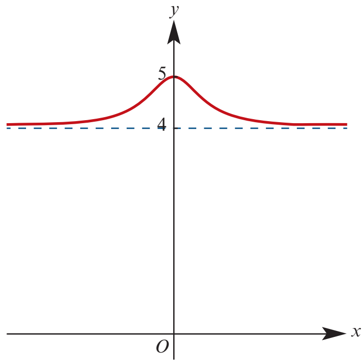
14 Asymptote $y = 4$; stat point $(0, 5)$



15 $\dfrac{dy}{dx} = -\tan t, \quad -1$

16 e

17 **a** $6\pi a$ **b** $\dfrac{56\pi a^2}{3}$

18 **a** $\dfrac{1}{2}(\sin(e^2) - \sin(1))$ **b** $\dfrac{4}{15}$ **c** $\log_e\left(\dfrac{27}{32}\right)$

d $-\dfrac{6\sqrt{5}}{5}\log_e(2 + \sqrt{5})$ **e** $\dfrac{38}{3}$ **f** $\dfrac{2\sqrt{2} - 1}{3}$

19 1.27

20 $8\pi a^5$

21 $\dfrac{1}{2}\log_e(1 + u^2)$

22 $A = \dfrac{5g}{2}, \ B = \dfrac{2}{5}$

23 $x = 10\,000\log_e\left(\dfrac{5}{6}\right) + 2000$

24 **b** $(0, 0), \left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right), \left(-\sqrt{3}, -\dfrac{\sqrt{3}}{2}\right)$

25 **a** $\boldsymbol{v}(t) = \cos t\,\boldsymbol{i} + \cos(2t)\boldsymbol{j}$

b $\boldsymbol{a}(t) = -\sin t\,\boldsymbol{i} - 2\sin(2t)\boldsymbol{j}$

c $d(t) = |\sin t|\sqrt{2 - \sin^2 t}$

d $s(t) = \sqrt{2 - 5\sin^2 t + 4\sin^4 t}$

e $y^2 = x^2(1 - x^2)$

26 **a** $\dfrac{x^2}{4} - 4y^2 = 1, \ x \geq 2, \ y \geq 0$

b $\boldsymbol{v}(t) = 2\tan t\sec t\,\boldsymbol{i} + 0.5\sec^2 t\,\boldsymbol{j}$

c $2\sqrt{13}$ m/s

27 $\boldsymbol{x}(\log_e 2) = \dfrac{5}{2}\boldsymbol{i} + \boldsymbol{j} - \dfrac{19}{8}\boldsymbol{k}$

28 **b** $y = \sqrt{3}x - \dfrac{g}{200}x^2$

29 **a** $\boldsymbol{r}(t) = (\cos(2t) + 1)\boldsymbol{i} + (\sin(2t) - 1)\boldsymbol{j}$

b $(x - 1)^2 + (y + 1)^2 = 1$ **c** $t = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

30 **a** $\dfrac{28}{g}$ seconds **b** $y = \dfrac{\sqrt{3}}{3}x - \dfrac{g}{1176}x^2$

c $\dfrac{98}{g} = 10$ metres

31 **a** $y = \dfrac{e^{-\frac{1}{2}(x-1)^2}}{2 - e^{-\frac{1}{2}(x-1)^2}}$ or $y = 0, -1$ **b** $(1, 1)$

32 $e - e^{-1}$

33 **a** 3.2 metres **b** 4 seconds

34 **b** $\dfrac{\pi}{4}$

Multiple-choice questions

1 D	**2** D	**3** B	**4** C	**5** A
6 C	**7** C	**8** A	**9** E	**10** B
11 A	**12** B	**13** E	**14** A	**15** A
16 A	**17** C	**18** C	**19** B	**20** D
21 B	**22** B	**23** B	**24** E	**25** E
26 E	**27** A	**28** C	**29** B	**30** A
31 D	**32** C	**33** C	**34** B	**35** C
36 A	**37** E	**38** D	**39** D	**40** A
41 E	**42** A	**43** C	**44** B	**45** B
46 A	**47** C	**48** B	**49** D	**50** C
51 D	**52** A	**53** E	**54** D	**55** A
56 E	**57** C	**58** B	**59** B	**60** D
61 C	**62** D	**63** C	**64** C	**65** E
66 E	**67** D			

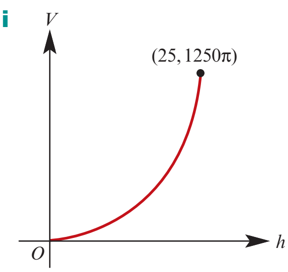
Extended-response questions

1 **a** 1250π

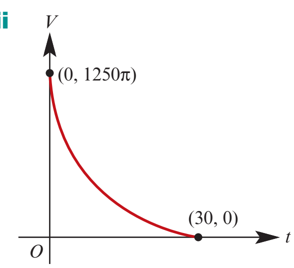
b **ii** $k = \dfrac{10\pi}{3}$ **iii** $h = -\dfrac{5t}{6} + 25$

iv $V = 2\pi\left(25 - \dfrac{5t}{6}\right)^2$

c **i**



ii



2 **b** $I_0 = \dfrac{\pi}{4} - \dfrac{1}{2}\log_e 2, \quad I_1 = \dfrac{\pi}{4} - \dfrac{1}{2}$

d **i** $I_2 = \dfrac{1}{12}(2\log_e 2 + \pi - 2)$

ii $I_3 = \dfrac{1}{6}$

iii $I_4 = \dfrac{1}{20}(-2\log_e 2 + \pi + 1)$

iv $I_5 = \dfrac{1}{180}(15\pi - 26)$

3 **a** $I_0 = 2\log_e 2 - 1$

4 **a** $(0, 0), (0, 2), (2, 0)$

b $(0, 0)$

c Interchanging x and y does not change the equation for the graph

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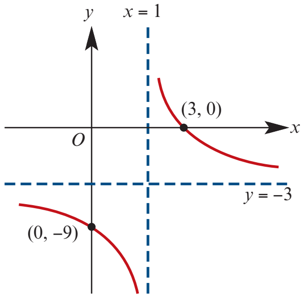
d **i** $\dfrac{dy}{dx} = \dfrac{x-y-1}{x-y+1}$, $\dfrac{d^2y}{dx^2} = \dfrac{4}{(x-y+1)^3}$

ii $x = \dfrac{3}{4}$, $y = -\dfrac{1}{4}$ **iii** $x = -\dfrac{1}{4}$, $y = \dfrac{3}{4}$

iv $\dfrac{1}{2}$

f $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

5 **a**



b (2, 3), (3, 0) **c** $4.5 - \log_e 64$

d $y = -3x + 6\sqrt{2}$, $y = -3x - 6\sqrt{2}$

6 **a** $k = 1180$ **b** 129 000

7 **e** **i** $\dfrac{dv}{dh} = \pi\left(\dfrac{25h}{3} + 100\right)$

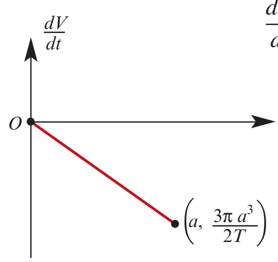
ii $\dfrac{dh}{dt} = \dfrac{-9\sqrt{h}}{625\pi^2(h+12)^2}$

f 65 days 19 hours

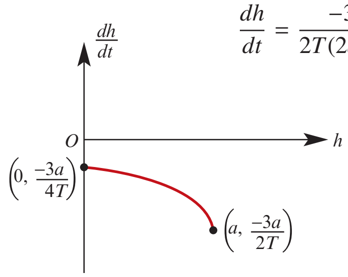
8 **a** **ii** 6.355 cm

d $k = 15.7$

e **i** $\dfrac{dV}{dt} = \dfrac{-3\pi a^2}{2T}h$



ii $\dfrac{dh}{dt} = \dfrac{-3a^2}{2T(2a-h)}$



f **i** $-\dfrac{a}{T}$ cm/s **ii** $-\dfrac{6a}{7T}$ cm/s

g −0.37 cm/s

9 **a** $N = 500\sqrt{5t+4}$

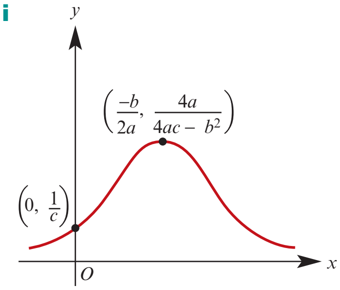
b **i** $N = \dfrac{4000}{5e^{-0.2t} - 1}$

ii After $5\log_e\left(\dfrac{15}{11}\right) \approx 1.55$ weeks

10 **a** $f'(x) = \dfrac{-2ax - b}{(ax^2 + bx + c)^2}$

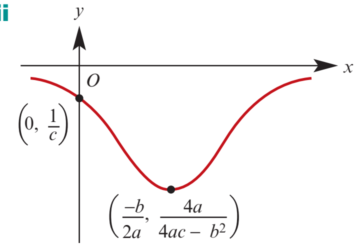
b $\left(-\dfrac{b}{2a}, \dfrac{4a}{4ac - b^2}\right)$ **i** max **ii** min

c **i**



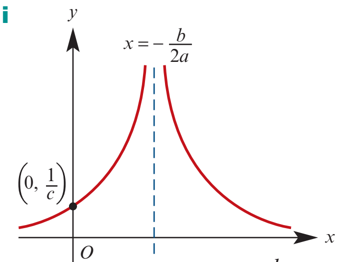
Asymptote $y = 0$

ii



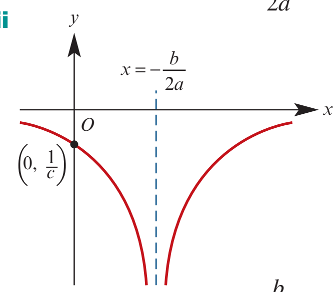
Asymptote $y = 0$

d **i**



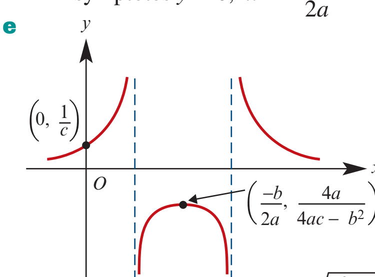
Asymptotes $y = 0$, $x = -\dfrac{b}{2a}$

ii



Asymptotes $y = 0$, $x = -\dfrac{b}{2a}$

e



Asymptotes $y = 0$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

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11 a $\dfrac{dy}{dx} = 2ax - \dfrac{2b}{x^3}$

b $\left(\dfrac{\sqrt[4]{a^3 b}}{a}, 2\sqrt{ab}\right), \left(\dfrac{-\sqrt[4]{a^3 b}}{a}, 2\sqrt{ab}\right)$;
both are minimum if $a, b \in \mathbb{R}^+$

12 a $\left\{\dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}\right\}$

b $e^{-2\pi}$

c Max: $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}e^{\frac{-\pi}{4}}\right), \left(\dfrac{9\pi}{4}, \dfrac{\sqrt{2}}{2}e^{\frac{-9\pi}{4}}\right)$;
Min: $\left(\dfrac{5\pi}{4}, -\dfrac{\sqrt{2}}{2}e^{\frac{-5\pi}{4}}\right), \left(\dfrac{13\pi}{4}, -\dfrac{\sqrt{2}}{2}e^{\frac{-13\pi}{4}}\right)$

d $\dfrac{1 + e^{\pi}}{2e^{\pi}}$

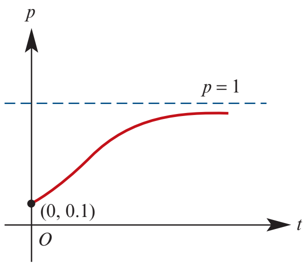
e $\dfrac{1 + e^{\pi}}{2e^{3\pi}}$

13 a $\dfrac{1}{5}$

14 b $p = \dfrac{9}{25}$ **c** $p = \dfrac{1}{9\left(\frac{2}{3}\right)^t + 1}$

d $t > 5.419$

e



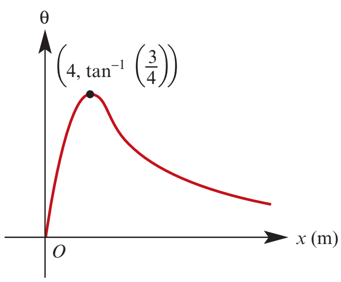
15 b $\dfrac{\sqrt[3]{k^2 p}}{k}$

16 a $\theta = \tan^{-1}\left(\dfrac{8}{x}\right) - \tan^{-1}\left(\dfrac{2}{x}\right),\ x > 0$

b $\dfrac{d\theta}{dx} = \dfrac{-8}{x^2 + 64} + \dfrac{2}{x^2 + 4}$

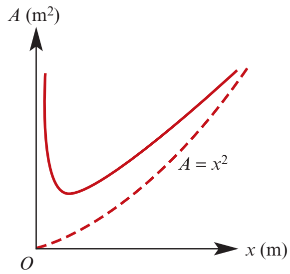
c $0 < \theta \le \tan^{-1}\left(\dfrac{3}{4}\right)$

d



e 0.23

18 b



c $x = 6.51$ or $x = 46.43$ **d** $x = 20$

19 288 cm^2

20 a $y = \dfrac{2}{5}x^2$

b $V = 40\sqrt{10}y^{\frac{3}{2}}$

c 252 mm

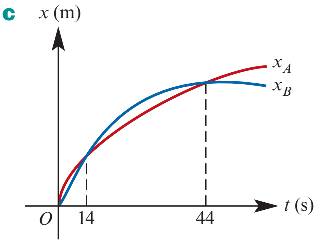
d $\dfrac{dy}{dt} = \dfrac{\sqrt{10y}}{10y}, \quad t = \dfrac{2\sqrt{10}}{3}y^{\frac{3}{2}}$

e i 3 mins 9 secs **ii** 5 mins 45 secs

21 a $v_A = \dfrac{20}{\sqrt{2t + 1}}, \quad v_B = \dfrac{100}{t + 10}$

b $x_A = 20(\sqrt{2t + 1} - 1),\ x_B = 100\log_e\left(\dfrac{t + 10}{10}\right)$

c

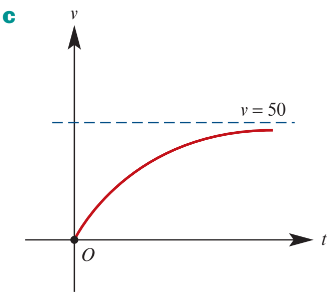


d 14 s and 44 s

22 a $v = 50 - 50e^{\frac{-t}{5}}$

b 49.9963

c



d i $x = 50\left(t + 5e^{\frac{-t}{5}} - 5\right)$ **ii** 125.2986

23 a $y = \dfrac{Ne^{2t}}{3 + e^{2t}}, \quad \dfrac{dy}{dt} = \dfrac{6Ne^{2t}}{(3 + e^{2t})^2}$

b N

c $\dfrac{dy}{dt} > 0$ for all t

d When population is $\dfrac{N}{2}$

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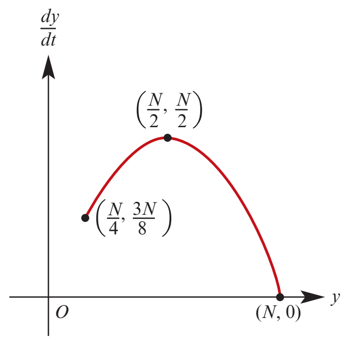
e **i**



ii At $t = \frac{1}{2}\log_e 3 \approx 0.549306$

24 **a** **i** $v^2 = \frac{2gR^2}{x} + u^2 - 2gR$

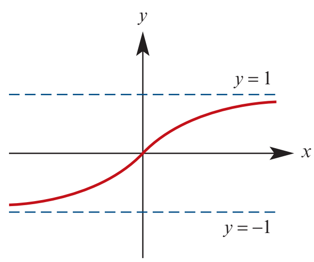
ii $x = \frac{2gR^2}{2gR - u^2}$

iii $u \geq \sqrt{2gR}$

b 40 320 km/h

25 **a** 0 **b** 1 **c** −1 **d** $f'(x) = \frac{4}{(e^x + e^{-x})^2}$

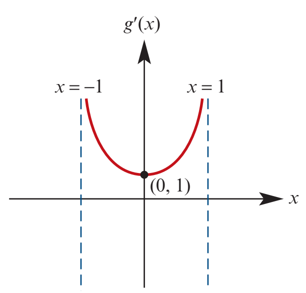
e



f $f^{-1}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right), \ -1 < x < 1$

g $g'(x) = \frac{1}{1-x^2}$

h



26 **a** **i** $y = 2r\sin\left(\frac{1}{2}\theta\right)$ **ii** $\cos\theta = \frac{r}{r+h}$

b **i** $\frac{dy}{d\theta} = r\cos\left(\frac{1}{2}\theta\right)$;

$$\frac{dy}{dt} = \frac{r\cos\left(\frac{1}{2}\theta\right)\cos^2\theta\ \sin t}{\sin\theta}$$

ii 6000 km

iii 1500 km/h

28 **a** $V = \frac{4}{3}\pi r^3$ **b** $4\pi r^2 \frac{dr}{dt} = -t^2$

c $r = \sqrt[3]{\frac{4000\pi - t^3}{4\pi}}$ **d** 23.2 mins

29 **a** $2\boldsymbol{i} - 10\boldsymbol{j}$ m/s **b** $\dot{\boldsymbol{r}}_1(t) = 2\boldsymbol{i} - 2t\boldsymbol{j}$

c $\boldsymbol{i} - 3\boldsymbol{j}$ **d** $t = 0$ **e** $t = 5$

f Yes; $t = 2$

30 **a** $\boldsymbol{r} = (\cos(4t) - 1)\boldsymbol{i} + (\sin(4t) + 1)\boldsymbol{j}$

b $-\boldsymbol{i} + \boldsymbol{j}$ **c** $\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = 0$

31 **a** 6π s

b **i** $-(3\sqrt{3}\boldsymbol{i} + 2.25\boldsymbol{j})$ **ii** $\boldsymbol{i} - \frac{3\sqrt{3}}{4}\boldsymbol{j}$

c **i** $1.5\sqrt{9 + 7\sin^2\left(\frac{t}{3}\right)}$

ii $t = 3\left(\frac{\pi}{2} + n\pi\right), \ n \in \mathbb{N} \cup \{0\}$

d $\ddot{\boldsymbol{r}} = -\frac{1}{9}\boldsymbol{r}, \ t = 3n\pi, \ n \in \mathbb{N} \cup \{0\}$

32 **a** **i** $\frac{3}{2}\sin(2t)\,\boldsymbol{i} - 2\cos(2t)\,\boldsymbol{j}$

ii $-6\sin(2t)\,\boldsymbol{i} + 8\cos(2t)\,\boldsymbol{j}$

iii $t = \frac{n\pi}{4}, \ n \in \mathbb{N} \cup \{0\}$

iv $16x^2 + 9y^2 = 36$

b $a = \frac{(2n+1)\pi}{4}, \ n \in \mathbb{N} \cup \{0\}$

33 **b** **i** $\boldsymbol{r}_2 = (0.2t - 1.2)\boldsymbol{i} + (-0.2t + 3.2)\boldsymbol{j} + \boldsymbol{k}$

ii $t = 16$ at $2\boldsymbol{i} + \boldsymbol{k}$

34 **a** **i** $h\boldsymbol{j}$, for $0\boldsymbol{i} + 0\boldsymbol{j}$ at the base of the cliff

ii $V\cos\alpha\,\boldsymbol{i} + V\sin\alpha\,\boldsymbol{j}$

b **i** $V\cos\alpha\,\boldsymbol{i} + (V\sin\alpha - gt)\boldsymbol{j}$

ii $Vt\cos\alpha\,\boldsymbol{i} + \left(h + Vt\sin\alpha - \frac{gt^2}{2}\right)\boldsymbol{j}$

c $t = \frac{V\sin\alpha}{g}$

35 **c** **i** $-(\boldsymbol{i} + \boldsymbol{j}), \boldsymbol{0}$ **iii** $-0.43\boldsymbol{i} - 0.68\boldsymbol{j}$

36 **a** **i** $0\boldsymbol{i} + 0\boldsymbol{j}$ **ii** $10\boldsymbol{i} + 10\sqrt{3}\boldsymbol{j}$, 20, 60°

iii $-9.8\boldsymbol{j}$

b **i** $\frac{x}{10}$ **ii** $x\boldsymbol{i} + (x\sqrt{3} - 0.049x^2)\boldsymbol{j}$

iii $10\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$

iv $-8\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$

c **i** $-8\boldsymbol{i} + (10\sqrt{3} - 0.98x - 9.8t_1)\boldsymbol{j}$

ii $\boldsymbol{r} = (x - 8t_1)\,\boldsymbol{i} + \big(x\sqrt{3} - 0.049x^2 + t_1(10\sqrt{3} - 0.98x - 4.9t_1)\big)\,\boldsymbol{j}$

d $\frac{20\sqrt{3} - 0.98x}{9.8}$

e $x = 15.71$

37 **a** $5\boldsymbol{i}$

b **i** $(5 - 3t_1)\boldsymbol{i} + 2t_1\boldsymbol{j} + t_1\boldsymbol{k}$,
$(5 - 3t_2)\boldsymbol{i} + 2t_2\boldsymbol{j} + t_2\boldsymbol{k}$,

ii $-3(t_2 - t_1)\boldsymbol{i} + 2(t_2 - t_1)\boldsymbol{j} + (t_2 - t_1)\boldsymbol{k}$

c $-3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

d **i** 36.70° **ii** 13.42

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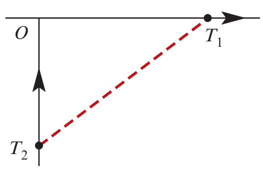
38 a i T_1 ii t_0

b ii $\frac{2\sqrt{5}}{5}Vt_0$

iii 

39 a $y = 5 - 2x, \ x \le 2$

b i $\boldsymbol{r}_1(t) = 2\boldsymbol{i} + \boldsymbol{j} + t(-\boldsymbol{i} + 2\boldsymbol{j})$

ii $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ is the starting position; $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j}$ is the velocity

c i $\boldsymbol{c} = -13\boldsymbol{i} + 6\boldsymbol{j}$ ii $5\sqrt{10}$

40 a $13\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$

b $\frac{\sqrt{14}}{14}(-3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}), \ \frac{\sqrt{6}}{6}(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

c 40.20° **d** $7\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k}$

e $13\boldsymbol{i} - \boldsymbol{j} - 8\boldsymbol{k} + t(-5\boldsymbol{i} + 3\boldsymbol{k})$ **f** $\frac{\sqrt{1190}}{34}$

Algorithms and pseudocode

See solutions supplement

Chapter 15

Exercise 15A

1 a $C = 450 + 0.5X$

b

c	950	1200	1450
$\Pr(C = c)$	0.05	0.15	0.35

c	1700	1950	2450
$\Pr(C = c)$	0.25	0.15	0.05

c 0.05

2 a $W = 2.5X - 5$

b

w	−5	−2.5	0	2.5
$\Pr(W = w)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

c $\frac{1}{8}$

3 a 0.027 **b** 0.125

4 a 0.3827 **b** 0.2929

5 a 0.5078 **b** 1

6 a $E(Y) = 77, \ Var(Y) = 81$

b $E(U) = -45, \ sd(U) = 6$

c $E(V) = -8.5, \ Var(V) = 2.25$

7 a $m = 2, \ n = -5$ **b** 35

8 a $E(X) = 0.4$ **b** $Var(X) = 0.2733$

c $E(4X + 2) = 3.6, \ sd(4X + 2) = 2.0913$

9 a 424.1 mL **b** 32.0 mL^2

10 a \$110 000 **b** \$1000

11 a \$5650 **b** \$4650 **c** \$537.63

Exercise 15B

1 0.45

2 a $E(X_1) = 3$ **b** $Var(X_1) = 2$

c $E(X_1 + X_2) = 6$ **d** $Var(X_1 + X_2) = 4$

3 a 20 **b** 18 **c** 4.243

4 a 35 **b** 20 **c** 4.472

5 a 1.7 **b** 0.287 **c** 0.535

6 a

s	3	4	5	6	7
$\Pr(S = s)$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{7}{18}$	$\frac{2}{9}$	$\frac{1}{9}$

b $\frac{2}{3}$

7 a $\frac{1}{6}$ **b** $\frac{1}{36}$

8 a $E(S) = 5$ **b** $sd(S) = 1.202$

9 Mean 49 mins, sd 8.5446 mins

10 a Mean 195 mL, sd 11.1803 mL

b Mean \$1.08, sd 6.40 cents

11 $a = 5, \ b = 3$

12 a Mean \$6.75, sd 3.52 cents

b Mean 4250 g, sd 13.23 g

Exercise 15C

1 0.1729 **2** 0.0548

3 0.3410 **4** 0.4466

5 0.0771 **6** 0.0512

7 7 people **8** 0.0127

9 a 0.0019 **b** 0.0062

10 0.6554

Exercise 15D

1 Mean 74, sd 4.6188

2 Mean 25.025, sd 0.0013

3 a 0.0478

b 0.0092

c Much smaller probability for the mean than for an individual

4 a 0.0912

b 0.0105

c Much smaller probability for the mean than for an individual

5 0.0103 **6** 0.0089

7 0.0478 **8** 0.0014

9 0.0786 **10** 0.0127

Exercise 15E

1 Answers will vary

2 Answers will vary

3 a Answers will vary **b** Mean 1, sd 0.002

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Exercise 15F

1 a 0.5 **b** 0.0288

2 0.0153

3 0.0228

4 a 0.7292 **b** 0.9998

5 0.0092

6 0.8426

7 0.000005

8 a 0.7745 **b** 0.7997

Chapter 15 review

Technology-free questions

1 a $E(Y) = 31$, $Var(Y) = 100$
b $E(U) = -35$, $sd(U) = 15$
c $E(V) = -39$, $Var(V) = 400$

2 a 0.45 **b** $\frac{9}{14}$

3 a 27 cm^3 **b** 0.0081 cm^6 **c** 54 cm^2

4 a Mean 200 mL, sd 13 mL
b Mean \$1.14

5 $a = 3$, $b = 1$

6 a Mean 125 g, sd 2.5 g
b Mean 65 g, sd 2 g
c Mean 3255 g, var 183 g^2

7 Mean 1.6

8 Mean 65, sd 1.4

9 At least 17 nails

Multiple-choice questions

1 D **2** C **3** A **4** E **5** A
6 B **7** B **8** B **9** D

Extended-response questions

1 a 0.3821
b $a = 20.8$, $b = 99.2$
c i 0.2512 **ii** 0.2512 **iii** 0.2870
d $c = 42.47$, $d = 77.53$

2 $\mu = 7.37$, $\sigma = 1.72$

3 a 0.0062 **b** 0.0000884
c 0.0000317 **d** 0.0075

4 a 0.1151 **b** 50 batteries

Chapter 16

Exercise 16A

1 (6.84, 7.96)

2 90%: (32.62, 38.78); 95%: (32.03, 39.37); 99%: (30.87, 40.53)

3 (66.84, 75.36) **4** (14.25, 14.95)

5 (25.54, 39.79) **6** (35.32, 43.68)

7 (3.10, 3.47)

8 a (127.23, 132.77) **b** (126.36, 133.64)
c Increasing the level of confidence results in a wider confidence interval

9 a (3.02, 5.03) **b** (2.82, 5.23)
c Increasing the level of confidence results in a wider confidence interval

10 a (28.18, 30.82) **b** (27.64, 31.36)
c Increasing the level of confidence results in a wider confidence interval

11 a 80 **b** 85 **c** 90

12 a 0.9025 **b** 0.9975

13 a (24.75, 26.05) **b** (25.01, 25.79)
c A larger sample results in a narrower confidence interval

14 a Increase by a factor of 4
b Increase by 56.25%
c Reduced by a factor of $\frac{2}{3}$
d Increased by a factor of 4

15 97 **16** 62

17 166 **18** 153

19 a 217 **b** 374

Exercise 16B

1 $H_0: \mu = 2.4$; $H_1: \mu < 2.4$

2 $H_0: \mu = 2.66$; $H_1: \mu > 2.66$

3 p-value = 0.000 02

4 p-value = 0.0924

5 a Good evidence against H_0
b Insufficient evidence against H_0
c Strong evidence against H_0
d Strong evidence against H_0
e Very strong evidence against H_0

6 Good evidence that the mean is less than 50

7 Insufficient evidence that the mean is greater than 10

8 Good evidence that the mean is less than 40

9 a $H_0: \mu = 2.9$; $H_1: \mu > 2.9$
b p-value = 0.003
c Yes, since the p-value is less than 0.05, we reject H_0 and conclude that the average monthly weight gain has increased.

10 a $H_0: \mu = 3.6$; $H_1: \mu < 3.6$
b p-value = 0.003
c Yes, since the p-value is less than 0.05, we reject H_0 and conclude that the mean number of residents per household has decreased.

11 a $H_0: \mu = 42\,150$; $H_1: \mu < 42\,150$
b p-value = 0.118
c No, since the p-value is not less than 0.05, there is insufficient evidence that the average income in this town is lower than for the rest of the state.

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12 a $H_0: \mu = 3.5$; $H_1: \mu > 3.5$
b p-value = 0.009
c Yes, since the p-value is less than 0.05, we reject H_0 and conclude that the average service time has increased.

13 $H_0: \mu = 20$; $H_1: \mu > 20$; p-value = 0.0003. Yes, since the p-value is less than 0.01, we reject H_0 and conclude that the average score is higher for students who sleep for 8 hours.

Exercise 16C

1 a $H_0: \mu = 0.5$; $H_1: \mu \neq 0.5$
b p-value = 0.012
c Yes, since the p-value is less than 0.05, we reject H_0 and conclude that the mean diameter of the ball bearings has changed.

2 $H_0: \mu = 2$; $H_1: \mu \neq 2$; p-value = 0.025. Yes, since the p-value is less than 0.05, we reject H_0 and conclude that the average weight of the bags has changed.

3 $H_0: \mu = 40$; $H_1: \mu \neq 40$; p-value = 0.025. Since the p-value is less than 0.05, we reject H_0 and conclude that the average length of stay in this hospital differs from other hospitals.

4 $H_0: \mu = 484$; $H_1: \mu \neq 484$; p-value = 0.0003. Yes, since the p-value is less than 0.01, we reject H_0 and conclude that the average number of visitors has changed.

5 $H_0: \mu = 2$; $H_1: \mu \neq 2$; p-value = 0.0015. Since the p-value is less than 0.05, we reject H_0 and conclude that the average hours that children watch television in this town has changed.

6 $H_0: \mu = 60$; $H_1: \mu \neq 60$; p-value = 0.0062. Yes, since the p-value is less than 0.05, we reject H_0 and conclude that the mean battery life has changed after the new process.

7 a p-value = 0.2636. No, insufficient evidence to conclude that the mean number of hours children sleep has changed.
b (7.62, 9.38)
c Do not reject H_0, since the hypothesised value (9) is in the confidence interval.

8 a p-value = 0.0279. Yes, conclude that the average starting salary for graduates of this university differs from the rest of the state.
b (52 059, 54 831)
c Reject H_0, since the hypothesised value (55 000) is not in the confidence interval.

Exercise 16D

1 a 0.3173 **b** 0.3829 **c** 0.0801
d 0.9643 **e** 0.3179

2 0.3173 **3** 0.1842
4 0.02145 **5** 0.3711
6 a 0.0149 **b** 0.5428
7 a 0.1148 **b** 0.0739
8 0.0321 **9** 0.1138
10 a 0.0736
b $H_0: \mu = 15$; $H_1: \mu \neq 15$. Do not reject H_0, since 0.0736 is greater than 0.05.
c More than 2.19 minutes

Exercise 16E

1 a Concluding that weight gain is higher on the special feed when in fact it is not
b Concluding that weight gain is the same when in fact it is higher on the special feed

2 a Type I error
b Showing that the patient did not have TB when in fact they did – Type II error

3 a 25.647 **b** 0.074
4 a 57.697 **b** 0.586
5 a $\bar{x} > 24.251$ **b** 0.187
6 a $\bar{x} > 2260$ **b** 0.188
7 a 0.0024 **b** 0.6804
c No, by requiring the sample mean to be more than 29.3 seconds, there is a very small probability of rejecting H_0, and hence there is a high probability that any increase in the mean time due to the blood alcohol content will be missed.

Chapter 16 review

Technology-free questions

1 a 160 **b** (140, 180)
2 a At least 226
b Decrease the width by a factor of $\sqrt{2}$
3 a $\bar{x} = 2480$, $n = 64$
b 100
4 a 57 **b** $(0.95)^{60}$
5 a i Do not reject H_0 **ii** Do not reject H_0
b i Reject H_0 **ii** Do not reject H_0
c i Reject H_0 **ii** Reject H_0
d i Reject H_0 **ii** Reject H_0
6 a H_0: time to complete the puzzle is the same when it is noisy as when it is not
H_1: time to complete the puzzle is longer when it is noisy
b p-value = 0.02. Since the p-value is less than 0.05, we reject H_0 and conclude that the time to complete the puzzle is longer when it is noisy.
c 2% of the time

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7 a $H_0: \mu = 4;\quad H_1: \mu > 4$
b p-value = 0.001
c Since the p-value is less than 0.01, we reject H_0 and conclude that children who receive praise are happier.

8 a $H_0: \mu = 50;\quad H_1: \mu < 50$
b p-value = 0.003
c Since the p-value is less than 0.05, we reject H_0 and conclude that the time to learn the new technology has reduced.

9 a Decrease **b** Decrease
c No effect **d** Increase

10 a 0.1336 **b** 0.9108

11 a 18 or 22 **b** p-value = 0.044
c Reject H_0 and conclude that the population mean is not 20.

Multiple-choice questions

1 D **2** A **3** E **4** C **5** B **6** D
7 E **8** C **9** E **10** A **11** B **12** B
13 C **14** A **15** E **16** B **17** C **18** E
19 D **20** B **21** D

Extended-response questions

1 a 0.8243
b i (11.45, 13.55) **ii** (12.83, 14.17)
iii (12.65, 13.78) **iv** At least 89

2 a (14.51, 16.09)
b i $H_0: \mu = 11.3;\quad H_1: \mu > 11.3$
ii p-value = 0.063
iii Since the p-value is greater than 0.05, we do not reject H_0. There is insufficient evidence to conclude that the mean job satisfaction score has increased.
c i $\bar{x} > 12.320$ **ii** 0.136

3 a (37.12, 42.88)
b i $H_0: \mu = 42;\quad H_1: \mu < 42$
ii p-value = 0.037
iii Since the p-value is less than 0.05, we reject H_0 and conclude that the assembly time for the new bookcase is quicker.
c 40.160
d 0.002

4 a $H_0: \mu = 70;\quad H_1: \mu > 70$
b p-value = 0.006
c Since the p-value is less than 0.05, we reject H_0 and conclude that the new batteries last longer between charges.
d 73.289 **e** $k = 75.0$

5 a i $H_0: \mu = 8.2;\quad H_1: \mu \neq 8.2$
ii p-value = 0.012
iii Since the p-value is less than 0.05, we reject H_0 and conclude that the mean plant growth has changed.
b i $c = 7.808$ **ii** $d = 8.592$ **iii** 0.516

Chapter 17

Technology-free questions

1 a $\mathrm{E}(Y) = 14,\ \mathrm{Var}(Y) = 144$
b $\mathrm{E}(U) = -7,\ \mathrm{sd}(U) = 8$
c $\mathrm{E}(V) = 42,\ \mathrm{Var}(V) = 832$

2 a 1 cm^3 **b** 0.0001 cm^6

3 Mean 30 g, sd 0.3 g

4 $a = 3,\ b = 4$

5 a Mean 100 g, var 5 g^2
b Mean 20 g, var 1.25 g^2
c Mean 1480 g, var 80 g^2

6 a $\mathrm{E}(Y) = 1,\ \mathrm{Var}(Y) = \frac{1}{2}$
b $\mathrm{E}(V) = 1,\ \mathrm{Var}(V) = \frac{1}{6}$

7 Mean 68, sd 2

8 At least 65

9 (80.08, 87.92)

10 a $\bar{x} = 440,\ n = 25$ **b** 625

11 a Reject H_0 and conclude that the population mean is less than 20.
b i p-value = 0.09
ii Do not reject H_0. There is insufficient evidence to conclude that the population mean is not 20.

12 a $H_0: \mu = 95;\quad H_1: \mu < 95$
b p-value = 0.023
c Since the p-value is less than 0.05, we reject H_0 and conclude that students who first meditate complete the puzzle more quickly.

Multiple-choice questions

1 C **2** B **3** A **4** A **5** B
6 A **7** D **8** D **9** E **10** B
11 D **12** D **13** D **14** E **15** B

Extended-response questions

1 a 0.0384 **b** 0.0256 **c** 50 **d** $\frac{100}{3}$
e 0.04

2 $\mu = 1.001,\ \sigma = 0.012$

3 a $k_1 = 40.8,\ k_2 = 119.2$
b $c_1 = 71.2,\ c_2 = 88.8$
c (76.2, 93.8)

4 a i $H_0: \mu = 1000;\quad H_1: \mu > 1000$
ii p-value = 0.074
iii Since the p-value is greater than 0.05, we do not reject H_0. There is insufficient evidence to conclude that the machine overfills the bags.
b 1000.91
c 0.024

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5 a **i** $H_0: \mu = 55; \quad H_1: \mu < 55$
ii p-value = 0.0008
iii Since the p-value is less than 0.05, we reject H_0 and conclude that the average riding time has decreased.
b 52.399
c 0.400

Algorithms and pseudocode

See solutions supplement

Chapter 18

Technology-free questions

2 a $\overrightarrow{CA} = 2(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}), \ \overrightarrow{CB} = 3(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
b $12(-\boldsymbol{j} + \boldsymbol{k})$ **c** $6\sqrt{2}$ **d** $12\sqrt{2}$

4 $\dfrac{4\sqrt{91}}{9}$

5 $\boldsymbol{r} = -\dfrac{3}{10}(2t - 115)\boldsymbol{i} + \dfrac{1}{10}(2t + 35)\boldsymbol{j} + t\boldsymbol{k}$

6 $\dfrac{3\sqrt{2}}{5}$

7 a If n is a perfect square, then n has an odd number of factors.
b If n is not a perfect square, then n has an even number of factors.
c n has an odd number of factors and n is not a perfect square.

9 a $\dfrac{1}{22}$ **b** 3π

10 a $[0, 1]$ **b** $[0, 4\pi]$ **c** 2π
d $\dfrac{1}{2}\left(1 - \dfrac{1}{\sqrt{2}}\right)$ **e** $y = 4 + 2\pi - 8x$

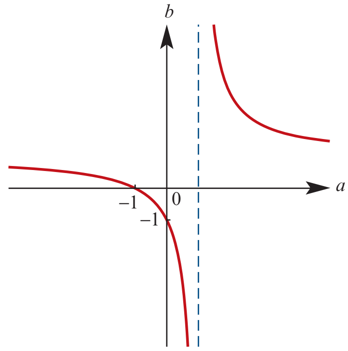
11 a 20 mins **b** $\dfrac{dm}{dt} = -\dfrac{3m}{20 - t}, \ m(0) = 10$
c $m = \dfrac{(20 - t)^3}{800}$ **d** $20 - 8\sqrt{5}$ mins

12 a $y = 0$
b $\left(-3 - 2\sqrt{3}, \dfrac{1}{2} - \dfrac{1}{\sqrt{3}}\right), \left(-3 + 2\sqrt{3}, \dfrac{1}{2} + \dfrac{1}{\sqrt{3}}\right)$
c $\dfrac{\pi}{\sqrt{3}} + \log_e 2$

13 a $\dfrac{85\sqrt{85} - 8}{243}$ **b** $\sqrt{10}$

14 a **i** $19 + 9i$ **ii** $-7 - \sqrt{3}i$
iii $-\dfrac{11}{8} - \dfrac{1}{4}i$ **iv** $1.48 + 0.8i$
b **i** $(ab - 1) + (a + b)i$
ii $b = \dfrac{a + 1}{a - 1}$

iii



15 a $\dfrac{1}{9}$ **b** $\dfrac{13}{36}$ **c** $\dfrac{1}{3}$ **d** $\mathrm{E}(Y) = \dfrac{1}{3}$

16 Mean 750 kg, sd $\sqrt{1220}$ kg

17 a $P\left(e, \dfrac{1}{e}\right), \ Q(1, 0)$ **b** $\dfrac{1}{2}$

18 At least 26

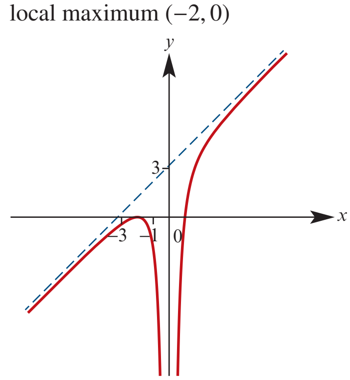
19 a $y = -\log_e(e + e^{-1} - e^x)$
b $(-\infty, \log_e(e + e^{-1}))$
c $y = \dfrac{x}{e + e^{-1} - 1} - \log_e(e + e^{-1} - 1)$

20 a $y = 2\tan\left(x^2 + \dfrac{\pi}{4}\right)$
b $\mathbb{R} \setminus \left\{\dfrac{\sqrt{(4n + 1)\pi}}{2}, n \in \mathbb{Z}\right\}$
c $y = -\dfrac{x}{8}\sqrt{\dfrac{3}{\pi}} + 2\sqrt{3} + \dfrac{1}{16}$

21 a $\dfrac{1}{(1 - x)^2} - \dfrac{1}{1 - x}$ **b** $\dfrac{2}{3} + \log_e 3$

22 b $\pi\left(\dfrac{1}{2}a^2 + a + \log_e(a - 1) - 4\right)$

23 Axis intercepts $(-2, 0), (1, 0)$;
asymptotes $x = 0, \ y = x + 3$;
local maximum $(-2, 0)$



24 a $\pm\dfrac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$
b $m + n = 1, \ \overrightarrow{OP} = m\boldsymbol{i} + (1 - m)\boldsymbol{j}$
c $m = \dfrac{3 \pm \sqrt{3}}{6}$

25 a $-\dfrac{2}{9}$ **b** -4

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26 a $a = 1,\ b = 1$ **b** $c = 3,\ d = 2$

27 a $m = 3,\ n = 5$ **b** $1 + 3i,\ 2 - i$

28 a $2(x - 4)e^x$ **b** $\frac{1}{4}x^2(2\log_e(2x) - 1)$

c $\frac{1}{3}x\tan(3x) + \frac{1}{9}\log_e|\cos(3x)|$

d $-\frac{1}{2}x^2 + x\tan x + \log_e|\cos x|$

30 b $\begin{bmatrix} 7\times 2^n - 6\times 3^n & 2\times 3^n - 2\times 2^n \\ 21\times 2^n - 21\times 3^n & 7\times 3^n - 6\times 2^n \end{bmatrix}$

31 a \$1.70 **b** (\$1.50, \$1.90)

32 a $H_0: \mu = 8.3;\quad H_1: \mu > 8.3$

b p-value = 0.067

c Since the p-value is greater than 0.05, we do not reject H_0. There is insufficient evidence to conclude that the new batteries last longer.

Multiple-choice questions

1 A	**2** C	**3** D	**4** C	**5** B
6 D	**7** B	**8** E	**9** B	**10** A
11 C	**12** D	**13** A	**14** D	**15** D
16 B	**17** D	**18** A	**19** E	**20** D
21 B	**22** A	**23** E	**24** C	**25** E
26 C	**27** E	**28** D	**29** E	**30** A
31 B	**32** B	**33** B		

Extended-response questions

2 a $-\frac{2}{5}$ **b** $2x + 5y = 17$ **c** $b = d = \sqrt{29}$

d i $29\pi(\sqrt{3} - 1) - \frac{2\sqrt{29}\pi^2}{3}$

ii $29\pi(\sqrt{3} - 1) - \frac{\sqrt{29}\pi^2}{3}$

e -5

3 b

```
n ← 0
sum ← 0
while sum < 10 000
   n ← n + 1
   sum ← n³(n² + 1)
end while
print n
```

4 a $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k} + t(\boldsymbol{j} - \boldsymbol{k})$

b $\boldsymbol{r}\cdot(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = 9$ **d** 1.43

e $2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$; 61.9°

5 b $(-2, 4, 3)$ **c** $-x + 2y + z = 13$

d $\left(\frac{4}{3}, \frac{7}{3}, \frac{29}{3}\right)$

e $\boldsymbol{r} = -2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$

6 a $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k} + t(2\boldsymbol{i} + 3\boldsymbol{j})$ **b** $(0, -2, 4)$

c 21.85° **d** $(0, -2, 7)$

7 a 0.655 **b** 0.314

8 a Mean 0.8 mm, sd 0.014 mm

b Mean 0.8 mm, sd 0.04 mm

9 a 0.5 **b** 0.5

c Mean 200 cm, variance 1.3 cm²

10 a $f'(x) = \log_e x - 2$ **b** $A(e^3, 0)$

c $y = x - e^3$ **d** $2 : 1$

11 a i $\frac{dy}{dx} = \frac{(b^2 - a^2)\cos x}{(b + a\sin x)^2}$

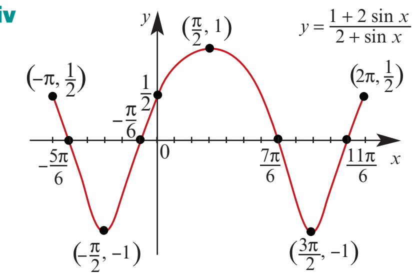
ii 1, −1

b i $\left(0, \frac{1}{2}\right)$

ii $\left(-\frac{5\pi}{6}, 0\right), \left(-\frac{\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$

iii $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -1\right)$

iv



v $2\pi(3 - \sqrt{3})$

12 a $r = 2,\ a = \frac{\pi}{3}$ **b** $[-2, 2]$ **c** $(0, 1)$

d $\left(\frac{5\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$ **e** $\frac{\pi}{12}, \frac{7\pi}{12}$

f $\frac{1}{4}\log_e(21 + 12\sqrt{3})$ **g** $(10\pi + 3\sqrt{3})\frac{\pi}{6}$

13 a i $\int_{10}^{5} \frac{-50}{v(1 + v^2)}\, dv$

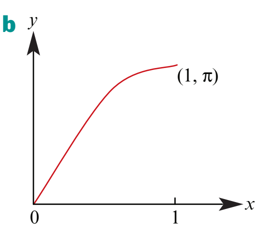
ii $25\log_e\left(\frac{104}{101}\right)$ seconds

b ii $x = 50(\tan^{-1}(10) - \tan^{-1} v)$

iv 74 m

14 a i $p = \pi$

b



c $\frac{(2\pi^2 + 15)\pi}{6}$

d $k = 2$ **e** 1.066 **f** 0.572

15 a $z^4 + z^3 + z^2 + z + 1$ **c** $\operatorname{cis}\left(-\frac{2\pi}{5}\right)$

d $\operatorname{cis}\left(\pm\frac{2\pi}{5}\right),\ \operatorname{cis}\left(\pm\frac{4\pi}{5}\right),\ 1$

e $\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$

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16 a $m = \sqrt{3}$

b i $\overrightarrow{OC} = -\overrightarrow{OA}$

c ii $2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k},\ \frac{8}{3}\boldsymbol{i} - \frac{1}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$

d $\frac{\pm 3}{\sqrt{18 - 2\sqrt{3}}}\Big((2 + \sqrt{3})\boldsymbol{i} + (-1 + \sqrt{3})\boldsymbol{j} + (2 - \sqrt{3})\boldsymbol{k}\Big)$

e $t = \frac{3}{4},\ k = \frac{1}{2},\ \ell = \frac{13\sqrt{3}}{12}$

f Particle lies outside the circle

17 a i $\frac{x^2}{9} + \frac{(y + a)^2}{36} = 1$ **ii** $\pm\frac{\sqrt{36 - a^2}}{2}$

b $f(x) = 2\sqrt{9 - x^2} - a$

c $\sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}}$

d i $A = 9$

e $\frac{1}{2}\left(x\sqrt{9 - x^2} + 9\arcsin\left(\frac{x}{3}\right)\right)$

f $18\arcsin\left(\frac{\sqrt{36 - a^2}}{6}\right) - \frac{a}{2}\sqrt{36 - a^2}$

g 18π

h 144π

18 a $y^2 = x\left(\frac{x}{3} - 1\right)^2$ **b** $\left(1, \frac{2}{3}\right), \left(1, -\frac{2}{3}\right)$

c $\frac{8\sqrt{3}}{5}$ **d** $\frac{3\pi}{4}$

19 a $y^2 = 16x^2(1 - x^2)(1 - 2x^2)^2$

b $\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = 4\cos(4t),\ \frac{dy}{dx} = \frac{4\cos(4t)}{\cos t}$

c i $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

ii $-\frac{1}{2}\sqrt{2 - \sqrt{2}},\ -\frac{1}{2}\sqrt{2 + \sqrt{2}},\ \frac{1}{2}\sqrt{2 - \sqrt{2}},\ \frac{1}{2}\sqrt{2 + \sqrt{2}}$

iii $(-\frac{1}{2}\sqrt{2 - \sqrt{2}}, 1),\quad (-\frac{1}{2}\sqrt{2 - \sqrt{2}}, -1),$
$(-\frac{1}{2}\sqrt{2 + \sqrt{2}}, 1),\quad (-\frac{1}{2}\sqrt{2 + \sqrt{2}}, -1),$
$(\frac{1}{2}\sqrt{2 - \sqrt{2}}, 1),\quad (\frac{1}{2}\sqrt{2 - \sqrt{2}}, -1),$
$(\frac{1}{2}\sqrt{2 + \sqrt{2}}, 1),\quad (-\frac{1}{2}\sqrt{2 + \sqrt{2}}, -1)$

iv $\frac{dy}{dx} = \pm 4$ when $x = 0$;
$\frac{dy}{dx} = \pm 4\sqrt{2}$ when $x = \pm\frac{1}{\sqrt{2}}$

d $\frac{16}{15}(\sqrt{2} + 1)$

e $\frac{64\pi}{63}$

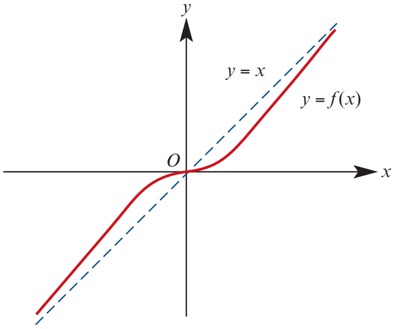
20 a $f'(x) = \frac{x^4 + 3ax^2}{(x^2 + a)^2},\ f''(x) = \frac{6a^2x - 2ax^3}{(x^2 + a)^3}$

b $(0, 0)$ stationary point of inflection

c $\left(-\sqrt{3a}, -\frac{3\sqrt{3a}}{4}\right), \left(\sqrt{3a}, \frac{3\sqrt{3a}}{4}\right)$

d $y = x$

e



f $a = 1$

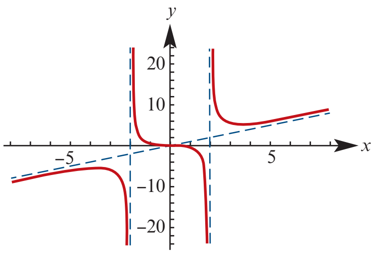
21 a $f'(x) = \frac{x^4 - 3ax^2}{(x^2 - a)^2},\ f''(x) = \frac{6a^2x + 2ax^3}{(x^2 - a)^3}$

b $\left(-\sqrt{3a}, -\frac{3\sqrt{3a}}{2}\right)$ local maximum,
$\left(\sqrt{3a}, \frac{3\sqrt{3a}}{2}\right)$ local minimum,
$(0, 0)$ stationary point of inflection

c $(0, 0)$

d $y = x,\ x = \sqrt{a},\ x = -\sqrt{a}$

e



f $a = 16$

22 a $f'(x) = \frac{x}{\sqrt{1 - x^2}} + \arcsin(x)$,
$(0, 0)$ local minimum
(Note: It is easy to see $f(x) \geq 0$ for all x, as x and $\arcsin(x)$ have the same sign for all x, and $f(x) = 0$ if and only if $x = 0$.)

b $f''(x) = \frac{x^2\sqrt{1 - x^2} + 2(1 - x^2)^{\frac{3}{2}}}{(x^2 - 1)^2} = \frac{\sqrt{1 - x^2}\,(2 - x^2)}{(x^2 - 1)^2} \geq 0$ for all $x \in (-1, 1)$

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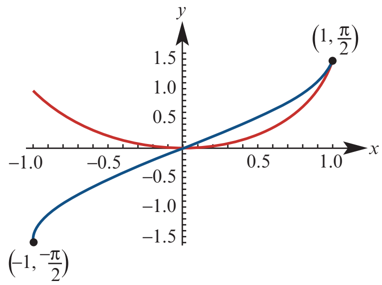
c $f(x) \geq 0$ for all x, as x and $\arcsin(x)$ have the same sign for all x

d $x = 0, 1$

e



f $\frac{3\pi}{8} - 1$

23 a $x = \frac{3}{4}\sin(2t),\ y = -\frac{1}{2}\cos(2t)$

b $\frac{16x^2}{9} + 4y^2 = 1$ **c** $\frac{2}{3}\tan(2t)$

d $y = -\frac{1}{2}\sec(2t),\ x = \frac{3}{4}\operatorname{cosec}(2t)$

e $\frac{3}{8}|\operatorname{cosec}(4t)|$, minimum area $= \frac{3}{8}$ when $t = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}, \dots$

f $x = \frac{3}{4}\sin(2t),\ y = \frac{3}{4}\cos(2t)$ (infinitely many possible answers)

g $\frac{5\pi}{16}$

24 a 0.0808

b $k_1 = 45.2,\ k_2 = 64.8$

c i 0.0008 **ii** 0.0289 **iii** 0.0598

d $c_1 = 51.90,\ c_2 = 58.10$

25 a $\mathrm{E}(X) = \frac{b}{2},\ \mathrm{sd}(X) = \frac{b}{\sqrt{12}}$

b $\mathrm{E}(\bar{X}) = \frac{b}{2},\ \mathrm{sd}(\bar{X}) = \frac{b}{\sqrt{12n}}$

c $(2.4 - 0.067b, 2.4 + 0.067b)$

d $4.23 < b < 5.54$ with 90% confidence

27 a 0.0679

b 0.5

c i (2.991, 3.009)

ii Machine A, since the confidence interval contains the mean for machine A but not machine B

28 a $\frac{dP}{dt} = k(P - 0.5P_0)$ **b** $P = 500\left(1 + \left(\frac{6}{5}\right)^t\right)$

c 144 goats **d** 6.03 years

29 a $\boldsymbol{r}_A(t) = 30\sqrt{3}t\,\boldsymbol{i} + (30t - \frac{1}{2}gt^2)\boldsymbol{j}$

$\boldsymbol{r}_B(t) = (100 - 50t\cos\beta)\boldsymbol{i} + (50t\sin\beta - \frac{1}{2}gt^2)\boldsymbol{j}$

b $\beta = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.87°$

c 1.09 seconds

d (56.90, 26.83)

30 a $\mu = 112.8$

b i $H_0: \mu = 112.8$; $H_1: \mu < 112.8$

ii p-value = 0.073

iii Since the p-value is greater than 0.05, we do not reject H_0. There is insufficient evidence that the mean lifetime of the light bulbs is less than that claimed by the manufacturer.

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