MATHSQUEST 12

SPECIALIST MATHEMATICS

VCE UNITS 3 AND 4

MATHSQUEST 12
SPECIALIST MATHEMATICS

VCE UNITS 3 AND 4

AUTHOR

RAYMOND ROZEN

CONTRIBUTING AUTHORS

CATHERINE SMITH | JO BRADLEY | SUE MICHELL
STEVEN MORRIS | MARGARET SWALE

SUPPORT MATERIAL

AILEEN TOLL | KATHRYN MARNELL

jacaranda
A Wiley Brand

First published 2016 by
John Wiley & Sons Australia, Ltd
42 McDougall Street, Milton, Qld 4064

Typeset in 12/14.5 pt Times LT Std

National Library of Australia Cataloguing-in-Publication entry

Creator:	Rozen, Raymond, author.
Title:	Maths quest. 12, Specialist mathematics, VCE units 3 & 4 / Raymond Rozen, Sue Michell, Steven Morris, Margaret Swale.
ISBN:	978 0 7303 2303 7 (set) 978 0 7303 2305 1 (ebook) 978 0 7303 2767 7 (paperback) 978 0 7303 2456 0 (studyON)
Notes:	Includes index.
Target Audience:	For secondary school age.
Subjects:	Mathematics—Australia—Textbooks. Mathematics—Problems, exercises, etc. Victorian Certificate of Education examination—Study guides.
Other Creators/ Contributors:	Michell, Sue, author. Morris, Steven P., author. Swale, Margaret, author.
Dewey Number:	510

Illustrated by diacriTech and Wiley Composition Services

Typeset in India by diacriTech

Printed in Singapore by
Markono Print Media Pte Ltd

10 9 8 7 6 5 4 3 2 1

Contents

Introduction

At Jacaranda, we are deeply committed to the ideal that learning brings life-changing benefits to all students. By continuing to provide resources for Mathematics of exceptional and proven quality, we ensure that all VCE students have the best opportunity to excel and to realise their full potential.

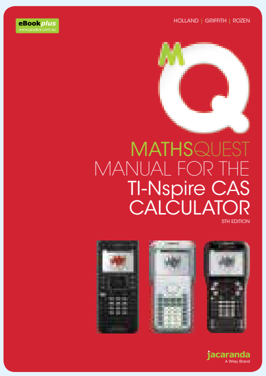
Maths Quest 12 Specialist Mathematics VCE Units 3 and 4 comprehensively covers the requirements of the revised Study Design 2016–2018.

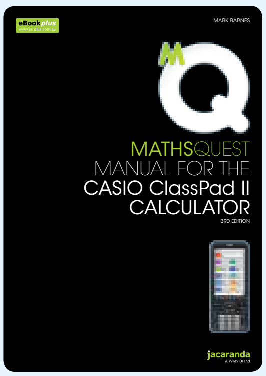
Features of the new *Maths Quest* series

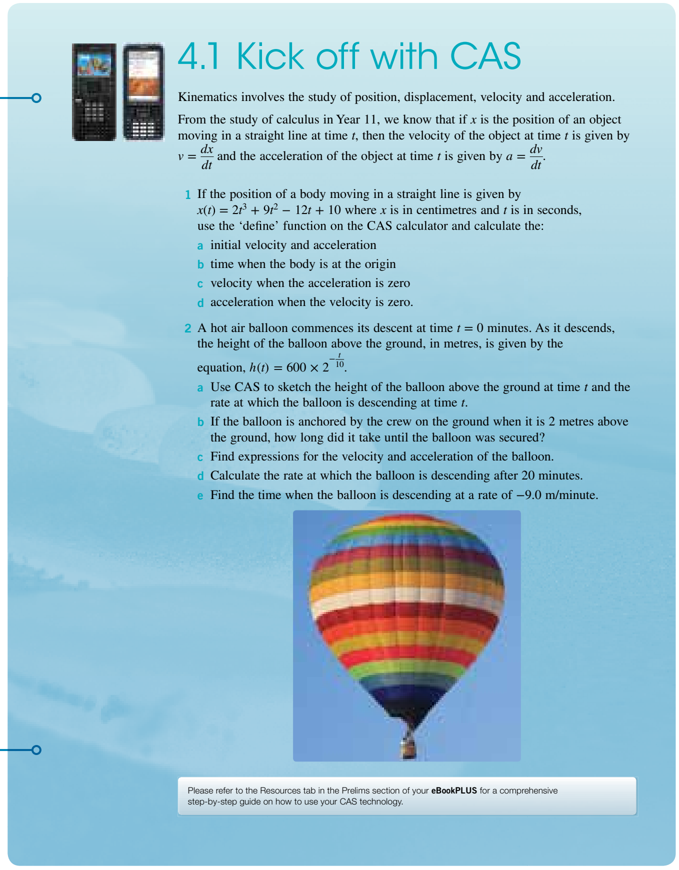
CAS technology

Each topic opens with an engaging **Kick off with CAS** activity designed to stimulate students' interest and curiosity and to highlight the important applications of CAS technology in developing deep understanding of the mathematical concepts presented.

For up-to-date, step-by-step instructions on how to use CAS technology, we have provided the *Manual for the TI-Nspire CAS calculator* and the *Manual for the Casio ClassPad II* in the Prelims section of the eBook.







4.1 Kick off with CAS

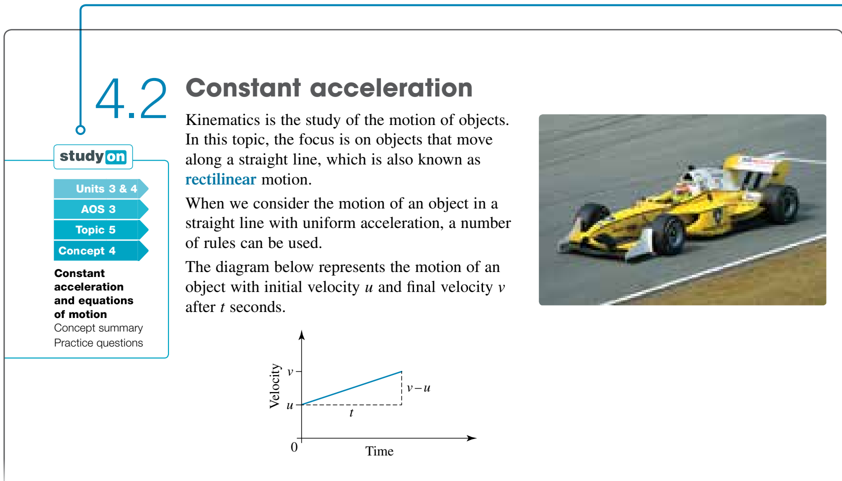
Kinematics involves the study of position, displacement, velocity and acceleration.

From the study of calculus in Year 11, we know that if x is the position of an object moving in a straight line at time t, then the velocity of the object at time t is given by $v = \frac{dx}{dt}$ and the acceleration of the object at time t is given by $a = \frac{dv}{dt}$.

1 If the position of a body moving in a straight line is given by $x(t) = 2t^3 + 9t^2 - 12t + 10$ where x is in centimetres and t is in seconds, use the 'define' function on the CAS calculator and calculate the:
a initial velocity and acceleration
b time when the body is at the origin
c velocity when the acceleration is zero
d acceleration when the velocity is zero.

2 A hot air balloon commences its descent at time $t = 0$ minutes. As it descends, the height of the balloon above the ground, in metres, is given by the equation, $h(t) = 600 \times 2^{-\frac{t}{10}}$.
a Use CAS to sketch the height of the balloon above the ground at time t and the rate at which the balloon is descending at time t.
b If the balloon is anchored by the crew on the ground when it is 2 metres above the ground, how long did it take until the balloon was secured?
c Find expressions for the velocity and acceleration of the balloon.
d Calculate the rate at which the balloon is descending after 20 minutes.
e Find the time when the balloon is descending at a rate of −9.0 m/minute.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.



4.2 Constant acceleration

Kinematics is the study of the motion of objects. In this topic, the focus is on objects that move along a straight line, which is also known as rectilinear motion.

When we consider the motion of an object in a straight line with uniform acceleration, a number of rules can be used.

The diagram below represents the motion of an object with initial velocity u and final velocity v after t seconds.

studyON links

Link to **studyON**, an interactive and highly visual study, revision and exam practice tool for instant feedback and on-demand progress reports.

Graded questions

A wide variety of questions at **Practise**, **Consolidate** and **Master** levels allow students to build, apply and extend their knowledge independently and progressively.

4 Substitute the quantities into the equation and solve.

$s = ut + \frac{1}{2}at^2$

$s = 0 + \frac{1}{2} \times 9.8 \times 3.5^2$

$s = 60.025$

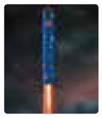
The rock travels a distance of 60.025 m to the bottom of the well.

EXERCISE 4.3 Motion under gravity

PRACTISE

1 WE3 A basketball is thrown vertically upwards with a velocity of 5 m/s. What is the maximum height of the basketball in the air?

2 A skyrocket is projected vertically upwards to a maximum height of 154.34 m. Find the velocity of projection.



3 WE4 A rock is dropped down a well. It reaches the bottom of the well in 3.2 seconds. How deep is the well?

4 A stone is dropped from the top of a cliff 122.5 m high. How long does it take for the stone to reach the bottom of the cliff?

CONSOLIDAT

5 A ball is thrown into the air with a velocity of 8 m/s. How long does the ball take to reach its maximum height?

6 A shot is fired vertically upward and attains a maximum height of 800 m. Find the initial velocity of the shot.

7 A boulder falls from the top of a cliff 45 metres high. Find the boulder's speed just before it hits the ground.

8 A ball is thrown vertically upwards with a velocity of 15 m/s from the top of a building 20 metres high and then lands on the ground below. Find the time of flight for the ball.

9 A skyrocket is projected vertically upwards from the ground. It runs out of fuel at a velocity of 52 m/s and a height of 35 m. From this point on it is subject only to acceleration due to gravity. Find its maximum height.

10 A stone is projected vertically upwards with a velocity of 12 m/s from the top of a cliff. If the stone reaches the bottom of the cliff in 8 seconds, find:

a the height of the cliff

b the velocity at which the stone must be projected to reach the bottom of the cliff in 4 s.

11 From a hot air balloon rising vertically upward with a speed of 8 m/s, a sandbag is dropped which hits the ground in 4 seconds. Determine the height of the balloon when the sandbag was dropped.

12 A missile is projected vertically upward with a speed of 73.5 m/s, and 3 seconds later a second missile is projected vertically upward from the same point with the same speed. Find when and where the two missiles collide.

13 A flare, A, is fired vertically upwards with a velocity of 35 m/s from a boat. Four seconds later, another flare, B, is fired vertically upwards from the same point with a velocity of 75 m/s. Find when and where the flares collide.

14 A skyrocket is launched upwards with a velocity of 60 m/s. Two seconds later, another skyrocket is launched upwards from the same point with the same initial velocity. Find when and where the two skyrockets meet.

MASTER

15 A stone, A, is projected vertically upwards with a velocity of 25 m/s. After stone A has been in motion for 3 s, another stone, B, is dropped from the same point. Find when and where the two stones meet.

16 A worker climbs vertically up a tower to a certain height and accidently drops a small bolt. The man ascends a further 45 m and drops another small bolt. The second bolt takes 1 second longer than the first to reach the ground. Find:

a the height above the ground at which the worker dropped the first bolt

b the time it took the first bolt to reach the ground.

4.4 Velocity–time graphs

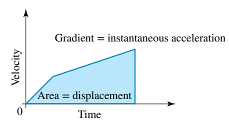
studyon

Units 3 & 4
AOS 3
Topic 5
Concept 3

Velocity–time graphs
Concept summary
Practice questions

Velocity–time graphs are a useful visual representation of the motion of an object in a straight line. We can use velocity–time graphs to solve kinematic problems. The following properties of velocity–time graphs make this possible.

- Because $a = \frac{dv}{dt}$, the gradient of the velocity–time graph at time t gives the instantaneous acceleration at time t.
- Because $v = \frac{dx}{dt}$, the displacement is found by evaluating the definite integral $\int_{t_1}^{t_2} v\,dt = x_2 - x_1$. The distance is found by determining the magnitude of the signed area under the curve bounded by the graph and the t axis, $\left|\int v(t)dt\right|$. Distance travelled cannot be a negative value.



Some useful formulas to assist in finding the displacement without using calculus are:

- area of a triangle: $A = \frac{1}{2}bh$
- area of a rectangle: $A = LW$
- area of a trapezium: $A = \frac{1}{2}(a + b)h$.

192 MATHS QUEST 12 SPECIALIST MATHEMATICS VCE Units 3 and 4 | Topic 4 KINEMATICS 193

Review

Each topic concludes with a customisable **Review**, available in the resources tab of the **eBookPLUS**, giving students the opportunity to revise key concepts covered throughout the topic. A variety of typical question types is available including short-answer, multiple-choice and extended response.

Summary

A comprehensive and fully customisable topic summary is available in the resources tab of the e**BookPLUS**, enabling students to add study notes and key information relevant to their personal study needs.

eBook*plus* ONLINE ONLY

4.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

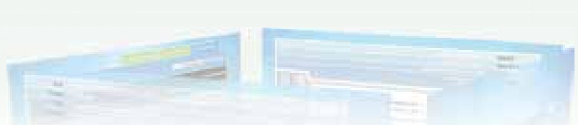
A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.



eBook*plus*

The **eBookPLUS** is available for students and teachers and contains:

- the full text online in HTML format, including PDFs of all topics
- the *Manual for the TI-Nspire CAS calculator* for step-by-step instructions
- the *Manual for the Casio ClassPad II calculator* for step-by-step instructions
- topic reviews in a customisable format
- topic summaries in a customisable format
- links to **studyON**.

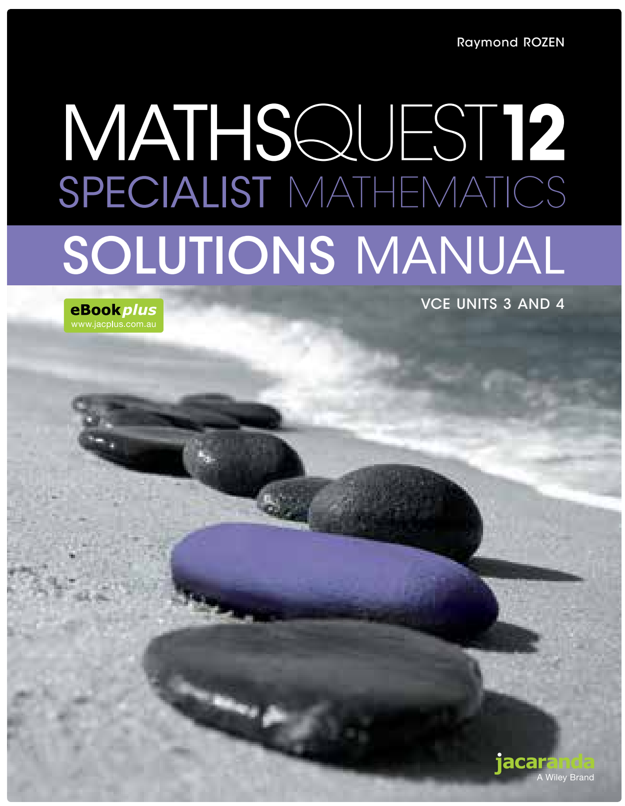
eGuide*plus*

The **eGuidePLUS** is available for teachers and contains:

- the full **eBookPLUS**
- a Work Program to assist with planning and preparation
- School-assessed Coursework — Application task and Modelling and Problem-solving tasks, including fully worked solutions
- two tests per topic with fully worked solutions.

Maths Quest 12 Specialist Mathematics Solutions Manual VCE Units 3 and 4

Available to students and teachers to purchase separately, the Solutions Manual provides fully worked solutions to every question in the corresponding student text. The Solutions Manual is designed to encourage student independence and to model best practice. Teachers will benefit by saving preparation and correction time.



About eBookPLUS and studyON

Access your online Jacaranda resources anywhere, anytime, from any device in three easy steps:

STEP 1 Go to **www.jacplus.com.au** and create a user account.
STEP 2 Enter your registration code.
STEP 3 Instant access!

eBookplus + studyon

eBookPLUS is an electronic version of the textbook, together with a targeted range of supporting multimedia resources.

eBookPLUS features:

- **eBook** — the entire textbook in electronic format
- **Digital documents** designed for easy customisation and editing
- **Interactivities** to reinforce and enhance students' learning
- **eLessons** — engaging video clips and supporting material
- **Weblinks** to relevant support material on the internet

eGuidePLUS features assessment and curriculum material to support teachers.

studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.

studyON features:

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access 1000+ past VCAA questions** or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback.
- **Sit past VCAA exams** (Units 3 and 4) or **topic tests** (Units 1 and 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 and 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

NEED HELP? Go to www.jacplus.com.au and select the Help link.

- Visit the JacarandaPLUS Support Centre at **http://jacplus.desk.com** to access a range of step-by-step user guides, ask questions or search for information.
- **Contact** John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
  Phone: 1800 JAC PLUS (1800 522 7587)

Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

Acknowledgements

The authors and publisher would like to thank the following copyright holders, organisations and individuals for their assistance and for permission to reproduce copyright material in this book.

Images

• Shutterstock: **3**/Lonely; **3**/topseller; **47**/Buslik; **111**/rook76; **112**/Menno Schaefer; **113**/ChameleonsEye; **114**/Matthew Cole; **114**/light poet; **116**/stockphoto mania; **117**/deb talan; **137**/MIKHAIL GRACHIKOV; **175**/Jakez; **184**/harikarn; **184**/Jag_cz; **185**/monticello; **185**/Mudryuk; **185**/monticello; **186**/Pal Teravagimov; **197**/Zaharia Bogdan Rares; **198**/Martynova Anna; **199**/Kzenon; **207**/saicle; **224**/Willyam Bradberry; **225**/Susan Flashman; **226**/Palo_ok; **226**/Johan Larson; **227**/TheFinalMiracle; **236**/Gordon Bell; **237**/gorillaimages; **237**/Pete Saloutos; **237**/Maxim Tupikov; **238**/Gordana Sermek; **249**/Stephen Gibson; **278**/Always Joy; **287**/Nattika; **287**/Leena Robinson; **288**/anekoho; **289**/pirita; **319**/Orla; **321**/Wildnerdpix; **321**/Miks Mihails Ignats; **327**/heromen30; **356**/Maria Sbytova; **357**/Dja65; **359**/Kim D. Lyman; **359**/del.Monaco; **360**/Tania Zbrodko; **360**/Umberto Shtanzman; **367**/heromen30; **387**/wavebreakmedia; **387**/Zeljko Radojko; **388**/Joanne Harris and Daniel Bubnich; **389**/StockLite; **397**/simez78; **399**/GOLFX; **400**/serhio; **404**/Sonulkaster; **405**/dien; **406**/alaaddin; **406**/**54613**; **413**/Scott Rothstein; **414**/v.s.anandhakrishna; **415**/Brittny; **416**/Butter45; **421**/pbombaert; **422**/gfdunt; **429**/Pavel Hlystov; **429**/Pavel Hlystov; **431**/Jalin; **432**/© Susan Flashman, **2011**. Used under license from Shutterstock.com; **432**/Anthony MonterottiUnknown; **433**/Alexey Losevich; **434**/Pincasso; **442**/seeyou; **442**/Tatuasha; **444**/TK Kurikawa; **444**/antoniodiaz; **445**/JNP; **447**/totallyPic.com; **453**/imredesiuk; **454**/Bulls-Eye Arts; **455**/schankz; **461**/gielmichal; **462**/Herbert Kratky; **486**/Simon Bratt; **487**/DAE Photo; **488**/M. Unal Ozmen; **497**/Ersin Kurtdal; **507**/VladKol; **508**/martiapunts; **509**/Odua Images; **513**/Tobik; **514**/mhatzapaUnknown; **515**/mkmakingphotos; **516**/ER_09; **516**/Elena Elisseeva; **517**/Dmitry Morgan; **518**/paulrommer; **518**/Iakov Filimonov; **523**/Dorottya Mathe; **523**/Erik Lam; **524**/Rawpixel; **525**/graphit; **527**/Ruth Black; **528**/Vorobyeva; **528**/Monkey Business Images; **528**/bonchan; **529**/Yulia Davidovich; **533**/FXQuadro; **534**/Tortoon Thodsapol; **536**/Daniel Korzeniewski; **537**/Africa Studio; **538**/Nattika; **538**/Suzanne Tucker; **538**/David P. Smith; **539**/wavebreakmedia; **539**/Neale Cousland

Text

Specialist Mathematics Units 3 and 4 Study Design content copyright of the Victorian Curriculum and Assessment Authority (VCAA); used with permission. The VCAA does not endorse this product and makes no warranties regarding the correctness or accuracy of its content. To the extent permitted by law, the VCAA excludes all liability for any loss or damage suffered or incurred as a result of accessing, using or relying on the content. VCE® is a registered trademark of the VCAA.

Every effort has been made to trace the ownership of copyright material. Information that will enable the publisher to rectify any error or omission in subsequent editions will be welcome. In such cases, please contact the Permissions Section of John Wiley & Sons Australia, Ltd.

1

Sketching graphs

1.1 Kick off with CAS

Exploring the absolute value function with CAS

We are familiar with the graphs of straight lines, for example, $f(x) = 2x + 1$, and functions such as $f(x) = x^2$, $f(x) = x^3$ and $f(x) = 2^x$, which are equations of curves. Can we sketch a function, other than a hybrid function, that has a sharp vertex?

1 Using CAS, sketch the absolute value function $f(x) = |x|$ ($f(x) = \text{abs}(x)$).

2 Sketch and describe the following transformations of $f(x) = |x|$.

a $f(x) = 2|x|$

b $f(x) = \frac{1}{2}|-x|$

c $f(x) = |x| - 1$

d $f(x) = -|3x|$

e $f(x) = |x - 2|$

f $f(x) = -\left|\frac{x}{2} + 1\right| - 2$

3 Solve the following for x using CAS.

a $|2x - 1| = 4$

b $|x^2 - 7x + 12| = 0$

c $\left(1 - |x|\right)\left(|x| - 4\right) = 0$

d $|x + 2| > |-x - 1|$

e $|x - 1| < -2$

f $\dfrac{|x + 2|}{|3x - 4|} > 1$

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

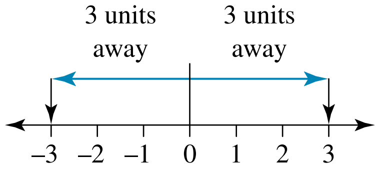
1.2 An introduction to the modulus function

studyon

Units 3 & 4
AOS 1
Topic 1
Concept 8

The modulus function
Concept summary
Practice questions

Any real number has both a magnitude and a sign. For example, the number −1 has a magnitude of 1 and its sign is negative; the number 612 has a magnitude of 612 and its sign is positive. (By convention, the + sign is not shown in front of positive numbers.)



Sometimes only the magnitude of a number is required. This is referred to as the **absolute value** (**modulus**) of a number. Consider the numbers −3 and 3: they are both 3 units from the origin. They both have an absolute value of 3.

More formally, $|x| = \sqrt{x^2}$. Considering 3 and −3 again: if $x = 3$, then $x^2 = 9$ and $\sqrt{9} = 3$. Additionally, if $x = -3$, then $x^2 = 9$ and $\sqrt{9} = 3$.

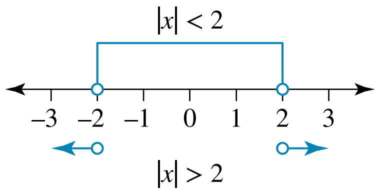
WORKED EXAMPLE 1

Solve $|4x| = 16$.

THINK	WRITE
1 $|16| = 16$ and $|-16| = 16$, so there are 2 possible values for $4x$: 16 and −16.	$4x = 16$ or $4x = -16$
2 Solve the equations.	$x = 4$ or $x = -4$

Using modulus notation to describe domains

Modulus notation can be used to describe intervals of variables. For example, $|x| < 2$ can be described as $-2 < x < 2$; $|x| > 2$ can be described as $x < -2$ or $x > 2$.



WORKED EXAMPLE 2

Describe the following intervals of x without the use of modulus signs.

a $|x| \le 3$ **b** $|x| > 1$

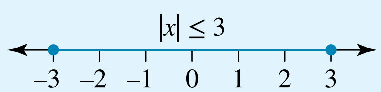
THINK

a Values between −3 and 3 inclusive will have a magnitude less than or equal to 3.

b Numbers greater than 1 or less than −1 will have a magnitude greater than 1.

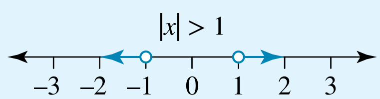
WRITE/DRAW

a



$-3 \le x \le 3$

b



$x > 1$ or $x < -1$

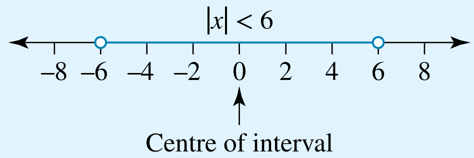
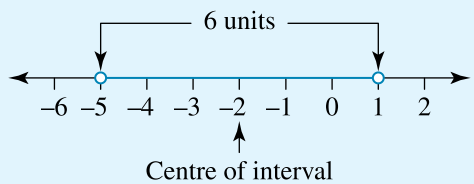
WORKED EXAMPLE 3

Describe the interval of x without the use of modulus signs where $|x + 1| < 5$.

THINK	WRITE
1 $x + 1$ must lie between −5 and 5.	$-5 < x + 1 < 5$
2 Solve by subtracting 1 from each term.	$-5 - 1 < x < 5 - 1$ $-6 < x < 4$

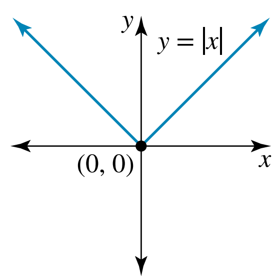
WORKED EXAMPLE 4 Use modulus notation to describe the following intervals.

a $-6 < x < 6$ **b** $-5 < x < 1$

THINK	WRITE/DRAW
a 1 Sketch the interval.	**a** 
2 0 is the centre of the interval. Therefore, no adjustments are necessary.	$-6 < x < 6$ $\lvert x \rvert < 6$
b 1 Sketch the interval.	**b** 
2 The interval is 6 units and the centre is -2. Shifting by 2 units will centre the interval at 0.	$-5 < x < 1$ $-5 + 2 < x + 2 < 1 + 2$ $-3 < x + 2 < 3$
3 Rewrite using modulus notation.	$\lvert x + 2 \rvert < 3$

Using the graph of $y = |x|$

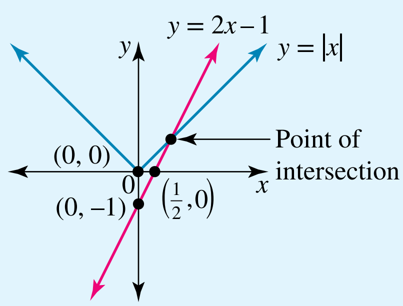
We have observed that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. Therefore, the graph of $|x|$ looks like this.



WORKED EXAMPLE 5

a Use the graph of $y = |x|$ to solve $|x| = 2x - 1$.

b Hence, solve $|x| > 2x - 1$.

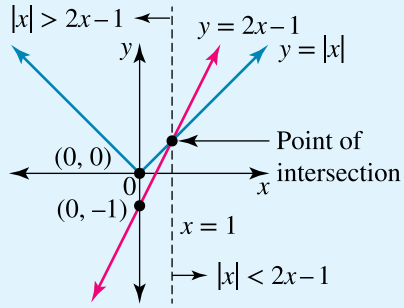
THINK	WRITE/DRAW
a 1 Sketch $y = \lvert x \rvert$ and $y = 2x - 1$.	**a** 
2 The intersection occurs in the $y = x$ section of the $y = \lvert x \rvert$ graph. Solve for x.	$x = 2x - 1$ $x = 1$

3 Check the solution.

If $x = 1$, $|x| = 1$ and $2x - 1 = 1$. Therefore, the solution is valid.

b The graph of $|x|$ is above $2x - 1$ for $x < 1$.

b $|x| > 2x - 1$ for $x < 1$



EXERCISE 1.2 An introduction to the modulus function

PRACTISE

1 WE1 Solve $|3x| = 12$.

2 Solve $|x + 1| = 3$.

3 WE2 Describe the following intervals of x without the use of modulus signs.

a $|x| < 5$ **b** $|x| > 3$

4 Describe the following intervals of x without the use of modulus signs.

a $|x| \geq 4$ **b** $|x| \leq 1$

5 WE3 Describe the domain of x without the use of modulus signs where $|x + 1| < 7$.

6 Describe the domain of x without the use of modulus signs where $|x - 4| < 5$.

7 WE4 Use modulus notation to describe the following intervals.

a $-3 < x < 3$ **b** $-4 < x < 4$

8 Use modulus notation to describe the following intervals.

a $-3 < x < 1$ **b** $-7 < x < 1$

9 WE5 **a** Use the graph of $y = |x|$ to solve $|x| = 2x - 3$.

b Hence, solve $|x| < 2x - 3$.

10 a Use the graph of $y = |x|$ to solve $|x| = 2x + 1$.

b Hence, solve $|x| > 2x + 1$.

CONSOLIDATE

11 Solve the following for x.

a $|5x| = 35$ **b** $|x - 5| = 9$ **c** $|2x + 1| = 7$ **d** $\left|\frac{1}{x}\right| = 1.5$

12 Describe the following intervals without using modulus signs.

a $|x| < 6$ **b** $|x + 2| < 7$ **c** $|x - 3| \geq 4$ **d** $\left|\frac{x}{2} - 1\right| \geq 1$

13 Use modulus notation to describe the following intervals.

a $-2 < y < 2$ **b** $-3 < y < 5$ **c** $-5 \leq y \leq 7$

d $y > 5$ or $y < -5$ **e** $y \geq 5$ or $y \leq 1$

14 Use modulus notation to describe the following intervals (x_0, a and b are positive constants).

a $-a \leq x \leq a$ **b** $x_0 - 3 \leq x \leq x_0 + 3$

c $a - b < x < b - a$ **d** $5 - a \leq x \leq a + 5$

15 a Use the graph of $y = |x|$ to solve $|x| = 3x + 2$.
b Hence, solve $|x| > 3x + 2$.

16 a Use the graph of $y = |x|$ to solve $|x| = 2x - 2$.
b Hence, solve $|x| < 2x - 2$.

17 a Use the graph of $y = |x|$ to solve $|x| = -x + 4$.
b Hence, solve $|x| > -x + 4$.

18 a Use the graph of $y = |x|$ to solve $|x| = -2x - 3$.
b Hence, solve $|x| < -2x - 3$.

19 Consider $|-a| = a$.
a Find a value for a where $|-a| \neq a$.
b For what values of a does $|-a| = a$ hold true?

20 When does $|1 - x| = 1 - x$ and when does $|1 - x| = x - 1$?

MASTER

The following questions may be attempted with a CAS calculator.

21 a Use the graph of $y = |x + 1|$ to solve $|x + 1| = 2x$.
b Hence, solve $|x + 1| \geq 2x$.

22 a By drawing appropriate graphs, solve $|x + 1| = |2x - 1|$.
b Hence, solve $|x + 1| > |2x - 1|$.

1.3 Sketching graphs of reciprocal functions

studyon

Units 3 & 4
AOS 1
Topic 1
Concept 3

Sketch graphs of reciprocal functions
Concept summary
Practice questions

Turning points and points of inflection will be studied in a later topic.

Consider $\frac{1}{f(x)}$.

- The function is undefined if $f(x) = 0$. On a graph, this is shown as a vertical asymptote.
- As $f(x) \to \infty$, $\frac{1}{f(x)} \to 0$, but $\frac{1}{f(x)} \neq 0$, therefore, $y = 0$ is the horizontal asymptote.

The graph of $y = f(x)$ can be used to sketch $y = \frac{1}{f(x)}$, as shown in the following worked example.

WORKED EXAMPLE 6

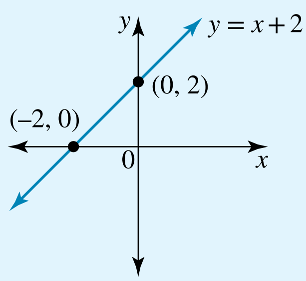
Use the graph of $y = x + 2$ to sketch $y = \frac{1}{x + 2}$.

THINK

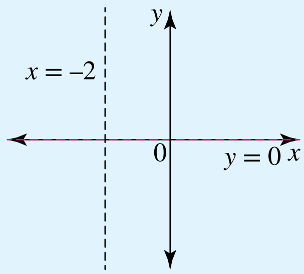
1 Sketch $y = x + 2$ and identify the intercepts.

WRITE/DRAW

When $x = 0$, $y = 2$.
When $y = 0$, $x = -2$.



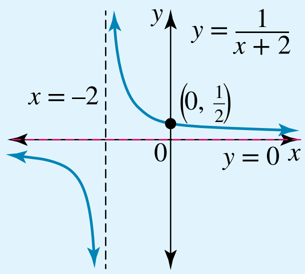
2 The x-intercept occurs at $x = -2$; this will be the vertical asymptote. The horizontal asymptote is $y = 0$.



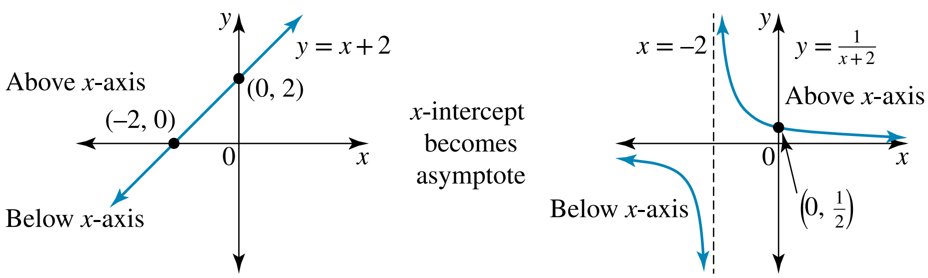
3 The y-intercept of $y = x + 2$ is $y = 2$.
Therefore, the y-intercept of $y = \dfrac{1}{x+2}$ will be $y = \dfrac{1}{2}$.

The y-intercept of $y = \dfrac{1}{x+2}$ will be $y = \dfrac{1}{2}$.

4 $y = x + 2$ is below the x-axis for $x < 2$; therefore, $y = \dfrac{1}{x+2}$ is also below the x-axis for $x < 2$.
$y = x + 2$ is above the x-axis for $x > 2$; therefore, $y = \dfrac{1}{x+2}$ is also above the x-axis for $x > 2$.



Compare the graphs of $y = x + 2$ and $y = \dfrac{1}{x+2}$.



study on

Units 3 & 4
AOS 1
Topic 1
Concept 4

Sketch graphs of reciprocal quadratic functions
Concept summary
Practice questions

Graphing the reciprocal functions for quadratic and cubic functions

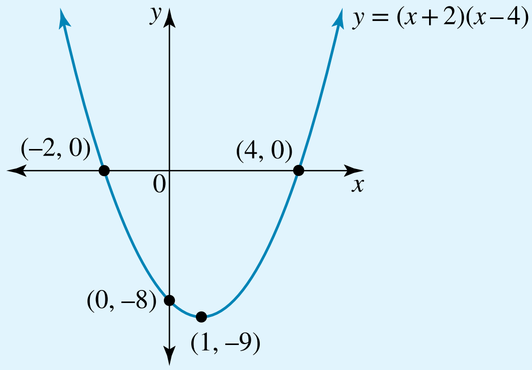
The same process can be used to graph the reciprocal functions of quadratic and cubic functions.

WORKED EXAMPLE 7

Use the graph of $y = (x + 2)(x - 4)$ to sketch $y = \dfrac{1}{(x+2)(x-4)}$.

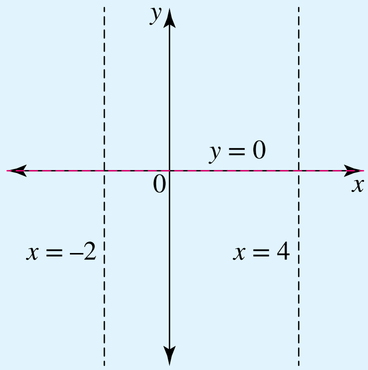
THINK	WRITE/DRAW
1 Identify the key points of $y = (x + 2)(x - 4)$.	y-intercept: $x = 0$, $y = -8$ x-intercepts: $y = 0$, $x = -2$ or $x = 4$ Turning point: $x = 1$, $y = -9$

2 Sketch $y = (x + 2)(x - 4)$.



3 The x-intercepts occur at $x = -2$ and $x = 4$, so these become the vertical asymptotes.

The horizontal asymptote is $y = 0$.



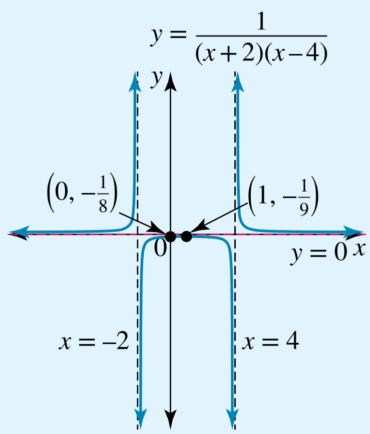
4 The y-intercept of $y = (x + 2)(x - 4)$ is -8. Therefore, the y-intercept of $y = \dfrac{1}{(x + 2)(x - 4)}$ is $\dfrac{-1}{8}$.

The y-intercept is $\left(0, \frac{-1}{8}\right)$.

5 The turning point of $y = (x + 2)(x - 4)$ is $(1, -9)$. Therefore, the turning point of $y = \dfrac{1}{(x + 2)(x - 4)}$ is $\left(1, \dfrac{-1}{9}\right)$

The turning point is $\left(1, \frac{-1}{9}\right)$.

6 The graph of $y = (x + 2)(x - 4)$ is above the x-axis for $x < -2$ and $x > 4$. Therefore, $y = \dfrac{1}{(x + 2)(x - 4)}$ is above the x-axis in the region. Similarly, the graph is below the x-axis for $-2 < x < 4$. Sketch the graph.



If $y = f(x)$ does not have any x-intercepts, then $y = \dfrac{1}{f(x)}$ does not have any vertical asymptotes. The horizontal asymptote is still $y = 0$. Notice that in the previous examples, at the extreme values of x ($x \to \infty$ and $x \to -\infty$), the graph approaches the horizontal asymptote. This is true for all graphs with a horizontal asymptote.

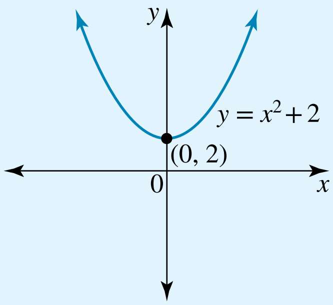
WORKED EXAMPLE 8 Use the graph of $y = x^2 + 2$ to sketch $y = \dfrac{1}{x^2 + 2}$.

THINK

1 Sketch $y = x^2 + 2$, identifying key points.

WRITE/DRAW

y-intercept: $x = 0$, $y = 2$
x-intercepts: There are no x-intercepts.
Turning point $x = 0$, $y = 2$



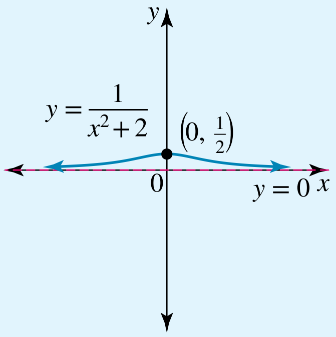
2 There are no x-intercepts and therefore no vertical asymptotes in the reciprocal function. The horizontal asymptote is still $y = 0$. The turning point (and x-intercept) (0, 2) will become $\left(0, \frac{1}{2}\right)$ in the reciprocal function.

No vertical asymptotes
Horizontal asymptote: $y = 0$
The turning point (and y-intercept) is $\left(0, \frac{1}{2}\right)$.

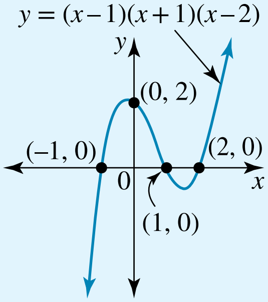
3 $y = x^2 + 2$ is always above the x-axis, so $y = \dfrac{1}{x^2 + 2}$ will always be above the x-axis.

The graph will approach the horizontal asymptote as $x \to \infty$ and $x \to -\infty$.

Sketch the reciprocal function.



WORKED EXAMPLE 9 Use the given sketch of $y = (x - 1)(x + 1)(x - 2)$ to sketch $y = \dfrac{1}{(x - 1)(x + 1)(x - 2)}$.



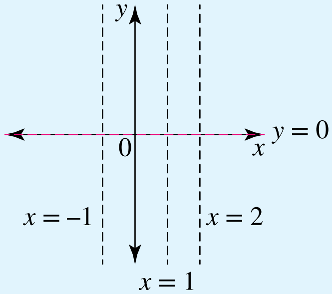
THINK

1 The x-intercepts are at $x = -1$, $x = 1$ and $x = 2$. These become the vertical asymptotes.

The horizontal asymptote is $y = 0$.

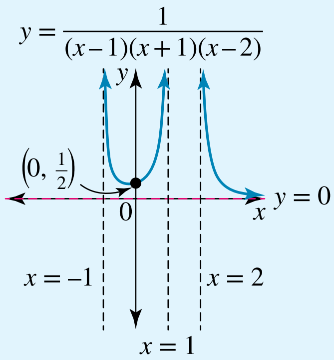
WRITE/DRAW

The vertical asymptotes are $x = -1$, $x = 1$ and $x = 2$. The horizontal asymptote is $y = 0$.

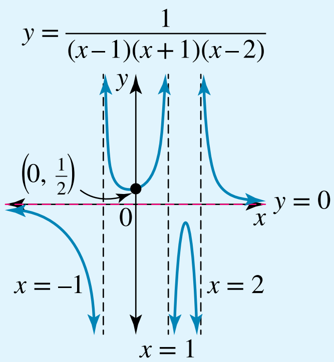


2 The graph of $y = (x - 1)(x + 1)(x - 2)$ is above the x-axis for $-1 < x < 1$ and $x > 2$. The graph of $y = \dfrac{1}{(x - 1)(x + 1)(x - 2)}$ will also be above the axis for these intervals.

The y-intercept (0, 2) will become $\left(0, \frac{1}{2}\right)$.



3 The graph of $y = (x - 1)(x + 1)(x - 2)$ is below the x-axis for $x < -1$ and $1 < x < 2$. The graph of $y = \dfrac{1}{(x - 1)(x + 1)(x - 2)}$ will also be below the axis for these intervals. There is not enough information to determine the turning points at this stage.



EXERCISE 1.3 Sketching graphs of reciprocal functions

PRACTISE

1 WE6 Use the graph of $y = x - 2$ to sketch $y = \dfrac{1}{x - 2}$.

2 Use the graph of $y = 2x - 1$ to sketch $y = \dfrac{1}{2x - 1}$.

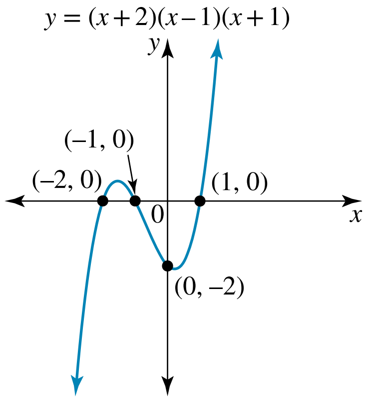
3 WE7 Use the graph of $y = (x + 1)(x - 3)$ to sketch $y = \dfrac{1}{(x + 1)(x - 3)}$.

4 Use the graph of $y = (x - 4)(x - 3)$ to sketch $y = \dfrac{1}{(x - 4)(x - 3)}$.

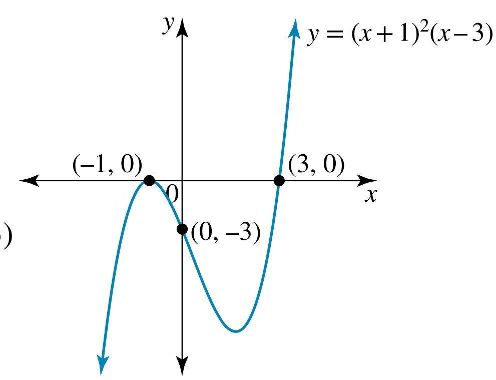
5 WE8 Use the graph of $y = x^2 + 4$ to sketch $y = \dfrac{1}{x^2 + 4}$.

6 Use the graph of $y = -x^2 - 4$ to sketch $y = \dfrac{1}{-x^2 - 4}$.

7 WE9 Use the sketch above right of $y = (x + 2)(x - 1)(x + 1)$ to sketch $y = \dfrac{1}{(x + 2)(x - 1)(x + 1)}$



8 Use the sketch at right of $y = (x + 1)^2(x - 3)$ to sketch $y = \dfrac{1}{(x + 1)^2(x - 3)}$.



CONSOLIDATE

9 Use the graph of $y = x + 1$ to sketch $y = \dfrac{1}{x+1}$.

10 Use the graph of $y = 3 - x$ to sketch $y = \dfrac{1}{3-x}$.

11 Use the graph of $y = (x-1)(x+3)$ to sketch $y = \dfrac{1}{(x-1)(x+3)}$.

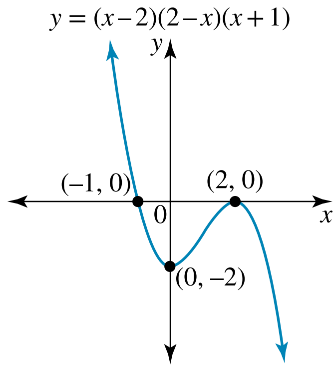
12 Use the graph of $y = (2x-1)(x+3)$ to sketch $y = \dfrac{1}{(2x-1)(x+3)}$.

13 Use the graph of $y = (x-2)^2$ to sketch $y = \dfrac{1}{(x-2)^2}$.

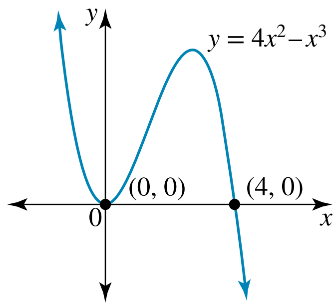
14 Use the graph of $y = x^2 + 3$ to sketch $y = \dfrac{1}{x^2+3}$.

15 Use the graph of $y = -x^2 - 2$ to sketch $y = \dfrac{-1}{x^2+2}$.

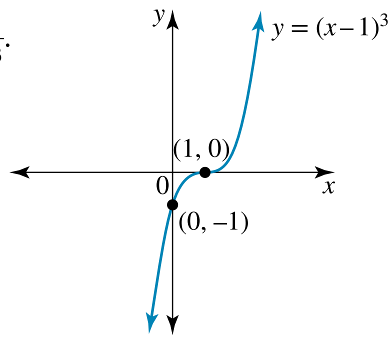
16 Use the graph of $y = (x-2)(2-x)(x+1)$ to sketch $y = \dfrac{1}{(x-2)(2-x)(x+1)}$.



17 Use the graph of $y = 4x^2 - x^3$ to sketch $y = \dfrac{1}{4x^2 - x^3}$.



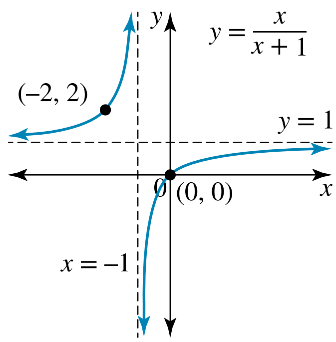
18 Use the graph of $y = (x-1)^3$ to sketch $y = \dfrac{1}{(x-1)^3}$.



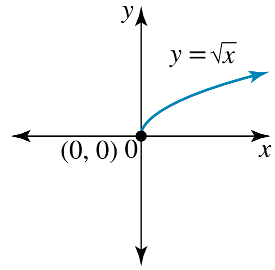
MASTER

The following questions may be attempted with a CAS calculator.

19 The graph of $y = \dfrac{x}{x+1}$ is shown. Use your understanding of reciprocal functions to sketch $y = \dfrac{x+1}{x}$. Justify your decisions. You may use technology to confirm your decisions.



20 The graph of $y = \sqrt{x}$ is shown. Use your understanding of reciprocal functions to sketch $y = \frac{1}{\sqrt{x}}$. Justify the decisions you make. You may use technology to confirm your decisions.



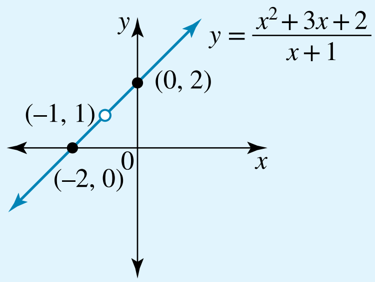
1.4 Sketching graphs of rational functions

Consider the equation $y = \frac{x^2 + 3x + 2}{x + 1}$. It is undefined for $x = -1$. However, it can be simplified by factorising the numerator.

$$y = \frac{(x + 1)(x + 2)}{x + 1},\ x \neq -1$$
$$= x + 2$$

This means that the sketch of $y = \frac{x^2 + 3x + 2}{x + 1}$ looks like the sketch of $y = x + 2$, but there is an undefined point at $x = -1$. This is shown in the following worked example.

WORKED EXAMPLE 10 Sketch $y = \frac{x^2 + 3x + 2}{x + 1}$.

THINK	WRITE/DRAW
1 This function is undefined at $x = -1$.	$x \neq -1$
2 Factorise the numerator.	$y = \frac{(x + 2)(x + 1)}{x + 1},\ x \neq -1$ $= x + 2$
3 Sketch the line with an undefined point at $x = -1$.	

studyon
Units 3 & 4
AOS 1
Topic 1
Concept 1
Sketch graphs of power functions defined by $f(x) = ax + bx^{-n} + c$ for $n = 1$ or 2
Concept summary
Practice questions

Graphing improper fractions

Recall that an **improper fraction** is any fraction in which the numerator is equal to or greater than the denominator, for example $\frac{3}{3}$ or $\frac{9}{4}$. (In other words, an improper fraction is any fraction with an absolute value equal to or greater than 1.) Algebraic fractions can also be improper fractions. One way to identify improper algebraic fractions is if the degree of the numerator is equal to or higher than the degree of the denominator, for example $\frac{x + 1}{x}$ or $\frac{2x + 5}{x + 1}$. Conversely, if the degree of the numerator is lower than the degree of the denominator, the rational expression is a **proper fraction**.

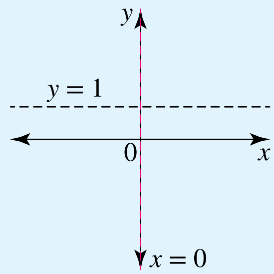
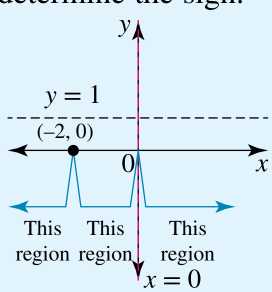
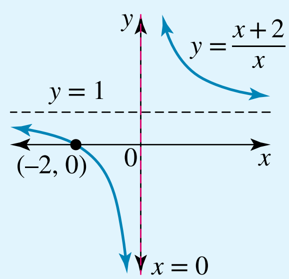
Improper fractions can be rewritten as mixed fractions. For example, $\frac{x+1}{x} = 1 + \frac{1}{x}$ and $\frac{2x+5}{x+1} = \frac{2(x+1)+3}{(x+1)} = 2 + \frac{3}{x+1}$. When graphing a rational expression that is also an improper fraction, it is not necessary to complete the division; we need only to determine the initial term, as this becomes the horizontal asymptote.

WORKED EXAMPLE 11

Sketch the following.

a $y = \frac{x+2}{x}$

b $y = \frac{2x}{x+2}$

THINK	WRITE/DRAW
a 1 The function is undefined at $x = 0$, so this is the vertical asymptote.	**a** $x \neq 0 \therefore$ Vertical asymptote is $x = 0$.
2 To find the horizontal asymptote, begin to simplify the expression.	Horizontal asymptote: $y = \frac{x}{x}$ $= 1$
3 Determine the x- and y-intercepts.	$x \neq 0 \therefore$ No y-intercept. $y = 0$: $0 = \frac{x+2}{x}$ $x = -2$
4 Sketch the asymptotes and intercepts.	
5 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign. 	(see table below)
6 Sketch the function.	

x	Sign of $y = \frac{x+2}{x}$	Position
-3	$\frac{(-)}{(-)} = (+)$	Above the x-axis
-1	$\frac{(+)}{(-)} = (-)$	Below the x-axis
1	$\frac{(+)}{(+)} = (+)$	Above the x-axis

b **1** The function is undefined at $x = -2$ so this is the vertical asymptote.

b $x \neq -2 \therefore$ Vertical asymptote is $x = -2$.

2 To find the horizontal asymptote, begin to simplify the expression.

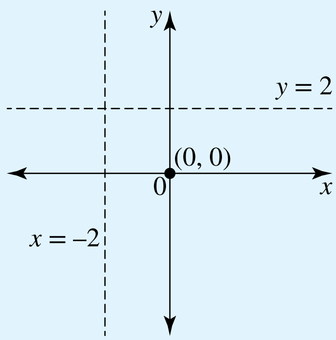
Horizontal asymptote: $y = \frac{2x}{x}$
$= 2$

3 Determine the x- and y-intercepts.

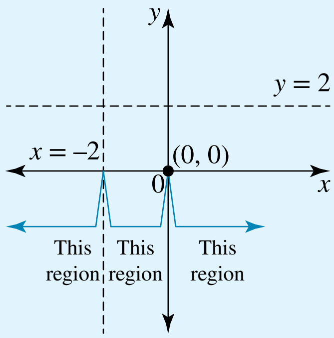
$x = 0$: $y = 0$

$y = 0$: $0 = \frac{2x}{x+2}$
$x = 0$

4 Sketch the asymptotes and intercepts.

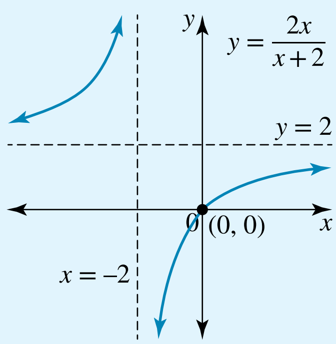


5 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.



x	Sign of $y = \frac{2x}{x+2}$	Position
−3	$\frac{(-)}{(-)} = (+)$	Above the x-axis
−1	$\frac{(-)}{(+)} = (-)$	Below the x-axis
2	$\frac{(+)}{(+)} = (+)$	Above the x-axis

6 Sketch the function.



WORKED EXAMPLE 12 Sketch the following: **a** $y = \frac{x^2 - x - 6}{x^2 - 1}$ **b** $y = \frac{2x^2 + 5}{x^2 - 25}$

THINK

a **1** Determine vertical asymptotes by solving $x^2 - 1 = 0$.

WRITE/DRAW

a $x^2 - 1 = 0$
$(x + 1)(x - 1) = 0$
$x = \pm 1$

The vertical asymptotes are $x = 1$ and $x = -1$.

2 To find the horizontal asymptote, begin to simplify the expression.

Horizontal asymptote: $y = \dfrac{x^2}{x^2}$

$= 1$

3 Determine the x- and y-intercepts.

$x = 0$: $y = \dfrac{-6}{-1}$

$= 6$

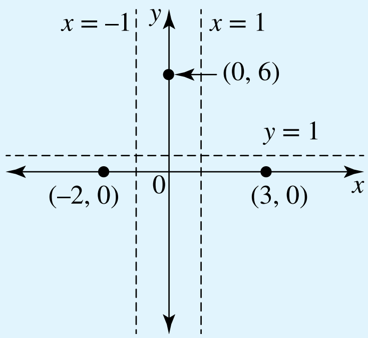
$y = 0$: $0 = \dfrac{x^2 - x - 6}{x^2 - 1}$

$0 = x^2 - x - 6$

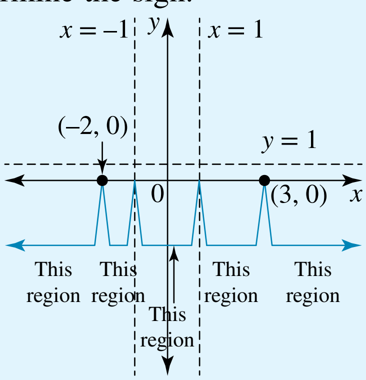
$= (x - 3)(x + 2)$

$x = 3, x = -2$

4 Sketch the asymptotes and intercepts.

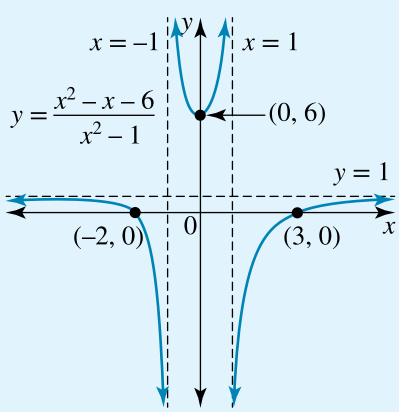


5 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.



x	Sign of $y = \dfrac{x^2 - x - 6}{x^2 - 1} = \dfrac{(x - 3)(x + 2)}{(x - 1)(x + 1)}$	Position
-3	$\dfrac{(-)(-)}{(-)(-)} = (+)$	Above the x-axis
-1.5	$\dfrac{(-)(+)}{(-)(-)} = (-)$	Below the x-axis
0	$\dfrac{(-)(+)}{(-)(+)} = (+)$	Above the x-axis
2	$\dfrac{(-)(+)}{(+)(+)} = (-)$	Below the x-axis
4	$\dfrac{(+)(+)}{(+)(+)} = (+)$	Above the x-axis

6 Sketch the function.



b **1** Determine vertical asymptotes by solving $x^2 - 25 = 0$.

b
$$x^2 - 25 = 0$$
$$(x + 5)(x - 5) = 0$$
$$x = \pm 5$$
The vertical asymptotes are $x = 5$ and $x = -5$.

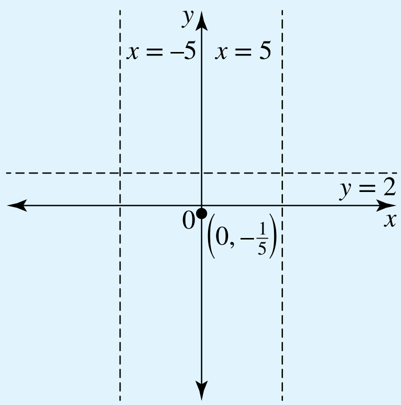
2 To find the horizontal asymptote, begin to simplify the expression.

Horizontal asymptote: $y = \dfrac{2x^2}{x^2}$
$$= 2$$

3 Determine the x- and y-intercepts.

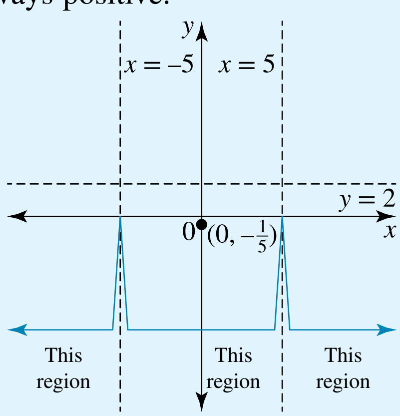
$x = 0$: $y = \frac{5}{-25}$
$$= \frac{-1}{5}$$
$y = 0$: $0 = \dfrac{2x^2 + 5}{x^2 - 25}$
$$0 = 2x^2 + 5$$
There is no x-intercept.

4 Sketch the asymptotes and intercepts.



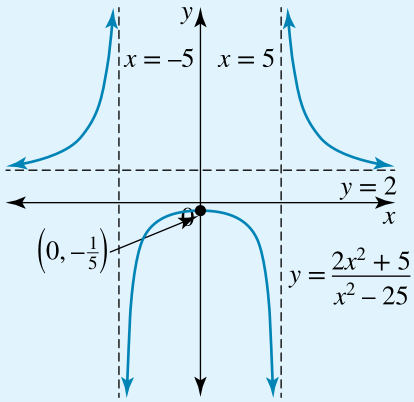
5 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

In this instance, the numerator is always positive.



x	Sign of $y = \dfrac{2x^2 + 5}{x^2 - 25} = \dfrac{2x^2 + 5}{(x + 5)(x - 5)}$	Position
−6	$\dfrac{(+)}{(-)(-)} = (+)$	Above the x-axis
0	$\dfrac{(+)}{(+)(-)} = (-)$	Below the x-axis
6	$\dfrac{(+)}{(+)(+)} = (+)$	Above the x-axis

6 Sketch the function.



Graphs that cross the horizontal asymptote

It is possible for a graph to cross the horizontal asymptote (but never the vertical asymptote). The horizontal asymptote is approached at the extreme ends of the graph ($x \to \infty$ and $x \to -\infty$) but may be crossed elsewhere in the graph.

In Worked example 13, because $y = \frac{x-3}{x^2 - 3x - 10}$ is a proper fraction, $y = 0$ is the horizontal asymptote. However, if $x = 3$, then $y = 0$, so there is a value for x that lies on the horizontal asymptote. The graph will approach $y = 0$ as $x \to \pm\infty$.

WORKED EXAMPLE 13 Sketch $y = \frac{x-3}{x^2 - 3x - 10}$.

THINK

1 Determine the vertical asymptotes by solving $x^2 - 3x - 10 = 0$.

2 The expression is a proper fraction, so $y = 0$ is the horizontal asymptote.

3 Determine the x- and y-intercepts.

4 Sketch the asymptotes and intercepts.

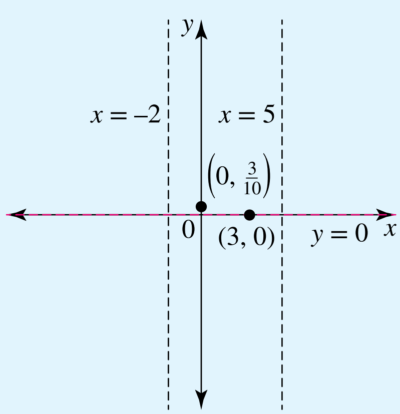
WRITE/DRAW

1
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
The vertical asymptotes are $x = 5$ and $x = -2$.

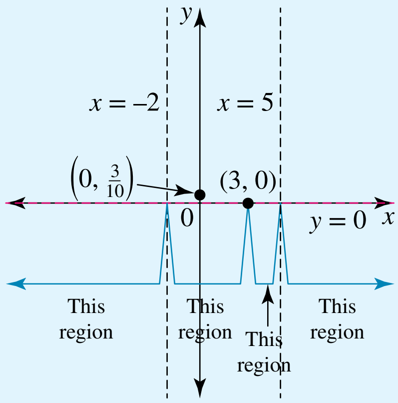
2 Horizontal asymptote: $y = 0$.

3
$$x = 0: y = \frac{-3}{-10} = \frac{3}{10}$$
$$y = 0: x - 3 = 0$$
$$x = 3$$

4

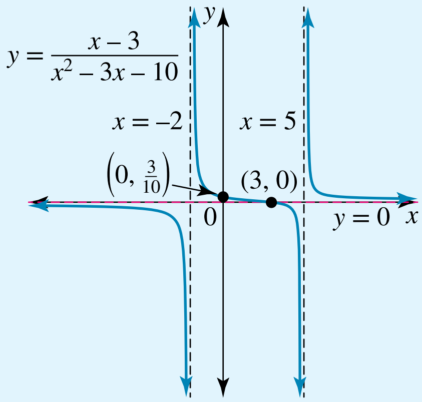


5 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual x-values, merely to determine the sign.



x	Sign of $y = \dfrac{x-3}{x^2-3x-10} = \dfrac{x-3}{(x-5)(x+2)}$	Position
−3	$\dfrac{(-)}{(-)(-)} = (-)$	Below the x-axis
−1	$\dfrac{(-)}{(-)(+)} = (+)$	Above the x-axis
4	$\dfrac{(+)}{(-)(+)} = (-)$	Below the x-axis
6	$\dfrac{(+)}{(+)(+)} = (+)$	Above the x-axis

6 Sketch the function.



Graphs without vertical asymptotes

Consider a rational function in the form $\dfrac{f(x)}{g(x)}$. If there is no solution for $g(x) = 0$, then there will be no vertical asymptotes. The horizontal asymptote is identified in the usual manner.

WORKED EXAMPLE 14

a Sketch $y = \dfrac{x+4}{x^2+2}$.

b Sketch $y = \dfrac{x^2-x-6}{x^2+1}$.

THINK	WRITE/DRAW
a 1 There are no solutions to $x^2 + 2 = 0$, so there are no vertical asymptotes.	**a** $x^2 + 2 \neq 0$ There are no vertical asymptotes.
2 The expression is a proper fraction, so $y = 0$ is the horizontal asymptote.	Horizontal asymptote: $y = 0$.

3 Determine the x- and y-intercepts.

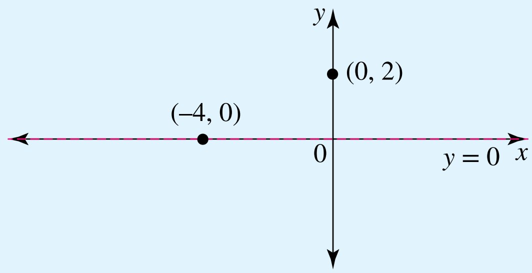
$x = 0$: $y = \frac{4}{2}$

$= 2$

$y = 0$: $x + 4 = 0$

$x = -4$

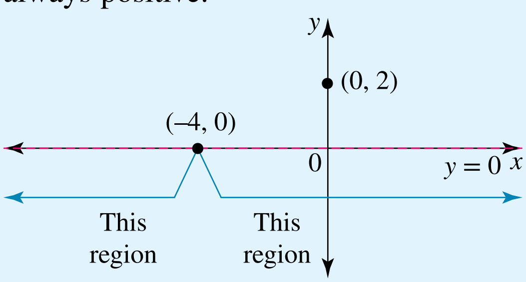
4 Sketch the asymptotes and intercepts.



5 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

In this instance, the denominator is always positive.

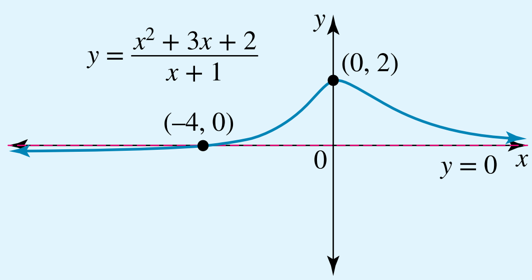
x	Sign of $y = \frac{x+4}{x^2+2}$	Position
5	$\frac{(-)}{(+)} = (-)$	Below the x-axis
0	$\frac{(+)}{(+)} = (+)$	Above the x-axis



6 Sketch the function.

The function crosses the x-axis when $x = -4$ but must approach the horizontal asymptote as $x \to -\infty$. This means that there is a local minimum in the interval $(-\infty, -4)$.

Similarly, the function passes through the points $(-4, 0)$ and $(0, 2)$ but approaches the horizontal asymptote as $x \to \infty$. If $x = -1$, $y = -1$. If $x = 1$, $y = \frac{5}{3}$. Both of these y-values are smaller than the y-intercept of 2. Therefore, there is a local maximum in the interval $(-1, 1)$.



b 1 There are no solutions to $x^2 + 1 = 0$, so there are no vertical asymptotes.

b $x^2 + 1 \neq 0$

There are no vertical asymptotes.

2 The expression is not a proper fraction. To find the horizontal asymptote, begin to simplify the expression.

Horizontal asymptote: $y = \frac{x^2}{x^2}$.

$= 1$

3 Determine the x- and y-intercepts.

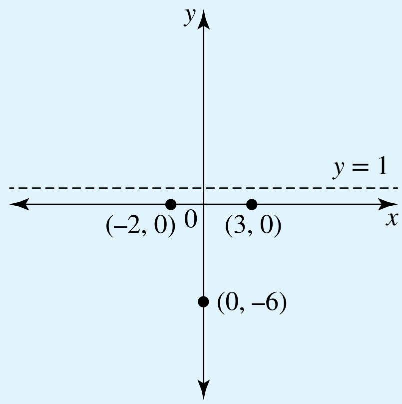
$$x = 0: \; y = \frac{-6}{1} = -6$$

$$y = 0: \quad x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3, \; x = -2$$

4 Check whether the graph crosses the horizontal asymptote $y = 1$.

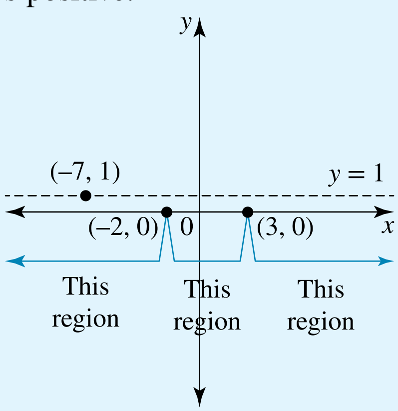
$$\frac{x^2 - x - 6}{x^2 + 1} = 1$$
$$x^2 - x - 6 = x^2 + 1$$
$$0 = x + 7$$
$$x = -7$$

5 Sketch the asymptotes and intercepts.



6 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual x-values, merely to determine the sign. If the function is above the x-axis, it may be useful to determine if it is above or below the horizontal asymptote.

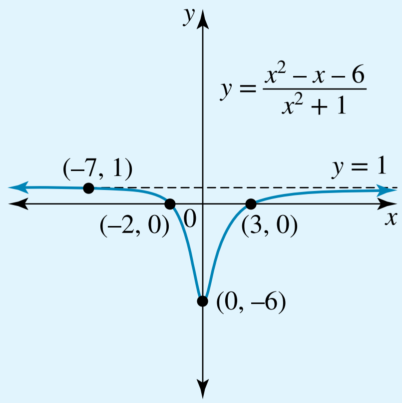
In this instance, the denominator is always positive.



x	Sign of $y = \frac{x^2 - x - 6}{x^2 + 1} = \frac{(x - 3)(x + 2)}{x^2 + 1}$	Position
-8	$\frac{(-)(-)}{(+)} = (+)$ $\frac{(-11)(-6)}{65} > 1$	Above the x-axis Above the horizontal asymptote
-3	$\frac{(-)(+)}{(+)} = (-)$	Below the x-axis
0	$\frac{(-)(+)}{(+)} = (-)$	Below the x-axis
4	$\frac{(+)(+)}{(+)} = (+)$ $\frac{(1)(6)}{17} < 1$	Above the x-axis Below the horizontal asymptote

7 Sketch the function.

The function crosses the horizontal asymptote when $x = -7$ but must approach the horizontal asymptote as $x \to -\infty$. This means that there is a local maximum in the interval $(-\infty, -7)$.



studyon

Units 3 & 4

AOS 1

Topic 1

Concept 2

Sketch graphs of power functions defined by $f(x) = ax^2 + bx^{-n} + c$ for $n = 1$ or 2

Concept summary

Practice questions

Oblique asymptotes

Sometimes, instead of a horizontal asymptote, the function may have an oblique asymptote (a line in the form $y = mx + c$). As with horizontal asymptotes, the function may cross an oblique asymptote, but as $x \to \pm\infty$, the function will approach the oblique asymptote.

For example, the function $y = \dfrac{x+1}{x}$ can be rewritten as $y = 1 + \dfrac{1}{x}$. In this instance, the horizontal asymptote is $y = 1$.

The function $y = \dfrac{x^2+1}{x}$ can be rewritten as $y = x + \dfrac{1}{x}$. In this instance, the oblique asymptote is $y = x$.

In the previous examples involving improper fractions, the degrees of the numerator and denominator were equal, so it was only necessary to begin the simplification to identify the horizontal asymptote. If the degree of the numerator is greater than that of the denominator, it is necessary to rewrite the expression as a mixed fraction. A CAS calculator may assist with this process.

WORKED EXAMPLE 15

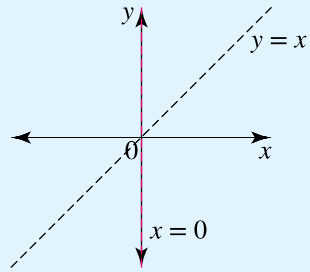
Sketch the following.

a $y = \dfrac{x^2+1}{x}$

b $y = \dfrac{x^3-8}{x^2+5x+6}$

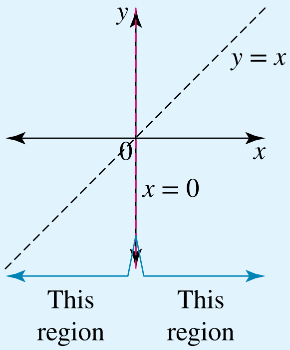
THINK	WRITE/DRAW
a 1 Determine the vertical asymptotes.	**a** The vertical asymptote is $x = 0$.
2 The expression is an improper fraction, so rewrite it (using a CAS calculator if needed).	$y = \dfrac{x^2+1}{x}$ $= x + \dfrac{1}{x}$
3 Identify the oblique asymptote.	The oblique asymptote is $y = x$.
4 Determine the x- and y-intercepts.	$x = 0$: undefined. There are no x-intercepts. $y = 0$: $x^2 + 1 = 0$. No solution. There are no y-intercepts.

5 Sketch the asymptotes.



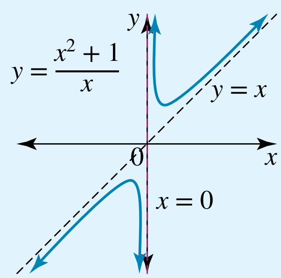
6 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

The numerator will always be positive.



x	Sign of $y = \frac{x^2 + 1}{x}$	Position
-1	$\frac{(+)}{(-)} = (-)$	Below the x-axis
1	$\frac{(+)}{(+)} = (+)$	Above the x-axis

7 Sketch the function.



b 1 Determine the vertical asymptotes by solving $x^2 + 5x + 6 = 0$.

b $x^2 + 5x + 6 = 0$

$(x + 3)(x + 2) = 0$

$x = -3, \; x = -2$

The vertical asymptotes are $x = -3$ and $x = -2$.

2 The function is an improper fraction, so rewrite it (using a CAS calculator if needed).

$$y = \frac{x^3 - 8}{x^2 + 5x + 6}$$

$$= x - 5 + \frac{19x + 22}{x^2 + 5x + 6}$$

3 Identify the oblique asymptote.

The oblique asymptote is $y = x - 5$.

4 Find where the graph crosses the oblique asymptote by solving $\frac{19x + 22}{x^2 + 5x + 6} = 0$.

Use the asymptote $y = x - 5$ to identify the x-value.

The graph will cross the oblique asymptote when $19x + 22 = 0$

$x = -1.16$

$y = -1.16 - 5$

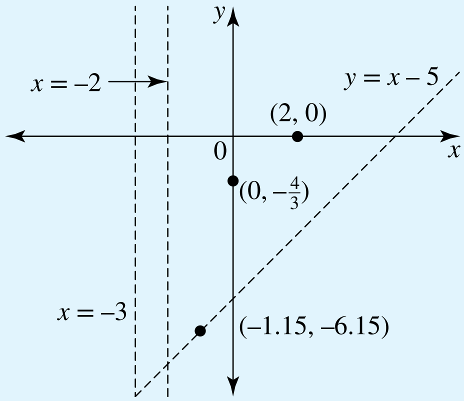
$= -6.16$

5 Determine the x- and y-intercepts.

$x = 0$: $y = \frac{-8}{6}$
$= \frac{-4}{3}$

$y = 0$: $x^3 - 8 = 0$
$x = 2$

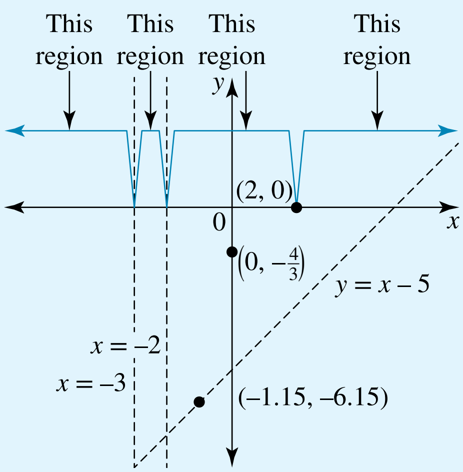
6 Sketch the asymptotes and intercepts.



7 Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

Although the numerator can be factorised as $(x - 2)(x^2 + 2x + 4)$ it is probably easier to use the form $x^3 - 8$.

If the oblique asymptote and the graph are on the same side of the x-axis, evaluate the sign of $\frac{19x + 22}{(x + 2)(x + 3)}$ to determine if the function is above or below the asymptote.



x	Sign of $y = \frac{x^3 - 8}{x^2 + 5x + 6} = \frac{x^3 - 8}{(x + 2)(x + 3)}$	Position
-4	$\frac{(-)}{(-)(-)} = (-)$ $\left(\frac{19x + 22}{(x + 2)(x + 3)}\right) = \frac{(-)}{(-)(-)} = (-)$	Below the x-axis Below the oblique asymptote
-2.5	$\frac{(-)}{(+)(-)} = (+)$	Above the x-axis
-1.5	$\frac{(-)}{(+)(+)} = (-)$ $\left(\frac{19x + 22}{(x + 2)(x + 3)}\right) = \frac{(-)}{(+)(+)} = (-)$	Below the x-axis Below the oblique asymptote
0	$\frac{(-)}{(+)(+)} = (-)$ $\left(\frac{19x + 22}{(x + 2)(x + 3)}\right) = \frac{(+)}{(+)(+)} = (+)$	Below the x-axis Above the oblique asymptote
3	$\frac{(+)}{(+)(+)} = (+)$	Above the x-axis

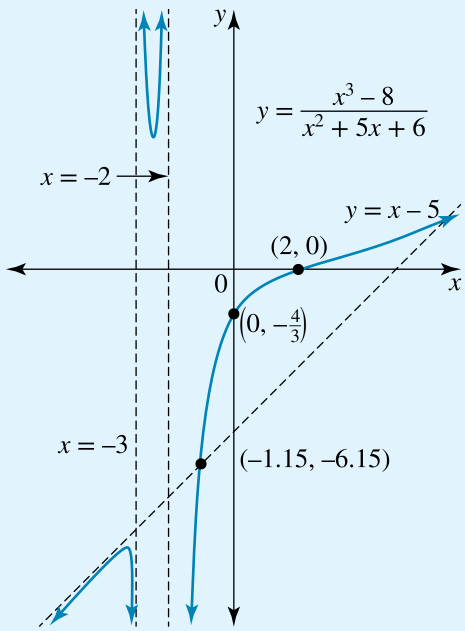
8 Sketch the function.

In the interval $(-\infty, -3)$, the function is below the oblique asymptote.

In the interval $(-3, -2)$, the function is above the x-axis. The point $(-2.5,\ 94.5)$ lies on the function in this interval.

In the interval $(-2,\ \infty)$, the function is initially below the oblique asymptote. It crosses over the asymptote at the point $(-1.15,\ -6.15)$ and then remains above the oblique asymptote.

The function will approach the oblique asymptote as $x \to \pm\infty$.



EXERCISE 1.4 Sketching graphs of rational functions

PRACTISE

1 **WE10** Sketch $y = \dfrac{x^2 - x - 6}{x - 3}$.

2 Sketch $y = \dfrac{x^2 - 4x + 4}{x - 2}$.

3 **WE11** Sketch the following.

a $y = \dfrac{x + 3}{x}$ b $y = \dfrac{2x}{x + 3}$

4 Sketch the following.

a $y = \dfrac{x - 2}{x}$ b $y = \dfrac{3x}{x - 2}$

5 **WE12** Sketch the following.

a $y = \dfrac{x^2 - 1}{x^2 - x - 6}$ b $y = \dfrac{2x^2}{x^2 - 1}$

6 Sketch the following.

a $y = \dfrac{x^2 - 2x - 3}{x^2 - 2x - 15}$ b $y = \dfrac{3x^2}{1 - x^2}$

7 **WE13** Sketch $y = \dfrac{x + 1}{x^2 - x - 6}$.

8 Sketch $y = \dfrac{3x}{x^2 + 2x - 8}$.

9 **WE14** Sketch the following.

a $y = \dfrac{x + 2}{x^2 + 1}$ b $y = \dfrac{x^2 - 1}{x^2 + 2}$

10 Sketch the following.

a $y = \frac{x}{x^2 + 3}$

b $y = \frac{x^2 - 4}{x^2 + 2}$

11 WE15 Sketch the following.

a $y = \frac{x^2 - 1}{x}$

b $y = \frac{x^3}{x^2 - 2x - 3}$

12 Sketch $y = \frac{1 - x^2}{x}$.

CONSOLIDATE

13 Sketch the following.

a $y = \frac{x^2 - x - 2}{x - 2}$

b $y = \frac{x^2 - 1}{x + 1}$

14 Sketch $y = \frac{x + 5}{x + 2}$.

15 Sketch $y = \frac{2x + 1}{x + 3}$.

16 Sketch the following.

a $y = \frac{x^2}{x^2 - 4}$

b $y = \frac{4x^2 - 1}{x^2 - 3x - 4}$

17 Sketch $y = \frac{x^2 - 2x - 3}{x^2 - 2x + 1}$.

18 Sketch the following.

a $y = \frac{x - 1}{x^2 - 2x + 1}$

b $y = \frac{2 - x}{x^2 - 3x - 4}$

c $y = \frac{x - 4}{4 - x^2}$

19 Sketch the following.

a $y = \frac{1 - x}{x^2 + 3}$

b $y = \frac{x + 2}{x^2 + 2}$

20 Sketch the following.

a $y = \frac{1 - x^2}{x^2 + 1}$

b $y = \frac{2x^2}{x^2 + 2}$

21 Sketch the following.

a $y = \frac{x^2 - 2}{x}$

b $y = \frac{x^2}{x - 1}$

22 Sketch the following.

a $y = \frac{x^2 - 2x + 1}{x^2 - x - 2}$

b $y = \frac{x^2 + x - 6}{x^2 - 9}$

MASTER

23 Sketch the following.

a $y = \frac{2x^2 - x}{x - 1}$

b $y = \frac{x^2 - 5x + 7}{x - 2}$

24 Sketch the following.

a $y = \frac{x^3 + x - 1}{x^2 - 1}$

b $y = \frac{2x^3 - 4x^2 - x + 5}{x^2 - x - 2}$

1.5 Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$

studyon

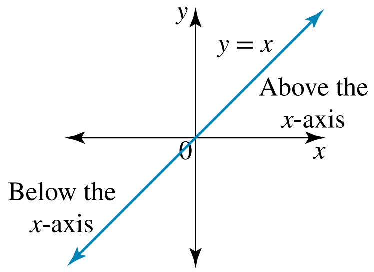
Units 3 & 4
AOS 1
Topic 1
Concept 9

Graphs of modulus functions
Concept summary
Practice questions

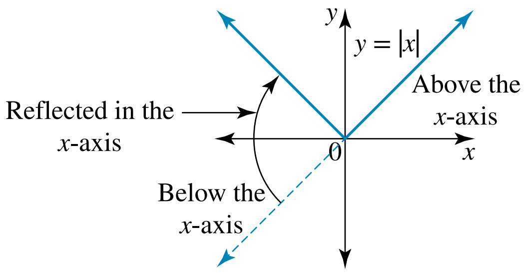
Graph of $y = |x|$

Consider the graph of $y = x$. When $x \geq 0$, $y \geq 0$, and when $x < 0$, $y < 0$.

x	−2	−1	0	1	2
$\lvert x \rvert$	2	1	0	1	2

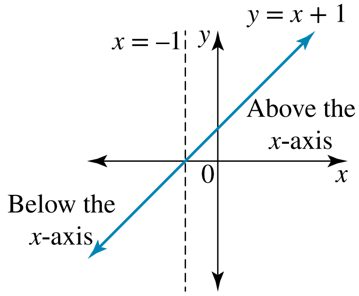


This means that $y = x$ and $y = |x|$ are the same for $x \geq 0$, but $y = |x|$ and $y = -x$ are the same for $x < 0$.



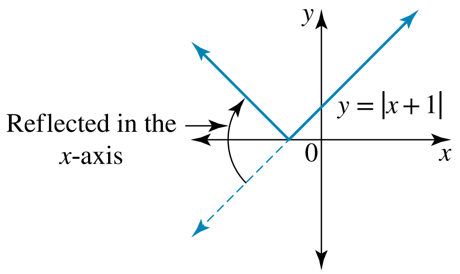
Graphing $y = |f(x)|$ from $y = f(x)$

Consider the function $y = x + 1$.



$|x + 1| = x + 1$ for $x \geq -1$ and $|x + 1| = -(x + 1)$ for $x < -1$.

The means that the graph of $y = |x + 1|$ looks like the following.



WORKED EXAMPLE 16

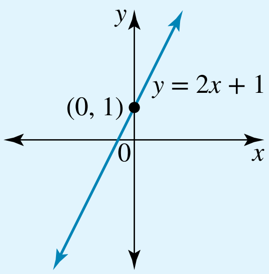
Use the graph of $y = 2x + 1$ to sketch $y = |2x + 1|$.

THINK

WRITE/DRAW

1 Sketch $y = 2x + 1$.

The gradient is 2 and the y-intercept is 1.

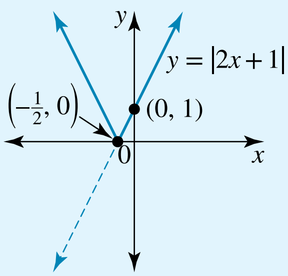


2 Identify where $2x + 1 \geq 0$.

$2x + 1 \geq 0$

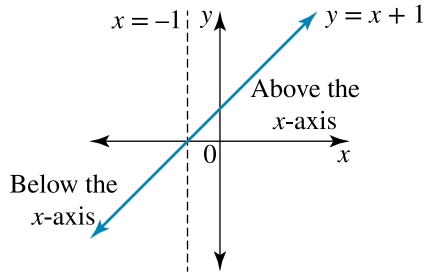
$x \geq -\frac{1}{2}$

3 Sketch $y = |2x + 1|$.



Graphing $y = f(|x|)$ from $y = f(x)$

If we again consider the graph of $y = x + 1$, how does it change if we sketch $y = |x| + 1$?

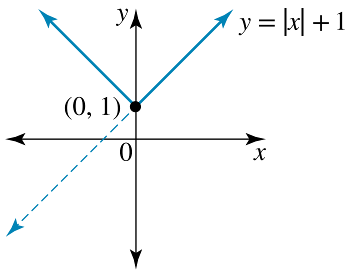


For $x \geq 0$, $|x| = x$. Therefore, the graph would be unchanged for this interval.

x	−2	−1	0	1	2
$\lvert x\rvert$	2	1	0	1	2
$y = \lvert x\rvert + 1$	3	2	1	2	3

This means that the graph for the interval $x \geq 0$ is reflected in the y-axis.

Note also that this is a vertical shift of 1 for the graph of $y = |x|$.



WORKED EXAMPLE 17

Use the graph of $y = 2x + 1$ to sketch $y = 2|x| + 1$.

THINK

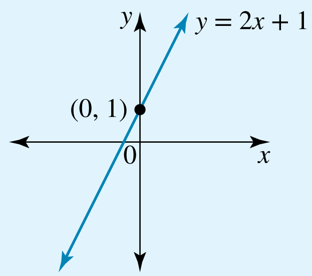
1 Sketch $y = 2x + 1$.

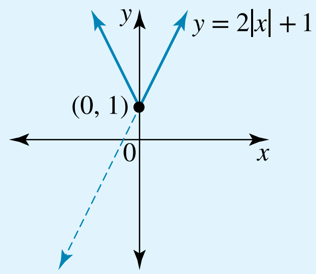
The gradient is 2 and the y-intercept is 1.

2 $|x| = x$ for $x \geq 0$; therefore, the graph is unchanged for $x \geq 0$.

The graph is symmetrical about the y-axis.

WRITE/DRAW





Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$

The patterns observed above are true for any graph involving the modulus function.

When graphing $y = |f(x)|$,

$$y = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0. \end{cases}$$

When graphing $y = f\left(|x|\right)$,

$$y = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0. \end{cases}$$

WORKED EXAMPLE 18

Use the graph of $y = (x - 4)(x - 2)$ to sketch:

a $y = |(x - 4)(x - 2)|$

b $y = \left(|x| - 4\right)\left(|x| - 2\right)$.

THINK

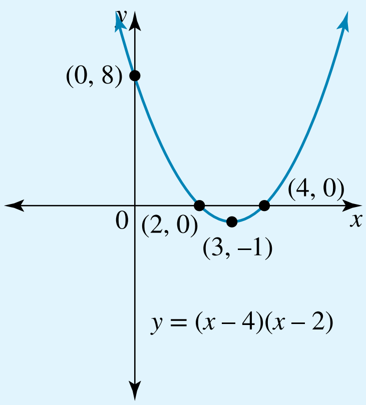
Sketch $y = (x - 4)(x - 2)$.

The x-intercepts are $x = 4$ and $x = 2$.

The x-value for the turning point is the midpoint of the intercepts, $(3, -1)$.

The y-intercept is $y = 8$.

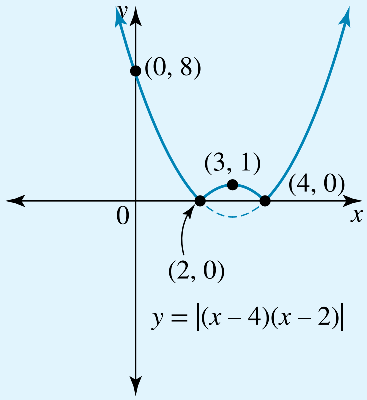
WRITE/DRAW



a **1** Identify where the graph is above and below the x-axis.

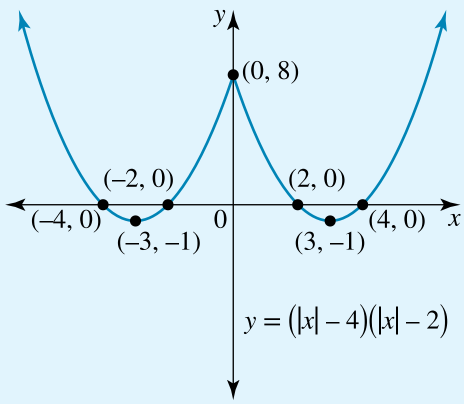
a The graph is above the x-axis for $x \leq 2$ and $x \geq 4$.
The graph is below the x-axis for $2 < x < 4$.

2 Sketch $y = |(x - 4)(x - 2)|$ by reflecting the graph for the interval (2, 4) in the x-axis.



b The graph for $y = \left(|x| - 4\right)\left(|x| - 2\right)$ is the same as the graph of $y = (x - 4)(x - 2)$ for $x \geq 0$.
For $x < 0$, reflect the graph in the y-axis.

b



Solving inequations involving the modulus function

When solving inequations involving a modulus function, sketching the function may assist in identifying the interval.

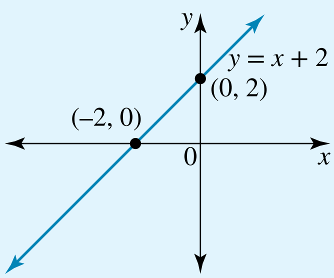
WORKED EXAMPLE 19

Use the graph of $y = |x + 2|$ to solve $|x + 2| < 1$.

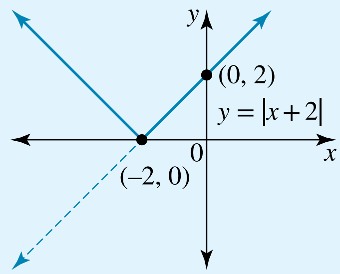
THINK

1 Sketch $y = x + 2$.
The gradient is 1 and the y-intercept is 2.

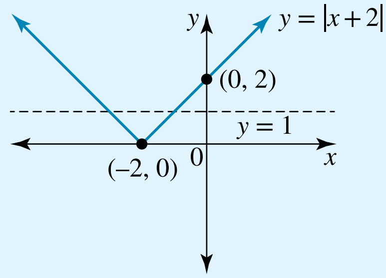
WRITE/DRAW



2 The graph of $y = |x + 2|$ is the same for $x \geq -2$ and reflected in the x-axis for $x < -2$.
Sketch $y = |x + 2|$.



3 Sketch the line $y = 1$.



4 Find the points of intersection.
One point occurs where $x \geq -2$, so $y = x + 2$.

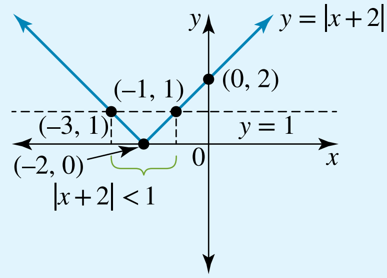
The second point occurs where $x < -2$, so $y = -(x + 2)$

Locating points of intersection:

$$x + 2 = 1$$
$$x = -1$$
$$-(x + 2) = 1$$
$$x + 2 = -1$$
$$x = -3$$

5 Use the diagram to identify when $|x + 2| < 1$.

$|x + 2| < 1$ for $-3 < x < -1$

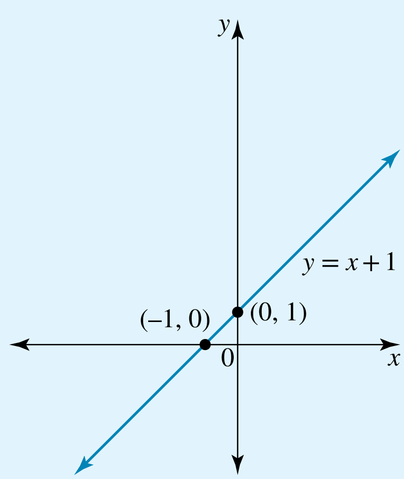


WORKED EXAMPLE 20 Use the graph of $y = x + 1$ to solve $|x + 1| < 2x + 8$.

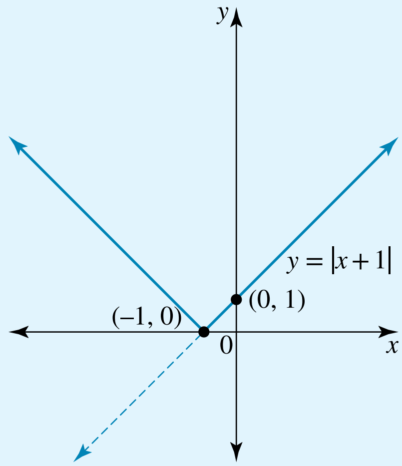
THINK

1 Sketch $y = x + 1$.
The gradient is 1 and the y-intercept is 1.

WRITE/DRAW

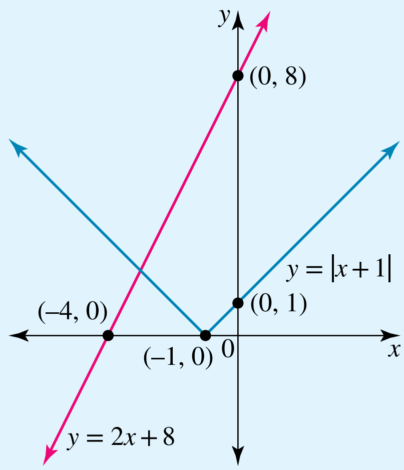


2 The graph of $y = |x + 1|$ is the same for $x \geq -1$ and reflected in the x-axis for $x < -1$.
Sketch $y = |x + 1|$.



3 Sketch the line $y = 2x + 8$.

The gradient is 2 and the y-intercept is 8.



4 Find the point of intersection.

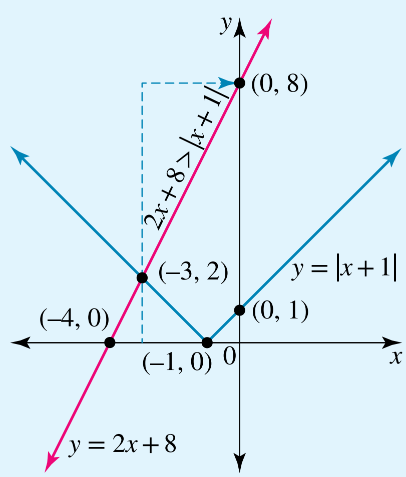
The point occurs where $x \leq -1$, so $y = -(x + 1)$.

Locating point of intersection:

$$\begin{aligned} -(x + 1) &= 2x + 8 \\ -x - 1 &= 2x + 8 \\ 3x &= -9 \\ x &= -3 \end{aligned}$$

5 Use the diagram to identify when $|x + 1| < 2x + 8$.

$|x + 1| < 2x + 8$ for $x > -3$



EXERCISE 1.5 Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$

PRACTISE

1 WE16 Use the graph of $y = 2x - 3$ to sketch $y = |2x - 3|$.

2 Use the graph of $y = 1 - 2x$ to sketch $y = |1 - 2x|$.

3 WE17 Use the graph of $y = 2x - 3$ to sketch $y = 2|x| - 3$.

4 Use the graph of $y = 1 - 2x$ to sketch $y = 1 - 2|x|$.

5 WE18 Use the graph of $y = (x - 1)(x + 3)$ to sketch:

a $y = |(x - 1)(x + 3)|$

b $y = \left(|x| - 1\right)\left(|x| + 3\right)$.

6 Use the graph of $y = (1 - x)(x + 5)$ to sketch:

a $y = |(1 - x)(x + 5)|$

b $y = \left(1 - |x|\right)\left(|x| + 5\right)$.

7 WE19 Use the graph of $y = |x + 1|$ to solve $|x + 1| < 2$.

8 Use the graph of $y = |x - 2|$ to solve $|x - 2| < 3$.

9 WE20 Use the graph of $y = x - 1$ to solve $|x - 1| > -2x + 8$.

10 Use the graph of $y = x + 2$ to solve $|x + 2| > 3x + 4$.

CONSOLIDATE

11 Use the graph of $y = 1 - 2x$ to sketch:

a $y = |1 - 2x|$

b $y = 1 - 2|x|$.

12 Use the graph of $y = 3x + 2$ to sketch:

a $y = |3x + 2|$

b $y = 3|x| + 2$.

13 Use the graph of $y = (x + 1)(x - 3)$ to sketch:

a $y = |(x + 1)(x - 3)|$

b $y = \left(|x| + 1\right)\left(|x| - 3\right)$.

14 Use the graph of $y = (x - 1)(x + 1)(x - 2)$ to sketch:

a $y = |(x - 1)(x + 1)(x - 2)|$

b $y = \left(|x| - 1\right)\left(|x| + 1\right)\left(|x| - 2\right)$.

15 Use the graph of $y = \dfrac{1}{x}$ to sketch:

a $y = \left|\dfrac{1}{x}\right|$

b $y = \dfrac{1}{|x|}$.

What do you notice about your answers for a and b?

16 Use the graph of $y = (x - 2)^2$ to sketch:

a $y = |(x - 2)^2|$

b $y = \left(|x| - 2\right)^2$.

What do you notice about your answer for a?

17 Use the graph of $y = \left|\dfrac{1}{x}\right|$ to solve $\left|\dfrac{1}{x}\right| \le 1$.

18 Use the graph of $y = \left|\dfrac{x}{2} - 1\right|$ to solve $\left|\dfrac{x}{2} - 1\right| \le 1$.

19 Use the graph of $y = |x + 2|$ to solve $|x + 2| < \frac{1}{2}x + 3$.

20 Use the graph of $y = x - 3$ to solve $|x - 3| > 3x + 8$.

MASTER

21 Use suitable graphs to solve $|x - 1| > |2x + 3|$.

22 Use suitable graphs to solve $\dfrac{|x - 1|}{|2x - 4|} > 1$.

1.6 Circles, ellipses and hyperbolas

If we consider equations in the form $Ax^2 + By^2 + Cx + Dy + E = 0$, depending on the values of A and B, the equation may be a circle, an ellipse or a hyperbola. If $A = B$, then the equation is a circle. If $A > 0$ and $B > 0$ but $A \neq B$, then the equation is an ellipse. If either $A > 0$ and $B < 0$ or $A < 0$ and $B > 0$, then the equation is a hyperbola.

studyon

Units 3 & 4
AOS 1
Topic 1
Concept 5

Sketch graphs of circles
Concept summary
Practice questions

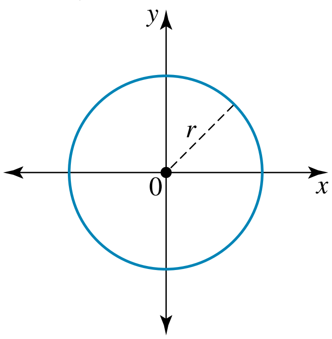
Each equation can be rewritten in a form that gives information to assist with sketching.

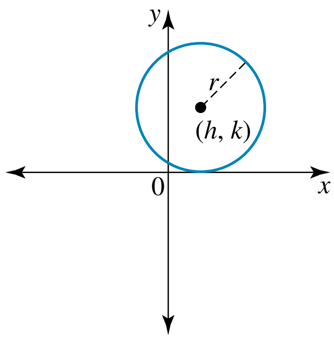
Note: The reciprocal functions investigated in previous sections of this topic include functions of the form $y = \dfrac{1}{x - h} + k$. These are examples of hyperbolas with horizontal and vertical asymptotes.

Circles

A circle with its centre at the origin and radius r is described by the equation $x^2 + y^2 = r^2$ as shown below left.

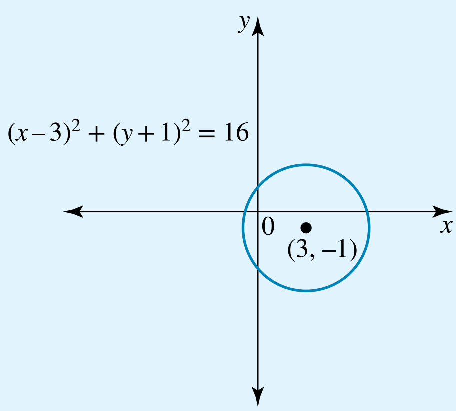
A circle with its centre at (h, k) and radius r is described by the equation $(x - h)^2 + (y - k)^2 = r^2$ as shown below right.





WORKED EXAMPLE 21

Sketch the circle $x^2 + y^2 - 6x + 2y = 6$.

THINK	WRITE/DRAW
1 The circle needs to be in the form $(x - h)^2 + (y - k)^2 = r^2$ to identify the centre and radius. Use completing the square to achieve this.	$x^2 + y^2 - 6x + 2y = 6$ $x^2 - 6x + y^2 + 2y = 6$ $x^2 - 6x + 9 + y^2 + 2y + 1 = 6 + 9 + 1$ $(x - 3)^2 + (y + 1)^2 = 16$
2 Identify the centre and radius of the circle.	$h = 3$, $k = -1$, $r = 4$ Centre $(3, -1)$, radius 4
3 Sketch the circle.	

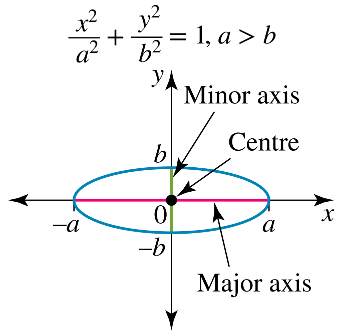
studyon

Units 3 & 4
AOS 1
Topic 1
Concept 6

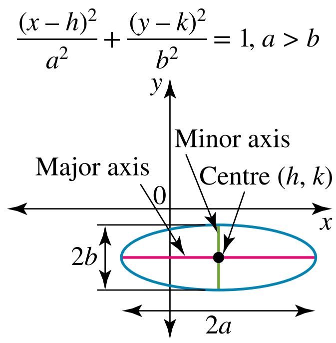
Sketch graphs of ellipses
Concept summary
Practice questions

Ellipses

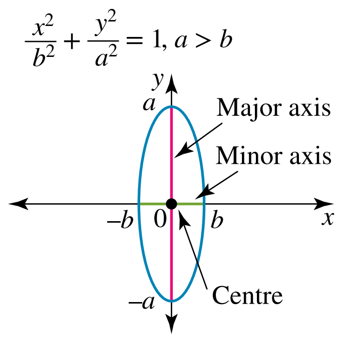
In the ellipse shown, the major axis is the x-axis (the longer axis) and it is $2a$ units long. The minor axis is the y-axis and it is $2b$ units long. The centre of this ellipse is the origin. The equation for this ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$.



This ellipse can be shifted so that the centre becomes the point (h, k). In this case, the equation becomes $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where $a > b$. The ellipse will look like the diagram shown.



The equation $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a > b$ will result in an ellipse with centre $(0, 0)$ and major axis $2a$ units long but along the y-axis. The minor axis will be $2b$ units long but along the x-axis.

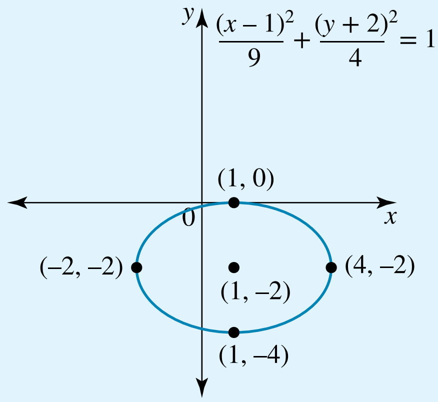


WORKED EXAMPLE 22

Sketch the ellipse $4x^2 + 9y^2 - 8x + 36y + 4 = 0$.

THINK	WRITE/DRAW
1 Use completing the square in order to identify the form of the ellipse. (Group the x and y terms together first).	$4x^2 + 9y^2 - 8x + 36y + 4 = 0$ $4x^2 - 8x + 9y^2 + 36y = -4$ $4(x^2 - 2x) + 9(y^2 + 4y) = -4$ $4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) = -4 + 4 \times 1 + 9 \times 4$ $4(x-1)^2 + 9(y+2)^2 = 36$ $\frac{4(x-1)^2}{36} + \frac{9(y+2)^2}{36} = 1$ $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$
2 This ellipse is in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The major axis is parallel to the x-axis. Identify the centre and the major and minor axes.	The ellipse is in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. $h = 1, k = -2, a = 3, b = 2$ Centre (h, k): $(1, -2)$ Major axis (parallel to the x-axis): length $2a = 6$ Minor axis: length $2b = 4$

3 Sketch the ellipse, labelling the important points.



studyon

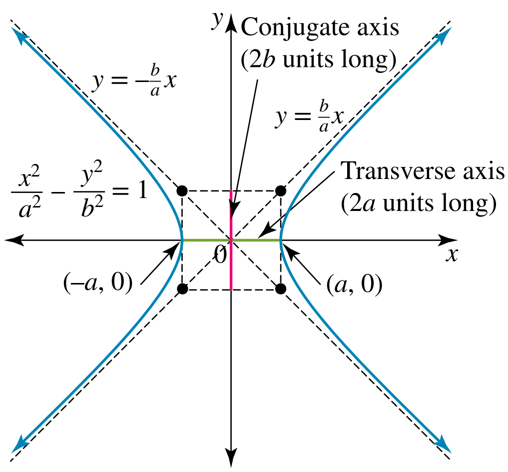
Units 3 & 4
AOS 1
Topic 1
Concept 6

Sketch graphs of hyperbolas
Concept summary
Practice questions

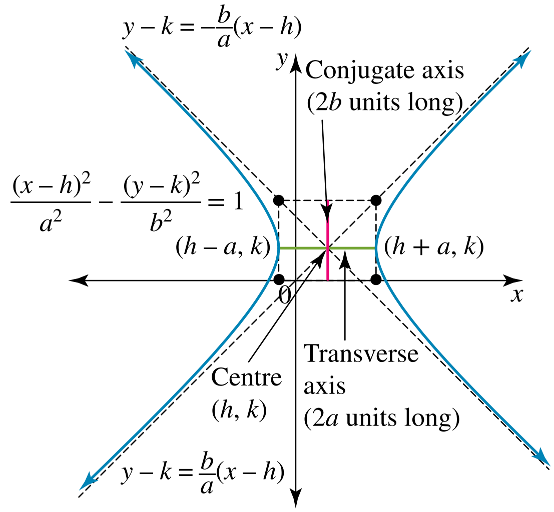
Hyperbolas

When sketching hyperbolas, it is necessary to find the centre, the asymptotes and the coordinates of the vertex. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the centre is the origin. To make the sketching easier, draw a rectangle $2a$ units wide and $2b$ units high with a centre at the origin. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ would fit inside this rectangle with its vertices on the sides. For the corresponding hyperbola, the asymptotes are the diagonals of the rectangle. The vertices occur when $y = 0$ and $x = \pm a$. The vertices fall on the transverse axis, which is $2a$ units long. The hyperbola also has an axis of symmetry along the conjugate axis, which is $2b$ units long.

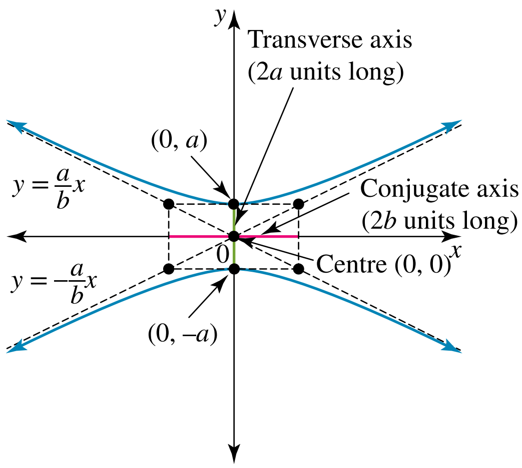
Alternatively, the asymptotes are $y = \pm\frac{b}{a}x$, the centre is (0, 0) and the vertices $(\pm a, 0)$, the transverse axis is horizontal and $2a$ units long, and the conjugate axis is vertical and $2b$ units long.



The hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, as shown in the figure on the next page, has been shifted so that its centre is now at (h, k). This means that the vertices are now $(h \pm a, k)$ and the asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.



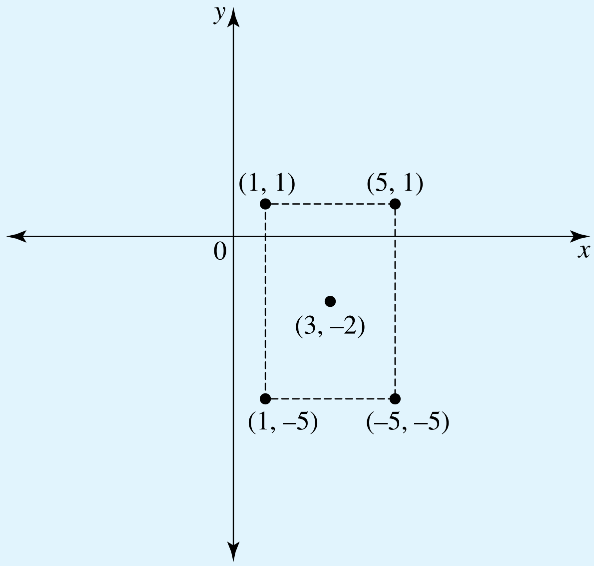
Hyperbolas can also be written in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. In this case, the centre is still the origin. The transverse axis is parallel to the y-axis and is $2a$ units long; the conjugate axis is parallel to the x-axis and is $2b$ units long. The vertices occur when $x = 0$ and $y = \pm a$. A rectangle can be drawn to find the asymptotes, or they can be sketched using the formula $y = \pm\frac{a}{b}x$.



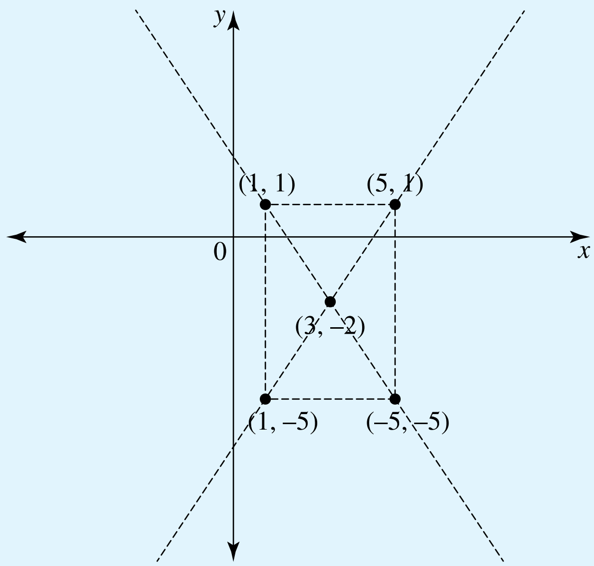
WORKED EXAMPLE 23 Sketch the hyperbola $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$.

THINK	WRITE/DRAW
1 The hyperbola is in the form $\frac{(x-h)^2}{a^2} - \frac{(h-k)^2}{b^2} = 1$. Find h, k, a and b.	The hyperbola is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. $h = 3$, $k = -2$, $a = 2$, $b = 3$
2 Identify the centre and the transverse axis.	The centre (h, k): $(3, -2)$ The transverse axis is parallel to the x-axis.

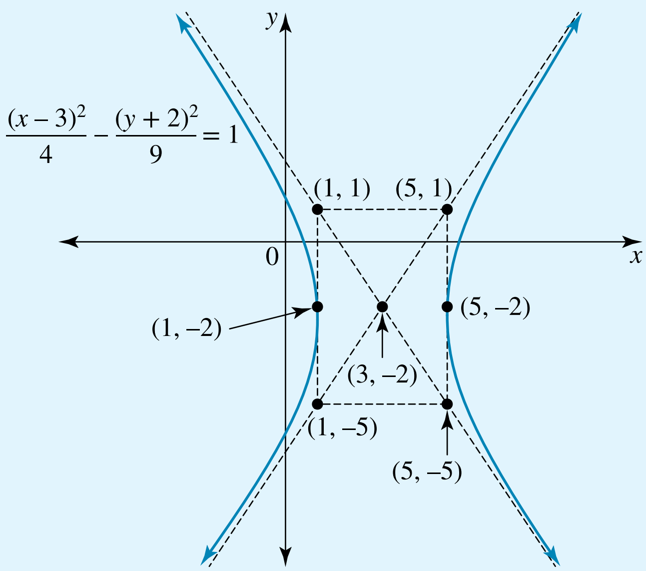
3 Sketch the rectangle: centre (3, −2), $2a = 4$ units wide and $2b = 6$ units high.



4 Sketch the asymptotes by drawing the diagonals. Alternatively, sketch $y + 2 = \pm\frac{3}{2}(x - 3)$.



5 The transverse axis is parallel to the x-axis. This means that the vertices are $(h \pm a, k)$. Sketch the hyperbola.



EXERCISE 1.6 Circles, ellipses and hyperbolas

PRACTISE

1 WE21 Sketch the circle $x^2 + y^2 + 6x - 2y = 15$.

2 Sketch the circle $x^2 + y^2 + 4x - 6y + 9 = 0$.

3 WE22 Sketch the ellipse $9x^2 + 16y^2 + 54x - 64y + 1 = 0$.

4 Sketch the ellipse $16x^2 + 4y^2 + 32x + 16y - 32 = 0$.

5 WE23 Sketch the hyperbola $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{4} = 1$

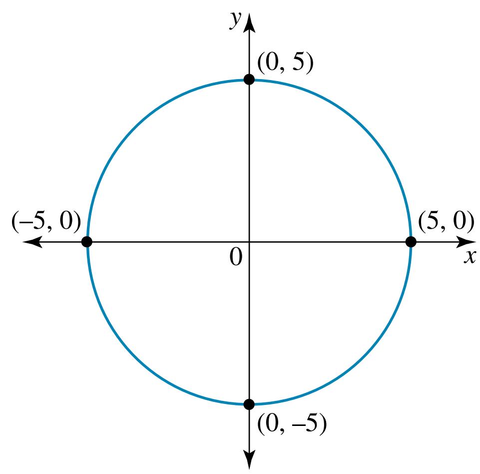
6 Sketch the hyperbola $\frac{(y+2)^2}{9} - \frac{(x-3)^2}{4} = 1$

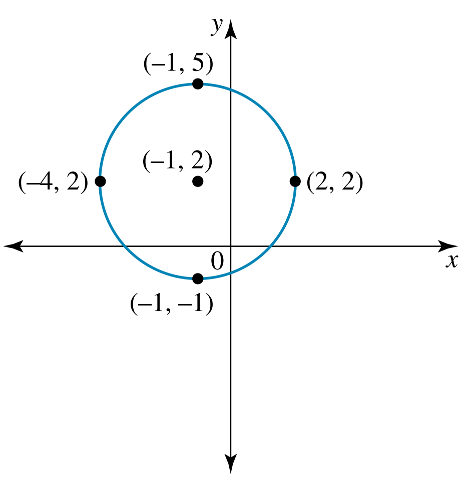
CONSOLIDATE

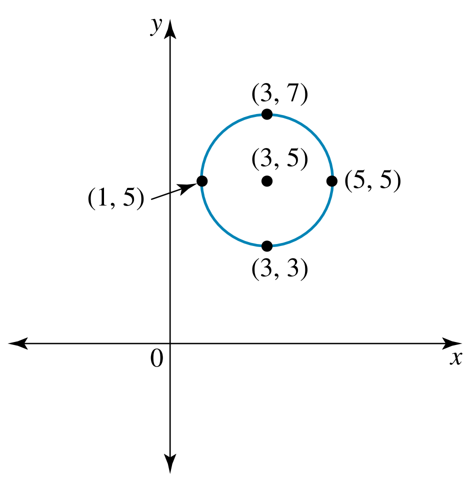
7 Sketch the following circles.

a $x^2 + y^2 = 16$
b $(x - 2)^2 + (y + 3)^2 = 25$
c $x^2 + y^2 + 2x - 4y = 20$
d $4x^2 + 4y^2 - 24x - 3y + 29 = 0$

8 Identify the equations of the following circles. Give your answers in the form $(x - h)^2 + (y - k)^2 = r^2$.

a


b


c


9 Sketch the following ellipses.

a $\frac{x^2}{9} + \frac{y^2}{4} = 1$

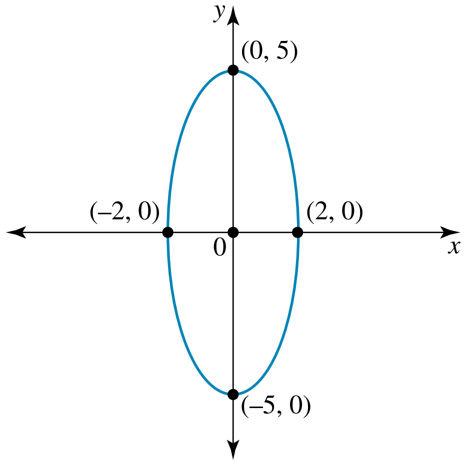
b $25x^2 + 16y^2 = 400$

c $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$

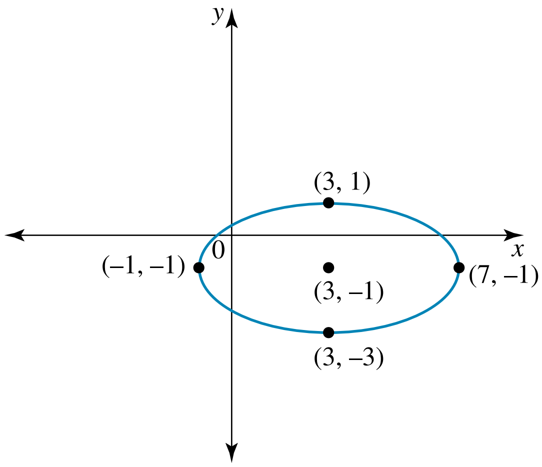
d $9x^2 + 16y^2 - 36x + 64y = 44$

10 Identify the equations of the following ellipses.

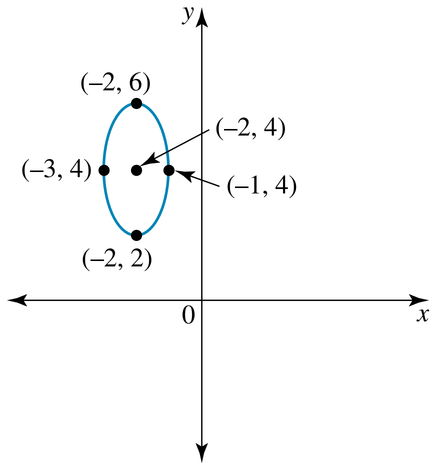
a



b



c



11 Sketch the following hyperbolas.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{x^2}{16} - \frac{y^2}{25} = 1$

c $\frac{y^2}{16} - \frac{x^2}{9} = 1$

d $\frac{y^2}{16} - \frac{x^2}{25} = 1$

12 Sketch the following hyperbolas.

a $\frac{(x-3)^2}{9} - \frac{(y+1)^2}{4} = 1$

b $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

c $\frac{(y+2)^2}{49} - \frac{(x+5)^2}{36} = 1$

d $\frac{(y-4)^2}{4} - \frac{(x-2)^2}{9} = 1$

13 Sketch the following hyperbolas.

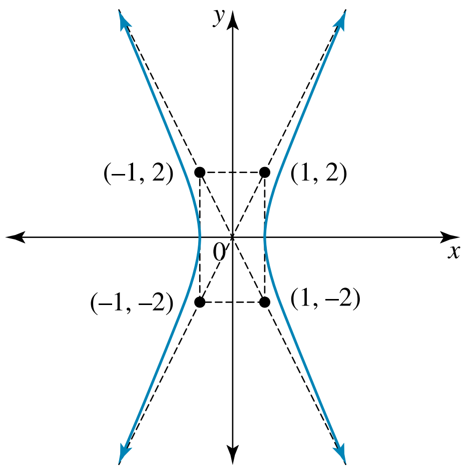
a $x^2 - y^2 - 4x + 6y = 14$

b $x^2 - y^2 - 6x + 6y = 4$

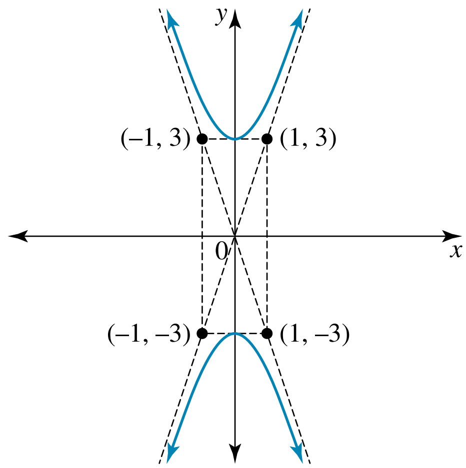
c $4x^2 - 9y^2 + 24x + 36y + 36 = 0$

14 Identify the equations of the following hyperbolas.

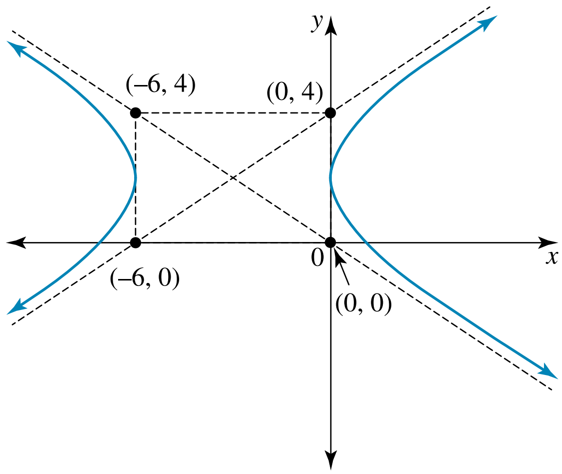
a



b



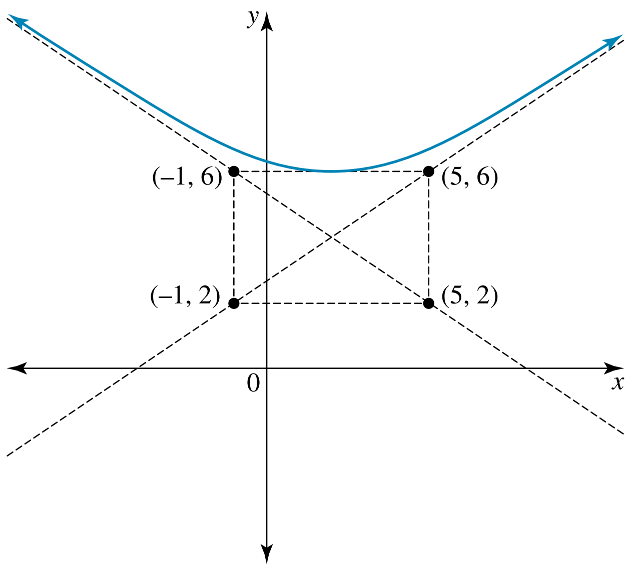
c



15 **a** Sketch the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{2} = 1$.

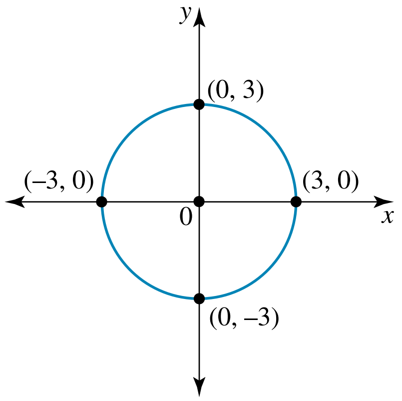
b The conjugate hyperbola has the same asymptote, but the transverse axis is now the y-axis. Find the equation of the conjugate hyperbola.

16 A hyperbola consists of two arms. For the following hyperbola, find the equation for the upper arm.



MASTER

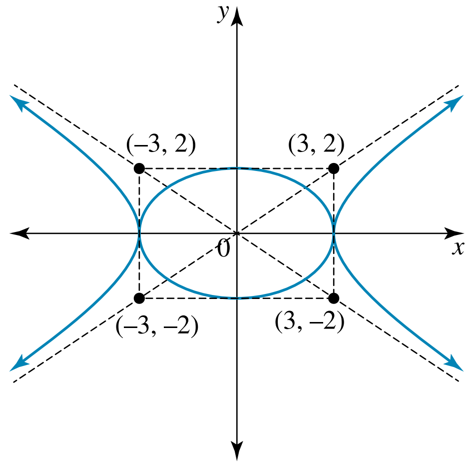
17 The circle shown, $x^2 + y^2 = 9$, has a radius of 3 units and an area of 9π.



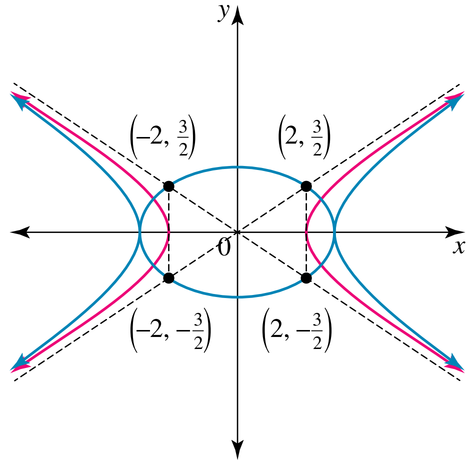
The ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ has a major axis of 8 units and a minor axis of 4 units.

a Determine the area of the ellipse (technology may be used).
b Find an ellipse with a major axis of 8 units along the x-axis and the same area as the circle.
c Sketch the circle and ellipse on the same set of axes.

18 Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. An ellipse can be drawn inside the hyperbola as shown.



a Determine the equation of the ellipse.
A second hyperbola is to be drawn. The asymptotes and centre of this hyperbola are the same as for the original hyperbola. The vertices have the same x-coordinate as the points where the ellipse intersects with the asymptotes.



b Find the equation of the second hyperbola.

eBook*plus* ONLINE ONLY

1.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

1 Answers

EXERCISE 1.2

1 $x = \pm 4$

2 $x = 2, x = -4$

3 a $-5 < x < 5$

b $x > 3$ or $x < -3$

4 a $x \geq 4$ or $x \leq -4$

b $-1 \leq x \leq 1$

5 $-8 < x < 6$

6 $-1 < x < 9$

7 a $|x| < 3$ **b** $|x| < 4$

8 a $|x + 1| < 2$ **b** $|x + 3| < 4$

9 a $x = 3$ **b** $x > 3$

10 a $x = \frac{-1}{3}$ **b** $x < \frac{-1}{3}$

11 a $x = \pm 7$ **b** $x = 14, x = -4$

c $x = 3, x = -4$ **d** $x = \pm\frac{2}{3}$

12 a $-6 < x < 6$ **b** $-9 < x < 5$

c $x \geq 7$ or $x \leq -1$ **d** $x \leq 0$ or $x \geq 4$

13 a $|y| < 2$ **b** $|y - 1| < 4$

c $|y - 1| \leq 6$ **d** $|y| > 5$

e $|y - 3| \geq 2$

14 a $|x| \leq a$ **b** $|x - x_0| \leq 3$

c $|x| < b - a$ **d** $|x - 5| \leq a$

15 a $x = -\frac{1}{2}$ **b** $x < -\frac{1}{2}$

16 a $x = 2$ **b** $x > 2$

17 a $x = 2$ **b** $x > 2$

18 a $x = -3$ **b** $x < -3$

19 a Check with your teacher.

b $a > 0$

20 $|1 - x| = 1 - x$ if $x \leq 1$; $|1 - x| = x - 1$ if $x > 1$

21 a $x = 1$

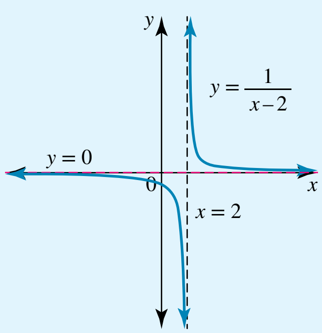
b $x \leq 1$

22 a $x = 0, x = 2$

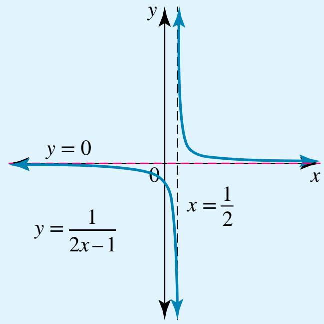
b $0 < x < 2$

EXERCISE 1.3

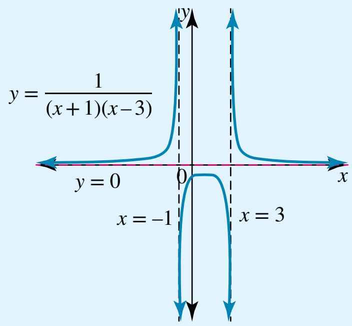
1



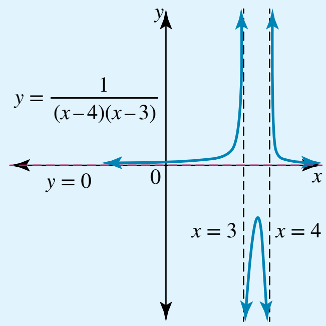
2



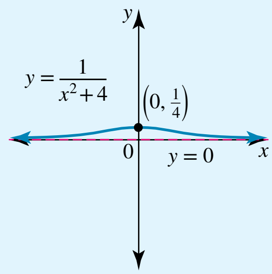
3



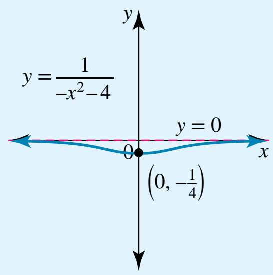
4



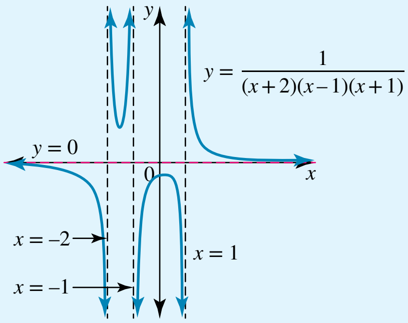
5



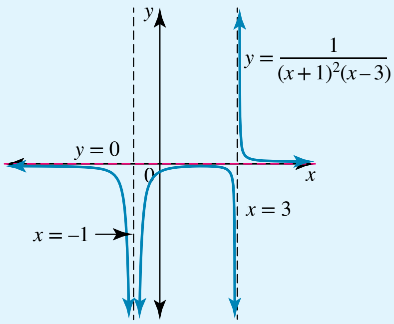
6



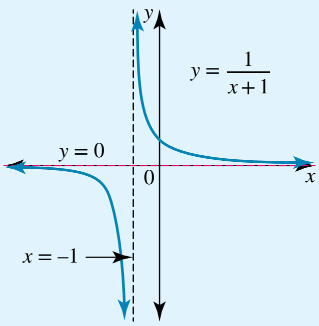
7



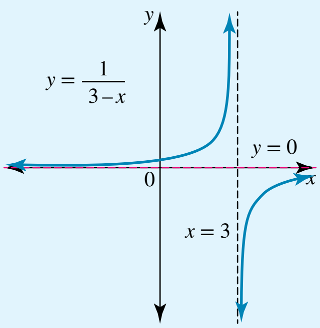
8



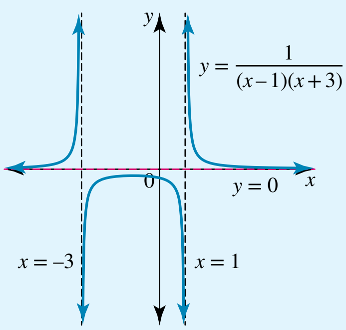
9



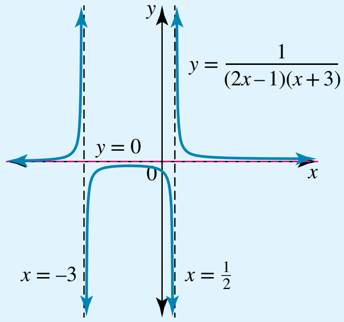
10



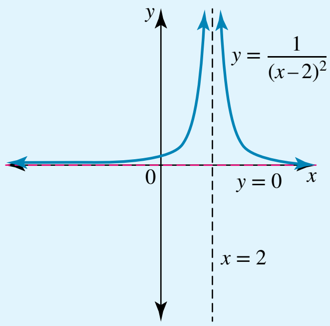
11



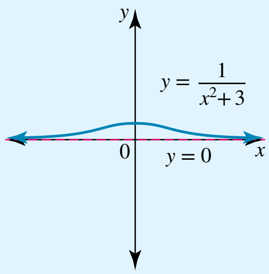
12



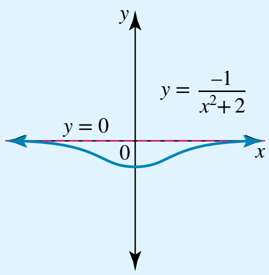
13



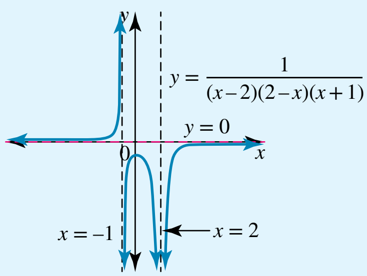
14



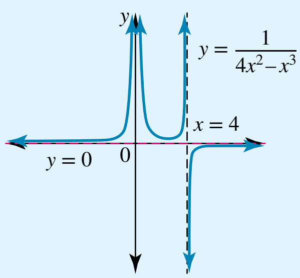
15



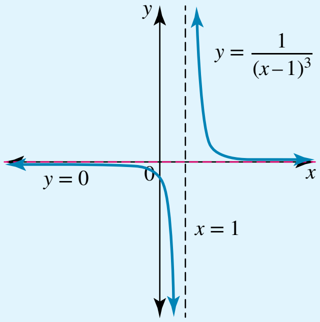
16



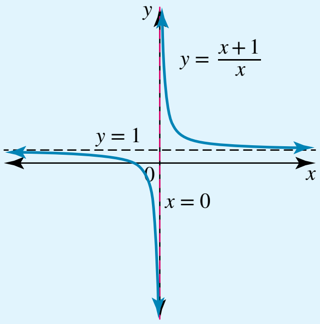
17



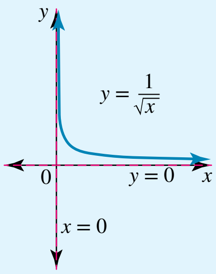
18



19

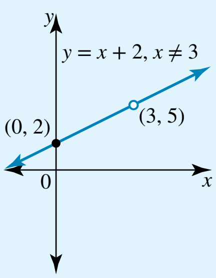


20

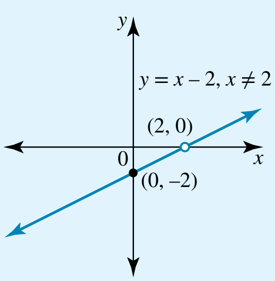


EXERCISE 1.4

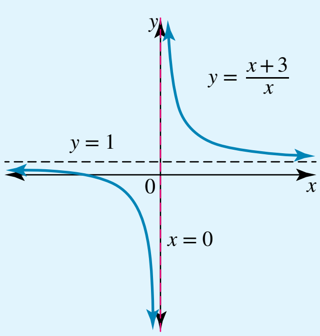
1



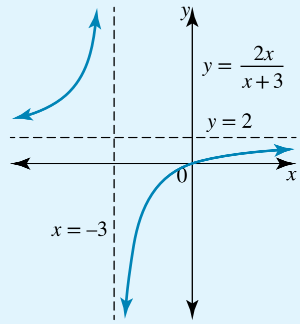
2



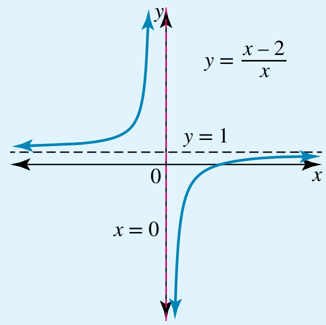
3 a



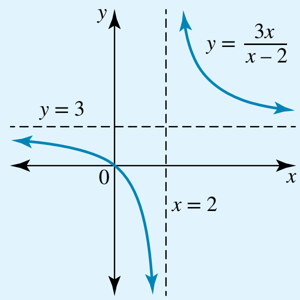
b



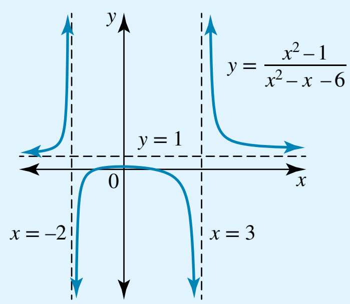
4 a



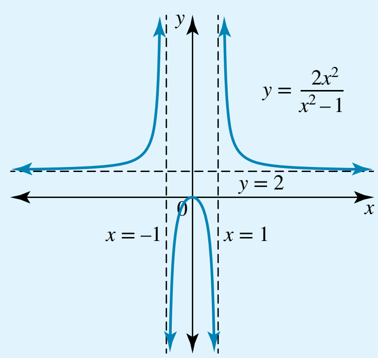
b



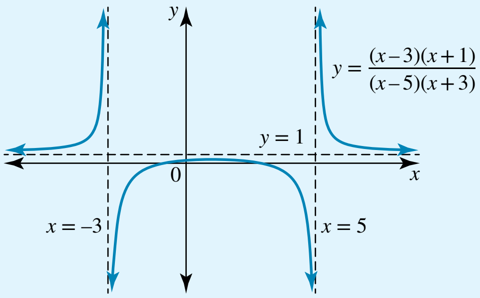
5 a



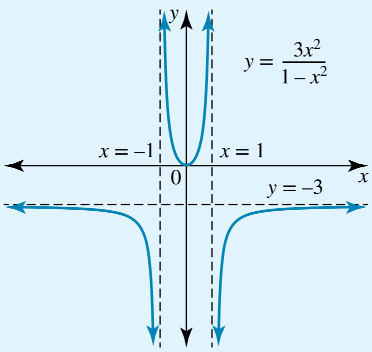
b



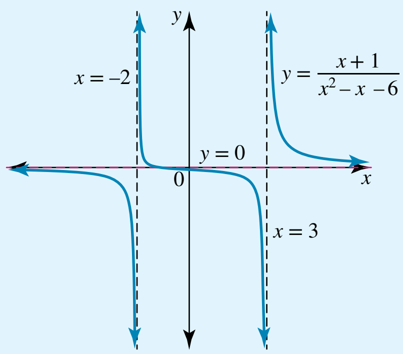
6 a



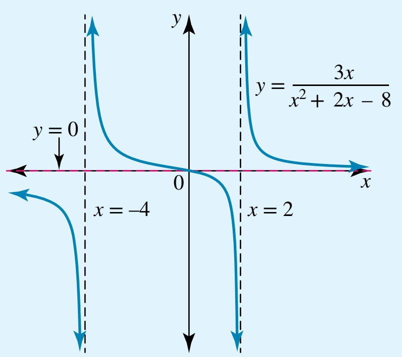
b



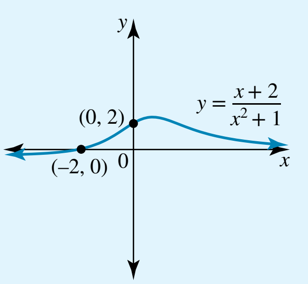
7



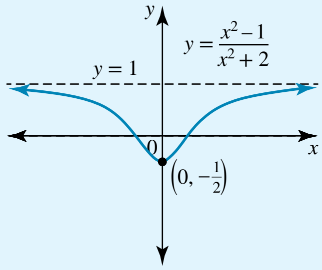
8



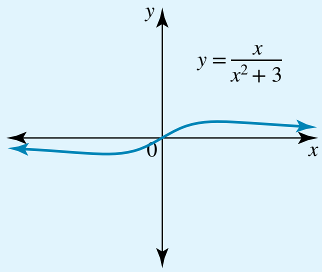
9 a



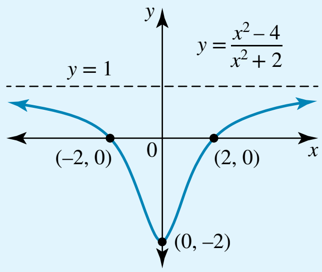
b



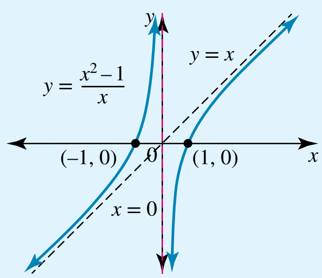
10 a



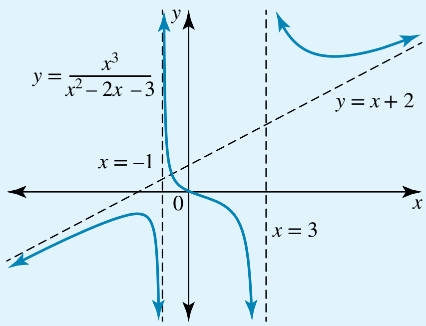
b



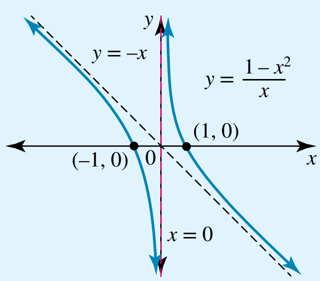
11 a



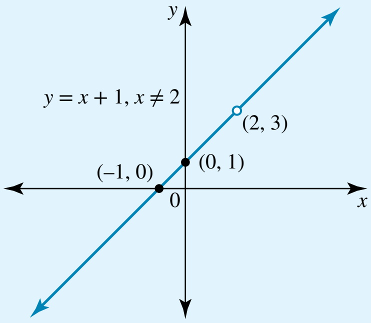
b



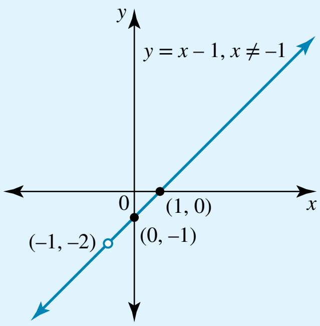
12



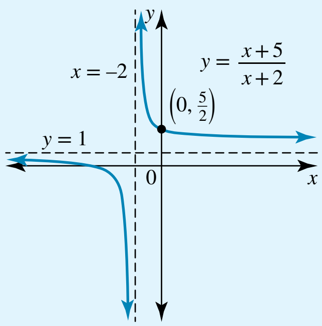
13 a



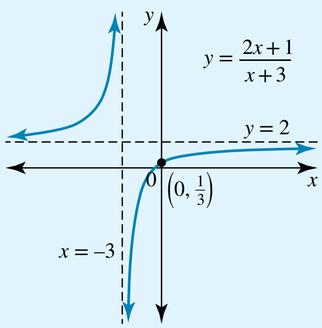
b



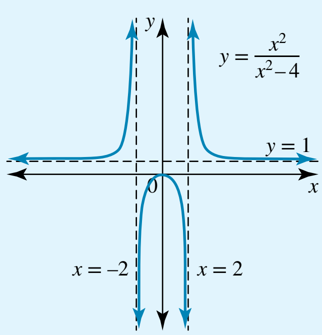
14



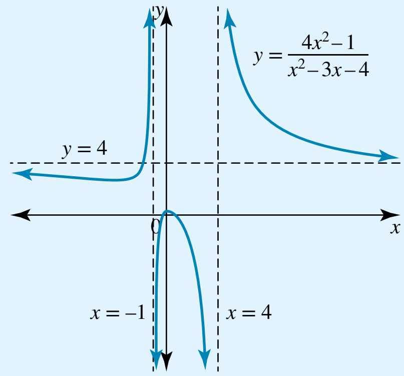
15



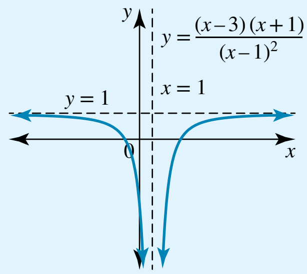
16 a



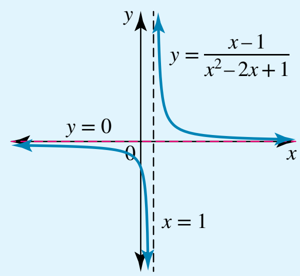
b



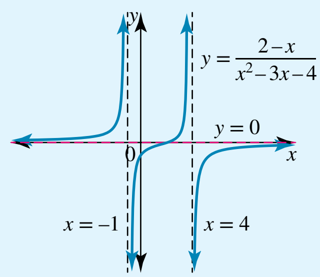
17



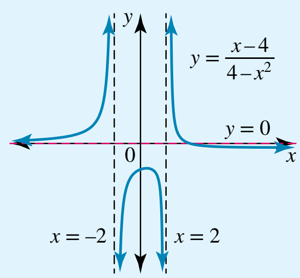
18 a



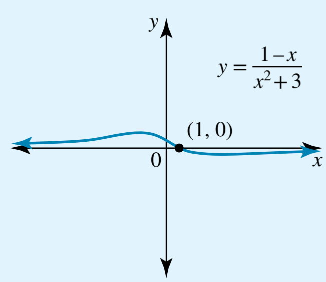
b



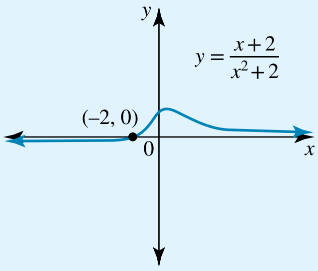
c



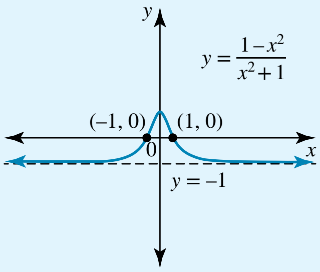
19 a



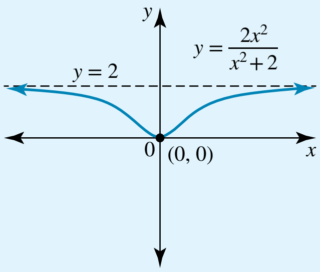
b



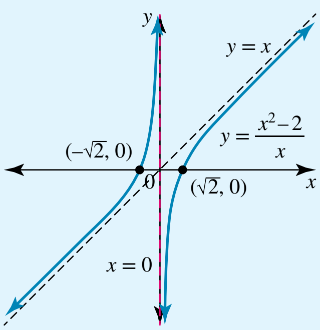
20 a



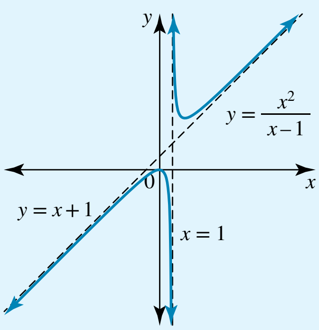
b



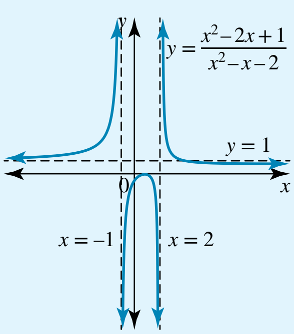
21 a



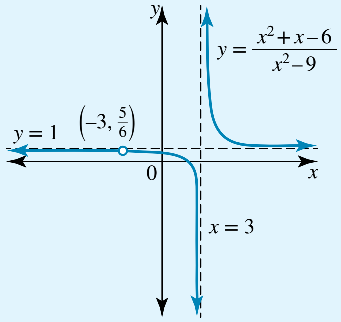
b



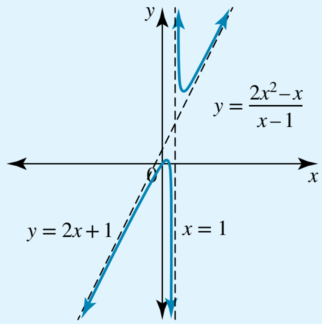
22 a



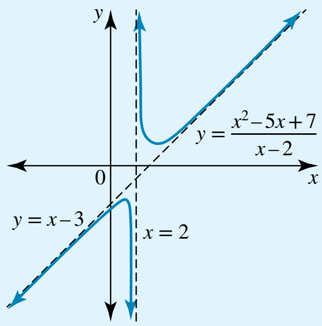
b



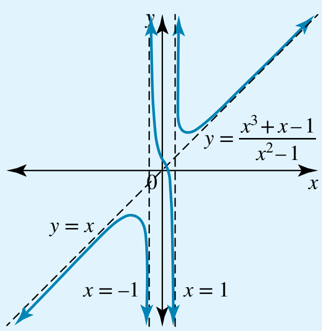
23 a



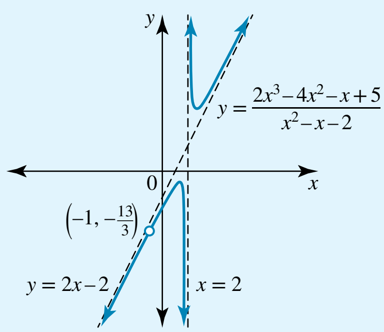
b



24 a

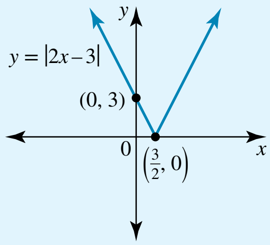


b

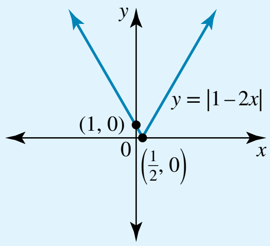


EXERCISE 1.5

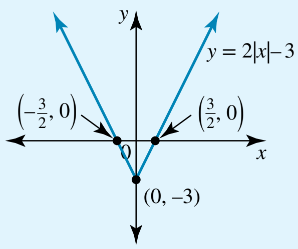
1



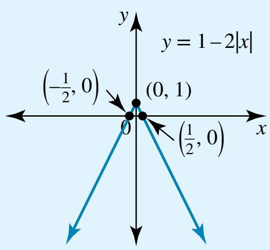
2



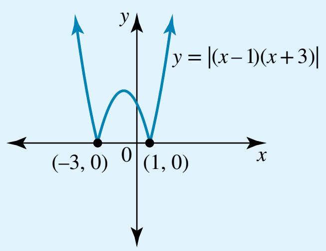
3



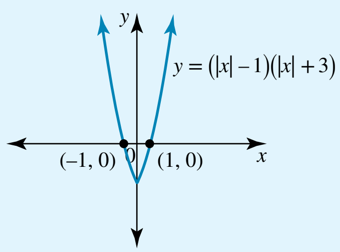
4



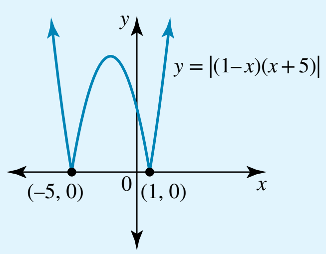
5 a



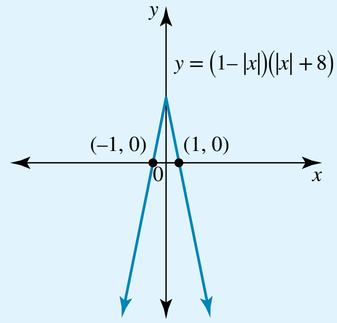
b



6 a



b



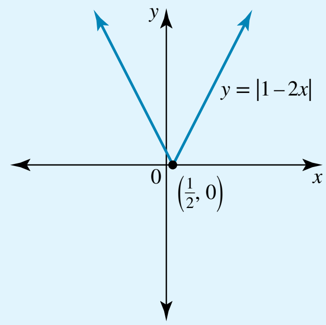
7 $-3 < x < 1$

8 $-1 < x < 5$

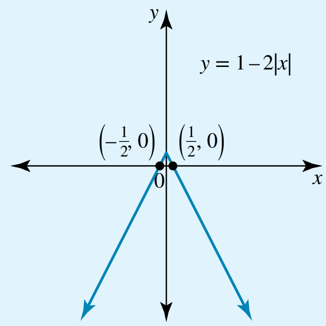
9 $x > 3$

10 $x < -1$

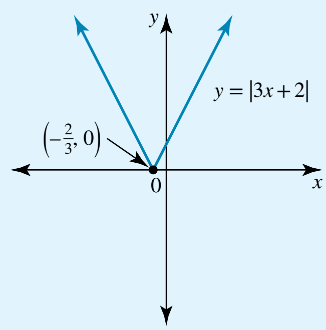
11 a



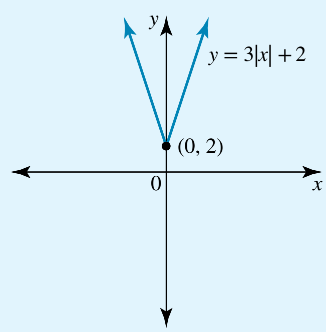
b



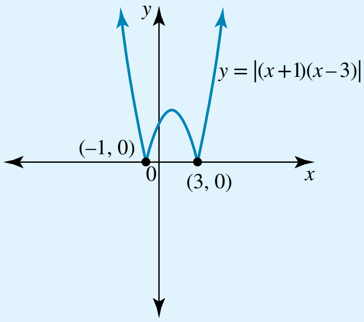
12 a



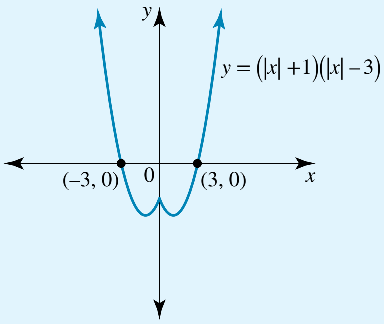
b



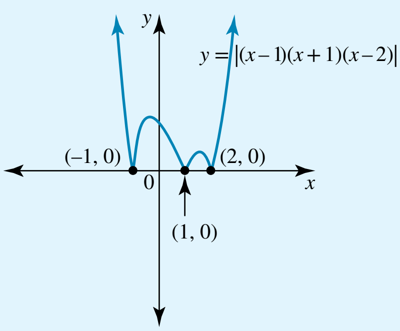
13 a



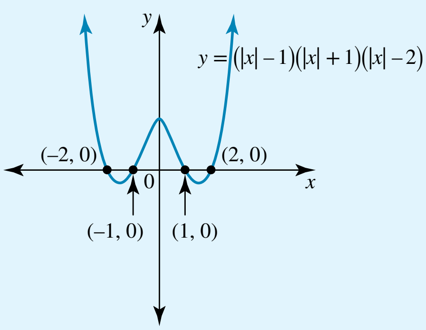
b



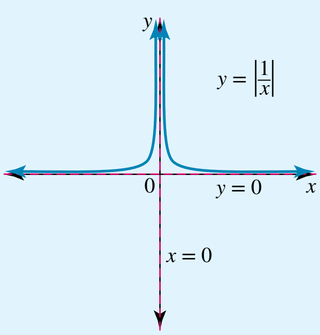
14 a



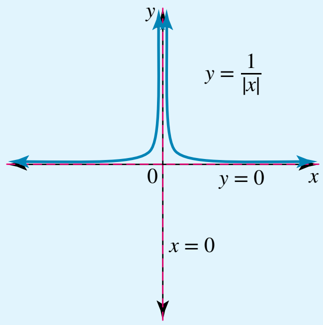
b



15 a

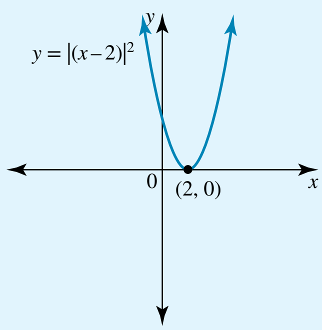


b

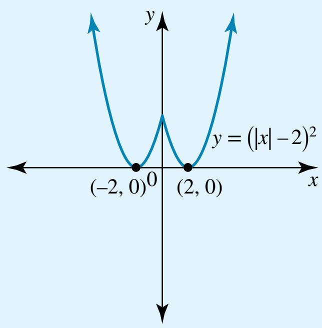


The graphs for **a** and **b** are identical.

16 a



b



The graph for **a** is identical to the original graph, $y = (x - 2)^2$.

17 $x \geq 1, x \leq -1$

18 $0 \leq x \leq 4$

19 $\frac{-10}{3} < x < 2$

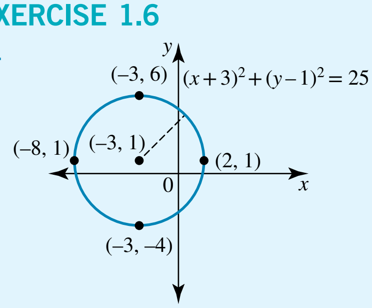
20 $x < \frac{-5}{4}$

21 $-4 < x < \frac{-2}{3}$

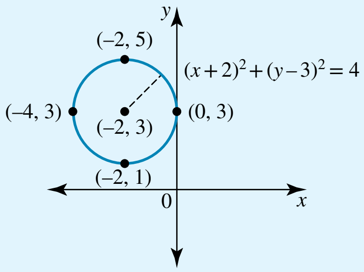
22 $\frac{5}{3} < x < 3$

EXERCISE 1.6

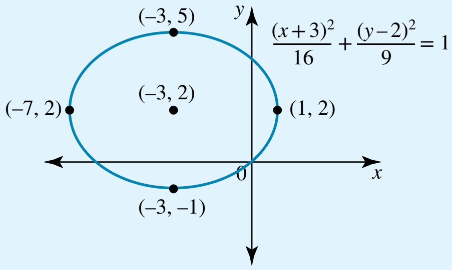
1



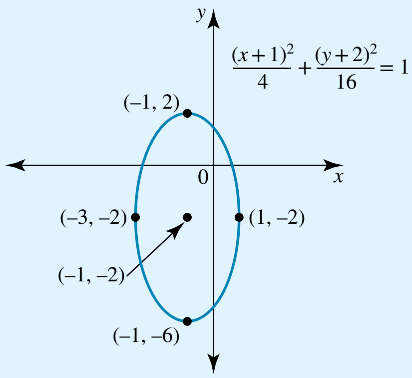
2



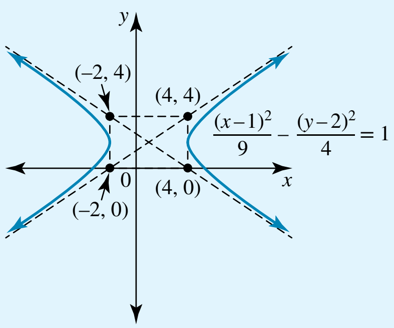
3



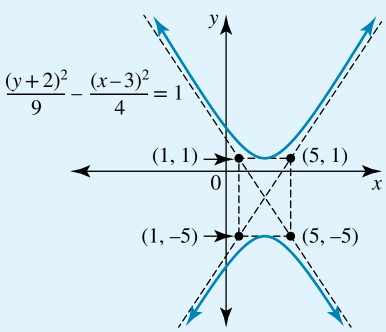
4



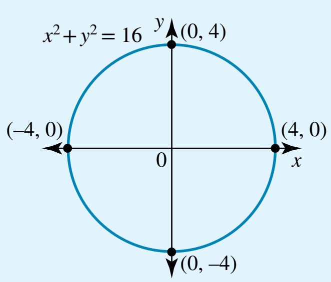
5



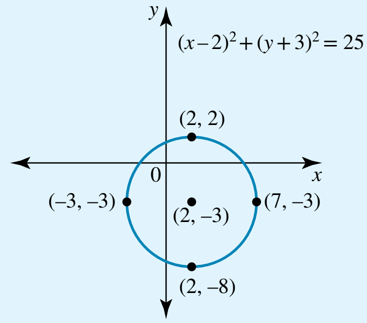
6



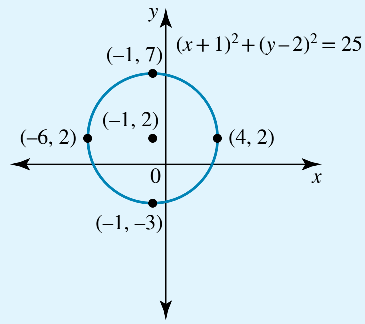
7 a



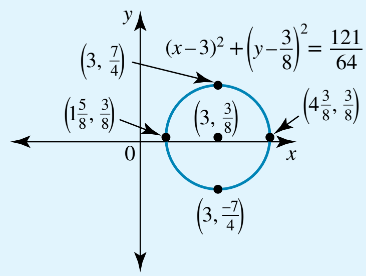
b



c



d

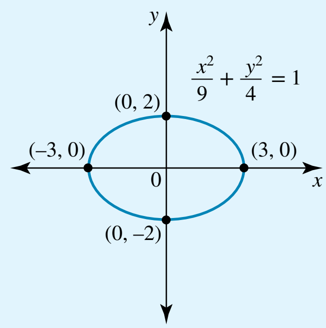


8 a $x^2 + y^2 = 25$

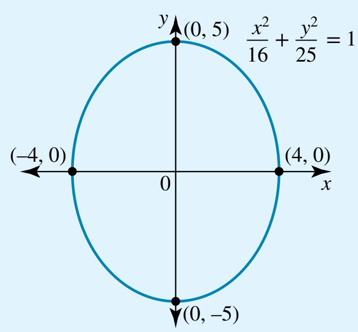
b $(x + 1)^2 + (y - 2)^2 = 9$

c $(x - 3)^2 + (y - 5)^2 = 4$

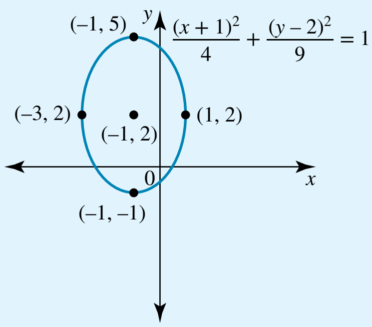
9 a



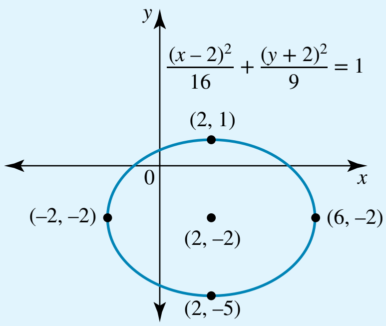
b



c



d

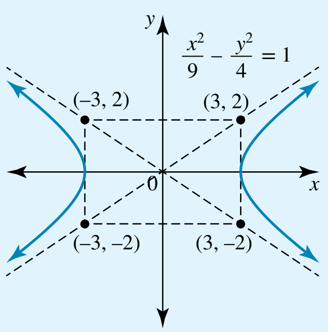


10 a $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

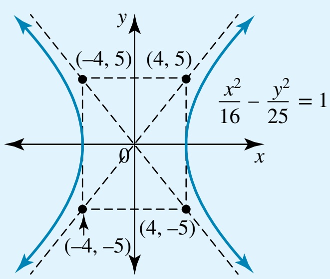
b $\dfrac{(x-3)^2}{16} + \dfrac{(y+1)^2}{4} = 1$

c $(x+2)^2 + \dfrac{(y-4)^2}{4} = 1$

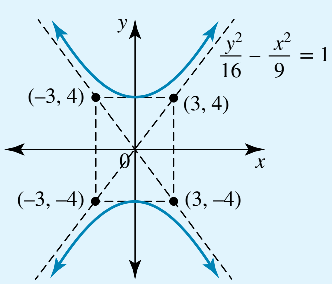
11 a



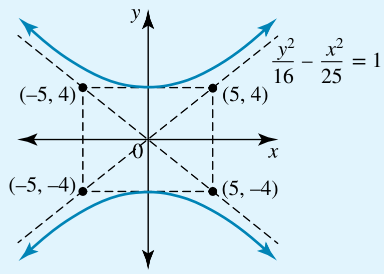
b



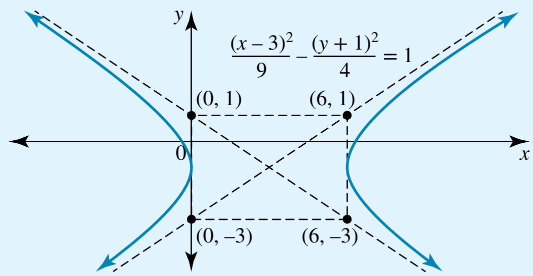
c



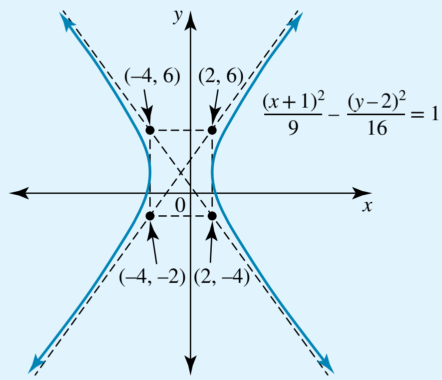
d



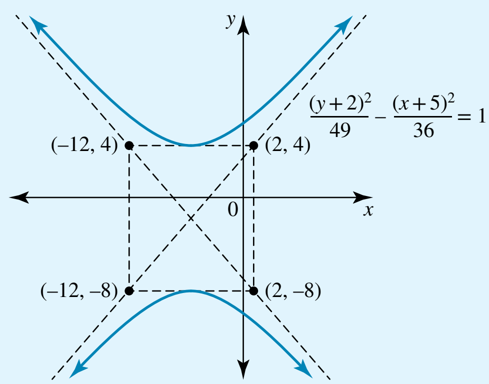
12 a



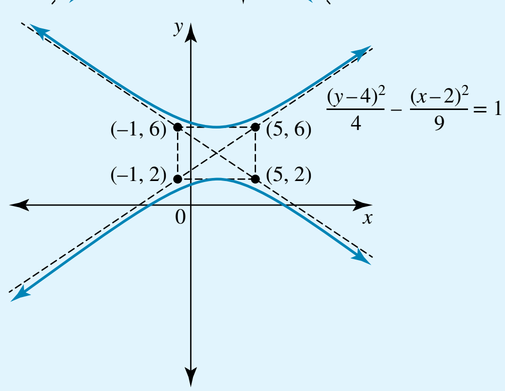
b



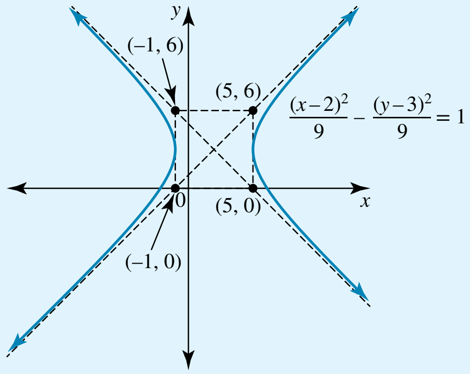
c



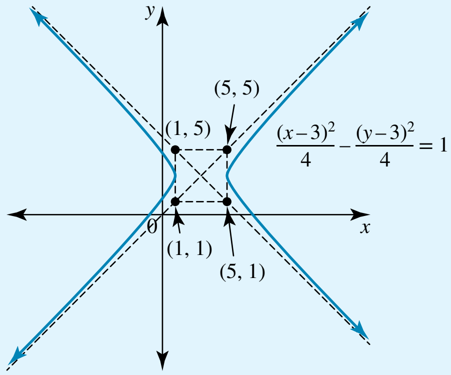
d



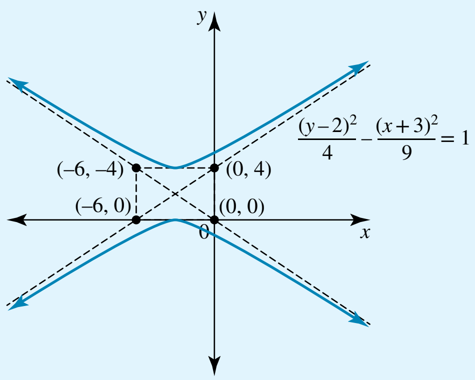
13 a



b



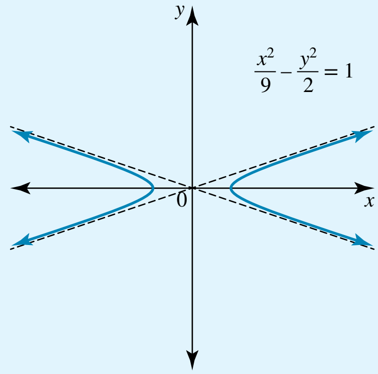
c



14 a $x^2 - \frac{y^2}{4} = 1$ **b** $\frac{y^2}{9} - x^2 = 1$

c $\frac{(x+3)^2}{9} - \frac{(y-2)^2}{4} = 1$

15 a



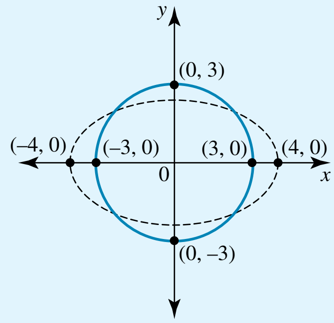
b $\frac{y^2}{2} - \frac{x^2}{9} = 1$

16 $y = 4 + 2\sqrt{1 + \frac{(x-2)^2}{9}}$

17 a 8π

b $\frac{x^2}{16} + \frac{16y^2}{81} = 1$

c



18 a $\frac{x^2}{9} + \frac{y^2}{4} = 1$

b $\frac{2x^2}{9} - \frac{y^2}{2} = 1$

2

Trigonometry

2.1 Kick off with CAS

Exploring inverse trigonometric functions

In this topic, we will investigate the inverse trigonometric functions.

1 Using CAS, determine each of the following. Remember to have the calculator in radians mode.

a $\cos^{-1}\left(\cos\left(\frac{2}{5}\right)\right)$

b $\cos^{-1}(\cos(3))$

c $\cos^{-1}(\cos(6))$

d $\cos^{-1}\left(\cos\left(\frac{\pi}{7}\right)\right)$

e $\cos^{-1}\left(\cos\left(\frac{8\pi}{7}\right)\right)$

f $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$

g $\cos^{-1}(\cos(-\pi))$

2 For what values of x is $\cos^{-1}(\cos(x)) = x$? Confirm your result using CAS.

3 Using CAS, determine each of the following.

a $\tan^{-1}\left(\tan\left(\frac{1}{3}\right)\right)$

b $\tan^{-1}\left(\tan\left(-\frac{4}{5}\right)\right)$

c $\tan^{-1}(\tan(6))$

d $\tan^{-1}\left(\tan\left(\frac{\pi}{5}\right)\right)$

e $\tan^{-1}\left(\tan\left(\frac{7\pi}{5}\right)\right)$

f $\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)$

g $\tan^{-1}\left(\tan\left(\frac{4\pi}{3}\right)\right)$

4 For what values of x is $\tan^{-1}(\tan(x)) = x$? Confirm your result using CAS.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

2.2 Reciprocal trigonometric functions

studyon

Units 3 & 4
AOS 1
Topic 2
Concept 1

Reciprocal circular functions
Concept summary
Practice questions

History of trigonometry

The word **trigonometry** is derived from the Greek words *trigonon* and *metron*, meaning 'triangle' and 'measure'. Trigonometry is the branch of mathematics that deals with triangles and the relationships between the angles and sides of a triangle. Trigonometry was originally devised in the third century BC to meet the needs of the astronomers of those times. Hipparchus was a Greek astronomer and mathematician and is considered to be the founder of trigonometry, as he compiled the first trigonometric tables in about 150 BC.

Definitions of trigonometric ratios

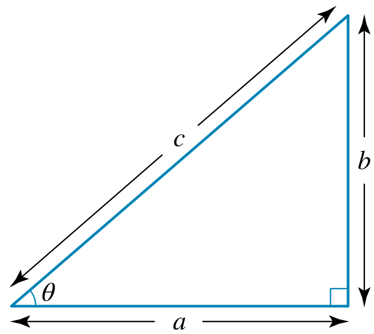
The following is a review of trigonometry, which is needed for the rest of this topic and subsequent work in this book.

The trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ are defined in terms of the ratio of the lengths of the sides of a right-angled triangle. Let the lengths of the three sides of the triangle be a, b and c, and let the angle between sides a and c be θ.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

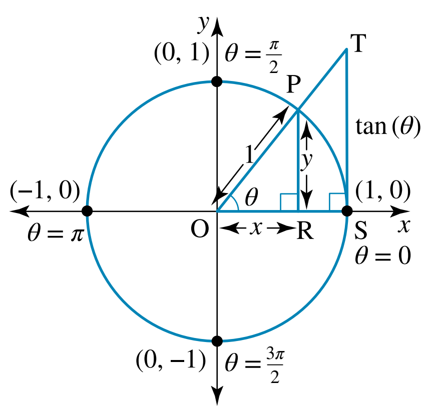


Pythagoras' theorem states that in any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. That is:

$$a^2 + b^2 = c^2.$$

The unit circle

An alternative definition of the trigonometric functions is based on the unit circle, which is a circle with radius one unit and centre at the origin. The unit circle has the equation $x^2 + y^2 = 1$. The coordinate of any point P (x, y) on the unit circle is defined in terms of the trigonometric functions OR $= x = \cos(\theta)$ and RP $= y = \sin(\theta)$, where θ is the angle measured as a positive angle, anticlockwise from positive direction of the x-axis. The **trigonometric functions** are also called circular functions as they are based on the unit circle.



By substituting $x = \cos(\theta)$ and $y = \sin(\theta)$ into the equation $x^2 + y^2 = 1$, we can derive the relationship $\sin^2(\theta) + \cos^2(\theta) = 1$.

Note that $\sin^2(\theta) = (\sin(\theta))^2$ and $\cos^2(\theta) = (\cos(\theta))^2$.

The vertical distance from S to T is defined as $\tan(\theta)$. As the triangles ΔORP and ΔOST are similar,

$$\frac{\text{RP}}{\text{OR}} = \frac{\text{ST}}{\text{OS}} = \frac{\tan(\theta)}{1}$$

$$\frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

Angles of any magnitude

In the diagram of the unit circle, consider the point (0, 1) on the y-axis. This point corresponds to the angle $\theta = 90°$ or $\frac{\pi}{2}$ radians rotated from the positive end of the x-axis. Since the sine of the angle is the y-coordinate, it follows that $\sin(90°) = \sin\left(\frac{\pi}{2}\right) = 1$. Since the cosine of the angle is the x-coordinate, it follows that $\cos(90°) = \cos\left(\frac{\pi}{2}\right) = 0$. The tangent is the value of sine divided by the cosine; because we cannot divide by zero, the tan of $\theta = 90°$ or $\frac{\pi}{2}$ radians is undefined.

Similarly for the point (–1, 0), where $\theta = 180°$ or π radians, it follows that $\cos(180°) = \cos(\pi) = -1$ and $\sin(180°) = \sin(\pi) = 0$.

The diagram can be used to obtain the trigonometric value of any multiple of 90°, and these results are summarised in the following table.

Angle (degrees)	0°	90°	180°	270°	360°
Angle (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin (θ)	0	1	0	-1	0
cos (θ)	1	0	-1	0	1
tan (θ)	0	Undefined	0	Undefined	0

Note: Whenever an angle measurement is shown without a degree symbol in this topic, assume that it is measured in radians.

The first quadrant

The angle in the first quadrant is $0° < \theta < 90°$ in degrees or $0 < \theta < \frac{\pi}{2}$ in radians.

In the first quadrant, $x > 0$ and $y > 0$, so $\cos(\theta) > 0$ and $\sin(\theta) > 0$; therefore, $\tan(\theta) > 0$. The following table shows values derived from triangles in the first quadrant using the trigonometric ratios. You should memorise these values, as they are used extensively in this topic.

Angle (degrees)	0°	30°	45°	60°	90°
Angle (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin(θ)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos(θ)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan(θ)	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

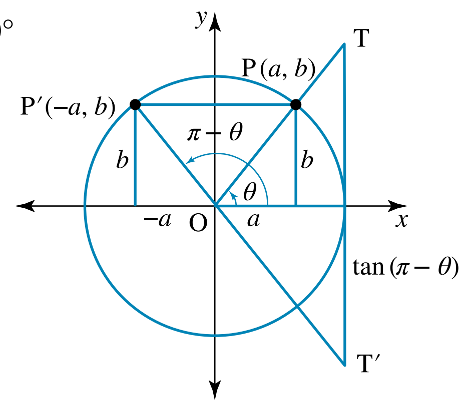
Note that $\sin(30°) + \sin(60°) \neq \sin(90°)$ and in general

$\sin(A + B) \neq \sin(A) + \sin(B)$,
$\cos(A + B) \neq \cos(A) + \cos(B)$ and
$\tan(A + B) \neq \tan(A) + \tan(B)$.

The formulas for $\sin(A + B)$ are called compound-angle formulas. They are studied in greater depth in Section 2.4.

The second quadrant

The angle in the second quadrant is $90° < \theta < 180°$ in degrees or $\frac{\pi}{2} < \theta < \pi$ in radians. In the second quadrant, $x < 0$ and $y > 0$, so $\cos(\theta) < 0$ and $\sin(\theta) > 0$; therefore, $\tan(\theta) < 0$.



Consider the point P (a, b) in the first quadrant. When this point is reflected in the y-axis, it becomes the point P′ $(-a, b)$. If P makes an angle of θ with the x-axis, then P′ makes an angle of $180 - \theta$ degrees or $\pi - \theta$ radians with the x-axis. From the definitions of sine and cosine, we obtain the following relationships.

$$\sin(180° - \theta) = \sin(\theta) \qquad \sin(\pi - \theta) = \sin(\theta)$$
$$\cos(180° - \theta) = -\cos(\theta) \qquad \cos(\pi - \theta) = -\cos(\theta)$$
$$\tan(180° - \theta) = -\tan(\theta) \qquad \tan(\pi - \theta) = -\tan(\theta)$$

For example:

$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

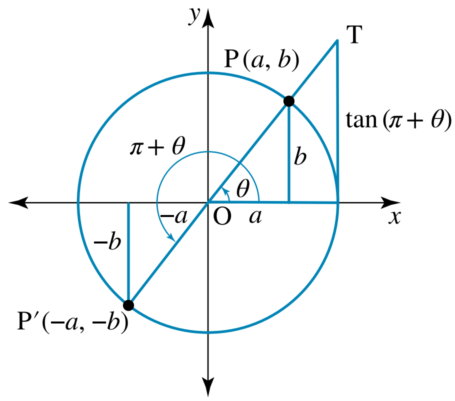
$$\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\tan\left(\frac{2\pi}{3}\right) = \tan\left(\pi - \frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$$

The third quadrant

The angle in the third quadrant is $180° < \theta < 270°$ in degrees or $\pi < \theta < \frac{3\pi}{2}$ in radians. In the third quadrant, $x < 0$ and $y < 0$, so $\cos(\theta) < 0$ and $\sin(\theta) < 0$. However, $\tan(\theta) > 0$.

Consider the point P (a, b) in the first quadrant. When this point is reflected in both the x- and y-axes, it becomes the point P′ $(-a, -b)$. If P makes an angle of θ with the x-axis, then P′ makes an angle of $180 + \theta$ degrees or $\pi + \theta$ radians with the positive end of the x-axis. From the definitions of sine and cosine, we obtain the following relationships.



$\sin(180° + \theta) = -\sin(\theta)$ $\quad \sin(\pi + \theta) = -\sin(\theta)$

$\cos(180° + \theta) = -\cos(\theta)$ $\quad \cos(\pi + \theta) = -\cos(\theta)$

$\tan(180° + \theta) = \tan(\theta)$ $\quad \tan(\pi + \theta) = \tan(\theta)$

For example:

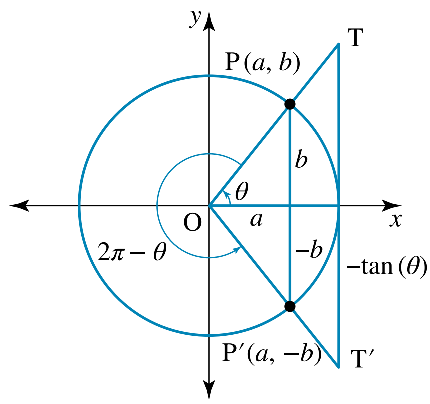
$$\sin\left(\frac{7\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

The fourth quadrant

The angle in the fourth quadrant is $270° < \theta < 360°$ in degrees or $\frac{3\pi}{2} < \theta < 2\pi$ in radians. In the fourth quadrant, $x > 0$ and $y < 0$, so $\cos(\theta) > 0$ and $\sin(\theta) < 0$; therefore, $\tan(\theta) < 0$.



Consider the point P (a, b) in the first quadrant. When this point is reflected in the x-axis, it becomes the point P′ $(a, -b)$. If P makes an angle of θ with the x-axis, then P′ makes an angle of $360 - \theta$ degrees or $2\pi - \theta$ radians with the x-axis. From the definitions of sine and cosine, we obtain the following relationships.

$\sin(360° - \theta) = -\sin(\theta)$ $\quad \sin(2\pi - \theta) = -\sin(\theta)$

$\cos(360° - \theta) = \cos(\theta)$ $\quad \cos(2\pi - \theta) = \cos(\theta)$

$\tan(360° - \theta) = -\tan(\theta)$ $\quad \tan(2\pi - \theta) = -\tan(\theta)$

For example:

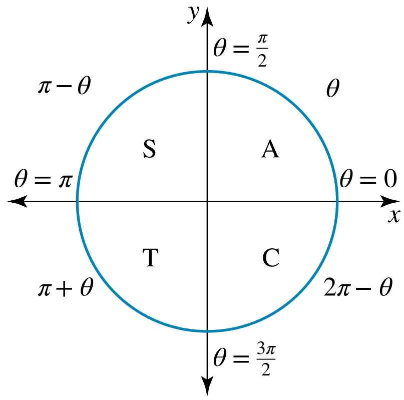
$$\sin\left(\frac{7\pi}{4}\right) = \sin\left(2\pi - \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos\left(\frac{5\pi}{3}\right) = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\tan\left(\frac{11\pi}{6}\right) = \tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

Summary

The trigonometric ratios $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ are all positive in the first quadrant. Only $\sin(\theta)$ is positive in the second quadrant; only $\tan(\theta)$ is positive in the third quadrant; and finally, only $\cos(\theta)$ is positive in the fourth quadrant. This is summarised in the diagram at right. The mnemonic CAST is often used as a memory aid.

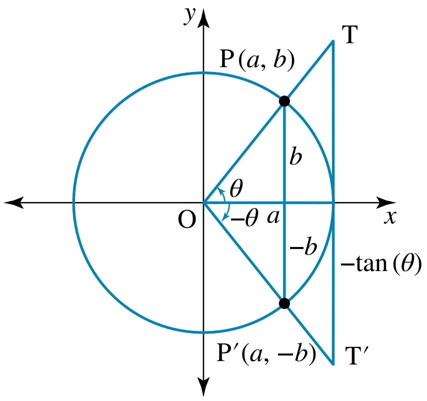


$$\sin(\theta) = \sin(\pi - \theta) = -\sin(\pi + \theta) = -\sin(2\pi - \theta)$$
$$\cos(\theta) = -\cos(\pi - \theta) = -\cos(\pi + \theta) = \cos(2\pi - \theta)$$
$$\tan(\theta) = -\tan(\pi - \theta) = \tan(\pi + \theta) = -\tan(2\pi - \theta)$$

Negative angles

A negative angle is one that is measured clockwise from the positive direction of the x-axis.

Consider the point P (a, b) in the first quadrant. When this point is reflected in the x-axis, it becomes the point P′ $(a, -b)$. If P makes an angle of θ with the x-axis, then P′ makes an angle of $-\theta$ with the x-axis. From the definitions of sine and cosine, we obtain the following relationships.



$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$

A negative angle $-\frac{\pi}{2} < \theta < 0$ is just the equivalent angle in the fourth quadrant.

For positive angles greater than 360° or 2π, we can just subtract multiples of 360° or 2π.

$$\sin(360° + \theta) = \sin(\theta) \qquad \sin(2\pi + \theta) = \sin(\theta)$$
$$\cos(360° + \theta) = \cos(\theta) \qquad \cos(2\pi + \theta) = \cos(\theta)$$
$$\tan(360° + \theta) = \tan(\theta) \qquad \tan(2\pi + \theta) = \tan(\theta)$$

For example:

$$\sin\left(-\frac{4\pi}{3}\right) = -\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\pi + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$\cos\left(-\frac{7\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right) = \cos\left(2\pi - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$\tan\left(-\frac{2\pi}{3}\right) = -\tan\left(\frac{2\pi}{3}\right) = -\tan\left(\pi - \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Reciprocal trigonometric functions

The reciprocal of the sine function is called the cosecant function, often abbreviated to cosec. It is defined as $\text{cosec}(x) = \frac{1}{\sin(x)}$, provided that $\sin(x) \neq 0$.

The reciprocal of the cosine function is called the secant function, often abbreviated to sec. It is defined as $\sec(x) = \dfrac{1}{\cos(x)}$, provided that $\cos(x) \neq 0$.

The reciprocal of the tangent function is called the cotangent function, often abbreviated to cot. It is defined as $\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$, provided that $\sin(x) \neq 0$.

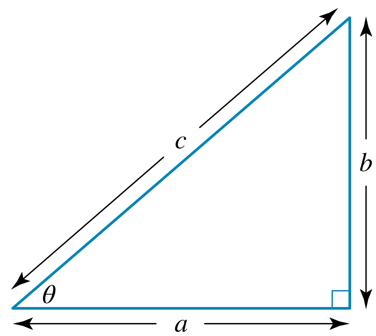
Note that these are **not** the inverse trigonometric functions. (The inverse trigonometric functions are covered in Section 2.6.)

The reciprocal trigonometric functions can also be defined in terms of the sides of a right-angled triangle.

$$\operatorname{cosec}(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}$$

$$\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}$$

$$\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$$



Exact values

The exact values for the **reciprocal trigonometric functions** for angles that are multiples of 30° and 45° can be found from the corresponding trigonometric values by finding the reciprocals. Often it is necessary to simplify the resulting expression or rationalise the denominator.

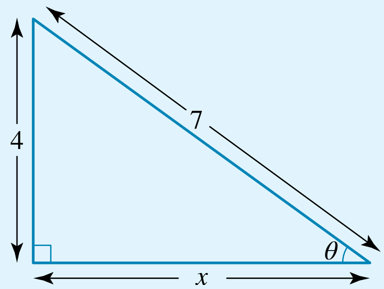
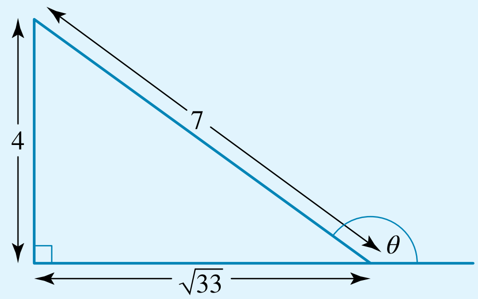
WORKED EXAMPLE 1 Find the exact value of $\operatorname{cosec}\left(\dfrac{5\pi}{4}\right)$.

THINK	WRITE
1 State the required identity.	$\operatorname{cosec}(\theta) = \dfrac{1}{\sin(\theta)}$ $\operatorname{cosec}\left(\dfrac{5\pi}{4}\right) = \dfrac{1}{\sin\left(\dfrac{5\pi}{4}\right)}$
2 Use the known results.	Use $\sin(\pi + \theta) = -\sin(\theta)$ with $\theta = \dfrac{\pi}{4}$. $\operatorname{cosec}\left(\dfrac{5\pi}{4}\right) = \dfrac{1}{\sin\left(\pi + \dfrac{\pi}{4}\right)}$ $= \dfrac{1}{-\sin\left(\dfrac{\pi}{4}\right)}$
3 Simplify the ratio and state the final answer.	$\operatorname{cosec}\left(\dfrac{5\pi}{4}\right) = \dfrac{1}{-\dfrac{1}{\sqrt{2}}}$ $= -\sqrt{2}$

Using triangles to find values

Triangles can be used to find the values of the required trigonometric ratios. Particular attention should be paid to the sign of the ratio.

WORKED EXAMPLE 2 **If $\text{cosec}(\theta) = \frac{7}{4}$ and $\frac{\pi}{2} < \theta < \pi$, find the exact value of $\cot(\theta)$.**

THINK	WRITE/DRAW
1 State the values of the sides of a corresponding right-angled triangle.	$\text{cosec}(\theta) = \frac{7}{4}$ $\frac{1}{\sin(\theta)} = \frac{7}{4}$ $\sin(\theta) = \frac{4}{7}$ The hypotenuse has a length of 7 and the opposite side length is 4.
2 Draw the triangle and label the side lengths using the definition of the trigonometric ratio. Label the unknown side length as x.	
3 Calculate the value of the third side using Pythagoras' theorem.	$x^2 + 4^2 = 7^2$ $x^2 + 16 = 49$ $x^2 = 49 - 16$ $x^2 = 33$ $x = \sqrt{33}$ 
4 State the value of a related trigonometric ratio.	Given that $\frac{\pi}{2} < \theta < \pi$, θ is in the second quadrant. Although $\sin(\theta)$ is positive in this quadrant, $\tan(\theta)$ is negative. $\tan(\theta) = -\frac{4}{\sqrt{33}}$
5 Calculate the value of the required trigonometric value.	$\cot(\theta) = \frac{1}{\tan(\theta)}$ $= \frac{1}{-\frac{4}{\sqrt{33}}}$ $= -\frac{\sqrt{33}}{4}$

EXERCISE 2.2 Reciprocal trigonometric functions

PRACTISE

1 WE1 Find the exact value of $\operatorname{cosec}\left(\frac{2\pi}{3}\right)$.

2 Find the exact value of $\sec\left(-\frac{7\pi}{6}\right)$.

3 WE2 If $\operatorname{cosec}(\theta) = \frac{5}{2}$ and $\frac{\pi}{2} < \theta < \pi$, find the exact value of $\cot(\theta)$.

4 If $\cot(\theta) = 4$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact value of $\sec(\theta)$.

CONSOLIDATE

5 Find the exact values of each of the following.

a $\sec\left(\frac{\pi}{6}\right)$ b $\sec\left(\frac{3\pi}{4}\right)$ c $\sec\left(\frac{4\pi}{3}\right)$ d $\sec\left(-\frac{7\pi}{4}\right)$

6 Find the exact values of each of the following.

a $\operatorname{cosec}\left(\frac{\pi}{3}\right)$ b $\operatorname{cosec}\left(\frac{5\pi}{6}\right)$ c $\operatorname{cosec}\left(\frac{7\pi}{4}\right)$ d $\operatorname{cosec}\left(\frac{5\pi}{3}\right)$

7 Find the exact values of each of the following.

a $\cot\left(\frac{\pi}{6}\right)$ b $\cot\left(\frac{2\pi}{3}\right)$ c $\cot\left(\frac{7\pi}{4}\right)$ d $\cot\left(\frac{11\pi}{6}\right)$

8 a If $\sin(x) = \frac{1}{3}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\sec(x)$.

b If $\operatorname{cosec}(x) = 4$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\cot(x)$.

9 a If $\cos(x) = -\frac{3}{7}$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of $\cot(x)$.

b If $\sec(x) = -\frac{5}{2}$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of $\operatorname{cosec}(x)$.

10 a If $\cos(x) = \frac{3}{7}$ and $\frac{3\pi}{2} < x < 2\pi$, find the exact value of $\operatorname{cosec}(x)$.

b If $\sec(x) = \frac{8}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find the exact value of $\cot(x)$.

11 a If $\operatorname{cosec}(x) = 4$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\tan(x)$.

b If $\cot(x) = -\frac{5}{6}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\operatorname{cosec}(x)$.

12 a If $\sec(x) = -7$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of $\cot(x)$.

b If $\cot(x) = 4$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of $\operatorname{cosec}(x)$.

13 a If $\sec(x) = 6$ and $\frac{3\pi}{2} < x < 2\pi$, find the exact value of $\operatorname{cosec}(x)$.

b If $\cot(x) = -\frac{5}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find the exact value of $\sec(x)$.

14 a If $\cot(x) = -\frac{\sqrt{6}}{3}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\text{cosec}(x)$.

b If $\sec(x) = \frac{2\sqrt{6}}{3}$ and $\frac{3\pi}{2} < x < 2\pi$, find the exact value of $\cot(x)$.

MASTER

15 If $\text{cosec}(x) = \frac{p}{q}$ where $p, q \in R^+$ and $\frac{\pi}{2} < x < \pi$, evaluate $\sec(x) - \cot(x)$.

16 If $\sec(x) = \frac{a}{b}$ where $a, b \in R^+$ and $\frac{3\pi}{2} < x < 2\pi$, evaluate $\cot(x) - \text{cosec}(x)$.

2.3 Trigonometric identities using reciprocal trigonometric functions

studyon

Units 3 & 4
AOS 1
Topic 2
Concept 3

Trigonometric identities
Concept summary
Practice questions

Identities

By mathematical convention, $(\sin(\theta))^2$ is written as $\sin^2(\theta)$, and similarly $(\cos(\theta))^2$ is written as $\cos^2(\theta)$.

Note that $\sin^2(\theta) + \cos^2(\theta) = 1$ is an identity, not an equation, since it holds true for all values of θ.

Similarly, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ holds for all values of θ for which $\tan(\theta)$ is defined, that is for all values where $\cos(\theta) \neq 0$, or $\theta \neq (2n + 1)\frac{\pi}{2}$ where $n \in Z$ or odd multiples of $\frac{\pi}{2}$.

Proving trigonometric identities

A trigonometric identity is verified by transforming one side into the other. Success in verifying trigonometric identities relies upon familiarity with known trigonometric identities and using algebraic processes such as simplifying, factorising, cancelling common factors, adding fractions and forming common denominators. The following identities must be known.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\text{cosec}(\theta) = \frac{1}{\sin(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

WORKED EXAMPLE 3

Prove the identity $\tan(\theta) + \cot(\theta) = \sec(\theta)\text{cosec}(\theta)$.

THINK	WRITE
1 Start with the left-hand side.	$\text{LHS} = \tan(\theta) + \cot(\theta)$
2 Substitute for the appropriate trigonometric identities.	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ $\text{LHS} = \frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}$

3 Add the fractions, forming the lowest common denominator.

$$\text{LHS} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos(\theta)\sin(\theta)}$$

4 Simplify the numerator.

Since $\sin^2(\theta) + \cos^2(\theta) = 1$,

$$\text{LHS} = \frac{1}{\cos(\theta)\sin(\theta)}$$

5 Write the expression as factors.

$$\text{LHS} = \frac{1}{\cos(\theta)} \times \frac{1}{\sin(\theta)}$$

6 Substitute for the appropriate trigonometric identities. The proof is complete.

$\sec(\theta) = \dfrac{1}{\cos(\theta)}$ and $\text{cosec}(\theta) = \dfrac{1}{\sin(\theta)}$

$$\begin{aligned}\text{LHS} &= \sec(\theta)\text{cosec}(\theta)\\ &= \text{RHS}\end{aligned}$$

Fundamental relations

If all terms of $\sin^2(\theta) + \cos^2(\theta) = 1$ are divided by $\sin^2(\theta)$, we obtain

$\dfrac{\sin^2(\theta)}{\sin^2(\theta)} + \dfrac{\cos^2(\theta)}{\sin^2(\theta)} = \dfrac{1}{\sin^2(\theta)}$ and hence obtain the trigonometric identity

$$1 + \cot^2(\theta) = \text{cosec}^2(\theta).$$

If all terms of $\sin^2(\theta) + \cos^2(\theta) = 1$ are divided by $\cos^2(\theta)$, we obtain

$\dfrac{\sin^2(\theta)}{\cos^2(\theta)} + \dfrac{\cos^2(\theta)}{\cos^2(\theta)} = \dfrac{1}{\cos^2(\theta)}$ and hence obtain the trigonometric identity

$$\tan^2(\theta) + 1 = \sec^2(\theta).$$

WORKED EXAMPLE 4

Prove the identity $\dfrac{1 + \tan^2(\theta)}{1 + \cot^2(\theta)} = \tan^2(\theta)$.

THINK

WRITE

1 Start with the left-hand side.

$$\text{LHS} = \frac{1 + \tan^2(\theta)}{1 + \cot^2(\theta)}$$

2 Substitute the appropriate trigonometric identities.

Replace $1 + \tan^2(\theta) = \sec^2(\theta)$ in the numerator and $1 + \cot^2(\theta) = \text{cosec}^2(\theta)$ in the denominator.

$$\text{LHS} = \frac{\sec^2(\theta)}{\text{cosec}^2(\theta)}$$

3 Use appropriate trigonometric identities to express the quotient in terms of sines and cosines.

$\sec^2(\theta) = \dfrac{1}{\cos^2(\theta)}$ and $\text{cosec}^2(\theta) = \dfrac{1}{\sin^2(\theta)}$

$$\text{LHS} = \frac{\dfrac{1}{\cos^2(\theta)}}{\dfrac{1}{\sin^2(\theta)}}$$

4 Simplify the quotient.

Use $\dfrac{\frac{1}{a}}{\frac{1}{b}} = \dfrac{b}{a}$.

$\text{LHS} = \dfrac{\sin^2(\theta)}{\cos^2(\theta)}$

5 Simplify and state the final result.

Since $\tan^2(\theta) = \dfrac{\sin^2(\theta)}{\cos^2(\theta)}$,

$\text{LHS} = \tan^2(\theta)$
$= \text{RHS}$

EXERCISE 2.3 Trigonometric identities using reciprocal trigonometric functions

PRACTISE

1 WE3 Prove the identity $\sec^2(\theta) + \text{cosec}^2(\theta) = \sec^2(\theta)\text{cosec}^2(\theta)$.

2 Prove the identity $\dfrac{\sin(\theta)}{1 + \cos(\theta)} + \dfrac{1 + \cos(\theta)}{\sin(\theta)} = 2\,\text{cosec}(\theta)$.

3 WE4 Prove the identity $\dfrac{1 + \cot^2(\theta)}{1 + \tan^2(\theta)} = \cot^2(\theta)$.

4 Prove the identity $(1 - \sin^2(\theta))(1 + \tan^2(\theta)) = 1$.

CONSOLIDATE

For questions 5–14, prove each of the given identities.

5 a $\cos(\theta)\text{cosec}(\theta) = \cot(\theta)$
b $\cos(\theta)\tan(\theta) = \sin(\theta)$

6 a $\sin(\theta)\sec(\theta)\cot(\theta) = 1$
b $\cos(\theta)\text{cosec}(\theta)\tan(\theta) = 1$

7 a $(\cos(\theta) + \sin(\theta))^2 + (\cos(\theta) - \sin(\theta))^2 = 2$
b $2 - 3\cos^2(\theta) = 3\sin^2(\theta) - 1$

8 a $\tan^2(\theta)\cos^2(\theta) + \cot^2(\theta)\sin^2(\theta) = 1$
b $\dfrac{\sin(\theta)}{\text{cosec}(\theta)} + \dfrac{\cos(\theta)}{\sec(\theta)} = 1$

9 a $\dfrac{1}{1 - \sin(\theta)} + \dfrac{1}{1 + \sin(\theta)} = 2\sec^2(\theta)$
b $\dfrac{1}{1 - \cos(\theta)} + \dfrac{1}{1 + \cos(\theta)} = 2\,\text{cosec}^2(\theta)$

10 a $\dfrac{1}{1 + \sec^2(\theta)} + \dfrac{1}{1 + \cos^2(\theta)} = 1$
b $(1 - \tan(\theta))^2 + (1 + \tan(\theta))^2 = 2\sec^2(\theta)$

11 a $(\tan(\theta) + \sec(\theta))^2 = \dfrac{1 + \sin(\theta)}{1 - \sin(\theta)}$
b $\sec^4(\theta) - \sec^2(\theta) = \tan^4(\theta) + \tan^2(\theta)$

12 a $\dfrac{\tan(\theta)}{\sec(\theta) - 1} + \dfrac{\tan(\theta)}{\sec(\theta) + 1} = 2\,\text{cosec}(\theta)$
b $\dfrac{1 + \cot(\theta)}{\text{cosec}(\theta)} - \dfrac{\sec(\theta)}{\tan(\theta) + \cot(\theta)} = \cos(\theta)$

13 a $\dfrac{\cos(\theta)}{1 - \sin(\theta)} = \sec(\theta) + \tan(\theta)$ **b** $\dfrac{\cos(\theta)}{1 + \sin(\theta)} = \sec(\theta) - \tan(\theta)$

14 a $\dfrac{1}{1 + \sin^2(\theta)} + \dfrac{1}{1 + \text{cosec}^2(\theta)} = 1$ **b** $\dfrac{1}{1 + \cot^2(\theta)} + \dfrac{1}{1 + \tan^2(\theta)} = 1$

MASTER

For questions **15** and **16**, prove each of the given identities.

15 a $\dfrac{a - b\cos^2(\theta)}{\sin^2(\theta)} = b + (a - b)\text{cosec}^2(\theta)$ **b** $\dfrac{a - b\sin^2(\theta)}{\cos^2(\theta)} = b + (a - b)\sec^2(\theta)$

16 a $\dfrac{a - b\tan^2(\theta)}{1 + \tan^2(\theta)} = (a + b)\cos^2(\theta) - b$ **b** $\dfrac{a - b\cot^2(\theta)}{1 + \cot^2(\theta)} = (a + b)\sin^2(\theta) - b$

2.4 Compound-angle formulas

The compound-angle formulas are also known as trigonometric addition and subtraction formulas.

studyon

Units 3 & 4
AOS 1
Topic 2
Concept 4

Compound- and double-angle formulas
Concept summary
Practice questions

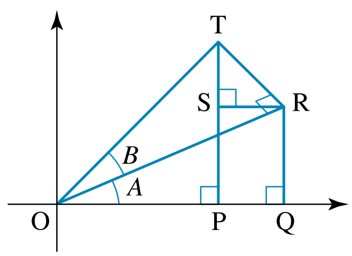
Proof of the compound-angle formulas

The compound addition formulas state that:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

It is interesting to consider one method of proving these identities.

Consider the triangle OQR with a right angle at Q, as shown in the diagram. The line segment TR is constructed so that TR is perpendicular to OR, and the line segment TP is constructed so that it is perpendicular to OP and SR. Let ∠ROQ = A and ∠TOR = B so that ∠TOP = $A + B$.



Using the properties of similar triangles in ΔTSR and ΔOQR, or the property that supplementary angles sum to 90°, it follows that ∠STR = A.

In triangle OQR, $\sin(A) = \dfrac{QR}{OR}$ and $\cos(A) = \dfrac{OQ}{OR}$.

In triangle RST, $\sin(A) = \dfrac{SR}{RT}$ and $\cos(A) = \dfrac{ST}{RT}$.

In triangle ORT, $\sin(B) = \dfrac{RT}{OT}$ and $\cos(B) = \dfrac{OR}{OT}$.

Now consider the triangle OPT.

$$\sin(A + B) = \frac{PT}{OT} = \frac{PS + ST}{OT} = \frac{PS}{OT} + \frac{ST}{OT}$$

PS = QR, so

$$\begin{aligned}\sin(A+B) &= \frac{\text{QR}}{\text{OT}} + \frac{\text{ST}}{\text{OT}} \\ &= \frac{\text{QR}}{\text{OT}} \times \frac{\text{OR}}{\text{OR}} + \frac{\text{ST}}{\text{OT}} \times \frac{\text{RT}}{\text{RT}} \\ &= \frac{\text{QR}}{\text{OR}} \times \frac{\text{OR}}{\text{OT}} + \frac{\text{ST}}{\text{RT}} \times \frac{\text{RT}}{\text{OT}}\end{aligned}$$

That is,

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

Also in the triangle OPT:

$$\cos(A+B) = \frac{\text{OP}}{\text{OT}} = \frac{\text{OQ} - \text{PQ}}{\text{OT}} = \frac{\text{OQ}}{\text{OT}} - \frac{\text{PQ}}{\text{OT}}$$

PQ = SR, so

$$\begin{aligned}\cos(A+B) &= \frac{\text{OQ}}{\text{OT}} - \frac{\text{SR}}{\text{OT}} \\ &= \frac{\text{OQ}}{\text{OT}} \times \frac{\text{OR}}{\text{OR}} - \frac{\text{SR}}{\text{OT}} \times \frac{\text{RT}}{\text{RT}} \\ &= \frac{\text{OQ}}{\text{OR}} \times \frac{\text{OR}}{\text{OT}} - \frac{\text{SR}}{\text{RT}} \times \frac{\text{RT}}{\text{OT}} \\ \cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)\end{aligned}$$

Proof of the compound-angle subtraction formulas

The compound subtraction formulas state that:

$$\mathbf{\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)}$$
$$\mathbf{\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)}$$

These formulas can obtained by replacing B with $-B$ and using $\cos(-B) = \cos(B)$ and $\sin(-B) = -\sin(B)$.

Substituting into the formula $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$, we derive $\sin(A+(-B)) = \sin(A)\cos(-B) + \cos(A)\sin(-B)$, so that $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$.

Similarly, in the formula $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$, we derive $\cos(A+(-B)) = \cos(A)\cos(-B) - \sin(A)\sin(-B)$, so that $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.

Proof of the compound-angle formulas involving tangents

Let us substitute the formulas for $\sin(A+B)$ and $\cos(A+B)$ into the identity for the tangent ratio.

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)}\end{aligned}$$

In order to simplify this fraction, divide each term by $\cos(A)\cos(B)$:

$$\tan(A+B) = \frac{\dfrac{\sin(A)\cos(B)}{\cos(A)\cos(B)} + \dfrac{\cos(A)\sin(B)}{\cos(A)\cos(B)}}{\dfrac{\cos(A)\cos(B)}{\cos(A)\cos(B)} - \dfrac{\sin(A)\sin(B)}{\cos(A)\cos(B)}}$$

$$= \frac{\dfrac{\sin(A)}{\cos(A)} + \dfrac{\sin(B)}{\cos(B)}}{1 - \dfrac{\sin(A)}{\cos(A)} \times \dfrac{\sin(B)}{\cos(B)}}$$

$$= \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

The corresponding formula for the tangent for the difference of two angles is obtained by replacing B with $-B$ and using $\tan(-B) = -\tan(B)$.

$$\tan(A + (-B)) = \frac{\tan(A) + \tan(-B)}{1 - \tan(A)\tan(-B)}$$

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

Summary of the compound-angle formulas

These results are called the compound-angle formulas or addition theorems. They can be summarised as:

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

$$\tan(A-B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

Using compound-angle formulas in problems

The compound-angle formulas can be used to simplify many trigonometric expressions. They can be used in both directions, for example $\sin(A)\cos(B) + \cos(A)\sin(B) = \sin(A+B)$.

WORKED EXAMPLE 5

Evaluate $\sin(22°)\cos(38°) + \cos(22°)\sin(38°)$.

THINK	WRITE
1 State an appropriate identity.	$\sin(A)\cos(B) + \cos(A)\sin(B) = \sin(A+B)$ Let $A = 22°$ and $B = 38°$. $\sin(22°)\cos(38°) + \cos(22°)\sin(38°) = \sin(22° + 38°)$

2 Simplify and use the exact values.

$$\sin(22°)\cos(38°) + \cos(22°)\sin(38°) = \sin(60°)$$
$$= \frac{\sqrt{3}}{2}$$

Expanding trigonometric expressions with phase shifts

The compound-angle formulas can be used to expand trigonometric expressions.

WORKED EXAMPLE 6 Expand $2\cos\left(\theta + \frac{\pi}{3}\right)$.

THINK

1 State an appropriate identity.

WRITE

$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Let $A = \theta$ and $B = \frac{\pi}{3}$.

$$2\cos\left(\theta + \frac{\pi}{3}\right) = 2\left(\cos(\theta)\cos\left(\frac{\pi}{3}\right) - \sin(\theta)\sin\left(\frac{\pi}{3}\right)\right)$$

2 Substitute for exact values.

Since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$,

$$2\cos\left(\theta + \frac{\pi}{3}\right) = 2\left(\cos(\theta) \times \frac{1}{2} - \sin(\theta) \times \frac{\sqrt{3}}{2}\right)$$

3 Simplify.

$$= \cos(\theta) - \sqrt{3}\sin(\theta)$$

4 State the answer.

$$2\cos\left(\theta + \frac{\pi}{3}\right) = \cos(\theta) - \sqrt{3}\sin(\theta)$$

Simplification of $\sin\left(\frac{n\pi}{2} \pm \theta\right)$ and $\cos\left(\frac{n\pi}{2} \pm \theta\right)$ for $n \in Z$

Recall that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$ as complementary angles.

Compound-angle formulas can be used to simplify and verify many of these results and similar formulas from earlier results, that is trigonometric expansions of the forms $\sin\left(\frac{n\pi}{2} \pm \theta\right)$ and $\cos\left(\frac{n\pi}{2} \pm \theta\right)$ where $n \in Z$.

WORKED EXAMPLE 7 Use compound-angle formulas to simplify $\cos\left(\frac{3\pi}{2} - \theta\right)$.

THINK

1 State an appropriate identity.

WRITE

$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

Let $A = \frac{3\pi}{2}$ and $B = \theta$.

$$\cos\left(\frac{3\pi}{2} - \theta\right) = \cos\left(\frac{3\pi}{2}\right)\cos(\theta) + \sin\left(\frac{3\pi}{2}\right)\sin(\theta)$$

2 Simplify and use exact values.	Since $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$, $\cos\left(\frac{3\pi}{2} - \theta\right) = 0 \times \cos(\theta) + -1 \times \sin(\theta)$
3 State the final answer.	$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta)$

Exact values for multiples of $\frac{\pi}{12}$

Exact values are known for the trigonometric ratios for all multiples of $\frac{\pi}{6}$ radians or 30°, and for all multiples $\frac{\pi}{4}$ radians or 45°. Using the compound-angle formulas the exact value can be found for a trigonometric ratio of an angle that is an odd multiple of $\frac{\pi}{12}$ radians or 15°. This can be obtained by rewriting the multiple of $\frac{\pi}{12}$ radians or 15° as a sum or difference of known fractions in terms of multiples of $\frac{\pi}{6}$ radians or 30° and $\frac{\pi}{4}$ radians or 45°.

WORKED EXAMPLE 8

Find the exact value of $\sin\left(\frac{13\pi}{12}\right)$.

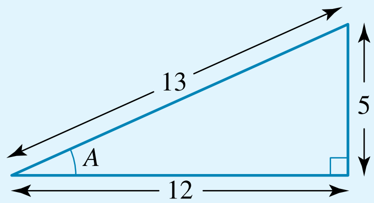
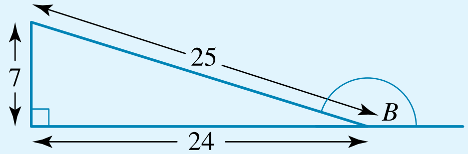
THINK	WRITE
1 Rewrite the argument as a sum or difference of fractions.	$\frac{5\pi}{6} + \frac{\pi}{4} = \frac{13\pi}{12}$, or in degrees, $150° + 45° = 195°$. $\sin\left(\frac{13\pi}{12}\right) = \sin\left(\frac{5\pi}{6} + \frac{\pi}{4}\right)$
2 State an appropriate identity.	$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ Let $A = \frac{5\pi}{6}$ and $B = \frac{\pi}{4}$. $\sin\left(\frac{13\pi}{12}\right) = \sin\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$
3 Simplify and use exact values.	Substitute $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. $\sin\left(\frac{13\pi}{12}\right) = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{-\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$ $= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$
4 Simplify and state the final answer.	$\sin\left(\frac{13\pi}{12}\right) = \frac{1}{4}(\sqrt{2} - \sqrt{6})$

Using triangles to find values

By drawing triangles to find the values of trigonometric ratios of a single angle and then using the compound-angle formulas, the trigonometric values of the addition or subtraction of two angles may be found.

WORKED EXAMPLE 9

If $\cos(A) = \frac{12}{13}$ and $\sin(B) = \frac{7}{25}$, where $0 < A < \frac{\pi}{2}$ and $\frac{\pi}{2} < B < \pi$, find the exact value of $\sin(A - B)$.

THINK	WRITE/DRAW
1 State the values of the sides of the required right-angled triangle.	$\cos(A) = \frac{12}{13} = \frac{\text{adjacent}}{\text{hypotenuse}}$ The adjacent side length is 12 and the hypotenuse is 13.
2 Use Pythagoras' theorem to calculate the third side length.	$\sqrt{13^2 - 12^2} = \sqrt{169 - 144}$ $= \sqrt{25}$ $= 5$
3 State the third side length of the triangle. Draw the triangle.	The other side length is 5. We know that 5, 12, 13 is a Pythagorean triad. 
4 State the value of the unknown trigonometric ratio.	Given that $0 < A < \frac{\pi}{2}$, so A is in the first quadrant, $\sin(A) = \frac{5}{13}$.
5 State the values of the sides of another required right-angled triangle.	$\sin(B) = \frac{7}{25} = \frac{\text{opposite}}{\text{hypotenuse}}$ The opposite side length is 7 and the hypotenuse is 25.
6 Use Pythagoras' theorem to calculate the third side length.	$\sqrt{25^2 - 7^2} = \sqrt{625 - 49}$ $= \sqrt{576}$ $= 24$
7 State the third side length of the triangle. Draw the triangle.	The other side length is 24. We know that 7, 24, 25 is a Pythagorean triad. 
8 Calculate the value of the unknown trigonometric ratio.	Since $\frac{\pi}{2} < B < \pi$, B is in the second quadrant, B is an obtuse angle and cosine is negative in the second quadrant. Therefore, $\cos(B) = -\frac{24}{25}$.

9 State and use an appropriate identity. $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

10 Substitute for the values and simplify. $\sin(A - B) = \frac{5}{13} \times \frac{-24}{25} - \frac{12}{13} \times \frac{7}{25}$

11 Simplify and state the final answer. $\sin(A - B) = -\frac{204}{325}$

EXERCISE 2.4 Compound-angle formulas

PRACTISE

1 WE5 Evaluate $\sin(51°)\cos(9°) + \cos(51°)\sin(9°)$.

2 Find the value of $\cos(37°)\cos(23°) - \sin(37°)\sin(23°)$.

3 WE6 Expand $4\cos\left(\theta + \frac{\pi}{6}\right)$.

4 Express $\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$ as a combination of sines and cosines.

5 WE7 Use compound-angle formulas to simplify $\cos(\pi - \theta)$.

6 Simplify $\sin(2\pi - \theta)$.

7 WE8 Find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

8 Find the exact value of $\tan\left(\frac{11\pi}{12}\right)$.

9 WE9 If $\cos(A) = \frac{3}{5}$ and $\sin(B) = \frac{8}{17}$ where $0 < A < \frac{\pi}{2}$ and $\frac{\pi}{2} < B < \pi$, find the exact value of $\sin(A - B)$.

10 Given that $\tan(A) = \frac{9}{40}$ and $\cos(B) = \frac{7}{25}$ where $\pi < A < \frac{3\pi}{2}$ and $0 < B < \frac{\pi}{2}$, find the exact value of $\cos(A + B)$.

CONSOLIDATE

11 Evaluate each of the following.

a $\sin(27°)\cos(33°) + \cos(27°)\sin(33°)$
b $\cos(47°)\cos(43°) - \sin(47°)\sin(43°)$
c $\cos(76°)\cos(16°) + \sin(76°)\sin(16°)$
d $\cos(63°)\sin(18°) - \sin(63°)\cos(18°)$

12 Evaluate each of the following.

a $\frac{\tan(52°) - \tan(22°)}{1 + \tan(52°)\tan(22°)}$
b $\frac{\tan(32°) + \tan(28°)}{1 - \tan(32°)\tan(28°)}$

13 Expand each of the following.

a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$
b $2\sin\left(\theta + \frac{\pi}{3}\right)$
c $2\cos\left(\theta - \frac{\pi}{6}\right)$
d $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

14 Use compound-angle formulas to simplify each of the following.

a $\sin\left(\frac{\pi}{2} - \theta\right)$ b $\cos\left(\frac{\pi}{2} - \theta\right)$ c $\sin(\pi + \theta)$ d $\cos(\pi - \theta)$

15 Use compound-angle formulas to simplify each of the following.

a $\sin\left(\frac{3\pi}{2} - \theta\right)$ b $\cos\left(\frac{3\pi}{2} + \theta\right)$ c $\tan(\pi - \theta)$ d $\tan(\pi + \theta)$

16 Simplify each of the following.

a $\sin\left(x + \frac{\pi}{3}\right) - \sin\left(x - \frac{\pi}{3}\right)$ b $\tan\left(x + \frac{\pi}{4}\right)\tan\left(x - \frac{\pi}{4}\right)$

c $\cos\left(\frac{\pi}{3} + x\right) - \cos\left(\frac{\pi}{3} - x\right)$ d $\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)$

17 Find each of the following in exact simplest surd form.

a $\cos\left(\frac{7\pi}{12}\right)$ b $\tan\left(\frac{\pi}{12}\right)$ c $\sin\left(\frac{11\pi}{12}\right)$ d $\tan\left(\frac{5\pi}{12}\right)$

18 Given that $\cos(A) = \frac{4}{5}$, $\sin(B) = \frac{12}{13}$, and A and B are both acute angles, find the exact value of:

a $\cos(A - B)$ b $\tan(A + B)$.

19 Given that $\sin(A) = \frac{5}{13}$ and $\tan(B) = \frac{24}{7}$ where A is obtuse and B is acute, find the exact value of:

a $\sin(A + B)$ b $\cos(A + B)$.

20 Given that $\sec(A) = \frac{7}{2}$, $\text{cosec}(B) = \frac{3}{2}$, and A is acute but B is obtuse, find the exact value of:

a $\cos(A + B)$ b $\sin(A - B)$.

MASTER

21 Given that $\text{cosec}(A) = \frac{1}{a}$, $\sec(B) = \frac{1}{b}$, A and B are both acute, and $0 < a < 1$ and $0 < b < 1$, evaluate $\tan(A + B)$.

22 Given that $\sin(A) = \frac{a}{a+1}$ and $\cos(B) = \frac{a}{a+2}$ where A and B are both acute, evaluate $\tan(A + B)$.

2.5 Double-angle formulas

In this section we consider the special cases of the addition formulas when $B = A$.

Double-angle formulas

In the formula $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$, let $B = A$.

$$\sin(2A) = \sin(A)\cos(A) + \sin(A)\cos(A)$$
$$\sin(2A) = 2\sin(A)\cos(A)$$

In the formula $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$, let $B = A$.

$$\cos(2A) = \cos(A)\cos(A) - \sin(A)\sin(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

Since $\sin^2(A) + \cos^2(A) = 1$, it follows that $\cos^2(A) = 1 - \sin^2(A)$. This formula can be rewritten in terms of $\sin(A)$ only.

$$\cos(2A) = (1 - \sin^2(A)) - \sin^2(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$

Alternatively, if we substitute $\sin^2(A) = 1 - \cos^2(A)$, then this formula can also be rewritten in terms of $\cos(A)$ only.

$$\cos(2A) = \cos^2(A) - (1 - \cos^2(A))$$
$$\cos(2A) = 2\cos^2(A) - 1$$

There are thus three equivalent forms of the double-angle formulas for $\cos(2A)$.

If we let $B = A$ in the formula $\tan(A + B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$, we obtain

$$\tan(2A) = \frac{\tan(A) + \tan(A)}{1 - \tan(A)\tan(A)}$$
$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

All of these formulas can be summarised as follows:

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\cos(2A) = 2\cos^2(A) - 1$$
$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

Using double-angle formulas in simplifying expressions

The double-angle formulas can be used to simplify many trigonometric expressions and can be used both ways; for example, $\sin(A)\cos(A) = \frac{1}{2}\sin(2A)$.

WORKED EXAMPLE 10

Find the exact value of $\sin\left(\dfrac{7\pi}{12}\right)\cos\left(\dfrac{7\pi}{12}\right)$.

THINK	WRITE
1 State an appropriate identity.	$\sin(A)\cos(A) = \dfrac{1}{2}\sin(2A)$ Let $A = \dfrac{7\pi}{12}$. $\sin\left(\dfrac{7\pi}{12}\right)\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1}{2}\sin\left(2 \times \dfrac{7\pi}{12}\right)$
2 Simplify.	Since $2 \times \dfrac{7\pi}{12} = \dfrac{7\pi}{6}$, $\sin\left(\dfrac{7\pi}{12}\right)\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1}{2}\sin\left(\dfrac{7\pi}{6}\right)$

3 Use the exact values to substitute into the expression.

Substitute $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$:

$$\frac{1}{2}\sin\left(\frac{7\pi}{6}\right) = \frac{1}{2} \times -\frac{1}{2}$$

4 State the answer.

$$\sin\left(\frac{7\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right) = -\frac{1}{4}$$

Finding trigonometric expressions involving double-angle formulas

We can use the double-angle formulas to obtain exact values for trigonometric expressions.

WORKED EXAMPLE 11

If $\cos(A) = \frac{1}{4}$, determine the exact values of:

a $\sin(2A)$ **b** $\cos(2A)$ **c** $\tan(2A)$.

THINK

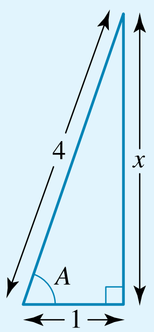
WRITE/DRAW

1 State the values of the sides of the required right-angled triangle.

$$\cos(A) = \frac{1}{4} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

The adjacent side length is 1 and the hypotenuse is 4.

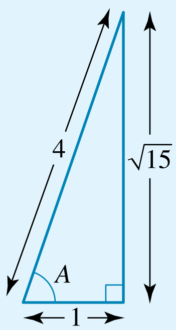
2 Draw the triangle and label the side lengths using the definition of the trigonometric ratio. Label the unknown side length as x.



3 Use Pythagoras' theorem to calculate the third unknown side length.

$$1^2 + x^2 = 4^2$$
$$x^2 = 16 - 1$$
$$x = \sqrt{15}$$

4 Redraw the triangle.



5 Apply the definitions of the sine and tangent functions.

$\cos(A) = \frac{1}{4}$, $\sin(A) = \frac{\sqrt{15}}{4}$ and $\tan(A) = \sqrt{15}$

a **1** Use the required identity.	**a** $\sin(2A) = 2\sin(A)\cos(A)$
2 Substitute the known values and simplify.	$= 2 \times \frac{\sqrt{15}}{4} \times \frac{1}{4}$ $= \frac{\sqrt{15}}{8}$
b **1** Using the required identity, choose any one of the three choices for cos(2*A*).	**b** $\cos(2A) = \cos^2(A) - \sin^2(A)$
2 Substitute the known values and simplify.	$\cos(2A) = \left(\frac{1}{4}\right)^2 - \left(\frac{\sqrt{15}}{4}\right)^2$ $= \frac{1}{16} - \frac{15}{16}$ $= -\frac{7}{8}$
c **1** State the required identity.	**c** $\tan(2A) = \frac{\sin(2A)}{\cos(2A)}$
2 Substitute the known values and simplify the ratio.	$= \frac{\frac{\sqrt{15}}{8}}{-\frac{7}{8}}$ $= -\frac{\sqrt{15}}{7}$
3 As an alternative method, use the double-angle formulas for tan.	$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$
4 Substitute for the known value and simplify.	$\tan(A) = \sqrt{15}$ $\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$ $= \frac{2\sqrt{15}}{1 - (\sqrt{15})^2}$ $= \frac{2\sqrt{15}}{-14}$ $= -\frac{\sqrt{15}}{7}$

Solving trigonometric equations involving double-angle formulas

Trigonometric equations are often solved over a given domain, usually $x \in [0, 2\pi]$. In this section we consider solving trigonometric equations that involve using the double-angle formulas.

WORKED EXAMPLE 12

Solve for x if $\sin(2x) + \sqrt{3}\cos(x) = 0$ for $x \in [0, 2\pi]$.

THINK	WRITE
1 Expand and write the equation in terms of one argument only.	Use $\sin(2x) = 2\sin(x)\cos(x)$ $\sin(2x) + \sqrt{3}\cos(x) = 0$ $2\sin(x)\cos(x) + \sqrt{3}\cos(x) = 0$

2 Factorise by taking out the common factor. $\cos(x)(2\sin(x)+\sqrt{3})=0$

3 Use the Null Factor Law. $\cos(x)=0$ or $2\sin(x)+\sqrt{3}=0$

$\sin(x)=-\frac{\sqrt{3}}{2}$

4 Solve the first equation. $\cos(x)=0$

$x=\frac{\pi}{2},\frac{3\pi}{2}$

5 Solve the second equation. $\sin(x)=-\frac{\sqrt{3}}{2}$

$x=\frac{4\pi}{3},\frac{5\pi}{3}$

6 State all solutions of the original equation. $x=\frac{\pi}{2},\frac{4\pi}{3},\frac{3\pi}{2},\frac{5\pi}{3}$

Trigonometric identities using double-angle formulas

Previously we used the fundamental trigonometric relationships to prove trigonometric identities using the reciprocal trigonometric functions. In this section we use the compound-angle formulas and the double-angle formulas to prove more trigonometric identities.

WORKED EXAMPLE 13 Prove the identity $\dfrac{\cos(2A)\cos(A)+\sin(2A)\sin(A)}{\sin(3A)\cos(A)-\cos(3A)\sin(A)}=\dfrac{1}{2}\operatorname{cosec}(A)$.

THINK | **WRITE**

1 Start with the left-hand side.

$$\text{LHS}=\frac{\cos(2A)\cos(A)+\sin(2A)\sin(A)}{\sin(3A)\cos(A)-\cos(3A)\sin(A)}$$

2 Simplify the numerator and denominator by recognising these as expansions of appropriate compound-angle identities.

$$=\frac{\cos(2A-A)}{\sin(3A-A)}$$

3 Simplify.

$$=\frac{\cos(A)}{\sin(2A)}$$

4 Expand the denominator using the double-angle formula.

$$=\frac{\cos(A)}{2\sin(A)\cos(A)}$$

5 Simplify by cancelling the common factor. The proof is complete.

$$=\frac{1}{2\sin(A)}$$

Since $\frac{1}{\sin(A)}=\operatorname{cosec}(A)$,

$$\text{LHS}=\frac{1}{2\sin(A)}$$

$$=\tfrac{1}{2}\operatorname{cosec}(A)$$

$=$ RHS as required.

Half-angle formulas

If we replace A with $\frac{A}{2}$, the double-angle formulas can be written as the half-angle formulas.

$$\sin(A) = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$\cos(A) = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$$

$$= 2\cos^2\left(\frac{A}{2}\right) - 1$$

$$= 1 - 2\sin^2\left(\frac{A}{2}\right)$$

These can also be rearranged and are often used as:

$$1 - \cos(A) = 2\sin^2\left(\frac{A}{2}\right)$$

$$1 + \cos(A) = 2\cos^2\left(\frac{A}{2}\right)$$

WORKED EXAMPLE 14 **Prove the identity $\operatorname{cosec}(A) - \cot(A) = \tan\left(\frac{A}{2}\right)$.**

THINK	WRITE
1 Start with the left-hand side.	$\text{LHS} = \operatorname{cosec}(A) - \cot(A)$
2 Use $\operatorname{cosec}(A) = \frac{1}{\sin(A)}$ and $\cot(A) = \frac{\cos(A)}{\sin(A)}$.	$= \frac{1}{\sin(A)} - \frac{\cos(A)}{\sin(A)}$
3 Form the common denominator.	$= \frac{1 - \cos(A)}{\sin(A)}$
4 Use appropriate half-angle formulas.	$= \frac{2\sin^2\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$
5 Simplify by cancelling the common factors. The proof is complete.	$= \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}$
	$= \tan\left(\frac{A}{2}\right)$
	$= \text{RHS}$

Multiple-angle formulas

There are many other trigonometric formulas for multiple angles. For example:

$$\sin(3A) = 3\sin(A) - 4\sin^3(A)$$
$$\cos(3A) = 4\cos^3(A) - 3\cos(A)$$
$$\tan(3A) = \frac{3\tan(A) - \tan^3(A)}{1 - 3\tan^2(A)}$$
$$\sin(4A) = \cos(A)(4\sin(A) - 8\sin^3(A))$$
$$\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$$
$$\tan(4A) = \frac{4\tan(A)(1 - \tan^2(A))}{1 - 6\tan^2(A) + \tan^4(A)}$$

Some of these proofs are provided in the next worked example; the remaining ones are left for the exercises.

WORKED EXAMPLE 15 Prove the identity $\cos(3A) = 4\cos^3(A) - 3\cos(A)$.

THINK	WRITE
1 Start with the left-hand side.	$\text{LHS} = \cos(3A)$
2 Expand the multiple argument.	$\cos(2A + A) = \cos(A)\cos(2A) - \sin(A)\sin(2A)$
3 Since we want the right-hand side in terms of cosines, substitute using appropriate trigonometric identities.	$\cos(2A) = 2\cos^2(A) - 1$ $\sin(2A) = 2\sin(A)\cos(A)$ $\cos(3A) = \cos(A)(2\cos^2(A) - 1) - \sin(A)(2\sin(A)\cos(A))$
4 Expand the brackets.	$\cos(3A) = 2\cos^3(A) - \cos(A) - 2\sin^2(A)\cos(A)$
5 Rearrange the expression in terms of cosines.	Substitute $\sin^2(A) = 1 - \cos^2(A)$: $\cos(3A) = 2\cos^3(A) - \cos(A) - 2(1 - \cos^2(A))\cos(A)$
6 Expand the brackets and simplify. The proof is complete.	$\cos(3A) = 2\cos^3(A) - \cos(A) - 2(\cos(A) - \cos^3(A))$ $= 4\cos^3(A) - 3\cos(A)$ $= \text{RHS}$

EXERCISE 2.5 Double-angle formulas

PRACTISE

1 WE10 Find the exact value of $\sin\left(\frac{5\pi}{8}\right)\cos\left(\frac{5\pi}{8}\right)$.

2 Find the exact value of $1 - 2\cos^2(157°30')$.

3 WE11 If $\cos(A) = \frac{1}{3}$, determine the exact values of:

a $\sin(2A)$ b $\cos(2A)$ c $\tan(2A)$.

4 Given that $\tan(A) = \frac{4}{7}$, determine the exact values of:

a $\sin(2A)$ b $\cos(2A)$ c $\tan(2A)$.

5 WE12 Solve for x if $\sin(2x) - \sqrt{3}\cos(x) = 0$ for $x \in [0, 2\pi]$.

6 Find the values of $x \in [0, 2\pi]$ if $\sin(2x) + \cos(x) = 0$.

7 WE13 Prove the identity $\dfrac{\sin(3A)\cos(A) - \cos(3A)\sin(A)}{\cos(2A)\cos(A) + \sin(2A)\sin(A)} = 2\sin(A)$.

8 Prove the identity $\tan\left(\dfrac{\pi}{4} + A\right) + \tan\left(\dfrac{\pi}{4} - A\right) = 2\sec(2A)$.

9 WE14 Prove the identity $\operatorname{cosec}(A) + \cot(A) = \cot\left(\dfrac{A}{2}\right)$.

10 Prove the identity $\sqrt{\dfrac{1 - \cos(A)}{1 + \cos(A)}} = \tan\left(\dfrac{A}{2}\right)$.

11 WE15 Prove the identity $\sin(3A) = 3\sin(A) - 4\sin^3(A)$.

12 Prove the identity $\tan(3A) = \dfrac{3\tan(A) - \tan^3(A)}{1 - 3\tan^2(A)}$.

CONSOLIDATE

13 Evaluate each of the following expressions, giving your answers in exact form.

a $\sin\left(\dfrac{\pi}{8}\right)\cos\left(\dfrac{\pi}{8}\right)$

b $\cos^2(112°30') - \sin^2(112°30')$

c $2\sin^2(22°30') - 1$

d $\dfrac{2\tan\left(\dfrac{\pi}{12}\right)}{1 - \tan^2\left(\dfrac{\pi}{12}\right)}$

14 Given that $\sec(A) = \frac{8}{3}$, find the exact values of:

a $\sin(2A)$

b $\cos(2A)$

c $\tan(2A)$.

15 Solve each of the following equations for $x \in [0, 2\pi]$.

a $\sin(2x) = \sin(x)$

b $\cos(x) = \cos(2x)$

c $\sin(x) = \cos(2x)$

d $\sin(2x) = \cos(x)$

16 Solve each of the following equations for $x \in [0, 2\pi]$.

a $\tan(x) = \sin(2x)$

b $\sin(2x) = \sqrt{3}\cos(x)$

c $\sin(4x) = \sin(2x)$

d $\cos(2x) = \sin(4x)$

For questions 17–21, prove each of the given identities.

17 a $\dfrac{\sin(2A)\cos(A) - \cos(2A)\sin(A)}{\cos(2A)\cos(A) + \sin(2A)\sin(A)} = \tan(A)$

b $\dfrac{\cos(2A)\cos(A) + \sin(2A)\sin(A)}{\sin(2A)\cos(A) - \cos(2A)\sin(A)} = \cot(A)$

c $\dfrac{\sin(3A)}{\sin(A)} - \dfrac{\cos(3A)}{\cos(A)} = 2$

d $\dfrac{\cos(3A)}{\sin(A)} + \dfrac{\sin(3A)}{\cos(A)} = 2\cot(2A)$

18 a $\dfrac{\tan(3A) + \tan(A)}{\tan(3A) - \tan(A)} = 2\cos(2A)$

b $\dfrac{\tan(A) - \tan(B)}{\tan(A) + \tan(B)} = \dfrac{\sin(A - B)}{\sin(A + B)}$

c $\dfrac{\sin(A) - \cos(A)}{\sin(A) + \cos(A)} - \dfrac{\sin(A) + \cos(A)}{\sin(A) - \cos(A)} = 2\tan(2A)$

d $\dfrac{\cos(A) + \sin(A)}{\cos(A) - \sin(A)} + \dfrac{\cos(A) - \sin(A)}{\cos(A) + \sin(A)} = 2\sec(2A)$

19 a $\dfrac{\sin(A)}{1 - \cos(A)} = \cot\left(\dfrac{A}{2}\right)$

b $\dfrac{\sin(A)}{1 + \cos(A)} = \tan\left(\dfrac{A}{2}\right)$

c $\dfrac{1 - \cos(2A) + \sin(2A)}{1 + \cos(2A) + \sin(2A)} = \tan(A)$

d $\dfrac{\sin(A) + \sin(2A)}{1 + \cos(2A) + \cos(A)} = \tan(A)$

20 a $\sin(A + B)\sin(A - B) = \sin^2(A) - \sin^2(B)$

b $\tan(A + B)\tan(A - B) = \dfrac{\tan^2(A) - \tan^2(B)}{1 - \tan^2(A)\tan^2(B)}$

c $\cot(A + B) = \dfrac{\cot(A)\cot(B) - 1}{\cot(A) + \cot(B)}$

d $\cot(A - B) = \dfrac{\cot(A)\cot(B) + 1}{\cot(B) - \cot(A)}$

21 a $\sin(2A) = \dfrac{2\tan(A)}{1 + \tan^2(A)}$

b $\cos(2A) = \dfrac{1 - \tan^2(A)}{1 + \tan^2(A)}$

c $\dfrac{\cos^3(A) - \sin^3(A)}{\cos(A) - \sin(A)} = 1 + \dfrac{1}{2}\sin(2A)$

d $\dfrac{\cos^3(A) + \sin^3(A)}{\cos(A) + \sin(A)} = 1 - \dfrac{1}{2}\sin(2A)$

22 In a triangle with side lengths a, b and c, where C is a right angle and c the hypotenuse, show that:

a $\sin(2A) = \dfrac{2ab}{c^2}$

b $\cos(2A) = \dfrac{b^2 - a^2}{c^2}$

c $\tan(2A) = \dfrac{2ab}{b^2 - a^2}$

d $\sin\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{c - b}{2c}}$

e $\cos\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{c + b}{2c}}$

f $\tan\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{c - b}{c + b}}$.

MASTER

23 Chebyshev (1821–1894) was a famous Russian mathematician. Although he is known more for his work in the fields of probability, statistics, number theory and differential equations, Chebyshev also devised recurrence relations for trigonometric multiple angles. One such result is
$\cos(nx) = 2\cos(x)\cos((n - 1)x) - \cos((n - 2)x)$.
Using this result, show that:

a $\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$

b $\cos(5A) = 16\cos^5(A) - 20\cos^3(A) + 5\cos(A)$

c $\cos(6A) = 32\cos^6(A) - 48\cos^4(A) + 18\cos^2(A) - 1$.

24 Chebyshev's recurrence formula for multiple angles of the sine function is $\sin(nx) = 2\cos(x)\sin((n-1)x) - \sin((n-2)x)$. Using this result, show that:

a $\sin(4A) = \cos(A)(4\sin(A) - 8\sin^3(A))$
b $\sin(5A) = 16\sin^5(A) - 20\sin^3(A) + 5\sin(A)$
c $\sin(6A) = \cos(A)(32\sin^5(A) - 32\sin^3(A) + 6\sin(A))$.

2.6 Inverse trigonometric functions

Inverse functions

studyon

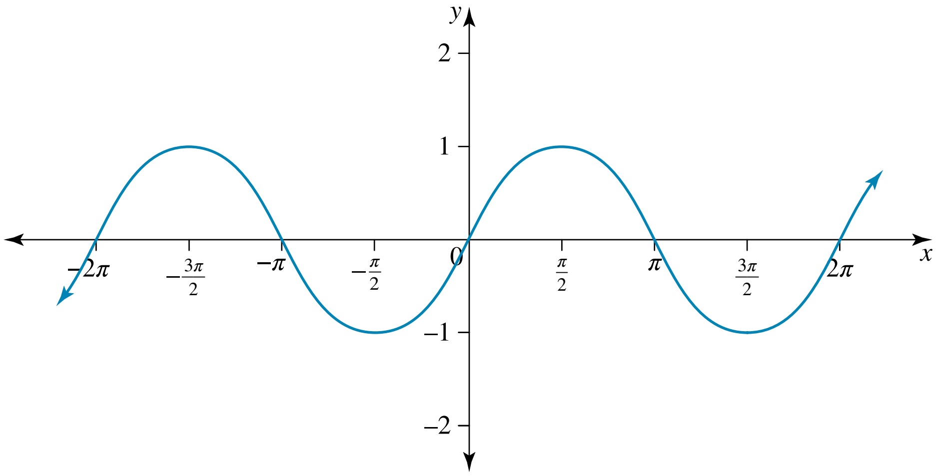
Units 3 & 4
AOS 1
Topic 2
Concept 5

Restricted circular functions
Concept summary
Practice questions

All circular functions are periodic and are many-to-one functions. Therefore, the inverses of these functions cannot be functions. However, if the domain is restricted so that the circular functions are one-to-one functions, then their inverses are functions.

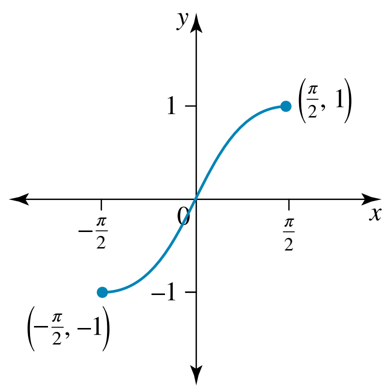
The inverse sine function

The sine function, $y = \sin(x)$, is a many-to-one function.



Therefore, its inverse does not exist as a function. However there are many restrictions of the domain, such as $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, that will ensure it is a one-to-one function. For convenience, let $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be the domain and [–1, 1] the range of the restricted sine function.

$$f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

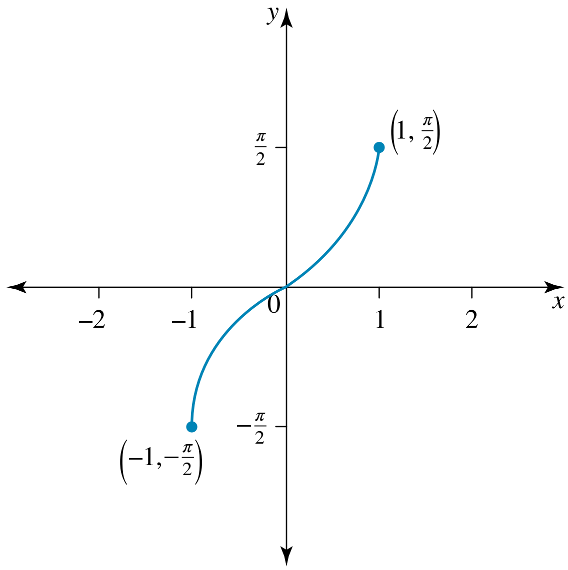


Therefore, it is a one-to-one function and its inverse exists.

The inverse of this function is denoted by $\sin^{-1}$. (An alternative notation is arcsin.)

The graph of $y = \sin^{-1}(x)$ is obtained from the graph of $y = \sin(x)$ by reflection in the line $y = x$.

$$f: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], f(x) = \sin^{-1}(x)$$



There are an infinite number of solutions to $\sin(x) = \frac{1}{2}$, for example $\frac{\pi}{6}$, $2\pi + \frac{\pi}{6}$ and $4\pi + \frac{\pi}{6}$, since we can always add any multiple of 2π to any angle and get the same result. However, $\sin^{-1}\left(\frac{1}{2}\right)$ means $\sin(x) = \frac{1}{2}$ and $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so there is only one solution in this case: $\frac{\pi}{6}$.

WORKED EXAMPLE 16

Find each of the following.

a $\sin^{-1}(2)$ **b** $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$ **c** $\sin(\sin^{-1}(0.5))$

THINK	WRITE
a **1** Write an equivalent statement.	**a** $x = \sin^{-1}(2)$ $\sin(x) = 2$
2 State the result.	This does not exist. There is no solution to $\sin(x) = 2$.
b **1** Use the known results.	**b** Since $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, $x = \sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right)$
2 Write an equivalent statement and state the result.	$\sin(x) = \frac{1}{2}$ and $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ The only solution is $x = \frac{\pi}{6}$. $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \frac{\pi}{6}$
c State the result.	**c** $\sin(\sin^{-1}(0.5)) = 0.5$

General results for the inverse sine function

In general, we have the following results for the inverse sine function:

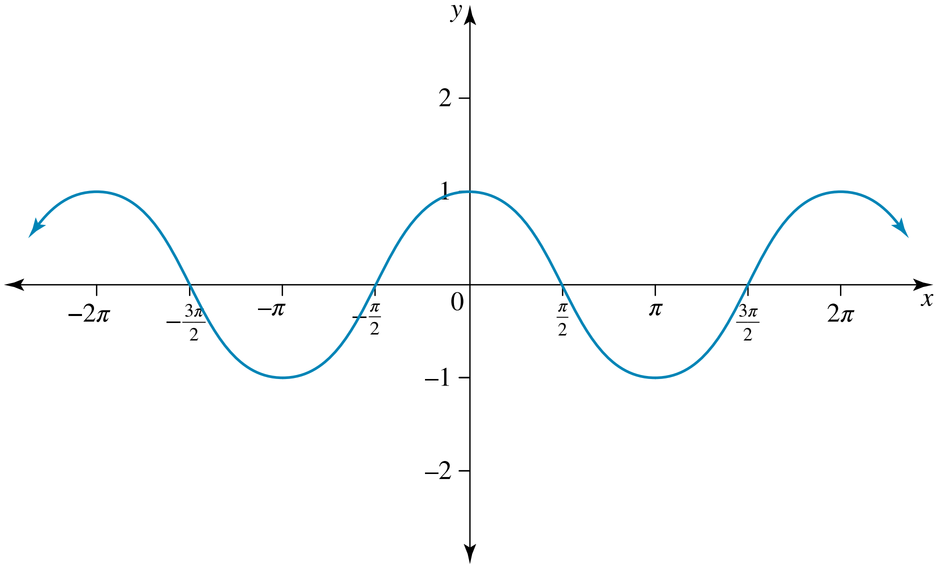
$$f:[-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ f(x) = \sin^{-1}(x)$$

$$\sin(\sin^{-1}(x)) = x \text{ if } x \in [-1, 1]$$

$$\sin^{-1}(\sin(x)) = x \text{ if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

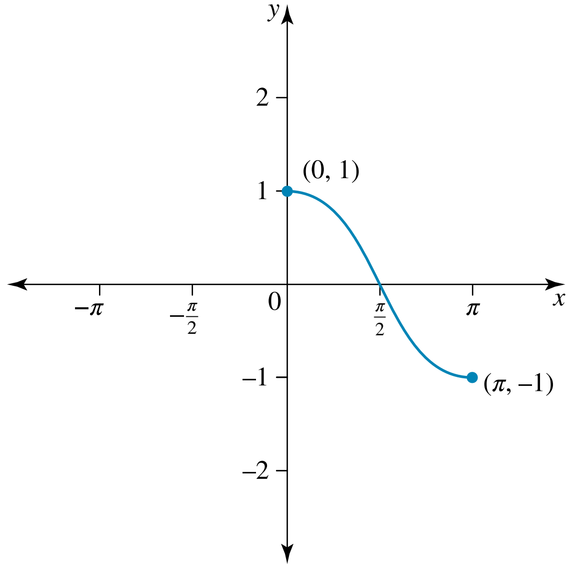
The inverse cosine function

The cosine function, $y = \cos(x)$, is a many-to-one function.



Therefore, its inverse does not exist as a function. However, there are many restrictions of the domain, such as $[-\pi, 0]$ or $[0, \pi]$ or $[\pi, 2\pi]$, that will ensure it is a one-to-one function. Let $[0, \pi]$ be the domain and $[-1, 1]$ the range of the restricted cosine function.

$$f:[0, \pi] \to [-1, 1] \text{ where } f(x) = \cos(x).$$

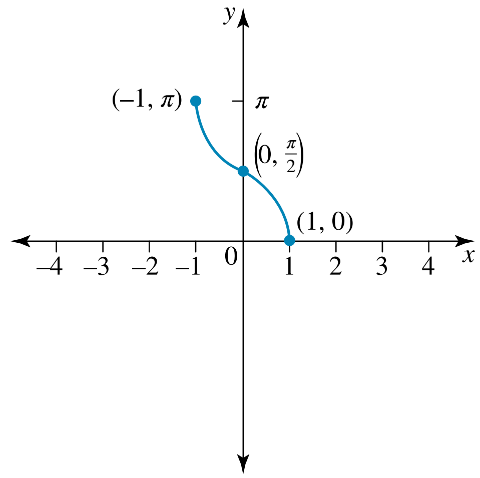


Therefore, it is a one-to-one function and its inverse exists.

The inverse of this function is denoted by $\cos^{-1}$. (An alternative notation is arccos.)

The graph of $y = \cos^{-1}(x)$ is obtained from the graph of $y = \cos(x)$ by reflection in the line $y = x$.

$$f : [-1, 1] \to [0, \pi],\ f(x) = \cos^{-1}(x)$$



There are an infinite number of solutions to $\cos(x) = \frac{\sqrt{2}}{2}$, for example $\frac{\pi}{4}$, $2\pi + \frac{\pi}{4}$, $4\pi + \frac{\pi}{4}$, $2\pi - \frac{\pi}{4}$ and $4\pi - \frac{\pi}{4}$, since we can always add any multiple of 2π to any angle. However, $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ means $\cos(x) = \frac{\sqrt{2}}{2}$ and $x \in [0\ ,\ \pi]$, so there is only one solution, namely $\frac{\pi}{4}$.

WORKED EXAMPLE 17

Find each of the following.

a $\cos^{-1}\left(\frac{3}{2}\right)$ **b** $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$ **c** $\cos\left(\cos^{-1}\left(\frac{\pi}{12}\right)\right)$

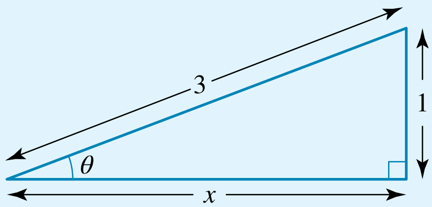
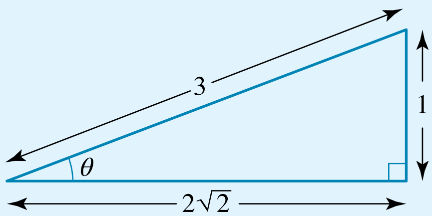
THINK	WRITE
a 1 Write an equivalent statement.	**a** $x = \cos^{-1}\left(\frac{3}{2}\right)$
2 State the result.	This does not exist. There is no solution to $\cos(x) = \frac{3}{2}$.
b 1 Use the known results.	**b** Since $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $x = \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
2 Write an equivalent statement and state the result.	$\cos(x) = -\frac{\sqrt{2}}{2}$ and $x \in [0, \pi]$ The only solution is $x = \frac{3\pi}{4}$. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) = \frac{3\pi}{4}$
c State the result.	**c** $\cos\left(\cos^{-1}\left(\frac{\pi}{12}\right)\right) = \frac{\pi}{12}$

General results for the inverse cosine function

In general, we find that:

$$f: [-1, 1] \to [0, \pi],\ f(x) = \cos^{-1}(x)$$
$$\cos(\cos^{-1}(x)) = x \text{ if } x \in [-1, 1]$$
$$\cos^{-1}(\cos(x)) = x \text{ if } [0, \pi]$$

WORKED EXAMPLE 18 **Find the exact value of $\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$.**

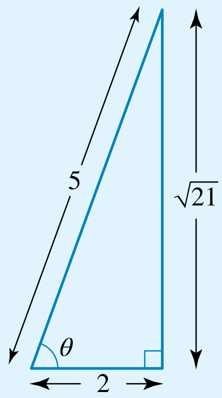
THINK	WRITE/DRAW
1 The inverse trigonometric functions are angles.	Let $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ so that $\sin(\theta) = \frac{1}{3}$.
2 Draw a right-angled triangle and label the side lengths using the definition of the trigonometric ratios.	
3 Calculate the value of the third side using Pythagoras.	$x^2 + 1^2 = 3^2$ $x^2 + 1 = 9$ $x^2 = 9 - 1$ $x^2 = 8$ $x = 2\sqrt{2}$
4 State the required value.	$\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \cos(\theta)$ $= \frac{2\sqrt{2}}{3}$ 

Double-angle formulas

Sometimes we may need to use the double-angle formulas.

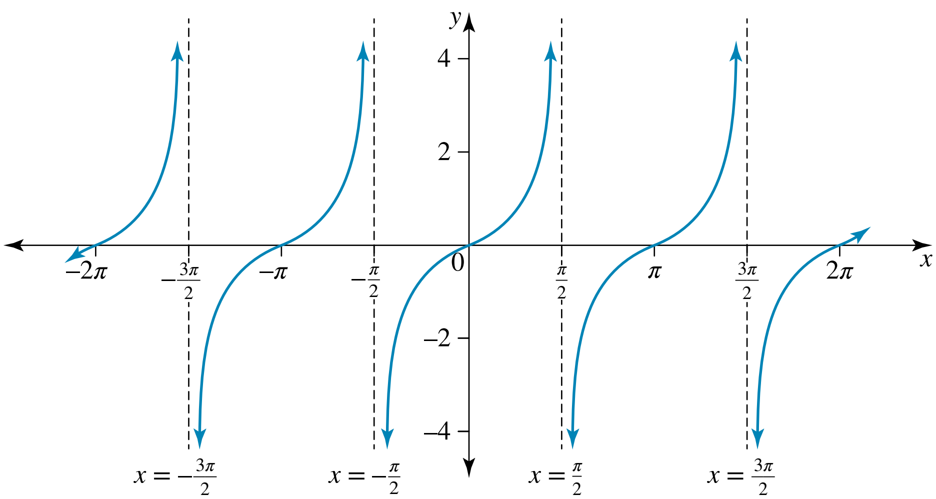
$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$= 2\cos^2(A) - 1$$
$$= 1 - 2\sin^2(A)$$

WORKED EXAMPLE 19 Find the exact value of $\sin\left(2\cos^{-1}\left(\frac{2}{5}\right)\right)$.

THINK	WRITE/DRAW
1 The inverse trigonometric functions are angles.	Let $\theta = \cos^{-1}\left(\frac{2}{5}\right)$ so that $\cos(\theta) = \frac{2}{5}$.
2 Draw a right-angled triangle and label the side lengths using the definition of the trigonometric ratios. Calculate the value of the third side using Pythagoras' theorem.	$x^2 + 2^2 = 5^2$ $x^2 + 4 = 25$ $x^2 = 25 - 4$ $x^2 = 21$ $x = \sqrt{21}$ 
3 Use an appropriate double-angle formula.	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ $= 2 \times \frac{\sqrt{21}}{5} \times \frac{2}{5}$
4 State what is required.	$\sin\left(2\cos^{-1}\left(\frac{2}{5}\right)\right) = \frac{4\sqrt{21}}{25}$

The inverse tangent function

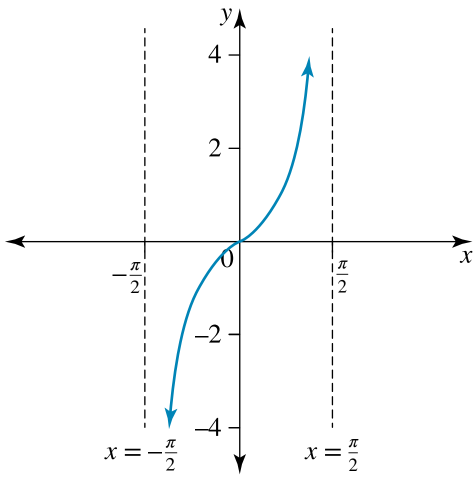
The tangent function, $y = \tan(x)$, is a many-to-one function.



Therefore, its inverse does not exist as a function. However, there are many restrictions of the domain, such as $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, that will ensure it is a one-to-one function.

Let $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be the domain and R the range of the restricted tangent function. Note that we must have an open interval, because the function is not defined at $x = \pm\frac{\pi}{2}$; at these points we have vertical asymptotes.

$$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R,\ f(x) = \tan(x)$$



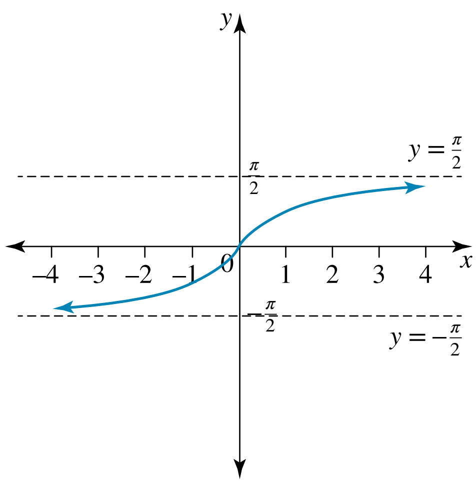
Therefore, it is a one-to-one function and its inverse exists.

The inverse of this function is denoted by $\tan^{-1}$. (An alternative notation is arctan.)

The graph of $y = \tan^{-1}(x)$ is obtained from the graph of $y = \tan(x)$ by reflection in the line $y = x$.

$$f : R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ where } f(x) = \tan^{-1}(x)$$

Note that there horizontal asymptotes at $y = \pm\frac{\pi}{2}$.



There are an infinite number of solutions to $\tan(x) = \sqrt{3}$, for example, $\frac{\pi}{3}$, $2\pi + \frac{\pi}{3}$ and $4\pi + \frac{\pi}{3}$, since we can always add any multiple of 2π to any angle. However, $\tan^{-1}(\sqrt{3})$ means $\tan(x) = \sqrt{3}$ and $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so there is only one solution: $\frac{\pi}{3}$.

WORKED EXAMPLE 20

Find:

a $\tan^{-1}\left(\tan\left(\frac{11\pi}{6}\right)\right)$

b $\tan(\tan^{-1}(2))$.

THINK

a 1 Use the known results.

2 Write an equivalent statement and state the result.

b 3 State the result.

WRITE

a $\tan\left(\frac{11\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

Let $\tan^{-1}\left(\tan\left(\frac{11\pi}{6}\right)\right) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = x$

$\tan(x) = -\frac{\sqrt{3}}{3}$ and $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The only solution is $x = -\frac{\pi}{6}$.

$\tan^{-1}\left(\tan\left(\frac{11\pi}{6}\right)\right) = -\frac{\pi}{6}$

b $\tan(\tan^{-1}(2)) = 2$

General results for the inverse tan function

In general, we find that:

$$f: R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ f(x) = \tan^{-1}(x)$$

$$\tan(\tan^{-1}(x)) = x \text{ if } x \in R$$

$$\tan^{-1}(\tan(x)) = x \text{ if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Double-angle formulas

It may be necessary to use the double-angle formulas, such as $\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$.

WORKED EXAMPLE 21

Find the exact value of $\tan\left(2\tan^{-1}\left(\frac{1}{2}\right)\right)$.

THINK

1 The inverse trigonometric functions are angles.

2 Use the double-angle formulas.

3 State the result.

WRITE

Let $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ so that $\tan(\theta) = \frac{1}{2}$.

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} = \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{1 - \frac{1}{4}} = \frac{1}{\frac{3}{4}}$$

$\tan\left(2\tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{4}{3}$

Compound-angle formulas

We may also need to use the compound-angle formulas:

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

WORKED EXAMPLE 22 Evaluate $\cos\left(\sin^{-1}\left(\frac{12}{13}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$.

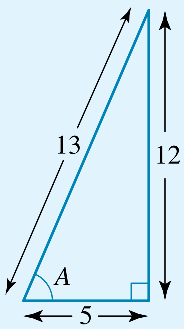
THINK / **WRITE/DRAW**

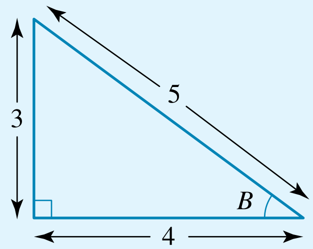
1. The inverse trigonometric functions are angles. Use the definitions of the inverse trigonometric functions.

Let $A = \sin^{-1}\left(\frac{12}{13}\right)$ and $B = \tan^{-1}\left(\frac{3}{4}\right)$.

Thus, $\sin(A) = \frac{12}{13}$ and $\tan(B) = \frac{3}{4}$.

2. Draw the right-angled triangle and state the unknown side length using well-known Pythagorean triads.





3. State the ratios from the triangles.

$\sin(A) = \frac{12}{13}$, $\cos(A) = \frac{5}{13}$

$\sin(B) = \frac{3}{5}$, $\cos(B) = \frac{4}{5}$

4. Substitute the ratios into the compound-angle formulas.

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$= \frac{5}{13}\times\frac{4}{5} + \frac{12}{13}\times\frac{3}{5}$$

5. State the required result.

$$\cos\left(\sin^{-1}\left(\frac{12}{13}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right) = \frac{56}{65}$$

Determining maximal domains and ranges

For $y = \sin^{-1}(x)$, the domain is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

For $y = \cos^{-1}(x)$, the domain is $[-1, 1]$ and the range is $[0, \pi]$.

For $y = \tan^{-1}(x)$, the domain is R and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For inverse trigonometric functions that have been dilated or translated, we can apply these dilations and translations to determine the domain and range of the transformed function.

WORKED EXAMPLE 23

State the domain and range of:

a $y = 2\cos^{-1}\left(\frac{3x-2}{5}\right) - 3$

b $y = 4\tan^{-1}\left(\frac{2x-7}{6}\right) + 1.$

THINK	WRITE
a 1 $y = \cos^{-1}(x)$ has a domain of $[-1, 1]$.	**a** $\left\lvert\frac{3x-2}{5}\right\rvert \le 1$
2 Use the definition of the modulus function.	$-1 \le \frac{3x-2}{5} \le 1$
3 Solve the inequality.	$-5 \le 3x - 2 \le 5$ $-3 \le 3x \le 7$
4 State the domain.	$y = 2\cos^{-1}\left(\frac{3x-2}{5}\right) - 3$ has a maximal domain of $-1 \le x \le \frac{7}{3}$ or $\left[-1, \frac{7}{3}\right]$.
5 $y = \cos^{-1}(x)$ has a range of $[0, \pi]$.	There is a dilation by a factor of 3 parallel to the y-axis and a translation of 2 units down parallel to the y-axis. The range is from $2 \times 0 - 3$ to $2 \times \pi - 3$.
6 State the range.	$y = 2\cos^{-1}\left(\frac{4x-3}{5}\right) - 3$ has a range of $[-3, 2\pi - 3]$.
b 1 $y = \tan^{-1}(x)$ has a domain of R.	**b** $y = 4\tan^{-1}\left(\frac{2x-7}{6}\right) + 1$ has a domain of R.
2 $y = \tan^{-1}(x)$ has a range of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.	There is a dilation by a factor of 4 parallel to the y-axis and a translation of 1 unit up parallel to the y-axis. The range is from $4 \times \frac{-\pi}{2} + 1$ to $4 \times \frac{\pi}{2} + 1$, not including the end points.
3 State the range.	$y = 4\tan^{-1}\left(\frac{2x-7}{6}\right) + 1$ has a range of $(-2\pi + 1, 2\pi + 1)$.

EXERCISE 2.6 Inverse trigonometric functions

PRACTISE

1 WE16 Find each of the following.

a $\sin^{-1}(1.1)$ b $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right)$ c $\sin(\sin^{-1}(0.9))$

2 Find each of the following.

a $\sin^{-1}\left(-\frac{6}{5}\right)$ b $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$ c $\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

3 WE17 Find each of the following.

a $\cos^{-1}(1.2)$ b $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$ c $\cos\left(\cos^{-1}\left(\frac{\pi}{6}\right)\right)$

4 Find each of the following.

a $\cos^{-1}\left(\frac{4}{3}\right)$ b $\cos^{-1}\left(\cos\left(\frac{11\pi}{3}\right)\right)$ c $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$

5 WE18 Find the exact value of $\cos\left(\sin^{-1}\left(\frac{1}{5}\right)\right)$.

6 Find the exact value of $\sin\left(\cos^{-1}\left(\frac{3}{7}\right)\right)$.

7 WE19 Find the exact value of $\sin\left(2\cos^{-1}\left(\frac{4}{7}\right)\right)$.

8 Find the exact value of $\cos\left(2\sin^{-1}\left(\frac{3}{8}\right)\right)$.

9 WE20 Find:

a $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$ b $\tan(\tan^{-1}(1.1))$.

10 Find:

a $\tan^{-1}\left(\tan\left(\frac{5\pi}{3}\right)\right)$ b $\tan\left(\tan^{-1}\left(\frac{5}{4}\right)\right)$.

11 WE21 Find the exact value of $\tan\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$

12 Find the exact value of $\cot\left(2\tan^{-1}\left(\frac{1}{4}\right)\right)$

13 WE22 Evaluate $\sin\left(\cos^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{5}{12}\right)\right)$.

14 Evaluate $\tan\left(\sin^{-1}\left(\frac{3}{5}\right) - \cot^{-1}\left(\frac{5}{12}\right)\right)$.

15 WE23 State the domain and range of:

a $y = 3\sin^{-1}\left(\frac{2x-5}{4}\right) - 2\pi$ b $y = \frac{6}{\pi}\tan^{-1}\left(\frac{3x-5}{4}\right) + 2$.

16 State the domain and range of:

a $y = \frac{4}{\pi}\cos^{-1}(3x+5) - 3$ b $y = \frac{8}{\pi}\tan^{-1}(10x) + 3$.

CONSOLIDATE

17 Evaluate each of the following.

a $\sin^{-1}(1)$ b $\sin^{-1}(1.3)$ c $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d $\cos^{-1}(-1)$ e $\cos^{-1}\left(-\frac{1}{2}\right)$ f $\cos^{-1}(-1.2)$

g $\tan^{-1}(\sqrt{3})$ h $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

18 Evaluate each of the following.

a $\sin^{-1}(\sin(1.2))$

b $\sin^{-1}\left(\sin\left(\frac{\pi}{5}\right)\right)$

c $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

d $\cos^{-1}(\cos(0.5))$

e $\cos^{-1}\left(\cos\left(\frac{\pi}{10}\right)\right)$

f $\cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right)$

g $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$

h $\tan^{-1}\left(\tan\left(\frac{4\pi}{3}\right)\right)$

19 Evaluate each of the following.

a $\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

b $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

c $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

d $\sin(\tan^{-1}(-1))$

e $\cos\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$

f $\tan\left(\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$

20 Evaluate each of the following.

a $\sin\left(\cos^{-1}\left(\frac{2}{9}\right)\right)$

b $\tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$

c $\tan\left(\sin^{-1}\left(-\frac{5}{6}\right)\right)$

d $\sin\left(\tan^{-1}\left(\frac{5}{8}\right)\right)$

e $\cos\left(\sin^{-1}\left(\frac{2}{5}\right)\right)$

f $\cos\left(\tan^{-1}\left(-\frac{7}{4}\right)\right)$

21 Evaluate each of the following.

a $\sin\left(2\cos^{-1}\left(\frac{1}{4}\right)\right)$

b $\tan\left(2\sin^{-1}\left(\frac{3}{4}\right)\right)$

c $\cos\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$

d $\sin\left(2\tan^{-1}\left(\frac{2}{3}\right)\right)$

e $\tan\left(2\cos^{-1}\left(\frac{1}{5}\right)\right)$

f $\cos\left(2\sin^{-1}\left(\frac{2}{5}\right)\right)$

22 Evaluate each of the following.

a $\sin\left(\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)\right)$

b $\cos\left(\cos^{-1}\left(\frac{4}{5}\right)-\sin^{-1}\left(\frac{5}{13}\right)\right)$

c $\cos\left(\tan^{-1}\left(\frac{15}{8}\right)+\cos^{-1}\left(\frac{9}{41}\right)\right)$

d $\sin\left(\tan^{-1}\left(\frac{8}{15}\right)-\sin^{-1}\left(\frac{60}{61}\right)\right)$

23 Show that:

a $\cos^{-1}\left(\frac{7}{25}\right)+\tan^{-1}\left(\frac{7}{24}\right)=\frac{\pi}{2}$

b $\sin^{-1}\left(\frac{12}{13}\right)+\tan^{-1}\left(\frac{5}{12}\right)=\frac{\pi}{2}$

c $\sin^{-1}\left(\frac{15}{17}\right)+\tan^{-1}\left(\frac{8}{15}\right)=\frac{\pi}{2}$

d $\tan^{-1}(3)-\tan^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{4}$

e $\tan^{-1}(4)-\tan^{-1}\left(\frac{3}{5}\right)=\frac{\pi}{4}$

f $\tan^{-1}(5)-\tan^{-1}\left(\frac{2}{3}\right)=\frac{\pi}{4}$.

24 Show that:

a $2\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{24}{25}\right)$

b $2\sin^{-1}\left(\frac{7}{25}\right)=\sin^{-1}\left(\frac{336}{625}\right)$

c $2\cos^{-1}\left(\frac{1}{4}\right)=\cos^{-1}\left(-\frac{7}{8}\right)$

d $2\cos^{-1}\left(\frac{2}{3}\right)=\cos^{-1}\left(-\frac{1}{9}\right)$

e $2\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}\left(\frac{3}{4}\right)$

f $2\tan^{-1}\left(\frac{1}{4}\right)=\tan^{-1}\left(\frac{8}{15}\right)$.

25 State the implied domain and range of each of the following.

a $y=2\sin^{-1}(x-1)$

b $y=3\cos^{-1}(x-2)$

c $y=4\tan^{-1}(x-3)$

d $y=5\sin^{-1}\left(\frac{x}{3}\right)$

e $y=6\cos^{-1}\left(\frac{x}{4}\right)$

f $y=7\tan^{-1}\left(\frac{x}{5}\right)$

26 State the implied domain and range of each of the following.

a $y = 2\sin^{-1}(3x - 1) + \pi$

b $y = 3\cos^{-1}(2x - 5) - \pi$

c $y = 5\tan^{-1}(4x + 3) - \dfrac{\pi}{2}$

d $y = \dfrac{4}{\pi}\sin^{-1}\left(\dfrac{3 - 4x}{5}\right) + 2$

e $y = \dfrac{5}{\pi}\cos^{-1}\left(\dfrac{4 - 3x}{7}\right) - 4$

f $y = \dfrac{8}{\pi}\tan^{-1}\left(\dfrac{5x - 3}{4}\right) + 3$

MASTER

27 a State a sequence of transformations that, when applied to $y = \sin^{-1}(x)$, produce the graph of $y = a + b\sin^{-1}\left(\dfrac{x}{c}\right)$. Hence, state the domain and range of $y = a + b\sin^{-1}\left(\dfrac{x}{c}\right)$.

b State a sequence of transformations that, when applied to $y = \cos^{-1}(x)$, produce the graph of $y = a + b\cos^{-1}(cx)$. Hence, state the domain and range of $y = a + b\cos^{-1}(cx)$.

c State a sequence of transformations that, when applied to $y = \tan^{-1}(x)$, produce the graph of $y = a + b\tan^{-1}\left(\dfrac{x}{c}\right)$. Hence, state the domain and range of $y = a + b\tan^{-1}\left(\dfrac{x}{c}\right)$.

28 Show that:

a $\sin^{-1}(x) = \cos^{-1}(\sqrt{1 - x^2})$ for $x \in [0, 1]$

b $\tan^{-1}(x) + \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$ for $x > 0$

c $\cos^{-1}(x) = \tan^{-1}\left(\dfrac{\sqrt{1 - x^2}}{x}\right)$ for $x \in (0, 1)$

d $\sin^{-1}\left(\dfrac{a}{\sqrt{a^2 + b^2}}\right) + \tan^{-1}\left(\dfrac{b}{a}\right) = \dfrac{\pi}{2}$ for $a > 0$ and $b > 0$

e $\cos^{-1}\left(\dfrac{a}{\sqrt{a^2 + b^2}}\right) + \tan^{-1}\left(\dfrac{a}{b}\right) = \dfrac{\pi}{2}$ for $a > 0$ and $b > 0$

f $\tan^{-1}(x) - \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right) = \dfrac{\pi}{4}$ for $x > -1$

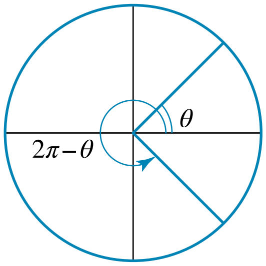
g $\sin^{-1}\left(\dfrac{x^2 - 1}{x^2 + 1}\right) = \cos^{-1}\left(\dfrac{2x}{x^2 + 1}\right) = \tan^{-1}\left(\dfrac{x^2 - 1}{2x}\right)$ for $x > 0$.

2.7 General solutions of trigonometric equations

In this section consideration is given to the general solutions of trigonometric equations, rather than finding the solutions over a specified domain.

Trigonometric equations can have an infinite number of solutions. To express the possible solutions mathematically, we derive formulas that will give the general solution in terms of any natural number n, where $n \in Z$.

General solutions involving cosines



Consider the equation $\cos(x) = a$. One answer is $x = \cos^{-1}(a)$.

If $0 < a < 1$, then $0 < x < \frac{\pi}{2}$, so x is in the first quadrant.

Because cosine is positive in the first and fourth quadrant, there is also another answer, $x = 2\pi - \cos^{-1}(a)$.

We can add or subtract any multiple of 2π to either answer and obtain an equivalent angle.

$$\cos(x) = a$$
$$x = \cos^{-1}(a), 2\pi + \cos^{-1}(a), 4\pi + \cos^{-1}(a), \ldots$$
$$x = 2\pi - \cos^{-1}(a), 4\pi - \cos^{-1}(a), 6\pi - \cos^{-1}(a), \ldots$$

The totality of solutions can be represented as $x = 2n\pi \pm \cos^{-1}(a)$, where $n \in Z$.

Although we have demonstrated this result for $0 < a < 1$, it is in fact true for $-1 \leq a \leq 1$.

The general solution of $\cos(x) = a$ where $-1 \leq a \leq 1$

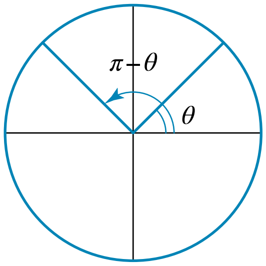
The general solution of $\cos(x) = a$ where $-1 \leq a \leq 1$ is given by

$$x = 2n\pi \pm \cos^{-1}(a), \text{ where } n \in Z.$$

WORKED EXAMPLE 24 Find the general solution to the equation $\cos(x) = \frac{1}{2}$.

THINK	WRITE
1 State one solution.	$x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$
2 State the general solution.	$x = 2n\pi \pm \frac{\pi}{3}$
3 Take out a common factor so that the general solution can be written in simplest form.	$x = \frac{\pi}{3}(6n \pm 1)$ where $n \in Z$

General solutions involving sines



Consider the equation $\sin(x) = a$. One answer is $x = \sin^{-1}(a)$, and if $0 < a < 1$, then $0 < x < \frac{\pi}{2}$, so x is in the first quadrant.

Since sine is positive in the first and second quadrants, there is also another answer, $x = \pi - \sin^{-1}(a)$.

We can add or subtract any multiple of 2π to either answer and obtain an equivalent angle.

$$\sin(x) = a$$
$$x = \sin^{-1}(a), 2\pi + \sin^{-1}(a), 4\pi + \sin^{-1}(a), \ldots$$
$$x = \pi - \sin^{-1}(a), 3\pi - \sin^{-1}(a), 5\pi - \sin^{-1}(a), \ldots$$

If n is any integer, then $2n$ is an even integer and $2n + 1$ is an odd integer.

The totality of solutions can be represented as $x = 2n\pi + \sin^{-1}(a)$ or $x = (2n + 1)\pi - \sin^{-1}(a)$, where $n \in Z$.

Although we have demonstrated this result for $0 < a < 1$, it is true for $-1 \leq a \leq 1$.

The general solution of $\sin(x) = a$ where $-1 \leq a \leq 1$

The general solution of $\sin(x) = a$ where $-1 \leq a \leq 1$ is given by

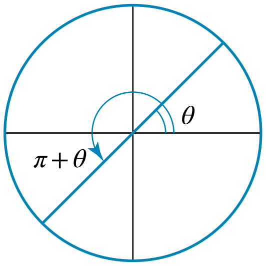
$$x = 2n\pi + \sin^{-1}(a),\ (2n + 1)\pi - \sin^{-1}(a), \text{ where } n \in Z.$$

WORKED EXAMPLE 25

Find the general solution to the equation $\sin(x) = \frac{\sqrt{3}}{2}$.

THINK	WRITE
1 State one solution.	$x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
2 State the general solution.	$x = 2n\pi + \frac{\pi}{3}$ or $x = (2n + 1)\pi - \frac{\pi}{3}$
3 Take out common factors in the first solution so that the general solution can be written in simplest form.	$x = 2n\pi + \frac{\pi}{3}$ $= \frac{\pi}{3}(6n + 1)$
4 Take out common factors in the second solution	$x = (2n + 1)\pi - \frac{\pi}{3}$ $= 2n\pi + \pi - \frac{\pi}{3}$ $= 2n\pi + \frac{2\pi}{3}$ $= \frac{2\pi}{3}(3n + 1)$
5 State the general solution.	$x = \frac{\pi}{3}(6n + 1), \frac{2\pi}{3}(3n + 1)$ where $n \in Z$

General solutions involving tangents



Consider the equation $\tan(x) = a$. One answer is $x = \tan^{-1}(a)$, and if $a > 0$, then $0 < x < \frac{\pi}{2}$, so x is in the first quadrant. Since tangent is positive in the first and third quadrants, there is also another answer, $x = \pi + \tan^{-1}(a)$.

We can add or subtract any multiple of 2π to either answer and obtain an equivalent angle.

$$\tan(x) = a$$
$$x = \tan^{-1}(a),\ 2\pi + \tan^{-1}(a),\ 4\pi + \tan^{-1}(a),\ \ldots$$
$$x = \pi + \tan^{-1}(a),\ 3\pi + \tan^{-1}(a),\ 5\pi + \tan^{-1}(a),\ \ldots$$

The totality of solutions can be represented as one solution: $x = n\pi + \tan^{-1}(a)$, where $n \in Z$.

The general solution of tan(x) = a

The general solution of $\tan(x) = a$ where $a \in R$ is given by

$$x = n\pi + \tan^{-1}(a), \text{ where } n \in Z.$$

Although we have demonstrated this result only for $a > 0$, it is true for $a \in R$.

WORKED EXAMPLE 26 **Find the general solution to the equation $\tan(x) = \sqrt{3}$.**

THINK	WRITE
1 State one solution.	$x = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
2 State the general solution.	$x = n\pi + \frac{\pi}{3}$
3 Take out a common factor so that the general solution can be written in simplest form.	$x = \frac{\pi}{3}(3n + 1)$ where $n \in Z$

When solving more complicated trigonometric equations, often multiple solutions exist. We may be required to find all solutions to each part of the equation being considered.

WORKED EXAMPLE 27 **Find the general solution to the equation $4\cos^2(2x) - 3 = 0$.**

THINK	WRITE
1 Make the trigonometric function the subject.	$4\cos^2(2x) - 3 = 0$ $\cos^2(2x) = \frac{3}{4}$ $\cos(2x) = \pm\frac{\sqrt{3}}{2}$ So that: (1) $\cos(2x) = \frac{\sqrt{3}}{2}$ or (2) $\cos(2x) = -\frac{\sqrt{3}}{2}$
2 Use the formula to find the general solution of the first equation.	$\cos(2x) = \frac{\sqrt{3}}{2}$ $2x = 2n\pi \pm \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ $2x = 2n\pi \pm \frac{\pi}{6}$ $2x = \frac{\pi}{6}(12n \pm 1)$ $x = \frac{\pi}{12}(12n \pm 1)$

3 Use the formula to find the general solution of the second equation.

$$\cos(2x) = -\frac{\sqrt{3}}{2}$$
$$2x = 2n\pi \pm \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$
$$2x = 2n\pi \pm \frac{5\pi}{6}$$
$$2x = \frac{\pi}{6}(12n \pm 5)$$
$$x = \frac{\pi}{12}(12n \pm 5)$$

4 State the final general solutions.

$$x = \frac{\pi}{12}(12n \pm 1) \text{ or } x = \frac{\pi}{12}(12n \pm 5) \text{ where } n \in Z$$

General solutions involving phase shifts

When solving trigonometric equations involving phase shifts, we must solve the resulting equations for the unknown values of x.

WORKED EXAMPLE 28

Find the general solution of $\sqrt{2}\sin\left(3x + \frac{\pi}{4}\right) + 1 = 0$.

THINK | WRITE

1 Make the trigonometric function the subject.

$$\sqrt{2}\sin\left(3x + \frac{\pi}{4}\right) + 1 = 0$$
$$\sqrt{2}\sin\left(3x + \frac{\pi}{4}\right) = -1$$
$$\sin\left(3x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

2 Use the formula to state the general solution.

$$(1)\quad 3x + \frac{\pi}{4} = 2n\pi + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) \text{ or}$$
$$(2)\quad 3x + \frac{\pi}{4} = (2n+1)\pi - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

3 Solve the first equation.

$$3x + \frac{\pi}{4} = 2n\pi + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$3x + \frac{\pi}{4} = 2n\pi - \frac{\pi}{4}$$
$$3x = 2n\pi - \frac{\pi}{2}$$
$$3x = \frac{\pi}{2}(4n - 1)$$
$$x = \frac{\pi}{6}(4n - 1)$$

4 Solve the second equation.

$$3x + \frac{\pi}{4} = (2n+1)\pi - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$3x + \frac{\pi}{4} = 2n\pi + \pi + \frac{\pi}{4}$$
$$3x = \pi(2n+1)$$
$$x = \frac{\pi}{3}(2n+1)$$

5 State the final solutions.

$$x = \frac{\pi}{6}(4n-1) \text{ or } \frac{\pi}{3}(2n+1)\ n \in Z$$

Equations reducible to quadratics

Equations can often be reduced to quadratics under a suitable substitution.

WORKED EXAMPLE 29 **Find the general solution of the equation $2\sin^2(2x) + \sin(2x) - 1 = 0$.**

THINK	WRITE
1 Use a substitution.	$2\sin^2(2x) + \sin(2x) - 1 = 0$ Let $u = \sin(2x)$.
2 Factorise.	$2u^2 + u - 1 = 0$ $(2u-1)(u+1) = 0$
3 Substitute back for u.	$(2\sin(2x) - 1)(\sin(2x) + 1) = 0$
4 Solve the trigonometric equation using the Null Factor Law.	(1) $\sin(2x) = \frac{1}{2}$ or (2) $\sin(2x) = -1$
5 Find the general solution of the first equation.	$\sin(2x) = \frac{1}{2}$ $2x = 2n\pi + \sin^{-1}\left(\frac{1}{2}\right)$ $2x = 2n\pi + \frac{\pi}{6}$ $2x = \frac{\pi}{6}(12n+1)$ $x = \frac{\pi}{12}(12n+1)$ $\sin(2x) = \frac{1}{2}$ $2x = (2n+1)\pi - \sin^{-1}\left(\frac{1}{2}\right)$ $2x = 2n\pi + \pi - \frac{\pi}{6}$ $2x = 2n\pi + \frac{5\pi}{6}$ $2x = \frac{\pi}{6}(12n+5)$ $x = \frac{\pi}{12}(12n+5)$

6 Find the general solution of the second equation.

$$\sin(2x) = -1$$
$$2x = 2n\pi + \sin^{-1}(-1)$$
$$2x = 2n\pi - \frac{\pi}{2}$$
$$2x = \frac{\pi}{2}(4n - 1)$$
$$x = \frac{\pi}{4}(4n - 1)$$
$$\sin(2x) = -1$$
$$2x = (2n + 1)\pi - \sin^{-1}(-1)$$
$$2x = 2n\pi + \pi + \frac{\pi}{2}$$
$$2x = 2n\pi + \frac{3\pi}{2}$$
$$2x = \frac{\pi}{2}(4n + 3)$$
$$x = \frac{\pi}{4}(4n + 3)$$

7 Sometimes some parts of the solution are already included in some other parts. Give n some values.

Let $n = 0, 1, 2, 3, 4$.

$$x = \frac{\pi}{4}(4n - 1)$$
$$\Rightarrow \quad x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$
$$x = \frac{\pi}{4}(4n + 3)$$
$$\Rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

We can see that the solution $x = \frac{\pi}{4}(4n + 3)$ incorporates all the solutions from $x = \frac{\pi}{4}(4n - 1)$.

8 State all the general solutions of the equation.

$x = \frac{\pi}{4}(4n - 1)$ or $\frac{\pi}{12}(12n + 1)$ or $\frac{\pi}{12}(12n + 5)$ where $n \in Z$.

Trigonometric equations involving multiple angles

We can find the general solutions to trigonometric equations involving multiple angles by applying the general solution formulas rather than expanding the multiple angles.

WORKED EXAMPLE 30

Find the general solution to $\cos(4x) = \sin(2x)$.

THINK

1 Rewrite using one trigonometric function. Convert sines into cosines, since the solution for cosine is easier to work with.

WRITE

Use $\sin\left(\frac{\pi}{2} - A\right) = \cos(A)$.

$$\cos(4x) = \sin(2x)$$
$$\cos(4x) = \cos\left(\frac{\pi}{2} - 2x\right)$$

2 Solve using an appropriate general solution.

$$4x = 2n\pi \pm \left(\frac{\pi}{2} - 2x\right)$$

$$(1) \quad 4x = 2n\pi + \left(\frac{\pi}{2} - 2x\right) \text{ or}$$

$$(2) \quad 4x = 2n\pi - \left(\frac{\pi}{2} - 2x\right)$$

3 Solve the first equation.

$$4x = 2n\pi + \frac{\pi}{2} - 2x$$

$$6x = 2n\pi + \frac{\pi}{2}$$

$$6x = \frac{\pi}{2}(4n + 1)$$

$$x = \frac{\pi}{12}(4n + 1)$$

4 Solve the second equation.

$$4x = 2n\pi - \frac{\pi}{2} + 2x$$

$$2x = 2n\pi - \frac{\pi}{2}$$

$$2x = \frac{\pi}{2}(4n - 1)$$

$$x = \frac{\pi}{4}(4n - 1)$$

5 State the general solutions of the equation.

$$x = \frac{\pi}{12}(4n + 1) \text{ or } \frac{\pi}{4}(4n - 1) \text{ where } n \in Z.$$

Comparison of examples

Note that the last two worked examples, 29 and 30, are in fact the same, as

$$\cos(4x) = \sin(2x)$$

$$\cos(2(2x)) = \sin(2x) \text{ by double-angle formulas}$$

$$1 - 2\,(\sin(2x))^2 = \sin(2x)$$

$$2\sin^2(2x) + \sin(2x) - 1 = 0$$

and therefore they should have the same general solution. The two given answers do not appear to be the same, although one answer, $x = \frac{\pi}{4}(4n - 1)$, is common to both.

This situation is very common in these types of problems. However, if we substitute values of n, the two results generate the same particular solutions.

When $n = 0, 1, 2, 3, 4, 5, 6$ from Worked example 29:

$$x = \frac{\pi}{4}(4n - 1) \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$$

$$x = \frac{\pi}{12}(12n + 1) \Rightarrow x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}$$

$$x = \frac{\pi}{12}(12n + 5) \Rightarrow x = \frac{5\pi}{12}, \frac{17\pi}{12}$$

When $n = 0, 1, 2, 3, 4, 5, 6$ from Worked example 30:

$$x = \frac{\pi}{4}(4n - 1) \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$$

$$x = \frac{\pi}{12}(4n + 1) \Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}, \frac{25\pi}{12}$$

It is interesting to compare these results to those obtained by CAS calculators. In some cases a calculator will not solve the equation for the general solution, and in other cases it will. The solution obtained by CAS may be in a different form to our answers above. The results may be given differently depending on the MODE, which could be set to either Exact or Auto.

EXERCISE 2.7 General solutions of trigonometric equations

PRACTISE

1 **WE24** Find the general solution to the equation $\cos(x) = \frac{\sqrt{2}}{2}$.

2 Determine the general solution of $2\cos(2x) + \sqrt{3} = 0$.

3 **WE25** Find the general solution to the equation $\sin(x) = \frac{1}{2}$.

4 Determine the general solution of $2\sin(2x) + \sqrt{3} = 0$.

5 **WE26** Find the general solution to the equation $\tan(x) = 1$.

6 Find the general solution to $\tan(2x) + \sqrt{3} = 0$.

7 **WE27** Find the general solution to the equation $4\cos^2(2x) - 1 = 0$.

8 Find the general solution to the equation $3\tan^2(2x) - 1 = 0$.

9 **WE28** Find the general solution of $2\sin\left(3x + \frac{\pi}{6}\right) - 1 = 0$.

10 Find the general solution of $2\cos\left(2x - \frac{\pi}{6}\right) + \sqrt{3} = 0$.

11 **WE29** Find the general solution to $2\sin^2(2x) - 3\sin(2x) + 1 = 0$.

12 Find the general solution to the equation $2\cos^2(2x) + \cos(2x) - 1 = 0$.

13 **WE30** Find the general solution to $\cos(3x) = \sin(2x)$.

14 Find the general solution to $\cos(4x) = \sin(3x)$.

CONSOLIDATE

15 Find the general solution to each of the following equations.

a $2\cos(3x) - \sqrt{3} = 0$

b $2\cos(2x) + 1 = 0$

c $\sqrt{2}\sin(2x) + 1 = 0$

d $2\sin(3x) + 1 = 0$

16 Find the general solution to each of the following equations.

a $4\sin^2(2x) - 3 = 0$

b $2\sin^2(2x) - 1 = 0$

c $4\cos^2(3x) - 1 = 0$

d $2\cos^2(3x) - 1 = 0$

17 Find the general solution to each of the following equations.

a $\tan(x) + \sqrt{3} = 0$

b $\sqrt{3}\tan(3x) - 1 = 0$

c $\tan^2(2x) - 3 = 0$

d $3\tan^2(2x) - 1 = 0$

18 Find the general solution to each of the following equations.

a $2\sin^2(2x) + \sin(2x) = 0$

b $\cos^2(2x) - \cos(2x) = 0$

c $2\cos^2(2x) + \sqrt{3}\cos(2x) = 0$

d $2\sin^2(2x) - \sqrt{3}\sin(2x) = 0$

19 Find the general solution to each of the following equations.

a $2\sin^2(2x) + 3\sin(2x) + 1 = 0$ b $2\cos^2(2x) - 3\cos(2x) + 1 = 0$

20 Find the general solution to each of the following equations.

a $\sqrt{2}\sin\left(3x - \frac{\pi}{4}\right) - 1 = 0$ b $2\sin\left(2\left(x + \frac{\pi}{6}\right)\right) + 1 = 0$

21 Find the general solution to each of the following equations.

a $2\cos\left(2x + \frac{\pi}{6}\right) + \sqrt{3} = 0$ b $2\cos\left(3\left(x - \frac{\pi}{12}\right)\right) - 1 = 0$

22 Find the general solution to each of the following equations.

a $\tan\left(3x + \frac{\pi}{4}\right) - 1 = 0$ b $\sqrt{3}\tan\left(2\left(x - \frac{\pi}{12}\right)\right) + 1 = 0$

23 Find the general solution to each of the following equations.

a $\tan^2(x) + (\sqrt{3} + 1)\tan(x) + \sqrt{3} = 0$

b $\tan^2(x) + (\sqrt{3} - 1)\tan(x) - \sqrt{3} = 0$

24 Find the general solution to each of the following equations.

a $\sin(2x) = \sin(x)$ b $\cos(x) = \cos(2x)$

MASTER

25 Find the general solution to each of the following equations.

a $2\sin^3(x) + \sin^2(x) - 2\sin(x) - 1 = 0$

b $2\cos^3(x) - \cos^2(x) - 2\cos(x) + 1 = 0$

26 Find the general solution to each of the following equations.

a $\tan^3(x) - \tan^2(x) - \tan(x) + 1 = 0$

b $\tan^4(x) - 4\tan^2(x) + 3 = 0$

2.8 Graphs of reciprocal trigonometric functions

studyon

Units 3 & 4

AOS 1

Topic 2

Concept 2

Sketch graphs of reciprocal circular functions

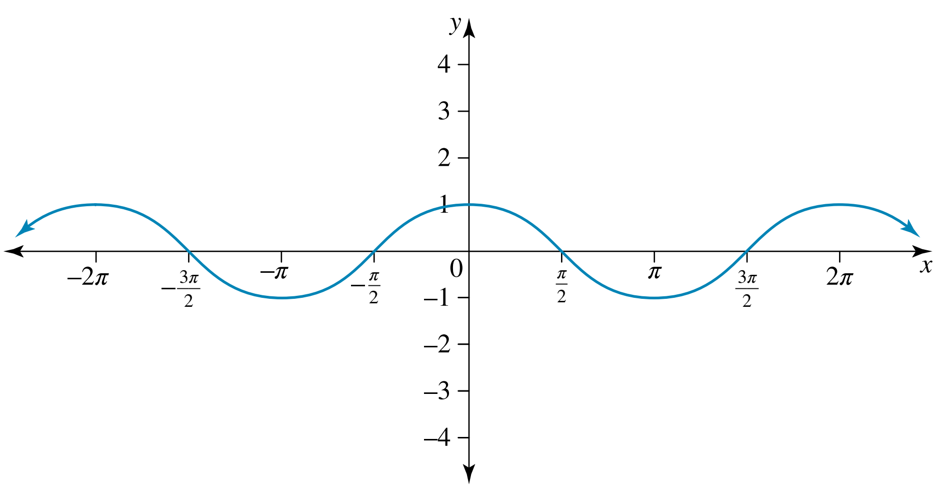
Concept summary

Practice questions

Topic 1 described how the graph of $f(x)$ can be used to find the graph of $\frac{1}{f(x)}$. We can use this method to graph $\sec(x) = \frac{1}{\cos(x)}$, $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$ and $\cot(x) = \frac{1}{\tan(x)}$.

The graph of $y = \sec(x)$

Consider the graph of $y = \cos(x)$.

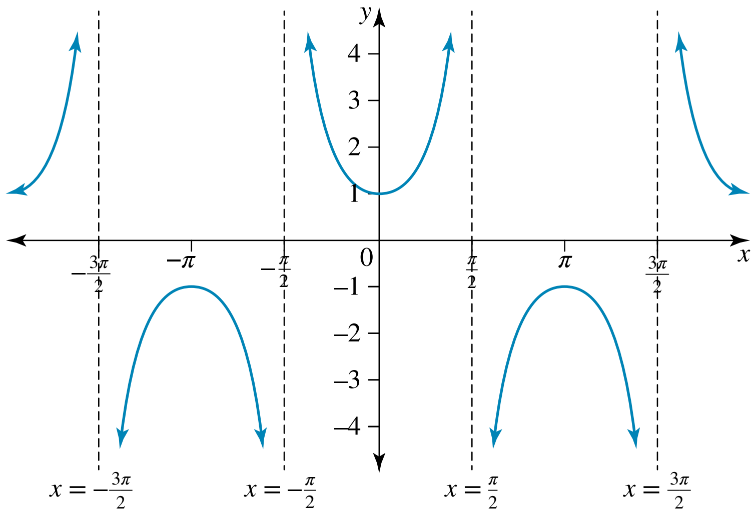


In the portion of the graph shown, the x-intercepts occur at $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. This means that the reciprocal function will have vertical asymptotes at $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. The horizontal asymptote will be $y = 0$.

The graph of $y = \cos(x)$ is below the x-axis for $-\frac{3\pi}{2} < x < -\frac{\pi}{2}$ and passes through the point $(-\pi, -1)$. This means that $y = \frac{1}{\cos(x)}$ will also be below the x-axis in this interval and will pass through the point $(-\pi, -1)$. It will follow a similar pattern in the region $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

In the region $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the graph of $y = \cos(x)$ is above the x-axis and passes through the point $(0, 1)$. This means that $y = \frac{1}{\cos(x)}$ will also be above the x-axis and will pass through $(0, 1)$.

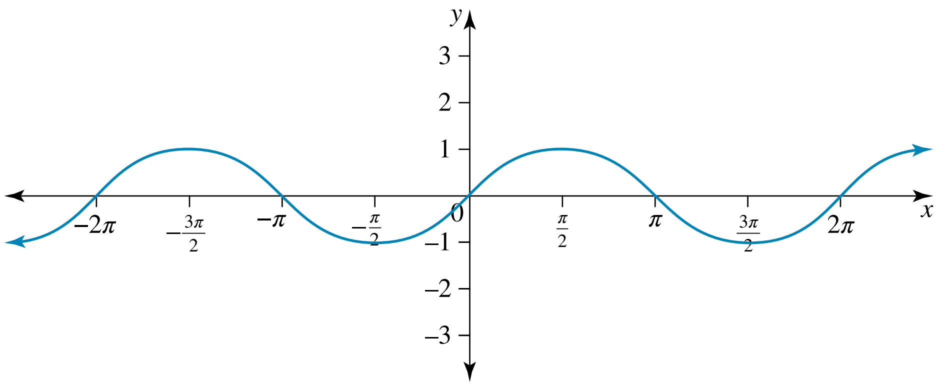
The graph of $y = \frac{1}{\cos(x)}$ (or $y = \sec(x)$) is shown below.



The graph of $y = \text{cosec}(x)$

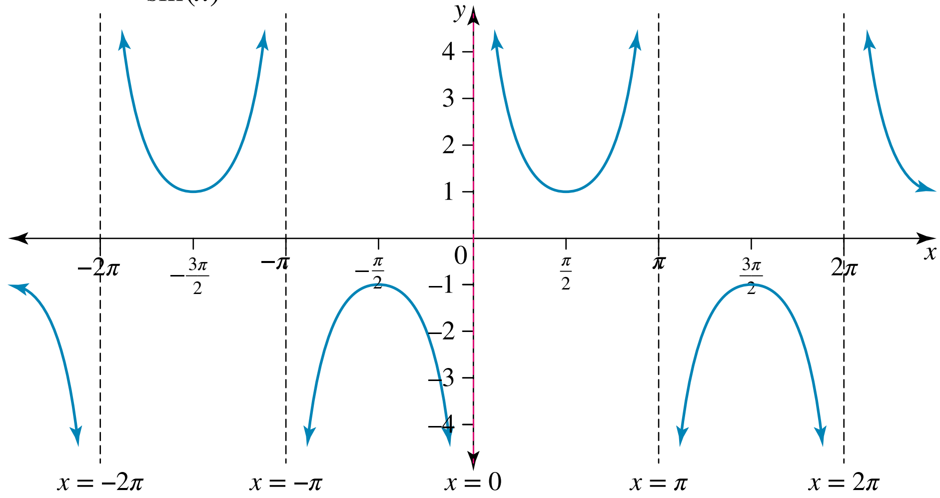
In a similar fashion, the graph of $y = \sin(x)$ can be used to determine the graph of $y = \frac{1}{\sin(x)}$ (or $y = \text{cosec}(x)$).

The graph of $y = \sin(x)$ is shown below.



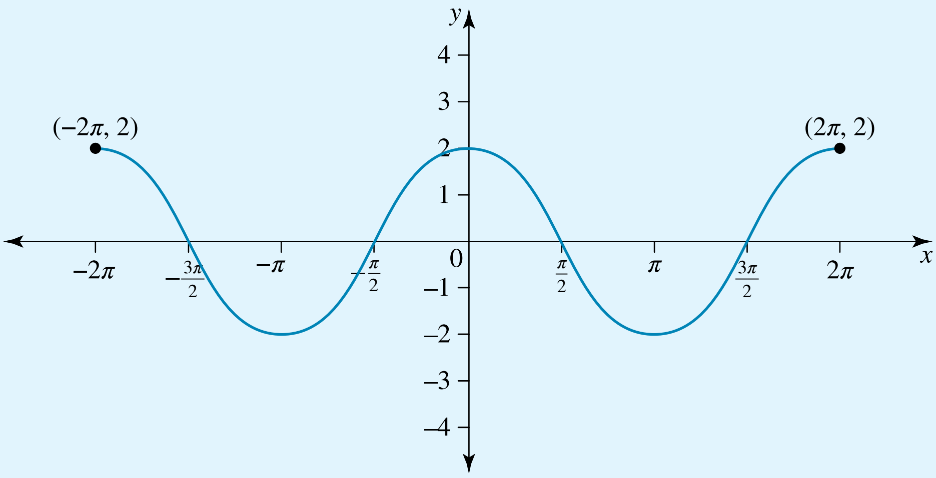
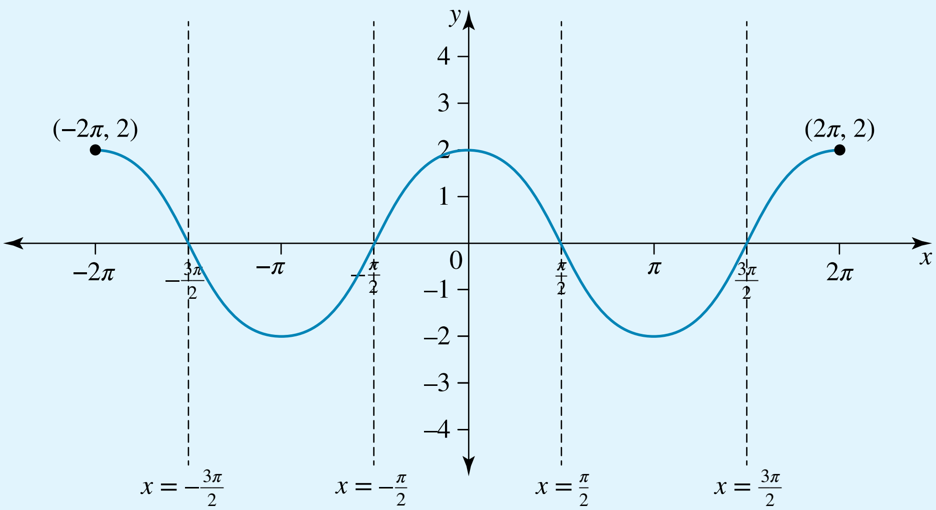
Note that in this instance, the x-intercepts occur at -2π, $-\pi$, 0, π and 2π.

The graph of $y = \dfrac{1}{\sin(x)}$ looks like the following.

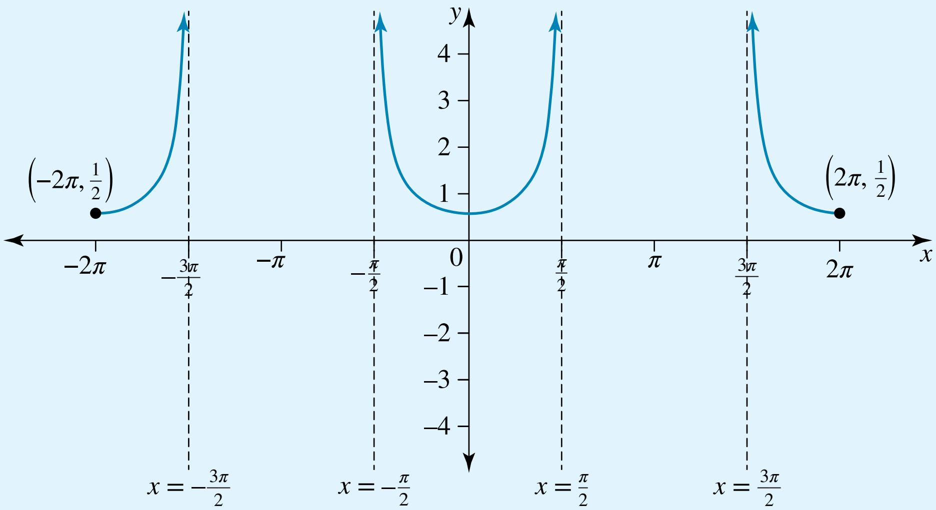


WORKED EXAMPLE 31

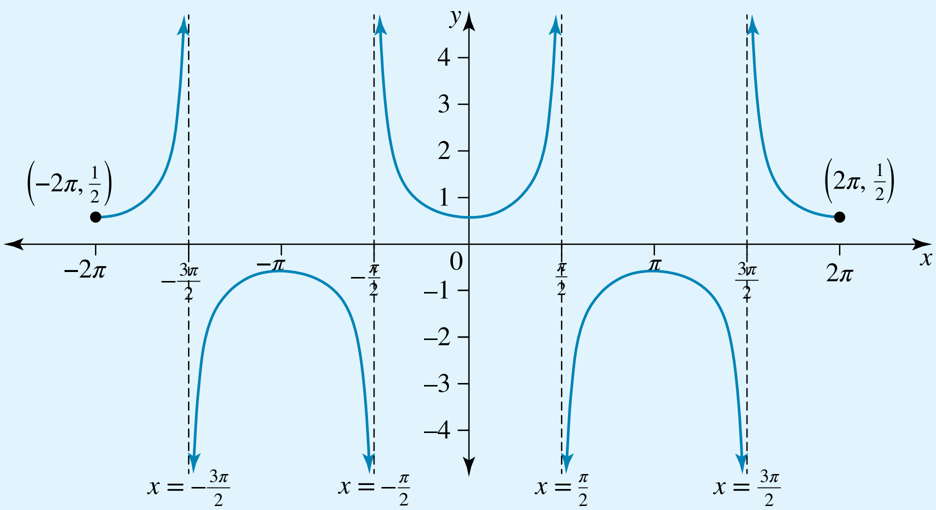
Use the graph of $y = 2\cos(x)$ to sketch $y = \dfrac{1}{2\cos(x)}$ over the domain $-2\pi \le x \le 2\pi$.

THINK	WRITE/DRAW
1 Sketch $y = 2\cos(x)$. Period: 2π Amplitude: 2 Horizontal shift: 0 Vertical shift: 0	
2 Find the x-intercepts and hence the vertical asymptotes for the reciprocal graph.	x-intercepts occur at $x = -\dfrac{3\pi}{2}$, $x = -\dfrac{\pi}{2}$, $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$. These will be the vertical asymptotes for the reciprocal function. 

3 The graph of $y = 2\cos(x)$ is above the x-axis in the regions $-2\pi \leq x < -\frac{3\pi}{2}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x \leq 2\pi$. The graph of $y = \frac{1}{2\cos(x)}$ will also be above the x-axis in these regions. A maximum value of $y = 2$ is reached in the original graph, meaning that a minimum of $y = \frac{1}{2}$ will be reached in the reciprocal graph.



4 The graph of $y = 2\cos(x)$ is below the x-axis in the regions $-\frac{3\pi}{2} < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Therefore, $y = \frac{1}{2\cos(x)}$ is also below the x-axis in these regions. The minimum of $y = -2$ will become a maximum of $y = -\frac{1}{2}$.



WORKED EXAMPLE 32

Use the graph of $y = \sin\left(\frac{x}{2}\right)$ to sketch $y = \frac{1}{\sin\left(\frac{x}{2}\right)}$ over the domain $-2\pi \leq x \leq 2\pi$.

THINK

1 Sketch $y = \sin\left(\frac{x}{2}\right)$.

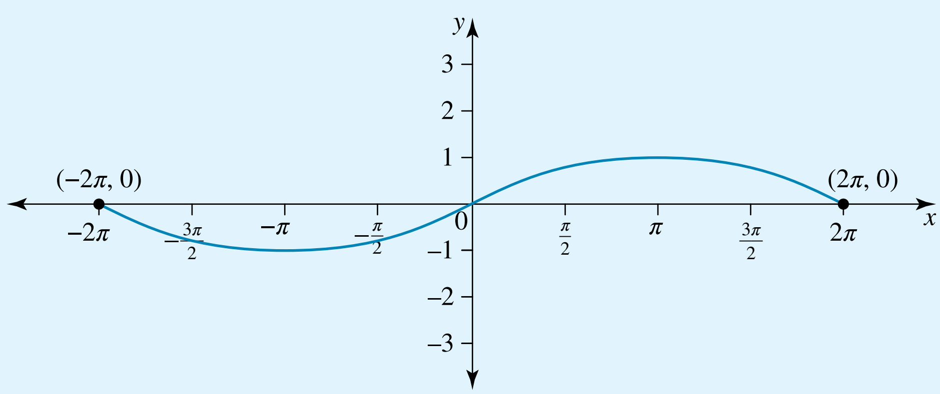
Period: 4π

Amplitude: 1

Horizontal shift: 0

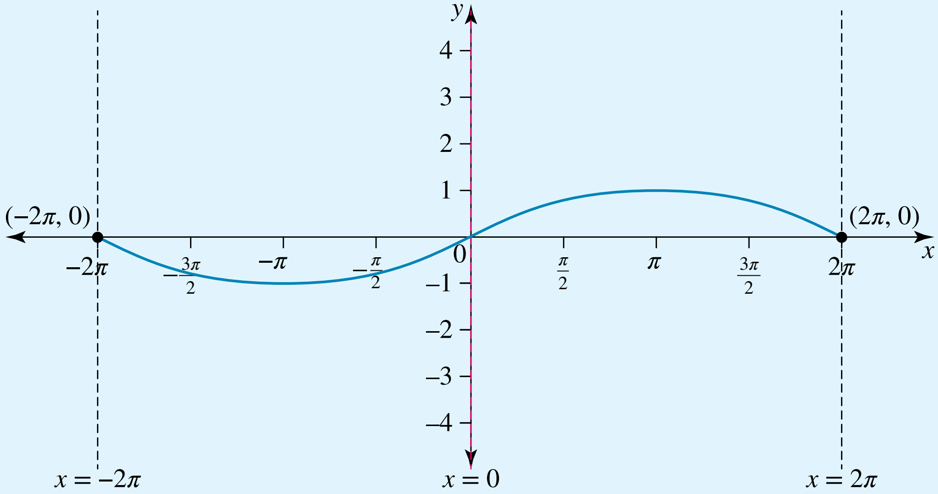
Vertical shift: 0

WRITE/DRAW

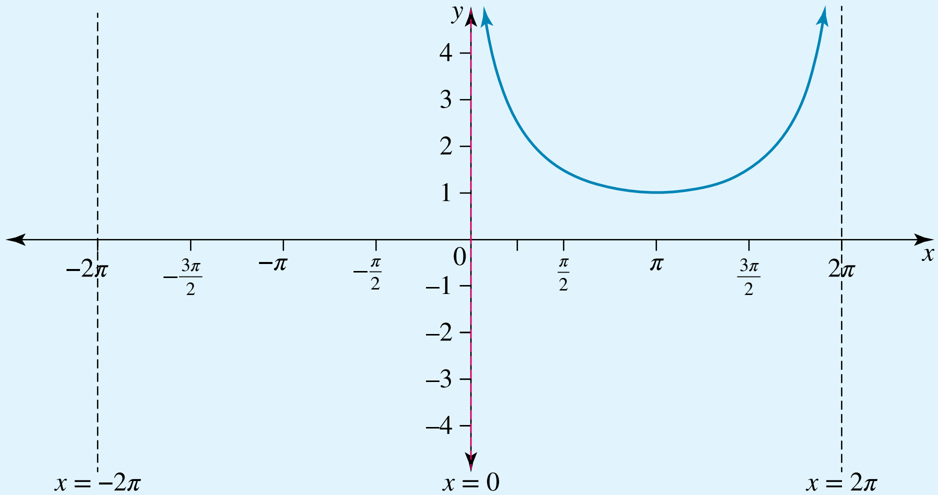


2 Find the x-intercepts and hence the vertical asymptotes for the reciprocal graph.

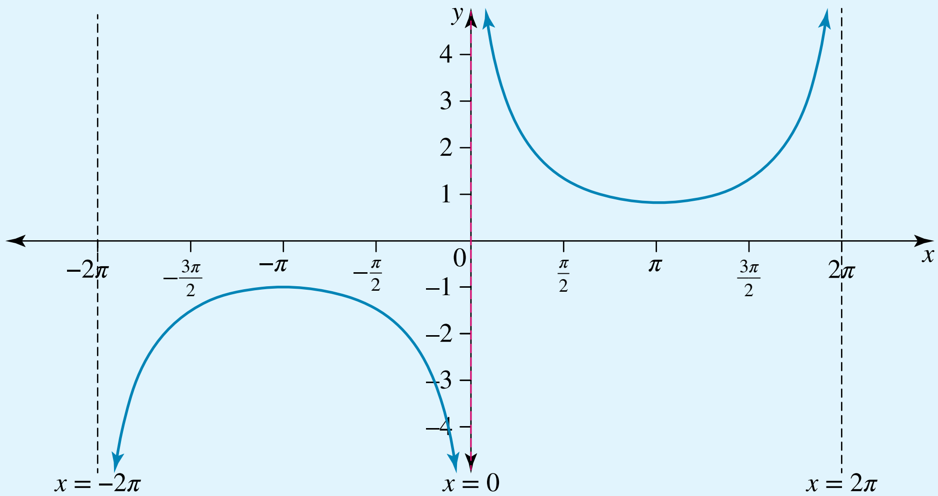
x-intercepts occur at $x = -2\pi$, $x = 0$ and $x = 2\pi$. These will be the vertical asymptotes for the reciprocal function.



3 The graph of $y = \sin\left(\frac{x}{2}\right)$ is above the x-axis in the region $0 < x \le 2\pi$. The graph of $y = \frac{1}{\sin\left(\frac{x}{2}\right)}$ will also be above the x-axis in this region. A maximum value of $y = 1$ is reached in the original graph, meaning that a minimum of $y = 1$ will be reached in the reciprocal graph.

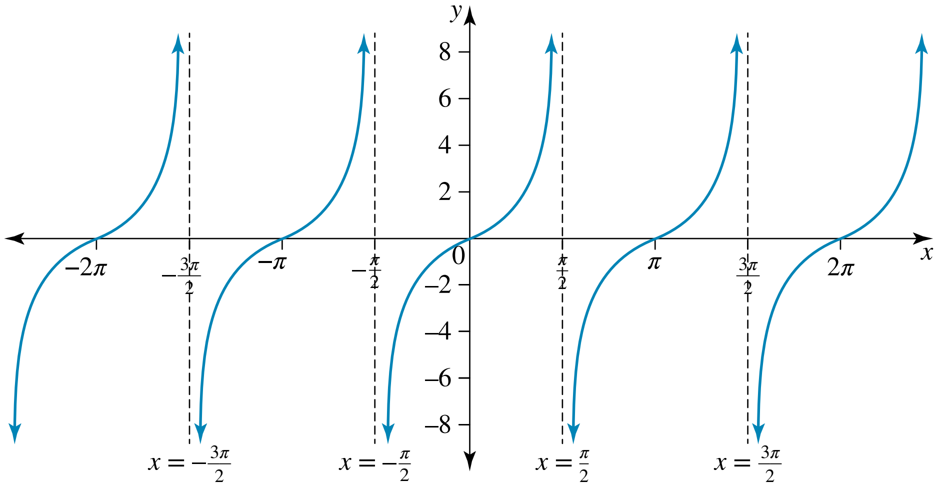


4 The graph of $y = \sin\left(\frac{x}{2}\right)$ is below the x-axis in the region $-2\pi \le x < 0$. The graph of $y = \frac{1}{\sin\left(\frac{x}{2}\right)}$ is also below the x-axis in this region. The minimum of $y = -1$ will become a maximum of $y = -1$.



The graph of $y = \cot(x)$

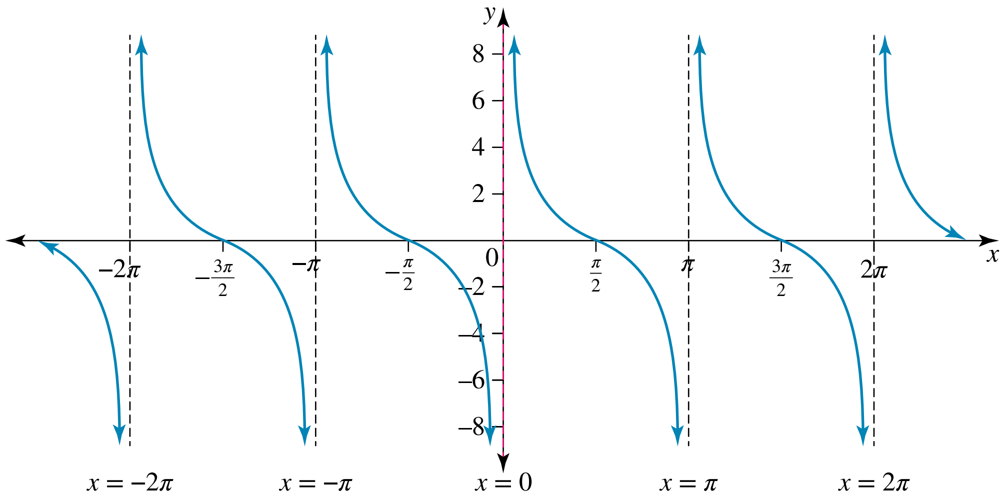
The graph of $y = \tan(x)$ can be used to find the graph of $y = \dfrac{1}{\tan(x)}$ (or $y = \cot(x)$). The graph of $y = \tan(x)$ is shown below.



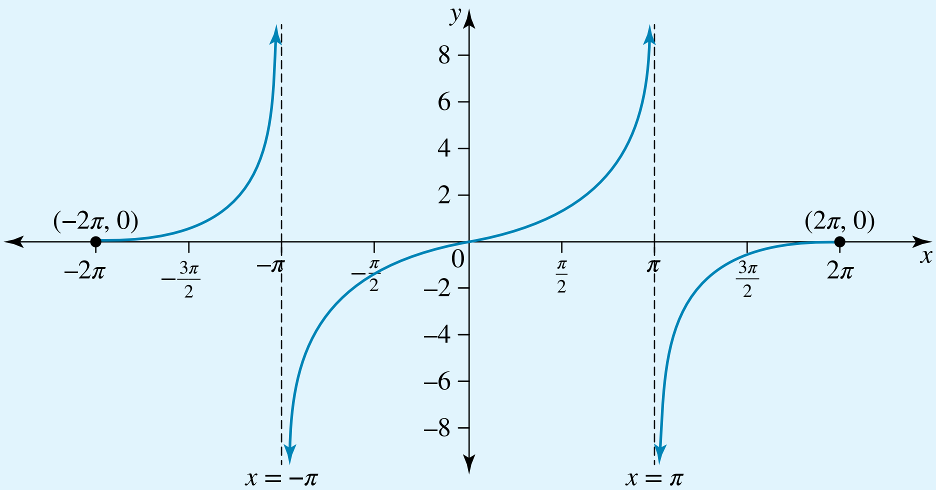
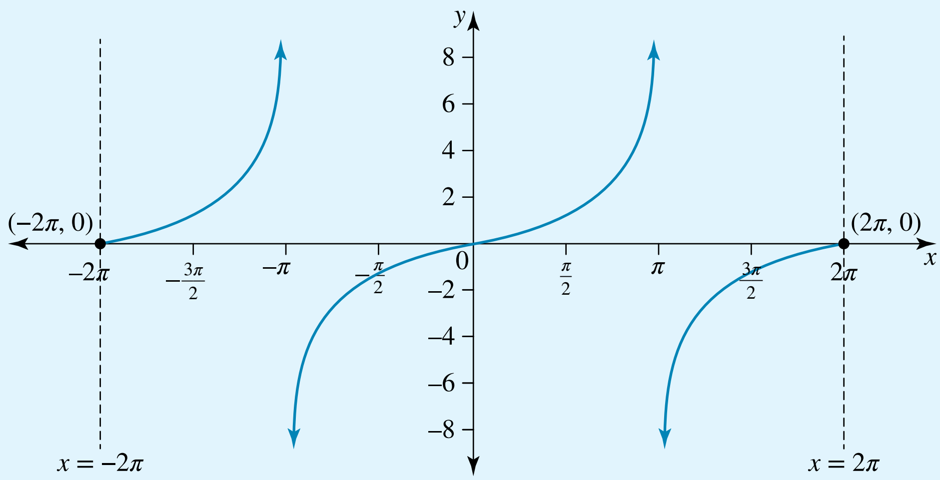
In the portion of the graph shown, the x-intercepts occur at -2π, $-\pi$, 0, π and 2π. This means that the reciprocal function will have vertical asymptotes at $x = -2\pi$, $x = -\pi$, $x = 0$, $x = \pi$ and $x = 2\pi$.

$y = \tan(x)$ has asymptotes at $x = \dfrac{-3\pi}{2}$, $x = \dfrac{-\pi}{2}$, $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$. Therefore, the reciprocal function will have x-intercepts at these positions.

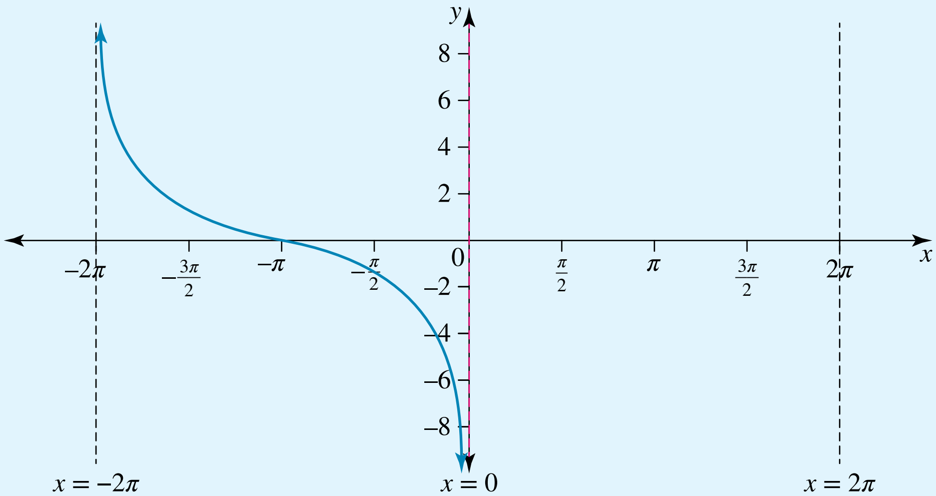
Remembering that sections of the graph that are above the x-axis for $y = \tan(x)$ will also be above the x-axis for $y = \dfrac{1}{\tan(x)}$ and similarly for sections below the x-axis, the graph of $y = \dfrac{1}{\tan(x)}$ (or $y = \cot(x)$) looks like this:



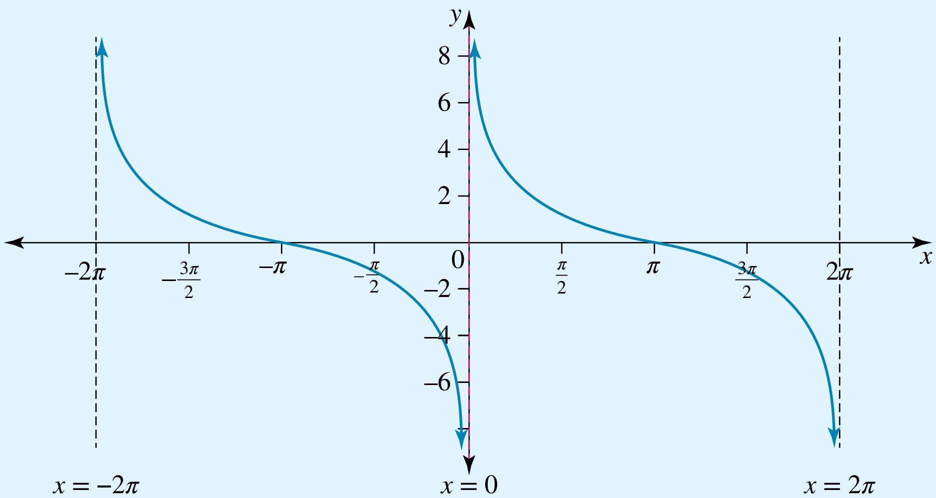
WORKED EXAMPLE 33 Use the graph of $y = \tan\left(\frac{x}{2}\right)$ to sketch $y = \frac{1}{\tan\left(\frac{x}{2}\right)}$ over the domain $-2\pi \le x \le 2\pi$.

THINK	WRITE/DRAW
1 Sketch $y = \tan\left(\frac{x}{2}\right)$. Period: 2π Dilation: 1 Horizontal shift: 0 Vertical shift: 0	
2 The graph of $y = \tan\left(\frac{x}{2}\right)$ has asymptotes at $x = -\pi$ and $x = \pi$. These will be the x-intercepts of the reciprocal function.	The x-intercepts will be $x = -\pi$ and $x = \pi$.
3 Find the x-intercepts for $y = \tan\left(\frac{x}{2}\right)$ and hence the vertical asymptotes for the reciprocal graph.	x-intercepts occur at $x = -2\pi$, $x = 0$ and $x = 2\pi$. These will be the vertical asymptotes for the reciprocal function. 

4 If we consider the region between $x = -2\pi$ and $x = 0$, the graph of $y = \tan\left(\frac{x}{2}\right)$ is initially above the x-axis between $x = -2\pi$ and $x = -\pi$ and is then below the x-axis. This will also be true for the reciprocal function.



5 In a similar fashion, the graph for $x = 0$ to $x = 2\pi$ can be obtained.



Transformations of reciprocal trigonometric graphs

Transformations can also be applied to the reciprocal trigonometric graphs.

WORKED EXAMPLE 34

Sketch the graph of $y = \dfrac{1}{\sin\left(x + \frac{\pi}{4}\right)} + 1$ over the domain $[-\pi, 2\pi]$.

THINK

1 Use the graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ to find the graph of $y = \dfrac{1}{\sin\left(x + \frac{\pi}{4}\right)}$.

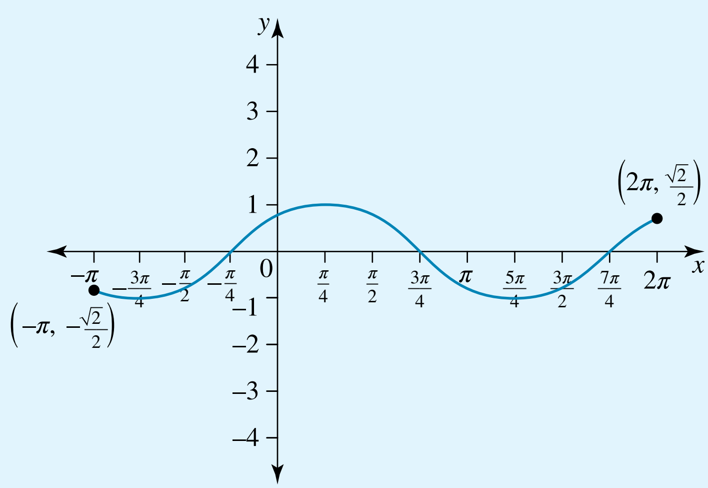
Amplitude: 1

Period: 2π

Horizontal shift: $\frac{\pi}{4}$ left

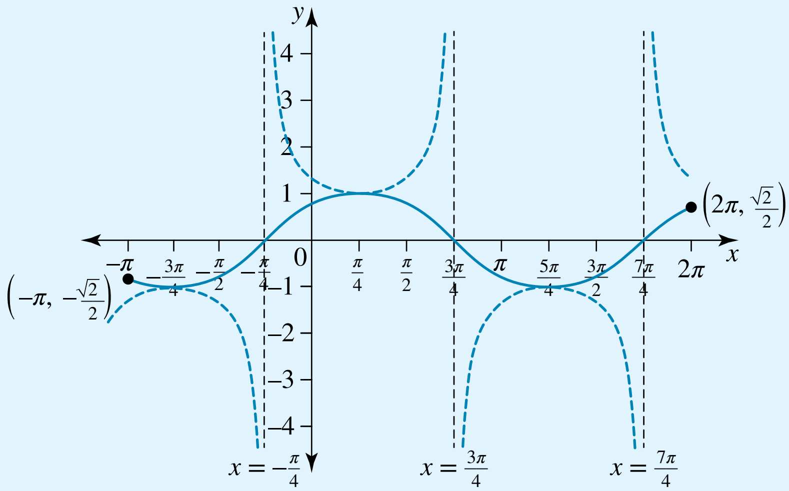
Vertical shift: 0

WRITE/DRAW



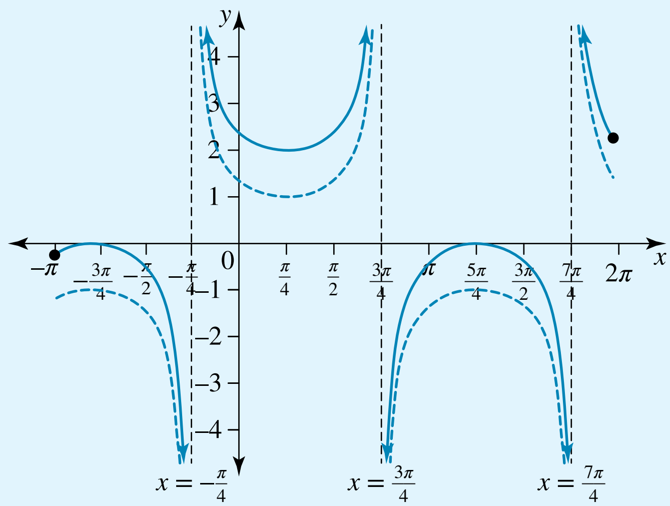
2 Consider the graph of $y = \dfrac{1}{\sin\left(x + \dfrac{\pi}{4}\right)}$.

The asymptotes will occur at $x = \dfrac{-\pi}{4}$, $x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$.



3 To graph $y = \dfrac{1}{\sin\left(x + \dfrac{\pi}{4}\right)} + 1$,

move $y = \dfrac{1}{\sin\left(x + \dfrac{\pi}{4}\right)}$ up 1.



EXERCISE 2.8 Graphs of reciprocal trigonometric functions

PRACTISE

1 WE31 Use the graph of $y = 4\cos(x)$ to sketch $y = \dfrac{1}{4\cos(x)}$ over the domain $-2\pi \le x \le 2\pi$.

2 Use the graph of $y = 2\sin(x)$ to sketch $y = \dfrac{1}{2\sin(x)}$ over the domain $-2\pi \le x \le 2\pi$.

3 WE32 Use the graph of $y = \sin(2x)$ to sketch $y = \dfrac{1}{\sin(2x)}$ over the domain $-2\pi \le x \le 2\pi$.

4 Use the graph of $y = \cos(2x)$ to sketch $y = \dfrac{1}{\cos(2x)}$ over the domain $-2\pi \le x \le 2\pi$.

5 WE33 Use the graph of $y = \tan(2x)$ to sketch $y = \dfrac{1}{\tan(2x)}$ over the domain $-2\pi \le x \le 2\pi$.

6 Use the graph of $y = \tan(3x)$ to sketch $y = \dfrac{1}{\tan(3x)}$ over the domain $-2\pi \le x \le 2\pi$.

7 WE34 Sketch the graph of $y = \cot\left(x + \dfrac{\pi}{4}\right) + 1$ over the domain $[-\pi, 2\pi]$.

8 Sketch the graph of $y = \frac{1}{2}\sec\left(x + \frac{\pi}{4}\right) - 1$ over the domain $[-\pi, 2\pi]$.

CONSOLIDATE

9 Use the graph of $y = 4\sin(x)$ to sketch $y = \frac{1}{4\sin(x)}$ over the domain $[-\pi, \pi]$.

10 Use the graph of $y = \cos\left(\frac{x}{2}\right)$ to sketch $y = \frac{1}{\cos\left(\frac{x}{2}\right)}$ over the domain $[-\pi, \pi]$.

11 Use the graph of $y = \tan\left(\frac{x}{3}\right)$ to sketch $y = \frac{1}{\tan\left(\frac{x}{3}\right)}$ over the domain $[-3\pi, 3\pi]$.

12 Sketch $y = \sec(x) + 1$ over the domain $[0, 2\pi]$.

13 Sketch $y = \frac{1}{2}\text{cosec}\left(\frac{x}{2}\right)$ over the domain $[0, 2\pi]$.

14 Sketch $y = \cot\left(\frac{x}{4}\right) - 2$ over the domain $[-2\pi, 2\pi]$

15 Sketch $y = 2\sec(x) - 1$ over the domain $[-2\pi, 2\pi]$.

16 Sketch $y = \frac{2}{\sin\left(x + \frac{\pi}{4}\right)}$ over the domain $[-\pi, \pi]$.

17 Sketch $y = 0.25\,\text{cosec}\left(x - \frac{\pi}{4}\right)$ over the domain $[-\pi, \pi]$.

18 Sketch $y = 3\sec\left(2x + \frac{\pi}{2}\right) - 2$ over the domain $[-\pi, \pi]$.

MASTER

19 Use the graph of $y = \sin(x) + 2$ to sketch $y = \frac{1}{\sin(x) + 2}$ over the domain $\left[\frac{-5\pi}{2}, \frac{5\pi}{2}\right]$. Sketch both graphs on the same set of axes. Check your graphs with CAS.

20 a Use the graph of $y = \cos^2(x)$ to sketch $y = \frac{1}{\cos^2(x)}$ over the domain $\left[\frac{-3\pi}{2}, \frac{3\pi}{2}\right]$.

Sketch both graphs on the same set of axes. Check your graphs with CAS.

b Hence, determine the graph of $y = \tan^2(x)$ for the same domain.

2.9 Graphs of inverse trigonometric functions

studyon

Units 3 & 4
AOS 1
Topic 2
Concept 6

Graphs of inverse circular functions
Concept summary
Practice questions

There are at least two possible approaches to sketching inverse trigonometric functions. The first method is to find the inverse of the function (which will be a trigonometric function) and use your knowledge of trigonometric functions to sketch the trigonometric function and its inverse.

Alternatively, you could use your knowledge about transforming equations to transform $y = \sin^{-1}(x)$, $y = \cos^{-1}(x)$ or $y = \tan^{-1}(x)$ as required. In the following worked examples, we will find the original trigonometric function and then sketch both functions.

WORKED EXAMPLE 35 Sketch $y = \sin^{-1}(2x)$.

THINK

WRITE/DRAW

1 Find the inverse of $y = \sin^{-1}(2x)$.

$y = \sin^{-1}(2x)$

$2x = \sin(y)$

$x = \frac{1}{2}\sin(y)$

Therefore, the inverse is $y = \frac{1}{2}\sin(x)$.

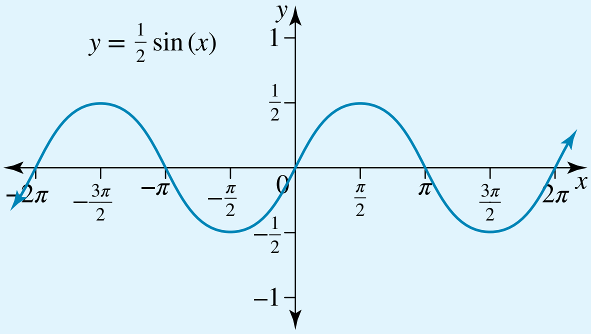
2 Sketch $y = \frac{1}{2}\sin(x)$.

Amplitude: $\frac{1}{2}$

Period: 2π

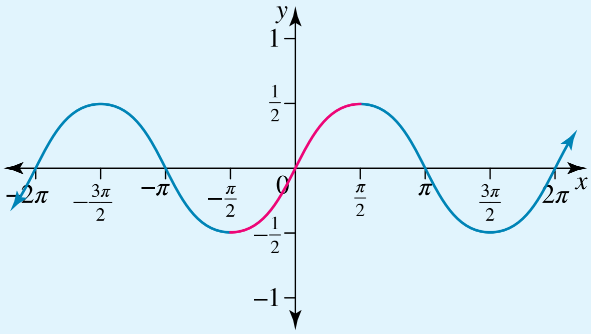
Horizontal shift: 0

Vertical shift: 0



3 The domain needs to be restricted so that the function is one-to-one. The domain becomes $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



4 The domain and range of $y = \frac{1}{2}\sin(x)$ become the range and domain of $y = \sin^{-1}(2x)$ respectively.

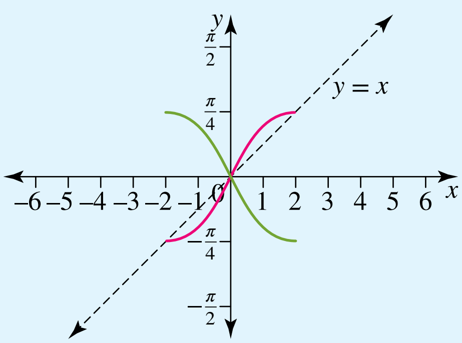
$y = \frac{1}{2}\sin(x)$:

Domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, range $\left[-\frac{1}{2}, \frac{1}{2}\right]$

$y = \sin^{-1}(2x)$:

Domain $\left[-\frac{1}{2}, \frac{1}{2}\right]$, range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

5 Use the graph of $y = \frac{1}{2}\sin(x)$ to sketch $y = \sin^{-1}(2x)$ by reflecting the graph in the line $y = x$.



WORKED EXAMPLE 36 Sketch $y = \cos^{-1}(x + 5)$

THINK

WRITE/DRAW

1 Find the inverse of $y = \cos^{-1}(x + 5)$.

$$y = \cos^{-1}(x + 5)$$
$$x + 5 = \cos(y)$$
$$x = \cos(y) - 5$$

Therefore, $y = \cos(x) - 5$.

2 Sketch $y = \cos(x) - 5$.

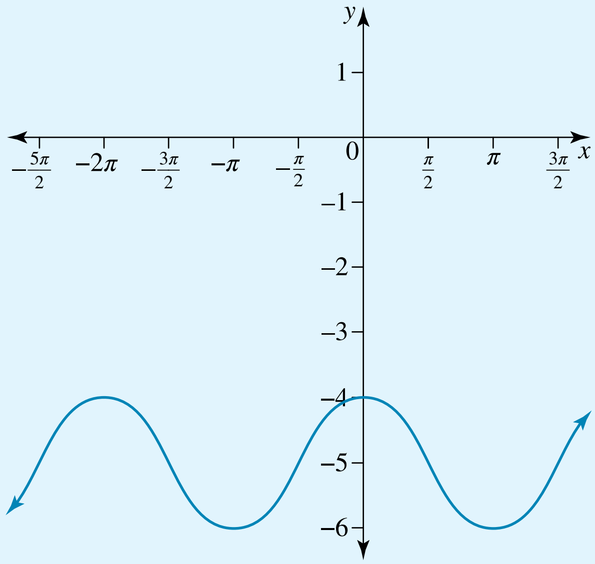
Amplitude: 1

Period: 2π

Horizontal shift: 0

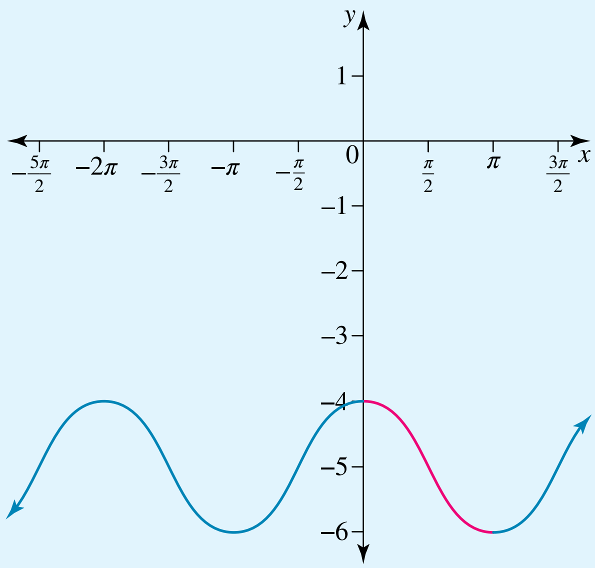
Vertical shift: 5 down

$y = \cos(x) - 5$



3 The domain needs to be restricted so that the function is one-to-one. The domain becomes $[0, \pi]$.

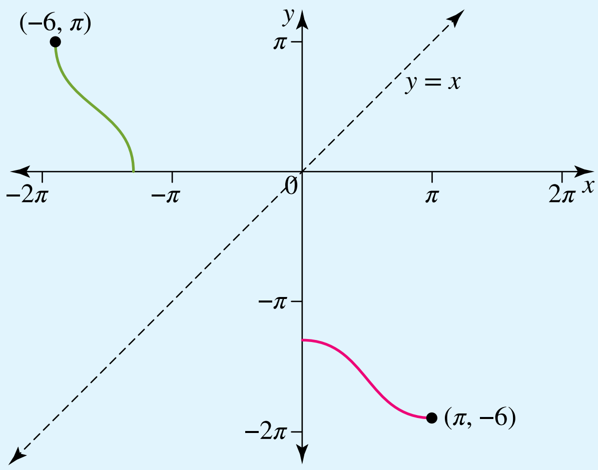
Restrict the domain to $[0, \pi]$.



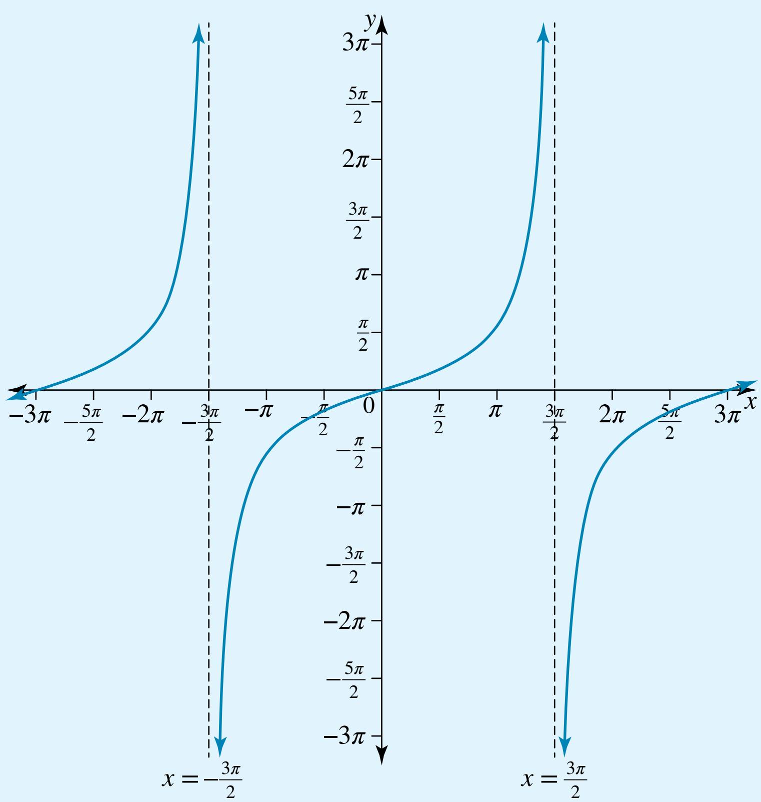
4 The domain and range of $y = \cos(x) - 5$ become the range and domain of $y = \cos^{-1}(x + 5)$ respectively.

$y = \cos(x) - 5$:
Domain $[0, \pi]$, range $[-6, -4]$
$y = \cos^{-1}(x + 5)$:
Domain $[-6, -4]$, range $[0, \pi]$

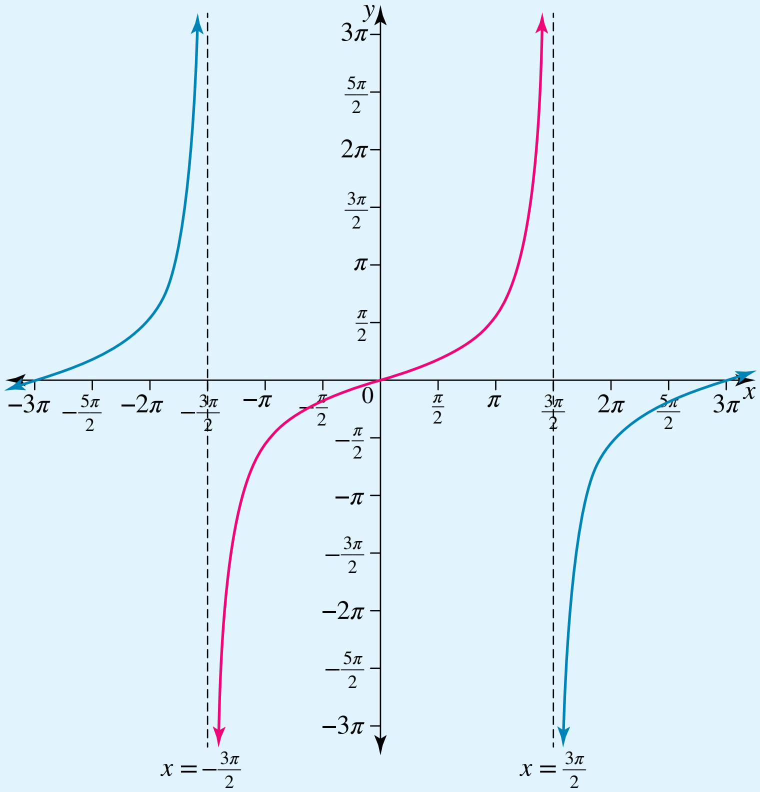
5 Use the graph of $y = \cos(x) - 5$ to sketch $y = \cos^{-1}(x + 5)$ by reflecting the graph in the line $y = x$.



WORKED EXAMPLE 37 Sketch $y = 3\tan^{-1}(x)$.

THINK	WRITE/DRAW
1 Find the inverse of $y = 3\tan^{-1}(x)$.	$y = 3\tan^{-1}(x)$ $\frac{y}{3} = \tan^{-1}(x)$ $x = \tan\left(\frac{y}{3}\right)$ Therefore $y = \tan\left(\frac{x}{3}\right)$.
2 Sketch $y = \tan\left(\frac{x}{3}\right)$. Period: 3π Horizontal shift: 0 Vertical shift: 0	

3 The domain needs to be restricted so that the function is one-to-one. The domain becomes $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

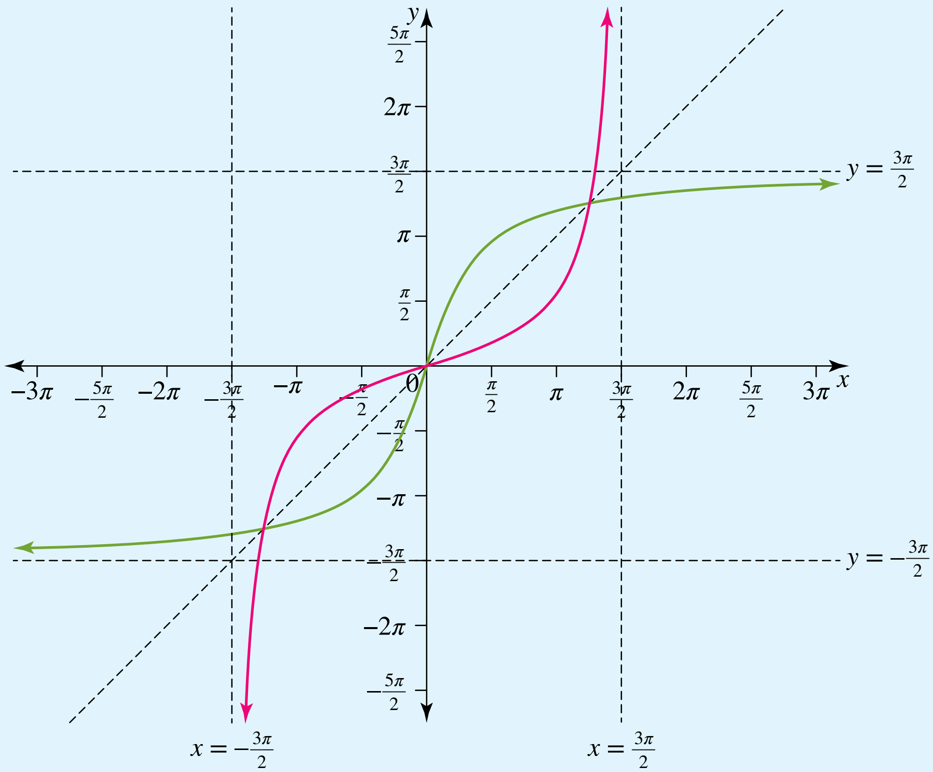


4 The domain and range of $y = \tan\left(\frac{x}{3}\right)$ become the range and domain of $y = 3\tan^{-1}(x)$ respectively.

$y = \tan\left(\frac{x}{3}\right)$: Domain $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$, range R

$y = 3\tan^{-1}(x)$: Domain R, range $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

5 Use the graph of $y = \tan\left(\frac{x}{3}\right)$ to sketch $y = 3\tan^{-1}(x)$ by reflecting the graph in the line $y = x$.



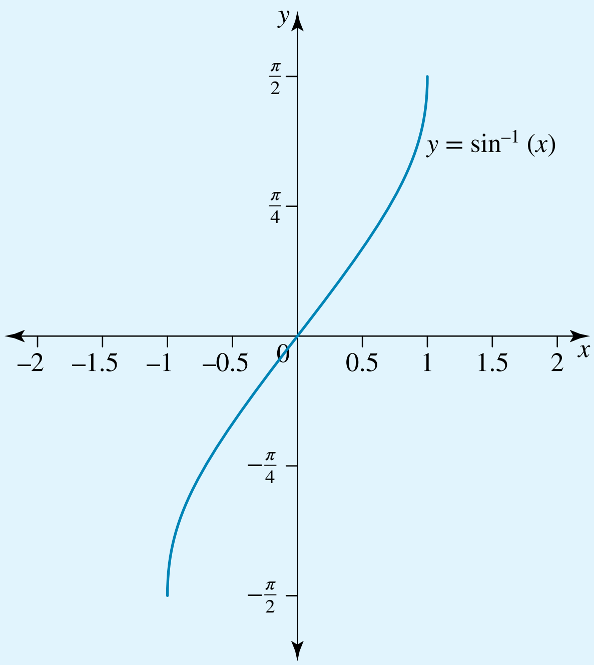
The next worked example is completed by transforming the inverse trigonometric function.

WORKED EXAMPLE 38 Sketch $y = \sin^{-1}(x) + \frac{\pi}{4}$.

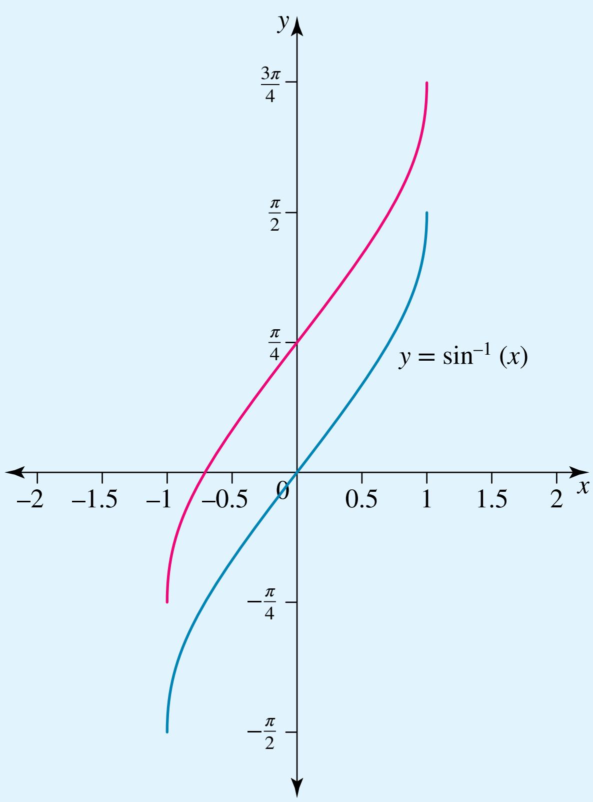
THINK

WRITE/DRAW

1 The graph of $y = \sin^{-1}(x) + \frac{\pi}{4}$ is the graph of $y = \sin^{-1}(x)$ raised by $\frac{\pi}{4}$ units. Sketch $y = \sin^{-1}(x)$.



2 Raise the graph by $\frac{\pi}{4}$. This means that the range is now $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$



EXERCISE 2.9 Graphs of inverse trigonometric functions

PRACTISE

1 WE35 Sketch $y = \cos^{-1}(2x)$.

2 Sketch $y = \tan^{-1}(2x)$.

3 WE36 Sketch $y = \sin^{-1}(x + 3)$.

4 Sketch $y = \tan^{-1}(x - 3)$.

5 WE37 Sketch $y = 2\sin^{-1}(x)$.

6 Sketch $y = 2\cos^{-1}(x)$.

7 WE38 Sketch $y = \cos^{-1}(x) - \frac{\pi}{4}$.

8 Sketch $y = \tan^{-1}(x) + \frac{\pi}{3}$.

CONSOLIDATE

9 Sketch $y = \sin^{-1}\left(\frac{x}{2}\right)$.

10 Sketch $y = \tan^{-1}\left(\frac{x}{3}\right)$.

11 Sketch $y = \cos^{-1}(4x)$.

12 Sketch $y = \tan^{-1}(x - 3)$.

13 Sketch $y = \sin^{-1}(2x + 1)$.

14 Sketch $y = \cos^{-1}(3x - 2)$.

15 Sketch $y = 3\cos^{-1}(x)$.

16 Sketch $y = 3\sin^{-1}(2x)$.

17 Sketch $y = 2\tan^{-1}(x) + \frac{\pi}{4}$.

18 Sketch $y = \frac{\cos^{-1}(2x - 3)}{\pi} + 1$.

MASTER

19 a Draw the graph of $y = \sec(x)$.

b Identify a suitable domain to make $y = \sec(x)$ a one-to-one function.

c Sketch the graph of $y = \sec^{-1}(x)$.

20 Sketch $y = \frac{\cot^{-1}(x + 1)}{\pi} - 2$.

studyon

Units 3 & 4
AOS 1
Topic 2
Concept 7

Transformations of inverse circular functions
Concept summary
Practice questions

eBookplus ONLINE ONLY

2.10 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

2 Answers

EXERCISE 2.2

1 $\frac{2\sqrt{3}}{3}$

2 $-\frac{2\sqrt{3}}{3}$

3 $-\frac{\sqrt{21}}{2}$

4 $-\frac{\sqrt{17}}{4}$

5 a $\frac{2\sqrt{3}}{3}$ b $-\sqrt{2}$ c -2 d $\sqrt{2}$

6 a $\frac{2\sqrt{3}}{3}$ b 2 c $-\sqrt{2}$ d $-\frac{2\sqrt{3}}{3}$

7 a $\sqrt{3}$ b $-\frac{\sqrt{3}}{3}$ c -1 d $-\sqrt{3}$

8 a $-\frac{3\sqrt{2}}{4}$ b $-\sqrt{15}$

9 a $\frac{3\sqrt{10}}{20}$ b $-\frac{5\sqrt{21}}{21}$

10 a $-\frac{7\sqrt{10}}{20}$ b $-\frac{5\sqrt{39}}{39}$

11 a $-\frac{\sqrt{15}}{15}$ b $\frac{\sqrt{61}}{6}$

12 a $\frac{\sqrt{3}}{12}$ b $-\sqrt{17}$

13 a $-\frac{6\sqrt{35}}{35}$ b $\frac{\sqrt{29}}{5}$

14 a $\frac{\sqrt{15}}{3}$ b $-\frac{\sqrt{15}}{5}$

15 $\dfrac{-pq + p^2 - q^2}{q\sqrt{p^2 - q^2}}$

16 $\dfrac{\sqrt{a^2 - b^2}}{a + b}$

EXERCISE 2.3

Refer to *Maths Quest 12 VCE Specialist Maths Solutions Manual.*

EXERCISE 2.4

1 $\frac{\sqrt{3}}{2}$

2 $\frac{1}{2}$

3 $2\sqrt{3}\cos(\theta) - 2\sin(\theta)$

4 $\sin(\theta) + \cos(\theta)$

5 $-\cos(\theta)$

6 $-\sin(\theta)$

7 $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

8 $\sqrt{3} - 2$

9 $-\frac{84}{85}$

10 $-\frac{64}{1025}$

11 a $\frac{\sqrt{3}}{2}$ b 0 c $\frac{1}{2}$ d $-\frac{\sqrt{2}}{2}$

12 a $\frac{\sqrt{3}}{3}$ b $\sqrt{3}$

13 a $\sin(\theta) - \cos(\theta)$ b $\sqrt{3}\cos(\theta) + \sin(\theta)$

c $\sqrt{3}\cos(\theta) + \sin(\theta)$ d $\cos(\theta) - \sin(\theta)$

14 a $\cos(\theta)$ b $\sin(\theta)$

c $-\sin(\theta)$ d $-\cos(\theta)$

15 a $-\cos(\theta)$ b $\sin(\theta)$

c $-\tan(\theta)$ d $\tan(\theta)$

16 a $\sqrt{3}\cos(x)$ b -1

c $-\sqrt{3}\sin(x)$ d $\sin(x)$

17 a $\frac{1}{4}(\sqrt{2} - \sqrt{6})$ b $2 - \sqrt{3}$

c $\frac{1}{4}(\sqrt{6} - \sqrt{2})$ d $2 + \sqrt{3}$

18 a $\frac{56}{65}$ b $-\frac{63}{16}$

19 a $-\frac{253}{325}$ b $-\frac{204}{325}$

20 a $-\frac{8\sqrt{5}}{21}$ b $-\frac{19}{21}$

21 $\dfrac{ab + \sqrt{1 - a^2}\sqrt{1 - b^2}}{b\sqrt{1 - a^2} - a\sqrt{1 - b^2}}$

22 $\dfrac{a^2 + 2\sqrt{a + 1}\sqrt{2a + 1}}{a\left(\sqrt{2a + 1} - 2\sqrt{a + 1}\right)}$

EXERCISE 2.5

1 $-\frac{\sqrt{2}}{4}$

2 $-\frac{\sqrt{2}}{2}$

3 a $\frac{4\sqrt{2}}{9}$ b $-\frac{7}{9}$ c $-\frac{4\sqrt{2}}{7}$

4 a $\frac{56}{65}$ b $\frac{33}{65}$ c $\frac{56}{33}$

5 $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$

6 $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

7–12 Check with your teacher.

13 a $\frac{\sqrt{2}}{4}$ b $-\frac{\sqrt{2}}{2}$ c $-\frac{\sqrt{2}}{2}$ d $\frac{\sqrt{3}}{3}$

14 a $\frac{3\sqrt{55}}{32}$ b $-\frac{23}{32}$ c $-\frac{3\sqrt{55}}{23}$

15 a $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ b $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

c $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ d $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

16 a $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

b $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$

c $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$

d $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{7\pi}{4}$

17–24 Check with your teacher.

EXERCISE 2.6

1 a Does not exist b $-\frac{\pi}{3}$ c 0.9

2 a Does not exist b $-\frac{\pi}{6}$ c $\frac{1}{3}$

3 a Does not exist b $\frac{5\pi}{6}$ c $\frac{\pi}{6}$

4 a Does not exist b $\frac{\pi}{3}$ c $\frac{1}{4}$

5 $\frac{2\sqrt{6}}{5}$

6 $\frac{2\sqrt{10}}{7}$

7 $\frac{8\sqrt{33}}{49}$

8 $\frac{23}{32}$

9 a $\frac{\pi}{6}$ b 1.1

10 a $-\frac{\pi}{3}$ b $\frac{5}{4}$

11 $\frac{3}{4}$

12 $\frac{15}{8}$

13 $\frac{33}{65}$

14 $-\frac{33}{56}$

15 a Domain $\left[\frac{1}{2}, \frac{9}{2}\right]$, range $\left[-\frac{7\pi}{2}, -\frac{\pi}{2}\right]$

b Domain R, range (–1, 5)

16 a Domain $\left[-2, -\frac{4}{3}\right]$, range [–3, 1]

b Domain R, range (–1, 7)

17 a $\frac{\pi}{2}$ b Does not exist

c $-\frac{\pi}{3}$ d π

e $\frac{2\pi}{3}$ f Does not exist

g $\frac{\pi}{3}$ h $-\frac{\pi}{6}$

18 a $\frac{6}{5}$ b $\frac{\pi}{5}$ c $\frac{\pi}{6}$ d $\frac{1}{2}$

e $\frac{\pi}{10}$ f $\frac{\pi}{3}$ g $\frac{\pi}{8}$ h $\frac{\pi}{3}$

19 a $\frac{\sqrt{3}}{2}$ b $\frac{1}{2}$ c $-\frac{\sqrt{3}}{3}$

d $-\frac{\sqrt{2}}{2}$ e $\frac{\sqrt{3}}{2}$ f -1

20 a $\frac{\sqrt{77}}{9}$ b $-\frac{\sqrt{5}}{2}$ c $-\frac{5\sqrt{11}}{11}$

d $\frac{5\sqrt{89}}{89}$ e $\frac{\sqrt{21}}{5}$ f $\frac{4\sqrt{65}}{65}$

21 a $\frac{\sqrt{15}}{8}$ b $-3\sqrt{7}$ c $\frac{4}{5}$

d $\frac{12}{13}$ e $-\frac{4\sqrt{6}}{23}$ f $\frac{17}{25}$

22 a $\frac{56}{65}$ b $\frac{63}{65}$ c $-\frac{528}{697}$ d $-\frac{812}{1037}$

23 Check with your teacher.

24 Check with your teacher.

25 a Domain [0, 2], range $[-\pi, \pi]$

b Domain [1, 3], range $[0, 3\pi]$

c Domain R, range $(-2\pi, 2\pi)$

d Domain [–3, 3], range $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$

e Domain [–4, 4], range $[0, 6\pi]$

f Domain R, range $\left(-\frac{7\pi}{2}, \frac{7\pi}{2}\right)$

26 a Domain $\left[0, \frac{2}{3}\right]$, range $[0, 2\pi]$

b Domain [2, 3], range $[-\pi, 2\pi]$

c Domain R, range $(-3\pi, 2\pi)$

d Domain $\left[-\frac{1}{2}, 2\right]$, range [0, 4]

e Domain $\left[-1, \frac{11}{3}\right]$, range [–4, 1]

f Domain R, range (–1, 7)

27 a Dilation by a factor of c units parallel to the x-axis (or away from the y-axis), dilation by a factor of b units parallel to the y-axis (or away from the x-axis), translation by a units up and parallel to the y-axis (or away from the x-axis)

Domain $[-c, c]$, range $\left[a - \frac{b\pi}{2}, a + \frac{b\pi}{2}\right]$

b Dilation by a factor of $\frac{1}{c}$ units parallel to the x-axis (or away from the y-axis), dilation by a factor of b units parallel to the y-axis (or away from the x-axis), translation by a units up and parallel to the y-axis (or away from the x-axis)

Domain $\left[-\frac{1}{c}, \frac{1}{c}\right]$, range $[a, a + b\pi]$

c Dilation by a factor of c units parallel to the x-axis (or away from the y-axis), dilation by a factor of b units parallel to the y-axis (or away from the x-axis), translation by a units up and parallel to the y-axis (or away from the x-axis)

Domain R, range $\left(a - \frac{b\pi}{2}, a + \frac{b\pi}{2}\right)$

28 Check with your teacher.

EXERCISE 2.7

Note that $n \in Z$.

1 $\frac{\pi}{4}(8n \pm 1)$

2 $\frac{\pi}{12}(12n \pm 5)$

3 $\frac{\pi}{6}(12n + 1), \frac{\pi}{6}(12n + 5)$

4 $\frac{\pi}{6}(6n - 1), \frac{\pi}{3}(3n + 2)$

5 $\frac{\pi}{4}(4n + 1)$

6 $\frac{\pi}{6}(3n - 1)$

7 $\frac{\pi}{6}(6n \pm 1), \frac{\pi}{3}(3n \pm 1)$

8 $\frac{\pi}{12}(6n \pm 1)$

9 $\frac{2n\pi}{3}, \frac{2\pi}{9}(3n+1)$

10 $\frac{\pi}{2}(2n+1), \frac{\pi}{3}(3n-1)$

11 $\frac{\pi}{12}(12n+1), \frac{\pi}{12}(12n+5), \frac{\pi}{4}(4n+1)$

12 $\frac{\pi}{6}(6n \pm 1), \frac{\pi}{2}(2n \pm 1)$

13 $\frac{\pi}{10}(4n+1), \frac{\pi}{2}(4n-1)$

14 $\frac{\pi}{14}(4n+1), \frac{\pi}{2}(4n-1)$

15 a $\frac{\pi}{18}(12n \pm 1)$

b $\frac{\pi}{3}(3n \pm 1)$

c $\frac{\pi}{8}(8n-1), \frac{\pi}{8}(8n+5)$

d $\frac{\pi}{18}(12n-1), \frac{\pi}{18}(12n+7)$

16 a $\frac{\pi}{6}(6n \pm 1), \frac{\pi}{3}(3n+1), \frac{\pi}{3}(3n+2)$

b $\frac{\pi}{8}(8n \pm 1), \frac{\pi}{8}(8n+3), \frac{\pi}{8}(8n+5)$

c $\frac{\pi}{9}(6n \pm 1), \frac{2\pi}{9}(3n \pm 1)$

d $\frac{\pi}{12}(8n \pm 1), \frac{\pi}{12}(8n \pm 3)$

17 a $\frac{\pi}{3}(3n-1)$ b $\frac{\pi}{18}(6n+1)$

c $\frac{\pi}{6}(3n \pm 1)$ d $\frac{\pi}{12}(6n \pm 1)$

18 a $\frac{n\pi}{2}, \frac{\pi}{12}(12n-1), \frac{\pi}{12}(12n+7)$

b $\frac{\pi}{4}(4n \pm 1), n\pi$

c $\frac{\pi}{4}(4n \pm 1), \frac{\pi}{12}(12n \pm 5)$

d $\frac{n\pi}{2}, \frac{\pi}{6}(6n+1), \frac{\pi}{3}(3n+1)$

19 a $\frac{\pi}{12}(12n-1), \frac{\pi}{12}(12n+7), \frac{\pi}{4}(4n-1)$

b $\frac{\pi}{6}(6n \pm 1), n\pi$

20 a $\frac{\pi}{6}(4n+1), \frac{\pi}{3}(2n+1)$ b $\frac{\pi}{4}(4n-1), \frac{\pi}{12}(12n+5)$

21 a $\frac{\pi}{3}(3n+1), \frac{\pi}{2}(2n-1)$

b $\frac{\pi}{36}(24n+7), \frac{\pi}{36}(24n-1)$

22 a $\frac{n\pi}{3}$ b $\frac{n\pi}{2}$

23 a $\frac{\pi}{3}(3n-1), \frac{\pi}{4}(4n-1)$ b $\frac{\pi}{3}(3n-1), \frac{\pi}{4}(4n+1)$

24 a $n\pi, \frac{\pi}{3}(6n \pm 1)$ b $2n\pi, \frac{2n\pi}{3}$

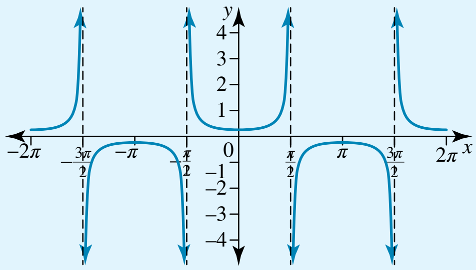
25 a $\frac{\pi}{2}(4n \pm 1), \frac{\pi}{6}(12n-1), \frac{\pi}{6}(12n+7)$

b $n\pi, \frac{\pi}{3}(6n \pm 1)$

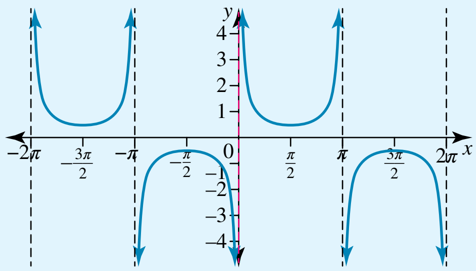
26 a $\frac{\pi}{4}(4n \pm 1)$ b $\frac{\pi}{4}(4n \pm 1), \frac{\pi}{3}(3n \pm 1)$

EXERCISE 2.8

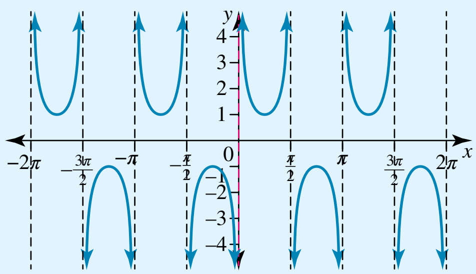
1



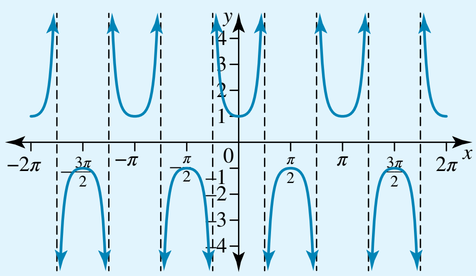
2



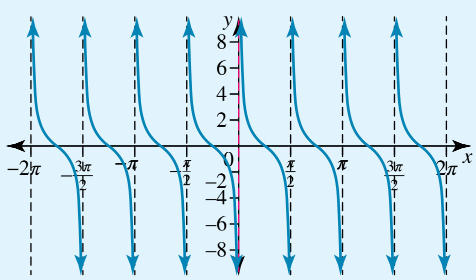
3



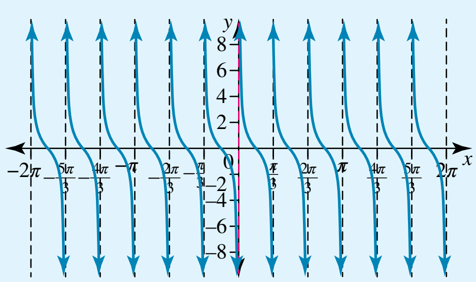
4



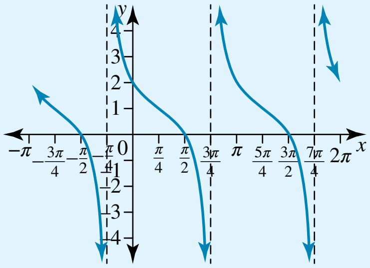
5



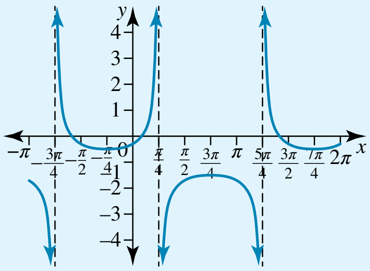
6



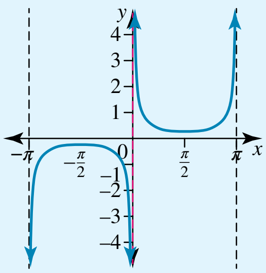
7



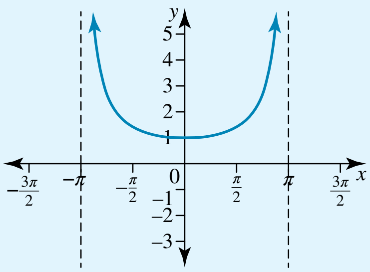
8



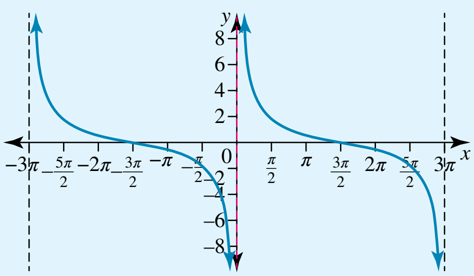
9



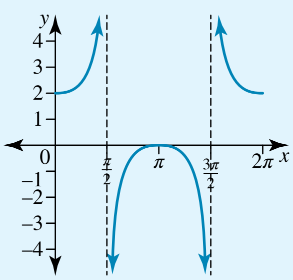
10



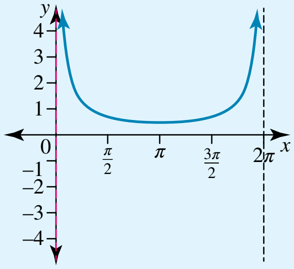
11



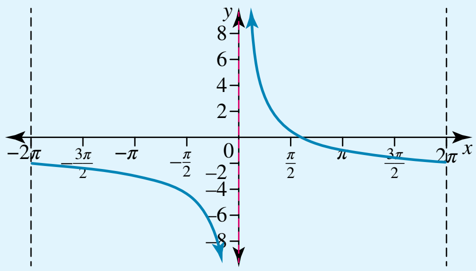
12



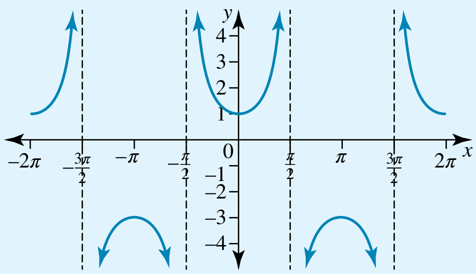
13



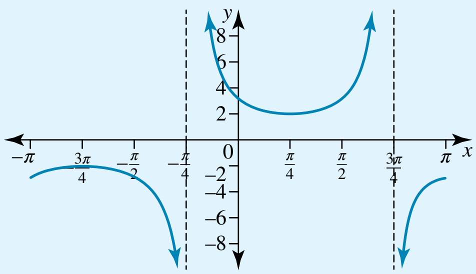
14



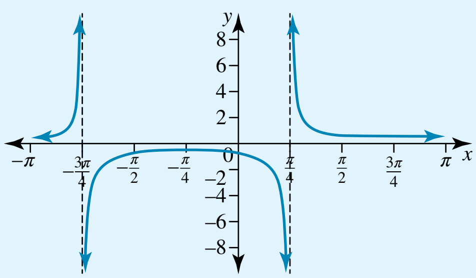
15



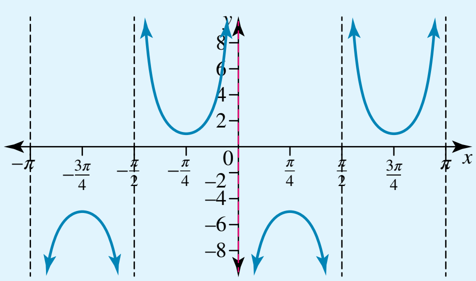
16



17

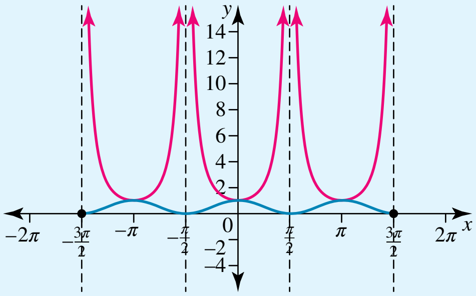


18

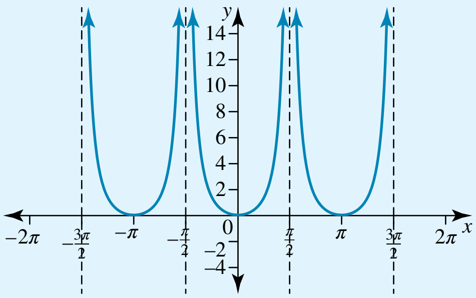


19 See figure at foot of page.*

20 a

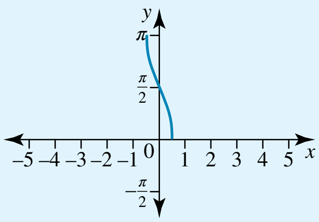


b

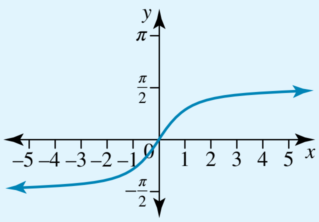


EXERCISE 2.9

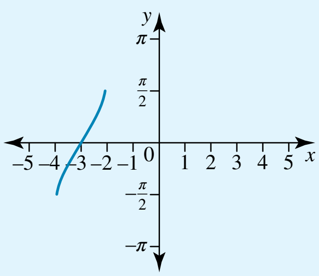
1



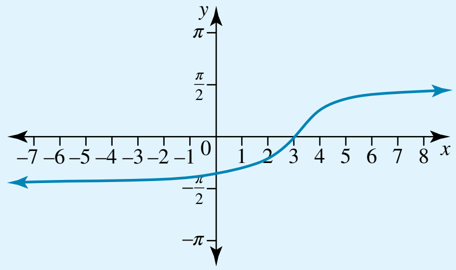
2



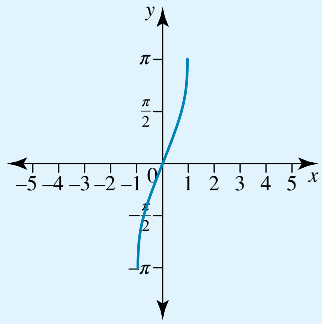
3



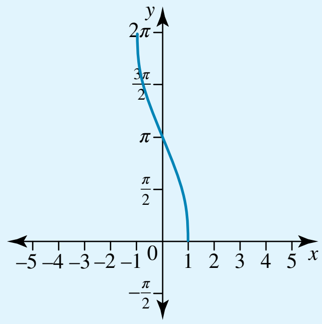
4



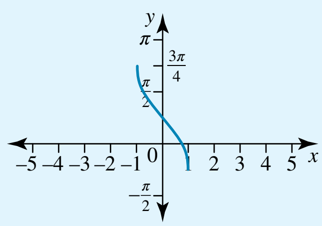
5



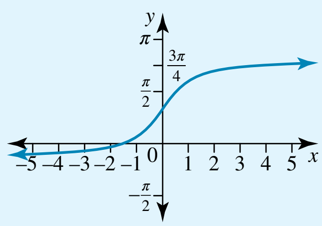
6



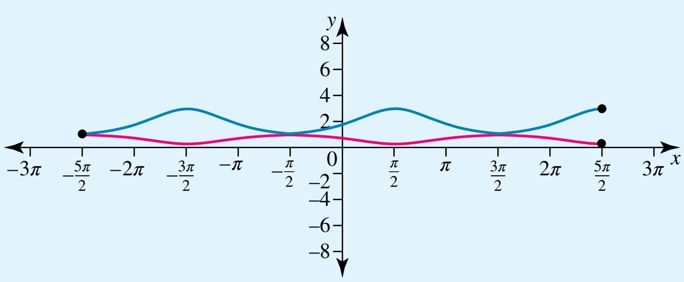
7



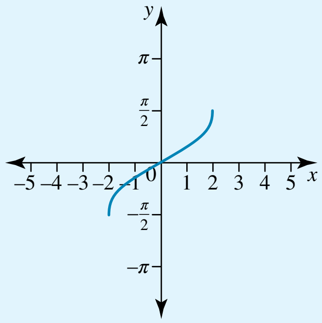
8



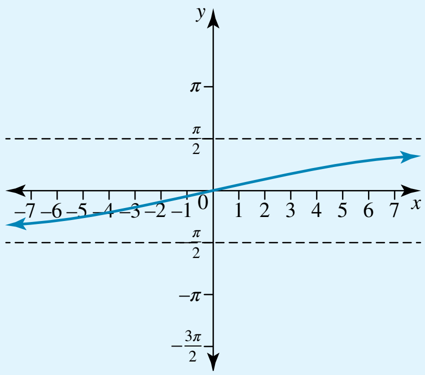
***19**



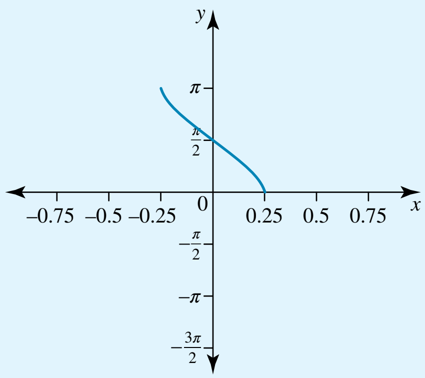
9



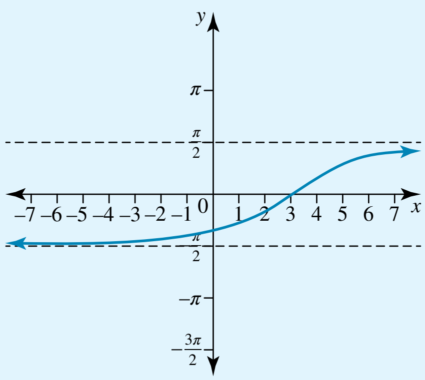
10



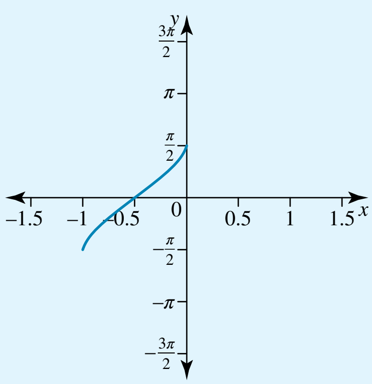
11



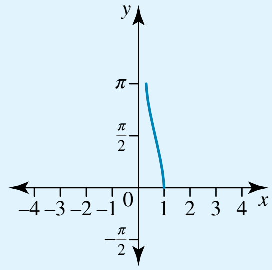
12



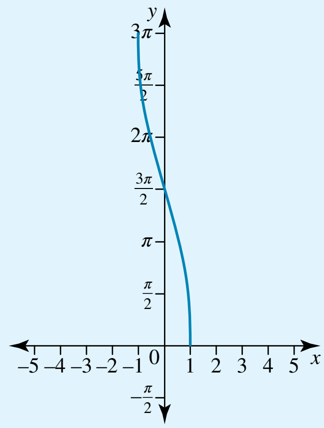
13



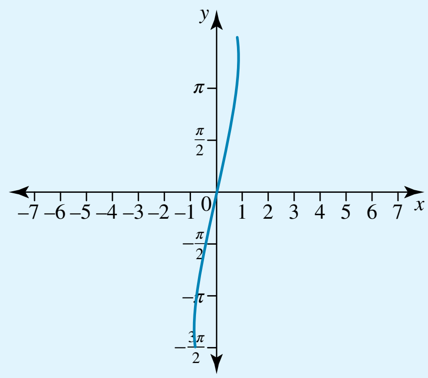
14



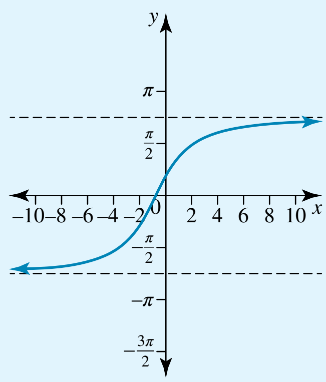
15



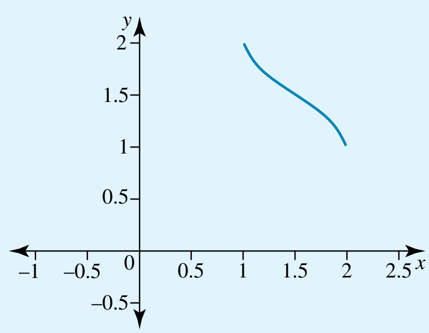
16



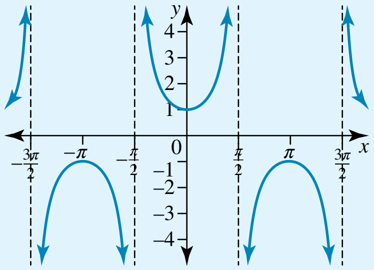
17



18

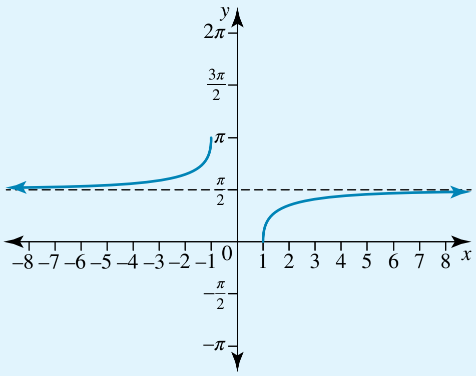


19 a

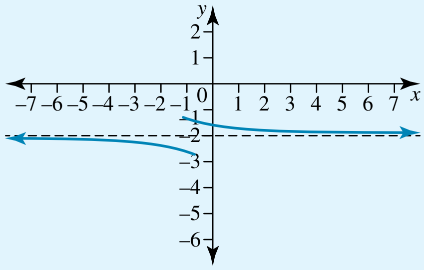


b $[0, \pi]/\left\{\frac{\pi}{2}\right\}$

c



20



3

Complex numbers

3.1 Kick off with CAS

Exploring complex numbers with CAS

Complex numbers will be studied in greater depth throughout this topic. Complex numbers arise when solving quadratic equations with negative discriminants.

1 Using CAS syntax csolve (equation $= 0$, z), solve each of the following quadratic equations, expressing the answers as complex numbers in the form $z = a + bi$. Can you predict the relationship between the roots as complex numbers and the coefficients of the quadratic equation?

a $z^2 - 2z + 2 = 0$

b $z^2 + 2z + 5 = 0$

c $z^2 - 4z + 5 = 0$

d $z^2 + 4z + 8 = 0$

e $z^2 - 6z + 10 = 0$

f $z^2 + 6z + 13 = 0$

g $z^2 - 8z + 17 = 0$

h $z^2 + 8z + 20 = 0$

i Determine the quadratic equation that has $z = 5 \pm i$ as its roots.

j Determine the quadratic equation that has $z = -5 \pm 2i$ as its roots.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

3.2 Complex numbers in rectangular form

You may have covered complex numbers in Year 11. This section is a review of the relevant content.

The **complex number system** is an extension of the real number system. Complex numbers are numbers that involve the number i, known as the imaginary unit.

The imaginary unit i is defined as $i^2 = -1$. This means that equations involving the solution to $x^2 = -1$ can now be found in terms of i.

WORKED EXAMPLE 1

a Find $\sqrt{-25}$.

b Solve the equation $z^2 + 25 = 0$ for z.

THINK	WRITE
a 1 Rewrite the surd as a product in terms of i.	**a** $\sqrt{-25} = \sqrt{-1 \times 25}$ $= \sqrt{25i^2}$
2 Simplify and state the answer. Note that there is only one solution.	$\sqrt{-25} = 5i$
b **Method 1** 1 We cannot factorise the sum of two squares. Rewrite the equation as the difference of two squares.	**b** $z^2 + 25 = 0$ $z^2 - (-25) = 0$
2 Substitute $i^2 = -1$.	$z^2 - 25i^2 = 0$ $z^2 - (5i)^2 = 0$
3 Factorise as the difference of two squares.	$(z + 5i)(z - 5i) = 0$
4 Use the Null Factor Theorem to state the two solutions.	$z = \pm 5i$
Method 2 1 Rearrange to make z the subject.	$z^2 = -25$ $= 25i^2$
2 Take the square root of both sides.	$z = \pm\sqrt{25i^2}$
3 State the two answers.	$z = \pm 5i$

studyon

Units 3 & 4

AOS 2

Topic 2

Concept 1

The complex number system, C

Concept summary

Practice questions

Solving quadratic equations

Consider the quadratic equation $az^2 + bz + c = 0$, where the coefficients a, b and c are real. Recall that the roots of a quadratic equation depend upon the value of the discriminant, $\Delta = b^2 - 4ac$.

If $\Delta > 0$, the equation has two distinct real roots.

If $\Delta = 0$, the equation has one real repeated root.

With the introduction of complex numbers, it can now be stated that if $\Delta < 0$, then the equation has one pair of **complex conjugate roots**.

WORKED EXAMPLE 2 Solve for z if $z^2 + 4z + 13 = 0$.

THINK	WRITE
Method 1	$z^2 + 4z + 13 = 0$
1 Complete the square.	$z^2 + 4z + 4 = -13 + 4$ $(z + 2)^2 = -9$
2 Substitute $i^2 = -1$.	$(z + 2)^2 = 9i^2$
3 Take the square root of both sides.	$z + 2 = \pm 3i$
4 State the two solutions.	$z = -2 \pm 3i$
Method 2	$z^2 + 4z + 13 = 0$
1 Determine the coefficients for the quadratic formula.	$a = 1, b = 4, c = 13$
2 Find the discriminant.	$\Delta = b^2 - 4ac$ $\Delta = 4^2 - 4 \times 1 \times 13$ $= -36$
3 Find the square root of the discriminant.	$\sqrt{\Delta} = \sqrt{-36} = \sqrt{36i^2}$ $= 6i$
4 Use the quadratic formula to solve for z.	$z = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ $z = \dfrac{-4 \pm 6i}{2}$
5 Simplify and state the two solutions.	$z = -2 \pm 3i$

Complete numbers in rectangular form

studyon
Units 3 & 4
AOS 2
Topic 2
Concept 3
Operations using complex numbers
Concept summary
Practice questions

A general complex number is represented by z and defined as $z = x + yi$, where x and $y \in R$, and $z \in C$, where C is used to denote the set of complex numbers (in the same way that R denotes the set of real numbers). Note that $z = x + yi$ is one single number but is composed of two parts: a real part and an imaginary complex part. The real part is written as $\text{Re}(z) = x$ and the imaginary part is written as $\text{Im}(z) = y$.

A complex number in the form $z = x + yi$, where both x and y are real numbers, is called the Cartesian form or rectangular form of a complex number. Throughout this topic, it is assumed that all equations are solved over C.

Operations on complex numbers in rectangular form

Addition and subtraction

When adding or subtracting complex numbers in rectangular form, add or subtract the real and imaginary parts separately.

Multiplication by a constant

When a complex number is multiplied by a constant, both the real and imaginary parts are multiplied by the constant.

If $z = x + yi$ and $k \in R$, then $kz = k(x + yi) = kx + kyi$.

WORKED EXAMPLE 3

Given the complex numbers $u = 2 - 5i$ and $v = -3 + 2i$, find the complex numbers:

a $u + v$ b $u - v$ c $2u - 3v$.

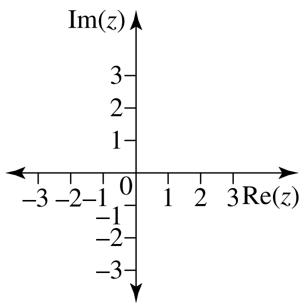
THINK	WRITE
a 1 Substitute for u and v.	**a** $u + v = (2 - 5i) + (-3 + 2i)$
2 Group the real and imaginary parts.	$= (2 - 3) + i(-5 + 2)$
3 Using the rules, state the final result.	$= -1 - 3i$
b 1 Substitute for u and v.	**b** $u - v = (2 - 5i) - (-3 + 2i)$
2 Group the real and imaginary parts.	$= (2 + 3) + i(-5 - 2)$
3 Using the rules, state the final result.	$= 5 - 7i$
c 1 Substitute for u and v.	**c** $2u - 3v = 2(2 - 5i) - 3(-3 + 2i)$
2 Expand by multiplying by the constants.	$= (4 - 10i) - (-9 + 6i)$
3 Group the real and imaginary parts.	$= (4 + 9) + i(-10 - 6)$
4 Using the rules, state the final result.	$= 13 - 16i$

Argand diagrams

studyon

Units 3 & 4
AOS 2
Topic 2
Concept 2

Argand diagrams
Concept summary
Practice questions



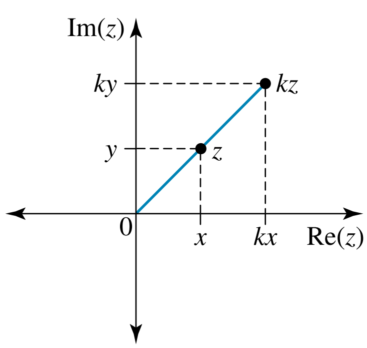
Complex numbers cannot be represented on a traditional Cartesian diagram because of their imaginary part. However, a similar plane was created by the Swiss mathematician Jean-Robert Argand (1768–1822). It is called an **Argand plane** or Argand diagram, and it allows complex numbers to be represented visually.

Because a complex number has two parts, a real part and an imaginary part, the horizontal axis is called the real axis and the vertical axis is called the imaginary axis. A complex number $z = x + yi$ is represented by the equivalent point (x, y) in a Cartesian coordinate system. Note that the imaginary axis is labelled 1, 2, 3 etc., not i, $2i$, $3i$ etc.

Geometrical representation of operations on complex numbers

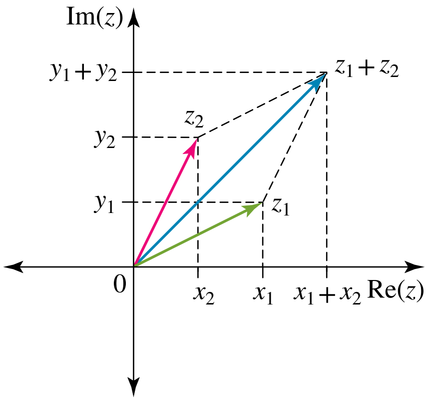
Scalar multiplication of complex numbers

If $z = x + yi$, then $kz = kx + kyi$, where $k \in R$. The diagram below shows the situation for $x > 0$, $y > 0$ and $k > 1$.



Addition of complex numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$. This can be represented by a directed line segment from the origin (the point $0 + 0i$) to the points z_1 and z_2. The addition of two complex numbers can be achieved using the same procedure as adding two vectors.

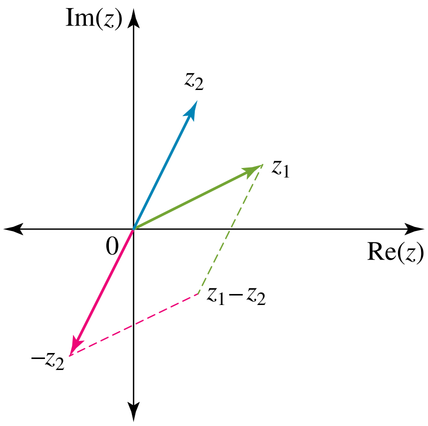


Subtraction of complex numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (x_1 + y_1 i) - (x_2 + y_2 i) \\ &= (x_1 - x_2) + (y_1 - y_2)i \end{aligned}$$

The subtraction of two complex numbers can be achieved using the same procedure as subtracting two vectors.



Multiplication of complex numbers

To multiply complex numbers in rectangular form, proceed as in conventional algebra and replace i^2 with –1 when it appears.

WORKED EXAMPLE 4

Given the complex numbers $u = 2 - 5i$ and $v = -3 + 2i$, find the complex numbers:

a uv **b** u^2.

THINK	WRITE
a 1 Substitute for u and v.	**a** $uv = (2 - 5i)(-3 + 2i)$
2 Expand the brackets using the distributive law.	$= -6 + 15i + 4i - 10i^2$
3 Simplify and replace i^2 by -1 and group the real and imaginary parts.	$= -6 + 10 + i(4 + 15)$
4 Simplify and state the final result.	$uv = 4 + 19i$
b 1 Substitute for u.	**b** $u^2 = (2 - 5i)^2$
2 Expand.	$= 4 - 20i + 25i^2$
3 Replace i^2 with -1.	$= 4 - 20i - 25$
4 Simplify and state the final result.	$u^2 = -21 - 20i$

Multiplication of complex numbers in general

In general, if $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c and $d \in R$, then:

$$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) \\ &= ac + bci + adi + bdi^2 \\ &= ac + (ad + bc)i - bd \\ &= ac - bd + (ad + bc)i \end{aligned}$$

WORKED EXAMPLE 5

Given the complex numbers $u = 2 - 5i$ and $v = 2 + 5i$, find:

a Re(uv) **b** Im(uv).

THINK	WRITE
1 Substitute for u and v.	$uv = (2 - 5i)(2 + 5i)$
2 Expand the brackets.	$= 4 - 10i + 10i - 25i^2$
3 Simplify and replace i^2 by -1.	$= 29$
a State the real part.	a Re(uv) = 29
b State the imaginary part.	b Im(uv) = 0

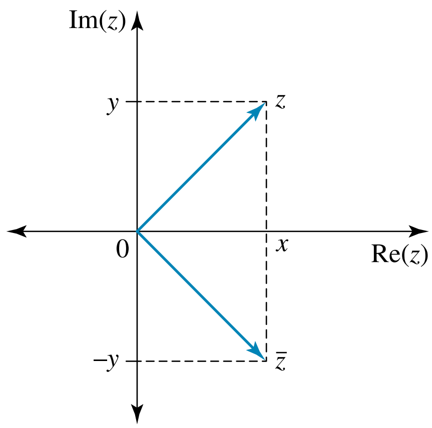
Complex conjugates

studyon

Units 3 & 4
AOS 2
Topic 2
Concept 4

Division of complex numbers and the complex conjugate
Concept summary
Practice questions

In Worked example 5, the complex numbers u and v have the property that the imaginary part of their products is zero. Such numbers are called complex conjugates of each other.



In general, if $z = x + yi$, the conjugate of z is denoted by $\bar{z}$ (read as z bar), and $\bar{z} = x - yi$. That is, the complex conjugate of a number is simply obtained by changing the sign of the imaginary part.

$$\begin{aligned} z\bar{z} &= (x + yi)(x - yi) \\ &= x^2 - xyi + xyi - y^2 i^2 \\ &= x^2 + y^2 \end{aligned}$$

so that Re($z\bar{z}$) = $x^2 + y^2$ and Im($z\bar{z}$) = 0.

From the diagram above right it can be seen that $\bar{z}$ is the reflection of the complex number z in the real axis.

Division of complex numbers

The conjugate is useful in division of complex numbers, because both the numerator and denominator can be multiplied by the conjugate of the denominator. Hence, the complex number can be replaced with a real number in the denominator. This process is similar to rationalising the denominator to remove surds.

WORKED EXAMPLE 6

Given the complex numbers $u = 2 - 5i$ and $v = -3 + 2i$, find the complex numbers:

a u^{-1} **b** $\dfrac{u}{v}$.

THINK	WRITE
a **1** Find the multiplicative inverse and substitute for u.	**a** $u^{-1} = \dfrac{1}{u}$
	$= \dfrac{1}{2 - 5i}$
2 Multiply both the numerator and the denominator by the conjugate of the denominator.	$= \dfrac{1}{2 - 5i} \times \dfrac{2 + 5i}{2 + 5i}$
3 Simplify the denominator.	$= \dfrac{2 + 5i}{4 - 25i^2}$
4 Replace i^2 with -1.	$= \dfrac{2 + 5i}{29}$
5 State the final answer in $x + yi$ form.	$= \dfrac{2}{29} + \dfrac{5}{29}i$
b **1** Substitute for u and v.	**b** $\dfrac{u}{v} = \dfrac{2 - 5i}{-3 + 2i}$
2 Multiply both the numerator and the denominator by the conjugate of the denominator.	$= \dfrac{2 - 5i}{-3 + 2i} \times \dfrac{-3 - 2i}{-3 - 2i}$
3 Expand the expression in both the numerator and the denominator.	$= \dfrac{-6 + 15i - 4i + 10i^2}{9 - 4i^2}$
4 Simplify and replace i^2 with -1.	$= \dfrac{-6 + 11i - 10}{9 + 4}$
5 State the final answer in $x + yi$ form.	$= -\dfrac{16}{13} + \dfrac{11}{13}i$

General division of complex numbers

In general, if $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c and $d \in R$, then:

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1}{z_2} \times \frac{\bar{z}_2}{\bar{z}_2} \\
&= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\
&= \frac{ac + bci - adi - bdi^2}{c^2 - d^2i^2} \\
&= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \\
&= \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i
\end{aligned}$$

Equality of complex numbers

Two complex numbers are equal if and only if their real parts and their imaginary parts are both equal. For example, if $5 + yi = x - 3i$, then from equating the real part, we find that $x = 5$, and from equating the imaginary part, we find that $y = -3$.

WORKED EXAMPLE 7

Find the values of x and y if $2x + 5iy - 3ix - 4y = 16 - 21i$.

THINK	WRITE
1 Group the real and imaginary parts.	$2x + 5iy - 3ix - 4y = 16 - 21i$ $2x - 4y + i(5y - 3x) = 16 - 21i$
2 Equate the real and imaginary components.	$2x - 4y = 16 \quad (1)$ $5y - 3x = -21 \quad (2)$
3 Solve the simultaneous equations by elimination.	$6x - 12y = 48 \quad 3 \times (1)$ $10y - 6x = -42 \quad 2 \times (2)$
4 Add the equations to eliminate x.	$3 \times (1) + 2 \times (2)$: $-2y = 6$
5 Solve for y.	$y = -3$
6 Substitute and solve for x.	$2x = 16 + 4y$ $2x = 16 - 12$ $2x = 4$ $x = 2$

Solving equations involving complex numbers

To solve an equation involving a complex number, rearrange the equation to find the unknown quantity, then use the same rules and strategies as when solving equations with real coefficients.

WORKED EXAMPLE 8

Find the complex number z if $\dfrac{3(z+2)}{z+2i} = 5 - 2i$.

THINK	WRITE
1 Multiply both sides by the expression in the denominator.	$\dfrac{3(z+2)}{z+2i} = 5 - 2i$ $3(z+2) = (5-2i)(z+2i)$
2 Expand the brackets on both sides of the equation.	$3z + 6 = 5z + 10i - 2iz - 4i^2$
3 Replace i^2 with -1.	$3z + 6 = 5z + 10i - 2iz + 4$
4 Collect all terms containing z on one side of the equation.	$2 - 10i = 2z - 2iz$
5 Isolate z by taking out the common factors.	$2(1 - 5i) = 2z(1 - i)$
6 Solve for z.	$z = \dfrac{1-5i}{1-i}$
7 Multiply both the numerator and the denominator by the conjugate of the denominator.	$z = \dfrac{1-5i}{1-i} \times \dfrac{1+i}{1+i}$

8 Expand the numerator and denominator. $z = \dfrac{1 + i - 5i - 5i^2}{1 - i^2}$

9 Replace i^2 with -1. $z = \dfrac{6 - 4i}{2}$

10 Express the complex number in $x + yi$ form. $z = 3 - 2i$

Powers of *i*

As $i^2 = -1$, it follows that

$$i^3 = i \times i^2 = -i,$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$

and $i^5 = i \times i^4 = i$.

A pattern can be seen for repetitions of the powers of i. Any even power of i will give ± 1, while any odd power of i will give $\pm i$.

WORKED EXAMPLE 9

Find $\text{Im}\left(\dfrac{26}{-3 + 2i} + i^{69}\right)$.

THINK / **WRITE**

1 Realise the denominator and group the power of i as multiples of i^4 using index laws.

$$\text{Im}\left(\frac{26}{-3 + 2i} + i^{69}\right) = \text{Im}\left(\frac{26}{-3 + 2i} \times \frac{-3 - 2i}{-3 - 2i} + i^{17\times4+1}\right)$$

2 Expand the expression in the denominator and use index laws on the power of i.

$$= \text{Im}\left(\frac{-26(3 + 2i)}{9 - 4i^2} + (i^4)^{17}i\right)$$

3 Simplify and replace i^2 with -1 and i^4 with 1.

$$= \text{Im}\left(\frac{-26(3 + 2i)}{13} + (1)^{17}i\right)$$

4 Simplify.

$$= \text{Im}(-2(3 + 2i) + i)$$
$$= \text{Im}(-6 - 4i + i)$$

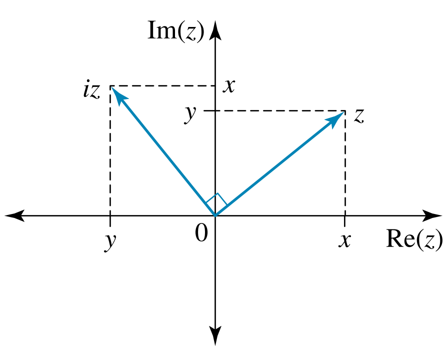
5 The imaginary part is the coefficient of the i term. State the final result.

$$= \text{Im}(-6 - 3i)$$
$$= -3$$

Multiplication by *i*

If $z = x + yi$, then iz is given by

$$\begin{aligned} iz &= i(x + yi) \\ &= ix + i^2y \\ &= -y + xi \end{aligned}$$



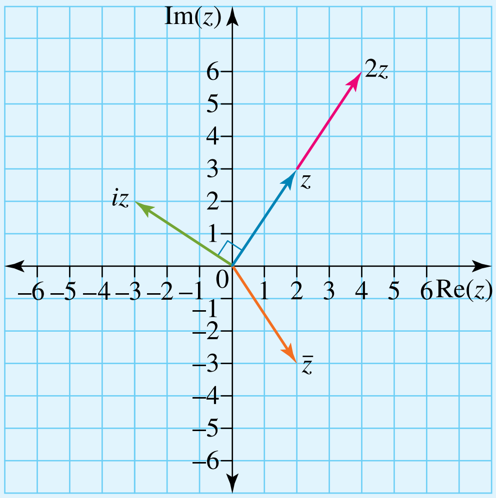
The complex number iz is a rotation of z by 90° anticlockwise.

WORKED EXAMPLE 10 Given the complex number $z = 2 + 3i$, represent the complex numbers z, $2z$, $\bar{z}$ and iz on one Argand diagram. Comment on their relative positions.

THINK

$z = 2 + 3i$
$2z = 4 + 6i$
$\bar{z} = 2 - 3i$
$iz = i(2 + 3i)$
$= 2i + 3i^2$
$= -3 + 2i$

WRITE/DRAW



The complex number $2z$ is twice the length of z, the complex conjugate $\bar{z}$ is the reflection of the complex number in the real axis, and the complex number iz is a rotation of 90° anticlockwise from z.

EXERCISE 3.2 Complex numbers in rectangular form

PRACTISE

1 **WE1** a Find $\sqrt{-64}$.
   b Solve the equation $z^2 + 64 = 0$ for z.

2 a Find $\sqrt{-18}$.
   b Solve the equation $z^2 + 18 = 0$ for z.

3 **WE2** Solve for z if $z^2 - 8z + 41 = 0$.

4 Solve for z if $(z - 4)(z + 1) + \frac{15}{2} = 0$.

5 **WE3** Given the complex numbers $u = 3 - i$ and $v = 4 - 3i$, find the complex numbers:
   a $u + v$ b $u - v$ c $3u - 2v$.

6 Solve for z if $2z + 3 - 4i = 7 + 6i$.

7 **WE4** Given the complex numbers $u = 3 - i$ and $v = 4 - 3i$, find the complex numbers:
   a uv b v^2.

8 Given the complex numbers $u = 1 + 3i$ and $v = 3 + 4i$, find the complex numbers:
   a $(u - v)^2$ b $(3u - 2v)^2$.

9 **WE5** Given the complex numbers $u = 3 - i$ and $v = 3 + i$, find:
   a $\text{Re}(uv)$ b $\text{Im}(uv)$.

10 If $v = -4 - 3i$ and $\text{Im}(uv) = 0$, find the complex number u.

11 WE6 Given the complex numbers $u = 3 - i$ and $v = 4 - 3i$, find the complex numbers:

a u^{-1}

b $\frac{u}{v}$.

12 Given the complex numbers $u = 1 + 3i$ and $v = 3 + 4i$, find the complex number $\frac{u+v}{u-v}$.

13 WE7 Find the values of x and y if $4x - 2iy + 3ix - 4y = -6 - i$.

14 Find the values of x and y if $(x + yi)(3 - 2i) = 6 - i$.

15 WE8 Find the complex number z if $\frac{z-i}{z+i} = 2 + i$.

16 Determine the complex number z if $\frac{5(z+2i)}{z-2} = 11 - 2i$.

17 WE9 Find $\text{Im}\left(\frac{25}{4-3i} + i^{77}\right)$.

18 Find $\text{Re}\left(\frac{10}{1+3i} + i^{96}\right)$.

19 WE10 Given the complex number $z = -2 - i$, represent the complex numbers z, $3z$, $\bar{z}$ and iz on one Argand diagram. Comment on their relative positions.

20 Given the complex numbers $u = 1 - 2i$ and $v = 2 + i$, represent the complex numbers $u + v$ and $u - v$ on one Argand diagram.

CONSOLIDATE

21 Simplify each of the following.

a $\frac{10\,i^3}{\sqrt{-25}}$

b $\frac{-7i^6}{\sqrt{-49}}$

c $\frac{\sqrt{-18}}{3i^9}$

d $\frac{-\sqrt{-75}}{i^{10}}$

e $\frac{1}{\sqrt{-8}}$

f $\frac{-\sqrt{-72}}{4i^6}$

22 If $z = 5 - 3i$, then find the following in $a + bi$ form.

a $\bar{z}$

b $z\,\bar{z}$

c z^{-1}

d z^2

e $(z - \bar{z})^2$

f $\frac{z+i}{z-3}$

23 Find the following.

a $\text{Im}(4(2 + 3i) + i^{13})$

b $\text{Re}(10(4 - 3i) + 2i^{18})$

c $\text{Re}(-5(3 + 2i)^2 - 4i^{28})$

d $\text{Im}(-3(7 + 4i)^2 - 5i^{15})$

e $\text{Im}\left(\frac{4}{2+3i} + i^{13}\right)$

f $\text{Re}\left(\frac{10}{4-3i} + 2i^{18}\right)$

24 Solve each of the following for z.

a $z^2 - 49 = 0$

b $z^2 + 49 = 0$

c $4z^2 + 9 = 0$

d $3z^2 + 25 = 0$

e $2z^2 + 81 = 0$

f $az^2 + b = 0$ if $ab > 0$

25 Find the roots of each of the following.

a $z^2 - 8z + 41 = 0$

b $z^2 - 4z + 9 = 0$

c $2z^2 - 8z + 11 = 0$

d $3z^2 - 2z + 1 = 0$

e $4z^2 - 6z + 3 = 0$

f $5z^2 + 4z + 2 = 0$

26 Solve for z given that:

a $(z + 5)(z - 1) + 10 = 0$

b $z = \frac{34}{10 - z}$

c $z + \frac{74}{z - 14} = 0$　　d $\frac{1}{z} + \frac{z + 16}{73} = 0.$

27 Given that $u = 2 - 5i$ and $v = 4 + 3i$, find each of the following, giving your answer in rectangular form.

a $3u - 2v$　b uv　c $\frac{u}{v}$　d $(v - u)^2$　e $\frac{u - 1}{v - i}$

28 Solve each of the following for z.

a $\frac{z + 3}{z - 3i} = 1 + 3i$　b $\frac{z + 2}{z - i} = 3 + i$

c $\frac{z - 3}{z + 2i} = 1 - i$　d $\frac{z - 4}{z - 2i} = 2 + i$

29 Find the values of x and y if:

a $x(1 - 2i) + y(3 + 5i) = -11i$　b $x(4 + 3i) + y(6 - 5i) = 38$

c $(x + yi)(3 + 5i) = 4$　d $(x + yi)(6 - 5i) = 2 - 3i.$

30 Solve each of the following for z.

a $z^2 + 4iz + 12 = 0$　b $z^2 - 6iz + 16 = 0$

c $z^2 - 3iz + 4 = 0$　d $z^2 + 5iz - 6 = 0$

MASTER

31 a Show that $(1 - 2i)^2 = -3 - 4i$ and hence solve $z^2 - 8z + 19 + 4i = 0.$

b Show that $(1 + 6i)^2 = -35 + 12i$ and hence solve $z^2 - 9z + 29 - 3i = 0.$

c Show that $(3 + 5i)^2 = -16 + 30i$ and hence solve $z^2 + (i - 7)z + 16 - 11i = 0.$

d Show that $(6 + 7i)^2 = -13 + 84i$ and hence solve $z^2 + (i - 8)z + 19 - 25i = 0.$

32 a If $u = 1 + i$, find:

i u^2　ii u^3　iii u^4.

b If $u = 3 - 4i$, find:

i u^2　ii u^3　iii u^4.

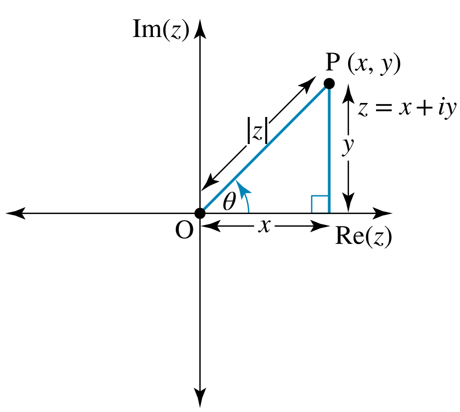
c If $u = a + bi$, find:

i u^2　ii u^3　iii u^4.

3.3 Complex numbers in polar form

Polar form

Modulus of a complex number



The form $z = x + yi$ is called the Cartesian form or rectangular form of a complex number. This is only one of several possible representations of a complex number.

Another way in which complex numbers can be represented is polar form. This form has two parts: the modulus and the argument.

To demonstrate this form, let us designate the point P as the complex number $z = x + yi$ with coordinates (x, y). The length, magnitude or modulus of the complex number is the distance from the origin, O (the point $0 + 0i$) to the point P. This distance is represented as $OP = |z| = |x + yi| = \sqrt{x^2 + y^2}$, using Pythagoras' theorem. It is also often given by $r = |z| = |x + yi| = \sqrt{x^2 + y^2}$. Note that this distance is always a positive real number.

Argument of a complex number

studyon

Units 3 & 4
AOS 2
Topic 2
Concept 5

Polar form (modulus and argument) of complex numbers
Concept summary
Practice questions

The angle θ that the line segment OP makes with the positive real axis is called the **argument of z** and is denoted by $\arg(z)$. The argument is also known as the phase angle. Usually this angle is given in radians, as multiples of π, although it can also be given in degrees and minutes. Angles measured anticlockwise from the real axis are positive angles, and angles measured clockwise from the real axis are negative angles.

One problem that arises when using angles is that the angle is not unique, because any integer multiple of 2π radians (or $360°$) can be added or subtracted to any angle to get the same result on the Argand plane. The principal value is defined to be the angle $\text{Arg}(z)$ where $-\pi < \text{Arg}(z) \leq \pi$. (The capital A is important here to distinguish the principal value from the general value.)

When determining the polar form of a complex number, we must carefully consider the quadrant in which the complex number lies to determine the correct angle and value of $\text{Arg}(z)$.

Recall that $r = \sqrt{x^2 + y^2}$.

Also recall that $x = r\cos(\theta)$ (1) and $y = r\sin(\theta)$ (2).

$$\frac{r\sin(\theta)}{r\cos(\theta)} = \frac{y}{x} \quad \frac{(2)}{(1)}$$

$$\tan(\theta) = \frac{y}{x}$$

Therefore, $\theta = \text{Arg}(z) = \tan^{-1}\left(\dfrac{y}{x}\right)$

This result only gives the correct angle in the first and fourth quadrants. It does not produce the correct angle in the second and third quadrants.

Expressing a complex number in terms of r and θ

To express a complex number in polar form:

$$z = r\cos(\theta) + ir\sin(\theta)$$
$$z = r(\cos(\theta) + i\sin(\theta))$$
$$z = r\,\text{cis}(\theta)$$

where 'cis' is just a mathematically commonly used shorthand for 'cos + i sin'.

Note: A complex number is a point; the modulus and argument are a means of locating the point.

Conversions

Rectangular form to polar form

Given a complex number in rectangular form, that is given the values of x and y, the values of r and θ must be found to convert the rectangular form to **polar form**, that is from $R \rightarrow P$ or $(x, y) \rightarrow [r, \theta]$.

WORKED EXAMPLE 11

Convert each of the following complex numbers to polar form.

a $1 + i$ b $-\sqrt{3} + i$ c $-4 - 4i$

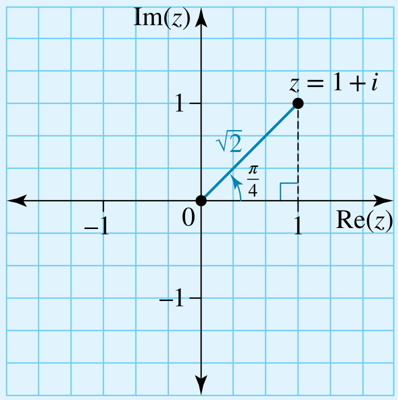
d $1 - \sqrt{3}i$ e -5 f $3i$

THINK | **WRITE/DRAW**

a **1** Draw the complex number on an Argand diagram.

a $z = 1 + i$

This complex number is in the first quadrant.



2 Identify the real and imaginary parts.

$x = \text{Re}(z) = 1$ and $y = \text{Im}(z) = 1$

3 Find the modulus.

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \end{aligned}$$

4 Find the argument.

$$\begin{aligned} \theta &= \text{Arg}(z) \\ &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}(1) \\ &= \frac{\pi}{4} \end{aligned}$$

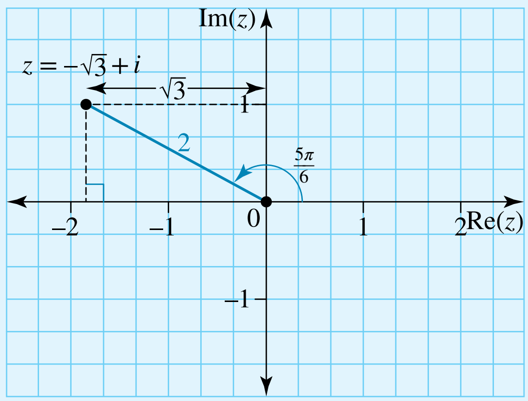
5 State the complex number in polar form.

$$\begin{aligned} z &= 1 + i \\ &= \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right) \end{aligned}$$

b **1** Draw the complex number on an Argand diagram.

b $z = -\sqrt{3} + i$

This complex number is in the second quadrant.



2 Identify the real and imaginary parts.

$x = \text{Re}(z) = -\sqrt{3}$ and $y = \text{Im}(z) = 1$

3 Find the modulus.

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-\sqrt{3})^2 + 1^2} \\ &= 2 \end{aligned}$$

4 Find the argument.

$$\begin{aligned}\theta &= \text{Arg}(z)\\ &= \pi + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\\ &= \pi - \frac{\pi}{6}\\ &= \frac{5\pi}{6}\end{aligned}$$

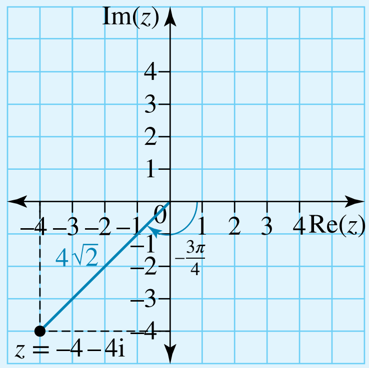
5 State the complex number in polar form.

$$z = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$$

c 1 Draw the complex number on an Argand diagram.

c $z = -4 - 4i$

This complex number is in the third quadrant.



2 Identify the real and imaginary parts.

$x = \text{Re}(z) = -4$ and $y = \text{Im}(z) = -4$

3 Find the modulus.

$$\begin{aligned}|z| &= \sqrt{x^2 + y^2}\\ &= \sqrt{(-4)^2 + (-4)^2}\\ &= 4\sqrt{2}\end{aligned}$$

4 Find the argument.

$$\begin{aligned}\theta &= \text{Arg}(z)\\ &= -\pi + \tan^{-1}(1)\\ &= -\pi + \frac{\pi}{4}\\ &= -\frac{3\pi}{4}\end{aligned}$$

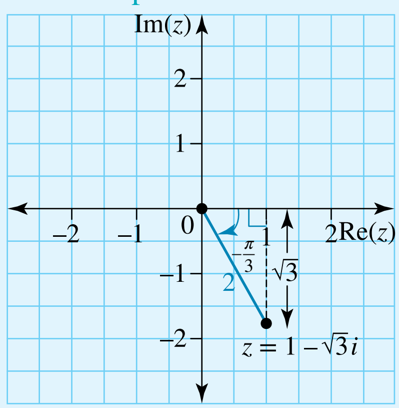
5 State the complex number in polar form.

$$z = 4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$$

d 1 Draw the complex number on an Argand diagram.

d $z = 1 - \sqrt{3}i$

This complex number is in the fourth quadrant.



2 Identify the real and imaginary parts.

$x = \text{Re}(z) = 1$ and $y = \text{Im}(z) = -\sqrt{3}$

3 Find the modulus.

$$\begin{aligned}|z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-\sqrt{3})^2} \\ &= 2\end{aligned}$$

4 Find the argument.

$$\begin{aligned}\theta &= \text{Arg}(z) \\ &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}(-\sqrt{3}) \\ &= -\frac{\pi}{3}\end{aligned}$$

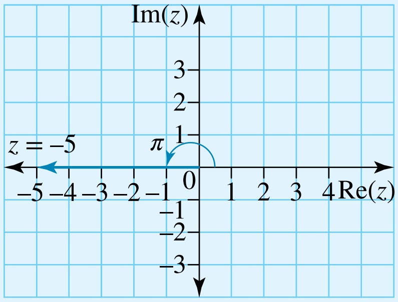
5 State the complex number in polar form.

$$\begin{aligned}z &= 1 - \sqrt{3}i \\ &= 2\,\text{cis}\left(-\frac{\pi}{3}\right)\end{aligned}$$

e 1 Draw the complex number on an Argand diagram.

e $z = -5$

This complex number is actually a real number and lies on the real axis.



2 Identify the real and imaginary parts.

$x = \text{Re}(z) = -5$ and $y = \text{Im}(z) = 0$

3 Find the modulus.

$$\begin{aligned}|z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-5)^2 + 0^2} \\ &= 5\end{aligned}$$

4 Find the argument.

$$\begin{aligned}\theta &= \text{Arg}(z) \\ &= \pi\end{aligned}$$

Note that $\theta = \text{Arg}(z) = -\pi$ is not correct, since $-\pi < \text{Arg}(z) \leq \pi$.

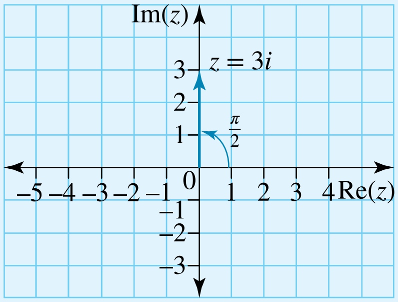
5 State the complex number in polar form.

$$\begin{aligned}z &= -5 \\ &= 5\,\text{cis}(\pi)\end{aligned}$$

f 1 Draw the complex number on an Argand diagram.

f $z = 3i$

This complex number lies on the imaginary axis.



2 Identify the real and imaginary parts. $x = \text{Re}(z) = 0$ and $y = \text{Im}(z) = 3$

3 Find the modulus.
$$|z| = \sqrt{x^2 + y^2} = \sqrt{0^2 + 3^2} = 3$$

4 Find the argument.
$$\theta = \frac{\pi}{2}$$

5 State the complex number in polar form.
$$z = 3i = 3\,\text{cis}\left(\frac{\pi}{2}\right)$$

Polar form to rectangular form

Now consider converting in the other direction: when given a complex number in polar form, that is using the values of r and θ, determine the values of x and y. To convert the polar form of a complex number to rectangular form, that is from $P \rightarrow R$ or $[r, \theta] \rightarrow (x, y)$, we expand the number using

$$r\,\text{cis}(\theta) = r\cos(\theta) + ir\sin(\theta).$$

WORKED EXAMPLE 12

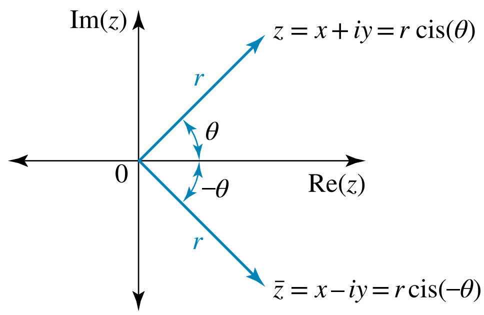
a **Convert $8\,\text{cis}\left(-\frac{\pi}{6}\right)$ into rectangular form.**

b **Convert $16\,\text{cis}\left(\frac{2\pi}{3}\right)$ into rectangular form.**

THINK	WRITE
a 1 Expand.	**a** $8\,\text{cis}\left(-\frac{\pi}{6}\right) = 8\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$
2 Use trigonometric results for functions of negative angles.	$\cos(-\theta) = \cos(\theta)$ $\sin(-\theta) = -\sin(\theta)$ $8\,\text{cis}\left(-\frac{\pi}{6}\right) = 8\left(\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$
3 Substitute for the exact trigonometric values. Note that the complex number is in the fourth quadrant.	$= 8\left(\frac{\sqrt{3}}{2} - i \times \frac{1}{2}\right)$
4 Simplify and write in $x + yi$ form.	$= 4\sqrt{3} - 4i$
b 1 Expand.	**b** $16\,\text{cis}\left(\frac{2\pi}{3}\right) = 16\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$
2 Substitute for the exact trigonometric values. Note that the complex number is in the second quadrant.	$= 16\left(-\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$
3 Simplify and write in $x + yi$ form.	$= -8 + 8\sqrt{3}\,i$

Conjugates in polar form

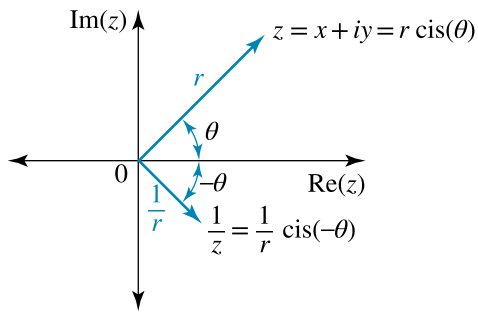
If $z = x + iy = r\,\text{cis}(\theta)$, then the conjugate of z is given by $\bar{z} = x - iy = r\,\text{cis}(-\theta)$. Furthermore, $\bar{z}z = |z|^2 = x^2 + y^2$.



Multiplicative inverses in polar form

If $z = x + iy = r\,\text{cis}(\theta)$, then the multiplicative inverse or reciprocal of z is given by

$$\begin{aligned} z^{-1} &= \frac{1}{z} \\ &= \frac{1}{x + yi} \\ &= \frac{x - yi}{x^2 + y^2} \\ &= \frac{1}{r}\text{cis}(-\theta) \end{aligned}$$



WORKED EXAMPLE 13 **If $u = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$, find $\bar{u}^{-1}$ giving your answer in rectangular form.**

THINK	WRITE
1 Use the conjugate rule.	$u = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$ $\bar{u} = 2\,\text{cis}\left(\frac{\pi}{6}\right)$
2 Find the multiplicative inverse.	$\bar{u}^{-1} = \frac{1}{\bar{u}}$ $= \frac{1}{2\,\text{cis}\left(\frac{\pi}{6}\right)}$
3 Use the results.	$= \frac{1}{2}\text{cis}\left(-\frac{\pi}{6}\right)$
4 Expand and use the trigonometric results for functions of negative angles.	$= \frac{1}{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ $= \frac{1}{2}\left(\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$
5 Substitute for the trigonometric values.	$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$
6 Simplify and write in $x + yi$ form.	$= \frac{1}{4}(\sqrt{3} - i)$ $= \frac{\sqrt{3}}{4} - \frac{1}{4}i$

Operations in polar form

Addition and subtraction

Polar form is not convenient for addition and subtraction. To add or subtract complex numbers that are given in polar form, they must be converted to rectangular form, and the real and imaginary parts added and subtracted separately.

Multiplication and division

studyon

Units 3 & 4

AOS 2

Topic 2

Concept 6

Multiplication and division in polar form

Concept summary

Practice questions

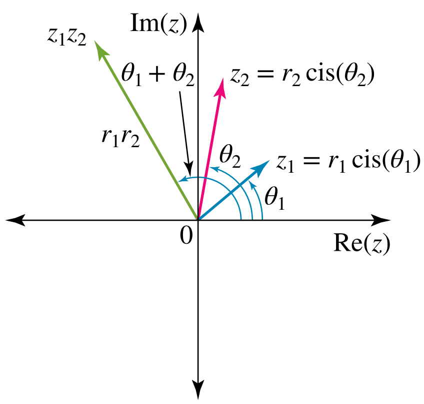
Two complex numbers can be multiplied together if they are both given in rectangular form. If they are both given in polar form, consider $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$. The rules for multiplication and division in polar form are given by $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$ Therefore, to multiply two complex numbers in polar form, multiply the moduli and add the arguments. To divide two complex numbers in polar form, divide the moduli and subtract the arguments. The proof of these results is as follows.

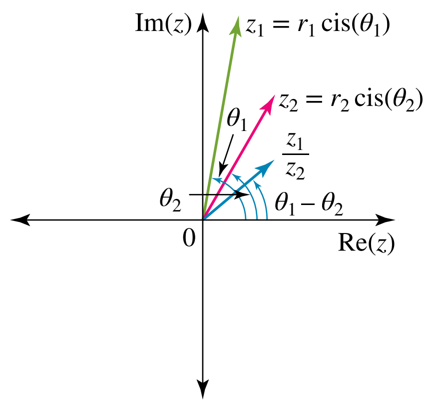
$$\begin{aligned} z_1 z_2 &= (r_1 \operatorname{cis}(\theta_1))(r_2 \operatorname{cis}(\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2 ((\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \sin(\theta_2)\cos(\theta_1))) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \text{ by compound-angle formulas} \\ &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= r_1 \operatorname{cis}(\theta_1) \times \frac{1}{r_2 \operatorname{cis}(\theta_2)} \\ &= \frac{r_1}{r_2} \operatorname{cis}(\theta_1) \operatorname{cis}(-\theta_2) \\ &= \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \end{aligned}$$

The diagrams below demonstrate this geometrically.





Note that if two complex numbers are given with one in polar form and one in rectangular form, they cannot be multiplied or divided until they are both in the same form.

WORKED EXAMPLE 14

If $u = 8\operatorname{cis}\left(-\frac{\pi}{4}\right)$ and $v = \frac{1}{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$, find each of the following, giving your answers in rectangular form.

a uv

b $\frac{u}{v}$

THINK	WRITE
a 1 Substitute for u and v.	a $uv = 8\operatorname{cis}\left(-\frac{\pi}{4}\right) \times \frac{1}{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$
2 Multiply the moduli and add the arguments.	$= 8 \times \frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{4} + \frac{3\pi}{4}\right)$
3 Simplify.	$= 4\operatorname{cis}\left(\frac{\pi}{2}\right)$
4 Express in $x + yi$ form.	$= 4i$
b 1 Substitute for u and v.	b $\frac{u}{v} = \dfrac{8\operatorname{cis}\left(-\frac{\pi}{4}\right)}{\frac{1}{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}$
2 Divide the moduli and subtract the arguments.	$= 16\operatorname{cis}\left(-\frac{\pi}{4} - \frac{3\pi}{4}\right)$
3 Simplify and expand the cis term.	$= 16\operatorname{cis}(-\pi)$ $= 16(\cos(-\pi) + i\sin(-\pi))$ $= 16(-1 + 0i)$
4 Express the final result in $x + yi$ form.	$= -16$

Powers in polar form

studyon

Units 3 & 4
AOS 2
Topic 2
Concept 7

De Moivre's theorem
Concept summary
Practice questions

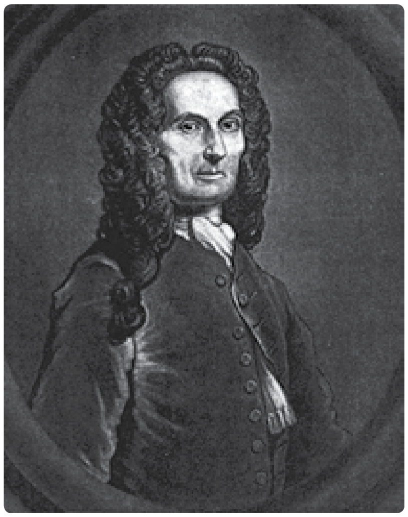
If $z = r\operatorname{cis}(\theta)$, then

$$z^2 = r\operatorname{cis}(\theta) \times r\operatorname{cis}(\theta)$$
$$= r^2\operatorname{cis}(2\theta)$$
$$z^3 = r^3\operatorname{cis}(3\theta) \ldots$$

In general,

$$z^n = r^n\operatorname{cis}(n\theta) \quad \text{for } n \in Z.$$

This result can be proved by the process of mathematical induction and is known as **de Moivre's theorem**.



Abraham de Moivre (1667–1754) was born in France, but he lived in England for most of his life. He was friends with several notable mathematicians of the time, including Isaac Newton and Edmond Halley.

WORKED EXAMPLE 15

If $u = -\sqrt{3} + i$, find:

a $\text{Arg}(u^{12})$ **b** u^{12}**, giving your answer in rectangular form.**

THINK	WRITE
a 1 Convert to polar form (see Worked example 11b).	**a** $u = -\sqrt{3} + i$ $= 2\,\text{cis}\left(\frac{5\pi}{6}\right)$
2 Use De Moivre's theorem.	$u^{12} = 2^{12}\,\text{cis}\left(12 \times \frac{5\pi}{6}\right)$
3 Simplify.	$u^{12} = 4096\,\text{cis}(10\pi)$
4 $-\pi < \text{Arg}(z) \leq \pi$ and is unique, but $\arg(z)$ is not unique.	$\arg(u^{12}) = 10\pi$ but $\text{Arg}(u^{12}) \neq 10\pi$
5 Add or subtract an appropriate multiple of 2π to the angle.	$\text{Arg}(u^{12}) = 10\pi - 10\pi$ $= 0$
6 State the answer.	$\text{Arg}(u^{12}) = 0$
b 1 Expand the cis term.	**b** $u^{12} = 4096\,\text{cis}(0)$ $= 4096(\cos(0) + i\sin(0))$ $= 4096(1 + 0i)$
2 State the answer.	$u^{12} = 4096$

Using trigonometric compound-angle formulas

Recall the trigonometric compound-angle formulas from Topic 2:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$
$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

These identities can be used in problems involving complex numbers to obtain or check certain required results.

WORKED EXAMPLE 16

a **Show that** $\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$.

b **Given** $u = 1 + (2 - \sqrt{3})i$**, find** iu **and hence find** $\text{Arg}(\sqrt{3} - 2 + i)$.

THINK	WRITE
a 1 Rewrite the argument as a sum or difference of fractions.	**a** $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$, or in degrees, 45° – 30° = 15°. $\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

2 State and use an appropriate identity.

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

Let $A = \frac{\pi}{4}$ and $B = \frac{\pi}{6}$.

$$\tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$$

3 Simplify and use exact values.

Substitute $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$:

$$\tan\left(\frac{\pi}{12}\right) = \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

4 Form common denominators in both the numerator and denominator, and cancel the factors.

$$\tan\left(\frac{\pi}{12}\right) = \frac{\frac{3 - \sqrt{3}}{3}}{\frac{3 + \sqrt{3}}{3}}$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

5 To rationalise, multiply both the numerator and denominator by the conjugate surd in the denominator.

$$\tan\left(\frac{\pi}{12}\right) = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \times \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$

6 Expand and simplify.

$$\tan\left(\frac{\pi}{12}\right) = \frac{9 - 6\sqrt{3} + 3}{9 - 3}$$

$$= \frac{12 - 6\sqrt{3}}{6}$$

$$= \frac{6(2 - \sqrt{3})}{6}$$

7 Simplify and state the final answer.

$$\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$$

b **1** State the complex number and its argument, as it is in the first quadrant.

b $u = 1 + (2 - \sqrt{3})i$

$$\text{Arg}(u) = \tan^{-1}(2 - \sqrt{3}) = \frac{\pi}{12}$$

2 Find the complex number iu, which is in the second quadrant. The complex number iu is a rotation of u by 90° anticlockwise.

$$iu = i + (2 - \sqrt{3})i^2 = \sqrt{3} - 2 + i$$

$$\text{Arg}(iu) = \frac{\pi}{12} + \frac{\pi}{2}$$

3 State the final result.

$$\text{Arg}(\sqrt{3} - 2 + i) = \frac{7\pi}{12}$$

Using de Moivre's theorem

De Moivre's theorem can be used to find values of powers of complex numbers of a certain form.

WORKED EXAMPLE 17 Find all values of n such that $(-\sqrt{3}+i)^n + (-\sqrt{3}-i)^n = 0$.

THINK	WRITE
1 Express the complex number $-\sqrt{3}+i$ in polar form (see Worked example 11b).	$-\sqrt{3}+i = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$
2 The complex number $-\sqrt{3}-i$ is the conjugate. Express $-\sqrt{3}-i$ in polar form.	$-\sqrt{3}-i = 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$
3 Express the equation in polar form.	$(-\sqrt{3}+i)^n + (-\sqrt{3}-i)^n = 0$ $\left(2\operatorname{cis}\left(\frac{5\pi}{6}\right)\right)^n + \left(2\operatorname{cis}\left(-\frac{5\pi}{6}\right)\right)^n = 0$
4 Use de Moivre's theorem.	$2^n\operatorname{cis}\left(\frac{5\pi n}{6}\right) + 2^n\operatorname{cis}\left(-\frac{5\pi n}{6}\right) = 0$
5 Take out the common factor and expand $\operatorname{cis}(\theta)$.	$2^n\left(\operatorname{cis}\left(\frac{5\pi n}{6}\right) + \operatorname{cis}\left(-\frac{5\pi n}{6}\right)\right) = 0$ $\cos\left(\frac{5\pi n}{6}\right) + i\sin\left(\frac{5\pi n}{6}\right) + \cos\left(-\frac{5\pi n}{6}\right) + i\sin\left(-\frac{5\pi n}{6}\right) = 0$
6 Use the trigonometric results for functions of negative angles and simplify.	Since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, $2\cos\left(\frac{5\pi n}{6}\right) = 0$
7 Use the formula for the general solutions of trigonometric equations.	$\cos\left(\frac{5\pi n}{6}\right) = 0$ $\frac{5\pi n}{6} = \frac{(2k+1)\pi}{2}$ where $k \in Z$
8 Solve for n and state the final answer.	$n = \frac{3(2k+1)}{5}$ where $k \in Z$

EXERCISE 3.3 Complex numbers in polar form

PRACTISE

1 **WE11** Convert each of the following complex numbers to polar form.
- a $1+\sqrt{3}i$
- b $-1+i$
- c $-2-2\sqrt{3}i$
- d $\sqrt{3}-i$
- e 4
- f $-2i$

2 Convert each of the following complex numbers to polar form.
- a $\sqrt{3}+i$
- b $-1+\sqrt{3}i$
- c $-\sqrt{3}-i$
- d $2-2i$
- e -7
- f $5i$

3 **WE12** a Convert $4\operatorname{cis}\left(-\frac{\pi}{3}\right)$ into rectangular form.
- b Convert $8\operatorname{cis}\left(-\frac{\pi}{2}\right)$ into rectangular form.

4 a Convert $6\sqrt{2}\,\text{cis}(-135°)$ into $x + yi$ form.

b Convert $5\,\text{cis}(126°52')$ into $a + bi$ form.

5 WE13 If $u = 6\,\text{cis}\left(-\dfrac{\pi}{3}\right)$, find $\bar{u}^{-1}$, giving your answer in rectangular form.

6 If $u = \dfrac{\sqrt{2}}{4}\text{cis}\left(\dfrac{3\pi}{4}\right)$, find $\dfrac{1}{u}$, giving your answer in rectangular form.

7 WE14 If $u = 4\sqrt{2}\,\text{cis}\left(-\dfrac{3\pi}{4}\right)$ and $v = \sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right)$, find each of the following, giving your answers in rectangular form.

a uv b $\dfrac{u}{v}$

8 If $u = 4\,\text{cis}\left(\dfrac{\pi}{3}\right)$ and $v = \dfrac{1}{2}\text{cis}\left(-\dfrac{2\pi}{3}\right)$ find each of the following, giving your answers in rectangular form.

a uv b $\dfrac{u}{v}$

9 WE15 If $u = -1 - i$, find:

a $\text{Arg}(u^{10})$ b u^{10}, giving your answer in rectangular form.

10 Simplify $\dfrac{(-1 + i)^6}{(\sqrt{3} - i)^4}$, giving your answer in rectangular form.

11 WE16 a Show that $\tan\left(\dfrac{5\pi}{12}\right) = \sqrt{3} + 2$.

b Given $u = 1 + (\sqrt{3} + 2)i$, find iu and hence find $\text{Arg}(-\sqrt{3} - 2 + i)$.

12 Show that $\tan\left(\dfrac{11\pi}{12}\right) = \sqrt{3} - 2$ and hence find $\text{Arg}(1 + (\sqrt{3} - 2)i)$.

13 WE17 Find all values of n such that $(1 + \sqrt{3}\,i)^n - (1 - \sqrt{3}\,i)^n = 0$.

14 Find all values of n such that $(1 + \sqrt{3}\,i)^n + (1 - \sqrt{3}\,i)^n = 0$.

CONSOLIDATE

15 Express each of the following in polar form, giving angles in degrees and minutes.

a $3 + 4i$ b $7 - 24i$ c $-5 + 12i$ d $-4 - 4i$

16 If $u = 6\,\text{cis}(12°)$ and $v = 3\,\text{cis}(23°)$, find each of the following in polar form, giving angles in degrees.

a uv b $\dfrac{v}{u}$ c u^2

d v^3 e u^5v^4 f $\dfrac{v^6}{u^3}$

17 Given $u = 3 + 2i$ and $v = 7\sqrt{2}\,\text{cis}\left(\dfrac{3\pi}{4}\right)$, find each of the following, expressing your answers in exact rectangular form.

a uv b $2u - 3v$ c $\dfrac{v}{u}$ d v^2

18 a If $z = 2 + 2i$, find each of the following.

i z^8 ii $\text{Arg}(z^8)$

b If $z = -3\sqrt{3} + 3i$, find each of the following.

i z^6 ii $\text{Arg}(z^6)$

c If $z = -\dfrac{5}{2} - \dfrac{5\sqrt{3}}{2}i$, find each of the following.

i z^9 ii $\text{Arg}(z^9)$

d If $z = 2\sqrt{3} - 2i$, find each of the following.

i z^7 ii $\text{Arg}(z^7)$

19 Let $u = \frac{1}{2}(\sqrt{3} - i)$.

a Find $\bar{u}$, $\frac{1}{u}$ and u^6, giving all answers in rectangular form.

b Find $\text{Arg}(\bar{u})$, $\text{Arg}\left(\frac{1}{u}\right)$ and $\text{Arg}(u^6)$.

c Is $\text{Arg}(\bar{u})$ equal to $-\text{Arg}(u)$?

d Is $\text{Arg}\left(\frac{1}{u}\right)$ equal to $-\text{Arg}(u)$?

e Is $\text{Arg}(u^6)$ equal to $6\,\text{Arg}(u)$?

20 a Let $u = -1 + \sqrt{3}i$ and $v = -2 - 2i$.

i Find $\text{Arg}(u)$.

ii Find $\text{Arg}(v)$.

iii Find $\text{Arg}(uv)$.

iv Find $\text{Arg}\left(\frac{u}{v}\right)$.

v Is $\text{Arg}(uv)$ equal to $\text{Arg}(u) + \text{Arg}(v)$?

vi Is $\text{Arg}\left(\frac{u}{v}\right)$ equal to $\text{Arg}(u) - \text{Arg}(v)$?

b Let $u = -\sqrt{3} + i$ and $v = -3 + 3i$.

i Find $\text{Arg}(u)$.

ii Find $\text{Arg}(v)$.

iii Find $\text{Arg}(uv)$.

iv Find $\text{Arg}\left(\frac{u}{v}\right)$.

v Is $\text{Arg}(uv)$ equal to $\text{Arg}(u) + \text{Arg}(v)$?

vi Is $\text{Arg}\left(\frac{u}{v}\right)$ equal to $\text{Arg}(u) - \text{Arg}(v)$?

21 a Let $u = \frac{1}{4}(\sqrt{3} - i)$ and $v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$.

i Find uv, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find uv, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\sin\left(\frac{\pi}{12}\right)$.

iv Using the formula $\sin(x - y)$, verify your exact value for $\sin\left(\frac{\pi}{12}\right)$.

b Let $u = \sqrt{2}(1 - i)$ and $v = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$.

i Find uv, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find uv, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\sin\left(\frac{5\pi}{12}\right)$.

iv Using the formula $\sin(x - y)$, verify your exact value for $\sin\left(\frac{5\pi}{12}\right)$.

22 a Let $u = -4 - 4\sqrt{3}\,i$ and $v = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$.

i Find $\frac{u}{v}$, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find $\frac{u}{v}$, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\cos\left(\frac{\pi}{12}\right)$.

iv Using the formula $\cos(x - y)$, verify your exact value for $\cos\left(\frac{\pi}{12}\right)$.

b Let $u = -1 - \sqrt{3}i$ and $v = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$.

i Find $\frac{u}{v}$, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find $\frac{u}{v}$, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\cos\left(\frac{7\pi}{12}\right)$.

iv Using the formula $\cos(x - y)$, verify your exact value for $\cos\left(\frac{7\pi}{12}\right)$.

23 a i Show that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$.

ii Let $u = 1 + (\sqrt{2} - 1)i$ and hence find $\text{Arg}(u)$.

iii Find iu and hence find $\text{Arg}((1 - \sqrt{2}) + i)$.

b i Show that $\tan\left(\frac{7\pi}{12}\right) = -(\sqrt{3} + 2)$.

ii Hence, find $\text{Arg}(-1 + (\sqrt{3} + 2)i)$.

iii Hence, find $\text{Arg}(1 - (\sqrt{3} + 2)i)$.

iv Hence, find $\text{Arg}((\sqrt{3} + 2) + i)$.

24 Find all values of n such that:

a $(1 + i)^n + (1 - i)^n = 0$

b $(1 + i)^n - (1 - i)^n = 0$

c $(\sqrt{3} + i)^n - (\sqrt{3} - i)^n = 0$

d $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = 0$.

MASTER

25 If $z = \text{cis}(\theta)$, show that:

a $|z + 1| = 2\cos\left(\frac{\theta}{2}\right)$

b $\text{Arg}(1 + z) = \frac{\theta}{2}$

c $\frac{1 + z}{1 - z} = i\cot\left(\frac{\theta}{2}\right)$.

26 Use de Moivre's theorem to show that:

a i $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

ii $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

b i $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$

ii $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$.

3.4 Polynomial equations

Quadratic equations

Recall the quadratic equation $az^2 + bz + c = 0$. If the coefficients a, b and c are all real, then the roots depend upon the discriminant, $\Delta = b^2 - 4ac$.

If $\Delta > 0$, then there are two distinct real roots.

If $\Delta = 0$, then there is one real root.

If $\Delta < 0$, then there is one pair of complex conjugate roots.

studyon

Units 3 & 4

AOS 2

Topic 3

Concept 3

Factorisation of polynomials over C

Concept summary

Practice questions

Relationship between the roots and coefficients

Given a quadratic equation with real coefficients, if the discriminant is negative, then the roots occur in complex conjugate pairs. A relationship can be formed between the roots and the coefficients.

Given a quadratic $az^2 + bz + c = 0$, if $a \neq 0$, then

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0.$$

Let the roots be α and β, so the factors are

$$(z - \alpha)(z - \beta).$$

Expanding gives

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0 \text{ or}$$
$$z^2 - (\text{sum of the roots}) + \text{product of the roots} = 0$$

so that

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

This provides a relationship between the roots and coefficients.

Rather than solving a quadratic equation, consider the reverse problem of forming a quadratic equation with real coefficients, given one of the roots.

WORKED EXAMPLE 18 **Find the equation of the quadratic $P(z)$ with real coefficients, given that $P(-2 + 5i) = 0$.**

THINK	WRITE
1 State one of the given roots.	Let $\alpha = -2 + 5i$.
2 The conjugate is also a root.	Let $\beta = \overline{\alpha} = -2 - 5i$
3 Find the sum of the roots.	$\alpha + \beta = -4$
4 Find the product of the roots.	$\alpha\beta = 4 - 25i^2$ $= 29$
5 State the linear factors and the quadratic equation.	$(z + 2 + 5i)(z + 2 - 5i) = z^2 + 4z + 29$

Solving cubic equations

Cubic equations with real coefficients

A cubic polynomial equation of the form $az^3 + bz^2 + cz + d = 0$ with $z \in C$, but with all the coefficients real, will have three linear factors. These may be repeated, but the cubic must have at least one real factor. When solving $az^3 + bz^2 + cz + d = 0$, the roots can be three real roots, not necessarily all distinct, or they can be one real root and one pair of complex conjugate roots.

WORKED EXAMPLE 19 **Find the roots of $z^3 + 2z^2 + 21z - 58 = 0$.**

THINK	WRITE
1 Use trial and error to find the one real root.	$P(z) = z^3 + 2z^2 + 21z - 58 = 0$ $P(1) = 1 + 2 + 21 - 58 \neq 0$ $P(2) = 8 + 8 + 42 - 58 = 0$
2 Use the factor theorem.	Therefore, $(z - 2)$ is a factor.

THINK	WRITE
3 We do not need to do long division. We can do short division to find the other quadratic factor.	$P(z) = z^3 + 2z^2 + 21z - 58 = 0$ $P(z) = (z - 2)(z^2 + 4z + 29) = 0$
4 State the linear factors (see Worked example 18).	$(z - 2)(z + 2 + 5i)(z + 2 - 5i) = 0$
5 Apply the Null Factor Theorem and state all the roots and their nature.	The roots are one real root and one pair of complex conjugates: $z = 2$ and $z = -2 \pm 5i$.

The conjugate root theorem

The preceding results are true not only for quadratic and cubic equations, but for any nth degree polynomial. In general, provided that all the coefficients of the polynomial are real, if the roots are complex, they must occur in conjugate pairs.

Note that if one of the coefficients is a complex number, then the roots do not occur in conjugate pairs.

Rather than formulating a problem such as solving a cubic equation, consider the reverse problem: determine some of the coefficients of a cubic equation with real coefficients, given one of the roots.

WORKED EXAMPLE 20

If $P(z) = z^3 + bz^2 + cz - 75 = 0$ where b and c are real, and $P(-4 + 3i) = 0$, find the values of b and c, and state all the roots of $P(z) = 0$.

THINK	WRITE
1 Apply the conjugate root theorem.	Let $\alpha = -4 + 3i$ and $\beta = -4 - 3i$.
2 Find the sum of the roots.	$\alpha + \beta = -8$
3 Find the product of the roots.	$\alpha\beta = 16 - 9i^2$ $= 25$
4 Find the quadratic factor.	$z^2 + 8z + 25$
5 Use short division.	$P(z) = z^3 + bz^2 + cz - 75 = 0$ $P(z) = (z^2 + 8z + 25)(z - 3) = 0$
6 Expand the brackets.	$P(z) = z^3 + 5z^2 + z - 75 = 0$
7 Equate coefficients.	From the z^2: $b = 5$ and from the coefficient of z: $c = 25 - 8 \times 3 = 1$.
8 State all the roots and their nature.	The roots are one real root and one pair of complex conjugates: $z = 3$ and $z = -4 \pm 3i$.

Cubic equations with complex coefficients

We can use the grouping technique to solve certain types of cubic equations with complex coefficients.

WORKED EXAMPLE 21

Solve for z if $z^3 + iz^2 + 5z + 5i = 0$.

THINK	WRITE
1 This cubic can be solved by grouping terms together.	$z^3 + iz^2 + 5z + 5i = 0$ $z^2(z + i) + 5(z + i) = 0$

2 Factorise. $(z^2+5)(z+i)=0$

3 Express the quadratic factor as the difference of two squares using $i^2=-1$. $(z^2-5i^2)(z+i)=0$

4 State the linear factors. $(z+\sqrt{5}i)(z-\sqrt{5}i)(z+i)=0$

5 Apply the Null Factor Theorem and state all the roots. $z=\pm\sqrt{5}i$ and $z=-i$.

General cubic equations with complex coefficients

If one of the roots of a cubic equation is given, the remaining roots can be determined. Note that if one of the coefficients in the cubic equation is a complex number, the roots do not all occur in conjugate pairs.

WORKED EXAMPLE 22

Show that $z=5-2i$ is a root of the equation $z^3+(-5+2i)z^2+4z+8i-20=0$, and hence find all the roots.

THINK	WRITE
1 Substitute $5-2i$ for z.	$P(5-2i)=(5-2i)^3+(-5+2i)(5-2i)^2+4(5-2i)+8i-20$
2 Simplify.	$P(5-2i)=(5-2i)^3-(5-2i)^3+20-8i+8i-20=0$ Therefore, $z=5-2i$ is a root of the cubic equation.
3 Since $z=5-2i$ is a root, $(z-5+2i)$ is a factor. Use short division.	$(z-5+2i)(z^2+4)=0$
4 Express as the difference of two squares using $i^2=-1$.	$(z-5+2i)(z^2-4i^2)=0$
5 State the linear factors.	$(z-5+2i)(z+2i)(z-2i)=0$
6 Apply the Null Factor Theorem and state all the roots.	$z=5-2i$ and $z=\pm 2i$.

The Fundamental Theorem of Algebra

This theorem states that an nth degree polynomial will always have exactly n roots, provided that multiple roots are counted accordingly.

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\ldots+a_1z+a_0=0$$
$$P(z)=a_n(z-z_1)(z-z_2)\ldots(z-z_n)=0$$

Note that the z_i are not necessarily all unique.

studyon
Units 3 & 4
AOS 2
Topic 3
Concept 4
Solving equations over *C*
Concept summary
Practice questions

Quartic equations with real coefficients

A quartic of the form $P(z)=az^4+bz^3+cz^2+dz+e$ with all real coefficients can have four linear factors. The roots can be either all real roots; two real roots and one pair of complex conjugate roots; or two pairs of complex conjugate roots.

WORKED EXAMPLE 23 Solve for z if $z^4 + 3z^2 - 28 = 0$.

THINK	WRITE
1 Use a suitable substitution to reduce the quartic equation to a quadratic equation.	Let $u = z^2$, then $u^2 = z^4$. $z^4 + 3z^2 - 28 = 0$ $u^2 + 3u - 28 = 0$
2 Factorise the expression.	$(u + 7)(u - 4) = 0$
3 Substitute z^2 for u.	$(z^2 + 7)(z^2 - 4) = 0$
4 Express as the difference of two squares using $i^2 = -1$.	$(z^2 - 7i^2)(z^2 - 4) = 0$
5 State the linear factors.	$(z + \sqrt{7}i)(z - \sqrt{7}i)(z + 2)(z - 2) = 0$
6 Apply the Null Factor Theorem and state all the four roots and their nature.	The equation $z^4 + 3z^2 - 28 = 0$ has two real roots and one pair of complex conjugate roots: $z = \pm\sqrt{7}i$ and $z = \pm 2$.

Solving quartic equations with real coefficients

Solving a general quartic is more difficult. To make it easier, one of the roots can be given; however, this makes it too easy, so one of the coefficients will be an unknown real number.

WORKED EXAMPLE 24 Given that $3 - 2i$ is a root of the equation $z^4 - 6z^3 + 18z^2 + pz + 65 = 0$, find all the roots and the real number p.

THINK	WRITE
1 Apply the conjugate root theorem.	Let $\alpha = 3 - 2i$ and $\beta = 3 + 2i$.
2 Find the sum and product of the roots.	$\alpha + \beta = 6$ and $\alpha\beta = 9 - 4i^2 = 13$
3 Write one quadratic factor.	$z^2 - 6z + 13$ is a factor of $z^4 - 6z^3 + 18z^2 + pz + 65 = 0$.
4 Use short division.	$z^4 - 6z^3 + 18z^2 + pz + 65 = (z^2 - 6z + 13)(z^2 + bz + 5)$
5 Expand and equate coefficients.	The coefficient of z^3: $-6 = b - 6$ The coefficient of z^2: $18 = 13 + 5 - 6b$ The coefficient of z: $p = 13b - 30$
6 Solve the equations to find the values of b and p.	$b = 0$ and $p = -30$
7 Substitute for b and p.	$z^4 - 6z^3 + 18z^2 - 30z + 65 = (z^2 - 6z + 13)(z^2 + 5)$
8 Complete the square.	$= (z^2 - 6z + 9 + 4)(z^2 + 5)$ $= ((z - 3)^2 + 4)(z^2 + 5)$
9 Express both quadratic factors as the difference of two squares using $i^2 = -1$.	$= ((z - 3)^2 - 4i^2)(z^2 - 5i^2)$
10 State the linear factors.	$(z - 3 - 2i)(z - 3 + 2i)(z + \sqrt{5}i)(z - \sqrt{5}i)$
11 Apply the Null Factor Theorem and state all the four roots and their nature.	There are two pairs of complex conjugate roots: $z = \pm\sqrt{5}i$ and $z = 3 \pm 2i$.

EXERCISE 3.4 Polynomial equations

PRACTISE

1 WE18 Find the equation of the quadratic $P(z)$ with real coefficients given that $P(-3 - 4i) = 0$.

2 Determine a quadratic $P(z)$ with real coefficients given that $P(-2i) = 0$.

3 WE19 Find the roots of $z^3 + 6z^2 + 9z - 50 = 0$.

4 Find the linear factors of $z^3 - 3z^2 + 4z - 12$.

5 WE20 If $P(z) = z^3 + bz^2 + cz - 39 = 0$ where b and c are real, and $P(-2 + 3i) = 0$, find the values of b and c, and state all the roots of $P(z) = 0$.

6 If $P(z) = z^3 + bz^2 + cz - 50 = 0$ where b and c are real, and $P(5i) = 0$, find the values of b and c, and state all the roots of $P(z) = 0$.

7 WE21 Solve for z if $z^3 - 2iz^2 + 4z - 8i = 0$.

8 Find the linear factors of $z^3 + 3iz^2 + 7z + 21i$.

9 WE22 Show that $z = 2 - 3i$ is a root of the equation $z^3 + (-2 + 3i)z^2 + 4z + 12i - 8 = 0$, and hence find all the roots.

10 Show that $z = \frac{3}{2} + 2i$ is a root of the equation $2z^3 - (4i + 3)z^2 + 10z - 20i - 15 = 0$, and hence find all the roots.

11 WE23 Solve for z if $z^4 - z^2 - 20 = 0$.

12 Solve for z if $2z^4 - 3z^2 - 9 = 0$.

13 WE24 Given that $5 - 6i$ is a root of the equation $z^4 + pz^3 + 35z^2 + 26z + 2074 = 0$, find all the roots and the real number p.

14 Given that $-2 + 3i$ is a root of the equation $z^4 - 4z^3 + pz^2 - 4z + 325 = 0$, find all the roots and the real number p.

CONSOLIDATE

15 Solve each of the following for z.

a $z^2 + 10z + 46 = 0$

b $z^2 + 50 = 0$

c $z(12 - z) = 85$

d $(z + 4)(12 - z) = 73$

16 Form quadratic equations with integer coefficients that have the following roots.

a -3 and $\frac{1}{2}$

b $3 - 5i$

c $2 + \sqrt{5}i$

d $-\frac{1}{2}$ and $-\frac{3}{4}$

17 Solve each of the following for z.

a $z^3 - 13z^2 + 52z - 40 = 0$

b $z^3 - 9z^2 + 19z + 29 = 0$

c $z^3 + 2z^2 + z - 18 = 0$

d $z^3 - 6z^2 + 9z + 50 = 0$

18 Form cubic equations with integer coefficients that have the following roots.

a $\frac{1}{2}$, -2 and 3

b 2 and $5 + 3i$

c -3 and $2 - \sqrt{7}i$

d $-\frac{1}{3}$ and $-3 - \sqrt{2}i$

19 a Given that $1 - 2i$ is a solution to the equation $z^3 + az^2 + bz - 10 = 0$ where a and b are real, find the values of a and b and determine all the roots.

b If $P(z) = z^3 + az^2 + bz + 68 = 0$ and $P(3 + 5i) = 0$, find the values of the real constants a and b and determine all the roots.

c If $P(z) = z^3 + az^2 + bz - 87 = 0$ and $P(2 - 5i) = 0$, find the values of the real constants a and b and determine all the roots.

d Given that $4 - 5i$ is a root of the equation $z^3 + az^2 + bz + 82 = 0$, find the values of the real constants a and b and determine all the roots.

20 Solve each of the following for z.

a $z^3 - 4iz^2 + 4z - 16i = 0$
b $z^3 - 3iz^2 + 7z - 21i = 0$
c $z^3 + 2iz^2 + 5z + 10i = 0$
d $z^3 - 4iz^2 + 3z - 12i = 0$

21 a i If $P(z) = z^3 - (2 - 5i)z^2 + 3z + 15i - 6 = 0$, show that $P(2 - 5i) = 0$.

ii Find all the values of z when $P(z) = 0$.

b i Given the equation $P(z) = z^3 + (-3 + 2i)z^2 + 4z + 8i - 12 = 0$, verify that $P(3 - 2i) = 0$.

ii Hence, find all values of z if $P(z) = 0$.

c i Show that $-2 - 3i$ is a solution of the equation $z^3 + (2 + 3i)z^2 + 5z + 10 + 15i = 0$.

ii Find all solutions of the equation $z^3 + (2 + 3i)z^2 + 5z + 10 + 15i = 0$.

d i If $P(z) = z^3 + (-3 + 4i)z^2 + 25z + 100i - 75$ and $P(ai) = 0$, find the value(s) of the real constant a.

ii Hence, find all values of z if $P(z) = 0$.

22 Solve each of the following for z.

a $z^4 + 5z^2 - 36 = 0$
b $z^4 + 4z^2 - 21 = 0$
c $z^4 - 3z^2 - 40 = 0$
d $z^4 + 9z^2 + 18 = 0$

23 a Given $P(z) = z^4 + az^3 + 34z^2 - 54z + 225$ and $P(3i) = 0$, find the value of the real constant a and find all the roots.

b Given $P(z) = z^4 + 6z^3 + 29z^2 + bz + 100 = 0$ and $P(-3 - 4i) = 0$, find the value of the real constant b and find all the roots.

24 a Given that $z = -2 + 3i$ is a root of the equation $2z^4 + 3z^3 + pz^2 - 77z - 39 = 0$, find the value of the real constant p and all the roots.

b Given that $z = ai$ is a root of the equation $z^4 + 6z^3 + 41z^2 + 96z + 400 = 0$, find the value of the real constant a and all the roots.

MASTER

25 Find a quartic polynomial with integer coefficients that has:

a $3i$ and $2 - 3i$ as the roots
b $-2i$ and $-4 + 3i$ as the roots.

26 Find a quintic polynomial with integer coefficients that has:

a $-4i$, $1 + 2i$ and -3 as the roots
b $5i$, $3 - 5i$ and 2 as the roots.

3.5 Subsets of the complex plane: circles, lines and rays

In previous sections, complex numbers have been used to represent points on the Argand plane. If we consider z as a complex variable, we can sketch subsets or regions of the Argand plane.

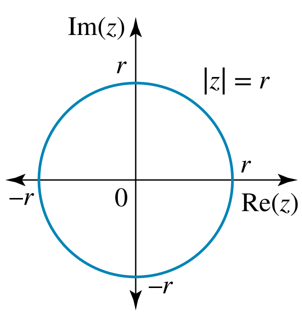
studyon

Units 3 & 4
AOS 2
Topic 4
Concept 2

Circles and ellipses
Concept summary
Practice questions

Circles

The equation $|z| = r$ where $z = x + yi$ is given by $|z| = \sqrt{x^2 + y^2} = r$. Expanding this produces $x^2 + y^2 = r^2$. This represents a circle with centre at the origin and radius r. Geometrically, $|z| = r$ represents the set of points, or what is called the locus of points, in the Argand plane that are at r units from the origin.



WORKED EXAMPLE 25

Determine the Cartesion equation and sketch the graph of $\{z : |z + 2 - 3i| = 4\}$.

THINK

1 Consider the equation.

2 Group the real and imaginary parts.

3 Use the definition of the modulus.

4 Square both sides.

5 Sketch and identify the graph of the Argand plane.

WRITE/DRAW

$|z + 2 - 3i| = 4$

Substitute $z = x + yi$:

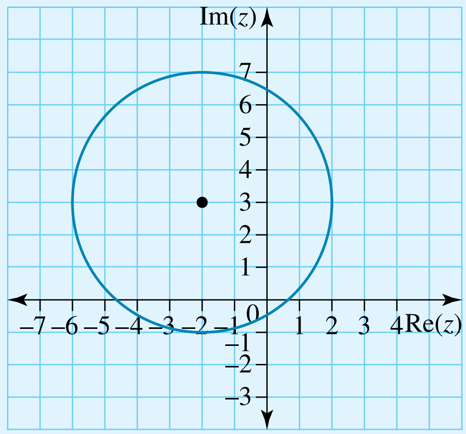
$|x + yi + 2 - 3i| = 4$

$|(x + 2) + i(y - 3)| = 4$

$\sqrt{(x + 2)^2 + (y - 3)^2} = 4$

$(x + 2)^2 + (y - 3)^2 = 16$

The equation represents a circle with centre at (–2, 3) and radius 4.



studyon

Units 3 & 4

AOS 2

Topic 4

Concept 1

Lines and rays

Concept summary

Practice questions

Lines

If $z = x + yi$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$. The equation $a\text{Re}(z) + b\text{Im}(z) = c$ where a, b and $c \in R$ represents the line $ax + by = c$.

WORKED EXAMPLE 26

Determine the Cartesion equation and sketch the graph defined by $\{z : 2\text{Re}(z) - 3\text{Im}(z) = 6\}$.

THINK

1 Consider the equation.

2 Find the axial intercepts.

WRITE/DRAW

$2\text{Re}(z) - 3\text{Im}(z) = 6$,

As $z = x + yi$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$.

This is a straight line with the Cartesian equation $2x - 3y = 6$.

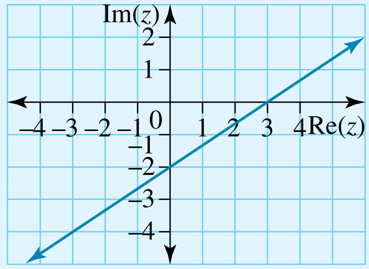
When $y = 0$, $2x = 6 \Rightarrow x = 3$.

(3, 0) is the intercept with the real axis.

When $x = 0$, $-3y = 6 \Rightarrow y = -2$.

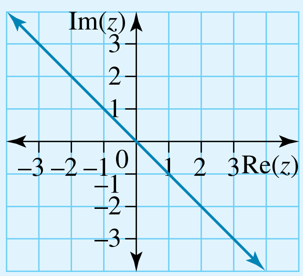
(0, –2) is the intercept with the imaginary axis.

3 Identify and sketch the equation. | The equation represents the line $2x - 3y = 6$.



Lines in the complex plane can also be represented as a set of points that are equidistant from two other fixed points. The equations of a line in the complex plane can thus have multiple representations.

WORKED EXAMPLE 27 Determine the Cartesion equation and sketch the graph defined by $\{z : |z - 2i| = |z + 2|\}$.

THINK	WRITE/DRAW
1 Consider the equation as a set of points.	$\|z - 2i\| = \|z + 2\|$ Substitute $z = x + yi$: $\|x + yi - 2i\| = \|x + yi + 2\|$
2 Group the real and imaginary parts together.	$\|x + (y - 2)i\| = \|(x + 2) + yi\|$
3 Use the definition of the modulus.	$\sqrt{x^2 + (y - 2)^2} = \sqrt{(x + 2)^2 + y^2}$
4 Square both sides, expand, and cancel like terms.	$x^2 + y^2 - 4y + 4 = x^2 + 4x + 4 + y^2$ $-4y = 4x$
5 Identify the required line.	$y = -x$
6 Identify the line geometrically.	The line is the set of points that is equidistant from the two points (0, 2) and (–2, 0).
7 Sketch the required line.	

Intersection of lines and circles

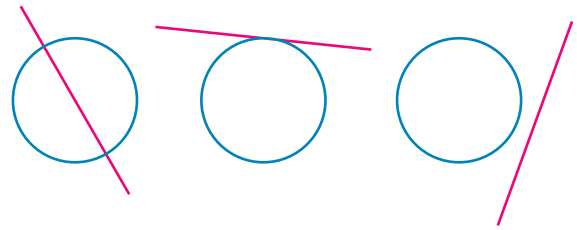
The coordinates of the points of intersection between a straight line and a circle can be found algebraically by solving the system of equations. If there are two solutions to the equations, the line intersects the circle at two points. If there is one solution

study on

Units 3 & 4
AOS 2
Topic 4
Concept 3

Curves in the complex plane
Concept summary
Practice questions

to the equation, the line and the circle touch at one point, and the line is a tangent to the circle at the point of contact. If there are no solutions to the equations, the line does not intersect the circle.



WORKED EXAMPLE 28

a **Two sets of points in the complex plane are defined by $S = \{z : |z| = 5\}$ and $T = \{z : 2\text{Re}(z) - \text{Im}(z) = 10\}$. Find the coordinates of the points of intersection between S and T.**

b **Two sets of points in the complex plane are defined by $S = \{z : |z| = 3\}$ and $T = \{z : 2\text{Re}(z) - \text{Im}(z) = k\}$. Find the values of k for which the line through T is a tangent to the circle S.**

THINK	WRITE
a 1 Find the Cartesian equation of S.	**a** $\|z\| = 5$ Substitute $z = x + yi$: $\|x + yi\| = 5$
2 Use the definition of the modulus.	$\sqrt{x^2 + y^2} = 5$
3 Square both sides and identify the boundary of S.	(1) $x^2 + y^2 = 25$ S is a circle with centre at the origin and radius 5.
4 Find the Cartesian equation of T.	Substitute $z = x + yi$: $\text{Re}(z) = x$ and $\text{Im}(z) = y$ $2\text{Re}(z) - \text{Im}(z) = 10$ (2) $2x - y = 10$ T is a straight line.
5 Solve equations (1) and (2) for x and y by substitution.	(2) $y = 2x - 10$ (1) $x^2 + (2x - 10)^2 = 25$
6 Expand and simplify.	$x^2 + 4x^2 - 40x + 100 = 25$ $5x^2 - 40x + 75 = 0$ $5(x^2 - 8x + 15) = 0$
7 Solve for x.	$x^2 - 8x + 15 = 0$ $(x - 5)(x - 3) = 0$ $x = 5$ or $x = 3$
8 Find the corresponding y-values.	From (2) $y = 2x - 10$, when $x = 5 \Rightarrow y = 0$ and when $x = 3 \Rightarrow y = -4$.
9 State the coordinates of the two points of intersection.	The points of intersection are (5, 0) and (3, –4).
b 1 Find the Cartesian equation of S.	**b** $\|z\| = 3$ Substitute $z = x + yi$: $\|x + yi\| = 3$

2 Use the definition of the modulus. — $\sqrt{x^2 + y^2} = 3$

3 Square both sides and identify the boundary of S. — (1) $x^2 + y^2 = 9$
S is a circle with centre at the origin and radius 3.

4 Find the Cartesian equation of T. — Substitute $z = x + yi$:
$\text{Re}(z) = x$ and $\text{Im}(z) = y$
$2\text{Re}(z) - \text{Im}(z) = k$
(2) $2x - y = k$
T is a straight line.

5 Solve equations (1) and (2) for x and y by substitution. — (2) $y = 2x - k$
(1) $x^2 + (2x - k)^2 = 9$

6 Expand and simplify. — $x^2 + 4x^2 - 4kx + k^2 = 9$
$5x^2 - 4kx + k^2 - 9 = 0$

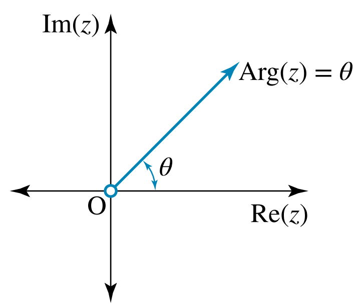
7 If the line through T is a tangent to the circle S, there will be only one solution for x. — The discriminant $\Delta = b^2 - 4ac = 0$, where $a = 5, b = -4k$ and $c = k^2 - 9$.
$\Delta = (-4k)^2 - 4 \times 5 \times (k^2 - 9)$
$= 16k^2 - 20(k^2 - 9)$
$= -4k^2 + 180$
$= 4(45 - k^2)$

8 Solve the discriminant equal to zero for k. — $45 - k^2 = 0$
$k = \pm\sqrt{45}$

9 State the value of k for which the line through T is a tangent to the circle S. — $k = \pm 3\sqrt{5}$

Rays

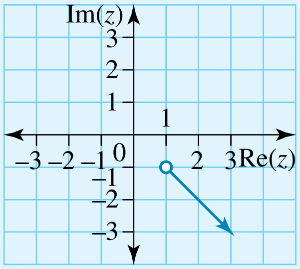
$\text{Arg}(z) = \theta$ represents the set of all points on the half-line or ray that has one end at the origin and makes an angle of θ with the positive real axis. Note that the endpoint, in this case the origin, is not included in the set. We indicate this by placing a small open circle at this point.



WORKED EXAMPLE 29

Determine the Cartesion equation and sketch the graph defined by $\left\{z : \text{Arg}(z - 1 + i) = -\frac{\pi}{4}\right\}$.

THINK	WRITE/DRAW
1 Find the Cartesian equation of the ray.	$\text{Arg}(z - 1 + i) = -\frac{\pi}{4}$ Substitute $z = x + yi$: $\text{Arg}(x + yi - 1 + i) = -\frac{\pi}{4}$
2 Group the real and imaginary parts.	$\text{Arg}((x - 1) + (y + 1)i) = -\frac{\pi}{4}$

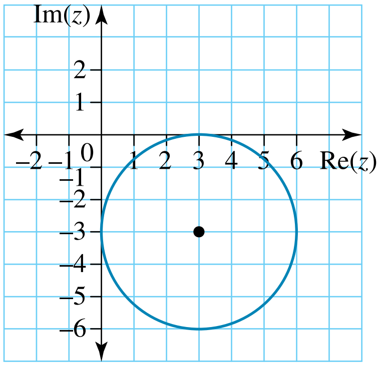
3 Use the definition of the argument.	$\tan^{-1}\left(\frac{y+1}{x-1}\right) = -\frac{\pi}{4}$ for $x > 1$
4 Simplify.	$\frac{y+1}{x-1} = \tan\left(-\frac{\pi}{4}\right)$ $= -1$ for $x > 1$ $y + 1 = -(x - 1)$ for $x > 1$
5 State the Cartesian equation of the ray.	$y = -x$ for $x > 1$.
6 Identify the point from which the ray starts.	The ray starts from the point (1, –1).
7 Determine the angle the ray makes.	The ray makes an angle of $-\frac{\pi}{4}$ with the positive real axis.
8 Describe the ray.	The point (1, –1) is not included. Alternatively, consider the ray from the origin making an angle of $-\frac{\pi}{4}$ with the positive real axis to have been translated one unit to the right parallel to the real axis, and one unit down parallel to the imaginary axis.
9 Sketch the required ray.	

EXERCISE 3.5 Subsets of the complex plane: circles, lines and rays

PRACTISE

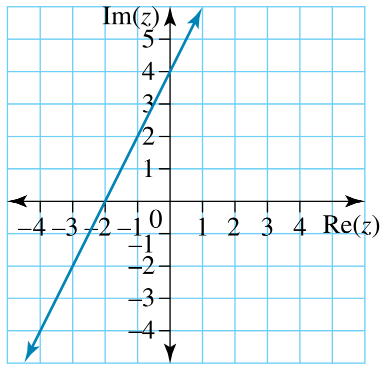
1 WE25 Sketch and describe the region of the complex plane defined by $\{z : |z - 3 + 2i| = 4\}$.

2 The region of the complex plane shown below can be described by $\{z : |z - (a + bi)| = r\}$. Find the values of a, b and r.



3 WE26 Sketch and describe the region of the complex plane defined by $\{z : 4\text{Re}(z) + 3\text{Im}(z) = 12\}$.

4 The region of the complex plane shown at right can be described by $\{z : a\text{Re}(z) + b\text{Im}(z) = 8\}$. Determine the values of a and b.



5 WE27 Sketch and describe the region of the complex plane defined by $\{z : |z + 3i| = |z - 3|\}$.

6 Sketch and describe the region of the complex plane defined by $\{z : |z - i| = |z + 3i|\}$.

7 WE28 **a** Two sets of points in the complex plane are defined by $S = \{z : |z| = 3\}$ and $T = \{z : 3\text{Re}(z) + 4\text{Im}(z) = 12\}$. Find the coordinates of the points of intersection between S and T.

b Two sets of points in the complex plane are defined by $S = \{z : |z| = 4\}$ and $T = \{z : 4\text{Re}(z) - 2\text{Im}(z) = k\}$. Find the values of k for which the line through T is a tangent to the circle S.

8 **a** Two sets of points in the complex plane are defined by $S = \{z : |z| = \sqrt{29}\}$ and $T = \{z : 3\text{Re}(z) - \text{Im}(z) = 1\}$. Find the coordinates of the points of intersection between S and T.

b Two sets of points in the complex plane are defined by $S = \{z : |z| = 5\}$ and $T = \{z : 2\text{Re}(z) - 3\text{Im}(z) = k\}$. Find the values of k for which the line through T is a tangent to the circle S.

9 WE29 Determine the Cartesion equation and sketch the graph defined by $\left\{z : \text{Arg}(z - 2) = \frac{\pi}{6}\right\}$.

10 Determine the Cartesion equation and sketch the graph defined by $\left\{z : \text{Arg}(z + 3i) = -\frac{\pi}{2}\right\}$.

CONSOLIDATE

11 For each of the following, sketch and find the Cartesian equation of the set, and describe the region.

a $\{z : |z| = 3\}$
b $\{z : |z| = 2\}$
c $\{z : |z + 2 - 3i| = 2\}$
d $\{z : |z - 3 + i| = 3\}$

12 Illustrate each of the following and describe the subset of the complex plane.

a $\{z : \text{Im}(z) = 2\}$
b $\{z : \text{Re}(z) + 2\text{Im}(z) = 4\}$
c $\{z : 3\text{Re}(z) + 2\text{Im}(z) = 6\}$
d $\{z : 2\text{Re}(z) - \text{Im}(z) = 6\}$

13 Sketch and describe each of the following sets, clearly indicating which boundaries are included.

a $\{z : |z - 2| = |z - 4|\}$
b $\{z : |z + 4i| = |z - 4|\}$
c $\{z : |z + 4| = |z - 2i|\}$
d $\{z : |z + 2 - 3i| = |z - 2 + 3i|\}$

14 **a** Show that the complex equation $\left\{z : \text{Im}\left(\frac{z - 2i}{z - 3}\right) = 0\right\}$ represents a straight line and find its equation.

b Show that the complex equation $\left\{z : \text{Re}\left(\frac{z - 2i}{z - 3}\right) = 0\right\}$ represents a circle and find its centre and radius.

15 a Find the Cartesian equation of $\{z : |z - 3| = 2|z + 3i|\}$.

b Find the locus of the set of points in the complex plane given by $\{z : |z + 3| = 2|z + 6i|\}$.

c Let $S = \{z : |z - 6| = 2|z - 3i|\}$ and $T = \{z : |z - (a + bi)| = r\}$. Given that $S = T$, find the values of a, b and r.

d Let $\{z : |z + 3| = 2|z - 3i|\}$ and $T = \{z : |z - (a + bi)| = r\}$. Given that $S = T$, state the values of a, b and r.

16 Four sets of points in the complex plane are defined by
$R = \{z : (z - 3 + 4i)(\bar{z} - 3 - 4i) = 25\}$, $S = \{z : |z - 3 + 4i| = 5\}$,
$T = \{z : 3\text{Re}(z) - 4\text{Im}(z) = 25\}$ and $U = \{z : |z| = |z - 6 + 8i|\}$.

a Find the Cartesian equations of T and U and show that $T = U$.

b Find the Cartesian equations of S and R and show that $S = R$.

c Sketch S and T on one Argand plane and find $u : S = R$ where $u \in C$.

17 a Two sets of points in the complex plane are defined by $S = \{z : |z| = 3\}$ and $T = \{z : 3\text{Re}(z) + 4\text{Im}(z) = 15\}$. Show that the line T is a tangent to the circle S and find the coordinates of the point of contact.

b Two sets of points in the complex plane are defined by $S = \{z : |z| = r\}$ and $T = \{z : 3\text{Re}(z) - 4\text{Im}(z) = 10\}$. Given that the line T is a tangent to the circle S, find the value of r.

c Two sets of points in the complex plane are defined by $S = \{z : |z| = 2\}$ and $T = \{z : 3\text{Im}(z) - 4\text{Re}(z) = 8\}$. Find the coordinates of the points of intersection between S and T.

d Two sets of points in the complex plane are defined by $S = \{z : |z| = 6\}$ and $T = \{z : 3\text{Re}(z) - 4\text{Im}(z) = k\}$. Find the values of k for which the line through T is a tangent to the circle S.

18 Sketch and describe the following subsets of the complex plane.

a $\left\{z : \text{Arg}(z) = \dfrac{\pi}{6}\right\}$

b $\left\{z : \text{Arg}(z + i) = \dfrac{\pi}{4}\right\}$

c $\left\{z : \text{Arg}(z - 2) = \dfrac{3\pi}{4}\right\}$

d $\left\{z : \text{Arg}(z + 2 - i) = -\dfrac{\pi}{2}\right\}$

19 a Let $S = \{z : |z| = 2\}$ and $T = \left\{z : \text{Arg}(z) = \dfrac{\pi}{4}\right\}$. Sketch the sets S and T on the same Argand diagram and find $z : S = T$.

b Let $S = \{z : |z| = 3\}$ and $T = \left\{z : \text{Arg}(z) = -\dfrac{\pi}{4}\right\}$. Sketch the sets S and T on the same Argand diagram and find $z : S = T$.

c Sets of points in the complex plane are defined by $S = \{z : |z + 3 + i| = 5\}$ and $R = \left\{z : \text{Arg}(z + 3) = -\dfrac{3\pi}{4}\right\}$

i Find the Cartesian equation of S.

ii Find the Cartesian equation of R.

iii If $u \in C$, find u where $S = R$.

20 a Show that the complex equation $|z - a|^2 - |z - bi|^2 = a^2 + b^2$, where a and b are real and $b \neq 0$, represents a line.

b Show that the complex equation $|z - a|^2 + |z - bi|^2 = a^2 + b^2$, where a and b are real, represents a circle, and find its centre and radius.

c Show that the complex equation $3z\bar{z} + 6z + 6\bar{z} + 2 = 0$ represents a circle, and find its centre and radius.

d Consider the complex equation $az\bar{z} + bz + b\bar{z} + c = 0$ where a, b and c are real.

i If $b^2 > ac$ and $a \neq 0$, what does the equation represent?

ii If $a = 0$ and $b \neq 0$, what does the equation represent?

e Show that the complex equation $z\bar{z} + (3 + 2i)z + (3 - 2i)\bar{z} + 4 = 0$ represents a circle, and find its centre and radius.

f Consider the complex equation $az\bar{z} + \bar{b}z + b\bar{z} + c = 0$ where a and c are real and $b = \alpha + \beta i$ is complex. Show that the equation represents a circle provided $b\bar{b} > ac$ and $a \neq 0$, and determine the circle's centre and radius.

MASTER

21 a Show that the complex equation $\left\{z : \text{Im}\left(\frac{z - ai}{z - b}\right) = 0\right\}$ where a and b are real represents a straight line if $ab \neq 0$.

b Show that the complex equation $\left\{z : \text{Re}\left(\frac{z - ai}{z - b}\right) = 0\right\}$ where a and b are real represents a circle if $ab \neq 0$. State the circle's centre and radius.

22 Given that $c = a + bi$ where a and b are real:

a show that the complex equation $(z - c)(\bar{z} - \bar{c}) = r^2$ represents a circle, and find its centre and radius

b show that the complex equation $|z - c| = 2|z - \bar{c}|$ represents a circle, and find its centre and radius.

3.6 Roots of complex numbers

Square roots of complex numbers

If $z^2 = x + yi = r\,\text{cis}(\theta)$, the complex number z can be found using two different methods: either a rectangular method or a polar method.

Square roots of complex numbers using rectangular form

WORKED EXAMPLE 30

If $z^2 = 2 + 2\sqrt{3}i$, find the complex number z using a rectangular method.

THINK	WRITE
1 Expand and replace i^2 with -1.	Let $z = a + bi$ where $a, b \in R$. $z^2 = a^2 + 2abi + b^2i^2$ $= a^2 - b^2 + 2abi$
2 Equate the real and imaginary parts.	$z^2 = a^2 - b^2 + 2abi$ $= 2 + 2\sqrt{3}i$ From the real part: $a^2 - b^2 = 2 \quad (1)$ From the imaginary part: $2ab = 2\sqrt{3} \quad (2)$
3 Solve for b and substitute into (1).	From (2): $b = \frac{\sqrt{3}}{a}$. Substitute into (1): $a^2 - \frac{3}{a^2} = 2$
4 Multiply by a^2.	$a^4 - 3 = 2a^2$ $a^4 - 2a^2 - 3 = 0$

5 Solve for a.	$(a^2 - 3)(a^2 + 1) = 0$ $a = \pm\sqrt{3}$ only, since $a \in R$.
6 Substitute to find b.	$b = \dfrac{\sqrt{3}}{a} \Rightarrow b = \pm 1$
7 State the two answers.	$z = \pm(\sqrt{3} + i)$

Square roots of complex numbers using polar form

To use the polar method to find z from z^2, express z in polar form and use de Moivre's theorem to find the roots. There will be two answers. Because any multiple of 2π can be added to an angle, the working is as follows:

$z^2 = r\,\text{cis}(\theta + 2k\pi)$ where $k \in Z$

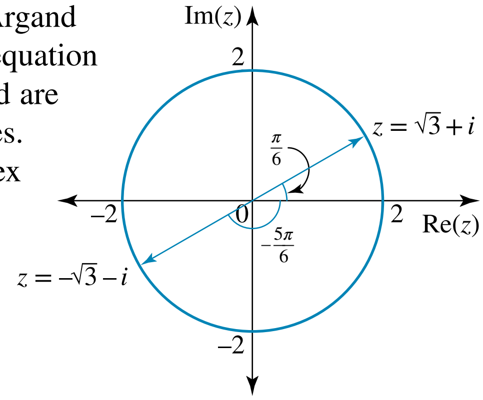
$z = \sqrt{r}\,\text{cis}\left(\dfrac{\theta}{2} + k\pi\right)$. Let $k = 0, -1$ to generate the two different answers.

WORKED EXAMPLE 31

If $z^2 = 2 + 2\sqrt{3}i$, find the complex number z using a polar method. Express the final answers in rectangular form.

THINK	WRITE
1 Express $2 + 2\sqrt{3}i$ in polar form.	$z^2 = 2 + 2\sqrt{3}i$ $= 4\,\text{cis}\left(\dfrac{\pi}{3} + 2k\pi\right)$
2 Use de Moivre's theorem.	$z = \sqrt{4}\,\text{cis}\left(\dfrac{\frac{\pi}{3} + 2k\pi}{2}\right)$ $= 2\,\text{cis}\left(\dfrac{\pi}{6} + k\pi\right)$
3 Let $k = 0$ and convert to rectangular form.	$z = 2\,\text{cis}\left(\dfrac{\pi}{6}\right)$ $= 2\left(\cos\left(\dfrac{\pi}{6}\right) + i\sin\left(\dfrac{\pi}{6}\right)\right)$ $= 2\left(\dfrac{\sqrt{3}}{2} + i \times \dfrac{1}{2}\right)$ $= \sqrt{3} + i$
4 Let $k = -1$ and convert to rectangular form.	$z = 2\,\text{cis}\left(-\dfrac{5\pi}{6}\right)$ $= 2\left(\cos\left(-\dfrac{5\pi}{6}\right) + i\sin\left(-\dfrac{5\pi}{6}\right)\right)$ $= 2\left(-\dfrac{\sqrt{3}}{2} + i \times -\dfrac{1}{2}\right)$ $= -\sqrt{3} - i$
5 State the two answers.	$z = \pm(\sqrt{3} + i)$

It is interesting to plot the two solutions on one Argand diagram. It can be seen that the two roots to the equation $z^2 = 2 + 2\sqrt{3}i$ both lie on a circle of radius 2 and are equally separated around the circle by 180 degrees. Although there are two roots, they are not complex conjugates of one another.



Cube roots of complex numbers

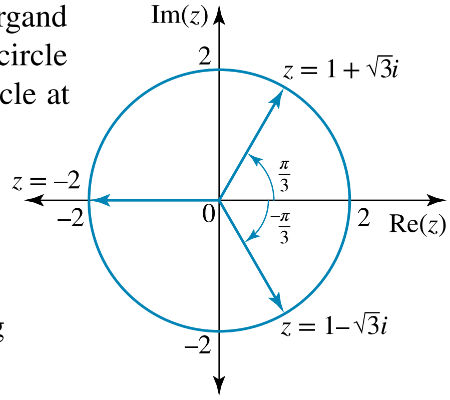
Cube roots using rectangular form

All three cube roots of a number can be found using complex numbers and rectangular form.

WORKED EXAMPLE 32 **If $z^3 + 8 = 0$, find the complex number z using a rectangular method.**

THINK	WRITE
1 Use the sum of two cubes. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$z^3 + 8 = 0$ $z^3 + (2)^3 = 0$ $(z + 2)(z^2 - 2z + 4) = 0$
2 Complete the square on the quadratic term.	$(z + 2)(z^2 - 2z + 1 + 3) = 0$ $(z + 2)((z - 1)^2 + 3) = 0$
3 Replace i^2 with -1.	$(z + 2)((z - 1)^2 - 3i^2) = 0$
4 Factorise as the difference of two squares and state the linear factors.	$(z + 2)(z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i) = 0$
5 From the Null Factor Theorem, state the three roots and their nature.	The roots are one real root and one pair of complex conjugate roots. $z = -2,\ 1 \pm \sqrt{3}i$

It is interesting to plot the three solutions on one Argand diagram. It can be seen that all three roots lie on a circle of radius 2 and are equally separated around the circle at 120° intervals. Because the coefficients are all real, the three roots consist of one real number and one pair of complex conjugates.



Cube roots using polar form

All three cube roots of a number can be found using complex numbers and polar form.

To use the polar method, express z in polar form and use de Moivre's theorem to find the roots. However, there will be three answers, so write:

$z^3 = r\,\text{cis}(\theta + 2k\pi)$ where $k \in Z$

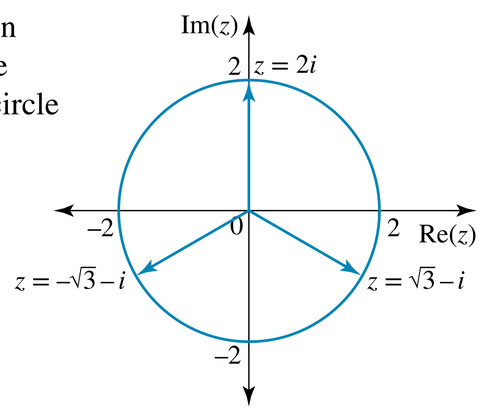
$z = \sqrt[3]{r}\,\text{cis}\left(\dfrac{\theta + 2k\pi}{3}\right)$ and let $k = 0, \pm 1$ to generate the three different answers.

Note that if different values for k were used, the roots would just repeat themselves.

WORKED EXAMPLE 33 **If $z^3 + 8i = 0$, find the complex number z using a polar method.**

THINK	WRITE
1 Express in polar form.	$z^3 = -8i$ $= 8\text{ cis}\left(-\frac{\pi}{2} + 2k\pi\right)$
2 Use de Moivre's theorem.	$z = \sqrt[3]{8}\text{ cis}\left(\frac{-\pi + 4k\pi}{6}\right)$
3 Let $k = 0$ and convert to rectangular form.	$z = 2\text{ cis}\left(-\frac{\pi}{6}\right)$ $= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ $= 2\left(\frac{\sqrt{3}}{2} + i \times -\frac{1}{2}\right)$ $= \sqrt{3} - i$
4 Let $k = 1$ and convert to rectangular form.	$z = 2\text{ cis}\left(\frac{\pi}{2}\right)$ $= 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ $= 2(0 + 1i) = 2i$
5 Let $k = -1$ and convert to rectangular form.	$z = 2\text{ cis}\left(-\frac{5\pi}{6}\right)$ $= 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$ $= 2\left(-\frac{\sqrt{3}}{2} + i \times -\frac{1}{2}\right)$ $= -\sqrt{3} - i$
6 State all three answers.	$z = \pm\sqrt{3} - i, 2i$

Again it is interesting to plot the three solutions on one Argand diagram. All three roots lie on a circle of radius 2 and are equally separated around the circle at 120° intervals. Although there are three roots, there is no conjugate pair, because the cubic has non-real coefficients.



studyon

Units 3 & 4
AOS 2
Topic 3
Concept 2

The *n*th root of a complex number
Concept summary
Practice questions

Fourth roots of complex numbers

When finding fourth roots of complex numbers, although it is possible to work in rectangular form, the preferred and easier method is to use polar form.

WORKED EXAMPLE 34 Solve for z if $z^4 + 16 = 0$.

THINK	WRITE
1 Express in polar form.	$z^4 = -16 = 16\,\text{cis}(\pi + 2k\pi)$
2 Use de Moivre's theorem.	$z = \sqrt[4]{16}\,\text{cis}\left(\dfrac{\pi + 2k\pi}{4}\right)$
3 Let $k = 0$ and convert to rectangular form.	$z = 2\,\text{cis}\left(\dfrac{\pi}{4}\right) = 2\left(\cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right)\right)$ $= 2\left(\dfrac{\sqrt{2}}{2} + i \times \dfrac{\sqrt{2}}{2}\right)$ $= \sqrt{2}(1 + i)$
4 Let $k = 1$ and convert to rectangular form.	$z = 2\,\text{cis}\left(\dfrac{3\pi}{4}\right)$ $= 2\left(\cos\left(\dfrac{3\pi}{4}\right) + i\sin\left(\dfrac{3\pi}{4}\right)\right)$ $= 2\left(-\dfrac{\sqrt{2}}{2} + i \times \dfrac{\sqrt{2}}{2}\right)$ $= \sqrt{2}(-1 + i)$
5 Let $k = -1$ and convert to rectangular form.	$z = 2\,\text{cis}\left(-\dfrac{\pi}{4}\right)$ $= 2\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$ $= 2\left(\dfrac{\sqrt{2}}{2} - i \times \dfrac{\sqrt{2}}{2}\right)$ $= \sqrt{2}(1 - i)$
6 Let $k = -2$ and convert to rectangular form.	$z = 2\,\text{cis}\left(-\dfrac{3\pi}{4}\right)$ $= 2\left(\cos\left(-\dfrac{3\pi}{4}\right) + i\sin\left(-\dfrac{3\pi}{4}\right)\right)$ $= 2\left(-\dfrac{\sqrt{2}}{2} - i \times \dfrac{\sqrt{2}}{2}\right)$ $= -\sqrt{2}(1 + i)$
7 State all four answers.	$z = \pm\sqrt{2}(1 + i), \pm\sqrt{2}(1 - i)$

studyon

Units 3 & 4

AOS 2

Topic 3

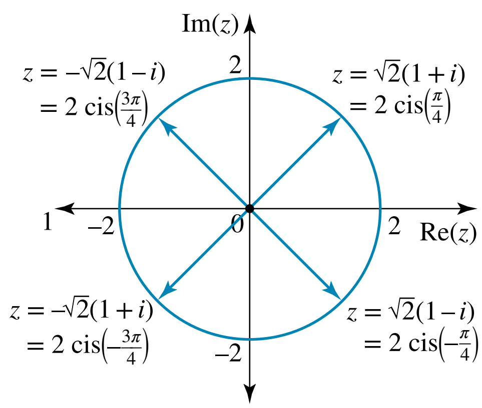
Concept 5

Conjugate root theorem

Concept summary

Practice questions

Again it is interesting to plot the four solutions on one Argand diagram. It can be seen that all four roots lie on a circle of radius 2 and are equally separated around the circle by 90°. The four roots consist of two pairs of complex conjugates, as all the coefficients in the quartic are real.



EXERCISE 3.6 Roots of complex numbers

PRACTISE

1 WE30 If $z^2 = 2 - 2\sqrt{3}i$, find the complex number z using a rectangular method.

2 If $z^2 + 2i = 0$, find the complex number z using a rectangular method.

3 WE31 If $z^2 = 2 - 2\sqrt{3}i$, find the complex number z using a polar method. Express the final answers in rectangular form.

4 If $z^2 + 2i = 0$, find the complex number z using a polar method. Express the final answers in rectangular form.

5 WE32 If $z^3 + 64 = 0$, find the complex number z using a rectangular method.

6 If $z^3 - 8 = 0$, find the complex number z using a rectangular method.

7 WE33 If $z^3 + 64i = 0$, find the complex number z using a polar method.

8 If $z^3 - 8i = 0$, find the complex number z using a polar method.

9 WE34 Solve for z if $z^4 - 16 = 0$.

10 Solve for z if $z^4 + 8 - 8\sqrt{3}i = 0$.

CONSOLIDATE

11 Find all the solutions for each of the following, giving your answers in rectangular form.

a $z^2 - 36 = 0$
b $z^2 + 36 = 0$
c $z^2 - 36i = 0$
d $z^2 + 36i = 0$

12 Find all the solutions for each of the following, giving your answers in rectangular form.

a $z^2 = 7 + 24i$
b $z^2 = 24 - 7i$
c $z^2 = -24 + 7i$
d $z^2 = -7 - 24i$

13 Find all the solutions for each of the following, giving your answers in both polar and rectangular form.

a $z^2 = 8(1 + \sqrt{3}i)$
b $z^2 = 8(1 - \sqrt{3}i)$
c $z^2 = 8(-1 + \sqrt{3}i)$
d $z^2 = -8(1 + \sqrt{3}i)$

14 a i Solve for z if $z^4 + 16z^2 - 225 = 0$.
ii Find all the real numbers a and b that satisfy $(a + bi)^2 = -16 + 30i$.
iii Hence, find the exact values of z if $\frac{1}{2}z^2 + 4iz - 15i = 0$.
b i Solve for z if $z^4 + 21z^2 - 100 = 0$.
ii Find all the real numbers a and b that satisfy $(a + bi)^2 = -21 - 20i$.
iii Hence, find the exact values of z if $z^2 + \sqrt{21}iz + 5i = 0$.
c i Solve for z if $z^4 - 18z^2 - 243 = 0$.
ii Find all the real numbers a and b that satisfy $(a + bi)^2 = 18 - 18\sqrt{3}i$.
iii Hence, find the exact values of z if $\frac{1}{2}z^2 + 3\sqrt{2}z + 9\sqrt{3}i = 0$.

15 If 1, u and v represent the three cube roots of unity, show that:

a $v = \bar{u}$
b $u^2 = v$
c $1 + u + v = 0$.

16 Find all the solutions for each of the following, giving your answers in rectangular form.

a $z^3 - 512 = 0$
b $z^3 + 512 = 0$
c $z^3 + 512i = 0$
d $z^3 - 512i = 0$

17 Find all the roots of the equation $z^3 + 125i = 0$, giving your answers in both rectangular and polar form. Plot all the roots on one Argand diagram and comment on their relative positions.

18 Find all the solutions for each of the following, giving your answers in both polar and rectangular form.

a $z^4 - 4 = 0$

b $z^4 + 8 + 8\sqrt{3}i = 0$

19 Let $u = 2 - 2i$.

a Find $\text{Arg}(u^4)$ and hence find u^4.

b Find all the solutions of $z^4 + 64 = 0$, giving your answers in both rectangular and polar form.

c Plot all the solutions of $z^4 + 64 = 0$ on one Argand diagram and comment on their relative positions.

20 Find all the roots of the equation $z^5 + 32 = 0$, giving your answers in polar form with angles in degrees. Plot all the roots on one Argand diagram and comment on their relative positions.

MASTER

21 Find all the solutions for each of the following, giving your answers in both polar and rectangular form.

a $z^6 - 64 = 0$

b $z^6 + 64 = 0$

22 a Solve $z^8 - 16 = 0$, giving your answers in both polar and rectangular form.

b Find all the roots of the equation $z^{12} - 4096 = 0$, giving your answers in both polar and rectangular form.

eBook*plus* ONLINE ONLY

3.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

3 Answers

EXERCISE 3.2

1 a $8i$ **b** $\pm 8i$

2 a $3\sqrt{2}i$ **b** $\pm 3\sqrt{2}i$

3 $4 \pm 5i$ **4** $\dfrac{3}{2} \pm \dfrac{\sqrt{5}i}{2}$

5 a $7 - 4i$ **b** $-1 + 2i$ **c** $1 + 3i$

6 $2 + 5i$

7 a $9 - 13i$ **b** $7 - 24i$

8 a $3 + 4i$ **b** $8 - 6i$

9 a 10 **b** 0

10 $-4 + 3i$

11 a $\frac{3}{10} + \frac{1}{10}i$ **b** $\frac{3}{5} + \frac{1}{5}i$

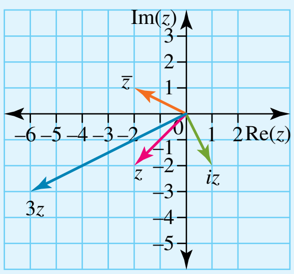
12 $-3 - 2i$

13 $x = 2, y = \frac{7}{2}$ **14** $x = \frac{20}{13}, y = \frac{9}{13}$

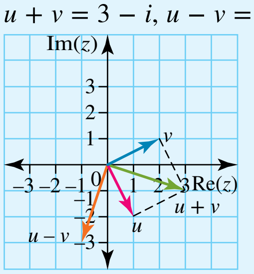
15 $-1 - 2i$ **16** $3 + 2i$

17 4 **18** 2

19 $3z$ has a length of three times z, $\bar{z} = -2 + i$ is the reflection in the real axis, and $iz = 1 - 2i$ is a rotation of 90° anticlockwise from z.



20 $u + v = 3 - i$, $u - v = -1 - 3i$



21 a -2 **b** $-i$ **c** $\sqrt{2}$

d $5\sqrt{3}i$ **e** $-\dfrac{\sqrt{2}i}{4}$ **f** $\dfrac{3\sqrt{2}i}{2}$

22 a $5 + 3i$ **b** 34 **c** $\frac{5}{34} + \frac{3}{34}i$

d $16 - 30i$ **e** -36 **f** $\frac{16}{13} + \frac{11}{13}i$

23 a 13 **b** 38 **c** -29

d -163 **e** $\frac{1}{13}$ **f** $-\frac{2}{5}$

24 a ± 7 **b** $\pm 7i$ **c** $\pm\dfrac{3i}{2}$

d $\pm\dfrac{5\sqrt{3}i}{3}$ **e** $\pm\dfrac{9\sqrt{2}i}{2}$ **f** $\pm\sqrt{\dfrac{b}{a}}\,i$

25 a $4 \pm 5i$ **b** $2 \pm \sqrt{5}i$ **c** $2 \pm \dfrac{\sqrt{6}i}{2}$

d $\dfrac{1 \pm \sqrt{2}i}{3}$ **e** $\dfrac{3 \pm \sqrt{3}i}{4}$ **f** $\dfrac{-2 \pm \sqrt{6}i}{5}$

26 a $-2 \pm i$ **b** $5 \pm 3i$ **c** $7 \pm 5i$ **d** $-8 \pm 3i$

27 a $-2 - 21i$ **b** $23 - 14i$ **c** $-\frac{7}{25} - \frac{26}{25}i$

d $-60 + 32i$ **e** $-\frac{3}{10} - \frac{11}{10}i$

28 a $1 + 2i$ **b** $1 + i$ **c** $2 - 5i$ **d** $-1 + 5i$

29 a $x = 3, y = -1$ **b** $x = 5, y = 3$

c $x = \frac{6}{17}, y = -\frac{10}{17}$ **d** $x = \frac{27}{61}, y = -\frac{8}{61}$

30 a $2i, -6i$ **b** $8i, -2i$

c $4i, -i$ **d** $-2i, -3i$

31 a $5 - 2i, 3 + 2i$ **b** $4 - 3i, 5 + 3i$

c $2 - 3i, 5 + 2i$ **d** $7 + 3i, 1 - 4i$

32 a i $2i$ **ii** $-2 + 2i$ **iii** -4

b i $-7 - 24i$ **ii** $-117 - 44i$ **iii** $-527 + 336i$

c i $a^2 - b^2 + 2abi$

ii $a^3 - 3ab^3 + (3a^2b - b^3)i$

iii $a^4 - 6a^2b^2 + b^4 + (4a^3b - 4ab^3)i$

EXERCISE 3.3

1 a $2\,\text{cis}\left(\dfrac{\pi}{3}\right)$ **b** $\sqrt{2}\,\text{cis}\left(\dfrac{3\pi}{4}\right)$ **c** $4\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$

d $2\,\text{cis}\left(-\dfrac{\pi}{6}\right)$ **e** $4\,\text{cis}(0)$ **f** $2\,\text{cis}\left(-\dfrac{\pi}{2}\right)$

2 a $2\,\text{cis}\left(\dfrac{\pi}{6}\right)$ **b** $2\,\text{cis}\left(\dfrac{2\pi}{3}\right)$ **c** $2\,\text{cis}\left(-\dfrac{5\pi}{6}\right)$

d $2\sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$ **e** $7\,\text{cis}(\pi)$ **f** $5\,\text{cis}\left(\dfrac{\pi}{2}\right)$

3 a $2 - 2\sqrt{3}i$ **b** $-8i$

4 a $-6 - 6i$ **b** $-3 + 4i$

5 $\frac{1}{12} - \frac{\sqrt{3}}{12}i$

6 $-2 + 2i$

7 a $-8i$ **b** -4

8 a $1 - \sqrt{3}i$ **b** -8

9 a $\dfrac{\pi}{2}$ **b** $32i$

10 $-\frac{\sqrt{3}}{4} - \frac{1}{4}i$

11 a Check with your teacher.

b $-\sqrt{3} - 2 + i, \dfrac{11\pi}{12}$

12 $-\dfrac{\pi}{12}$

13 $n = 3k$ where $k \in Z$

14 $n = 3k + \frac{3}{2}$ where $k \in Z$

15 a $5\text{ cis}(53°8')$ b $25\text{ cis}(-73°44')$
c $13\text{ cis}(112°37')$ d $4\sqrt{2}\text{ cis}(-135°)$

16 a $18\text{ cis}(35°)$ b $\frac{1}{2}\text{cis}(11°)$
c $36\text{ cis}(24°)$ d $27\text{ cis}(69°)$
e $629\,856\text{ cis}(152°)$ f $\frac{27}{8}\text{cis}(102°)$

17 a $-35+7i$ b $27-17i$
c $-\frac{7}{13}+\frac{35}{13}i$ d $-98i$

18 a i 4096 ii 0
b i −46 656 ii π
c i 1 953 125 ii 0
d i $-8192\sqrt{3}+8192i$ ii $\frac{5\pi}{6}$

19 a $\frac{1}{2}(\sqrt{3}+i), \frac{1}{2}(\sqrt{3}+i), -1$ b $\frac{\pi}{6}, \frac{\pi}{6}, \pi$
c Yes d Yes
e No

20 a i $\frac{2\pi}{3}$ ii $-\frac{3\pi}{4}$
iii $-\frac{\pi}{12}$ iv $-\frac{7\pi}{12}$
v In this case yes but not in general vi No
b i $\frac{5\pi}{6}$ ii $\frac{3\pi}{4}$
iii $-\frac{5\pi}{12}$ iv $\frac{\pi}{12}$
v No
vi In this case yes but not in general

21 a i $\frac{1}{4}(\sqrt{3}+1)+\frac{1}{4}(\sqrt{3}-1)i$ ii $\frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{12}\right)$
iii $\frac{1}{4}(\sqrt{6}-\sqrt{2})$ iv $\frac{1}{4}(\sqrt{6}-\sqrt{2})$
b i $(\sqrt{6}-\sqrt{2})+(\sqrt{6}+\sqrt{2})i$
ii $4\text{ cis}\left(\frac{5\pi}{12}\right)$
iii $\frac{1}{4}(\sqrt{6}+\sqrt{2})$
iv Check with your teacher.

22 a i $2(\sqrt{3}+1)+2(\sqrt{3}-1)i$
ii $4\sqrt{2}\text{ cis}\left(\frac{\pi}{12}\right)$
iii $\frac{1}{4}(\sqrt{6}+\sqrt{2})$
iv Check with your teacher.
b i $\frac{1}{2}(1-\sqrt{3})+\frac{1}{2}(\sqrt{3}+1)i$
ii $\sqrt{2}\text{ cis}\left(\frac{7\pi}{12}\right)$
iii $\frac{1}{4}(\sqrt{2}-\sqrt{6})$
iv Check with your teacher.

23 a i Check with your teacher.
ii $\frac{\pi}{8}$
iii $\frac{5\pi}{8}$
b i Check with your teacher. ii $\frac{7\pi}{12}$
iii $-\frac{5\pi}{12}$ iv $\frac{\pi}{12}$

24 a $n=2(2k+1)$ where $k\in Z$
b $n=4k$ where $k\in Z$
c $n=6k$ where $k\in Z$
d $n=3(2k+1)$ where $k\in Z$

25, 26 Answers may vary.

EXERCISE 3.4

1 $z^2+6z+25=0$

2 $z^2+4=0$

3 $z=-4\pm 3i, 2$

4 $(z+2i)(z-2i)(z-3)$

5 $-2\pm 3i, 3, b=1, c=1$

6 $\pm 5i, 2, b=-2, c=25$

7 $\pm 2i$

8 $(z+\sqrt{7}i)(z-\sqrt{7}i)(z+3i)$

9 $\pm 2i, 2-3i$

10 $\frac{3}{2}+2i, \pm\sqrt{5}i$

11 $\pm 2i, \pm\sqrt{5}$

12 $\pm\frac{\sqrt{6}i}{2}, \pm\sqrt{3}$

13 $5\pm 6i, -3\pm 5i, p=-4$

14 $4\pm 3i, -2\pm 3i, p=6$

15 a $-5\pm\sqrt{21}i$ b $\pm 5\sqrt{2}i$
c $6\pm 7i$ d $4\pm 3i$

16 a $2z^2+5z-3$ b $z^2-6z+34$
c z^2-4z+9 d $8z^2+10z+3$

17 a $1, 6\pm 2i$ b $-1, 5\pm 2i$
c $2, -2\pm\sqrt{5}i$ d $-2, 4\pm 3i$

18 a $2z^3-3z^2-11z+6$ b $z^3-12z^2+54z-68$
c z^3-z^2-z+33 d $3z^3+19z^2+39z+11$

19 a $a=-4, b=9, 1\pm 2i, 2$
b $a=-4, b=22, 3\pm 5i, -2$
c $a=-7, b=41, 2\pm 5i, 3$
d $a=-6, b=25, 4\pm 5i, -2$

20 a $\pm 2i, 4i$ b $\pm\sqrt{7}i, 3i$
c $\pm\sqrt{5}i, -2i$ d $\pm\sqrt{3}i, 4i$

21 a i Check with your teacher.
ii $2-5i, \pm\sqrt{3}i$
b i Check with your teacher.
ii $3-2i, \pm 2i$
c i Check with your teacher.
ii $-2-3i, \pm\sqrt{5}i$
d i $a=\pm 5$
ii $3-4i, \pm 5i$

22 a $\pm 3i, \pm 2$ b $\pm\sqrt{7}i, \pm\sqrt{3}$
c $\pm\sqrt{5}i, \pm 2\sqrt{2}$ d $\pm\sqrt{6}i, \pm\sqrt{3}i$

23 a $a=-6, \pm 3i, 3\pm 4i$ b $b=24, -3\pm 4i, \pm 2i$

24 a $p=3, -2\pm 3i, -\frac{1}{2}, 3$ b $a=\pm 4, \pm 4i, -3\pm 4i$

25 a $z^4 - 4z^3 + 22z^2 - 36z + 117 = 0$

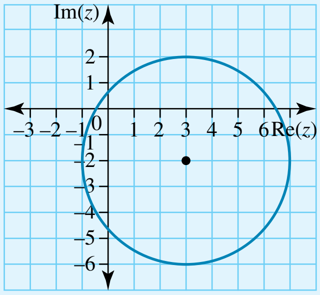
b $z^4 + 8z^3 + 29z^2 + 32z + 100 = 0$

26 a $z^5 + z^4 + 15z^3 + 31z^2 - 16z + 240 = 0$

b $z^5 - 8z^4 + 71z^3 - 268z^2 + 1150z - 1700 = 0$

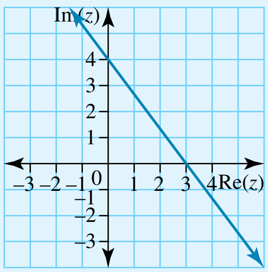
EXERCISE 3.5

1 Circle $(x - 3)^2 + (y + 2)^2 = 16$, with centre (3, –2) and radius 4



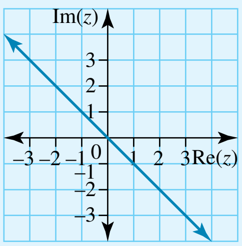
2 $a = 3$, $b = -3$, $r = 3$

3 The line $4x + 3y = 12$

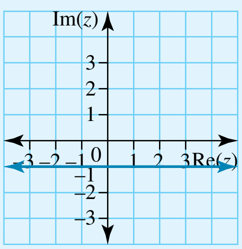


4 $a = -4$, $b = 2$

5 Line $y = -x$; the set of points equidistant from (0, –3) and (3, 0)



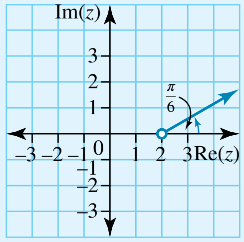
6 Line $y = -1$; the set of points equidistant from (0, 1) and (0, –3)



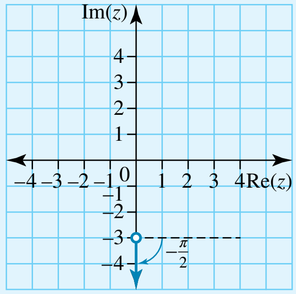
7 a $(0, 3), \left(\frac{72}{25}, \frac{21}{25}\right)$ b $\pm 8\sqrt{5}$

8 a $(2, 5), \left(-\frac{7}{5}, -\frac{26}{5}\right)$ b $\pm 5\sqrt{13}$

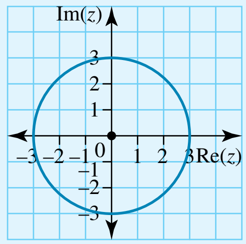
9 A ray from (2, 0) making an angle of $\frac{\pi}{6}$ or 30° with the real axis



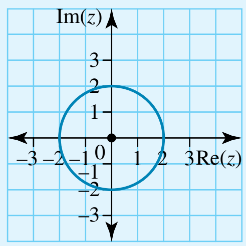
10 A ray from (0, –3) making an angle of $-\frac{\pi}{2}$ or 90° with the real axis



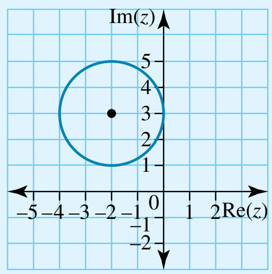
11 a $x^2 + y^2 = 9$; circle with centre (0, 0), radius 3



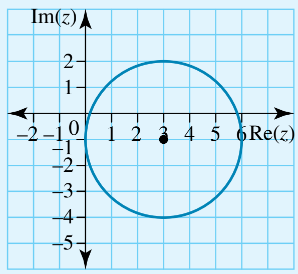
b $x^2 + y^2 = 4$; circle with centre (0, 0), radius 2



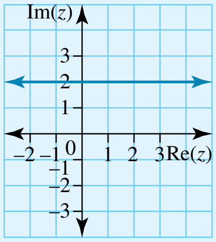
c $(x + 2)^2 + (y - 3)^2 = 4$; circle with centre (–2, 3), radius 2



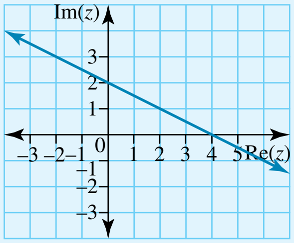
d $(x - 3)^2 + (y + 1)^2 = 9$; circle with centre (3, –1), radius 3



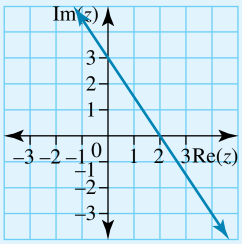
12 a $y = 2$; line



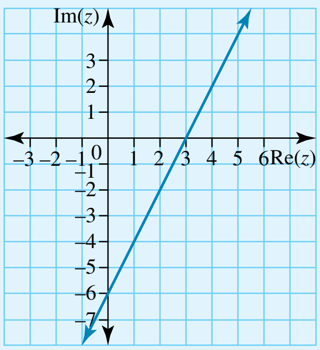
b $x + 2y = 4$; line



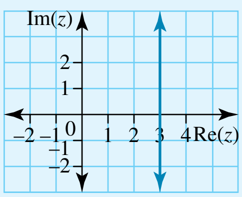
c $3x + 2y = 6$



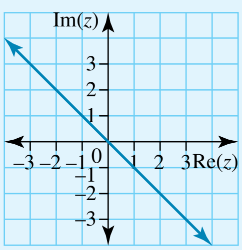
d $y = 2x - 6$



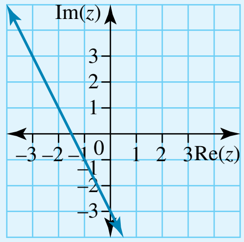
13 a $x = 3$; line



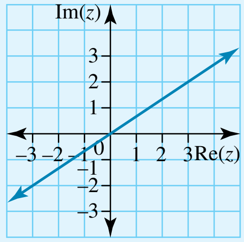
b $y = -x$



c $y = -2x - 3$



d $y = \dfrac{2x}{3}$



14 a $y = -\dfrac{2x}{3} + 2$; line

b $\left(x - \frac{3}{2}\right)^2 + (y - 1)^2 = \frac{13}{4}$; circle with centre $\left(\frac{3}{2}, 1\right)$, radius $\frac{\sqrt{13}}{2}$

15 a $(x + 1)^2 + (y + 4)^2 = 8$; circle with centre $(-1, -4)$, radius $2\sqrt{2}$

b $(x - 1)^2 + (y + 8)^2 = 20$; circle with centre $(1, -8)$, radius $2\sqrt{5}$

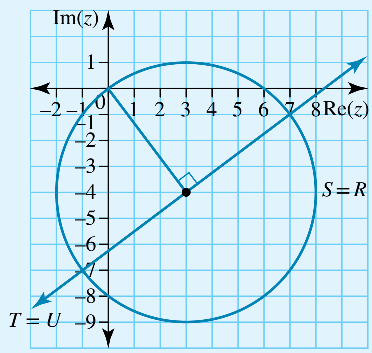
c $a = -2$, $b = 4$, $r = 2\sqrt{5}$

d $a = 1$, $b = 4$, $r = 2\sqrt{2}$

16 a $3x - 4y = 25$; line

b $(x - 3)^2 + (y + 4)^2 = 25$; circle with centre $(3, -4)$, radius 5

c $7 - i, -1 - 7i$



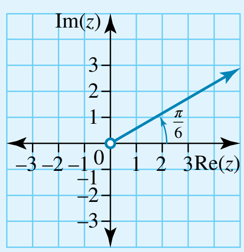
17 a $\left(\frac{9}{5}, \frac{12}{5}\right)$

b 2

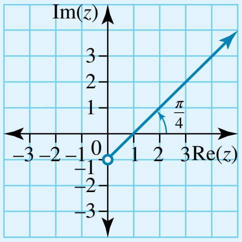
c $(-2, 0)$, $\left(-\frac{14}{25}, \frac{48}{25}\right)$

d ± 30

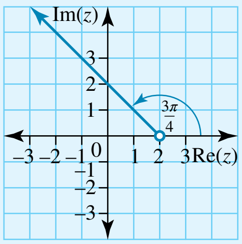
18 a $y = \dfrac{x}{\sqrt{3}}$ for $x > 0$; a ray from $(0, 0)$ making an angle of 30° with the real axis



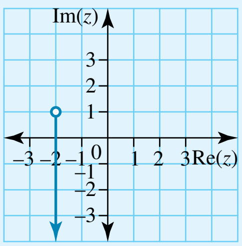
b $y = x - 1$ for $x > 0$; a ray from (0, –1) making an angle of 45° with the real axis



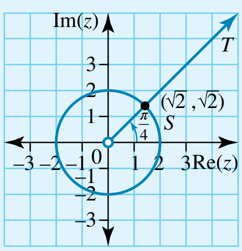
c $y = 2 - x$ for $x < 2$; a ray from (2, 0) making an angle of 135° with the real axis

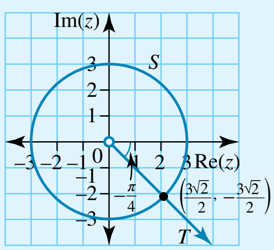


d $x = -2$ for $y < 1$; a ray from (–2, 1) going down parallel to the imaginary axis.



19 a $(\sqrt{2}, \sqrt{2})$ **b** $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$





c i S is the circle with centre (–3, –1) and radius 5: $(x + 3)^2 + (y + 1)^2 = 25$.

ii T is $y = x + 3$ for $x < -3$, the ray from (–3, 0) making an angle of –135° with the real axis.

iii $u = -7 - 4i$

20 a $y = \frac{ax}{b} + b$; line

b $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}$; circle with centre $\left(\frac{a}{2}, \frac{b}{2}\right)$, radius $\frac{\sqrt{a^2 + b^2}}{2}$

c $(x + 2)^2 + y^2 = \frac{10}{3}$; circle with centre (–2, 0), radius $\frac{\sqrt{30}}{3}$

d i $\left(x + \frac{b}{a}\right)^2 + y^2 = \frac{b^2 - ac}{a^2}$; circle with centre $\left(-\frac{b}{a}, 0\right)$, radius $\frac{\sqrt{b^2 - ac}}{a}$

ii If $a = 0$ and $b \neq 0$, the equation is $x = -\frac{c}{2b}$, a line.

e $(x + 3)^2 + (y - 2)^2 = 9$; circle with centre (–3, 2), radius 3

f $\left(x + \frac{\alpha}{a}\right)^2 + \left(y + \frac{\beta}{a}\right)^2 = \frac{b\bar{b} - ac}{a^2}$; circle with centre $\left(-\frac{\alpha}{a}, -\frac{\beta}{a}\right)$, radius $\frac{\sqrt{b\bar{b} - ac}}{a}$

21 a $y = -\frac{ax}{b} + a$, $ab \neq 0$; line

b $\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$; circle with centre $\left(\frac{b}{2}, \frac{a}{2}\right)$, radius $\frac{\sqrt{a^2 + b^2}}{2}$

22 a $(x - a)^2 + (y - b)^2 = r^2$; circle with centre (a, b), radius r

b $(x - a)^2 + \left(y + \frac{5b}{3}\right)^2 = \frac{16b^2}{9}$; circle with centre $\left(a, \frac{-5b}{3}\right)$, radius $\frac{4b}{3}$

EXERCISE 3.6

1 $\pm(\sqrt{3} - i)$ **2** $\pm(1 - i)$ **3** $\pm(\sqrt{3} - i)$

4 $\pm(1 - i)$ **5** $-4, 2 \pm 2\sqrt{3}i$ **6** $2, -1 \pm \sqrt{3}i$

7 $2\sqrt{3} - 2i, -2\sqrt{3} - 2i, 4i$

8 $-\sqrt{3} + i, \sqrt{3} + i, -2i$

9 $\pm 2i, \pm 2$

10 $\pm(\sqrt{3} + i), \pm(1 - \sqrt{3}i)$

11 a ± 6 **b** $\pm 6i$

c $\pm 3\sqrt{2}(1 + i)$ **d** $\pm 3\sqrt{2}(1 - i)$

12 a $\pm(4 + 3i)$ **b** $\pm\frac{\sqrt{2}}{2}(7 - i)$

c $\pm\frac{\sqrt{2}}{2}(1 + 7i)$ **d** $\pm(3 - 4i)$

13 a $\pm 2(\sqrt{3} + i)$ **b** $\pm 2(\sqrt{3} - i)$

c $\pm 2(1 + \sqrt{3}i)$ **d** $\pm 2(1 - \sqrt{3}i)$

14 a i $\pm 5i \pm 3$ **ii** $a = \pm 3, b = \pm 5$

iii $3 + i, -3 - 9i$

b i $\pm 5i \pm 2$ **ii** $a = \pm 2, b = \mp 5$

iii $1 - \left(\frac{\sqrt{21} + 5}{2}\right)i, -1 - \left(\frac{\sqrt{21} - 5}{2}\right)i$

c i $\pm 3i \pm 3\sqrt{3}$ **ii** $a = \pm 3\sqrt{3}, b = \mp 3$

iii $3(\sqrt{3} - \sqrt{2}) - 3i, -3(\sqrt{3} + \sqrt{2}) + 3i$

15 Check with your teacher.

16 a $-4+4\sqrt{3}i, -4-4\sqrt{3}i, 8$

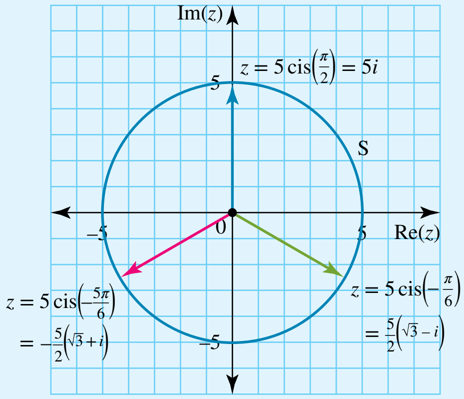
b $4+4\sqrt{3}i, 4-4\sqrt{3}i, -8$

c $4\sqrt{3}-4i, -4\sqrt{3}-4i, 8i$

d $4\sqrt{3}+4i, -4\sqrt{3}+4i, -8i$

17 $\dfrac{5\sqrt{3}}{2}-\dfrac{5}{2}i, -\dfrac{5\sqrt{3}}{2}-\dfrac{5}{2}i, 5i$ or $5\,\text{cis}\left(-\dfrac{5\pi}{6}\right), 5\,\text{cis}\left(-\dfrac{\pi}{6}\right),$
$5\,\text{cis}\left(\dfrac{\pi}{2}\right)$

All the roots are on a circle of radius 5 and are equally spaced around the circle at 120° intervals.



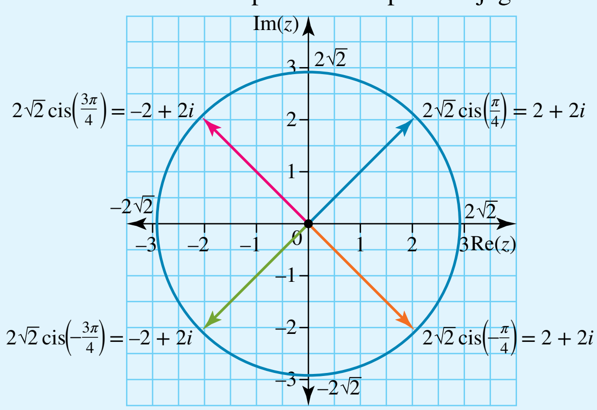
18 a $\pm\sqrt{2}i \pm \sqrt{2}$

b $\pm(\sqrt{3}-i), \pm(1+\sqrt{3}i)$

19 a $-64, \pi$

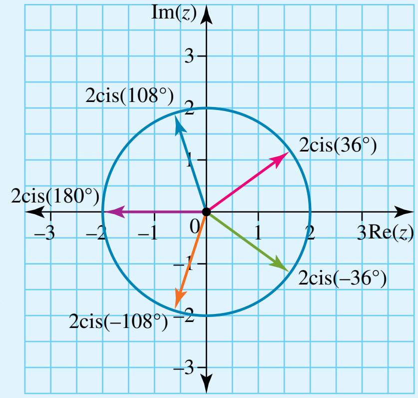
b $\pm2(1+i), \pm2(1-i)$ or $2\sqrt{2}\,\text{cis}\left(\pm\dfrac{\pi}{4}\right),$
$2\sqrt{2}\,\text{cis}\left(\pm\dfrac{3\pi}{4}\right)$

c All 4 roots are on a circle of radius $2\sqrt{2}$ and are equally spaced around the circle at 90° intervals. The roots consist of 2 pairs of complex conjugates.



20 $2\,\text{cis}(\pm36°)$,
$2\,\text{cis}(\pm108°)$,
$2\,\text{cis}(180°)$

All 5 roots are on a circle of radius 2 and are equally spaced around the circle at 72° intervals. The roots consist of 2 pairs of complex conjugates and 1 real root.



21 a $1\pm\sqrt{3}\,i, -1\pm\sqrt{3}\,i, \pm2$ or $2\,\text{cis}\left(\pm\dfrac{\pi}{3}\right), 2\,\text{cis}\left(\pm\dfrac{2\pi}{3}\right),$
$\pm2\,\text{cis}(\pi)$

b $-\sqrt{3}\pm i, \sqrt{3}\pm i, \pm2i$ or $2\,\text{cis}\left(\pm\dfrac{\pi}{6}\right), 2\,\text{cis}\left(\pm\dfrac{5\pi}{6}\right),$
$2\,\text{cis}\left(\pm\dfrac{\pi}{2}\right)$

22 a $1\pm i, -1\pm i, \pm\sqrt{2}i, \pm\sqrt{2}$ or
$\sqrt{2}\,\text{cis}\left(\pm\dfrac{\pi}{2}\right), \pm\sqrt{2}\,\text{cis}(0), \sqrt{2}\,\text{cis}\left(\pm\dfrac{3\pi}{4}\right)$

All 8 roots are on a circle of radius $\sqrt{2}$ and are equally spaced around the circle at 45° intervals. The roots consist of 3 pairs of complex conjugates and 2 real roots.

b $-\sqrt{3}\pm i, \sqrt{3}\pm i, -1\pm\sqrt{3}i, 1\pm\sqrt{3}i, \pm2, \pm2i$ or
$2\,\text{cis}\left(\pm\dfrac{\pi}{3}\right), 2\,\text{cis}\left(\pm\dfrac{\pi}{2}\right), 2\,\text{cis}\left(\pm\dfrac{5\pi}{6}\right), 2\,\text{cis}\left(\pm\dfrac{2\pi}{3}\right),$
$\pm2\,\text{cis}(\pi)$

All 12 roots are on a circle of radius 2 and are equally spaced around the circle at 30° intervals. The roots consist of 5 pairs of complex conjugates and 2 real roots.

4

Kinematics

4.1 Kick off with CAS

Kinematics involves the study of position, displacement, velocity and acceleration.

From the study of calculus in Year 11, we know that if x is the position of an object moving in a straight line at time t, then the velocity of the object at time t is given by $v = \frac{dx}{dt}$ and the acceleration of the object at time t is given by $a = \frac{dv}{dt}$.

1 If the position of a body moving in a straight line is given by $x(t) = 2t^3 + 9t^2 - 12t + 10$ where x is in centimetres and t is in seconds, use the 'define' function on the CAS calculator and calculate the:
   a initial velocity and acceleration
   b time when the body is at the origin
   c velocity when the acceleration is zero
   d acceleration when the velocity is zero.

2 A hot air balloon commences its descent at time $t = 0$ minutes. As it descends, the height of the balloon above the ground, in metres, is given by the equation, $h(t) = 600 \times 2^{-\frac{t}{10}}$.
   a Use CAS to sketch the height of the balloon above the ground at time t and the rate at which the balloon is descending at time t.
   b If the balloon is anchored by the crew on the ground when it is 2 metres above the ground, how long did it take until the balloon was secured?
   c Find expressions for the velocity and acceleration of the balloon.
   d Calculate the rate at which the balloon is descending after 20 minutes.
   e Find the time when the balloon is descending at a rate of −9.0 m/minute.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Constant acceleration

study on

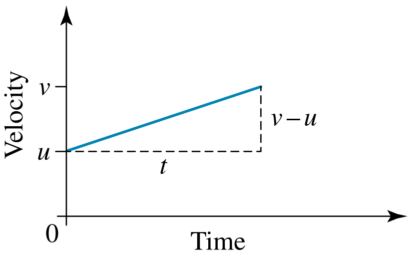
Units 3 & 4
AOS 3
Topic 5
Concept 4

Constant acceleration and equations of motion
Concept summary
Practice questions

Kinematics is the study of the motion of objects. In this topic, the focus is on objects that move along a straight line, which is also known as **rectilinear** motion.

When we consider the motion of an object in a straight line with uniform acceleration, a number of rules can be used.

The diagram below represents the motion of an object with initial velocity u and final velocity v after t seconds.



The gradient of the line is calculated by $\dfrac{v-u}{t}$. On a velocity–time graph, the gradient is the acceleration of the object. Hence, $a = \dfrac{v-u}{t}$.

Transposing $a = \dfrac{v-u}{t}$, we get $v = u + at$ (equation 1).

Alternatively, by antidifferentiating $\dfrac{dv}{dt} = a$ with respect to t:

$$\int \frac{dv}{dt}dt = \int a\, dt$$
$$v = at + c$$

When $t = 0$, $c = u$ (initial velocity):

$$v = at + u$$

The area under a velocity–time graph gives the displacement of the object. Therefore, using the rule for the area of a trapezium, we get $s = \frac{1}{2}(u + v)t$ (equation 2), where $\frac{1}{2}(u + v)$ is the average velocity of the object. Displacement is the average velocity of an object multiplied by time.

Substituting equation 1, $v = u + at$, into equation 2, $s = \frac{1}{2}(u + v)t$, gives:

$$s = \tfrac{1}{2}(u + u + at)t$$
$$s = \tfrac{1}{2}(2u + at)t$$
$$s = ut + \tfrac{1}{2}at^2 \quad \text{(equation 3)}$$

Alternatively, by antidifferentiating $\dfrac{dx}{dt} = v = u + at$ with respect to t,

$x = ut + \frac{1}{2}at^2 + d$, where d is the initial position.

When $s = x - d$, which is the change in position of the particle from its starting point, $s = ut + \frac{1}{2}at^2$.

Using $v = u + at$ (equation 1) and $s = ut + \frac{1}{2}at^2$ (equation 3):

$$v^2 = (u + at)^2$$
$$v^2 = u^2 + 2uat + a^2t^2$$
$$v^2 = u^2 + 2a\left(ut + \tfrac{1}{2}at^2\right)$$
$$v^2 = u^2 + 2as \quad \text{(equation 4)}$$

If we substitute $u = v - at$ into $s = ut + \frac{1}{2}at^2$, we get

$$s = (v - at)t + \tfrac{1}{2}at^2,$$

which simplifies to

$$s = vt - \tfrac{1}{2}at^2 \quad \text{(equation 5).}$$

In summary, when considering rectilinear motion of an object in which the acceleration is constant, the following rules can be used.

- $v = u + at$
- $s = ut + \frac{1}{2}at^2$
- $v^2 = u^2 + 2as$
- $s = \frac{1}{2}(u + v)t$
- $s = vt - \frac{1}{2}at^2$

where v = velocity (m/s), u = initial velocity (m/s), a = acceleration (m/s^2), t = time (s) and s = displacement (m)

In applying these rules, we must consider the following conditions.

- These quantities only apply when the acceleration is constant.
- Retardation or deceleration implies that acceleration is negative.
- The variable s is the displacement of an object. It is not necessarily the distance travelled by the object.

In solving constant acceleration problems it is important to list the quantities given to determine what is required and which equation is appropriate. It is also important to check units. All quantities must be converted to compatible units such as m and m/s or km and km/h.

Remember that to convert km/h to m/s, we multiply by $\frac{1000}{3600}$ or $\frac{5}{18}$; to convert m/s to km/h, we multiply by $\frac{18}{5}$ or 3.6.

WORKED EXAMPLE 1

A tram uniformly accelerates from a velocity of 10 m/s to a velocity of 15 m/s in a time of 20 s. Find the distance travelled by the tram.

THINK

1 List all the quantities given and check the units.

2 Determine the appropriate equation that will solve for distance, given all the quantities.

3 Substitute the quantities into the equation and solve.

WRITE

Units: m, s

$u = 10$, $v = 15$, $t = 20$

$s = \frac{1}{2}(u + v)t$

$s = \frac{1}{2}(10 + 15) \times 20$

$= 250$

The distance travelled by the tram is 250 m.

WORKED EXAMPLE 2

A triathlete on a bicycle reduces her speed from 10 m/s to 4 m/s over 28 m. Assuming the deceleration is constant, determine the deceleration and how long the triathlete will travel on her bicycle before she comes to rest.

THINK

1 List all the quantities given and check the units.

2 Determine the appropriate equation that will solve for deceleration given all the quantities.

3 Substitute the quantities into the equation and solve.

4 List all the quantities available to determine how long it will take for the triathlete to come to rest.

5 Determine the appropriate equation that will solve for deceleration given all the quantities.

6 Substitute the quantities into the equation and solve.

WRITE

Units: m, s

$u = 10$, $v = 4$, $s = 28$

$v^2 = u^2 + 2as$

$v^2 = u^2 + 2as$

$4^2 = 10^2 + 2 \times a \times 28$

$a = -1.5$

The deceleration of the triathlete is 1.5 m/s^2.

$u = 10$, $v = 0$, $a = -1.5$

$v = u + at$

$v = u + at$

$0 = 10 - 1.5 \times t$

$t = 6\frac{2}{3}$

The triathlete will come to rest after $6\frac{2}{3}$ seconds.

EXERCISE 4.2 Constant acceleration

PRACTISE

1 WE1 A train uniformly accelerates from a velocity of 30 m/s to a velocity of 40 m/s in a time of 15 s. Find the distance travelled by the train.

2 A train uniformly accelerates from a velocity of 20 m/s to a velocity of 50 m/s in a time of 20 s. Find the acceleration of the train.

3 WE2 A triathlete on a bicycle reduces his speed from 10 m/s to 6 m/s over 100 m. Assuming the deceleration is constant, determine the deceleration.

4 A triathlete on a bicycle reduces her speed from 8 m/s to 5 m/s over 150 m. Assuming the deceleration is constant, determine the time the triathlete will travel on her bicycle before she comes to rest.

CONSOLIDATE

5 A train uniformly accelerates from a velocity of 10 m/s to a velocity of 30 m/s in a time of 25 s. Find the distance travelled by the train.

6 A motorcyclist reduces her speed from 20 m/s to 5 m/s over 200 m. Assuming the deceleration is constant, determine:

a the deceleration
b the time the motorcyclist will travel on her motorbike before she comes to rest
c how much further she will travel on her motorbike before she comes to rest.

7 A skateboarder starting from rest rolls down a skate ramp. After 10 seconds, his velocity is 15 m/s. Assuming constant acceleration, find:

a the distance travelled by the skateboarder
b the acceleration of the skateboarder.

8 A snowboarder starting from rest travels down a ski slope. After 15 seconds, her velocity is 20 m/s. Find the time taken for her to travel 200 m.

9 A truck initially travelling at a constant speed is subject to a constant deceleration of 2 m/s^2, bringing it to rest in 6 seconds. Find:

a the initial speed of the truck
b the distance covered before the truck comes to rest.

10 A train is travelling at 25 m/s when the brakes are applied, reducing the speed to 10 m/s in 2 minutes. Assuming constant acceleration, find how far the train will travel in total before stopping.

11 A jet plane lands at one end of a runway of length 1200 m. It takes 15 seconds to come to rest and its deceleration is 4.2 m/s^2. Is the runway long enough for the landing of the plane?

12 A motorcycle moving with uniform acceleration is observed to cover 60 m in 5 seconds and 22 m in the next second. Find the motorcycle's:

a acceleration b initial velocity.

13 A cyclist travelling in a straight line passes a certain point at 5 m/s and is accelerating uniformly at 2 m/s^2 for a distance of 60 m. Find the time the cyclist takes to travel this distance.

14 A truck moving with constant acceleration is seen to move 21.5 m in 2 seconds and 15 m in the next second. What is the acceleration of the truck?

MASTER

15 A tram moves for 3 seconds with constant acceleration. During this time it travels 26.5 m. It then travels with uniform velocity, and during the next 3 seconds it travels 21.8 m. Find the tram's:

a acceleration b initial velocity.

16 An eagle is moving with uniform acceleration. In the third second of its motion, it covers 30.2 m, and in the seventh second, it covers 17.8 m. Find the eagle's:

a acceleration b initial velocity.

4.3 Motion under gravity

Vertical motion is when an object is projected vertically into the air, causing a one-dimensional movement along a straight line of force. The most common force around us is the Earth's gravity, so vertical motion in relation to that gravity is towards or away from the centre of the Earth.

Acceleration due to gravity is the form of acceleration that is directed towards the centre of the Earth.

The **acceleration due to gravity** is approximately constant when an object's motion is near the Earth's surface. When considering vertical motion of an object in a straight line in which the acceleration is constant, the equations for constant acceleration can be applied.

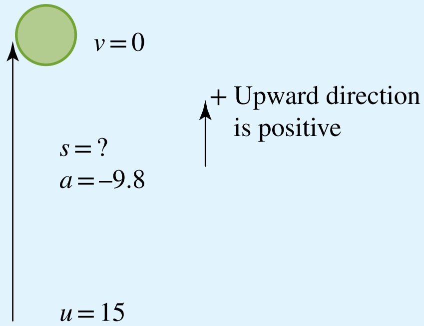
In vertical motion problems, acceleration due to gravity is generally preferred as g = 9.8 m/s^2 vertically downwards (towards the centre of Earth) rather than $g = -9.8$ m/s^2 upwards.

In solving vertical motion problems, we can take either the upward or downward direction as positive. It also helps to sketch the situation.

- If the upward direction is taken as positive, velocity is upwards and positive, and acceleration is downwards and negative.
- If the downward direction is taken as positive, velocity is upwards and negative, and acceleration is downwards and positive.

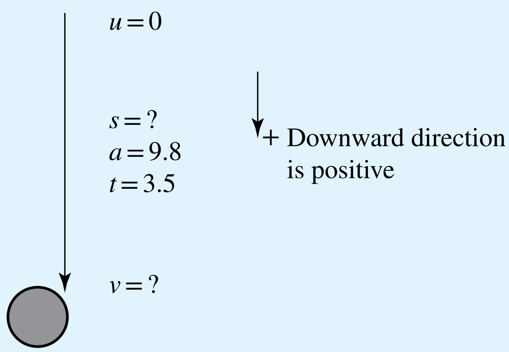
WORKED EXAMPLE 3

A tennis ball is thrown vertically upwards with a velocity of 15 m/s. What is the maximum height reached by the ball?

THINK	WRITE/DRAW
1 Draw a diagram and label the quantities. Let the direction be positive upwards. State the units.	Units: m, s 
2 List all the quantities given.	$u = 15$, $v = 0$, $a = g = -9.8$
3 Determine the appropriate constant acceleration equation that will solve for the height of the tennis ball.	$v^2 = u^2 + 2as$
4 Substitute the quantities into the equation and solve.	$v^2 = u^2 + 2as$ $0^2 = 15^2 + 2 \times -9.8 \times s$ $s = 11.48$ The tennis ball travels a distance of 11.48 m to its maximum height.

WORKED EXAMPLE 4

A rock is dropped down a well. It reaches the bottom of the well in 3.5 seconds. How deep is the well?

THINK	WRITE/DRAW
1 Draw a diagram and label the quantities. Let the direction be positive downwards. State the units.	Units: m, s 
2 List all the quantities given.	$t = 3.5$, $u = 0$, $a = g = 9.8$
3 Determine the appropriate constant acceleration equation that will solve for the depth of the well.	$s = ut + \frac{1}{2}at^2$

4 Substitute the quantities into the equation and solve.

$s = ut + \frac{1}{2}at^2$

$s = 0 + \frac{1}{2} \times 9.8 \times 3.5^2$

$s = 60.025$

The rock travels a distance of 60.025 m to the bottom of the well.

EXERCISE 4.3 Motion under gravity

PRACTISE

1 WE3 A basketball is thrown vertically upwards with a velocity of 5 m/s. What is the maximum height of the basketball in the air?

2 A skyrocket is projected vertically upwards to a maximum height of 154.34 m. Find the velocity of projection.

3 WE4 A rock is dropped down a well. It reaches the bottom of the well in 3.2 seconds. How deep is the well?

4 A stone is dropped from the top of a cliff 122.5 m high. How long does it take for the stone to reach the bottom of the cliff?

CONSOLIDATE

5 A ball is thrown into the air with a velocity of 8 m/s. How long does the ball take to reach its maximum height?

6 A shot is fired vertically upward and attains a maximum height of 800 m. Find the initial velocity of the shot.

7 A boulder falls from the top of a cliff 45 metres high. Find the boulder's speed just before it hits the ground.

8 A ball is thrown vertically upwards with a velocity of 15 m/s from the top of a building 20 metres high and then lands on the ground below. Find the time of flight for the ball.

9 A skyrocket is projected vertically upwards from the ground. It runs out of fuel at a velocity of 52 m/s and a height of 35 m. From this point on it is subject only to acceleration due to gravity. Find its maximum height.

10 A stone is projected vertically upwards with a velocity of 12 m/s from the top of a cliff. If the stone reaches the bottom of the cliff in 8 seconds, find:

a the height of the cliff

b the velocity at which the stone must be projected to reach the bottom of the cliff in 4 s.

11 From a hot air balloon rising vertically upward with a speed of 8 m/s, a sandbag is dropped which hits the ground in 4 seconds. Determine the height of the balloon when the sandbag was dropped.

12 A missile is projected vertically upward with a speed of 73.5 m/s, and 3 seconds later a second missile is projected vertically upward from the same point with the same speed. Find when and where the two missiles collide.

13 A flare, A, is fired vertically upwards with a velocity of 35 m/s from a boat. Four seconds later, another flare, B, is fired vertically upwards from the same point with a velocity of 75 m/s. Find when and where the flares collide.

14 A skyrocket is launched upwards with a velocity of 60 m/s. Two seconds later, another skyrocket is launched upwards from the same point with the same initial velocity. Find when and where the two skyrockets meet.

MASTER

15 A stone, A, is projected vertically upwards with a velocity of 25 m/s. After stone A has been in motion for 3 s, another stone, B, is dropped from the same point. Find when and where the two stones meet.

16 A worker climbs vertically up a tower to a certain height and accidently drops a small bolt. The man ascends a further 45 m and drops another small bolt. The second bolt takes 1 second longer than the first to reach the ground. Find:

a the height above the ground at which the worker dropped the first bolt

b the time it took the first bolt to reach the ground.

4.4 Velocity–time graphs

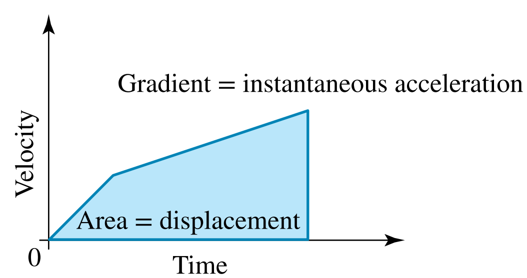
studyon

Units 3 & 4
AOS 3
Topic 5
Concept 3

Velocity–time graphs
Concept summary
Practice questions

Velocity–time graphs are a useful visual representation of the motion of an object in a straight line. We can use velocity–time graphs to solve kinematic problems. The following properties of velocity–time graphs make this possible.

- Because $a = \frac{dv}{dt}$, the gradient of the velocity–time graph at time t gives the instantaneous acceleration at time t.
- Because $v = \frac{dx}{dt}$, the displacement is found by evaluating the definite integral $\int_{t_1}^{t_2} v\,dt = x_2 - x_1$. The distance is found by determining the magnitude of the signed area under the curve bounded by the graph and the t axis, $\left|\int v(t)dt\right|$. Distance travelled cannot be a negative value.



Some useful formulas to assist in finding the displacement without using calculus are:

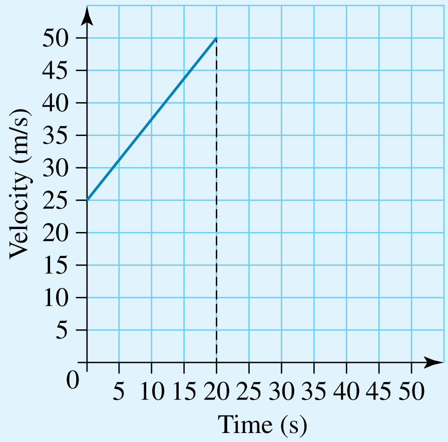
- area of a triangle: $A = \frac{1}{2}bh$
- area of a rectangle: $A = LW$
- area of a trapezium: $A = \frac{1}{2}(a + b)h$.

WORKED EXAMPLE 5

A train uniformly accelerates from a velocity of 25 m/s to a velocity of 50 m/s in a time of 20 seconds. Find the acceleration of the train and distance travelled by the train.

THINK | **WRITE/DRAW**

1 Draw a velocity–time graph.



2 To determine the acceleration of the train, we need to calculate the gradient of the graph.

$a = \frac{50 - 25}{20 - 0}$

$= 1.25$

The acceleration of the train is 1.25 m/s^2.

3 To determine the distance travelled we calculate the area under the graph. We can use the area of a trapezium.

$s = \frac{1}{2}(25 + 50) \times 20$

$= 750$

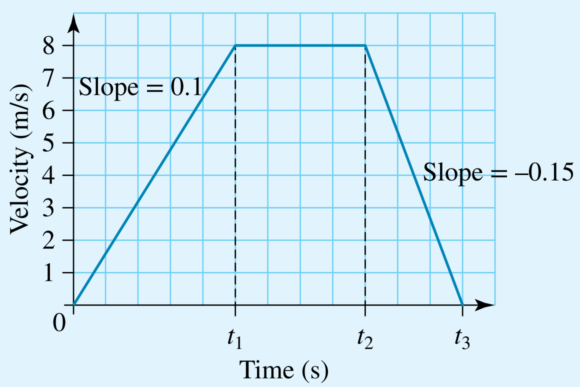
The distance travelled by the train is 750 m.

WORKED EXAMPLE 6

Puffing Billy runs on a straight track between Belgrave and Lakeside stations, which are 10 km apart. It accelerates at 0.10 m/s^2 from rest at Belgrave station until it reaches its maximum speed of 8 m/s. It maintains this speed for a time, then decelerates at 0.15 m/s^2 to rest at Lakeside station. Find the time taken for the Puffing Billy journey.

THINK | **WRITE/DRAW**

1 Draw a velocity–time graph.



2 To determine the time taken for Puffing Billy's journey, we need to use the distance travelled and area under the graph.

Total distance:

$s = \frac{1}{2} \times 8 \times t_1 + (t_2 - t_1) \times 8 + \frac{1}{2} \times 8 \times (t_3 - t_2)$

$= 10\,000$

3 To determine t_1, the time taken for Puffing Billy to accelerate to 8 m/s, we can use the acceleration and slope equation.	$a = 0.1$ $= \frac{8}{t_1}$ $t_1 = 80$
4 To determine the time taken to decelerate, $t_3 - t_2$, we can use the deceleration value and slope equation.	$a = 0.15$ $= \frac{8}{t}$ $t_1 = 53\frac{1}{3}$
5 Substituting the time values for the train to accelerate and decelerate, we can find the time taken for the train to move at a constant speed.	Let $T = t_2 - t_1$. $\frac{1}{2} \times 8 \times 80 + T \times 8 + \frac{1}{2} \times 8 \times 53\frac{1}{3} = 10000$ $T = 1183.33$
6 Calculating the total time.	$\text{Total time} = 80 + 1183.33 + 53.33$ $= 1316.66$ The time taken is 1316.66 seconds, or 21 minutes and 57 seconds.

EXERCISE 4.4 Velocity–time graphs

(Draw velocity–time graphs for all questions.)

PRACTISE

1 WE5 A train uniformly accelerates from a velocity of 15 m/s to a velocity of 40 m/s in a time of 15 seconds. Find the distance travelled by the train.

2 A train uniformly accelerates from a velocity of 25 m/s to a velocity of 55 m/s in a time of 10 seconds. Find the acceleration of the train and the distance travelled.

3 WE6 Puffing Billy runs on a straight track between Emerald and Lakeside stations, which are 12 km apart. It accelerates at 0.05 m/s^2 from rest at Emerald station until it reaches its maximum speed of 8 m/s. It maintains this speed for a time, then decelerates at 0.10 m/s^2 to rest at Lakeside station. Find the time taken for Puffing Billy's journey.

4 A tram runs on a straight track between two stops, a distance of 4 km. It accelerates at 0.1 m/s^2 from rest at stop A until it reaches its maximum speed of 15 m/s. It maintains this speed for a time, then decelerates at 0.05 m/s^2 to rest at stop B. Find the time taken for the tram journey.

CONSOLIDATE

5 A train uniformly accelerates from a velocity of 10 m/s to a velocity of 40 m/s in a time of 20 seconds. Find the distance travelled by the train.

6 A tram slows down to rest from a velocity of 15 m/s at a constant deceleration of 0.5 m/s^2. Find the distance travelled by the tram.

7 A car starting from rest speeds up uniformly to a velocity of 12 m/s in 15 seconds, continues at this velocity for 20 seconds and then slows down to a stop in 25 seconds. How far has the car travelled?

8 A rocket is travelling with a velocity of 75 m/s. The engines are switched on for 8 seconds and the rocket accelerates uniformly at 40 m/s^2. Calculate the distance travelled over the 8 seconds.

9 The current world record for the 100-metre men's sprint is 9.58 seconds, run by Usain Bolt in 2009. Assuming that the last 40 m was run at a constant speed and that the acceleration during the first 60 m was constant, calculate Usain's acceleration.

10 A tourist train runs on a straight track between Belvedere and Eureka stations, which are 8 km apart. It accelerates at 0.07 m/s^2 from rest at Belvedere until it reaches its maximum speed of 8 m/s. It maintains this speed for a time, then decelerates at 0.05 m/s^2 to rest at Eureka. Find the time taken for the journey.

11 Two racing cars are travelling along the same straight road. At time $t = 0$, car A passes car B. Car A is travelling at a constant velocity of 60 m/s. Car B accelerates from rest until it reaches 80 m/s after 20 seconds, and then it maintains that speed. What distance does car B travel before it overtakes car A?

12 A stationary unmarked police car is passed by a speeding car travelling at a constant velocity of 70 km/h. The police car accelerates from rest until it reaches 30 m/s after 15 seconds, which speed it then maintains. Find the time taken for the unmarked police car to catch up to the speeding car.

13 A bus takes 100 seconds to travel between two bus stops 1.5 km apart. It starts from rest and accelerates uniformly to a speed of 25 m/s, then maintains that speed until the brakes are applied to decelerate. If the time taken for acceleration is the same as deceleration, find the acceleration.

14 A particle travels in a straight line with a constant velocity of 40 m/s for 10 seconds. It is then subjected to a constant acceleration in the opposite direction for 20 seconds, which returns the particle to its original position. Sketch a velocity–time graph to represent the motion of the particle.

MASTER

15 During a fireworks display, a skyrocket accelerates from rest to 30 m/s after 8 seconds. It is then subjected to a constant acceleration in the opposite direction for 4 seconds. It reaches its maximum height after 10 seconds.

a Sketch a velocity–time graph to represent the motion of the skyrocket.

b Find:

i the maximum height the skyrocket will reach

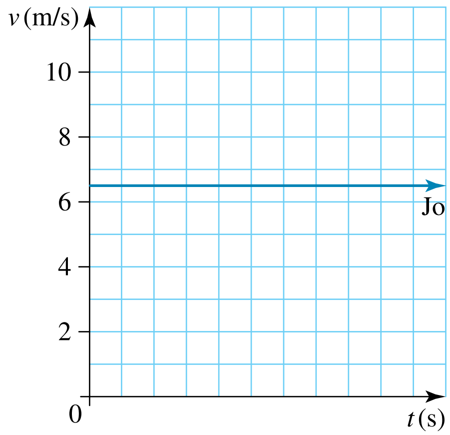
ii the time at which the skyrocket will hit the ground on its return.

16 At time $t = 0$, Jo is cycling on her bike at a speed of 6.5 m/s along a straight bicycle path and passes her friend Christina, who is stationary on her own bike.

Four seconds later, Christina accelerates in the direction of Jo for 8 seconds so that her speed, v m/s, is given by $v = (t-4)\tan\left(\frac{\pi}{48}t\right)$. Christina then maintains her speed of 8 m/s.

a Show that Christina accelerates to a speed of 8 m/s.

b i On the velocity–time graph shown, draw a graph representing Christina's speed.



ii Find the time at which Christina passes Jo on her bike.

4.5 Variable acceleration

There are many different types of motion in which acceleration is not constant. If we plot the velocity of an object against time, then the acceleration, $\frac{dv}{dt}$, can be estimated by drawing the tangent to the graph at that time and finding the slope of the tangent. Alternatively, we can use calculus.

studyon
Units 3 & 4
AOS 3
Topic 5
Concept 5
Expressions for acceleration
Concept summary
Practice questions

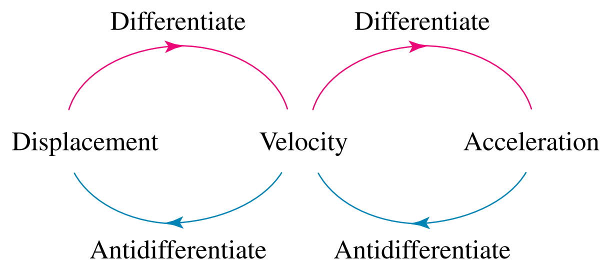
We know that instantaneous acceleration at time t can be found by $a = \frac{dv}{dt}$. We also know that the distance covered between two time intervals is the area under a velocity–time graph, which can be calculated by the integral $x = \int_{t_1}^{t_2} v\,dt$.

Furthermore, if we start with displacement, we know that velocity is the rate at which displacement varies and acceleration is the rate at which velocity varies, so we can write $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$.

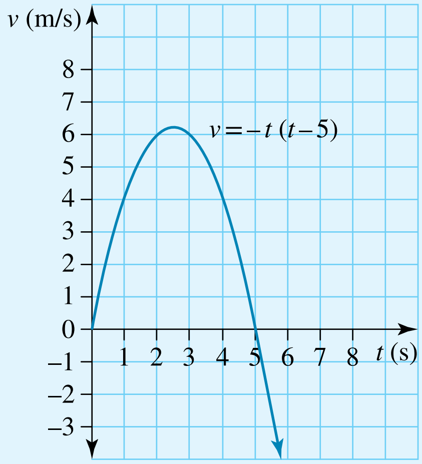
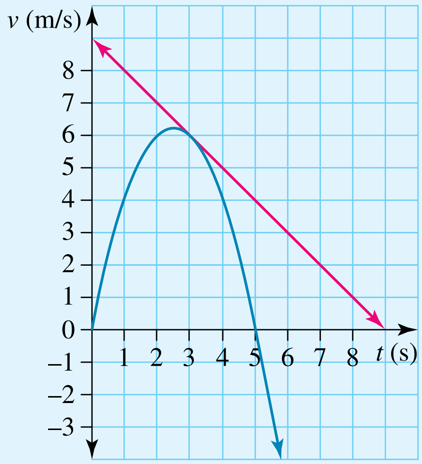
If we start with acceleration of a moving object as a function of time, velocity can be found by integrating a with respect to t, and displacement can be found by integrating v with respect to t. Therefore, we can write

$$v = \int a\,dt = \int \frac{dv}{dt}\,dt \text{ and } x = \int v\,dt = \int \frac{dx}{dt}\,dt.$$

In summary:

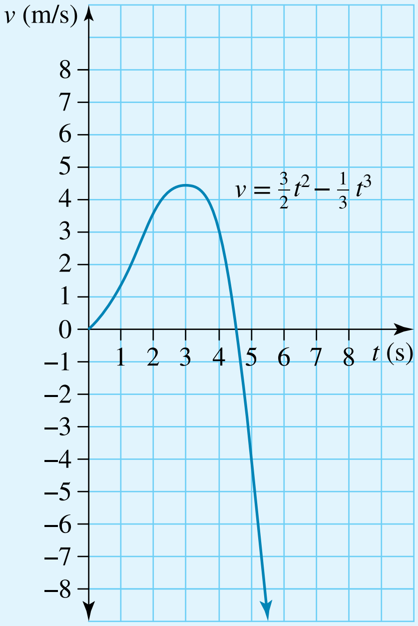
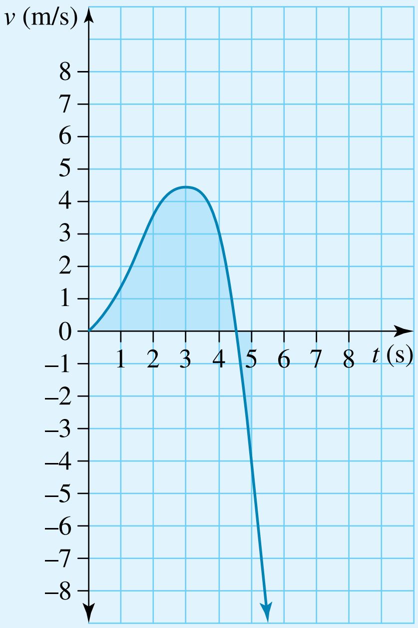


WORKED EXAMPLE 7 The motion of an object along a straight line is modelled by the equation $v = -t(t - 5)$ where v m/s is the velocity and time is t seconds. What is the acceleration at $t = 3$?

THINK	WRITE/DRAW
1 Draw a velocity–time graph.	
2 If we draw a tangent at $t = 3$, we can approximate the acceleration. We can see the acceleration is negative and approximately -1 m/s^2 by using the coordinates (3, 6) and (7, 2) to calculate the slope.	
3 To calculate the instantaneous acceleration at time t, we need to use $a = \frac{dv}{dt}$.	$v = -t(t - 5)$ $v = -t^2 + 5t$ $\frac{dv}{dt} = -2t + 5$
4 To calculate the acceleration at $t = 3$, substitute into $a = -2t + 5$.	$a = -2t + 5$ $a = -2 \times 3 + 5$ $a = -6 + 5$ $a = -1$ The acceleration at $t = 3$ for the motion of an object modelled by $v = -t(t - 5)$ is -1 m/s^2.

WORKED EXAMPLE 8

A dog starts from rest to go on a walk. Its acceleration can be modelled by the equation $a = 3t - t^2$, where the units are metres and seconds. What is the distance the dog has travelled in the first 5 seconds of its walk?

THINK	WRITE/DRAW
1 Given $a = 3t - t^2$, velocity can be found by integrating a with respect to t.	$\int a\,dt = \int 3t - t^2\,dt$ $v = \frac{3}{2}t^2 - \frac{1}{3}t^3 + c$ When $t = 0$, $v = 0$, so $c = 0$. $\therefore v = \frac{3}{2}t^2 - \frac{1}{3}t^3$
2 Draw a velocity–time graph of $v = \frac{3}{2}t^2 - \frac{1}{3}t^3$, representing the motion of the dog.	
3 The distance travelled is the area between the curve and the t-axis on a velocity–time graph.	

4 Calculate the distance by setting up and solving the definite integral equations.

Find when $v = 0$ to determine the limits for integration.

When $v = 0$,

$$\frac{3}{2}t^2 - \frac{1}{3}t^3 = 0$$

$$t^2\left(\frac{3}{2} - \frac{1}{3}t\right) = 0$$

$$t = 0 \text{ or } t = \frac{9}{2}$$

$$x = \int_0^{4.5} \frac{3}{2}t^2 - \frac{1}{3}t^3\,dt - \int_{4.5}^{5} \frac{3}{2}t^2 - \frac{1}{3}t^3\,dt$$

$$= \left[\frac{t^3}{2} - \frac{t^4}{12}\right]_0^{4.5} - \left[\frac{t^3}{2} - \frac{t^4}{12}\right]_{4.5}^{5}$$

$$= 11.39 + 0.97$$

$$= 12.36$$

The distance travelled by the dog in the first 5 seconds is 12.36 m.

Acceleration can be expressed in four different ways.

As stated earlier, acceleration is the rate at which velocity varies as a function of time. When acceleration is given as a function of time, $a = f(t)$, we use the expression $a = \frac{dv}{dt}$.

Knowing that $a = \frac{d}{dt}(v)$ and $v = \frac{dx}{dt}$, we have $a = \frac{d}{dt}\left(\frac{dx}{dt}\right)$, which gives another expression, $a = \frac{d^2x}{dt^2}$.

When acceleration is written as a function of displacement, $a = f(x)$, we know that $a = \frac{dv}{dt}$. Using the chain rule, we get $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$.

Since $v = \frac{dx}{dt}$, we have $a = \frac{dv}{dx} \times v$, which simplifies to the expression $a = v\frac{dv}{dx}$.

We use this expression when a relationship between velocity and displacement is required.

Knowing that $a = \frac{dv}{dx} \times v$ and $v = \frac{d}{dv}\left(\frac{1}{2}v^2\right)$, we can derive $a = \frac{dv}{dx} \times \frac{d}{dv}\left(\frac{1}{2}v^2\right)$, which simplifies to the expression $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$. We use this expression when acceleration is given a function of displacement.

So acceleration can be expressed in any of these forms:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

The form to use in a particular mathematical problem will depend on the form of the equation defining acceleration.

- If $a = f(t)$, then use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$.
- If $a = f(x)$, then use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.
- If $a = f(v)$, then use $\frac{dv}{dt}$ if the initial conditions are given in terms of t and v, or $a = v\frac{dv}{dx}$ if the initial conditions relate to v and x.

WORKED EXAMPLE 9

A particle moves in a straight line. When the particle's displacement from a fixed origin is x m, its velocity is v m/s and its acceleration is a m/s^2. Given that $a = 4x$ and that $v = -3$ when $x = 0$, find v in terms of x.

THINK	WRITE
1 Decide which acceleration form to use to find the relationship between v and x. $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	$a = 4x$ $v\frac{dv}{dx} = 4x$
2 Solve for v in terms of x by using integration.	$\int\left(v\frac{dv}{dx}\right)dx = \int(4x)\,dx$ $\int(v)\,dv = \int(4x)\,dx$ $\frac{1}{2}v^2 = 2x^2 + c$ $v^2 = 4x^2 + d$ $v = \pm\sqrt{4x^2 + d}$
3 Calculate the constant by using the initial conditions.	When $v = -3$ and $x = 0$, $v = \pm\sqrt{4x^2 + d}$ $-3 = -\sqrt{d}$ $d = 9$
4 Determine the relationship between v and x.	The relationship between v and x is given by $v = -\sqrt{4x^2 + 9}$.

EXERCISE 4.5 Variable acceleration

Throughout this exercise, units are metres and seconds unless otherwise stated.

PRACTISE

1 **WE7** The motion of an object along a straight line is modelled by the equation $v = -t(t - 6)$, where v m/s is the velocity and time is t seconds. What is the acceleration at $t = 2$?

2 The motion of an object along a straight line is modelled by the equation $v = -\frac{1}{2}t^2 - 4t$, where v m/s is the velocity and time is t seconds. What is the acceleration at $t = 7$?

3 **WE8** An object starting from rest accelerates according to the equation $a = 4 - t$, where the units are metres and seconds. What is the distance the object has travelled in the first 3 seconds?

4 A sprinter starting from rest accelerates according to the equation $a = 3t - t^2$, where the units are metres and seconds. What is the distance the sprinter has travelled in the first 2 seconds?

5 WE9 A particle moves in a straight line. When the particle's displacement from a fixed origin is x m, its velocity is v m/s and its acceleration is a m/s^2. Given that $a = 16x$ and that $v = 4$ when $x = 0$, find the relationship between v and x.

6 A particle moves in a straight line. When the particle's displacement from a fixed origin is x m, its velocity is v m/s and its acceleration is a m/s^2. Given that $a = 2v^3$ and that $v = 2$ when $x = 0$, find the relationship between x and v.

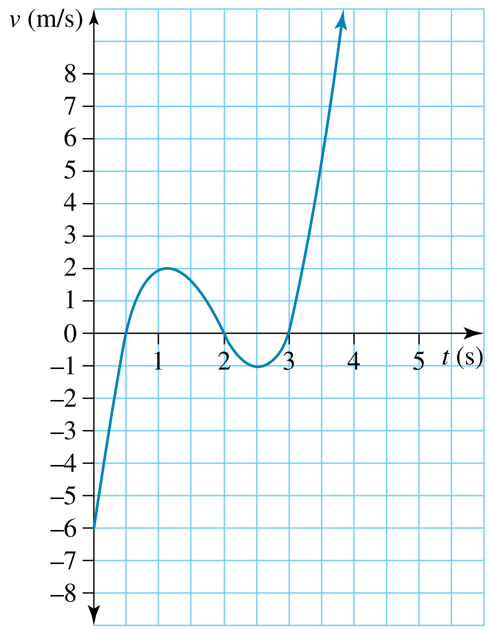
CONSOLIDATE

7 The motion of an object along a straight line is modelled by the equation $v = -t(t - 6)$, where v is the velocity and t is the time. What is the acceleration at $t = 3.5$?

8 The motion of an object along a straight line is modelled by the equation $v = 10\left(\frac{1}{2}t^2 + e^{-0.1t}\right)$, where v is the velocity and t is the time. What is the acceleration at $t = 2$?

9 A jet plane starting from rest accelerates according to the equation $a = 30 - 2t$. What is the distance the jet has travelled in the first 15 seconds?

10 The diagram below shows the motion of an object along a straight line.



a When is the object at rest?
b When is the acceleration equal to zero?
c Approximate the acceleration at $t = 1.5$, correct to 1 decimal place.

11 A golf ball is putted on the green with an initial velocity of 6 m/s and decelerates uniformly at a rate of 2 m/s^2. If the hole is 12 m away, will the golf ball reach it?

12 An object travels in a line so that the velocity, v m/s, is given by $v^2 = 10 - 2x^2$. Find the acceleration at $x = 2$.

13 The velocity, v m/s, and the acceleration, a m/s^2, of a particle at time t seconds after the particle is dropped from rest are given by $a = \frac{1}{50}(490 - v)$, $0 \le v < 490$. Express v in terms of t.

14 If $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - 3x^2$ and $v = 2$ when $x = 0$, find v in terms of x.

15 A particle moves in a straight line. At time t its displacement from a fixed origin is x.

a If $v = 2x + 5$, find a in terms of x.

b If $v = \dfrac{(x-1)^2}{2}$, and $x = 0$ when $t = 0$, find x when $t = 2$.

c If $a = \dfrac{1}{(x+4)^2}$, and $v = 0$ when $x = 0$, find x when $v = \dfrac{1}{2}$.

d If $a = 3 - v$, and $v = 0$ when $x = 0$, find x when $v = 2$.

16 A truck travelling at 20 m/s passes a stationary speed camera and then decelerates so that its velocity, v m/s, at time t seconds after passing the speed camera is given by $v = 20 - 2\tan^{-1}(t)$.

a After how many seconds will the truck's speed be 17 m/s?

b Explain why v will never equal 16.

c **i** Write a definite integral that gives the distance, x metres, travelled by the truck after T seconds.

ii Find the distance travelled by the truck at $t = 8$.

MASTER

17 An object falls from a hovering surf-lifesaving helicopter over Port Phillip Bay at 500 m above sea level. Find the velocity of the object when it hits the water when the acceleration of the object is $0.2v^2 - g$.

18 **a** A particle moves from rest at the origin, O, with an acceleration of $v^3 + \pi^2 v$ m/s^2, where v is the particle's velocity measured in m/s. Find the velocity of the particle when it is 0.75 m to the right of O.

b A particle travels in a straight line with velocity v m/s at time t s. The acceleration of the particle, a m/s^2, is given by $a = -2 + \sqrt{v^2 + 5}$. Find the time it takes for the speed of the particle to increase from 2 m/s to 8 m/s.

eBook*plus* ONLINE ONLY

4.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

4 Answers

EXERCISE 4.2

1 525 m

2 1.5 m/s^2

3 –0.32 m/s^2

4 61.5 s

5 500 m

6 a –0.94 m/s^2 b 21.33 s c 13.33 m

7 a 75 m b 1.5 m/s^2

8 17.32 s

9 a 12 m/s b 36 m

10 2500 m

11 Yes; 472.5 m to rest

12 a $3\frac{1}{3}$ m/s^2 b $3\frac{2}{3}$ m/s

13 5.64 s

14 2.83 m/s^2

15 a –1.02 m/s^2 b 10.4 m/s

16 a –3.1 m/s^2 b 37.95 m/s

EXERCISE 4.3

1 1.28 m

2 55 m/s

3 50.176 m

4 5 s

5 0.816 s

6 125.22 m/s

7 29.7 m/s

8 4.06 s

9 172.96 m

10 a 217.6 m b 48.3 m/s downward

11 46.4 m

12 6 s after missile 2 is projected; 264.6 m above the point of projection

13 55.37 m; 4.78 s after flare A is projected

14 7.122 s; 178.73 m above point

15 7.022 s after B is dropped; 241.66 m below the point of projection

16 a 32.14 m b 2.56 s

EXERCISE 4.4

1 412.5 m

2 3 m/s^2; 400 m

3 15 min

4 $491\frac{2}{3}$ s or 8.19 min

5 500 m

6 225 m

7 480 m

8 1880 m

9 2.3245 m/s^2

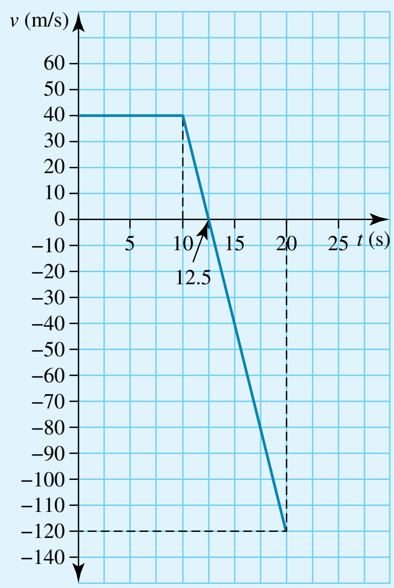
10 18.95 min or 1137.14 s

11 2.4 km

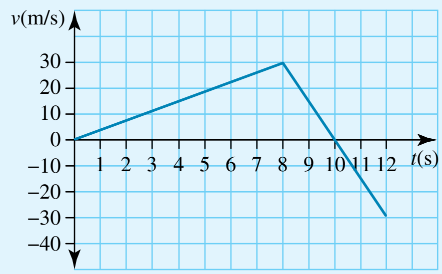
12 21.3158 s

13 0.625 m/s^2

14



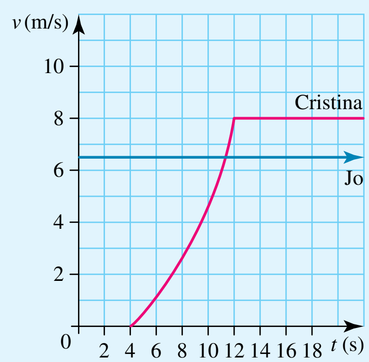
15 a



b i 150 m ii 15.5 s

16 a 8 m/s

b i



ii 48.74 s

EXERCISE 4.5

1 2 m/s^2

2 -11 m/s^2

3 13.5 m

4 $2\frac{2}{3}$ m

5 $v = 4\sqrt{x^2 + 1}$

6 $x = -\dfrac{1}{2v} + \dfrac{1}{4}$

7 -1 m/s^2

8 19.18 m/s^2

9 2250 m

10 a 0.5 s, 2 s, 3 s

b 1 s and 2.5 s

c Approximately -3 m/s^2

11 No, only travels 9 m

12 -4 m/s^2

13 $v = 490\left(1 - e^{-\frac{t}{50}}\right)$

14 $v^2 = -2x^3 + 2x^2 + 4$

15 a $a = 4x + 10$

b $\frac{1}{2}$ m

c 4 m

d 1.296 m or $3 \log_e 3 - 2$ m

16 a 14.1 s

b As $t \to \infty$, $v \to 20 - \pi$

c i $\displaystyle\int_0^T 20 - 2\tan^{-1}(t)\, dt$

ii 141.03 m

17 7 m/s

18 a -3.142 m/s

b 2.19 s

5

Vectors in three dimensions

5.1 Kick off with CAS

Exploring the lengths of vectors with CAS

A vector in three dimensions from the origin to the terminal point P(x, y, z) is represented as $\overrightarrow{OP} = \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$. We often need to calculate the length of the vector. With CAS, this can be done using the 'norm' command and representing a vector as 1×3 matrix: norm ($[x \quad y \quad z]$)

1 Use CAS to find the length of each of the following vectors.

a $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

b $\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

c $\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

d $3\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k}$

e $3\underset{\sim}{i} + 4\underset{\sim}{j} - 12\underset{\sim}{k}$

f $3\underset{\sim}{i} - 4\underset{\sim}{j} - 12\underset{\sim}{k}$

2 Does it matter if some of the values are negative?

3 Determine a general formula for the length of the vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

4 Find the value of t:

a if the vector $\underset{\sim}{a} = 2\underset{\sim}{i} + t\underset{\sim}{j} - 3\underset{\sim}{k}$ has a length of 5

b if the two vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + t\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2t\underset{\sim}{i} - \underset{\sim}{j}$ are equal in length.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

5.2 Vectors

Review of vectors

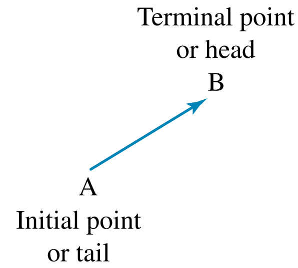
studyon
Units 3 & 4
AOS 4
Topic 1
Concept 1
Operations with vectors
Concept summary
Practice questions

A quantity that can be completely described by its magnitude or size expressed in a particular unit is called a **scalar quantity**.

A quantity that can be completely described by stating both its magnitude or size expressed in a particular unit and its direction is called a **vector quantity**.

Vector notation

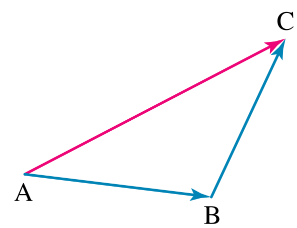
To represent a vector geometrically, a directed line segment is used. If the vector goes from point A to point B, then A is called the initial point or tail and B is called the terminal point or head. The vector from A to B is denoted by $\overrightarrow{AB}$; the arrow indicates the direction. A vector can also be denoted as a variable, for example $\underset{\sim}{a}$; the line underneath (called a tilde) indicates that it is a vector.



Addition of two vectors

If a particle moves in a straight line from a point A to another point B, and then in another straight line from point B to a final point C, the final position is found by adding the two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. This is equivalent to $\overrightarrow{AC}$.

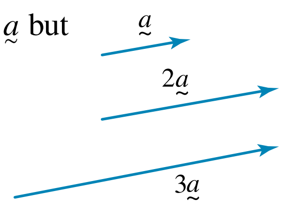
Vector addition follows the triangle rule for addition. If two vectors are placed head to tail, then their **resultant** or sum is the vector joining the head of the first to the tail of the second: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



Scalar multiplication of vectors

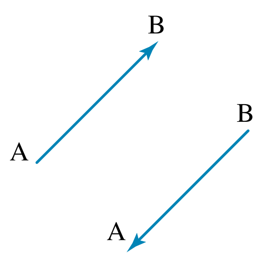
If $\underset{\sim}{a}$ is a vector, then the vector $2\underset{\sim}{a} = \underset{\sim}{a} + \underset{\sim}{a}$ is a vector parallel to $\underset{\sim}{a}$ but twice its length.

The vector $3\underset{\sim}{a} = 2\underset{\sim}{a} + \underset{\sim}{a}$ is a vector parallel to $\underset{\sim}{a}$ but three times its length.



The negative of a vector

If the head and tail of a vector are reversed, the vector will have the same length but will point in the opposite direction, so $\overrightarrow{AB} = -\overrightarrow{BA}$.

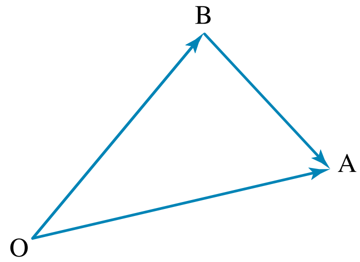


Subtraction of vectors

To subtract two vectors, use $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$.

Consider the vectors $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ where O is the origin.

$$\begin{aligned}\underset{\sim}{a} - \underset{\sim}{b} &= \underset{\sim}{a} + (-\underset{\sim}{b}) \\ &= \overrightarrow{OA} - \overrightarrow{OB} \\ &= \overrightarrow{BO} + \overrightarrow{OA} \\ &= \overrightarrow{BA}\end{aligned}$$



The zero or null vector

Note that $\underset{\sim}{a} + (-\underset{\sim}{a}) = \underset{\sim}{0}$. This is a vector of no magnitude and no direction. In fact, if all the sides of the triangle above are added, then

$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BO} &= (\overrightarrow{OA} + \overrightarrow{AB}) + \overrightarrow{BO} \\ &= \overrightarrow{OB} + \overrightarrow{BO} \\ &= \overrightarrow{OB} - \overrightarrow{OB} \\ &= \underset{\sim}{0}.\end{aligned}$$

This is the rule for the addition of three vectors placed head to tail. It is justified by the associative law.

The algebra of vectors

Vectors satisfy the field laws using vector addition. Let $\underset{\sim}{a}, \underset{\sim}{b} \in V$, where V is the set of vectors; then under the operation of vector addition, these five field laws are also satisfied.

1. Closure:	$\underset{\sim}{a} + \underset{\sim}{b} \in V$
2. Commutative law:	$\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a}$
3. Associative law:	$(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c} = \underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$
4. Additive identity law:	$\underset{\sim}{a} + \underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{a} = \underset{\sim}{a}$
5. Inverse law:	$\underset{\sim}{a} + (-\underset{\sim}{a}) = (-\underset{\sim}{a}) + \underset{\sim}{a} = \underset{\sim}{0}$

Using these laws, vector expressions can be simplified.

WORKED EXAMPLE 1

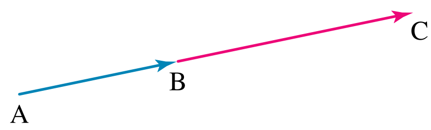
Simplify the vector expression $4\overrightarrow{AB} - \overrightarrow{CB} - 4\overrightarrow{AC}$.

THINK	WRITE
1 As we can sum vectors, express the negative as sums of vectors.	$4\overrightarrow{AB} - \overrightarrow{CB} - 4\overrightarrow{AC}$ $= 4\overrightarrow{AB} + \overrightarrow{BC} + 4\overrightarrow{CA}$
2 Group the terms with the common factor.	$= 4\overrightarrow{AB} + 4\overrightarrow{CA} + \overrightarrow{BC}$
3 Reorder and take out the common factor so we can add the vectors.	$= 4\overrightarrow{CA} + 4\overrightarrow{AB} + \overrightarrow{BC}$ $= 4(\overrightarrow{CA} + \overrightarrow{AB}) + \overrightarrow{BC}$
4 Perform the vector addition.	$= 4\overrightarrow{CB} + \overrightarrow{BC}$
5 Express $\overrightarrow{BC} = -\overrightarrow{CB}$	$= 4\overrightarrow{CB} - \overrightarrow{CB}$
6 State the final simplified vector expression.	$= 3\overrightarrow{CB}$

Note that there may be many ways to arrive at this simplified vector expression.

Collinear points

Three points are said to be collinear if they all lie on the same straight line. In vector terms, if the vector $\overrightarrow{AB}$ is parallel to the vector $\overrightarrow{BC}$, then these two parallel vectors have the point B in common, and so they must lie in a straight line. Hence A, B and C are collinear if $\overrightarrow{AB} = \lambda \overrightarrow{BC}$ where $\lambda \in R \setminus \{0\}$.



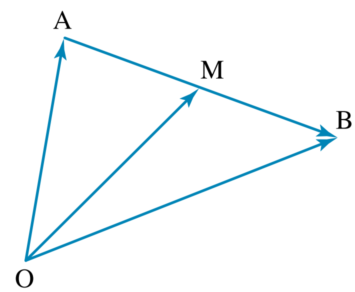
WORKED EXAMPLE 2 If $3\overrightarrow{OA} - 2\overrightarrow{OB} - \overrightarrow{OC} = \underset{\sim}{0}$, show that the points A, B and C are collinear.

THINK	WRITE
1 Use the negative of a vector, in the middle term, $-\overrightarrow{BO} = \overrightarrow{OB}$.	$3\overrightarrow{OA} - 2\overrightarrow{OB} - \overrightarrow{OC} = \underset{\sim}{0}$ $3\overrightarrow{OA} + 2\overrightarrow{BO} - \overrightarrow{OC} = \underset{\sim}{0}$
2 Let $3\overrightarrow{OA} = 2\overrightarrow{OA} + \overrightarrow{OA}$.	$2\overrightarrow{OA} + \overrightarrow{OA} + 2\overrightarrow{BO} - \overrightarrow{OC} = \underset{\sim}{0}$
3 Reorder the terms.	$2\overrightarrow{BO} + 2\overrightarrow{OA} + \overrightarrow{OA} - \overrightarrow{OC} = \underset{\sim}{0}$
4 Take out the common factor in the first two terms and use vector addition.	$2(\overrightarrow{BO} + \overrightarrow{OA}) + \overrightarrow{OA} - \overrightarrow{OC} = \underset{\sim}{0}$ $2\overrightarrow{BA} + \overrightarrow{OA} - \overrightarrow{OC} = \underset{\sim}{0}$
5 Take the last two terms across to the right-hand side.	$2\overrightarrow{BA} = \overrightarrow{OC} - \overrightarrow{OA}$
6 Use the negative of a vector. This statement shows that A, B and C are collinear, since the point A is common.	$2\overrightarrow{BA} = \overrightarrow{AO} + \overrightarrow{OC}$ $2\overrightarrow{BA} = \overrightarrow{AC}$

Midpoints

If O is the origin and A and B are points, the midpoint M of the line segment AB is given by

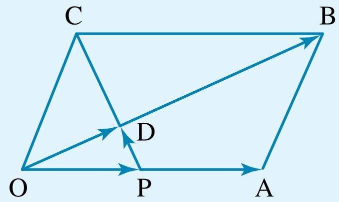
$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB}\\ &= \overrightarrow{OA} + \tfrac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})\\ &= \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})\end{aligned}$$



Applications to geometry

Vectors can be used to prove many of the theorems of geometry. These vector proofs involve the properties of vectors discussed so far.

WORKED EXAMPLE 3 OABC is a parallelogram. P is the midpoint of OA, and the point D divides PC in the ratio 1 : 2. Prove that O, D and B are collinear.

THINK	WRITE/DRAW
1 Draw a parallelogram and label it as OABC. Mark in the point P as the midpoint of OA. Mark in the point D dividing PC into thirds, with the point D closer to P on PC since $d(PC) = 3d(DP)$.	
2 P is the midpoint of $\overrightarrow{OA}$.	$\overrightarrow{OP} = \overrightarrow{PA} = \frac{1}{2}\overrightarrow{OA}$
3 Since $d(PC) = 3d(DP)$, and D is on the line segment PC, $\overrightarrow{PD} = \frac{1}{3}\overrightarrow{PC}$.	$\overrightarrow{PD} = \frac{1}{3}\overrightarrow{PC}$
4 Consider an expression for $\overrightarrow{OD}$.	$\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD}$
5 Substitute for the expression above.	$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{PC}$
6 Use $\overrightarrow{PC} = \overrightarrow{PO} + \overrightarrow{OC}$, or $\overrightarrow{PC} = \overrightarrow{OC} - \overrightarrow{OP}$	$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}(\overrightarrow{PO} + \overrightarrow{OC})$ $= \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}(\overrightarrow{OC} - \overrightarrow{OP})$
7 Substitute again for $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA}$.	$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\left(\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA}\right)$
8 Take out the common factor of and simplify the fraction.	$\overrightarrow{OD} = \left(\frac{1}{2} - \frac{1}{6}\right)\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OC}$ $= \frac{1}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OC}$
9 However, since OABC is a parallelogram, $\overrightarrow{OC} = \overrightarrow{AB}$.	$\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$ $= \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{AB})$
10 As $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$, this vector equation implies that the points O, D and B are collinear.	$\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ O, D and B are collinear.

EXERCISE 5.2 Vectors

PRACTISE

1 **WE1** Simplify the vector expression $4\overrightarrow{CB} - \overrightarrow{AB} + 4\overrightarrow{AC}$.

2 Simplify the vector expression $2\overrightarrow{BO} - 5\overrightarrow{AO} - 2\overrightarrow{BA}$.

3 **WE2** If $\overrightarrow{AO} + \overrightarrow{OB} - 2\overrightarrow{BO} - 2\overrightarrow{OC} = \underset{\sim}{0}$, show that the points A, B and C are collinear.

4 If $\overrightarrow{PO} - 4\overrightarrow{RO} + 3\overrightarrow{QO} = \underset{\sim}{0}$, show that the points P, Q and R are collinear.

5 **WE3** OABC is a parallelogram in which P is the midpoint of CB, and D is a point on AP such that $d(AD) = \frac{2}{3}d(AP)$. Prove that $\overrightarrow{OD} = \frac{2}{3}\overrightarrow{OB}$ and that O, D and B are collinear.

6 OABC is a parallelogram in which the point P divides OA in the ratio 1 : 2, and the point D divides PC in the ratio 1 : 3. Prove that $\overrightarrow{OD} = \frac{1}{4}\overrightarrow{OB}$ and that O, D and B are collinear.

CONSOLIDATE

7 Simplify each of the following vector expressions.

a $2\overrightarrow{AC} - \overrightarrow{CB} + \overrightarrow{AB}$

b $5\overrightarrow{CA} + \overrightarrow{BC} + 4\overrightarrow{OC} - \overrightarrow{BO}$

c $\overrightarrow{OC} + 6\overrightarrow{AB} + \overrightarrow{CA} + 5\overrightarrow{OA}$

d $3\overrightarrow{OB} + \overrightarrow{AB} - 3\overrightarrow{AC} + 4\overrightarrow{BC} - \overrightarrow{OC}$

8 For each of the following vector expressions, show that the points A, B and C are collinear.

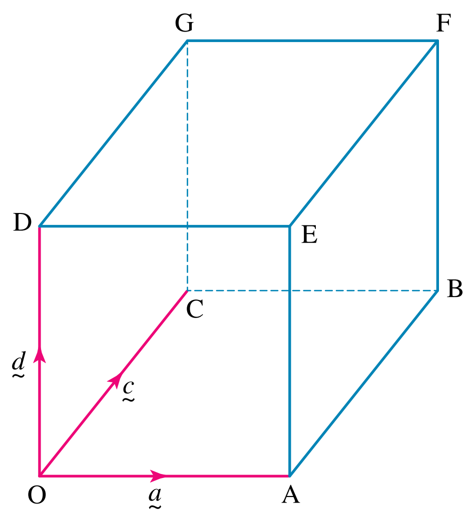
a $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$

b $3\overrightarrow{OA} - 2\overrightarrow{OB} = \overrightarrow{OC}$

c $\overrightarrow{BO} + 4\overrightarrow{AO} - 5\overrightarrow{CO} = \underset{\sim}{0}$

d $3\overrightarrow{BO} - 5\overrightarrow{CO} + 2\overrightarrow{AO} = \underset{\sim}{0}$

9 OABCGDEF is a cuboid with $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{OD} = \underset{\sim}{d}$.



a List all the vectors equal to $\overrightarrow{OA} = \underset{\sim}{a}$.

b List all the vectors equal to $\overrightarrow{OC} = \underset{\sim}{c}$.

c List all the vectors equal to $\overrightarrow{OD} = \underset{\sim}{d}$.

d Express each of the following in terms of $\underset{\sim}{a}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$.

i $\overrightarrow{DF}$ ii $\overrightarrow{EB}$

iii $\overrightarrow{FO}$ iv $\overrightarrow{DB}$

10 a ABC is a triangle. The points P and Q are the midpoints of AB and BC respectively. Show that PQ is parallel to AC and that the length of PQ is half the length of AC.

b ABC is a triangle. The point P divides AB in the ratio 2 : 1, and the point Q divides BC in the ratio 1 : 2. Show that PQ is parallel to AC and that the length of PQ is one-third the length of AC.

11 a OAB is a triangle with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. Let M be the midpoint of AB. Show that $\overrightarrow{OM} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$.

b OPQ is a triangle. The point M divides PQ in the ratio 1 : 2. Show that $\overrightarrow{OM} = \frac{1}{3}(2\overrightarrow{OP} + \overrightarrow{OQ})$.

c OAB is a triangle with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. The point M divides PQ in the ratio 1 : 3. Show that $\overrightarrow{OM} = \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$.

12 a ABCD is a quadrilateral. The points P and Q are the midpoints of AB and BC respectively. Show that $\overrightarrow{PQ} = \overrightarrow{AP} + \overrightarrow{QC}$.

b ABCD is a quadrilateral. P is a point on DB such that $\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PD} = \overrightarrow{PC}$. Show that ABCD is a parallelogram.

13 a ABCD is a quadrilateral. The points M and N are the midpoints of AB and CD respectively. Show that $2\overrightarrow{MN} = \overrightarrow{BC} + \overrightarrow{AD}$.

b ABCD is a trapezium with sides AB and DC parallel. P and Q are the midpoints of the sides AD and BC respectively. Show that PQ is parallel to both AB and DC and that the distance PQ is one half the sum of the distances AB and DC.

14 **a** ABC is a triangle. The points D, E and F are the midpoints of the sides AB, BC and CA respectively. Show that $\overrightarrow{AE} + \overrightarrow{BF} + \overrightarrow{CD} = \underset{\sim}{0}$.

b ABC is a triangle. The points P, Q and R are the midpoints of the sides AB, BC and CA respectively. If O is any other point, show that $\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.

15 **a** ABCD is a rectangle. The points P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively. Show that PQRS is a parallelogram.

b ABCD is a parallelogram. The points M and N are the midpoints of the diagonals AC and DB respectively. Show that M and N are coincident.

16 ABCD is a quadrilateral. The points M and N are the midpoints of AC and BD respectively. Show that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{MN}$.

MASTER

17 ABC is a triangle with $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

a Let D be the midpoint of AB and G_1 be the point which divides CD in the ratio 2 : 1. Show that $\overrightarrow{OG_1} = \frac{1}{3}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$.

b Let E be the midpoint of BC, and G_2 be the point which divides AE in the ratio 2 : 1. Show that $\overrightarrow{OG_2} = \frac{1}{3}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$.

c Let F be the midpoint of AC, and G_3 be the point which divides BF in the ratio 2 : 1. Show that $\overrightarrow{OG_3} = \frac{1}{3}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$.

d What can be deduced about the points G_1, G_2 and G_3?

18 OABC is a quadrilateral, with $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

a The points P and Q are the midpoints of OA and BC respectively. Let M be the midpoint of PQ. Express $\overrightarrow{OM}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

b The points R and S are the midpoints of AB and OC respectively. Let N be the midpoint of SR. Express $\overrightarrow{ON}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

c The points U and V are the midpoints of AC and OB respectively. Let T be the midpoint of UV. Express $\overrightarrow{OT}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

d What can be deduced about the points M, N and T?

5.3 $\underset{\sim}{i}$ $\underset{\sim}{j}$ $\underset{\sim}{k}$ notation

Unit vectors

studyon

Units 3 & 4
AOS 4
Topic 1
Concept 5

Unit vectors
Concept summary
Practice questions

A unit vector is a vector that has a magnitude of 1. Unit vectors are direction givers. A hat or circumflex above the vector, for example $\hat{\underset{\sim}{a}}$, is used to indicate that it is a unit vector.

If $\underset{\sim}{a}$ is a vector, then the length or magnitude of the vector is denoted by $|\underset{\sim}{a}|$. Dividing the vector by its length makes it a unit vector, so that $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

A unit vector in the positive direction parallel to the x-axis is denoted by $\underset{\sim}{i}$.

A unit vector in the positive direction parallel to the y-axis is denoted by $\underset{\sim}{j}$.

A unit vector in the positive direction parallel to the z-axis is denoted by $\underset{\sim}{k}$.

Since we understand that $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors, we do not need to place the hat or circumflex above these vectors, although some other notations do use this notation and write $\hat{\underset{\sim}{i}}$, $\hat{\underset{\sim}{j}}$ and $\hat{\underset{\sim}{k}}$.

If a vector is expressed in terms of $\underset{\sim}{i}$, then the coefficient of $\underset{\sim}{i}$ represents the magnitude parallel to the x-axis and the $\underset{\sim}{i}$ indicates that this vector is parallel to the x-axis.

If a vector is expressed in terms of $\underset{\sim}{j}$, then the coefficient of $\underset{\sim}{j}$ represents the magnitude parallel to the y-axis and the $\underset{\sim}{j}$ indicates that this vector is parallel to the y-axis.

If a vector is expressed in terms of $\underset{\sim}{k}$, then the coefficient of $\underset{\sim}{k}$ represents the magnitude parallel to the z-axis and the $\underset{\sim}{k}$ indicates that this vector is parallel to the z-axis.

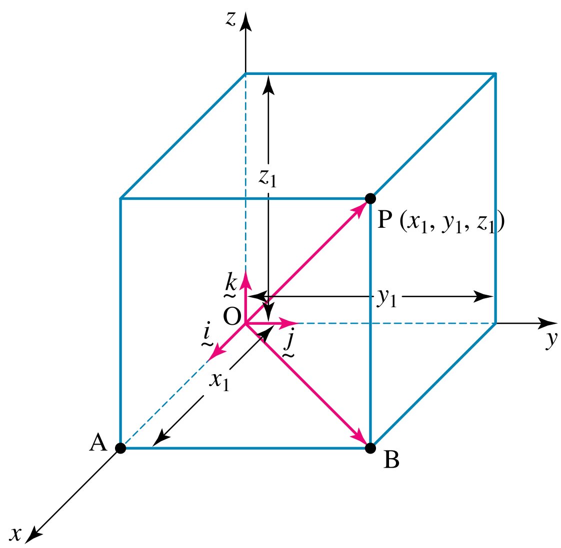
Position vectors

studyon

Units 3 & 4
AOS 4
Topic 1
Concept 2

Position vectors
Concept summary
Practice questions

In two-dimensional Cartesian coordinates, a point P is represented by (x_1, y_1); that is, we need two coordinates to represent it. In three dimensions we need three coordinates to represent a point. The convention for showing three dimensions on a page is to show the x-axis coming out of the page and the y- and z-axes flat on the page.



Consider a point P with coordinates (x_1, y_1, z_1) relative to the origin, O (0, 0, 0). Use $\underset{\sim}{r}$ as a notation for the position vector. The vector $\underset{\sim}{r} = \overrightarrow{OP}$ can be expressed in terms of three other vectors: one parallel to the x-axis, one parallel to the y-axis and one parallel to the z-axis. This method of splitting a vector up into its components is called resolution of vectors.

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$$

but $d(OA) = x_1$, $d(AB) = y_1$ and $d(BP) = z_1$,

so the vectors $\overrightarrow{OA} = x_1\underset{\sim}{i}$, $\overrightarrow{AB} = y_1\underset{\sim}{j}$ and $\overrightarrow{BP} = z_1\underset{\sim}{k}$.

Therefore, $\underset{\sim}{r} = \overrightarrow{OP} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$.

A vector expressed in $\underset{\sim}{i}\ \underset{\sim}{j}\ \underset{\sim}{k}$ form is said to be in component form. We can now give answers to questions in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

Magnitude of a vector

studyon

Units 3 & 4
AOS 4
Topic 1
Concept 4

Magnitude of a vector
Concept summary
Practice questions

In the diagram above, the distance between the origin O and the point P is the length or magnitude of the vector. By Pythagoras' theorem using the triangle in the xy plane, $d^2(OB) = d^2(OA) + d^2(AB) = x_1^2 + y_1^2$ and using triangle OPB, $d^2(OP) = d^2(OB) + d^2(BP) = x_1^2 + y_1^2 + z_1^2$.

The magnitude of the position vector is thus given by

$$|\underset{\sim}{r}| = |\overrightarrow{OP}| = d(OP) = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

WORKED EXAMPLE 4

If the point P has coordinates (3, 2, –4), find:

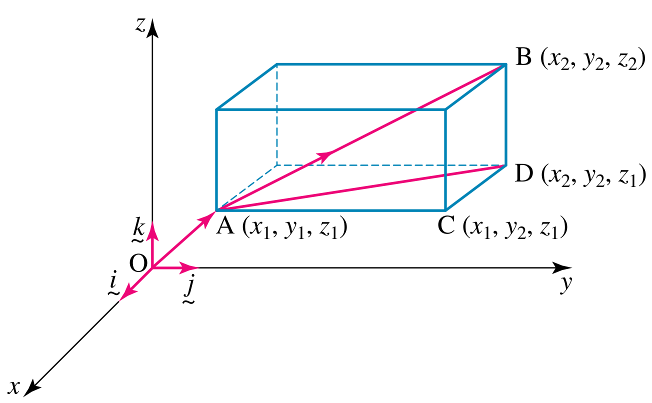
a the vector $\overrightarrow{OP}$ **b** a unit vector parallel to $\overrightarrow{OP}$.

THINK	WRITE
a There is no need to draw a three-dimensional diagram. The vector $\overrightarrow{OP}$ has the x-coordinate of P as the $\underset{\sim}{i}$ component, the y-coordinate of P as the $\underset{\sim}{j}$ component and the z-coordinate of P as the $\underset{\sim}{k}$ component.	**a** The point (3, 2, –4) has coordinates $x = 3$, $y = 2$ and $z = -4$. $\overrightarrow{OP} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$
b 1 First find the magnitude or length of the vector using $\lvert\overrightarrow{OP}\rvert = d(OP) = \sqrt{x^2 + y^2 + z^2}$. Leave the answer as an exact answer, that is as a surd.	**b** $\lvert\overrightarrow{OP}\rvert = \sqrt{3^2 + 2^2 + (-4)^2}$ $= \sqrt{9 + 4 + 16}$ $= \sqrt{29}$
2 A unit vector parallel to $\overrightarrow{OP}$ is $\hat{\overrightarrow{OP}} = \dfrac{\overrightarrow{OP}}{\lvert OP\rvert}$. There is no need to rationalise the denominator.	$\hat{\overrightarrow{OP}} = \frac{1}{\sqrt{29}}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k})$

Addition and subtraction of vectors in three dimensions

If A is the point (x_1, y_1, z_1) and B is the point (x_2, y_2, z_2) relative to the origin O, then $\overrightarrow{OA} = \underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$.

To add two vectors in component form, add the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components separately. So $\overrightarrow{OA} + \overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{b} = (x_1 + x_2)\underset{\sim}{i} + (y_1 + y_2)\underset{\sim}{j} + (z_1 + z_2)\underset{\sim}{k}$.
To subtract two vectors in component form, subtract the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components separately. So $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{b} - \underset{\sim}{a} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j} + (z_2 - z_1)\underset{\sim}{k}$.
The vector $\overrightarrow{AB}$ represents the position vector of B relative to A, that is B as seen from A.
The distance between the points A and B, $d(AB)$, is the magnitude of the vector: $d(AB) = \lvert\overrightarrow{AB}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.



WORKED EXAMPLE 5

Two points, A and B, have the coordinates (1, –2, 1) and (3, 4, –2) respectively. Find a unit vector parallel to $\overrightarrow{AB}$.

THINK	WRITE
1 Write the vector $\overrightarrow{OA}$.	$\overrightarrow{OA} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

2 Write the vector $\overrightarrow{OB}$.

$\overrightarrow{OB} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$

3 The vector $\overrightarrow{AB}$ is obtained by subtracting the two vectors $\overrightarrow{OB}$ and $\overrightarrow{OA}$.

Substitute for the two vectors.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}) - (\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$$

4 Use the rules for subtraction of vectors.

$$\overrightarrow{AB} = (3-1)\underset{\sim}{i} + (4-(-2))\underset{\sim}{j} + (-2-1)\underset{\sim}{k}$$
$$= 2\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}$$

5 Find the magnitude of the vector $\overrightarrow{AB}$.

$$|\overrightarrow{AB}| = \sqrt{2^2 + 6^2 + (-3)^2}$$
$$= \sqrt{4 + 36 + 9}$$
$$= \sqrt{49}$$
$$= 7$$

6 Write the unit vector parallel to $\overrightarrow{AB}$, that is $\hat{\overrightarrow{AB}} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$.

$$\hat{\overrightarrow{AB}} = \tfrac{1}{7}(2\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k})$$

Equality of two vectors

Given the vectors $\overrightarrow{OA} = \underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, the two vectors are equal, $\underset{\sim}{a} = \underset{\sim}{b}$, if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

Scalar multiplication of vectors

For scalar multiplication of vectors, $\lambda\underset{\sim}{a} = \lambda x_1\underset{\sim}{i} + \lambda y_1\underset{\sim}{j} + \lambda z_1\underset{\sim}{k}$. That is, each coefficient is multiplied by the scalar.

For example, if $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, then $2\underset{\sim}{a} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 6\underset{\sim}{k}$ and $-\underset{\sim}{a} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

The following worked examples further illustrate scalar multiplication and equality of vectors in component forms.

WORKED EXAMPLE 6

If $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + z\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k}$, find the value of z if the vector $\underset{\sim}{c} = 3\underset{\sim}{a} - 2\underset{\sim}{b}$ is parallel to the xy plane.

THINK / **WRITE**

1 Substitute for the given vectors.

$$\underset{\sim}{c} = 3\underset{\sim}{a} - 2\underset{\sim}{b}$$
$$= 3(2\underset{\sim}{i} - 3\underset{\sim}{j} + z\underset{\sim}{k}) - 2(4\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k})$$

2 Use the rules for multiplying a vector by a scalar.

$$\underset{\sim}{c} = (6\underset{\sim}{i} - 9\underset{\sim}{j} + 3z\underset{\sim}{k}) - (8\underset{\sim}{i} - 10\underset{\sim}{j} - 4\underset{\sim}{k})$$

3 Use the rules for subtraction of vectors.

$$\underset{\sim}{c} = (6-8)\underset{\sim}{i} + (-9+10)\underset{\sim}{j} + (3z+4)\underset{\sim}{k}$$
$$\underset{\sim}{c} = -2\underset{\sim}{i} + \underset{\sim}{j} + (3z+4)\underset{\sim}{k}$$

4 If this vector is parallel to the xy plane, then its $\underset{\sim}{k}$ component must be zero.

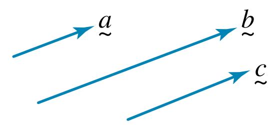
$$3z + 4 = 0$$

5 Solve for the unknown value in this case.

$$3z = -4$$
$$z = -\tfrac{4}{3}$$

Parallel vectors

When solving vector problems, it is sometimes necessary to recall some of the other properties of vectors.



In the diagram at right, the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are all parallel, as they are pointing in the same direction; however, they have different lengths. Two vectors are parallel if one is a scalar multiple of the other. That is, $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$ if $\underset{\sim}{a} = \lambda \underset{\sim}{b}$ where $\lambda \in R$.

WORKED EXAMPLE 7

Given the vectors $\underset{\sim}{r} = \underset{\sim}{i} - 3\underset{\sim}{j} + z\underset{\sim}{k}$ and $\underset{\sim}{s} = -2\underset{\sim}{i} + 6\underset{\sim}{j} - 7\underset{\sim}{k}$, find the value of z in each case if:

a **the length of the vector $\underset{\sim}{r}$ is 8**

b **the vector $\underset{\sim}{r}$ is parallel to the vector $\underset{\sim}{s}$.**

THINK	WRITE
a 1 Determine the magnitude of the vector in terms of the unknown value.	**a** $\underset{\sim}{r} = \underset{\sim}{i} - 3\underset{\sim}{j} + z\underset{\sim}{k}$ $\lvert\underset{\sim}{r}\rvert = \sqrt{1^2 + (-3)^2 + z^2}$ $= \sqrt{1 + 9 + z^2}$ $= \sqrt{10 + z^2}$
2 Equate the length of the vector to the given value.	Since $\lvert\underset{\sim}{r}\rvert = 8$, $\sqrt{10 + z^2} = 8$
3 Square both sides.	$10 + z^2 = 64$
4 Solve for the unknown value. Both answers are acceptable values.	$z^2 = 54$ $z = \pm\sqrt{54}$ $= \pm\sqrt{9 \times 6}$ $= \pm 3\sqrt{6}$
b 1 If two vectors are parallel, then one is a scalar multiple of the other. Substitute for the given vectors and expand.	**b** $\underset{\sim}{r} = \lambda \underset{\sim}{s}$ $\underset{\sim}{i} - 3\underset{\sim}{j} + z\underset{\sim}{k} = \lambda(-2\underset{\sim}{i} + 6\underset{\sim}{j} - 7\underset{\sim}{k})$ $= -2\lambda\underset{\sim}{i} + 6\lambda\underset{\sim}{j} - 7\lambda\underset{\sim}{k}$
2 For the two vectors to be equal, all components must be equal.	From the $\underset{\sim}{i}$ component, $-2\lambda = 1$, and from the $\underset{\sim}{j}$ component, $6\lambda = -3$, so $\lambda = -\frac{1}{2}$. From the $\underset{\sim}{k}$ component, $z = -7\lambda$.
3 Solve for the unknown value in this case.	$z = -7\lambda$ and $\lambda = -\frac{1}{2}$, so $z = \frac{7}{2}$.

studyon

Units 3 & 4
AOS 4
Topic 1
Concept 3

Linear dependence and independence
Concept summary
Practice questions

Linear dependence

Consider the three non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. The vectors are said to be **linearly dependent** if there exist non-zero scalars α, β and γ such that $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} + \gamma\underset{\sim}{c} = \underset{\sim}{0}$.

If $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} + \gamma\underset{\sim}{c} = \underset{\sim}{0}$, then we can write $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$. That is, one of the vectors is a linear combination of the other two, where $m = -\frac{\alpha}{\gamma}$ and $n = -\frac{\beta}{\gamma}$; since $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$, it follows that $m \neq 0$ and $n \neq 0$.

The set of non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are said to be **linearly independent** if $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} + \gamma\underset{\sim}{c} = \underset{\sim}{0}$, only if $\alpha = 0$, $\beta = 0$ and $\gamma = 0$.

For example, the two vectors $\underset{\sim}{a} = 4\underset{\sim}{i} - 8\underset{\sim}{j} + 12\underset{\sim}{k}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j} - 9\underset{\sim}{k}$ are linearly dependent, since $3\underset{\sim}{a} + 4\underset{\sim}{b} = \underset{\sim}{0}$ or alternatively $\underset{\sim}{a} = -\frac{4}{3}\underset{\sim}{b}$; in fact, the vectors are parallel.

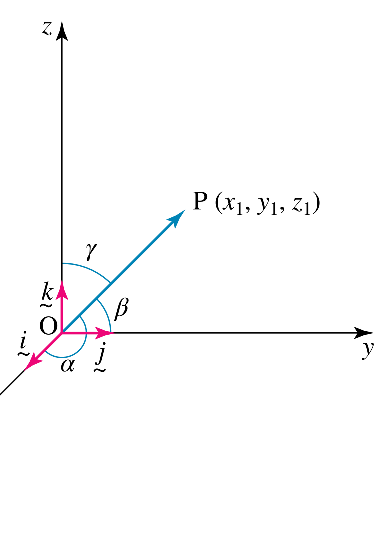
WORKED EXAMPLE 8

Show that the vectors $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = 4i - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{c} = 5i - \underset{\sim}{j} + z\underset{\sim}{k}$ are linearly dependent, and determine the value of z.

THINK	WRITE
1 Since the vectors are linearly dependent, they can be written as a linear combination.	$\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$
2 Substitute for the given vectors.	$5i - \underset{\sim}{j} + z\underset{\sim}{k} = m(\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}) + n(4\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})$
3 Use the rules for scalar multiplication.	$5i - \underset{\sim}{j} + z\underset{\sim}{k} = (m\underset{\sim}{i} - m\underset{\sim}{j} + 4m\underset{\sim}{k}) + (4n\underset{\sim}{i} - 2n\underset{\sim}{j} + 3n\underset{\sim}{k})$
4 Use the rules for addition of vectors.	$5i - \underset{\sim}{j} + z\underset{\sim}{k} = (m + 4n)\underset{\sim}{i} - (m + 2n)\underset{\sim}{j} + (4m + 3n)\underset{\sim}{k}$
5 Since the two vectors are equal, their components are equal.	$\underset{\sim}{i}$: $m + 4n = 5$ (1) $\underset{\sim}{j}$: $-m - 2n = -1$ (2) $\underset{\sim}{k}$: $4m + 3n = z$ (3)
6 Solve equations (1) and (2) for m and n.	(1) + (2): $2n = 4$ $n = 2$
7 Substitute and find the value of m.	$m = 5 - 4n$ $= 5 - 8$ $= -3$
8 Use the values of m and n to solve for z.	Substitute $n = 2$ and $m = -3$ into (3): $z = -12 + 6$ $= -6$
9 State the result.	The vectors are linearly dependent since $\underset{\sim}{c} = 2\underset{\sim}{b} - 3\underset{\sim}{a}$.

Direction cosines

Let α be the angle that the vector $\overrightarrow{OP} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ makes with the positive x-axis, let β be the angle that the vector $\overrightarrow{OP}$ makes with the positive y-axis, and let γ be the angle $\overrightarrow{OP}$ makes with the positive z-axis.



Generalising from the two-dimensional case,

$$\cos(\alpha) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{|\underset{\sim}{r}|},$$

$$\cos(\beta) = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{|\underset{\sim}{r}|} \text{ and}$$

$$\cos(\gamma) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{|\underset{\sim}{r}|}.$$

$\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$ are called the direction cosines. Also,

$\hat{\underset{\sim}{a}} = \cos(\alpha)\, \underset{\sim}{i} + \cos(\beta)\, \underset{\sim}{j} + \cos(\gamma)\, \underset{\sim}{k}$

so $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$.

WORKED EXAMPLE 9

Find the angle to the nearest degree that the vector $2\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k}$ makes with the z-axis.

THINK	WRITE
1 Give the vector a name.	Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k}$.
2 Find the magnitude of the vector.	$\begin{aligned}\lvert\underset{\sim}{a}\rvert &= \sqrt{2^2 + (-3)^2 + (-4)^2} \\ &= \sqrt{4 + 9 + 16} \\ &= \sqrt{29}\end{aligned}$
3 The angle that the vector makes with the z-axis is given by $\cos(\gamma) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{\lvert\underset{\sim}{a}\rvert}$	$\cos(\gamma) = \frac{-4}{\sqrt{29}}$
4 Find the angle using a calculator, making sure the angle mode is set to degrees format.	$\begin{aligned}\gamma &= \cos^{-1}\left(\frac{-4}{\sqrt{29}}\right) \\ &= 137.97^\circ\end{aligned}$
5 State the result, giving the answer in decimal degrees.	The vector makes an angle of 138° with the z-axis.

Application problems

As in the two-dimensional case, we can find the position vector of moving objects in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$. Usually $\underset{\sim}{i}$ is a unit vector in the east direction, $\underset{\sim}{j}$ is a unit vector in the north direction and $\underset{\sim}{k}$ is a unit vector vertically upwards.

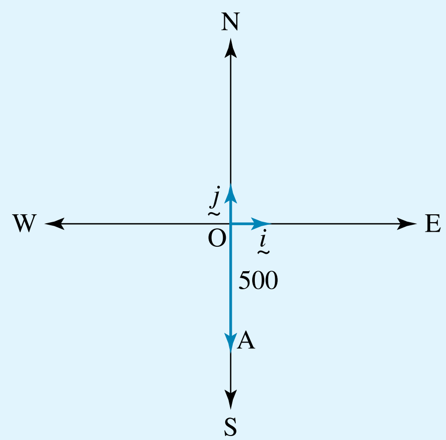
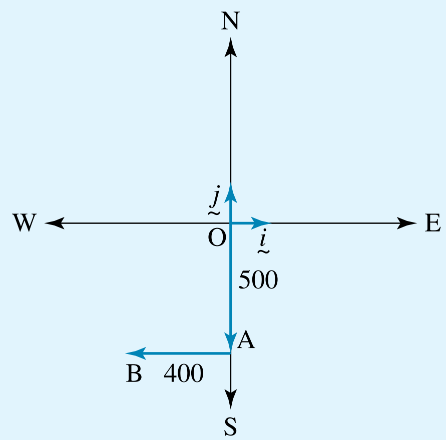
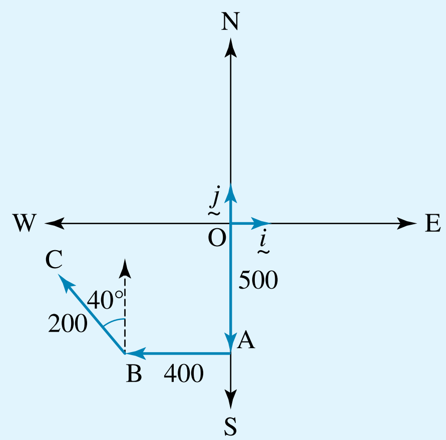
WORKED EXAMPLE 10

Mary walks 500 metres due south, turns and moves 400 metres due west, and then turns again to move in a direction N40°W for a further 200 metres. In all three of those movements she is at the same altitude. At this point, Mary enters a building and travels 20 metres vertically upwards in a lift.

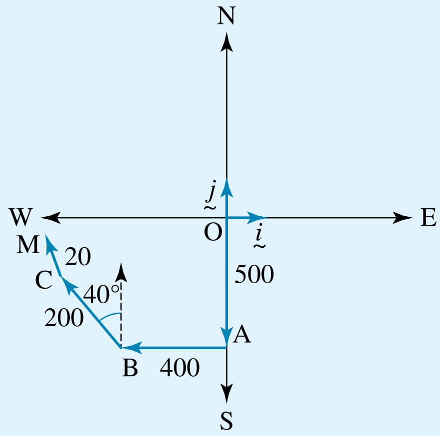
Let $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ represent unit vectors of length 1 metre in the directions of east, north and vertically upwards respectively.

a Find the position vector of Mary when she leaves the lift, relative to her initial position.

b Find her displacement correct to 1 decimal place in metres from her initial point.

THINK	WRITE/DRAW
a 1 Consider the two-dimensional north–east situation. First Mary walks 500 metres due south, from the initial point O to a point A.	**a**  $\overrightarrow{OA} = -500\underset{\sim}{j}$
2 Next Mary walks 400 metres due west, from A to B.	 $\overrightarrow{AB} = -400\underset{\sim}{i}$
3 Mary now walks in a direction N40°W for a further 200 metres, from B to a point C.	 Resolve the vector $\overrightarrow{BC}$ into components: $\overrightarrow{BC} = -200\sin(40°)\underset{\sim}{i} + 200\cos(40°)\underset{\sim}{j}$

4 Now Mary goes up in the lift from C to her final point, M.



$\overrightarrow{CM} = 20\underset{\sim}{k}$

5 To find the position vector of Mary, sum the vectors.

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM}\\ &= -500\underset{\sim}{j} - 400\underset{\sim}{i} + (-200\sin(40°)\underset{\sim}{i}\\ &\quad + 200\cos(40°)\underset{\sim}{j}) + 20\underset{\sim}{k}\end{aligned}$$

6 Add and subtract the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components.

$$\begin{aligned}\overrightarrow{OM} &= -(400 + 200\sin(40°))\underset{\sim}{i}\\ &\quad + (200\cos(40°) - 500)\underset{\sim}{j} + 20\underset{\sim}{k}\end{aligned}$$

7 Use a calculator, giving the results to 2 decimal places.

The position vector of Mary is

$\overrightarrow{OM} = -528.56\underset{\sim}{i} - 346.79\underset{\sim}{j} + 20\underset{\sim}{k}$

b Her total displacement is the magnitude of the vector.

b $$\begin{aligned}|\overrightarrow{OM}| &= \sqrt{(-528.56)^2 + (-346.79)^2 + 20^2}\\ &= 632.5 \text{ metres}\end{aligned}$$

Column vector notation

Because vectors have similar properties to matrices, it is common to represent vectors as column matrices. Vectors in three dimensions can be represented by the unit vectors $\underset{\sim}{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\underset{\sim}{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\underset{\sim}{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

The vector from the origin O to a point P with coordinates (x_1, y_1, z_1) can be expressed as $\overrightarrow{OP} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$. Thus, this is represented as a column matrix by $\overrightarrow{OP} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + y_1\begin{bmatrix}0\\1\\0\end{bmatrix} + z_1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$.

Operations such as addition, subtraction and scalar multiplication are performed on matrices in similar ways to those performed on vectors. For this reason, the sets of vectors and column matrices are called isomorphic, a Greek word meaning having the same structure.

WORKED EXAMPLE 11

Given the vectors represented as $A = \begin{bmatrix}4\\5\\-5\end{bmatrix}$, $B = \begin{bmatrix}5\\3\\-2\end{bmatrix}$ and $C = \begin{bmatrix}3\\7\\-8\end{bmatrix}$, show that the points A, B and C are linearly dependent.

THINK	WRITE
1 Consider the matrix A and vector $\underset{\sim}{a}$.	$A = \begin{bmatrix} 4 \\ 5 \\ -5 \end{bmatrix}$, $\underset{\sim}{a} = 4\underset{\sim}{i} + 5\underset{\sim}{j} - 5\underset{\sim}{k}$
2 Consider the matrix B and vector $\underset{\sim}{b}$.	$B = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$, $\underset{\sim}{b} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$
3 Consider the matrix C and vector $\underset{\sim}{c}$.	$C = \begin{bmatrix} 3 \\ 7 \\ -8 \end{bmatrix}$, $\underset{\sim}{c} = 3\underset{\sim}{i} + 7\underset{\sim}{j} - 8\underset{\sim}{k}$
4 Find the matrix C as a linear combination of A and B.	$C = \alpha A + \beta B$
5 Write the equation in matrix form.	$\begin{bmatrix} 3 \\ 7 \\ -8 \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 5 \\ -5 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$
6 Use the properties of scalar multiplication and addition of matrices.	$\begin{bmatrix} 3 \\ 7 \\ -8 \end{bmatrix} = \begin{bmatrix} 4\alpha \\ 5\alpha \\ -5\alpha \end{bmatrix} + \begin{bmatrix} 5\beta \\ 3\beta \\ -2\beta \end{bmatrix} = \begin{bmatrix} 4\alpha + 5\beta \\ 5\alpha + 3\beta \\ -5\alpha - 2\beta \end{bmatrix}$
7 Use the properties of equality of matrices.	$3 = 4\alpha + 5\beta \quad (1)$ $7 = 5\alpha + 3\beta \quad (2)$ $-8 = -5\alpha - 2\beta \quad (3)$
8 Solve the simultaneous equations using elimination.	Add (2) and (3) to eliminate α so that $\beta = -1$.
9 Substitute the value of β into (2) to find the value of α.	$5\alpha = 7 - 3\beta$ $5\alpha = 10$ $\alpha = 2$
10 Since we have not used (1), we must check that this equation is valid.	$3 = 4\alpha + 5\beta \quad (1)$ Substitute $\alpha = 2$, $\beta = -1$: RHS $= 8 - 5$ $=$ LHS
11 Write the equation relating the matrices.	$C = 2A - B$ or $\underset{\sim}{c} = 2\underset{\sim}{a} - \underset{\sim}{b}$
12 State the conclusion.	The points A, B and C are linearly dependent.

EXERCISE 5.3 $\underset{\sim}{i}\ \underset{\sim}{j}\ \underset{\sim}{k}$ notation

PRACTISE

1 WE4 If the point P has coordinates (2, –2, –1), find:

a the vector $\overrightarrow{OP}$

b a unit vector parallel to $\overrightarrow{OP}$.

2 If $\underset{\sim}{a} = 4\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k}$, find $\hat{\underset{\sim}{a}}$.

3 WE5 Two points are given by P (2, 1, –3) and Q (4, –1, 2) respectively. Find a unit vector parallel to $\overrightarrow{PQ}$.

4 Two points are given by A (–1, 2, –4) and B (2, 6, 8). Find the distance between the points A and B.

5 **WE6** If $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - z\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k}$, find the value of z if the vector $\underset{\sim}{c} = \underset{\sim}{a} - 2\underset{\sim}{b}$ is parallel to the xy plane.

6 Find the values of x, y and z, given the points C (x, –2, 4) and D (2, y –3) and the vector $\overrightarrow{CD} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + z\underset{\sim}{k}$.

7 **WE7** Given the vectors $\underset{\sim}{r} = 2\underset{\sim}{i} - \underset{\sim}{j} + z\underset{\sim}{k}$ and $\underset{\sim}{s} = -4\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$, find the value of z if:

a the length of the vector $\underset{\sim}{r}$ is 5

b the vector $\underset{\sim}{r}$ is parallel to the vector $\underset{\sim}{s}$.

8 Given the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + y\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{b} = -6\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k}$, find the value of y if:

a the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are equal in length

b the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$.

9 **WE8** Show that the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = -2i + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = 2i + \underset{\sim}{j} + z\underset{\sim}{k}$ are linearly dependent, and determine the value of z.

10 Given the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = 2i - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{c} = xi + 2\underset{\sim}{j}$, show that they are linearly dependent and determine the value of x.

11 **WE9** Find the angle in decimal degrees that the vector $\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ makes with the z-axis.

12 The vector $\underset{\sim}{i} + z\underset{\sim}{k}$ makes an angle of 150° with the z-axis. Find the value of z.

13 **WE10** Peter is training for a fun run. First he runs 900 metres in a direction S25°E, then a further 800 metres due south and finally 300 metres due east. In all three of these movements he maintains the same altitude. Finally Peter enters a building and runs up the stairs, climbing to a height of 150 metres vertically upwards above ground level.

Let $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ represent unit vectors of length 1 metre in the directions of east, north and vertically upwards respectively.

a Find the position vector of Peter at the top of the stairs relative to his initial position.

b Find his displacement correct to 1 decimal place in metres from his initial position.

14 A helicopter takes off from a helipad and moves 800 metres vertically upwards. It then turns and moves 2 km due west, and finally it turns again to move in a direction S20°W for a further 2 km. In both of the latter two movements it maintains the same altitude. Let $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ represent unit vectors of 1 metre in the directions of east, north and vertically upwards respectively.

a Find the position vector of the helicopter relative to its initial position.

b Find the displacement to the nearest metre of the helicopter from its initial position.

15 WE11 Given the vectors represented as $A = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ -7 \\ 8 \end{bmatrix}$, show that A, B and C are linearly dependent.

16 Given the vectors represented as $A = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ -7 \\ 8 \end{bmatrix}$, show that A, B and C are linearly independent.

CONSOLIDATE

17 a Given the point A (–2, 4, 1), find:
   i a unit vector parallel to OA
   ii the angle that the vector $\overrightarrow{OA}$ makes with the x-axis.
 b Given the point B (3, 5, –2), find:
   i a unit vector parallel to OB
   ii the angle that the vector $\overrightarrow{OB}$ makes with the y-axis.
 c Given the point C (4, 6, –8), find:
   i a unit vector parallel to OC
   ii the angle that the vector $\overrightarrow{OC}$ makes with the z-axis.

18 a Find a vector of magnitude 6 parallel to the vector $\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$.
 b Find a vector of magnitude 26 in the opposite direction to the vector $-3\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k}$.
 c Find a vector of magnitude $10\sqrt{2}$ in the opposite direction to the vector $-5\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$.

19 a Given the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, show that the vector $2\underset{\sim}{a} + \underset{\sim}{b}$ is parallel to the xy plane.
 b Given the vectors $\underset{\sim}{p} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$ and $\underset{\sim}{q} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, show that the vector $2\underset{\sim}{p} - 3\underset{\sim}{q}$ is parallel to the yz plane.
 c Given the vectors $\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{s} = \underset{\sim}{i} + y\underset{\sim}{j} - 2\underset{\sim}{k}$, find the value of y if the vector $4\underset{\sim}{r} + 3\underset{\sim}{s}$ is parallel to the xz plane.

20 a Given the points A (3, 5, –2) and B (2, –1, 3), find the distance between the points A and B.
 b Given the points P (–2, 4, 1) and Q (3, –5, 2), find the angle the vector $\overrightarrow{PQ}$ makes with the y-axis.
 c Given the points R (4, 3, –1) and S (6, 1, –7), find a unit vector parallel to $\overrightarrow{SR}$.

21 a Show that the points A (2, –1, 3), B (8, –7, 15) and C (4, –3, 7) are collinear.
 b Show that the points P (2, 1, 4), Q (1, –2, 3) and R (–1, –8, 1) are collinear.
 c Find the values of x and y if the points A (x, 1, 2), B (2, y, –1) and C (3 –4, 5) are collinear.

22 a Given the points A (3, 1, –2) and B (5, 3, 4), find the position vector of P where P is the midpoint of AB.
 b Given the points C (–2, 4, 1) and D (–5, 1, 4), find the position vector of P where P divides CD in the ratio 1 : 2.
 c Given the points R (1, –2, –3) and S (–3, 2, 5), find the position vector of P where P divides RS in the ratio 3 : 1.

23 a Show that the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 4\underset{\sim}{j} - 6\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + 6\underset{\sim}{j} - 9\underset{\sim}{k}$ and $\underset{\sim}{c} = 3\underset{\sim}{i} + 10\underset{\sim}{j} - 9\underset{\sim}{k}$ are linearly dependent.

b Find the value of y if the vectors $\underset{\sim}{p} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{q} = 2\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{r} = 2\underset{\sim}{i} + y\underset{\sim}{j} + 8\underset{\sim}{k}$ are linearly dependent.

c Find the value of z if the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -7\underset{\sim}{i} + 8\underset{\sim}{j} + z\underset{\sim}{k}$ are linearly dependent.

24 a Given the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{b} = x\underset{\sim}{i} + 6\underset{\sim}{j} - 12\underset{\sim}{k}$, find the value of x if:

i $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

ii the length of the vector $\underset{\sim}{b}$ is $10\sqrt{2}$.

b Given the vector $\underset{\sim}{r} = 3\underset{\sim}{i} + y\underset{\sim}{j} + \underset{\sim}{k}$, find the value of y if:

i the length of the vector $\underset{\sim}{r}$ is 10

ii the vector $\underset{\sim}{r}$ makes an angle of $\cos^{-1}\left(-\frac{1}{3}\right)$ with the y-axis.

c Given the vector $\underset{\sim}{u} = 2\sqrt{2}\underset{\sim}{i} + 2\underset{\sim}{j} + z\underset{\sim}{k}$, find the value of z if:

i the length of the vector $\underset{\sim}{u}$ is 6

ii the vector $\underset{\sim}{u}$ makes an angle of 120° with the z-axis.

25 a A mountain climber walks 2 km due west to a point A, then 1 km due south to a point B. At this point he ascends a vertical cliff to a point C, which is 500 metres above ground level. If $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ represent unit vectors of 1 kilometre in the directions of east, north and vertically upwards, find the position vector and displacement of the mountain climber from his initial position.

b A student walks on level ground in a north-westerly direction a distance of 200 m to a point P. She then walks 50 m due north to a point Q. At this point she turns southward and ascends a set of stairs inclined at an angle of 50° to the horizontal, moving 5 m along the stairs to a point R. If $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ represent unit vectors of 1 metre in the directions of east, north and vertically upwards, find the position vector and displacement of the student from her initial position.

26 a A plane takes off and flies upwards facing east for a distance of 30 km at an angle of elevation of 35°. It then moves horizontally east at 300 km/h. If $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ represent unit vectors of 1 kilometre in the directions of east, north and vertically upwards, find the position vector and displacement of the plane after it has flown horizontally for 5 minutes.

b A helicopter moves 600 metres vertically upwards and then moves S60°W for 500 m parallel to the ground. It then moves south-west, moving upwards at an angle of elevation of 50° and a speed of 120 km/h for 1 minute. If $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ represent unit vectors of one metre in the directions of east, north and vertically upwards, find the position vector and displacement of the helicopter from its initial position.

MASTER

27 a Find the value of m if the length of the vector $(m-1)\underset{\sim}{i} + (m+1)\underset{\sim}{j} + m\underset{\sim}{k}$ is $\sqrt{17}$.

b A unit vector makes an angle of 45° with the x-axis and an angle of 120° with the y-axis. Find the angle it makes with the z-axis if it is known that this angle is acute.

c A unit vector makes an angle of 60° with the x-axis and an angle of 120° with the z-axis. Find the angle it makes with the y-axis if it is known that this angle is obtuse.

28 a Given the vectors $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{c} = 4\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{d} = 7\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k}$, find the values of the scalars p, q and r if $\underset{\sim}{d} = p\underset{\sim}{a} + q\underset{\sim}{b} + r\underset{\sim}{c}$.

b Find the values of x and y if the two vectors $x\underset{\sim}{i} + 2y\underset{\sim}{j} + 3\underset{\sim}{k}$ and $2x\underset{\sim}{i} + y\underset{\sim}{j} + 4\underset{\sim}{k}$ both have a length of 5.

5.4 Scalar product and applications

Multiplying vectors

studyon

Units 3 & 4
AOS 4
Topic 1
Concept 7

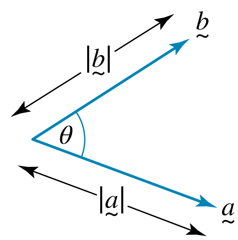
Scalar product
Concept summary
Practice questions

When a vector is multiplied by a scalar, the result is a vector. What happens when two vectors are multiplied together? Is the result a vector or a scalar? What are the applications of multiplying two vectors together? In fact, vectors can be multiplied together in two different ways: either by using the scalar or dot product and obtaining a scalar as the result, or by using the vector product and obtaining a vector as the result. The vector product is not covered in this course.

Definition of the scalar product

The scalar or dot product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined by

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta).$$

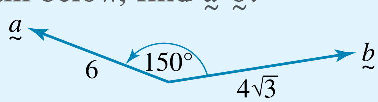


This is read as $\underset{\sim}{a}$ dot $\underset{\sim}{b}$, where θ is the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Note that the angle between two vectors must be the angle between the tails of the vectors. The scalar product or dot product is also known as the inner product of two vectors.

WORKED EXAMPLE 12

Given the diagram below, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.



THINK	WRITE
1 Write the magnitudes or lengths of the two vectors.	$\lvert\underset{\sim}{a}\rvert = 6$ and $\lvert\underset{\sim}{b}\rvert = 4\sqrt{3}$
2 Write the angle between the two vectors.	$\theta = 150°$
3 Apply the formula from the definition to calculate the value of the scalar product.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos(\theta)$ $= 6 \times 4\sqrt{3}\cos(150°)$ $= 24\sqrt{3} \times \frac{-\sqrt{3}}{2}$
4 State the final result, which is negative because $\cos(150°) < 0$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = -36$

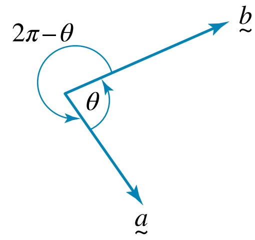
Properties of the scalar product

- The scalar product always gives a number as the result, hence its name. This number can be positive, negative or zero.
- Because $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$ for non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, both $|\underset{\sim}{a}| > 0$ and $|\underset{\sim}{b}| > 0$, so the sign of $\underset{\sim}{a}\cdot\underset{\sim}{b}$ depends upon the sign of $\cos(\theta)$. Hence, it follows that:

  $\underset{\sim}{a}\cdot\underset{\sim}{b} > 0$ if $\cos(\theta) > 0$; that is, θ is an acute angle or $0 < \theta < \dfrac{\pi}{2}$.

  $\underset{\sim}{a}\cdot\underset{\sim}{b} < 0$ if $\cos(\theta) < 0$; that is, θ is an obtuse angle or $\dfrac{\pi}{2} < \theta < \pi$.

  $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ if $\cos(\theta) = 0$; that is, $\theta = \dfrac{\pi}{2}$. This means that $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$ unless either $\underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$.
- The scalar product is commutative. That is, $\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{b}\cdot\underset{\sim}{a}$.
- This follows because the angle between $\underset{\sim}{b}$ and $\underset{\sim}{a}$ is $2\pi - \theta$, and $\cos(2\pi - \theta) = \cos(\theta)$.



- The scalar product of a vector with itself is the square of the magnitude of the vector. That is, $\underset{\sim}{a}\cdot\underset{\sim}{a} = |\underset{\sim}{a}|^2$, as $\theta = 0$ and $\cos(0) = 1$.
- Scalars or common factors in a vector are merely multiples. That is, if $\lambda \in R$, then $\underset{\sim}{a}\cdot(\lambda\underset{\sim}{b}) = (\lambda\underset{\sim}{a})\cdot\underset{\sim}{b} = \lambda(\underset{\sim}{a}\cdot\underset{\sim}{b})$.

Component forms

The vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are all unit vectors; that is, $|\underset{\sim}{i}| = 1$, $|\underset{\sim}{j}| = 1$ and $|\underset{\sim}{k}| = 1$. The angle between the vectors $\underset{\sim}{i}$ and $\underset{\sim}{i}$ is zero, as $\cos(0) = 1$. It follows from the definition and properties of the scalar product that $\underset{\sim}{i}\cdot\underset{\sim}{i} = 1$. Similarly, $\underset{\sim}{j}\cdot\underset{\sim}{j} = 1$ and $\underset{\sim}{k}\cdot\underset{\sim}{k} = 1$.

The unit vectors are mutually perpendicular; that is, the angles between $\underset{\sim}{i}$ and $\underset{\sim}{j}$, $\underset{\sim}{i}$ and $\underset{\sim}{k}$, and $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are all 90°, as $\cos(90°) = 0$. From that fact and the commutative law, it follows that $\underset{\sim}{i}\cdot\underset{\sim}{j} = \underset{\sim}{j}\cdot\underset{\sim}{i} = 0$, $\underset{\sim}{i}\cdot\underset{\sim}{k} = \underset{\sim}{k}\cdot\underset{\sim}{i} = 0$ and $\underset{\sim}{j}\cdot\underset{\sim}{k} = \underset{\sim}{k}\cdot\underset{\sim}{j} = 0$.

In general, if $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, then

$$\begin{aligned}\underset{\sim}{a}\cdot\underset{\sim}{b} &= (x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k})\cdot(x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k})\\ &= x_1x_2\underset{\sim}{i}\cdot\underset{\sim}{i} + y_1y_2\underset{\sim}{j}\cdot\underset{\sim}{j} + z_1z_2\underset{\sim}{k}\cdot\underset{\sim}{k}\\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

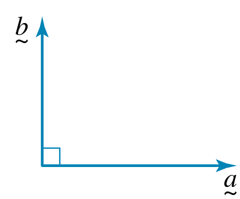
WORKED EXAMPLE 13

If $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k}$, find $\underset{\sim}{a}\cdot\underset{\sim}{b}$.

THINK	WRITE
1 Substitute for the given vectors.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = (2\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k})\cdot(4\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k})$
2 Use the result for multiplying vectors in component form.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = 2 \times 4 + 3 \times -5 + -5 \times -2$ $= 8 - 15 + 10$
3 State the final result.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = 3$

Orthogonal vectors

Two vectors are said to be orthogonal or perpendicular if the angle between them is 90°. In this case, the dot product of the two vectors is zero.



WORKED EXAMPLE 14

If the two vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 5\underset{\sim}{j} + z\underset{\sim}{k}$ are orthogonal, find the value of z.

THINK	WRITE
1 First calculate the value of the scalar product between the two vectors.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = (3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k})\cdot(2\underset{\sim}{i} - 5\underset{\sim}{j} + z\underset{\sim}{k})$ $= 3 \times 2 + -2 \times -5 - 4z$ $= 6 + 10 - 4z$ $= 16 - 4z$
2 The vectors are orthogonal. Equate their dot product to zero.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = 16 - 4z$ $= 0$
3 Solve for z.	$4z = 16$ $z = 4$

Angle between two vectors

Previously, the angle that a single vector makes with the x-, y- or z-axis was found using direction cosines. Now, by using the scalar product, the angle between two vectors can be found.

The formula $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$ can be rearranged to find the angle θ between the two vectors, giving $\cos(\theta) = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$ or $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$.

WORKED EXAMPLE 15

Given the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k}$, find the angle in decimal degrees between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

THINK	WRITE
1 First calculate the value of the scalar product between the two vectors.	$\underset{\sim}{a}\cdot\underset{\sim}{b} = (\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})\cdot(2\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k})$ $= 1 \times 2 + -2 \times -3 + 3 \times -4$ $= 2 + 6 - 12$ $= -4$
2 Find the magnitude of the first vector.	$\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ $|\underset{\sim}{a}| = \sqrt{1^2 + (-2)^2 + 3^2}$ $= \sqrt{1 + 4 + 9}$ $= \sqrt{14}$
3 Find the magnitude of the second vector.	$\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k}$ $|\underset{\sim}{b}| = \sqrt{2^2 + (-3)^2 + (-4)^2}$ $= \sqrt{4 + 9 + 16}$ $= \sqrt{29}$
4 Apply the formula to find the angle between the vectors.	$\cos(\theta) = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$ $\cos(\theta) = \dfrac{-4}{\sqrt{14}\sqrt{29}}$

5 Determine the angle between the two vectors. $\theta = \cos^{-1}\left(\frac{-4}{\sqrt{14}\sqrt{29}}\right)$

6 Evaluate the angle using a calculator, giving the answer in decimal degrees. Make sure the angle mode is set to degrees format. Since the dot product is negative, the angle between the vectors will be obtuse.
$\theta = 101.45°$

Finding magnitudes of vectors

The magnitude and the sum or difference of vectors can be found using the properties of the scalar product: $\underset{\sim}{a}\cdot\underset{\sim}{a} = |\underset{\sim}{a}|^2$ and $\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{b}\cdot\underset{\sim}{a}$.

WORKED EXAMPLE 16 **If $|\underset{\sim}{a}| = 3$, $|\underset{\sim}{b}| = 5$ and $|\underset{\sim}{a}\cdot\underset{\sim}{b}| = -4$, find $|\underset{\sim}{a} + \underset{\sim}{b}|$.**

THINK	WRITE
1 The magnitude of a vector squared is obtained by finding the dot product of the vector with itself.	$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = (\underset{\sim}{a} + \underset{\sim}{b})\cdot(\underset{\sim}{a} + \underset{\sim}{b})$
2 Expand the brackets.	$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{b}$
3 Using the properties of the scalar product, $\underset{\sim}{a}\cdot\underset{\sim}{a} = |\underset{\sim}{a}|^2$, $\underset{\sim}{b}\cdot\underset{\sim}{b} = |\underset{\sim}{b}|^2$ and $\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{b}\cdot\underset{\sim}{a}$.	$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$
4 Substitute for the given values.	$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = 3^2 + 2 \times -4 + 5^2$ $= 9 - 8 + 25$ $= 26$
5 State the answer.	$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{26}$

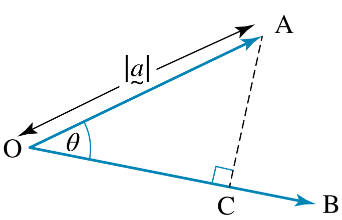
studyon
Units 3 & 4
AOS 4
Topic 1
Concept 6
Resolution of vectors
Concept summary
Practice questions

Projections

As seen previously, a vector can be resolved parallel and perpendicular to the x- or y-axis. In this section, a generalisation of this process will be considered in which one vector is resolved parallel and perpendicular to another vector.

Scalar resolute

The projection of the vector $\underset{\sim}{a} = \overrightarrow{OA}$ onto the vector $\underset{\sim}{b} = \overrightarrow{OB}$ is defined by dropping the perpendicular from the end of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$, that is at the point C. The projection is defined as this distance along $\underset{\sim}{b}$ in the direction of $\underset{\sim}{b}$. This distance, OC, is called the scalar resolute of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$, or the scalar resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$.



Because $\cos(\theta) = \dfrac{d(OC)}{|\underset{\sim}{a}|}$, it follows that

$d(OC) = |\underset{\sim}{a}|\cos(\theta)$.

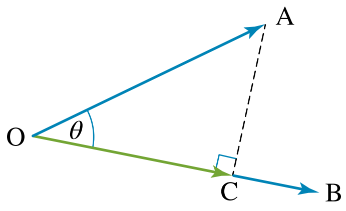
But $\cos(\theta) = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$, so

$$d(OC) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|} = \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}.$$

Parallel vector resolute

studyon
Units 3 & 4
AOS 4
Topic 1
Concept 8
Scalar and vector resolutes
Concept summary
Practice questions

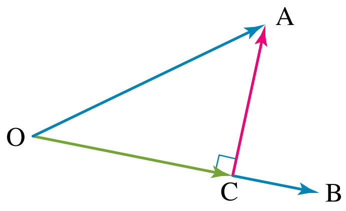
The vector resolute of $\underset{\sim}{a} = \overrightarrow{OA}$ onto the vector $\underset{\sim}{b} = \overrightarrow{OB}$ is defined as the vector along $\underset{\sim}{b}$. This green vector $\overrightarrow{OC}$ has a length of $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$ and its direction is in the direction of the unit vector $\hat{\underset{\sim}{b}}$. This vector resolute is given by $\overrightarrow{OC} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$ and is also called the component of the vector $\underset{\sim}{a}$ onto the vector $\underset{\sim}{b}$ or parallel to the vector $\underset{\sim}{b}$.



Perpendicular vector resolute

The vector resolute or component of $\underset{\sim}{a} = \overrightarrow{OA}$ perpendicular to the vector $\underset{\sim}{b} = \overrightarrow{OB}$ is the red vector $\overrightarrow{CA}$.

Since $\overrightarrow{OC} + \overrightarrow{CA} = \overrightarrow{OA}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$. Thus, $\overrightarrow{CA}$ is obtained by subtracting the vectors: $\overrightarrow{CA} = \underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.



WORKED EXAMPLE 17

Given the vectors $\underset{\sim}{u} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$, find:

a **the scalar resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$**

b **the vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$**

c **the vector resolute of $\underset{\sim}{u}$ perpendicular to $\underset{\sim}{v}$.**

THINK	WRITE
a 1 First find the magnitude of the second vector, that is $\underset{\sim}{v}$.	**a** $\underset{\sim}{v} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$ $\lvert\underset{\sim}{v}\rvert = \sqrt{2^2 + (-3)^2 + (-1)^2}$ $= \sqrt{4 + 9 + 1}$ $= \sqrt{14}$
2 Write a unit vector parallel to the second given vector, that is $\hat{\underset{\sim}{v}}$.	$\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{14}}(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})$
3 Find the scalar product of the two vectors, $\underset{\sim}{u} \cdot \underset{\sim}{v}$.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = (3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) \cdot (2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})$ $= 6 + 3 - 2$ $= 7$
4 The scalar resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is given by $\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}}$.	$\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}} = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\lvert\underset{\sim}{v}\rvert}$ $= \frac{7}{\sqrt{14}}$

5 Rationalise the denominator.

$$\underset{\sim}{u} \cdot \underset{\sim}{\hat{v}} = \frac{7}{\sqrt{14}} \times \frac{\sqrt{14}}{\sqrt{14}}$$
$$= \frac{\sqrt{14}}{2}$$

b The vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$ is given by $(\underset{\sim}{u} \cdot \underset{\sim}{\hat{v}})\underset{\sim}{\hat{v}}$. Substitute for the scalar resolute $\underset{\sim}{u} \cdot \underset{\sim}{\hat{v}}$ and the unit vector $\underset{\sim}{\hat{v}}$.

b $$(\underset{\sim}{u} \cdot \underset{\sim}{\hat{v}})\underset{\sim}{\hat{v}} = \frac{\sqrt{14}}{2}\underset{\sim}{\hat{v}}$$
$$= \frac{\sqrt{14}}{2} \times \frac{1}{\sqrt{14}}(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})$$
$$= \frac{1}{2}(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})$$

c 1 The vector resolute of $\underset{\sim}{u}$ perpendicular to $\underset{\sim}{v}$ is given by $\underset{\sim}{u} - (\underset{\sim}{u} \cdot \underset{\sim}{\hat{v}})\underset{\sim}{\hat{v}}$. Substitute for the given vectors.

c $$\underset{\sim}{u} - (\underset{\sim}{u} \cdot \underset{\sim}{\hat{v}})\underset{\sim}{\hat{v}} = (3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) - \left(\frac{1}{2}(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})\right)$$

2 Form a common denominator to subtract the vectors.

$$= \frac{1}{2}[2(3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) - (2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})]$$
$$= \frac{1}{2}[(6\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}) - (2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})]$$

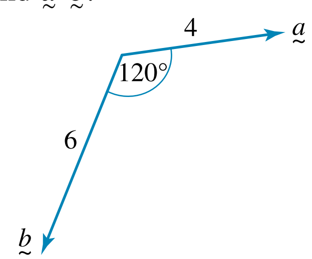
3 State the final result.

$$= \frac{1}{2}(4\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k})$$

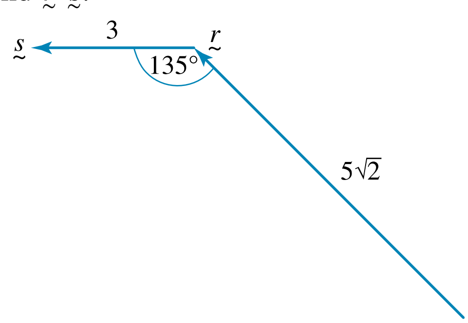
EXERCISE 5.4 Scalar product and applications

PRACTISE

1 **WE12** Given the diagram, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.



2 Given the diagram, find $\underset{\sim}{r} \cdot \underset{\sim}{s}$.



3 **WE13** Given the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

4 Given the vectors $\underset{\sim}{r} = \underset{\sim}{i} + y\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{s} = -2\underset{\sim}{i} + 6\underset{\sim}{j} - 7\underset{\sim}{k}$, find the value of y if $\underset{\sim}{r} \cdot \underset{\sim}{s} = 12$.

5 **WE14** If the two vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + z\underset{\sim}{k}$ are orthogonal, find the value of z.

6 The vectors $\underset{\sim}{a} = x\underset{\sim}{i} - \underset{\sim}{j} + x\underset{\sim}{k}$ and $\underset{\sim}{b} = x\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k}$ are orthogonal. Find the value of x.

7 **WE15** Given the vectors $\underset{\sim}{a} = 4\underset{\sim}{i} - 5\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, find the angle in decimal degrees between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

8 The angle between the vectors $\underset{\sim}{u} = x\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ is 120°. Find the value of x.

9 **WE16** If $|\underset{\sim}{a}| = 7$, $|\underset{\sim}{b}| = 5$ and $\underset{\sim}{a}\cdot\underset{\sim}{b} = -6$, find $|\underset{\sim}{a} - \underset{\sim}{b}|$.

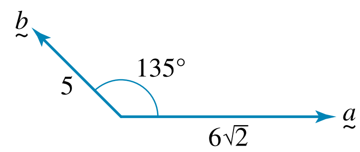
10 Given that $|\underset{\sim}{r}| = 2$, $|\underset{\sim}{s}| = 4$ and $\underset{\sim}{r}\cdot\underset{\sim}{s} = 8$, find $|2\underset{\sim}{s} - \underset{\sim}{r}|$.

11 **WE17** Given the vectors $\underset{\sim}{u} = 5\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$, find:
a the scalar resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$
b the vector resolute of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$
c the vector resolute of $\underset{\sim}{u}$ perpendicular to $\underset{\sim}{v}$.

12 Given the vectors $\underset{\sim}{r} = 2\underset{\sim}{i} + 4\underset{\sim}{k}$ and $\underset{\sim}{s} = \underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}$, find:
a the vector component of $\underset{\sim}{r}$ parallel to $\underset{\sim}{s}$
b the vector component of $\underset{\sim}{r}$ perpendicular to $\underset{\sim}{s}$.

CONSOLIDATE

13 a Two vectors have lengths of 8 and 3 units and are inclined at an angle 60°. Find the value of their scalar product.
b Given the diagram at right, find the value of $\underset{\sim}{a}\cdot\underset{\sim}{b}$.



c Two vectors have lengths of 7 and 3 units. The vectors are parallel but point in opposite directions. Find the value of their scalar product.

14 a Given the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, find the value of $\underset{\sim}{u}\cdot\underset{\sim}{v}$.
b Find the value of $(\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k})\cdot(2\underset{\sim}{i} + \underset{\sim}{j})$.
c Given the vectors $\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{s} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$, find the value of $\underset{\sim}{r}\cdot\underset{\sim}{s}$.

15 a Given the vectors $\underset{\sim}{p} = 2\underset{\sim}{i} - \underset{\sim}{j} + z\underset{\sim}{k}$ and $\underset{\sim}{q} = -6\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$, find the value of z if:
i $\underset{\sim}{p}$ is parallel to the vector $\underset{\sim}{q}$
ii $\underset{\sim}{p}$ is perpendicular to the vector $\underset{\sim}{q}$
iii the length of the vector $\underset{\sim}{p}$ is 5.
b Given the vectors $\underset{\sim}{r} = 6\underset{\sim}{i} + y\underset{\sim}{j} - 9\underset{\sim}{k}$ and $\underset{\sim}{s} = -4\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$, find the value of y if:
i $\underset{\sim}{r}$ is parallel to the vector $\underset{\sim}{s}$
ii $\underset{\sim}{r}$ is perpendicular to the vector $\underset{\sim}{s}$
iii the length of the vector $\underset{\sim}{r}$ is twice the length the vector $\underset{\sim}{s}$.
c Given the vectors $\underset{\sim}{a} = x\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}$, find the value of x if:
i $2\underset{\sim}{a} - 3\underset{\sim}{b}$ is parallel to the yz plane
ii $2\underset{\sim}{a} - 3\underset{\sim}{b}$ is perpendicular to the vector $\underset{\sim}{c}$
iii the length of the vector $2\underset{\sim}{a} - 3\underset{\sim}{b}$ is 11.

16 Consider the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{c} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$.
a Show that $\underset{\sim}{a}\cdot(\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c}$.
b Show that $\underset{\sim}{a}\cdot(\underset{\sim}{b} - \underset{\sim}{c}) = \underset{\sim}{a}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{c}$.
c What property do **a** and **b** illustrate?
d Can you give meaning to $\underset{\sim}{a}\cdot\underset{\sim}{b}\cdot\underset{\sim}{c}$?

17 a Given the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$, find:
i a unit vector parallel to $\underset{\sim}{b}$
ii the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$
iii the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$
iv the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$
v the angle in decimal degrees between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

b Given the vectors $\underset{\sim}{p} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$ and $\underset{\sim}{q} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, find:

i a unit vector parallel to $\underset{\sim}{q}$

ii the scalar resolute of $\underset{\sim}{p}$ in the direction of $\underset{\sim}{q}$

iii the component of $\underset{\sim}{p}$ in the direction of $\underset{\sim}{q}$

iv the component of $\underset{\sim}{p}$ perpendicular to $\underset{\sim}{q}$

v the angle in decimal degrees between the vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$.

c Given the vectors $\underset{\sim}{r} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{s} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$:

i resolve the vector $\underset{\sim}{r}$ into two components, one parallel to $\underset{\sim}{s}$ and one perpendicular to $\underset{\sim}{s}$

ii find the angle in decimal degrees between the vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$.

18 a Given the points A (3, –2, 5), B (–1, 0, 4) and C (2, –1, 3), find:

i a unit vector parallel to $\overrightarrow{BC}$

ii the vector resolute of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$

iii the vector resolute of $\overrightarrow{AB}$ perpendicular to $\overrightarrow{BC}$

iv the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

b Given the points A (2, 3, 2), B (4, p, 0), C (–1, –1, 0) and D (–2, 2, 1), find the value of p if:

i $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

ii $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{DC}$

iii the length of the vector $\overrightarrow{AB}$ is equal to the length of the vector $\overrightarrow{DC}$

iv the scalar resolute of $\overrightarrow{AB}$ parallel to $\overrightarrow{DC}$ is equal to $\frac{4}{\sqrt{11}}$.

c Find the value of p if the points P (4, p, –3), Q (–1, –4, –6) and R (1, 6, –1) form a right-angled triangle at P.

19 a Given the points A (2, –1, 3), B (1, –2, 4) and C (4, 3, –1), by finding the lengths of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, determine the exact area of the triangle ABC.

b Given the points P (3, 1, –1), Q (1, 0, 2) and R (2, 1, –3), by finding the lengths of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, determine the exact area of the triangle PQR.

c Given the points A (–4, 2, –1), B (–2, 1, –3) and C (–3, 2, 1), find the exact area of the triangle ABC.

20 a The angle between the vectors $\underset{\sim}{a} = x\underset{\sim}{i} - 3\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ is 150°. Determine the value of x.

b The angle between the vectors $\underset{\sim}{p} = 6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{q} = \underset{\sim}{i} + y\underset{\sim}{j} - 2\underset{\sim}{k}$ is equal to $\cos^{-1}\left(\frac{4}{21}\right)$. Determine the value of y.

c Determine the value of z if the cosine of the angle between the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 4\underset{\sim}{j} + z\underset{\sim}{k}$ is equal to $-\frac{1}{3}$.

21 a If $\underset{\sim}{a} = \cos(\alpha)\underset{\sim}{i} + \sin(\alpha)\underset{\sim}{j}$ and $\underset{\sim}{b} = \cos(\beta)\underset{\sim}{i} + \sin(\beta)\underset{\sim}{j}$, find the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and hence show that $\cos(\beta - \alpha) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

b When a force $\underset{\sim}{F}$ moves a point from A to B producing a displacement $\underset{\sim}{s} = \overrightarrow{AB}$, the work done is given by $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$. Find the work done when the force $\underset{\sim}{F} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$ moves a point from A (1, –2, 2) to the point B (2, –1, 4).

c Given the vectors $|\underset{\sim}{a}| = 3$, $|\underset{\sim}{b}| = 4$ and $|\underset{\sim}{c}| = 5$ such that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$, find the value of $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

d Given the vectors $|\underset{\sim}{p}| = 5$, $|\underset{\sim}{q}| = 12$ and $|\underset{\sim}{r}| = 13$ and $\underset{\sim}{p} \cdot \underset{\sim}{q} = 0$, determine $\underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r}$.

e If $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c}$, what can be deduced about the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$?

22 a If $|\underset{\sim}{u}| = 3$, $|\underset{\sim}{v}| = 4$ and $\underset{\sim}{u} \cdot \underset{\sim}{v} = 6$, find:

- **i** $|\underset{\sim}{u} + \underset{\sim}{v}|$
- **ii** $|\underset{\sim}{u} - \underset{\sim}{v}|$
- **iii** $|3\underset{\sim}{u} - 2\underset{\sim}{v}|$.

b If $|\underset{\sim}{r}| = 4\sqrt{2}$, $|\underset{\sim}{s}| = 5\sqrt{3}$ and $\underset{\sim}{r} \cdot \underset{\sim}{s} = -6$, find:

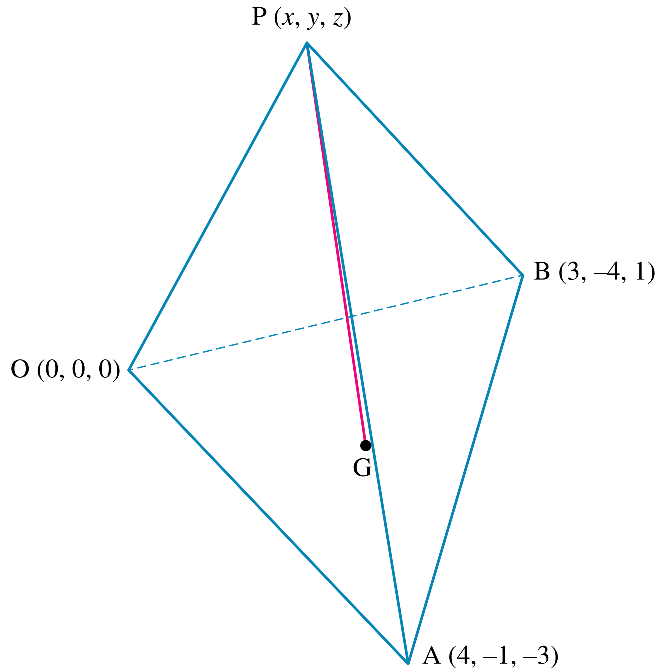
- **i** $|\underset{\sim}{r} + \underset{\sim}{s}|$
- **ii** $|\underset{\sim}{r} - \underset{\sim}{s}|$
- **iii** $|4\underset{\sim}{r} - 3\underset{\sim}{s}|$.

c **i** Show that $\underset{\sim}{u} \cdot \underset{\sim}{v} = \frac{1}{4}|\underset{\sim}{u} + \underset{\sim}{v}|^2 - \frac{1}{4}|\underset{\sim}{u} - \underset{\sim}{v}|^2$ and $|\underset{\sim}{u} + \underset{\sim}{v}|^2 + |\underset{\sim}{u} - \underset{\sim}{v}|^2 = 2\left(|\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2\right)$.

ii If $|\underset{\sim}{u} + \underset{\sim}{v}| = \sqrt{17}$ and $|\underset{\sim}{u} - \underset{\sim}{v}| = \sqrt{13}$ and the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is $\cos^{-1}\left(\frac{1}{\sqrt{50}}\right)$, find $|\underset{\sim}{u}|$ and $|\underset{\sim}{v}|$.

MASTER

23 a OABP is a pyramid, where O is the origin. The coordinates of the points are A (4, –1, –3), B (3, –4, 1) and P (x, y, z). The height of the pyramid is the length of GP, where G is a point on the base OAB such that GP is perpendicular to the base.



i Show using vectors that OAB is an equilateral triangle.

ii Let M be the midpoint of AB. Given that the point G is such that $\overrightarrow{OG} = \frac{2}{3}\overrightarrow{OM}$, find the vector $\overrightarrow{OG}$.

iii Find the vector $\overrightarrow{GP}$ and, using the fact that $\overrightarrow{GP}$ is perpendicular to $\overrightarrow{OG}$, show that $7x - 5y - 2z = 26$.

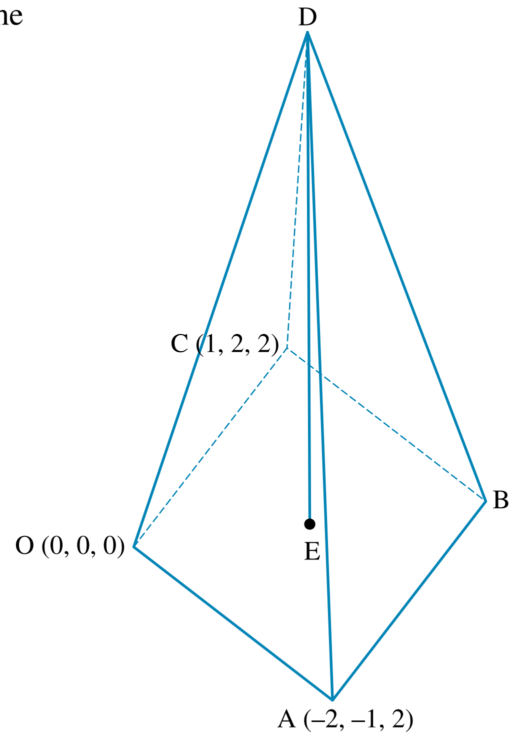
iv The faces of the pyramid, OAP, ABP and OBP, are all similar isosceles triangles, with distance $OP = AP = BP = 5\sqrt{14}$. Write another set of equations expressing this relationship in terms of x, y and z.

v Find the coordinates of P.

vi Determine the height of the pyramid.

vii Determine the angle that the sloping edge makes with the base.

b OABCD is a right pyramid, where O is the origin. The coordinates of the points are A (–2, –1, 2), C (1, 2, 2) and D (x, y, z). The height of the pyramid is the length of ED, where E is a point on the base of OABC such that E is the midpoint of OB.



i If OABC is a square, show that the coordinates of B are (–1, 1, 4).

ii Find the coordinates of the point E.

iii If the vector $\overrightarrow{ED}$ is perpendicular to $\overrightarrow{OE}$, show that $-x + y + 4z = 9$.

iv The faces of the pyramid, OAD, ABD, BCD and OCD, are all similar isosceles triangles, with sloping edges OD, AD, BD and CD all equal in length to $\frac{9\sqrt{22}}{2}$. Write another set of equations that can be used to solve for x, y and z.

v Find the coordinates of D and hence determine the height of the pyramid.

vi Find the angle in degrees that the sloping edge makes with the base.

24 a Find a unit vector perpendicular to both $3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $4\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$.

b Find a unit vector perpendicular to both $5\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $-2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$.

c Find a unit vector perpendicular to both $-5\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$.

5.5 Vector proofs using the scalar product

Geometrical shapes

studyon

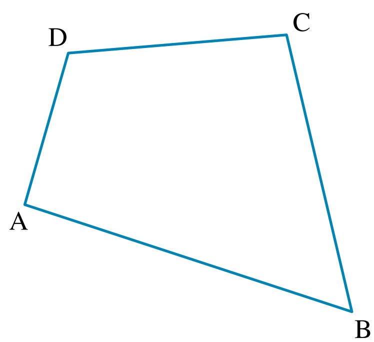
Units 3 & 4
AOS 4
Topic 1
Concept 9

Vector proofs
Concept summary
Practice questions

Before using vectors to prove geometrical theorems, it is necessary to recall the definitions of some geometrical shapes and state the vector properties associated with these shapes.

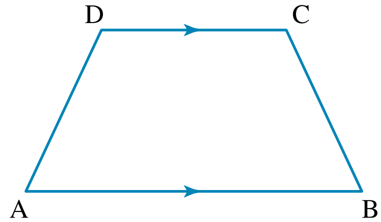
Quadrilaterals

A general **quadrilateral** is a plane four-sided figure with no two sides necessarily parallel nor equal in length.



Trapeziums

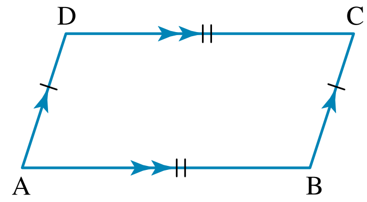
A **trapezium** is a plane four-sided figure with one pair of sides parallel, but not equal.



In the trapezium ABCD, since AB is parallel to DC, $\overrightarrow{AB} = \lambda\overrightarrow{DC}$.

Parallelograms

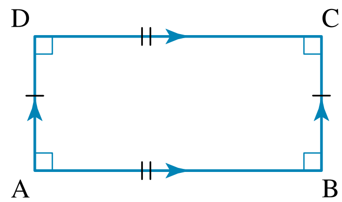
A **parallelogram** is a plane four-sided figure with two sets of sides parallel and equal in length.



In the parallelogram ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Rectangles

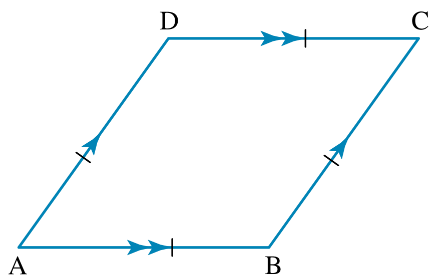
A **rectangle** is a parallelogram with all angles being 90°.



In the rectangle ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$ and thus $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{BC} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{CD} \cdot \overrightarrow{DA} = 0$ and $\overrightarrow{DA} \cdot \overrightarrow{AB} = 0$, since all these sides are perpendicular.

Rhombuses

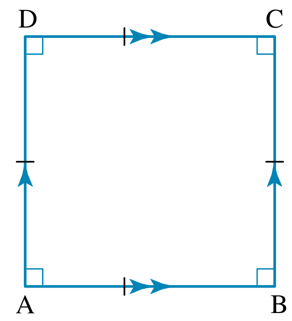
A **rhombus** is a parallelogram with all sides equal in length.



In the rhombus ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$; also $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{DC}| = |\overrightarrow{AD}|$, since all sides are equal in length.

Squares

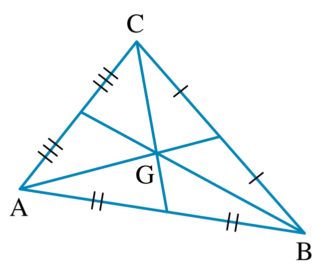
A **square** is a rhombus with all angles 90°.



In the square ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$ and thus $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{BC} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{CD} \cdot \overrightarrow{DA} = 0$, $\overrightarrow{DA} \cdot \overrightarrow{AB} = 0$ and $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{DC}| = |\overrightarrow{AD}|$

Triangles

A median of a **triangle** is the line segment from a vertex to the midpoint of the opposite side. The centroid of a triangle is the point of intersection of the three medians. If G is the centroid of the triangle ABC and O is the origin, then it can be shown that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$

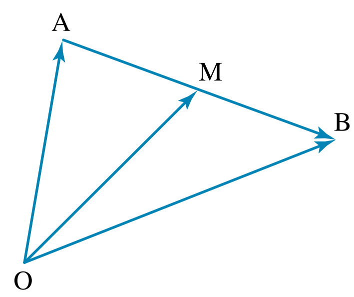


Using vectors to prove geometrical theorems

The properties of vectors can be used to prove many geometrical theorems. The following statements are useful in proving geometrical theorems.

1. If O is the origin and A and B are points, the midpoint M of the line segment AB is given by

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB}\\ &= \overrightarrow{OA} + \tfrac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})\\ &= \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})\end{aligned}$$

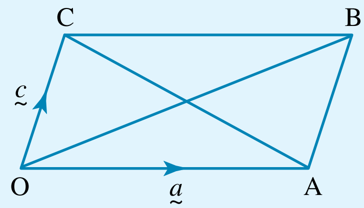


2. If two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, then $\overrightarrow{AB} = \lambda\overrightarrow{CD}$ where $\lambda \in R$ is a scalar.
3. If two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular, then $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$.
4. If two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$; furthermore, these two vectors are equal in length, so that $\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow |\overrightarrow{AB}| = |\overrightarrow{CD}|$.
5. If $\overrightarrow{AB} = \lambda\overrightarrow{BC}$, then the points A, B and C are collinear; that is, A, B and C all lie on a straight line.

WORKED EXAMPLE 18 Prove that if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

THINK / **WRITE/DRAW**

1 Let OABC be a parallelogram.



Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

2 State the properties of the parallelogram.

Because OABC is parallelogram, $\underset{\sim}{a} = \overrightarrow{OA} = \overrightarrow{CB}$ and $\underset{\sim}{c} = \overrightarrow{OC} = \overrightarrow{AB}$.

3 Find a vector expression for the diagonal OB in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= \overrightarrow{OA} + \overrightarrow{OC}\\ &= \underset{\sim}{a} + \underset{\sim}{c}\end{aligned}$$

4 Find a vector expression for the diagonal AC in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= \overrightarrow{OC} - \overrightarrow{OA}\\ &= \underset{\sim}{c} - \underset{\sim}{a}\end{aligned}$$

5 The dot product of the diagonals is zero, since it was given that they are perpendicular.

$$\begin{aligned}\overrightarrow{OB}\cdot\overrightarrow{AC} &= (\underset{\sim}{a} + \underset{\sim}{c})\cdot(\underset{\sim}{c} - \underset{\sim}{a})\\ &= 0\end{aligned}$$

6 Expand the brackets

$$\begin{aligned}\overrightarrow{OB}\cdot\overrightarrow{AC} &= \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{c}\cdot\underset{\sim}{a} - \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot\underset{\sim}{c}\\ &= 0\end{aligned}$$

7 Use the properties of the dot product: $\underset{\sim}{c}\cdot\underset{\sim}{a} = \underset{\sim}{a}\cdot\underset{\sim}{c}$ and $\underset{\sim}{a}\cdot\underset{\sim}{a} = |\underset{\sim}{a}|^2$.

$$\begin{aligned}\overrightarrow{OB}\cdot\overrightarrow{AC} &= |\underset{\sim}{a}|^2 - |\underset{\sim}{c}|^2\\ &= 0\end{aligned}$$

8 State the conclusion.

$\overrightarrow{OB}\cdot\overrightarrow{AC} = 0 \Rightarrow |\underset{\sim}{a}|^2 = |\underset{\sim}{c}|^2$ so that $|\overrightarrow{OA}| = |\overrightarrow{OC}|$.
The length of $\overrightarrow{OA}$ is equal to the length of $\overrightarrow{OC}$; therefore, OABC is a rhombus.

EXERCISE 5.5 Vector proofs using the scalar product

PRACTISE

1 **WE18** Prove that the diagonals of a rhombus are perpendicular.

2 Prove that if the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle.

CONSOLIDATE

3 a Given the points A (8, 3, –1), B (4, 5, –2) and C (7, 9, –6), show that ABC forms a right-angled triangle at B, and hence find the area of the triangle.

b Given the points A (–3, 5, 4), B (2, 3, 5) and C (4, 6, 1), show that ABC forms a right-angled triangle at B, and hence find the area of the triangle.

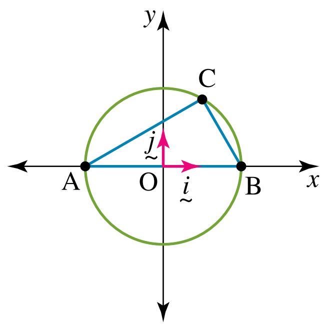
4 a Given the points A (4, 7, 3), B (8, 7, 1) and C (6, 5, 2), show that ABC forms an isosceles triangle. Let M be the midpoint of AB, and show that MC is perpendicular to AB.

b Given the points A (3, –3, 4), B (5, 3, 6) and C (3, 1, 3), show that ABC forms an isosceles triangle. Let M be the midpoint of AB, and show that MC is perpendicular to AB.

5 Prove Pythagoras' theorem.

6 **a** Prove that the angle inscribed in a semicircle is a right angle.

b The diagram at right shows a circle of radius r with centre at the origin O on the x- and y-axes. The points A and B lie on the diameter of the circle and are on the x-axis; their coordinates are $(-r, 0)$ and $(r, 0)$ respectively. The point C has coordinates (a, b) and lies on the circle, where a, b and r are all positive real constants. Show that CA is perpendicular to CB.



7 OABC is a square. The points P, Q, R and S are the midpoints of the sides OA, AB, BC and OC respectively. Prove that PQRS is a square.

8 OABC is a rhombus. The points P, Q, R and S are the midpoints of the sides OA, AB, BC and OC respectively. Prove that PQRS is a rectangle.

9 Prove that the line segments joining the midpoints of consecutive sides of a rectangle form a rhombus.

10 AB and CD are two diameters of a circle with centre O. Letting $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$, prove that ACBD is a rectangle.

11 OAB is a right-angled isosceles triangle with $|\overrightarrow{OA}| = |\overrightarrow{OB}|$. Let M be the midpoint of AB, and let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

a Express $\overrightarrow{OM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$

b Prove that $\overrightarrow{OM}$ is perpendicular to $\overrightarrow{AB}$.

c Show that $|\overrightarrow{OM}| = \frac{1}{2}|\overrightarrow{AB}|$.

12 Prove that the vector $\hat{\underset{\sim}{a}} + \hat{\underset{\sim}{b}}$ bisects the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

MASTER

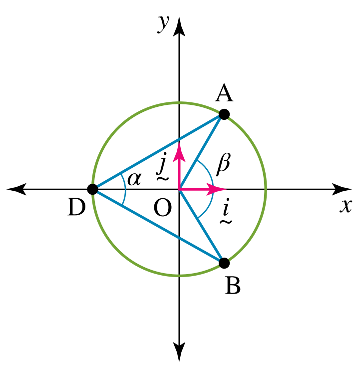
13 ABC is a triangle. P, Q and R are the midpoints of the sides AC, BC and AB respectively. Perpendicular lines are drawn through the points P and Q and intersect at the point O. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

a Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

b Show that $|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}|$.

c Prove that $\overrightarrow{OR}$ is perpendicular to $\overrightarrow{AB}$.

14 The diagram at right shows a circle of radius r with centre at the origin O on the x- and y-axes. The three points A, B and D all lie on the circle and have coordinates A (a, b), B $(a, -b)$ and D $(-r, 0)$, where a, b and r are all positive real constants.



a Let β be the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Show that $\cos(\beta) = \dfrac{a^2 - b^2}{a^2 + b^2}$.

b Let α be the angle between the vectors $\overrightarrow{DA}$ and $\overrightarrow{DB}$. Show that $\cos(\alpha) = \dfrac{a}{\sqrt{a^2 + b^2}}$.

c Hence show that $2\alpha = \beta$.

5.6 Parametric equations

A locus is a set of points traced out in the plane, satisfying some geometrical relationship. The path described by a moving particle forms a locus and can be described by a Cartesian equation. However, the Cartesian equation does not tell us where the particle is at any particular time.

The path traced out by the particle can be defined in terms of another or third variable. Here we will use the variable t as the parameter. For example, the unit circle can be described by the use of a parameter θ, $x = \cos(\theta)$, $y = \sin(\theta)$. In the two-dimensional case there are two parametric equations, as both the x- and y- coordinates depend upon the parameter t.

$$x = x(t) \qquad (1)$$
$$y = y(t) \qquad (2)$$

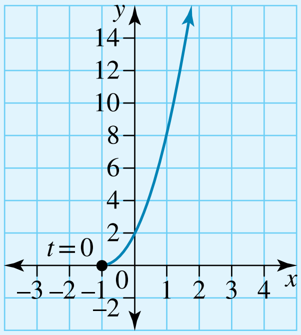
Because a position vector is given by $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the x and y directions, this is also called the vector equation of the path. If we can eliminate the parameter from these two parametric equations and obtain an equation of the form $y = f(x)$, then this is called an explicit relationship and is the equation of the path.

Often we may be unable to obtain an explicit relationship but can find an implicit relationship of the form $f(x, y) = 0$. Either way, the relationship between x and y is called the Cartesian equation of the path.

WORKED EXAMPLE 19

Given the vector equation $\underset{\sim}{r}(t) = (t - 1)\underset{\sim}{i} + 2t^2\underset{\sim}{j}$, for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

THINK	WRITE/DRAW
1 Write the vector equation: $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$.	$\underset{\sim}{r}(t) = (t - 1)\underset{\sim}{i} + 2t^2\underset{\sim}{j}$, $t \geq 0$
2 State the parametric equations.	$x = t - 1 \quad (1)$ $y = 2t^2 \quad (2)$
3 Eliminate the parameter. Express t in terms of x.	From (1), $t = x + 1$.
4 Substitute into the second parametric equation to obtain the Cartesian equation of the path.	Substitute (1) $t = x + 1$ into (2) $y = 2t^2$: $y = 2(x + 1)^2$
5 From the restriction on t, determine the domain and range.	Since $t \geq 0$, it follows from the parametric equations that $x \geq -1$ and $y \geq 0$.
6 The graph is a restricted domain function.	The graph is not the whole parabola; it is a parabola on a restricted domain, with an endpoint at (–1, 0).



7 Although it is not required, a table of values can show the points as they are plotted and can give the direction of a particle as it moves along the curve.

t	$x = t - 1$	$y = 2t^2$
0	–1	0
1	0	2
2	1	8
3	2	18
4	3	32
5	4	50

Eliminating the parameter

Eliminating the parameter is not always an easy task. Sometimes direct substitution will work; other times it is necessary to use trigonometric formulas and simple ingenuity.

WORKED EXAMPLE 20

Given the vector equation $\underset{\sim}{r}(t) = (2 + 5\cos(t))\underset{\sim}{i} + (4\sin(t) - 3)\underset{\sim}{j}$ for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

THINK / WRITE/DRAW

1 From the vector equation, $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$.

$\underset{\sim}{r}(t) = (2 + 5\cos(t))\underset{\sim}{i} + (4\sin(t) - 3)\underset{\sim}{j}$

2 State the parametric equations.

$x = 2 + 5\cos(t)$ (1)
$y = 4\sin(t) - 3$ (2)

3 Express the trigonometric ratios cos(t) and sin(t) in terms of x and y respectively.

$(1) \Rightarrow \cos(t) = \dfrac{x-2}{5}$

$(2) \Rightarrow \sin(t) = \dfrac{y+3}{4}$

4 Eliminate the parameter to find the Cartesian equation of the path. In this case the expression is given as an implicit equation.

Since $\cos^2(t) + \sin^2(t) = 1$, it follows that
$$\frac{(x-2)^2}{25} + \frac{(y+3)^2}{16} = 1$$
This is an ellipse, with centre at (2, –3) and semi-major and semi-minor axes 5 and 4.

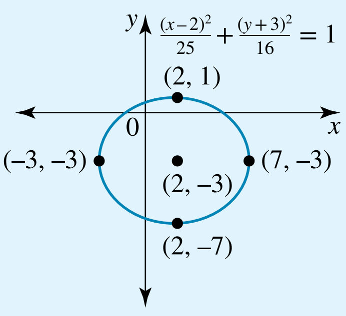
5 Determine the domain.

Since $-1 \leq \cos(t) \leq 1$, it follows from the parametric equation $x(t) = 2 + 5\cos(t)$ that the domain is $-3 \leq x \leq 7$, that is $[-3, 7]$.

6 Determine the range.

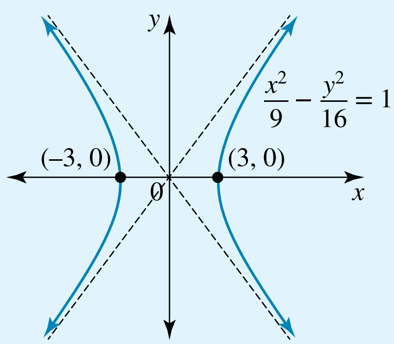
Since $-1 \leq \sin(t) \leq 1$, it follows from the parametric equation $y(t) = 4\sin(t) - 3$ that the range is $-7 \leq y \leq 1$, that is $[-7, 1]$.

7 The graph is the whole ellipse. The exact ordinates of the x- and y-intercepts are not required in this case.



WORKED EXAMPLE 21

Given the vector equation $\underset{\sim}{r}(t) = 3\sec(2t)\underset{\sim}{i} + 4\tan(2t)\underset{\sim}{j}$ for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

THINK	WRITE/DRAW
1 From the vector equation, $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$.	$\underset{\sim}{r}(t) = 3\sec(2t)\underset{\sim}{i} + 4\tan(2t)\underset{\sim}{j}$
2 State the parametric equations.	$x = 3\sec(2t)$ (1) $y = 4\tan(2t)$ (2)
3 Express the trigonometric ratios $\sec(2t)$ and $\tan(2t)$ in terms of x and y respectively.	(1) $\Rightarrow \sec(2t) = \frac{x}{3}$ (2) $\Rightarrow \tan(2t) = \frac{y}{4}$
4 Eliminate the parameter using an appropriate trigonometric identity to find the Cartesian equation of the path. In this case the expression is given as an implicit equation.	Since $\sec^2(2t) - \tan^2(2t) = 1$, it follows that $\frac{x^2}{9} - \frac{y^2}{16} = 1$ This is a hyperbola with centre at the origin. It has asymptotes when $\frac{x^2}{9} - \frac{y^2}{16} = 0$; that is, when $y = \pm\frac{4x}{3}$.
5 Determine the domain.	It follows from the parametric equation $x(t) = 3\sec(2t)$ that the domain is $(-\infty, -3] \cup [3, \infty)$.
6 Determine the range.	It follows from the parametric equation $y(t) = 4\tan(2t)$ that the range is R.
7 The graph is the whole hyperbola.	

Parametric representation

The parametric representation of a curve is not necessarily unique.

WORKED EXAMPLE 22

Show that the parametric equations $x(t) = \frac{3}{2}\left(t + \frac{1}{t}\right)$ and $y(t) = 2\left(t - \frac{1}{t}\right)$ where $t \in R \setminus \{0\}$ represent the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

THINK	WRITE
1 State the parametric equations.	$x = \frac{3}{2}\left(t + \frac{1}{t}\right)$ (1) $y = 2\left(t - \frac{1}{t}\right)$ (2)

2 Express the equations in a form to eliminate the parameter.

$$\frac{2x}{3} = t + \frac{1}{t} \quad (1)$$
$$\frac{y}{2} = t - \frac{1}{t} \quad (2)$$

3 Square both equations.

$$\frac{4x^2}{9} = t^2 + 2 + \frac{1}{t^2} \quad (1)$$
$$\frac{y^2}{4} = t^2 - 2 + \frac{1}{t^2} \quad (2)$$

4 Subtract the equations to eliminate the parameter.

(1) − (2):

$$\frac{4x^2}{9} - \frac{y^2}{4} = 4$$

5 Divide by 4.

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

This gives the hyperbola as required.

Sketching parametric curves

Graphing calculators and CAS calculators can draw the Cartesian equation of the path from the two parametric equations, even if the parameter cannot be eliminated.

EXERCISE 5.6

Parametric equations

PRACTISE

1 WE19 Given the vector equation $\underset{\sim}{r}(t) = (t + 1)\underset{\sim}{i} + (t - 1)^2\underset{\sim}{j}$ for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

2 Given the vector equation $\underset{\sim}{r}(t) = \sqrt{t}\underset{\sim}{i} + (2t + 3)\underset{\sim}{j}$ for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

3 WE20 Given the vector equation $\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 4\sin(t)\underset{\sim}{j}$ for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

4 Given the vector equation $\underset{\sim}{r}(t) = (5 - 2\cos(t))\underset{\sim}{i} + (3\sin(t) - 4)\underset{\sim}{j}$, for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

5 WE21 Given the vector equation $\underset{\sim}{r}(t) = 5\sec(2t)\underset{\sim}{i} + 3\tan(2t)\underset{\sim}{j}$ for $t \geq 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

6 Given the vector equation $\underset{\sim}{r}(t) = 4\cot\left(\frac{t}{2}\right)\underset{\sim}{i} + 3\,\text{cosec}\left(\frac{t}{2}\right)\underset{\sim}{j}$ for $t > 0$, find and sketch the Cartesian equation of the path, and state the domain and range.

7 WE22 Show that the parametric equations $x = \frac{5}{2}\left(t + \frac{1}{t}\right)$ and $y = \frac{3}{2}\left(t - \frac{1}{t}\right)$ where $t \in R \setminus \{0\}$ represent the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

8 Show that the parametric equations $x = \frac{6t}{1 + t^2}$ and $y = \frac{3(1 - t^2)}{1 + t^2}$ represent the circle $x^2 + y^2 = 9$.

CONSOLIDATE

For questions **9–15**, find and sketch the Cartesian equation of the path for each of the given vector equations, and state the domain and range.

9 a $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 4t^2\underset{\sim}{j}$ for $t \geq 0$

b $\underset{\sim}{r}(t) = (t - 1)\underset{\sim}{i} + 3t\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 8t^3\underset{\sim}{j}$ for $t \geq 0$

10 a $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + \frac{1}{t}\underset{\sim}{j}$ for $t > 0$

b $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + (t^2 - 4t)\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j}$ for $t > 0$

11 a $\underset{\sim}{r}(t) = e^{-2t}\underset{\sim}{i} + e^{2t}\underset{\sim}{j}$ for $t \geq 0$

b $\underset{\sim}{r}(t) = e^{-t}\underset{\sim}{i} + (2 + e^{2t})\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = e^{t}\underset{\sim}{i} + (2 + e^{2t})\underset{\sim}{j}$ for $t \geq 0$

12 a $\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{j}$ for $t \geq 0$

b $\underset{\sim}{r}(t) = 4\cos(t)\underset{\sim}{i} + 3\sin(t)\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = 4\sec(t)\underset{\sim}{i} + 3\tan(t)\underset{\sim}{j}$ for $t \geq 0$

13 a $\underset{\sim}{r}(t) = (1 + 3\cos(t))\underset{\sim}{i} + (3\sin(t) - 2)\underset{\sim}{j}$ for $t \geq 0$

b $\underset{\sim}{r}(t) = (4 + 3\cos(t))\underset{\sim}{i} + (2\sin(t) - 3)\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = (2 - 3\sec(t))\underset{\sim}{i} + (5\tan(t) - 4)\underset{\sim}{j}$ for $t \geq 0$

14 a $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \sin^2(t)\underset{\sim}{j}$ for $t \geq 0$

b $\underset{\sim}{r}(t) = \cos^3(t)\underset{\sim}{i} + \sin^3(t)\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = \cos^4(t)\underset{\sim}{i} + \sin^4(t)\underset{\sim}{j}$ for $t \geq 0$

15 a $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$ for $t \geq 0$

b $\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$ for $t \geq 0$

16 If a and b are positive real numbers, show that the following vector equations give the same Cartesian equation.

a $\underset{\sim}{r}(t) = a\cos(t)\underset{\sim}{i} + a\sin(t)\underset{\sim}{j}$ and $\underset{\sim}{r}(t) = \left(\frac{2at}{1 + t^2}\right)\underset{\sim}{i} + \left(\frac{a(1 - t^2)}{1 + t^2}\right)\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = a\cos(t)\underset{\sim}{i} + b\sin(t)\underset{\sim}{j}$ and $\underset{\sim}{r}(t) = \left(\frac{2at}{1 + t^2}\right)\underset{\sim}{i} + \left(\frac{b(1 - t^2)}{1 + t^2}\right)\underset{\sim}{j}$

c $\underset{\sim}{r}(t) = a\sec(t)\underset{\sim}{i} + b\tan(t)\underset{\sim}{j}$, $\underset{\sim}{r}(t) = \frac{a}{2}\left(t + \frac{1}{t}\right)\underset{\sim}{i} + \frac{b}{2}\left(t - \frac{1}{t}\right)\underset{\sim}{j}$ and

$\underset{\sim}{r}(t) = \frac{a}{2}(e^{2t} + e^{-2t})\underset{\sim}{i} + \frac{b}{2}(e^{2t} - e^{-2t})\underset{\sim}{j}$

17 a The position vector of a moving particle is given by $\underset{\sim}{r}(t) = 2\sin(t)\underset{\sim}{i} + 2\sin(t)\tan(t)\underset{\sim}{j}$, for $t > 0$. Show that the particle moves along the curve $y = \frac{x^2}{\sqrt{4 - x^2}}$.

b A curve called the Witch of Agnesi is defined by the parametric equations $x = at$ and $y = \frac{a}{1 + t^2}$. Show that Cartesian equation is given by $y = \frac{a^3}{a^2 + x^2}$.

c **i** Show that $\cos(3A) = 4\cos^3(A) - 3\cos(A)$.

ii A curve is defined by the parametric equations $x = 2\cos(t)$ and $y = 2\cos(3t)$. Find the Cartesian equation of the curve.

d **i** Show that $\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$.

ii A curve is defined by the parametric equations $x = 2\cos^2(t)$ and $y = \cos(4t)$. Find the Cartesian equation of the curve.

18 **a** The position vector of a moving particle is given by $\underset{\sim}{r}(t) = 2\tan(t)\underset{\sim}{i} + 2\operatorname{cosec}(2t)\underset{\sim}{j}$ for $t > 0$. Show that the particle moves along the curve $y = \dfrac{x^2 + 4}{2x}$.

b A curve is defined by the parametric equations $x = \cos(t)(\sec(t) + a\cos(t))$ and $y = \sin(t)(\sec(t) + a\cos(t))$ for $t \in [0, 2\pi]$. Show that the curve satisfies the implicit equation $(x - 1)(x^2 + y^2) - ax^2 = 0$.

c A curve is defined by the parametric equations $x = 4\cos(t)$ and $y = \dfrac{4\sin^2(t)}{2 + \sin(t)}$ for $t \in [0, 2\pi]$. Show that the curve satisfies the implicit equation $y^2(16 - x^2) = (x^2 + 8y - 16)^2$.

MASTER

19 For each of the following vector equations, sketch the equation of the path, using CAS.

a $\underset{\sim}{r}(t) = \cos(2t)\underset{\sim}{i} + \sin(4t)\underset{\sim}{j}$ for $t \geq 0$

b $\underset{\sim}{r}(t) = \cos(2t)\underset{\sim}{i} + \sin(6t)\underset{\sim}{j}$ for $t \geq 0$

c $\underset{\sim}{r}(t) = \cos(3t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$ for $t \geq 0$

d $\underset{\sim}{r}(t) = \cos(3t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$ for $t \geq 0$

20 For each of the following vector equations, sketch the equation of the path, using CAS.

a The cycloid $\underset{\sim}{r}(t) = 2(t - \sin(t))\underset{\sim}{i} + 2(1 - \cos(t))\underset{\sim}{j}$ for $t \geq 0$

b The cardioid $\underset{\sim}{r}(t) = 2\cos(t)(1 + \cos(t))\underset{\sim}{i} + 2\sin(t)(1 + \cos(t))\underset{\sim}{j}$ for $t \geq 0$

c The deltoid $\underset{\sim}{r}(t) = (2\cos(t) + \cos(2t))\underset{\sim}{i} + (2\sin(t) - \sin(2t))\underset{\sim}{j}$ for $t \geq 0$

d The hypercycloid $\underset{\sim}{r}(t) = (5\cos(t) + \cos(5t))\underset{\sim}{i} + (5\sin(t) - \sin(5t))\underset{\sim}{j}$

eBook*plus* ONLINE ONLY

5.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

5 Answers

EXERCISE 5.2

1 $3\overrightarrow{AB}$

2 $3\overrightarrow{OA}$

3 $\overrightarrow{AB} = 2\overrightarrow{BC}$

4 $\overrightarrow{PR} = 3\overrightarrow{PQ}$

5 Check with your teacher.

6 Check with your teacher.

7 a $3\overrightarrow{AC}$ b $5\overrightarrow{OA}$
c $6\overrightarrow{OB}$ d $2\overrightarrow{OA}$

8 Check with your teacher.

9 a $\underset{\sim}{a} = \overrightarrow{OA} = \overrightarrow{CB} = \overrightarrow{DE} = \overrightarrow{GF}$
b $\underset{\sim}{c} = \overrightarrow{OC} = \overrightarrow{AB} = \overrightarrow{EF} = \overrightarrow{DG}$
c $\underset{\sim}{d} = \overrightarrow{OD} = \overrightarrow{AE} = \overrightarrow{CG} = \overrightarrow{BF}$
d i $\underset{\sim}{a} + \underset{\sim}{c}$ ii $\underset{\sim}{c} - \underset{\sim}{d}$
iii $-(\underset{\sim}{a} + \underset{\sim}{c} + \underset{\sim}{d})$ iv $\underset{\sim}{a} + \underset{\sim}{c} - \underset{\sim}{d}$

10 Check with your teacher.

11 Check with your teacher.

12 Check with your teacher.

13 Check with your teacher.

14 Check with your teacher.

15 Check with your teacher.

16 Check with your teacher.

17 a–c Check with your teacher.
d All are coincident at the centroid of the triangle.

18 a $\frac{1}{4}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$ b $\frac{1}{4}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$
c $\frac{1}{4}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$ d All are coincident.

EXERCISE 5.3

1 a $2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ b $\frac{1}{3}(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$

2 $\frac{1}{\sqrt{21}}(2\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k})$

3 $\frac{1}{\sqrt{33}}(2\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k})$

4 13

5 4

6 $x = -1, y = 2, z = -7$

7 a $\pm 2\sqrt{5}$ b $-\frac{5}{2}$

8 a $\pm 2\sqrt{21}$ b $-\frac{3}{2}$

9 7

10 7

11 143.3°

12 $-\sqrt{3}$

13 a $680.36\underset{\sim}{i} - 1615.68\underset{\sim}{j} + 150\underset{\sim}{k}$ b 1759.5 metres

14 a $-2684\underset{\sim}{i} - 1879.4\underset{\sim}{j} + 800\underset{\sim}{k}$ b 3373 metres

15 Check with your teacher.

16 Check with your teacher.

17 a i $\frac{1}{\sqrt{21}}(-2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k})$ ii 115.9°
b i $\frac{1}{\sqrt{38}}(3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k})$ ii 35.8°
c i $\frac{1}{\sqrt{29}}(2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k})$ ii 138°

18 a $2\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$
b $6\underset{\sim}{i} - 8\underset{\sim}{j} - 24\underset{\sim}{k}$
c $10\underset{\sim}{i} + 6\underset{\sim}{j} - 8\underset{\sim}{k}$

19 a, b Check with your teacher.
c 4

20 a $\sqrt{62}$
b 150.5°
c $\frac{1}{\sqrt{11}}(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k})$

21 a, b Check with your teacher.
c $x = \frac{5}{2}, y = 6$

22 a $4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$
b $-3\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$
c $-2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

23 a $\underset{\sim}{c} = \frac{4}{3}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$ b 5 c 6

24 a i -3 ii $\pm 2\sqrt{10}$
b i $\pm 3\sqrt{10}$ ii $-\frac{\sqrt{5}}{2}$
c i $\pm 2\sqrt{6}$ ii -2

25 a $-2\underset{\sim}{i} - \underset{\sim}{j} + 0.5\underset{\sim}{k}$, 2.29 km
b $-141.42\underset{\sim}{i} + 188.21\underset{\sim}{j} + 3.83\underset{\sim}{k}$, 235.45 m

26 a $49.575\underset{\sim}{i} + 17.207\underset{\sim}{k}$, 52.48 km
b $-1342.05\underset{\sim}{i} - 1159.03\underset{\sim}{j} + 2132.09\underset{\sim}{k}$, 2773.13 m

27 a $\pm\sqrt{5}$ b 60° c 135°

28 a $p = -4, q = -3, r = 5$
b $x = \pm\frac{2\sqrt{3}}{3}, y = \pm\frac{\sqrt{33}}{3}$

EXERCISE 5.4

1 –12

2 15

3 –21

4 7

5 –1

6 –1, 5

7 54.74°

8 $-\sqrt{6}$

9 $\sqrt{86}$

10 6

11 a 5

b $\frac{5}{3}(\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k})$

c $\frac{1}{3}(10\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})$

12 a $\frac{2}{7}(-\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k})$

b $\frac{2}{7}(8\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k})$

13 a 12   b –30   c –21

14 a –16   b 3   c 0

15 a i $-\frac{5}{3}$ ii 3   iii $\pm 2\sqrt{5}$

b i –3   ii 39   iii $\pm\sqrt{107}$

c i $\frac{3}{2}$ ii $-\frac{49}{4}$ iii $\frac{3 \pm 2\sqrt{3}}{2}$

16 a 27

b -49

c The dot product is distributive over addition and subtraction.

d It is meaningless; the dot product of a scalar and vector cannot be found.

17 a i $\frac{1}{\sqrt{26}}(3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k})$ ii $\frac{6}{\sqrt{26}}$

iii $\frac{3}{13}(3\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k})$ iv $\frac{1}{13}(17\underset{\sim}{i} - 49\underset{\sim}{j} + 25\underset{\sim}{k})$

v 75.12°

b i $\frac{1}{3}(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$ ii 6

iii $2(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$ iv $-\underset{\sim}{i} - \underset{\sim}{k}$

v 13.26°

c i $\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}, 2\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

ii 42.79°

18 a i $\frac{1}{\sqrt{11}}(3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k})$ ii $-\frac{13}{11}(3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k})$

iii $\frac{1}{11}(-5\underset{\sim}{i} + 9\underset{\sim}{j} - 24\underset{\sim}{k})$ iv 148.8°

b i –3   ii $\frac{13}{3}$

iii $3 \pm \sqrt{3}$ iv 3

c 5, –3

19 a $\sqrt{2}$ b $\frac{3\sqrt{6}}{2}$ c $\frac{1}{2}\sqrt{41}$

20 a $5, \frac{5}{7}$ b 2   c 3

21 a Check with your teacher.

b 13

c 0

d $\underset{\sim}{0}$

e It is possible that $\underset{\sim}{a} = \underset{\sim}{c}$. It is possible that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a} - \underset{\sim}{c}$.

22 a i $\sqrt{37}$ ii $\sqrt{13}$ iii $\sqrt{73}$

b i $\sqrt{95}$ ii $\sqrt{119}$ iii $\sqrt{1331}$

c i Check with your teacher.

ii $|\underset{\sim}{u}| = \sqrt{5}$ and $|\underset{\sim}{v}| = \sqrt{10}$ or $|\underset{\sim}{u}| = \sqrt{10}$ and $|\underset{\sim}{v}| = \sqrt{5}$

23 a i Check with your teacher.

ii $\frac{1}{3}(7\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k})$

iii Check with your teacher.

iv $x^2 + y^2 + z^2 = 350$
$-8x + 2y + 6z = -26$
$-6x + 8y - 2z = -26$

v $(13, 9, 10)$ or $\left(-\frac{25}{3}, -\frac{37}{3}, -\frac{34}{3}\right)$

vi $\frac{32\sqrt{3}}{3}$

vii 80.95°

b i Check with your teacher.

ii $\left(-\frac{1}{2}, \frac{1}{2}, 2\right)$

iii Check with your teacher.

iv $x^2 + y^2 + z^2 = \frac{891}{2}$
$4x + 2y - 4z = -9$
$2x - 2y - 8z = -18$
$2x + 4y + 4z = 9$

v $\left(-\frac{29}{2}, \frac{29}{2}, -5\right)$ or $\left(\frac{27}{2}, -\frac{27}{2}, 9\right)$, 21

vi 84.23°

24 a $\frac{\pm\sqrt{6}}{18}(\underset{\sim}{i} + 7\underset{\sim}{j} + 2\underset{\sim}{k})$

b $\frac{\pm\sqrt{19}}{57}(7\underset{\sim}{i} + \underset{\sim}{j} + 11\underset{\sim}{k})$

c $\frac{\pm\sqrt{2}}{6}(\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k})$

EXERCISE 5.5

1 Check with your teacher.

2 Check with your teacher.

3 a $\frac{1}{2}\sqrt{861}$

b $\frac{1}{2}\sqrt{870}$

4–10 Check with your teacher.

11 a $\overrightarrow{OM} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

b, c Check with your teacher.

12 Check with your teacher.

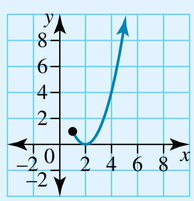
13 a $\overrightarrow{OP} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c}), \overrightarrow{OQ} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c})$

b, c Check with your teacher.

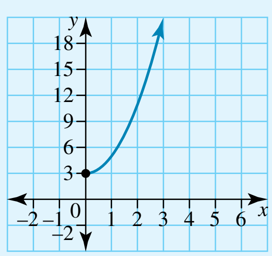
14 Check with your teacher.

EXERCISE 5.6

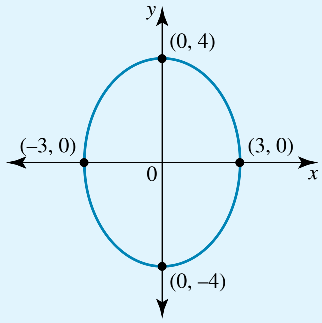
1 Parabola $y = (x - 2)^2$; domain $[1, \infty)$, range $[0, \infty)$



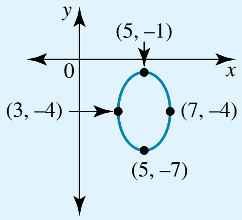
2 Parabola $y = 2x^2 + 3$; domain $[0, \infty)$, range $[3, \infty)$



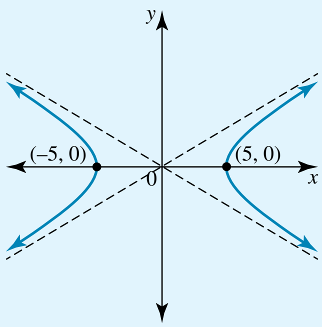
3 Ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$; domain $[-3, 3]$, range $[-4, 4]$



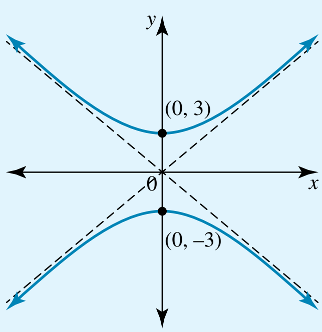
4 Ellipse $\frac{(x-5)^2}{4} + \frac{(y+4)^2}{9} = 1$; domain $[3, 7]$, range $[-7, -1]$



5 Hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$, asymptotes $y = \pm\frac{3x}{5}$; domain $(-\infty, -5] \cup [5, \infty)$, range R



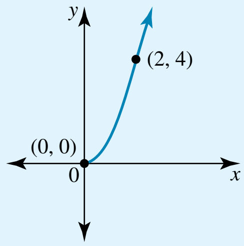
6 Hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$, asymptotes $y = \pm\frac{3x}{4}$; domain R, range $(-\infty, -3] \cup [3, \infty)$



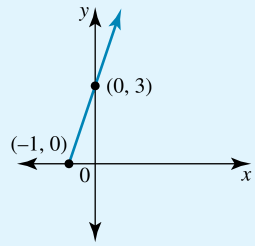
7 Check with your teacher.

8 Check with your teacher.

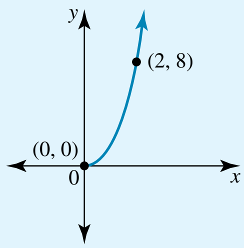
9 a Part of a parabola, $y = x^2$; domain $[0, \infty)$, range $[0, \infty)$



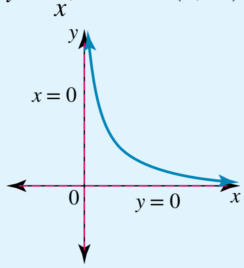
b Part of a straight line, $y = 3x + 3$; domain $[-1, \infty)$, range $[0, \infty)$



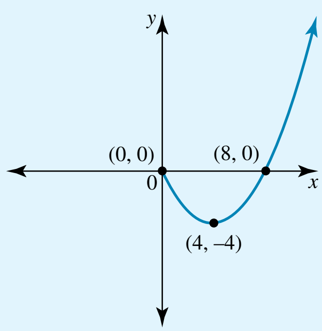
c Part of a cubic, $y = x^3$; domain $[0, \infty)$, range $[0, \infty)$



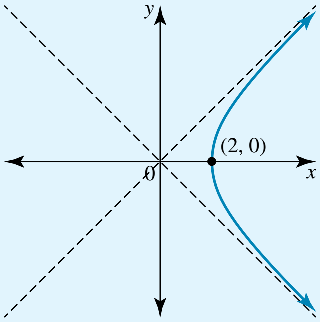
10 a $y = \frac{2}{x}$; domain $(0, \infty)$, range $(0, \infty)$



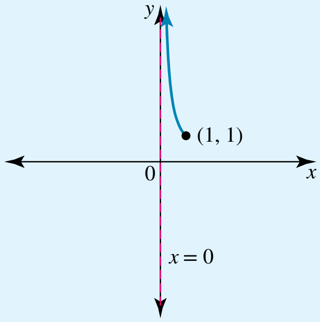
b Part of a parabola, $y = \frac{1}{4}(x^2 - 8x)$; domain $[0, \infty)$, range $[-4, \infty)$



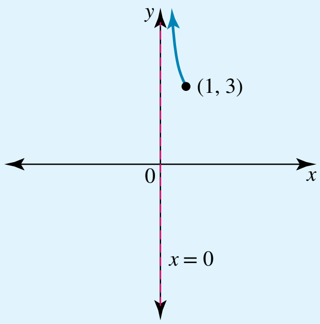
c Part of a hyperbola, $y = \sqrt{x^2 - 4}$; domain $[2, \infty)$, range R



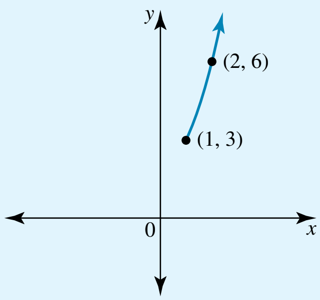
11 a Part of a hyperbola, $y = \frac{1}{x}$; domain $(0, 1]$, range $[1, \infty)$



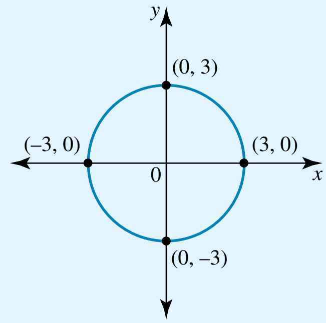
b Part of a truncus, $y = 2 + \frac{1}{x^2}$; domain $(0, 1]$, range $[3, \infty)$



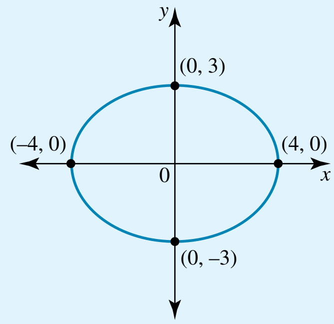
c Part of a parabola, $y = 2 + x^2$; domain $[1, \infty)$, range $[3, \infty)$



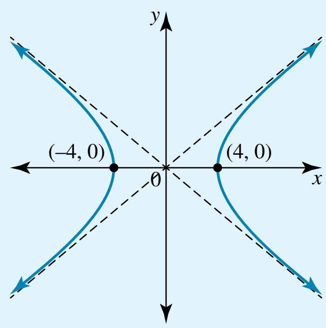
12 a Circle with centre at the origin, radius 3, $x^2 + y^2 = 9$; domain $[-3, 3]$, range $[-3, 3]$



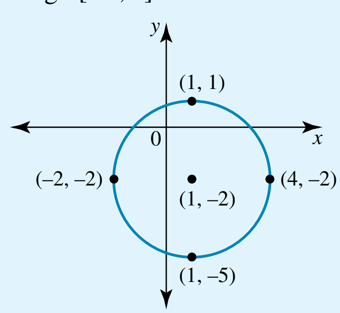
b Ellipse with centre at the origin, $\frac{x^2}{16} + \frac{y^2}{9} = 1$; domain $[-4, 4]$, range $[-3, 3]$



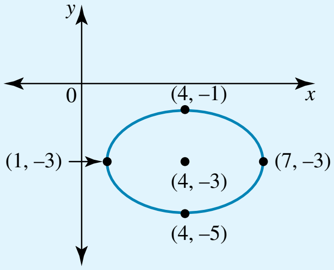
c Hyperbola with centre at the origin, $\frac{x^2}{16} - \frac{y^2}{9} = 1$, asymptotes $y = \pm\frac{3x}{4}$; domain $(-\infty, -4] \cup [4, \infty)$, range R



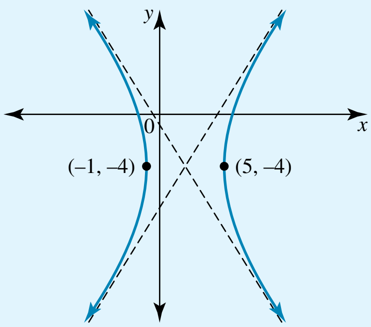
13 a Circle with centre at $(1, -2)$, radius 3, $(x - 1)^2 + (y + 2)^2 = 9$; domain $[-2, 4]$, range $[-5, 1]$



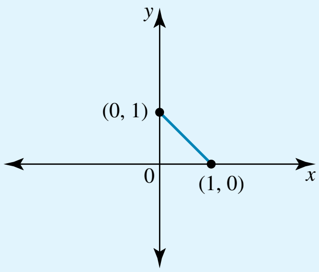
b Ellipse with centre at (4, –3), $\dfrac{(x-4)^2}{9}+\dfrac{(y+3)^2}{4}=1$; domain [1, 7], range [−5, −1]



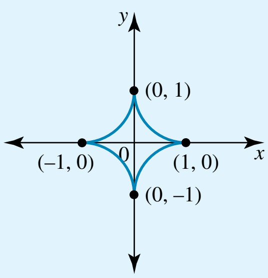
c Hyperbola with centre at (2, –4), $\dfrac{(x-2)^2}{9}-\dfrac{(y+4)^2}{25}=1$, asymptotes $y=\dfrac{5x}{3}-\dfrac{22}{3}$, $y=-\dfrac{5x}{3}-\dfrac{2}{3}$; domain $(-\infty,-1]\cup[5,\infty)$, range R



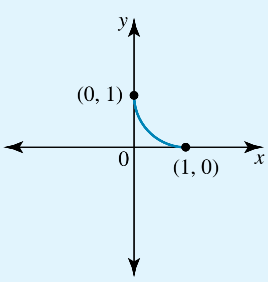
14 a Part of a straight line, $y=1-x$; domain [0, 1], range [0, 1]



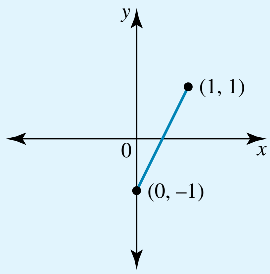
b $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$; domain [−1, 1], range [−1, 1]



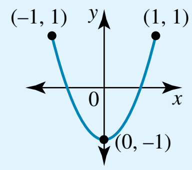
c $\sqrt{y}+\sqrt{x}=1$; domain [0, 1], range [0, 1]



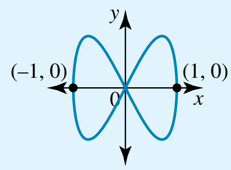
15 a Part of a straight line, $y=2x-1$; domain [0, 1], range [−1, 1]



b Part of a parabola, $y=2x^2-1$; domain [−1, 1], range [−1, 1]



c $y=\pm 2x\sqrt{1-x^2}$; domain [−1, 1], range [−1, 1]



16 a Circle, centre at the origin, radius a, $x^2+y^2=a^2$

b Ellipse, centre at the origin, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

c Hyperbola, centre at the origin, $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

17 a, b Check with your teacher

c i Check with your teacher.

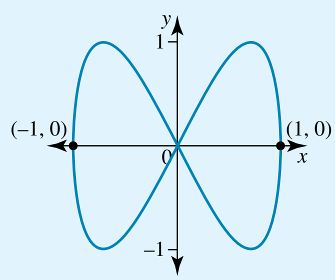
ii $y=x^3-3x$

d i Check with your teacher.

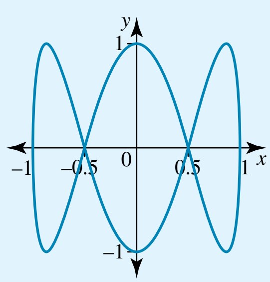
ii $y=2x^2-4x+1$

18 Check with your teacher.

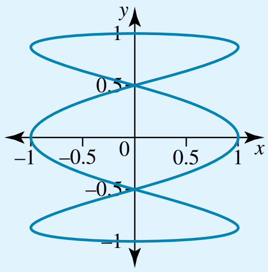
19 a



b

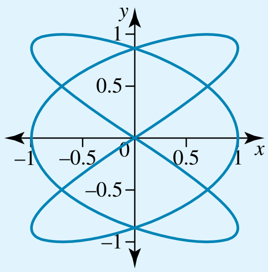


c



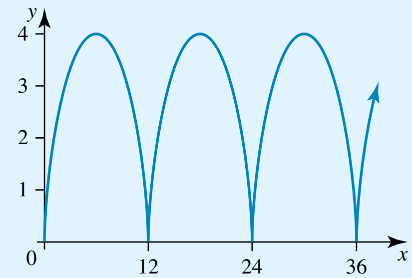
y
x
1
0.5
–0.5
–1
–1
–0.5
0
0.5
1

d



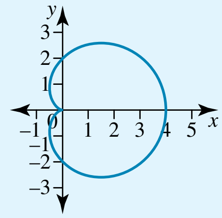
y
x
1
0.5
–0.5
–1
–1
–0.5
0
0.5
1

20 a



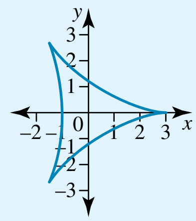
y
x
4
3
2
1
0
12
24
36

b



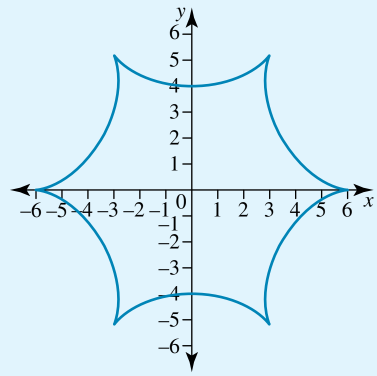
y
x
3
2
1
–1
–2
–3
–1
0
1
2
3
4
5

c



y
x
3
2
1
–1
–2
–3
–2
0
1
2
3

d



y
x
6
5
4
3
2
1
–1
–2
–3
–4
–5
–6
–6
–5
–4
–3
–2
–1
0
1
2
3
4
5
6

6

Mechanics

6.1 Kick off with CAS

Solving equations with CAS

Algebra can be used to solve many equations.

For example, the solution of the polynomial $x^3 - 4x^2 + x + 6 = 0$ can be found by determining a linear factor using the Null Factor Law, dividing the polynomial by the linear factor and factorising the resulting quadratic quotient.

The solution to $27^{4-x} = 9^{2x+1}$ can be found by expressing both sides of the equation to the base 3 and then equating the coefficients.

A trigonometric equation such as $\sqrt{2}\cos(x) = -1$ for $x \in [0, 2\pi]$ can be solved without the use of a calculator as a knowledge of certain values for sin, cos and tan is expected.

There are many equations that cannot be solved using algebra and must be solved either numerically (using iteration) or graphically. If a question requires a solution to a set number of decimal places, then a numeric or graphical solution may be the only, or shortest, means of obtaining the solution.

1 Use CAS to solve the following equations, giving your answers correct to 3 decimal places.

a $2^x = x + 4$

b $2x^3 - 10x^2 + 8x + 4 = 0$

c $2x^2 + 1 = \frac{1}{x}$

d $5\cos(x) = x$

e $x + \log_e(x) = 0$

f $3^x = x^2 + 2x$

g $\sin(x) = \log(x) + x^2$

h $x^5 = 4x - 2$

2 In a narrow passage between two walls there are two ladders: one ladder 3 metres long and the other 2 metres long. The floor is horizontal and the walls are vertical. Each ladder has its foot at the bottom of one wall and its top resting against the other wall. The 3 metre ladder slopes up from left to right and the 2 metre ladder slopes up from right to left. The ladders cross 1 metre above the ground. How wide is the passage? (Give your answer correct to 3 decimal places.)

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

6.2 Statics of a particle

Statics

Statics is the study of the equilibrium of a particle under the action of concurrent coplanar forces.

Some definitions

Particle

A particle is a body or an object with a mass such that all of the mass acts through one point. Large objects such as cars, trains or desks are still regarded as point particles for the purposes of this topic. It will always be assumed that a particle is uniform or a point particle, so that in problems a square or a rectangle can be drawn to represent the object.

Forces

A force is merely a push or a pull. A force is an action which tries to alter the state of the motion of a particle. For example, pushing a car with a certain force may move the car, but if the push is not strong enough, it will not move.

A force has both magnitude and direction and is a vector quantity. The magnitude of the vector $\underset{\sim}{F}$ is expressed as F (that is, without the vector tilde). The standard unit of force is the newton, N. This is named after Sir Isaac Newton (1642–1727) who postulated several laws of motion.

Equilibrium

A system of concurrent coplanar forces acting on a particle is a set of two-dimensional forces all passing through the same point. The equilibrium of a particle means that the vector sum of all the forces acting on the particle is zero. The forces cancel each other out; the particle is said to be in static equilibrium, and the acceleration of the particle is zero.

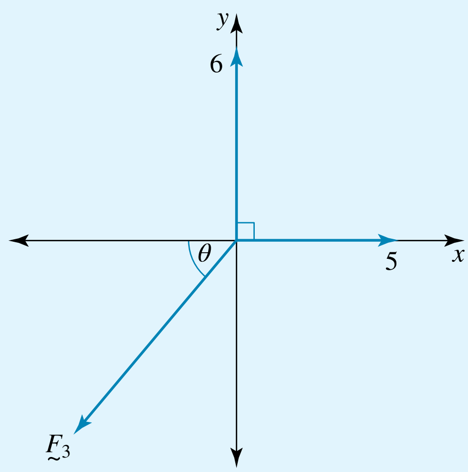
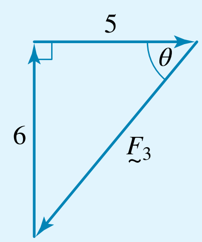
When a particle is in equilibrium and is acted upon by only two forces, $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$, then $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 = \underset{\sim}{0}$ and $\underset{\sim}{F}_1 = -\underset{\sim}{F}_2$, so the two forces act in a straight line but in opposite directions. If a particle is in equilibrium and has two forces acting on it that are not in a straight line, then there must be another, or third, force that balances these forces. So when a particle is acted upon by three forces, $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$, and the particle is in equilibrium, the vector sum of these three forces must be zero: $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$. In this case these three concurrent forces can be represented as the sides of a closed triangle, and the vector sum when the forces are placed head to tail is zero.

Trigonometry and Pythagoras' theorem can be used to solve many force problems.

Throughout this topic answers should be given as exact values where appropriate; otherwise, they should be given correct to 2 decimal places. For calculations involving inexact values, retain the decimal places throughout the working, otherwise the final answer will be affected by the rounding process.

WORKED EXAMPLE 1

A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 5 newtons; another force acts at right angles to the first force and has a magnitude of 6 newtons. Find the third force.

THINK	WRITE/DRAW
1 Draw a force diagram with the three forces acting through a point. Since the particle is in equilibrium, the third force, $\underset{\sim}{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.	
2 Because the particle is in equilibrium, the vector sum of three forces is zero and they form a closed triangle.	Redraw the diagram. 
3 Use Pythagoras' theorem to find the magnitude of the third force.	$\lvert\underset{\sim}{F}_3{}^2\rvert = 6^2 + 5^2$ $= 61$
4 State the magnitude of the third force.	$\lvert\underset{\sim}{F}_3\rvert = \sqrt{61}$
5 Find the direction of the third force.	$\tan(\theta) = \frac{6}{5}$
6 Solve for θ.	$\theta = \tan^{-1}\left(\frac{6}{5}\right)$ $= 50.19°$
7 State the final result.	The third force has a magnitude of $\sqrt{61}$ newtons and acts at an angle of 129.81° with the force of 5 newtons.

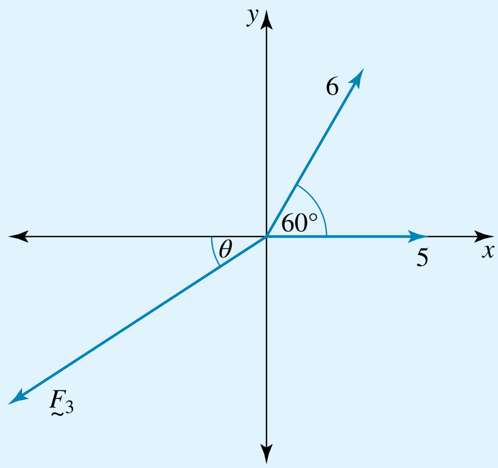
Angles other than right angles

If the forces are not at right angles, then the sine and cosine rules can be used to solve the force problems.

WORKED EXAMPLE 2 **A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 5 newtons; the second force, of magnitude 6 newtons, acts at 60° with the first force. Find the third force.**

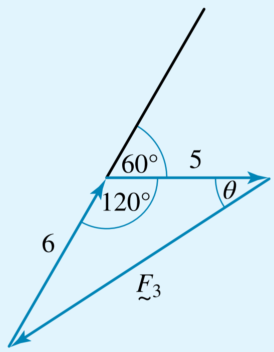
THINK / **WRITE/DRAW**

1 Draw a force diagram with the three forces acting through a point. Because the particle is in equilibrium, the third force, $\underset{\sim}{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.



2 Because the particle is in equilibrium, the vector sum of three forces is zero and they form a closed triangle.

Redraw the diagram.



3 Use the cosine rule to find the magnitude of the third force.

$$\left|\underset{\sim}{F}_3{}^2\right| = 6^2 + 5^2 - 2 \times 6 \times 5\cos(120°)$$
$$= 91$$

4 State the magnitude of the third force.

$$\left|\underset{\sim}{F}_3\right| = \sqrt{91}$$

5 To find the direction of the third force, use the sine rule.

$$\frac{F_3}{\sin(120°)} = \frac{6}{\sin(\theta)}$$

6 Solve for $\sin(\theta)$.

$$\sin(\theta) = \frac{6\sin(120°)}{F_3}$$
$$= \frac{3\sqrt{3}}{\sqrt{91}}$$

7 Find the angle θ.

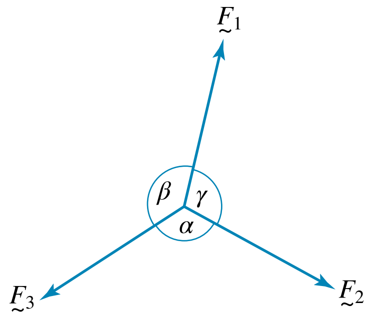
$$\theta = \sin^{-1}\left(\frac{3\sqrt{3}}{\sqrt{91}}\right)$$
$$= 33°$$

8 State the final result.

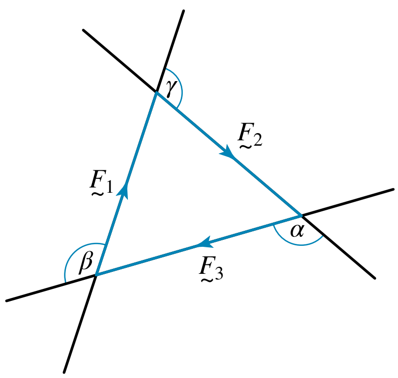
The third force has a magnitude of $\sqrt{91}$ newtons and acts at an angle of 147° with the force of 5 newtons.

Lami's theorem

Lami's theorem states that if three concurrent forces act on a body in equilibrium, then the magnitude of each force is proportional to the sine of the angle between the other two forces.



Redraw the diagram with the forces forming a closed triangle. The magnitude of each force is proportional to the sine of the angle between the other two forces.



The sine rule gives $\dfrac{F_1}{\sin(180° - \alpha)} = \dfrac{F_2}{\sin(180° - \beta)} = \dfrac{F_3}{\sin(180° - \gamma)}$, and

$\sin(180° - \theta) = \sin(\theta)$. Thus, $\dfrac{F_1}{\sin(\alpha)} = \dfrac{F_2}{\sin(\beta)} = \dfrac{F_3}{\sin(\gamma)}$, which is Lami's theorem.

studyon

Units 3 & 4
AOS 5
Topic 1
Concept 2

Types of forces
Concept summary
Practice questions

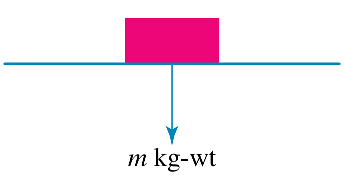
Other types of forces

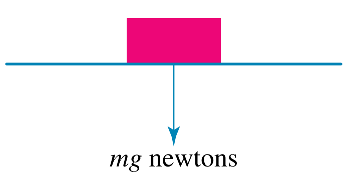
Weight force

The mass and weight of a particle are different. The mass of a particle is defined as the amount of substance in the body; mass is a scalar quantity. When an object is placed on a set of bathroom scales, the reading is the amount of mass in the object, typically in kilograms. When an object is suspended from a spring balance, the reading gives a measure of the gravitational force exerted; this is the object's weight.

Weight is a vector quantity and is defined to be the downwards force exerted in a gravitational field on a particle. For a particle of mass m kg, this weight force is m kg-wt or mg newtons and always acts vertically downwards, where $g = 9.8$ m/s^2 is the acceleration due to gravity in the Earth's gravitational field.

A weight force acts on all bodies near the Earth's surface. The units for a weight force can be either kilograms-weight or newtons; the SI unit for force is the newton.





studyon

Units 3 & 4

AOS 5

Topic 2

Concept 2

Friction and tension

Concept summary

Practice questions

Tension

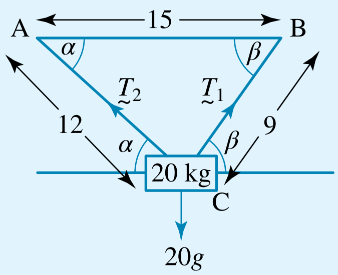
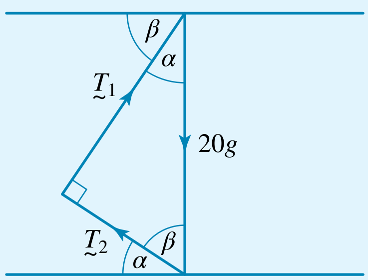
A string has a tension in it denoted by T.

The tension is the same at all points in the string and this tension is unaltered if the string passes over a smooth hook or pulley.

Note: In this discussion, only light strings are considered, so that the mass of the string can be ignored. Also, strings are assumed to be inextensible, so that when the string is pulled tight it is taut.

WORKED EXAMPLE 3

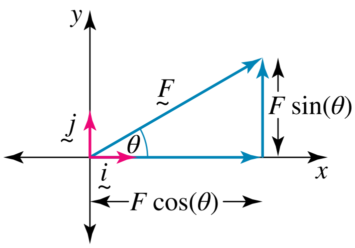
A particle of mass 20 kg is supported by two strings of lengths 12 cm and 9 cm. The ends of the string are attached to two fixed points 15 cm apart on the same horizontal level. Find the tensions in the strings.

THINK	WRITE/DRAW
1 Draw the force diagram. Let the two fixed points be A and B and the position of the particle be point C. Let ∠BAC = α and ∠ABC = β. Let the tension in the 12 cm string be T_2 and let the tension in the 9 cm string be T_1. Include the weight force, $20g$. Note that all the forces in the diagram are in newtons.	
2 The triangle is a right-angled triangle.	Because $d(AC) = 12$, $d(BC) = 9$ and $d(AB) = 15$, and $12^2 + 9^2 = 15^2$, ΔACB is right angled. $\angle ACB = 90°$ and $\alpha + \beta = 90°$.
3 Find the trigonometric ratios.	$\sin(\beta) = \cos(\alpha) = \frac{12}{15}$ and $\cos(\beta) = \sin(\alpha) = \frac{9}{15}$
4 Since the particle is in equilibrium, the vector sum of three forces is zero and they form a closed triangle.	Redraw the diagram. 
5 Use Lami's theorem.	$\dfrac{T_1}{\sin(\beta)} = \dfrac{T_2}{\sin(\alpha)} = \dfrac{20g}{\sin(90°)}$
6 Since $\sin(90°) = 1$, find the tension in the shorter string.	$T_1 = 20g\sin(\beta)$
7 Substitute for the trigonometric ratio and evaluate the tension in the shorter string.	$T_1 = \dfrac{20g \times 12}{15}$ $= 16g$ $= 16 \times 9.8$ $= 156.8$

8 Find the tension in the longer string.	$T_2 = 20g\sin(\alpha)$
9 Substitute for the trigonometric ratio and evaluate the tension in the longer string.	$T_2 = \dfrac{20g \times 9}{15}$ $= 12g$ $= 12 \times 9.8$ $= 117.6$
10 Note that the lengths of the strings do not represent the magnitudes of the tensions in the strings; however, the tensions act along the line and direction of the strings. State the final result.	The tension in the 9 cm string is 156.8 newtons and the tension in the 12 cm string is 117.6 newtons.

Resolving forces

An alternative method to Lami's theorem and using trigonometry is the method of resolving forces. Consider a force $\underset{\sim}{F}$ making an angle of θ with the positive x-axis.

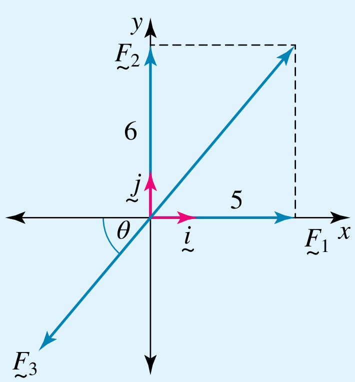


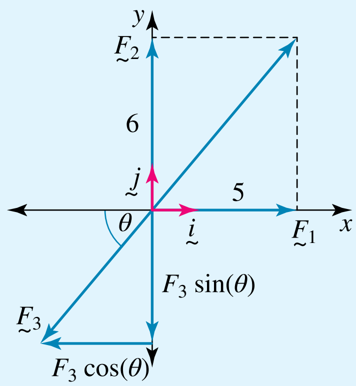
Resolving a vector means splitting the vector up into its horizontal and vertical components. Because $\underset{\sim}{i}$ is a unit vector in the x-direction and $\underset{\sim}{j}$ is a unit vector in the y-direction, $\underset{\sim}{F} = F\cos(\theta)\underset{\sim}{i} + F\sin(\theta)\underset{\sim}{j}$.

In two dimensions, the equilibrium of a particle means that the vector sum of all the resolved horizontal components of the forces is equal to zero. In the $\underset{\sim}{i}$ direction, $\sum F_x = 0$. The vector sum of all the resolved vertical components of the forces in a perpendicular direction is equal to zero, from the $\underset{\sim}{j}$ component: $\sum F_y = \underset{\sim}{0}$. Now we will repeat the first three worked examples, but this time, we will solve them using the method of resolution of vectors.

WORKED EXAMPLE 4

A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 5 newtons; another force acts at right angles to the first force and has a magnitude of 6 newtons. By resolving the forces, find the third force.

THINK	WRITE/DRAW
1 Draw a force diagram with the three forces acting through a point. Since the particle is in equilibrium, the third force, $\underset{\sim}{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.	
2 Express the given horizontal force in terms of $\underset{\sim}{i}$.	$\underset{\sim}{F}_1 = 5\underset{\sim}{i}$

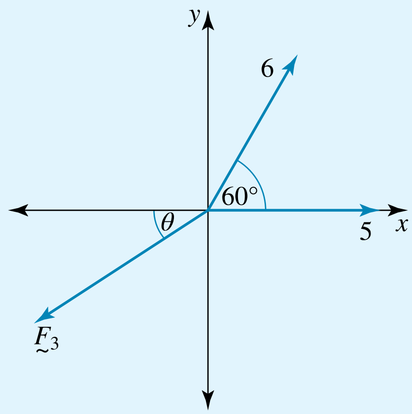
3 Express the given vertical force in terms of $\underset{\sim}{j}$.	$\underset{\sim}{F}_2 = 6\underset{\sim}{j}$
4 Resolve the force $\underset{\sim}{F}_3$ into its horizontal and vertical components.	 $\underset{\sim}{F}_3 = -F_3\cos(\theta)\underset{\sim}{i} - F_3\sin(\theta)\underset{\sim}{j}$
5 Because the particle is in equilibrium and by addition of vectors, $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$.	$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$ $(5 - F_3\cos(\theta))\underset{\sim}{i} + (6 - F_3\sin(\theta))\underset{\sim}{j} = \underset{\sim}{0}$
6 By resolving horizontally, the $\underset{\sim}{i}$ component must be zero.	$5 - F_3\cos(\theta) = 0$ $F_3\cos(\theta) = 5 \quad (1)$
7 By resolving vertically, the $\underset{\sim}{j}$ component must be zero.	$6 - F_3\sin(\theta) = 0$ $F_3\sin(\theta) = 6 \quad (2)$
8 To find F_3 eliminate θ from (1) and (2).	Square both equations. $(1)^2$: $F_3^2\cos^2(\theta) = 25$ $(2)^2$: $F_3^2\sin^2(\theta) = 36$
9 Now add these equations, since $\cos^2(\theta) + \sin^2(\theta) = 1$.	$F_3^2\cos^2(\theta) + F_3^2\sin^2(\theta) = 25 + 36$ $F_3^2(\cos^2(\theta) + \sin^2(\theta)) = 61$ $F_3^2 = 61$
10 State the magnitude of the third force.	$F_3 = \sqrt{61}$
11 To find θ, the direction of the third force, divide the equations to eliminate F_3.	$\frac{(2)}{(1)} \Rightarrow \tan(\theta) = \frac{6}{5}$
12 Solve for θ.	$\theta = \tan^{-1}\left(\frac{6}{5}\right)$ $= 50.19°$
13 State the final result.	The third force has a magnitude of $\sqrt{61}$ newtons and acts at an angle of 129.81° with the force of 5 newtons.

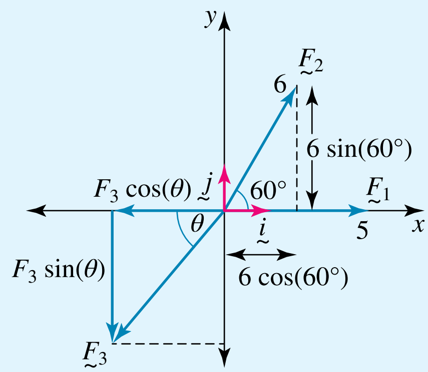
WORKED EXAMPLE 5

A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 5 newtons; the second force, of magnitude 6 newtons, acts at 60° with the first force. By resolving the forces, find the third force.

THINK | **WRITE/DRAW**

1 Draw a force diagram and resolve the forces into their horizontal and vertical components.





2 Express the given horizontal force $\underset{\sim}{F}_1$ in terms of $\underset{\sim}{i}$.

$\underset{\sim}{F}_1 = 5\underset{\sim}{i}$

3 $\underset{\sim}{F}_2$ has components $F_2\cos(60°)$ horizontally to the right and $F_2\sin(60°)$ vertically upwards.

$$\begin{aligned}\underset{\sim}{F}_2 &= 6\cos(60°)\underset{\sim}{i} + 6\sin(60°)\underset{\sim}{j}\\ &= 3\underset{\sim}{i} + 3\sqrt{3}\underset{\sim}{j}\end{aligned}$$

4 Resolve the force $\underset{\sim}{F}_3$ into its horizontal and vertical components, noting that both are in the opposite directions to $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

$\underset{\sim}{F}_3 = -F_3\cos(\theta)\underset{\sim}{i} - F_3\sin(\theta)\underset{\sim}{j}$

5 Since the particle is in equilibrium, use addition of vectors.

$$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$$
$$(8 - F_3\cos(\theta))\underset{\sim}{i} + (3\sqrt{3} - F_3\sin(\theta))\underset{\sim}{j} = \underset{\sim}{0}$$

6 By resolving horizontally, the $\underset{\sim}{i}$ component must be zero.

$$\begin{aligned}8 - F_3\cos(\theta) &= 0\\ F_3\cos(\theta) &= 8 \quad (1)\end{aligned}$$

7 By resolving vertically, the $\underset{\sim}{j}$ component must be zero.

$$\begin{aligned}3\sqrt{3} - F_3\sin(\theta) &= 0\\ F_3\sin(\theta) &= 3\sqrt{3} \quad (2)\end{aligned}$$

8 To find F_3 eliminate θ from (1) and (2).

Square both equations.

$(1)^2$: $F_3^2\cos^2(\theta) = 64$

$(2)^2$: $F_3^2\sin^2(\theta) = 27$

9 Now add these equations, since $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$\begin{aligned}F_3^2\cos^2(\theta) + F_3^2\sin^2(\theta) &= 64 + 27\\ F_3^2(\cos^2(\theta) + \sin^2(\theta)) &= 91\\ F_3^2 &= 91\end{aligned}$$

10 State the magnitude of the force.

$F_3 = \sqrt{91}$

11 To find θ, the direction of the third force, divide the equations to eliminate F_3.

$\frac{(2)}{(1)} \Rightarrow \tan(\theta) = \frac{3\sqrt{3}}{8}$

12 Solve for θ.

$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{8}\right)$$
$$= 33°$$

13 State the final result.

The third force has a magnitude of $\sqrt{91}$ newtons and acts at an angle of 147° with the force of 5 newtons.

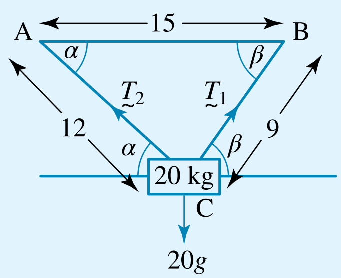
WORKED EXAMPLE 6

A particle of mass 20 kg is supported by two strings of lengths 12 cm and 9 cm. The ends of the string are attached to two fixed points 15 cm apart on the same horizontal level. By resolving the forces, find the tensions in the strings.

THINK

WRITE/DRAW

1 Draw the force diagram. Let the two fixed points be A and B and the position of the particle be point C. Let $\angle BAC = \alpha$ and $\angle ABC = \beta$. Let the tension in the 12 cm string be T_2 and let the tension in the 9 cm string be T_1. Include the weight force $20g$. Note that all the forces in the diagram are in newtons.



2 The triangle is a right-angled triangle.

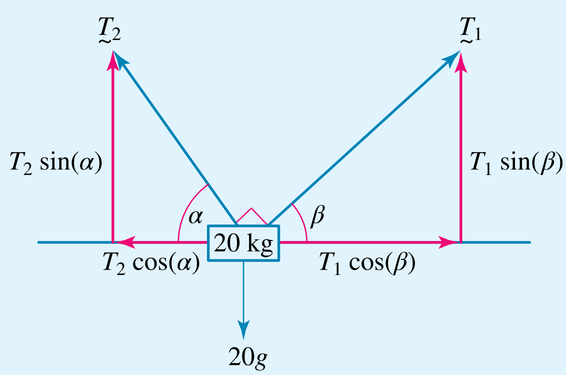
Because $d(AC) = 12$, $d(BC) = 9$ and $d(AB) = 15$, and $12^2 + 9^2 = 15^2$, ΔACB is right angled.
$\angle ACB = 90°$ and $\alpha + \beta = 90°$.

3 Find the trigonometric ratios.

$\sin(\beta) = \cos(\alpha) = \frac{12}{15}$ and $\cos(\beta) = \sin(\alpha) = \frac{9}{15}$

4 $\underset{\sim}{T}_1$ has components $T_1\cos(\beta)$ horizontally to the right and $T_1\sin(\beta)$ vertically upwards.

$\underset{\sim}{T}_2$ has components $T_2\cos(\alpha)$ horizontally to the left and $T_2\sin(\alpha)$ vertically upwards.



5 Resolving horizontally, all the forces horizontally to the right minus all the forces horizontally to the left add to zero.

$T_1\cos(\beta) - T_2\cos(\alpha) = 0 \quad (1)$

6 Resolving vertically, all the forces vertically upward minus all the forces vertically downwards sum to zero.	$T_2 \sin(\alpha) + T_1 \sin(\beta) - 20g = 0 \qquad (2)$
7 Substitute for the trigonometric ratios into (1) and (2).	$\frac{9T_1}{15} - \frac{12T_2}{15} = 0 \qquad (1)$ $\frac{9T_2}{15} + \frac{12T_1}{15} - 20g = 0 \qquad (2)$
8 From (1), express T_2 in terms of T_1.	$T_2 = \frac{3T_1}{4}$
9 Simplify (2).	$9T_2 + 12T_1 = 300g$
10 Substitute for T_2 and solve for T_1.	$9\left(\frac{3T_1}{4}\right) + 12T_1 = 300g$ $\frac{T_1(27 + 48)}{4} = 300g$ $75T_1 = 1200g$
11 State the value of T_1.	$T_1 = \frac{1200g}{75}$ $= 16g$ $= 16 \times 9.8$ $= 156.8$
12 Substitute for T_1 and evaluate T_2.	$T_2 = \frac{3T_1}{4}$ $= \frac{3}{4} \times 16g$ $= 12g$ $= 12 \times 9.8$ $= 117.6$
13 State the final result, noting that the shorter string carries more tension.	The tension in the 9 cm string is 156.8 newtons and the tension in the 12 cm string is 117.6 newtons.

Resolving all the forces

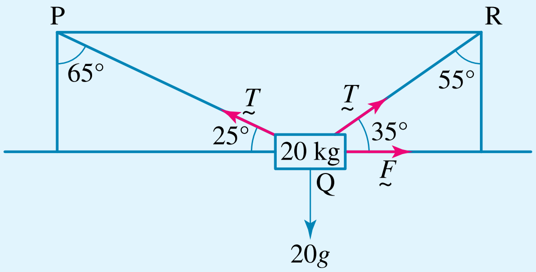
Lami's theorem can only be used when dealing with problems involving three forces. When there are more than three forces acting, resolve the forces.

WORKED EXAMPLE 7

A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of 20 kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of *F* newtons until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 65° and 55° to the vertical respectively. Find the value of *F* and the tension in the strings.

THINK | **WRITE/DRAW**

1 Because the hook is smooth, the tension is unaltered when passing over it, so let the tension in the strings PQ and QR both be T. We need to find the values of T and F. Note that all the forces in the diagram are in newtons.



2 The tension in the string QR has a horizontal component $T\cos(35°)$ and a vertical component $T\sin(35°)$, both in the positive direction.

QR: horizontal component $T\cos(35°)$, vertical component $T\sin(35°)$

3 The tension in the string PQ has a horizontal component $T\cos(25°)$ in the negative direction and a vertical component $T\sin(25°)$ in the positive direction.

PQ: horizontal component $T\cos(25°)$ in the negative direction, vertical component $T\sin(25°)$ in the positive direction.

4 Resolve the forces horizontally.

$$F + T\cos(35°) - T\cos(25°) = 0 \quad (1)$$

5 Resolve the forces vertically.

$$T\sin(35°) + T\sin(25°) - 20g = 0 \quad (2)$$

6 Use (2) to factor and solve for T.

$$T(\sin(35°) + \sin(25°)) = 20g$$

$$T = \frac{20g}{\sin(35°) + \sin(25°)} = 196.75$$

7 State the value of the tension in the strings.

The tension in the strings PQ and QR is 196.75 newtons.

8 Use (1) to find F.

$$F = T\cos(25°) - T\cos(35°) = T(\cos(25°) - \cos(35°))$$

9 Substitute for T.

$$F = \frac{20g(\cos(25°) - \cos(35°))}{(\sin(35°) + \sin(25°))}$$

10 State the value of F.

$F = 17.15$ N

EXERCISE 6.2 Statics of a particle

PRACTISE

1 **WE1** A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 7 newtons; another force acts at right angles to this force and has a magnitude of 8 newtons. Find the third force.

2 A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of F newtons; another force acts at right angles to this force and has a magnitude of $2F$ newtons. Find the third force.

3 **WE2** A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 7 newtons; a second force of magnitude 8 newtons acts at 30° with the first force. Find the third force.

4 A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of F newtons; a second force of magnitude $2F$ newtons acts at 45° with the first force. Find the third force.

5 WE3 A particle of mass 10 kg is supported by two strings of lengths 5 cm and 12 cm. The ends of the string are attached to two fixed points 13 cm apart on the same horizontal level. Find the tensions in the strings.

6 A particle of mass 5 kg is supported by two strings of lengths 7 cm and 24 cm. The ends of the string are attached to two fixed points 25 cm apart on the same horizontal level. Find the tensions in the strings.

7 WE4 A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 7 newtons; another force acts at right angles to this force and has a magnitude of 8 newtons. By resolving the forces, find the third force.

8 A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of F newtons; another force acts at right angles to this force and has a magnitude of $2F$ newtons. By resolving the forces, find the third force.

9 WE5 A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 7 newtons; a second force of magnitude 8 newtons acts at 30° with the first force. By resolving the forces, find the third force.

10 A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of F newtons; a second force of magnitude $2F$ newtons acts at 45° with the first force. By resolving the forces, find the third force.

11 WE6 A particle of mass 10 kg is supported by two strings of lengths 5 cm and 12 cm. The ends of the string are attached to two fixed points 13 cm apart on the same horizontal level. By resolving the forces, find the tensions in the strings.

12 A particle of mass 5 kg is supported by two strings of lengths 7 cm and 24 cm. The ends of the string are attached to two fixed points 25 cm apart on the same horizontal level. By resolving the forces, find the tensions in the strings.

13 WE7 A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of 5 kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of F newtons, until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 60° and 30° to the vertical respectively. Find the value of F and the tension in the strings.

14 A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of m kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of 8 newtons, until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 40° and 20° to the vertical respectively. Find the value of m and the tension in the strings.

CONSOLIDATE

15 a A body is in equilibrium under the action of three forces, $\underset{\sim}{F}_1 = 3\underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{F}_2 = x\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{F}_3 = 6\underset{\sim}{i} + y\underset{\sim}{j}$. Find the values of x and y.

b A body is in equilibrium under the action of three forces, $\underset{\sim}{F}_1 = -2\underset{\sim}{i} + 5\underset{\sim}{j}$, $\underset{\sim}{F}_2 = 6\underset{\sim}{i} - 8\underset{\sim}{j}$ and $\underset{\sim}{F}_3 = x\underset{\sim}{i} + y\underset{\sim}{j}$. Find the values of x and y and the magnitude of the force $\underset{\sim}{F}_3$.

16 a A particle is in equilibrium and is acted upon by three forces. One acts horizontally and has a magnitude of 50 newtons; another acts at right angles to this force and has a magnitude of 120 newtons. Find the third force.

b A particle is in equilibrium and is acted upon by three forces. One acts horizontally and has a magnitude of 30 newtons; another acts at right angles to the 30 newton force and has a magnitude of F_1 newtons. The second force $\underset{\sim}{F}_2$ acts at an angle of 120° with the 30 newton force. Find the magnitudes of the forces $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$.

17 a A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 30 newtons; the second force, with magnitude 60 newtons, acts at 40° with the first force. Find the third force.

b A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 120 newtons; the second force, with magnitude 50 newtons, acts at 140° with the first force. Find the third force.

18 a A particle of mass 4 kg is supported by a string AB attached to a fixed point A and is pulled aside from the vertical by a horizontal force of F newtons acting on the particle at B. If AB makes an angle of 35° with the vertical in the equilibrium position, find:

i the value of F

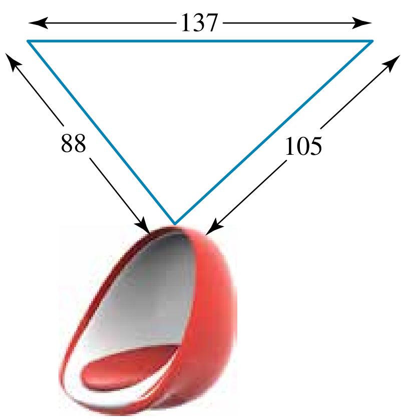
ii the magnitude of the tension in the string.

b A child of mass m kg is sitting on a swing that is held in equilibrium by a horizontal force of 200 newtons applied by his father. At this time, the rope attached to the swing makes an angle of 40° with the vertical. Find the tension in the rope and the mass of the child.

19 a A painting of mass 600 g is to be hung on a wall. It is supported by a string whose ends are attached to two fixed points on the wall at the same horizontal level a distance of 80 cm apart. A hook is attached to the painting. When the painting is hanging in equilibrium, the hook is 30 cm directly below the midpoint of the two fixed points. Find the tension in the string around the hook.

b A string of length $2L$ has its ends fixed to two points at the same horizontal level a distance of L apart. A bead of mass m kg is threaded on the string and hangs at the midpoint of the string. Find the tension in the string.

20 a An egg chair is hanging by two inextensible cables of lengths 88 cm and 105 cm. The other ends of the cable are attached to the ceiling at a distance of 137 cm apart. The shorter cable has a tension of T_1 newtons and the longer cable has a tension of T_2 newtons.



i Show that $T_2 = \dfrac{88\,T_1}{105}$.

ii The longer cable is strong, but the shorter cable will break if the tension in it exceeds

980 newtons. The egg chair has a mass of 10 kg, and a person of mass m kg sits in it. Find the maximum value of m for which the eggchair remains in equilibrium.

b A downlight of mass 250 g is hanging from two cables of lengths 48 cm and 55 cm. The other ends of the cables are attached to the ceiling a distance of 73 cm apart. Show that the tensions in the 48 cm and 55 cm cables have magnitudes of $\frac{55g}{292}$ and $\frac{48g}{292}$ newtons respectively.

21 a A particle of mass 10 kg is attached to one end of a string, and the other end of the string is attached to a fixed point. When in equilibrium, the mass is pulled aside by a horizontal force of $2T$ newtons and the tension in the string is $3T$ newtons. The string makes an angle of θ with the vertical. Find the angle θ and the value of T.

b A particle of mass $2m$ kg is attached to one end of a string, and the other end of the string is attached to a fixed point. When in equilibrium, the mass is pulled aside by a horizontal force of $3mg$ newtons and the tension in the string is T newtons. The string makes an angle of θ with the vertical. Find the angle θ and express T in terms of m.

22 a A string ABC is tied to two fixed points, A and C, at the same horizontal level. A mass of 10 kg is suspended at B. When in equilibrium, the strings AB and BC make angles of θ and 70° with AC, and the magnitudes of the tensions in the strings AB and BC are T and $2T$ newtons respectively. Find the angle θ and the value of T.

b A string PQR is tied to two fixed points, P and R, at the same horizontal level. A mass of 4 kg is suspended at Q. When in equilibrium, the strings PQ and QR make angles of 50° and θ with PR, and the magnitude of the tension in the string PQ is 12 newtons. Find the angle θ and the tension in the string QR.

23 a Two points A and B are at the same horizontal level and are a distance of $5L$ apart. A mass of m kg at point C is connected by two strings, such that the lengths AC and BC are $3L$ and $4L$ respectively. Find the tensions in the two strings AC and BC.

b A particle is supported by two strings at right angles to one another and attached to two points at the same horizontal line. Prove that the tensions in the strings are inversely proportional to the lengths of the strings.

24 a A string is tied to two points at the same horizontal level. A mass of m kg is suspended from a smooth hook and is pulled across horizontally by a force of 10 newtons. When the string makes angles of 55° and 35° to the vertical, the system is in equilibrium. Find the tension in the string and the mass, m.

b A string is tied to two points at the same horizontal level. A mass of m kg is suspended from a smooth hook and is pulled across horizontally by a force of F newtons. When in equilibrium, the string makes angles of α and β to the horizontal, where $\beta > \alpha$. Show that

$$\frac{F}{mg} = \frac{\cos(\alpha) - \cos(\beta)}{\sin(\alpha) + \sin(\beta)}.$$

MASTER

25 a A string PQR is tied to two fixed points P and R at the same horizontal level. A mass of 5 kg is suspended at Q. When in equilibrium, the strings PQ and QR make angles of α and β with PR respectively. The magnitude of the tensions in the strings PQ and QR are 30 and 35 newtons respectively. Find the angles α and β.

b A string ABC is tied to two fixed points A and C at the same horizontal level. A mass of $7m$ kg is suspended at B. When in equilibrium, the strings AB and BC make angles of α and β with AC respectively. The magnitude of the tensions in the strings AB and BC are $5mg$ and $4mg$ newtons respectively. Find the angles α and β.

26 a A string is tied to two points at the same horizontal level. A mass of 1.5 kg is suspended from a smooth hook and is pulled across horizontally by a force of 4 newtons. When in equilibrium, the string makes angles of α and β to the horizontal, where $\beta > \alpha$ and the tension in the string is 10 newtons. Find the angles α and β.

b A string ABC is tied to two points A and C at the same horizontal level. A mass of 5 kg is suspended from a smooth hook at B and is pulled across horizontally by a force of 10 newtons. When in equilibrium, the strings AB and BC make angles of α and 30° with the vertical respectively. Find the angle α and the tension in the string.

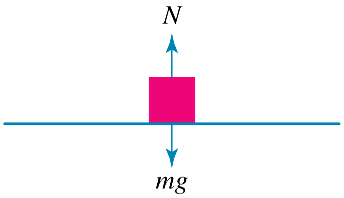
6.3 Inclined planes and connected particles

Other types of forces

Action and reaction

If we think of a computer resting on a desk, the weight force is not the only force acting on the computer, for if the desk were not there, the computer would fall vertically downwards.

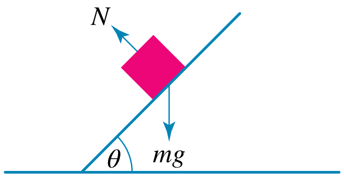
Newton's Third Law of Motion states that for every action there is an equal and opposite reaction. That is, if a body exerts a force by resting on a desk, then the desk exerts a force of equal magnitude but in the opposite direction. This type of force is called the normal reaction and will be denoted by N. The upward force is the reaction of the desk on the computer or object.



The normal reaction is always perpendicular to the surface and only acts when a body is in contact with another surface.

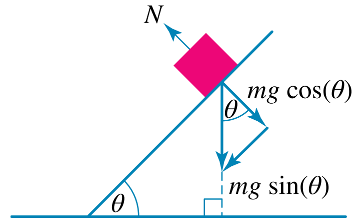
Inclined planes

Consider a particle of mass m kg on a smooth plane inclined at angle of θ to the horizontal. A smooth plane means that there are no frictional forces acting. The particle is being acted on by two external forces: the normal reaction N, which is perpendicular to the plane, and the weight force, which acts vertically downwards.



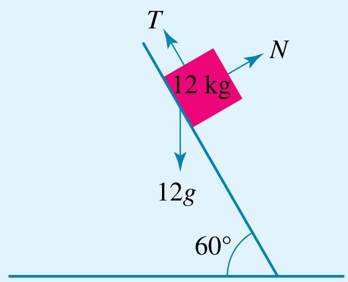
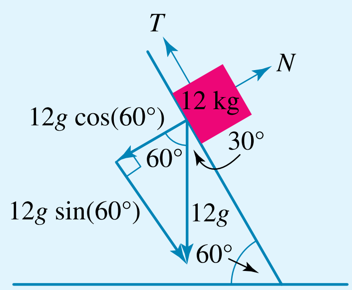
When dealing with problems involving inclined planes, it is usually easier to resolve parallel and perpendicular to the plane rather than horizontally and vertically.

The weight force has components $mg\sin(\theta)$ parallel to and down the plane and $mg\cos(\theta)$ perpendicular to the plane. Note that if these were the only forces acting on the particle, then it could not be in equilibrium. In that situation the particle would slide down the plane.



WORKED EXAMPLE 8

A suitcase of mass 12 kg is maintained in equilibrium on a smooth plane inclined at 60° to the horizontal by means of a strap. Find the tension in the strap and the reaction of the plane if the strap is parallel and acting up the line of greatest slope.

THINK	WRITE/DRAW
1 Draw the diagram and mark in the forces on the suitcase. Let T be the tension in the strap, acting up the line of the slope. $12g$ is the weight force and N is the normal reaction. Note that the normal reaction is perpendicular to the plane. All forces in the diagram are in newtons.	
2 Resolve the weight force into its components parallel and perpendicular to the plane.	
3 Resolve the forces up the plane. The tension force is up the plane and the component of the weight force acts down the plane.	$T - 12g\sin(60°) = 0 \qquad (1)$
4 Resolve the forces perpendicular to the plane. The normal reaction and the component of the weight force act perpendicular to the plane.	$N - 12g\cos(60°) = 0 \qquad (2)$
5 Use (1) to solve for T.	From (1): $T = 12g\sin(60°)$ $= 12g \times \frac{\sqrt{3}}{2}$
6 State the value of T.	The tension in the strap is $6g\sqrt{3}$ newtons.
7 Use (2) to solve for N.	From (2): $N = 12g\cos(60°)$ $= 12g \times \frac{1}{2}$
8 State the value of N.	The normal reaction is $6g$ newtons.

Resolve all the forces

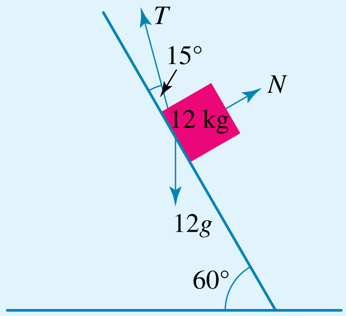
When two or more forces have components, it is a common mistake to forget to find the resolved components of *all* the forces both parallel and perpendicular to the plane.

WORKED EXAMPLE 9

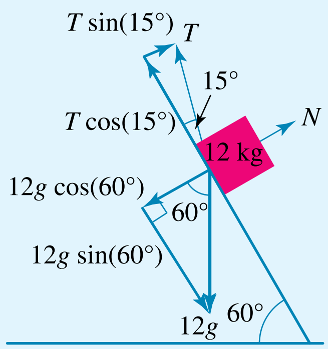
A suitcase of mass 12 kg is maintained in equilibrium on a smooth plane inclined at 60° to the horizontal by means of a strap. Find the tension in the strap and the reaction of the plane if the strap is inclined at an angle of 15° to the plane.

THINK | **WRITE/DRAW**

1 Draw the diagram and mark in the forces on the suitcase. Let T be the tension in the strap, acting at an angle to the plane. $12g$ is the weight force and N is the normal reaction. All forces in the diagram are in newtons.



2 Resolve the weight force and the tension into their components parallel and perpendicular to the plane.



3 Resolve the forces up the plane. The component of the tension force is up the plane and the component of the weight force acts down the plane.

$$T\cos(15°) - 12g\sin(60°) = 0 \qquad (1)$$

4 Resolve the forces perpendicular to the plane. The normal reaction and the component of the tension act up and the component of the weight force acts down, perpendicular to the plane.

$$N + T\sin(15°) - 12g\cos(60°) = 0 \qquad (2)$$

5 Use (1) to solve for T.

From (1):

$$T\cos(15°) = 12g\sin(60°)$$

$$T = \frac{12g\sin(60°)}{\cos(15°)}$$

6 State the value of T.

The tension in the strap is 105.44 newtons.

7 Use (2) to solve for N.

From (2):

$$N = 12g\cos(60°) - T\sin(15°)$$

$$= 12g \times \tfrac{1}{2} - 105.44 \times \sin(15°)$$

8 State the value of N.

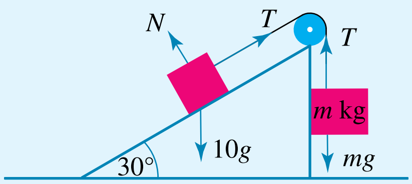
The normal reaction is 31.51 newtons.

Note that, in Worked examples 8 and 9, the normal reaction and tension forces have different values.

Connected particles

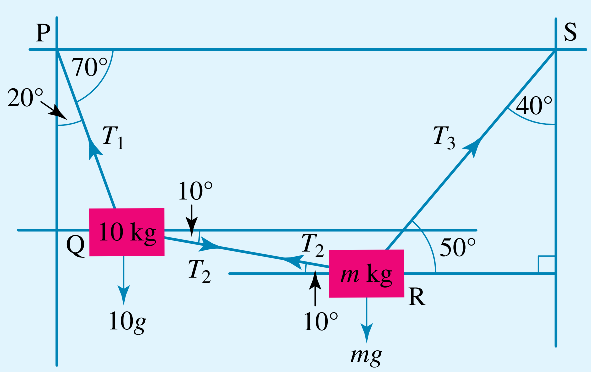
When dealing with connected particles or a system involving two or more particles, isolate the particles and resolve around each particle independently.

WORKED EXAMPLE 10 **A mass of 10 kg lies at rest on a smooth plane inclined at 30° to the horizontal. The mass is held in place by a string that is parallel to the plane and passes over a smooth pulley at the top of the plane. The other end of the string is connected to a mass of *m* kg that hangs vertically. Find the value of *m*.**

THINK	WRITE/DRAW
1 Draw the diagram and mark in all the forces. The pulley is smooth, meaning the tensions on each side are equal. Let the tension in the strings be T. All forces in the diagram are in newtons.	
2 Resolve vertically upwards around the mass m hanging vertically.	$T - mg = 0$
3 Resolve parallel to the plane around the 10 kg mass.	$T - 10g\sin(30°) = 0$
4 Equate the tensions.	$mg = 10g\sin(30°)$
5 Solve for the unknown mass.	$m = 10\sin(30°)$ $= 10 \times \frac{1}{2}$
6 State the value of the mass.	The mass is 5 kg.

If the system has two connected particles in a situation similar to that depicted in Worked example 11, then resolve horizontally and vertically around each particle. This can give four equations, so in these types of problems, there can be up to four unknowns that need to be found.

WORKED EXAMPLE 11 **PQRS is a rope fastened to a horizontal beam at P and S. A mass of *m* kg is suspended from R and a mass of 10 kg is suspended from Q. The section of the rope PQ makes an angle of 70° with PS, and section QR is inclined at 10° below the horizontal, while section RS is inclined at an angle of 40° to the vertical. Find the tensions in all the sections of the rope and the value of *m*.**

THINK	WRITE/DRAW
1 Draw the diagram and mark in all appropriate angles. Let the tensions in the strings PQ, QR and RS be T_1, T_2 and T_3 respectively. We have 4 unknowns: T_1, T_2, T_3 and m.	

2 Resolve horizontally around Q. $T_2\cos(10°) - T_1\sin(20°) = 0 \quad (1)$

3 Resolve vertically around Q. $T_1\cos(20°) - T_2\sin(10°) - 10g = 0 \quad (2)$

4 Resolve horizontally around R. $T_3\cos(50°) - T_2\cos(10°) = 0 \quad (3)$

5 Resolve vertically around R. $T_3\sin(50°) + T_2\sin(10°) - mg = 0 \quad (4)$

6 Use (1) to express T_2 in terms of T_1.

$$(1) \Rightarrow T_2\cos(10°) = T_1\sin(20°)$$
$$T_2 = \frac{T_1\sin(20°)}{\cos(10°)}$$

7 Substitute for T_2 into (2).

$$(2) \Rightarrow T_1\cos(20°) - T_2\sin(10°) = 10g$$
$$T_1\cos(20°) - \frac{T_1\sin(20°)\sin(10°)}{\cos(10°)} = 10g$$

8 Solve for T_1 by taking a common denominator.

$$\frac{T_1\cos(20°)\cos(10°) - T_1\sin(20°)\sin(10°)}{\cos(10°)} = 10g$$
$$\frac{T_1[\cos(20°)\cos(10°) - \sin(20°)\sin(10°)]}{\cos(10°)} = 10g$$

9 Use a trigonometric compound-angle formula.

$$T_1\cos(30°) = 10g\cos(10°)$$
$$T_1 = \frac{10g\cos(10°)}{\cos(30°)}$$
$$T_1 = 111.44$$

10 State the value of T_1. The tension in the string PQ is 111.44 newtons.

11 Back substitute and solve for T_2.

$$T_2 = \frac{10g\cos(10°)}{\cos(30°)} \times \frac{\sin(20°)}{\cos(10°)}$$
$$T_2 = \frac{10g\sin(20°)}{\cos(30°)}$$
$$T_2 = 38.70$$

12 State the value of T_2. The tension in the string QR is 38.70 newtons.

13 Use (3) to solve for T_3.

$$(3) \Rightarrow T_3\cos(50°) - T_2\cos(10°) = 0$$
$$T_3 = \frac{T_2\cos(10°)}{\cos(50°)}$$

14 Substitute for T_2.

$$T_3 = \frac{10g\sin(20°)}{\cos(30°)} \times \frac{\cos(10°)}{\cos(50°)}$$
$$T_3 = 59.30$$

15 State the value of T_3. The tension in the string RS is 59.30 newtons.

16 Use (4) to solve for m.

$$(4) \Rightarrow mg = T_3\sin(50°) + T_2\sin(10°)$$

17 Substitute for T_2 and T_3.

$$mg = 59.30\sin(50°) + 38.70\sin(10°)$$
$$mg = 52.147$$
$$m = \frac{52.147}{9.8}$$
$$= 5.32$$

18 State the value of m. The mass m is 5.32 kg.

EXERCISE 6.3 Inclined planes and connected particles

PRACTISE

1 WE8 A parcel of mass 5 kg is maintained in equilibrium on a smooth plane inclined at 40° to the horizontal by means of a force acting up and parallel to the plane. Find the magnitude of this force and the reaction of the plane.

2 A book of mass 200 g is maintained in equilibrium on a smooth inclined plane by a force of 1 N acting up and parallel to the plane. Find the inclination of the plane to the horizontal and the reaction of the plane on the book.

3 WE9 A parcel of mass 5 kg is maintained in equilibrium on a smooth plane inclined at 40° to the horizontal by means of a force acting at an angle of 20° to the plane. Find the magnitude of this force and the reaction of the plane on the parcel.

4 A book of mass 200 g is maintained in equilibrium on a smooth inclined plane by a force of 1 N acting at an angle 20° with the plane. Find the inclination of the plane to the horizontal and the reaction of the plane on the book.

5 WE10 A mass of m kg is held at rest on a smooth plane inclined at 60° to the horizontal by a string that is parallel to the plane and passes over a smooth pulley at the top of the plane. The other end of the string is connected to a mass of 4 kg that hangs vertically. Find the value of m.

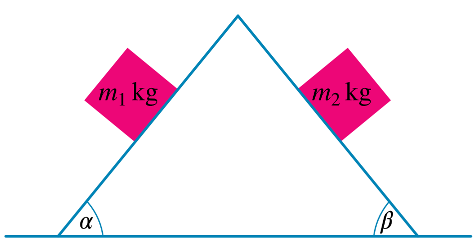
6 A mass of M kg is held at rest on a smooth plane inclined at θ to the horizontal by a string that is parallel to the plane and passes over a smooth pulley at the top of the plane. The other end of the string is connected to a mass of m kg that hangs vertically. If $M > m$, show that $\sin(\theta) = \dfrac{m}{M}$.

7 WE11 PQRS is a rope fastened to a horizontal beam at P and S. A mass of 5 kg is suspended from R and a mass of m kg is suspended from Q. The section of the rope PQ makes an angle of 60° with PS, and section QR is inclined at 45° below the horizontal, while section RS is inclined at an angle of 30° to the vertical. Find the tensions in all the sections of the rope and the value of m.

8 PQRS is a rope fastened to a horizontal beam at P and S. A mass of m_1 kg is suspended from Q and a mass of m_2 kg is suspended from R. The section of the rope PQ makes an angle of 55° with PS, and section QR is inclined at 25° below the horizontal. The tensions in the string PQ and RS are 30 and 40 newtons respectively. Find the tension in QR, the inclination of RS to PS, and the masses m_1 and m_2.

CONSOLIDATE

9 a Two particles are at rest on two smooth inclined planes which are placed back to back. The planes are inclined at angles of α and β to the horizontal, and the particles are connected by a string which passes over a smooth pulley at the top of the planes. The particle on the plane inclined at an angle of α has a mass of m_1 kg and the particle on the plane inclined at an angle of β has a mass of m_2 kg. Show that



$$\frac{m_1}{m_2} = \frac{\sin(\beta)}{\sin(\alpha)}.$$

b Two bodies are at rest on two smooth inclined planes which are placed back to back. The planes are inclined at angles of 60° and 30° to the horizontal, and the bodies are connected by a string passing over a smooth pulley at the top of the planes. The body on the steeper plane has a mass of 4 kg and the other body has a mass of m kg. Find the tension in the string and the value of m.

10 Masses of m kg and 5 kg are suspended from two points of a string that has one end attached to a beam. The lower mass of 5 kg is pulled aside by a horizontal force of F newtons until the upper and lower parts of the string are inclined at 20° and 40° respectively to the vertical. Find the values of:

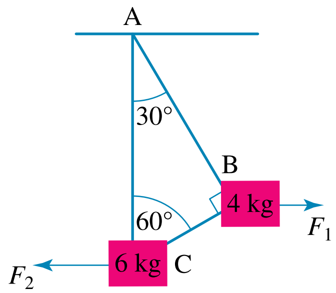
a F

b the mass m

c the tensions in both sections of the string.

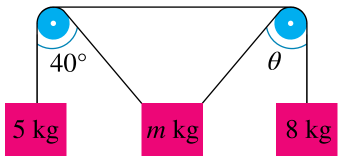
11 Two masses are attached to two different points on a string that hangs at one end from a support. The higher mass is 3 kg; the lower mass is 4 kg and is acted upon by a horizontal force of 20 newtons to the right. Find the tension in each portion of the string and the angle that each section of the string makes with the vertical.

12 Two masses are attached to two different points of a string that hangs from a fixed point, A. The higher mass of 4 kg at point B is acted upon by a horizontal force of F_1 newtons acting to the right, and the lower mass of 6 kg at point C is acted upon by a horizontal force of magnitude F_2 newtons to the left, such that the lower mass is vertically below A. If AB is inclined at the angle of 30° to the vertical and angle ABC is 90° when the system is in equilibrium, find the tension in each portion of the string and the magnitudes of F_1 and F_2.



13 Two masses are attached to two different points of a string that hangs from a fixed point A. The higher mass m_1 kg at point B is acted upon by a horizontal force of 15 newtons acting to the right so that the string AB makes an angle of 40° with the vertical. The lower mass m_2 kg at point C is acted upon by a horizontal force of magnitude 5 newtons to the left such that the lower mass is vertically below A. The angle ABC is 90° when the system is in equilibrium. Find the tension in each portion of the string and the values of m_1 and m_2.

14 Two bodies of mass 5 kg and 8 kg are at the same horizontal level, with the 5 kg mass to the left of the 8 kg mass. Both are hanging vertically and are connected by a light string that passes over two pulleys vertically above both particles and at the same horizontal level. A third body of mass m kg is suspended between the pulleys by the string such that the string on the left makes an angle of 40° to the vertical. If the system is in equilibrium, find the angle that the string on the right makes with the vertical and the mass of the third body.



15 A string passes over two smooth pegs on the same horizontal level and supports masses of 5 kg at both ends hanging vertically. A mass of 6 kg is then hung from the string midway between the pegs, and this moves down 3 cm.

a If the system is now in equilibrium, find the distance between the pegs.

b If instead the 6 kg mass was replaced with a mass of m kg, find an expression in terms of m for the distance of the mass below the pegs.

c What would happen if $m = 10$ kg?

16 PQRS is a rope fastened to a horizontal beam at P and S. A mass of 5 kg is suspended from R and a mass of m kg is suspended from Q. If the section of the rope PQ makes an angle of 50° with PS, and QR is 25° below the horizontal while RS is inclined at 40° to the vertical, find all the tensions in the strings and the value of m.

17 ABCD is a rope fastened to a horizontal beam at A and D. A mass of m kg is suspended from B and a mass of M kg is suspended from C. If the tensions in the strings AB and CD are equal and twice the tension in the string BC, show that the strings AB and CD make equal angles with the vertical. If the string BC is inclined at 60° below the horizontal, find the angle that AB makes with the vertical.

18 a Two masses are attached to two different points of a string that hangs from a fixed point A. The higher mass of $2m$ kg at point B is acted upon by a horizontal force of $3F$ newtons acting to the right, and the lower mass of m kg at point C is acted upon by a horizontal force of magnitude F newtons to the left, such that the lower mass is vertically below A. If the angle ABC is 90° when the system is in equilibrium, find the angle that AB makes with the vertical.

b Two masses are attached to two different points of a string that hangs from a fixed point A. The higher mass m_1 kg at point B is acted upon by a horizontal force of F_1 newtons acting to the right. The lower mass m_2 kg at point C is acted upon by a horizontal force of magnitude F_2 newtons to the left such that the lower mass is vertically below A. If the angle ABC is 90° and AB is inclined at angle of α to the vertical, show that

$$\frac{m_1}{m_2} = \left(\frac{F_1}{F_2} - 1\right)\cot^2(\alpha) - 1.$$

MASTER

19 PQR is a rope fastened to a beam at P. A mass of m_1 kg is suspended at Q and a mass of m_2 kg is suspended at R so that m_2 is below and to the right of m_1. A horizontal force of magnitude F newtons acts to the right at the point R. The section of the rope PQ makes an angle of α with the horizontal, and the section of the rope QR is inclined at an angle of β below the horizontal where $\alpha > \beta$.

Show that $\dfrac{m_2}{m_1} = \dfrac{\cos(\alpha)\sin(\beta)}{\sin(\alpha - \beta)}$ and $F = \dfrac{m_1 g\cos(\alpha)\cos(\beta)}{\sin(\alpha - \beta)}$.

20 PQRS is a rope fastened to a horizontal beam at P and S. A mass of m_1 kg is suspended from Q and a mass of m_2 kg is suspended from R. The sections of the rope PQ and RS make angles of α and γ respectively with PS, and QR makes an angle of β below the horizontal, where $\alpha > \beta > \gamma$.

Show that $\dfrac{m_2}{m_1} = \dfrac{\cos(\alpha)\sin(\beta + \gamma)}{\cos(\gamma)\sin(\alpha - \beta)}$.

6.4 Dynamics

Newton's laws of motion

In contrast to kinematics, which is the study of the motion of particles without consideration of the causes of the motion, dynamics is the study of the causes of the motion of particles.

studyon

Units 3 & 4
AOS 5
Topic 1
Concept 4

Equilibrium and Newton's First Law of Motion
Concept summary
Practice questions

The study of dynamics is based upon Newton's three laws of motion. Isaac Newton (1642–1727) worked in many areas of mathematics and physics. It is believed that he first started studying the effects of gravity after watching an apple fall from a tree.

He wondered why the apple fell, and he wanted to calculate the speed at which it hit the ground. He was also curious about the stars and the movement of the planets in the skies: he asked, were they falling too? He developed his theory of gravitation in 1666, and in 1686 he published his three laws of motion in the book *Principia Mathematica Philosophiae Naturalis*. These laws can be stated in modern English as follows.

Newton's First Law of Motion: Every body continues in a state of rest or uniform motion in a straight line unless acted upon by an external force.

Newton's Second Law of Motion: The resultant force acting on a body is proportional to the rate of change of momentum.

Newton's Third Law of Motion: For every action there is an equal but opposite reaction.

Momentum

The concept of momentum is fundamental to Newton's Second Law of Motion. The momentum $\underset{\sim}{p}$ of a body of mass m moving with velocity $\underset{\sim}{v}$ is defined by

$$\text{momentum} = \text{mass} \times \text{velocity}$$
$$\underset{\sim}{p} = m\underset{\sim}{v}.$$

The standard unit for momentum is kilogram metres/second (kg m/s).

Note that momentum is a vector quantity.

WORKED EXAMPLE 12

Find the magnitude of the momentum of a car of mass 1.4 tonnes moving at 72 km/h.

THINK	WRITE
1 Determine the mass of the car in SI units; that is, express the mass in kg.	$m = 1.4$ tonnes $m = 1400$ kg
2 Convert the speed into m/s. Multiply by 1000 and divide by 60 and 60 again, since there are 60 minutes in an hour and 60 seconds in a minute.	$v = 72$ km/h $v = \dfrac{72 \times 1000}{60 \times 60}$ $= 20$ m/s
3 Find the product of the mass and speed.	$p = mv$ $= 1400 \times 20$
4 State the magnitude of the momentum.	The magnitude of the momentum is 2800 kg m/s.

Resultant force, $\underset{\sim}{R} = m\underset{\sim}{a}$

Newton's second law also involves the resultant force. Let $\underset{\sim}{R} = \sum \underset{\sim}{F}$ be the vector resultant sum of all the forces acting on a body. If the resultant force acting on a body

studyon

Units 3 & 4
AOS 5
Topic 1
Concept 1

Inertial mass and momentum
Concept summary
Practice questions

is proportional to the rate of change of momentum, then $\underset{\sim}{R} \propto \frac{d}{dt}(\underset{\sim}{p})$. Taking k as the proportionality constant, $\underset{\sim}{R} = k\frac{d}{dt}(\underset{\sim}{p})$.

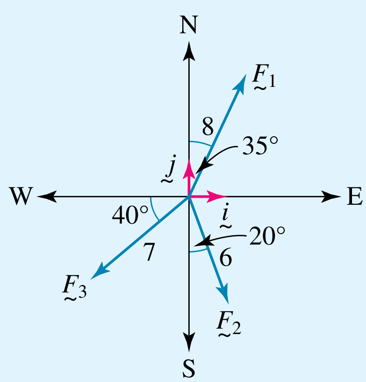
Now substituting for the momentum, $\underset{\sim}{R} = k\frac{d}{dt}(m\underset{\sim}{v})$. Differentiating using the product rule gives $\underset{\sim}{R} = k\left[m\frac{d\underset{\sim}{v}}{dt} + \underset{\sim}{v}\frac{dm}{dt}\right]$. But since the mass is constant, $\frac{dm}{dt} = 0$, we have $\underset{\sim}{R} = km\frac{d\underset{\sim}{v}}{dt} = km\underset{\sim}{a}$, since the acceleration $\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt}$.

We can make $k = 1$ if we use correct and appropriate units. If the mass m is measured in kg, the forces in N and the acceleration in m/s^2, then $k = 1$. So a force of 1 newton is defined as the force which, when applied to a mass of 1 kilogram, produces an acceleration of 1 m/s^2.

Newton's laws imply that a force is needed to make an object move. By considering the forces acting on an object we can analyse its motion.

WORKED EXAMPLE 13

A particle of mass 2 kg on a smooth horizontal plane is acted upon by three forces in that horizontal plane. $\underset{\sim}{F}_1$ has a magnitude of 8 newtons and acts in direction N35°E, $\underset{\sim}{F}_2$ has a magnitude of 6 newtons and acts in direction S20°E, and $\underset{\sim}{F}_3$ has a magnitude of 7 newtons and acts in direction W40°S. If $\underset{\sim}{i}$ is a unit vector of magnitude 1 newton acting east and $\underset{\sim}{j}$ is a unit vector of magnitude 1 newton acting north, find the magnitude of the acceleration of the particle.

THINK	WRITE/DRAW
1 Draw the force diagram.	
2 Resolve the force $\underset{\sim}{F}_1$ into its components.	$\underset{\sim}{F}_1 = 8\sin(35°)\underset{\sim}{i} + 8\cos(35°)\underset{\sim}{j}$
3 Resolve the force $\underset{\sim}{F}_2$ into its components.	$\underset{\sim}{F}_2 = 6\sin(20°)\underset{\sim}{i} - 6\cos(20°)\underset{\sim}{j}$
4 Resolve the force $\underset{\sim}{F}_3$ into its components.	$\underset{\sim}{F}_3 = -7\cos(40°)\underset{\sim}{i} - 7\sin(40°)\underset{\sim}{j}$
5 Find the resultant force, which is the vector sum of all the forces acting on the body.	$\underset{\sim}{R} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$ $\underset{\sim}{R} = (8\sin(35°)\underset{\sim}{i} + 8\cos(35°)\underset{\sim}{j}) + (6\sin(20°)\underset{\sim}{i} - 6\cos(20°)\underset{\sim}{j}) - (7\cos(40°)\underset{\sim}{i} + 7\sin(40°)\underset{\sim}{j})$
6 Use the properties of vector addition.	$\underset{\sim}{R} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$ $\underset{\sim}{R} = (8\sin(35°) + 6\sin(20°) - 7\cos(40°))\underset{\sim}{i} + (8\cos(35°) - 6\cos(20°) - 7\sin(40°))\underset{\sim}{j}$

7 Evaluate the resultant force. $\underset{\sim}{R} = 1.278\underset{\sim}{i} - 3.584\underset{\sim}{j}$

8 Use Newton's Second Law of Motion to find the acceleration, which is a vector.

Since the mass $m = 2$,

$\underset{\sim}{R} = 1.278\underset{\sim}{i} - 3.584\underset{\sim}{j} = 2\underset{\sim}{a}$

$\underset{\sim}{a} = 0.639\underset{\sim}{i} - 1.792\underset{\sim}{j}$

9 Find the magnitude of the acceleration.

$|\underset{\sim}{a}| = \sqrt{0.639^2 + 1.792^2}$

$= 1.90$

10 State the result. The magnitude of the acceleration is 1.90 m/s^2.

studyon

Units 3 & 4
AOS 5
Topic 1
Concept 3

Resultant force
Concept summary
Practice questions

Constant acceleration formulas

When considering constant forces, this implies that the acceleration is constant. The constant acceleration formulas developed in earlier topics can be used for rectilinear motion. Rectilinear motion is motion in a straight line, either horizontally or vertically.

The constant acceleration formulas are $v = u + at$, $s = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2as$ and $s = \left(\frac{u+v}{2}\right)t$.

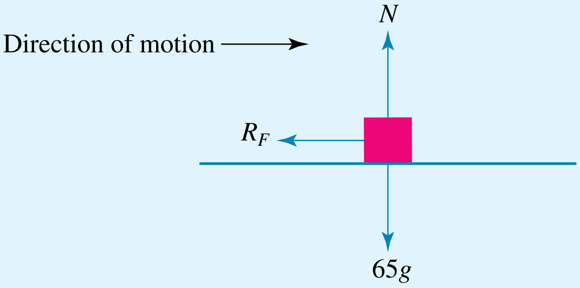
WORKED EXAMPLE 14

A boy of mass 63 kg on a skateboard of mass 2 kg is freewheeling on a level track. His speed decreases from 10 m/s to 5 m/s as he moves a distance of 30 m. Find the magnitude of the resistant force, which is assumed to be constant.

THINK

WRITE/DRAW

1 Draw a force diagram. The only horizontal force acting on the boy and the skateboard is the resistance force R_F, which opposes the direction of motion.



2 Use the constant acceleration formulas.

$u = 10$, $v = 5$, $s = 30$, $a = ?$

Use $v^2 = u^2 + 2as$.

3 Find the acceleration, a, of the boy on the skateboard.

$$\begin{aligned} 25 &= 100 + 2a \times 30 \\ 60a &= -75 \\ a &= -1.25 \text{ m/s}^2 \end{aligned}$$

The acceleration is negative as he is slowing down.

4 Find the resistance force using Newton's second law.

$R_F = ma$
$m = 65$ and $a = -1.25$
$R_F = 65 \times -1.25$
$= -81.25$

5 State the magnitude of the resistance force.

The resistance force is 81.25 newtons.

studyon

Units 3 & 4
AOS 5
Topic 2
Concept 1

Newton's Second and Third Laws of Motion
Concept summary
Practice questions

Newton's Third Law of Motion

Action and reaction pairs are equal and opposite and act on different bodies.

This law has already been considered when including the normal reaction on particles when they are at rest on, or in contact with, a surface. Note that only the external forces acting on a body will be considered in this course, and the effects of compression or the rotational aspects of bodies when forces are applied to them will be ignored.

WORKED EXAMPLE 15

A 60 kg student stands in a lift. Find the reaction of the lift floor on the student if the lift is:

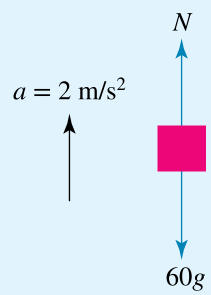
a accelerating upwards at 2 m/s²
b accelerating downwards at 2 m/s²
c moving with constant velocity.

THINK

WRITE/DRAW

a 1 Draw a force diagram, including the direction of motion. Let N newtons be the reaction of the lift floor on the student when the lift travels upwards.

a



2 Resolve the forces in the direction of motion using Newton's second law.

Consider the upwards direction as positive.
The lift is accelerating upwards at 2 m/s² and $m = 60$.
$N - 60g = 60 \times 2$

3 Find the magnitude of the reaction force.

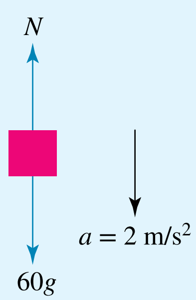
$N = 60g + 60 \times 2$
$= 60(9.8 + 2)$
$= 708$

4 State the value of the reaction force.

The reaction force is 708 N.

b 1 Draw a force diagram, including the direction of motion. Let N newtons now be the reaction of the lift floor on the student when the lift travels downwards. This value of N is different from the case in part a.

b

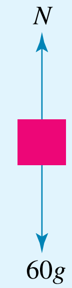


2 Resolve the forces in the direction of motion using Newton's second law. | Consider the downwards direction as positive. The lift is accelerating downwards at 2 m/s^2.
$60g - N = 60 \times 2$

3 Find the magnitude of the reaction force. | $N = 60g - 60 \times 2$
$= 60(9.8 - 2)$
$= 468$

4 State the value of the reaction force. | The reaction force is 468 N.

c 1 When the lift moves with constant velocity, its acceleration is zero. | **c** The weight force is equal to the reaction force.
$N = mg$
$= 60 \times 9.8$
$= 588$



2 State the value of the reaction force in this case. | The reaction force is 588 N.

studyon
Units 3 & 4
AOS 5
Topic 2
Concept 4
Motion on an inclined plane
Concept summary
Practice questions

Motion on inclined planes

When a particle is moving along a slope, the forces are resolved parallel and perpendicular to the slope. For inclined plane problems, consider the $\underset{\sim}{i}$ direction as parallel to the slope. Thus, Newton's Second Law of Motion states that $\sum F_x = ma$. That is, the vector sum of all the resolved forces in the direction of motion is equal to ma. The vector sum of all the resolved forces in the direction perpendicular to the motion in the $\underset{\sim}{j}$ direction, $\sum F_y$, is equal to zero, because the particle is not moving in that direction.

In this section, the term 'tractive force' is used. This is a force that moves a body on a surface.

Slopes

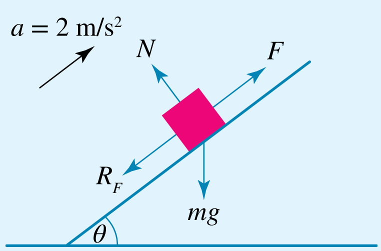
When a particle moves up or down a slope, the inclination of the slope is often stated as the vertical distance raised over the distance travelled along the slope. So a slope of 1 in n means that $\sin(\theta) = \frac{1}{n}$, where θ is the inclination of the slope to the horizontal.

WORKED EXAMPLE 16

A boy of mass 70 kg rides a 10 kg bicycle up a slope of 1 in 5. If the resistance to the motion is one-quarter of the weight force, find the tractive force of the tyres when he ascends with an acceleration of 2 m/s^2.

THINK | **WRITE/DRAW**

1 Draw a force diagram. Let F newtons be the tractive force exerted by the tyres that makes the bike go up the hill. The resistance force R_F is one-quarter of the weight force. Consider the $\underset{\sim}{i}$ direction as the positive direction going up and parallel to the slope.



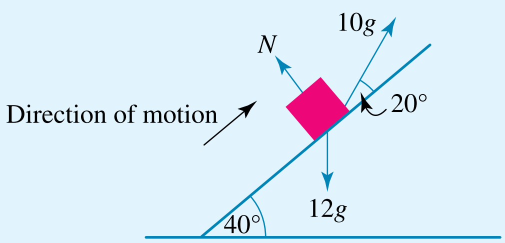
2 Resolve the forces upwards in the direction of motion.	The boy and the bike move up the slope with an acceleration of a. $F - R_F - mg\sin(\theta) = ma$
3 Solve for the tractive force.	$F = R_F + mg\sin(\theta) + ma$
4 Substitute for the given values.	$R_F = \frac{1}{4}mg$ and a slope of 1 in 5 is defined to mean $\sin(\theta) = \frac{1}{5}$. $F = \frac{mg}{4} + \frac{mg}{5} + ma$
5 Evaluate and find the value of the required force.	$F = m\left(\frac{9g}{20} + a\right)$ The total mass of the boy and the bike is 80 kg and the acceleration is 2 m/s^2. $F = 80\left(\frac{9 \times 9.8}{20} + 2\right)$ $= 512.80$
6 State the result.	The tractive force is 512.80 N.

Resolution of all the forces

When two or more forces have components, it is important to remember to find the resolved components of *all* the forces parallel and perpendicular to the plane.

WORKED EXAMPLE 17

A certain strap on a suitcase will break when the tension in it exceeds 10 kg-wt. Find the greatest acceleration with which a 12 kg suitcase can be dragged up a smooth plane inclined at an angle of 40° to the horizontal when the strap is inclined at 20° to the plane.

THINK	WRITE/DRAW
1 Draw the force diagram. Let T newtons be the tension in the strap. Note that $T = 10g$, as all the forces must be in newtons.	
2 Find the component of the tension force up and parallel to the plane.	The component of the tension in the strap parallel to the plane is $10g\cos(20°)$.
3 Find the component of the weight force parallel to the plane.	The component of the weight force is $12g\sin(40°)$, which acts down the plane.
4 Use Newton's second law to find the acceleration.	The suitcase moves up the plane with an acceleration of a. $10g\cos(20°) - 12g\sin(40°) = 12a$
5 Resolve all the forces, perpendicular to the plane. However, this equation is not needed in this case.	$10g\sin(20°) + N - 12g\cos(40°) = 0$

6 Solve to find the maximum acceleration.

$a = \frac{9.8}{12}(10\cos(20°) - 12\sin(40°))$
$= 1.37$

7 State the acceleration that occurs when the tension is $10g$ newtons.

The acceleration is 1.37 m/s^2.

EXERCISE 6.4 Dynamics

PRACTISE

1 WE12 Find the magnitude of the momentum of a truck of mass 6 tonnes moving at 96 km/h.

2 Find the magnitude of the momentum of the Earth as it orbits the Sun. Use the internet to obtain the relevant values.

3 WE13 A particle of mass 3 kg on a smooth horizontal plane is acted upon by three forces in that horizontal plane. $\underset{\sim}{F}_1$ has a magnitude of 9 newtons and acts in direction N55°E, $\underset{\sim}{F}_2$ has a magnitude of 7 newtons and acts in direction S40°E, and $\underset{\sim}{F}_3$ has a magnitude of 6 newtons and acts in direction W35°S. If $\underset{\sim}{i}$ is a unit vector of magnitude 1 newton acting east and $\underset{\sim}{j}$ is a unit vector of magnitude 1 newton acting north, find the magnitude of the acceleration of the particle.

4 A particle of mass 4 kg on a smooth horizontal plane is acted upon by four forces in that horizontal plane. $\underset{\sim}{F}_1$ has a magnitude of 12 newtons and acts due north, $\underset{\sim}{F}_2$ has a magnitude of 8 newtons and acts in direction S60°W, $\underset{\sim}{F}_3$ has a magnitude of 8 newtons and acts in direction E30°S, and $\underset{\sim}{F}_4$ has a magnitude of 10 newtons and acts due east. If $\underset{\sim}{i}$ is a unit vector of magnitude 1 newton acting east and $\underset{\sim}{j}$ is a unit vector of magnitude 1 newton acting north, find the magnitude of the acceleration of the particle.

5 WE14 A car of mass 1200 kg moves on a horizontal road and brakes. Its speed decreases from 16 m/s to 6 m/s while it moves a distance of 25 m. Find the magnitude of the resistant breaking force which is assumed to be constant.

6 A woman is riding a bicycle and is moving at a speed of 6 km/h. She freewheels, coming to rest after 5 seconds. If the magnitude of the constant resistance force is 20 newtons, find the mass of the woman and the bicycle.

7 WE15 An 80 kg man stands in a lift. Find the reaction of the lift floor on the man if the lift is:

a accelerating upwards at 3 m/s^2
b accelerating downwards at 3 m/s^2
c moving with constant velocity.

8 A girl stands in a lift. When the lift accelerates up at 1.5 m/s^2, the reaction of the lift floor on the girl is 734.5 N. When the lift accelerates down at 1.5 m/s^2, find the reaction of the lift floor on the girl.

9 WE16 A car of mass 1200 kg ascends a hill of slope 1 in 4. If the resistance to the motion is one-fifth of the weight force, find the tractive force of the tyres when the car ascends the hill with an acceleration of 2 m/s^2.

10 A truck of mass 1.5 tonnes is moving at a speed of 36 km/h down a hill of slope 1 in 6. The driver applies the brakes and the truck comes to rest after 2 seconds. Find the braking force, which is assumed to be constant, ignoring any other resistance forces.

11 WE17 A parcel of mass 5 kg is pushed up a slope inclined at an angle of 40° to the horizontal by a force of F N that is inclined at an angle of 15° to the plane. The parcel accelerates at 2 m/s^2 up the slope. Find the value of F.

12 A crate of mass 8 kg sits on a smooth plane inclined at an angle of 25° to the horizontal. It is acted upon by a force of F N that is inclined at an angle of 20° to the plane.

a If the crate accelerates at 1 m/s^2 up the slope, find the value of F.

b Find the maximum value of F that can be applied without the crate leaving the plane.

CONSOLIDATE

13 a Find the resultant force acting on a body of mass 3 kg moving with an acceleration of 2 m/s^2.

b Find the magnitude of the acceleration when a mass of 2.5 kg is acted upon by a force of $\underset{\sim}{F} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ N.

c Find the magnitude of the acceleration when a mass of 2 kg is acted upon by two forces of $\underset{\sim}{F}_1 = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{F}_2 = -7\underset{\sim}{i} + 7\underset{\sim}{j}$ N.

14 a A particle of mass 2 kg is acted upon by two forces. One force has a magnitude of $3\sqrt{2}$ N acting in the north-west direction; the other force has magnitude of $4\sqrt{2}$ N acting in the south-west direction. Find the magnitude of the acceleration of the particle.

b A particle of mass m kg is acted upon by two forces. One force has a magnitude of b N acting in the north-east direction; the other force has magnitude of c N acting in the south-east direction. Find the magnitude of the acceleration acting on the body.

15 a A particle of mass 5 kg on a smooth horizontal plane is acted upon by three forces. $\underset{\sim}{F}_1$ has a magnitude of 9 N and acts in direction N55°E, $\underset{\sim}{F}_2$ has a magnitude of 8 N and acts in direction S40°E, and $\underset{\sim}{F}_3$ has a magnitude of 7 N and acts in direction W35°S. Find the magnitude of the acceleration of the particle.

b A body on a smooth horizontal plane is acted upon by four forces. $\underset{\sim}{F}_1$ has a magnitude of 6 N and acts in direction N30°E, $\underset{\sim}{F}_2$ has a magnitude of $8\sqrt{2}$ N and acts in direction S45°E, $\underset{\sim}{F}_3$ has a magnitude of 4 N and acts in direction W30°N, and $\underset{\sim}{F}_4$ has a magnitude of $5\sqrt{2}$ N and acts in direction S45°W. If $\underset{\sim}{i}$ is a unit vector of magnitude 1 N acting east and $\underset{\sim}{j}$ is a unit vector of magnitude 1 N acting north, find the resultant force acting on the body.

16 a A constant force of 12 newtons acts on a body of mass 4 kg originally at rest. Find:

i the speed of the body after 3 seconds

ii the distance travelled by the body in 3 seconds.

b Find the average resistance force exerted by the brakes on a car of mass 1.2 tonnes during braking if its speed decreases from 60 km/h to rest:

i in a time of 1.5 seconds

ii in travelling a distance of 15 metres.

17 **a** A package of mass 4 kg lies on the floor of a lift that is accelerating upwards at 2 m/s^2. Find the reaction of the lift floor on the package.

b A parcel of mass 5 kg rests on the floor of a lift. If the reaction between the box and the floor is 48 N, find the downwards acceleration of the lift.

18 A man of mass 75 kg stands in a lift. What is the reaction between his feet and the floor when the lift is:

a ascending with a constant acceleration of 0.5 m/s^2
b descending with a constant acceleration of 0.5 m/s^2
c moving with a constant speed of 0.5 m/s?

19 A boy stands in a lift. When the lift accelerates upwards with a constant acceleration of a m/s^2, the reaction of the floor on the boy is 502.5 N. When the lift accelerates downwards with a constant acceleration of a m/s^2, the reaction of the floor on the boy is 477.5 N. Find the mass of the boy and the value of a.

20 A child of mass 40 kg is on a 2 kg sled on horizontal snow-covered ground. The child and sled are being pulled along the snow by a force of 100 N. Find the acceleration of the child and the magnitude of the normal reaction of the child on the snow if the force is:

a horizontal
b at an angle of 35° to the horizontal.

21 **a** A child of mass 40 kg is on a 2 kg sled which is being pulled up a snow-covered slope by a force of 300 N. The slope is inclined at an angle of 40° to the horizontal. Find the acceleration of the child and the magnitude of the normal reaction of the child on the snow if the force is:

i parallel to the slope
ii acting at an angle of 25° to the slope.

b A child of mass 40 kg is on a 2 kg sled sliding down a snow-covered slope which is inclined at an angle of 40° to the horizontal. Given that the initial speed is zero and that the friction is negligible, calculate the speed attained when the child and sled have travelled a distance of 20 m down the slope.

22 Calculate the tension in a rope if a 15 kg mass is moved with an acceleration of 2 m/s^2:

a vertically upwards
b along a smooth horizontal plane parallel to the rope
c along a smooth horizontal plane if the rope makes an angle of 15° with the plane
d up along a smooth plane that is inclined at an angle of 20° to the horizontal, and the rope is parallel to the plane
e up along a smooth plane that is inclined at an angle of 20°, and the rope makes an angle of 15° with the plane.

MASTER

23 a A parcel of mass m kg is on a smooth slope inclined at an angle of θ to the horizontal. When a force of 25 N is applied acting up and parallel to the slope, the parcel accelerates up the slope at 0.8 m/s^2. When the force is increased to 30 N acting up and parallel to the slope, the parcel accelerates up the slope at 1.8 m/s^2. Find the values of m and θ.

b A parcel of mass m kg is on a smooth slope that is initially inclined at an angle of θ to the horizontal. When a force of 16 N is applied acting up and parallel to the slope, the parcel accelerates up the slope at 0.6 m/s^2. When the inclination of the slope is increased to 2θ to the horizontal and the force is increased to 30 N acting up and parallel to the slope, the parcel accelerates up the slope at 1.2 m/s^2. Find the values of m and θ.

24 a A package of mass m kg is on a smooth horizontal surface. When a force of 5 newtons is applied to the parcel acting at an angle of α to the horizontal, the parcel accelerates at 0.5 m/s^2. When the force is increased to 10 N and is acting at an angle of 2α to the horizontal, the package accelerates at 0.9 m/s^2. Find the values of m and α.

b A package of mass 5 kg is on a smooth slope inclined at an angle of θ to the horizontal. When a force of 40 N is applied acting at an angle of α to the slope, the package accelerates up the slope at 1.2 m/s^2. When the force is increased to 50 N, the package accelerates at 3.1 m/s^2. Find the values of θ and α.

6.5 Dynamics with connected particles

Two or more particles

When dealing with connected particles or a system involving two or more particles, isolate the particles and resolve around each independently. In the direction of motion, equate the resultant force to ma where m is the mass of the particle and a the acceleration of the system.

studyon

Units 3 & 4
AOS 5
Topic 2
Concept 5

Motion with connected bodies
Concept summary
Practice questions

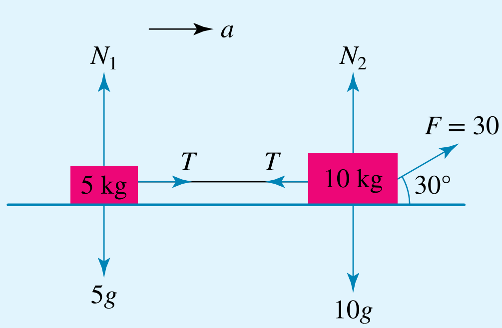
WORKED EXAMPLE 18

Two boxes of masses 5 kg and 10 kg are connected by a string and lie on a smooth horizontal surface. A force of 30 newtons acts on the 10 kg box at an angle of 30° to the surface. Find the acceleration of the system and the tension in the string connecting the boxes.

THINK

1 Draw the force diagram. The force pulling the box is 30 N. Let T be the tension in newtons in the connecting string and let a m/s^2 be the acceleration of the system.

WRITE/DRAW



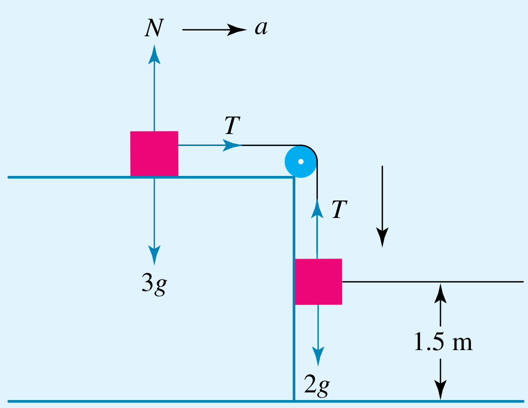
2 Isolate the particles and consider the motion of the first box. It is not necessary to resolve vertically around the boxes in this situation.	Resolving horizontally around the 10 kg box: $30\cos(30°) - T = 10a \quad (1)$
3 Consider the motion of the second box.	Resolving horizontally around the 5 kg box: $T = 5a \quad (2)$
4 To find the acceleration, eliminate the tension in the string.	Substitute $T = 5a$ into (1): $30\cos(30°) - 5a = 10a$ $30 \times \frac{\sqrt{3}}{2} = 15a$
5 Find the common acceleration of the system.	$a = \sqrt{3}$ The acceleration of the system is $\sqrt{3}$ m/s^2.
6 Find the tension in the string.	From (2), $T = 5a$. The tension is $5\sqrt{3}$ N.

Particles connected by smooth pulleys

Often particles may be connected by a smooth pulley. In these situations, the tension in the string is unaltered when the string passes over the pulley.

WORKED EXAMPLE 19

A block of mass 3 kg rests on a smooth table, 2 metres from the edge. It is connected by a light inextensible string that passes over a smooth pulley at the edge of the table to a second block of mass 2 kg hanging vertically. Initially the 2 kg block is 1.5 metres above the ground. Find the time taken for the 2 kg block to hit the ground after the system is released from rest.

THINK	WRITE/DRAW
1 Draw the force diagram and let T newtons be the tension in string.	
2 Isolate the particles and consider the motion of the particle hanging vertically.	The 2 kg block moves vertically downwards with an acceleration of a m/s^2. $2g - T = 2a \quad (1)$
3 Consider the motion of the particle on the table.	The 3 kg block moves horizontally along the smooth table with an acceleration of a m/s^2. $T = 3a \quad (2)$
4 To find the acceleration, eliminate the tension in the string.	Substitute $T = 3a$ into (1): $2g - 3a = 2a$

5 Find the common acceleration of the system.

$$5a = 2g$$
$$a = \frac{2g}{5}$$

6 Use constant acceleration formulas to find the time for the particle hanging vertically to hit the ground.

$a = \frac{2g}{5}$, $u = 0$, $s = 1.5$, $t = ?$

Use $s = ut + \frac{1}{2}at^2$.

7 Find the required time.

$$\frac{3}{2} = 0 + \frac{1}{2} \times \frac{2g}{5}t^2$$
$$t^2 = \frac{15}{2g}$$
$$t = \sqrt{\frac{15}{2 \times 9.8}}$$
$$= 0.87$$

8 State the final result.

The time taken is 0.87 seconds.

Atwood's machine

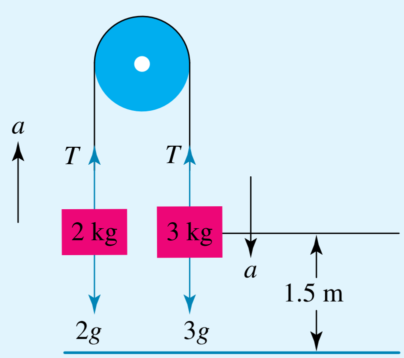
George Atwood (1745–1807) was an English mathematician who invented a machine consisting of masses attached to pulleys to illustrate Newton's laws of motion. He was also a highly skilled chess player.

WORKED EXAMPLE 20

Two particles of masses 3 kg and 2 kg are both hanging vertically and are connected by a light inextensible string which passes over a smooth pulley. Initially the 3 kg block is 1.5 metres above the ground. Find the time taken for the 3 kg block to hit the ground once the system is released from rest.

THINK / **WRITE**

1 Draw the force diagram and let T newtons be the tension in string. Because the 3 kg mass is heavier, it moves down and the 2 kg mass moves up. Because they are connected, they both move with the same acceleration a m/s^2.



2 Isolate the particles and consider the motion of the 3 kg particle hanging vertically.

The 3 kg particle moves vertically downwards with an acceleration of a m/s^2.

$3g - T = 3a \quad (1)$

3 Consider the motion of the 2 kg particle hanging vertically.

The 2 kg particle moves vertically upwards with an acceleration of a m/s^2.

$T - 2g = 2a \quad (2)$

4 To find the acceleration, eliminate the tension in the string.

Add (1) and (2):
$g = 5a$

5 Find the common acceleration of the system.

$a = \frac{g}{5}$ m/s^2

6 Use constant acceleration formulas to find the time for the particle hanging vertically to hit the ground.

$a = \frac{g}{5}, u = 0, s = 1.5, t = ?$

Use $s = ut + \frac{1}{2}at^2$.

7 Find the required time.

$\frac{3}{2} = 0 + \frac{1}{2} \times \frac{g}{5}t^2$

$t^2 = \frac{15}{g}$

$t = \sqrt{\frac{15}{9.8}}$
$= 1.24$

8 State the final result.

The time taken is 1.24 seconds.

Connected vehicles

When cars, trucks, trains and carriages are connected, they are typically connected by tow bars. It is the tension in the tow bar that enables the secondary object to move with the same acceleration as the primary object, which exerts the force to make the connected objects move.

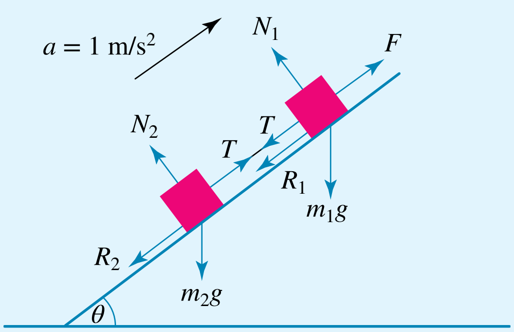
WORKED EXAMPLE 21

A car of mass 1400 kg is pulling a caravan of mass 2850 kg up an incline of slope 1 in 4. The resistances to the motion of the car and the caravan are both one-quarter of their weight forces. Find the tension in the tow bar and the tractive force exerted by the engine in the car if the car moves up the incline with an acceleration of 1 m/s^2.

THINK

1 Draw the force diagram. Let the tension in the tow bar be T newtons and let the tractive force of the car be F newtons. Let the resistances to the car and the caravan be R_1 and R_2 respectively.

WRITE/DRAW



$m_1 = 1400$ and $m_2 = 2850$

2 Isolate the particles and resolve around each independently.	Resolving upwards parallel to the slope around the car: $F - R_1 - T - m_1 g \sin(\theta) = m_1 a \quad (1)$
3 Isolate the particles and resolve around each independently.	Resolving upwards parallel to the slope for the caravan: $T - R_2 - m_2 g \sin(\theta) = m_2 a \quad (2)$
4 The values of F and T are to be found.	The resistance forces are one-quarter of the weight forces. $R_1 = \frac{m_1 g}{4}$ and $R_2 = \frac{m_2 g}{4}$
5 Find the tension in the tow bar. Recall that a slope of 1 in 4 means $\sin(\theta) = \frac{1}{4}$. The system accelerates up the slope at 1 m/s^2, so $a = 1$ m/s^2.	Substitute the known values into (2): $T = R_2 + m_2 g \sin(\theta) + m_2 a$ $T = \frac{m_2 g}{4} + \frac{m_2 g}{4} + 1 \times m_2$ $T = m_2\left(\frac{g}{2} + 1\right)$ $T = 2850(4.9 + 1)$ $T = 16815$
6 State the value of the tension in the tow bar. Note that kN is the unit representation of 1000 newtons.	The tension in the tow bar is 16 815 newtons or 16.815 kN.
7 Find the tractive force.	Substitute the known values into (1): $F = R_1 + T + m_1 g \sin(\theta) + m_1 a$ $F = \frac{m_1 g}{4} + 16815 + \frac{m_1 g}{4} + 1 \times m_1$ $F = m_1\left(\frac{g}{2} + 1\right) + 16815$ $F = 1400(4.9 + 1) + 16815$ $F = 25075$
8 State the value of the tractive force.	The tractive force exerted by the car is 25.075 kN.
9 Alternative methods are possible.	If we add equations (1) and (2), we can eliminate the tension in the tow bar. $F - (R_1 + R_1) - (m_1 + m_2) g \sin(\theta) = (m_1 + m_2) a$ Doing this is equivalent to resolving around the system as a whole unit rather than considering individual particles.
10 Use the system to find the value of the tractive force.	$F = (R_1 + R_1) + (m_1 + m_2) g \sin(\theta) + (m_1 + m_2) a$ $F = \frac{1}{4}(m_1 + m_2) g + (m_1 + m_2) g \sin(\theta) + (m_1 + m_2) a$ $F = (m_1 + m_2)\left[\frac{g}{4} + g \sin(\theta) + a\right]$ $F = 4250\left[\frac{9.8}{4} + \frac{9.8}{4} + 1\right]$ $F = 25075$ N as before.

EXERCISE 6.5 Dynamics with connected particles

PRACTISE

1 WE18 Two boxes of mass 3 kg and 4 kg are connected by a string and lie on a smooth horizontal surface. A force of F N acts on the 4 kg box at angle of 60° to the surface. The system accelerates at 1.5 m/s^2. Find the value of F.

2 Two boxes of mass m_1 kg and m_2 kg are connected by a string and lie on a smooth horizontal surface. A force of 18 N acts on the m_1 kg box at angle of 30° to the surface, and the tension in the string joining the two masses is $4\sqrt{3}$ N. The system accelerates at $\sqrt{3}$ m/s^2. Find the values of m_1 and m_2.

3 WE19 A block of mass 4 kg rests on a smooth table, 1.5 m from the edge. It is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a second block of mass 3 kg hanging vertically over the edge. Initially the 3 kg block is 1 m above the ground. Find the speed with which the 3 kg block hits the ground when the system is released.

4 A block of mass m_2 kg rests on a smooth table. It is connected by a light inextensible string that passes over a smooth pulley at the edge of the table to a second block of mass m_1 kg hanging vertically over the edge of the table. When the system is released from rest, the tension in the string is $\frac{24g}{5}$ N and the mass m_1 accelerates down at $\frac{2g}{5}$ m/s^2. Find the values of m_1 and m_2.

5 WE20 Two particles with masses 4 kg and 3 kg are both hanging vertically and are connected by a light inextensible string that passes over a smooth pulley. Initially the 4 kg block is 2 m above the ground. Find the speed with which the 4 kg block hits the ground when the system is released from rest.

6 Two particles with masses m kg and $2m$ kg are both hanging vertically and are connected by a light inextensible string that passes over a smooth pulley. Find the tension in the string when the system is released from rest.

7 WE21 A car of mass 1200 kg is pulling a trailer of mass 950 kg up an incline of slope 1 in 5. The resistances to the motion of the car and the trailer are both one-third of their weight forces. Find the tractive force exerted by the engine in the car and the tension in the tow bar if the car moves up the incline with an acceleration of 0.8 m/s^2.

8 A train of mass 10 tonnes is pulling a carriage of mass 5 tonnes up an incline of slope 1 in 3. The resistances to the motion of the train and the carriage are both one-fifth of their weight forces. The tractive force exerted by the engine in the train is 90 kN. Find the tension in the tow bar connecting the train and the carriage and the acceleration of the system.

CONSOLIDATE

9 Two suitcases of mass 12 kg and 15 kg are connected by a string and lie on a smooth horizontal surface. A force of 20 newtons acts on the 12 kg suitcase. Find the acceleration of the system and the tension in the string connecting the suitcases if:

a the force is horizontal
b the force acts at angle of 40° to the surface.

10 a Two particles with masses 3 kg and 5 kg both initially at rest are hanging vertically and are connected by a light inextensible string that passes over a smooth fixed pulley. Find the acceleration of the system when released and the tension in the string.

b Two particles with masses m_1 kg and m_2 kg are hanging vertically and are connected by a light inextensible string that passes over a smooth fixed pulley.

If $m_2 > m_1$, show that the acceleration of the system is given by $\dfrac{(m_2 - m_1)g}{m_1 + m_2}$ m/s^2

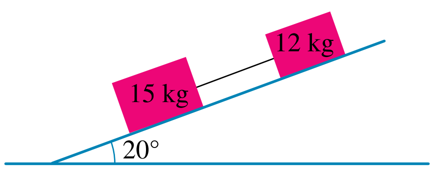
and the tension in the string is equal to $\dfrac{2m_1m_2g}{m_1 + m_2}$ N.

11 a A mass of 3 kg is lying on a smooth horizontal table and is joined by a light inextensible string to a mass of 5 kg hanging vertically. This string passes over a smooth pulley at the edge of the table. The system is released from rest. Find the system's acceleration and the tension in the string connecting the masses.

b Two particles with masses m_1 kg and m_2 kg are connected by a light inextensible string. Mass m_1 lies on a smooth horizontal table and m_2 hangs vertically. The connecting string passes over a smooth pulley at the table's edge.

Show that the tension in the string is $\dfrac{m_1m_2g}{m_1 + m_2}$ N.

12 Two suitcases of mass 12 kg and 15 kg are connected by a string. They both lie on a smooth ramp inclined at an angle of 20° to the horizontal. A force of 120 N acts on the 12 kg suitcase, pulling it up the ramp.

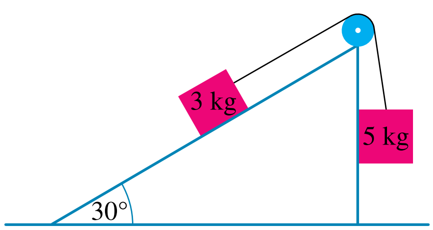


Find the acceleration of the system and the tension in the string connecting the suitcases if:

a the force is parallel to the ramp

b the force acts at an angle of 40° to the ramp.

13 a A particle of mass 3 kg is placed on the surface of a smooth plane inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a particle of mass 5 kg hanging freely. Find the acceleration of the system and the tension force in the string.

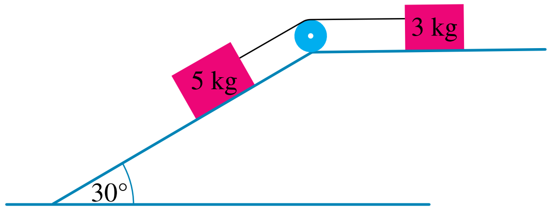


b A particle of mass m_1 kg is placed on the surface of a smooth plane inclined at an angle of θ to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a particle of mass m_2 kg hanging vertically. Show that the acceleration of the system is given by

$\dfrac{g(m_2 - m_1\sin(\theta))}{m_1 + m_2}$ m/s^2 and that the tension force in the string is

$\dfrac{m_2m_1g(1 + \sin(\theta))}{m_1 + m_2}$ N, provided $m_2 > m_1\sin(\theta)$.

14 a A particle of mass 5 kg is placed on a slope inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass 3 kg, which is on a smooth horizontal table. Find the tension in the string.

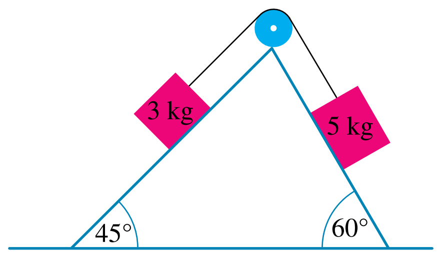


b A particle of mass m_2 kg is placed on a slope inclined at an angle of θ to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass m_2 kg, which is on a smooth horizontal table. Show that the tension in the string is given by

$$\frac{m_2 m_1 g \sin(\theta)}{m_1 + m_2} \text{ N.}$$

15 a Two masses of 3 kg and 5 kg are placed on two smooth inclines of 45° and 60° respectively placed back to back. The masses are connected by a light string that passes over a smooth pulley at the top of the inclines. Show that the acceleration of the system is given by

$$\frac{g}{16}(5\sqrt{3} - 3\sqrt{2}) \text{ m/s}^2.$$



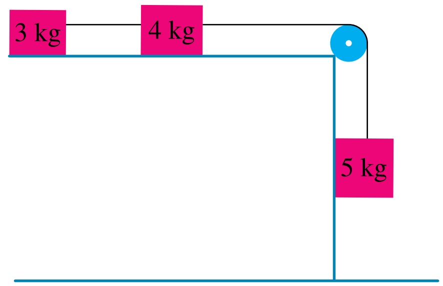
b A wedge consists of two smooth inclined planes placed back to back. A particle of mass m_1 rests on the slope inclined at an angle of α to the horizontal, and a particle of mass m_2 rests on the slope inclined at an angle of β to the horizontal. The particles are connected by a light string that passes over a smooth pulley at the common vertex of the two planes. If $\beta > \alpha$ and the mass m_2 moves down the plane, show that the tension in the string is given by

$$\frac{m_1 m_2 g(\sin(\alpha) + \sin(\beta))}{m_1 + m_2} \text{ N.}$$

16 a A truck of mass 8 tonnes pulls a trailer of mass 2 tonnes with an acceleration of magnitude 2 m/s^2 along a level road. The truck exerts a tractive force of magnitude 60 000 newtons, and the total resistance due to wind and other factors of the trailer is 2000 newtons. Find the resistance to motion of the truck and the tension in the coupling.

b A truck of mass 8 tonnes is pulling a trailer of mass 2 tonnes up a hill of slope 1 in 3. The tractive force exerted by the engine is 60 000 newtons, and the resistances to both the truck and the trailer are one-tenth of the respective weight forces. Find the acceleration of the system and the tension in the coupling between the truck and the trailer.

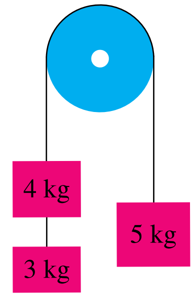
17 Two masses of 3 kg and 4 kg are connected by a light string and both lie on a smooth horizontal table. The 4 kg mass is connected to another mass of 5 kg hanging vertically by another light inextensible string that passes over a smooth fixed pulley at the edge of the table. The system is released from rest and the 5 kg mass moves downwards.



Find the tension in the string connecting the:

a 3 kg and 4 kg masses
b 4 kg and 5 kg masses.

18 Two masses of 3 kg and 4 kg are hanging vertically and are connected by a light string with the 4 kg mass above the 3 kg mass. The 4 kg mass is attached to another string, which passes over a smooth pulley and supports a third mass of 5 kg hanging vertically. The system is released from rest.

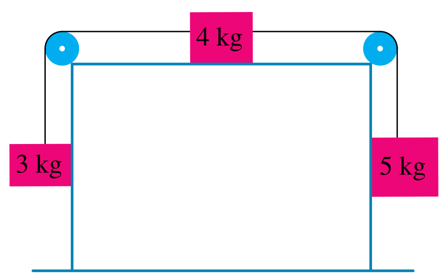


Find the tension in the string connecting the:

a 3 kg and 4 kg masses
b 4 kg and 5 kg masses.

MASTER

19 a Two masses of 3 kg and 5 kg are hanging vertically over two different edges of a table. They are connected by two light strings that pass over two smooth fixed pulleys at the edges of the table to another 4 kg mass which lies on a smooth horizontal table. The system is released from rest and the 5 kg mass moves downwards.



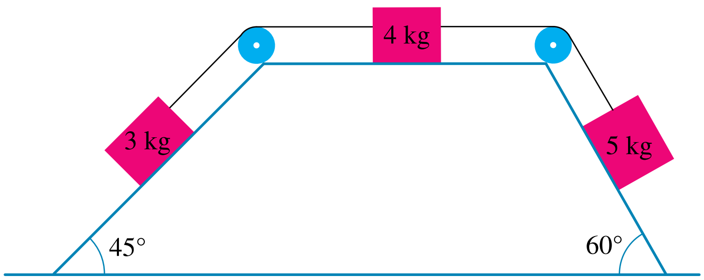
Find the tension in the string connecting the:

i 3 kg and 4 kg masses **ii** 4 kg and 5 kg masses.

b Two masses of m_2 and m_1 kg are hanging vertically over two different edges of a table. They are connected by two light strings that pass over two smooth fixed pulleys at the edges of the table to another mass of M kg which lies on a smooth horizontal table. The system is released from rest. If $m_2 > m_1$, so the mass m_2 moves downwards, show that the acceleration of the system is given by

$$\frac{g(m_2 - m_1)}{m_1 + m_2 + M} \text{ m/s}^2.$$

20 a A particle of mass 3 kg is placed on a slope inclined at an angle of 45° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass 4 kg which is on a smooth horizontal table. Another light inextensible string passing over a smooth pulley at a different edge of the table connects the 4 kg mass to 5 kg mass on a slope inclined at an angle of 60° to the horizontal. If the 5 kg mass moves down the slope, show that the tension connecting the 3 kg and 4 kg masses is $\frac{g}{8}(5\sqrt{3} + 9\sqrt{2})$ N and the tension connecting the 4 kg and 5 kg masses is $\frac{g}{24}(35\sqrt{3} + 15\sqrt{2})$ N.



b A particle of mass m_1 kg is placed on a slope inclined at an angle of α to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass M kg which is on a smooth horizontal table. Another light inextensible string passing over a smooth pulley at the edge of the table connects the M kg mass to another mass, m_2 kg, on a slope inclined at an angle of β to the horizontal. Assuming that $m_2\sin(\beta) > m_1\sin(\alpha)$ so that the mass m_2 moves down the plane, show that the acceleration of the system is given by

$$\frac{g(m_2\sin(\beta) - m_1\sin(\alpha))}{m_1 + m_2 + M}\text{ m/s}^2.$$

eBook*plus* ONLINE ONLY

6.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

6 Answers

EXERCISE 6.2

1 $\sqrt{113}$ N acting at angle of 131.19° with the 7 newton force

2 $\sqrt{5}F$ N acting at angle of 116.57° with the F newton force

3 14.49 N acting at angle of 163.98° with the 7 newton force

4 2.80F N acting at angle of 149.64° with the F newton force

5 90.46 N in the 5 cm string and 37.69 N in the 12 cm string

6 47.04 N in the 7 cm string and 13.72 N in the 24 cm string

7 $\sqrt{113}$ N acting at angle of 131.19° with the 7 newton force

8 $\sqrt{5}F$ N acting at angle of 116.57° with the F newton force

9 14.49 N acting at angle of 163.98° with the 7 newton force

10 2.80F N acting at angle of 149.64° with the F newton force

11 90.46 N in the 5 cm string and 37.69 N in the 12 cm string

12 47.04 N in the 7 cm string and 13.72 N in the 24 cm string

13 $49(2-\sqrt{3}), 49(\sqrt{3}-1)$ N

14 4.63, 26.60 N

15 a $x=-9, y=-3$

b $x=-4, y=3 \; |\underset{\sim}{F}_3| = 5$

16 a 130 N acting at angle of 112.62° with the 50 newton force

b $|\underset{\sim}{F}_1| = 30\sqrt{3}, |\underset{\sim}{F}_2| = 60$

17 a 85.19 N acting at angle of 153.08° with the 30 newton force

b 87.79 N acting at angle of 158.53° with the 120 newton force

18 a i $F = 27.45$ ii $T_{AB} = 47.85$ N

b 311.14 N, 24.32 kg

19 a 4.9 N b $\dfrac{\sqrt{3}mg}{3}$ N

20 a 120.48

b Check with your teacher.

21 a 41.81°, 43.83 b 56.31°, $\sqrt{13}\,mg$

22 a 46.84°, 37.56 b 75.58°, 30.98 N

23 a $T_{AC} = \dfrac{4mg}{5}, T_{BC} = \dfrac{3mg}{5}$ N

b Check with your teacher.

24 a 40.72 N, 5.79

25 a $\alpha = 44.92°, \beta = 52.63°$ b $\alpha = 55.95°, \beta = 45.58°$

26 a $\alpha = 34.39°, \beta = 64.84°$ b 53.07°, 33.40 N

EXERCISE 6.3

1 31.50, 37.54 N

2 30.68°, 1.69 N

3 33.52, 26.07 N

4 28.65°, 1.38 N

5 $\dfrac{8\sqrt{3}}{3}$

6 Check with your teacher.

7 $T_{PQ} = T_{RS} = 35.86, T_{QR} = 25.36$ N, 1.34

8 18.99 N, 64.52°, $m_1 = 1.69, m_2 = 4.5$

9 a Check with your teacher.

b $2\sqrt{3}\,g, m = 4\sqrt{3}$

10 a 41.12 N b 6.53 kg

c $T_{upper} = 120.21$ N, $T_{lower} = 63.96$ N

11 $T_{upper} = 71.45$ N at 16.25°

$T_{lower} = 44.01$ N at 27.03°

12 $T_{AB} = \dfrac{20g\sqrt{3}}{3}, T_{BC} = 12g$ N

$F_1 = \dfrac{28\sqrt{3}g}{3}, F_2 = 6\sqrt{3}g$

13 $T_{AB} = 15.56, T_{BC} = 6.53$ N

$m_1 = 0.79, m_2 = 0.43$

14 23.69°, 11.16

15 a 8 cm b $\dfrac{4m}{\sqrt{100-m^2}}$

c The system cannot be in equilibrium; therefore, the string breaks.

16 $T_{PQ} = 45.98, T_{QR} = 32.61, T_{RS} = 45.98$ N, $m = 2.19$

17 14.48°

18 a 39.23° b Check with your teacher.

19 Check with your teacher.

20 Check with your teacher.

EXERCISE 6.4

1 160 000 kg m/s

2 1.79×10^{29} kg m/s

3 2.62 m/s^2

4 $\frac{1}{2}\sqrt{29}$ m/s^2

5 5280 N

6 60 kg

7 a 1024 N b 544 N c 784 N

8 539.5 N

9 7692 N

10 17 450 N

11 42.96 N

12 a 43.77 N b 207.75 N

13 a 6 N b 2 m/s^2 c 2.5 m/s^2

14 a $\frac{5\sqrt{2}}{2} \text{ m/s}^2$ b $\frac{1}{m}\sqrt{b^2 + c^2} \text{ m/s}^2$

15 a 1.68 m/s^2

b $(6 - 2\sqrt{3})\underset{\sim}{i} + (3\sqrt{3} - 11)\underset{\sim}{j}$ N

16 a i 9 m/s ii 13.5 m

b i $13\,333\frac{1}{3}$ N ii $11\,111\frac{1}{9}$ N

17 a 47.2 N b 0.2 m/s^2

18 a 772.5 N b 697.5 N c 735 N

19 $m = 50$ kg, 0.25

20 a 2.38 m/s^2, 411.60 N b 1.95 m/s^2, 354.24 N

21 a i 0.84 m/s^2, 315.31 N ii 0.17 m/s^2, 188.52 N

b 15.87 m/s

22 a 177 N b 30 N c 31.06 N

d 80.28 N e 83.11 N

23 a $m = 5, \theta = 25.38°$ b $m = 3.75, \theta = 21.99°$

24 a $m = 9.67, \alpha = 14.75°$ b $\theta = 40.77°, \alpha = 18.19°$

EXERCISE 6.5

1 21 N

2 $m_1 = 5, m_2 = 4$

3 2.90 m/s

4 $m_1 = 8, m_2 = 12$

5 2.37 m/s

6 $\frac{4mg}{3}$ N

7 Tractive force 12 957.33 N, tension 5725.33 N

8 0.77m/s^2, 30 kN

9 a 0.74 m/s^2, 11.11 N b 0.57 m/s^2, 8.51 N

10 a 2.45 m/s^2, 36.75 N

b Check with your teacher.

11 a 6.13 m/s^2, 18.38 N

b Check with your teacher.

12 a 1.09 m/s^2, 66.67 N b 0.053 m/s^2, 51.07 N

13 a $\frac{7g}{16} \text{ m/s}^2, \frac{45g}{16}$ N

b Check with your teacher.

14 a $\frac{15g}{16}$ N

b Check with your teacher.

15 a Check with your teacher.

b Check with your teacher.

16 a $R_{\text{truck}} = 38$ kN, $T = 6$ kN b 1.75 m/s^2, $T = 12$ kN

17 a $\frac{5g}{4}$ N b $\frac{35g}{12}$ N

18 a $\frac{5g}{2}$ N b $\frac{35g}{6}$ N

19 a i $\frac{7g}{2}$ N ii $\frac{25g}{6}$ N

b Check with your teacher.

20 Check with your teacher.

7

Differential calculus

7.1 Kick off with CAS

Exploring differentiation with CAS

In this topic we will be revising differentiation techniques and differentiating the inverse trigonometric functions.

1 Use CAS to determine:

a $\frac{d}{dx}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]$ b $\frac{d}{dx}\left[\sin^{-1}\left(\frac{3x}{4}\right)\right]$ c $\frac{d}{dx}\left[\sin^{-1}\left(\frac{4x}{5}\right)\right]$.

2 Can you predict $\frac{d}{dx}\left[\sin^{-1}\left(\frac{ax}{b}\right)\right]$ where a and b are positive real numbers?

3 Use CAS to determine:

a $\frac{d}{dx}\left[\tan^{-1}\left(\frac{2x}{3}\right)\right]$ b $\frac{d}{dx}\left[\tan^{-1}\left(\frac{3x}{4}\right)\right]$ c $\frac{d}{dx}\left[\tan^{-1}\left(\frac{4x}{5}\right)\right]$.

4 Can you predict $\frac{d}{dx}\left[\tan^{-1}\left(\frac{ax}{b}\right)\right]$ where a and b are positive real numbers?

5 Use CAS to determine:

a $\frac{d}{dx}\left[\cos^{-1}\left(\frac{2}{x}\right)\right]$ b $\frac{d}{dx}\left[\cos^{-1}\left(\frac{3}{x}\right)\right]$

6 Can you predict $\frac{d}{dx}\left[\cos^{-1}\left(\frac{a}{x}\right)\right]$?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

7.2 Review of differentiation techniques

Introduction

In the Mathematical Methods course, the **derivatives** of functions and the standard rules for differentiation are covered. If $y = f(x)$ is the equation of a curve, then the gradient function is given by $\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Using this formula to obtain the gradient function or the first derivative is called using the method of first principles.

Usually the standard rules for differentiation are used to obtain the gradient function. In this section these fundamental techniques of differentiation are revised and extended. The table below shows the basic functions for $y = f(x)$ and the corresponding gradient functions for $\frac{dy}{dx} = f'(x)$, where k and n are constants and $x \in R$.

Function: $y = f(x)$	Gradient function: $\frac{dy}{dx} = f'(x)$
x^n	nx^{n-1}
$\sin(kx)$	$k\cos(kx)$
$\cos(kx)$	$-k\sin(kx)$
e^{kx}	ke^{kx}
$\log_e(x)$	$\frac{1}{x}$

The chain rule

The chain rule is used to differentiate functions of functions. It states that if $y = f(x) = g(h(x)) = g(u)$, where $u = h(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = g'(u)h'(x)$.

A quick proof is as follows. Let δx be the increment in x and δy, and δu be the corresponding increments in y and u. Provided that $\delta u \neq 0$ and $\delta x \neq 0$, $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$

Now in the limit as $\delta x \to 0$, $\delta u \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\frac{\delta y}{\delta u} \to \frac{dy}{du}$ and $\frac{\delta u}{\delta x} \to \frac{du}{dx}$, so that $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

WORKED EXAMPLE 1

a If $f(x) = \sqrt{4x^2 + 9}$, find the value of $f'(-1)$.

b Differentiate $2\cos^4(3x)$ with respect to x.

THINK	WRITE
a 1 Write the equation in index form.	**a** $f(x) = \sqrt{4x^2 + 9}$ $= (4x^2 + 9)^{\frac{1}{2}}$
2 Express y in terms of u and u in terms of x.	$y = u^{\frac{1}{2}}$ where $u = 4x^2 + 9$

3 Differentiate y with respect to u and u with respect to x.

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ and $\frac{du}{dx} = 8x$

$= \frac{1}{2\sqrt{u}}$

4 Find $f'(x)$ using the chain rule.

$f'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}} \times 8x$

5 Substitute back for u and cancel factors.

$f'(x) = \frac{4x}{\sqrt{4x^2 + 9}}$

6 Evaluate at the indicated point.

$f'(-1) = \frac{-4}{\sqrt{(4 + 9)}}$

7 In this case we need to rationalise the denominator.

$f'(-1) = -\frac{4}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}}$

8 State the final result. This represents the gradient of the curve at the indicated point.

$f'(-1) = -\frac{4\sqrt{13}}{13}$

b 1 Write the equation, expressing it in index notation.

b $y = 2\cos^4(3x)$

$= 2(\cos(3x))^4$

2 Express y in terms of u and u in terms of x.

$y = 2u^4$ where $u = \cos(3x)$

3 Differentiate y with respect to u and u with respect to x.

$\frac{dy}{du} = 8u^3$ and $\frac{du}{dx} = -3\sin(3x)$

4 Find $\frac{dy}{dx}$ using the chain rule.

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 8u^3 \times -3\sin(3x)$

5 Substitute back for u and state the final result.

$\frac{dy}{dx} = -24\cos^3(3x)\sin(3x)$

The chain rule can also be used in conjunction with other mixed types of functions.

WORKED EXAMPLE 2

a If $f(x) = 4\sin\left(\frac{2}{x}\right)$, find $f'\left(\frac{6}{\pi}\right)$.

b Differentiate $e^{\cos(4x)}$ with respect to x.

THINK | **WRITE**

a 1 Write the equation.

a $f(x) = 4\sin\left(\frac{2}{x}\right)$

2 Express y in terms of u and u in terms of x.

Let $y = 4\sin(u)$ where $u = \frac{2}{x} = 2x^{-1}$.

3 Differentiate y with respect to u and u with respect to x.

$\frac{dy}{du} = 4\cos(u)$ and $\frac{du}{dx} = -2x^{-2} = -\frac{2}{x^2}$

4 Find $f'(x)$ using the chain rule.

$$f'(x) = \frac{dy}{du}\frac{du}{dx} = 4\cos(u) \times -\frac{2}{x^2}$$
$$= -\frac{8}{x^2}\cos(u)$$

5 Substitute back for u.

$$f'(x) = -\frac{8}{x^2}\cos\left(\frac{2}{x}\right)$$

6 Evaluate at the indicated point and simplify.

$$f'\left(\frac{6}{\pi}\right) = -\frac{8}{\left(\frac{6}{\pi}\right)^2}\cos\left(\frac{2}{\frac{6}{\pi}}\right)$$
$$= -\frac{8\pi^2}{36}\cos\left(\frac{2\pi}{6}\right)$$

7 Substitute for the trigonometric values.

$$f'\left(\frac{6}{\pi}\right) = -\frac{2\pi^2}{9} \times \frac{1}{2}$$

8 State the final result.

$$f'\left(\frac{6}{\pi}\right) = -\frac{\pi^2}{9}$$

b 1 Write the equation.

b Let $y = e^{\cos(4x)}$.

2 Express y in terms of u and u in terms of x.

$y = e^u$ where $u = \cos(4x)$

3 Differentiate y with respect to u and u with respect to x.

$$\frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = -4\sin(4x)$$

4 Find $\frac{dy}{dx}$ using the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \times -4\sin(4x)$$

5 Substitute back for u and state the final result.

$$\frac{dy}{dx} = -4\sin(4x)e^{\cos(4x)}$$

The product rule

The product rule states that if $u = u(x)$ and $v = v(x)$ are two differentiable functions of x, and $y = u \cdot v$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. A quick proof is as follows.

Let δx be the increment in x and δu, δv and δy be the corresponding increments in u, v and y. Then

$$y + \delta y = (u + \delta u)(v + \delta v)$$

Expanding, $$y + \delta y = uv + u\delta v + v\delta u + \delta u\delta v$$

Since $y = u \cdot v$, $$\delta y = u\delta v + v\delta u + \delta u\delta v$$

Divide each term by δx: $\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u \cdot \delta v}{\delta x}$

Now in the limit as $\delta x \to 0$, $\delta u \to 0$, $\delta v \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$, $\frac{\delta u}{\delta x} \to \frac{du}{dx}$ and and $\frac{\delta u \cdot \delta v}{\delta x} \to 0$.

Hence, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

WORKED EXAMPLE 3

a If $f(x) = x^4 \sin(2x)$, find $f'(x)$.

b Find $\frac{d}{dx}[x^5 e^{-2x}]$.

THINK	WRITE
a 1 Write the equation.	a $y = f(x) = x^4 \sin(2x) = u \cdot v$
2 State the functions u and v.	$u = x^4$ and $v = \sin(2x)$
3 Differentiate u and v with respect to x.	$\frac{du}{dx} = 4x^3$ and $\frac{dv}{dx} = 2\cos(2x)$
4 Find $f'(x)$ using the product rule. Substitute for u, $\frac{dv}{dx}$, v and $\frac{du}{dx}$.	$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$ $= 2x^4\cos(2x) + 4x^3\sin(2x)$
5 Simplify by taking out the common factors.	$f'(x) = 2x^3(x\cos(2x) + 2\sin(2x))$
b 1 Write the equation.	b Let $y = x^5 e^{-2x} = u \cdot v$.
2 State the functions u and v.	$u = x^5$ and $v = e^{-2x}$
3 Differentiate u and v with respect to x.	$\frac{du}{dx} = 5x^4$ and $\frac{dv}{dx} = -2e^{-2x}$
4 Find $\frac{dy}{dx}$ using the product rule. Substitute for u, $\frac{dv}{dx}$, v and $\frac{du}{dx}$.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $\frac{dy}{dx} = x^5 \times -2e^{-2x} + e^{-2x} \times 5x^4$
5 Simplify by taking out the common factors.	$\frac{dy}{dx} = x^4 e^{-2x}(5 - 2x)$
Alternative notation The product rule can be used without explicitly writing the u and v. The setting out is similar, but an alternative notation is used.	
1 Write the equation.	$\frac{d}{dx}[x^5 e^{-2x}]$
2 Use the product rule.	$= x^5\frac{d}{dx}(e^{-2x}) + e^{-2x}\frac{d}{dx}(x^5)$
3 Find the derivatives.	$= x^5 \times -2e^{-2x} + e^{-2x} \times 5x^4$
4 Simplify by taking out the common factors and state the final result.	$\frac{d}{dx}[x^5 e^{-2x}] = x^4 e^{-2x}(5 - 2x)$

The quotient rule

The quotient rule states that if $u = u(x)$ and $v = v(x)$ are two differentiable functions of x and $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. A quick proof is as follows.

Let $y = \frac{u}{v} = u \cdot v^{-1}$.

Now by the product rule, $\frac{dy}{dx} = u\frac{d}{dx}(v^{-1}) + v^{-1}\frac{du}{dx}$.

Using the chain rule,
$$\frac{dy}{dx} = u\frac{d}{dv}(v^{-1})\frac{dv}{dx} + v^{-1}\frac{du}{dx}$$
$$= -uv^{-2}\frac{dv}{dx} + v^{-1}\frac{du}{dx}$$
$$= \frac{v\frac{du}{dx} - u\frac{dv}{dz}}{v^2}$$

WORKED EXAMPLE 4

a Differentiate $\frac{3\sin(2x)}{2x^4}$ with respect to x.

b If $f(x) = \frac{3x}{\sqrt{4x^2+9}}$, find $f'(x)$.

THINK	WRITE
a 1 Write the equation.	**a** Let $y = \frac{3\sin(2x)}{2x^4} = \frac{u}{v}$.
2 State the functions u and v.	$u = 3\sin(2x)$ and $v = 2x^4$
3 Differentiate u and v with respect to x.	$\frac{du}{dx} = 6\cos(2x)$ and $\frac{dv}{dx} = 8x^3$
4 Find $\frac{dy}{dx}$ using the quotient rule. Substitute for u, $\frac{dv}{dx}$, v and $\frac{du}{dx}$.	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $\frac{dy}{dx} = \frac{2x^4 \times 6\cos(2x) - 3\sin(2x) \times 8x^3}{(2x^4)^2}$
5 Expand and simplify by taking out the common factors.	$\frac{dy}{dx} = \frac{12x^3(x\cos(2x) - 2\sin(2x))}{4x^8}$
6 Cancel the common factors and state the final result in simplest form.	$\frac{dy}{dx} = \frac{3}{x^5}(x\cos(2x) - 2\sin(2x))$
b 1 Write the equation.	**b** Let $f(x) = \frac{3x}{\sqrt{4x^2+9}} = \frac{u}{v}$.
2 State the functions u and v.	$u = 3x$ and $v = \sqrt{4x^2+9}$
3 Differentiate u and v with respect to x, using the chain rule to find $\frac{dv}{dx}$. See Worked example 1a for $\frac{dv}{dx}$.	$\frac{du}{dx} = 3$ and $\frac{dv}{dx} = \frac{4x}{\sqrt{4x^2+9}}$

4 Find $f'(x)$ using the quotient rule. Substitute for u, $\frac{dv}{dx}$, v and $\frac{du}{dx}$.

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$f'(x) = \frac{3\sqrt{4x^2+9} - \frac{4x}{\sqrt{4x^2+9}} \times 3x}{(\sqrt{4x^2+9})^2}$$

5 Simplify by forming a common denominator in the numerator.

$$f'(x) = \frac{\frac{3(4x^2+9) - 12x^2}{\sqrt{4x^2+9}}}{4x^2+9}$$

6 Expand and simplify the terms in the numerator.

$$f'(x) = \frac{\frac{12x^2 + 27 - 12x^2}{\sqrt{4x^2+9}}}{4x^2+9}$$

7 Simplify using $\frac{\frac{a}{b}}{c} = \frac{a}{b} \times \frac{1}{c}$.

$$f'(x) = \frac{27}{\sqrt{4x^2+9}} \times \frac{1}{4x^2+9}$$

$$= \frac{27}{(4x^2+9)^{\frac{3}{2}}}$$

8 Use index laws to simplify and state the final result in simplest form.

$$f'(x) = \frac{27}{\sqrt{(4x^2+9)^3}}$$

Derivative of tan(*kx*)

The quotient rule can be used to find the derivative of $\tan(kx)$.

Let $y = \tan(kx) = \frac{\sin(kx)}{\cos(kx)} = \frac{u}{v}$, where $u = \sin(kx)$ and $v = \cos(kx)$.

Then $\frac{du}{dx} = k\cos(kx)$ and $\frac{dv}{dx} = -k\sin(kx)$.

Using the quotient rule, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\cos(kx) \times k\cos(kx) - \sin(kx) \times -k\sin(kx)}{(\cos(kx))^2}$$

$$= \frac{k(\cos^2(kx) + \sin^2(kx))}{\cos^2(kx)} \text{ since } \cos^2(kx) + \sin^2(kx) = 1$$

$$= \frac{k}{\cos^2(kx)}$$

$$= k\sec^2(kx)$$

Hence, $\frac{d}{dx}(\tan(kx)) = k\sec^2(kx) = \frac{k}{\cos^2(kx)}$.

WORKED EXAMPLE 5

a Differentiate $2\tan^5\left(\frac{x}{2}\right)$ with respect to x.

b If $f(x) = x^2\tan\left(\frac{2x}{3}\right)$, find $f'\left(\frac{\pi}{2}\right)$.

THINK	WRITE
a 1 Write the equation.	**a** $y = 2\tan^5\left(\frac{x}{2}\right)$
2 Express y in terms of u and u in terms of x.	$y = 2u^5$ where $u = \tan\left(\frac{x}{2}\right)$
3 Differentiate y with respect to u and u with respect to x using $\frac{d}{dx}(\tan(kx)) = k\sec^2(kx)$ with $k = \frac{1}{2}$.	$\frac{dy}{du} = 10u^4$ and $\frac{du}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 10u^4 \times \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$
5 Substitute back for u and state the final result.	$\frac{dy}{dx} = 5\tan^4\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)$
b 1 Write the equation.	**b** $f(x) = x^2\tan\left(\frac{2x}{3}\right) = u \cdot v$
2 State the functions u and v.	$u = x^2$ and $v = \tan\left(\frac{2x}{3}\right)$
3 Differentiate u and v with respect to x.	$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{2}{3}\sec^2\left(\frac{2x}{3}\right)$
4 Find $f'(x)$ using the product rule. Substitute for u, $\frac{dv}{dx}$, v and $\frac{du}{dx}$.	$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$ $f'(x) = x^2 \times \frac{2}{3}\sec^2\left(\frac{2x}{3}\right) + \tan\left(\frac{2x}{3}\right) \times 2x$
5 Substitute $x = \frac{\pi}{2}$.	$f'\left(\frac{\pi}{2}\right) = \frac{2}{3}\left(\frac{\pi}{2}\right)^3 \times \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} + \tan\left(\frac{\pi}{3}\right) \times 2 \times \frac{\pi}{2}$
6 Substitute for the trigonometric values and simplify.	$f'\left(\frac{\pi}{2}\right) = \frac{\pi^2}{6 \times \left(\frac{1}{2}\right)^2} + \sqrt{3}\pi$ $= \frac{2\pi^2}{3} + \sqrt{3}\pi$
7 Express the final result in simplest form by taking a common denominator and the common factor out in the numerator.	$f'\left(\frac{\pi}{2}\right) = \frac{\pi(2\pi + 3\sqrt{3})}{3}$

Derivatives involving logarithms

If $y = \log_e(x) = \ln(x)$ where $x > 0$, then $x = e^y$ so that $\frac{dx}{dy} = e^y$.

Since $\frac{dx}{dy} = 1 / \frac{dy}{dx}$, $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

In general, if a and b are constants and $y = \log_e(ax + b)$, then $\frac{dy}{dx} = \frac{a}{ax + b}$.

Furthermore, if $y = \log_e(f(x))$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$.

This is the rule that we will now use to differentiate logarithmic functions.

WORKED EXAMPLE 6

a Differentiate $\log_e(\cos(3x))$ with respect to x.

b Find $\frac{d}{dx}\left[\log_e\left(\frac{3x+5}{3x-5}\right)\right]$.

THINK	WRITE
a 1 Write the equation.	**a** $y = \log_e(\cos(3x))$
2 Use the general result.	$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos(3x))}{\cos(3x)}$
3 State the derivative in the numerator.	$\frac{dy}{dx} = \frac{-3\sin(3x)}{\cos(3x)}$
4 Simplify and state the final result.	$\frac{dy}{dx} = -3\tan(3x)$
b 1 Write the equation.	**b** $y = \log_e\left(\frac{3x+5}{3x-5}\right)$
2 Simplify using log laws.	$y = \log_e(3x+5) - \log_e(3x-5)$
3 Differentiate each term.	$\frac{dy}{dx} = \frac{3}{3x+5} - \frac{3}{3x-5}$
4 Form a common denominator.	$\frac{dy}{dx} = \frac{3(3x-5) - 3(3x+5)}{(3x+5)(3x-5)}$
5 Expand the numerator and denominator and simplify.	$\frac{dy}{dx} = \frac{9x - 15 - (9x+15)}{9x^2 - 25}$
6 State the final answer in simplest form.	$\frac{dy}{dx} = \left[\log_e\left(\frac{3x+5}{3x-5}\right)\right]$ $= \frac{-30}{9x^2 - 25}$

EXERCISE 7.2 Review of differentiation techniques

PRACTISE

1 WE1 **a** If $f(x) = \frac{2}{3x^2 + 5}$, find the value of $f'(-1)$.

b Differentiate $5\sin^3(2x)$ with respect to x.

2 a Find $\frac{d}{dx}\left[\frac{1}{2x+5}\right]$.

b If $f(x) = 6\sqrt{\cos(4x)}$, find the value of $f'\left(\frac{\pi}{12}\right)$.

3 WE2 a If $f(x) = 6\cos\left(\frac{3}{x}\right)$, find $f'\left(\frac{18}{\pi}\right)$.

b Differentiate $e^{\sin(2x)}$ with respect to x.

4 a Differentiate $\sin(\sqrt{x})$ with respect to x.

b If $f(x) = e^{\cos(2x)}$, find the value of $f'\left(\frac{\pi}{6}\right)$.

5 WE3 a If $f(x) = x^3\cos(4x)$, find $f'(x)$.

b Find $\frac{d}{dx}[x^4 e^{-3x}]$.

6 a Differentiate $e^{-3x}\cos(2x)$ with respect to x.

b If $f(x) = x^2 e^{-x^2}$, find $f'(2)$.

7 WE4 a Differentiate $\frac{3\cos(3x)}{2x^3}$ with respect to x.

b If $f(x) = \frac{x}{\sqrt{4x+9}}$, find $f'(x)$.

8 a If $f(x) = \frac{1}{3xe^{2x}}$, find $f'(x)$.

b Find $\frac{d}{dx}\left[\left(\frac{3x^2+5}{3x^2-5}\right)^2\right]$.

9 WE5 a Differentiate $6\tan^4\left(\frac{x}{3}\right)$ with respect to x.

b If $f(x) = x^4\tan\left(\frac{x}{4}\right)$, find $f'(\pi)$.

10 a If $f(x) = \frac{\tan(3x)}{x}$, find $f'\left(\frac{\pi}{9}\right)$.

b If $f(x) = 5\tan\left(\frac{2}{x}\right)$, find $f'\left(\frac{12}{\pi}\right)$.

11 WE6 a Differentiate $\log_e\left(\sin\left(\frac{x}{2}\right)\right)$ with respect to x.

b Find $\frac{d}{dx}\left[\log_e\left(\frac{4x^2+9}{4x^2-9}\right)\right]$.

12 a If $f(x) = \log_e(\sqrt{4x^2+9})$, find $f'(-1)$.

b Differentiate $\cos\left(\log_e\left(\frac{x}{2}\right)\right)$ with respect to x.

CONSOLIDATE

13 Differentiate each of the following with respect to x.

a $2\sin^4(3x)$
b $5\cos^3(4x)$
c $x\sqrt{2x^2+9}$
d $\frac{x}{\sqrt{3x^2+5}}$

14 Find $\frac{dy}{dx}$ for each of the following.

a $y = x^3\sin(5x)$
b $y = x^4\cos(4x)$
c $y = \frac{3\sin(3x)}{2x^3}$
d $y = \frac{4\cos(2x)}{3x^4}$

15 Find $f'(x)$ for each of the following.

a $f(x) = e^{-\frac{1}{2}x^2}$
b $f(x) = e^{\cos(2x)}$
c $f(x) = \cos(e^{2x})$
d $f(x) = e^{\sqrt{x}}$

16 Find each of the following.

a $\frac{d}{dx}[x^3e^{-4x}]$

b $\frac{d}{dx}[e^{-3x}\sin(2x)]$

c $\frac{d}{dx}\left[\frac{e^{3x}}{x^2}\right]$

d $\frac{d}{dx}\left[\frac{1}{x^3e^{2x}}\right]$

17 Find $\frac{dy}{dx}$ for each of the following.

a $y = x^2\log_e(5x + 4)$

b $y = \log_e(\sqrt{2x^2 + 9})$

c $y = \log_e\left(\frac{4x - 9}{4x + 9}\right)$

d $y = \log_e\left(\frac{3x^2 + 5}{3x^2 - 5}\right)$

18 a If $f(x) = \log_e(\sin(3x))$, find the exact value of $f'\left(\frac{\pi}{12}\right)$.

b If $f(x) = \log_e(\tan(2x))$, find the exact value of $f'\left(\frac{\pi}{6}\right)$.

c If $f(x) = 4\cos\left(\frac{2}{x}\right)$, find $f'\left(\frac{3}{2\pi}\right)$.

d If $f(x) = 2\tan\left(\frac{3}{x}\right)$, find $f'\left(\frac{18}{\pi}\right)$.

19 a Use the chain rule to show that $\frac{d}{dx}(\sec(kx)) = k\sec(kx)\tan(kx)$.

b Use the chain rule to show that $\frac{d}{dx}(\cos(kx)) = -k\operatorname{cosec}(kx)\cot(kx)$.

c Use the quotient rule to show that $\frac{d}{dx}(\cot(kx)) = -k\operatorname{cosec}^2(kx)$.

d If $y = \log_e(\tan(3x) + \sec(3x))$, show that $\frac{dy}{dx} = 3\sec(3x)$.

e If $y = \log_e\left(\cot\left(\frac{x}{2}\right) + \operatorname{cosec}\left(\frac{x}{2}\right)\right)$, show that $\frac{dy}{dx} = -\frac{1}{2}\operatorname{cosec}\left(\frac{x}{2}\right)$.

20 If n, k, b and α are all constants, verify each of the following.

a $\frac{d}{dx}[\sin(nx + \alpha)] = n\cos(nx + \alpha)$

b $\frac{d}{dx}[\cos(nx + \alpha)] = -n\sin(nx + \alpha)$

c $\frac{d}{dx}[\sin^n(kx)] = nk\sin^{n-1}(kx)\cos(kx)$

d $\frac{d}{dx}[\cos^n(kx)] = -nk\cos^{n-1}(kx)\sin(kx)$

e $\frac{d}{dx}[e^{kx}\sin(bx)] = e^{kx}(b\cos(bx) + k\sin(bx))$

f $\frac{d}{dx}[e^{kx}\cos(bx)] = e^{kx}(k\cos(bx) - b\sin(bx))$

21 If n and k are both constants, verify each of the following.

a $\frac{d}{dx}[x^n\sin(kx)] = x^{n-1}(n\sin(kx) + kx\cos(kx))$

b $\frac{d}{dx}[x^n\cos(kx)] = x^{n-1}(n\cos(kx) - kx\sin(kx))$

c $\frac{d}{dx}\left[\frac{\sin(kx)}{x^n}\right] = \frac{1}{x^{n+1}}(kx\cos(kx) - n\sin(kx))$

d $\frac{d}{dx}\left[\frac{\cos(kx)}{x^n}\right] = \frac{-1}{x^{n+1}}(kx\sin(kx) - n\cos(kx))$

e $\frac{d}{dx}[x^n e^{kx}] = x^{n-1}e^{kx}(n + kx)$

f $\frac{d}{dx}\left[\frac{e^{kx}}{x^n}\right] = \frac{e^{kx}(kx - n)}{x^{n+1}}$

22 If a, b, c, d and n are all constants, verify each of the following.

a $\frac{d}{dx}[\log_e(\sqrt{ax^2 + b})] = \frac{ax}{ax^2 + b}$

b $\frac{d}{dx}\left[\log_e\left(\frac{ax + b}{cx + d}\right)\right] = \frac{ad - bc}{(ax + b)(cx + d)}$

c $\frac{d}{dx}\left[\log_e\left(\frac{ax^2 + b}{cx^2 + d}\right)\right] = \frac{2(ad - bc)x}{(ax^2 + b)(cx^2 + d)}$

d $\frac{d}{dx}[\log_e(\sin^n(bx))] = \frac{nb}{\tan(bx)}$

e $\frac{d}{dx}[\log_e(\cos^n(bx))] = -nb\tan(bx)$

f $\frac{d}{dx}[\log_e(\tan^n(bx))] = \frac{2nb}{\sin(2bx)}$

MASTER

23 a If $u(x)$, $v(x)$ and $w(x)$ are all functions of x and $y = u(x)v(x)w(x)$, use the product rule to show that $\frac{dy}{dx} = v(x)w(x)\frac{du}{dx} + u(x)w(x)\frac{dv}{dx} + u(x)v(x)\frac{dw}{dx}$.

b Hence, find $\frac{dy}{dx}$ for $y = x^3e^{-4x}\cos(2x)$.

24 Using the fundamental limit $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$, verify each of the following, using the method of first principles where k is a constant and $x \in R$.

a $\frac{d}{dx}(\sin(kx)) = k\cos(kx)$

b $\frac{d}{dx}(\cos(kx)) = -k\sin(kx)$

c $\frac{d}{dx}(\tan(kx)) = k\sec^2(kx)$

d $\frac{d}{dx}(\sec(kx)) = k\tan(kx)\sec(kx)$

e $\frac{d}{dx}(\text{cosec}(kx)) = -k\cot(kx)\text{cosec}(kx)$

f $\frac{d}{dx}(\cot(kx)) = -k\,\text{cosec}^2(kx)$

7.3 Applications of differentiation

Introduction

There are many mathematical applications of differential calculus, including:

- finding tangents and normals to curves
- rates of change
- maxima and minima problems
- curve sketching
- related rate problems
- kinematics.

Some of these topics are revised in this section. Related rate problems are covered in more detail later in the topic. Maxima and minima problems have already been covered in the Mathematical Methods course. Later topics include a more detailed coverage of the applications of differentiation to kinematics.

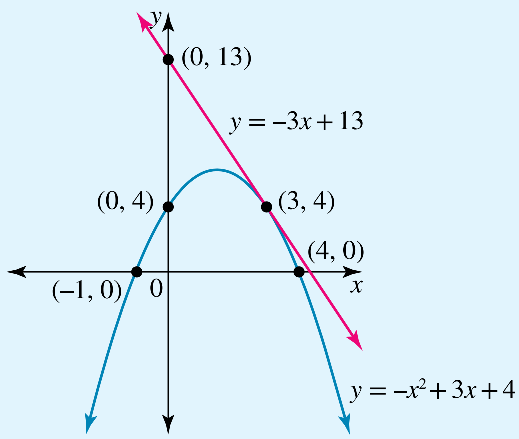
Tangents and normal to curves

Tangents to curves

A tangent to a curve is a straight line that touches a curve at the point of contact. Furthermore, the gradient of the tangent is equal to the gradient of the curve at the point of contact.

WORKED EXAMPLE 7 **Find the equation of the tangent to the curve $y = -x^2 + 3x + 4$ at the point where $x = 3$. Sketch the curve and the tangent.**

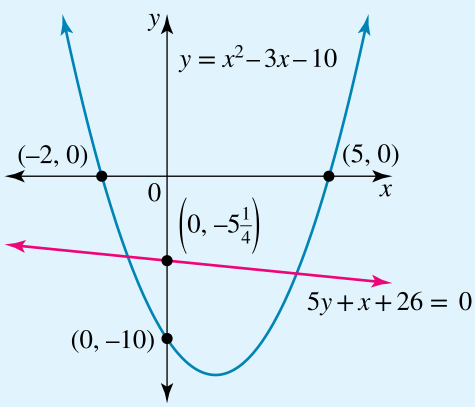
THINK	WRITE/DRAW
1 Find the y-coordinate corresponding to the given x-value.	$y = -x^2 + 3x + 4$ When $x = 3$, $y = -9 + 9 + 4$ $= 4$ The point is (3, 4).
2 Find the gradient of the curve at the given x-value. This will be denoted by m_T.	$\frac{dy}{dx} = -2x + 3$ When $x = 3$, $\left.\frac{dy}{dx}\right\|_{x=3} = m_T$ $= -3$
3 Find the equation of the tangent, that is, the straight line passing through the given point with the given gradient. Use $y - y_1 = m_T(x - x_1)$.	$x_1 = 3, y_1 = 4$ and $m_T = -3$ $y - 4 = -3(x - 3)$ $y = -3x + 9 + 4$
4 State the equation of the tangent and sketch the graph.	$y = -3x + 13$



Normals to curves

The normal to a curve is a straight line perpendicular to the tangent to the curve. If two lines are perpendicular, the product of their gradients is -1. If m_T is the gradient of the tangent and m_N is the gradient of the normal, then $m_T \cdot m_N = -1$.

WORKED EXAMPLE 8 **Determine the equation of the normal to the curve $y = x^2 - 3x - 10$ at the point where $x = 4$. Sketch the curve and the normal.**

THINK	WRITE/DRAW
1 Find the y-coordinate corresponding to the given x-value.	$y = x^2 - 3x - 10$ When $x = 4$. $y = 16 - 12 - 10$ $= -6$ The point is (4, –6).
2 Find the gradient of the curve at the given x-value. This is denoted by m_T.	$\frac{dy}{dx} = 2x - 3$ When $x = 4$, $\left.\frac{dy}{dx}\right\|_{x=3} = m_T$ $= 5$
3 Find the gradient of the normal, which will be denoted by m_N. Since the normal line is perpendicular to the tangent, the product of their gradients is –1.	$m_N m_T = -1$ $m_T = 5$ $\Rightarrow m_N = -\frac{1}{5}$ The normal has a gradient $m_N = -\frac{1}{5}$.
4 Find the equation of the normal, that is, the straight line passing through the given point with the given gradient. Use $y - y_1 = m_N(x - x_1)$.	$x_1 = 4, y_1 = -6$ and $m_N = -\frac{1}{5}$ $y + 6 = -\frac{1}{5}(x - 4)$ $5y + 30 = -x + 4$
5 State the equation of the normal. To avoid fractions, give the result in the form $ax + by + k = 0$. Sketch the graph.	$5y + x + 26 = 0$ 

General results for finding tangents and normals to curves

In general, to find the equation of the tangent to the curve $y = f(x)$ at the point where $x = a$, the y-value is $y = f(a)$, so the coordinates of the point are $(a, f(a))$. The gradient of the curve at this point is $m_T = f'(a)$, so the equation of the tangent is given by $y - f(a) = f'(a)(x - a)$. The normal has a gradient of $m_N = -\frac{1}{f'(a)}$, so the equation of the normal to the curve is given by $y - f(a) = -\frac{1}{f'(a)}(x - a)$.

WORKED EXAMPLE 9

The tangent to the curve $y = \sqrt{x}$ at a point is given by $6y - x + c = 0$. Find the value of c.

THINK | **WRITE**

1 Find the gradient of the curve.

$y = \sqrt{x} = x^{\frac{1}{2}}$

Then $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

2 Find the gradient of the tangent line.

$6y - x + c = 0$,

Rearrange to make y the subject:

$6y = x - c$

$y = \frac{x}{6} - \frac{c}{6}$

3 State the gradient of the tangent line.

$m_T = \frac{1}{6}$

4 At the point of contact these gradients are equal.

$\frac{1}{2\sqrt{x}} = \frac{1}{6}$

5 Solve the equation to find the x-value at the point of contact.

$\sqrt{x} = 3$

$x = 9$

6 Find the coordinate at the point of contact. Substitute the x-value into the curve.

When $x = 9$,

$y = \sqrt{x} = \sqrt{9} = 3$.

The point is (9, 3).

7 This point also lies on the tangent.

$6y - x + c = 0$

$c = x - 6y$

$= 9 - 18$

8 State the answer.

$c = -9$

Finding tangents and normals to other functions

When finding the equation of the tangent and normal to trigonometric functions or other types of functions, we must use exact values and give exact answers. That is, give answers in terms of π or surds such as $\sqrt{3}$, and do not give answers involving decimals.

WORKED EXAMPLE 10

Determine the equation of the tangent and normal to the curve $y = 4\cos(2x)$ at the point where $x = \frac{\pi}{8}$. Sketch the tangent and the normal.

THINK | **WRITE/DRAW**

1 Find the y-coordinate corresponding to the given x-value.

When $x = \frac{\pi}{8}$,

$y = 4\cos\left(\frac{\pi}{4}\right)$

$= 2\sqrt{2}$

The point is $\left(\frac{\pi}{8}, 2\sqrt{2}\right)$.

2 Find the gradient of the curve at the given x-value.

$$\frac{dy}{dx} = -8\sin(2x)$$

When $x = \frac{\pi}{8}$,

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{8}} = m_T$$

$$= -8\sin\left(\frac{\pi}{4}\right)$$

$$= -4\sqrt{2}$$

3 Find the equation of the tangent, that is, the straight line passing through the given point with the given gradient. Use $y - y_1 = m_T(x - x_1)$

$x_1 = \frac{\pi}{8}$, $y_1 = 2\sqrt{2}$ and $m_T = -4\sqrt{2}$

$$y - 2\sqrt{2} = -4\sqrt{2}\left(x - \frac{\pi}{8}\right)$$

4 State the equation of the tangent.

$$y = -4\sqrt{2}x + \frac{\sqrt{2}\pi}{2} + 2\sqrt{2}$$

5 Find the gradient of the normal, m_N. Since the normal line is perpendicular to the tangent, the product of their gradients is –1.

$$m_N m_T = -1$$

$$m_T = -4\sqrt{2}$$

$$\Rightarrow m_N = \frac{1}{4\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{8}$$

The normal has a gradient $m_N = \frac{\sqrt{2}}{8}$.

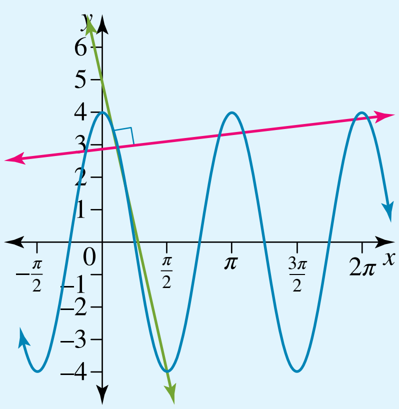
6 Find the equation of the normal, that is, the straight line passing through the given point with the given gradient. Use $y - y_1 = m_N(x - x_1)$.

$x_1 = \frac{\pi}{8}$, $y_1 = 2\sqrt{2}$ and $m_N = \frac{\sqrt{2}}{8}$

$$y - 2\sqrt{2} = \frac{\sqrt{2}}{8}\left(x - \frac{\pi}{8}\right)$$

7 State the equation of the normal and sketch the graph, showing the tangent and normal.

$$y = \frac{\sqrt{2}x}{8} - \frac{\sqrt{2}\pi}{64} + 2\sqrt{2}$$



Rates of change

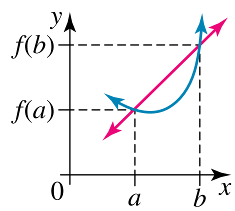
The first derivative or gradient function, $\frac{dy}{dx}$, is also a measure of the instantaneous rate of change of y with respect to x.

WORKED EXAMPLE 11 **Find the rate of change of the area of a circle with respect to the radius.**

THINK	WRITE
1 Define the variables and state the area of a circle.	If the radius of the circle is r and the area is A, then $A = \pi r^2$.
2 We require the rate of change of the area of a circle with respect to the radius.	$\frac{dA}{dr}$
3 State the required result.	$\frac{dA}{dr} = 2\pi r$

Average rates of change

The average rate of change is not to be confused with the instantaneous rate of change. The average rate of change of a function $y = f(x)$ over $x \in [a, b]$ is the gradient of the line segment joining the points: $\frac{f(b) - f(a)}{b - a}$.



The following worked example illustrates these different ideas.

WORKED EXAMPLE 12 **The tides in a certain bay can be modelled by**

$$D(t) = 9 + 3\sin\left(\frac{\pi t}{6}\right)$$

where D is the depth of water in metres and t is the time in hours after midnight on a particular day.

a What is the depth of water in the bay at 2 am?

b Sketch the graph of $D(t)$ against t for $0 \le t \le 24$.

c Find when the depth of water is below 7.5 metres.

d Find the rate of change of the depth at 2 am.

e Over the first 4 hours, find the average rate of change of the depth.

THINK	WRITE/DRAW
a 2 am corresponds to $t = 2$. Find $D(2)$.	a $D(2) = 9 + 3\sin\left(\frac{2\pi}{6}\right)$ $= 9 + 3 \times \frac{\sqrt{3}}{2}$ $= 11.598$ metres

b **1** Find the period and amplitude of the graph. State the maximum and minimum values of the depth.

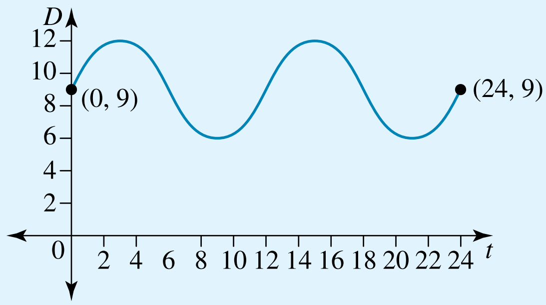
b $n = \frac{\pi}{6}$

The period is $\frac{2\pi}{n}$ or $\frac{2\pi}{\frac{\pi}{6}} = 12$.

Over $0 \le t \le 24$, there are two cycles.

The maximum depth is $9 + 3 = 12$ metres and the minimum depth is $9 - 3 = 6$ metres.

2 Sketch the graph on the restricted domain.



c **1** Solve an appropriate equation to find the times when the depth of water is below 7.5 metres.

c $D(t) = 7.5 = 9 + 3\sin\left(\frac{\pi t}{6}\right)$

$\sin\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$

$\frac{\pi t}{6} = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{7\pi}{6} + 2\pi, \frac{11\pi}{6} + 2\pi$

$t = 7, 11, 19, 23$

2 State when the depth is below 7.5 metres.

The depth is below 7.5 metres between 7 am and 11 am and between 7 pm and 11 pm.

d **1** Find the rate of change of depth with respect to time.

d $\frac{dD}{dt} = \frac{\pi}{2}\cos\left(\frac{\pi t}{6}\right)$

2 Evaluate this rate at 2 am, giving the correct units.

When $t = 2$, $\left.\frac{dD}{dT}\right|_{t=2} = \frac{\pi}{2}\cos\left(\frac{\pi}{3}\right)$

$= \frac{\pi}{4}$ m/h

e **1** Find the average rate of change over the first 4 hours.

e $t \in [0, 4]$

$D(0) = 9 + 3\sin(0) = 9$

$D(4) = 9 + 3\sin\left(\frac{4\pi}{6}\right) = 9 + \frac{3\sqrt{3}}{2}$

$\frac{D(4) - D(0)}{4 - 0} = \frac{9 + \frac{3\sqrt{3}}{2} - 9}{4 - 0}$

2 State the required average rate, giving the correct units.

The average rate is $\frac{3\sqrt{3}}{8}$ m/h.

EXERCISE 7.3 Applications of differentiation

PRACTISE

1 WE7 Find the equation of the tangent to the curve $y = x^2 - 2x - 8$ at the point where $x = 3$.

2 Find the equation of the tangent to the curve $y = \sqrt{4x + 5}$ at the point where $x = 1$.

3 WE8 Determine the equation of the normal to the curve $y = -x^2 + 2x + 15$ at the point where $x = 2$.

4 Determine the equation of the normal to the curve $y = \dfrac{4}{3x - 2}$ at the point where $x = 1$.

5 WE9 The tangent to the curve $y = x^2 - 6x + 5$ at a point is given by $y = 2x + c$. Find the value of c.

6 The normal to the curve $y = x^2 + 4x + 12$ at a point is given by $2y - x + c = 0$. Find the value of c.

7 WE10 Determine the equation of the tangent and normal to the curve $y = 4\sin\left(\dfrac{x}{2}\right)$ at the point where $x = \dfrac{\pi}{3}$.

8 Determine the equation of the tangent and normal to the curve $y = -3e^{-2x} + 4$ at the point where it crosses the y-axis.

9 WE11 For a sphere, find the rate of change of volume with respect to the radius.

10 For a cone, find the rate of change of volume with respect to:

a the radius, assuming the height remains constant
b the height, assuming the radius remain constant.

11 WE12 The tides in a certain bay can be modelled by

$$D(t) = 6 + 4\cos\left(\frac{\pi t}{12}\right)$$

where D is the depth of water in metres and t is the time in hours after midnight on a particular day.

a What is the depth of water in the bay at 6 am?
b Sketch the graph of $D(t)$ against t for $0 \le t \le 24$.
c Find when the depth of water is below 8 metres.
d Find the rate of change of the depth at 3 am.
e Over the first 6 hours, find the average rate of change of the depth.

12 The population number, $N(t)$, of a certain city can be modelled by the equation $N(t) = N_0 e^{kt}$, where N_0 and k are constants and t is the time in years after the year 2000. In the year 2000, the population number was 500 000 and in 2010 the population had grown to 750 000.

a Find the values of N_0 and k.
b What is the predicted population in 2015?

c Find the rate of change of population in 2015.
d Find the average rate of growth over the first 30 years.
e Sketch the graph of $N(t)$.

CONSOLIDATE

13 Find the equation of the tangent to each of the following curves at the point indicated.
a $y = 9 - x^2$ at $x = 2$
b $y = 5\sin(2x)$ at $x = \frac{\pi}{3}$
c $y = \sqrt{2x + 1}$ at $x = 4$
d $y = 4\cos(3x)$ at $x = \frac{\pi}{12}$

14 Find the equation of the normal to each of the following curves at the point indicated.
a $y = 16 - x^2$ at $x = 3$
b $y = \tan(3x)$ at $x = \frac{\pi}{9}$
c $y = \log_e(4x - 3)$ at $x = 1$
d $y = \frac{3}{(2x - 3)^2}$ at $x = 3$

15 a Find the value of c if $y = -12x + c$ is a tangent to the curve $y = x^2 - 8x - 9$.
b Find the value of c if $y = 8x + c$ is a tangent to the curve $y = \frac{2}{(2x - 3)^2}$.
c Find the value of c if $3y - 4x + c = 0$ is a normal to the curve $y = \frac{3}{x^2}$.
d Find the value of c if $2y + x + c = 0$ is a normal to the curve $y = \sqrt{4x - 3}$.

16 Given the function $f: R \to R, f(x) = 5 - 3e^{-2x}$:
a show that the function is always increasing
b sketch the graph, stating any axis intercepts and the equations of any asymptotes
c find the equation of the tangent to the curve at $x = \frac{1}{2}$.

17 Given the function $f: R \to R, f(x) = 4 - 3e^{2x}$:
a show that the function is always decreasing
b sketch the graph, stating any axis intercepts and the equations of any asymptotes
c find the equation of the normal to the curve at $x = \frac{1}{2}$.

18 a Sketch the graph of $y = 3\log_e(4x - 5)$, stating the axis intercepts, the equations of any asymptotes, and the domain and range.
b Show that there are no stationary points.
c Find the equation of the tangent to the curve $y = 3\log_e(4x - 5)$ at the point where $x = 2$.

19 a Sketch the graph of $f(x) = 2\log_e(5 - 3x)$, stating the axis intercepts, the equations of any asymptotes, and the domain and range.
b Show that the function is a one-to-one decreasing function.
c Find the equation of the normal to the curve $y = 2\log_e(5 - 3x)$ at the point where $x = 1$.

20 Given the function $f: R \to R, f(x) = x^3e^{-2x}$:
a find the equations of the tangents to the curve $y = x^3e^{-2x}$ at the points where $x = 1$ and $x = 0$
b show that the tangents above are the only two tangents to the curve that pass through the origin.

21 a The tangent to the curve $y = \frac{1}{x}$ at the point where $x = a$ and $a > 0$ crosses the x-axis at B and crosses the y-axis at C. If O is the origin, find the area of the triangle OAB.

b If the normal to the curve $y = \frac{1}{x}$ at the point where $x = a$ and $a > 0$ passes through the origin O, find the value of a and hence find the closest distance of the curve $y = \frac{1}{x}$ to the origin.

c i The tangent to the curve $y = k - x^2$, where $k > 0$, at the point where $x = a$ and $a > 0$ crosses the x-axis at B and crosses the y-axis at C. If O is the origin, find the area of the triangle OAB.

ii The normal to the curve $y = k - x^2$ at the point where $x = a$ and $a > 0$ passes through the origin. Show that $k = a^2 + \frac{1}{2}$.

22 The population number, $P(t)$, of ants in a certain area is given by

$$P(t) = \frac{520}{0.3 + e^{-0.15t}}$$

where $t \geq 0$ is the time in months.

a Find the initial population of the ants.

b Find the rate at which the ant population is increasing with respect to time and evaluate the rate after 10 months.

c Over the first 10 months find the average rate at which the ant population is increasing.

MASTER

23 The current, i milliamps, in a circuit after a time t milliseconds is given by $i = 120e^{-3t}\cos(10t)$ for $t \geq 0$.

a Find the rate of change of current with respect to time and evaluate it after 0.01 milliseconds, giving your answer correct to 3 decimal places.

b Over the time interval from $t = 0$ to $t = 0.02$, find the average rate of change of current.

24 The amount of a drug, D milligrams, in the bloodstream at a time t hours after it is administered is given by $D(t) = 30te^{-\frac{t}{3}}$.

a Find the average amount of the drug present in the bloodstream over the time from $t = 1$ to $t = 2$ hours after it is administered.

b Find the instantaneous amount of the drug after 1.5 hours.

c Find the time when the amount of drug is a maximum and find the maximum amount of the drug in the body.

d For how long is the amount of the drug in the body more than 10 milligrams?

7.4 Implicit and parametric differentiation

Introduction

study on

Units 3 & 4
AOS 3
Topic 1
Concept 4

Implicit differentiation
Concept summary
Practice questions

Up until now the relationship between x and y has always been explicit: y is the dependent variable and x the independent variable, with y in terms of x written as $y = f(x)$. It is said that y depends upon x and $\frac{dy}{dx}$ can be found directly. For example:

$$\text{If } y = x^2 + 4x + 13, \text{ then } \frac{dy}{dx} = 2x + 4.$$

$$\text{If } y = 3\sin(2x) + 4e^{-2x}, \text{ then } \frac{dy}{dx} = 6\cos(2x) - 8e^{-2x}.$$

$$\text{If } y = \sqrt{16 - x^2}, \text{ then } \frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}}.$$

$$\text{If } y = \log_e(5x + 3), \text{ then } \frac{dy}{dx} = \frac{5}{5x + 3}.$$

There are times, however, when y is not expressed explicitly in terms of x. In these cases there is a functional dependence or a so-called implicit relationship between x and y of the form $f(x, y) = c$, where c is a constant, or $g(x, y) = 0$. For example:

$$x^2 + y^2 = 16$$

$$4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$$

$$e^{-2xy} + 3\sin(2x - 3y) = c$$

These represent curves as an implicit relation and are not graphs of functions.

In some cases it may be possible to rearrange to make y the subject, but in most cases this is simply not possible. In this implicit form an expression for $\frac{dy}{dx}$ can still be obtained, but it will be in terms of both x and y. This can be obtained by differentiating each term in turn.

For example, $\frac{d}{dx}(x^2) = 2x$, and in general, if n is a constant, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Consider $\frac{d}{dx}(y^2)$. To find this, use the chain rule and write $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$; furthermore, in general $\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\frac{dy}{dx} = ny^{n-1}\frac{dy}{dx}$.

This last result is known as **implicit differentiation**; it is really just an application of the chain rule. Basically, when it is necessary to differentiate a function of y with respect to x, differentiate with respect to y and multiply by $\frac{dy}{dx}$.

WORKED EXAMPLE 13

Given $x^2 + y^2 = 16$, find an expression for $\frac{dy}{dx}$ in terms of both x and y.

THINK	WRITE/DRAW
1 Write the equation.	$x^2 + y^2 = 16$
2 Take $\frac{d}{dx}(\ )$ of each term in turn.	$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$

3 Use the results above together with $\frac{d}{dx}(c) = 0$, since the derivative of a constant is zero.

$$2x + 2y\frac{dy}{dx} = 0$$

4 Transpose the equation to make $\frac{dy}{dx}$ the subject.

$$2x = -2y\frac{dy}{dx}$$

5 Cancel the common factor and divide by x. State the final result, giving $\frac{dy}{dx}$ in terms of both x and y.

$$\frac{dy}{dx} = -\frac{x}{y}$$

Alternatively, notice that in this particular example it is possible to rearrange the equation to make y the subject.

1 Write the equation.

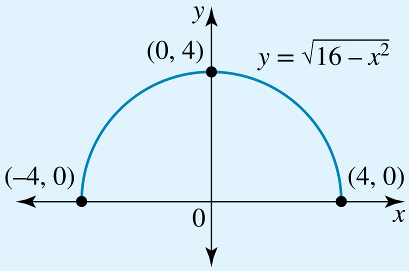
$$x^2 + y^2 = 16$$

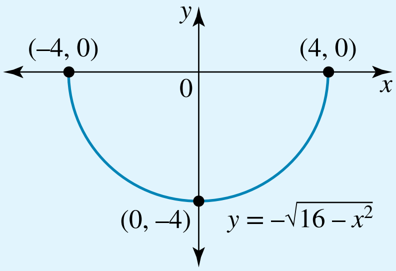
2 Rearrange to make y the subject.

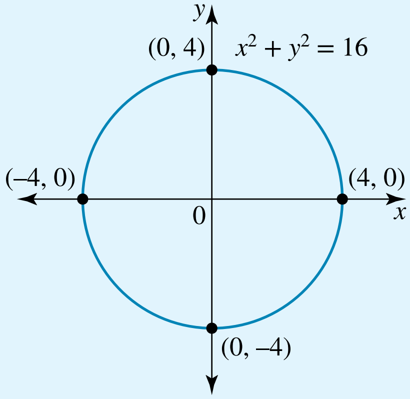
$$y^2 = 16 - x^2$$

3 There are two branches to the relation, which is a circle with centre at the origin and radius 4. Consider the branch or top half of the circle, which by itself is a function.

$$y = \pm\sqrt{16 - x^2}$$







4 Find $\frac{dy}{dx}$ in terms of x, differentiating using the chain rule.

Consider $y = \sqrt{16 - x^2}$.

$$\frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}}$$

5 However, since $y = \sqrt{16 - x^2}$, express $\frac{dy}{dx}$ in terms of both x and y as before.

$$\frac{dy}{dx} = -\frac{x}{y}$$

Further examples involving implicit differentiation

Sometimes it may be necessary to use the product rule to obtain the required derivatives.

WORKED EXAMPLE 14

a Find $\frac{d}{dx}(3xy)$.

b Given $4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$, find an expression for $\frac{dy}{dx}$ in terms of both x and y.

THINK	WRITE/DRAW
a 1 Write the expression.	**a** $\frac{d}{dx}(3xy)$
2 Use the product rule.	$\frac{d}{dx}(3x \times y) = \frac{d}{dx}(u \cdot v)$ Let $u = 3x$ and $v = y$.
3 Find the derivatives.	$\frac{du}{dx} = 3$ and $\frac{dv}{dx} = \frac{d}{dy}(y)\frac{dy}{dx} = 1 \times \frac{dy}{dx} = \frac{dy}{dx}$
4 Use the product rule.	$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$
5 State the final result.	$\frac{d}{dx}(3xy) = 3x\frac{dy}{dx} + 3y$
b 1 Write the equation.	**b** $4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$
2 Take $\frac{d}{dx}(\ )$ of each term in turn.	$\frac{d}{dx}(4x^2) + \frac{d}{dx}(3xy) - \frac{d}{dx}(2y^2) + \frac{d}{dx}(5x) - \frac{d}{dx}(7y) + \frac{d}{dx}(8) = 0$
3 Substitute for the result from the second term from part **a** above and use implicit differentiation on each term.	$8x + 3x\frac{dy}{dx} + 3y - 4y\frac{dy}{dx} + 5 - 7\frac{dy}{dx} + 0 = 0$
4 Transpose the equation to get all terms involving $\frac{dy}{dx}$ on one side of the equation.	$8x + 3y + 5 = 7\frac{dy}{dx} + 4y\frac{dy}{dx} - 3x\frac{dy}{dx}$
5 Factor the terms involving $\frac{dy}{dx}$ on the right-hand side of the equation.	$8x + 3y + 5 = \frac{dy}{dx}(7 + 4y - 3x)$
6 Divide and state the final result, giving $\frac{dy}{dx}$ in terms of both x and y.	$\frac{dy}{dx} = \frac{8x + 3y + 5}{7 + 4y - 3x}$

Implicit differentiation with exponential or trigonometric functions

Sometimes it may be necessary to use the derivatives of exponential or trigonometric functions together with implicit differentiation techniques to obtain the required derivative.

WORKED EXAMPLE 15

Given $e^{-2xy} + 3\sin(2x - 3y) = c$, where c is a constant, find an expression for $\frac{dy}{dx}$ in terms of both x and y.

THINK	WRITE
1 Write the equation.	$e^{-2xy} + 3\sin(2x - 3y) = c$
2 Take $\frac{d}{dx}(\ )$ of each term in turn.	$\frac{d}{dx}(e^{-2xy}) + \frac{d}{dx}(3\sin(2x - 3y)) = \frac{d}{dx}(c)$
3 Consider just the first term and use implicit differentiation.	$\frac{d}{dx}(e^{-2xy}) = \frac{d}{dx}(e^{-u})$ where $u = 2xy$ $\frac{d}{du}(e^{-u})\frac{du}{dx}$
4 Use the product rule.	$-e^{-u}\frac{d}{dx}(2xy) = -e^{-u}\left(2y + 2x\frac{dy}{dx}\right)$
5 Substitute back for u.	$\frac{d}{dx}(e^{-2xy}) = -\left(2y + 2x\frac{dy}{dx}\right)e^{-2xy}$
6 Consider just the second term and use implicit differentiation.	$\frac{d}{dx}(3\sin(2x - 3y)) = \frac{d}{dv}(3\sin(v))\frac{dv}{dx}$ where $v = 2x - 3y$ so that $\frac{dv}{dx} = 2 - 3\frac{dy}{dx}$. $\frac{d}{dx}(3\sin(2x - 3y)) = 3\cos(v)\left(2 - 3\frac{dy}{dx}\right)$
7 Substitute back for v.	$\frac{d}{dx}(3\sin(2x - 3y)) = 3\cos(2x - 3y)\left(2 - 3\frac{dy}{dx}\right)$
8 Substitute for the first and second terms.	$-\left(2y + 2x\frac{dy}{dx}\right)e^{-2xy} + 3\cos(2x - 3y)\left(2 - 3\frac{dy}{dx}\right) = 0$
9 Expand the brackets.	$-2ye^{-2xy} - 2x\frac{dy}{dx} + 6\cos(2x - 3y) - 9\frac{dy}{dx}\cos(2x - 3y) = 0$
10 Transpose the equation to get all terms involving $\frac{dy}{dx}$ on one side of the equation.	$-2ye^{-2xy} + 6\cos(2x - 3y)$ $= 2xe^{-2xy}\frac{dy}{dx} + 9\cos(2x - 3y)\frac{dy}{dx}$
11 Factor the terms involving $\frac{dy}{dx}$ on the right-hand side of the equation.	$-2ye^{-2xy} + 6\cos(2x - 3y)$ $= (2xe^{-2xy} + 9\cos(2x - 3y))\frac{dy}{dx}$
12 Divide and state the final result, giving $\frac{dy}{dx}$ in terms of both x and y.	$\frac{dy}{dx} = \frac{6\cos(2x - 3y) - 2ye^{-2xy}}{2xe^{-2xy} + 9\cos(2x - 3y)}$

Parametric differentiation

In the previous section, we saw that whether the variables x and y are given in the form $y = f(x)$ as an explicit relationship or the form $f(x, y) = 0$ as an implicit relationship, an expression for $\frac{dy}{dx}$ can be found.

In this section, the two variables x and y are connected or related in terms of another variable, called a linking variable or a parameter. Often t or θ are used as parameters. A parameter is a variable that changes from case to case but in each particular case or instant it remains the same. For example, an expression can be found for $\frac{dy}{dx}$; however, it will be in terms of the parameter by using the chain rule, since $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$, and noting that $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$. Alternatively, it may also be possible to eliminate the parameter from the two parametric equations and obtain an implicit relationship between the two variables x and y. The following examples will illustrate these concepts.

WORKED EXAMPLE 16

a **If $x = 3\cos(t)$ and $y = 4\sin(t)$, find the gradient $\frac{dy}{dx}$ in terms of t.**

b **For the parametric equations $x = 3\cos(t)$ and $y = 4\sin(t)$, find an implicit relationship between x and y, and find the gradient. Hence, verify your answer to part a.**

THINK	WRITE
a 1 Differentiate x with respect to t. The dot notation is used for the derivative with respect to t.	**a** $x = 3\cos(t)$ $\frac{dx}{dt} = \dot{x} = -3\sin(t)$
2 Differentiate y with respect to t.	$y = 4\sin(t)$ $\frac{dy}{dt} = \dot{y} = 4\cos(t)$
3 Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$
4 Substitute for the derivatives.	$\frac{dy}{dx} = \frac{4\cos(t)}{-3\sin(t)}$
5 State the gradient in simplest form.	$\frac{dy}{dx} = -\frac{4}{3}\cot(t)$
b 1 Write the parametric equations.	**b** $x = 3\cos(t)$ (1) $y = 4\sin(t)$ (2)
2 Express the trigonometric ratios on their own.	$\cos(t) = \frac{x}{3}$ (1) $\sin(t) = \frac{y}{4}$ (2)
3 Eliminate the parameter to find the implicit relationship.	Since $\cos^2(1) + \sin^2(t) = 1$, $\frac{x^2}{9} + \frac{y^2}{16} = 1$

4 Use implicit differentiation on the implicit form.	$\frac{1}{9}\frac{d}{dx}(x^2) + \frac{1}{16}\frac{d}{dx}(y^2) = \frac{d}{dx}(1)$
5 Perform the implicit differentiation.	$\frac{2x}{9} + \frac{2y}{16}\frac{dy}{dx} = 0$
6 Transpose the equation.	$\frac{2x}{9} = -\frac{y}{8}\frac{dy}{dx}$
7 Make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} = -\frac{16x}{9y}$
8 From the implicit differentiation, substitute for the parametric equations.	$\frac{dy}{dx} = -\frac{16 \times 3\cos(t)}{9 \times 4\sin(t)}$
9 Simplify to verify the given result, as above.	$\frac{dy}{dx} = -\frac{4}{3}\cot(t)$

EXERCISE 7.4 Implicit and parametric differentiation

PRACTISE

1 WE13 Given $x^3 + y^3 = 27$, find an expression for $\frac{dy}{dx}$ in terms of both x and y.

2 Given $\sqrt{x} + \sqrt{y} = 4$, find an expression for $\frac{dy}{dx}$ in terms of both x and y.

3 WE14 **a** Find $\frac{d}{dx}(4x^2y^2)$.

b Given $9x^3 + 4x^2y^2 - 3y^3 + 2x - 5y + 12 = 0$, find an expression for $\frac{dy}{dx}$ in terms of both x and y.

4 Find the gradient of the normal to the curve $x^2 - 4xy + 2y^2 - 3x + 5y - 7 = 0$ at the point where $x = 2$ in the first quadrant.

5 WE15 Given $2xy + e^{-(x^2+y^2)} = c$, where c is a constant, find an expression for $\frac{dy}{dx}$ in terms of both x and y.

6 If $\sin(3x + 2y) + x^2 + y^2 = c$, where c is a constant, find $\frac{dy}{dx}$ in terms of both x and y.

7 WE16 **a** If $x = 4\cos(t)$ and $y = 4\sin(t)$, find the gradient $\frac{dy}{dx}$ in terms of t.

b For the parametric equations $x = 4\cos(t)$ and $y = 4\sin(t)$, find an implicit relationship between x and y, and find the gradient. Hence, verify your answer to part **a**.

8 **a** Given $x = t^2$ and $y = 2t - t^4$, find $\frac{dy}{dx}$ in terms of t.

b For the parametric equations $x = t^2$ and $y = 2t - t^4$, express y in terms of x and find $\frac{dy}{dx}$. Hence, verify your answer to part **a**.

CONSOLIDATE

9 For each of the following implicitly defined relations, find an expression for $\frac{dy}{dx}$ in terms of x and y.

a $y^2 - 2x = 3$

b $\frac{x^2}{4} + \frac{y^2}{9} = 1$

c $\frac{x^2}{16} - \frac{y^2}{9} = 1$

d $4x - 2y - 3x^2 + y^2 = 10$

10 For each of the following implicitly defined relations, find an expression for $\frac{dy}{dx}$ in terms of x and y.

a $2x^2 + 3xy - 3y^2 + 8 = 0$
b $x^2 + x^2y^2 - 6x + 5 = 0$
c $x^3 - 3x^2y + 3xy^2 - y^3 - 27 = 0$
d $y^3 - y^2 - 3x - x^2 + 9 = 0$

11 For each of the following implicitly defined relations, find an expression for $\frac{dy}{dx}$ in terms of x and y.

a $\frac{y^2 - 2x}{3x^2 + 4y} = 6x$
b $\frac{x^3 + 8y^3}{x^2 - 2xy + 4y^2} = x^2$
c $e^{x+y} + \cos(y) - y^2 = 0$
d $e^{xy} + \cos(xy) + x^2 = 0$

12 For each of the following implicitly defined relations, find an expression for $\frac{dy}{dx}$ in terms of x and y.

a $\log_e(2x + 3y) + 4x - 5y = 10$
b $\log_e(3xy) + x^2 + y^2 - 9 = 0$
c $\frac{1}{x} + 2xy + \frac{1}{y} - 6 = 0$
d $2y^3 - \frac{\sin(3x)}{\sec(2y)} + x^2 = 0$

13 a Find the gradient of the tangent to the curve $y^2 = x^3(2 - x)$ at the point (1, 1).
b Find the equation of the tangent to the curve $x^3 + 3xy + y^3 + 13 = 0$ at the point (1, –2).
c For the circle $x^2 + 4x + y^2 - 3y + 6xy - 4 = 0$, find the gradient of the tangent to the circle at the point (2, –1).
d A certain ellipse has the equation $3x^2 + 2xy + 4y^2 + 5x - 10y - 8 = 0$. Find the gradient of the normal to the ellipse at the point (1, 2).

14 a Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point in the fourth quadrant where $x = 3$.
b Find the gradient of the tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at the point in the first quadrant where $x = 1$.
c Find the gradient of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point in the fourth quadrant where $x = 5$.
d Find the gradient of the normal to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point (3, 1).

15 For each of the following, find $\frac{dy}{dx}$ in terms of the parameter.

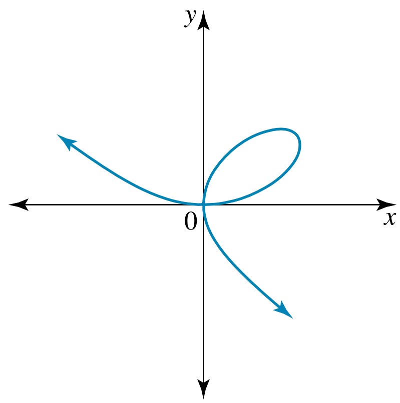
a The parabola $x = 2t^2$ and $y = 4t$
b The ellipse $x = 3\sin(2t)$ and $y = 4\cos(2t)$
c The rectangular hyperbola $x = 5t$ and $y = \frac{5}{t}$
d The hyperbola $x = 3\sec(t)$ and $y = 4\tan(t)$

16 In each of the following, a and b are constants. Find $\frac{dy}{dx}$ in terms of the parameter.

a $x = at^2$ and $y = 2at$
b $x = a\cos(t)$ and $y = b\sin(t)$
c $x = at$ and $y = \frac{a}{t}$
d $x = a\sec(t)$ and $y = b\tan(t)$

17 Check your answers to question **16a–d** by eliminating the parameter and determining $\frac{dy}{dx}$ by another method.

18 Rene Descartes (1596–1650) was a French mathematician and philosopher who lived in the Dutch republic. He is noted for introducing the Cartesian coordinate system in two dimensions and is credited as the father of analytical geometry, the link between algebra and geometry.

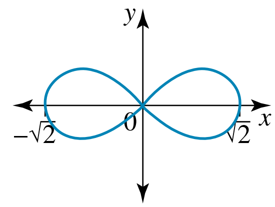


a A curve called the folium of Descartes has the equation $x^3 - 3axy + y^3 = 0$. Its graph is shown at right. Find $\dfrac{dy}{dx}$ for this curve, given that a is a constant.

b Show that the folium of Descartes can be represented by the parametric equations $x = \dfrac{3at}{1 + t^3}$ and $y = \dfrac{3at^2}{1 + t^3}$, and find the gradient of the curve at the point t.

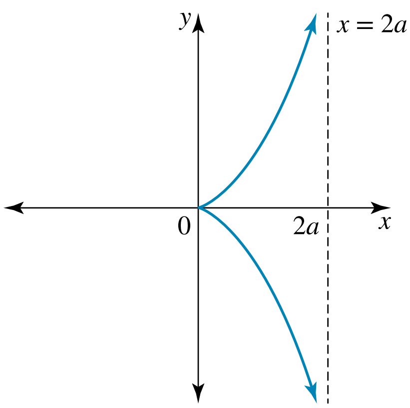
MASTER

19 a A curve called a lemniscate has the implicit equation $(x^2 + y^2)^2 = 2(x^2 - y^2)$. Its graph is shown at right. Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.



b Show that the curve can be represented by the parametric equations $x = \dfrac{\sqrt{2}\cos(t)}{\sin^2(t) + 1}$ and $y = \dfrac{\sqrt{2}\sin(t)\cos(t)}{\sin^2(t) + 1}$, and find $\dfrac{dy}{dx}$ in terms of t.

20 A curve known as the cissoid of Diocles has the equation $y^2 = \dfrac{x^3}{2a - x}$. Its graph is shown at right.



a Find $\dfrac{dy}{dx}$ for this curve, given that a is constant.

b Show that a parametric representation of the cissoid curve is given by the equations $x = \dfrac{2at^2}{1 + t^2}$ and $y = \dfrac{2at^3}{1 + t^2}$, and find $\dfrac{dy}{dx}$ in terms of t.

c Show that another alternative parametric representation of the cissoid curve is given by the equations $x = 2a\sin^2(t)$ and $y = \dfrac{2a\sin^3(t)}{\cos(t)}$, and find $\dfrac{dy}{dx}$ in terms of t.

7.5 Second derivatives

Introduction

If $y = f(x)$ is the equation of the curve, then the first derivative is $\dfrac{dy}{dx} = f'(x)$ and it is the gradient of the curve. It is also the rate of change of y with respect to x, often abbreviated to 'wrt x'. This is still a function of x, so if the first derivative is differentiated again, the **second derivative** or the rate of change of the gradient is obtained. This is denoted by $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} = f''(x)$. Notice the position of the 2s in this notation and that the two dashes after f indicate the second derivative with respect to x.

study on

Units 3 & 4
AOS 3
Topic 1
Concept 2

Second derivatives
Concept summary
Practice questions

Similarly, the third derivative is denoted by $\frac{d^3y}{dx^3} = f'''(x)$.

In general, the n^{th} derivative is denoted by $\frac{d^ny}{dx^n} = f^{(n)}(x)$.

For example, consider the general cubic equation $y = f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are constants. The first derivative or gradient function is $\frac{dy}{dx} = f'(x) = 3ax^2 + 2bx + c$. The second derivative or rate of change of gradient function is $\frac{d^2y}{dx^2} = 6ax + 2b$. The third derivative is $\frac{d^3y}{dx^3} = 6a$, and all further derivatives are zero.

WORKED EXAMPLE 17

If $f(x) = \frac{4\sqrt{x^5}}{3x^2}$, find $f''(9)$.

THINK	WRITE
1 Express the function in simplified form using index laws.	$f(x) = \frac{4\sqrt{x^5}}{3x^2} = \frac{4x^{\frac{5}{2}}}{3x^2} = \frac{4}{3}x^{\frac{5}{2}-2} = \frac{4}{3}x^{\frac{1}{2}}$
2 Find the first derivative, using the basic laws for differentiation.	$f'(x) = \frac{4}{3} \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{2}{3}x^{-\frac{1}{2}}$
3 Find the second derivative, using the basic laws for differentiation, by differentiating the first derivative again.	$f''(x) = \frac{2}{3} \times \frac{-1}{2}x^{-\frac{3}{2}} = \frac{-1}{3\sqrt{x^3}}$
4 Substitute in the value for x.	$f''(9) = \frac{-1}{3\sqrt{9^3}} = \frac{-1}{3 \times 27}$
5 State the final result.	$f''(9) = -\frac{1}{81}$

Using product and quotient rules

Often when we differentiate we may need to use rules such as the product and quotient rules.

WORKED EXAMPLE 18

Find $\frac{d^2y}{dx^2}$ for $y = x^2 \log_e(3x + 5)$.

THINK	WRITE
1 Write the equation.	Let $y = x^2 \log_e(3x + 5)$

2 Use the product rule.

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\log_e(3x+5)) + \frac{d}{dx}(x^2)\log_e(3x+5)$$

3 State the result for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{3x^2}{3x+5} + 2x\log_e(3x+5)$$

4 Differentiate with respect to x again.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3x^2}{3x+5}\right) + \frac{d}{dx}(2x\log_e(3x+5))$$

5 Use the quotient rule on the first term and the product rule again on the second term.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}(3x^2)(3x+5) - \frac{d}{dx}(3x+5)(3x^2)}{(3x+5)^2}$$
$$+ \frac{d}{dx}(2x)(\log_e(3x+5)) + 2x\frac{d}{dx}(\log_e(3x+5))$$

6 Perform the required derivatives.

$$\frac{d^2y}{dx^2} = \frac{6x(3x+5) - 3(3x^2)}{(3x+5)^2} + 2\log_e(3x+5) + \frac{6x}{3x+5}$$

7 Simplify the numerator in the first expression and state the final result.

$$\frac{d^2y}{dx^2} = \frac{9x^2+30x}{(3x+5)^2} + 2\log_e(3x+5) + \frac{6x}{3x+5}$$

8 Add the first and last terms by forming a common denominator.

$$\frac{d^2y}{dx^2} = 2\log_e(3x+5) + \frac{9x^2+30x+6x(3x+5)}{(3x+5)^2}$$
$$\frac{d^2y}{dx^2} = 2\log_e(3x+5) + \frac{9x^2+30x+18x^2+30x}{(3x+5)^2}$$

9 State the final result in simplest form.

$$\frac{d^2y}{dx^2} = 2\log_e(3x+5) + \frac{3x(9x+20)}{(3x+5)^2}$$

Finding second derivatives using implicit differentiation

Implicit differentiation techniques can be used to find relationships between the first and second derivatives. Equations involving the function y and its first and second derivatives are called differential equations. They are explored in greater depth in later topics.

WORKED EXAMPLE 19

If a and b are constants and $y = x\sqrt{a+bx^2}$, show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = a + 6bx^2$.

THINK / **WRITE**

1 Write the equation and square both sides.

$$y = x\sqrt{a+bx^2}$$
$$y^2 = x^2(a+bx^2)$$

2 Expand the brackets.

$$y^2 = ax^2 + bx^4$$

3 Take $\frac{d}{dx}(\ )$ of each term in turn.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx^4)$$

4 Use implicit differentiation to find the first derivative.

$$2y\frac{dy}{dx} = 2ax + 4bx^3$$

5 Take $\frac{d}{dx}(\ )$ of each term in turn again.

$$\frac{d}{dx}\left(2y\frac{dy}{dx}\right)=\frac{d}{dx}(2ax)+\frac{d}{dx}(4bx^3)$$

6 Use the product rule on the first term.

$$2y\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\frac{d}{dx}(2y)=2a+12bx^2$$

7 Use implicit differentiation.

$$2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=2a+12bx^2$$

8 Divide each term by 2 and state the final result.

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=a+6bx^2$$

Using parametric differentiation to find second derivatives

When finding $\frac{d^2y}{dx^2}$ with parametric differentiation, often our first thought might be to use the chain rule as $\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}\cdot\frac{d^2t}{dx^2}$. However, this rule is *incorrect* and cannot be used, as dt^2 and d^2t are not equivalent; furthermore, $\frac{d^2y}{dx^2}\neq 1\,/\,\frac{d^2x}{dy^2}$. To correctly obtain $\frac{d^2y}{dx^2}$, we must use implicit differentiation in conjunction with the parametric differentiation. Because $\frac{dy}{dx}$ is itself a function of t, we obtain the second derivative using

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}.$$

WORKED EXAMPLE 20

If $x=3\cos(t)$ and $y=4\sin(t)$, find $\frac{d^2y}{dx^2}$ in terms of t.

THINK	WRITE
1 Differentiate x with respect to t. The dot notation is used for the derivative with respect to t.	$x=3\cos(t)$ $\frac{dx}{dt}=\dot{x}=-3\sin(t)$
2 Differentiate y with respect to t.	$y=4\sin(t)$ $\frac{dy}{dt}=\dot{y}=4\cos(t)$
3 Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{\dot{y}}{\dot{x}}$
4 Substitute for the derivatives.	$\frac{dy}{dx}=\frac{4\cos(t)}{-3\sin(t)}$
5 State the gradient in simplest form.	$\frac{dy}{dx}=-\frac{4}{3}\cot(t)$
6 Take $\frac{d}{dx}(\ )$ of each term.	$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{4}{3}\cot(t)\right)$

7 Use implicit differentiation and $\cot(t) = \dfrac{\cos(t)}{\sin(t)}$.	$\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(-\dfrac{4\cos(t)}{3\sin(t)}\right)\dfrac{dt}{dx}$
8 Use the quotient rule.	$\dfrac{d^2y}{dx^2} = -\left(\dfrac{\frac{d}{dt}(4\cos(t))(3\sin(t)) - \frac{d}{dt}(3\sin(t))(4\cos(t))}{(3\sin(t))^2}\right)\dfrac{dt}{dx}$
9 Perform the derivatives in the numerator.	$\dfrac{d^2y}{dx^2} = -\left(\dfrac{-12\sin^2(t) - 12\cos^2(t)}{(3\sin(t))^2}\right)\dfrac{dt}{dx}$
10 Simplify using $\sin^2(t) + \cos^2(t) = 1$.	$\dfrac{d^2y}{dx^2} = \dfrac{12}{9\sin^2(t)}\dfrac{dt}{dx}$
11 Substitute for $\dfrac{dt}{dx} = 1 \,/\, \dfrac{dx}{dt}$.	$\dfrac{d^2y}{dx^2} = \dfrac{4}{3\sin^2(t)} \times \dfrac{-1}{3\sin(t)}$
12 State the final result.	$\dfrac{d^2y}{dx^2} = \dfrac{-4}{9\sin^3(t)}$ $= -\frac{4}{9}\operatorname{cosec}^3(t)$

EXERCISE 7.5 Second derivatives

PRACTISE

1 **WE17** If $f(x) = \dfrac{8\sqrt{x^3}}{3x}$, find $f''(4)$.

2 If $f(x) = 8\cos\left(\dfrac{x}{2}\right)$, find $f''\left(\dfrac{\pi}{3}\right)$.

3 **WE18** Find $\dfrac{d^2y}{dx^2}$ for $y = x^3\log_e(2x^2 + 5)$.

4 Find $\dfrac{d^2y}{dx^2}$ for $y = \dfrac{x^4}{e^{3x}}$.

5 **WE19** If a and b are constants and $y^2 = a + bx^2$, show that $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = b$.

6 If a and b are constants and $y^3 = x(a + bx^3)$, show that $y^2\dfrac{d^2y}{dx^2} + 2y\left(\dfrac{dy}{dx}\right)^2 = 4bx^2$.

7 **WE20** If $x = 4\cos(t)$ and $y = 4\sin(t)$, find $\dfrac{d^2y}{dx^2}$ in terms of t.

8 Given $x = t^2$ and $y = 2t - t^4$, find the rate of change of gradient and evaluate when $t = 2$.

CONSOLIDATE

9 **a** If $f(x) = \dfrac{4x^2}{3\sqrt{x}}$, find $f''(4)$.

b If $f(x) = \dfrac{2}{3x - 5}$, find $f''(1)$.

c If $f(x) = 4\log_e(2x - 3)$, find $f''(3)$.

d If $f(x) = e^{x^2}$, find $f''(1)$.

e If $f(x) = 4\tan\left(\dfrac{x}{2}\right)$, find $f''\left(\dfrac{\pi}{3}\right)$.

f If $f(x) = e^{\cos(2x)}$, find $f''\left(\dfrac{\pi}{6}\right)$.

10 Find $\frac{d^2y}{dx^2}$ if:

a $y = \log_e(x^2 + 4x + 13)$
b $e^{3x}\cos(4x)$
c $y = x^3e^{-2x}$
d $x^2\cos(3x)$
e $y = x\log_e(6x + 7)$
f $y = \log_e(x + \sqrt{x^2 + 16})$.

11 If a and b are constants, find $\frac{d^2y}{dx^2}$ for each of the following.

a $y^2 = 4ax$ **b** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **c** $xy = a^2$

12 In each of the following, a and b are constants. Find $\frac{d^2y}{dx^2}$ in terms of the parameter.

a $x = at^2$ and $y = 2at$
b $x = at$ and $y = \frac{a}{t}$
c $x = a\cos(t)$ and $y = b\sin(t)$

13 Let a, b and k be constants.

a If $y^2 = a + bx$, prove that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$.
b Given that $y = x\sqrt{a + bx}$, verify that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = a + 3bx$.
c If $y = \frac{a\cos(kx) + b\sin(kx)}{x}$, show that $y\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + k^2y = 0$.

14 If a, b and n are constants, show that:

a $\frac{d^2}{dx^2}[(ax + b)^n] = a^2n(n - 1)(ax + b)^{n-2}$
b $\frac{d^2}{dx^2}[\log_e(ax + b)^n] = \frac{-a^2n}{(ax + b)^2}$
c $\frac{d^2}{dx^2}[\log_e(ax^2 + b)^n] = \frac{-2an(ax^2 - b)}{(ax^2 + b)^2}$
d $\frac{d^2}{dx^2}[(ax^2 + b)^n] = 2an(ax^2 + b)^{n-2}(a(2n - 1)x^2 + b)$.

15 A circle of radius a with centre at the origin has the equation $x^2 + y^2 = a^2$.

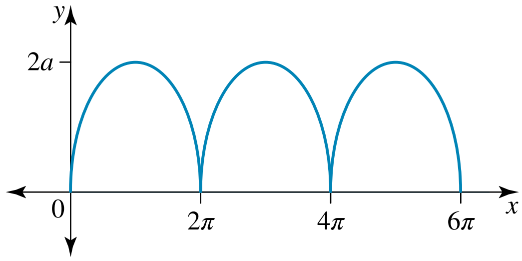
a Show that $\frac{d^2y}{dx^2} = -\frac{a^2}{y^3}$.
b The radius of curvature ρ of a plane curve is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Show that the radius of curvature of a circle has a magnitude of a.

16 An involute of a circle has the parametric equations $x = \cos(\theta) + \theta\sin(\theta)$ and $y = \sin(\theta) - \theta\cos(\theta)$. Show that the radius of curvature is θ.

17 A cycloid is the curve obtained by a point on a circle of radius a rolling along the x-axis. Its graph is shown below.



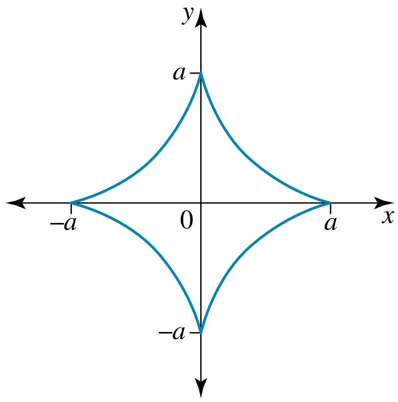
It has the parametric equations $x = a(\theta - \sin(\theta))$ and $y = a(1 - \cos(\theta))$.
Show that:

a $\dfrac{dy}{dx} = \cot\left(\dfrac{\theta}{2}\right)$

b $\dfrac{d^2y}{dx^2} = \dfrac{-1}{4a\sin^4\left(\dfrac{\theta}{2}\right)}$

c the radius of curvature of the cycloid is $-4a\sin\left(\dfrac{\theta}{2}\right)$.

18 a Given the implicit equation for an astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, where a is a constant, express $\dfrac{dy}{dx}$ in terms of both x and y.



b Show that the astroid can be expressed in parametric form as $x = a\cos^3(\theta)$ and $y = a\sin^3(\theta)$, and find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of θ.

MASTER

19 a If $y = e^{-2x}\sin(3x)$, find $\dfrac{d^2y}{dx^2}$.

b If $y = u(x)v(x)$ where $u(x)$ and $v(x)$ are differentiable functions, show that
$\dfrac{d^2y}{dx^2} = u(x)\dfrac{d^2v}{dx^2} + 2\dfrac{du}{dx}\dfrac{dv}{dx} + v(x)\dfrac{d^2u}{dx^2}$.
Use this result to verify $\dfrac{d^2y}{dx^2}$ for $y = e^{-2x}\sin(3x)$.

20 a If x and y are both expressed in terms of a parameter t, that is, $x = f(t)$ and $y = g(t)$, show that $\dfrac{d^2y}{dx^2} = \dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$.

b Show that $\dfrac{d^2y}{dx^2} = -\left(\dfrac{dy}{dx}\right)^3\dfrac{d^2x}{dy^2}$.

7.6 Curve sketching

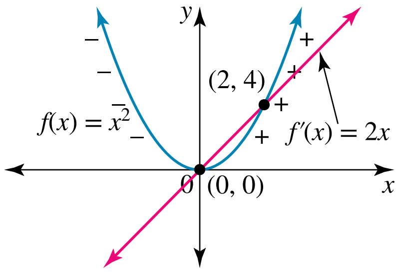
Curve sketching and derivatives

We can sketch a curve from its main characteristic or critical points and axial intercepts.

We will use the derivatives as an aid to sketching curves. Because the first derivative, $\frac{dy}{dx} = f'(x)$, represents the slope or gradient of a curve at any x-value, we say that a curve is increasing when its gradient is positive, $\{x : f'(x) > 0\}$; that is, the graph slopes upwards to the right. A curve is decreasing when its gradient is negative, $\{x : f'(x) < 0\}$; that is, the graph slopes upwards to the left.

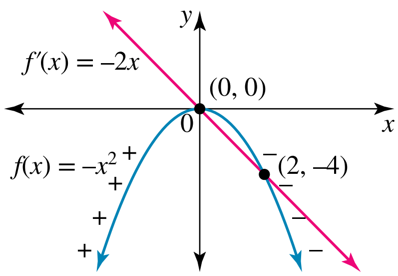
The second derivative, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$, is the rate of change of the gradient function or the gradient of the first derivative and is also useful in sketching graphs.

Consider the graph of $y = f(x) = x^2$.



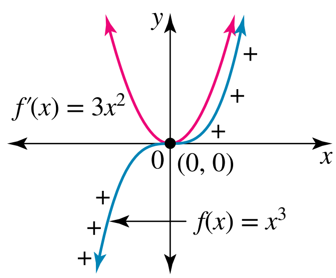
The graph has a minimum turning point at $x = 0$, that is at the origin, (0, 0). This is also where the derivative function $\frac{dy}{dx} = f'(x)$ intersects the x-axis. Note that $f'(x) = 2x$ and $f''(x) = 2$. When $x < 0$, to the left of the minimum turning point, the gradient is negative; when $x > 0$, to the right of the minimum turning point, the gradient is positive. So the gradient changes sign from negative to zero to positive, and the rate of change of the gradient is positive.

Consider the graph of $y = f(x) = -x^2$.



The graph has a maximum turning point at $x = 0$, that is at the origin, (0, 0). This is also where the derivative function $\frac{dy}{dx} = f'(x)$ intersects the x-axis. Note that $f'(x) = -2x$ and $f''(x) = -2$. When $x < 0$, to the left of the maximum turning point, the gradient is positive; when $x > 0$, to the right of the maximum turning point, the gradient is negative. So the gradient changes sign from positive to zero to negative, and the rate of change of the gradient is negative.

Consider the graph of $y = f(x) = x^3$.



For this function, $f'(x) = 3x^2$ and $f''(x) = 6x$. The origin, (0, 0), is a special point on the graph. This is where the derivative function $\frac{dy}{dx} = f'(x)$ intersects the x-axis. When $x < 0$, to the left of the origin, the gradient is positive; when $x > 0$, to the right of the origin, the gradient is positive. At the origin $f'(0) = 0$ and $f''(0) = 0$. So the gradient changes from positive to zero to positive, and the rate of change of the gradient is zero at this point. In this case the point at the origin is a horizontal stationary point of inflection. The tangent to the curve $y = x^3$ at the origin is the x-axis, and the tangent crosses the curve.

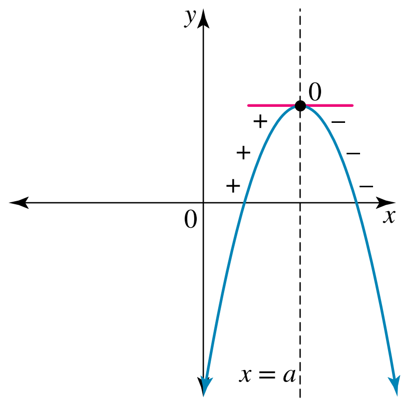
Stationary points

Although we have used the previous graphs to make some important observations about the first and second derivatives, the results are true in general and can be used as an aid to sketching graphs.

A **stationary point** on a curve is defined as a point on the curve at which the slope of the tangent to the curve is zero; that is, where the gradient $\frac{dy}{dx} = f'(x) = 0$. All turning points are stationary points.

There are two types of turning points: **maximum turning points** and **minimum turning points**.

A maximum turning point is a point on the curve at which the y-coordinate has its highest value within a certain interval. At such a point the slope of the curve changes from positive to zero to negative as x increases. The rate of change of the gradient is negative at such a point. If there are higher y-values outside the immediate neighbourhood of this maximum, it is called a local maximum. If it is the highest y-value on the whole domain, it is called an absolute maximum. A maximum turning point is like the top of a hill.

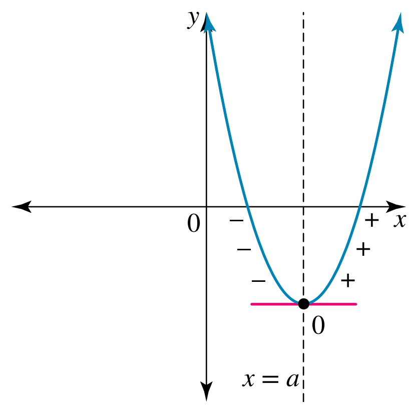


At $x = a$, $f'(a) = 0$ and $f''(a) < 0$.

If $x < a$, then $f'(x) > 0$.

If $x > a$, then $f'(x) < 0$.

A minimum turning point is a point on the curve at which the y-coordinate has its lowest value within a certain interval. At such a point the slope of the curve changes from negative to zero to positive as x increases. The rate of change of the gradient is positive at such a point. If there are lower y-values outside the immediate neighbourhood of this minimum, it is called a local minimum. If it is the lowest y-value on the whole domain, it is called an absolute minimum. A minimum turning point is like at the bottom of a valley.



At $x = a$, $f'(a) = 0$ and $f''(a) > 0$.

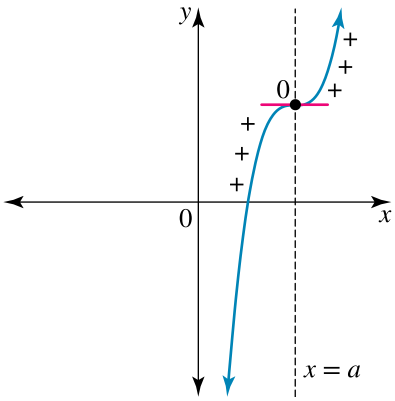
If $x < a$, then $f'(x) < 0$.

If $x > a$, then $f'(x) > 0$.

Note that for both a maximum turning point and a minimum turning point, the first derivative is zero. To distinguish between a maximum and a minimum, we use the sign test or the second derivative test.

On a continuous curve, between a local maximum and a local minimum there is another critical point called an **inflection** point. At such a point the curve changes from being concave to convex or vice versa. The tangent to the curve at such a point crosses the graph. The rate of change of the gradient is zero at a point of inflection. There are two types of inflection points.

Horizontal stationary point of inflection

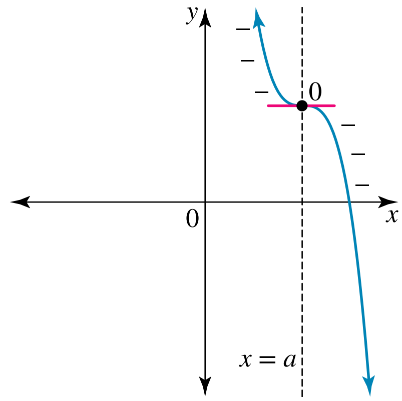


$x < a, f'(a) > 0$

$x = a, f'(a) = 0$

$x > a, f'(a) > 0$

$f''(a) = 0$



$x < a, f'(a) < 0$

$x = a, f'(a) = 0$

$x > a, f'(a) < 0$

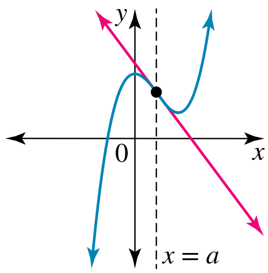
$f''(a) = 0$

Inflection point

At $x = a$,

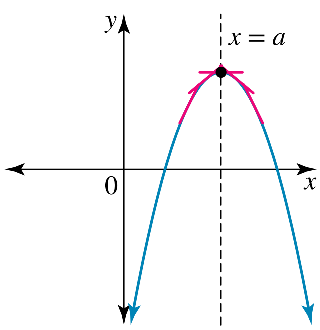
$f'(a) \neq 0$

$f''(a) = 0$

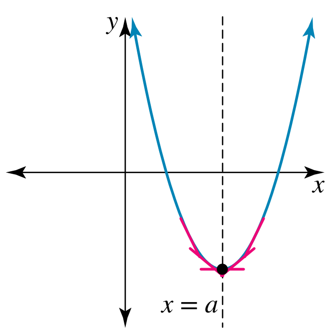


Concavity

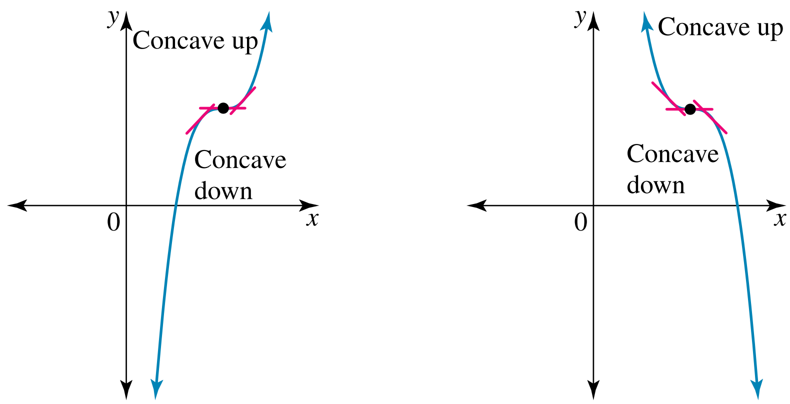
A non-stationary point of inflection is where the tangent to the curve moves from being above the curve to below the curve or vice versa. Curves are also defined as being **concave up** or **concave down**. When a tangent is above the curve at each point, then the derivative function is decreasing and $f''(x) < 0$.



If the tangent to the curve is below the curve at each point, then the derivative function is increasing and $f''(x) > 0$.



At a point of inflection, the tangent will pass through the curve, and the curve will change from concave down to concave up or vice versa on either side of the point of inflection.



Sketching the graphs of cubic functions

A general cubic function is of the form $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$. A cubic function is a polynomial of degree 3.

When sketching the graphs of cubic functions, we need to find where the graph crosses the x-axis. To solve the cubic equation $ax^3 + bx^2 + cx + d = 0$, we factorise or use the factor theorem. To find the stationary points we solve when the first derivative $\frac{dy}{dx} = 3ax^2 + 2bx + c = 0$. To find the inflection point we solve when the second derivative $\frac{d^2y}{dx^2} = 6ax + 2b = 0$.

We can use the second derivative to determine the nature of the stationary points. The values for x obtained for these equations are not necessarily integers, and there are several different possibilities for the shapes of the graphs. For example, the cubic may cross the x-axis at three distinct points and may have both a maximum and a minimum turning point as well as a non-horizontal point of inflection.

WORKED EXAMPLE 21

Sketch the graph of the function $f: R \to R$, $f(x) = x^3 + 6x^2 + 9x$ by finding the coordinates of all axis intercepts and stationary points and establishing their nature. Also find the coordinates of the point of inflection, and find and draw the tangent to the curve at the point of inflection.

THINK	WRITE/DRAW
1 Factorise the function.	$f(x) = x^3 + 6x^2 + 9x$ $= x(x^2 + 6x + 9)$ $= x(x + 3)^2$
2 Determine and find the axis intercepts.	The graph cross the x-axis when $y = 0$, at $x = 0$ or $x = -3$. The intercepts are (0, 0) and (–3, 0).
3 Find the first derivative and factorise.	$f(x) = x^3 + 6x^3 + 9x$ $f'(x) = 3x^2 + 12x + 9$ $= 3(x^2 + 4x + 3)$ $= 3(x + 3)(x + 1)$
4 Find the stationary points by equating the first derivative to zero.	$f'(x) = 3(x + 3)(x + 1) = 0$ when $x = -1$ or $x = -3$

5 Find the y-values at the stationary points.

$f(x) = x(x + 3)^2$
$f(-1) = -1(-1 + 3)^2 = -4$
$f(-3) = 0$
The stationary points are (–3, 0) and (–1, –4).

6 Find the second derivative.

$f'(x) = 3x^2 + 12x + 9$
$f''(x) = 6x + 12$
$= 6(x + 2)$

7 Test the two stationary points by using the second derivative to determine their nature.

When $x = -3$, $f''(-3) = -6 < 0$.
The point (–3, 0) is a local maximum.
When $x = -1$, $f''(-1) = 6 > 0$.
The point (–1, 4) is a local minimum.

8 Find the point of inflection by equating the second derivative to zero and find the y-value.

$f''(x) = 6(x + 2) = 0$ when $x = -2$.
$f(-2) = -2(-2 + 3)^2 = -2$
The point (–2, –2) is the point of inflection.

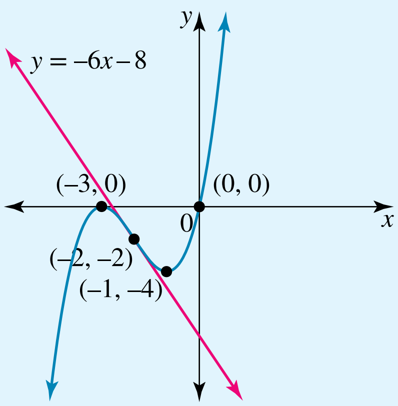
9 Determine the gradient at the point of inflection.

$f'(-2) = 3(-2)^2 + 12 \times -2 + 9$
$= -3$

10 Determine the equation of the tangent at the point of inflection.

Use $y - y_1 = m(x - x_1)$:
P (–2, –2), $m = -3$
$y + 2 = -3(x + 2)$
$y + 2 = -3x - 6$
$y = -6x - 8$

11 Sketch the graph and the tangent using an appropriate scale. We see that the tangent crosses the curve at the point of inflection.



Sketching the graphs of quartic functions

A general quartic function is of the form $y = ax^4 + bx^3 + cx^2 + dx + e$ where $a \neq 0$. A quartic function is a polynomial of degree 4.

When sketching the graphs of quartic functions, we need to find where the graph crosses the x-axis. To solve the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, we factorise or use the factor theorem. To find the stationary points, we solve when the first derivative $\frac{dy}{dx} = 4ax^3 + 3bx^2 + 2cx + d = 0$, and to find the points of inflection, we solve when the second derivative $\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 2c = 0$.

The graph of a quartic function will cross the x-axis four times at most, and will have at most three stationary points and possibly two inflection points.

WORKED EXAMPLE 22 **Sketch the graph of $y = 6x^2 - x^4$ by finding the coordinates of all axis intercepts, stationary points and points of inflection, and establishing their nature.**

THINK	WRITE/DRAW
1 Factorise.	$y = 6x^2 - x^4$ $= x^2(6 - x^2)$ $= x^2(\sqrt{6} - x)(\sqrt{6} + x)$
2 Determine and find the axis intercepts.	The graph crosses the x-axis when $y = 0$, at $x = 0$ or $x = \pm\sqrt{6}$. The x-intercepts are $(0, 0)$, $(\sqrt{6}, 0)$, $(-\sqrt{6}, 0)$. $\sqrt{6} \approx 2.45$
3 Find the first derivative and factorise.	$y = 6x^2 - x^4$ $\frac{dy}{dx} = 12x - 4x^3$ $= 4x(3 - x^2)$ $= 4x(\sqrt{3} - x)(\sqrt{3} + x)$
4 Find the stationary points by equating the first derivative to zero.	$\frac{dy}{dx} = 4x(\sqrt{3} - x)(\sqrt{3} + x) = 0$ $x = 0$ or $x = \pm\sqrt{3}$.
5 Find the y-values at the stationary points.	When $x = \sqrt{3}$, $y = 6(\sqrt{3})^2 - (\sqrt{3})^4 = 9$. When $x = -\sqrt{3}$, $y = 6(-\sqrt{3})^2 - (-\sqrt{3})^4 = 9$. The stationary points are $(0, 0)$, $(\sqrt{3}, 9)$ and $(-\sqrt{3}, 9)$.
6 Find the second derivative and factorise.	$\frac{dy}{dx} = 12x - 4x^3$ $\frac{d^2y}{dx^2} = 12 - 12x^2$ $= 12(1 - x^2)$ $= 12(1 - x)(1 + x)$
7 Test the stationary points using the second derivative to determine their nature.	When $x = 0$, $\frac{d^2y}{dx^2} = 12 > 0$. The point $(0, 0)$ is a local minimum. There are y-values lower than zero in the range of the function. When $x = \sqrt{3}$, $\frac{d^2y}{dx^2} = -24 < 0$. The point $(\sqrt{3}, 9)$ is an absolute maximum. When $x = -\sqrt{3}$, $\frac{d^2y}{dx^2} = -24 < 0$. The point $(\sqrt{3}, 9)$ is an absolute maximum.

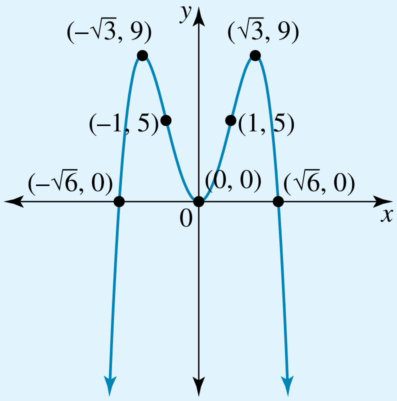
8 Find the points of inflection by equating the second derivative to zero and find the y-value.

$\frac{d^2y}{dx^2} = 12(1 - x)(1 + x) = 0$
$x = \pm 1$
When $x = \pm 1$, $y = 5$.
The points $(-1, 5)$ and $(1, 5)$ are both points of inflection.

9 List any other features about the graph.

Let $f(x) = 6x^2 - x^4$, then $f(-x) = f(x)$, so the graph is an even function and the graph is symmetrical about the y-axis.

10 Sketch the graph, showing all the critical points using an appropriate scale.



Sketching the graphs of reciprocal functions using calculus

In this section, we sketch graphs of the form $y = \frac{1}{ax^2 + bx + c}$.

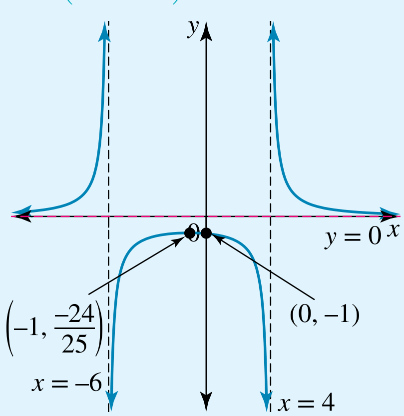
If $b^2 - 4ac > 0$, the graph will have two vertical asymptotes and one turning point. It can be shown that the graph will not have any inflection points.

We can deduce the graph of the function $f(x) = \frac{1}{g(x)}$ from the graph of the function $g(x)$ by noting that:

- the x-intercepts of $g(x)$ become the equations for the vertical asymptotes of $f(x)$
- the reciprocal of a positive number is positive, so the parts of the graph where $g(x)$ is above the x-axis remain above the x-axis for $f(x)$
- the reciprocal of a negative number is negative, so the parts of the graph where $g(x)$ is below the x-axis remain below the x-axis for $f(x)$
- if $g(x)$ crosses the y-axis at $(0, a)$, then $f(x)$ will cross the y-axis at $\left(0, \frac{1}{a}\right)$
- if $g(x) \to \infty$, then $f(x) \to 0^+$ (from above)
- if $g(x) \to -\infty$, then $f(x) \to 0^-$ (from below)
- if $g(x)$ has a local maximum at (p, q), then $f(x)$ will have a local minimum at $\left(p, \frac{1}{q}\right)$
- if $g(x)$ has a local minimum at (p, q), then $f(x)$ will have a local maximum at $\left(p, \frac{1}{q}\right)$
- if $g(x) = 1$, $\frac{1}{g(x)} = 1$.

WORKED EXAMPLE 23

Sketch the graph of $y = \frac{24}{x^2 + 2x - 24}$. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any turning points. State the maximal domain and range.

THINK	WRITE/DRAW
1 Factorise the denominator.	$y = f(x) = \dfrac{24}{x^2 + 2x - 24}$ $= \dfrac{24}{(x+6)(x-4)}$
2 Vertical asymptotes occur when the denominator is zero.	The lines $x = -6$ and $x = 4$ are both vertical asymptotes.
3 Determine axis intercepts. First find the x-intercepts.	The graph does not cross the x-axis, as the numerator is never zero.
4 Find the y-intercepts.	The graph crosses the y-axis when $x = 0$, $f(0) = -1$ at $(0, -1)$
5 Find the equations of any other asymptotes.	As $x \to \pm\infty$, $y \to 0^+$. The plus indicates that the graph approaches from above the asymptote. The line $y = 0$ or the x-axis is a horizontal asymptote.
6 Use the chain rule to differentiate the function.	$f(x) = \dfrac{24}{x^2 + 2x - 24}$ $= 24(x^2 + 2x - 24)^{-1}$ $f'(x) = -24(2x + 2)(x^2 + 2x - 24)^{-2}$ $f'(x) = -\dfrac{24(2x+2)}{(x^2 + 2x - 24)^2}$
7 Stationary points occur when the gradient is zero. Equate the gradient function to zero and solve for x.	$f'(x) = -\dfrac{24(2x+2)}{(x^2 + 2x - 24)^2} = 0$ $\Rightarrow 2x + 2 = 0$ $x = -1.$
8 Determine the y-value of the turning point. The second derivative will be complicated; however, we can determine the nature of the turning point.	Substitute $x = -1$: $f(-1) = \dfrac{24}{1 - 2 - 24} = -\dfrac{24}{25}$ Since the graph of $y = x^2 + 2x - 24$ has a local minimum at $x = -1$, the graph of $y = \dfrac{24}{x^2 + 2x - 24}$ has $\left(-1, -\frac{24}{25}\right)$ as a local maximum.
9 Using all of the above information, we can sketch the graph using an appropriate scale. Draw the asymptotes as dotted lines and label the graph with all the important features.	

10 From the graph we can state the domain and range. | The domain is $R\backslash\{-6, 4\}$ and the range is $\left(-\infty, -\frac{24}{25}\right] \cup (0, \infty)$.

Sketching rational functions

A **rational function** is defined as $f(x) = \dfrac{P(x)}{Q(x)}$ where both $P(x)$ and $Q(x)$ are polynomials. In this section we will sketch the graphs of simple rational functions where $P(x)$ is a linear, quadratic or cubic function and $Q(x)$ is a simple linear or quadratic function. To sketch the graphs of these types of rational functions, we consider the following main points.

Axis intercepts of rational functions

The curve may cross the x-axis where $y = 0$, that is, at values of x where the numerator $P(x) = 0$.

The curve may cross the y-axis where $x = 0$, that is $\dfrac{P(0)}{Q(0)}$.

Asymptotic behaviour of rational functions

A function is not defined when the denominator is zero. We thus have a vertical asymptote for each value of x where $Q(x) = 0$. A vertical asymptote is never crossed.

To obtain the equations of other asymptotes, if the degree of $P(x) \geq Q(x)$, we divide the denominator into the numerator to obtain an expression of the form $S(x) + \dfrac{R(x)}{Q(x)}$, so that $R(x)$ is of a lower degree than $Q(x)$.

Now as $x \to \infty$, then $y \to S(x)$, so $y = S(x)$ is the equation of an asymptote.

$S(x)$ may be of the form $y = c$ (a constant), so we get a horizontal line as an asymptote, or it may be of the form $y = ax + b$, in which case we have an oblique asymptote. It is also possible that $S(x) = ax^2$, so we can even get quadratics as asymptotes.

Stationary points of rational functions

It is easiest to find the gradient function from the divided form of these types of functions. Equate the gradient function to zero and solve for x.

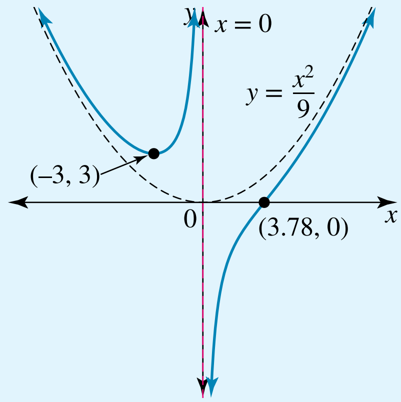
Note that these stationary points may be local minima or local maxima. We are not concerned with finding the inflection points of these types of graphs. However, we can find the second derivative and use it to determine the nature of the stationary points. Using this information we can sketch the curve. Sometimes graphs of these types may not cross the x- or y-axes, or they may not have any turning points.

WORKED EXAMPLE 24

Sketch the graph of $y = \dfrac{x^3 - 54}{9x}$. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any stationary points, and establish their nature.

THINK	WRITE/DRAW
1 Determine axis intercepts. First, find the x-intercepts.	The graph crosses the x-axis when the numerator is zero. Solve $x^3 - 54 = 0$. $x^3 = 54$; $x = \sqrt[3]{54} \approx 3.78$. The graph crosses the x-axis at $(\sqrt[3]{54}, 0)$ or $(3.78, 0)$.
2 Vertical asymptotes occur when the denominator is zero.	$y = \dfrac{x^3 - 54}{9x}$. The line $x = 0$ or the y-axis is a vertical asymptote.
3 Simplify the expression by dividing the denominator into the numerator.	$y = \dfrac{x^3 - 54}{9x} = \dfrac{x^3}{9x} - \dfrac{54}{9x} = \dfrac{x^2}{9} - \dfrac{6}{x}$
4 Find the equations of any other asymptotes.	As $x \to \infty$, $y \to \dfrac{x^2}{9}$ from below. As $x \to \infty$, $y \to \dfrac{x^2}{9}$ from above. The quadratic $y = \dfrac{x^2}{9}$ is an asymptote.
5 Find the first derivative by differentiating the divided form.	$y = \dfrac{x^2}{9} - 6x^{-1}$; $\dfrac{dy}{dx} = \dfrac{2x}{9} + 6x^{-2} = \dfrac{2x}{9} + \dfrac{6}{x^2}$
6 Stationary points occur when the gradient is zero. Equate the gradient function to zero and solve for x.	$\dfrac{dy}{dx} = \dfrac{2x}{9} + \dfrac{6}{x^2} = 0 \Rightarrow \dfrac{2x}{9} = -\dfrac{6}{x^2}$; $x^3 = -27$; $x = \sqrt[3]{-27} = -3$
7 Determine the y-value of the turning point.	When $x = -3$, $y = \dfrac{-27 - 54}{-27} = 3$.
8 Find the second derivative.	$\dfrac{dy}{dx} = \dfrac{2x}{9} + 6x^{-2}$; $\dfrac{d^2y}{dx^2} = \dfrac{2}{9} - 12x^{-3} = \dfrac{2}{9} - \dfrac{12}{x^3}$
9 Determine the sign of the second derivative to determine the nature of the turning point.	When $x = -3$, $\dfrac{d^2y}{dx^2} = \dfrac{2}{9} + \dfrac{12}{27} = \dfrac{2}{3} > 0$. The point $(-3, 3)$ is a local minimum turning point.

10 Using all of the above information, we can sketch the graph using an appropriate scale. Draw the asymptotes as dotted lines, and label the graph with all the important features.



EXERCISE 7.6 Curve sketching

PRACTISE

1 WE21 Sketch the graph of $y = x^3 - 4x^2 + 4x$ by finding the coordinates of all axis intercepts and stationary points, and establishing their nature. Find the coordinates of the point of inflection. Find and draw the tangent to the curve at the point of inflection.

2 Show that the graph of $y = x^3 - 2ax^2 + a^2x$, where $a \in R\backslash\{0\}$, crosses the x-axis at $(a, 0)$ and $(0, 0)$, has turning points at $(a, 0)$ and $\left(\frac{a}{3}, \frac{4a^3}{27}\right)$, and has an inflection point at $\left(\frac{2a}{3}, \frac{2a^3}{27}\right)$. Show that the equation of the tangent to the curve at the point of inflection is given by $y = \frac{8a^3}{27} - \frac{a^2x}{3}$.

3 WE22 Sketch the graph of $y = x^4 - 24x^2 + 80$ by finding the coordinates of all axis intercepts and stationary points, and establishing their nature. Find the coordinates of the point of inflection.

4 The graph of $y = ax^4 + bx^2 + c$ has one stationary point at $(0, -8)$ and points of inflection at $(\pm\sqrt{2}, 12)$. Determine the values of a, b and c. Find and determine the nature of any other stationary points.

5 WE23 Sketch the graph of $y = \frac{16}{16 - x^2}$. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any turning points. State the maximal domain and range.

6 Sketch the graph of $y = \frac{12}{x^2 - 4x - 12}$ by finding the equations of all straight line asymptotes. Find the coordinates of any axis intercepts and turning points. State the maximal domain and range.

7 WE24 Sketch the graph of $y = \frac{16 - x^3}{4x}$. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any stationary points, and establish their nature.

8 Sketch the graph of $y = \frac{x^2 + 9}{2x}$. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any stationary points, and establish their nature.

CONSOLIDATE

9 Sketch the graphs of each of the following by finding the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also find the coordinates of the point(s) of inflection.

a $y = x^3 - 27x$
b $y = 9x - x^3$
c $y = x^3 + 12x^2 + 36x$
d $y = -x^3 + 10x^2 - 25x$

10 Sketch the graphs of each of the following by finding the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also find the coordinates of the point(s) of inflection.

a $y = x^3 - 3x^2 - 9x - 5$
b $y = -x^3 + 9x^2 - 15x - 25$
c $y = x^3 - x^2 - 16x + 16$
d $y = -x^3 - 5x^2 + 8x + 12$

11 Sketch the graphs of each of the following by finding the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also find the coordinates of the point(s) of inflection.

a $y = x^4 - 4x^3$
b $y = 4x^2 - x^4$
c $y = x^4 + 4x^3 - 16x - 16 = (x - 2)(x + 2)^3$
d $y = x^4 - 6x^2 + 8x - 3 = (x - 1)^3(x + 3)$

12 Sketch the graphs of each of the following. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any turning points. State the domain and range.

a $y = \dfrac{18}{x^2 - 9}$
b $y = \dfrac{12}{x^2 + 4x}$
c $y = \dfrac{18}{8 + 2x - x^2}$
d $y = \dfrac{8}{7 - 6x - x^2}$

13 Sketch the graphs of each of the following. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any stationary points, and establish their nature.

a $y = \dfrac{x^2 + 4}{2x}$
b $y = \dfrac{x^2 - 4}{2x}$
c $y = \dfrac{x^3 - 8}{4x}$
d $y = \dfrac{x^3 + 16}{2x}$

14 Sketch the graphs of each of the following. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any stationary points, and establish their nature.

a $y = \dfrac{x^3 - 32}{2x^2}$
b $y = \dfrac{x^3 + 32}{2x^2}$
c $y = \dfrac{27 - 2x^3}{3x^2}$
d $y = \dfrac{-(x^3 + 4)}{x^2}$

15 Sketch the graphs of each of the following. State the equations of any asymptotes. Find the coordinates of any axis intercepts and any stationary points, and establish their nature.

a $y = \dfrac{x^4 - 81}{2x^2}$
b $y = \dfrac{x^4 + 256}{8x^2}$
c $y = \dfrac{16 - x^4}{x^2}$
d $y = \dfrac{-(x^4 + 16)}{2x^2}$

16 a The function $f(x) = x^3 + bx^2 + cx + d$ has a stationary point of inflection at (1, –2). Find the values of b, c and d.

b A cubic polynomial $y = x^3 + bx^2 + cx + d$ crosses the y-axis at $y = 5$ and has a point of inflection at (1, –21). Find the equation of the tangent at the point of inflection.

c The function $f(x) = x^3 + bx^2 + cx + d$ crosses the x-axis at $x = 3$ and has a point of inflection at $(2, -4)$. Find the values of b, c and d.

d A cubic polynomial $y = ax^3 + bx^3 + cx$ has a point of inflection at $x = -2$. The tangent at the point of inflection has the equation $y = 21x + 8$. Find the values of a, b and c.

17 a The curve $y = \dfrac{A}{x^2 + bx + 7}$ has a local maximum at $\left(-4, -\dfrac{4}{3}\right)$. Determine the values of A and b. State the equations of all straight-line asymptotes and the domain and range.

b The curve $y = \dfrac{A}{bx + c - x^2}$ has a local minimum at $(3, 2)$ and a vertical asymptote at $x = 8$. Determine the values of A, b and c. State the equations of all straight-line asymptotes and the domain and range.

c State conditions on k such that the graph of $y = \dfrac{1}{kx - 4x^2 - 25}$ has:

i two vertical asymptotes

ii only one vertical asymptote

iii no vertical asymptotes.

d Sketch the graph of $y = \dfrac{18}{x^2 + 9}$ by finding the coordinates of any stationary points and establishing their nature. Also find the coordinates of the points of inflection, and find the equation of the tangent to the curve at the point of inflection where $x > 0$.

18 a Show that the graph of $y = x^3 - a^2x$, where $a \in R\backslash\{0\}$, crosses the x-axis at $(\pm a, 0)$ and $(0, 0)$ and has turning points at $\left(\dfrac{\sqrt{3}a}{3}, -\dfrac{2a^3\sqrt{3}}{9}\right)$ and $\left(-\dfrac{\sqrt{3}a}{3}, \dfrac{2a^3\sqrt{3}}{9}\right)$. Show that $(0, 0)$ is also an inflection point.

b Show that the graph of $y = (x - a)^2(x - b)$, where $a, b \in R\backslash\{0\}$, has turning points at $(a, 0)$ and $\left(\dfrac{a + 2b}{3}, \dfrac{4(a - b)^3}{27}\right)$ and an inflection point at $\left(\dfrac{2a + b}{3}, \dfrac{2(a - b)^3}{27}\right)$.

c Show that the graph of $y = (x - a)^3(x - b)$, where $a, b \in R\backslash\{0\}$, has a stationary point of inflection at $(a, 0)$, a turning point at $\left(\dfrac{a + 3b}{4}, \dfrac{-27(a - b)^4}{256}\right)$ and an inflection point at $\left(\dfrac{a + b}{2}, \dfrac{-(a - b)^4}{16}\right)$.

MASTER

19 a Consider the function $f(x) = \dfrac{ax^2 + b}{x}$ where $a, b \in R\backslash\{0\}$. State the equations of all the asymptotes and show that if $ab > 0$, then the graph has two turning points at $\left(\sqrt{\dfrac{b}{a}}, 2\sqrt{ab}\right)$ and $\left(-\sqrt{\dfrac{b}{a}}, -2\sqrt{ab}\right)$ and does not cross the x-axis. However, if $ab < 0$, then there are no turning points, but the graph crosses the x-axis at $x = \pm\sqrt{-\dfrac{b}{a}}$.

b Consider the function $f(x) = \dfrac{ax^3 + b}{x^2}$ where $a, b \in R\backslash\{0\}$. State the equations of all the asymptotes and show that the graph has a turning point at $\left(\sqrt[3]{\dfrac{2b}{a}}, \dfrac{3\sqrt[3]{2a^2b}}{2}\right)$ and an x-intercept at $x = \sqrt[3]{-\dfrac{b}{a}}$.

c Consider the function $f(x) = \dfrac{ax^3 + b}{x}$ where $a, b \in R\backslash\{0\}$. State the equations of all the asymptotes and show that the graph has a turning point at $\left(\sqrt[3]{\dfrac{b}{2a}}, \dfrac{3\sqrt[3]{2ab^2}}{2}\right)$ and an x-intercept at $x = \sqrt[3]{-\dfrac{b}{a}}$.

d Consider the function $f(x) = \dfrac{ax^4 + b}{x^2}$ where $a, b \in R\backslash\{0\}$. State the equations of all the asymptotes and show that if $ab > 0$, the graph has two turning points at $\left(\pm\sqrt[4]{\dfrac{b}{a}}, 2\sqrt{ab}\right)$ and does not cross the x-axis. However, if $ab < 0$, then there are no turning points, but the graph crosses the x-axis at $x = \pm\sqrt[4]{-\dfrac{b}{a}}$.

20 a The amount of a certain drug, A milligrams, in the bloodstream at a time t hours after it is administered is given by

$$A(t) = 30te^{-\frac{t}{3}}, \text{ for } 0 \le t \le 12.$$

i Find the time when the amount of the drug, A, in the body is a maximum, and find the maximum amount.

ii Find the point of inflection on the graph of A versus t.

b The amount of a drug, B milligrams, in the bloodstream at a time t hours after it is administered is given by

$$B(t) = 15t^2e^{-\frac{t}{2}}, \text{ for } 0 \le t \le 12.$$

i Find the time when the amount of the drug, B, in the body is a maximum, and find the maximum amount.

ii Find the points of inflection correct to 2 decimal places on the graph of B versus t.

c Sketch the graphs of $A(t)$ and $B(t)$ on one set of axes and determine the percentage of the time when the amount of drug B is greater than drug A in the bloodstream.

7.7 Derivatives of inverse trigonometric functions

studyon

Units 3 & 4
AOS 3
Topic 1
Concept 1

Derivatives of inverse circular functions
Concept summary
Practice questions

Introduction

In this section, the derivatives of the inverse trigonometric functions are determined. These functions have already been studied in earlier topics; recall the definitions and alternative notations.

The inverse function $y = \sin^{-1}(x)$ or $y = \arcsin(x)$ has a domain of $[-1, 1]$ and a range of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. It is equivalent to $x = \sin(y)$ and $\sin(\sin^{-1}(x)) = x$ if $x \in [-1, 1]$, and $\sin^{-1}(\sin(x)) = x$ if $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

The inverse function $y = \cos^{-1}(x)$ or $y = \arccos(x)$ has a domain of $[-1, 1]$ and a range of $[0, \pi]$. It is equivalent to $x = \cos(y)$ and $\cos(\cos^{-1}(x)) = x$ if $x \in [-1, 1]$, and $\cos^{-1}(\cos(x)) = x$ if $x \in [0, \pi]$.

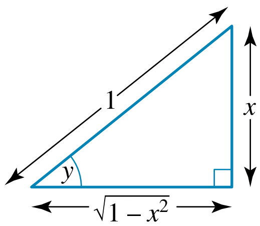
The inverse function $y = \tan^{-1}(x)$ or $y = \arctan(x)$ has a domain of R and a range of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It is equivalent to $x = \tan(y)$ and $\tan(\tan^{-1}(x))$ if $x \in R$, and $\tan^{-1}(\tan(x)) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The derivative of $\sin^{-1}(x)$

First we must determine the derivative of $\sin^{-1}(x)$.

Let $y = \sin^{-1}(x)$. From the definition of the inverse function, $x = \sin(y)$, so $\frac{dx}{dy} = \cos(y)$ and $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

However, we need to express the result back in terms of x.



Using $\sin(y) = \frac{x}{1}$, draw a right-angled triangle. Label the angle y with the opposite side length being x, so the hypotenuse has a length of 1 unit. From Pythagoras' theorem, the adjacent side length is $\sqrt{1 - x^2}$.

Thus, $\cos(y) = \sqrt{1 - x^2}$. It follows that $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1 - x^2}}$.

A better result is if $y = \sin^{-1}\left(\frac{x}{a}\right)$, where a is a positive real constant; then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$. Note however that the maximal domain of $y = \sin^{-1}\left(\frac{x}{a}\right)$ is $|x| \le a$, whereas the domain of the derivative is $|x| < a$. Although the function is defined at the endpoints, the gradient is not defined at the endpoints; for this reason the domain of the derivative is required.

WORKED EXAMPLE 25

Determine $\frac{dy}{dx}$ for each of the following, stating the maximal domain for which the derivative is defined.

a $y = \sin^{-1}\left(\frac{x}{2}\right)$

b $y = \sin^{-1}(2x)$

THINK	WRITE
a 1 State the function and the maximal domain.	**a** $y = \sin^{-1}\left(\frac{x}{2}\right)$ is defined for $\left\lvert\frac{x}{2}\right\rvert \le 1$; that is, $x \in [-2, 2]$
2 Use the result $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}$ with $a = 2$.	$\frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}$ for $x \in (-2, 2)$
b 1 State the function and the maximal domain.	**b** $y = \sin^{-1}(2x)$ is defined for $\lvert 2x\rvert \le 1$; that is, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

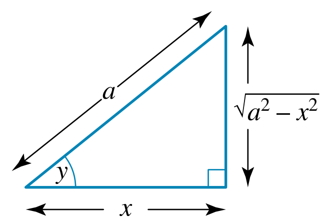
2 Express y in terms of u.	$y = \sin^{-1}(u)$ where $u = 2x$
3 Differentiate y with respect to u and u with respect to x.	$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = 2$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{2}{\sqrt{1-u^2}}$
5 Substitute back for u and state the final result.	$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$ for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

The derivative of $\cos^{-1}\left(\frac{x}{a}\right)$

The derivative of $\cos^{-1}\left(\frac{x}{a}\right)$ where a is a real positive constant is required.

Let $y = \cos^{-1}\left(\frac{x}{a}\right)$. From the definition of the inverse function, $\frac{x}{a} = \cos(y)$, so $x = a\cos(y)$ and $\frac{dx}{dy} = -a\sin(y)$, and $\frac{dy}{dx} = \frac{-1}{a\sin(y)}$.

We need to express the result back in terms of x. Using $\cos(y) = \frac{x}{a}$, draw a right-angled triangle. Label the angle y, with the adjacent side length being x and the length of the hypotenuse being a. From Pythagoras' theorem, the opposite side length is $\sqrt{a^2 - x^2}$.



Therefore, $\sin(y) = \frac{\sqrt{a^2 - x^2}}{a}$.

It follows that $\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - x^2}}$,

and if $a = 1$, we obtain $\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1 - x^2}}$.

WORKED EXAMPLE 26

Determine $\frac{dy}{dx}$ for each of the following, stating the maximal domain for which the derivative is defined.

a $y = \cos^{-1}\left(\frac{2x}{3}\right)$ **b** $y = \cos^{-1}\left(\frac{5x-2}{6}\right)$

THINK	WRITE
a 1 State the function and the maximal domain.	**a** $y = \cos^{-1}\left(\frac{2x}{3}\right)$ is defined for $\left\lvert\frac{2x}{3}\right\rvert \le 1$; that is, $-1 \le \frac{2x}{3} \le 1$ or $x \in \left[-\frac{3}{2}, \frac{3}{2}\right]$
2 Express y in terms of u.	$y = \cos^{-1}\left(\frac{u}{3}\right)$ where $u = 2x$

3 Differentiate y with respect to u using $\frac{d}{du}\left(\cos^{-1}\left(\frac{u}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - u^2}}$ with $a = 3$, and differentiate u with respect to x.

$\frac{dy}{du} = \frac{-1}{\sqrt{9 - u^2}}$ and $\frac{du}{dx} = 2$

4 Find $\frac{dy}{dx}$ using the chain rule.

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-2}{\sqrt{9 - u^2}}$

5 Substitute back for u and state the final result.

$\frac{dy}{dx} = \frac{-2}{\sqrt{9 - 4x^2}}$ for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

b 1 State the function and solve the inequality to find the maximal domain of the function.

b $y = \cos^{-1}\left(\frac{5x - 2}{6}\right)$ is defined for $\left|\frac{5x - 2}{6}\right| \le 1$

$-1 \le \frac{5x - 2}{6} \le 1$

$-6 \le 5x - 2 \le 6$

$-4 \le 5x \le 8$

$-\frac{4}{5} \le x \le \frac{8}{5}$ or $x \in \left[-\frac{4}{5}, \frac{8}{5}\right]$

2 Express y in terms of u.

$y = \cos^{-1}\left(\frac{u}{6}\right)$ where $u = 5x - 2$

3 Differentiate y with respect to u and u with respect to x.

$\frac{dy}{du} = \frac{-1}{\sqrt{36 - u^2}}$ and $\frac{du}{dx} = 5$

4 Find $\frac{dy}{dx}$ using the chain rule.

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-5}{\sqrt{36 - u^2}}$

5 Substitute back for u.

$\frac{dy}{dx} = \frac{-5}{\sqrt{36 - (5x - 2)^2}}$

6 Simplify the denominator using the difference of two squares.

$\frac{dy}{dx} = \frac{-5}{\sqrt{(6 + (5x - 2))(6 - (5x - 2))}}$

7 State the final result.

$\frac{dy}{dx} = \frac{-5}{\sqrt{(4 + 5x)(8 - 5x)}}$ for $x \in \left(-\frac{4}{5}, \frac{8}{5}\right)$

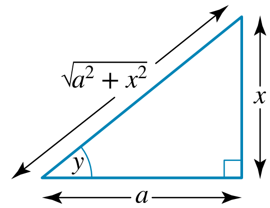
The derivative of $\tan^{-1}\left(\frac{x}{a}\right)$

We also need to determine the derivative of $\tan^{-1}\left(\frac{x}{a}\right)$, where a is a real positive constant.

Let $y = \tan^{-1}\left(\frac{x}{a}\right)$. From the definition of the inverse function, $\frac{x}{a} = \tan(y)$, so $x = a\tan(y)$, $\frac{dx}{dy} = a\sec^2(y)$, and $\frac{dy}{dx} = \frac{1}{a\sec^2(y)}$.

However, we need to express the result in terms of x.

Using $\tan(y) = \frac{x}{a}$, draw a right-angled triangle. Label the angle y, with the opposite side length being x and the adjacent side length being a. From Pythagoras' theorem, the length of the hypotenuse is $\sqrt{a^2 + x^2}$.



We know that $\sec^2(y) = \frac{1}{\cos^2(y)}$, and from above, $\cos(y) = \frac{a}{\sqrt{a^2 + x^2}}$.

It follows that $\frac{dy}{dx} = \frac{\cos^2(y)}{a} = \frac{1}{a}\left(\frac{a}{\sqrt{a^2 + x^2}}\right)^2$.

Thus, $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}$, and if $a = 1$, then $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1 + x^2}$.

The domain is defined for all $x \in R$.

WORKED EXAMPLE 27

Find $\frac{dy}{dx}$ for each of the following.

a $y = \tan^{-1}\left(\frac{3x}{4}\right)$ **b** $y = \tan^{-1}\left(\frac{5x + 4}{7}\right)$

THINK	WRITE
a 1 State the function.	**a** $y = \tan^{-1}\left(\frac{3x}{4}\right)$
2 Express y in terms of u.	$y = \tan^{-1}\left(\frac{u}{4}\right)$ where $u = 3x$
3 Differentiate y with respect to u and u with respect to x, using $\frac{d}{du}\left(\tan^{-1}\left(\frac{u}{a}\right)\right) = \frac{a}{a^2 + u^2}$ with $a = 4$.	$\frac{dy}{du} = \frac{4}{16 + u^2}$ and $\frac{du}{dx} = 3$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{12}{16 + u^2}$
5 Substitute back for u and state the final result.	$\frac{dy}{dx} = \frac{12}{16 + 9x^2}$
b 1 State the function.	**b** $y = \tan^{-1}\left(\frac{5x + 4}{7}\right)$
2 Express y in terms of u.	$y = \tan^{-1}\left(\frac{u}{7}\right)$ where $u = 5x + 4$
3 Differentiate y with respect to u and u with respect to x, using $\frac{d}{du}\left(\tan^{-1}\left(\frac{u}{a}\right)\right) = \frac{a}{a^2 + u^2}$ with $a = 7$.	$\frac{dy}{du} = \frac{7}{49 + u^2}$ and $\frac{du}{dx} = 5$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{35}{49 + u^2}$

5 Substitute back for u.

$$\frac{dy}{dx} = \frac{35}{49 + (5x + 4)^2}$$

6 Expand the denominator, and simplify and take out common factors.

$$\frac{dy}{dx} = \frac{35}{49 + 25x^2 + 40x + 16}$$
$$= \frac{35}{25x^2 + 40x + 65}$$
$$= \frac{35}{5(5x^2 + 8x + 13)}$$

7 State the final result.

$$\frac{dy}{dx} = \frac{7}{5x^2 + 8x + 13}$$

Finding second derivatives

Recall that the second derivative $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is also the rate of change of the gradient function.

WORKED EXAMPLE 28

Find $\frac{d^2y}{dx^2}$ if $y = \sin^{-1}\left(\frac{3x}{5}\right)$.

THINK / **WRITE**

1 State the function and the maximal domain.

$y = \sin^{-1}\left(\frac{3x}{5}\right)$ is defined for $\left|\frac{3x}{5}\right| \leq 1$; that is $-1 \leq \frac{3x}{5} \leq 1$ or $x \in \left[-\frac{5}{3}, \frac{5}{3}\right]$.

2 Express y in terms of u.

$y = \sin^{-1}\left(\frac{u}{5}\right)$ where $u = 3x$

3 Differentiate y with respect to u and u with respect to u.

$\frac{dy}{du} = \frac{1}{\sqrt{25 - u^2}}$ and $\frac{du}{dx} = 3$

4 Find $\frac{dy}{dx}$ using the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{3}{\sqrt{25 - u^2}}$$

5 Substitute back for u.

$$\frac{dy}{dx} = \frac{3}{\sqrt{25 - 9x^2}}$$

6 Write in index notation.

$$\frac{dy}{dx} = 3(25 - 9x^2)^{-\frac{1}{2}}$$

7 Differentiate again using the chain rule.

$$\frac{d^2y}{dx^2} = -3 \times \frac{1}{2} \times -18x(25 - 9x^2)^{-\frac{3}{2}}$$

8 Simplify the terms.

$$\frac{d^2y}{dx^2} = \frac{27x}{(25 - 9x^2)^{\frac{3}{2}}}$$

9 State the final result in simplest form and the domain.

$$\frac{d^2y}{dx^2} = \frac{27x}{\sqrt{(25 - 9x^2)^3}} \text{ for } x \in \left(-\frac{5}{3}, \frac{5}{3}\right)$$

Further examples

Because $\sec(x) = \dfrac{1}{\cos(x)}$, $\operatorname{cosec}(x) = \dfrac{1}{\sin(x)}$ and $\cot(x) = \dfrac{1}{\tan(x)}$, it follows that

$\sec^{-1}(x) = \cos^{-1}\left(\dfrac{1}{x}\right)$, $\operatorname{cosec}^{-1}(x) = \sin^{-1}\left(\dfrac{1}{x}\right)$ and $\cot^{-1}(x) = \tan^{-1}\left(\dfrac{1}{x}\right)$.

WORKED EXAMPLE 29

Find $\dfrac{dy}{dx}$ if $y = \cos^{-1}\left(\dfrac{3}{x}\right)$, stating the maximal domain for which the derivative is defined.

THINK	WRITE
1 State the function and the maximal domain.	$y = \cos^{-1}\left(\dfrac{3}{x}\right)$ for $\left\lvert\dfrac{3}{x}\right\rvert \le 1$
2 Solve the inequality to find the maximal domain of the function.	$-1 \le \dfrac{3}{x} \le 1$ is equivalent to $\dfrac{3}{x} \le 1 \quad \Rightarrow \dfrac{x}{3} \ge 1 \quad \Rightarrow x \ge 3$ $\dfrac{3}{x} \ge -1 \Rightarrow \dfrac{x}{3} \le -1 \Rightarrow x \le -3$
3 Express y in terms of u.	$y = \cos^{-1}(u)$ where $u = \dfrac{3}{x} = 3x^{-1}$
4 Differentiate y with respect to u and u with respect to x.	$\dfrac{dy}{du} = \dfrac{-1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dx} = -3x^{-2} = -\dfrac{3}{x^2}$
5 Find $\dfrac{dy}{dx}$ using the chain rule.	$\dfrac{dy}{dx} = \dfrac{3}{x^2\sqrt{1-u^2}}$
6 Substitute back for u.	$\dfrac{dy}{dx} = \dfrac{3}{x^2\sqrt{1-\dfrac{9}{x^2}}}$
7 Simplify by taking a common denominator in the square root in the denominator.	$\dfrac{dy}{dx} = \dfrac{3}{x^2\sqrt{\dfrac{x^2-9}{x^2}}}$
8 Simplify, noting that $\sqrt{x^2} = \lvert x\rvert$.	$\dfrac{dy}{dx} = \dfrac{3\lvert x\rvert}{x^2\sqrt{x^2-9}}$
9 State the final result and the domain.	$\dfrac{dy}{dx} = \dfrac{3}{\lvert x\rvert\sqrt{x^2-9}}$ for $x < -3$ or $x > 3$.

EXERCISE 7.7 Derivatives of inverse trigonometric functions

PRACTISE

1 WE25 Determine $\dfrac{dy}{dx}$ for each of the following, stating the maximal domain for which the derivative is defined.

a $y = \sin^{-1}\left(\dfrac{x}{5}\right)$

b $y = \sin^{-1}(5x)$

2 Given $f(x) = \sin^{-1}(4x)$, find $f'\left(\frac{1}{8}\right)$.

3 WE26 Determine $\frac{dy}{dx}$ for each of the following, stating the maximal domain for which the derivative is defined.

a $y = \cos^{-1}\left(\frac{3x}{4}\right)$

b $y = \cos^{-1}\left(\frac{2x-3}{5}\right)$

4 Determine the gradient of the curve $y = \cos^{-1}\left(\frac{3x-4}{5}\right)$ at the point where $x = \frac{4}{3}$.

5 WE27 Find $\frac{dy}{dx}$ for each of the following.

a $y = \tan^{-1}(4x)$

b $y = \tan^{-1}\left(\frac{2x-3}{5}\right)$

6 Find the gradient of the curve $y = \tan^{-1}\left(\frac{3x-5}{7}\right)$ at the point where $x = 4$.

7 WE28 Find $\frac{d^2y}{dx^2}$ if $y = \cos^{-1}\left(\frac{2x}{3}\right)$.

8 Find $\frac{d^2y}{dx^2}$ if $y = \tan^{-1}\left(\frac{4x}{3}\right)$.

9 WE29 Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left(\frac{2}{x}\right)$, stating the maximal domain for which the derivative is defined.

10 Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left(\frac{4}{\sqrt{x}}\right)$, stating the maximal domain for which the derivative is defined.

CONSOLIDATE

11 Determine the derivative of each of the following, stating the maximal domain.

a $\sin^{-1}\left(\frac{x}{3}\right)$

b $\sin^{-1}(3x)$

c $\sin^{-1}\left(\frac{4x}{3}\right)$

d $\sin^{-1}\left(\frac{4x+3}{5}\right)$

12 Determine the derivative of each of the following, stating the maximal domain.

a $\cos^{-1}\left(\frac{x}{4}\right)$

b $\cos^{-1}(4x)$

c $\cos^{-1}\left(\frac{3x}{4}\right)$

d $\cos^{-1}\left(\frac{3x+5}{7}\right)$

13 Determine the derivative of each of the following.

a $\tan^{-1}\left(\frac{x}{6}\right)$

b $\tan^{-1}(6x)$

c $\tan^{-1}\left(\frac{5x}{6}\right)$

d $\tan^{-1}\left(\frac{6x+5}{4}\right)$

14 Determine the derivative of each of the following, stating the maximal domain.

a $\sin^{-1}\left(\frac{\sqrt{x}}{3}\right)$

b $\sin^{-1}\left(\frac{3}{4x}\right)$

c $\cos^{-1}\left(\frac{e^{2x}}{4}\right)$

d $\cos^{-1}\left(\frac{4}{3x}\right)$

e $\tan^{-1}\left(\frac{x^2}{3}\right)$

f $\tan^{-1}\left(\frac{6}{5x}\right)$

15 Find $\frac{d^2y}{dx^2}$ if:

a $y = \sin^{-1}\left(\frac{5x}{4}\right)$

b $y = \cos^{-1}\left(\frac{6x}{5}\right)$

c $y = \tan^{-1}\left(\frac{7x}{6}\right)$.

16 For each of the following:

i sketch the graph of the function, and state the domain and the range
ii find the exact equation of the tangent to the curve at the point indicated.

a $y = 4\sin^{-1}\left(\frac{3x}{4}\right), x = \frac{2}{3}$

b $y = 6\cos^{-1}\left(\frac{4x}{5}\right), x = \frac{5\sqrt{2}}{8}$

c $y = 8\tan^{-1}\left(\frac{5x}{6}\right), x = \frac{6}{5}$

17 a If $f(x) = x^2\sin^{-1}\left(\frac{x}{4}\right)$, find $f'(2)$.

b If $f(x) = x\cos^{-1}\left(\frac{x}{6}\right)$, find $f'(3)$.

c If $f(x) = x^3\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$, find $f'(1)$.

18 Consider the function $T(x) = \tan^{-1}\left(\frac{x}{3}\right) + \tan^{-1}\left(\frac{3}{x}\right)$.

a Find the values of $T(3)$ and $T(-3)$.
b Find $T'(x)$. What can you deduce about the function $T(x)$?
c State the domain and range of $T(x)$, and sketch its graph.

19 If a and b are positive constants, verify the following.

a $\frac{d}{dx}\left[\sin^{-1}\left(\frac{bx}{a}\right)\right] = \frac{b}{\sqrt{a^2 - b^2x^2}}$

b $\frac{d^2}{dx^2}\left[\sin^{-1}\left(\frac{bx}{a}\right)\right] = \frac{b^3x}{\sqrt{(a^2 - b^2x^2)^3}}$

20 If a and b are positive constants, verify the following.

a $\frac{d}{dx}\left[\cos^{-1}\left(\frac{bx}{a}\right)\right] = \frac{-b}{\sqrt{a^2 - b^2x^2}}$

b $\frac{d^2}{dx^2}\left[\cos^{-1}\left(\frac{bx}{a}\right)\right] = \frac{-b^3}{\sqrt{(a^2 - b^2x^2)^3}}$

MASTER

21 If a and b are positive constants, verify the following.

a $\frac{d}{dx}\left[\tan^{-1}\left(\frac{bx}{a}\right)\right] = \frac{ab}{a^2 + b^2x^2}$

b $\frac{d^2}{dx^2}\left[\tan^{-1}\left(\frac{bx}{a}\right)\right] = \frac{-2ab^3x}{(a^2 + b^2x^2)^2}$

22 If a and b are positive constants, verify the following.

a $\frac{d}{dx}\left[\sin^{-1}\left(\frac{a}{bx}\right)\right] = \frac{-a}{|x|\sqrt{b^2x^2 - a^2}}$

b $\frac{d}{dx}\left[\cos^{-1}\left(\frac{a}{bx}\right)\right] = \frac{a}{|x|\sqrt{b^2x^2 - a^2}}$

c $\frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{bx}\right)\right] = \frac{-ab}{a^2 + b^2x^2}$

7.8 Related rate problems

Introduction

studyon

Units 3 & 4
AOS 3
Topic 1
Concept 3

Related rates of change
Concept summary
Practice questions

When two or more quantities vary with time and are related by some condition, their rates of change are also related. The steps involved in solving these related rate problems are listed below.

1. Define the variables
2. Write down the rate that is provided in the question.
3. Establish the relationship between the variables.
4. Write down the rate that needs to found.
5. Use a chain rule or implicit differentiation to relate the variables.

WORKED EXAMPLE 30

A circular metal plate is being heated and its radius is increasing at a rate of 2 mm/s. Find the rate at which the area of the plate is increasing when the radius is 30 millimetres.

THINK	WRITE
1 Define the variables.	Let r mm be the radius of the metal at time t seconds. Let A mm^2 be the area of the plate at time t seconds.
2 The radius is increasing at a rate of 2 mm/s. Note that the units also help to find the given rate.	$\frac{dr}{dt} = 2$ mm/s.
3 The plate is circular. Write the formula for the area of a circle.	$A = \pi r^2$.
4 Find the rate at which the area of the plate is increasing.	$\frac{dA}{dt} = ?$ and evaluate this rate when $r = 30$.
5 Form a chain rule for the required rate in terms of the given variables.	$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$
6 Since one rate, $\frac{dr}{dt}$, is known, we need to find $\frac{dA}{dr}$.	$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$
7 Substitute the given rates into the required equations.	$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ $= 2\pi r \times 2$
8 Evaluate the required rate when $r = 30$.	$\frac{dA}{dt} = 4\pi r$ when $r = 30$
9 State the final result with the required units, leaving the answer in terms of π.	$\left.\frac{dA}{dr}\right\|_{r=30} = 120\pi$ mm^2/s The area is increasing at 120π mm^2/s.

Decreasing rates

If a quantity has a rate decreasing with respect to time, then the required rate is given as a negative quantity.

WORKED EXAMPLE 31

A spherical balloon has a hole in it, and the balloon's volume is decreasing at a rate of 2 cm^3/s. At what rate is the radius changing when the radius of the balloon is 4 cm?

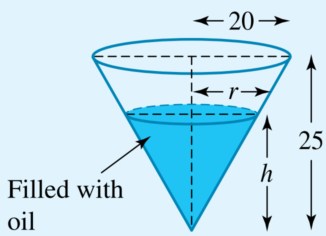
THINK	WRITE
1 Define the variables.	Let r cm be the radius of the balloon at time t seconds. Let V cm^3 be the volume of the balloon at time t seconds.
2 The volume is decreasing at a rate of 2 cm^3/s. Notice that the units also help find the given rate.	$\frac{dV}{dt} = -2\ cm^3/s$
3 The ballon is spherical. Write the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$
4 We need to find the rate at which the radius changing.	$\frac{dr}{dt} = ?$ and evaluate this rate when $r = 4$.
5 Form a chain rule for the required rate in terms of the given variables.	$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$
6 Since one rate, $\frac{dV}{dt}$, is known, we need to find $\frac{dr}{dV}$.	Since $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$ and $\frac{dr}{dV} = 1 / \frac{dV}{dr} = \frac{1}{4\pi r^2}$
7 Substitute the given rates into the required equations.	$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ $= \frac{1}{4\pi r^2} \times -2$ $= -\frac{1}{2\pi r^2}$
8 Evaluate the required rate when $r = 4$.	When $r = 4$, $\left.\frac{dr}{dt}\right\vert_{r=4} = -\frac{1}{32\pi}$.
9 State the final result in the required units, leaving the answer in terms of π.	The radius is decreasing at a rate of $\frac{1}{32\pi}$ cm/s.

Relating the variables

It is often necessary to express a required expression in terms of only one variable instead of two. This can be achieved by finding relationships between the variables.

WORKED EXAMPLE 32

A conical funnel has a height of 25 centimetres and a radius of 20 centimetres. It is positioned so that its axis is vertical and its vertex is downwards. Oil leaks out through an opening in the vertex at a rate of 4 cubic centimetres per second. Find the rate at which the oil level is falling when the height of the oil in the funnel is 5 centimetres. (Ignore the cylindrical section of the funnel.)

THINK	WRITE/DRAW
1 Define the variables.	Let r cm be the radius of the oil in the funnel. Let h cm the height of the oil in the funnel.
2 Write the rate given in the question.	$\frac{dV}{dt} = -4 \text{ cm}^3/\text{s}$
3 Write which rate is required to be found.	$\frac{dh}{dt} = ?$ when $h = 5$
4 Make up a chain rule for the required rate, in terms of the given variables.	$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$
5 Find the relationship between the variables. The variables r and h change; however, the height and radius of the funnel are constant, and this can be used to find a relationship between h and r.	 $V = \frac{1}{3}\pi r^2 h$ Using similar triangles, $\frac{20}{25} = \frac{r}{h}$ $r = \frac{4h}{5}$
6 We need to express the volume in terms of h only.	Substitute into $V = \frac{1}{3}\pi r^2 h$: $V = \frac{1}{3}\pi\left(\frac{4h}{5}\right)^2 h$ $= \frac{16\pi h^3}{75}$
7 Find the rate to substitute into the chain rule.	$\frac{dV}{dh} = \frac{16\pi h^2}{25}$ $\frac{dh}{dV} = \frac{25}{16\pi h^2}$
8 Substitute for the required rates.	$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{25}{16\pi h^2} \times -4$
9 Evaluate the required rate when $h = 5$.	$\left.\frac{dh}{dt}\right\|_{h=5} = \frac{25}{16\pi 5^2} \times -4 = -\frac{1}{4\pi}$
10 State the final result with the required units, leaving the answer in terms of π.	The height is decreasing at a rate of $\frac{1}{4\pi}$ cm/s.

Determining the required variables

An alternative method to solving related rate problems is to use implicit differentiation.

WORKED EXAMPLE 33

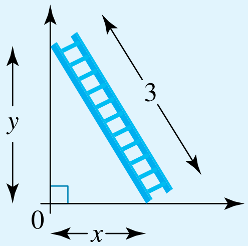
A ladder 3 metres long has its top end resting against a vertical wall and its lower end on horizontal ground. The top end of the ladder is slipping down at a constant speed of 0.1 metres per second. Find the rate at which the lower end is moving away from the wall when the lower end is 1 metre from the wall.

THINK | **WRITE/DRAW**

Method 1: Using the chain rule

1 Define the variables.

Let x metres be the distance of the base of the ladder from the wall, and let y metres be the distance of the top of the ladder from the ground.



2 The top end of the ladder is slipping down at a constant speed of 0.1 m/s.

$\frac{dy}{dt} = -0.1$ m/s

3 Apply Pythagoras' theorem.

$x^2 + y^2 = 3^2 = 9$

4 We need to find the rate at which the lower end is moving away from the wall when the lower end is 1 m from the wall.

$\frac{dx}{dt} = ?$ when $x = 1$.

5 Construct a chain rule for the required rate in terms of the given variables.

$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt}$

6 Express y in terms of x.

$x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9 - x^2}$

$y = \sqrt{9 - x^2}$ since $y > 0$

7 Since one rate, $\frac{dy}{dt}$, is known, we need to find $\frac{dx}{dy}$.

$\frac{dy}{dx} = -\frac{x}{\sqrt{9 - x^2}}$

$\frac{dx}{dy} = -\frac{\sqrt{9 - x^2}}{x}$

8 Substitute the given rates into the required equation.

$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt}$

$= -\frac{\sqrt{9 - x^2}}{x} \times -0.1$

9 Evaluate the required rate when $x = 1$.

$$\left.\frac{dx}{dt}\right|_{x=1} = \frac{\sqrt{8}}{10} = \frac{2\sqrt{2}}{10}$$

10 State the final result with the required units, giving an exact answer.

$$\left.\frac{dx}{dt}\right|_{x=1} = \frac{\sqrt{2}}{5} \text{ m/s}$$

The lower end is moving away from the wall at a rate of $\frac{\sqrt{2}}{5}$ m/s.

Method 2: Using implicit differentiation

The first 4 steps are identical to method 1 above. From this point, we use implicit differentiation to find the required rate.

5 Take $\frac{d}{dt}(\ )$ of each term in turn.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(9)$$

6 Use implicit differentiation.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

7 Rearrange to make the required rate the subject.

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

8 Find the appropriate values.

When $x = 1$, $y = \sqrt{8}$ and $\frac{dy}{dt} = -0.1 = -\frac{1}{10}$

9 Substitute in the appropriate values of x and y.

$$\left.\frac{dx}{dt}\right|_{x=1} = \frac{\sqrt{8}}{10} = \frac{2\sqrt{2}}{10}$$

10 State the final result as before.

$$\left.\frac{dx}{dt}\right|_{x=1} = \frac{\sqrt{2}}{5} \text{ m/s}$$

The lower end is moving away from the wall at a rate of $\frac{\sqrt{2}}{5}$ m/s.

EXERCISE 7.8 Related rate problems

PRACTISE

1 WE30 A circular oil slick is expanding so that its radius increases at a rate of 0.5 m/s. Find the rate at which the area of the oil slick is increasing when the radius of the slick is 20 metres.

2 A circular disc is expanding so that its area increases at a rate of 40π cm^2/s. Find the rate at which the radius of the disc is increasing when the radius is 10 cm.

3 WE31 A spherical basketball has a hole in it and its volume is decreasing at a rate of 6 cm^3/s. At what rate is the radius changing when the radius of the basketball is 6 cm?

4 A metal ball is dissolving in an acid bath. Its radius is decreasing at a rate of 3 cm/s. At what rate is the ball's surface area changing when the radius of the ball is 2 cm?

5 WE32 A conical vase has a height of 40 cm and a radius of 8 cm. The axis of the vase is vertical and its vertex is downwards. Initially it is filled with water which leaks out through a small hole in the vertex at a rate of 6 cm^3/s. Find the rate at which the water level is falling when the height of the water is 16 cm.

6 A cone is such that its radius is always equal to half its height. If the radius is decreasing at a rate of 2 cm/s, find the rate at which the volume of the cone is decreasing when the radius is 4 cm.

7 WE33 A ladder 5 metres long has its top end resting against a vertical wall and its lower end on horizontal ground. The bottom end of the ladder is pushed closer to the wall at a speed of 0.3 metres per second. Find the rate at which the top end of the ladder is moving up the wall when the lower end is 3 metres from the wall.

8 A kite is 30 metres above the ground and is moving horizontally away at a speed of 2 metres per second from the boy who is flying it. When the length of the string is 50 metres, at what rate is the string being released?

CONSOLIDATE

9 a A square has its sides increasing at a rate of 2 centimetres per second. Find the rate at which the area is increasing when the sides are 4 centimetres long.

b A stone is dropped into a lake, sending out concentric circular ripples. The area of the disturbed water region increases at a rate of 2 m^2/s. Find the rate at which the radius of the outermost ripple is increasing when its radius is 4 metres.

10 A spherical bubble is blown so that its radius is increasing at a constant rate of 2 millimetres per second. When its radius is 10 millimetres, find the rate at which:

a its volume is increasing

b its surface area is increasing.

11 A mothball has its radius decreasing at a constant rate of 0.2 millimetres per week. Assume it remains spherical.

a Show that the volume is decreasing at a rate that is proportional to its surface area.

b If its initial radius is 30 millimetres, find how long it takes to disappear.

12 The sides of an equilateral triangle are increasing at a rate of 2 centimetres per second. When the sides are $2\sqrt{3}$ centimetres, find the rate at which:

a its area is increasing

b its height is increasing.

13 a Show that the formula for the volume, V, of a right circular cone with height h is given by $V = \frac{1}{3}\pi h^3 \tan^2(\alpha)$, where α is the semi-vertex angle.

b Falling sand forms a heap in the shape of a right circular cone whose semi-vertex angle is 71.57°. If its height is increasing at 2 centimetres per second when the heap is 5 centimetres high, find the rate at which its volume is increasing.

14 a The volume, V cm^3, of water in a hemispherical bowl of radius r cm when the depth of the water is h cm is given by $V = \frac{1}{3}\pi h^2(3r - h)$. A hemispherical bowl of radius 10 centimetres is being filled with water at a constant rate of 3 cubic centimetres per second. At what rate is the depth of the water increasing when the depth is 5 cm?

b A drinking glass is in the shape of a truncated right circular cone. When the glass is filled to a depth of h cm, the volume of liquid in the glass, V cm^3, is given by $V = \dfrac{\pi}{432}(h^3 + 108h^2 + 388h)$. Lemonade is leaking out from the glass at a rate of 7 cm^3/s. Find the rate at which the depth of the lemonade is falling when the depth is 6 cm.

15 a A rubber flotation device is being pulled into a wharf by a rope at a speed of 26 metres per minute. The rope is attached to a point on the wharf 1 metre vertically above the dingy. At what rate is the rope being drawn in when the dingy is 10 metres from the wharf?

b A car approaches the ground level of a 30-metre-tall building at a speed of 54 kilometres per hour. Find the rate of change of the distance from the car to the top of the building when it is 40 metres from the foot of the building.

16 The distance, q cm, between the image of an object and a certain lens in terms of p cm, the distance of the object from the lens, is given by $q = \dfrac{10p}{p - 10}$.

a Show that the rate of change of distance that an image is from the lens with respect to the distance of the object from the lens is given by $\dfrac{dq}{dp} = \dfrac{-100}{(p - 10)^2}$.

b If the object distance is increasing at a rate 0.2 cm/s, how fast is the image distance changing, when the distance from the object is 12 cm?

17 a When a gas expands without a change of temperature, the pressure P and volume V are given by the relationship $PV^{1.4} = C$, where C is a constant. At a certain instant, the pressure is 1.01×10^5 pascals and the volume is 22.4×10^{-3} cubic metres. The volume is increasing at a rate of 0.005 cubic metres per second. Find the rate at which the pressure is changing at this instant.

b The pressure P and volume V of a certain fixed mass of gas during an adiabatic expansion are connected by the law $PV^n = C$, where n and C are constants. Show that the time rate of change of volume satisfies $\dfrac{dV}{dt} = -\dfrac{V}{nP}\dfrac{dP}{Dt}$.

18 a A jet aircraft is flying horizontally at a speed of 300 km/h at an altitude of 1 km. It passes directly over a radar tracking station located at ground level. Find the rate in degrees per second at which the radar beam to the aircraft is turning when the jet is at a horizontal distance of 30 kilometres from the station.

b A helicopter is flying horizontally at a constant height of 300 metres. It passes directly over a light source located at ground level. The light source is always directed at the helicopter. If the helicopter is flying at 108 kilometres per hour, find the rate in degrees per second at which the light source to the helicopter is turning when the helicopter is at a horizontal distance of 40 kilometres from the light source.

MASTER

19 A man 2 metres tall is walking at 1.5 m/s. He passes under a light source 6 metres above the ground. Find:

a the rate at which his shadow is lengthening

b the speed at which the end point of his shadow is increasing

c the rate at which his head is receding from the light source when he is 8 metres from the light.

20 **a** A train leaves a point O and travels south at a constant speed of 30 km/h. A car that was initially 90 km east of O travels west at a constant speed of 40 km/h. Let s be the distance in km of the car from the train after a time of t hours.

i Show that $s = 10\sqrt{25t^2 - 72t + 81}$.

ii Find when the train and the car are closest, and find the closest distance in kilometres between the car and the train.

iii Find the rate at which the distance between the train and the car is changing after 1 hour.

b Two railway tracks intersect at 60°. One train is 100 km from the junction and moves towards it at 80 km/h, while another train is 120 km from the junction and moves towards the junction at 90 km/h.

i Show that the trains do not collide.

ii Find the rate at which the trains are approaching each other after 1 hour.

7.9 Review

eBook*plus* ONLINE ONLY

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyON

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

7 Answers

EXERCISE 7.2

1 a $\frac{3}{16}$

b $30\sin^2(2x)\cos(2x)$

2 a $\dfrac{-2}{(2x+5)^2}$ **b** $-6\sqrt{6}$

3 a $\dfrac{\pi^2}{36}$ **b** $2\cos(2x)e^{\sin(2x)}$

4 a $\dfrac{\cos(\sqrt{x})}{2\sqrt{x}}$ **b** $-\sqrt{3e}$

5 a $x^2(3\cos(4x) - 4x\sin(4x))$

b $x^3e^{-3x}(4-3x)$

6 a $-e^{-3x}(2\sin(2x) + 3\cos(2x))$

b $-12e^{-4}$

7 a $-\dfrac{9}{2x^4}(x\sin(3x) + \cos(3x))$

b $\dfrac{2x+9}{\sqrt{(4x+9)^3}}$

8 a $\dfrac{-(1+2x)e^{-2x}}{3x^2}$ **b** $\dfrac{-120x(3x^2+5)}{(3x^2-5)^3}$

9 a $8\tan^3\left(\dfrac{x}{3}\right)\sec^2\left(\dfrac{x}{3}\right)$ **b** $\dfrac{\pi^3(\pi+8)}{2}$

10 a $\dfrac{27(4\pi - 3\sqrt{3})}{\pi^2}$ **b** $-\dfrac{5\pi^2}{54}$

11 a $\dfrac{1}{2}\cot\left(\dfrac{x}{2}\right)$ **b** $\dfrac{-144x}{16x^4-81}$

12 a $-\frac{4}{13}$ **b** $-\dfrac{1}{x}\sin\left(\log_e\left(\dfrac{x}{2}\right)\right)$

13 a $24\sin^3(3x)\cos(3x)$ **b** $-60\cos^2(4x)\sin(4x)$

c $\dfrac{4x^2+9}{\sqrt{2x^2+9}}$ **d** $\dfrac{5}{\sqrt{(3x^2+5)^3}}$

14 a $x^2(5x\cos(5x) + 3\sin(5x))$

b $4x^3(\cos(4x) - x\sin(4x))$

c $\dfrac{9}{2x^4}(x\cos(3x) - \sin(3x))$

d $-\dfrac{8}{3x^5}(2\cos(2x) + x\sin(2x))$

15 a $-xe^{-\frac{1}{2}x^2}$ **b** $-2\sin(2x)e^{\cos(2x)}$

c $-2e^{2x}\sin(e^{2x})$ **d** $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

16 a $x^2e^{-4x}(3-4x)$

b $e^{-3x}(2\cos(2x) - 3\sin(2x))$

c $\dfrac{e^{3x}(3x-2)}{x^3}$

d $\dfrac{-(2x+3)}{x^4e^{2x}}$

17 a $2x\log_e(5x+4) + \dfrac{5x^2}{5x+4}$

b $\dfrac{2x}{2x^2+9}$

c $\dfrac{72}{16x^2-81}$

d $\dfrac{60x}{25-9x^4}$

18 a 3

b $\frac{8\sqrt{3}}{3}$

c $-\dfrac{16\pi^2\sqrt{3}}{9}$

d $-\dfrac{2\pi^2}{81}$

19–22 Check with your teacher.

23 a Check with your teacher.

b $e^{-4x}[(3x^2 - 4x^3)\cos(2x) - 2x^3\sin(2x)]$

24 Check with your teacher.

EXERCISE 7.3

1 $y = 4x - 17$

2 $y = \dfrac{2x}{3} + \dfrac{7}{3}$

3 $2y - x - 28 = 0$

4 $12y - x - 47 = 0$

5 -11

6 -21

7 $y_T = \sqrt{3}x - \dfrac{\sqrt{3}\pi}{3} + 2,\ y_N = -\dfrac{\sqrt{3}x}{3} - \dfrac{\sqrt{3}\pi}{9} + 2$

8 $y = 6x + 1,\ 6y + x - 6 = 0$

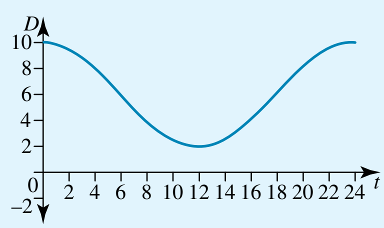
9 $4\pi r^2$

10 a $\frac{2}{3}\pi rh$

b $\frac{1}{3}\pi r^2$

11 a 6 m

b



c Between 4 am and 8 am

d $-\dfrac{\sqrt{2}\pi}{6}$ m/s

e $-\frac{2}{3}$ m/s

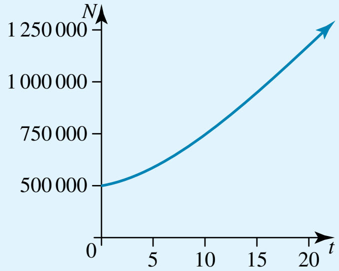
12 a $N_0 = 500\,000, k = \frac{1}{10}\log_e\left(\frac{3}{2}\right)$

b 918 559

c 37 244.3

d 39 583.3

e



13 a $y = -4x + 13$

b $y = -5x + \dfrac{5\pi}{3} + \dfrac{5\sqrt{3}}{2}$

c $3y - x - 5 = 0$

d $y = -6\sqrt{2}x + \dfrac{\sqrt{2}\pi}{2} + 2\sqrt{2}$

14 a $6y - x - 39 = 0$

b $12y + x - 12\sqrt{3} - \dfrac{\pi}{9} = 0$

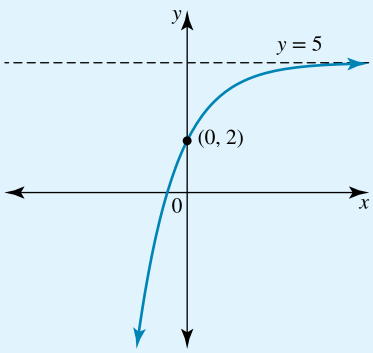
c $4y + x - 1 = 0$

d $12y - 27x + 77 = 0$

15 a –13 **b** –6 **c** $\frac{23}{4}$ **d** –3

16 a Check with your teacher.

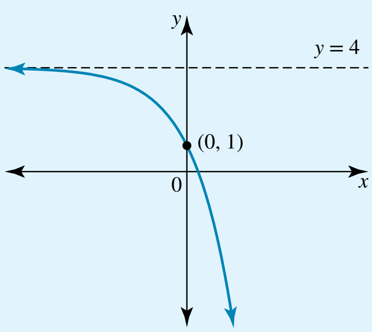
b $(0, 2), \left(\frac{1}{2}\log_e\left(\frac{3}{5}\right), 0\right)$; $y = 5$ is a horizontal asymptote



c $y = \dfrac{6x}{e} + 5 - \dfrac{6}{e}$

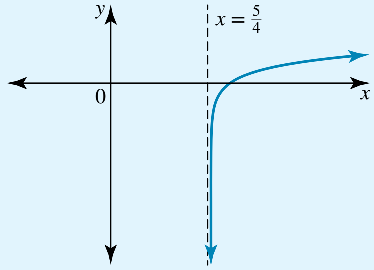
17 a Check with your teacher.

b $(0, 1), \left(\frac{1}{2}\log_e\left(\frac{4}{3}\right), 0\right)$; $y = 4$ is a horizontal asymptote



c $y = \dfrac{x}{6e} - \dfrac{1}{12e} - 3e + 4$

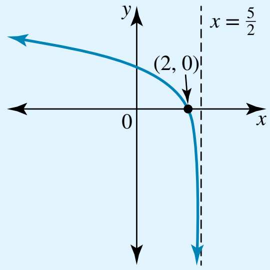
18 a Domain $\left(\frac{5}{4}, \infty\right)$, range R, $\left(\frac{3}{2}, 0\right)$; doesn't cross the y-axis;
$x = \frac{5}{4}$ is a vertical asymptote



b Check with your teacher.

c $y = 4x - 8 + \log_e(27)$

19 a Domain $\left(-\infty, \frac{5}{2}\right)$, range R;
intercepts $(2, 0)$, $(0, \log_e(25))$;
$x = \frac{5}{2}$ is a vertical asymptote



b Check with your teacher.

c $y = \dfrac{3x}{4} - \dfrac{3}{4} + \log_e(9)$

20 a $y = xe^{-2}, y = 0$

b Check with your teacher.

21 a 2

b $a = 1, \sqrt{2}$

c i $\dfrac{(a^2 + k)^2}{4a}$ **ii** Check with your teacher.

22 a 400 **b** 63.6 **c** 59.4

23 a –463.875 mA/s **b** –462.048 mA/s

24 a 9.31 mg/hour **b** 9.1 mg/hour

c 3 hours, 33.11 mg **d** 9.91 hours

EXERCISE 7.4

1 $-\dfrac{x^2}{y^2}$

2 $-\sqrt{\dfrac{y}{x}}$

3 a $8x^2y\dfrac{dy}{dx} + 8xy^2$ **b** $\dfrac{27x^2 + 8xy^2 + 2}{5 + 9y^2 - 8x^2y}$

4 $-\frac{9}{11}$

5 $\dfrac{y - xe^{-(x^2+y^2)}}{ye^{-(x^2+y^2)} - x}$

6 $\dfrac{-(3\cos(3x + 2y) + 2x)}{2(y + \cos(3x + 2y))}$

7 a $-\cot(t)$ b Check with your teacher.

8 a $\frac{1-2t^3}{t}$ b Check with your teacher.

9 a $\frac{1}{y}$ b $-\frac{9x}{4y}$

c $\frac{9x}{16y}$ d $\frac{3x-2}{y-1}$

10 a $\frac{4x+3y}{6y-3x}$ b $\frac{3-x-xy^2}{x^2y}$

c 1 d $\frac{2x+3}{y(3y-2)}$

11 a $\frac{27x^2+12y+1}{y-12x}$

b $\frac{4x^3-6x^2y-3x^2+8xy^2}{24y^2+2x^3-8x^2y}$

c $\frac{e^{x+y}}{2y+\sin(y)+e^{x+y}}$

d $\frac{y\sin(xy)-ye^{xy}-2x}{xe^{xy}-x\sin(xy)}$

12 a $\frac{2+8x+12y}{15y+10x-3}$ b $-\frac{y(2x^2+1)}{x(2y^2+1)}$

c $\frac{y^2(2x^2y-1)}{x^2(1-2xy^2)}$ d $\frac{3\cos(3x)\cos(2y)-2x}{6y^2+2\sin(3x)\sin(2y)}$

13 a 1 b $5y-x+11=0$

c $-\frac{2}{7}$ d $\frac{8}{15}$

14 a $4y-3x+25=0$ b $-\frac{\sqrt{3}}{2}$

c $\frac{4}{5}$ d $\frac{13}{9}$

15 a $\frac{1}{t}$ b $-\frac{4}{3}\tan(2t)$

c $-\frac{1}{t^2}$ d $\frac{4}{3}\operatorname{cosec}(t)$

16 a $\frac{1}{t}$ b $-\frac{b}{a}\cot(t)$

c $-\frac{1}{t^2}$ d $\frac{b}{a}\operatorname{cosec}(t)$

17 a $\frac{1}{t}$ b $-\frac{b}{a}\cot(t)$

c $-\frac{1}{t^2}$ d $\frac{b}{a}\operatorname{cosec}(t)$

18 a $\frac{x^2-ay}{ax-y^2}$ b $\frac{t(t^3-2)}{2t^3-1}$

19 a $-\frac{x(x^2+y^2-1)}{y(x^2+y^2+1)}$ b $\frac{2-3\cos^2(t)}{\sin(t)(2+\cos^2(t))}$

20 a $\frac{y^2+3x^2}{2y(2a-x)}$

b $\frac{t(t^2+3)}{2}$

c $\frac{\sin(t)(2\cos^2(t)+1)}{2\cos^3(t)}$

EXERCISE 7.5

1 $-\frac{1}{12}$

2 $-\sqrt{3}$

3 $6x\log_e(2x^2+5)+\frac{20x^3(2x^2+7)}{(2x^2+5)^2}$

4 $3x^2(3x^2-8x+4)e^{-3x}$

5 Check with your teacher.

6 Check with your teacher.

7 $\frac{-1}{4\sin^3(t)}$

8 $\frac{-(1+4t^3)}{2t^3}, -\frac{33}{16}$

9 a $\frac{1}{2}$ b $-\frac{9}{2}$ c $-\frac{16}{9}$

d $6e$ e $\frac{8\sqrt{3}}{9}$ f $\sqrt{e}$

10 a $\frac{-2(x^2+4x-5)}{(x^2+4x+13)^2}$

b $-e^{3x}(7\cos(4x)+24\sin(4x))$

c $2xe^{-2x}(2x^2-6x+3)$

d $(2-9x^2)\cos(3x)-12\boldsymbol{x}\sin(3x)$

e $\frac{12(3x+7)}{(6x+7)^2}$

f $\frac{-x}{\sqrt{(x^2+16)^3}}$

11 a $-\frac{4a^2}{y^3}$

b $-\frac{b^4}{a^2y^3}$

c $\frac{2a^2}{x^3}$

12 a $-\frac{1}{2at^3}$

b $\frac{2}{at^3}$

c $\frac{b}{a\sin^3(t)}$

13–17 Check with your teacher.

18 a $-\sqrt[3]{\frac{y}{x}}$

b $\frac{dy}{dx}=-\tan(\theta)$

$\frac{d^2y}{dx^2}=\frac{1}{3a\cos^4(\theta)\sin(\theta)}$

19 a $-e^{-2x}(5\sin(3x)+12\cos(3x))$

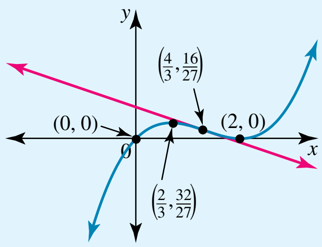
b Check with your teacher.

20 Check with your teacher.

EXERCISE 7.6

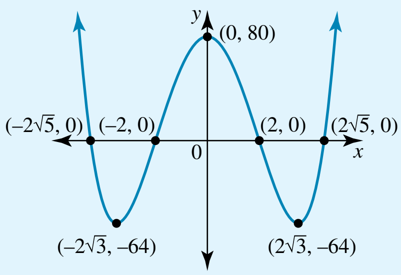
1 (0, 0), (2, 0) local min., $\left(\frac{2}{3}, \frac{32}{27}\right)$ local max., $\left(\frac{4}{3}, \frac{16}{27}\right)$ inflection

$y = -\dfrac{4x}{3} + \dfrac{64}{27}$



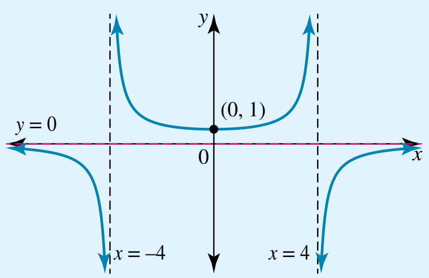
2 Check with your teacher.

3 $(\pm 2\sqrt{5}, 0)$, $(\pm 2, 0)$, (0, 80) local max., $(\pm 2\sqrt{3}, -64)$ min., $(\pm 2, 0)$ inflection

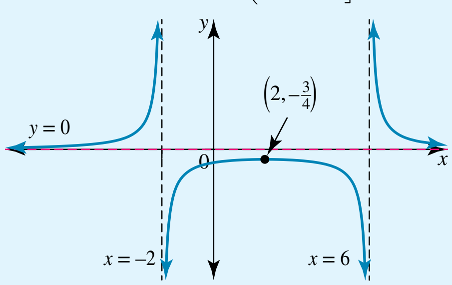


4 $a = -1, b = 12, c = -8$; $(\pm\sqrt{6}, 28)$ max.

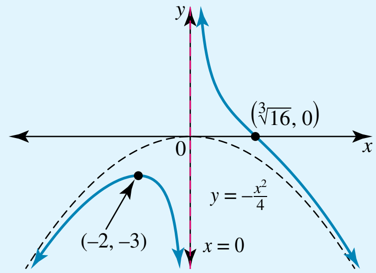
5 $y = 0, x = \pm 4$; (0, –1) local min.; domain $x \in R\backslash\{\pm 4\}$, range $(-\infty, \theta) \cup [1, \infty)$



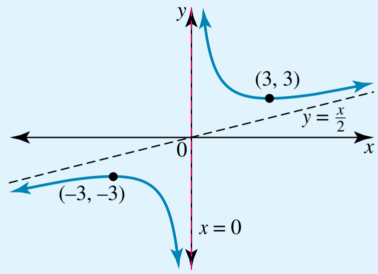
6 $y = 0, x = -2, x = 6$; $\left(2, -\frac{3}{4}\right)$ local max.; domain $x \in R\backslash\{-2, 6\}$, range $\left(-\infty, -\frac{3}{4}\right] \cup (0, \infty)$



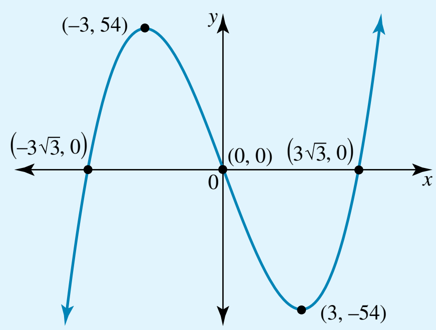
7 $x = 0, y = -\dfrac{x^2}{4}$; $(\sqrt[3]{16}, 0)$, (–2, –3) max.



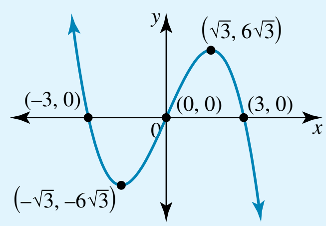
8 $x = 0, y = \dfrac{x}{2}$; (3, 3) min., (3, –3) max.



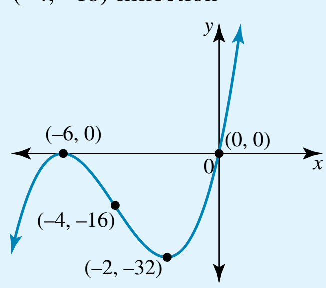
9 a (0, 0), $(\pm 3\sqrt{3}, 0)$, (3, –54) local min., (–3, 54) local max., (0, 0) inflection



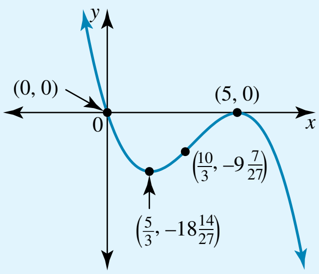
b (0, 0), $(\pm 3, 0)$, $(-\sqrt{3}, -6\sqrt{3})$ local min., $(\sqrt{3}, 6\sqrt{3})$ local max., (0, 0) inflection



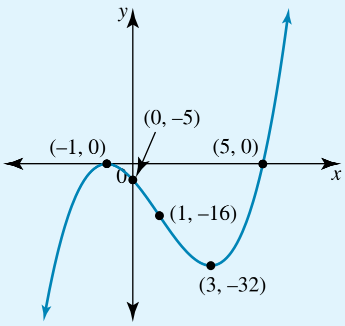
c (0, 0), (–6, 0) local max., (–2, –32) local min., (–4, –16) inflection



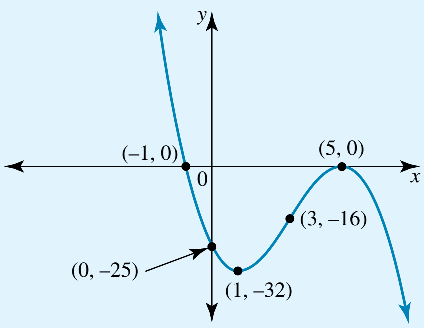
d (0, 0), (5, 0) local max., $\left(\frac{5}{3}, -18\frac{14}{27}\right)$ local min., $\left(\frac{10}{3}, -9\frac{7}{27}\right)$ inflection



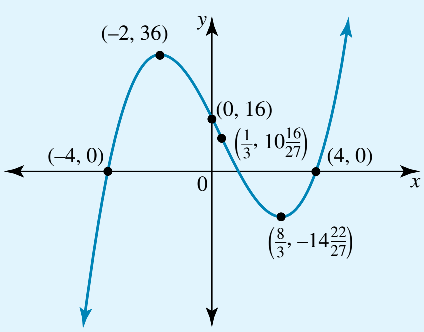
10 a (0, –5), (5, 0), (–1, 0) local max., (3, –32) local min., (1, –16) inflection



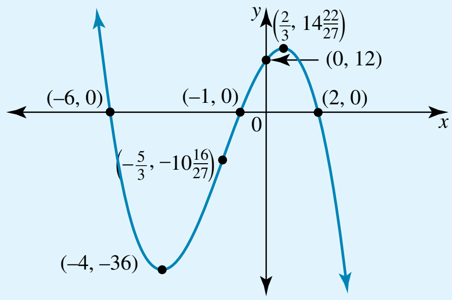
b (0, –25), (–1, 0), (5, 0) local max., (1, –32) local min., (3, –16) inflection



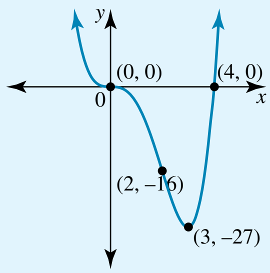
c (0, 16), (±4, 0), (–2, 36) local max., $\left(\frac{8}{3}, -14\frac{22}{27}\right)$ local min., $\left(\frac{1}{3}, 10\frac{16}{27}\right)$ inflection



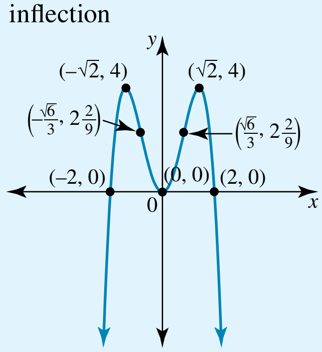
d (0, 12), (–6, 0), (–1, 0), (2, 0), $\left(\frac{2}{3}, 14\frac{22}{27}\right)$ local max., (–4, –36) local min., $\left(-\frac{5}{3}, -10\frac{16}{27}\right)$ inflection



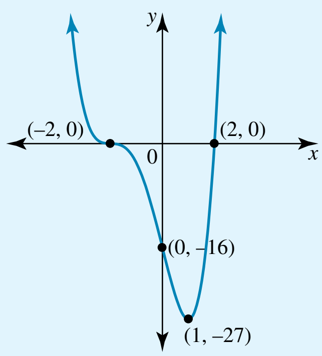
11 a (0, 0), (4, 0) (3, –27) min., (2, –16) inflection, (0, 0) stationary point of inflection



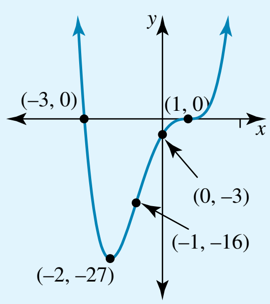
b (±2, 0), (0, 0) local min., $(\pm\sqrt{2}, 4)$ max., $\left(\pm\frac{\sqrt{6}}{3}, 2\frac{2}{9}\right)$ inflection



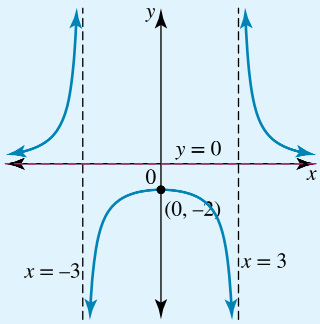
c (0, –16), (2, 0), (1, –27) min., (–2, 0) stationary point of inflection



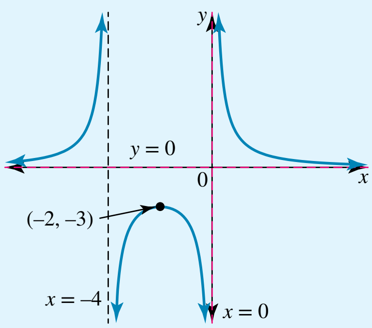
d (0, –3), (–3, 0), (–2, –27) min., (–1, –16) inflection, (1, 0) stationary point of inflection



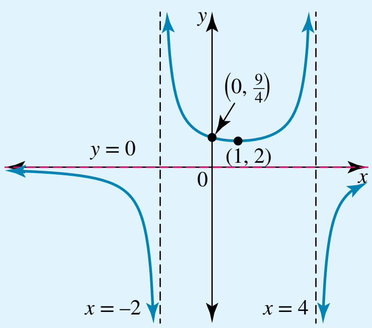
12 a $y = 0, x = \pm 3$; (0, –2) local max.;
domain $x \in R \setminus \{\pm 3\}$, range $(-\infty, -2] \cup (0, \infty)$



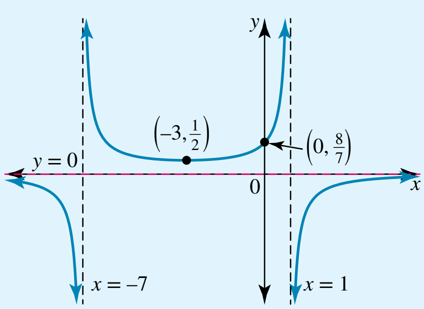
b $y = 0, x = -4, x = 0$; (–2, –3) local max.;
domain $x \in R \setminus \{-4, 0\}$, range $(-\infty, -3] \cup (0, \infty)$



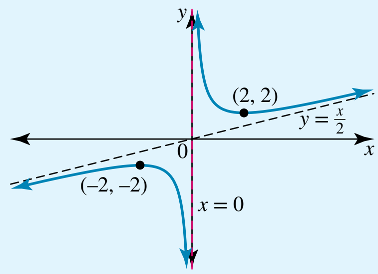
c $y = 0, x = -2, x = 4$; $\left(0, \frac{9}{4}\right)$, (1, 2) local min.;
domain $x \in R \setminus \{-2, 4\}$, range $(-\infty, 0) \cup [1, \infty)$



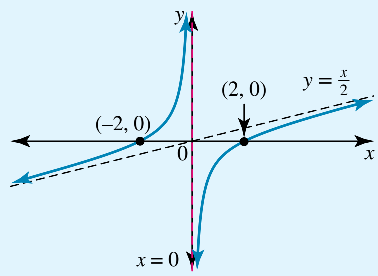
d $y = 0, x = -7, x = 1$; $\left(0, \frac{7}{8}\right)$, $\left(-3, \frac{1}{2}\right)$ local min.;
domain $x \in R \setminus \{-7, 1\}$, range $(-\infty, 0) \cup \left[\frac{1}{2}, \infty\right)$



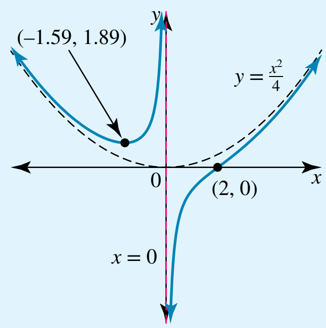
13 a $x = 0, y = \frac{x}{2}$; (2, 2) local min., (–2, –2) local max.



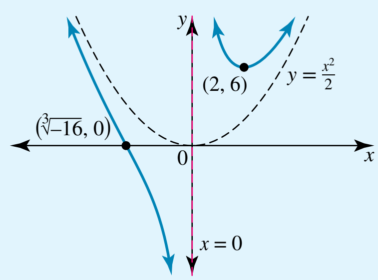
b $x = 0, y = \frac{x}{2}$; (±2, 0), no turning points



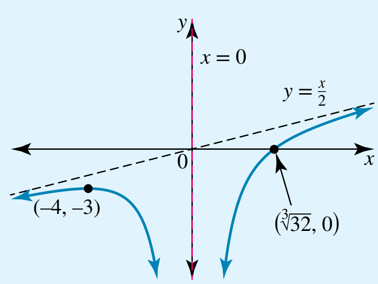
c $x = 0, y = \frac{x^2}{4}$; (2, 0), (–1.59, 1.89) local min.



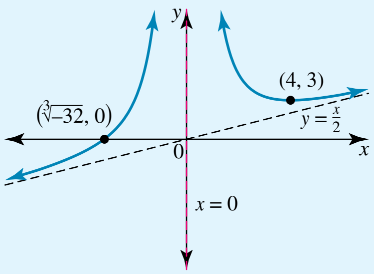
d $x = 0, y = \frac{x^2}{2}$; $(\sqrt[3]{-16}, 0)$, (2, 6) local min.



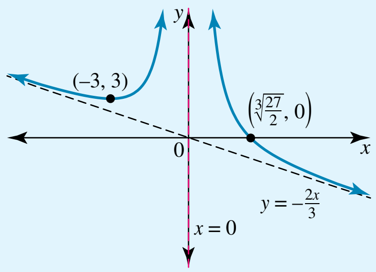
14 a $x = 0, y = \frac{x}{2}$; $(\sqrt[3]{32}, 0)$, (–4, –3) local max.



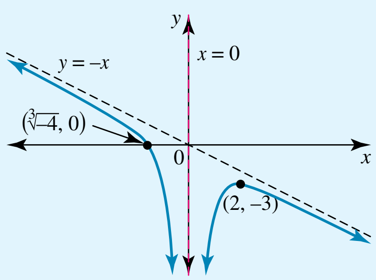
b $x = 0, y = \dfrac{x}{2}$; $(\sqrt[3]{-32}, 0)$, (4, 3) local min.



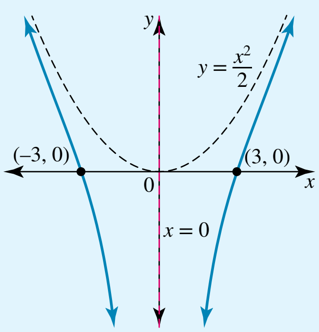
c $x = 0, y = -\dfrac{2x}{3}$; $\left(\sqrt[3]{\dfrac{27}{2}}, 0\right)$, (–3, 3) local min.



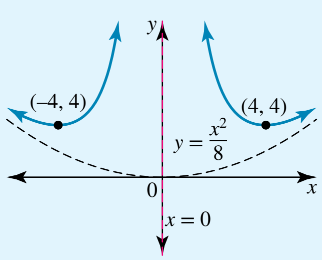
d $x = 0, y = -x$; $(\sqrt[3]{-4}, 0)$, (2, –3) local max.



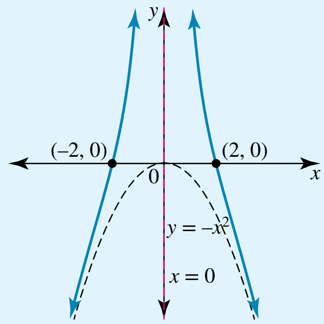
15 a $x = 0, y = \dfrac{x^2}{2}$; $(\pm 3, 0)$, no turning points



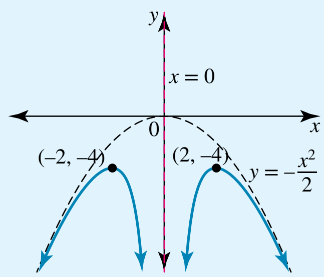
b $x = 0, y = \dfrac{x^2}{8}$; no axis intercepts, $(\pm 4, 4)$ min.



c $x = 0, y = -x^2$, $(\pm 2, 0)$, no turning points



d $x = 0, y = -\dfrac{x^2}{2}$, no axis intercepts, $(\pm 2, 4)$ max.



16 a $b = -3, c = 3, d = -3$

b $y = 6 - 27x$

c $b = -6, c = 15, d = -18$

d $a = -1, b = -6, c = 9$

17 a $A = 12, b = 8, y = 0, x = -7, x = -1$;

domain $x \in R\backslash\{-7, -1\}$, range $\left(-\infty, -\frac{4}{3}\right] \cup (0, \infty)$

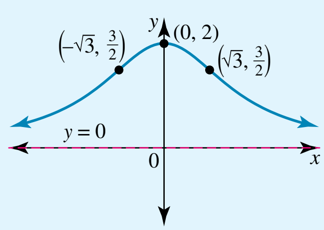
b $A = 50, b = 6, c = 16, y = 0, x = -2, x = 8$;
domain $x \in R\backslash\{-2, 8\}$, range $(-\infty, 0) \cup [2, \infty)$

c **i** $(-\infty, -20) \cup (20, \infty)$

ii ± 20

iii (–20, 20)

d (0, 2) max., $\left(\pm\sqrt{3}, \dfrac{3}{2}\right)$ inflection, $y = -\dfrac{\sqrt{3}x}{4} + \dfrac{9}{4}$



18 Check with your teacher.

19 a $x = 0, y = ax$

b $x = 0, y = ax$

c $x = 0, y = ax^2$

d $x = 0, y = ax^2$

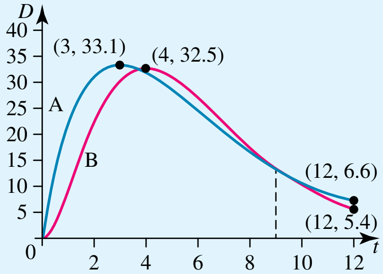
20 a **i** $(3, 90e^{-1})$

ii $(6, 180e^{-2})$

b **i** $(4, 240e^{-2})$

ii (1.17, 11.46), (6.83, 23.01)

c 44.7%



EXERCISE 7.7

1 a $\dfrac{1}{\sqrt{25-x^2}}\ x \in (-5, 5)$ b $\dfrac{5}{\sqrt{1-25x^2}}\ x \in \left(-\dfrac{1}{5}, \dfrac{1}{5}\right)$

2 $\dfrac{8\sqrt{3}}{3}$

3 a $\dfrac{-3}{\sqrt{16-9x^2}}\ x \in \left(-\dfrac{4}{3}, \dfrac{4}{3}\right)$

b $\dfrac{-1}{\sqrt{(x+1)(4-x)}}\ x \in (-1, 4)$

4 $-\frac{3}{5}$

5 a $\dfrac{4}{1+16x^2}$ b $\dfrac{5}{2x^2-6x+17}$

6 $\frac{3}{14}$

7 $\dfrac{-8x}{\sqrt{(9-4x^2)^3}}$ 8 $\dfrac{-384x}{(9+16x^2)^2}$

9 $\dfrac{-2}{|x|\sqrt{x^2-4}}, |x| > 2$

10 $\dfrac{-2}{\sqrt{x}(x+16)}, x > 0$

11 a $\dfrac{1}{\sqrt{9-x^2}}, |x| < 3$

b $\dfrac{3}{\sqrt{1-9x^2}}, |x| < \dfrac{1}{3}$

c $\dfrac{4}{\sqrt{9-16x^2}}, |x| < \dfrac{3}{4}$

d $\dfrac{\sqrt{2}}{\sqrt{(x+2)(1-2x)}}, -2 < x < \dfrac{1}{2}$

12 a $\dfrac{-1}{\sqrt{16-x^2}}, |x| < 4$

b $\dfrac{-4}{\sqrt{1-16x^2}}, |x| < \dfrac{1}{4}$

c $\dfrac{-3}{\sqrt{16-9x^2}}, |x| < \dfrac{4}{3}$

d $\dfrac{-\sqrt{3}}{\sqrt{(x+4)(2-3x)}}, -4 < x < \dfrac{2}{3}$

13 a $\dfrac{6}{x^2+36}$ b $\dfrac{6}{36x^2+1}$

c $\dfrac{30}{25x^2+36}$ d $\dfrac{24}{36x^2+60x+41}$

14 a $\dfrac{1}{2\sqrt{x(9-x)}}, 0 < x < 9$ b $\dfrac{-3}{|x|\sqrt{16x^2-9}}, |x| > \dfrac{3}{4}$

c $\dfrac{-2e^{2x}}{\sqrt{16-e^{4x}}}, x < \log_e 2$ d $\dfrac{4}{|x|\sqrt{9x^2-16}}, |x| > \dfrac{4}{3}$

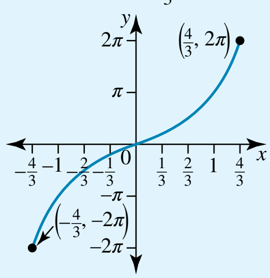
e $\dfrac{6x}{x^4+9}$ f $\dfrac{-30}{25x^2+36}, x \neq 0$

15 a $\dfrac{125x}{(16-25x^2)^{\frac{3}{2}}}, |x| < \dfrac{4}{5}$

b $\dfrac{-216x}{(25-36x^2)^{\frac{3}{2}}}, |x| < \dfrac{5}{6}$

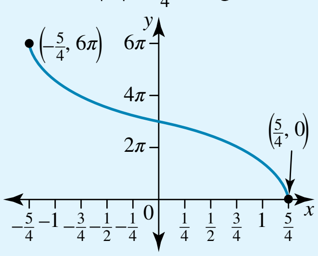
c $\dfrac{-4116x}{(49x^2+36)^2}$

16 a i Domain $|x| < \frac{4}{3}$, range $[-2\pi, 2\pi]$



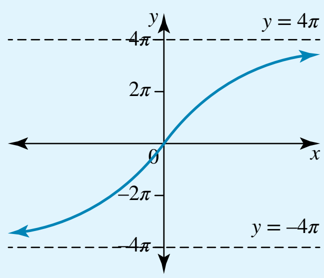
ii $y = 2\sqrt{3}x - \dfrac{4\sqrt{3}}{3} + \dfrac{2\pi}{3}$

b i Domain $|x| < \frac{5}{4}$, range $[0, 6\pi]$



ii $y = -\dfrac{24\sqrt{2}x}{5} + \dfrac{3\pi}{2} + 6$

c i Domain R, range $[-4\pi, 4\pi]$



ii $y = \dfrac{10x}{3} + 2\pi - 4$

17 a $\dfrac{2(\pi + \sqrt{3})}{3}$ **b** $\dfrac{\pi - \sqrt{3}}{3}$ **c** $\dfrac{2\pi + \sqrt{3}}{4}$

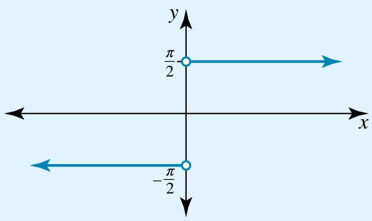
18 a $T(3) = \dfrac{\pi}{2}, T(-3) = -\dfrac{\pi}{2}$

b $T'(x) = 0$ if $x \neq 0$

c Domain $x \in R \backslash \{0\}$

$T(x)$ is a constant function, $T(x) = \begin{cases} \dfrac{\pi}{2} & \text{if } x > 0 \\ -\dfrac{\pi}{2} & \text{if } x < 0 \end{cases}$

Range $y = \pm\dfrac{\pi}{2}$



19–22 Check with your teacher.

EXERCISE 7.8

1 $20 \text{ m}^2\text{/s}$

2 2 cm/s

3 $\dfrac{1}{24\pi}$ cm/s

4 $48\pi \text{ cm}^2\text{/s}$

5 Decreasing at $\dfrac{75}{128\pi}$ cm/s

6 Decreasing at $64\pi \text{ cm}^3\text{/s}$

7 0.225 m/s

8 1.6 m/s

9 a $16 \text{ cm}^2\text{/s}$ **b** $\dfrac{1}{4\pi}$ m/s

10 a $800\pi \text{ mm}^3\text{/s}$ **b** $160\pi \text{ mm}^2\text{/s}$

11 a Check with your teacher.

b 150 weeks

12 a $6 \text{ cm}^2\text{/s}$ **b** $\sqrt{3}$ cm/s

13 a Check with your teacher.

b $450\pi \text{ cm}^3\text{/s}$

14 a $\dfrac{1}{25\pi}$ cm/s **b** $\dfrac{4}{7\pi}$ cm/s

15 a 25.87 m/s **b** 12 m/s

16 a Check with your teacher.

b Decreases by 5 cm/s

17 a Decreases by 31 562 Pa/s

b Check with your teacher.

18 a 0.005°/s **b** –10.3°/s

19 a 0.75 m/s **b** 2.25 m/s **c** 1.6 m/s

20 a **i** Check with your teacher.

ii 1.44 h, 54 km

iii 18.86 km/h

b **i** Check with your teacher.

ii $\dfrac{220\sqrt{7}}{7}$ km/h

8

Integral calculus

8.1 Kick off with CAS

Exploring integration with CAS

In this topic, we will be integrating various types of functions.

1 Use CAS to find each of the following.

a $\int (3x + 4)^5 dx$ b $\int \frac{1}{(3x + 4)^2} dx$ c $\int \frac{1}{\sqrt{3x + 4}} dx$ d $\int \frac{1}{3x + 4} dx$

2 Can you predict $\int (ax + b)^n dx$? What happens when $n = -1$?

3 Use CAS to find each of the following.

a $\int \frac{1}{\sqrt{4 - 9x^2}} dx$ b $\int \frac{1}{\sqrt{9 - 16x^2}} dx$ c $\int \frac{1}{\sqrt{16 - 25x^2}} dx$

4 Can you predict $\int \frac{1}{\sqrt{a^2 - b^2x^2}} dx$, where a and b are positive constants?

5 Use CAS to find each of the following.

a $\int \frac{1}{4 + 9x^2} dx$ b $\int \frac{1}{9 + 16x^2} dx$ c $\int \frac{1}{16 + 25x^2} dx$

6 Can you predict $\int \frac{1}{a^2 + b^2x^2} dx$, where a and b are positive constants?

7 Use CAS to find each of the following.

a $\int \frac{1}{4 - 9x^2} dx$ b $\int \frac{1}{9 - 16x^2} dx$ c $\int \frac{1}{16 - 25x^2} dx$

8 Can you predict $\int \frac{1}{a^2 - b^2x^2} dx$, where a and b are positive constants?

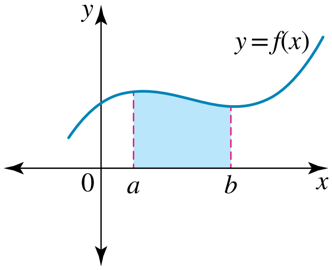
Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

8.2 Areas under and between curves

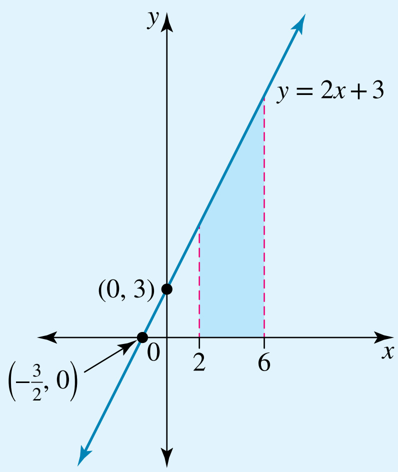
Area bounded by a curve and the *x*-axis

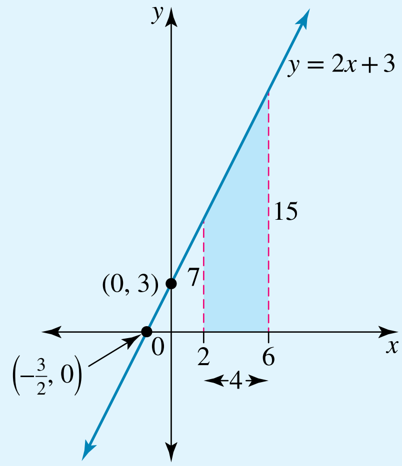
Basic integration techniques and evaluating areas bounded by curves and the *x*-axis, have been covered in the Mathematical Methods course. The examples and theory presented here are a review of this material.

Recall that the definite integral, $A = \int_a^b f(x)dx$, gives a measure of the area A bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. This result is known as the Fundamental Theorem of Calculus.



WORKED EXAMPLE 1 **Find the area bounded by the line $y = 2x + 3$, the x-axis and the lines $x = 2$ and $x = 6$.**

THINK	WRITE/DRAW
1 Draw a diagram to identify the required area and shade this area.	
2 The required area is given by a definite integral.	$A = \int_2^6 (2x + 3)dx$
3 Perform the integration using square bracket notation.	$A = \left[x^2 + 3x\right]_2^6$
4 Evaluate.	$A = (6^2 + 3 \times 6) - (2^2 + 3 \times 2)$ $= (36 + 18) - (4 + 6)$ $= 44$
5 State the value of the required area region.	The area is 44 square units.
6 The shaded area is a trapezium. As a check on the result, find the area using the formula for a trapezium.	The width of the trapezium is $h = 6 - 2 = 4$, and since $y = 2x + 3$: when $x = 2$, $y = 7$ and when $x = 6$, $y = 15$.



The area of a trapezium is $\frac{h}{2}(a + b)$.

The area is $\frac{4}{2}(7 + 15) = 44$ square units.

Using symmetry

Sometimes symmetry can be used to simplify the area calculation.

WORKED EXAMPLE 2

Find the area bounded by the curve $y = 16 - x^2$, the x-axis and the lines $x = \pm 3$.

THINK

1 Factorise the quadratic to find the x-intercepts.

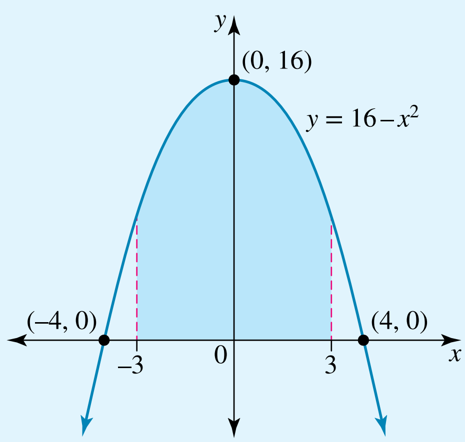
WRITE/DRAW

$y = 16 - x^2$

$y = (4 - x)(4 + x)$

The graph crosses the x-axis at $x = \pm 4$ and crosses the y-axis at $y = 16$.

2 Draw a diagram to identify the required area and shade this area.



3 The required area is given by a definite integral; however, we can use symmetry.

$$A = \int_{-3}^{3} (16 - x^2)dx$$

$$= \int_{-3}^{0} (16 - x^2)dx + \int_{0}^{3} (16 - x^2)dx$$

However, $\int_{-3}^{0} (16 - x^2)dx = \int_{0}^{3} (16 - x^2)dx$

$A = 2\int_{0}^{3} (16 - x^2)dx$

4 Perform the integration using square bracket notation.

$A = 2\left[16x - \frac{1}{3}x^3\right]_0^3$

5 Evaluate.

$A = 2\left[\left(16 \times 3 - \frac{1}{3}(3)^3\right) - 0\right]$

$A = 2(48 - 9)$

6 State the value of the required area.

The area is 78 square units.

Areas involving basic trigonometric functions

For integrals and area calculations involving the basic trigonometric functions, we use the results $\int \cos(kx)dx = \frac{1}{k}\sin(kx) + c$ and $\int \sin(kx)dx = -\frac{1}{k}\cos(kx) + c$, where $x \in R$, $k \neq 0$, and k and c are constants.

WORKED EXAMPLE 3

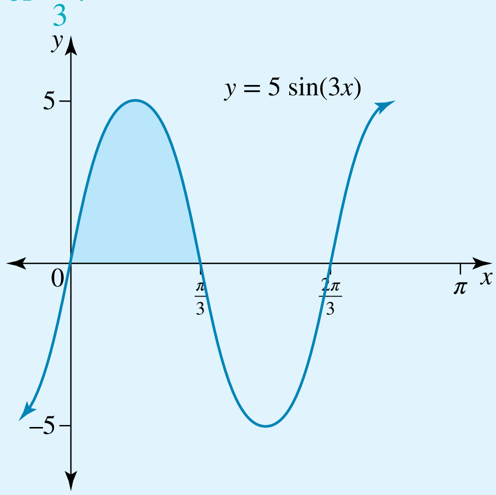
Find the area under one arch of the sine curve $y = 5\sin(3x)$.

THINK

WRITE/DRAW

1 Draw a diagram to identify the required area and shade this area.

$y = 5\sin(3x)$ has an amplitude of 5 and a period of $\frac{2\pi}{3}$.



2 One arch is defined to be the area under one half-cycle of the sine wave.

The graph crosses the x-axis at $\sin(3x) = 0$, when $x = 0$, $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. The required area is

$A = \int_{0}^{\frac{\pi}{3}} 5\sin(3x)dx$

3 Perform the integration using square bracket notation. $A = 5\left[-\frac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{3}}$

4 Evaluate, taking the constant factors outside the brackets. $A = -\frac{5}{3}[\cos(\pi) - \cos(0)]$
$= -\frac{5}{3}[-1 - 1]$

5 State the value of the required area. The area is $\frac{10}{3}$ square units.

Areas involving basic exponential functions

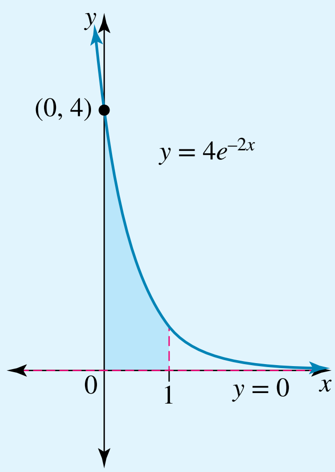
For integrals and area calculations involving the basic exponential functions, we use the result $\int e^{kx}dx = \frac{1}{k}e^{kx} + c$, where $x \in R$, $k \neq 0$, and k and c are constants.

WORKED EXAMPLE 4

Find the area bounded by the coordinate axes, the graph of $y = 4e^{-2x}$ and the line $x = 1$.

THINK | **WRITE/DRAW**

1 Draw a diagram to identify the required area and shade this area.



2 The required area is given by a definite integral. $A = \int_0^1 4e^{-2x}dx$

3 Perform the integration using square bracket notation. $A = 4\left[-\frac{1}{2}e^{-2x}\right]_0^1$
$= -2\left[e^{-2x}\right]_0^1$

4 Evaluate. $A = -2[e^{-2} - e^{0}]$
$= -2(e^{-2} - 1)$

5 State the value of the required area. The exact area is $2(1 - e^{-2})$ square units.

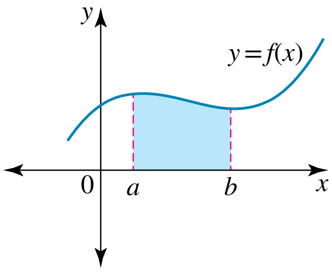
Areas involving signed areas

When evaluating a definite integral, the result is a number; this number can be positive or negative. A definite integral which represents an area is a signed area; that is, it may also be positive or negative. However, areas cannot be negative.

Areas above the x-axis

When a function is such that $f(x) \geq 0$ for $a \leq x \leq b$, where $b > a$, that is, the function lies above the x-axis, then the definite integral that represents the area A is positive:

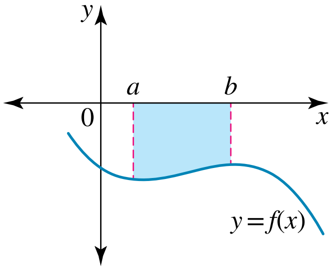
$$A = \int_a^b f(x)dx > 0.$$



Areas below the x-axis

When a function is such that $f(x) \leq 0$ for $a \leq x \leq b$, where $b > a$, that is, the function lies below the x-axis, then the definite integral that represents the area A is negative:

$$\int_a^b f(x)dx < 0.$$

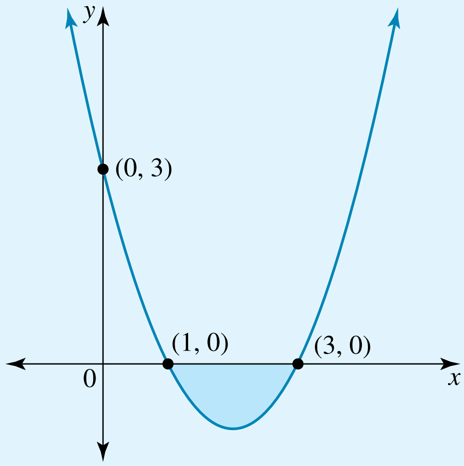


So, when an area is determined that is bounded by a curve that is entirely below the x-axis, the result will be a negative number. Because areas cannot be negative, the absolute value of the integral must be used.

$$A = \left| \int_a^b f(x)dx \right| = -\int_a^b f(x)dx = \int_b^a f(x)dx$$

WORKED EXAMPLE 5

Find the area bounded by the curve $y = x^2 - 4x + 3$ and the x-axis.

THINK	WRITE/DRAW
1 Factorise the quadratic to find the x-intercepts.	$y = x^2 - 4x + 3$ $y = (x - 3)(x - 1)$ The graph crosses the x-axis at $x = 1$ and $x = 3$ and crosses the y-axis at $y = 3$.
2 Sketch the graph, shading the required area.	

3 The required area is below the x-axis and will evaluate to a negative number. The area must be given by the absolute value or the negative of this definite integral.

$$A = \left|\int_1^3 (x^2 - 4x + 3)dx\right|$$

$$= -\int_1^3 (x^2 - 4x + 3)dx$$

4 Perform the integration using square bracket notation.

$$A = -\left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$$

5 Evaluate the definite integral.

$$A = -\left[\left(\frac{1}{3} \times 3^3 - 2 \times 3^2 + 3 \times 3\right) - \left(\frac{1}{3} \times 1^3 - 2 \times 1^2 + 3 \times 1\right)\right]$$

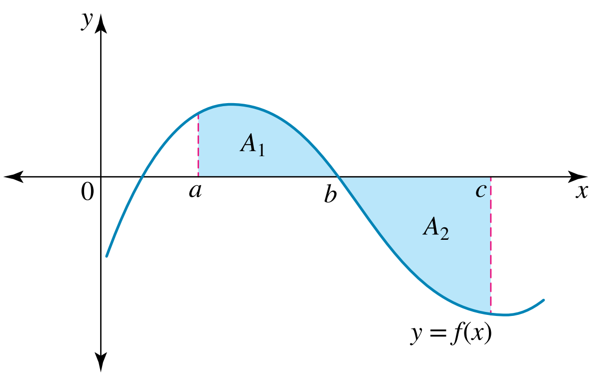
$$= \frac{4}{3}$$

6 State the value of the required area.

The area is $\frac{4}{3}$ square units.

Areas both above and below the x-axis

When dealing with areas that are both above and below the x-axis, each area must be evaluated separately.



Since $A_1 = \int_a^b f(x)dx > 0$ and

$A_2 = \int_b^c f(x)dx < 0$, the required area is

$$A = A_1 + |A_2| = \int_a^b f(x)dx + \left|\int_b^c f(x)dx\right|$$

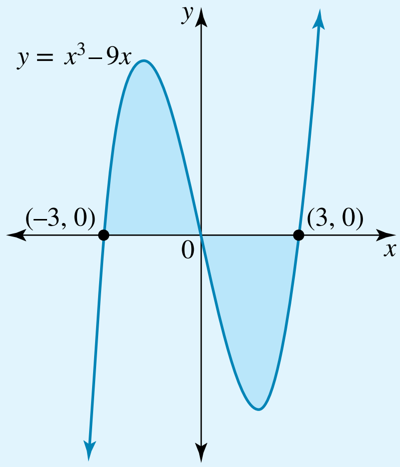
$$= A_1 - A_2 = \int_a^b f(x)dx - \int_b^c f(x)dx = \int_a^b f(x)dx + \int_c^b f(x)dx$$

WORKED EXAMPLE 6

Find the area bounded by the curve $y = x^3 - 9x$ and the x-axis.

THINK	WRITE/DRAW
1 Factorise the cubic to find the x-intercepts.	$y = x^3 - 9x$ $y = x(x^2 - 9)$ $y = x(x + 3)(x - 3)$ The graph crosses the x-axis at $x = 0$ and $x = \pm 3$.

2 Sketch the graph, shading the required area.



3 The required area is given by a definite integral.

If we work out $A = \int_{-3}^{3} (x^3 - 9x)dx$ it comes to zero, as the positive and negative area have cancelled out.

Let $A_1 = \int_{-3}^{0} (x^3 - 9x)dx$ and $A_2 = \int_{0}^{3} (x^3 - 9x)dx$, so that $A_1 > 0$ and $A_2 < 0$, but $A_1 = |A_2|$ by symmetry.

4 Perform the integration, using square bracket notation.

$$A_1 = \int_{-3}^{0} (x^3 - 9x)dx$$

$$A_1 = \left[\tfrac{1}{4}x^4 - \tfrac{9}{2}x^2\right]_{-3}^{0}$$

5 Evaluate the definite integral.

$$A_1 = \left[\left(0 - \tfrac{1}{4} \times (-3)^4 + \tfrac{9}{2} \times (-3)^2\right)\right]$$

$$A_1 = \tfrac{81}{4}$$

$$= 20\tfrac{1}{4}$$

$$A_2 = \int_{0}^{3} (x^3 - 9x)dx = -\tfrac{81}{4} = -20\tfrac{1}{4}$$

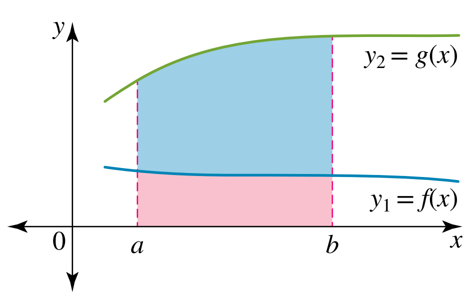
$$A_1 + |A_2| = 2 \times \tfrac{81}{4} = \tfrac{81}{2} = 40\tfrac{1}{2}$$

6 State the value of the required area.

The area is $40\frac{1}{2}$ square units.

Area between curves

If $y_1 = f(x)$ and $y_2 = g(x)$ are two continuous curves that do not intersect between $x = a$ and $x = b$, then the area between the curves is obtained by simple subtraction.



studyon

Units 3 & 4
AOS 3
Topic 2
Concept 9

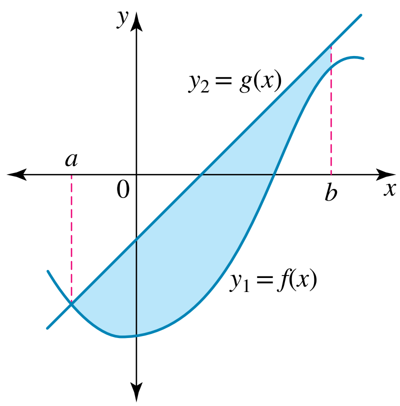
Areas of bounded regions
Concept summary
Practice questions

The area A_1 is the entire shaded area bounded by the curve $y_2 = g(x)$, the x-axis and the lines $x = a$ and $x = b$, so $A_1 = \int_a^b g(x)dx$. The red area, A_2, is the area bounded by the curve $y_1 = f(x)$, the x-axis and the lines $x = a$ and $x = b$, so $A_2 = \int_a^b f(x)dx$.

The required area is the blue area, which is the area between the curves.

$$A = A_1 - A_2 = \int_a^b g(x)dx - \int_a^b f(x)dx,$$ and by the properties of definite integrals

$$A = A_1 - A_2 = \int_a^b (g(x) - f(x))dx = \int_a^b (y_2 - y_1)dx.$$



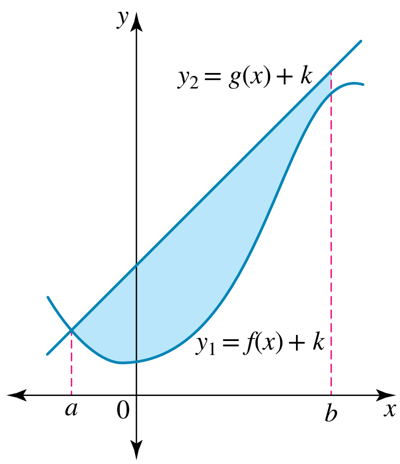
Note that when finding areas between curves, it does not matter if some of the area is above or below the x-axis.

We can translate both curves up k units parallel to the y-axis so that the area between the curves lies entirely above the x-axis as shown below right.

$$A = \int_a^b (g(x) + k)dx - \int_a^b (f(x) + k)dx$$

$$A = \int_a^b (g(x) - f(x))dx$$

$$A = \int_a^b (y_2 - y_1)dx$$



Provided that $y_2 \geq y_1$ for $a < x < b$, it does not matter if some or all of the area is above or below the x-axis, as the required area between the curves will be a positive number. Note that only one definite integral is required, that is $y_2 - y_1 = g(x) - f(x)$. Evaluate this as a one definite integral.

WORKED EXAMPLE 7

Find the area between the parabola $y = x^2 - 2x - 15$ and the straight line $y = 2x - 3$.

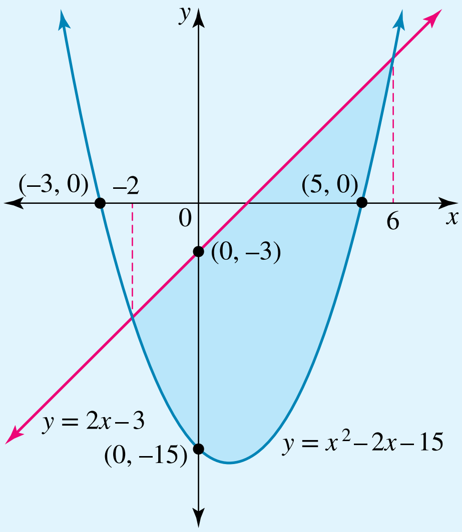
THINK	WRITE/DRAW
1 Factorise the quadratic to find the x-intercepts.	$y = x^2 - 2x - 15$ $y = (x - 5)(x + 3)$ The parabola crosses the x-axis at $x = 5$ and $x = -3$ and crosses the y-axis at $y = -15$. The straight line crosses the x-axis at $x = \frac{3}{2}$ and crosses the y-axis at $y = -3$.

2 Find the x-values of the points of intersection between the parabola and the straight line.

Let $y_1 = x^2 - 2x - 15$ and $y_2 = 2x - 3$.
To find the points of intersection, solve $y_1 = y_2$.

$$\begin{aligned} x^2 - 2x - 15 &= 2x - 3 \\ x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0 \\ x &= 6, -2 \end{aligned}$$

3 Sketch the graph of the parabola and the straight line on one set of axes, shading the required area.



4 The required area is given by a definite integral.

$$A = \int_a^b (y_2 - y_1)dx \text{ with } a = -2,\ b = 6,$$

$y_1 = x^2 - 2x - 15$ and $y_2 = 2x - 3$.

$y_2 - y_1 = -x^2 + 4x + 12$

$$A = \int_{-2}^{6} (-x^2 + 4x + 12)dx$$

5 Perform the integration using square bracket notation.

$$A = \left[-\tfrac{1}{3}x^3 + 2x^2 + 12x\right]_{-2}^{6}$$

6 Evaluate the definite integral.

$$A = \left[\left(-\tfrac{1}{3} \times 6^3 + 2 \times 6^2 + 12 \times 6\right) - \left(\left(-\tfrac{1}{3} \times (-2)^3 + 2 \times (-2)^2 + 12 \times (-2)\right)\right)\right]$$

$A = 85\tfrac{1}{3}$

7 State the value of the required area between the parabola and the straight line.

The area between the straight line and the parabola is $85\frac{1}{3}$ square units.

EXERCISE 8.2 Areas under and between curves

PRACTISE

1 WE1 Find the area bounded by the line $y = 4x + 5$, the x-axis, and the lines $x = 1$ and $x = 3$. Check your answer algebraically.

2 Find the area bounded by the line $y = 4 - \frac{3x}{2}$ and the coordinate axes. Check your answer algebraically.

3 WE2 Find the area bounded by the curve $y = 9 - x^2$, the x-axis and the lines $x = \pm 2$.

4 The area bounded by the curve $y = b - 3x^2$, the x-axis and the lines $x = \pm 1$ is equal to 16. Given that $b > 3$, find the value of b.

5 WE3 Find the area under one arch of the sine curve $y = 4\sin(2x)$.

6 Find the area under one arch of the curve $y = 3\cos\left(\frac{x}{2}\right)$.

7 WE4 Find the area bounded by the coordinate axes, the graph of $y = 6e^{3x}$ and the line $x = 2$.

8 Find the area bounded by the graph of $y = 6(e^{-2x} + e^{-2x})$, the x-axis and $x = \pm 1$.

9 WE5 Find the area bounded by the curve $y = x^2 - 5x + 6$ and the x-axis.

10 Find the area bounded by the curve $y = 3x^2 - 10x - 8$ and the x-axis.

11 WE6 Find the area bounded by the curve $y = x^3 - 4x$ and the x-axis.

12 Find the area bounded by the curve $y = 16x - x^3$, the x-axis, $x = -2$ and $x = 4$.

13 WE7 Find the area between the parabola $y = x^2 - 3x - 18$ and the straight line $y = 4x - 10$.

14 Find the area corresponding to the region $\{y \geq x^2 - 2x - 8\} \cap \{y \leq 1 - 2x\}$.

CONSOLIDATE

15 a Find the area between the line $y = 6 - 2x$ and the coordinate axis. Check your answer algebraically.

b Find the area between the line $y = 3x + 5$, the x-axis, $x = 1$ and $x = 4$. Check your answer algebraically.

16 a Calculate the area bounded by:

i the curve $y = 12 - 3x^2$ and the x-axis

ii the curve $y = 12 - 3x^2$, the x-axis and the lines $x = \pm 1$.

b Determine the area bounded by:

i the graph of $y = x^2 - 25$ and the x-axis

ii the graph of $y = x^2 - 25$, the x-axis and the lines $x = \pm 3$.

17 a Find the area under one arch of the sine curve $y = 6\sin\left(\frac{\pi x}{3}\right)$.

b Find the area under one arch of the curve $y = 4\cos\left(\frac{\pi x}{2}\right)$.

c Find the area under one arch of the sine curve $y = a\sin(nx)$.

18 a Find the area under the graph of $y = \frac{1}{x}$ between the x-axis and:

i $x = 1$ and $x = 4$

ii $x = 1$ and $x = e$

iii $x = 1$ and $x = a$, where $a > e$.

b Find the area under the graph of $y = \frac{1}{x^2}$ between the x-axis and:

i $x = 1$ and $x = 3$

ii $x = 1$ and $x = a$, where $a > 1$.

c Find the area bounded by the curve $y = x^2 - 2x - 15$ and the x-axis.

19 a Find the area bounded by the curve $y = x^2 + 3x - 18$, the x-axis and the lines $x = -3$ and $x = 6$.

b Find the area bounded by the curve $y = x^2 - 2x - 24$, the x-axis and the lines $x = 2$ and $x = 8$.

c Find the area bounded by:

i the curve $y = x^3 - 36x$ and the x-axis

ii the curve $y = x^3 - 36x$, the x-axis and the lines $x = -3$ and $x = 6$.

20 a If a is a constant, find the area bounded by the curve $y = x^2 - a^2$ and the x-axis.

b If a is a constant, find the area bounded by the curve $y = x^3 - a^2x$ and the x-axis.

21 Find the area between the curves:

a $y = x^2$ and $y = x$

b $y = x^3$ and $y = x$

c $y = x^4$ and $y = x$

d $y = x^5$ and $y = x$.

22 a i Find the area between the parabola $y = x^2 - 2x - 35$ and the x-axis.

ii Find the area between the parabola $y = x^2 - 2x - 35$ and the straight line $y = 4x - 8$.

b i Find the area between the parabola $y = x^2 + 5x - 14$ and the x-axis.

ii Find the area corresponding to the region $\{y \geq x^2 + 5x - 14\} \cap \{y \leq 2x + 4\}$.

23 a Find the area between the line $2y + x - 5 = 0$ and the hyperbola $y = \frac{2}{x}$.

b Find the area between the line $9y + 3x - 10 = 0$ and the hyperbola $y = \frac{1}{3x}$.

24 a Prove using calculus methods that the area of a right-angled triangle of base length a and height b is given by $\frac{1}{2}ab$.

b Prove using calculus that the area of a trapezium of side lengths a and b, and width h is equal to $\frac{h}{2}(a + b)$.

MASTER

25 a Consider the graphs of $y = \frac{x^2}{3}$ and $y = 4\sin\left(\frac{x}{2}\right)$.

i Find the coordinates of the point of intersection between the graphs.

ii Determine the area between the graphs, the origin and this point of intersection, giving your answer correct to 4 decimal places.

b Consider the graphs of $y = 5e^{-\frac{x}{4}}$ and $y = \frac{x}{2}$.

i Find the coordinates of the point of intersection between the graphs.

ii Determine the area between the curves, the y-axis and this point of intersection, giving your answer correct to 4 decimal places.

26 **a** Consider the graphs of $y = 23e^{\frac{x}{2}}$ and $y = 45\sin\left(\frac{2x}{3}\right) + 42$ for $x \geq 0$.

i Find the coordinates of the point of intersection between the graphs.

ii Determine the area between the curves, the y-axis and this point of intersection, giving your answer correct to 4 decimal places.

b Consider the graphs of $y = \frac{190}{x^2} - 5$ and $y = -32\cos\left(\frac{x}{5}\right)$ for $x \geq 0$.

i Find the coordinates of the first two points of intersection.

ii Determine the area between the curves and these first two points of intersection, giving your answer correct to 4 decimal places.

8.3 Linear substitutions

A linear substitution is of the form $u = ax + b$.

studyon

Units 3 & 4
AOS 3
Topic 2
Concept 5

Antiderivatives using linear substitution
Concept summary
Practice questions

Finding integrals of the form $\int (ax+b)^n dx$ where $n \in Z$

Integrals of the form $\int (ax + b)^n dx$, where a and b are non-zero real numbers and n is a positive integer, can be performed using a linear substitution with $u = ax + b$. The derivative $\frac{du}{dx} = a$ is a constant, and this constant factor can be taken outside the integral sign by the properties of indefinite and definite integrals. The integration process can then be completed in terms of u. Note that since u has been introduced in this solution process, the final answer must be given back in terms of the original variable, x.

WORKED EXAMPLE 8

Find $\int (2x - 5)^4 dx$.

THINK	WRITE
1 Although we could expand and integrate term by term, it is preferable and easier to use a linear substitution.	Let $u = 2x - 5$. $\int (2x-5)^4 dx = \int u^4 dx$
2 Differentiate u with respect to x.	$u = 2x - 5$ $\frac{du}{dx} = 2$
3 Express dx in terms of du by inverting both sides.	$\frac{dx}{du} = \frac{1}{2}$ $dx = \frac{1}{2}du$
4 Substitute for dx.	$\int u^4 dx = \int u^4 \frac{1}{2} du$
5 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$= \frac{1}{2}\int u^4 du$

6 Perform the integration using $\int u^n du = \frac{1}{n+1}u^{n+1}$ with $n = 4$ so that $n + 1 = 5$, and add in the constant $+c$.

$\frac{1}{2} \times \frac{1}{5}u^5 + c$

$= \frac{1}{10}u^5 + c$

7 Substitute back for $u = 2x - 5$ and express the final answer in terms of x only and an arbitrary constant $+c$.

$\int (2x - 5)^4 dx = \frac{1}{10}(2x - 5)^5 + c$

Finding particular integrals of the form $\int (ax + b)^n dx$ where $n \in Q$

Integrals of the form $\int (ax + b)^n dx$, where a and b are non-zero real numbers, and n is a rational number, can also be performed using a linear substitution with $u = ax + b$. First express the integrand (the function being integrated) as a power using the index laws.

WORKED EXAMPLE 9

The gradient of a curve is given by $\frac{1}{\sqrt{4x+9}}$. Find the particular curve that passes through the origin.

THINK

1 Recognise that the gradient of a curve is given by $\frac{dy}{dx}$.

WRITE

$\frac{dy}{dx} = \frac{1}{\sqrt{4x+9}}$

2 Integrate both sides to give an expression for y.

$y = \int \frac{1}{\sqrt{4x+9}} dx$

3 Use index laws to express the integrand as a function to a power and use a linear substitution.

Let $u = 4x + 9$.

$y = \int (4x + 9)^{-\frac{1}{2}} dx$

$y = \int u^{-\frac{1}{2}} dx$

4 The integral cannot be done in this form, so differentiate.

$u = 4x + 9$

$\frac{du}{dx} = 4$

5 Express dx in terms of du by inverting both sides.

$\frac{dx}{du} = \frac{1}{4}$

$dx = \frac{1}{4}du$

6 Substitute for dx.

$y = \int u^{-\frac{1}{2}} \frac{1}{4} du$

7 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$y = \frac{1}{4}\int u^{-\frac{1}{2}} du$

8 Perform the integration process using $\int u^n du = \frac{1}{n+1}u^{n+1}$ with $n = -\frac{1}{2}$ so that $n + 1 = \frac{1}{2}$, and add in the constant $+c$.	$y = \frac{1}{4} \times \frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + c$ $y = \frac{2}{4}u^{\frac{1}{2}} + c$
	$y = \frac{1}{2}\sqrt{u} + c$
9 Simplify and substitute for $u = 4x + 9$ to express the answer in terms of x and an arbitrary constant $+c$.	$y = \frac{1}{2}\sqrt{4x+9} + c$
10 The arbitrary constant $+c$ in this particular case can be found using the given condition that the curve passes through the origin.	Substitute $y = 0$ and $x = 0$ to find c: $0 = \frac{1}{2}\sqrt{0+9} + c$ $c = -\frac{3}{2}$
11 Substitute back for c.	$y = \frac{1}{2}\sqrt{4x+9} - \frac{3}{2}$
12 State the equation of the particular curve in a factorised form.	$y = \frac{1}{2}(\sqrt{4x+9} - 3)$

Finding integrals of the form $\int (ax + b)^n dx$ when $n = -1$

Integrals of the form $\int (ax + b)^n dx$, when $n = -1$, $a \neq 0$ and $b \in R$, involve the logarithm function, since $\int \frac{1}{x} dx = \log_e\left(|x|\right) + c$.

WORKED EXAMPLE 10 Antidifferentiate $\frac{1}{5x+4}$.

THINK	WRITE
1 Write the required integral.	$\int \frac{1}{5x+4} dx$
2 Use a linear substitution.	Let $u = 5x + 4$. $\int \frac{1}{5x+4} dx = \int \frac{1}{u} dx$
3 The integral cannot be done in this form, so differentiate.	$u = 5x + 4$ $\frac{du}{dx} = 5$
4 Express dx in terms of du by inverting both sides.	$\frac{dx}{du} = \frac{1}{5}$ $dx = \frac{1}{5} du$
5 Substitute for dx.	$\int \frac{1}{5x+4} dx = \int \frac{1}{u} \times \frac{1}{5} du$
6 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$= \frac{1}{5}\int \frac{1}{u} du$

7 Perform the integration process using $\int \frac{1}{u} du = \log_e |u| + c$.

$\int \frac{1}{5x+4} dx = \frac{1}{5} \log_e \left(|u| \right) + c$

8 Simplify and substitute for $u = 5x + 4$ to express the final answer in terms of x only and an arbitrary constant $+c$, in simplest form.

$\int \frac{1}{5x+4} dx = \frac{1}{5} \log_e \left(|5x + 4| \right) + c$

studyon

Units 3 & 4
AOS 3
Topic 2
Concept 3

Antiderivatives involving logarithms
Concept summary
Practice questions

Finding integrals of the form $\int (ax + b)^n dx$ where $n \in Q$

We can generalise the results from the last three examples to state:

$$\int (ax+b)^n dx = \begin{cases} \frac{1}{a(n+1)}(ax+b)^{n+1} + c & n \neq -1 \\ \frac{1}{a} \log_e \left(|ax + b| \right) + c & n = -1 \end{cases}$$

Evaluating definite integrals using a linear substitution

When we evaluate a definite integral, the result is a number. This number is also independent of the original variable used. When using a substitution, change the terminals to the new variable and evaluate this definite integral in terms of the new variable with new terminals. The following worked example clarifies this process.

WORKED EXAMPLE 11

Evaluate $\int_0^1 \frac{4}{(3x+2)^2} dx$.

THINK

1 Write the integrand as a power using index laws and transfer the constant factor outside the front of the integral sign.

WRITE

$\int_0^1 \frac{4}{(3x+2)^2} dx$

$= 4 \int_0^1 (3x + 2)^{-2} dx$

2 Use a linear substitution. Note that the terminals in the definite integral refer to x-values.

Let $u = 3x + 2$.

$4 \int_0^1 (3x+2)^{-2} dx = 4 \int_{x=0}^{x=1} u^{-2} dx$

3 The integral cannot be done in this form, so differentiate. Express dx in terms of du by inverting both sides.

$u = 3x + 2$

$\frac{du}{dx} = 3$

$\frac{dx}{du} = \frac{1}{3}$

$dx = \frac{1}{3} du$

4 Change the terminals to the new variable.	When $x = 0 \Rightarrow u = 2$ and when $x = 1 \Rightarrow u = 5$.
5 Substitute for dx and the new terminals.	$4\int_{u=2}^{u=5} u^{-2}\frac{1}{3}du$
6 Transfer the constant factor outside the front of the integral sign.	$= \frac{4}{3}\int_{2}^{5} u^{-2}du$
7 The value of this definite integral has the same value as the original definite integral. There is no need to substitute back for x, and there is no need for the arbitrary constant when evaluating a definite integral.	$= \frac{4}{3}\left[-\frac{1}{u}\right]_2^5$
8 Evaluate this definite integral.	$= \frac{4}{3}\left[-\frac{1}{5} - \left(-\frac{1}{2}\right)\right]$ $= \frac{4}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$ $= \frac{4}{3}\left(\frac{5-2}{10}\right)$
9 State the final result.	$\int_0^1 \frac{4}{(3x+2)^2}dx = \frac{2}{5}$

Finding integrals using a back substitution

Integrals of the form $\int x(ax+b)^n dx$ can also be performed using a linear substitution with $u = ax + b$. Since the derivative $\frac{du}{dx} = a$ is a constant, this constant can be taken outside the integral sign. However, we must express the integrand in terms of u only before integrating. We can do this by expressing x in terms of u; that is, $x = \frac{1}{a}(u - b)$. However, the final result for an indefinite integral must be given in terms of the original variable, x.

WORKED EXAMPLE 12

Find:

a $\int x(2x-5)^4 dx$

b $\int \frac{6x-5}{4x^2-12x+9}dx$.

THINK	WRITE
a 1 Use a linear substitution.	**a** Let $u = 2x - 5$. $\int x(2x-5)^4 dx = \int xu^4 dx$

2 The integral cannot be done in this form, so differentiate and express dx in terms of du by inverting both sides.

$$u = 2x - 5$$
$$\frac{du}{dx} = 2$$
$$\frac{dx}{du} = \frac{1}{2}$$
$$dx = \tfrac{1}{2}du$$

3 Express x in terms of u.

$$2x = u + 5$$
$$x = \tfrac{1}{2}(u + 5)$$

4 Substitute for x and dx.

$$\int x(2x-5)^4 dx = \int \tfrac{1}{2}(u+5)u^4\,\tfrac{1}{2}du$$

5 Use the properties of indefinite integrals to transfer the constant factors outside the front of the integral sign and expand the integrand.

$$= \tfrac{1}{4}\int (u^5 + 5u^4)du$$

6 Perform the integration, integrating term by term and adding in the constant.

$$= \tfrac{1}{4} \times \left(\tfrac{1}{6}u^6 + u^5\right) + c$$

7 Simplify the result by expanding.

$$= \tfrac{1}{24}u^6 + \tfrac{1}{4}u^5 + c$$

8 Substitute $u = 2x - 5$ and express the final answer in terms of x only and an arbitrary constant $+c$.

$$\int x(2x-5)^4 dx = \tfrac{1}{24}(2x-5)^6 + \tfrac{1}{4}(2x-5)^5 + c$$

9 Alternatively, the result can be expressed in a simplified form by taking out the common factors.

$$\frac{1}{24}u^6 + \frac{1}{4}u^5 + c = \frac{u^5}{24}(u + 6) + c$$

10 Substitute back for $u = 2x - 5$ and simplify.

$$= \frac{(2x-5)^5}{24}(2x - 5 + 6) + c$$

11 Express the final answer in terms of x only and an arbitrary constant $+c$.

$$\int x(2x-5)^4 dx = \tfrac{1}{24}(2x-5)^5(2x+1) + c$$

b 1 Factorise the denominator as a perfect square.

b $$\int \frac{6x-5}{4x^2 - 12x + 9}dx = \int \frac{6x-5}{(2x-3)^2}dx$$

2 Use a linear substitution.

Differentiate and express dx in terms of du by inverting both sides.

Let $u = 2x - 3$.

$$\frac{du}{dx} = 2$$
$$\frac{dx}{du} = \frac{1}{2}$$
$$dx = \tfrac{1}{2}du$$

3 Express the numerator $6x - 5$ in terms of u.

$$2x = u + 3$$
$$6x = 3(u + 3)$$
$$6x = 3u + 9$$
$$6x - 5 = 3u + 4$$

4 Substitute for $6x-5$, u and dx.

$$\int \frac{6x-5}{(2x-3)^2}dx = \int \frac{3u+4}{u^2} \times \frac{1}{2}du$$

5 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$= \frac{1}{2}\int \left(\frac{3u+4}{u^2}\right)du$$

6 Simplify the integrand.

$$= \frac{1}{2}\int \left(\frac{3}{u} + \frac{4}{u^2}\right)du$$

7 Write in index form.

$$= \frac{1}{2}\int \left(\frac{3}{u} + 4u^{-2}\right)du$$

8 Perform the integration, adding in the constant. The first term is a log, but in the second term, we use $\int u^n du = \frac{1}{n+1}u^{n+1}$ with $n = -2$, so that $n + 1 = -1$.

$$= \frac{1}{2}\left(3\log_e\left(|u|\right) - 4u^{-1}\right) + c$$

$$= \frac{1}{2}\left(3\log_e\left(|u|\right) - \frac{4}{u}\right) + c$$

9 Substitute $u = 2x - 3$ and express the final answer in terms of x only and an arbitrary constant $+c$, as before.

$$\int \frac{6x-5}{4x^2-12x+9}dx = \frac{3}{2}\log_e\left(|2x-3|\right) - \frac{2}{2x-3} + c$$

Definite integrals using a back substitution

WORKED EXAMPLE 13

Evaluate $\int_0^8 \frac{x}{\sqrt{2x+9}}dx$.

THINK / **WRITE**

1 Write the integrand as a power, using index laws.

$$\int_0^8 \frac{x}{\sqrt{2x+9}}dx = \int_0^8 x(2x+9)^{-\frac{1}{2}}dx$$

2 Use a linear substitution,

Let $u = 2x + 9$.

$$\int_0^8 x(2x+9)^{-\frac{1}{2}}dx = \int_{x=0}^{x=8} xu^{-\frac{1}{2}}dx$$

3 The integral cannot be done in this form, so differentiate. Express dx in terms of du by inverting both sides.

$$\frac{du}{dx} = 2$$

$$\frac{dx}{du} = \frac{1}{2}$$

$$dx = \tfrac{1}{2}du$$

4 Express x back in terms of u.

$$u = 2x + 9$$

$$2x = u - 9$$

$$x = \tfrac{1}{2}(u-9)$$

5 Change the terminals to the new variable.

When $x = 0$, $u = 9$, and when $x = 8$, $u = 25$.

6 Substitute for dx, x and the new terminals.

$$\int_{x=0}^{x=8} xu^{-\frac{1}{2}}dx$$

$$= \int_{u=9}^{u=25} \tfrac{1}{2}(u-9)u^{-\frac{1}{2}}\tfrac{1}{2}du$$

7 Transfer the constant factors outside the front of the integral sign.

$$= \frac{1}{4}\int_{9}^{25}(u-9)u^{-\frac{1}{2}}du$$

8 Expand the integrand.

$$= \frac{1}{4}\int_{9}^{25}\left(u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}\right)du$$

9 Perform the integration.

$$= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}} - 18u^{\frac{1}{2}}\right]_9^{25}$$

10 Evaluate the definite integral.

$$= \frac{1}{4}\left[\left(\frac{2}{3}(25)^{\frac{3}{2}} - 18(25)^{\frac{1}{2}}\right) - \left(\frac{2}{3}(9)^{\frac{3}{2}} - 18(9)^{\frac{1}{2}}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}\times 125 - 18\times 5\right) - \left(\frac{2}{3}\times 27 - 18\times 3\right)\right]$$

11 State the final result.

$$\int_0^8 \frac{x}{\sqrt{2x+9}}dx = \frac{22}{3}$$

WORKED EXAMPLE 14 Find $\int \frac{2x}{4x-3}dx$.

THINK / **WRITE**

Method 1

1 Use a linear substitution. Differentiate and express dx in terms of du by inverting both sides.

$$u = 4x - 3$$
$$\frac{du}{dx} = 4$$
$$\frac{dx}{du} = \frac{1}{4}$$
$$dx = \frac{1}{4}du$$

2 Express the numerator $2x$ in terms of u.

$$4x = u + 3$$
$$2x = \frac{1}{2}(u+3)$$

3 Substitute for $2x$, u and dx.

$$\int \frac{2x}{4x-3}dx = \int \frac{\frac{1}{2}(u+3)}{u}\times\frac{1}{4}du$$

4 Use the properties of indefinite integrals to transfer the constant factors outside the front of the integral sign.

$$= \frac{1}{8}\int\left(\frac{u+3}{u}\right)du$$

5 Simplify the integrand.

$$= \frac{1}{8}\int\left(1 + \frac{3}{u}\right)du$$

6 Perform the integration, adding in the constant.

$$= \frac{1}{8}\left(u + 3\log_e\left(|u|\right)\right) + c$$

$$= \frac{u}{8} + \frac{3}{8}\log_e\left(|u|\right) + c$$

7 Substitute $u = 4x - 3$ and express the final answer in terms of x only and an arbitrary constant $+c$.

$$\int\frac{2x}{4x-3}dx = \frac{4x-3}{8} + \frac{3}{8}\log_e\left(|4x-3|\right) + c$$

Method 2

1 Express the numerator as a multiple of the denominator (in effect, use long division to divide the denominator into the numerator).

$$\int\frac{2x}{4x-3}dx = \frac{1}{2}\int\frac{4x}{4x-3}dx$$

$$= \frac{1}{2}\int\frac{(4x-3)+3}{4x-3}dx$$

2 Simplify the integrand.

$$= \frac{1}{2}\int\left(1 + \frac{3}{4x-3}\right)dx$$

3 Perform the integration, adding in the constant.

$$= \tfrac{1}{2}\left(x + \tfrac{3}{4}\log_e\left(|4x-3|\right)\right) + c$$

4 State the final answer.

$$\int\frac{2x}{4x-3}dx = \frac{x}{2} + \frac{3}{8}\log_e\left(|4x-3|\right) + c$$

5 Although the two answers do not appear to be the same, the log terms are identical. However, since $\frac{4x-3}{8} = \frac{x}{2} - \frac{3}{8}$, the two answers are equivalent in x and differ in the constant only, $c_1 = -\frac{3}{8} + c$.

$$= \frac{4x-3}{8} + \frac{3}{8}\log_e\left(|4x-3|\right) + c$$

$$= \frac{x}{2} + \frac{3}{8}\log_e\left(|4x-3|\right) + c_1$$

The situation above can often happen when evaluating indefinite integrals. Answers may not appear to be identical, but after some algebraic or trigonometric simplification, they are revealed to be equivalent and may differ by a constant only.

EXERCISE 8.3 Linear substitutions

PRACTISE

1 WE8 Find $\int(5x-9)^6\,dx$.

2 Find $\int(3x+4)^7dx$.

3 WE9 A particular curve has a gradient equal to $\frac{1}{\sqrt{16x+25}}$. Find the particular curve that passes through the origin.

4 Given that $f'(x) = \frac{1}{(3x-7)^2}$ and $f(2) = 3$, find the value of $f(1)$.

5 WE10 Antidifferentiate $\frac{1}{3x-5}$.

6 Find an antiderivative of $\frac{1}{7-2x}$.

7 WE11 Evaluate $\int_{-1}^{0} \frac{9}{(2x+3)^3}dx$.

8 Find the area bounded by the graph of $y = \frac{6}{\sqrt{3x+4}}$, the coordinate axes and $x = 4$.

9 WE12 Find:

a $\int x(5x-9)^5 dx$

b $\int \frac{2x-1}{9x^2-24x+16}dx$.

10 Find:

a $\int \frac{x}{(2x+7)^3}dx$

b $\int \frac{x}{\sqrt{6x+5}}dx$.

11 WE13 Evaluate $\int_{0}^{5} \frac{x}{\sqrt{3x+1}}dx$.

12 Evaluate $\int_{-1}^{1} \frac{15x}{(3x+2)^2}dx$.

13 WE14 Find $\int \frac{6x}{3x+4}dx$.

14 Evaluate $\int_{0}^{1} \frac{4x}{2x-5}dx$.

CONSOLIDATE

15 Integrate each of the following with respect to x.

a $(3x+5)^6$

b $\frac{1}{(3x+5)^2}$

c $\frac{1}{(3x+5)^3}$

d $\frac{1}{\sqrt[3]{3x+5}}$

16 Find each of the following.

a $\int (6x+7)^8 dx$

b $\int \frac{1}{\sqrt{6x+7}}dx$

c $\int \frac{1}{6x+7}dx$

d $\int \frac{1}{(6x+7)^2}dx$

17 Integrate each of the following with respect to x.

a $x(3x+5)^6$

b $\frac{x}{(3x+5)^2}$

c $\frac{x}{(3x+5)^3}$

d $\frac{x}{\sqrt[3]{3x+5}}$

18 Find each of the following.

a $\int x(6x+7)^8 dx$

b $\int \frac{x}{\sqrt{6x+7}}dx$

c $\int \frac{x}{6x+7}dx$

d $\int \frac{x}{(6x+7)^2}dx$

19 **a** Given that $\frac{dx}{dt} = \frac{1}{(2-5t)^2}$ and $x(0) = 0$, express x in terms of t.

b A certain curve has its gradient given by $\frac{1}{\sqrt{3-2x}}$ for $x < \frac{3}{2}$. If the point $\left(-\frac{1}{2}, -2\right)$ lies on the curve, find the equation of the curve.

c Given that $f'(x) = \frac{3}{3-2x}$ and that $f(0) = 0$, find $f(1)$.

d A certain curve has a gradient of $\frac{x}{\sqrt{2x+9}}$. Find the particular curve that passes through the origin.

20 Evaluate each of the following.

a $\int_1^2 (3x-4)^5 dx$ **b** $\int_1^2 x(3x-4)^5 dx$ **c** $\int_0^{13} \frac{1}{\sqrt[3]{2t+1}} dt$ **d** $\int_0^{13} \frac{t}{\sqrt[3]{2t+1}} dt$

21 **a** Sketch the graph of $y = \sqrt{2x+1}$. Find the area between the curve, the coordinates axes and the line $x = 4$.

b Sketch the graph of $y = \frac{1}{3x+5}$. Determine the area bounded by the curve, the coordinate axes and $x = 3$.

c Sketch the curve $y = \frac{1}{(2x+3)^2}$. Find the area bounded by this curve and the x-axis between $x = 1$ and $x = 2$.

d Find the area bounded by the curve $y = \frac{x}{\sqrt{16-3x}}$, the coordinate axes and $x = 5$.

22 **a** Find the area of the region enclosed by the curves with the equations:

i $y = 4\sqrt{x-1}$ and $y = 4\sqrt{3-x}$

ii $y^2 = 16(x-1)$ and $y^2 = 16(3-x)$.

b Find the area between the curve $y^2 = 4 - x$ and the y-axis.

c Determine the area of the loop with the equation $y^2 = x^2(4-x)$.

d **i** Find the area between the curve $y^2 = a - x$ where $a > 0$ and the y-axis.

ii Determine the area of the loop with the equation $y^2 = x^2(a-x)$, where $a > 0$.

23 Given that a and b are non-zero real constants, find each of the following.

a $\int \sqrt{ax+b}\, dx$ **b** $\int x\sqrt{ax+b}\, dx$ **c** $\int \frac{1}{ax+b} dx$ **d** $\int \frac{x}{ax+b} dx$

24 Given that a and b are non-zero real constants, find each of the following.

a $\int \frac{1}{\sqrt{ax+b}} dx$ **b** $\int \frac{x}{\sqrt{ax+b}} dx$ **c** $\int \frac{1}{(ax+b)^2} dx$ **d** $\int \frac{x}{(ax+b)^2} dx$

MASTER

25 Given that a, b, c and d are non-zero real constants, find each of the following.

a $\int \frac{cx+d}{ax+b} dx$ **b** $\int \frac{cx+d}{(ax+b)^2} dx$ **c** $\int \frac{cx^2+d}{(ax+b)^2} dx$ **d** $\int \frac{cx^2+d}{ax+b} dx$

26 Given that a and b are non-zero real constants, find each of the following.

a $\int \frac{x^2}{ax+b} dx$ **b** $\int \frac{x^2}{(ax+b)^2} dx$ **c** $\int \frac{x^2}{(ax+b)^3} dx$ **d** $\int \frac{x^2}{\sqrt{ax+b}} dx$

8.4 Other substitutions

Non-linear substitutions

studyon

Units 3 & 4
AOS 3
Topic 2
Concept 3

Integration by substitution
Concept summary
Practice questions

The basic idea of a non-linear substitution is to reduce the integrand to one of the standard u forms shown in the table below. Remember that after making a substitution, x or the original variable should be eliminated. The integral must be entirely in terms of the new variable u.

$f(u)$	$\int f(u)du$
$u^n, n \neq -1$	$\dfrac{u^{n+1}}{n+1}$
$\dfrac{1}{u}$	$\log_e(\lvert u \rvert)$
e^u	e^u
$\cos(u)$	$\sin(u)$
$\sin(u)$	$-\cos(u)$
$\sec^2(u)$	$\tan(u)$

WORKED EXAMPLE 15

Find $\int \dfrac{3x}{(x^2+9)^2}dx$.

THINK	WRITE
1 Write the integrand as a power using index laws.	$\int 3x(x^2+9)^{-2}dx$
2 Use a non-linear substitution.	Let $u = x^2 + 9$. $\int 3x(x^2+9)^{-2}dx = \int 3xu^{-2}dx$
3 The integral cannot be done in this form, so differentiate. Express dx in terms of du by inverting both sides.	$\dfrac{du}{dx} = 2x$; $\dfrac{dx}{du} = \dfrac{1}{2x}$; $dx = \dfrac{1}{2x}du$
4 Substitute for dx, noting that the terms involving x will cancel.	$\int 3xu^{-2}dx = \int 3xu^{-2} \times \dfrac{1}{2x}du$
5 Transfer the constant factors outside the front of the integral sign.	$= \frac{3}{2}\int u^{-2}du$
6 The integral can now be done. Antidifferentiate using $\int u^n du = \dfrac{u^{n+1}}{n+1}$ with $n = -2$, so that $n + 1 = -1$.	$= -\frac{3}{2}u^{-1} + c$

7 Write the expression with positive indices.

$$= -\frac{3}{2u} + c$$

8 Substitute back for x, and state the final result.

$$\int \frac{3x}{(x^2+9)^2}dx = -\frac{3}{2(x^2+9)} + c$$

Integrals involving the logarithm function

The result $\int u^n du = \frac{u^{n+1}}{n+1}$ is true provided that $n \neq -1$. When $n = -1$ we have the special case $\int \frac{1}{u}du = \log_e\left(|u|\right) + c$.

WORKED EXAMPLE 16 Find $\int \frac{x-3}{x^2-6x+13}dx$.

THINK	WRITE/DRAW
Method 1	
1 Use a non-linear substitution.	Let $u = x^2 - 6x + 13$. $\frac{du}{dx} = 2x - 6$ $= 2(x-3)$ $\frac{dx}{du} = \frac{1}{2(x-3)}$ $dx = \frac{1}{2(x-3)}du$
2 Substitute for dx and u, noting that the terms involving x cancel.	$\int \frac{x-3}{x^2-6x+13}dx = \int \frac{x-3}{u} \times \frac{1}{2(x-3)}du$
3 Transfer the constant outside the front of the integral sign.	$= \frac{1}{2}\int \frac{1}{u}du$
4 The integration can now be done. Antidifferentiate using $\int \frac{1}{u}du = \log_e\left(|u|\right)$.	$= \frac{1}{2}\log_e\left(|u|\right) + c$
5 In this case, since $x^2 - 6x + 13 = (x-3)^2 + 4 > 0$, for all values of x, the modulus is not needed. Substitute back for x, and state the final result. Note that since $\frac{d}{dx}[\log_e(f(x))] = \frac{f'(x)}{f(x)}$, it follows that $\int \frac{f'(x)}{f(x)}dx = \log_e\left(|f(x)|\right) + c$.	$\int \frac{x-3}{x^2-6x+13}dx = \frac{1}{2}\log_e(x^2-6x+13) + c$

Method 2

1 To make the numerator the derivative of the denominator, multiply both the numerator and the denominator by 2, and take the constant factor outside the front of the integral sign.

$$\int \frac{x-3}{x^2-6x+13}dx$$
$$= \int \frac{2(x-3)}{2(x^2-6x+13)}dx$$
$$= \frac{1}{2}\int \frac{2x-6}{x^2-6x+13}dx$$

2 Use the result $\int \frac{f'(x)}{f(x)}dx = \log_e\left(|f(x)|\right) + c$, with $f(x) = x^2 - 6x + 13$.

$$\int \frac{x-3}{x^2-6x+13}dx = \frac{1}{2}\log_e\left(x^2-6x+13\right) + c$$

Examples involving trigonometric functions

For trigonometric functions we use the results $\int \cos(u)du = \sin(u) + c$ and $\int \sin(u)du = -\cos(u) + c$.

WORKED EXAMPLE 17 Find $\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$.

THINK

1 Use a non-linear substitution. Let $u = \frac{1}{x}$.

We choose this as the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$, which is present in the integrand.

WRITE

$$u = \frac{1}{x}$$
$$= x^{-1}$$
$$\frac{du}{dx} = -x^{-2}$$
$$= -\frac{1}{x^2}$$
$$\frac{dx}{du} = -x^2$$
$$dx = -x^2 dx$$

2 Substitute for u and dx, noting that the x^2 terms cancel.

$$\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$$
$$= \int \frac{1}{x^2}\cos(u) \times -x^2 du$$

3 Transfer the negative sign outside the integral sign.

$$= -\int \cos(u)du$$

4 Antidifferentiate, using $\int \cos(u)du = \sin(u) + c$.

$$= -\sin(u) + c$$

5 Substitute back for x and state the answer.

$$\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx = -\sin\left(\frac{1}{x}\right) + c$$

Examples involving exponential functions

For exponential functions we use the result $\int e^u du = e^u + c$.

WORKED EXAMPLE 18

Find:

a $\int \sin(2x)\, e^{\cos(2x)} dx$

b $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$.

THINK	WRITE
a 1 Use a non-linear substitution. Let $u = \cos(2x)$. Choose this as the derivative of $\cos(2x)$ is $-2\sin(2x)$, which is present in the integrand. Note that the substitution $u = \sin(2x)$ will not work.	**a** $u = \cos(2x)$ $\frac{du}{dx} = -2\sin(2x)$ $\frac{dx}{du} = \frac{-1}{2\sin(2x)}$ $dx = \frac{-1}{2\sin(2x)} du$
2 Substitute for u and dx.	$\int \sin(2x) e^{\cos(2x)} dx$ $= \int \sin(2x) e^u \frac{-1}{2\sin(2x)} du$
3 Transfer the constant factor outside the integral sign.	$= -\frac{1}{2}\int e^u du$
4 Antidifferentiate using $\int e^u du = e^u + c$.	$= -\frac{1}{2}e^u + c$
5 Substitute $u = \cos(2x)$ and state the answer.	$\int \sin(2x) e^{\cos(2x)} dx = -\frac{1}{2}e^{\cos(2x)} + c$
b 1 Use a non-linear substitution. Let $u = \sqrt{x}$, but only replace u in the numerator. Express dx in terms of du by inverting both sides.	**b** $u = \sqrt{x} = x^{\frac{1}{2}}$ $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $\frac{dx}{du} = 2\sqrt{x}$ $dx = 2\sqrt{x}\, du$
2 Substitute for u and dx, noting that the $\sqrt{x}$ terms cancel.	$\int \frac{\sin(\sqrt{x})}{(\sqrt{x})} dx = \int \sin(\sqrt{x}) \times \frac{1}{\sqrt{x}} dx$ $= \int \sin(u) \times \frac{1}{\sqrt{x}} dx \times 2\sqrt{x} du$
3 Transfer the constant factor outside the integral sign.	$= 2\int \sin(u) du$
4 Antidifferentiate using $\int \sin(u) du = -\cos(u) + c$.	$= -2\cos(u) + c$
5 Substitute $u = \sqrt{x}$ and state the answer.	$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx = -2\cos(\sqrt{x}) + c$

Definite integrals involving non-linear substitutions

When evaluating a definite integral, recall that the result is a number independent of the original or dummy variable. In these cases, instead of substituting the original variable back into the integral, change the terminals and work with the new definite integral obtained.

WORKED EXAMPLE 19

Evaluate $\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2 + 4}} dt$.

THINK	WRITE
1 Write the integrand as a power using index laws.	$\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2 + 4}} dt$ $\int_0^{\sqrt{5}} t(t^2 + 4)^{-\frac{1}{2}} dt$
2 Use a non-linear substitution.	Let $u = t^2 + 4$. $\int_0^{\sqrt{5}} t(t^2 + 4)^{-\frac{1}{2}} dt = \int_{t=0}^{t=\sqrt{5}} tu^{-\frac{1}{2}} dt$
3 The integral cannot be done in this form, so differentiate. Express *dt* in terms of *du* by inverting both sides.	$\frac{du}{dt} = 2t$ $\frac{dt}{du} = \frac{1}{2t}$ $dt = \frac{1}{2t} du$
4 Change the terminals to the new variable.	When $t = 0$, $u = 4$, and when $t = \sqrt{5}$, $u = 9$.
5 Substitute for *dt* and the new terminals, noting that the dummy variable *t* cancels.	$\int_{t=0}^{t=\sqrt{5}} tu^{-\frac{1}{2}} dt = \int_{u=4}^{u=9} tu^{-\frac{1}{2}} \frac{1}{2t} du$
6 Transfer the multiplying constant outside the front of the integral sign.	$= \frac{1}{2}\int_4^9 u^{-\frac{1}{2}} du$
7 Perform the integration using $\int u^n du = \frac{u^{n+1}}{n + 1}$ with $n = -\frac{1}{2}$, so that $n + 1 = \frac{1}{2}$.	$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_4^9$ $= \left[u^{\frac{1}{2}}\right]_4^9$

8 Evaluate the definite integral.	$= [\sqrt{9} - \sqrt{4}]$ $= 3 - 2$ $= 1$
9 State the answer.	$\displaystyle\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2+4}}\,dt = 1$

Definite integrals involving inverse trigonometric functions

Recall that $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}}$, $\dfrac{d}{dx}\left(\cos^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{-1}{\sqrt{a^2 - x^2}}$ for $a > 0$ and $|x| < a$ and $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2 + x^2}$ for $x \in R$.

WORKED EXAMPLE 20

Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{\cos^{-1}(2x)}{\sqrt{1-4x^2}}\,dx$.

THINK	WRITE
1 Use a non-linear substitution.	Let $u = \cos^{-1}(2x)$. $\displaystyle\int_0^{\frac{1}{2}} \frac{\cos^{-1}(2x)}{\sqrt{1-4x^2}}\,dx = \int_{x=0}^{x=\frac{1}{2}} \frac{u}{\sqrt{1-4x^2}}\,dx$
2 The integral cannot be done in this form, so differentiate. Express dx in terms of du by inverting both sides.	$\dfrac{du}{dx} = \dfrac{-2}{\sqrt{1-4x^2}}$ $\dfrac{dx}{du} = -\frac{1}{2}\sqrt{1-4x^2}$ $dx = -\frac{1}{2}\sqrt{1-4x^2}\,du$
3 Change the terminals to the new variable.	When $x = \frac{1}{2}$, $u = \cos^{-1}(1) = 0$, and when $x = 0$, $u = \cos^{-1}(0) = \dfrac{\pi}{2}$.
4 Substitute for dx and the new terminals, noting that the x terms cancel. Transfer the constant multiple outside the front of the integral sign.	$\displaystyle\int_{x=0}^{x=\frac{1}{2}} \frac{u}{\sqrt{1-4x^2}}\,dx$ $\displaystyle= -\frac{1}{2}\int_{u=\frac{\pi}{2}}^{u=0} \frac{1}{\sqrt{1-4x^2}}\,u\sqrt{1-4x^2}\,du$ $\displaystyle= -\frac{1}{2}\int_{\frac{\pi}{2}}^{0} u\,du$

5 Using the properties of the definite integral, swap the terminals to change the sign.

$$\int_a^b f(t)dt = -\int_b^a f(t)dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} u\,du$$

6 Perform the integration.

$$= \frac{1}{2}\left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[u^2\right]_0^{\frac{\pi}{2}}$$

7 Evaluate the definite integral.

$$= \frac{1}{4}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right]$$

8 State the answer.

$$\int_0^{\frac{1}{2}} \frac{\cos^{-1}(2x)}{\sqrt{1-4x^2}}dx = \frac{\pi^2}{16}$$

EXERCISE 8.4 Other substitutions

PRACTISE

1 WE15 Find $\int \frac{8x}{(x^2+16)^3}dx$.

2 Find $\int \frac{5x}{\sqrt{2x^2+3}}dx$.

3 WE16 Find $\int \frac{x+2}{x^2+4x+29}dx$.

4 Find $\int \frac{x^2}{x^3+9}dx$.

5 WE17 Find $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$.

6 Find $\int x\cos(x^2)dx$.

7 WE18 Find:

a $\int \cos(3x)e^{\sin(3x)}dx$

b $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}dx$.

8 a Find $\int \sec^2(2x)e^{\tan(2x)}dx$.

b Find $\int \frac{\log_e(3x)}{4x}dx$.

9 WE19 Evaluate $\int_0^{2\sqrt{2}} \frac{s}{\sqrt{2s^2+9}}ds$.

10 Evaluate $\int_0^1 \frac{p}{(3p^2+5)^2}\,dp$.

11 WE20 Evaluate $\int_0^{\frac{1}{3}} \frac{\sin^{-1}(3x)}{\sqrt{1-9x^2}}\,dx$.

12 Evaluate $\int_0^4 \frac{\tan^{-1}\left(\frac{x}{4}\right)}{16+x^2}\,dx$.

CONSOLIDATE

For questions **13–17**, find each of the indefinite integrals shown.

13 a $\int x(x^2+4)^5\,dx$

b $\int \frac{x}{x^2+4}\,dx$

c $\int \frac{x}{(x^2+9)^2}\,dx$

d $\int \frac{x}{\sqrt{x^2+9}}\,dx$

14 a $\int \frac{x^2}{(x^3+27)^3}\,dx$

b $\int \frac{x^2}{\sqrt{x^3+27}}\,dx$

c $\int \frac{x^2}{x^3+8}\,dx$

d $\int x^2(x^3+8)^3\,dx$

15 a $\int (x-2)(x^2-4x+13)^3\,dx$

b $\int \frac{x-2}{(x^2-4x+13)^2}\,dx$

c $\int \frac{4-x}{\sqrt{x^2-8x+25}}\,dx$

d $\int \frac{4-x}{x^2-8x+25}\,dx$

16 a $\int \frac{e^{2x}}{4e^{2x}+5}\,dx$

b $\int \frac{e^{-3x}}{(2e^{-3x}-5)^2}\,dx$

c $\int \frac{e^{-2x}}{(3e^{-2x}+4)^3}\,dx$

d $\int \frac{2e^{2x}+1}{(e^{2x}+x)^2}\,dx$

17 a $\int \frac{1}{x}\sin(\log_e(4x))\,dx$

b $\int \frac{1}{x}\cos(\log_e(3x))\,dx$

c $\int \frac{\sin^{-1}\left(\frac{x}{2}\right)}{\sqrt{4-x^2}}\,dx$

d $\int \frac{\tan^{-1}(2x)}{1+4x^2}\,dx$

18 a A certain curve has a gradient given by $x\sin(x^2)$. Find the equation of the particular curve that passes through the origin.

b If $\frac{dy}{dx} = \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)$ and when $x = \frac{4}{\pi}$, $y = 0$, find y when $x = \frac{3}{\pi}$.

c Given that $f'(x) = \frac{5-x}{x^2-10x+29}$ and $f(0) = 0$, find $f(1)$.

d If $\frac{dy}{dx} = \sin(2x)e^{\cos(2x)}$ and when $x = \frac{\pi}{4}$, $y = 0$, find y when $x = 0$.

For questions **19–21**, evaluate each of the definite integrals shown.

19 **a** $\displaystyle\int_3^5 \frac{3t}{\sqrt{t^2-9}}\,dt$ **b** $\displaystyle\int_0^2 \frac{x}{4x^2+9}\,dx$

c $\displaystyle\int_1^2 \frac{x}{(2x^2+1)^2}\,dx$ **d** $\displaystyle\int_{-1}^1 \frac{s}{\sqrt{2s^2+3}}\,ds$

20 **a** $\displaystyle\int_4^5 \frac{3-x}{x^2-6x+34}\,dx$ **b** $\displaystyle\int_2^3 \frac{2-x}{(x^2-4x+5)^2}\,dx$

c $\displaystyle\int_1^4 \frac{e^{\sqrt{p}}}{\sqrt{p}}\,dp$ **d** $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2\theta)e^{\tan(2\theta)}\,d\theta$

21 **a** $\displaystyle\int_0^{\frac{\pi}{4}} \sin(2\theta)e^{\cos(2\theta)}\,d\theta$ **b** $\displaystyle\int_{\frac{1}{2}}^{\frac{e}{2}} \frac{\log_e(2t)}{3t}\,dt$

c $\displaystyle\int_{\frac{6}{\pi}}^{\frac{3}{\pi}} \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$ **d** $\displaystyle\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

22 **a** The graph of $y = \dfrac{x}{x^2-c}$ has vertical asymptotes at $x = \pm 2$. Find the value of c and determine the area bounded by the curve, the x-axis and the lines $x = 3$ and $x = 5$.

b Find the area bounded by the curve $y = x\cos(x^2)$, the x-axis and the lines $x = 0$ and $x = \dfrac{\sqrt{\pi}}{2}$.

c Find the area bounded by the curve $y = xe^{-x^2}$, the x-axis and the lines $x = 0$ and $x = 2$.

d Find the area bounded by the curve $y = \dfrac{x}{\sqrt{x^2+4}}$, the x-axis and the lines $x = 0$ and $x = 2\sqrt{3}$.

MASTER

23 If $a, b \in R\backslash\{0\}$, then find each of the following.

a $\displaystyle\int \frac{x}{\sqrt{ax^2+b}}\,dx$ **b** $\displaystyle\int \frac{x}{(ax^2+b)^2}\,dx$

c $\displaystyle\int x(ax^2+b)^n\,dx \quad n \neq -1$ **d** $\displaystyle\int \frac{x}{ax^2+b}\,dx$

24 Deduce the following indefinite integrals, where $f(x)$ is any function of x.

a $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ **b** $\displaystyle\int \frac{f'(x)}{(f(x))^2}\,dx$ **c** $\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}}\,dx$ **d** $\displaystyle\int f'(x)e^{f(x)}\,dx$

8.5 Integrals of powers of trigonometric functions

studyon

Units 3 & 4
AOS 3
Topic 2
Concept 6

Antiderivatives with trigonometric identities
Concept summary
Practice questions

Introduction

In this section we examine indefinite and definite integrals involving powers of trigonometric functions of the form $\int \sin^m(kx)\cos^n(kx)dx$ where $m, n \in J$.

Integrals involving $\sin^2(kx)$ and $\cos^2(kx)$

The trigonometric double-angle formulas

$$(1) \quad \sin(2A) = 2\sin(A)\cos(A)$$

and

$$(2) \quad \cos(2A) = \cos^2(A) - \sin^2(A)$$
$$= 2\cos^2(A) - 1$$
$$= 1 - 2\sin^2(A)$$

are useful in integrating certain powers of trigonometric functions.
Rearranging (2) gives

$$(3) \quad \sin^2(A) = \tfrac{1}{2}(1 - \cos(2A)) \text{ and}$$

$$(4) \quad \cos^2(A) = \tfrac{1}{2}(1 + \cos(2A))$$

To integrate $\sin^2(kx)$ use (3); to integrate $\cos^2(kx)$ use (4).

WORKED EXAMPLE 21

Find:

a $\int \cos^2(3x)dx$

b $\int \sin^2(3x)\cos^2(3x)dx$.

THINK	WRITE
a 1 Use the double-angle formula $\cos^2(A) = \frac{1}{2}(1 + \cos(2A))$ with $A = 3x$. Transfer the constant factor outside the front of the integral sign.	**a** $\int \cos^2(3x)dx$ $= \frac{1}{2}\int(1 + \cos(6x))dx$
2 Integrate term by term, using $\int \cos(kx) = \frac{1}{k}\sin(kx) + c$ with $k = 6$.	$= \frac{1}{2}\left[x + \frac{1}{6}\sin(6x)\right] + c$
3 Expand and state the final result.	$\int \cos^2(3x)dx = \frac{x}{2} + \frac{1}{12}\sin(6x) + c$
b 1 Use the double-angle formula $2\sin(A)\cos(A) = \sin(2A)$ with $A = 3x$, so that $\sin^2(A)\cos^2(A) = \frac{1}{4}\sin^2(2A)$.	**b** $\int \sin^2(3x)\cos^2(3x)dx$ $= \frac{1}{4}\int(2\sin(3x)\cos(3x))^2dx$ $= \frac{1}{4}\int \sin^2(6x)dx$

2 Use the double-angle formula $\sin^2(A) = \frac{1}{2}(1 - \cos(2A))$ with $A = 6x$.

$$= \frac{1}{4}\int \frac{1}{2}(1 - \cos(12x))dx$$

$$= \frac{1}{8}\int (1 - \cos(12x))dx$$

3 Integrate term by term, using $\int \cos(kx) = \frac{1}{k}\sin(kx) + c$ with $k = 12$.

$$= \frac{1}{8}\left[x - \frac{1}{12}\sin(12x)\right] + c$$

4 Expand and state the final result.

$$\int \sin^2(3x)\cos^2(3x)dx = \frac{x}{8} - \frac{1}{96}\sin(12x) + c$$

Note that as well as the double-angle formulas, there are many other relationships between trigonometric functions, for example $\sin^2(A) + \cos^2(A) = 1$.

Often answers to integrals involving trigonometric functions can be expressed in several different ways, for example, as powers or multiple angles. Answers derived from CAS calculators may appear different, but often they are actually identical and differ in the constant term only.

Integrals involving $\sin(kx)\cos^m(kx)$ and $\cos(kx)\sin^m(kx)$ where $m > 1$

Integrals of the forms $\sin(kx)\cos^m(kx)$ and $\cos(kx)\sin^m(kx)$ where $m > 1$ can be performed using non-linear substitution, as described in the previous section.

WORKED EXAMPLE 22 Find $\int \sin(3x)\cos^4(3x)dx$.

THINK

1 Use a non-linear substitution. Let $u = \cos(3x)$. We choose this as the derivative of $\cos(3x)$ is $-3\sin(3x)$, which is present in the integrand.

WRITE

$$u = \cos(3x)$$

$$\frac{du}{dx} = -3\sin(3x)$$

$$\frac{dx}{du} = \frac{-1}{3\sin(3x)}$$

$$dx = \frac{-1}{3\sin(3x)}du$$

2 Substitute for u and dx, noting that the x terms cancel.

$$\int \sin(3x)\cos^4(3x)dx$$

$$= \int \sin(3x)u^4 \times \frac{-1}{3\sin(3x)}du$$

3 Transfer the constant factor outside the integral sign.

$$= -\frac{1}{3}\int u^4 du$$

4 Antidifferentiate.

$$= -\frac{1}{3} \times \frac{u^5}{5} + c$$

$$= -\frac{1}{15}u^5 + c$$

5 Substitute back for x and state the final result.

$$\int \sin(3x)\cos^4(3x)dx = -\frac{1}{15}\cos^5(3x) + c$$

Integrals involving odd powers

Antidifferentiation of $\sin^m(kx)\cos^n(kx)$ when at least one of m or n is an odd power can be performed using a non-linear substitution and the formula $\sin^2(A) + \cos^2(A) = 1$.

WORKED EXAMPLE 23 **Find an antiderivative of $\sin^3(2x)\cos^4(2x)$.**

THINK	WRITE
1 Write the required antiderivative.	$\int \sin^3(2x)\cos^4(2x)dx$
2 Break the odd power. $\sin^3(2x) = \sin(2x)\sin^2(2x)$, and $\sin^2(A) = 1 - \cos^2(A)$.	$\int \sin(2x)\sin^2(2x)\cos^4(2x)dx$ $= \int \sin(2x)(1 - \cos^2(2x))\cos^4(2x)dx$
3 Use a non-linear substitution. Let $u = \cos(2x)$. We choose this as the derivative of $\cos(2x)$ is $-2\sin(2x)$, which is present in the integrand and will cancel.	$u = \cos(2x)$ $\frac{du}{dx} = -2\sin(2x)$ $\frac{dx}{du} = \frac{-1}{2\sin(2x)}$ $dx = \frac{-1}{2\sin(2x)}du$
4 Substitute for u and dx, noting that the $\sin(2x)$ terms cancel.	$= \int \sin(2x)(1 - u^2)u^4 \times \frac{-1}{2\sin(2x)}du$
5 Transfer the constant factor outside the integral sign and expand.	$= -\frac{1}{2}\int (1 - u^2)u^4 du$ $= -\frac{1}{2}\int (u^4 - u^6)du$
6 Antidifferentiate.	$= -\frac{1}{2} \times \left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) + c$ $= \frac{1}{14}u^7 - \frac{1}{10}u^5 + c$
7 Substitute back for x and state the final result.	$\int \sin^3(2x)\cos^4(2x)dx$ $= \frac{1}{14}\cos^7(2x) - \frac{1}{10}\cos^5(2x) + c$

Integrals involving even powers

Antidifferentiation of $\sin^m(kx)\cos^n(kx)$ when both m and n are even powers must be performed using the double-angle formulas.

WORKED EXAMPLE 24 **Find an antiderivative of $\cos^4(2x)$.**

THINK	WRITE
1 Write the required antiderivative.	$\int \cos^4(2x)dx$

2 Since there is no odd power, we must use the double-angle formula $\cos^2(A) = \frac{1}{2}(1 + \cos(2A))$ with $A = 2x$.

$$\int (\cos^2(2x))^2 dx$$
$$= \int \left(\tfrac{1}{2}(1 + \cos(4x))\right)^2 dx$$

3 Expand the integrand

$$= \frac{1}{4}\int (1 + 2\cos(4x) + \cos^2(4x))dx$$

4 Replace $\cos^2(A) = \frac{1}{2}(1 + \cos(2A))$ with $A = 4x$ and expand. The integrand is now in a form that we can integrate term by term.

$$= \frac{1}{4}\int \left(1 + 2\cos(4x) + \tfrac{1}{2}(1 + \cos(8x))\right)dx$$
$$= \frac{1}{4}\int \left(\tfrac{3}{2} + 2\cos(4x) + \tfrac{1}{2}\cos(8x)\right)dx$$

5 Antidifferentiate term by term.

$$= \frac{1}{4}\left[\frac{3x}{2} + \frac{2}{4}\sin(4x) + \frac{1}{16}\sin(8x)\right] + c$$

6 State the final result.

$$\int \cos^4(2x)dx = \frac{3x}{8} + \frac{1}{8}\sin(4x) + \frac{1}{64}\sin(8x) + c$$

Integrals involving powers of tan (kx)

In this section we examine indefinite and definite integrals involving powers of the tangent function, that is, integrals of the form $\int \tan^n(kx)dx$ where $n \in J$.

The result $\tan(A) = \dfrac{\sin(A)}{\cos(A)}$ is used to integrate $\tan(A)$.

WORKED EXAMPLE 25

Find $\int \tan(2x)dx$.

THINK

1 Use $\tan(A) = \dfrac{\sin(A)}{\cos(A)}$ with $A = 2x$.

2 Use a non-linear substitution. Let $u = \cos(2x)$. We choose this as the derivative of $\cos(2x)$ is $-2\sin(2x)$, which is present in the numerator integrand and will cancel.

3 Substitute for u and dx, noting that the x terms cancel.

4 Transfer the constant factor outside the integral sign.

WRITE

1 $$\int \tan(2x)dx = \int \frac{\sin(2x)}{\cos(2x)}dx$$

2 $$u = \cos(2x)$$
$$\frac{du}{dx} = -2\sin(2x)$$
$$\frac{dx}{du} = \frac{-1}{2\sin(2x)}$$
$$dx = \frac{-1}{2\sin(2x)}du$$

3 $$\int \tan(2x)dx = \int \frac{\sin(2x)}{u} \times \frac{-1}{2\sin(2x)}du$$

4 $$= -\frac{1}{2}\int \frac{1}{u}du$$

5 Antidifferentiate.	$= -\frac{1}{2}\log_e\left(\lvert u\rvert\right) + c$
6 Substitute back for x and state the final result. Again note that there are different answers using log laws and trigonometric identities.	$\int \tan(2x)dx = -\frac{1}{2}\log_e\left(\lvert\cos(2x)\rvert\right) + c$ $= \frac{1}{2}\log_e\left(\lvert\cos(2x)\rvert\right)^{-1} + c$ $= \frac{1}{2}\log_e\left(\lvert\sec(2x)\rvert\right) + c$

Integrals involving $\tan^2(kx)$

To find $\int \tan^2(kx)dx$, we use the results $1 + \tan^2(A) = \sec^2(A)$ and $\frac{d}{dx}(\tan(kx)) = k\sec^2(kx)$, so that $\int \sec^2(kx) = \frac{1}{k}\tan(kx) + c$.

WORKED EXAMPLE 26 Find $\int \tan^2(2x)dx$.

THINK	WRITE
1 Use $1 + \tan^2(A) = \sec^2(A)$ with $A = 2x$.	$\int \tan^2(2x)dx = \int (\sec^2(2x) - 1)dx$
2 Use $\int \sec^2(kx) = \frac{1}{k}\tan(kx) + c$ and integrate term by term. State the final result.	$\int \tan^2(2x)dx = \frac{1}{2}\tan(2x) - x + c$

EXERCISE 8.5 Integrals of powers of trigonometric functions

PRACTISE

1 WE21 Find:

a $\int \sin^2\left(\frac{x}{4}\right)dx$

b $\int \sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)dx$.

2 Evaluate:

a $\int_0^{\frac{\pi}{6}} 4\sin^2(2x)dx$

b $\int_0^{\frac{\pi}{3}} \sin^2(2x)\cos^2(2x)dx$.

3 WE22 Find $\int \cos(4x)\sin^5(4x)dx$.

4 Evaluate $\int_0^{\frac{\pi}{4}} \sin(2x)\cos^3(2x)dx$.

5 WE23 Find an antiderivative of $\cos^5(4x)\sin^2(4x)$.

6 Evaluate $\int_0^{\frac{\pi}{12}} \sin^3(3x)dx$.

7 WE24 Find an antiderivative of $\sin^4(2x)$.

8 Evaluate $\int_0^{\frac{\pi}{12}} \cos^4(3x)dx$.

9 WE25 Find $\int \tan\left(\frac{x}{2}\right)dx$.

10 Evaluate $\int_0^{\frac{\pi}{12}} \tan(4x)dx$.

11 WE26 Find $\int \tan^2\left(\frac{x}{3}\right)dx$.

12 Evaluate $\int_0^{\frac{\pi}{16}} \tan^2(4x)dx$.

CONSOLIDATE

13 Find each of the following.

a $\int \cos(2x)\sin(2x)dx$

b $\int (\cos^2(2x) + \sin^2(2x))dx$

c $\int \cos^3(2x)\sin(2x)dx$

d $\int \cos(2x)\sin^3(2x)dx$

14 Evaluate each of the following.

a $\int_0^{\frac{\pi}{4}} \cos(2x)\sin^4(2x)dx$

b $\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin^3(2x)dx$

c $\int_0^{\frac{\pi}{4}} \cos^2(2x)\sin^2(2x)dx$

d $\int_0^{\frac{\pi}{4}} \cos^3(2x)\sin^2(2x)dx$

15 Find each of the following.

a $\int \cos^2(4x)\sin^2(4x)dx$

b $\int \cos^2(4x)\sin^3(4x)dx$

c $\int \cos^3(4x)\sin^2(4x)dx$

d $\int \cos^3(4x)\sin^4(4x)dx$

16 Find an antiderivative of each of the following.

a $\text{cosec}^2(2x)\cos(2x)$

b $\sec^2(2x)\sin(2x)$

c $\dfrac{\sin(2x)}{\cos^3(2x)}$

d $\dfrac{\cos(2x)}{\sin^3(2x)}$

17 Evaluate each of the following.

a $\int_0^{\frac{\pi}{6}} \sin^2(3x)dx$

b $\int_0^{\frac{\pi}{6}} \cos^3(3x)dx$

c $\int_0^{\frac{\pi}{6}} \sin^4(3x)dx$

d $\int_0^{\frac{\pi}{6}} \cos^5(3x)dx$

18 a Find $\int (\cos(2x) + \sin(2x))^2 dx$.

b Find $\int \cos^3(2x) + \sin^3(2x)dx$.

c Consider $\int \sin^3(2x)\cos^3(2x)dx$. Show that this integration can be done using:

i a double-angle formula

ii the substitution $u = \cos(2x)$

iii the substitution $u = \sin(2x)$.

19 Find each of the following.

a $\int \tan(3x)dx$

b $\int \cot(3x)dx$

c $\int \tan(3x)\sec^2(3x)dx$

d $\int \tan^2(3x)\sec^2(3x)dx$

20 Evaluate each of the following.

a $\int_0^{\frac{\pi}{20}} \tan(5x)dx$

b $\int_0^{\frac{\pi}{20}} \tan^2(5x)dx$

c $\int_0^{\frac{\pi}{20}} \tan^3(5x)\sec^2(5x)dx$

d $\int_0^{\frac{\pi}{20}} \tan^2(5x)\sec^4(5x)dx$

21 Given that $n \neq -1$ and $a \in R$, find each of the following.

a $\int \sin(ax)\cos^n(ax)dx$

b $\int \cos(ax)\sin^n(ax)dx$

c $\int \tan^n(ax)\sec^2(ax)dx$

22 Integrate each of the following where $a \in R \backslash \{0\}$.

a $\tan(ax)$ b $\tan^2(ax)$ c $\tan^3(ax)$ d $\tan^4(ax)$

MASTER

23 Integrate each of the following where $a \in R \backslash \{0\}$.

a $\sin^2(ax)$ b $\sin^3(ax)$ c $\sin^4(ax)$ d $\sin^5(ax)$

24 Integrate each of the following where $a \in R \backslash \{0\}$.

a $\cos^2(ax)$ b $\cos^3(ax)$ c $\cos^4(ax)$ d $\cos^5(ax)$

8.6 Integrals involving inverse trigonometric functions

studyon

Units 3 & 4

AOS 3

Topic 2

Concept 1

Antiderivatives involving inverse circular functions

Concept summary

Practice questions

Integrals involving the inverse sine function

Since $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}$, it follows that $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ for $a > 0$ and $|x| < a$.

WORKED EXAMPLE 27

Find:

a $\int \frac{12}{\sqrt{36 - x^2}}dx$

b $\int \frac{4}{\sqrt{49 - 36x^2}}dx$.

THINK	WRITE
a Use $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ with $a = 6$.	**a** $\int \frac{12}{\sqrt{36 - x^2}}dx$ $= 12\int \frac{1}{\sqrt{36 - x^2}}dx$ $= 12\sin^{-1}\left(\frac{x}{6}\right) + c$
b 1 Use a linear substitution with $u = 6x$ and express dx in terms of du by inverting both sides.	**b** $u = 6x$ $u^2 = 36x^2$ $\frac{du}{dx} = 6$ $\frac{dx}{du} = \frac{1}{6}$ $dx = \frac{1}{6}du$
2 Substitute for dx and u.	$\int \frac{4}{\sqrt{49 - 36x^2}}dx$ $= \int \frac{4}{\sqrt{49 - u^2}} \times \frac{1}{6}du$
3 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$= \frac{2}{3}\int \frac{1}{\sqrt{49 - u^2}}du$
4 Use $\int \frac{1}{\sqrt{a^2 - u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + c$ with $a = 7$.	$= \frac{2}{3}\sin^{-1}\left(\frac{u}{7}\right) + c$
5 Substitute back for u and express the final answer in terms of x only and an arbitrary constant $+c$.	$\int \frac{4}{\sqrt{49 - 36x^2}}dx = \frac{2}{3}\sin^{-1}\left(\frac{6x}{7}\right) + c$

Integrals involving the inverse cosine function

Since $\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - x^2}}$, it follows that $\int \frac{-1}{\sqrt{a^2 - x^2}}dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ where $a > 0$ and $|x| < a$.

WORKED EXAMPLE 28

On a certain curve the gradient is given by $\frac{-4}{\sqrt{81 - 25x^2}}$. Find the equation of the curve that passes through the origin.

THINK	WRITE
1 Recognise that the gradient of a curve is given by $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{-4}{\sqrt{81 - 25x^2}}$
2 Integrating both sides gives an expression for y.	$y = \int \frac{-4}{\sqrt{81 - 25x^2}}dx$
3 Use a linear substitution with $u = 5x$ and express dx in terms of du by inverting both sides.	$u = 5x$ $u^2 = 25x^2$ $\frac{du}{dx} = 5$ $\frac{dx}{du} = \frac{1}{5}$ $dx = \frac{1}{5}du$
4 Substitute for dx and u.	$y = \int \frac{-4}{\sqrt{81 - u^2}} \times \frac{1}{5}du$
5 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$y = \frac{4}{5}\int \frac{-1}{\sqrt{81 - u^2}}du$
6 Use $\int \frac{-1}{\sqrt{a^2 - u^2}}du = \cos^{-1}\left(\frac{u}{a}\right) + c$ with $a = 9$.	$y = \frac{4}{5}\cos^{-1}\left(\frac{u}{9}\right) + c$
7 Substitute back for u and express the final answer in terms of x only and an arbitrary constant $+c$.	$y = \frac{4}{5}\cos^{-1}\left(\frac{5x}{9}\right) + c$
8 Since the curve passes through the origin, we can let $y = 0$ when $x = 0$ to find c.	$0 = \frac{4}{5}\cos^{-1}(0) + c$ $0 = \frac{4}{5} \times \frac{\pi}{2} + c$ $c = -\frac{2\pi}{5}$
9 Substitute back for c.	$y = \frac{4}{5}\cos^{-1}\left(\frac{5x}{9}\right) - \frac{2\pi}{5}$
10 State the equation of the particular curve in a factored form.	$y = \frac{4}{5}\left(\cos^{-1}\left(\frac{5x}{9}\right) - \frac{\pi}{2}\right)$
11 Note that an alternative answer is possible, since $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$.	$y = -\frac{4}{5}\sin^{-1}\left(\frac{5x}{9}\right)$

Integrals involving the inverse tangent function

Since $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}$, it follows that $\int \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ for $x \in R$.

WORKED EXAMPLE 29

Antidifferentiate each of the following with respect to x.

a $\frac{12}{36 + x^2}$ **b** $\frac{4}{49 + 36x^2}$

THINK	WRITE
a 1 Write the required antiderivative.	**a** $\int \frac{12}{36 + x^2}dx$
2 Use $\int \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ with $a = 6$.	$12\int \frac{1}{36 + x^2}dx$ $= 12 \times \frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right) + c$ $= 2\tan^{-1}\left(\frac{x}{6}\right) + c$
b 1 Write the required antiderivative.	**b** $\int \frac{4}{49 + 36x^2}dx$
2 Use a linear substitution with $u = 6x$ and express dx in terms of du by inverting both sides.	$u = 6x$ $u^2 = 36x^2$ $\frac{du}{dx} = 6$ $dx = \frac{1}{6}du$
3 Substitute for dx and u.	$\int \frac{4}{49 + 36x^2}dx = \int \frac{4}{49 + u^2} \times \frac{1}{6}du$
4 Transfer the constant factor outside the front of the integral sign.	$= \frac{2}{3}\int \frac{1}{49 + u^2}du$
5 Use $\int \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ with $a = 7$.	$= \frac{2}{3} \times \frac{1}{7}\tan^{-1}\left(\frac{u}{7}\right) + c$
6 Substitute back for u and express the final answer in terms of x only and an arbitrary constant $+c$.	$\int \frac{4}{49 + 36x^2}dx = \frac{2}{21}\tan^{-1}\left(\frac{6x}{7}\right) + c$

Definite integrals involving inverse trigonometric functions

WORKED EXAMPLE 30

Evaluate:

a $\int_0^{\frac{2}{9}} \frac{-2}{\sqrt{16 - 81x^2}}dx$ **b** $\int_0^{\frac{4}{9}} \frac{2}{16 + 81x^2}dx.$

THINK	WRITE
a **1** Use a linear substitution with $u = 9x$ and express dx in terms of du by inverting both sides.	**a** $u = 9x$ $u^2 = 81x^2$ $\frac{du}{dx} = 9$ $\frac{dx}{du} = \frac{1}{9}$ $dx = \frac{1}{9}du$
2 Change the terminals to the new variable.	When $x = \frac{2}{9}$, $u = 2$, and when $x = 0$, $u = 0$.
3 Substitute for dx, u and the new terminals.	$\int_0^{\frac{2}{9}} \frac{-2}{\sqrt{16 - 81x^2}}dx$ $= \int_0^2 \frac{-2}{\sqrt{16 - u^2}} \times \frac{1}{9}du$
4 Transfer the constant factor outside the front of the integral sign.	$= \frac{2}{9}\int_0^2 \frac{-1}{\sqrt{16 - u^2}}du$
5 Perform the integration using $\int \frac{-1}{\sqrt{a^2 - u^2}}du = \cos^{-1}\left(\frac{u}{a}\right) + c$ with $a = 4$.	$= \frac{2}{9}\left[\cos^{-1}\left(\frac{u}{4}\right)\right]_0^2$
6 Evaluate the definite integral.	$= \frac{2}{9}\left[\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(0)\right]$ $= \frac{2}{9}\left[\frac{\pi}{3} - \frac{\pi}{2}\right]$ $= \frac{2}{9}\left(\frac{\pi(2 - 3)}{6}\right)$
7 State the final result. Note that since this is just evaluating a definite integral and not finding an area, the answer stays as a negative number.	$\int_0^{\frac{2}{9}} \frac{-2}{\sqrt{81 - 16x^2}}dx = -\frac{\pi}{27}$
b **1** Use a linear substitution with $u = 9x$ and express dx in terms of du by inverting both sides.	**b** $u = 9x$ $u^2 = 81x^2$ $\frac{du}{dx} = 9$ $\frac{dx}{du} = \frac{1}{9}$ $dx = \frac{1}{9}du$
2 Change the terminals to the new variable.	When $x = \frac{4}{9}$, $u = 4$, and when $x = 0$, $u = 0$.

3 Substitute for dx, u and the new terminals.

$$\int_0^{\frac{4}{9}} \frac{2}{16+81x^2}\,dx$$
$$= \int_0^4 \frac{2}{16+u^2} \times \frac{1}{9}\,du$$

4 Transfer the constant factor outside the front of the integral sign.

$$= \frac{2}{9}\int_0^4 \frac{1}{16+u^2}\,du$$

5 Perform the integration using $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ with $a = 4$.

$$= \frac{2}{9}\left[\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right)\right]_0^4$$
$$= \frac{1}{18}\left[\tan^{-1}\left(\frac{u}{4}\right)\right]_0^4$$

6 Evaluate this definite integral.

$$= \frac{1}{18}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$
$$= \frac{1}{18}\left[\frac{\pi}{4} - 0\right]$$

7 State the final result.

$$\int_0^{\frac{4}{9}} \frac{2}{16+81x^2}\,dx = \frac{\pi}{72}$$

Integrals involving completing the square

A quadratic expression in the form $ax^2 + bx + c$ can be expressed in the form $a(x+h)^2 + k$ (the completing the square form). This can be used to integrate expressions of the form $\frac{1}{ax^2+bx+c}$ with $a > 0$ and $\Delta = b^2 - 4ac < 0$, which will then involve the inverse tangent function.

Integrating expressions of the form $\frac{1}{\sqrt{ax^2+bx+c}}$ with $a < 0$ and $\Delta = b^2 - 4ac > 0$ will involve the inverse sine or cosine functions.

WORKED EXAMPLE 31

Find each of the following with respect to x.

a $\int \frac{1}{9x^2+12x+29}\,dx$

b $\int \frac{1}{\sqrt{21-12x-9x^2}}\,dx$

THINK

a 1 Express the quadratic factor in the denominator in the completing the square form by making it into a perfect square.

WRITE

a $9x^2 + 12x + 29$
$= (9x^2 + 12x + 4) + 25$
$= (3x + 2)^2 + 25$

2 Write the denominator as the sum of two squares.

$$\int \frac{1}{9x^2 + 12x + 29} dx$$

$$= \int \frac{1}{(3x+2)^2 + 25} dx$$

3 Use a linear substitution with $u = 3x + 2$ and express dx in terms of du by inverting both sides.

$u = 3x + 2$

$\frac{du}{dx} = 3$

$\frac{dx}{du} = \frac{1}{3}$

$dx = \frac{1}{3} du$

4 Substitute for dx and u.

$$\int \frac{1}{(3x+2)^2 + 25} dx = \int \frac{1}{u^2 + 25} \times \frac{1}{3} du$$

5 Transfer the constant factor outside the front of the integral sign.

$$= \frac{1}{3} \int \frac{1}{u^2 + 25} du$$

6 Use $\int \frac{1}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c$ with $a = 5$.

$$= \frac{1}{3} \times \frac{1}{5} \tan^{-1}\left(\frac{u}{5}\right) + c$$

$$= \frac{1}{15} \tan^{-1}\left(\frac{u}{5}\right) + c$$

7 Substitute back for u and express the final answer in terms of x only and an arbitrary constant $+c$.

$$\int \frac{1}{9x^2 + 12x + 29} dx = \frac{1}{15} \tan^{-1}\left(\frac{3x+2}{5}\right) + c$$

b **1** Express the quadratic factor in the denominator in the completing the square form by making it into the difference of two squares.

b $21 - 12x - 9x^2$

$= 21 - (9x^2 + 12x)$

$= 21 - (9x^2 + 12x + 4) + 4$

$= 25 - (3x + 2)^2$

2 Write the denominator as the difference of two squares under the square root.

$$\int \frac{1}{\sqrt{21 - 12x - 9x^2}} dx$$

$$= \int \frac{1}{\sqrt{25 - (3x+2)^2}} dx$$

3 Use a linear substitution with $u = 3x + 2$ and express dx in terms of du by inverting both sides.

$u = 3x + 2$

$\frac{du}{dx} = 3$

$\frac{dx}{du} = \frac{1}{3}$

$dx = \frac{1}{3} du$

4 Substitute for dx and u.

$$\int \frac{1}{\sqrt{25 - (3x+2)^2}} dx$$

$$= \int \frac{1}{\sqrt{25 - u^2}} \times \frac{1}{3} du$$

5 Transfer the constant factor outside the front of the integral sign.

$$= \frac{1}{3}\int \frac{1}{\sqrt{25 - u^2}}du$$

6 Use $\int \frac{1}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + c$ with $a = 5$.

$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{5}\right) + c$$

7 Substitute back for u and express the final answer in terms of x only and an arbitrary constant $+c$.

$$\int \frac{1}{\sqrt{21 - 12x - 9x^2}}dx = \frac{1}{3}\sin^{-1}\left(\frac{3x + 2}{5}\right) + c$$

Integrals involving substitutions and inverse trigonometric functions

We can break up complicated integrals into two manageable integrals using the following property of indefinite integrals: $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$.

WORKED EXAMPLE 32 Find $\int \frac{4x + 5}{\sqrt{25 - 16x^2}}dx$.

THINK / **WRITE**

1 If the x was not present in the numerator, the integral would involve an inverse sine function. If the 5 was not present in the numerator, the integral would involve a non-linear substitution. Break the integral into two distinct problems: one involving a non-linear substitution and one involving the inverse trigonometric function.

$$\int \frac{4x + 5}{\sqrt{25 - 16x^2}}dx$$
$$= \int \frac{4x}{\sqrt{25 - 16x^2}}dx + \int \frac{5}{\sqrt{25 - 16x^2}}dx$$

2 Write the first integral as a power using index laws.

$$\int \frac{4x}{\sqrt{25 - 16x^2}}dx$$
$$= \int 4x(25 - 16x^2)^{-\frac{1}{2}}dx$$

3 Use a non-linear substitution with $u = 25 - 16x^2$ and express dx in terms of du by inverting both sides.

$$u = 25 - 16x^2$$
$$\frac{du}{dx} = -32x$$
$$\frac{dx}{du} = -\frac{1}{32x}$$
$$dx = -\frac{1}{32x}du$$

4 Substitute for dx and u, noting that the x terms cancel.

$$\int 4x(25 - 4x^2)^{-\frac{1}{2}}dx$$
$$= \int 4xu^{-\frac{1}{2}} \times -\frac{1}{32x}du$$

5 Transfer the constant factor outside the integral sign.	$= -\frac{1}{8}\int u^{-\frac{1}{2}}du$
6 Perform the integration using $\int u^n du = \frac{1}{n+1}u^{n+1}$ with $n = -\frac{1}{2}$, so that $n + 1 = \frac{1}{2}$.	$= -\frac{1}{8} \times \frac{1}{\frac{1}{2}}u^{\frac{1}{2}}$ $= -\frac{1}{4}\sqrt{u}$
7 Substitute back for u.	$\int \frac{x}{\sqrt{25 - 4x^2}}dx = -\frac{1}{4}\sqrt{25 - 16x^2}$
8 Consider the second integral. Use a linear substitution with $v = 4x$ and express dx in terms of dv by inverting both sides.	$\int \frac{5}{\sqrt{25 - 16x^2}}dx$ $v = 4x$ $\frac{dv}{dx} = 4$ $\frac{dx}{dv} = \frac{1}{4}$ $dx = \frac{1}{4}dv$
9 Substitute for dx and v.	$\int \frac{5}{\sqrt{25 - 15x^2}}dx$ $= \int \frac{5}{\sqrt{25 - v^2}} \times \frac{1}{4}dv$
10 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$= \frac{5}{4}\int \frac{1}{\sqrt{25 - v^2}}dv$
11 Use $\int \frac{1}{\sqrt{a^2 - v^2}}dv = \sin^{-1}\left(\frac{v}{a}\right) + c$ with $a = 5$.	$= \frac{5}{4}\sin^{-1}\left(\frac{v}{5}\right)$
12 Substitute back for v.	$\int \frac{5}{\sqrt{25 - 4x^2}}dx = \frac{5}{4}\sin^{-1}\left(\frac{4x}{5}\right)$
13 Express the original integral as the sum as the two integrals, adding in only one arbitrary constant $+c$.	$\int \frac{4x + 5}{\sqrt{25 - 16x^2}}dx$ $= \frac{5}{4}\sin^{-1}\left(\frac{4x}{5}\right) - \frac{1}{4}\sqrt{25 - 16x^2} + c$

EXERCISE 8.6 Integrals involving inverse trigonometric functions

PRACTISE

1 WE27 Find:

a $\int \frac{1}{\sqrt{100 - x^2}}dx$

b $\int \frac{12}{\sqrt{64 - 9x^2}}dx.$

2 Integrate each of the following with respect to x.

a $\frac{1}{\sqrt{36 - 25x^2}}$

b $\frac{x}{\sqrt{36 - 25x^2}}$

3 WE28 On a certain curve the gradient is given by $\frac{-2}{\sqrt{25-16x^2}}$. Find the equation of the curve that passes through the origin.

4 On a certain curve the gradient is given by $\frac{-2}{\sqrt{4-x^2}}$. Find the equation of the curve that passes through the point (2, 3).

5 WE29 Antidifferentiate each of the following with respect to x.

a $\frac{1}{100+x^2}$

b $\frac{12}{64+9x^2}$

6 Integrate each of the following with respect to x.

a $\frac{1}{36+25x^2}$

b $\frac{5x}{64+25x^2}$

7 WE30 Evaluate:

a $\int_0^{\frac{9}{8}} \frac{1}{\sqrt{81-16x^2}}dx$

b $\int_0^{\frac{9}{4}} \frac{1}{81+16x^2}dx$.

8 Evaluate:

a $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{3+2x^2}dx$

b $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2-3x^2}}dx$.

9 WE31 Find each of the following with respect to x.

a $\int \frac{1}{4x^2-12x+25}dx$

b $\int \frac{1}{\sqrt{7+12x-4x^2}}dx$

10 Find each of the following with respect to x.

a $\int \frac{1}{25x^2-20x+13}dx$

b $\int \frac{1}{\sqrt{12+20x-25x^2}}dx$

11 WE32 Find $\int \frac{5-3x}{\sqrt{25-9x^2}}dx$.

12 Find $\int \frac{3x+5}{9x^2+25}dx$.

CONSOLIDATE

13 Integrate each of the following with respect to x.

a $\frac{1}{\sqrt{16-x^2}}$

b $\frac{1}{\sqrt{1-16x^2}}$

c $\frac{10}{\sqrt{49-25x^2}}$

d $\frac{10x}{\sqrt{49-25x^2}}$

14 Find each of the following.

a $\int \frac{-1}{\sqrt{4-x^2}}dx$

b $\int \frac{-2}{\sqrt{1-4x^2}}dx$

c $\int \frac{-3x}{\sqrt{36-49x^2}}dx$

d $\int \frac{-3}{\sqrt{36-49x^2}}dx$

15 Find a set of antiderivatives for each of the following.

a $\dfrac{1}{16 + x^2}$ **b** $\dfrac{x}{49 + 36x^2}$ **c** $\dfrac{21}{49 + 36x^2}$ **d** $\dfrac{8}{1 + 16x^2}$

16 Evaluate each of the following.

a $\displaystyle\int_0^3 \frac{x}{\sqrt{9 - x^2}}dx$ **b** $\displaystyle\int_0^3 \frac{1}{\sqrt{9 - x^2}}dx$ **c** $\displaystyle\int_0^3 \frac{1}{9 + x^2}dx$ **d** $\displaystyle\int_0^3 \frac{x}{9 + x^2}dx$

17 a On a certain curve the gradient is given by $\dfrac{1}{\sqrt{4 - x^2}}$. Find the equation of the curve that passes through the point $(\sqrt{3}, \pi)$.

b On a certain curve the gradient is given by $\dfrac{1}{1 + 4x^2}$. Find the equation of the curve that passes through the point $\left(\frac{1}{2}, \pi\right)$.

c If $\dfrac{dy}{dx} + \dfrac{1}{3 + x^2} = 0$, and when $x = 1$, $y = 0$, find y when $x = 0$.

d If $\dfrac{dy}{dx} + \dfrac{1}{\sqrt{6 - x^2}} = 0$, and when $x = \sqrt{3}$, $y = 0$, find y when $x = 0$.

18 Evaluate each of the following.

a $\displaystyle\int_0^{\frac{5}{6}} \frac{1}{\sqrt{25 - 9x^2}}dx$ **b** $\displaystyle\int_0^{\frac{5}{3}} \frac{x}{25 + 9x^2}dx$

c $\displaystyle\int_0^{\frac{5}{3}} \frac{x}{\sqrt{25 - 9x^2}}dx$ **d** $\displaystyle\int_0^{\frac{5}{3}} \frac{1}{25 + 9x^2}dx$

19 Evaluate each of the following.

a $\displaystyle\int_0^1 \frac{1}{\sqrt{4 - 3x^2}}dx$ **b** $\displaystyle\int_0^1 \frac{1}{1 + 3x^2}dx$ **c** $\displaystyle\int_0^1 \frac{x}{1 + 3x^2}dx$ **d** $\displaystyle\int_0^1 \frac{x}{\sqrt{4 - 3x^2}}dx$

20 Find each of the following.

a $\displaystyle\int \frac{2}{\sqrt{5 - 4x - x^2}}dx$ **b** $\displaystyle\int \frac{2}{x^2 + 4x + 13}dx$

c $\displaystyle\int \frac{6}{\sqrt{24 - 30x - 9x^2}}dx$ **d** $\displaystyle\int \frac{6}{74 + 30x + 9x^2}dx$

21 Find each of the following.

a $\displaystyle\int \frac{3x - 4}{\sqrt{9 - 16x^2}}dx$ **b** $\displaystyle\int \frac{3 + 4x}{9 + 16x^2}dx$

c $\displaystyle\int \frac{5 - 2x}{\sqrt{5 - 2x^2}}dx$ **d** $\displaystyle\int \frac{5 - 2x}{25 + 2x^2}dx$

22 If a, b, p and q are positive real constants, find each of the following.

a $\displaystyle\int \frac{1}{\sqrt{p^2 - q^2x^2}}dx$

b $\displaystyle\int \frac{1}{p^2 + q^2x^2}dx$

c $\displaystyle\int \frac{ax + b}{\sqrt{p^2 - q^2x^2}}dx$

d $\displaystyle\int \frac{ax + b}{p^2 + q^2x^2}dx$

MASTER

23 a i Use the substitution $x = 4\sin(\theta)$ to find $\displaystyle\int \sqrt{16 - x^2}\,dx$.

ii Evaluate $\displaystyle\int_0^4 \sqrt{16 - x^2}\,dx$. What area does this represent?

b i Use the substitution $x = \frac{6}{5}\cos(\theta)$ to find $\displaystyle\int \sqrt{36 - 25x^2}\,dx$.

ii Evaluate $\displaystyle\int_0^{\frac{6}{5}} \sqrt{36 - 25x^2}\,dx$. What area does this represent?

c Prove that the total area inside the circle $x^2 + y^2 = r^2$ is given by πr^2.

d Prove that the total area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by πab.

24 Evaluate each of the following.

a $\displaystyle\int_3^5 \frac{x - 4}{x^2 - 6x + 13}dx$

b $\displaystyle\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{2x + 1}{4x^2 - 12x + 25}dx.$

c $\displaystyle\int_1^4 \frac{3x + 2}{\sqrt{8 + 2x - x^2}}dx$

d $\displaystyle\int_{-2}^1 \frac{2x - 3}{\sqrt{12 - 8x - 4x^2}}dx$

8.7 Integrals involving partial fractions

Integration of rational functions

studyon

Units 3 & 4

AOS 3

Topic 2

Concept 7

Antiderivatives with partial fractions

Concept summary

Practice questions

A rational function is a ratio of two functions, both of which are polynomials. For example, $\dfrac{mx + k}{ax^2 + bx + c}$ is a rational function; it has a linear function in the numerator and a quadratic function in the denominator.

In the preceding section, we integrated certain expressions of this form when $a > 0$ and $\Delta = b^2 - 4ac < 0$, meaning that the quadratic function was expressed as the sum of two squares. In this section, we examine cases when $a \neq 0$ and $\Delta = b^2 - 4ac > 0$. This means that the quadratic function in the denominator can now be factorised into linear factors. Integrating expressions of this kind does not involve a new integration technique, just an algebraic method of expressing the integrand into its partial fractions decomposition.

Converting expressions into equivalent forms is useful for integration. We have seen this when expressing a quadratic in the completing the square form or converting trigonometric powers into multiple angles.

study on

Units 3 & 4

AOS 2

Topic 1

Concept 1

Partial fractions with a quadratic denominator that has linear factors
Concept summary
Practice questions

However, if the derivative of the denominator is equal to the numerator or a constant multiple of it, that is, if $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$, then $\int \frac{2ax + b}{ax^2 + bx + c}dx = \log_e\left(|ax^2 + bx + c|\right)$.

The method of partial fractions involves factorising the denominator, for example, $\frac{mx + k}{ax^2 + bx + c} = \frac{mc + k}{(ax + \alpha)(x + \beta)}$, and then expressing as $\frac{A}{(ax + \alpha)} + \frac{B}{(x + \beta)}$, where A and B are constants to be found. We can find A and B by equating coefficients and then solving simultaneous equations, or by substituting in specific values of x. See the following worked examples.

Equating coefficients means, for example, that if $Ax + B = 3x - 4$, then $A = 3$ from equating the coefficient of x, since $Ax = 3x$, and $B = -4$ from the term independent of x (the constant term).

WORKED EXAMPLE 33

Find:

a $\int \frac{12}{36 - x^2}dx$ **b** $\int \frac{4}{49 - 36x^2}dx$.

THINK	WRITE
a 1 Factorise the denominator into linear factors using the difference of two squares.	**a** $\frac{12}{36 - x^2} = \frac{12}{(6 + x)(6 - x)}$
2 Write the integrand in its partial fractions decomposition, where A and B are constants to be found.	$= \frac{A}{6 + x} + \frac{B}{6 - x}$
3 Add the fractions by forming a common denominator.	$= \frac{A(6 - x) + B(6 + x)}{(6 + x)(6 - x)}$
4 Expand the numerator, factor in x and expand the denominator.	$= \frac{6A - 6Ax + 6B + 6Bx}{(6 + x)(6 - x)}$ $= \frac{6x(B - A) + 6(A + B)}{36 - x^2}$
5 Because the denominators are equal, the numerators are also equal. Equate the coefficients of x and the term independent of x. This gives two simultaneous equations for the two unknowns, A and B.	$6x(B - A) + 6(A + B) = 12$ (1) $B - A = 0$ (2) $6(A + B) = 12$
6 Solve the simultaneous equations.	(1) $\Rightarrow A = B$; substitute into (2) $12A = 12 \Rightarrow A = B = 1$
7 An alternative method can be used to find the unknowns A and B. Equate the numerators from the working above.	$12 = A(6 - x) + B(x + 6)$
8 Substitute an appropriate value of x.	Substitute $x = 6$: $12 = 12B$ $B = 1$

9 Substitute an appropriate value of x.

Substitute $x = -6$:
$12 = 12A$
$A = 1$

10 Express the integrand as its partial fractions decomposition.

$$\frac{12}{36 - x^2} = \frac{1}{6 + x} + \frac{1}{6 - x}$$

11 Instead of integrating the original expression, we integrate the partial fractions expression, since these expressions are equal.

$$\int \frac{12}{36 - x^2}dx = \int\left(\frac{1}{6 + x} + \frac{1}{6 - x}\right)dx$$

12 Integrate term by term, using the result $\int \frac{1}{ax + b}dx = \frac{1}{a}\log_e\left(|ax + b|\right)$.

$$= \log_e\left(|6 + x|\right) - \log_e\left(|6 - x|\right) + c$$

13 Use the log laws to express the final answer as a single log term.

$$\int \frac{12}{36 - x^2}dx = \log_e\left(\left|\frac{6 + x}{6 - x}\right|\right) + c$$

b 1 Factorise the denominator into linear factors using the difference of two squares.

b $$\frac{4}{49 - 36x^2} = \frac{4}{(7 - 6x)(7 + 6x)}$$

2 Write the integrand in its partial fractions decomposition, where A and B are constants to be found.

$$= \frac{A}{7 - 6x} + \frac{B}{7 + 6x}$$

3 Add the fractions by forming a common denominator.

$$= \frac{A(7 + 6x) + B(7 - 6x)}{(7 - 6x)(7 + 6x)}$$

4 Expand the numerator, factor in x and expand the denominator.

$$= \frac{7A + 6Ax + 7B - 6Bx}{(7 - 6x)(7 + 6x)}$$
$$= \frac{6x(A - B) + 7(A + B)}{49 - 36x^2}$$

5 Because the denominators are equal, the numerators are also equal. Equate coefficients of x and the term independent of x. This gives two simultaneous equations for the two unknowns, A and B.

$6x(A - B) + 7(A - B) = 4$
(1) $A - B = 0$
(2) $7(A + B) = 4$

6 Solve the simultaneous equations

(1) $\Rightarrow A = B$: substitute into
(2) $14A = 4 \Rightarrow A = B = \frac{2}{7}$

7 Express the integrand as its partial fractions decomposition.

$$\frac{4}{49 - 36x^2} = \frac{2}{7(7 - 6x)} + \frac{2}{7(7 + 6x)}$$

8 Instead of integrating the original expression, we integrate the partial fractions expression, since these expressions are equal. Transfer the constant factors outside the integral sign.

$$\int \frac{4}{49 - 36x^2}dx$$
$$= \int\left(\frac{2}{7(7 - 6x)} + \frac{2}{7(7 + 6x)}\right)dx$$
$$= \frac{2}{7}\int\left(\frac{1}{7 - 6x} + \frac{1}{7 + 6x}\right)dx$$

9 Integrate term by term, using the result $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e\left(\lvert ax+b \rvert\right)$.	$= \frac{2}{7}\left[-\frac{1}{6}\log_e\left(\lvert 7-6x \rvert\right) + \frac{1}{6}\log_e\left(\lvert 7+6x \rvert\right)\right] + c$
10 Use the log laws to express the final answer as a single log term.	$\int \frac{4}{49-36x^2}dx = \frac{1}{21}\log_e\left(\left\lvert\frac{7+6x}{7-6x}\right\rvert\right) + c$

Compare the previous worked example to Worked examples 27 and 29.

WORKED EXAMPLE 34 Find $\int \frac{x+10}{x^2-x-12}dx$.

THINK	WRITE
1 Factorise the denominator into linear factors.	$\frac{x+10}{x^2-x-12} = \frac{x+10}{(x-4)(x+3)}$
2 Write the integrand in its partial fractions decomposition, where A and B are constants to be found.	$= \frac{A}{x-4} + \frac{B}{x+3}$
3 Add the fractions by forming a common denominator.	$= \frac{A(x+3)+B(x-4)}{(x-4)(x+3)}$
4 Expand the numerator, factor in x and expand the denominator.	$= \frac{Ax+3A+Bx-4B}{(x-4)(x+3)}$ $= \frac{x(A+B)+3A-4B}{x^2-x-12}$
5 Because the denominators are equal, the numerators are also equal. Equate the coefficients of x and the term independent of x to give two simultaneous equations for the two unknowns, A and B.	$x(A+B)+3A-4B = x+10$ (1) $A+B=1$ (2) $3A-4B=10$
6 Solve the simultaneous equations by elimination. Add the two equations to eliminate B and solve for A.	$4\times(1)\quad 4A+4B=4$ $(2)\quad 3A-4B=10$ $4\times(1)+(2)\quad 7A=14$ $A=2$
7 Substitute back into (1) to find B.	$2+B=1$ $B=-1$
8 An alternative method can be used to find the unknowns A and B. Equate the numerators from the working above.	$A(x+3)+B(x-4)=x+10$
9 Let $x=4$.	$7A=14$ $A=2$
10 Let $x=-3$.	$-7B=7$ $B=-1$

11 Express the integrand as its partial fractions decomposition.

$$\int \frac{x+10}{x^2-x-12}dx = \int \left(\frac{2}{x-4} - \frac{1}{x+3}\right)dx$$

12 Integrate term by term using the result $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e\left(|ax+b|\right)$, and state the final simplified answer using log laws.

$$\int \frac{x+10}{x^2-x-12}dx$$
$$= 2\log_e\left(|x-4|\right) - \log_e\left(|x+3|\right) + c$$
$$= \log_e\left(\frac{(x-4)^2}{|x+3|}\right)$$

study on

Units 3 & 4
AOS 2
Topic 1
Concept 2

Partial fractions with perfect square denominators
Concept summary
Practice questions

Perfect squares

If an expression is of the form $\frac{mx+k}{ax^2+bx+c}$ when $a \neq 0$ and $\Delta = b^2 - 4ac = 0$, the quadratic function in the denominator is now a perfect square.

Integrating by the method of partial fractions in this case involves writing $\frac{mx+k}{ax^2+bx+c} = \frac{mc+k}{(px+\alpha)^2}$ and then expressing it as $\frac{A}{(px+\alpha)} + \frac{B}{(px+\alpha)^2}$, where A and B are constants to be found. We can find A and B by equating coefficients and then solving the simultaneous equations, as before.

WORKED EXAMPLE 35

Find $\int \frac{6x-5}{4x^2-12x+9}dx.$

THINK	WRITE
1 Factorise the denominator as a perfect square.	$\frac{6x-5}{4x^2-12x+9} = \frac{6x-5}{(2x-3)^2}$
2 Write the integrand in its partial fractions decomposition, where A and B are constants to be found.	$= \frac{A}{2x-3} + \frac{B}{(2x-3)^2}$
3 Add the fractions by forming the lowest common denominator and expanding the numerator.	$= \frac{A(2x-3)+B}{(2x-3)^2}$ $= \frac{2Ax+B-3A}{(2x-3)^2}$
4 Because the denominators are equal, the numerators are also equal. Equate coefficients of x and the term independent of x to give two simultaneous equations for the two unknowns, A and B.	$2Ax + B - 3A = 6x - 5$ (1) $2A = 6$ (2) $B - 3A = -5$
5 Solve the simultaneous equations.	(1) $\Rightarrow$ $A = 3$: substitute into (2): $B - 9 = -5 \Rightarrow B = 4$
6 Express the integrand as its partial fractions decomposition.	$\int \frac{6x-5}{4x^2-12x+9}dx$ $= \int \left(\frac{3}{2x-3} + \frac{4}{(2x-3)^2}\right)dx$

7 Integrate term by term, using the result $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e\left(|ax+b|\right)$ in the first term and $\int \frac{1}{(ax+b)^2}dx = -\frac{1}{a(ax+b)}$ for the second term.

$$= \frac{3}{2}\log_e\left(|2x-3|\right) - \frac{2}{2x-3} + c$$

Note that this example can also be done using a linear substitution with a back substitution. See Worked example 12b.

Rational functions involving ratios of two quadratic functions

When the degree of the polynomial in the numerator is greater than or equal to the degree of the polynomial in the denominator, the rational function is said to be a non-proper rational function. In this case we have to divide the denominator into the numerator to obtain a proper rational function.

For example, when we have quadratic functions in both the numerator and denominator, that is, the form $\frac{rx^2+sx+t}{ax^2+bx+c}$ where $r \neq 0$ and $a \neq 0$, we can use long division to divide the denominator into the numerator to express the function as $\frac{rx^2+sx+t}{ax^2+bx+c} = q + \frac{mx+k}{ax^2+bx+c}$ where $q = \frac{r}{a}$.

WORKED EXAMPLE 36 Find $\int \frac{2x^2+5x+3}{x^2+3x-4}dx$.

THINK	WRITE
1 Use long division to express the rational function as a proper rational function.	$\frac{2x^2+5x+3}{x^2+3x-4}$ $= \frac{2(x^2+3x-4)+11-x}{x^2+3x-4}$ $= 2 + \frac{11-x}{x^2+3x-4}$
2 Factorise the denominator into linear factors.	$= 2 + \frac{11-x}{(x-1)(x+4)}$
3 Write the integrand in its partial fractions decomposition, where A and B are constants to be found.	$= 2 + \frac{A}{x-1} + \frac{B}{x+4}$
4 Add the fractions by forming a common denominator and expanding the numerator.	$= 2 + \frac{A(x+4)+B(x-1)}{(x-1)(x+4)}$ $= 2 + \frac{x(A+B)+4A-B}{x^2+3x-4}$
5 Because the denominators are equal, the numerators are also equal. Equate coefficients of x and the term independent of x to give two simultaneous equations for the two unknowns, A and B.	$x(A+B)+4A-B = 11-x$ (1) $A+B=-1$ (2) $4A-B=11$

6 Solve the simultaneous equations by elimination. Add the two equations to elimiate B.

$5A = 10$
$A = 2 \Rightarrow B = -3$

7 Express the integrand as its partial fractions decomposition.

$$\int \frac{2x^2 + 5x + 3}{x^2 + 3x - 4}dx = \int \left(2 + \frac{2}{x-1} - \frac{3}{x+4}\right)dx$$

8 Integrate term by term using the result $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e\left(|ax+b|\right)$ and state the final answer.

$$\int \frac{2x^2 + 5x + 3}{x^2 + 3x - 4}dx = 2x + 2\log_e\left(|x-1|\right) - 3\log_e\left(|x+4|\right) + c$$
$$= 2x + \log_e\left(\frac{(x-1)^2}{|x+4|^3}\right) + c$$

study on

Units 3 & 4
AOS 2
Topic 1
Concept 3

Partial fractions with quadratic factors in the denominator
Concept summary
Practice questions

Rational functions involving non-linear factors

If the denominator does not factorise into linear factors, then we proceed with $\frac{px^2 + qx + r}{(x+\alpha)(x^2+a^2)} = \frac{A}{x+\alpha} + \frac{Bx + C}{x^2 + a^2}$. Note that we will need three simultaneous equations to solve for the three unknowns: A, B and C.

WORKED EXAMPLE 37

Find $\int \frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8}dx$.

THINK

1 First try to factorise the cubic in the denominator using the factor theorem.

WRITE

$f(x) = x^3 + 2x^2 + 4x + 8$
$f(1) = 1 + 2 + 4 + 8 \neq 0$
$f(2) = 8 + 8 + 8 + 8 \neq 0$
$f(-2) = -8 + 8 - 8 + 8 = 0$
$(x+2)$ is a factor.

2 Factorise the denominator into factors.

$x^3 + 2x^2 + 4x + 8 = (x+2)(x^2+4)$

3 Write the integrand in its partial fractions decomposition, where A, B and C are constants to be found.

$$\frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} = \frac{A}{x+2} + \frac{Bx + C}{x^2 + 4}$$

4 Add the fractions by forming a common denominator and expanding the numerator.

$$= \frac{A(x^2+4) + (x+2)(Bx+C)}{(x+2)(x^2+4)}$$
$$= \frac{Ax^2 + 4A + Bx^2 + 2Bx + Cx + 2C}{(x+2)(x^2+4)}$$
$$= \frac{x^2(A+B) + x(2B+C) + 4A + 2C}{x^3 + 2x^2 + 4x + 8}$$

5 Because the denominators are equal, the numerators are also equal. Equate the coefficients of x^2 and x and the term independent of x to give three simultaneous equations for the three unknowns: A, B and C.

(1) $A + B = 1$
(2) $2B + C = 7$
(3) $4A + 2C = 2$

6 Solve the simultaneous equations by elimination.

$2 \times$ (1) $2A + 2B = 2$
(2) $2B + C = 7$
Subtracting gives
(4) $C - 2A = 5$
$2 \times$ (4) $2C - 4A = 10$
(3) $4A + 2C = 2$
Adding gives $4C = 12$
$C = 2$

7 Back substitute to find the remaining unknowns.

Substitute into (2):
$2B + 3 = 7 \Rightarrow B = 2$
Substitute into (1): $\Rightarrow A = -1$

8 Express the integrand as its partial fractions decomposition.

$$\frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} = \frac{-1}{x + 2} + \frac{2x + 3}{x^2 + 4}$$

9 Separate the last term into two expressions.

$$\int \frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} dx$$
$$= \int \left(\frac{-1}{x + 2} + \frac{2x + 3}{x^2 + 4}\right) dx$$
$$= \int \left(\frac{2x}{x^2 + 4} - \frac{1}{x + 2} + \frac{3}{x^2 + 4}\right) dx$$

10 Integrate term by term, using the results $\int \frac{f'(x)}{f(x)} dx = \log_e\left(|f(x)|\right)$, $\int \frac{1}{ax + b} dx = \frac{1}{a} \log_e\left(|ax + b|\right)$ and $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$, and state the final answer.

$$\int \frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} dx$$
$$= \log_e (x^2 + 4) - \log_e\left(|x + 2|\right) + \frac{3}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

EXERCISE 8.7 Integrals involving partial fractions

PRACTISE

1 **WE33** Find:

a $\int \frac{1}{100 - x^2} dx$

b $\int \frac{12}{64 - 9x^2} dx.$

2 Find:

a $\int \frac{1}{36 - 25x^2} dx$

b $\int \frac{x}{36 - 25x^2} dx.$

3 WE34 Find $\int \frac{x + 13}{x^2 + 2x - 15} dx$.

4 Find $\int \frac{x - 11}{x^2 + 3x - 4} dx$.

5 WE35 Find $\int \frac{2x + 1}{x^2 + 6x + 9} dx$.

6 Find $\int \frac{2x - 1}{9x^2 - 24x + 16} dx$.

7 WE36 Find $\int \frac{3x^2 + 10x - 4}{x^2 + 3x - 10} dx$.

8 Find $\int \frac{-2x^2 - x + 20}{x^2 + x - 6} dx$.

9 WE37 Find $\int \frac{19 - 3x}{x^3 - 2x^2 + 9x - 18} dx$.

10 Find $\int \frac{25}{x^3 + 3x^2 + 16x + 48} dx$.

CONSOLIDATE

11 Integrate each of the following with respect to x.

a $\frac{x + 11}{x^2 + x - 12}$ b $\frac{5x - 9}{x^2 - 2x - 15}$ c $\frac{2x - 19}{x^2 + x - 6}$ d $\frac{11}{x^2 - 3x - 28}$

12 Find an antiderivative of each of the following.

a $\frac{1}{x^2 - 4}$ b $\frac{2}{16 + x^2}$ c $\frac{x}{x^2 - 25}$ d $\frac{2x - 3}{x^2 - 36}$

13 Integrate each of the following with respect to x.

a $\frac{2x + 3}{x^2 - 6x + 9}$ b $\frac{2x - 5}{x^2 + 4x + 4}$

c $\frac{4x}{4x^2 + 12x + 9}$ d $\frac{6x - 19}{9x^2 - 30x + 25}$

14 Find an antiderivative of each of the following.

a $\frac{x^2 - 4x - 11}{x^2 + x - 12}$ b $\frac{-3x^2 - 4x - 5}{x^2 + 2x - 3}$

c $\frac{4x^2 - 17x - 26}{x^2 - 4x - 12}$ d $\frac{-2x^3 + 12x^2 - 17x}{x^2 - 6x + 8}$

15 a Determine $\int \frac{1}{x^2 + kx + 25} dx$ for the cases when:

i $k = 0$ ii $k = 26$ iii $k = -10$.

b Determine $\int \frac{1}{4x^2 - 12x + k} dx$ for the cases when:

i $k = 8$ ii $k = 9$ iii $k = 25$.

16 Evaluate each of the following.

a $\displaystyle\int_1^2 \frac{1}{x^2 + 4x}\,dx$

b $\displaystyle\int_5^6 \frac{1}{x^2 - 16}\,dx$

c $\displaystyle\int_{-1}^1 \frac{3x + 8}{x^2 + 6x + 8}\,dx$

d $\displaystyle\int_3^4 \frac{x - 6}{x^2 - 4x + 4}\,dx$

17 **a** Find the area bounded by the curve $y = \dfrac{5}{x^2 + x - 6}$, the coordinate axes and $x = -1$.

b Find the area bounded by the curve $y = \dfrac{2x - 3}{4 + 3x - x^2}$, the x-axis and the lines $x = 2$ and $x = 3$.

c Find the area bounded by the curve $y = \dfrac{21}{40 - 11x - 2x^2}$, the coordinate axes and $x = -5$.

d Find the area bounded by the curve $y = \dfrac{x^3 - 9x + 9}{x^2 - 9}$, the x-axis and the lines $x = 4$ and $x = 6$.

18 **a** Find the value of a if $\displaystyle\int_1^2 \frac{2}{4x - x^2}\,dx = \log_e(a)$.

b Find the value of b if $\displaystyle\int_1^2 \frac{2}{\sqrt{4x - x^2}}\,dx = \pi b$.

c Find the value of c if $\displaystyle\int_3^4 \frac{3x}{x^2 - x - 2}\,dx = \log_e(c)$.

d Find the value of d if $\displaystyle\int_{\frac{1}{2}}^2 \frac{3}{x^2 - x + 1}\,dx = \pi d$.

19 **a** If $y = \dfrac{16x^2 + 25}{16x^2 - 25}$, find:

i $\displaystyle\int y\,dx$

ii $\displaystyle\int \frac{1}{y}\,dx$.

b If $y = \dfrac{36x^2 - 49}{36x^2 + 49}$, find:

i $\displaystyle\int y\,dx$

ii $\displaystyle\int \frac{1}{y}\,dx$.

20 Show that $\displaystyle\int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\,dx = \frac{22}{7} - \pi$.

MASTER

21 Evaluate each of the following.

a $\displaystyle\int_0^1 \frac{27x^2}{16 - 9x^2}dx$

b $\displaystyle\int_0^2 \frac{20x^2}{4x^2 + 4x + 1}dx$

c $\displaystyle\int_{\sqrt{3}}^3 \frac{x^2 - 2x + 9}{x^3 + 9x}dx$

d $\displaystyle\int_2^3 \frac{4x^2 - 16x + 19}{(2x - 3)^3}dx$

22 If a, b, p and q are all non-zero real constants, find each of the following.

a $\displaystyle\int \frac{1}{b^2x^2 - a^2}dx$

b $\displaystyle\int \frac{x}{b^2x^2 - a^2}dx$

c $\displaystyle\int \frac{x}{(ax - b)^2}dx$

d $\displaystyle\int \frac{1}{(px + a)(qx + b)}dx$

eBook*plus* ONLINE ONLY

8.8 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyON

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

8 Answers

EXERCISE 8.2

1 26

2 $\frac{16}{3}$

3 $\frac{92}{3}$

4 16

5 4

6 12

7 $2(e^6 - 1)$

8 $6(e^2 - e^{-2})$

9 $\frac{1}{6}$

10 $\frac{1372}{27}$

11 8

12 128

13 $121\frac{1}{2}$

14 36

15 a 9 b $37\frac{1}{2}$

16 a i 32 ii 22

b i $166\frac{2}{3}$ ii 132

17 a $\frac{36}{\pi}$ b $\frac{16}{\pi}$ c $\frac{2a}{n}$

18 a i $\log_e(4)$ ii 1 iii $\log_e(a)$

b i $\frac{2}{3}$ ii $1 - \frac{1}{a}$

c $85\frac{1}{3}$

19 a $139\frac{1}{2}$ b $81\frac{1}{3}$

c i 648 ii $465\frac{3}{4}$

20 a $\frac{4a^3}{3}$ b $\frac{1}{2}a^4$

21 a $\frac{1}{6}$ b $\frac{1}{2}$ c $\frac{3}{10}$ d $\frac{2}{3}$

22 a i 288 ii 288

b i $121\frac{1}{2}$ ii $121\frac{1}{2}$

23 a $\frac{15}{4} - 2\log_e(4) \approx 0.9774$

b $\frac{40}{27} - \frac{1}{3}\log_e(9) \approx 0.7491$

24 Check with your teacher.

25 a i (3.4443, 3.9543) ii 4.6662

b i (3.8343, 1.9172) ii 8.6558

26 a i (2.6420, 86.1855) ii 64.8779

b i (7.5882, –1.7003), (24.2955, –4.6781)

ii 384.3732

EXERCISE 8.3

1 $\frac{1}{35}(5x-9)^7 + c$

2 $\frac{1}{24}(3x+4)^8 + c$

3 $y = \frac{1}{8}(\sqrt{16x+25} - 5)$

4 $\frac{11}{4}$

5 $\frac{1}{3}\log_e\left(|3x-5|\right) + c$

6 $-\frac{1}{2}\log_e\left(|7-2x|\right) + c$

7 2

8 8

9 a $\frac{1}{175}(5x-9)^7 + \frac{3}{50}(5x-9)^6 + c$
$= \frac{1}{350}(10x+3)(5x-9)^6$

b $\frac{2}{9}\log_e\left(|3x-4|\right) - \frac{5}{9(3x-4)} + c$

10 a $\frac{-(4x+7)}{8(2x+7)^2} + c$ b $\frac{1}{27}(3x-5)\sqrt{6x+5} + c$

11 4

12 $\frac{5}{3}\log_e\left(\frac{5}{2}\right) - 1$

13 $2x - \frac{8}{3}\log_e\left(|3x+4|\right) + c$

14 $2 + 5\log_e\left(\frac{3}{5}\right)$

15 a $\frac{1}{21}(3x+5)^7 + c$ b $\frac{-1}{3(3x+5)} + c$

c $\frac{-1}{6(3x+5)^2} + c$ d $\frac{1}{2}\sqrt[3]{(3x+5)^2} + c$

16 a $\frac{1}{54}(6x+7)^9 + c$ b $\frac{1}{3}\sqrt{6x+7} + c$

c $\frac{1}{6}\log_e\left(|6x+7|\right) + c$ d $\frac{-1}{6(6x+7)} + c$

17 a $\frac{1}{72}(3x+5)^8 - \frac{5}{63}(3x+5)^7 + c$
$= \frac{1}{504}(21x-5)(3x+5)^7 + c$

b $\frac{1}{9}\log_e\left(|3x+5|\right) + \frac{5}{9(3x+5)} + c$

c $\frac{-(6x+5)}{18(3x+5)^2} + c$

d $\frac{1}{10}(2x-5)(3x+5)^{\frac{2}{3}} + c$

18 a $\frac{1}{360}(6x+7)^{10} - \frac{7}{324}(6x+7)^9 + c$
$= \frac{1}{3240}(54x-7)(6x+7)^9 + c$

b $\frac{1}{27}(3x-7)\sqrt{6x+7} + c$

c $\frac{x}{6} - \frac{7}{36}\log_e\left(|6x+7|\right) + c$

d $\frac{7}{36(6x+7)} + \frac{1}{36}\log_e\left(|6x+7|\right) + c$

19 a $\frac{t}{2(2-5t)}$ b $y = -\sqrt{3-2x}$

c $\frac{3}{2}\log_e(3)$ d $y = \frac{1}{3}(x-9)\sqrt{2x+9} + 9$

20 a $\frac{7}{2}$ b $\frac{47}{7}$ c 6 d $\frac{333}{10}$

21 a $\frac{26}{3}$ b $\frac{1}{3}\log_e\left(\frac{14}{5}\right)$ c $\frac{1}{35}$ d 6

22 a i $\frac{16}{3}$

ii $\frac{32}{3}$

b $\frac{32}{3}$

c $\frac{256}{15}$

d i $\frac{4}{3}\sqrt{a^3}$

ii $\frac{8}{15}\sqrt{a^5}$

23 a $\frac{2}{3a}(ax+b)^{\frac{3}{2}}+c$

b $\frac{2}{15a^2}(3ax-2b)(ax+b)^{\frac{3}{2}}+c$

c $\frac{1}{a}\log_e\left(|ax+b|\right)+c$

d $\frac{x}{a}-\frac{b}{a^2}\log_e\left(|ax+b|\right)+c$

24 a $\frac{2}{a}\sqrt{ax+b}+c$

b $\frac{2}{3a^2}(ax-2b)\sqrt{ax+b}+c$

c $\frac{-1}{a(ax+b)}+c$

d $\frac{b}{a^2(ax+b)}+\frac{1}{a^2}\log_e\left(|ax+b|\right)+c$

25 a $\frac{ad-bc}{a^2}\log_e\left(|ax+b|\right)+\frac{cx}{a}+k$

b $\frac{c}{a^2}\log_e\left(|ax+b|\right)-\frac{ad-bc}{a^2(ax+b)}+k$

c $\frac{cx}{a^2}-\frac{2bc}{a^3}\log_e\left(|ax+b|\right)-\frac{a^2d+cb^2}{a^3(ax+b)}+k$

d $\frac{a^2d+cb^2}{a^3}\log_e\left(|ax+b|\right)+\frac{cx^2}{2a}-\frac{bcx}{a^2}+k$

26 a $\frac{1}{2a^3}(ax-3b)(ax+b)+\frac{b^2}{a^3}\log_e\left(|ax+b|\right)+c$

b $\frac{x(xa+2b)}{a^2(ax+b)}-\frac{2b}{a^3}\log_e\left(|ax+b|\right)+c$

c $\frac{1}{a^3}\log_e\left(|ax+b|\right)+\frac{b(4ax+3b)}{2a^3(ax+b)^2}+c$

d $\frac{2}{15a^3}(3a^2x^2-4abx+8b^2)\sqrt{ax+b}+c$

EXERCISE 8.4

1 $\frac{-2}{(x^2+16)^2}+c$

2 $\frac{5}{2}\sqrt{2x^2+3}+c$

3 $\frac{1}{2}\log_e(x^2+4x+29)+c$

4 $\frac{1}{3}\log_e\left(|x^3+9|\right)+c$

5 $\cos\left(\frac{1}{x}\right)+c$

6 $\frac{1}{2}\sin(x^2)+c$

7 a $\frac{1}{3}e^{\sin(3x)}+c$ b $2\sin(\sqrt{x})+c$

8 a $\frac{1}{2}e^{\tan(2x)}+c$ b $\frac{1}{8}(\log_e(3x))^2+c$

9 1

10 $\frac{1}{80}$

11 $\frac{\pi^2}{24}$

12 $\frac{\pi^2}{128}$

13 a $\frac{1}{12}(x^2+4)^6+c$ b $\frac{1}{2}\log_e(x^2+4)+c$

c $\frac{-1}{2(x^2+9)}+c$ d $\sqrt{x^2+9}+c$

14 a $\frac{-1}{6(x^3+27)^2}+c$ b $\frac{2}{3}\sqrt{x^3+27}+c$

c $\frac{1}{3}\log_e\left(|x^3+8|\right)+c$ d $\frac{1}{12}(x^3+8)^4+c$

15 a $\frac{1}{8}(x^2-4x+13)^4+c$ b $\frac{-1}{2(x^2-4x+13)}+c$

c $-\sqrt{x^2-8x+25}+c$

d $-\frac{1}{2}\log_e(x^2-8x+25)+c$

16 a $\frac{1}{8}\log_e|4e^{2x}+5|+c$ b $\frac{1}{6(2e^{-3x}-5)}+c$

c $\frac{1}{12(3e^{-2x}+4)^2}+c$ d $\frac{-1}{e^{2x}+x}+c$

17 a $-\cos(\log_e(4x))+c$ b $\sin(\log_e(3x))+c$

c $\frac{1}{2}\left(\sin^{-1}\left(\frac{x}{2}\right)\right)^2+c$ d $\frac{1}{4}(\tan^{-1}(2x))^2+c$

18 a $y=\frac{1}{2}(1-\cos(x^2))$ b $1-\sqrt{3}$

c $\frac{1}{2}\log_e\left(\frac{29}{20}\right)$ d $\frac{1}{2}(1-e)$

19 a 12 b $\frac{1}{4}\log_e\left(\frac{5}{3}\right)$ c $\frac{1}{18}$ d 0

20 a $\frac{1}{2}\log_e\left(\frac{26}{29}\right)$ b $-\frac{1}{4}$

c $2e(e-1)$ d $\frac{1}{2}(e-1)$

21 a $\frac{1}{2}(e-1)$ b $\frac{1}{6}$

c $\frac{1}{2}(1-\sqrt{3})$ d $2(\cos(1)-\cos(2))$

22 a $c=4, \frac{1}{2}\log_e\left(\frac{21}{5}\right)$ b $\frac{\sqrt{2}}{4}$

c $\frac{1}{2}(1-e^{-4})$ d 2

23 a $\frac{1}{a}\sqrt{ax^2+b}+c$ b $\frac{-1}{2a(ax^2+b)}+c$

c $\frac{1}{2a(n+1)}(ax^2+b)^{n+1}+c$ d $\frac{1}{2a}\log_e\left(|ax^2+b|\right)+c$

24 a $\log_e\left(|f(x)|\right)+c$ b $-\frac{1}{f(x)}+c$

c $2\sqrt{f(x)}+c$ d $e^{f(x)}+c$

EXERCISE 8.5

1 a $\frac{x}{2} - \sin\left(\frac{x}{2}\right) + c$ **b** $-\cos\left(\frac{x}{2}\right) + c$

2 a $\frac{4\pi - 3\sqrt{3}}{12}$ **b** $\frac{\pi}{24} - \frac{\sqrt{3}}{128}$

3 $\frac{1}{24}\sin^6(4x) + c$

4 $\frac{1}{8}$

5 $\frac{1}{12}\sin^3(4x) - \frac{1}{10}\sin^5(4x) + \frac{1}{28}\sin^7(4x)$

6 $\frac{8 - 5\sqrt{2}}{36}$

7 $\frac{3x}{8} - \frac{1}{8}\sin(4x) + \frac{1}{64}\sin(8x) + c$

8 $\frac{\pi}{32} + \frac{1}{12}$

9 $2\log_e\left(\left|\sec\left(\frac{x}{2}\right)\right|\right) + c$

10 $\frac{1}{4}\log_e(2)$

11 $3\tan\left(\frac{x}{3}\right) - x + c$

12 $\frac{4 - \pi}{16}$

13 a $-\frac{1}{8}\cos(4x) + c$ **b** $x + c$

c $-\frac{1}{8}\cos^4(2x) + c$ **d** $\frac{1}{8}\sin^4(2x) + c$

14 a $\frac{1}{10}$ **b** $\frac{1}{15}$ **c** $\frac{\pi}{32}$ **d** $\frac{1}{15}$

15 a $\frac{x}{8} - \frac{1}{128}\sin(16x) + c$

b $\frac{1}{20}\cos^5(4x) - \frac{1}{12}\cos^3(4x) + c$

c $\frac{1}{12}\sin^3(4x) - \frac{1}{20}\sin^5(4x) + c$

d $\frac{1}{20}\sin^5(4x) - \frac{1}{28}\sin^7(4x) + c$

16 a $-\frac{1}{2}\operatorname{cosec}(2x) + c$ **b** $\frac{1}{2}\sec(2x) + c$

c $\frac{1}{4}\sec^2(2x) + c$ **d** $-\frac{1}{4}\operatorname{cosec}^2(2x) + c$

17 a $\frac{\pi}{12}$ **b** $\frac{2}{9}$

c $\frac{\pi}{16}$ **d** $\frac{8}{45}$

18 a $x - \frac{1}{4}\cos(4x) + c$

b $\frac{1}{2}(\sin(2x) - \cos(2x)) + \frac{1}{6}(\cos^3(2x) - \sin^3(2x)) + c$

c i $\frac{1}{96}\cos^3(4x) - \frac{1}{32}\cos(4x) + c_1$

ii $\frac{1}{12}\cos^6(2x) - \frac{1}{8}\cos^4(2x) + c_3$

iii $\frac{1}{8}\sin^4(2x) - \frac{1}{12}\sin^6(2x) + c_3$

19 a $\frac{1}{3}\log_e\left(|\sec(3x)|\right) + c$ **b** $\frac{1}{3}\log_e\left(|\sin(3x)|\right) + c$

c $\frac{1}{6}\tan^2(3x) + c$ **d** $\frac{1}{9}\tan^3(3x) + c$

20 a $\frac{1}{10}\log_e(2)$ **b** $\frac{4 - \pi}{20}$

c $\frac{1}{20}$ **d** $\frac{8}{75}$

21 a $\frac{-1}{a(n+1)}\cos^{n+1}(ax) + c$

b $\frac{1}{a(n+1)}\sin^{n+1}(ax) + c$

c $\frac{1}{a(n+1)}\tan^{n+1}(ax) + c$

22 a $-\frac{1}{a}\log_e\left(|\cos(ax)|\right) + c$

b $\frac{1}{a}\tan(ax) - x + c$

c $\frac{1}{2a}\tan^2(ax) + \frac{1}{a}\log_e\left(|\cos(ax)|\right) + c$

d $\frac{1}{3a}\tan^3(ax) - \frac{1}{a}\tan(ax) + x + c$

23 a $\frac{x}{2} - \frac{1}{4a}\sin(2ax) + c$

b $\frac{1}{3a}\cos^3(ax) - \frac{1}{a}\cos(ax) + c$

c $\frac{3x}{8} - \frac{1}{4a}\sin(2ax) + \frac{1}{32a}\sin(4ax) + c$

d $-\frac{1}{a}\cos(ax) + \frac{2}{3a}\cos^3(ax) - \frac{1}{5a}\cos^5(ax) + c$

24 a $\frac{x}{2} + \frac{1}{4a}\sin(2ax) + c$

b $\frac{1}{a}\sin(ax) - \frac{1}{3a}\sin^3(ax) + c$

c $\frac{3x}{8} + \frac{1}{4a}\sin(2ax) + \frac{1}{32a}\sin(4ax) + c$

d $\frac{1}{a}\sin(ax) - \frac{2}{3a}\sin^3(ax) + \frac{1}{5a}\sin^5(ax) + c$

EXERCISE 8.6

1 a $\sin^{-1}\left(\frac{x}{10}\right) + c$ **b** $4\sin^{-1}\left(\frac{3x}{8}\right) + c$

2 a $\frac{1}{5}\sin^{-1}\left(\frac{5x}{6}\right) + c$ **b** $-\frac{1}{25}\sqrt{36 - 25x^2} + c$

3 $-\frac{1}{2}\sin^{-1}\left(\frac{4x}{5}\right) = \frac{1}{4}\left(2\cos^{-1}\left(\frac{4x}{5}\right) - \pi\right)$

4 $2\cos^{-1}\left(\frac{x}{2}\right) + 3$

5 a $\frac{1}{10}\tan^{-1}\left(\frac{x}{10}\right) + c$ **b** $\frac{1}{2}\tan^{-1}\left(\frac{3x}{8}\right) + c$

6 a $\frac{1}{30}\tan^{-1}\left(\frac{5x}{6}\right) + c$ **b** $\frac{1}{10}\log_e(64 + 25x^2) + c$

7 a $\frac{\pi}{24}$ **b** $\frac{\pi}{144}$

8 a $\frac{\pi\sqrt{6}}{36}$ **b** $\frac{\pi\sqrt{3}}{9}$

9 a $\frac{1}{8}\tan^{-1}\left(\frac{2x - 3}{4}\right) + c$ **b** $\frac{1}{2}\sin^{-1}\left(\frac{2x - 3}{4}\right) + c$

10 a $\frac{1}{15}\tan^{-1}\left(\frac{5x-2}{3}\right)+c$ b $\frac{1}{5}\sin^{-1}\left(\frac{5x-2}{3}\right)+c$

11 $\frac{5}{3}\sin^{-1}\left(\frac{3x}{5}\right)+\frac{1}{3}\sqrt{25-9x^2}+c$

12 $\frac{1}{6}\log_e(9x^2+25)+\frac{1}{3}\tan^{-1}\left(\frac{3x}{5}\right)+c$

13 a $\sin^{-1}\left(\frac{x}{4}\right)+c$ b $\frac{1}{4}\sin^{-1}(4x)+c$

c $2\sin^{-1}\left(\frac{5x}{7}\right)+c$ d $-\frac{2}{5}\sqrt{49-25x^2}+c$

14 a $\cos^{-1}\left(\frac{x}{2}\right)+c$ b $\cos^{-1}(2x)+c$

c $\frac{3}{49}\sqrt{36-49x^2}+c$ d $\frac{3}{7}\cos^{-1}\left(\frac{7x}{6}\right)+c$

15 a $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$ b $\frac{1}{72}\log_e(36x^2+49)+c$

c $\frac{1}{2}\tan^{-1}\left(\frac{6x}{7}\right)+c$ d $2\tan^{-1}(4x)+c$

16 a 3 b $\frac{\pi}{2}$

c $\frac{\pi}{12}$ d $\frac{1}{2}\log_e(2)$

17 a $y=\sin^{-1}\left(\frac{x}{2}\right)+\frac{2\pi}{3}$ b $y=\frac{1}{2}\tan^{-1}(2x)+\frac{7\pi}{8}$

c $\frac{\pi\sqrt{3}}{18}$ d $\frac{\pi}{4}$

18 a $\frac{\pi}{18}$ b $\frac{1}{18}\log_e(2)$

c $\frac{5}{9}$ d $\frac{\pi}{60}$

19 a $\frac{\sqrt{3}\pi}{9}$ b $\frac{\sqrt{3}\pi}{9}$

c $\frac{1}{3}\log_e(2)$ d $\frac{1}{3}$

20 a $2\sin^{-1}\left(\frac{x+2}{3}\right)+c$ b $\frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$

c $2\sin^{-1}\left(\frac{3x+5}{7}\right)+c$ d $\frac{2}{7}\tan^{-1}\left(\frac{3x+5}{7}\right)+c$

21 a $\cos^{-1}\left(\frac{4x}{3}\right)-\frac{3}{16}\sqrt{9-16x^2}+c$

b $\frac{1}{4}\tan^{-1}\left(\frac{4x}{3}\right)+\frac{1}{8}\log_e(16x^2+9)+c$

c $\sqrt{5-2x^2}+\frac{5\sqrt{2}}{2}\sin^{-1}\left(\frac{\sqrt{10}\,x}{5}\right)+c$

d $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}x}{5}\right)-\frac{1}{2}\log_e(2x^2+25)+c$

22 a $\frac{1}{q}\sin^{-1}\left(\frac{qx}{p}\right)+c$ b $\frac{1}{pq}\tan^{-1}\left(\frac{qx}{p}\right)+c$

c $\frac{b}{q}\sin^{-1}\left(\frac{qx}{p}\right)-\frac{a}{q^2}\sqrt{p^2-q^2x^2}+c$

d $\frac{b}{pq}\tan^{-1}\left(\frac{qx}{p}\right)+\frac{a}{2q^2}\log_e(q^2x^2+p^2)+c$

23 a i $8\sin^{-1}\left(\frac{x}{4}\right)+\frac{x}{2}\sqrt{16-x^2}+c$

ii 4π; one-quarter of the area of a circle of radius 4

b i $\frac{x}{2}\sqrt{36-25x^2}-\frac{18}{5}\cos^{-1}\left(\frac{5x}{6}\right)+c$

ii $\frac{9\pi}{5}$; one-quarter of the area of an ellipse with semi-minor and major axes of $\frac{6}{5}$ and 6

c Check with your teacher.

d Check with your teacher.

24 a $\frac{1}{2}\log_e(2)-\frac{\pi}{8}$ b $\frac{1}{4}\log_e(2)+\frac{\pi}{8}$

c $9+\frac{5\pi}{2}$ d $\sqrt{3}-\frac{5\pi}{3}$

EXERCISE 8.7

1 a $\frac{1}{20}\log_e\left(\left|\frac{x+10}{x-10}\right|\right)+c$

b $\frac{1}{4}\log_e\left(\left|\frac{3x+8}{3x-8}\right|\right)+c$

2 a $\frac{1}{60}\log_e\left(\left|\frac{5x+6}{5x-6}\right|\right)+c$

b $-\frac{1}{50}\log_e\left(|25x^2-36|\right)+c$

3 $\log_e\left(\frac{(x-3)^2}{|x+5|}\right)+c$

4 $\log_e\left(\frac{|x+4|^3}{(x-1)^2}\right)+c$

5 $\frac{5}{x+3}+2\log_e\left(|x+3|\right)+c$

6 $\frac{2}{9}\log_e\left(|3x-4|\right)-\frac{5}{9(3x-4)}+c$

7 $3x+\log_e\left(\frac{(x-2)^4}{|x+5|^3}\right)+c$

8 $\log_e\left(\frac{(x-2)^2}{|x+3|}\right)-2x+c$

9 $\log_e\left(\frac{|x-2|}{\sqrt{x^2+9}}\right)-\frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$

10 $\log_e\left(\frac{|x+3|}{\sqrt{x^2+16}}\right)+\frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$

11 a $\log_e\left(\frac{(x-3)^2}{|x+4|}\right)+c$

b $\log_e\left((x-5)^2|x+3|^3\right)+c$

c $\log_e\left(\frac{|x+3|^5}{|x-2|^3}\right)+c$

d $\log_e\left(\left|\frac{x-7}{x+4}\right|\right)+c$

12 a $\frac{1}{4}\log_e\left(\left|\frac{x-2}{x+2}\right|\right)+c$

b $\frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right)+c$

c $\frac{1}{2}\log_e\left(\left|x^2-25\right|\right)+c$

d $\frac{1}{4}\log_e\left(\left|(x-6)^3(x+6)^5\right|\right)+c$

13 a $2\log_e\left(\left|x-3\right|\right)-\frac{9}{x-3}+c$

b $2\log_e\left(\left|x+2\right|\right)+\frac{9}{x+2}+c$

c $\log_e\left(\left|2x+3\right|\right)+\frac{3}{2x+3}+c$

d $\frac{2}{3}\log_e\left(\left|3x-5\right|\right)+\frac{3}{3x-5}+c$

14 a $x-\log_e\left(\left|(x-3)^2(x+4)^3\right|\right)+c$

b $-3x+\log_e\left(\left|\frac{(x+3)^5}{(x-1)^3}\right|\right)+c$

c $4x+\log_e\left(\frac{(x-6)^2}{|x+2|^3}\right)+c$

d $-x^2+\log_e\left(\frac{|x-2|}{(x-4)^2}\right)+c$

15 a **i** $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+c$

ii $\frac{1}{24}\log_e\left(\left|\frac{x+1}{x+25}\right|\right)+c$

iii $-\frac{1}{x-5}+c$

b **i** $\frac{1}{4}\log_e\left(\left|\frac{x-2}{x-1}\right|\right)+c$

ii $\frac{1}{2(3-2x)}+c$

iii $\frac{1}{8}\tan^{-1}\left(\frac{2x-3}{4}\right)+c$

16 a $\frac{1}{4}\log_e\left(\frac{5}{3}\right)$ **b** $\frac{1}{8}\log_e\left(\frac{9}{5}\right)$

c $\log_e\left(\frac{25}{3}\right)$ **d** $\log_e(2)-2$

17 a $\log_e\left(\frac{9}{4}\right)$ **b** $\log_e\left(\frac{3}{2}\right)$

c $\log_e(8)$ **d** $10+\frac{3}{2}\log_e\left(\frac{7}{3}\right)$

18 a $\sqrt{3}$ **b** $\frac{1}{3}$ **c** 5 **d** $\frac{2\sqrt{3}}{3}$

19 a **i** $x+\frac{5}{4}\log_e\left(\left|\frac{4x-5}{4x+5}\right|\right)+c$

ii $x-\frac{5}{2}\tan^{-1}\left(\frac{4x}{5}\right)+c$

b **i** $x-\frac{7}{3}\tan^{-1}\left(\frac{6x}{7}\right)+c$

ii $x+\frac{7}{6}\log_e\left(\left|\frac{6x-7}{6x+7}\right|\right)+c$

20 Check with your teacher.

21 a $2\log_e(7)-3$ **b** $12-5\log_e(5)$

c $\frac{1}{2}\log_e(3)-\frac{\pi}{18}$ **d** $\frac{2}{9}+\frac{1}{2}\log_e(3)$

22 a $\frac{1}{2ab}\log_e\left(\frac{|bx-a|}{|bx+a|}\right)+c$

b $\frac{1}{2b^2}\log_e\left(\left|b^2x^2-a^2\right|\right)+c$

c $\frac{1}{a^2}\log_e\left(\left|ax-b\right|\right)-\frac{b}{a^2(ax-b)}+c$

d $\frac{1}{aq-bp}\log_e\left(\frac{|qx+b|}{|px+a|}\right)+c$

9

Differential equations

9.1 Kick off with CAS

Solving differential equations with CAS

This topic looks at differential equations. Applications of differential equations will be studied in later topics. CAS can be used to solve the differential equation $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ using the syntax desolve $\left(y' = f(x,y) \text{ and } y(x_0) = y_{0,} x, y\right)$.

1 Solve each of the following differential equations using CAS.

a $\frac{dy}{dx} = 3y, y(0) = 4$

b $\frac{dy}{dx} + 5y = 0, y(0) = 6$

2 The differential equation $\frac{dN}{dt} = kN, N(0) = N_0$ will be studied in a later topic. Determine the solution of this differential equation.

3 Solve each of the following differential equations using CAS.

a $\frac{dy}{dx} = 2(y - 3), y(0) = 6$

b $\frac{dy}{dx} = 3(y + 4), y(0) = 7$

c $\frac{dy}{dx} = 4(y - 5), y(0) = 8$

d $\frac{dy}{dx} = 5(y - 6), y(0) = 9$

4 The differential equation $\frac{dN}{dt} = k(P - N), N(0) = N_0$ will be studied in a later topic. Determine the solution of this differential equation.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

9.2 Verifying solutions to a differential equation

Classification of differential equations

studyon

Units 3 & 4
AOS 3
Topic 3
Concept 1

Verification of solutions
Concept summary
Practice questions

A differential equation is an equation involving derivatives. It is of the form $g\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right) = 0$.

A differential equation contains the function $y = f(x)$ as the dependent variable, x as the independent variable, and various derivatives. In this topic, only differential equations that contain functions of one variable, $y = f(x)$, are considered.

Differential equations can be classified according to their order and degree. The **order** of a differential equation is the order of the highest derivative present. The **degree** of a differential equation is the degree of the highest power of the highest derivative.

A linear differential equation is one in which all variables including the derivatives are raised to the power of 1.

Some examples of differential equations are:

(a) $\frac{dy}{dx} = ky$

(b) $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

(c) $\ddot{x} - t\dot{x} + 2x = t$

(d) $\frac{d^2y}{dx^2} + n^2y = 0$

(e) $x\left(\frac{dy}{dx}\right)^3 + 3\frac{dy}{dx} + 5y = 0$

(f) $D_t^{\,3}x = \sqrt{x^2 + 1}$

Note that (a) and (e) are first order; (b), (c) and (d) are second order; and (f) is third order. Equation (e) has a degree of 3, whereas all the others have a degree of 1. Equations (a), (b), (c) and (d) are linear; (e) and (f) are non-linear. Note also that there are many different notations for derivatives; for example, second-order derivatives can be expressed as $\ddot{x} = \frac{d^2x}{dt^2}$ and $D_t^{\,2}x = \frac{d^2x}{dt^2}$.

Differential equations are extremely important in the study of mathematics and appear in almost every branch of science. Sir Isaac Newton (1642–1727) and Gottfried Wilheim Leibniz (1646–1716) are both credited with the discovery of calculus and differential equations in the 1670s and 1680s.

Gottfried Leibniz

Verifying solutions to differential equations

To check that a given solution satisfies the differential equation, use the process of differentiation and substitution. Generally only first- or second-order differential equations will be considered in this topic.

When setting out a proof, it is necessary to show that the left-hand side (LHS) of the equation is equal to the right-hand side (RHS).

WORKED EXAMPLE 1

Verify that $y = x^3$ is a solution of the differential equation $x^3\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + 3xy = 0$.

THINK	WRITE
1 Use basic differentiation to find the first derivative.	$y = x^3$ $\dfrac{dy}{dx} = 3x^2$
2 Find the second derivative.	$\dfrac{d^2y}{dx^2} = 6x$
3 Substitute for y, the first derivative and second derivative into the LHS of the differential equation.	$\text{LHS} = x^3\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + 3xy$ $= x^3 \times (6x) - (3x^2)^2 + 3x \times (x^3)$
4 Simplify and expand, so that LHS = RHS = 0, thus proving the given solution does satisfy the differential equation.	$= 6x^4 - 9x^4 + 3x^4$ $= 0$ $= \text{RHS}$

Differential equations involving unknowns

When we verify a given solution to a differential equation involving algebraic, trigonometric or exponential functions, there may also be an unknown value that must be determined for which the given solution satisfies the differential equation.

WORKED EXAMPLE 2

Given that $y = e^{kx}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 0$, find the values of the real constant k.

THINK	WRITE
1 Use the rule for differentiation of exponential functions, $\dfrac{d}{dx}(e^{kx}) = ke^{kx}$, to find the first derivative.	$y = e^{kx}$ $\dfrac{dy}{dx} = ke^{kx}$
2 Differentiate again to find the second derivative.	$\dfrac{d^2y}{dx^2} = k^2e^{kx}$
3 Substitute for y, the first derivative $\dfrac{dy}{dx}$ and the second derivative $\dfrac{d^2y}{dx^2}$ into the given differential equation.	$\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 0$ $k^2e^{kx} - 2ke^{kx} - 8e^{kx} = 0$
4 Take out the common factor.	$e^{kx}(k^2 - 2k - 8) = 0$
5 Factorise the quadratic equation involving the unknown.	$e^{kx} \neq 0 \Rightarrow k^2 - 2k - 8 = 0$ $(k - 4)(k + 2) = 0$
6 Solve the resulting equation for the unknown and state the answer.	When $k = 4$ or $k = -2$, then $y = e^{kx}$ is a solution of $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 0$.

Solutions to a differential equation involving products

When verifying solutions to a differential equation involving a mixture of algebraic, trigonometric or exponential functions, it may be necessary to use the product or quotient rules for differentiation.

WORKED EXAMPLE 3

Verify that $y = xe^{-2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$.

THINK	WRITE
1 Use the product rule for differentiation to find the first derivative.	$y = xe^{-2x}$ $\frac{dy}{dx} = x\frac{d}{dx}(e^{-2x}) + e^{-2x}\frac{d}{dx}(x)$ $= -2xe^{-2x} + e^{-2x}$
2 Simplify the first derivative by taking out the common factor.	$\frac{dy}{dx} = e^{-2x}(1 - 2x)$
3 Find the second derivative, using the product rule again.	$\frac{d^2y}{dx^2} = e^{-2x}\frac{d}{dx}(1 - 2x) + (1 - 2x)\frac{d}{dx}(e^{-2x})$ $= -2e^{-2x} - 2(1 - 2x)e^{-2x}$
4 Simplify the second derivative by taking out the common factor.	$\frac{d^2y}{dx^2} = e^{-2x}(-2 - 2(1 - 2x))$ $= e^{-2x}(4x - 4)$
5 Substitute for y, the first derivative $\frac{dy}{dx}$ and the second derivative $\frac{d^2y}{dx^2}$ into the LHS of the differential equation.	$\text{LHS} = \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y$ $= e^{-2x}(4x - 4) + 4e^{-2x}(1 - 2x) + 4xe^{-2x}$
6 Take out the common factor and simplify, so that LHS = RHS = 0, thus proving the given solution does satisfy the differential equation.	$= e^{-2x}[(4x - 4) + 4(1 - 2x) + 4x]$ $= e^{-2x}[4x - 4 + 4 - 8x + 4x]$ $= 0$ $= \text{RHS}$

EXERCISE 9.2 Verifying solutions to a differential equation

PRACTISE

1 **WE1** Verify that $y = x^2$ is a solution of the differential equation

$$x^2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 2y = 0.$$

2 For the differential equation $x^4\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4x^2y = 0$, show that $y = x^4$ is a solution.

3 **WE2** Given that $y = e^{kx}$ is a solution of the differential equation

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 0$, find the values of the real constant k.

4 If $y = \cos(kx)$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 9y = 0$, find the values of the real constant k.

5 WE3 Verify that $y = xe^{3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

6 Given that $y = Ax\cos(2x)$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 8\sin(2x)$, find the value of the real constant A.

CONSOLIDATE

7 a Verify that $y = x^4$ satisfies the differential equation $x^4\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4x^2y = 0$.

b If $y = 2x^2 - 3x + 5$, show that $\left(\frac{dy}{dx}\right)^2 - 8y + 31 = 0$.

c Given the differential equation $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y + 6x^2 = 0$, show that $y = x^3 - 3x^2 - \frac{3x}{2} + 1$ is a solution.

d If $y = ax^3 + bx^2$ where a and b are constants, show that $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$.

8 a Find the constants a, b and c if $y = a + bx + cx^2$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 4x^2$.

b Determine the constants a, b, c and d if $y = ax^3 + bx^2 + cx + d$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^3$.

c Show that $y = x^n$ is a solution of the differential equation $x^2y\frac{d^2y}{dx^2} - x^2\left(\frac{dy}{dx}\right)^2 + ny^2 = 0$.

d The differential equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 10y = 0$ has a solution $y = x^n$ find the possible values of n.

9 a Given that $x = e^{3t} + e^{-4t}$ show that $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 12x = 0$.

b If $y = Ae^{3x} + Be^{-3x}$ where A and B are constants, show that $\frac{d^2y}{dx^2} - 9y = 0$.

c Find the values of real constant k such that $y = e^{kx}$ satisfies $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$.

d Find the values of m where $m \in C$ if $y = e^{mx}$ satisfies $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$.

10 a If $y = 3\sin(2x) + 4\cos(2x)$, show that $\dfrac{d^2y}{dx^2} + 4y = 0$.

b Show that $y = A\sin(3x) + B\cos(3x)$, where A and B are constants, is a solution of the differential equation $\dfrac{d^2y}{dx^2} + 9y = 0$.

c If $y = a\sin(nx) + b\cos(nx)$, where n is a positive real number, show that $\dfrac{d^2y}{dx^2} + n^2y = 0$.

d Given that $x = a\sin(pt)$ satisfies $\dfrac{d^2x}{dt^2} + 9x = 0$, find the value of p.

11 a Show that $y = e^{x^2}$ satisfies the differential equation $\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 2y = 0$.

b Verify that $y = \cos(x^2)$ satisfies the differential equation $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4x^3y = 0$.

c If $y = ax + b\sqrt{x^2 + 1}$ where a and b are constants, show that $(x^2 + 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$.

d Given that $y = \log_e(x + \sqrt{x^2 - 9})$, show that $(x^2 - 9)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$.

12 a Show that $y = \tan(ax)$, where $a \in R\backslash\{0\}$, satisfies the differential equation $\dfrac{d^2y}{dx^2} = 2a^2y(1 + y^2)$.

b Verify that $y = \tan^2(ax)$, where $a \in R\backslash\{0\}$, is a solution of the differential equation $\dfrac{d^2y}{dx^2} = 2a^2(3y^2 + 4y + 1)$.

c Show that $y = \log_e(ax + b)$, where $a, b \in R\backslash\{0\}$, is a solution of the differential equation $\dfrac{d^2y}{dx^2} + a^2e^{-2y} = 0$.

13 a Verify that $y = \tan^{-1}(2x)$ is a solution of the differential equation $(1 + 4x^2)\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} = 0$.

b Show that $y = \sin^{-1}(3x)$ is a solution of the differential equation $(1 - 9x^2)\dfrac{d^2y}{dx^2} - 9x\dfrac{dy}{dx} = 0$.

c Verify that $y = \cos^{-1}\left(\dfrac{x}{4}\right)$ is a solution of the differential equation $(16 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$.

14 a A parachutist of mass m falls from rest in the Earth's gravitational field and is subjected to air resistance. The velocity v is given by

$$v = \frac{mg}{k}\left(1 - e^{-kt}\right)$$

at a time t where g and k are constants. Show that $\frac{dv}{dt} + kv = mg$.

b In a transient circuit, the current i amperes at a time t seconds is given by $i = 3e^{-2t}\sin(3t)$.

Show that $\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 13i = 0$.

15 a Verify that $y = e^{3x}\cos(2x)$ satisfies the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$.

b Find the real constants a and b if $x = t(a\cos(3t) + b\sin(3t))$ is a solution of the differential equation $\frac{d^2x}{dt^2} + 9x = 6\cos(3t)$.

16 a Given that $y = xe^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$, find the values of the real constants a and b.

b Show that $y = e^{kx}(Ax + B)$, where A, B and k are all real constants, is a solution of the differential equation $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0$.

MASTER

17 a Given that $y = Ax^2e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 10e^{-3x}$, find the value of A.

b If $y = Ax^2e^{-kx}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + k^2y = Be^{-kx}$, show that $B = 2A$.

c Show that $y = \sqrt{\frac{\pi}{x}}\sin(x)$ is a solution of Bessel's equation,

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (4x^2 - 1)y = 0.$$

18 Adrien-Marie Legendre (1752–1833) was a famous French mathematician. He made many mathematical contributions in the areas of elliptical integrals, number theory and the calculus of variations. He is also known for the differential equation named after him. Legendre's differential equation of order n is given by $(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0$ for $|x| < 1$, and the solutions of the differential equation are given by the polynomials $P_n(x)$. The first few polynomials are given by

$$\begin{aligned}P_0(x) &= 1\\ P_1(x) &= x\\ P_2(x) &= \frac{1}{2}(3x^2 - 1)\\ P_3(x) &= \frac{1}{2}(5x^3 - 3x)\\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3)\end{aligned}$$

a Verify the solution of the Legendre's differential equation for the cases when $n = 3$ and $n = 4$.

b The Legendre polynomials also satisfy many other mathematical properties. One such relation is $P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}[(x^2 - 1)^n]$. Use this result to obtain $P_2(x)$ and $P_3(x)$.

c The Legendre polynomials also satisfy $\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0$ when $m \neq n$ and $\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n + 1}$. Verify these results for $P_2(x)$ and $P_3(x)$.

9.3 Solving Type 1 differential equations, $\frac{dy}{dx} = f(x)$

studyon

Units 3 & 4
AOS 3
Topic 3
Concept 2

First-order differential equations
Concept summary
Practice questions

Classifying solutions to a differential equation

The solution of a differential equation is usually obtained by the process of integration. Because the integration process produces an arbitrary constant of integration, the solutions of a differential equation are classified as follows.

A **general solution** is one which contains arbitrary constants of integration and satisfies the differential equation.

A **particular solution** is one which satisfies the differential equation and some other initial value condition, also known as a boundary value, that enables the constant(s) of integration to be found.

In general, the number of arbitrary constants of integration to be found is equal to the order of the differential equation. Throughout this course we study and solve special types of first- and second-order differential equations.

Type 1 differential equations, $\frac{dy}{dx} = f(x)$

Direct integration

In this section we solve first-order differential equations of the form $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$. Differential equations of this form can be solved by direct integration. Hence, it is necessary to be familiar with all the integration techniques studied so far.

Antidifferentiating both sides gives $y = \int f(x)\, dx + c$. This is the general solution, which can be thought of as a family of curves. If we use the given condition $x = x_0$ when $y = y_0$, we can determine the value of the constant of integration c in this particular case, which thus gives us the particular solution.

WORKED EXAMPLE 4

a Find the general solution to $\frac{dy}{dx} + 12x = 0$.

b Find the particular solution of $\frac{dy}{dx} + 6x^2 = 0,\ y(1) = 2$.

THINK	WRITE
a 1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	**a** $\frac{dy}{dx} + 12x = 0$ $\frac{dy}{dx} = -12x$
2 Antidifferentiate to obtain y.	$y = -\int 12x\,dx$
3 Write the general solution in terms of a constant.	$y = -6x^2 + c$
b 1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	**b** $\frac{dy}{dx} + 6x^2 = 0$ $\frac{dy}{dx} = -6x^2$
2 Antidifferentiate to obtain y.	$y = -\int 6x^2 dx$
3 Express y in terms of x with an arbitrary constant.	$y = -2x^3 + c$
4 Substitute and use the given conditions to determine the value of the constant.	$y(1) = 2$: $\Rightarrow x = 1$ when $y = 2$ $2 = -2(1)^3 + c$ $c = 4$
5 Substitute back for c and state the particular solution.	$y = 4 - 2x^3$

Finding particular solutions

In Topic 8, linear substitutions were used to integrate linear expressions. The example presented in Worked example 5 is a review of this process.

WORKED EXAMPLE 5

Solve the differential equation $(4 - 3x)^2\frac{dy}{dx} + 1 = 0,\ y(1) = 2$.

THINK	WRITE
1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$(4 - 3x)^2\frac{dy}{dx} + 1 = 0$ $(4 - 3x)^2\frac{dy}{dx} = -1$ $\frac{dy}{dx} = \frac{-1}{(4 - 3x)^2}$

2 Antidifferentiate to obtain y.

$$y = \int \frac{-1}{(4-3x)^2}dx$$

3 Use index laws to express the integrand as a function to a power.

$$y = -\int (4-3x)^{-2}dx$$

4 Use a linear substitution. Express dx in terms of du by inverting both sides.

Let $u = 4 - 3x$.

$$\frac{du}{dx} = -3$$

$$\frac{dx}{du} = -\frac{1}{3}$$

$$dx = -\frac{1}{3}du$$

5 Substitute for u and dx.

$$y = -\int u^{-2}\frac{-1}{3}du$$

6 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$y = \frac{1}{3}\int u^{-2}du$$

7 Perform the integration process, using $\int u^n\,du = \frac{1}{n+1}u^{n+1}$ with $n = -2$ so that $n + 1 = -1$, and add in the constant $+c$.

$$y = -\frac{1}{3}u^{-1} + c$$

$$y = -\frac{1}{3u} + c$$

8 Substitute back for x.

$$y = -\frac{1}{3(4-3x)} + c$$

9 Substitute and use the given conditions to determine the value of the constant.

$y(1) = 2$

$\Rightarrow x = 1$ when $y = 2$

$$2 = -\frac{1}{3} + c$$

$$c = 2 + \frac{1}{3}$$

$$c = \frac{7}{3}$$

10 Substitute back for c, and state the particular solution. Although this is a possible answer, this result can be simplified.

$$y = \frac{-1}{3(4-3x)} + \frac{7}{3}$$

11 Form the lowest common denominator.

$$y = \frac{-1 + 7(4-3x)}{3(4-3x)}$$

12 Expand the brackets in the numerator; do not expand the brackets in the denominator.

$$y = \frac{-1 + 28 - 21x}{3(4-3x)}$$

13 Simplify and take out common factors which cancel.

$$y = \frac{27 - 21x}{3(4-3x)}$$

$$= \frac{3(9-7x)}{3(4-3x)}$$

14 State the final answer in simplified form. Note the maximal domain for which the solution is valid.

$y = \frac{9 - 7x}{4 - 3x}$ for $x \neq \frac{4}{3}$

15 Note that as a check, we can use the given condition to check the value of y.

Substitute $x = 1$:

$y = \frac{9 - 7}{4 - 3} = 2$

Stating the domain for which the solution is valid

As seen in the last example, the maximal domain for which the solution is valid is important. When solving differential equations, unless the solution is defined for all values of x, that is for $x \in R$, we are required to state the largest subset of R for which the given differential equation and solution are valid.

WORKED EXAMPLE 6

Solve the differential equation $\sqrt{3x - 5}\,\frac{dy}{dx} + 6 = 0$, $y(7) = 2$, stating the largest domain for which the solution is valid.

THINK

1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.

WRITE

$$\sqrt{3x - 5}\,\frac{dy}{dx} + 6 = 0$$

$$\sqrt{3x - 5}\,\frac{dy}{dx} = -6$$

$$\frac{dy}{dx} = \frac{-6}{\sqrt{3x - 5}}$$

2 Antidifferentiate to obtain y.

$$y = \int \frac{-6}{\sqrt{3x - 5}}\,dx$$

3 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$y = -6\int \frac{1}{\sqrt{3x - 5}}\,dx$$

4 Use index laws to express the integrand as a function to a power.

$$y = -6\int (3x - 5)^{-\frac{1}{2}}\,dx$$

5 Use a linear substitution. Express dx in terms of du by inverting both sides.

Let $u = 3x - 5$.

$$\frac{du}{dx} = 3$$

$$\frac{dx}{du} = \frac{1}{3}$$

$$dx = \frac{1}{3}\,du$$

6 Substitute for u and dx.

$$y = -6\int u^{-\frac{1}{2}}\,\frac{1}{3}\,du$$

7 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$y = -2\int u^{-\frac{1}{2}}\,du$$

8 Perform the integration process using $\int u^n\,du = \frac{1}{n+1}u^{n+1}$ with $n = -\frac{1}{2}$, so that $n + 1 = \frac{1}{2}$, and add in the constant $+c$.

$y = -4\sqrt{u} + c$

9 Substitute back for x.

$y = -4\sqrt{3x-5} + c$

10 Substitute and use the given conditions to determine the value of the constant.

$y(7) = 2$
$\Rightarrow x = 7$ when $y = 2$
$2 = -4\sqrt{16} + c$
$c = 18$

11 Substitute back for c and state the particular solution.

$y = 18 - 4\sqrt{3x-5}$

12 Determine the domain for which the solution is valid from the differential equation.

$\frac{dy}{dx} = \frac{-6}{\sqrt{3x-5}}$ for $3x - 5 > 0$

13 Solve the inequality for x to state the largest domain for which the solution is valid for the given differential equation. State the answer.

$3x > 5$
$x > \frac{5}{3}$
The solution $y = 18 - 4\sqrt{3x-5}$ is valid for $x > \frac{5}{3}$.

Solving first-order differential equations involving inverse trigonometric functions

The results $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, $\int \frac{-1}{\sqrt{a^2 - x^2}}dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ and $\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ are used throughout this topic.

WORKED EXAMPLE 7

Solve the differential equation $\sqrt{16 - x^2}\frac{dy}{dx} + 2 = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

THINK

WRITE

1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.

$\sqrt{16 - x^2}\frac{dy}{dx} + 2 = 0$, $y(0) = 0$
$\sqrt{16 - x^2}\frac{dy}{dx} = -2$
$\frac{dy}{dx} = \frac{-2}{\sqrt{16 - x^2}}$

2 Antidifferentiate to obtain y.

$y = \int \frac{-2}{\sqrt{16 - x^2}}dx$

3 Perform the integration process using $\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$.

$y = 2\cos^{-1}\left(\frac{x}{4}\right) + c$

4 Substitute and use the given conditions to determine the value of the constant.

$y(0) = 0$
$\Rightarrow x = 0$ when $y = 0$
$0 = 2\cos^{-1}(0) + c$
$c = -2\cos^{-1}(0)$
$c = -2 \times \frac{\pi}{2}$
$c = -\pi$

5 Substitute back for c and state the particular solution.

$y = 2\cos^{-1}\left(\frac{x}{4}\right) - \pi$

6 Determine the domain for which the solution is valid from the differential equation.

$y = \int \frac{-2}{\sqrt{16 - x^2}} dx$
$\sqrt{16 - x^2} > 0$
$x^2 < 16$

7 Solve the inequality for x to state the largest domain for which the solution and the differential equation is valid. State the answer.

$|x| < 4$
The solution $y = 2\cos^{-1}\left(\frac{x}{4}\right) - \pi$ is valid for $|x| < 4$.

EXERCISE 9.3 Solving Type 1 differential equations, $\frac{dy}{dx} = f(x)$

PRACTISE

1 a WE4 Find the general solution of $\frac{dy}{dx} + 12x^3 = 0$.

b Find the particular solution of $\frac{dy}{dx} + 6x = 0$, $y(2) = 1$.

2 a Find the general solution of $\frac{dy}{dx} + 12\cos(2x) = 0$.

b Solve the differential equation $\frac{dy}{dx} + 6\sin(3x) = 0$, $y(0) = 0$, and express y in terms of x.

3 WE5 Solve the differential equation $(5 - 4x)^2 \frac{dy}{dx} + 1 = 0$, $y(1) = 2$.

4 Solve the differential equation $(7 - 4x)\frac{dy}{dx} + 2 = 0$, $y(2) = 3$.

5 WE6 Solve the differential equation $\sqrt{2x - 5}\frac{dy}{dx} + 1 = 0$, $y(3) = 0$, stating the largest domain for which the solution is valid.

6 Solve the differential equation $\sqrt{x}\frac{dy}{dx} + 2 = 0$, $y(4) = 3$, expressing y in terms of x, and state the largest domain for which the solution is valid.

7 WE7 Solve the differential equation $\sqrt{64 - x^2}\frac{dy}{dx} - 6 = 0$, $y(4) = 0$, stating the largest domain for which the solution is valid.

8 Solve the differential equation $(16 + x^2)\frac{dy}{dx} + 4 = 0$, $y(4) = \frac{\pi}{4}$, stating the largest domain for which the solution is valid.

CONSOLIDATE

9 Find the general solution to each of the following.

a $\frac{dy}{dx} - 4x = 3$

b $\frac{dy}{dx} - (3x - 5)(x + 4) = 0$

c $e^{2x}\frac{dy}{dx} + 6 = 2e^{4x}$

d $\sqrt{x^2 + 9}\frac{dy}{dx} - x = 0$

For questions 10–18, solve each of the differential equations given and state the maximal domain for which the solution is valid.

10 a $3x\frac{dy}{dx} - 2x^2 = 5$, $y(1) = 3$

b $\frac{dy}{dx} = 6(e^{-3x} + e^{3x})$, $y(0) = 0$

11 a $\frac{dy}{dx} - 4\sin(2x) = 0$, $y(0) = 2$

b $\frac{dy}{dx} + 6\cos(3x) = 0$, $y\left(\frac{\pi}{2}\right) = 5$

12 a $\frac{dy}{dx} - 8\sin^2(2x) = 0$, $y(0) = 0$

b $\frac{dy}{dx} - 12\cos^2(3x) = 0$, $y(0) = 0$

13 a $\frac{dy}{dx} = \frac{1}{\sqrt{4x + 9}}$, $y(0) = 0$

b $\frac{dy}{dx} + \frac{1}{3 - 2x} = 0$, $y(2) = 1$

14 a $\frac{dy}{dx} = \frac{1}{(3x - 5)^2}$, $y(2) = 3$

b $\frac{dy}{dx} = \frac{8}{7 - 4x}$, $y(2) = 5$

15 a $(x^2 + 9)\frac{dy}{dx} - 3x = 0$, $y(0) = 0$

b $\sqrt{x^2 + 4}\frac{dy}{dx} + x = 0$, $y(0) = 0$

16 a $(x^2 + 6x + 13)\frac{dy}{dx} - x = 3$, $y(0) = 0$

b $(x^2 - 4x + 9)\frac{dy}{dx} + x = 2$, $y(0) = 0$

17 a $\sec(2x)\frac{dy}{dx} + \sin^3(2x) = 0$, $y(0) = 0$

b $\operatorname{cosec}(3x)\frac{dy}{dx} + 9\cos^2(3x) = 0$, $y(0) = 0$

18 a $\frac{dy}{dx} + \log_e(2x) = 4$, $y\left(\frac{1}{2}\right) = 1$

b $e^x\frac{dy}{dx} + x = 5$, $y(0) = 0$

MASTER

19 Use CAS to solve the following differential equations and state the maximal domain for which the solution is valid.

a $(4x^2 + 9)\frac{dy}{dx} + 2x = 3$, $y(0) = 0$

b $\sqrt{9 - 4x^2}\frac{dy}{dx} + 2x = 3$, $y(0) = 0$

20 a If $a > 0$ and $b \neq 0$, use CAS to solve the following differential equations, stating the maximal domains for which the solution is valid.

i $\sqrt{a^2 - x^2}\frac{dy}{dx} + b = 0$, $y(0) = 0$

ii $(a^2 - x^2)\frac{dy}{dx} + b = 0$, $y(0) = 0$

iii $(a + bx)^2\frac{dy}{dx} + 1 = 0$, $y(0) = 0$

b Using CAS, solve the differential equation $e^{2x}\frac{dy}{dx} + \cos(3x) = 0$, $y(0) = 0$.

9.4 Solving Type 2 differential equations, $\frac{dy}{dx} = f(y)$

studyon

Units 3 & 4
AOS 3
Topic 3
Concept 4

First-order differential equations of the type $\frac{dy}{dx} = g(y)$
Concept summary
Practice questions

Invert, integrate and transpose

Solving first-order differential equations of the form $\frac{dy}{dx} = f(y)$, $y(x_0) = y_0$ is studied in this section. In this situation it is not possible to integrate directly. The first step in the solution process is to invert both sides.

From $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, we obtain $\frac{dx}{dy} = \frac{1}{f(y)}$.

Integrate both sides with respect to y to obtain

$$x = \int \frac{1}{f(y)}dy + c.$$

This gives the general solution. The initial condition can be used to find the value of the constant c. The resulting equation must be rearranged to express y in terms of x, which gives the particular solution.

Finding general solutions

Finding a general solution means finding the solution in terms of an arbitrary constant.

WORKED EXAMPLE 8

Find the general solution to the differential equation $\frac{dy}{dx} - 4\sqrt{y} = 0$.

THINK	WRITE
1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} - 4\sqrt{y} = 0$ $\frac{dy}{dx} = 4\sqrt{y}$
2 Invert both sides.	$\frac{dx}{dy} = \frac{1}{4\sqrt{y}}$
3 Antidifferentiate to obtain x in terms of y.	$x = \int \frac{1}{4\sqrt{y}}dy$
4 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$x = \frac{1}{4}\int \frac{1}{\sqrt{y}}dy$
5 Use index laws to express the integrand as a power.	$x = \frac{1}{4}\int y^{-\frac{1}{2}}dy$

6 Perform the integration process using $\int u^n du = \frac{1}{n+1}u^{n+1}$ with $n = -\frac{1}{2}$, so that $n + 1 = \frac{1}{2}$, and add in the constant of integration.	$x = \frac{1}{4} \times \frac{2}{1}y^{\frac{1}{2}} + c$
7 Simplify.	$x = \frac{1}{2}y^{\frac{1}{2}} + c$ $= \frac{1}{2}\sqrt{y} + c$
8 Transpose to make y the subject.	$\frac{1}{2}\sqrt{y} = x - c$ $\sqrt{y} = 2x - 2c$
9 Since c is a constant, $2c$ is also a constant.	Let $A = 2c$. $\sqrt{y} = 2x - A$
10 Square both sides and state the answer in terms of an arbitrary constant A.	$y = (2x - A)^2$

Finding particular solutions

Finding particular solutions involves solving the differential equation and expressing y in terms of x, and finding the value of the constant of integration.

WORKED EXAMPLE 9 **Solve the differential equation $\frac{dy}{dx} + (4 - 3y)^2 = 0$, $y(2) = 1$.**

THINK	WRITE
1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} + (4 - 3y)^2 = 0$ $\frac{dy}{dx} = -(4 - 3y)^2$
2 Invert both sides.	$\frac{dx}{dy} = -\frac{1}{(4 - 3y)^2}$
3 Express x as an integral of y.	$x = \int \frac{-1}{(4 - 3y)^2} dy$
4 Use index laws to express the integrand as a function to a power.	$x = -\int (4 - 3y)^{-2} dy$
5 Use a linear substitution. Express dy in terms of du by inverting both sides.	Let $u = 4 - 3y$. $\frac{du}{dy} = -3$ $\frac{dy}{du} = -\frac{1}{3}$ $dy = -\frac{1}{3}du$

6 Substitute for u and dy.

$$x = -\int u^{-2}\frac{-1}{3}du$$

7 Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$x = \frac{1}{3}\int u^{-2}du$$

8 Perform the integration process using $\int u^n du = \frac{1}{n+1}u^{n+1}$ with $n = -2$, so that $n + 1 = -1$, and add in the constant $+c$.

$$x = -\frac{1}{3}u^{-1} + c$$

$$x = -\frac{1}{3u} + c$$

9 Substitute back for y.

$$x = -\frac{1}{3(4-3y)} + c$$

10 Substitute and use the given conditions to determine the value of the constant.

$y(2) = 1$

$\Rightarrow x = 2$ when $y = 1$

$$2 = -\frac{1}{3} + c$$

$$c = 2 + \frac{1}{3}$$

$$c = \frac{7}{3}$$

11 Substitute back for c.

$$x = -\frac{1}{3(4-3y)} + \frac{7}{3}$$

12 To begin making y the subject, transpose the equation.

$$\frac{1}{3(4-3y)} = \frac{7}{3} - x$$

13 Form a common denominator on the right-hand side.

$$\frac{1}{3(4-3y)} = \frac{7-3x}{3}$$

14 Cancel the common factor and invert both sides.

$$4 - 3y = \frac{1}{7-3x}$$

15 Rearrange to make y the subject.

$$3y = 4 - \frac{1}{7-3x}$$

16 Express the right-hand side of the equation with a common denominator.

$$3y = \frac{4(7-3x)-1}{7-3x}$$

17 Expand the brackets in the numerator.

$$3y = \frac{28-12x-1}{4-3x}$$

18 Simplify and take out the common factor.

$$3y = \frac{27-12x}{7-3x}$$

$$3y = \frac{3(9-4x)}{7-3x}$$

19 State the final answer in a simplified form and state the maximal domain.

$$y = \frac{9-4x}{7-3x} \text{ for } x \neq \frac{7}{3}$$

Find c or rearrange to make y the subject?

When solving these types of differential equations, it is necessary to find the constant of integration and also rearrange to make y the subject. Sometimes the order in which we do these operations can make the processes simpler.

WORKED EXAMPLE 10 Solve the differential equation $\frac{dy}{dx} + 4y = 0,\ y(0) = 3$.

THINK	WRITE
1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} + 4y = 0$ $\frac{dy}{dx} = -4y$
2 Invert both sides.	$\frac{dx}{dy} = -\frac{1}{4y}$
3 Integrate both sides.	$x = -\int \frac{1}{4y}\,dy$
4 Take the constant factor outside the front of the integral sign.	$x = -\frac{1}{4}\int \frac{1}{y}\,dy$
5 Use $\int \frac{1}{u}\,du = \log_e\left(\lvert u\rvert\right) + c$ to express x in terms of y and the constant of integration c.	$x = -\frac{1}{4}\log_e\left(\lvert y\rvert\right) + c$
From this point forward, we have two processes to complete: find c, and transpose the equation to make y the subject. **Method 1:** Find c first, then transpose to make y the subject.	
6 Substitute and use the given conditions to determine the value of the constant.	$y(0) = 3$ $\Rightarrow x = 0$ when $y = 3$ $0 = -\frac{1}{4}\log_e\left(\lvert 3\rvert\right) + c$ $c = \frac{1}{4}\log_e(3)$
7 Substitute back for c and take out the common factor.	$x = -\frac{1}{4}\log_e\left(\lvert y\rvert\right) + \frac{1}{4}\log_e(3)$ $x = \frac{1}{4}\left[\log_e(3) - \log_e\left(\lvert y\rvert\right)\right]$
8 Use the logarithm laws to simplify the expression.	$x = \frac{1}{4}\log_e\left(\frac{3}{\lvert y\rvert}\right)$ $4x = \log_e\left(\frac{3}{\lvert y\rvert}\right)$
9 Use the definition of the logarithm.	$e^{4x} = \frac{3}{\lvert y\rvert}$

10 Invert both sides again in attempting to make y the subject.

$$\frac{|y|}{3} = \frac{1}{e^{4x}} = e^{-4x}$$

11 Because $e^{-4x} > 0$, the modulus is not needed. State the particular solution to the differential equation.

$$y = 3e^{-4x}$$

Method 2: Make y the subject and then find the constant c.

6 Rearrange to make y the subject.

$$x = -\frac{1}{4}\log_e\left(|y|\right) + c$$

$$\frac{1}{4}\log_e\left(|y|\right) = c - x$$

$$\log_e\left(|y|\right) = 4c - 4x$$

7 Since c is a constant, $4c$ is also a constant.

Let $B = 4c$.

$$\log_e\left(|y|\right) = B - 4x$$

8 Use the definition of the logarithm.

$$|y| = e^{B-4x} = e^B e^{-4x}$$

9 Since B is a constant, e^B is also a constant.

Let $A = e^B$.

$$|y| = Ae^{-4x}$$

10 Substitute and use the given condition to determine the value of the constant.

$y(0) = 3$

$\Rightarrow x = 0$ when $y = 3$

$3 = Ae^{-0}$

$3 = A$

11 Because $e^{-4x} > 0$, the modulus is not needed. Substitute for A and state the particular solution to the differential equation.

$$y = 3e^{-4x}$$

Stating the domain for which the solution is valid

As discussed in the previous section, the solution to a differential equation should include the largest domain for which the solution is valid.

WORKED EXAMPLE 11

Solve the differential equation $2\dfrac{dy}{dx} + \sqrt{16 - y^2} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

THINK

1 Rewrite the equation to make $\dfrac{dy}{dx}$ the subject.

WRITE

$$2\frac{dy}{dx} + \sqrt{16 - y^2} = 0$$

$$2\frac{dy}{dx} = -\sqrt{16 - y^2}$$

$$\frac{dy}{dx} = \frac{-\sqrt{16 - y^2}}{2}$$

2 Invert both sides.

$$\frac{dx}{dy} = \frac{-2}{\sqrt{16 - y^2}}$$

3 Integrate with respect to y.

$$x = \int \frac{-2}{\sqrt{16 - y^2}}\,dy$$

4 Perform the integration process using $\int \frac{-1}{\sqrt{a^2 - x^2}}dx = \cos^{-1}\left(\frac{x}{a}\right) + c$.

$$x = 2\cos^{-1}\left(\frac{y}{4}\right) + c$$

5 Substitute and use the given conditions to determine the value of the constant.

$$y(0) = 0$$
$$\Rightarrow x = 0 \text{ when } y = 0$$
$$0 = 2\cos^{-1}(0) + c$$
$$c = -2\cos^{-1}(0)$$
$$c = -2 \times \frac{\pi}{2}$$
$$c = -\pi$$

6 Substitute back for c.

$$x = 2\cos^{-1}\left(\frac{y}{4}\right) - \pi$$

7 Rewrite the equation.

$$2\cos^{-1}\left(\frac{y}{4}\right) = x + \pi$$
$$\cos^{-1}\left(\frac{y}{4}\right) = \frac{x + \pi}{2}$$

8 Take the cosine of both sides to make y the subject.

$$\frac{y}{4} = \cos\left(\frac{x}{2} + \frac{\pi}{2}\right)$$
$$y = 4\cos\left(\frac{x}{2} + \frac{\pi}{2}\right)$$

9 Expand using trigonometric compound-angle formulas.

$$y = 4\left(\cos\left(\frac{x}{2}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{x}{2}\right)\sin\left(\frac{\pi}{2}\right)\right)$$
$$= 4\left(\cos\left(\frac{x}{2}\right) \times 0 - \sin\left(\frac{x}{2}\right) \times 1\right)$$

10 State the particular solution.

$$y = -4\sin\left(\frac{x}{2}\right)$$

11 Determine the domain for which the solution is valid.

$$\cos^{-1}\left(\frac{y}{4}\right) = \frac{x + \pi}{2}$$

The range of $y = \cos^{-1}(x)$ is $[0, \pi]$, but $|y| < 4$, so

$$0 < \frac{x + \pi}{2} < \pi$$

12 Solve the inequality for x to state the largest domain for which the solution is valid. State the answer.

$$0 < x + \pi < 2\pi$$
$$-\pi < x < \pi$$

The solution $y = -4\sin\left(\frac{x}{2}\right)$ is valid for $-\pi < x < \pi$.

EXERCISE 9.4 Solving Type 2 differential equations, $\frac{dy}{dx} = f(y)$

PRACTISE

1 WE8 Find the general solution to the differential equation $\sqrt{y}\frac{dy}{dx} + 4 = 0$.

2 Find the general solution to the differential equation $\frac{dy}{dx} - \tan(2y) = 0$.

3 WE9 Solve the differential equation $\frac{dy}{dx} + (5 - 4y)^2 = 0$, $y(1) = 2$.

4 Solve the differential equation $\frac{dy}{dx} + 4y - 7 = 0$, $y(0) = 3$.

5 WE10 Solve the differential equation $\frac{dy}{dx} + 3y = 0$, $y(0) = 5$.

6 Given the differential equation $\frac{dy}{dx} - 5y = 0$, $y(0) = 3$, express y in terms of x.

7 WE11 Solve the differential equation $\sqrt{64 - y^2} - 6\frac{dy}{dx} = 0$, $y(0) = 8$, stating the largest domain for which the solution is valid.

8 Solve the differential equation $16 + y^2 - 4\frac{dy}{dx} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

CONSOLIDATE

9 Find the general solution to each of the following.

a $\frac{dy}{dx} = \frac{y^2}{4}$ b $\frac{dy}{dx} = y + 4$ c $\frac{dy}{dx} = \frac{y}{4}$ d $\frac{dy}{dx} = \frac{4}{y^2}$

10 Solve the following differential equations, expressing y in terms of x.

a $\frac{dy}{dx} + 5y = 0$, $y(0) = 4$ b $\frac{dy}{dx} - 3y = 0$, $y(1) = 2$

11 Solve the following differential equations, expressing y in terms of x.

a $\frac{dy}{dx} + 2y = 5$, $y(0) = 3$ b $\frac{dy}{dx} - 3y + 4 = 0$, $y(0) = 2$

For questions 12–18, solve each of the differential equations given, and where appropriate state the largest domain for which the solution is valid.

12 a $\frac{dy}{dx} = \sqrt{y}$, $y(1) = 4$ b $\frac{dy}{dx} = y^2$, $y(1) = 3$

13 a $\frac{dy}{dx} = 4e^{2y}$, $y(2) = 0$ b $\frac{dy}{dx} + 6e^{3y} = 0$, $y(1) = 0$

14 a $\frac{dy}{dx} = (5 - 2y)^2$, $y(1) = 3$ b $\frac{dy}{dx} + (7 - 3y)^2 = 0$, $y(3) = 2$

15 a $\frac{dy}{dx} + 6\operatorname{cosec}\left(\frac{y}{2}\right) = 0$, $y\left(\frac{1}{3}\right) = 0$ b $\frac{dy}{dx} = 2\sec(2y)$, $y\left(\frac{1}{8}\right) = \frac{\pi}{12}$

16 a $\frac{dy}{dx} - \sqrt{4y + 9} = 0$, $y(0) = 0$ b $\frac{dy}{dx} - 4y^2 = 9$, $y(0) = 0$

17 a $\frac{dy}{dx} + 4y = y^2$, $y(0) = 3$ b $\frac{dy}{dx} - 3y = y^2$, $y(0) = 6$

18 a $\frac{dy}{dx} + 7y = y^2 + 12$, $y(0) = 0$ b $\frac{dy}{dx} - 6y - y^2 = 8$, $y(0) = 0$

MASTER

19 a If k and y_0 are constants, solve the differential equation $\frac{dy}{dx} + ky = 0$, $y(0) = y_0$.

b Given that a, b and c are non-zero real constants, solve the differential equations:

i $\frac{dy}{dx} + ay = b$, $y(0) = c$

ii $\frac{dy}{dx} + ay = by^2$, $y(0) = c$.

20 a Given that a and b are non-zero real constants, solve the differential equations:

i $\frac{dy}{dx} = (ay + b)^2$, $y(0) = 0$

ii $\frac{dy}{dx} = b^2y^2 + a^2$, $y(0) = 0$.

b If a and b are constants with $a > b > 0$:

i solve the differential equation $\frac{dy}{dx} = (y + a)(y + b)$, $y(0) = 0$

ii find $\lim\limits_{x\to\infty} y(x)$.

9.5 Solving Type 3 differential equations, $\frac{dy}{dx} = f(x)g(y)$

Separation of variables

Differential equations of the form $\frac{dy}{dx} = f(x)g(y)$, $y(x_0) = y_0$ are called variables separable equations, as it is possible to separate all the x terms onto one side of the equation and all the y terms onto the other side of the equation.

For $\frac{dy}{dx} = f(x)g(y)$, divide both sides by $g(y)$, since $g(y) \neq 0$. This gives

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

Integrate both sides of the equation with respect to x.

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\text{Thus, } \int \frac{1}{g(y)}dy + c_1 = \int f(x)\,dx + c_2.$$

$$\int \frac{1}{g(y)}dy = \int f(x)\,dx + c, \text{ since } c = c_2 - c_1.$$

After performing the integration, an implicit relationship between x and y is obtained. However, in specific cases it may be possible to rearrange to make y the subject.

WORKED EXAMPLE 12

Find the general solution to the differential equation $\frac{dy}{dx} = \frac{x + 4}{y^2 + 4}$.

THINK	WRITE
1 Write the differential equation.	$\frac{dy}{dx} = \frac{x + 4}{y^2 + 4}$

2 Separate the variables and integrate both sides.	$\int (y^2 + 4)\, dy = \int (x + 4)\, dx$
3 Perform the integration and add the constant on one side only.	$\frac{1}{3}y^3 + 4y = \frac{1}{2}x^2 + 4x + c$
4 The general solution is given as an implicit equation, as in this case it is impossible to solve this equation explicitly for y.	$\frac{1}{3}y^3 + 4y - \frac{1}{2}x^2 - 4x = c$

Finding particular solutions

Finding particular solutions involves solving the differential equation, expressing y in terms of x where possible, and finding the value of the constant of integration.

WORKED EXAMPLE 13 **Solve the differential equation $\frac{dy}{dx} + y = 6x^2y$, $y(0) = 1$.**

THINK	WRITE
1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} + y = 6x^2y$ $\frac{dy}{dx} = 6x^2y - y$
2 Factor the RHS.	$\frac{dy}{dx} = y(6x^2 - 1)$
3 Separate the variables and integrate both sides.	$\int \frac{1}{y}\, dy = \int (6x^2 - 1) dx$
4 Perform the integration and add in the constant on one side only.	$\log_e\left(\lvert y \rvert\right) = 2x^3 - x + c$
5 Substitute and use the given conditions to determine the value of the constant.	$y(0) = 1$ $\Rightarrow x = 0$ when $y = 1$ $\log_e\left(\lvert 1 \rvert\right) = 0 + c$ $c = 0$
6 Substitute back for c and use definition of a logarithm to state the solution explicitly as y in terms of x. Note that the modulus is not needed, as $e^{2x^3 - x} > 0$.	$\log_e\left(\lvert y \rvert\right) = 2x^3 - x$ $y = e^{2x^3 - x}$

Stating the domain for which the solution is valid

As previously stated, when solving differential equations it is necessary to state the largest domain for which the solution is valid.

WORKED EXAMPLE 14 Solve the differential equation $\frac{dy}{dx} + 2x\sqrt{16 - y^2} = 0$, $y(0) = 4$, stating the largest domain for which the solution is valid.

THINK	WRITE
1 Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} + 2x\sqrt{16 - y^2} = 0,\ y(0) = 4$ $\frac{dy}{dx} = -2x\sqrt{16 - y^2}$
2 Separate the variables and integrate both sides.	$\int \frac{-1}{\sqrt{16 - y^2}}\,dy = \int 2x\,dx$
3 Perform the integration and add the constant on one side only.	$\cos^{-1}\left(\frac{y}{4}\right) = x^2 + c$
4 Substitute and use the given conditions to determine the value of the constant.	$y(0) = 4$ $\Rightarrow x = 0$ when $y = 4$ $\cos^{-1}(1) = c$ $c = 0$
5 Substitute back for c.	$\cos^{-1}\left(\frac{y}{4}\right) = x^2$
6 Take the cosine of both sides to make y the subject.	$\frac{y}{4} = \cos(x^2)$ $y = 4\cos(x^2)$
7 Determine the domain for which the solution is valid.	$\cos^{-1}\left(\frac{y}{4}\right) = x^2$ The range of $y = \cos^{-1}(x)$ is $[0, \pi]$, but $x \neq 0$ and $\frac{1}{\sqrt{16 - y^2}}$ is defined for $\lvert y \rvert < 4$, so $0 < x^2 < \pi$
8 Solve the inequality for x to state the largest domain for which the solution is valid. State the answer.	The solution $y = 4\cos(x^2)$ is valid for $0 < x < \sqrt{\pi}$.

EXERCISE 9.5 Solving Type 3 differential equations, $\frac{dy}{dx} = f(x)g(y)$

PRACTISE

1 WE12 Find the general solution to the differential equation $\frac{dy}{dx} = \frac{x + 2}{y^3 + 8}$.

2 Obtain an implicit relationship of the form $f(x, y) = c$ for $\frac{dy}{dx} = \frac{y^2 + 4}{x^2y^2}$.

3 WE13 Solve the differential equation $\frac{dy}{dx} - y = 3x^2y$, $y(0) = 1$.

4 Given the differential equation $\frac{dy}{dx} + y^2 = 2xy^2$, $y(2) = 1$, express y in terms of x.

5 WE14 Solve the differential equation $\frac{dy}{dx} - 2x\sqrt{64 - y^2} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

6 Solve the differential equation $2\frac{dy}{dx} - x(16 + y^2)$, $y(0) = 0$, stating the largest domain for which the solution is valid.

CONSOLIDATE

7 Obtain an implicit relationship of the form $f(x, y) = c$ for each of the following differential equations.

a $\frac{dy}{dx} = \frac{x^2 + 4}{y^2 + 4}$ b $\frac{dy}{dx} = \frac{xy}{y^2 + 4}$ c $\frac{dy}{dx} = \frac{x^2y^2}{y^2 + 4}$ d $\frac{dy}{dx} = \frac{xy^2e^{x^2}}{y^3 + 8}$

For questions 8–16, solve each of the given differential equations and express y in terms of x.

8 a $\frac{dy}{dx} - \frac{y^2}{x} = 0$, $y(1) = 1$ b $\frac{dy}{dx} + 12y^2\sin(4x) = 0$, $y(\pi) = 1$

9 a $\frac{dy}{dx} + \frac{x}{y} = 0$, $y(1) = 2$ b $\frac{dy}{dx} + 6y^2x^2 = 0$, $y(1) = 3$

10 a $\frac{dy}{dx} + 18x^3y^2 = 0$, $y(-1) = 2$ b $\frac{dy}{dx} - \frac{y^2}{x^2} = 0$, $y(1) = 2$

11 a $\frac{dy}{dx} = y^2e^{2x}$, $y(0) = 1$ b $\frac{dy}{dx} + 12x^5y^2 = 0$, $y(1) = 2$

12 a $\frac{dy}{dx} + y = 3x^2y$, $y(0) = 1$ b $\frac{dy}{dx} + 6x^2y^2 = y^2$, $y(-1) = 2$

13 a $\frac{dy}{dx} + 2xy^2 = y^2$, $y(1) = 2$ b $\frac{dy}{dx} + 8x^3y^4 = y^4$, $y(0) = 1$

14 a $x\frac{dy}{dx} + 2y = y^2$, $y(1) = 1$ b $x\frac{dy}{dx} - 4y = y^2$, $y(1) = 1$

15 a $(4 + x^2)\frac{dy}{dx} - 2xy = 0$, $y(0) = 1$ b $\frac{y^2 + 4}{x^2 + 9} - \frac{y}{x}\frac{dy}{dx} = 0$, $y(0) = 2$

16 a $\frac{dy}{dx} - x(25 + y^2) = 0$, $y(0) = 0$ b $\frac{dy}{dx} + 4x\sqrt{25 - y^2} = 0$, $y(0) = 5$

MASTER

17 For each of the following, use the substitution $v = \frac{y}{x}$ to show that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, and hence reduce to a separable differential equation and find the solution.

a $x\frac{dy}{dx} + 3y = 4x$, $y(2) = 1$ b $x\frac{dy}{dx} - y = 4x$, $y(1) = 2$

18 Use the substitution $v = \frac{y}{x}$ to show that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence, reduce the differential equation $x\frac{dy}{dx} + ay = bx$ to a separable differential equation and find its general solution for the cases when:

a $a = -1$ b $a \neq -1$.

9.6 Solving Type 4 differential equations, $\frac{d^2y}{dx^2} = f(x)$

studyon

Units 3 & 4
AOS 3
Topic 3
Concept 3

Second-order differential equations
Concept summary
Practice questions

Integrate twice

In this section, solutions of second-order differential equations of the form $\frac{d^2y}{dx^2} = f(x)$ are required. This type of differential equation can be solved by direct integration, since $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$. Integrating both sides with respect to x gives $\frac{dy}{dx} = \int f(x)\,dx + c_1$. This is now in the Type 1 form and can be solved by direct integration.

Finding a general solution involves giving the solution in terms of two arbitrary constants, which we usually denote as c_1 and c_2.

WORKED EXAMPLE 15 Find the general solution to the differential equation $\frac{d^2y}{dx^2} + 36x^2 = 0$.

THINK	WRITE
1 Rewrite the equation to make $\frac{d^2y}{dx^2}$ the subject.	$\frac{d^2y}{dx^2} + 36x^2 = 0$ $\frac{d^2y}{dx^2} = -36x^2$
2 Integrate both sides with respect to x.	$\frac{dy}{dx} = \int -36x^2 dx$
3 Perform the integration.	$\frac{dy}{dx} = -12x^3 + c_1$
4 Integrate both sides again with respect to x.	$y = \int (-12x^3 + c_1)\,dx$
5 Perform the integration and state the general solution in terms of two arbitrary constants.	$y = -3x^4 + c_1x + c_2$

Finding particular solutions

To solve $\frac{d^2y}{dx^2} = f(x)$ and obtain a particular solution, we need two sets of initial conditions to find the two constants of integration. These are usually of the form $y(x_0) = y_0$ and $y'(x_1) = y_1$.

WORKED EXAMPLE 16 Solve the differential equation $\frac{d^2y}{dx^2} + 36x = 0$, $y(1) = 3$, $y'(1) = 2$.

THINK	WRITE
1 Rewrite the equation to make $\frac{d^2y}{dx^2}$ the subject.	$\frac{d^2y}{dx^2} + 36x = 0$ $\frac{d^2y}{dx^2} = -36x$

2 Integrate both sides with respect to x.

$$\frac{dy}{dx} = \int -36x\,dx$$
$$= -18x^2 + c_1$$

3 Substitute and use the given condition to determine the value of the first constant of integration.

$y'(1) = 2$

$\Rightarrow \frac{dy}{dx} = 2$ when $x = 1$

$2 = -18 + c_1$

$c_1 = 20$

4 Substitute back for c_1.

$$\frac{dy}{dx} = -18x^2 + 20$$

5 Integrate both sides again with respect to x.

$$y = \int (-18x^2 + 20)\,dx$$

6 Perform the integration.

$$y = -6x^3 + 20x + c_2$$

7 Substitute and use the given condition to determine the value of the second constant of integration.

$y(1) = 3$

$\Rightarrow y = 3$ when $x = 1$

$3 = -6 + 20 + c_2$

$c_2 = -11$

8 Substitute back for c_2 and state the particular solution.

$$y = -6x^3 + 20x - 11$$

Simplifying the answer

We have seen earlier that answers can often be given in a simplified form.

WORKED EXAMPLE 17

Solve the differential equation $\frac{d^2y}{dx^2} + \frac{2}{(2x+9)^3} = 0,\ y(0) = 0,\ y'(0) = 0.$

THINK

WRITE

1 Rewrite the equation to make $\frac{d^2y}{dx^2}$ the subject.

$$\frac{d^2y}{dx^2} + \frac{2}{(2x+9)^3} = 0$$
$$\frac{d^2y}{dx^2} = \frac{-2}{(2x+9)^3}$$

2 Integrate both sides with respect to x.

$$\frac{dy}{dx} = \int \frac{-2}{(2x+9)^3}\,dx$$

3 Transfer the constant factor outside the front of the integral and use index laws to express the integrand as a function to a power.

$$\frac{dy}{dx} = -2\int (2x+9)^{-3}\,dx$$

4 Use a linear substitution. Express dx in terms of du by inverting both sides.

Let $u = 2x + 9$.

$$\frac{du}{dx} = 2$$
$$\frac{dx}{du} = \frac{1}{2}$$
$$dx = \frac{1}{2}\,du$$

5 Substitute for u and dx, and simplify.

$$\frac{dy}{dx} = -2\int u^{-3}\frac{1}{2}du$$
$$= -\int u^{-3}du$$

6 Perform the integration, adding in the first constant of integration and substitute back for x.

$$\frac{dy}{dx} = \frac{1}{2}u^{-2} + c_1$$
$$= \frac{1}{2(2x+9)^2} + c_1$$

7 Use the given condition to find the value of the first constant of integration.

$$y'(0) = 0$$
$$\Rightarrow \text{when } x = 0, \frac{dy}{dx} = 0$$
$$0 = \frac{1}{162} + c_1$$
$$c_1 = -\frac{1}{162}$$

8 Substitute back for c_1.

$$\frac{dy}{dx} = \frac{1}{2(2x+9)^2} - \frac{1}{162}$$

9 Integrate both sides again with respect to x.

$$y = \int\left(\frac{1}{2(2x+9)^2} - \frac{1}{162}\right)dx$$

10 Simplify the integrand.

$$y = \int\left(\frac{1}{2(2x+9)^2}\right)dx - \frac{x}{162}$$

11 Use the substitution $u = 2x + 9$ again.

$$y = \int\left(\frac{1}{2}u^{-2}\right)\frac{1}{2}du - \frac{x}{162}$$
$$= \frac{1}{4}\int u^{-2}du - \frac{x}{162}$$

12 Perform the integration and add in the second constant of integration.

$$y = -\frac{1}{4}u^{-1} - \frac{x}{162} + c_2$$

13 Substitute back for x.

$$y = -\frac{1}{4(2x+9)} - \frac{x}{162} + c_2$$

14 Substitute and use the given condition to determine the value of the second constant of integration.

$$y(0) = 0$$
$$\Rightarrow y = 0 \text{ when } x = 0$$
$$0 = -\frac{1}{36} + c_2$$
$$c_2 = \frac{1}{36}$$

15 Substitute back for c_2 and state the particular solution. Although this is a possible answer, this result can be simplified.

$$y = -\frac{1}{4(2x+9)} - \frac{x}{162} + \frac{1}{36}$$

16 Form the lowest common denominator.

$$y = \frac{-81 - 2x(2x+9) + 9(2x+9)}{324(2x+9)}$$

17 Expand and simplify the numerator.

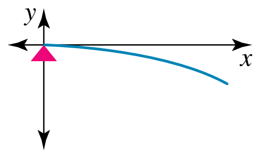
$$y = \frac{-81 - 4x^2 - 18x + 18x + 81}{324(2x + 9)}$$

$$y = \frac{-4x^2}{324(2x + 9)}$$

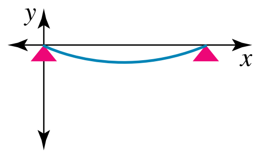
18 State the particular solution in simplest form.

$$y = \frac{-x^2}{81(2x + 9)}$$

Beam deflections



One application of the Type 4 differential equations, $\frac{d^2y}{dx^2} = f(x)$, is beam deflections. A cantilever or a beam can be fixed at one end and have a weight at the other end. The weight at the unfixed end causes the beam to bend so that the downwards deflection, y, at a distance, x, measured along the beam from the fixed point satisfies a differential equation of this type. In this situation the maximum deflection occurs at the end of the beam.



Another type of beam deflection is the case of a beam fixed at both ends. The weight of the beam causes the beam to bend so that the downwards deflection, y, at a distance, x, measured along the beam from the fixed point satisfies a differential equation of this type. In this situation we can show that the maximum deflection occurs in the middle of the beam.

WORKED EXAMPLE 18

A beam of length $2L$ rests with its end on two supports at the same horizontal level. The downward deflection, y, from the horizontal satisfies the differential equation $\frac{d^2y}{dx^2} = kx(x - 2L)$ for $0 \le x \le 2L$, where x is the horizontal distance from one end of the beam and k is a constant related to the stiffness and bending moment of the beam.

a **Find the deflection, y, in terms of x and show that the maximum deflection occurs in the middle of the beam.**

b **Find the maximum deflection of the beam.**

THINK

a 1 Expand.

2 Integrate both sides with respect to x.

3 Perform the integration.

WRITE

a $$\frac{d^2y}{dx^2} = kx(x - 2L)$$
$$= k(x^2 - 2Lx)$$

$$\frac{dy}{dx} = k\int (x^2 - 2Lx)\,dx$$

$$\frac{dy}{dx} = k\left(\frac{x^3}{3} - Lx^2 + c_1\right)$$

4 Since the beam is fixed at both ends, $x = 0$ when $y = 0$, and $y = 0$ when $x = 2L$. We cannot determine the first constant of integration at this stage. Integrate both sides with respect to x again.

$$y = k\int\left(\frac{x^3}{3} - Lx^2 + c_1\right)dx$$

5 Perform the integration.

$$y = k\left(\frac{x^4}{12} - \frac{Lx^3}{3} + c_1x + c_2\right)$$

6 To find the second constant of integration, c_2, use $x = 0$ when $y = 0$.

Substitute $x = 0$ when $y = 0$:

$$c_2 = 0$$

7 To find the first constant of integration, c_1, use $y = 0$ when $x = 2L$ and simplify.

Substitute $y = 0$ when $x = 2L$:

$$0 = k\left(\frac{(2L)^4}{12} - \frac{L(2L)^3}{3} + 2Lc_1\right)$$

$$0 = k\left(\frac{16L^4}{12} - \frac{8L^4}{3} + 2Lc_1\right)$$

8 Solve for the first constant and substitute back. Simplify the result by taking a common denominator. This gives the deflection, y, in terms of x.

$$c_1 = \frac{2L^3}{3}$$

$$y = k\left(\frac{x^4}{12} - \frac{Lx^3}{3} + \frac{2L^3x}{3}\right)$$

$$= \frac{k}{12}(x^4 - 4Lx^3 + 8L^2x)$$

9 Find the first derivative.

$$\frac{dy}{dx} = \frac{k}{12}(4x^3 - 12Lx^2 + 8L^2)$$

$$= \frac{k}{3}(x^3 - 3Lx^2 + 2L^3)$$

10 To show that the maximum deflection occurs in the middle of the beam, show that $\frac{dy}{dx} = 0$ when $x = L$.

Substitute $x = L$:

$$\frac{dy}{dx} = \frac{k}{3}(L^3 - 3L^3 + 2L^3)$$

$$= 0$$

So the maximum deflection occurs in the middle of the beam.

b 1 To find the maximum deflection, substitute $x = L$ into the result for y.

b $y_{max} = y(L)$

$$= \frac{k}{12}(L^3 - 4L^3 + 8L^3)$$

$$= \frac{5L^3k}{12}$$

2 State the maximum deflection of the beam.

The maximum deflection occurs in the middle of the beam and is $\frac{5L^3k}{12}$.

EXERCISE 9.6 Solving Type 4 differential equations, $\frac{d^2y}{dx^2} = f(x)$

PRACTISE

1 WE15 Find the general solution to the differential equation $\frac{d^2y}{dx^2} + 30x^4 = 0$.

2 Find the general solution to the differential equation $\frac{d^2y}{dx^2} + 36\sin(3x) = 0$.

3 WE16 Solve the differential equation $\frac{d^2y}{dx^2} + 24x^2 = 0$, $y(-1) = 2$, $y'(-1) = 3$.

4 Solve the differential equation $\frac{d^2y}{dx^2} + 12\sin(2x) = 0$, $y\left(\frac{\pi}{4}\right) = 4$, $y'\left(\frac{\pi}{4}\right) = 6$.

5 WE17 Solve the differential equation $\frac{d^2y}{dx^2} + \frac{12}{(3x+16)^3} = 0$, $y(0) = 0$, $y'(0) = 0$.

6 Solve the differential equation $\frac{d^2y}{dx^2} + \frac{12}{\sqrt{(2x+9)^3}} = 0$, $y(0) = 0$, $y'(0) = 1$.

7 WE18 A beam of length L has both ends simply supported at the same horizontal level and the downward deflection, y, satisfies the differential equation $\frac{d^2y}{dx^2} = k(x^2 - Lx)$ for $0 \le x \le L$ where k is a constant.

a Find the deflection, y, in terms of x and show that the maximum deflection occurs in the middle of the beam.

b Find the maximum deflection of the beam.

8 A cantilever of length L is rigidly fixed at one end and is horizontal in the unstrained position. If a load is added at the free end of the beam, the downward deflection, y, at a distance, x, along the beam satisfies the differential equation $\frac{d^2y}{dx^2} = k(L - x)$ for $0 \le x \le L$ where k is a constant. Find the deflection, y, in terms of x and hence find the maximum deflection of the beam.

CONSOLIDATE

9 Find the general solution to each of the following.

a $x^3\frac{d^2y}{dx^2} + 4 = 0$

b $\frac{d^2y}{dx^2} + (x+4)(2x-5) = 0$

c $x^3\frac{d^2y}{dx^2} + 2x - 5 = 0$

d $e^{3x}\frac{d^2y}{dx^2} + 5 = 2e^{2x}$

For questions 10–14, solve each of the given differential equations.

10 a $\frac{d^2y}{dx^2} + 6x = 0$, $y(1) = 2$, $y(2) = 3$

b $\frac{d^2y}{dx^2} + 24x^2 = 0$, $y(1) = 2$, $y(2) = 3$

11 a $\frac{d^2y}{dx^2} + 8(e^{2x} + e^{-2x}) = 0$, $x = 0$, $\frac{dy}{dx} = 0$, $y = 0$

b $e^x\frac{d^2y}{dx^2} + 4e^{-2x} = 5$, $x = 0$, $\frac{dy}{dx} = 0$, $y = 0$

12 a $\dfrac{d^2y}{dx^2} + 64\sin(4x) = 0,\ y(0) = 4,\ y'(0) = 8$

b $\dfrac{d^2y}{dx^2} + 27\cos(3x) = 0,\ y\left(\dfrac{\pi}{6}\right) = 3,\ y'\left(\dfrac{\pi}{6}\right) = 9$

13 a $\dfrac{d^2y}{dx^2} + 32\sin^2(2x) = 0,\ y(0) = 0,\ y'(0) = 0$

b $\dfrac{d^2y}{dx^2} + 16\cos^2(4x) = 0,\ y(0) = 0,\ y'(0) = 0$

14 a $\dfrac{d^2y}{dx^2} = \dfrac{1}{(3x+2)^3},\ y(0) = 0,\ y'(0) = 0$

b $\dfrac{d^2y}{dx^2} + \dfrac{1}{\sqrt{(2x+9)^3}} = 0,\ y(0) = 0,\ y'(0) = 0$

15 a At all points on a certain curve, the rate of change of gradient is constant. Show that the family of curves with this property are parabolas.

b At all points on a certain curve, the rate of change of the gradient is −12. If the curve has a turning point at (−2, 4), find the equation of the particular curve.

16 a At all points on a certain curve, the rate of change of the gradient is proportional to the x-coordinate. Show that this family of curves are cubics.

b At all points on a certain curve, the rate of change of the gradient is $18x$. If the curve has a turning point at (–2, 0), find the equation of the particular curve.

17 a Solve $\dfrac{d^2y}{dx^2} + \dfrac{20}{\sqrt{4x+9}} = 0,\ y(0) = 0$ and $y'(0) = 0$.

b Solve $\dfrac{d^2y}{dx^2} + \dfrac{16}{(4x+9)^2} = 0,\ y(0) = 0$ and $y'(0) = 0$.

18 a A diving board of length L is rigidly fixed at one end and has a girl of weight W standing at the free end. The downward deflection, y, measured at a distance, x, along the beam satisfies the differential equation

$$EI\frac{d^2y}{dx^2} = \frac{W}{2}(L-x)^2 \text{ for } 0 \le x \le L.$$

The deflection and inclination to the horizontal are both zero at the fixed end, and the product EI is a constant related to the stiffness of the beam. Find the formula for y in terms of x and determine the maximum deflection of the beam.

b A uniform beam of length L carries a load of W per unit length and has both ends clamped horizontally at the same horizontal level. The downward deflection, y, measured at any distance, x, from one end along the beam satisfies the differential equation

$$EI\frac{d^2y}{dx^2} = \frac{W}{2}\left(x^2 - Lx + \frac{L^2}{6}\right) \text{ for } 0 \le x \le L$$

where W, E and I are constants. Prove that the maximum deflection occurs in the middle of the beam, and determine the maximum deflection of the beam.

MASTER

19 If a and b are positive real constants, find the particular solution to each of the following differential equations.

a $\frac{d^2y}{dx^2} + \frac{1}{(ax+b)^3} = 0$, $y(0) = 0$ and $y'(0) = 0$

b $\frac{d^2y}{dx^2} + \frac{1}{(ax+b)^2} = 0$, $y(0) = 0$ and $y'(0) = 0$

20 a i Show that $\frac{d}{dx}\left[\frac{x}{\sqrt{9+4x^2}}\right] = \frac{9}{\sqrt{(9+4x^2)^3}}$.

ii Hence, find the general solution to $\frac{d^2y}{dx^2} + \frac{9}{\sqrt{(9+4x^2)^3}} = 0$.

b i If a and b are positive real constants, show that

$$\frac{d}{dx}\left[\frac{x}{\sqrt{a+bx^2}}\right] = \frac{a}{\sqrt{(a+bx^2)^3}}.$$

ii Hence, find the general solution to $\frac{d^2y}{dx^2} + \frac{1}{\sqrt{(a+bx^2)^3}} = 0$.

eBook*plus* ONLINE ONLY

9.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

9 Answers

EXERCISE 9.2

1 Check with your teacher.

2 Check with your teacher.

3 $-5, 2$

4 ± 3

5 Check with your teacher.

6 -2

7 Check with your teacher.

8 a $a = 0, b = -1, c = 1$

b $a = 1, b = -6, c = 18, d = -24$

c Check with your teacher.

d $-2, 5$

9 a, b Check with your teacher.

c $-6, 1$

d $-2 \pm 3i$

10 a–c Check with your teacher.

d ± 3

11–14 Check with your teacher.

15 a Check with your teacher.

b $a = 0, b = 1$

16 a $a = 6, b = 9$

b Check with your teacher.

17 a 5

b, c Check with your teacher.

18 Check with your teacher.

EXERCISE 9.3

1 a $y = c - 3x^4$ b $y = 13 - 3x^2$

2 a $y = c - 6\sin(2x)$ b $y = 2(\cos(3x) - 1)$

3 $y = \dfrac{9x - 11}{4x - 5}, x \neq \dfrac{5}{4}$

4 $y = 3 + \frac{1}{2}\log_e\left(|4x - 7|\right), x \neq \frac{7}{4}$

5 $y = 1 - \sqrt{2x - 5}, x > \frac{5}{2}$

6 $y = 11 - 4\sqrt{x}, x > 0$

7 $y = 6\sin^{-1}\left(\dfrac{x}{8}\right) - \pi, |x| < 8$

8 $y = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{x}{4}\right)$

9 a $y = 2x^2 + 3x + c$ b $y = x^3 + \frac{7}{2}x^2 - 20x + c$

c $y = e^{2x} + 3e^{-2x} + c$ d $y = \sqrt{x^2 + 9} + c$

10 a $y = \frac{1}{3}\left[5\log_e\left(|x|\right) + x^2 + 8\right], x \neq 0$

b $y = 2(e^{3x} - e^{-3x})$

11 a $y = 4 - 2\cos(2x)$ b $y = 3 - 2\sin(3x)$

12 a $y = 4x - \sin(4x)$ b $y = 6x + \sin(6x)$

13 a $y = \frac{1}{2}(\sqrt{4x + 9} - 3), x > -\frac{9}{4}$

b $y = 1 + \frac{1}{2}\log_e\left(|2x - 3|\right), x \neq \frac{3}{2}$

14 a $y = \dfrac{10x - 17}{3x - 5}, x \neq \dfrac{5}{3}$

b $y = 5 - 2\log_e\left(|7 - 4x|\right), x \neq \frac{7}{4}$

15 a $y = \frac{3}{2}\log_e\left(\dfrac{x^2 + 9}{9}\right)$

b $y = 2 - \sqrt{x^2 + 4}$

16 a $y = \frac{1}{2}\log_e\left(\dfrac{x^2 + 6x + 13}{13}\right)$

b $y = \log_e\left(\dfrac{3}{\sqrt{x^2 - 4x + 9}}\right)$

17 a $y = -\frac{1}{8}\sin^4(2x)$

b $y = \cos^3(3x) - 1$

18 a $y = 5x - x\log_e\left(2|x|\right) - \frac{3}{2}, x \neq 0$

b $y = (x - 4)e^{-x} + 4$

19 a $y = \frac{1}{2}\tan^{-1}\left(\dfrac{2x}{3}\right) + \dfrac{1}{4}\log_e\left(\dfrac{9}{4x^2 + 9}\right)$

b $y = \frac{3}{2}\left(\sin^{-1}\left(\dfrac{2x}{3}\right) - 1\right) + \dfrac{\sqrt{9 - 4x^2}}{2}, |x| < \frac{2}{3}$

20 a i $y = -b\sin^{-1}\left(\dfrac{x}{a}\right), |x| < a$

ii $y = \dfrac{b}{2a}\log_e\left(\left|\dfrac{a - x}{a + x}\right|\right), |x| < a$

iii $y = \dfrac{-x}{a(a + bx)}, x \neq -\dfrac{a}{b}$

b $y = \dfrac{e^{-2x}}{13}(2\cos(3x) - 3\sin(3x) - 2)$

EXERCISE 9.4

1 $y = \sqrt[3]{(B - 6x)^2}$

2 $y = \frac{1}{2}\sin^{-1}(Be^{2x})$

3 $y = \dfrac{15x - 13}{12x - 11}, x \neq \frac{11}{12}$

4 $y = \frac{1}{4}(7 + 5e^{-4x})$

5 $y = 5e^{-3x}$

6 $y = 3e^{5x}$

7 $y = 8\cos\left(\dfrac{x}{6}\right), -6\pi < x < 0$

8 $y = 4\tan(x), -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

9 a $y = \frac{4}{c-x}$

b $y = Ae^x - 4$

c $y = Ae^{\frac{x}{4}}$

d $y = \sqrt[3]{12x + A}$

10 a $y = 4e^{-5x}$

b $y = 2e^{3x-3}$

11 a $y = \frac{1}{2}(5 + e^{-2x})$

b $y = \frac{2}{3}(2 + e^{3x})$

12 a $y = \frac{1}{4}(x + 3)^2$

b $y = \frac{3}{4 - 3x}, x \neq \frac{4}{3}$

13 a $y = -\frac{1}{2}\log_e(17 - 8x), x < \frac{17}{8}$

b $y = -\frac{1}{3}\log_e(18x - 17), x > \frac{17}{18}$

14 a $y = \frac{5x-8}{2x-3}, x \neq \frac{3}{2}$

b $y = \frac{7x - 23}{3x - 10}, x \neq \frac{10}{3}$

15 a $y = 2\cos^{-1}(3x), |x| \leq \frac{1}{3}$

b $y = \frac{1}{2}\sin^{-1}(4x), |x| \leq \frac{1}{4}$

16 a $y = x^2 + 3x$

b $y = \frac{3}{2}\tan(6x), -\frac{\pi}{12} < x < \frac{\pi}{12}$

17 a $y = \frac{12}{3 + e^{4x}}$

b $y = \frac{6}{3e^{-3x} - 2}$

18 a $y = \frac{12(e^x + 1)}{4e^x + 3}$

b $y = \frac{4(1 - e^{2x})}{e^{2x} - 2}, x \neq \log_e(\sqrt{2})$

19 a $y = y_0 e^{-kx}$

b i $y = \left(c - \frac{b}{a}\right)e^{-ax} + \frac{b}{a}$

ii $y = \frac{ac}{(a - bc)e^{ax} + bc}$

20 a i $y = \frac{b^2 x}{1 - abx}, x \neq \frac{1}{ab}$

ii $y = \frac{a}{b}\tan(abx)$

b i $y = \frac{ab(1 - e^{-(a-b)x})}{ae^{-(a-b)x} - b}$

ii $-a$

EXERCISE 9.5

1 $\frac{y^4}{4} + 8y - \frac{x^2}{2} - 2x = c$

2 $y - 2\tan^{-1}\left(\frac{y}{2}\right) + \frac{1}{x} = c$

3 $y = e^{x^3 + x}$

4 $y = \frac{1}{3 + x - x^2}, x \neq \frac{1 \pm \sqrt{13}}{2}$

5 $y = 8\sin(x^2), 0 < x < \frac{\sqrt{2\pi}}{2}$

6 $y = 4\tan(x^2), 0 < x < \frac{\sqrt{2\pi}}{2}$

7 a $\frac{1}{3}y^3 + 4y - \frac{1}{3}x^3 - 4x = c$

b $\frac{1}{2}y^2 + 4\log_e(|y|) - \frac{1}{2}x^2 = c$

c $y - \frac{4}{y} - \frac{1}{3}x^3 = c$

d $\frac{1}{2}y^2 - \frac{8}{y} - \frac{1}{2}e^{x^2} = c$

8 a $y = \frac{1}{1 - \log_e(|x|)}, x \neq 0$

b $y = \frac{1}{4 - 3\cos(4x)}$

9 a $y = \sqrt{5 - x^2}, |x| \leq \sqrt{5}$

b $y = \frac{3}{6x^3 - 5}$

10 a $y = \frac{2}{9x^4 - 8}$

b $y = \frac{2x}{2 - x}$

11 a $y = \frac{2}{3 - e^{2x}}, x \neq \log_e(\sqrt{3})$

b $y = \frac{2}{4x^6 - 3}$

12 a $y = e^{x^3 - x}$

b $y = \frac{2}{4x^3 - 2x + 3}$

13 a $y = \frac{2}{2x^2 - 2x + 1}$

b $y = \frac{1}{\sqrt[3]{6x^4 - 3x + 1}}$

14 a $y = \frac{2}{1 - x^2}, x \neq \pm 1$

b $y = \frac{4x^4}{5 - x^4}, x \neq \pm\sqrt[4]{5}$

15 a $y = \frac{1}{4}(x^2 + 4)$

b $y = \frac{2}{3}\sqrt{2x^2 + 9}$

16 a $y = 5\tan\left(\frac{5x^2}{2}\right), |x| \leq \frac{\sqrt{5\pi}}{5}$

b $y = 5\cos(2x^2), |x| \leq \frac{\sqrt{2\pi}}{2}$

17 a $y = x - \frac{8}{x^3}, x \neq 0$

b $y = 2x\left(1 + 2\log_e\left(|x|\right)\right), x \neq 0$

18 a $y = x\left(c + b\log_e\left(|x|\right)\right), x \neq 0$

b $y = \frac{bx}{a+1} + \frac{c}{x^a}$

EXERCISE 9.6

1 $y = c_2 + c_1 x - x^6$

2 $y = c_2 + c_1 x + 4\sin(3x)$

3 $y = -2x^4 - 5x - 1$

4 $y = 3\sin(2x) + 6x + 1 - \frac{3\pi}{2}$

5 $y = \frac{-3x^2}{128(3x + 16)}, x \neq -\frac{16}{3}$

6 $y = 12\sqrt{2x + 9} - 3x - 36, x > -\frac{9}{2}$

7 $y = \frac{k}{12}(x^4 - 2Lx^3 + L^3 x), \frac{5kL^4}{192}$

8 $y = \frac{k}{6}(3Lx^2 - x^3), \frac{kL^3}{3}$

9 a $y = c_2 + c_1 x - \frac{2}{x}, x \neq 0$

b $y = c_2 + c_1 x + 10x^2 - \frac{x^3}{2} - \frac{x^4}{6}$

c $y = c_2 + c_1 x + 2\log_e\left(|x|\right) + \frac{5}{2x}, x \neq 0$

d $y = c_2 + c_1 x + 2e^{-x} - \frac{5}{9}e^{-3x}$

10 a $y = -x^3 + 8x - 5$

b $y = -2x^4 + 31x - 27$

11 a $y = 4 - 2e^{2x} - 2e^{-2x}$

b $y = 5e^{-x} - \frac{4}{9}e^{-3x} + \frac{11x}{3} - \frac{41}{9}$

12 a $y = 4\sin(4x) - 8x + 4$

b $y = 3\cos(3x) + 18x - 3\pi + 3$

13 a $y = 1 - \cos(4x) - 8x^2$

b $y = \frac{1}{8}\cos(8x) - 4x^2 - \frac{1}{8}$

14 a $y = \frac{x^2}{8(3x+2)}, x \neq -\frac{2}{3}$

b $y = \sqrt{2x+9} - \frac{x}{3} - 3, x > -\frac{9}{2}$

15 a Check with your teacher.

b $y = -6x^2 - 24x - 20$

16 a Check with your teacher.

b $y = 3x^3 - 36x - 48$

17 a $y = 30x + 45 - \frac{5}{3}\sqrt{(4x+9)^3}, x > -\frac{9}{4}$

b $y = \log_e\left(\frac{|4x+9|}{9}\right) - \frac{4x}{9}, x \neq -\frac{9}{4}$

18 a $y = \frac{W}{24EI}(6L^2x^2 - 4Lx^3 + x^4), \frac{WL^4}{8EI}$

b $y = \frac{Wx^2}{24EI}(x-L)^2, \frac{WL^4}{384EI}$

19 a $y = \frac{-x^2}{2b^2(ax+b)}, x \neq -\frac{b}{a}$

b $y = \frac{1}{a^2}\log_e\left(\frac{|ax+b|}{b}\right) - \frac{x}{ab}, x \neq -\frac{b}{a}$

20 a i Check with your teacher.

ii $y = c_2 + c_1 x - \frac{1}{4}\sqrt{9+4x^2}$

b i Check with your teacher.

ii $y = c_2 + c_1 x - \frac{1}{ab}\sqrt{a+bx^2}$

10

Further applications of integration

10.1 Kick off with CAS

Using CAS to solve area under curve problems

1 The area bounded by the curve $y = ax^2 + b - x^2$, the x-axis, $x = 1$ and $x = 2$ is $2\frac{1}{6}$ square units.
The area bounded by the curve $y = ax^2 + b - x^2$, the x-axis, $x = 3$ and $x = 5$ is $11\frac{1}{3}$ square units.
Find the values of a and b.

2 The curve $y = x^3 + bx^2 + cx + d$ has a turning point at (2, 34). The area bounded by the curve, the coordinate axes and the line $x = 6$ is 120 square units.
Find the values of b, c and d.

3 The curve $y = x^4 + ax^3 + bx^2 + cx + d$ has a turning point at (2, 1) and crosses the y-axis at $y = 17$. The area bounded by the curve, the coordinate axes and the line $x = 2$ is $8\frac{2}{5}$ square units.
Find the values of a, b, c and d.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

10.2 Integration by recognition

Deducing an antiderivative

Because differentiation and integration are inverse processes, differentiating a function $f(x)$ with respect to x will result in another function $g(x)$. It follows that an antiderivative of the function $g(x)$ with respect to x is the function $f(x)$ with the addition of an arbitrary constant c. In mathematical notation, if $\frac{d}{dx}(f(x)) = g(x)$, then $\int g(x)\,dx = f(x) + c$. However, functions may not be exactly the derivatives of other functions, but may differ by a constant multiple.

Let k be a non-zero constant. If $\frac{d}{dx}(f(x)) = kg(x)$, it follows that $\int kg(x)\,dx = k\int g(x)\,dx = f(x)$, since constant multiples can be taken outside integral signs. Then, dividing both sides by the constant multiple k, we obtain $\int g(x)\,dx = \frac{1}{k}f(x) + c$, as we can add in the arbitrary constant at the end. This is known as integration by recognition.

WORKED EXAMPLE 1 Differentiate $\sqrt{25x^2 + 16}$ and hence find $\int \frac{x}{\sqrt{25x^2+16}}\,dx$.

THINK	WRITE
1 Write the equation in index form.	Let $y = \sqrt{25x^2 + 16}$. $y = (25x^2 + 16)^{\frac{1}{2}}$
2 Express y in terms of u, and u in terms of x.	$y = u^{\frac{1}{2}}$ where $u = 25x^2 + 16$
3 Differentiate y with respect to u, and u with respect to x.	$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ and $\frac{du}{dx} = 50x$ $= \frac{1}{2\sqrt{u}}$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ $= \frac{1}{2\sqrt{u}} \times 50x$
5 Substitute back for u and cancel factors.	$\frac{dy}{dx} = \frac{25x}{\sqrt{25x^2+16}}$
6 Write the result as a derivative of one function.	$\frac{d}{dx}\left[\sqrt{25x^2+16}\right] = \frac{25x}{\sqrt{25x^2+16}}$
7 Use integration by recognition.	$\int \frac{25x}{\sqrt{25x^2+16}}\,dx = \sqrt{25x^2+16}$

8 Take the constant factor outside the integral sign.	$25\int \frac{x}{\sqrt{25x^2+16}}\,dx = \sqrt{25x^2+16}$
9 Divide by the constant factor, add in the arbitrary constant $+c$, and state the final result.	$\int \frac{x}{\sqrt{25x^2+16}}\,dx = \frac{1}{25}\sqrt{25x^2+16}+c$

Integrating other types of functions

For the worked example above, integration techniques developed in earlier topics could have been used to find the antiderivative. However, there are functions for which an antiderivative could not have been obtained using the rules developed so far. Also, this method of deducing an antiderivative can be used to evaluate definite integrals or find areas.

To understand the examples that follow, it is necessary to review the differentiation techniques developed in earlier topics. Recall the results for the derivatives of the inverse trigonometric functions. If a is a positive constant, then:

- $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2-x^2}}$ **for** $|x| < a$
- $\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2-x^2}}$ **for** $|x| < a$
- $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2}$ **for** $x \in R$.

WORKED EXAMPLE 2

Differentiate $\cos^{-1}\left(\frac{\sqrt{x}}{4}\right)$ and hence evaluate $\int_0^8 \frac{1}{\sqrt{16x-x^2}}\,dx$.

THINK	WRITE
1 Express y in terms of u, and u in terms of x.	Let $y = \cos^{-1}\left(\frac{\sqrt{x}}{4}\right)$ $= \cos^{-1}\left(\frac{u}{4}\right)$ where $u = \sqrt{x} = x^{\frac{1}{2}}$
2 Differentiate y with respect to u, and u with respect to x.	$\frac{dy}{du} = \frac{-1}{\sqrt{16-u^2}}$ and $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
3 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{x}} \times \frac{-1}{\sqrt{16-u^2}}$
4 Substitute back for u.	$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}\sqrt{16-x}}$ since $0 < x < 16$ $= \frac{-1}{2\sqrt{16x-x^2}}$

THINK	WRITE
5 Write the result as a derivative of one function.	$\frac{d}{dx}\left[\cos^{-1}\left(\frac{\sqrt{x}}{4}\right)\right] = \frac{-1}{2\sqrt{16x - x^2}}$
6 Use integration by recognition.	$\int \frac{-1}{2\sqrt{16x - x^2}}\,dx = \cos^{-1}\left(\frac{\sqrt{x}}{2}\right)$
7 Take the constant factor outside the integral sign.	$-\frac{1}{2}\int \frac{1}{\sqrt{16x - x^2}}\,dx = \cos^{-1}\left(\frac{\sqrt{x}}{4}\right)$
8 Multiply by the constant factor and add in the arbitrary constant $+c$.	$\int \frac{1}{\sqrt{16x - x^2}}\,dx = -2\cos^{-1}\left(\frac{\sqrt{x}}{4}\right) + c$
9 However, in this case we are required to evaluate a definite integral.	$\int_0^8 \frac{1}{\sqrt{16x - x^2}}\,dx = \left[-2\cos^{-1}\left(\frac{\sqrt{x}}{4}\right)\right]_0^8$
10 Substitute in the upper and lower terminals and simplify.	$\left(-2\cos^{-1}\left(\frac{\sqrt{8}}{4}\right)\right) - (-2\cos^{-1}(0))$ $= \left(-2\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) - (-2\cos^{-1}(0))$
11 Evaluate.	$= -2 \times \frac{\pi}{4} + 2 \times \frac{\pi}{2}$
12 State the final value of the definite integral.	$\int_0^8 \frac{1}{\sqrt{16x - x^2}}\,dx = \frac{\pi}{2}$

Using the product rule

Some problems require a combination of integration by recognition and the product rule. This is typical when integrating products of mixed types of functions. Recall that the product rule states that if $u = u(x)$ and $v = v(x)$ are two differentiable functions of x and $y = u \cdot v$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

WORKED EXAMPLE 3 Differentiate $x\cos(4x)$ and hence find $\int x\sin(4x)\,dx$.

THINK	WRITE
1 Write the equation.	Let $y = x\cos(4x) = u \cdot v$
2 State the functions u and v.	$u = x$ and $v = \cos(4x)$
3 Differentiate u and v with respect to x.	$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -4\sin(4x)$

4 Find $\frac{dy}{dx}$ using the product rule.

Substitute for u, $\frac{dv}{dx}$, v and $\frac{du}{dx}$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = -4x\sin(4x) + \cos(4x)$$

5 Write the result as a derivative of one function.

$$\frac{d}{dx}[x\cos(4x)] = -4x\sin(4x) + \cos(4x)$$

6 Use integration by recognition.

$$\int(-4x\sin(4x) + \cos(4x))\,dx = x\cos(4x)$$

7 Separate the integral on the left into two separate integrals and take the constant factor outside the integral sign.

$$-4\int x\sin(4x)\,dx + \int\cos(4x)\,dx = x\cos(4x)$$

8 Transfer one part of the integral which can be performed on the right-hand side.

$$-4\int x\sin(4x)dx = x\cos(4x) - \int\cos(4x)\,dx$$

9 Perform the integration on the term on the right-hand side.

$$-4\int x\sin(4x)\,dx = x\cos(4x) - \frac{1}{4}\sin(4x)$$

10 Divide by the constant factor, add in the arbitrary constant $+c$, and state the final result.

$$\int x\sin(4x)\,dx = -\frac{1}{4}x\cos(4x) + \frac{1}{16}\sin(4x) + c$$

Integrating inverse trigonometric functions

The technique used in the last worked example can also be used to find antiderivatives of inverse trigonometric functions.

WORKED EXAMPLE 4

Find $\frac{d}{dx}[x\cos^{-1}(2x)]$ and hence find $\int\cos^{-1}(2x)\,dx$.

THINK

1 Write the equation.

2 State the functions u and v.

3 Differentiate u and v with respect to x.

4 Find $\frac{dy}{dx}$ using the product rule. Substitute for u, $\frac{dv}{dx}$, v and $\frac{du}{dx}$.

WRITE

Let $y = x\cos^{-1}(2x) = u \cdot v$.

$u = x$ and $v = \cos^{-1}(2x)$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{-2}{\sqrt{1-4x^2}}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= \frac{-2x}{\sqrt{1-4x^2}} + \cos^{-1}(2x)$$

5 Write the result as a derivative of one function.

$$\frac{d}{dx}\left[x\cos^{-1}(2x)\right] = \frac{-2x}{\sqrt{1-4x^2}} + \cos^{-1}(2x)$$

6 Use integration by recognition.

$$\int\left(\frac{-2x}{\sqrt{1-4x^2}} + \cos^{-1}(2x)\right)dx = x\cos^{-1}(2x)$$

7 Separate the integral on the left into two separate integrals.

$$-\int\frac{2x}{\sqrt{1-4x^2}}\,dx + \int\cos^{-1}(2x)\,dx = x\cos^{-1}(2x)$$

8 Transfer one part of the integral which can be performed on the right-hand side.

$$\int\cos^{-1}(2x)\,dx = x\cos^{-1}(2x) + \int\frac{2x}{\sqrt{1-4x^2}}\,dx$$

9 Consider now just the integral on the right.

$$\int\frac{2x}{\sqrt{1-4x^2}}\,dx$$

10 Write the integrand as a power using index laws.

$$= \int 2x(1-4x^2)^{-\frac{1}{2}}\,dx$$

11 Use a non-linear substitution. Let $t = 1 - 4x^2$.

$$= \int 2xt^{-\frac{1}{2}}\,dx$$

12 The integral cannot be done in this form, so differentiate. Express dx in terms of dt by inverting both sides.

$$t = 1 - 4x^2$$

$$\frac{dt}{dx} = -8x$$

$$\frac{dx}{dt} = \frac{-1}{8x}$$

$$dx = \frac{-1}{8x}\,dt$$

13 Substitute for dx, noting that the terms involving x will cancel.

$$= \int 2xt^{-\frac{1}{2}} \times \frac{-1}{8x}\,dt$$

14 Transfer the constant factors outside the front of the integral sign.

$$= -\frac{1}{4}\int t^{-\frac{1}{2}}\,dt$$

15 The integral can now be done. Antidifferentiate using $\int t^n\,dt = \frac{t^{n+1}}{n+1}$ with $n = -\frac{1}{2}$, so that $n + 1 = \frac{1}{2}$.

$$= -\frac{1}{2}t^{\frac{1}{2}} + c$$

16 Write the expression with positive indices.

$$= -\frac{1}{2}\sqrt{t} + c$$

17 Substitute back for t and state the result for this part of the integral.

$$\int \frac{2x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{2}\sqrt{1-4x^2}$$

18 Substitute for the integral, add in the arbitrary constant $+c$, and state the final result.

$$\int \cos^{-1}(2x)\,dx = x\cos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2} + c$$

EXERCISE 10.2 Integration by recognition

PRACTISE

1 WE1 Differentiate $\sqrt{(9x^2+16)^3}$ and hence find $\int x\sqrt{9x^2+16}\,dx$.

2 Determine $\frac{d}{dx}[\cos^5(3x)]$ and hence find $\int \sin(3x)\cos^4(3x)\,dx$.

3 WE2 Differentiate $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)$ and hence evaluate $\int_0^4 \frac{1}{\sqrt{4x-x^2}}\,dx$.

4 If $f(x) = \arctan\left(\frac{4}{\sqrt{x}}\right)$, find $f'(x)$ and hence evaluate $\int_{16}^{48} \frac{1}{\sqrt{x}(x+16)}\,dx$.

5 WE3 Find $\frac{d}{dx}[x\sin(3x)]$ and hence find $\int x\cos(3x)\,dx$.

6 Differentiate xe^{-2x} and hence find $\int xe^{-2x}\,dx$.

7 WE4 Find $\frac{d}{dx}[x\sin^{-1}(5x)]$ and hence find $\int \sin^{-1}(5x)\,dx$.

8 Differentiate $x\tan^{-1}(4x)$ and hence evaluate $\int_0^{\frac{1}{4}} \arctan(4x)\,dx$.

CONSOLIDATE

9 a Differentiate $x\sin\left(\frac{x}{2}\right)$ and hence find $\int x\cos\left(\frac{x}{2}\right)dx$.

b Differentiate $x\cos(2x)$ and hence find $\int x\sin(2x)\,dx$.

c Differentiate $xe^{-\frac{x}{2}}$ and hence find $\int xe^{-\frac{x}{2}}\,dx$.

10 a Determine $\frac{d}{dx}\left[x\sin^{-1}\left(\frac{x}{3}\right)\right]$ and hence find $\int \sin^{-1}\left(\frac{x}{3}\right)dx$.

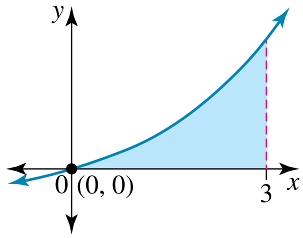
b Determine $\frac{d}{dx}[x\cos^{-1}(4x)]$ and hence find $\int \cos^{-1}(4x)\,dx$.

c Determine $\frac{d}{dx}\left[x\tan^{-1}\left(\frac{x}{5}\right)\right]$ and hence find $\int \tan^{-1}\left(\frac{x}{5}\right)dx$.

11 a If $f(x) = x\sin(4x)$, find $f'(x)$ and hence evaluate $\int_0^{\frac{\pi}{8}} x\cos(4x)\,dx$.

b If $f(x) = x\cos(6x)$, find $f'(x)$ and hence evaluate $\int_0^{\frac{\pi}{12}} x\sin(3x)\cos(3x)dx$.

c If $f(x) = xe^{\frac{x}{3}}$, find $f'(x)$. Hence, determine the shaded area bounded by the graph of $y = xe^{\frac{x}{3}}$, the origin, the x-axis and the line $x = 3$, as shown at right.

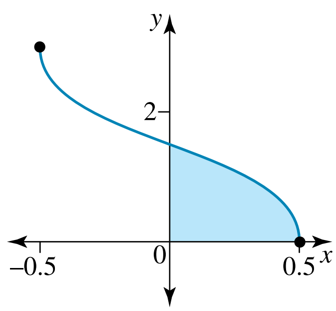


12 a Determine $\frac{d}{dx}\left[\cos^{-1}\left(\frac{4}{x^2}\right)\right]$ and hence evaluate $\int_2^{2\sqrt{2}} \frac{1}{x\sqrt{x^4 - 16}}dx$.

b Determine $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\sqrt{x}}{3}\right)\right]$ and hence evaluate $\int_9^{27} \frac{1}{\sqrt{x}(9 + x)}dx$.

c Determine $\frac{d}{dx}\left[\sin^{-1}\left(\frac{2}{x}\right)\right]$ and hence evaluate $\int_2^{4} \frac{1}{x\sqrt{x^2 - 4}}dx$.

13 a Differentiate $x\cos^{-1}(2x)$. Hence, determine the shaded area bounded by the graph of $y = \cos^{-1}(2x)$, the coordinate axes and $x = \frac{1}{2}$, as shown at right.



b Differentiate $x\sin^{-1}(3x)$ and hence find the area bounded by the curve $y = \sin^{-1}(3x)$, the x-axis, the origin and the line $x = \frac{1}{3}$.

c Differentiate $x\tan^{-1}\left(\frac{x}{4}\right)$ and hence find the area bounded by the graph of $y = \tan^{-1}\left(\frac{x}{4}\right)$, the x-axis, the origin and the line $x = 4$.

14 Assuming that a is a positive real constant:

a differentiate xe^{ax} and hence find $\int xe^{ax}\,dx$

b differentiate $x\sin(ax)$ and hence find $\int x\cos(ax)\,dx$

c differentiate $x\cos(ax)$ and hence find $\int x\sin(ax)\,dx$.

15 a Use your results from question 14a to evaluate $\int_0^{\frac{\pi}{8}} x\cos^2(2x)dx$.

b Use your results from question 14b to evaluate $\int_0^{\frac{\pi}{9}} x\sin^2(3x)dx$.

c Determine $\frac{d}{dx}[(2x+1)\log_e(2x+1)]$ and hence evaluate $\int_0^2 \log_e(2x+1)\,dx$.

16 Assuming that a is a positive real constant:

a differentiate $x\sin^{-1}\left(\frac{x}{a}\right)$ and hence find $\int \sin^{-1}\left(\frac{x}{a}\right)dx$

b differentiate $x\cos^{-1}\left(\frac{x}{a}\right)$ and hence find $\int \cos^{-1}\left(\frac{x}{a}\right)dx$

c differentiate $x\tan^{-1}\left(\frac{x}{a}\right)$ and hence find $\int \tan^{-1}\left(\frac{x}{a}\right)dx$.

17 a Determine $\frac{d}{dx}[x\log_e(4x)]$ and hence find $\int \log_e(4x)\,dx$.

b Determine $\frac{d}{dx}[x^2\log_e(4x)]$ and hence find $\int x\log_e(4x)\,dx$.

c Determine $\frac{d}{dx}[x^3\log_e(4x)]$ and hence find $\int x^2\log_e(4x)\,dx$.

d Hence, deduce $\int x^n\log_e(4x)\,dx$. What happens if $n = -1$?

18 a If $f(x) = e^{2x}(2\cos(3x) + 3\sin(3x))$, find $f'(x)$ and hence find $\int e^{2x}\cos(3x)\,dx$.

b Determine $\frac{d}{dx}[e^{-3x}(2\cos(2x) + 3\sin(2x))]$ and hence find $\int e^{-3x}\sin(2x)\,dx$.

c Let $f(x) = e^{ax}(a\cos(bx) + b\sin(bx))$ and find $f'(x)$. Hence, find $\int e^{ax}\cos(bx)\,dx$.

d Let $g(x) = e^{ax}(a\sin(bx) - b\cos(bx))$ and find $g'(x)$. Hence, find the area enclosed between the curve $y = e^{-2x}\sin(3x)$, the x-axis, the origin and the first intercept the curve makes with the positive x-axis.

MASTER

19 Assuming that a is a positive real constant:

a find the derivative of $x^2\tan^{-1}\left(\frac{x}{a}\right)$ and hence find $\int x\tan^{-1}\left(\frac{x}{a}\right)dx$

b **i** use the substitution $x = a\sin(\theta)$ to show that

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2 - x^2}$$

ii find the derivative of $x^2\sin^{-1}\left(\frac{x}{a}\right)$ and hence show that

$$\int x\sin^{-1}\left(\frac{x}{a}\right)dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{4}x\sqrt{a^2 - x^2} + c.$$

20 **a** Find $\frac{d}{dx}[\log_e(\tan(x) + \sec(x))]$. Hence, find the area enclosed between the curve $y = \sec(x)$, the coordinate axes and the line $x = \frac{\pi}{4}$.

b Given that a is a positive real constant, find $\frac{d}{dx}[\log_e(\operatorname{cosec}(ax) + \cot(ax))]$.

Hence, find the area enclosed between the curve $y = \operatorname{cosec}(2x)$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$.

c If $y = \log_e\sqrt{\frac{\sin(2x) + \cos(2x)}{\sin(2x) - \cos(2x)}}$, show that $\frac{dy}{dx} = 2\sec(4x)$. Hence, find the area enclosed between the curve $y = \sec(4x)$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

10.3 Solids of revolution

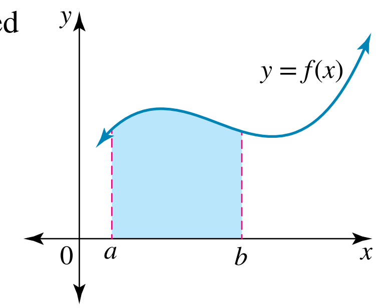
Rotations around the *x*-axis

study on

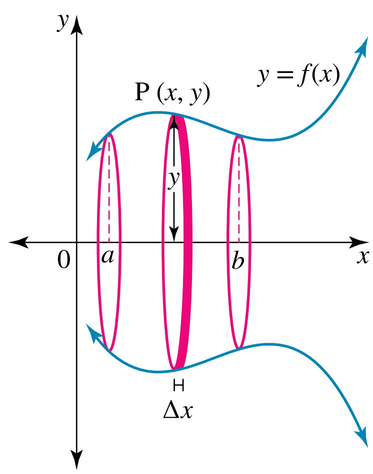
Units 3 & 4
AOS 3
Topic 2
Concept 10

Volumes of solids of revolution
Concept summary
Practice questions

Suppose that the curve $y = f(x)$ is continuous on the closed interval $a \le x \le b$. Consider the area bounded by the curve, the x-axis and the ordinates $x = a$ and $x = b$.



If this area is rotated 360° about the x-axis, it forms a solid of revolution and encloses a volume V.



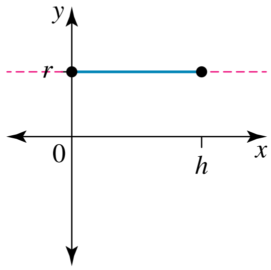
Consider a point on the curve with coordinates P (x, y). When rotated, it forms a circular disc with a radius of y and a cross-sectional area of $A(y) = \pi y^2$. If the disc has a width of Δx, then the volume of the disc is the cross-sectional area times the width, that is $A(y)\Delta x$. The total volume, V, is found by adding all such discs between $x = a$ and $x = b$ and taking the limit as $\Delta x \to 0$. This is given by

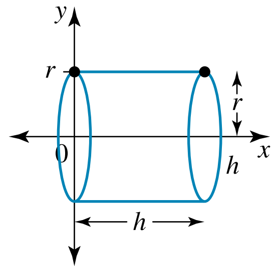
$$V = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \Delta x = \pi \int_a^b y^2\, dx.$$

We can use the above results to verify the volumes of some common geometrical shapes.

Volume of a cylinder

The volume of a cylinder of height h and radius r is given by $\pi r^2 h$. To verify this result, consider a cylinder lying on its side.





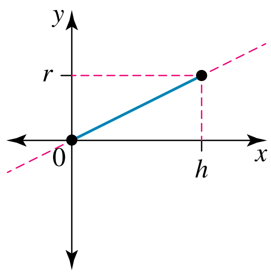
To form the cylinder, rotate the line $y = r$ by 360° about the x-axis between $x = h$ and the origin. Since h is a constant and r is a constant, they can be taken outside the integral sign.

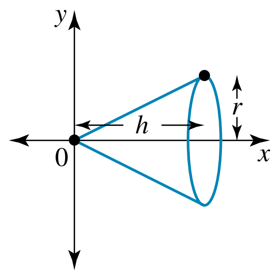
$$\begin{aligned} V &= \pi \int_0^h y^2 dx \\ &= \pi \int_0^h r^2 dx \\ &= \pi r^2 \int_0^h 1\, dx \\ &= \pi r^2 [x]_0^h \\ &= \pi r^2 (h - 0) \\ &= \pi r^2 h \end{aligned}$$

Volume of a cone

The volume of a cone of height h and radius r is given by $\frac{1}{3}\pi r^2 h$. To verify this result, it is easiest to consider the cone lying on its side.

Consider a line which passes through the origin, so its equation is given by $y = mx$. Now rotate the line 360° about the x-axis, between $x = h$ and the origin, so that it forms a cone.



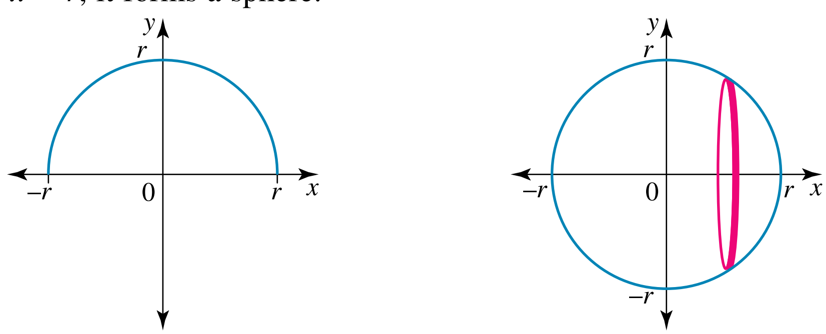


The gradient of the line is $m = \frac{r}{h}$, so the line has the equation $y = \frac{rx}{h}$. The volume is given by

$$\begin{aligned}V &= \pi\int_0^h y^2 dx \\ &= \pi\int_0^h \frac{r^2x^2}{h^2}\,dx \\ &= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx \\ &= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\ &= \frac{\pi r^2}{h^2}\left(\frac{h^3}{3} - 0\right) \\ &= \tfrac{1}{3}\pi r^2 h\end{aligned}$$

Volume of a sphere

The volume of a sphere of radius r is given by $\frac{4}{3}\pi r^3$. To verify this result, consider the circle $x^2 + y^2 = r^2$ with centre at the origin and radius r. If we rotate this circle by 360° or rotate the top half of the circle, $y = \sqrt{r^2 - x^2}$, about the x-axis, between $x = -r$ and $x = r$, it forms a sphere.



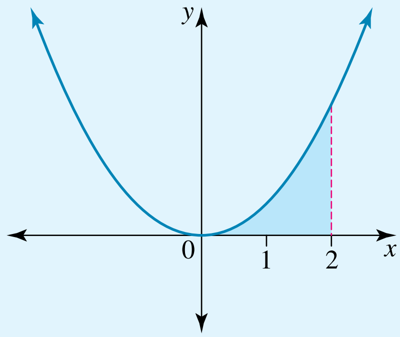
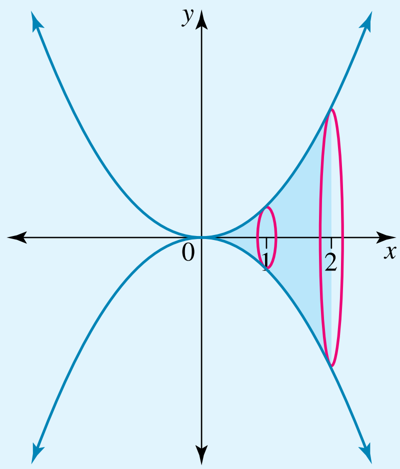
$$\begin{aligned}V &= \pi\int_{-r}^{r} y^2 dx \\ &= \pi\int_{-r}^{r} (r^2 - x^2)dx \text{ by symmetry} \\ &= 2\pi\int_0^r (r^2 - x^2)dx \\ &= 2\pi\left[r^2x - \tfrac{1}{3}x^3\right]_0^r \\ &= 2\pi\left(r^3 - \tfrac{1}{3}r^3 - 0\right) \\ &= \tfrac{4}{3}\pi r^3\end{aligned}$$

Solid volumes formed by rotating curves about the *x*-axis

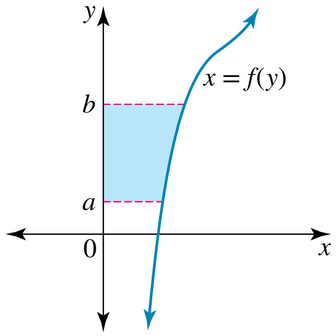
When rotating the curve $y = f(x)$ about the x-axis between $x = a$ and $x = b$, the volume formed is given by $V = \pi\int_a^b y^2\,dx$. The integrand must be in terms of constants and x-values only.

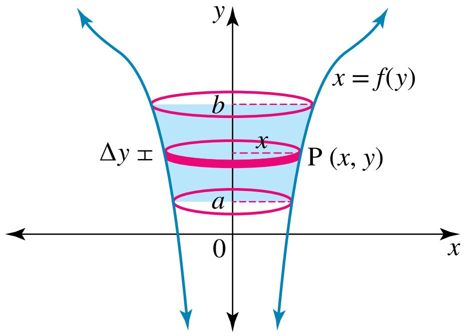
WORKED EXAMPLE 5

The area bounded by the curve $y = x^2$, the x-axis, the origin and the line $x = 2$ is rotated about the x-axis to form a solid of revolution. Find the volume formed.

THINK	WRITE/DRAW
1 Sketch the graph and identify the area to rotate.	 
2 Write a definite integral which gives the volume.	$V = \pi\int_a^b y^2\,dx$ $a = 0,\ b = 2,\ y = x^2$ so $y^2 = x^4$ $V = \pi\int_0^2 x^4\,dx$
3 Antidifferentiate using the rules.	$V = \pi\left[\frac{x^5}{5}\right]_0^2$
4 Evaluate the definite integral.	$V = \pi\left[\frac{32}{5} - 0\right]$ $= \frac{32\pi}{5}$
5 State the volume.	The volume formed is $\frac{32\pi}{5}$ cubic units.

Solid volumes formed by rotating curves about the y-axis

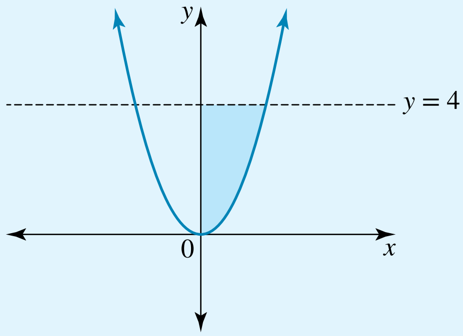
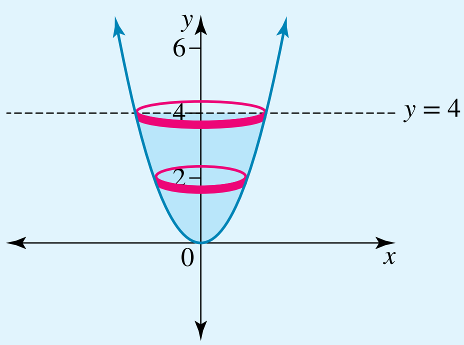




When the curve $x = f(y)$ is rotated about the y-axis between $y = a$ and $y = b$, the volume formed is given by $V = \pi \int_a^b x^2\,dy$. The integrand must be in terms of constants and y-values only.

WORKED EXAMPLE 6

The area bounded by the curve $y = x^2$, the y-axis, the origin and the line $y = 4$ is rotated about the y-axis to form a solid of revolution. Find the volume formed.

THINK	WRITE/DRAW
1 Sketch the graph and identify the area to rotate.	
2 Write a definite integral that gives the volume.	$V = \pi \int_a^b x^2\,dy$ $a = 0,\ b = 4,\ y = x^2$ $V = \pi \int_0^4 y\,dy$
3 Antidifferentiate using the rules.	$V = \pi \left[\dfrac{y^2}{2}\right]_0^4$

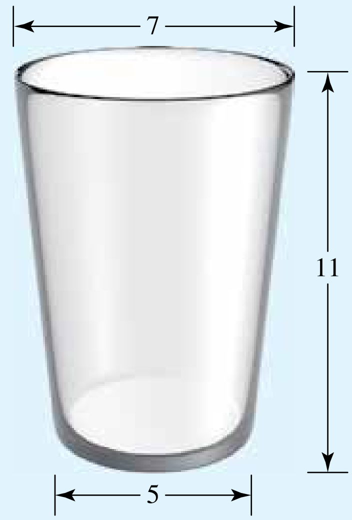
4 Evaluate the definite integral.	$V = \pi[8 - 0]$ $V = 8\pi$
5 State the volume.	The volume formed is 8π cubic units.

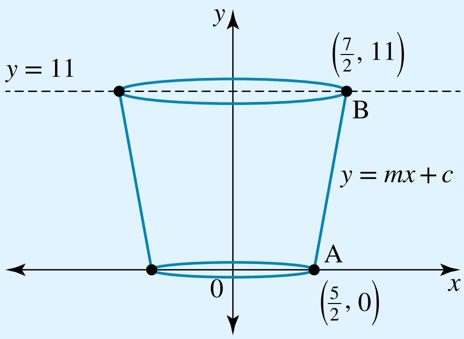
Applications

The volumes of some common geometrical objects can now be found using calculus, by rotating lines or curves about the x- or y-axis and using the above techniques.

WORKED EXAMPLE 7

A drinking glass has a base diameter of 5 cm, a top diameter of 7 cm and a height of 11 cm. Find the volume of the glass to the nearest mL.



THINK	WRITE/DRAW
1 Sketch the graph and identify the area to rotate. Write the coordinates of the points A and B.	
2 Establish the gradient of the line segment AB joining the points A $\left(\frac{5}{2}, 0\right)$ and B $\left(\frac{7}{2}, 11\right)$.	$m = \dfrac{11 - 0}{\frac{7}{2} - \frac{5}{2}} = 11$
3 Establish the equation of the line segment joining the points A and B.	$y - 0 = 11\left(x - \frac{5}{2}\right)$ $y = 11x - \frac{55}{2}$ The glass is formed when the line $y = 11x - \frac{55}{2}$ for $\frac{5}{2} \le x \le \frac{7}{2}$ is rotated about the y-axis.
4 Rotate this line about the y-axis to form the required volume.	$V = \pi\int_a^b x^2\,dy$ $a = 0, b = 11$

5 Rearrange to make x the subject.

$$y = 11x - \frac{55}{2}$$
$$11x = y + \frac{55}{2}$$
$$= \frac{1}{2}(2y + 55)$$
$$x = \frac{1}{22}(2y + 55)$$

6 Write a definite integral which gives the volume.

$$V = \frac{\pi}{484}\int_0^{11}(2y + 55)^2\,dy$$

7 Antidifferentiate using the rules.

$$V = \frac{\pi}{2 \times 3 \times 484}\left[(2y + 55)^3\right]_0^{11}$$

8 Evaluate the definite integral.

$$V = \frac{\pi}{2904}(77^3 - 55^3)$$
$$V \approx 99.92\pi$$
$$\approx 314 \text{ cm}^3$$

9 Since 1 cm^3 = 1 mL, state the volume of the glass. The volume of the glass is 314 mL.

EXERCISE 10.3 Solids of revolution

PRACTISE

1 WE5 The area bounded by the curve $y = \sqrt{x}$, the x-axis, the origin and the line $x = 4$ is rotated about the x-axis to form a solid of revolution. Find the volume formed.

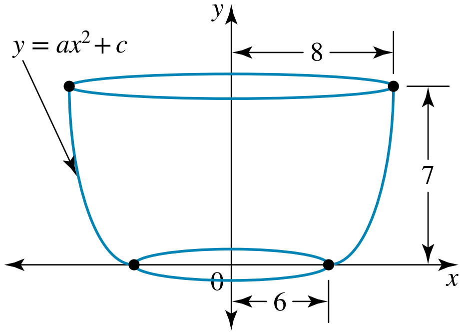
2 The area bounded by the curve $y = 4 - x^2$ and the coordinate axis is rotated about the x-axis to form a solid of revolution. Find the volume formed.

3 WE6 The area bounded by the curve $y = \sqrt{x}$, the y-axis, the origin and the line $y = 2$ is rotated about the x-axis to form a solid of revolution. Find the volume formed.

4 The area bounded by the curve $y = 4 - x^2$ and the coordinate axes is rotated about the y-axis to form a solid of revolution. Find the volume formed.

5 WE7 A plastic bucket has a base diameter of 20 cm, a top diameter of 26 cm and a height of 24 cm. Find the volume of the bucket to the nearest litre.

6 A soup bowl has a base radius of 6 cm, a top radius of 8 cm, and a height of 7 cm. The edge of the bowl is a parabola of the form $y = ax^2 + c$. Find the capacity of the soup bowl.



CONSOLIDATE

7 a Find the volume of the solid of revolution formed when the area between the line $y = 3x$, the x-axis, the origin and $x = 5$ is rotated 360° about the x-axis.

b Find the volume of the solid of revolution formed when the area between the line $y = 3x$, the y-axis, the origin and $y = 5$ is rotated 360° about the y-axis.

c A cone is formed by rotating the line segment of $2x + 3y = 6$ cut off by the coordinate axes about:

i the x-axis

ii the y-axis.

Find the volume in each case.

8 a If the region bounded by the curve $y = 3\sin(2x)$, the origin, the x-axis and the first intercept the curve makes with the x-axis is rotated 360° about the x-axis, find the volume formed.

b If the region bounded by the curve $y = 4\cos(3x)$, the coordinate axes and the first intercept the curve makes with the x-axis is rotated 360° about the x-axis, find the volume formed.

c If the region bounded by the curve $y = \sec(2x)$, the coordinate axes and $x = \frac{\pi}{8}$ is rotated 360° about the x-axis, find the volume formed.

d If the region bounded by the curve $y = 2e^{\frac{x}{2}}$, the coordinate axes and $x = 2$ is rotated 360° about the x-axis, find the volume formed.

9 a i If the area between the curve $y = 3x^2 + 4$, the coordinate axes and the line $x = 2$ is rotated 360° about the x-axis, find the volume formed.

ii If the area between the curve $y = 3x^2 + 4$, $y = 4$ and $y = 10$ is rotated about the y-axis, find the volume formed.

b i For the curve $y = x^2 - 9$, find the area between the curve and the x-axis.

ii If the area described in **i** is rotated 360° about the x-axis, find the volume formed.

iii If the area described in **i** is rotated 360° about the y-axis, find the volume formed.

10 a Find the volume formed by rotating the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ about:

i the x-axis **ii** the y-axis.

b i Determine the volume formed if the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, having semi-major and semi-minor axes a and b respectively, is rotated about the x-axis.

ii Determine the volume if the ellipse from part **bi** is rotated about the y-axis.

iii If $a = b$, verify that your results from **bi** and **ii** give the volume of a sphere.

c An egg can be regarded as an ellipsoid. The egg has a total length of 57 mm and its diameter at the centre is 44 mm. Find its volume in cubic millimetres.

11 a The drive shaft of an industrial spinning machine is 2 metres long and has the form of the curve $y = e^{\frac{x}{20}}$, measured in metres, rotated 360° about the x-axis between the y-axis.

i Find the volume in cubic metres.

ii What is the length of a similar shaft if it encloses a volume of $3.5\pi \text{ m}^3$?

b A piece of plastic tubing has its boundary in the form of the curve

$$y = \frac{1}{\sqrt{4 + 9x^2}}.$$

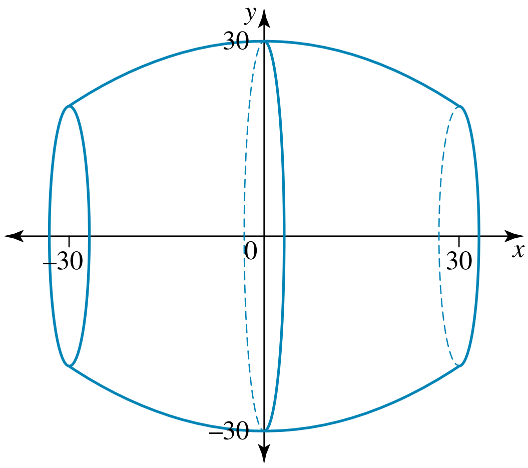
When this curve is rotated 360° about the x-axis between the y-axis and the line $x = 5$, find its volume.

12 a The diagram below shows a wine barrel. The barrel has a total length of 60 cm, a total height of 60 cm at the middle and a total height of 40 cm at the ends.

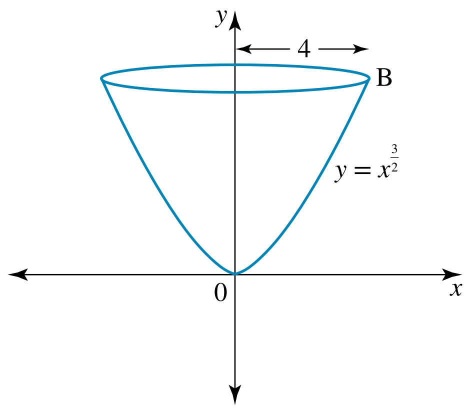
i If the upper arc can be represented by a parabolic boundary, show that its equation is given by

$$y = -\frac{x^2}{90} + 30 \text{ for } -30 \le x \le 30.$$

ii If the arc is rotated about the x-axis, find the capacity of the wine barrel to the nearest litre.

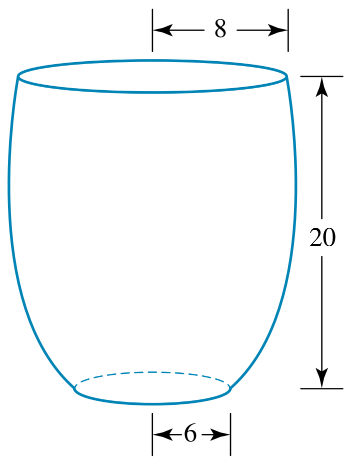


b A wine glass is formed when the arc OB with the equation $y = x^{\frac{3}{2}}$ is rotated about the y-axis, as shown below. The dimensions of the glass are given in cm. Find the volume of the glass in mL.



13 a A vase has a base radius of 6 cm, its top radius is 8 cm and its height is 20 cm. Find the volume of the vase in cubic centimetres if the side is modelled by:

i a straight line of the form $y = ax + c$
ii a parabola of the form $y = ax^2 + c$
iii a cubic $y = ax^3 + c$
iv a quartic of the form $y = ax^4 + c$.



b Another vase is modelled by rotating the curve with the equation

$$\frac{x^2}{4} - \frac{(y - 10)^2}{12} = 1$$

about the y-axis between the x-axis and $y = 20$. Find the volume of water needed to completely fill this vase.

14 a i For the curve $y = e^{-ax}$ where $a > 0$, find the area A bounded by the curve, the coordinate axes and $x = n$.
ii If the region in **i** is rotated 360° about the x-axis, find the volume formed, V.
iii Determine $\lim_{n \to \infty} A$ and $\lim_{n \to \infty} V$.

b The region bounded by the rectangular hyperbola $xy = 1$, the x-axis and the lines $x = 1$ and $x = a$ has an area of A and a volume of V when rotated about the x-axis.
i Find A and V.
ii Find $\lim_{a \to \infty} A$ and $\lim_{a \to \infty} V$ if they exist.

15 a If the region bounded by the curve $y = a\sin(nx)$, the origin, the x-axis and the first intercept the curve makes with the x-axis is rotated about the x-axis, find the volume formed.

b If the region bounded by the curve $y = a\cos(nx)$, the coordinate axes and the first intercept the curve makes with the x-axis is rotated about the x-axis, find the volume formed.

16 a Prove that the volume of a right truncated cone of inner and outer radii r_1 and r_2 respectively and height h is given by

$$\frac{\pi h\left(r_2^3 - r_1^3\right)}{3(r_2 - r_1)}.$$

b i A hemispherical bowl of radius r contains water to a depth of h. Show that the volume of water is given by

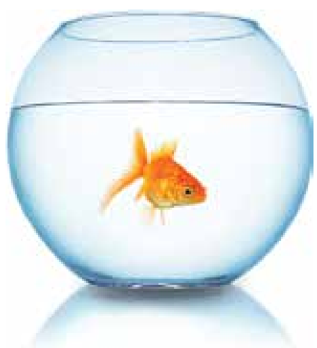
$$\frac{\pi h^2}{3}(3r - h).$$

ii If the bowl is filled to a depth of $\frac{r}{2}$, what is the volume of water in the bowl?

MASTER

17 a The region bounded by the curve $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$, the x-axes, and $x = 0$ and $x = 2$ is rotated about the x-axis. Find the volume formed.

b The region bounded by the curve $\sqrt{x} + \sqrt{y} = 2$ and the coordinate axes is rotated about the x-axis. Find the volume formed.



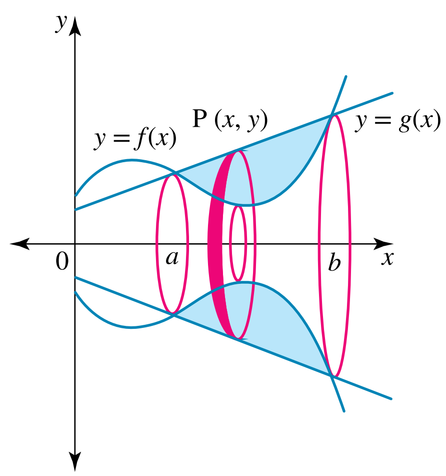
18 A fish bowl consists of a portion of a sphere of radius 20 cm. The bowl is filled with water so that the radius of the water at the top is 16 cm and the base of the bowl has a radius of 12 cm. If the total height of the water in the bowl is 28 cm, find the volume of water in the bowl.

10.4 Volumes

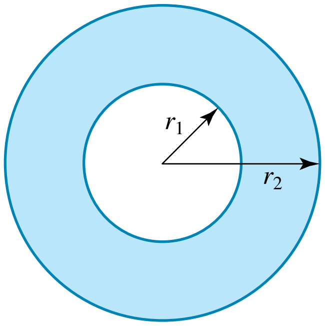
Volumes of revolution

Volumes around the *x*-axis

Consider $y = g(x)$ and $y = f(x)$ as two continuous non-intersecting curves on $a \le x \le b$ and $g(x) \ge f(x)$. If the area between the curves is rotated 360° about the x-axis, it forms a volume of revolution.



However, the volume is not solid and has a hole in it. Consider a cross-sectional area shaped like a circular washer, with inner radius $r_1 = y_1 = f(x)$ and outer radius $r_2 = y_2 = g(x)$.



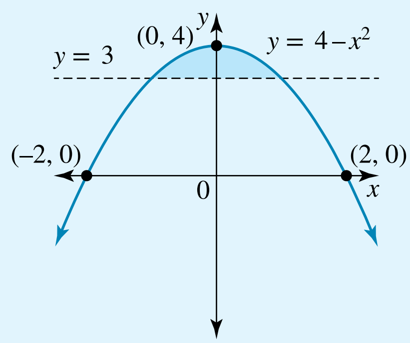
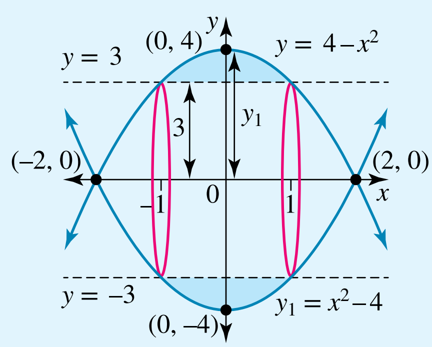
The volume formed is $V = \pi \int_a^b \left(r_2^2 - r_1^2\right) dx$. Note that we must express the inner and outer radii in terms of constants and x-values only.

WORKED EXAMPLE 8

Find the volume formed when the area bounded by the curve $y = 4 - x^2$ and the line $y = 3$ is rotated about:

a the x-axis

b the y-axis.

THINK	WRITE/DRAW
a 1 Find the points of intersection between the curve and the line.	**a** Let $y_1 = 4 - x^2$ and $y_2 = 3$. $y_1 = y_2$ $4 - x^2 = 3$ $x^2 = 1$ $x = \pm 1$
2 Sketch the graph and identify the area to rotate about the x-axis.	
3 Identify the inner and outer radii and the terminals of integration. The volume formed has a hole in it.	 $V = \pi \int_a^b \left(r_2^2 - r_1^2\right) dx$ $a = -1,\ b = 1,\ r_2 = y_1,\ r_1 = 3$
4 Write a definite integral for the volume.	$V = \pi \int_{-1}^{1} \left((4 - x^2)^2 - 9\right) dx$

5 Expand and simplify the integrand. $V = \pi \int_{-1}^{1} (16 - 8x^2 + x^4 - 9)\,dx$

6 Use symmetry to write the volume as a definite integral. $V = 2\pi \int_{0}^{1} (7 - 8x^2 + x^4)\,dx$

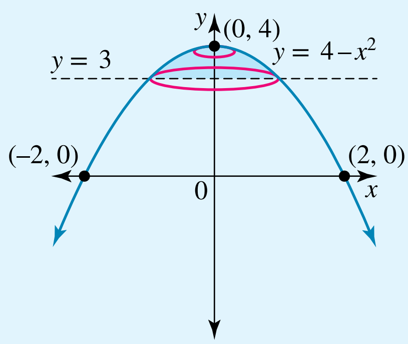
7 Perform the integration. $V = 2\pi \left[7x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^1$

8 Evaluate the definite integral. $V = 2\pi \left[\left(7 - \frac{8}{3} + \frac{1}{5}\right) - 0\right]$

9 State the value of the volume. The volume is $\frac{136\pi}{15}$ cubic units.

b 1 Sketch the graph and identify the area to rotate about the y-axis. This is a solid volume of revolution.

b



2 Write a definite integral for the volume. $V = \pi \int_{a}^{b} x^2\,dy$

$a = 3,\ b = 4,\ x^2 = 4 - y$

$V = \pi \int_{3}^{4} (4 - y)\,dy$

3 Perform the integration. $V = \pi \left[4y - \frac{1}{2}y^2\right]_3^4$

4 Evaluate the definite integral. $V = \pi \left[\left(4 \times 4 - \frac{1}{2} \times 4^2\right) - \left(4 \times 3 - \frac{1}{2} \times 3^2\right)\right]$

5 State the value of the volume. The volume is $\frac{\pi}{2}$ cubic units.

Volumes around the y-axis

In a similar way, if $x = g(y)$ and $x = f(y)$ are two continuous non-intersecting curves on $a \le y \le b$ where $g(y) \ge f(y)$ and the area between the curves is rotated 360° about the y-axis, it forms a volume of revolution.

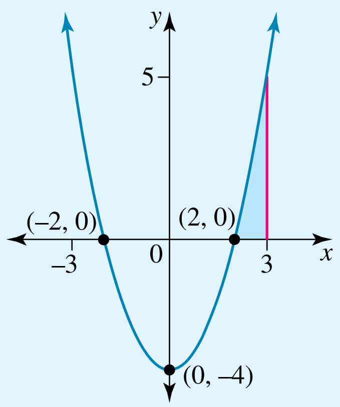
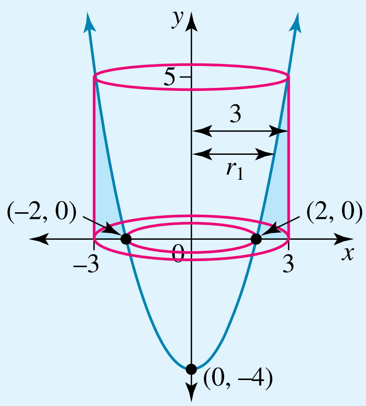
However, the volume is not solid and has a hole in it. Consider a typical cross-sectional area with inner radius $r_1 = x_1 = f(y)$ and outer radius $r_2 = x_2 = g(y)$.

The volume formed is $V = \pi\int_a^b \left(r_2^2 - r_1^2\right) dy$. Note that we must express the inner and outer radii in terms of constants and y-values only.

WORKED EXAMPLE 9

Find the volume formed when the area bounded by the curve $y = x^2 - 4$, the x-axis and the line $x = 3$ is rotated about:

a the y-axis **b** the x-axis.

THINK	WRITE/DRAW
a 1 Find the points of intersection between the curve and the line.	**a** Let $y_1 = x^2 - 4$ and $x_2 = 3$ $y = 9 - 4$ $= 5$
2 Sketch the graph and identify the area to rotate about the y-axis.	
3 Identify the inner and outer radii and the terminals of integration. The volume formed has a hole in it.	 $V = \pi\int_a^b \left(r_2^2 - r_1^2\right) dy$ $a = 0,\ b = 5,\ r_2 = x_2 = 3,\ r_1 = x$
4 Write a definite integral for the volume.	$V = \pi\int_0^5 (9 - x^2)\, dy$ $= \pi\int_0^5 (9 - (y + 4))\, dy$

5 Simplify the integrand.

$$V = \pi \int_0^5 (5 - y)\,dy$$

6 Perform the integration.

$$V = \pi \left[5y - \frac{1}{2}y^2 \right]_0^5$$

7 Evaluate the definite integral.

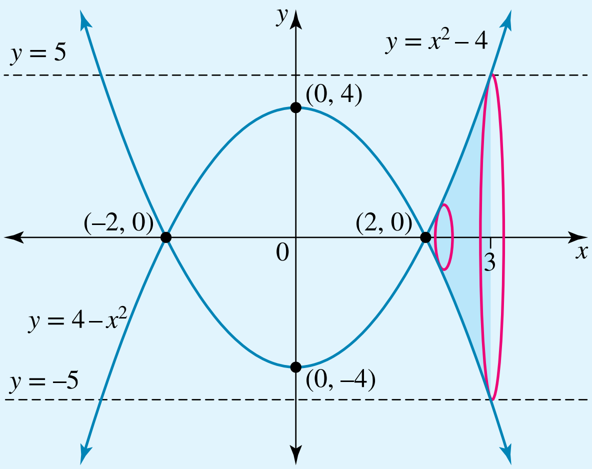
$$V = \pi \left[5 \times 5 - \frac{1}{2} \times 5^2 - 0 \right]$$

8 State the value of the volume.

The volume is $\frac{25\pi}{2}$ cubic units.

b 1 Sketch the graphs and identify the area to rotate about the x-axis. This is a solid volume ofrevolution.

b



2 Write a definite integral for the volume.

$$V = \pi \int_a^b y^2\,dx$$

$a = 2,\ b = 3,\ y^2 = (x^2 - 4)^2$

$$V = \pi \int_2^3 (x^2 - 4)^2\,dx$$

3 Expand the integrand.

$$V = \pi \int_2^3 (x^4 - 8x^2 + 16)\,dx$$

4 Perform the integration.

$$V = \pi \left[\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x \right]_2^3$$

5 Evaluate the definite integral.

$$V = \pi \left[\left(\frac{1}{5} \times 3^5 - \frac{8}{3} \times 3^3 + 16 \times 3 \right) - \left(\frac{1}{5} \times 2^5 - \frac{8}{3} \times 2^3 + 16 \times 2 \right) \right]$$

6 State the value of the volume.

The volume is $\frac{113\pi}{15}$ cubic units.

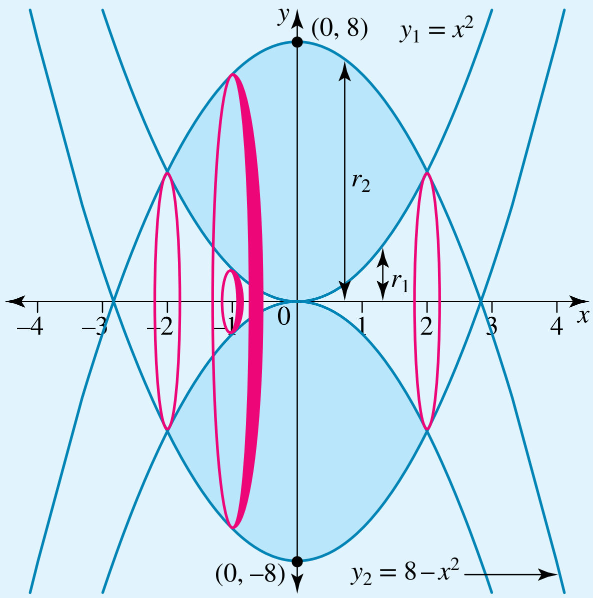
Composite figures

Sometimes we need to carefully consider the diagram and identify the regions that are rotated about the axes.

WORKED EXAMPLE 10

Find the volume formed when the area between the curves $y = x^2$ and $y = 8 - x^2$ is rotated about:

a **the x-axis** **b** **the y-axis.**

THINK	WRITE/DRAW
a **1** Find the points of intersection between the two curves.	**a** Let $y_1 = x^2$ and $y_2 = 8 - x^2$. $y_1 = y_2$ $x^2 = 8 - x^2$ $2x^2 = 8$ $x^2 = 4$ $x = \pm 2$ When $x = \pm 2$, $y = 4$.
2 Sketch the graphs and identify the area to rotate about the x-axis.	
3 Identify the inner and outer radii and the terminals of integration.	$V = \pi \int_a^b \left(r_2^2 - r_1^2\right) dx$ $a = -2,\ b = 2,\ r_2 = y_2,\ r_1 = y_1$
4 Write a definite integral for the volume.	$V = \pi \int_{-2}^{2} ((8 - x^2)^2 - (x^2)^2)\, dx$
5 Expand and simplify the integrand.	$V = \pi \int_{-2}^{2} (64 - 16x^2 + x^4 - x^4)\, dx$
6 Use symmetry to write the volume as a definite integral in simplest form.	$V = 2\pi \int_{0}^{2} (64 - 16x^2)\, dx$

7 Perform the integration.

$$V = 2\pi\left[64x - \frac{16}{3}x^3\right]_0^2$$

8 Evaluate the definite integral.

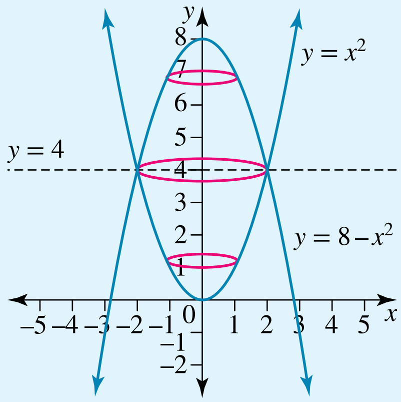
$$V = 2\pi\left[\left(64 \times 2 - \frac{16}{3} \times 2^3\right) - 0\right]$$

9 State the value of the volume.

The volume is $\frac{512\pi}{3}$ cubic units.

b 1 Sketch the graph and identify the area to rotate about the y-axis.

b



2 The region comprises two sections. Write a definite integral for the total volume.

$$V = \pi\int_0^4 x_1^2\,dy + \pi\int_4^8 x_2^2\,dy$$

$$= \pi\int_0^4 y\,dy + \pi\int_4^8 (8 - y)\,dy$$

3 Perform the integration.

$$V = \pi\left[\frac{1}{2}y^2\right]_0^4 + \pi\left[8y - \frac{1}{2}y^2\right]_4^8$$

4 Evaluate the definite integral.

$$V = \pi\left[\left(\frac{1}{2} \times 4^2\right) - 0\right] + \pi\left[\left(8 \times 8 - \frac{1}{2} \times 8^2\right) - \left(8 \times 4 - \frac{1}{2} \times 4^2\right)\right]$$

5 State the value of the volume.

The volume is 16π cubic units.

EXERCISE 10.4 Volumes

PRACTISE

1 **WE8** Find the volume formed when the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$ is rotated about:

a the x-axis **b** the y-axis.

2 Find the volume formed when the area bounded by the curve $y = \sqrt{x}$, the y-axis and the line $y = 2$ is rotated about:

a the x-axis **b** the y-axis.

3 **WE9** Find the volume formed when the area bounded by the curve $y = x^2 - 9$, the x-axis and the line $x = 4$ is rotated about:

a the y-axis **b** the x-axis.

4 Find the volume formed when the area bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is rotated about:

a the y-axis
b the x-axis.

5 WE10 Find the volume formed when the area between the curves $y = x^2$ and $y = 18 - x^2$ is rotated about:

a the x-axis
b the y-axis.

6 When the area between the curves $y = x^2 - 4x + 4$ and $y = 4 + 4x - x^2$ is rotated about the x-axis, it forms a volume of revolution. Find the volume.

CONSOLIDATE

7 Consider the area bounded by the curve $y = 16 - x^2$ and the line $y = 12$.

a If this area is rotated about the x-axis, it forms a volume of revolution. Find the volume formed.
b If this area is rotated about the y-axis, it forms a volume of revolution. Find the volume formed.

8 Consider the area bounded by the curve $y = x^2 - 16$, the x-axis and the line $x = 5$.

a If this area is rotated about the y-axis, it forms a volume of revolution. Find the volume formed.
b If this area is rotated about the x-axis, it forms a volume of revolution. Find the volume formed.

9 Let R be the region for the area bounded by the curve $y = x^3$, the x-axis and the line $x = 3$.

a If the region R is rotated about the y-axis, it forms a volume of revolution. Find the volume formed.
b If the region R is rotated about the x-axis, it forms a volume of revolution. Find the volume formed.

10 The area between the curve $y = \sqrt{2x - 8}$, the x-axis and the line $x = 6$ is rotated about:

a the y-axis
b the x-axis.

Determine the volume of the resulting solid of revolution in each case.

11 a Find the volume formed when the area bounded by the curve $y = 4\cos^{-1}\left(\frac{x}{2}\right)$ and the lines $x = 2$ and $y = 2\pi$ is rotated about the y-axis.

b Find the volume formed when the area bounded by the curve $y = 2\sin^{-1}\left(\frac{x}{3}\right)$, the x-axis and the line $x = 3$ is rotated about the y-axis.

12 a Find the volume formed when the area between the curves $y = 2x$ and $y = x^2$ is rotated about:

i the x-axis
ii the y-axis.

b Find the volume formed when the area between the curves $y = x^2$ and $y = x^3$ is rotated about:

i the x-axis
ii the y-axis.

13 a If the area between the two curves $y = 2\sqrt{x}$ and $y = \frac{x^2}{4}$ is rotated about the x-axis, find the volume formed.

b If the area between the two curves $y = \sqrt{ax}$ and $y = \frac{x^2}{a}$ where $a > 0$ is rotated about the y-axis, find the volume formed.

14 a Find the volume formed when the area in the first quadrant bounded by the curve $y = 12\sin\left(\frac{\pi x}{6}\right)$ and the line $y = 4x$ is rotated about the x-axis.

b Find the volume formed when the area in the first quadrant bounded by the curve $y = 4\sin\left(\frac{\pi x}{8}\right)$ and the line $y = 4 - x$ is rotated about the x-axis.

15 a Find the volume formed when the area between the curves $y = x^2$ and $y = 32 - x^2$ is rotated about:

i the x-axis ii the y-axis.

b Find the volume formed when the area in the first quadrant bounded by the curve $y = 8\sin\left(\frac{\pi x}{4}\right)$ and the parabola $y = 2x^2$ is rotated about the x-axis.

16 a A cylindrical hole of radius 2 cm is cut vertically through the centre of a solid sphere of cheese of radius 4 cm. Find the volume of the cheese remaining.

b A cylindrical hole of radius $\frac{a}{2}$ is drilled horizontally through the centre of a sphere of radius a. Show that the volume of the remaining solid is given by $\frac{\sqrt{3}}{2}a^3\pi$.

MASTER

17 a Find the volume of a torus (a doughnut-shaped figure) formed when the area bounded by the circle $x^2 + (y - 8)^2 = 16$ is rotated about the x-axis.

b Find the volume of a torus (a doughnut-shaped figure) formed when the area bounded by the circle $(x - a)^2 + y^2 = r^2$ where $a > r > 0$ is rotated about the y-axis.

18 a Find the volume formed when the area bounded by the curve $y = a^2 - x^2$ and the line $y = \frac{a^2}{4}$, where $a > 0$, is rotated about:

i the x-axis ii the y-axis.

b Find the volume formed when the area bounded by the curve $y = x^2 - a^2$, the x-axis and the line $x = 2a$, where $a > 0$, is rotated about:

i the y-axis ii the x-axis.

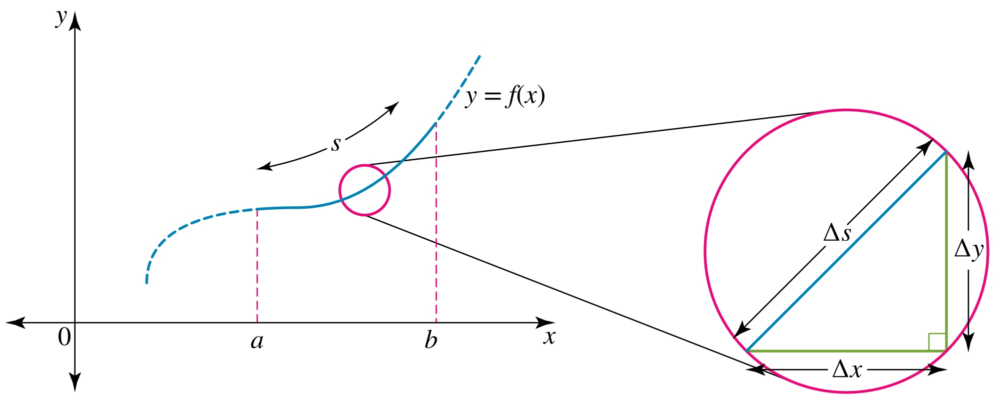
c Find the volume formed when the area between the curves $y = x^2$ and $y = 2a^2 - x^2$, where $a > 0$, is rotated about:

i the x-axis ii the y-axis.

10.5 Arc length, numerical integration and graphs of antiderivatives

Length of a curve

In this section integration will be used to determine the arc length, s, of a plane curve. Suppose that the curve $y = f(x)$ is a continuous curve on the closed interval $a \le x \le b$. The curve can be thought of as being made up of infinitely many short line segments as shown.



studyon
Units 3 & 4
AOS 3
Topic 2
Concept 4
Definite integrals and arc length
Concept summary
Practice questions

By Pythagoras' theorem, the length of a typical small segment Δs is equal to $\sqrt{(\Delta x)^2 + (\Delta y)^2}$. The total length of the curve s from $x = a$ to $x = b$ is obtained by summing all such segments and taking the limit as $\Delta x \to 0$.

$$s = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \sqrt{(\Delta x)^2 + (\Delta y)^2} = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

WORKED EXAMPLE 11 **Find the length of the curve $y = \dfrac{x^3}{2} + \dfrac{1}{6x}$ from $x = 1$ to $x = 5$.**

THINK	WRITE
1 Find the gradient function $\frac{dy}{dx}$ by differentiating and express back with positive indices.	$y = \frac{x^3}{2} + \frac{1}{6x}$ $= \frac{x^3}{2} + \frac{1}{6}x^{-1}$ $\frac{dy}{dx} = \frac{3x^2}{2} - \frac{1}{6}x^{-2}$ $= \frac{3x^2}{2} - \frac{1}{6x^2}$
2 Substitute into the formula and write a definite integral which gives the required length.	$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ with $a = 1$ and $b = 5$ $s = \int_1^5 \sqrt{1 + \left(\frac{3x^2}{2} - \frac{1}{6x^2}\right)^2}\,dx$
3 Expand using $(a - b)^2 = a^2 - 2ab + b^2$.	$s = \int_1^5 \sqrt{1 + \left(\left(\frac{3x^2}{2}\right)^2 - 2 \times \frac{3x^2}{2} \times \frac{1}{6x^2} + \left(\frac{1}{6x^2}\right)^2\right)}\,dx$
4 Cancel terms and simplify.	$s = \int_1^5 \sqrt{1 + \left(\frac{9x^4}{4} - \frac{1}{2} + \frac{1}{36x^4}\right)}\,dx$

5 Simplify the integrand.

$$s = \int_1^5 \sqrt{\left(\frac{9x^4}{4} + \frac{1}{2} + \frac{1}{36x^4}\right)}dx$$

6 Recognise the integrand as a perfect square.

$$s = \int_1^5 \sqrt{\left(\frac{3x^2}{2} + \frac{1}{6x^2}\right)^2} dx$$

7 Express the integrand in a form which can be integrated.

$$s = \int_1^5 \left(\frac{3x^2}{2} + \frac{1}{6x^2}\right)dx$$

$$= \int_1^5 \left(\frac{3x^2}{2} + \frac{1}{6}x^{-2}\right)dx$$

8 Perform the integration.

$$s = \left[\frac{x^3}{2} - \frac{1}{6}x^{-1}\right]_1^5 = \left[\frac{x^3}{2} - \frac{1}{6x}\right]_1^5$$

9 Evaluate the definite integral.

$$s = \left(\frac{5^3}{2} - \frac{1}{6 \times 5}\right) - \left(\frac{1^3}{2} - \frac{1}{6 \times 1}\right)$$

10 State the final result.

$$s = \frac{932}{15} \text{ units}$$

Numerical integration

The worked example above is somewhat contrived, because for many simple curves, the arc length formula results in a definite integral that cannot be evaluated by techniques of integration. In these situations we must resort to numerical methods, such as using calculators, which can give numerical approximations to the definite integrals obtained.

WORKED EXAMPLE 12

Express the length of the curve $y = x^3$ from $x = 0$ to $x = 2$ as a definite integral and hence find the length, giving your answer correct to 4 decimal places.

THINK

WRITE

1 Find the gradient function $\frac{dy}{dx}$ by differentiating.

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

2 Substitute into the formula and write a definite integral which gives the required length.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ with } a = 0 \text{ and } b = 2$$

$$s = \int_0^2 \sqrt{1 + 9x^4} dx$$

3 This definite integral must be evaluated using a calculator. State the final result.

$$s = 8.6303$$

studyon

Units 3 & 4
AOS 3
Topic 3
Concept 5

Numerical solutions of differential equations
Concept summary
Practice questions

Using numerical integration in differential equations

For many first-order differential equations, the integral cannot be found by techniques of integration. In these situations calculators can give a numerical approximation.

Consider the differential equation $\frac{dy}{dx} = f(x)$ and $y(x_0) = y_0$. We want to find the value of y_1 when $x = x_1$. We obtain $y = \int_0^x f(t)dy + c$, where we have arbitrarily used zero as the lower terminal and t as a dummy variable.

Note: x is maintained as the independent variable of the solution function.

We use the given initial condition, $y = y_0$ when $x = x_0$, then substitute $y_0 = \int_0^{x_0} f(t)dt + c$ to find that the constant of integration $c = y_0 - \int_0^{x_0} f(t)\,dt$. Substituting back for c gives $y = \int_0^x f(t)dt + y_0 - \int_0^{x_0} f(t)dt$.

$$y = y_0 + \int_0^x f(t)dt - \int_0^{x_0} f(t)dt$$

By properties of the definite integral,

$$\begin{aligned} y &= y_0 + \int_0^x f(t)dt + \int_{x_0}^0 f(t)dt \\ &= y_0 + \int_{x_0}^x f(t)dt \end{aligned}$$

When $x = x_1$,

$$y_1 = y_0 + \int_{x_0}^{x_1} f(t)dt.$$

WORKED EXAMPLE 13

Given that $\frac{dy}{dx} = e^{x^2}$, $y(1) = 2$:

a find a definite integral for y in terms of x

b determine the value of y correct to 4 decimal places when $x = 1.5$.

THINK	WRITE
a 1 Antidifferentiate the differential equation.	**a** $\frac{dy}{dx} = e^{x^2}$ $y = \int_0^x e^{t^2}dt + c$

2 Use the given initial conditions to find the value of the constant of integration.	Substitute $x = 1$ when $y = 2$: $2 = \int_0^1 e^{t^2}dt + c$ $c = 2 - \int_0^1 e^{t^2}dt$
3 Substitute back for the constant and simplify using the properties of definite integrals.	$y = \int_0^x e^{t^2}dt + 2 - \int_0^1 e^{t^2}dt$ $= 2 + \int_0^x e^{t^2}dt + \int_1^0 e^{t^2}dt$
4 State the solution for y as a definite integral involving x.	$y = 2 + \int_1^x e^{t^2}dt$
b 1 Find the value of y at the required x-value.	**b** Substitute $x = 1.5$: $y = 2 + \int_1^{1.5} e^{t^2}dt$
2 This definite integral must be evaluated using a calculator. State the final result.	$y = 4.6005$

Approximating volumes

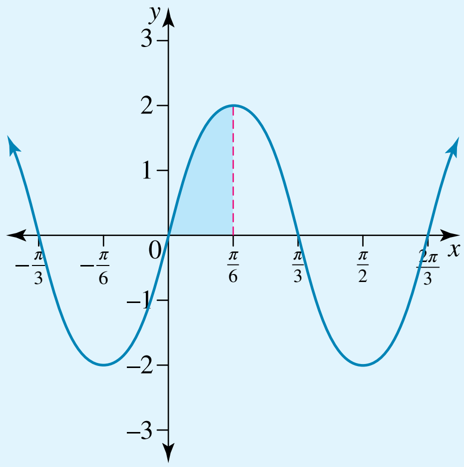
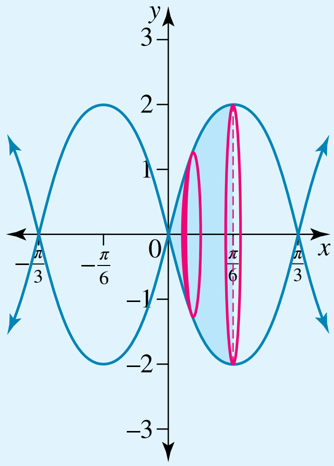
For many curves, the volume obtained also results in definite integrals that cannot be evaluated by any techniques of integration. Other volumes can only be evaluated by techniques that we have not as yet covered. In either of these situations we must again resort to numerical methods, such as using calculators, to find numerical approximations to the definite integrals obtained.

When a question asks for an answer correct to a specified number of decimal places, a calculator can be used to obtain a decimal approximation to the definite integral.

WORKED EXAMPLE 14

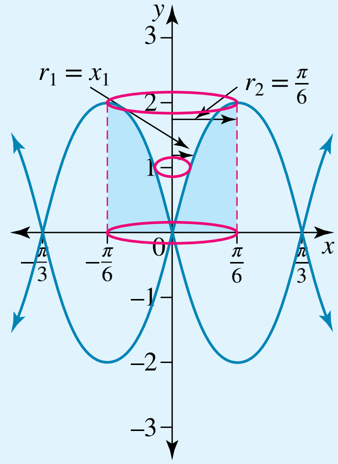
a **Find the exact volume formed when the area bounded by the curve $y = 2\sin(3x)$, the x-axis, the origin and the line $x = \frac{\pi}{6}$ is rotated about the x-axis.**

b **Set up a definite integral for the volume formed when the area bounded by the curve $y = 2\sin(3x)$, the x-axis, the origin and the line $x = \frac{\pi}{6}$ is rotated about the y-axis. Hence, find the volume correct to 4 decimal places.**

THINK	WRITE/DRAW
a 1 Sketch the graph and identify the area to rotate about the x-axis.	**a** 
2 Write a definite integral for the volume.	 $V = \pi\int_a^b y^2 dx$ $a = 0,\ b = \frac{\pi}{6},\ y = 2\sin(3x)$ $V = \pi\int_0^{\frac{\pi}{6}} 4\sin^2(3x)dx$
3 Use the double-angle formula $\sin^2(A) = \frac{1}{2}(1 - \cos(2A))$.	$V = 2\pi\int_0^{\frac{\pi}{6}} (1 - \cos(6x))dx$
4 Perform the integration.	$V = 2\pi\left[x - \frac{1}{6}\sin(6x)\right]_0^{\frac{\pi}{6}}$
5 Evaluate the definite integral.	$V = 2\pi\left[\left(\frac{\pi}{6} - \frac{1}{6}\sin(\pi)\right) - \left(0 - \frac{1}{6}\sin(0)\right)\right]$
6 State the value of the volume.	$V = \frac{\pi^2}{3}$ units^3

b **1** Sketch the graph and identify the area to rotate about the y-axis.

b When $x = \frac{\pi}{6}$, $y = 2\sin\left(\frac{\pi}{2}\right) = 2$.



2 Identify the inner and outer radii and the terminals of integration.

$$V = \pi\int_a^b \left(r_2^2 - r_1^2\right)dy$$

$$a = 0,\ b = 2,\ r_2 = x_2 = \frac{\pi}{6},\ r_1 = x_1$$

$$y = 2\sin(3x)$$

$$\frac{y}{2} = \sin(3x)$$

$$3x = \sin^{-1}\left(\frac{y}{2}\right)$$

$$x_1 = \frac{1}{3}\sin^{-1}\left(\frac{y}{2}\right)$$

3 Write a definite integral for the volume.

$$V = \pi\int_0^2 \left(\frac{\pi^2}{36} - \frac{1}{9}\left(\sin^{-1}\left(\frac{y}{2}\right)\right)^2\right)dy$$

4 This definite integral cannot be evaluated by integration techniques. Find a numerical value for the definite integral using a calculator, and state the final result.

$$V = 1.3963 \text{ units}^3$$

Graphs of antiderivatives of functions

Given a function $f(x)$, we can sketch the graph of the antiderivative $F(x) = \int f(x)dx$ by noting key features and considering $F'(x) = f(x)$. The table on the next page shows the relationships between the graphs. Note that the graph of the antiderivative cannot be completely determined as it includes a constant of integration, which is a vertical translation of the graph of $F(x)$ parallel to the y-axis.

studyon

Units 3 & 4
AOS 3
Topic 2
Concept 8

Graphs of antiderivative functions
Concept summary
Practice questions

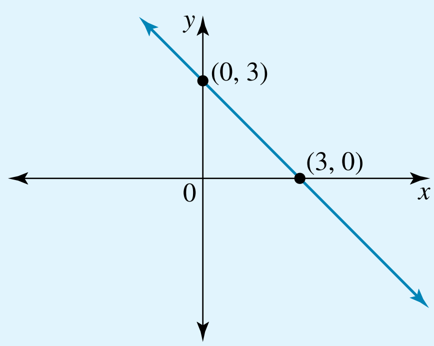
Graph of function f	Graph of antiderivative F
Negative, $f(x) < 0$ for $x \in (a, b)$	$F(x)$ has a negative gradient, or is decreasing for $x \in (a, b)$.
Positive, $f(x) > 0$ for $x \in (a, b)$	$F(x)$ has a positive gradient, or is increasing for $x \in (a, b)$.
$f(x)$ cuts the x-axis at $x = a$ from negative to positive.	$F(x)$ has a local minimum at $x = a$.
$f(x)$ cuts the x-axis at $x = a$ from positive to negative.	$F(x)$ has a local maximum at $x = a$.
$f(x)$ touches the x-axis at $x = a$.	$F(x)$ has stationary point of inflection at $x = a$.
$f(x)$ has a turning point at $x = a$.	$F(x)$ has a point of inflection at $x = a$ (non-stationary unless $f(x) = 0$).

In particular:

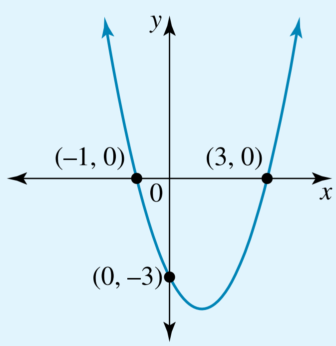
- If $f(x)$ is a linear function, then the graph of the antiderivative $F(x)$ will be a quadratic function.
- If $f(x)$ is a quadratic function, then the graph of the antiderivative $F(x)$ will be a cubic function.
- If $f(x)$ is a cubic function, then the graph of the antiderivative $F(x)$ will be a quartic function.

WORKED EXAMPLE 15

a **Given the graph below, sketch a possible graph of the antiderivative.**



b **The graph of the gradient function is shown below. Sketch a possible graph of the function.**



THINK

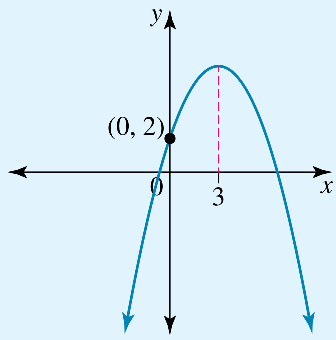
WRITE/DRAW

a 1 The given graph crosses the x-axis at $x = 3$, so the graph of the antiderivative has a stationary point at $x = 3$.

2 At $x = 3$ the given graph changes from positive to negative as x increases, so the stationary point is a maximum turning point.

3 No further information is provided, so we cannot determine the y-value of the turning point or any values of the axis intercepts. The graph of the antiderivative could be translated parallel to the y-axis.

a



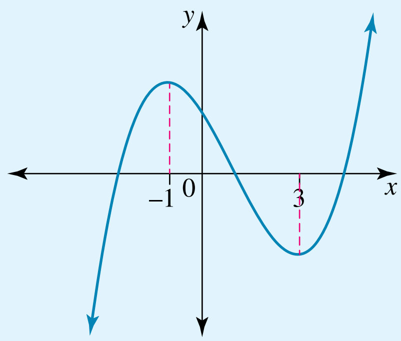
b 1 The given graph crosses the x-axis at $x = -1$ and $x = 3$, so the graph of the antiderivative has stationary points at these values.

2 At $x = -1$ the gradient changes from positive to negative as x increases, so the stationary point is a maximum turning point. At $x = 3$ the gradient changes from negative to positive as x increases, so the stationary point is a minimum turning point.

3 The gradient function has a turning point at $x = 1$ so the graph of the antiderivative has a point of inflection at $x = 1$.

4 No further information is provided. We cannot determine the y-values of the stationary points or the point of inflection, or any values of the axis intercepts. The graph could be translated parallel to the y-axis.

b



EXERCISE 10.5 Arc length, numerical integration and graphs of antiderivatives

PRACTISE

1 WE11 Find the length of the curve $y = \frac{x^3}{12} + \frac{1}{x}$ from $x = 1$ to $x = 6$.

2 Find the length of the curve $y = \frac{3x^4 + 4}{12x}$ from $x = 2$ to $x = 8$.

3 WE12 Express the length of the curve $y = x^2$ from $x = 0$ to $x = 3$ as a definite integral and hence find the length, giving your answer correct to 4 decimal places.

4 Express the length of the curve $y = \frac{2}{x^2}$ from $x = 1$ to $x = 4$ as a definite integral and hence find the length, giving your answer correct to 4 decimal places.

5 WE13 Given that $\frac{dy}{dx} = \sin(x^2)$, $y(0.5) = 3$:

a find a definite integral for y in terms of x

b determine the value of y correct to 4 decimal places when $x = 1$.

6 Given the differential equation $\frac{dy}{dx} = \tan\left(\frac{1}{x}\right)$, $y\left(\frac{\pi}{8}\right) = \frac{1}{2}$, find a definite integral for y in terms of x.

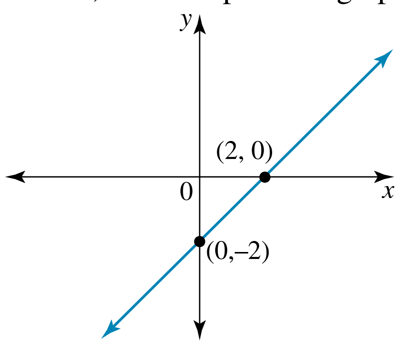
7 WE14 a Find the exact volume formed when the area bounded by the curve $y = 3\tan(2x)$, the x-axis, the origin and the line $x = \frac{\pi}{8}$ is rotated about the x-axis.

b Set up a definite integral for the volume formed when the area bounded by the curve $y = 3\tan(2x)$, the x-axis, the origin and the line $x = \frac{\pi}{8}$ is rotated about the y-axis. Hence, find the volume correct to 4 decimal places.

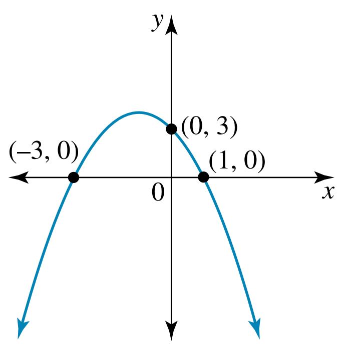
8 a Set up a definite integral for the volume formed when the area bounded by the curve $y = 3\sin^{-1}(2x)$, the x-axis, the origin and the line $x = \frac{1}{2}$ is rotated about the x-axis. Hence, find the volume correct to 4 decimal places.

b Find the exact volume formed when the area bounded by the curve $y = 3\sin^{-1}(2x)$, the x-axis, the origin and the line $x = \frac{1}{2}$ is rotated about the y-axis.

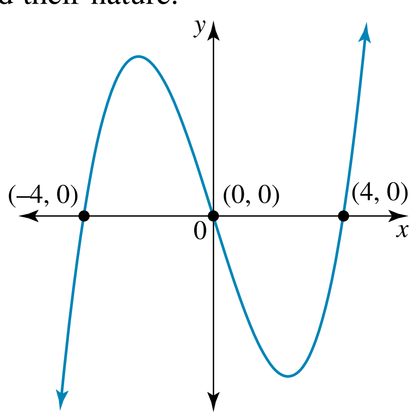
9 WE15 a Given the graph below, sketch a possible graph of the antiderivative.



b The graph of the gradient function is shown below. Sketch a possible graph of the function.



10 The graph of $y = f'(x)$ is shown. For the graph of $y = f(x)$, state the x-values of the stationary points and their nature.



CONSOLIDATE

11 a Find the length of $y = 3x + 5$ from $x = 1$ to $x = 6$.

b Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ from $x = 1$ to $x = 2$.

c Find the length of the curve $y = \frac{x^4 + 48}{24x}$ from $x = 2$ to $x = 4$.

d Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 1$.

12 a Determine the length of the curve $y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$ from $x = 0$ to $x = 1$.

b For the curve $27y^2 = 4(x - 2)^3$, find the length of the curve from $x = 3$ to $x = 8$.

c Find the length of the curve $y = \frac{2}{3}\sqrt{(x - 1)^3}$ from $x = 1$ to $x = 9$.

d Find the length of the curve $y = \frac{2}{3}\sqrt{(2x - 3)^3}$ from $x = \frac{5}{2}$ to $x = \frac{9}{2}$.

13 a Find the length of the curve $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$.

b Find the length of the curve $y = \sqrt{9 - x^2}$ from $x = 0$ to $x = 3$. What length does this represent?

14 Set up definite integrals for the lengths of the following curves, and hence determine the arc length in each case. Give your answers correct to 4 decimal places.

a $y = 3x^2 + 5$ from $x = 1$ to $x = 6$

b $y = 4\cos(2x)$ from $x = 0$ to $x = \frac{\pi}{4}$

c $y = 3e^{-2x}$ from $x = 0$ to $x = 1$

d $y = \log_e(2x + 1)$ from $x = 0$ to $x = 3$

15 a Set up a definite integral for the total length of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Find this length, giving your answer correct to 4 decimal places.

b Set up a definite integral for the total length of the curve $y = 3\cos^{-1}\left(\frac{x}{2}\right)$. Find this length correct to 4 decimal places.

16 a Find the exact volume formed when the area bounded by the curve $y = 3\cos(2x)$, the coordinate axes and $x = \frac{\pi}{4}$ is rotated about the x-axis.

b Set up a definite integral for the volume formed when the area bounded by the curve $y = 3\cos(2x)$, the coordinate axes and $x = \frac{\pi}{4}$ is rotated about the y-axis.

Determine this volume correct to 4 decimal places.

17 a Given that $\frac{dy}{dx} = e^{\frac{1}{x}}$, $y(1) = 3$:

i find a definite integral for y in terms of x

ii determine the value of y when $x = 2$.

b Given that $\frac{dy}{dx} = \sin^{-1}(x^2)$, $y(0.1) = 1$:

i find a definite integral for y in terms of x

ii determine the value of y when $x = 0.5$.

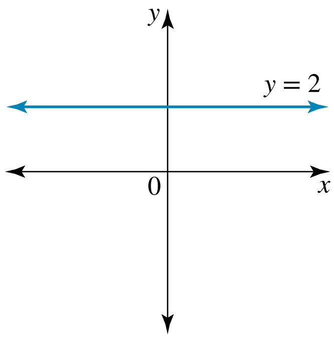
c Given that $\frac{dy}{dx} = \frac{1}{\sqrt{x^3 + 8}}$, $y(1) = \frac{1}{3}$:

i find a definite integral for y in terms of x

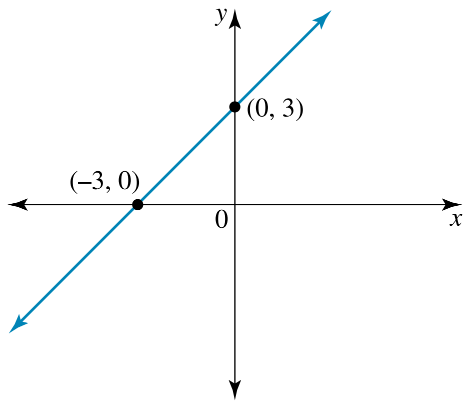
ii determine the value of y when $x = 2$.

18 The following graphs are of gradient functions. In each case sketch a possible graph of the original function.

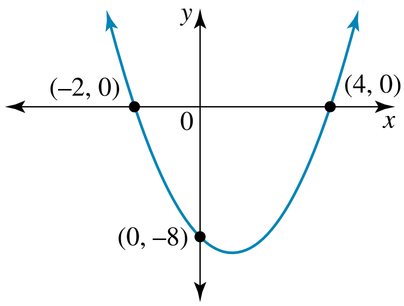
a



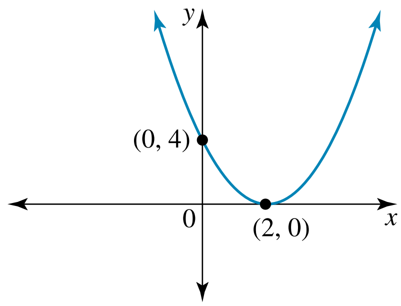
b



c



d



19 Let A be the area bounded by the graph of $y = x^2 - 6x$ and the x-axis.

a Find the value of A.

b If the area A is rotated about the x-axis, find the volume formed.

c Find the length of the curve $y = x^2 - 6x$ from $x = 0$ to $x = 6$, giving your answer correct to 4 decimal places.

d If the area A is rotated about the y-axis, find the volume formed.

20 Let A_1 be the area bounded by the graph of $y = 2\sin^{-1}\left(\frac{x}{4}\right)$, the x-axis, the origin and the line $x = 4$. Let A_2 be the area bounded by the graph of $y = 2\sin^{-1}\left(\frac{x}{4}\right)$, the y-axis, the origin and the line $y = \pi$.

a Differentiate $x\sin^{-1}\left(\frac{x}{4}\right)$ and hence find the value of A_1.

b Find the value of A_2.

c If the area A_1 is rotated about the x-axis, find the volume formed, giving your answer correct to 4 decimal places.

d If the area A_2 is rotated about the y-axis, find the exact volume formed.

e If the area A_1 is rotated about the y-axis, find the exact volume formed.

f If the area A_2 is rotated about the x-axis, find the volume formed, giving your answer correct to 4 decimal places.

MASTER

21 **a** **i** Write a definite integral which gives the length of the curve $y = x^n$ from $x = a$ to $x = b$.

ii Hence, find the length of the curve $y = \sqrt{x}$ from $x = 1$ to $x = 9$ correct to 4 decimal places.

b **i** Write a definite integral which gives the length of the curve $y = \sin(kx)$ from $x = a$ to $x = b$.

ii Hence, find the length of the curve $y = \sin(2x)$ from $x = 0$ to $x = \pi$ correct to 4 decimal places.

c **i** Write a definite integral which gives the length of the curve $y = e^{kx}$ from $x = a$ to $x = b$.

ii Hence, find the length of the curve $y = e^{2x}$ from $x = 0$ to $x = 1$ correct to 4 decimal places.

22 **a** For the line $y = mx + c$, verify that the arc length formula gives the distance along the line between the points $x = a$ and $x = b$.

b If a and p are positive real constants, show that the length of the curve $y = ax^3 + \dfrac{1}{12ax}$ from $x = 1$ to $x = p$ is given by

$$\frac{(p-1)(12a^2p(p^2+p+1)+1)}{12ap}.$$

c Prove that the circumference of a circle of radius r is $2\pi r$.

d Show that the total length of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by the definite integral

$$s = \frac{4}{a}\int_0^a \sqrt{\frac{a^4 + x^2(b^2 - a^2)}{a^2 - x^2}}\,dx.$$

10.6 Water flow

Torricelli's theorem

Evangelista Torricelli (1608–1647) was an Italian scientist interested in mathematics and physics. He invented the barometer to measure atmospheric pressure and was also one of the first to correctly describe what causes the wind. He also designed telescopes and microscopes. Modern weather forecasting owes much to the work of Torricelli. His main achievement is the theorem named after him, Torricelli's theorem, which describes the relationship between fluid leaving a container through a small hole and the height of the fluid in the container. Basically, the theorem states that the rate at which the volume of fluid leaves the container is proportional to the square root of the height of the fluid in the tank. This theorem applies for all types of containers.

Problem solving

In solving problems involving fluid flow, we use the techniques of finding volumes and use related rate problems to set up and solve differential equations. Sometimes we use numerical methods to evaluate a definite integral.

WORKED EXAMPLE 16

A vase has a circular base and top with radii of 4 cm and 9 cm respectively, and a height of 16 cm. The origin, O, is at the centre of the base. The vase is formed when the curve $y = a\sqrt{x} + b$ is rotated about the y-axis. Initially the vase is filled with water, but the water leaks out at a rate equal to $4\sqrt{h}$ cm^3/min, where h cm is the height of the water remaining in the vase after t minutes. Set up the differential equation for h and t, and determine how much time it takes for the vase to become empty.

THINK

1 Set up simultaneous equations which can be solved for a and b. The height of the vase is 16 and the height of the water in the vase is h, so $0 \leq h \leq 16$. Note all dimensions are in centimetres.

2 Determine the values of a and b.

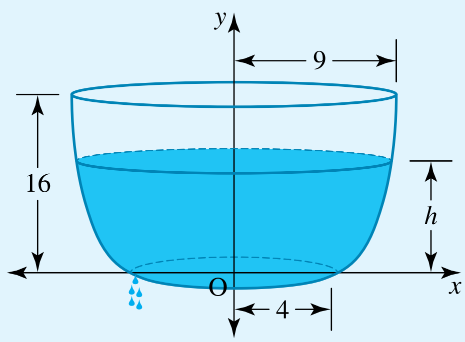
3 Determine the volume of the vase.

4 Rearrange the equation to make x the subject.

5 Determine a definite integral for the volume of water when the vase is filled to a height of h cm.

6 Determine the given rates in terms of time, t. The rate is negative as it is a decreasing rate.

WRITE/DRAW



The curve $y = a\sqrt{x} + b$ passes through the points (4, 0) and (9, 16). Substituting:

$(4, 0) \quad \Rightarrow \quad 0 = 2a + b \qquad (1)$

$(9, 16) \quad \Rightarrow \quad 16 = 3a + b \qquad (2)$

$(2) - (1) \quad \Rightarrow \quad a = 16$, so $b = -32$.

The vase is formed when $y = 16\sqrt{x} - 32$ for $4 \leq x \leq 9$ is rotated about the y-axis.

When a curve is rotated about the y-axis, the volume is $V = \pi \int_0^h x^2 dy$.

$$y = 16\sqrt{x} - 32$$

$$16\sqrt{x} = y + 32$$

$$\sqrt{x} = \frac{1}{16}(y + 32)$$

$$x = \frac{1}{256}(y + 32)^2$$

$$x^2 = \frac{(y + 32)^4}{65\,536}$$

$$V = \pi \int_0^h \frac{(y + 32)^4}{65\,536} dy$$

Since the water leaks out at a rate proportional to the square root of the remaining height of the water, $\frac{dV}{dt} = -4\sqrt{h}$.

7 Note the result used is from the numerical techniques described earlier.

$$V = \pi \int_0^h \frac{(y+32)^4}{65\,536} dy$$

$$\frac{dV}{dh} = \frac{\pi(h+32)^4}{65\,536}$$

8 Use related rates and a chain rule.

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$

9 Set up the differential equation for the height h at time t.

Substitute for the rates and use $\frac{dh}{dV} = 1 \,/\, \frac{dV}{dh}$:

$$\frac{dh}{dt} = \frac{-65\,536 \times 4\sqrt{h}}{\pi(h+32)^4}$$

10 To solve this type of differential equation, invert both sides.

$$\frac{dt}{dh} = \frac{-\pi(h+32)^4}{262\,144\sqrt{h}}$$

11 Set up a definite integral for the time for the vase to empty.

$$t = \int_{16}^{0} \frac{-\pi(h+32)^4}{262\,144\sqrt{h}} dh$$

Note the order of the terminals from $h = 16$ to $h = 0$.

12 Use a calculator to numerically evaluate the definite integral.

$t = 205.59$; note that the time is positive.

13 State the result.

The tank is empty after a total time of 205.59 minutes.

WORKED EXAMPLE 17

Another vase is formed when part of the curve $\frac{x^2}{16} - \frac{y^2}{500} = 1$ for $4 \le x \le 6$, $y \ge 0$, is rotated about the y-axis to form a volume of revolution. The x- and y-coordinates are measured in centimetres. The vase has a small crack in the base, and the water leaks out at a rate proportional to the square root of the remaining height of the water. Initially the vase was full, and after 10 minutes the height of the water in the vase is 16 cm. Find how much longer it will be before the vase is empty.

THINK | **WRITE/DRAW**

1 The vase is formed when the given curve is rotated about the y-axis.

$$V = \pi \int_a^b x^2 dy$$

2 Determine the height of the vase.

Find the value of y when $x = 6$.

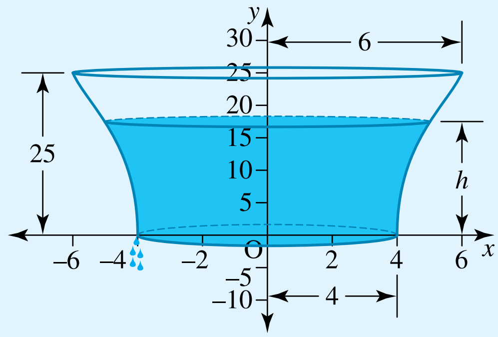
$$\frac{y^2}{500} = \frac{6^2}{16} - 1$$

$$= \frac{5}{4}$$

$$y^2 = 625$$

$$y = 25 \text{ as } y > 0$$

3 Sketch the region of the hyperbola which forms the vase. The height of the vase is 25 cm and the height of the water in the vase is h, so $0 \leq h \leq 25$.



4 Transpose the equation to make x^2 the subject.

$$\frac{x^2}{16} - \frac{y^2}{500} = 1$$

$$\frac{x^2}{16} = 1 + \frac{y^2}{500}$$

$$x^2 = \frac{16(500 + y^2)}{500}$$

$$x^2 = \frac{4(500 + y^2)}{125}$$

5 Find a definite integral for the volume of water when the vase is filled to a height of h cm, where $0 \leq h \leq 25$.

$$V = \frac{4\pi}{125}\int_0^h (500 + y^2)\,dy$$

6 Note the result used is from the numerical techniques described earlier.

$$\frac{dV}{dh} = \frac{4\pi(500 + h^2)}{125}$$

7 Determine the given rates in terms of time t in minutes.

Since the water leaks out at a rate proportional to the square root of the remaining height of the water, $\frac{dV}{dt} = -k\sqrt{h}$, where k is a positive constant.

8 Use related rates and a chain rule.

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$

9 Set up the differential equation for the height h at time t.

Substitute for the related rates, using $\frac{dh}{dV} = 1 \Big/ \frac{dV}{dh}$

$$\frac{dh}{dt} = \frac{-125k\sqrt{h}}{4\pi(h^2 + 500)}$$

10 Incorporate the constants into one constant. To solve this type 2 differential equation, invert both sides.

$$\frac{dt}{dh} = \frac{-A(h^2 + 500)}{\sqrt{h}} \text{ where } A = \frac{4\pi}{125k}$$

$$\frac{dt}{dh} = -A\left(h^{\frac{3}{2}} + 500h^{-\frac{1}{2}}\right)$$

11 Integrate with respect to h.

$$t = -A\int\left(h^{\frac{3}{2}} + 500h^{-\frac{1}{2}}\right)dh$$

12 Perform the integration.

$t = -A\left[\frac{2}{5}h^{\frac{5}{2}} + 1000h^{\frac{1}{2}}\right] + c$, where c is the constant of integration.

13 Two sets of conditions are required to find the values of the two unknowns.

Initially, when $t = 0$, $h = 25$, since the vase was full. Substitute $t = 0$ and $h = 25$:

$$0 = -A\left[\frac{2}{5}(25)^{\frac{5}{2}} + 1000 \times 25^{\frac{1}{2}}\right] + c$$

14 Simplify the relationship.

$$0 = -A\left[\frac{2}{5} \times 3125 + 1000 \times 5\right] + c$$

$$c - 6250A = 0$$

$$A = \frac{c}{6250}$$

15 Find another relationship between the unknowns.

Since after 10 minutes the height of the water in the vase is 16 cm, substitute $t = 10$ and $h = 16$:

$$10 = -A\left[\frac{2}{5}(16)^{\frac{5}{2}} + 1000 \times 16^{\frac{1}{2}}\right] + c$$

16 Simplify the relationship and solve for the constant of integration.

Substitute for A:

$$10 = -\frac{c}{6250}\left[\frac{2}{5} \times 1024 + 1000 \times 4\right] + c$$

$$10 = c\left(1 - \frac{1}{6250}\left[\frac{2}{5} \times 1024 + 4000\right]\right)$$

$$c = \frac{156\,250}{4601}$$

$$c = 33.96$$

17 Determine when the vase will be empty.

Since $t = -A\left[\frac{2}{5}h^{\frac{5}{2}} + 1000h^{\frac{1}{2}}\right] + c$, the vase is empty when $h = 0$, that is at time when $t = c$.

18 State the final result.

The vase is empty after a total time of 33.96 minutes, so it takes another 23.96 minutes to empty.

EXERCISE 10.6 Water flow

PRACTISE

1 WE16 A vase has a circular base and top with radii of 4 cm and 9 cm respectively, and a height of 16 cm. The origin, O, is at the centre of the base of the vase. The vase is formed when the line $y = ax + b$ is rotated about the y-axis. Initially the vase is filled with water, but the water leaks out at a rate equal to $2\sqrt{h}$ cm^3/min, where h cm is the height of the water remaining in the vase after t minutes. Set up the differential equation for h and t, and determine how long it will be before the vase is empty.

2 A vase has a circular base and top with radii of 4 cm and 9 cm respectively, and a height of 16 cm. The origin, O, is at the centre of the base of the vase. The vase is formed when the curve $y = ax^2 + b$ is rotated about the y-axis. Initially the vase is

filled with water, but the water leaks out at a rate equal to $2\sqrt{h}$ cm^3/min, where h cm is the height of the water remaining in the vase after t minutes. Set up the differential equation for h and t, and determine how long before the vase is empty.

3 WE17 A vase is formed when part of the curve $\frac{x^2}{16} - \frac{65y^2}{4096} = 1$ for $4 \le x \le 9$, $y \ge 0$, is rotated about the y-axis to form a volume of revolution. The x- and y-coordinates are measured in centimetres. The vase has a small crack in the base, and the water leaks out at a rate proportional to the square root of the remaining height of the water. Initially the vase was full, and after 10 minutes the height of the water in the vase is 9 cm. Find how much time it will take for the vase to become empty.

4 A vase has a base radius of 4 cm, a top radius is 9 cm, and a height of 16 cm. The origin, O, is at the centre of the base of the vase. Initially the vase is filled with water, but the water leaks out at a rate proportional to the square root of the remaining height of the water. The side of the vase is modelled by a quadratic $y = ax^2 + b$. Initially the vase was full, and after 10 minutes the height of the water in the vase is 9 cm. Find how much time it will take for the vase to become empty.

CONSOLIDATE

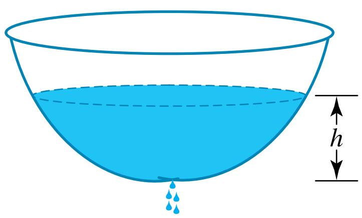
5 a A cylindrical coffee pot has a base radius of 10 cm, a height of 49 cm and is initially filled with hot coffee. Coffee is removed from the pot at a rate equal to $2\sqrt{h}$ cm^3/sec, where h cm is the height of the coffee remaining in the coffee pot after t seconds. Set up the differential equation for h and t, and determine how long it will be before the coffee pot is empty.

b A small cylindrical teapot with a base radius of 5 cm and a height of 16 cm is initially filled with hot water. The hot water is removed from the teapot at a rate proportional to $\sqrt{h}$ cm^3/min, where h cm is the height of the hot water remaining in the teapot after t minutes. Set up the differential equation for h and t. If after 10 minutes the height of the hot water is 9 cm, what further time elapses before the teapot is empty?

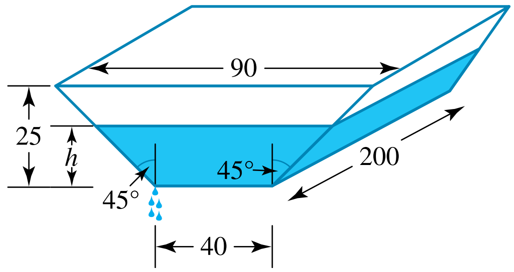
6 a A rectangular bathtub has a length of 1.5 metres and is 0.6 metres wide. It is filled with water to a height of 1 metre. When the plug is pulled, the water flows out of the bath at a rate equal to $2\sqrt{h}$ m^3/min, where h is the height of the water in metres in the bathtub at a time t minutes after the plug is pulled. How long will it take for the bathtub to empty?

b A cylindrical hot water tank with a capacity of 160 litres is 169 cm tall and is filled with hot water. Hot water starts leaking out through a crack in the bottom of the tank at a rate equal to $k\sqrt{h}$ cm^3/min, where h cm is the depth of water remaining in the tank after t minutes. If the tank is empty after 90 minutes, find the value of k.

7 a The volume of a hemispherical bowl is given by $\frac{\pi h^2}{3}(30 - h)$ cm^3, where h cm is the depth of the water in the bowl. Initially the bowl has water to a depth of 9 cm. The water starts leaking out through a small hole in the bowl at a rate equal to $k\sqrt{h}$ cm^3/min. After 1314 minutes the bowl is empty. Find the value of k.



b A drinking trough has a length of 2 m. Its cross-sectional face is in the shape of a trapezium with a height of 25 cm and with lengths 40 cm and 90 cm. Both sloping edges are at an angle of 45° to the vertical, as shown in the diagram.



The trough contains water to a height of h cm. The water leaks out through a crack in the base of the trough at a rate proportional to $\sqrt{h}$ cm^3/min. Initially the trough is full, and after 20 minutes the height of the water in the trough is 16 cm. How long will it be before the trough is empty?

8 a A plastic coffee cup has a base diameter of 5 cm, a top diameter of 8 cm and a height of 9 cm. Initially the cup is filled with coffee. However, the coffee leaks at a rate equal to $k\sqrt{h}$ cm^3/min, where h cm is the height of the coffee in the cup. If the cup is empty after 3 minutes, determine the value of k.

b A plastic bucket has a base diameter of 20 cm, a top diameter of 26 cm and a height of 24 cm. The side of the bucket is straight. Initially the bucket is filled with water to a height of 16 cm. However, there is a small hole in the bucket, and the water leaks out at a rate proportional to $\sqrt{h}$ cm^3/min, where h cm is the height of the water in the bucket. When the height of the water in the bucket is 16 cm, the height is decreasing at a rate of 0.1 cm/min. Determine how long it will be before the bucket is empty.

9 a A conical vessel with its vertex downwards has a height of 20 cm and a radius of 10 cm. Initially it contains water to a depth of 16 cm. Water starts flowing out through a hole in the vertex at a rate proportional to the square root of the remaining depth of the water in the vessel. If after 10 minutes the depth is 9 cm, how much longer will it be before the vessel is empty?

b A conical funnel with its vertex downwards has a height of 25 cm and a radius of 20 cm. Initially the funnel is filled with oil. The oil flows out through a hole in the vertex at a rate equal to $k\sqrt{h}$ cm^3/s, where h cm is the height of the oil in the funnel. If the funnel is empty after 40 seconds, determine the value of k.

10 When filled to a depth of h metres, a fountain contains V litres of water, where $V = 500\left(h^2 - \frac{h^4}{4}\right)$.

Initially the fountain is empty. Water is pumped into the fountain at a rate of 300 litres per hour and spills out at a rate equal to $2\sqrt{h}$ litres per hour.

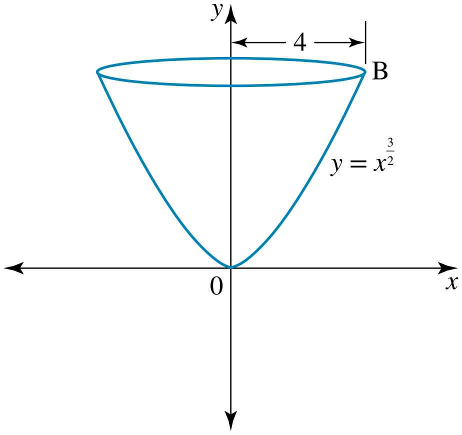
a Find the rate in metres per hour at which the water level is rising when the depth is 0.5 metres.

b The fountain is considered full when the height of the water in the fountain is 1 metre. How long does it take to fill the fountain?

c When the fountain is full, water is no longer pumped into the fountain, but water still spills out at the same rate. How long will it be before the fountain is empty again?

11 a A wine glass is formed when the arc OB with the equation $y = x^{\frac{3}{2}}$ is rotated about the y-axis. The dimensions of the glass are given in centimetres.

The wine leaks out through a crack in the base of the glass at a rate proportional to $\sqrt{h}$ cm^3/min. Initially the glass is full, and after 3 minutes the height of the wine in the glass is 1 cm. What further time elapses before the wine glass is empty?



b A large beer glass is formed when a portion of the curve with the equation $y = x^{\frac{4}{3}}$ is rotated about the y-axis between the origin and $y = 16$. The dimensions of this glass are given in centimetres. Beer leaks out through a crack in the base at a rate proportional to $\sqrt{h}$ cm^3/min. Initially the glass is full, and after 3 minutes the height of the beer in this glass is 12 cm. How much longer will it be before the glass is empty?

12 a An ornamental vase has a circular top and base, both with radii of 4 centimetres, and a height of 16 cm. The origin, O, is at the centre of the base of the vase, and the vase is formed when the hyperbola

$$\frac{25x^2}{144} - \frac{(y-8)^2}{36} = 1$$

is rotated about the y-axis, with dimensions in centimetres. Initially the vase is filled with water, but the water leaks out at a rate equal to $2\sqrt{h}$ cm^3/s, where h cm is the height of the water in the vase. Determine the time taken for the vase to empty and the capacity of the vase.

b A different ornamental vase has a circular top and base, both with radii of 3.6 cm, and a height of 16 cm. The origin, O, is at the centre of the base of the vase, and the vase is formed when the ellipse

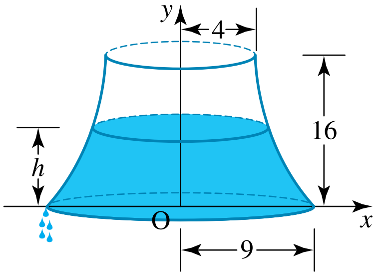
$$\frac{x^2}{36} + \frac{(y-8)^2}{100} = 1$$

is rotated about the y-axis, with dimensions in centimetres. Initially this vase is filled with water, but the water leaks out at a rate equal to $2\sqrt{h}$ cm^3/s, where h cm is the height of the water in the vase. Determine the time taken for this vase to empty and the capacity of the vase.

13 a A cylindrical vessel is initially full of water. Water starts flowing out through a hole in the bottom of the vessel at a rate proportional to the square root of the remaining depth of the water. After a time of T, the depth of the water is half its initial height. Show that the vessel is empty after a time of $\frac{2T}{2-\sqrt{2}}$.

b A conical tank is initially full of water. Water starts flowing out through a hole in the bottom of the tank at a rate proportional to the square root of the remaining depth of the water. After a time of T, the depth of the water is half its initial height. Show that the tank is empty after a time of $\dfrac{T}{1 - \sqrt{2^{-5}}}$.

14 The diagram at right shows a vase. The base and the top are circular with radii of 9 cm and 4 cm respectively, and the height is 16 cm. The origin, O, is at the centre of the base of the vase, with the coordinate axes as shown.



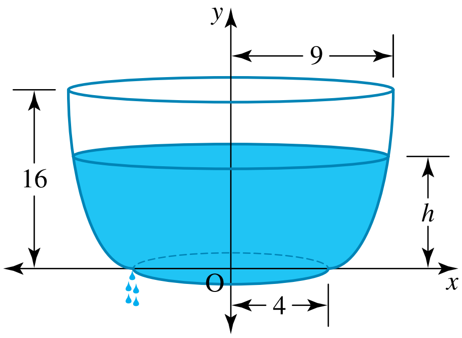
Initially the vase is filled with water, but the water leaks out at a rate equal to $2\sqrt{h}$ cm^3/s, where h cm is the height of the water in the vase. Determine the time taken for the vase to empty and the capacity of the vase if the side of the vase is modelled by:

a a hyperbola, $y = \dfrac{a}{x} + b$

b a truncus, $y = \dfrac{a}{x^2} + b$.

MASTER

15 A vase has a base radius of 4 cm, a top radius of 9 cm and a height of 16 cm. The origin, O, is at the centre of the base of the vase, and the vase can be represented by a curve rotated about the y-axis. Initially the vase is filled with water, but the water leaks out at a rate equal to $2\sqrt{h}$ cm^3/s, where h cm is the height of the water in the vase.



Determine the time taken for the vase to empty and the capacity of the vase if the side of the vase is modelled by:

a a cubic of the form $y = ax^3 + b$

b a quartic of the form $y = ax^4 + b$.

16 A hot water tank has a capacity of 160 litres and initially contains 100 litres of water. Water flows into the tank at a rate of $12\sqrt{t}\sin^2\left(\dfrac{\pi t}{4}\right)$ litres per hour over the time $0 \le t \le T$ hours, where $4 < T < 8$. During the time interval $0 \le t \le 4$, water flows out of the tank at a rate of $5\sqrt{t}$ litres per hour.

a After 3 hours, is the water level in the tank increasing or decreasing?

b After 4 hours, is the water level in the tank increasing or decreasing?

c At what time is the water level 100 litres?

d After 4 hours, find the volume of water in the tank.

e After 4 hours, no water flows out of the tank. However, the inflow continues at the same rate until the tank is full. Find the time, T, required to fill the tank.

eBook*plus*
ONLINE ONLY

10.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

10 Answers

EXERCISE 10.2

1 $\frac{1}{27}\sqrt{(9x^2+16)^3}+c$

2 $-\frac{1}{15}\cos^5(3x)+c$

3 π

4 $\dfrac{\pi}{24}$

5 $\frac{1}{3}x\sin(3x)+\frac{1}{9}\cos(3x)+c$

6 $-\frac{1}{4}(2x+1)e^{-2x}+c$

7 $x\sin^{-1}(5x)+\frac{1}{5}\sqrt{1-25x^2}+c$

8 $\frac{1}{16}(\pi-\log_e(4))$

9 a $2x\sin\left(\dfrac{x}{2}\right)+4\cos\left(\dfrac{x}{2}\right)+c$

b $\dfrac{1}{4}\sin(2x)-\dfrac{x}{2}\cos(2x)+c$

c $-(2x+4)e^{-\frac{x}{2}}+c$

10 a $x\sin^{-1}\left(\dfrac{x}{3}\right)+\sqrt{9-x^2}+c$

b $x\cos^{-1}(4x)-\frac{1}{4}\sqrt{1-16x^2}+c$

c $x\tan^{-1}\left(\dfrac{x}{5}\right)-\dfrac{5}{2}\log_e(x^2+25)+c$

11 a $4x\cos(4x)+\sin(4x),\ \frac{1}{32}(\pi-2)$

b $-6x\sin(6x)+\cos(6x),\ \frac{1}{72}$

c $\dfrac{x}{3}e^{\frac{x}{3}}+e^{\frac{x}{3}}$, 9 units^2

12 a $\dfrac{\pi}{24}$ **b** $\dfrac{\pi}{18}$ **c** $\dfrac{\pi}{6}$

13 a $\cos^{-1}(2x)-\dfrac{2x}{\sqrt{1-4x^2}},\ \dfrac{1}{2}$ units^2

b $\sin^{-1}(3x)+\dfrac{3x}{\sqrt{1-9x^2}},\ \dfrac{1}{6}(\pi-2)$ units^2

c $\tan^{-1}\left(\dfrac{x}{4}\right)+\dfrac{4x}{x^2+16},\ \pi-\log_e(4)$ units^2

14 a $e^{ax}(ax+1),\ \dfrac{1}{a^2}e^{ax}(ax-1)+c$

b $ax\cos(ax)+\sin(ax),\ \dfrac{1}{a^2}(ax\sin(ax)+\cos(ax))+c$

c $\cos(ax)-ax\sin(ax),\ \dfrac{1}{a^2}(\sin(ax)-ax\cos(ax))+c$

15 a $\frac{1}{256}(\pi^2+4\pi-8)$

b $\frac{1}{1296}(4\pi^2-6\pi\sqrt{3}+27)$

c $\frac{5}{2}\log_e(5)-2$

16 a $\sin^{-1}\left(\dfrac{x}{a}\right)+\dfrac{x}{\sqrt{a^2-x^2}},\ x\sin^{-1}\left(\dfrac{x}{a}\right)+\sqrt{a^2-x^2}+c$

b $\cos^{-1}\left(\dfrac{x}{a}\right)-\dfrac{x}{\sqrt{a^2-x^2}},\ x\cos^{-1}\left(\dfrac{x}{a}\right)-\sqrt{a^2-x^2}+c$

c $\tan^{-1}\left(\dfrac{x}{a}\right)+\dfrac{ax}{a^2+x^2}\ x\tan^{-1}\left(\dfrac{x}{a}\right)-\dfrac{a}{2}\log_e(x^2+a^2)+c$

17 a $x(\log_e(4x)-1)+c$

b $\dfrac{x^2}{4}(2\log_e(4x)-1)+c$

c $\dfrac{x^3}{9}(3\log_e(4x)-1)+c$

d $\dfrac{x^{n+1}}{(n+1)^2}((n+1)\log_e(4x)-1)+c,\ n\neq-1$

If $n=-1$, the result is $\dfrac{1}{2}(\log_e(4x))^2+c$.

18 a $13e^{2x}\cos(3x),\ \dfrac{e^{2x}}{13}(2\cos(3x)+3\sin(3x))+c$

b $-\dfrac{e^{-3x}}{13}(2\cos(2x)+3\sin(2x))+c$

c $(a^2+b^2)e^{ax}\cos(bx)$,

$\dfrac{e^{ax}}{a^2+b^2}(a\cos(bx)+b\sin(bx))+c$

d $(a^2+b^2)e^{ax}\sin(bx),\ \frac{3}{13}\left(1+e^{-\frac{2\pi}{3}}\right)$ units^2

19 a $\dfrac{1}{2}(x^2+a^2)\tan^{-1}\left(\dfrac{x}{a}\right)-\dfrac{ax}{2}+c$

b Check with your teacher.

20 a $\sec(x),\ \log_e(1+\sqrt{2})$ units^2

b $-a\,\mathrm{cosec}(ax),\ \log_e(\sqrt{3})$ units^2

c $\frac{1}{4}\log_e(\sqrt{3}+2)$ units^2

EXERCISE 10.3

All answers are in cubic units unless otherwise stated.

1 8π

2 $\dfrac{512\pi}{15}$

3 $\dfrac{32\pi}{5}$

4 8π

5 10 litres

6 350π cm^3

7 a 375π

b $\frac{125\pi}{27}$

c i 4π

ii 6π

8 a $\frac{9\pi^2}{4}$ **b** $\frac{4\pi^2}{3}$

c $\frac{\pi}{2}$ **d** $4\pi(e^2 - 1)$

9 a i $153\frac{3}{5}\pi$ **ii** 6π

b i 36 **ii** $259\frac{1}{5}\pi$ **iii** $40\frac{1}{2}\pi$

10 a i $106\frac{2}{3}\pi$ **ii** $133\frac{1}{3}\pi$

b i $\frac{4}{3}\pi ab^2$

ii $\frac{4}{3}\pi a^2 b$

iii Check with your teacher.

c $18\,392\pi$ mL^3

11 a i 6.96 m^3 **ii** 3.0 m

b 0.24π m^3

12 a i Check with your teacher. **ii** 136 L

b 172.34 mL

13 a i 3099.70 cm^3 **ii** 3141.59 cm^3

iii 3183.03 cm^3 **iv** 3223.69 cm^3

b $\frac{2720\pi}{9}$

14 a i $A = \frac{1}{a}(1 - e^{-na})$

ii $V = \frac{\pi}{2a}(1 - e^{-2na})$

iii $\frac{1}{a}, \frac{\pi}{2a}$

b i $A = \log_e(a),\ V = \pi\left(1 - \frac{1}{a}\right)$

ii Does not exist, π

15 a $\frac{\pi^2 a^2}{2n}$ **b** $\frac{\pi^2 a^2}{4n}$

16 a Check with your teacher.

b i Check with your teacher.

ii $\frac{5\pi r^3}{24}$

17 a $32\pi(3 - 4\log_e(2))$

b $\frac{64\pi}{15}$

18 $\frac{27\,776\pi}{3}$

EXERCISE 10.4

All answers are in cubic units unless otherwise stated.

1 a $\frac{704\pi}{5}$ **b** 8π

2 a 8π **b** $\frac{32\pi}{5}$

3 a $\frac{49\pi}{2}$ **b** $\frac{76\pi}{5}$

4 a $\frac{128\pi}{5}$ **b** 8π

5 a 1296π **b** 81π

6 $\frac{512\pi}{3}$

7 a $\frac{4352\pi}{15}$ **b** 8π

8 a $\frac{81\pi}{2}$ **b** $\frac{383\pi}{15}$

9 a $\frac{486\pi}{5}$ **b** $\frac{2187\pi}{7}$

10 a $\frac{416\pi}{15}$ **b** 4π

11 a $4\pi^2$ **b** $\frac{9\pi^2}{2}$

12 a i $\frac{64\pi}{15}$ **ii** $\frac{8\pi}{3}$

b i $\frac{2\pi}{35}$ **ii** $\frac{\pi}{10}$

13 a $\frac{96\pi}{5}$ **b** $\frac{3\pi a^3}{10}$

14 a 72π **b** $\frac{32\pi}{3}$

15 a i $\frac{16\,384\pi}{3}$ **ii** 256π

b $\frac{192\pi}{5}$

16 a $32\pi\sqrt{3}$ cm^3

b Check with your teacher.

17 a 256π **b** $2\pi^2 a r^2$

18 a i $\frac{11\sqrt{3}\pi a^5}{40}$ **ii** $\frac{9\pi a^4}{32}$

b i $\frac{9\pi a^2}{2}$ **ii** $\frac{38\pi a^5}{15}$

c i $\frac{16\pi a^5}{3}$ **ii** πa^4

EXERCISE 10.5

1 $\frac{75}{4}$ units

2 $\frac{1009}{8}$ units

3 9.7471 units

4 3.9140 units

5 a $\int_{\frac{1}{2}}^{x} \sin(t^2)\,dt + 3$

b 3.2688

6 $\displaystyle\int_{\frac{\pi}{8}}^{x} \tan\left(\frac{1}{t}\right) dt + \frac{1}{2}$

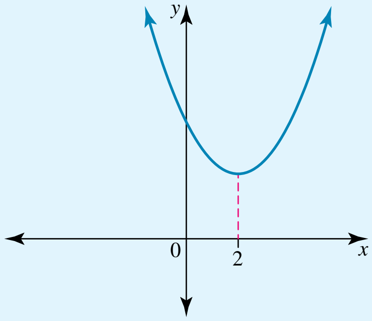
7 a $\dfrac{9\pi(4-\pi)}{8}$ cubic units

b 0.8755 cubic units

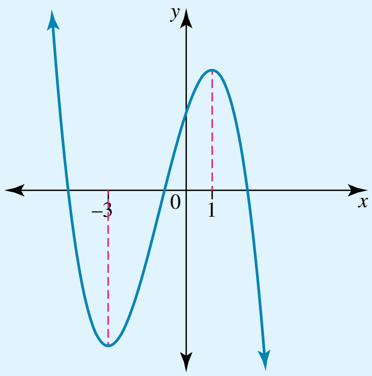
8 a 6.6077 cubic units

b $\dfrac{3\pi^2}{16}$ cubic units

9 a $x = 2$ is a minimum turning point.



b $x = -3$ is a minimum turning point,
$x = 1$ is a maximum turning point,
$x = -1$ is an inflection point.



10 $x = \pm 4$ are minimum turning points,
$x = 0$ is a maximum turning point.

11 a $5\sqrt{10}$ units **b** $\frac{17}{12}$ units

c $\frac{17}{6}$ units **d** $\dfrac{1}{2}\left(e - \dfrac{1}{e}\right)$ units

12 a $\frac{13}{6}$ units **b** $\frac{38\sqrt{3}}{9}$ units

c $\frac{52}{3}$ units **d** $\frac{49}{6}$ units

13 a $\dfrac{\pi}{2}$ units

b $\dfrac{3\pi}{2}$, one-quarter of the circumference of a circle of radius 3

14 a 105.1490 units **b** 4.1238 units

c 2.8323 units **d** 3.6837 units

15 a 15.8654 units **b** 10.3978 units

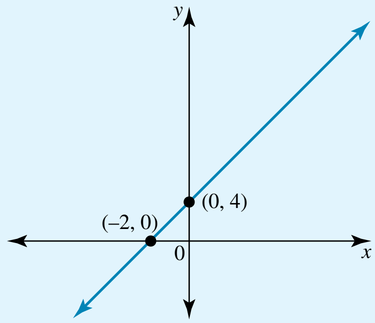
16 a $\dfrac{9\pi^2}{8}$ cubic units **b** 2.6898 cubic units

17 a i $\displaystyle\int_{1}^{x} e^{\frac{1}{2}}\, dt + 3$ **ii** 5.020

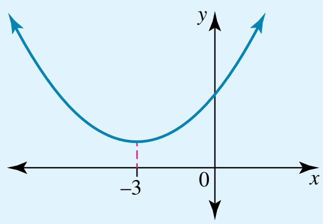
b i $\displaystyle\int_{0.1}^{x} \sin^{-1}(t^2)\, dt + 1$ **ii** 1.042

c i $\displaystyle\int_{1}^{x} \frac{1}{\sqrt{t^3+8}} dt + \frac{1}{3}$ **ii** 0.628

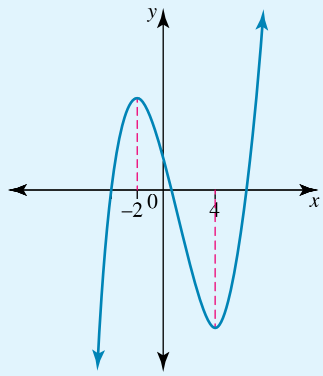
18 a A straight line with a gradient of 2



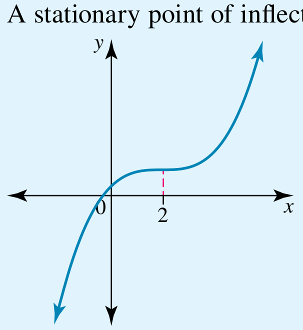
b $x = -3$ is a minimum turning point.



c A maximum turning point at $x = -2$, a minimum turning point at $x = 4$ and a point of inflection at $x = 1$



d A stationary point of inflection at $x = 2$



19 a 36 square units

b $\frac{1296\pi}{5}$ cubic units

c 19.4942 units

d 216π cubic units

20 a $\sin^{-1}\left(\frac{x}{4}\right) + \frac{x}{\sqrt{16 - x^2}}$, $4\pi - 8$ units^2

b 8 square units

c 23.4941 cubic units

d $8\pi^2$ cubic units

e $8\pi^2$ cubic units

f 100.5310 cubic units

21 a i $\int_a^b \frac{\sqrt{n^2x^{2n} + x^2}}{|x|}dx$

ii 8.2681 units

b i $\int_a^b \sqrt{1 + k^2\cos^2(kx)}\,dx$

ii 5.2704 units

c i $\int_a^b \sqrt{1 + k^2e^{2kx}}\,dx$

ii 6.4947 units

22 Check with your teacher.

EXERCISE 10.6

1 431.4 s

2 473.3 s

3 10.8 min

4 12.8 min

5 a 36.65 min b 20 min

6 a 54 s b 273.5

7 a $\frac{\pi}{5}$ b 50.25 min

8 a $\frac{92\pi}{5}$ b 253.63 s

9 a 3.11 min b 20π

10 a 0.68 m/h b 1.26 h c 261.9 h

11 a 2.07 min b 3.86 min

12 a 132.4 s, 461.1 cm^3 b 317.3 s, 1423.5 cm^3

13 Check with your teacher.

14 a 609.1 s, 1809.6 cm^3 b 560.7 s, 1625.5 cm^3

15 a 515.3 s, 2631.6 cm^3 b 555.6 s, 2802.8 cm^3

16 a Increasing b Decreasing c 0.89, 3.11 h

d 106.65 L e 7.04 h

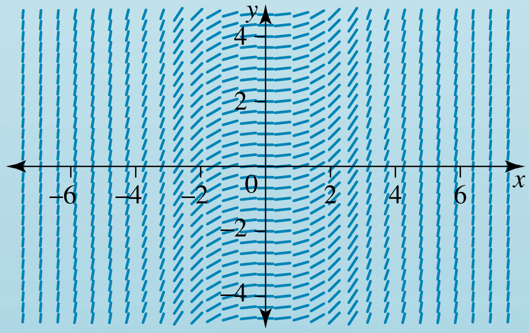
11

Applications of first-order differential equations

11.1 Kick off with CAS

1 A set of possible solutions to the differential equation $\frac{dy}{dx} = \frac{x^2}{4}$ is shown in this graph.

As $y = \frac{x^3}{12} + c$ is the general solution to this equation, the graphs are of a standard cubic with different y-intercepts. This graph is called a slope field.



Use CAS technology to match each of the following differential equations (**a**–**f**) to a slope field diagram (**i**–**vi**).

a $\frac{dy}{dx} = \cos(x)$

b $\frac{dy}{dx} = \frac{1}{y}$

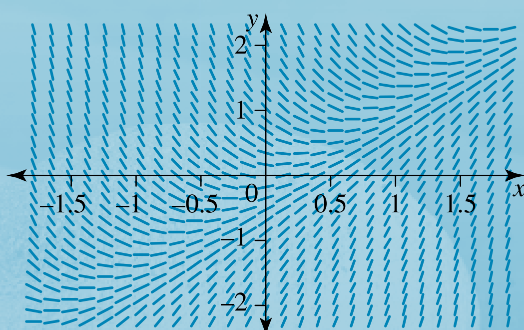
c $\frac{dy}{dx} = 3x - 2y$

d $\frac{dy}{dx} = e^{x^2}$

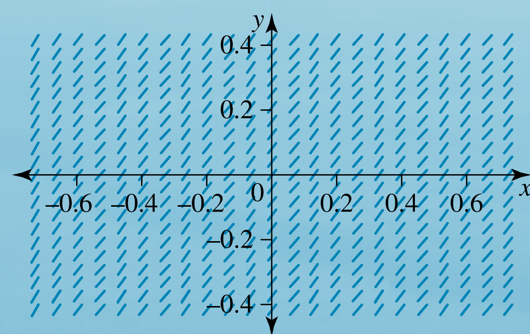
e $\frac{dy}{dx} = 3(4 - y)$

f $\frac{dy}{dx} = 2x - 1$

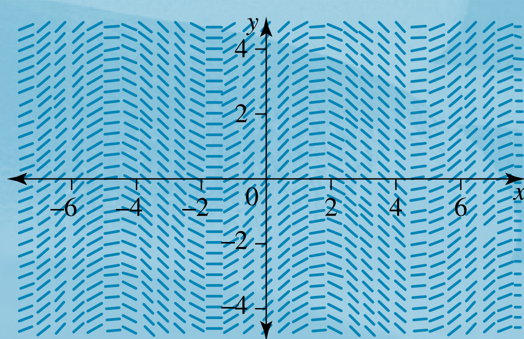
i



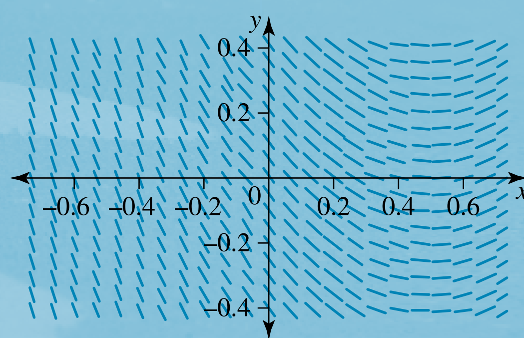
ii



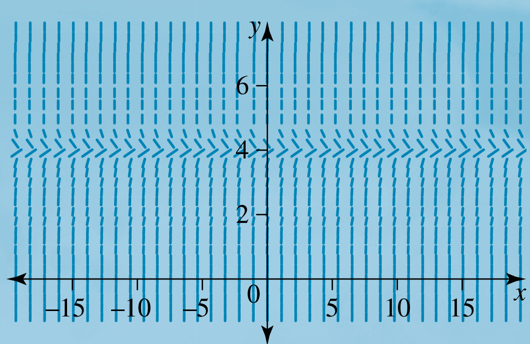
iii



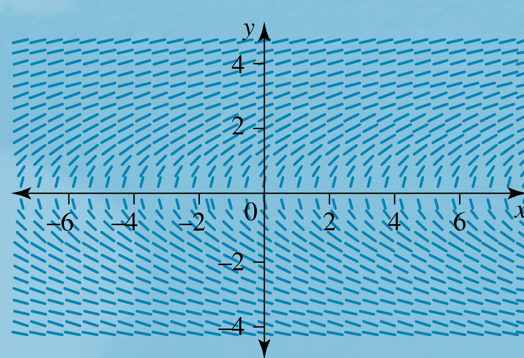
iv



v



vi



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

11.2 Growth and decay

Introduction

In Topic 10, methods of solving first- and second-order differential equations were discussed. In this topic some applications of first-order differential equations are explored. There are many applications of differential equations in business and science; in fact, they are useful whenever a rate of change needs to be considered.

studyon
Units 3 & 4
AOS 3
Topic 4
Concept 1
Modelling growth with differential equations
Concept summary
Practice questions

The law of natural growth

Consider the differential equation $\frac{dy}{dx} = ky$, where k is a constant.

To solve this equation:

Invert both sides of the equation: $\frac{dx}{dy} = \frac{1}{ky}$

$$\int \frac{dx}{dy} dy = \int \frac{1}{ky} dy$$

Separate the variables: $x = \int \frac{1}{ky} dy$

$$x = \frac{1}{k}\int \frac{1}{y} dy$$

Integrate: $x = \frac{1}{k}\log_e(y) + c$, where c is a constant of integration

Rearrange: $\log_e(y) = kx + A$, where $A = -kc$ is another constant

Set both sides to base e: $y = e^{kx+A}$

$y = Be^{kx}$, where $B = e^A$

Assume that $y(0) = y_0$, then $B = y_0$.

Therefore, the particular solution of the differential equation $\frac{dy}{dx} = ky$, $y(0) = y_0$ is given by $y = y_0 e^{kx}$. This can easily be verified using differentiation.

If $y = y_0 e^{kx}$, then

$$\begin{aligned}\frac{dy}{dx} &= ky_0 e^{kx} \\ &= k(y_0 e^{kx}) \\ &= ky\end{aligned}$$

Thus, there is a function, $y = y_0 e^{kx}$, whose derivative is proportional to itself. This result gives rise to several applications.

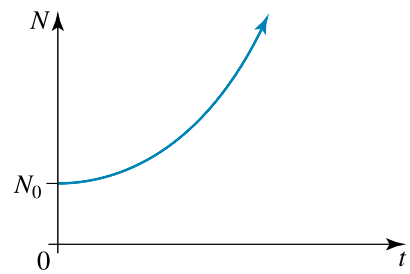
Population growth

The law of natural growth states that the rate of increase of population is proportional to the current population at that time. Let $N = N(t)$ represent the population number of a certain quantity at a time t, where $t \geq 0$. This leads to the equation $\frac{dN}{dt} \propto N$, or $\frac{dN}{dt} = kN$, where k is a positive constant.

Assuming that the initial population number is $N_0 = N(0)$, the solution of this differential equation is given by $N = N_0 e^{kt}$. This is obtained by simply substituting N for y and t for x.

This equation has been found to model population growth, although it applies over only a limited time frame. Also, it does not include factors for population-changing events such as war or famine.

The equation $N = N_0 e^{kt}$ is called the law of natural growth or exponential growth.



The constant k can be interpreted as the excess birth rate over the death rate, and is called the annual growth rate.

Note that mathematically, N is treated as a continuous variable. However, when these equations are used to model populations, N is really a discrete quantity, the number of individuals in the population. In these situations the final answer may need to be rounded to the nearest whole number.

WORKED EXAMPLE 1

The population of a certain town increases at a rate proportional to the current population. At the start of 2000 the population was 250 000, and at the start of 2008 the population was 750 000.

a **Express the population number in millions, N, in terms of t, the time in years after the start of 2000.**

b **What is the predicted population at the start of 2016?**

c **In which year does the population reach 5 million?**

THINK	WRITE
a 1 Let $N = N(t)$ represent the population number (in millions) of the town after 2000.	**a** $\frac{dN}{dt} = kN \Rightarrow N = N_0 e^{kt}$
2 In the year 2000, i.e. at $t = 0$, the population was 250 000, or one-quarter of a million. Use this to determine the value of N_0.	$N_0 = N(0) = \frac{1}{4}$
3 In the year 2008 ($t = 8$), the population was 750 000, which is three-quarters of a million. Use this to determine the value of k.	Substitute $N_0 = \frac{1}{4}$: $N(t) = \frac{1}{4}e^{kt}$ Substitute $t = 8$ and $N = \frac{3}{4}$: $\frac{3}{4} = \frac{1}{4}e^{8k}$
4 Solve the equation for k by taking the natural logarithms of both sides.	$e^{8k} = 3$ $8k = \log_e(3)$ $k = \frac{1}{8}\log_e(3)$ $k \approx 0.1373$
5 Express the population number in millions at a time t years after 2000.	$N(t) = 0.25e^{0.1373t}$

b **1** Determine the population in 2016, that is, when $t = 16$.

b Substitute $t = 16$:

$$N(10) = \frac{1}{4}e^{0.1373\times 16}$$

2 Use exact values, not the rounded numbers.

$$\begin{aligned} N(10) &= \frac{1}{4}e^{\frac{1}{8}\log_e(3)\times 16} \\ &= \frac{1}{4}e^{\log_e(9)} \\ &= \frac{1}{4}\times 9 \\ &= 2.25 \end{aligned}$$

3 State the result.

The population in 2016 is exactly 2.25 million.

c **1** Determine the year by finding the value of t when $N = 5$.

c $$5 = 0.25e^{0.1373t}$$

2 Solve the equation for t by taking the natural logarithms of both sides.

$$\begin{aligned} 20 &= e^{0.1373t} \\ 0.1373t &= \log_e(20) \\ t &= \frac{1}{0.1373}\log_e(20) \\ &\approx 21.815 \end{aligned}$$

3 State the result.

The population reaches 5 million late in the year 2021.

Radioactive decay

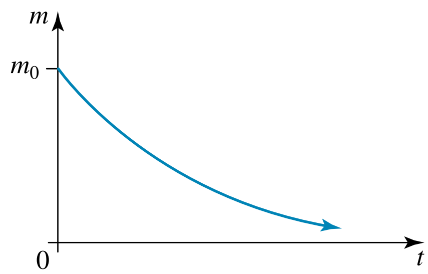
Chemical experiments have shown that when a radioactive substance decays, it changes into another element or an isotope of the same element. Scientists have determined that the rate of decay is proportional to the mass of the radioactive substance present at that time.

Let $m = m(t)$ represent the mass of the radioactive substance at a time, t, where $t \geq 0$.

Then $\frac{dm}{dt} \propto -m$, or $\frac{dm}{dt} = -km$, where $k > 0$ and is called the decay constant.

The solution of this differential equation is given by $m = m_0e^{-kt}$, where $m_0 = m(0)$.

This equation is the called the law of exponential decay.



WORKED EXAMPLE 2

The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. Initially 30 milligrams of a radioactive substance is present. After 2 hours the experimenter observes that 20% has disintegrated. Determine the amount remaining after a further 3 hours, correct to 3 decimal places.

THINK	WRITE
1 Let $m = m(t)$ represent the amount of the radioactive substance present in milligrams (mg) after a time t hours.	$\frac{dm}{dt} = -km \Rightarrow m = m_0e^{-kt}$ (1)
2 Initially, when $t = 0$, the mass is 30 mg. Use this to determine the value of m_0.	$m_0 = m(0) = 30$
3 After 2 hours, 20% has disintegrated, so 80% remains of the initial 30 mg. Use this to determine the amount left after 2 hours.	When $t = 2$, $m = 0.8 \times m_0$: $m(2) = 0.8 \times 30 = 24$
4 Substitute $m_0 = 30$, $t = 2$ and $m = 24$ into the law of exponential decay, $m = m_0e^{-kt}$, where $m_0 = m(0)$.	Substitute $m_0 = 30$, $t = 2$ and $m = 24$ into (1): $24 = 30e^{-2k}$
5 Solve the equation for k by taking the natural logarithms of both sides.	$e^{-2k} = \frac{24}{30}$ $-2k = \log_e\left(\frac{4}{5}\right)$ $k = -\frac{1}{2}\log_e\left(\frac{4}{5}\right)$ $k \approx 0.1116$
6 Express the mass remaining.	$m(t) = 30e^{-0.1116t}$
7 Determine the amount remaining after a further 3 hours, that is, when $t = 5$.	Substitute $t = 5$: $m(5) = 30e^{-0.1116\times5} \approx 17.173$
8 State the result.	17.173 mg remains after 5 hours.

studyon

Units 3 & 4

AOS 3

Topic 4

Concept 2

Modelling decay with differential equations

Concept summary
Practice questions

Half-lives

For $m = m_0e^{-kt}$ where $k > 0$, as $t \to \infty$, $m \to 0$.

An infinite time is required for all of the radioactive material to disintegrate. For this reason the rate of disintegration is often measured in terms of the **half-life** of the radioactive material. The half-life, T, is the time it takes for half of the original mass to disintegrate.

To determine the half-life, the value of T when $m = \frac{1}{2}m_0$ is required.

Substitute $m = \frac{1}{2}m_0$ into the equation $m = m_0e^{-kt}$.

This gives:

$$\frac{1}{2}m_0 = m_0e^{-kT}$$

$$\frac{1}{2} = e^{-kT}$$

$$e^{kT} = 2$$

Solving for T gives: $T = \frac{1}{k}\log_e(2)$

Notice that this half-life formula does not depend upon the initial mass m_0 and is thus independent of the time when observations began. Half-lives for radioactive substances can range from milliseconds to billions of years.

WORKED EXAMPLE 3 It takes 2 years for 30% of a particular radioactive substance to disintegrate. Determine the half-life of the substance, correct to 3 decimal places.

THINK	WRITE
1 Let $m = m(t)$ represent the amount of the radioactive substance present after time t in years.	$m = m_0 e^{-kt}$
2 Since 30% has disintegrated, 70% remains.	When $t = 2$, $m = 0.7m_0$.
3 Substitute $m = 0.7m_0$ and $t = 2$.	$0.7m_0 = m_0 e^{-2k}$ $0.7 = e^{-2k}$
4 Solve the equation for k by taking the natural logarithms of both sides.	$-2k = \log_e(0.7)$ $k = -\frac{1}{2}\log_e(0.7)$ $k \approx 0.1783$
5 Determine the half-life.	$T = \frac{1}{k}\log_e(2)$ $T = \frac{1}{0.1783}\log_e(2)$ $T = 3.887$
6 State the final result.	The half-life of the substance is 3.887 years.

EXERCISE 11.2 Growth and decay

PRACTISE

1 WE1 The population of a city increases at a rate proportional to the current population. In 2005 the population was 1.79 million, and in 2009 the population was 2 million.
   a Express the population number, N, in terms of t, the time in years after 2005.
   b What is the predicted population in 2020?
   c When does the population reach 5 million?

2 The number of frogs in a colony increases at a rate proportional to the current number. After 15 months there are 297 frogs present, and this number grows to 523 after 26 months. Determine the initial number of frogs present in the colony.

3 WE2 The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. Initially 80 milligrams of a radioactive substance is present, and after 1 hour it is observed that 10% has disintegrated. Determine the amount remaining after a further 2 hours.

4 The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. After 2 hours, 64 milligrams of a radioactive substance is present, and after 4 hours, 36 milligrams is present. Determine the initial amount of the substance present.

5 WE3 For a radioactive substance, it takes 3 years for 15% to disintegrate. Determine the half-life of the substance.

6 An isotope of radium has a half-life of 1601 years. What percentage remains after 1000 years?

CONSOLIDATE

7 a The rate of growth of a population of insects is proportional to its present size. A colony of insects initially contains 600 individuals and is found to contain 1300 after 2 weeks. Determine the number of insects after a further 3 weeks.

b Initially there are 10 000 fish in a reservoir. The number of fish in the reservoir grows continuously at a rate of 4% per year. Determine the number of fish in the reservoir after 3 years.

8 The number of possums in a certain area grows at a rate proportional to the current number. In 4 months the number of possums has increased from 521 to 678. Determine the number of possums in the area after a further 5 months.

9 The population of Australia in 2005 was approximately 20 177 000. In 2010 it had grown to 22 032 000.

a What is the predicted population in 2015?

b In what year will the population first exceed 30 million?

10 In 1999 the world population was estimated to be 6 billion. In 2011 it reached 7 billion.

a What is the estimated world population in 2017?

b In what year will the world population reach 10 billion?

11 a The rate of decay of a radioactive substance is proportional to the amount of the substance present at any time. If 1 gram of radioactive material decomposes to half a gram in 22 652 years, how much remains after 40 000 years?

b If 15% of a radioactive element disintegrates in 6 years, determine the half-life of this element.

12 Iodine-131 is present in radioactive waste and decays exponentially to form a substance that is not radioactive. The half-life of iodine-131 is 8 days. If 120 milligrams is considered a safe level, how long will it take 2 grams of iodine-131 to decay to a safe level?

13 Strontium-90 is an unpleasant radioactive isotope that is a byproduct of a nuclear explosion. Strontium-90 has a half-life of 28.9 years. How many years elapse before 60% of the strontium-90 in a sample has decayed?

14 Marie Curie (1867–1934) was a French scientist. She coined the term ‘radioactivity’ and was also the first female scientist to win a Nobel Prize. Unfortunately she died of radiation poisoning as a result of her experiments. In 1944 the element Curium was named after her.

a Curium-243 has a half-life of 29.1 years. It decays into plutonium-239 through alpha decay. Determine the percentage of curium-243 that remains in a sample after 10 years.

b Plutonium-239 is a silvery metal that is used for the production of nuclear weapons. If 3 micrograms of plutonium-239 decomposes to 1 microgram in 38 213 years, determine the half-life of plutonium-239.

15 Uranium-238 has a half-life of 4.468 billion years. Estimate the percentage of the original amount of uranium-238 that remains in the universe, assuming we began with that original amount 13.8 billion years ago.

16 A biologist is investigating two different types of bacteria. Both types grow at a rate proportional to the number present. Initially 900 type A bacteria are present, and after 10 hours 3000 are present. The number of type B bacteria doubles after 4 hours, and there are 2500 type B bacteria after 8 hours. Determine the time at which equal numbers of type A and type B bacteria are present.

MASTER

17 The table below shows the comparative population in millions of Sydney and Melbourne, in certain years.

City	2001	2009
Melbourne	3.339	3.996
Sydney	3.948	4.504

Assume the growth rates are proportional to the current population and these growth rates continue.

a Show that Melbourne's population growth rate is greater than Sydney's population growth rate.

b In what year will the population of Melbourne first exceed 8 million?

c In what year will the population of Sydney first exceed 8 million?

d In what year will the population of Melbourne exceed the population of Sydney?

18 In the 1950s, W F Libby and others at the University of Chicago devised a method of estimating the age of organic material based on the decay rate of carbon-14, a radioactive isotope of carbon. Carbon-14 dating can be used on objects ranging from a few hundred years old to 40 000 years old. Carbon-14 obeys the law of radioactive decay. If Q denotes the amount of carbon-14 present at time t years, then $\frac{dQ}{dt} = -kQ$.

a If Q_0 represents the amount of carbon-14 present at time t_0 and k is a constant, express Q in terms of Q_0, t_0 and k.

b Given that the half-life of carbon-14 is 5730 years, determine the value of k.

In residual amounts of carbon-14 it is possible to measure $\frac{Q}{Q_0}$ for some wood and plant remains, and hence determine the elapsed time since the death of these remains, that is, the period during which decay has been taking place. This technique of radiocarbon dating is of great value to archaeologists.

c For a particular specimen, $\frac{Q}{Q_0} = \frac{2}{7}$. Determine the age of the specimen.

11.3 Other applications of first-order differential equations

Miscellaneous types

There are many other applications of first-order differential equations that are similar to the growth and decay models.

Investments and money matters

When an amount of money, $\$P$, is invested with continuously compounding interest at $R\%$, the rate at which the accumulated amount grows is proportional to the amount invested. In fact, the proportionality constant is the interest, so $\frac{dP}{dt} = Pr$, where t is the time ($t \geq 0$) and $r = \frac{R}{100}$.

Pressure and height in the atmosphere

The rate of decrease of pressure in the atmosphere, P, with respect to the height above sea level, h, is proportional to the pressure at that height. For a proportionality constant k (such that $k > 0$), $\frac{dP}{dh} = -kP$.

Drug disappearance in the body

For many drugs, the rate of disappearance from the body is proportional to the amount, D, of the drug still present in the body at time t. If k is the proportionality constant ($k > 0$), $\frac{dD}{dt} = -kD$.

Light intensity and depth

When a beam of light passes through a medium, it loses its intensity as it penetrates more deeply. If I is the intensity of light at depth x, the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth. If k is the proportionality constant ($k > 0$), then $\frac{dI}{dx} = -kI$.

Other applications

There are also applications related to electrical circuits. Applications to kinematics, dynamics and variable forces are considered in Topic 13.

WORKED EXAMPLE 4

When money is invested in a bank at a constant rate of $R\%$ with continuously compounding interest, the accumulated amount $\$P$ at a time t years after the start of the investment satisfies the differential equation $\frac{dP}{dt} = Pr$.

a **Assuming an initial investment of $\$P_0$, solve the differential equation to determine the amount $\$P$ after a time t years.**

b **Determine the amount to which \$10 000 will grow in 6 years if invested at 4.5%.**

THINK	WRITE
a 1 To solve the differential equation, invert both sides. The initial investment is $\$P_0$.	**a** $\frac{dP}{dt} = Pr,\ P(0) = P_0$ $\frac{dt}{dP} = \frac{1}{Pr}$
2 Because r is a constant, it can be taken outside the integral sign.	$t = \frac{1}{r}\int \frac{1}{P}\, dP$

3 Integrate.

$$t = \frac{1}{r}\log_e\left(|P|\right) + c$$
$$= \frac{1}{r}\log_e(P) + c, \text{ as } P > 0$$

4 Substitute $t = 0$, $P = P_0$.

$$0 = \frac{1}{r}\log_e(P_0) + c$$
$$c = -\frac{1}{r}\log_e(P_0)$$

5 Rewrite the equation, substituting $c = -\frac{1}{r}\log_e(P_0)$, and use logarithm laws to simplify.

$$t = \frac{1}{r}\log_e(P) - \frac{1}{r}\log_e(P_0)$$
$$t = \frac{1}{r}\log_e\left(\frac{P}{P_0}\right)$$

6 Rearrange and write as an exponential.

$$rt = \log_e\left(\frac{P}{P_0}\right)$$
$$\frac{P}{P_0} = e^{rt}$$

7 Solve for P and state the solution of the differential equation.

$$P = P_0e^{rt}$$

b 1 Substitute $P_0 = 10\,000$ and $r = 4.5\% = \frac{4.5}{100}$.

b $P(t) = 10\,000e^{\frac{4.5t}{100}}$

2 Determine the amount after 6 years.

$$P(6) = 10\,000e^{\frac{4.5\times 6}{100}}$$
$$= 13\,099.64$$

3 State the total amount.

The principal has grown to \$13 099.64.

Other population models

In the natural growth model, the growth rate of a population is proportional to the current population. However, population models can have other growth rates. For example, the growth rate may be proportional to a power of the current population other than 1. In these models, the same approaches can be used as for natural growth.

WORKED EXAMPLE 5

The population of a certain town increases at a rate proportional to the square root of the current population. In 2000 the population was 250 000, and in 2008 the population was 640 000.

Express the population number, N, in terms of t, the time in years after 2000. What is the predicted population in 2016?

THINK	WRITE
1 Let $N = N(t)$ represent the population number of the town after 2000.	$\frac{dN}{dt} = k\sqrt{N}$, $N(0) = 250\,000$
2 To solve the differential equation, invert both sides and write in index notation.	$\frac{dt}{dN} = \frac{1}{k\sqrt{N}}$ $= \frac{1}{k}N^{-\frac{1}{2}}$

3 Multiply by the constant k and separate the variables.

$$kt = \int N^{-\frac{1}{2}} dN$$

4 Integrate.

$$kt = 2N^{\frac{1}{2}} + c$$
$$kt = 2\sqrt{N} + c$$

5 Substitute $t = 0$ and $N = 250\,000$ into the equation.

$$0 = 2\sqrt{250\,000} + c$$
$$c = -1000$$

6 Substitute $c = -1000$.

$$kt = 2\sqrt{N} - 1000$$

7 Substitute $t = 8$ and $N = 640\,000$ to solve for k.

$$8k = 2\sqrt{640\,000} - 1000$$
$$8k = 600$$
$$k = 75$$

8 Substitute $k = 75$ into the equation $kt = 2\sqrt{N} - 1000$ and rearrange to make N the subject.

$$75t = 2\sqrt{N} - 1000$$
$$2\sqrt{N} = (75t + 1000)$$
$$\sqrt{N} = \frac{1}{2}(75t + 1000)$$

9 State the particular solution of the differential equation.

$$N(t) = \frac{1}{4}(75t + 1000)^2$$

10 Substitute $t = 16$.

$$N(16) = \frac{1}{4}(75 \times 16 + 1000)^2$$
$$= 1\,210\,000$$

11 State the final result.

The predicted population in 2016 is 1 210 000.

Population models with regular removal

Another model that can limit but not bound the population is to remove some of the population at regular intervals. This has applications for situations such as people emigrating from a country or leaving a certain city yearly. The growth rate k still represents the excess of births over deaths and is often given as a percentage.

WORKED EXAMPLE 6

A farm initially has 200 sheep. The number of sheep on the farm grows at a rate of 10% per year. Each year the farmer sells 15 sheep.

a Write the differential equation modelling the number of sheep, N, on the farm after t years.

b Solve the differential equation to express the number of sheep, N, on the farm after t years.

c Hence, determine the number of sheep on the farm after 5 years.

THINK	WRITE
a 1 Let $N(t)$ be the number of sheep on the farm at time t years. The growth rate is 10%, and 15 are sold each year.	**a** $k = 0.1$ $\frac{dN}{dt} = 0.1N - 15$ $\frac{dN}{dt} = \frac{N}{10} - 15$
2 Write the differential equation in simplest form, along with the initial condition. The initial number of sheep is 200.	$\frac{dN}{dt} = \frac{N - 150}{10}, N(0) = 200$
b 1 To solve the differential equation, invert both sides.	**b** $\frac{dt}{dN} = \frac{10}{N - 150}$
2 Integrate both sides.	$t = \int \frac{10}{N - 150}\, dN$
3 Perform the integration.	$t = 10 \log_e\left(\lvert N - 150\rvert\right) + c$ $= 10 \log_e (N - 150) + c$, as $N > 150$
4 Substitute $t = 0$ and $N = 200$.	$0 = 10 \log_e (200 - 150) + c$ $c = -10 \log_e (50)$
5 Substitute $c = -10 \log_e (50)$ and use log laws to simplify.	$t = 10 \log_e (N - 150) - 10 \log_e (50)$ $t = 10[\log_e (N - 150) - \log_e (50)]$ $\frac{t}{10} = \log_e\left(\frac{N - 150}{50}\right)$
6 Rearrange and solve for N.	$\frac{N - 150}{50} = e^{\frac{t}{10}}$ $N - 150 = 50e^{\frac{t}{10}}$
7 State the solution of the differential equation.	$N = 150 + 50e^{\frac{t}{10}}$ $N = 50\left(3 + e^{\frac{t}{10}}\right)$
c 1 Substitute $t = 5$.	**c** $N(5) = 50\left(3 + e^{\frac{1}{2}}\right) \approx 232.44$
2 State the number of sheep present after 5 years. Note that we always round down, and express the answer as an integer.	There are 232 sheep present after 5 years.

EXERCISE 11.3 Other applications of first-order differential equations

PRACTISE

1 WE4 When money is invested in a bank at a constant rate of $R\%$ with continuously compounding interest, the accumulated amount $\$P$ at a time t years after the start of the investment satisfies the differential equation $\frac{dP}{dt} = P \times r$. What initial investment is required if \$10 000 is the target in 2 years' time and the interest rate is 5%?

2 The price of houses in a certain area grows at a rate proportional to their current value. In 1998 a particular house was purchased for \$315 000; in 2014 the house was sold for \$1 260 000. What was the value of the house in 2006?

3 **WE5** The population of a country increases at a rate proportional to the square root of the current population. In 2005 the population was 4 million, and in 2010 the population was 9 million. Express the population number, N, in terms of t, the time in years after 2005. What is the predicted population in 2020?

4 The number of insects in a colony increases at a rate inversely proportional to the current number. At first there are 20 insects present, and this number grows to 80 after 5 months. Determine the number of insects present in the colony after a further 11 months.

5 **WE6** A certain area initially contains 320 rabbits. The number of rabbits in the area grows at a rate of 25% per year. Each year 40 rabbits are culled to try to limit the population.

a Write the differential equation modelling the number of rabbits in the area, N, after t years.
b Solve the differential equation to express the number of rabbits in the area after t years.
c Hence, determine the number of rabbits in the area after 8 years.

6 Another area initially contains N_0 rabbits. The number of rabbits in this area grows at a rate of 20% per year. Each year K rabbits are culled to limit the population. After 10 years, there are $6K$ rabbits present. Show that $N_0 = K\left(5 + \frac{1}{e^2}\right)$.

CONSOLIDATE

7 a The rate of decrease of air pressure, P, with respect to the height above sea level, h, is proportional to the pressure at that height. The constant of proportionality is k such that $k > 0$. If P_0 is the air pressure at sea level, solve the differential equation to determine the pressure at height h.
b If the air pressure at sea level is 76 cm of mercury, and the pressure at a height of 1 kilometre above sea level is 62.2235 cm of mercury, determine the pressure at a height of 2 kilometres above sea level.

8 a For many drugs, the rate of disappearance from the body is proportional to the amount of the drug still present in the body. If $D = D(t)$ is the amount of the drug present at time t and $k > 0$ is the constant of proportionality, solve the differential equation expressing D in terms of t given the initial dosage of D_0.
b The elimination time for alcohol (the time it takes for a person's body to remove alcohol from their bloodstream) is measured as $T = \frac{1}{k}$. The process varies from person to person, so the value of k varies. For one person the elimination time is 3 hours. For this person, how long will it take the excess level of alcohol in their bloodstream to be reduced from 0.10% to the legal adult driving level of 0.05%?

9 When light passes through a medium, it loses its intensity as it penetrates deeper. If I is the intensity of light at depth x, the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth, so that $\frac{dI}{dx} = -kI$, where k is a positive constant.

a Solve the differential equation to show that $I = I_0 e^{-kx}$.

b If 5% of light is lost in penetrating 25% of a glass slab, what percentage is lost in penetrating the whole slab?

10 When a rope is wrapped around a pole, the rate of change of tension, T (in newtons), in the rope with respect to the angle, θ (in radians), is proportional to the tension at that instant. The constant of proportionality is the coefficient of friction, μ.

a Given that $\dfrac{dT}{d\theta} = \mu T$ and the tension when the angle is zero is T_0, show that $T = T_0 e^{\mu\theta}$.

b A man holds one end of a rope with a pull of 80 newtons. The rope goes halfway around a tree, and the coefficient of friction in this case is 0.3. What force can the other end sustain?

11 **a** The variation of resistance, R ohms, of a copper conductor with temperature T degrees Celsius satisfies the differential equation $\dfrac{dR}{dT} = \alpha R$.

i If the resistance of copper is R_0 at 0°C, show that $R = R_0 e^{\alpha T}$.

ii If $\alpha = 0.004$ per degree Celsius and the resistance at 60°C is 40 ohms, determine the resistance at 30°C.

b The charge, Q units, on a plate of a condenser t seconds after it starts to discharge is proportional to the charge at that instant. If the charge is 500 units after a half-second and falls to 250 units after 1 second, determine:

i the original charge

ii the time needed for the charge to fall to 125 units.

12 **a** In an electrical circuit consisting of a resistance of R ohms and an inductance of L henries, the current, i amperes, after a time t seconds satisfies the differential equation $L\dfrac{di}{dt} + Ri = 0$. The initial current is i_0. Solve the differential equation to determine the current at any time t.

b In an electrical circuit consisting of a resistance of R ohms and a capacitance of C farads, the charge, Q coulombs, at a time t seconds decays according to the differential equation $\dfrac{dQ}{dt} + \dfrac{Q}{RC} = 0$. Assuming the initial charge is Q_0, solve the differential equation to determine the charge at any time t after discharging.

13 A farm initially has 200 cows. The number of cows on the farm grows at a rate of 5% per year. Each year the farm sells 5 cows.

a Write the differential equation modelling the number of cows, N, on the farm after t years.

b Solve the differential equation to express the number of cows on the farm after t years.

c Hence, determine the number of cows on the farm after 3 years.

14 A certain country has a 4% population growth rate, and additionally every year 10 000 immigrate to the country. If in 2010 the population of the country was half a million, estimate the population in 2015.

15 Koalas on a plantation have a population growth rate of 10%. Initially 400 koalas are present.

a Every year 50 koalas are sold to zoos. How many years elapse before the number of koalas is reduced to 200?

b After the population has reached 200, what is the maximum number of koalas the managers of the plantation can sell to zoos each year so that the number of koalas at the plantation will increase above 200?

16 Solve the differential equation $\frac{dN}{dt} = kN^{\alpha}$, $N(0) = N_0$ for when:

a $\alpha = \frac{1}{2}$ b $\alpha = -1$ c $\alpha = \frac{3}{2}$ d $\alpha = 2$.

MASTER

17 a Show that the solution of the differential equation $\frac{dN}{dt} = kN - c$, where k and c are positive constants and $N(0) = N_0$, is given by $N(t) = \left(N_0 - \frac{c}{k}\right)e^{kt} + \frac{c}{k}$.

b Show that:

i N increases if $N_0 > \frac{c}{k}$ ii N decreases if $N_0 < \frac{c}{k}$

iii if $N_0 = \frac{c}{k}$, then N remains stable.

18 The number of bacteria in a culture increases at a rate proportional to the number of bacteria present in the culture at any time. By natural increase, the culture will double in $\log_e(8)$ days. If the initial number of bacteria present is N_0 and bacteria are removed from the colony at a constant rate of Q per day, show that the number of bacteria:

a will increase if $N_0 > 3Q$
b will remain stationary if $N_0 = 3Q$
c will decrease if $N_0 < 3Q$.

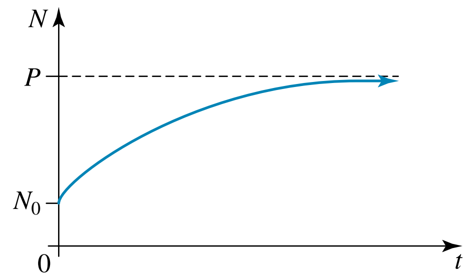
11.4 Bounded growth and Newton's Law of Cooling

Bounded growth models

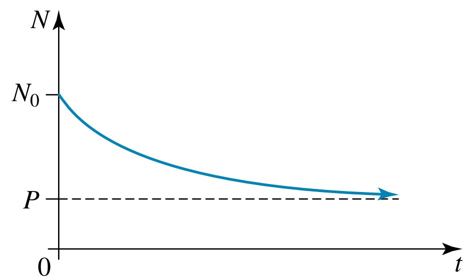
If $N = N(t)$ is the population, then the law of exponential growth states that $\frac{dN}{dt} = kN$, where k is a positive constant. Assuming that the initial population number is $N_0 = N(0)$, the solution of this differential equation is given by $N = N_0e^{kt}$. This model predicts that, as time increases, the population number increases without bounds; that is, as $t \to \infty$, $N \to \infty$. Usually the growth of a quantity is not unlimited but is instead bounded; that is, the quantity approaches some maximum value or equilibrium value, and growth cannot occur above this value. This is called bounded growth. In such cases, the growth rate is proportional to the difference between the population number and the equilibrium value.

The differential equation to model this type of behaviour is $\frac{dN}{dt} = k(P - N)$, where $k > 0$ and $P > 0$. P is called the carrying capacity or the equilibrium value.

Assuming an initial condition $N_0 = N(0)$, if $N_0 < P$, then as $t \to \infty$, $N \to P$ from below.



However, if $N_0 > P$, then as $t \to \infty$, $N \to P$ from above.



WORKED EXAMPLE 7

The weights of medium-sized dogs are found to follow a bounded growth model. The initial weight of a typical newborn puppy is 0.5 kg, and after 16 weeks its weight has increased to 4.5 kg. It is known that the dog's weight will never exceed 15 kg.

a Determine the weight of the dog after one year.

b Sketch the graph of weight versus time for this model.

THINK	WRITE/DRAW
a 1 Let $m = m(t)$ be the weight in kg of the dog after t weeks. The equilibrium value is $P = 15$ and the initial weight is 0.5 kg. State the differential equation to model the weight.	**a** $\frac{dm}{dt} = k(P - m)$ $\frac{dm}{dt} = k(15 - m),\ m(0) = 0.5$
2 To solve this differential equation, invert both sides.	$\frac{dt}{dm} = \frac{1}{k(15 - m)}$
3 Separate the variables and multiply by the constant k.	$t = \frac{1}{k}\int \frac{1}{15 - m}\,dm$ $kt = \int \frac{1}{15 - m}\,dm$
4 Perform the integration.	$kt = -\log_e(\lvert 15 - m\rvert) + c$ $kt = -\log_e(15 - m) + c$ as $0.5 < m < 15$
5 The initial mass of a newborn puppy is 0.5 kg. Use this to determine the value of c.	$m(0) = 0.5$: $0 = -\log_e(15 - 0.5) + c$ $c = \log_e(14.5)$

6 Substitute $c = \log_e(14.5)$ and simplify.

$kt = -\log_e(15 - m) + \log_e(14.5)$

$kt = \log_e\left(\frac{14.5}{15 - m}\right)$

7 After 16 weeks the mass is 4.5 kg. Use this to determine the value of k.

$m(16) = 4.5$:

$16k = \log_e\left(\frac{14.5}{15 - 4.5}\right)$

$k = \frac{1}{16}\log_e\left(\frac{29}{21}\right)$

$\approx 0.020\,17$

8 Leave the equation in terms of k and use the definition of the logarithm to rearrange.

$kt = \log_e\left(\frac{14.5}{15 - m}\right)$

$e^{kt} = \frac{14.5}{15 - m}$

9 Rearrange to make m the subject.

$(15 - m)e^{kt} = 14.5$

$15 - m = 14.5e^{-kt}$

$m = 15 - 14.5e^{-kt}$

10 State the particular solution of the differential equation, substituting $k = 0.020\,17$.

$m(t) = 15 - 14.5e^{-0.020\,17t}$

11 Now determine the weight of the dog after one year, that is $t = 52$ weeks.

$m(52) = 15 - 14.5e^{-0.020\,17 \times 52}$

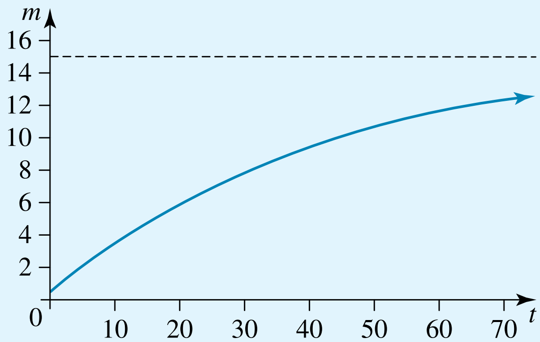
$= 9.92$

12 State the final result.

The weight of the dog after one year is 9.92 kg.

b Sketch the graph of m versus t.

b



Newton's Law of Cooling

studyon

Units 3 & 4

AOS 3

Topic 4

Concept 3

Newton's Law of Cooling
Concept summary
Practice questions

Newton's Law of Cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the surrounding medium. Let T denote the temperature of a body after a time t and T_m denote the temperature of the surrounding medium, which is assumed to be constant. Newton's Law of Cooling can be written as $\frac{dT}{dt} \propto (T - T_m)$ or $\frac{dT}{dt} = k(T - T_m)$, where k is a constant of proportionality. If the body is cooling, then $k < 0$.

To solve this differential equation, let θ be the difference in temperature, so that $\theta = T - T_m$. Because T_m is a constant, $\frac{dT}{dt} = \frac{d\theta}{dt}$; therefore, $\frac{d\theta}{dt} = k\theta$. The particular solution of the differential equation is $\theta = \theta_0 e^{kt}$, where $\theta(0) = \theta_0 = T_0 - T_m$ and T_0 is the initial temperature of the body. Newton's Law of Cooling is another example of a bounded growth or decay model.

WORKED EXAMPLE 8

A metal ball is heated to a temperature of 200°C and is then placed in a room that is maintained at a constant temperature of 30°C. After 5 minutes the temperature of the ball has dropped to 150°C. Assuming Newton's Law of Cooling applies, determine:

a the equation for the difference between the temperature of the ball and the temperature of the room

b the temperature of the ball after a further 10 minutes

c how long it will take for the temperature of the ball to reach 40°C.

THINK	WRITE
a 1 Let T denote the temperature of the ball after a time t minutes. The constant room temperature is $T_m = 30$°C.	a $\frac{dT}{dt} = k(T - T_m)$ $\frac{dT}{dt} = k(T - 30)$
2 Let θ be the difference in temperature between the ball and the room.	$\theta = T - 30$ $\frac{d\theta}{dt} = k\theta$
3 Write the solution of the particular differential equation.	$\theta = \theta_0 e^{kt}$
4 To determine the value of θ_0, use the given initial conditions.	Initially, $t = 0$ and $T_0 = 200$°C. $\theta_0 = T_0 - T_m$ $= 200 - 30$ $= 170$
5 To determine the value of k, use the other given condition.	When $t = 5$, $T = 150$°C, so $\theta(5) = 150 - 30$ $= 120$
6 Substitute in the given values.	$\theta_0 = 170$, $\theta = 120$ and $t = 5$: $120 = 170e^{5k}$
7 Rearrange and use the definition of the logarithm.	$\frac{120}{170} = e^{5k}$ $5k = \log_e\left(\frac{12}{17}\right)$
8 Solve for k.	$k = \frac{1}{5}\log_e\left(\frac{12}{17}\right)$ ≈ -0.06966
9 Write the solution of the differential equation.	$\theta(t) = 170e^{-0.06966t}$

b 1 Substitute $t = 15$ to determine the temperature after a further 10 minutes.

b $\theta(15) = 170e^{-0.06966 \times 15}$
$= 59.79$

2 State the temperature of the ball at this time.

Since $T = \theta + 30$, the temperature of the ball is 89.79°C.

c 1 To determine when the temperature of the ball reaches 40°C, use $T = 40$.

c $T = \theta + 30$
$40 = \theta + 30$
$\theta = 40 - 30$
$\theta = 10$

2 Write an equation involving the unknown time.

Substitute $\theta = 10$:
$$10 = 170e^{-0.06966t}$$

3 Solve the equation for t, using the definition of the logarithm.

$$\frac{10}{170} = e^{-0.06966t}$$
$$-0.06966t = \log_e\left(\frac{10}{170}\right)$$
$$t = -\frac{1}{0.06966}\log_e\left(\frac{1}{17}\right)$$
$$= 40.67$$

4 State the required result.

The temperature of the ball reaches 40°C after 40.67 minutes.

EXERCISE 11.4 Bounded growth and Newton's Law of Cooling

PRACTISE

1 WE7 The mass of a bird is found to follow a bounded growth model. The initial mass of a baby bird is 30 grams, and after 10 weeks its mass is 100 grams. If it is known that the mass of the bird will never exceed 200 grams, determine the mass of the bird after 30 weeks.

2 The number of birds in a flock follows a bounded growth model. Initially there are 200 birds in the flock. After 5 months the number has grown to 800. The number of birds in the flock can never exceed 3000. After how many months are there 1500 birds in the flock?

3 WE8 A warm can of soft drink at a temperature of 25°C is chilled by placing it in a refrigerator at a constant temperature of 2°C. If the temperature of the drink falls to 22°C in 5 minutes, determine how long it will take for the temperature of the drink to reach 13°C.

4 An iron is preheated to a temperature of 250°C. After 6 minutes the temperature of the iron is 210°C. Brent is ironing in a room that is kept at a constant temperature of 20°C. Brent knows that it is best to iron a cotton T-shirt when the temperature of the iron is between 195°C and 205°C. How long does Brent have to iron the T-shirt?

CONSOLIDATE

5 **a** The weights of cats are found to follow a bounded growth model. The initial weight of a typical newborn cat is 0.1 kg, and after 30 weeks the cat's weight has increased to 4 kg. If it is known that the cat's weight will never exceed 5 kg, determine the weight of the cat after 40 weeks.

b The weights of toy poodle dogs are found to follow a bounded growth model. The initial weight of a puppy toy poodle is 0.2 kg, and after 14 weeks its weight has increased to 1.3 kg. If it is known that the dog's weight will never exceed 4.5 kg, determine the weight of the toy poodle after one year.

6 The number of fish in a lake follows a bounded growth model. Initially there are 50 fish in the lake, and after 10 months the number of fish has increased to 500. The number of fish in the lake can never exceed 1000.

a Determine the number of fish in the lake after 20 months.

b After how many months are there 900 fish in the lake?

7 A woman is on a diet. Her initial weight is 84 kg, and she knows that her weight will always be above 70 kg. After 10 weeks she has lost 7 kg. Assuming a bounded decay model, what is her total weight loss after 20 weeks?

8 **a** A cold can of soft drink is taken from a refrigerator at a temperature of 3°C and placed in a sunroom at a temperature of 30°C. If after 2 minutes the temperature of the can is 4°C, determine its temperature after a further 3 minutes.

b A body at an unknown temperature is placed in a room that has a temperature of 18°C. If after 10 minutes the body has a temperature of 22°C, and after a further 10 minutes its temperature is 20°C, determine the initial temperature of the body.

9 A mother is giving her baby a bath. The bathtub contains hot water that is initially at a temperature of 42°C in a room where the temperature is constant at 20°C. The water cools, and after 2 minutes its temperature is 40°C. The mother knows that babies like to be bathed when the temperature of the water is between 38°C and 34°C, otherwise the bath water is either too hot or too cold. How long should the baby stay in the bath?

10 The temperature of a room is 25°C. A thermometer which was in the room is taken outdoors and in five minutes it reads 15°C. Five minutes later the thermometer reads 10°C. What is the temperature outdoors?

11 A frozen chicken should be thawed before cooking. A freezer is maintained at a constant temperature of –18°C. A chicken is taken out of the freezer at 8:00 am and placed on a kitchen bench. At 11:00 am the temperature of the chicken is 0°C, and at 2:00 pm its temperature is 10°C. At 5:00 pm the chicken has thawed and is placed in an oven which has been preheated to a temperature of 200°C. At 6:00 pm the temperature of the chicken in the oven is 70°C. Determine the temperature of the chicken when it is removed from the oven at 7:00 pm.

12 a An RL series circuit consists of a resistance, R ohms, and an inductance, L henries, connected to a voltage source, E volts. The rise of current, i amperes, after a time t seconds satisfies the differential equation $L\frac{di}{dt} + Ri = E$.

i Assuming the initial current is zero, determine the current at any time t.

ii Show that the time required for the current to reach half its ultimate value is given by $\frac{L}{R}\log_e(2)$.

b A capacitor of C farads is charged by applying a steady voltage of E volts through a resistance of R ohms. The potential difference, v volts, between the plates satisfies the differential equation $RC\frac{dv}{dt} + v = E$. Assuming the initial voltage is zero, determine the voltage v at any time t.

13 Show that the particular solution of the differential equation $\frac{dN}{dt} = k(P - N)$, $N(0) = N_0 > 0$, where $P > 0$, is given by $N(t) = P + (N_0 - P)e^{-kt}$. Consider both cases when $N_0 > P$ and $N_0 < P$.

14 a Show that the solution of the differential equation $\frac{dT}{dt} = k(T - T_m)$, where k and T_m are positive constants and $T(0) = T_0$, is given by $T(t) = T_m + (T_0 - T_m)e^{kt}$.

b Police discover a dead body in a hotel room at 8:00 am. At that time its temperature is 30.4°C. One hour later the temperature of the body is 29.9°C. The room was kept at a constant temperature of 17°C overnight. If normal body temperature is 37°C, determine the time of death.

MASTER

15 A refrigerator is kept at a constant temperature of 3°C. A baby's bottle is taken from the fridge, but the bottle is too cold to give to the baby. The mother places the bottle in a saucepan full of boiling water (maintained at 95°C) for 90 seconds. When the bottle is removed from the saucepan, its temperature is 45°C.

a Let $H = H(t)$ be the temperature of the bottle at a time t minutes while it is being heated. If $\frac{dH}{dt} = k_1(H - T_h)$, where $k_1 < 0$ for $0 \le t \le t_1$, state the values of t_1 and T_h.

b The mother knows that the bottle should be at a temperature of 35°C for the baby to drink it. She realises that the bottle is now unfortunately too hot to give to the baby, so she places it back in the fridge to cool. She takes the bottle out of the fridge after 45 seconds. It is now at the correct temperature to give to the baby. Let $C = C(t)$ be the temperature of the bottle at a time t minutes after it is placed back in the fridge. If $\frac{dC}{dt} = k_2(C - T_c)$, where $k_2 < 0$ for $0 \le t \le t_2$, state the values of t_2 and T_c.

c Determine the ratio of $\frac{k_1}{k_2}$.

16 Let $\$P_0$ be the initial amount of money borrowed from a lender at an annual interest rate of $r\%$. Assume that the interest is compounded continuously and that the borrower makes equal payments of $\$m$ per year to reduce the amount borrowed. If $\$P$ is the amount of money owing at time t, the differential equation modelling the repayments is given by $\frac{dP}{dt} = rP - m$.

a If the loan is completely paid off after a time T years, show that $\frac{m}{m - rP_0} = e^{rT}$.

b If Jared borrows $300 000 for a house loan at 6% per annum and pays off the loan after 20 years, determine his monthly repayments and the total amount of interest paid.

c Sharon can afford to pay $350 a month over 5 years at 8% per annum compounded continuously. What is the maximum amount she can borrow for a car loan?

d Ashley wants to buy a block of land and will have to borrow $120 000 to do it. He can afford to make monthly repayments of $1200 on the loan, compounded continuously at 6.3%. How long will it take Ashley to pay off the loan?

e Ryan pays off a loan of $8000 by paying back $150 per month over 6 years. Assuming the interest is compounded continuously, determine the annual interest rate.

11.5 Chemical reactions and dilution problems

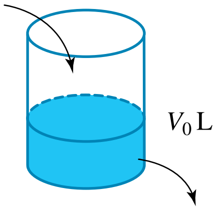
Input–output mixing problems

studyon

Units 3 & 4
AOS 3
Topic 4
Concept 3

Mixing problems with differential equations
Concept summary
Practice questions

Consider a tank that initially holds V_0 litres of a solution in which a kilograms of salt have been dissolved. Another solution containing b kilograms of salt per litre is poured into the tank at a rate of f litres per minute. The well-stirred mixture leaves the tank at a rate of g litres per minute. The problem is to determine the amount of salt in the tank at any time t.



Let Q denote the amount of salt in kilograms in the tank at a time t minutes. The time rate of change of Q is equal to the rate at which the salt flows into the tank minus the rate at which the salt flows out of the tank; that is,

$$\frac{dQ}{dt} = \text{inflow} - \text{outflow}.$$

We know b kilograms of salt per litre is being poured into the tank at a rate of f litres per minute; multiplying these together gives the inflow rate of the salt as bf kilograms per minute.

To determine the outflow rate, first we determine the volume $V(t)$ of the tank at time t. The initial volume is V_0, but f litres per minute flow in, while g litres per minute flow out. Therefore, the volume at time t minutes is given by $V(t) = V_0 + (f - g)t$. To get the required units, we multiply the solution's outflow rate of g litres per minute by the mass and divide by the volume to get the outflow rate of the salt as $\dfrac{gQ}{V_0 + (f - g)t}$ kilograms per minute. Therefore, the differential equation is $\dfrac{dQ}{dt} = bf - \dfrac{Qg}{V_0 + (f - g)t}$, and $Q(0) = a$.

(Note that concentration, c, is the mass divided by the volume, $c = \dfrac{\text{mass}}{\text{volume}}$.)

When $g \neq f$, this equation cannot be solved by the techniques of integration studied in this course. However, if we are given a solution, differentiation and substitution can be used to verify the solution.

WORKED EXAMPLE 9

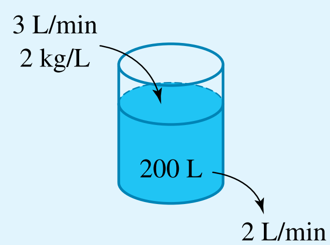
A tank has a capacity of 300 litres and contains 200 litres of water in which 100 kilograms of salt have been dissolved. A salt solution of concentration 2 kilograms per litre is poured into the tank at a rate of 3 litres per minute, and the well-stirred mixture flows out at a rate of 2 litres per minute.

a **State the differential equation for Q, the amount of salt in kilograms in the tank after t minutes.**

b **Verify that $Q = 2(200 + t) + \dfrac{C}{(200 + t)^2}$ is a general solution of the differential equation.**

c **Determine the value of C.**

d **Determine the concentration of salt in kilograms per litre when the tank overflows.**



THINK	WRITE
a 1 The initial volume in the tank is 200 litres; 3 litres per minute flows in and 2 litres per minute flows out. Let $V(t)$ be the volume in litres at time t minutes.	**a** $V(t) = 200 + (3 - 2)t$ $= 200 + t$
2 The inflow rate is the concentration × the inflow volume rate.	Inflow rate $= 3 \times 2$ kg/min
3 The output flow rate is the outflow concentration × the outflow volume rate.	Outflow rate $= \dfrac{2Q}{V(t)}$ kg/min
4 State the required differential equation along with the initial condition.	$\dfrac{dQ}{dt}$ = inflow rate − output rate $= 6 - \dfrac{2Q}{200 + t}$ (1) $Q(0) = 100$
b 1 To verify the given solution, first write the solution in index form.	**b** $Q = 2(200 + t) + \dfrac{C}{(200 + t)^2}$ (2) $= 2(200 + t) + C(200 + t)^{-2}$

2 Now differentiate the given solution as the left-hand side.

$$\text{LHS} = \frac{dQ}{dt}$$
$$= 2 - 2C(200 + t)^{-3}$$
$$= 2 - \frac{2C}{(200 + t)^3}$$

3 Substitute for Q into the right-hand side of (1).

$$\text{RHS} = 6 - \frac{2Q}{200 + t}$$
$$= 6 - \frac{2}{200 + t}\left[2(200 + t) + \frac{C}{(200 + t)^2}\right]$$

4 Expand the right-hand side.

$$= 6 - \frac{2 \times 2(200 + t)}{200 + t} - \frac{2C}{(200 + t)^3}$$

5 Simplify the right-hand side. Because the left-hand side is equal to the right-hand side, we have verified that the given solution does satisfy the differential equation.

$$= 6 - 4 - \frac{2C}{(200 + t)^3}$$
$$= 2 - \frac{2C}{(200 + t)^3}$$

c 1 To determine the value of C, use the given initial conditions.

c Substitute $t = 0$ and $Q = 100$ into (2).

2 Substitute the given values in the given solution.

$$100 = 2 \times 200 + \frac{C}{200^2}$$

3 Solve for C.

$$100 = 400 + \frac{C}{200^2}$$
$$\frac{C}{200^2} = -300$$
$$C = -12 \times 10^6$$

d 1 Determine the time when the tank overflows. The capacity of the tank is 300 litres.

d $300 = 200 + t$

The tank overflows when $t = 100$ minutes.

2 Determine the value of Q when the tank overflows.

Substitute for C:

$$Q(t) = 2(200 + t) - \frac{12 \times 10^6}{(200 + t)^2}$$
$$Q(100) = 600 - \frac{12 \times 10^6}{300^2}$$
$$= 466\tfrac{2}{3}\text{ kg}$$

3 Determine the concentration when the tank overflows.

$$\text{Concentration} = \frac{\text{mass}}{\text{volume}}$$
$$\frac{Q(100)}{V(100)} = \frac{466\frac{2}{3}}{300}$$
$$= 1.556\text{ kg/litre}$$

4 State the answer.

When the tank overflows, the concentration of salt in the solution is 1.556 kg/litre.

Equal input and output flow rates

In the special case when the inflow rate is equal to the outflow rate, that is, when $f = g$ so the volume remains constant, the differential equation can be solved using the separable integration techniques learnt in a previous topic. In fact, in this case the differential equation is just another application of a bounded growth or decay model.

WORKED EXAMPLE 10

A tank contains 5 litres of water in which 10 grams of salt have been dissolved. A salt solution containing 4 grams per litre is poured into the tank at a rate of 2 litres per minute, and the mixture is kept uniform by stirring. The mixture leaves the tank at a rate of 2 litres per minute.

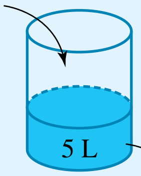
a Set up the differential equation for the amount of salt, Q grams, in the tank at time t minutes.

b Solve the differential equation to determine Q at any time t.

c How much salt is in the tank after 5 minutes? What is its concentration then?

d Show that the salt solution can never exceed 20 grams.

e Sketch the graph of Q versus t.



THINK	WRITE/DRAW
a 1 The inflow rate is equal to the outflow rate. The volume in the tank remains constant at 5 litres.	**a** $V(t) = 5$
2 The inflow rate is inflow concentration × the inflow volume rate. The output flow rate is the outflow concentration × the outflow volume rate.	$\frac{dQ}{dt} = \text{inflow rate} - \text{output rate}$ $= 4 \times 2 - \frac{2Q}{V(t)}$
3 State the differential equation.	$\frac{dQ}{dt} = 8 - \frac{2Q}{5}$
4 State the differential equation in simplest form, along with the initial condition.	$\frac{dQ}{dt} = \frac{2(20 - Q)}{5}, \; Q(0) = 10$
b 1 To solve the differential equation, invert both sides.	**b** $\frac{dt}{dQ} = \frac{5}{2(20 - Q)}$
2 Use the separation of variables technique, taking the constant factors to one side.	$\int \frac{2}{5}\, dt = \int \frac{1}{20 - Q}\, dQ$
3 Perform the integration.	$\frac{2t}{5} = -\log_e\left(\lvert 20 - Q \rvert\right) + C$
4 Since $Q(0) = 10$, the modulus signs are not needed. Use the initial condition to determine the constant of integration.	Substitute $t = 0$ when $Q = 10$: $0 = -\log_e(20 - 10) + C$ $C = \log_e(10)$

5 Substitute back for C and simplify using log laws.

$$\frac{2t}{5} = -\log_e(20 - Q) + \log_e(10)$$

$$\frac{2t}{5} = \log_e\left(\frac{10}{20 - Q}\right)$$

6 Use the definition of the logarithm.

$$e^{\frac{2t}{5}} = \frac{10}{20 - Q}$$

7 Invert both sides.

$$\frac{20 - Q}{10} = e^{-\frac{2t}{5}}$$

8 Rearrange and solve for Q.

$$20 - Q = 10e^{-\frac{2t}{5}}$$

$$Q = 20 - 10e^{-\frac{2t}{5}}$$

9 State the solution of the differential equation in simplest form.

$$Q(t) = 10\left(2 - e^{-\frac{2t}{5}}\right)$$

c Determine the concentration after 5 minutes.

c Substitute $t = 5$:

$$Q(5) = 10(2 - e^{-2})$$
$$= 18.646 \text{ grams}$$

The concentration is

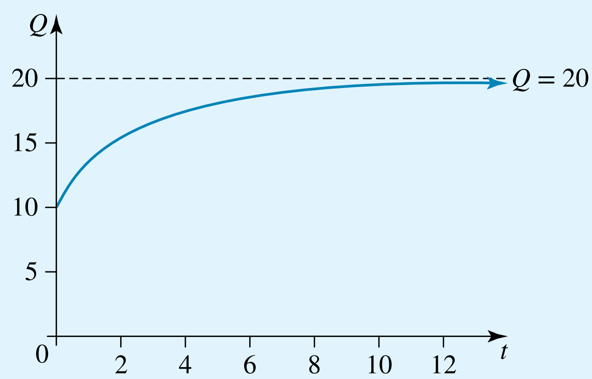
$$c = \frac{Q}{V}$$
$$= \frac{18.646}{5}$$
$$= 3.73 \text{ grams per litre}$$

d Determine the limit as t approaches infinity.

d As $t \to \infty$, $Q \to 20$.
Therefore, the amount of salt can never exceed 20 grams.

e The graph starts at $Q(0) = 10$ and approaches the horizontal asymptote of $Q = 20$.

e



Chemical reaction rates

In a chemical reaction, the rate at which a new substance is formed is proportional to the unused amounts of the reacting substances. Differential equations can be set up and solved to model the amount of the new substance as it forms over time.

WORKED EXAMPLE 11

In a biomolecular chemical reaction between 3 grams of substance A and 6 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B respectively. A and B combine in equal parts to form substance X. Initially no X is present, but after 3 minutes, 3 grams of X have formed.

a **Set up the differential equation for the amount of substance X at time t minutes.**

b **Determine the amount of X present after a further 3 minutes.**

THINK	WRITE
a 1 Let $x = x(t)$ be the amount in grams of substance X formed at time t minutes.	**a** $A + B \rightarrow X$
2 The velocity of the reaction is proportional to the product of the unused amounts of A and B, so $\frac{x}{2}$ is used from both A and B.	$\frac{dx}{dt} \propto \left(3 - \frac{x}{2}\right)\left(6 - \frac{x}{2}\right)$ $\frac{dx}{dt} = k(6 - x)(12 - x), \quad x(0) = 0$ where k is a constant to be found.
b 1 To solve the differential equation, invert both sides.	**b** $\frac{dt}{dx} = \frac{1}{k(6 - x)(12 - x)}$
2 Take the constant k to the other side and use separation of variables.	$kt = \int \frac{1}{(6 - x)(12 - x)}\,dx$
3 To determine the integral on the right-hand side, we need to use partial fractions.	$\frac{1}{(6 - x)(12 - x)} = \frac{A}{6 - x} + \frac{B}{12 - x}$
4 Add the partial fractions and simplify.	$\frac{1}{(6 - x)(12 - x)} = \frac{A(12 - x) + B(6 - x)}{(6 - x)(12 - x)}$ $= \frac{12A + 6B - x(A + B)}{(6 - x)(12 - x)}$
5 Equate the coefficients and solve for A and B.	$12A + 6B = 1 \quad (1)$ $A + B = 0 \Rightarrow A = -B \quad (2)$ $A = \frac{1}{6},\ B = -\frac{1}{6}$
6 Substitute back for the partial fraction decomposition.	$kt = \frac{1}{6}\int \left(\frac{1}{6 - x} + \frac{1}{12 - x}\right) dx$
7 Perform the integration.	$kt = -\frac{1}{6}\log_e(\lvert 6 - x\rvert) + \frac{1}{6}\log_e(\lvert 12 - x\rvert) + C$
8 Take out the common factors and use log laws.	$kt = \frac{1}{6}\log_e\left(\left\lvert\frac{12 - x}{6 - x}\right\rvert\right) + C$
9 Since $x(0) = 0$ and $0 \le x < 6$, the modulus signs are not needed. Use the initial condition to determine the constant of integration.	$0 = \frac{1}{6}\log_e(2) + C$ $C = -\frac{1}{6}\log_e(2)$

10 Substitute back for C and use log laws again.	$kt = \frac{1}{6}\log_e\left(\frac{12-x}{6-x}\right) - \frac{1}{6}\log_e(2)$ $kt = \frac{1}{6}\log_e\left(\frac{12-x}{2(6-x)}\right)$
11 We can now determine the value of k by substituting $t = 3$ when $x = 3$.	Substitute $t = 3$ when $x = 3$: $3k = \frac{1}{6}\log_e\left(\frac{12-3}{2(6-3)}\right)$ $3k = \frac{1}{6}\log_e\left(\frac{9}{6}\right)$ $k = \frac{1}{18}\log_e\left(\frac{3}{2}\right)$
12 Attempt to make x the subject, but leave the result in terms of k, since k has a known value.	$6kt = \log_e\left(\frac{12-x}{2(6-x)}\right)$
13 Use the definition of a logarithm.	$\frac{12-x}{2(6-x)} = e^{6kt}$
14 Invert both sides.	$\frac{12-2x}{12-x} = e^{-6kt}$
15 Remove the denominator.	$12 - 2x = (12-x)e^{-6kt}$
16 Expand the brackets.	$12 - 2x = 12e^{-6kt} - xe^{-6kt}$
17 Transfer the x to one side and factorise.	$12 - 12e^{-6kt} = 2x - xe^{-6kt}$ $12(1 - e^{-6kt}) = x(2 - e^{-6kt})$
18 State the solution of the differential equation.	$x = x(t) = \frac{12(1-e^{-6kt})}{2-e^{-6kt}}$ where $k = \frac{1}{18}\log_e\left(\frac{3}{2}\right)$
19 Determine the value of x after a further 3 minutes.	$x(6) = \frac{12(1-e^{-36k})}{2-e^{-36k}}$
20 Use exact values.	$36k = 2\log_e\left(\frac{3}{2}\right)$ $36k = \log_e\left(\frac{9}{4}\right)$ $e^{-36k} = \frac{4}{9}$
21 There is no need for a calculator.	$x(6) = \frac{12\left(1-\frac{4}{9}\right)}{2-\frac{4}{9}}$ $= \frac{30}{7}$
22 State the final result. Note that as $t \to \infty$, $x \to 6$.	After 6 minutes, $4\frac{2}{7}$ grams of X have formed.

EXERCISE 11.5 Chemical reactions and dilution problems

PRACTISE

1 WE9 A vessel initially contains 200 litres of a salt solution with a concentration of 0.1 kg/litre. The salt solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt of concentration 1.5 kg/litre is added to the vessel at a rate of 2 litres per minute. The contents of the vessel are kept well stirred.

a Set up the differential equation for Q, the amount of salt in kilograms in the vessel after t minutes.

b Verify that $Q = \frac{3}{2}(200 - t) + C(200 - t)^3$ is a general solution of the differential equation.

c Determine the value of C.

d Determine the concentration of salt after 100 minutes.

2 A vat initially contains 50 litres of a sugar solution. More sugar solution containing b grams per litre is poured into the vat at a rate of 6 litres per minute, and simultaneously the well-stirred mixture leaves the vat at a rate of 3 litres per minute.

a Set up the differential equation for Q, the amount of sugar in grams in the vat after t minutes.

b Verify that $Q = 2(50 + 3t) + \frac{C}{50 + 3t}$ is a general solution of the differential equation, and determine the value of b.

3 WE10 A tank initially contains 20 litres of a salt solution that has a concentration of 0.25 grams per litre. The salt solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt of concentration 4 grams per litre is added to the tank at a rate of 3 litres per minute. The contents of the tank are kept well stirred.

a Set up the differential equation for Q, the amount of salt in grams in the tank after t minutes.

b Solve the differential equation to determine Q at any time t.

c Determine the concentration of salt after 1.6 minutes.

d Show that the amount of salt can never exceed 80 grams.

e Sketch the graph of Q versus t.

4 A vat contains 15 litres of pure water. A sugar solution containing 5 grams per litre is poured into the vat at a rate of 4 litres per minute, and simultaneously the well-stirred mixture leaves the vat at the same rate.

a Set up the differential equation for the amount of sugar, Q grams, in the vat at time t minutes.

b Solve the differential equation to determine Q at any time t.

c After how long is the sugar concentration 4 grams per litre?

5 WE11 In a chemical reaction between 1 gram of substance A and 3 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X, and initially no X is present. After 3 minutes, 1 gram of X has formed.

a Set up the differential equation to determine the amount of substance X at time t minutes.

b Solve the differential equation to determine the amount of substance X at time t minutes.

c Determine the amount of X present after a further 3 minutes.

6 In a chemical reaction between 4 grams of substance A and 4 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X. Initially, no X is present, but after 2 minutes, 3 grams of X has formed.

a Set up the differential equation for the amount of substance X at time t minutes.
b Solve the differential equation to determine the amount of X in grams present after a time t minutes.
c How much more time passes before 6 grams of X are present?
d What is the maximum amount of X that could eventually be formed?

CONSOLIDATE

7 a A tank initially contains 50 litres of water. A chemical solution is drawn off at a rate of 5 litres per minute, and at the same time a mixture containing the chemical at a concentration of 3 grams per litre is added to the tank at a rate of 4 litres per minute. The contents of the tank are kept well stirred.

i Set up the differential equation for Q, the amount of the chemical in grams in the tank after t minutes.
ii Verify that $Q(t) = 3(50 - t) + C(50 - t)^5$ is a general solution of the differential equation.

b A container initially contains 30 litres of water. A brine solution of concentration 4 grams per litre is added to the container at a rate of 1 litre per minute. The well-stirred mixture is drawn off at 2 litres per minute.

i Set up the differential equation for Q, the amount of brine in grams in the container after t minutes.
ii Verify that $Q(t) = \frac{2t}{15}(30 - t)$ is the particular solution of the differential equation.

8 a A vessel initially contains 20 litres of water. A sugar solution with a concentration of 3 grams per litre is added at a rate of 2 litres per minute. The well-stirred mixture is drawn off at 1 litre per minute.

i Set up the differential equation for Q, the amount of sugar in grams in the vessel after t minutes.
ii Verify that $Q(t) = \frac{3t(t + 40)}{20 + t}$ is the particular solution of the differential equation.

b A trough initially contains 20 litres of water. A dye solution with a concentration of 4 grams per litre is added to the trough at a rate of 2 litres per minute. The well-stirred mixture is sent down a drain at a rate of 3 litres per minute.

i Set up the differential equation for Q, the amount of dye in grams in the trough after t minutes.
ii Verify that $Q(t) = \frac{t}{100}(t - 40)(t - 20)$ is a particular solution of the differential equation.

9 a A vessel initially contains 64 litres of a salt solution with a concentration of 2 grams per litre. The solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt at a concentration of 4 grams per litre is added to the vessel at a rate of 5 litres per minute. The contents of the vessel are kept well stirred.

i Set up the differential equation for Q, the amount of salt in grams in the vessel after t minutes.

ii Verify that $Q = 4(64 + 2t) + C(64 + 2t)^{-\frac{3}{2}}$ is a general solution of the differential equation.

iii Determine the value of C.

iv Determine the concentration when the volume of the vessel is 100 litres.

b A vat initially contains 40 litres of water. A brine solution containing b grams per litre is poured into the vat at a rate of 2 litres per minute, and simultaneously the well-stirred mixture leaves the vat at a rate of 5 litres per minute. If Q is the amount of brine in grams in the vessel after a time t minutes and $Q = 3(40 - 3t) + C(40 - 3t)^{\frac{5}{3}}$ is a general solution of the differential equation, determine the value of b.

10 a A tank contains 600 litres of water in which 30 kilograms of salt have been dissolved. Water is poured into the tank at a rate of 5 litres per minute, and the mixture is kept uniform by stirring. The mixture leaves the tank at a rate of 5 litres per minute.

i Determine the amount of salt in kilograms in the tank at any time t minutes.

ii Determine the amount of salt in kilograms after 2 hours.

iii How long before there are 15 kilograms of salt in the tank?

b A sink contains 50 litres of water for washing dishes, in which 50 grams of detergent have been dissolved. A dishwashing solution containing 3 grams per litre of detergent is poured into the sink at a rate of 4 litres per minute and the mixture is kept uniform. The mixture leaves the sink down the drain at the same rate. Determine the concentration of detergent after 5 minutes.

11 a A 10 litre urn is full of boiling water. Caterers pour the entire contents of a 300 gram jar of coffee into the urn and mix them thoroughly. While the caterers are making cups of coffee, the coffee is drawn out of the urn at a rate of 0.2 litres per minute, and at the same time boiling water is added at the same rate.

i Lilly likes to drink her coffee when the concentration of the coffee is 25 grams per litre. How long after the process starts should she wait to get her cup of coffee?

ii When the concentration of coffee in the urn falls below 15 grams per litre, more coffee must be added to the urn. How long after the process starts must more coffee be added?

b The urn now contains hot water at a temperature of 93°C. Water is poured from the urn to make a cup of coffee. The coffee cup is placed in a room where the temperature is constant at 17°C. After 1 minute the temperature of the coffee is 88°C. Lilly likes to drink her coffee when its temperature is between 50°C and 65°C. How long does Lilly have to drink the coffee?

12 In a chemical reaction between 2 grams of substance A and 4 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X. Initially, no X is present, and after 2 minutes, 1 gram of X has formed.

a Set up the differential equation to determine the amount of substance X at time t minutes.
b Solve the differential equation to determine the amount of substance X at time t minutes.
c Determine the amount of X present after a further 2 minutes.
d What is the ultimate amount of substance X that can eventually be formed?

13 In a chemical reaction between 5 grams of substance A and 2 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form a substance X. Initially, no X is present, and after 3 minutes, 2 grams of X have formed.

a How long before 3 grams of X have formed?
b After 6 minutes, how much of X has formed?
c What is the ultimate amount of substance X that can eventually be formed?

14 a Many chemical reactions follow Wilhelmy's Law, which states that the velocity of the reaction is proportional to the concentration of the reacting substance. In such a reaction containing a grams of a reagent, the amount of substance X transformed, x grams, after a time t minutes is given by $\frac{dx}{dt} = k\,(a - x)$, where a and k are both positive constants.

i If initially there is no X present, show that $x(t) = a(1 - e^{-kt})$.
ii If $a = 5$ and after 4 minutes 2 grams of X is present, determine the amount of X present after 10 minutes.

b In a chemical reaction, equal amounts of A and B combine to form substance X. Initially there are b grams of both A and B present, and no X is present. If x grams of X have formed after t minutes, the velocity of the reaction is proportional to the product of the unused amounts of A and B.

i Given that $\frac{dx}{dt} = k\left(b - \frac{x}{2}\right)^2$ where b and k are both positive constants, show that $x(t) = \frac{2b^2kt}{2 + bkt}$.
ii If $b = 5$ and after 4 minutes 2 grams of X is present, determine the amount of X present after 10 minutes.
iii What is the ultimate amount of substance X that can eventually be formed?

15 a A vat initially contains V_0 litres of water. A brine solution of concentration b grams per litre is added to the vat at a rate of f litres per minute. The well-stirred mixture is drawn off at the same rate.

i Set up the differential equation for Q, the amount of brine in grams in the vat after t minutes.
ii Use integration to show that $Q(t) = bV_0\left(1 - e^{-\frac{ft}{V_0}}\right)$ is a particular solution of the differential equation.

b A container initially contains V_0 litres of a brine solution in which q_0 grams of brine have been dissolved. A brine solution of concentration b grams per litre is added to the container at a rate of f litres per minute. The well-stirred mixture leaves the container at the same rate.

i Set up the differential equation for Q, the amount of brine in grams in the container after t minutes.

ii Use integration to show that $Q(t) = bV_0 + (q_0 - bV_0)e^{-\frac{ft}{V_0}}$ is a particular solution of the differential equation.

16 a In a trimolecular chemical reaction, three substances, A, B and C, react to form a single substance X. The rate of the reaction is proportional to the product of the unreacted amounts of A, B and C. Substances A, B and C combine in equal parts to form substance X, and initially no X is present. Given that $\frac{dx}{dt} = k\left(a - \frac{x}{3}\right)^3$ where a is the initial amount of all substances A, B and C present and $k > 0$, show that $t = \frac{3x(6a - x)}{2a^2k(3a - x)^2}$ for $0 \le x < 3a$.

b In a chemical reaction between a grams of substance A and b grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. Substances A and B combine in equal parts to form substance X, and initially no X is present. Given that $\frac{dx}{dt} = k\left(a - \frac{x}{2}\right)\left(b - \frac{x}{2}\right)$ where $k > 0$ is a constant, x is the amount of substance X formed at a time t minutes, and $a > b > 0$, show that

$x(t) = \dfrac{2ab\left(1 - e^{-\frac{(a-b)kt}{2}}\right)}{a - be^{-\frac{(a-b)kt}{2}}}$ and deduce that the limiting amount of X present is $2b$.

MASTER

17 A pond initially contains 200 litres of water with a mineral concentration of 0.01 grams per litre. A mineral solution of variable concentration of $2 + \sin\left(\frac{t}{6}\right)$ grams per litre is added to the pond at a rate of 2 litres per minute, where $t \ge 0$ is the time in minutes. The mixed water spills off at the same rate.

a Set up the differential equation for Q, the amount of minerals in grams in the pond at a time t minutes.

b Using CAS, solve the differential equation to express Q in terms of t.

c After 100 minutes of this process, determine the concentration of the minerals in the pond.

18 A tank has a capacity of 325 litres and initially contains 25 litres of water. A chemical solution of variable concentration of $3e^{-\frac{t}{2}}$ grams per litre is added to the tank at a rate of 10 litres per minute, where $t \ge 0$ is the time in minutes. The mixed water leaves the tank at a rate of 5 litres per minute.

a Set up the differential equation for Q, the amount of the chemical in grams in the tank at a time t minutes.

b Using CAS, solve the differential equation to express Q in terms of t.
c Determine the time when the amount of the chemical in the tank is a maximum, and determine the maximum concentration of the chemical.
d Determine the concentration of the chemical in the tank when the tank overflows.
e From the start until the time when the tank overflows, determine the total amount of the chemical which has flowed out of the tank.

11.6 The logistic equation

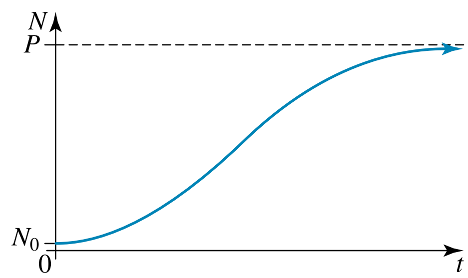
Introduction

In previous sections we considered growth when it is directly proportional to itself; that is, the law of natural growth, $\frac{dN}{dt} = kN$; and also when the growth rate is proportional to the difference between the number and the equilibrium value; that is, bounded growth, $\frac{dN}{dt} = k(P - N)$. In this section we consider growth rates that are proportional to the product of both of these: $\frac{dN}{dt} \propto N(P - N)$.

Logistic growth

This equation can be written in the form $\frac{dN}{dt} = cN(P - N)$ or $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$, where k is the growth rate and P is called the carrying capacity, the equilibrium value or the ultimate maximum value. For an initial number of $N(0) = N_0$, assuming $k > 0$ and $P > N_0 > 0$, it can be shown that $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$. (Showing this is left as an exercise for you to do yourself.) In this relationship, as $t \to \infty$, $N \to P$.

The graph of the logistic equation always has the S shape shown at right.



The growth pattern modelled by this type of differential equation is called logistic growth. It was devised by the Belgian mathematician Pierre François Verhulst (1804–1849). His idea was a response to the work of the English scholar Thomas Robert Malthus, who published a paper in 1798, *An Essay on the Principle of Population*, predicting unlimited population growth. Verhulst disagreed with Malthus's model and used his own model to show the characteristics of bounded population growth.

WORKED EXAMPLE 12

A rumour is spreading in a neighbourhood that contains 800 people. The rumour spreads at a rate proportional to the product of the number who have heard the rumour and the number who have not yet heard the rumour. At first only 2 people in the area have heard the rumour, but after 4 days, 15% of the residents of the neighbourhood have heard it.

a **Let $N = N(t)$ be the number who have heard the rumour after t days. Write and solve the differential equation.**

b Determine the number of residents in the neighbourhood who have heard the rumour after 5 more days.

c After how many days have 50% of the residents in the area heard the rumour?

d Sketch the graph.

THINK	WRITE/DRAW
a 1 The growth rate is proportional to the product of N and $(P - N)$.	**a** The maximum number is $P = 800$. $\frac{dN}{dt} \propto N(800 - N)$
2 Write the differential equation and the initial condition.	$\frac{dN}{dt} = kN(800 - N),\ N(0) = 2$
3 To solve the differential equation, invert both sides.	$\frac{dt}{dN} = \frac{1}{kN(800 - N)}$
4 Use the separation of variables.	$kt = \int \frac{1}{N(800 - N)}\, dN$
5 Use partial fractions on the integrand on the right-hand side.	$\frac{1}{N(800 - N)} = \frac{A}{N} + \frac{B}{800 - N}$
6 Add the partial fractions.	$\frac{1}{N(800 - N)} = \frac{A(800 - N) + BN}{N(800 - N)}$ $= \frac{N(B - A) + 800A}{N(800 - N)}$
7 Write equations that can be solved to determine the values of A and B.	From the coefficient of N: $B - A = 0 \qquad (1)$ From the term independent of N: $800A = 1 \qquad (2)$
8 Solve the equations and state the values of A and B.	From (1), $B = A$. From (2), $A = \frac{1}{800}$. $A = B = \frac{1}{800}$
9 Substitute for A and B in the integration equation.	$kt = \frac{1}{800}\int \left(\frac{1}{N} + \frac{1}{800 - N}\right) dN$
10 Perform the integration.	$800kt = \log_e\left(\lvert N\rvert\right) - \log_e\left(\lvert 800 - N\rvert\right) + c$
11 Because $2 \le N < 800$, the modulus signs are not needed.	$800kt = \log_e(N) - \log_e(800 - N) + c$
12 Use the initial condition to determine the constant of integration.	Substitute $N = 2$ when $t = 0$: $0 = \log_e(2) - \log_e(800 - 2) + c$

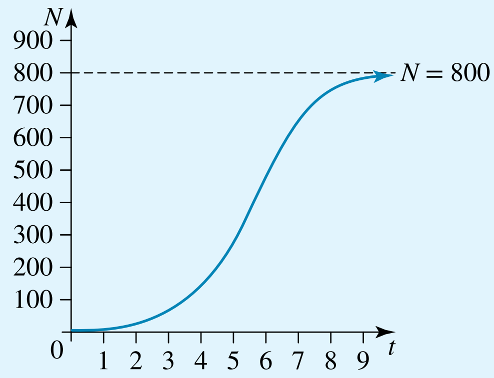
13 Solve for c and apply log laws.	$c = \log_e(798) - \log_e(2)$ $= \log_e\left(\frac{798}{2}\right)$ $= \log_e(399)$
14 Substitute back for c and use log laws again.	$800kt = \log_e(N) - \log_e(800 - N) + \log_e(399)$ $= \log_e\left(\frac{399N}{800 - N}\right)$
15 After 4 days, 15% of the residents have heard the rumour. This is 120 people. Use this to determine k.	Substitute $N = 120$ when $t = 4$: $3200k = \log_e\left(\frac{399 \times 120}{800 - 120}\right)$
16 Use a calculator to solve for k.	$k = \frac{1}{3200}\log_e\left(\frac{1197}{17}\right)$ $= 0.001\,329\ldots$ $800k = 1.0636\ldots$
17 Use the definition of the logarithm but leave the result in terms of k, as k has a known value.	$\frac{399N}{800 - N} = e^{800kt}$
18 Rearrange to make N the subject.	$399N = (800 - N)e^{800kt}$ $800 - N = 399Ne^{-800kt}$ $800 = N + 399Ne^{-800kt}$
19 Take out the common factor of N.	$800 = N(1 + 399e^{-800kt})$
20 State the solution of the differential equation.	$N(t) = \frac{800}{1 + 399e^{-1.0636t}}$
b 1 Determine the value of N after 5 more days (i.e. when $t = 9$).	**b** Substitute $t = 9$: $N(9) = \frac{800}{1 + 399e^{-1.0636 \times 9}}$ $= 778.3767$
2 State the required number, rounding down.	778 people have heard the rumour.
c 1 Determine t when 50% of 800 have heard the rumour; that is, when $N = 400$. It is easier to use an earlier equation.	**c** $800kt = 1.0636t$ $= \log_e\left(\frac{399N}{800 - N}\right)$ Substitute $N = 400$: $1.0636t = \log_e\left(\frac{399 \times 400}{800 - 400}\right)$
2 Solve for t.	$t = \frac{1}{1.0636}\log_e\left(\frac{399 \times 400}{400}\right)$ $= \frac{1}{1.0636}\log_e(399)$ $= 5.63$

3 State the answer.

After 5.63 days, 50% of the residents have heard the rumour.

d Sketch the graph. (Note that as $t \to \infty$, $N \to 800$.)

d



Solving for the parameters

We can see from the equation that in a logistic growth model, there are three parameters: the initial value, $N(0) = N_0$; the maximum value or ultimate value, P; and the constant of proportionality, k. If these parameters are not known, three conditions need to be given and the values substituted into the logistic growth model:

$$N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$$

$$= \frac{P}{1 + \left(\frac{P}{N_0} - 1\right)e^{-kt}}$$

The equations can be solved using algebra or technology.

WORKED EXAMPLE 13

The table shows the population, in millions, of Adelaide.

Year	2001	2005	2009
Population (millions)	1.066	1.129	1.187

Assuming the population follows a logistic growth rate, determine:

a the maximum population of Adelaide

b the year in which the population of Adelaide will first exceed 1.5 million.

THINK

a 1 Let $N = N(t)$ be the population of Adelaide, in millions, after 2001.

2 In 2005, the population was 1.129 million.

3 In 2009, the population was 1.187 million.

WRITE

a $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$

In 2001, $t = 0$ and $N(0) = N_0 = 1.066$.

Substitute $N_0 = 1.066$, $t = 4$ and $N = 1.129$:

$$1.129 = \frac{1.066P}{1.066 + (P - 1.066)e^{-4k}} \qquad (1)$$

Substitute $N_0 = 1.066$, $t = 8$ and $N = 1.187$:

$$1.187 = \frac{1.066P}{1.066 + (P - 1.066)e^{-8k}} \qquad (2)$$

4 Solve equations (1) and (2) using a CAS calculator.

The solution is $P = 1.572\,64\ldots$, $k = 0.047\,552\ldots$

5 The maximum or limiting population of Adelaide is the value of P. Give the answer to an appropriate number of decimal places (i.e. matching the values in the table).

The maximum population of Adelaide is 1.573 million.

b 1 Write the solution for $N(t)$.

b Substitute $P = 1.572\,64\ldots$, $k = 0.047\,552\ldots$ and $N_0 = 1.066$ into the formula:

$$N(t) = \frac{P}{N_0 + \left(\frac{P}{N_0} - 1\right)e^{-kt}}$$

$$N(t) = \frac{1.572\,64}{1 + 0.4743e^{-0.0476t}}$$

2 Determine the value of t when $N = 1.5$.

Substitute $N = 1.5$:

$$1.5 = \frac{1.572\,64}{1 + 0.4743e^{-0.0476t}}$$

3 Solve the equation for t.

$$1 + 0.4743e^{-0.0476t} = \frac{1.572\,64}{1.5}$$

$$0.4743e^{-0.0476t} = 0.048\,427$$

$$e^{-0.0476t} = \frac{0.048\,427}{0.4743}$$

$$e^{-0.0476t} = 0.101\,89$$

4 Use the definition of the logarithm.

$$-0.0476t = \log_e(0.101\,89)$$

5 Determine the value of t.

$$t = \frac{\log_e(0.101\,89)}{-0.0475}$$

$$= 48.1$$

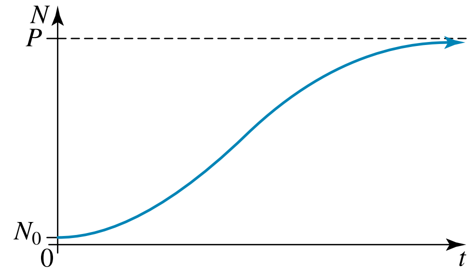
6 State the answer.

The population of Adelaide will first exceed 1.5 million in the year 2049.

Analysis of the logistic solution

The differential equation for the logistic curve in general is given by $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$.

For an initial number $N(0) = N_0$, assuming $k > 0$ and $P > N_0 > 0$, the solution is given by $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$ for $t \geq 0$.



The graph of the logistic curve always has this so-called S-shaped curve. The graph is relatively flat at the ends, so the gradient is close to zero as $N \to 0$ and $N \to P$. These lines, $N = 0$ and $N = P$, are in fact horizontal asymptotes.

Because $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$, we can write $\frac{dN}{dt} = \frac{k}{P}(PN - N^2)$. Taking the derivative again with respect to t, because k and P are constants, and using implicit differentiation or the chain rule, then

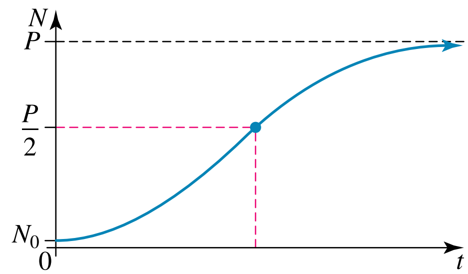
$$\begin{aligned}\frac{d^2N}{dt^2} &= \frac{d}{dt}\left(\frac{dN}{dt}\right)\\ &= \frac{k}{P}\frac{d}{dt}(PN - N^2)\\ &= \frac{k}{P}\frac{d}{dN}(PN - N^2)\frac{dN}{dt}\\ &= \frac{k}{P}(P - 2N)\frac{dN}{dt}\end{aligned}$$

There is a point on the graph at which the gradient or rate of change of growth is a maximum. Assume $k > 0$, $N_0 > 0$ and $\frac{dN}{dt} > 0$. This means that in the equation $\frac{d^2N}{dt^2} = \frac{k}{P}(P - 2N)\frac{dN}{dt}$, the expressions $\frac{k}{P}$ and $\frac{dN}{dt}$ are both positive.

To keep the two sides of the equation equal, if $\frac{d^2N}{dt^2} > 0$, then $P - 2N > 0$. Therefore, $0 < N < \frac{P}{2}$ is where the curve is concave up.

Conversely, if $\frac{d^2N}{dt^2} < 0$, then $P - 2N < 0$. Therefore, $\frac{P}{2} < N < P$ is where the curve is concave down.

Recall that if the second derivative on a curve is zero, there is a point of inflection. For a logistic curve, the point of inflection occurs when $\frac{d^2N}{dt^2} = 0$; that is, for $N = \frac{P}{2}$, or halfway up the curve.



The point of inflection is the point of maximum growth rate, or where the value is increasing most rapidly. It can be shown in general that when $N = \frac{P}{2}$, the corresponding value for t is given by $\frac{1}{k}\log_e\left(\frac{P}{N_0} - 1\right)$.

WORKED EXAMPLE 14

A logistic equation has the solution $y(x) = \frac{200}{1 + 99e^{-\frac{x}{2}}}$.

a **Using differentiation, show that the solution satisfies the differential equation $\frac{dy}{dx} = \frac{y(200 - y)}{400}$.**

b **Show that $\frac{d^2y}{dx^2} = \frac{y(100 - y)(200 - y)}{80\,000}$.**

c **Hence determine the coordinates of the point of inflection.**

d **Sketch the graph of y versus x.**

THINK — **WRITE/DRAW**

a 1 Write the given solution in index form.

a $y = 200\left(1 + 99e^{-\frac{x}{2}}\right)^{-1}$

2 Use the chain rule for differentiation.

$\frac{dy}{dx} = 200 \times -1 \times -\frac{1}{2} \times 99e^{-\frac{x}{2}}\left(1 + 99e^{-\frac{x}{2}}\right)^{-2}$

3 Simplify the expression for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{100 \times 99e^{-\frac{x}{2}}}{\left(1 + 99e^{-\frac{x}{2}}\right)^2}$

4 We need to express x in terms of y.

$\frac{200}{1 + 99e^{-\frac{x}{2}}} = y \qquad (1)$

$\frac{1}{1 + 99e^{-\frac{x}{2}}} = \frac{y}{200} \qquad (2)$

5 Express the exponential in terms of y.

$1 + 99e^{-\frac{x}{2}} = \frac{200}{y}$

$99e^{-\frac{x}{2}} = \frac{200}{y} - 1$

$99e^{-\frac{x}{2}} = \frac{200 - y}{y} \qquad (3)$

6 Rewrite the expression for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{100 \times 99e^{-\frac{x}{2}}}{\left(1 + 99e^{-\frac{x}{2}}\right)^2}$

7 Replace $99e^{-\frac{x}{2}}$ with $\frac{200 - y}{y}$.

Substitute (3) into $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{100 \times \left(\frac{200 - y}{y}\right)}{\left(1 + \frac{200 - y}{y}\right)^2}$

8 Simplify.

$\frac{dy}{dx} = \frac{y(200 - y)}{400}$

b 1 To determine the second derivative, differentiate with respect to x again. However, since $\frac{dy}{dx}$ is in terms of y, use implicit differentiation.

b $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) \times \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{y(200 - y)}{400}\right) \times \left(\frac{y(200 - y)}{400}\right)$

2 Expand the brackets.

$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{1}{400}(200y - y^2)\right) \times \left(\frac{y(200 - y)}{400}\right)$

3 Take the derivative and leave the constant factor.

$\frac{d^2y}{dx^2} = \frac{1}{400}(200 - 2y) \times \left(\frac{y(200 - y)}{400}\right)$

4 Take out the common factor of 2 and simplify.

$$\frac{d^2y}{dx^2} = \frac{y(100 - y)(200 - y)}{80\,000}$$

c 1 The inflection point occurs when the second derivative is zero.

c When $\frac{d^2y}{dx^2} = 0$, $y = 0, 100, 200$. However, $y \neq 0$, so the graph never crosses the x-axis, and $y \neq 200$. The lines $y = 200$ and $y = 0$ are horizontal asymptotes. Therefore, $y = 100$ is the only possible result.

2 Determine the corresponding x-value.

Substitute $y = 100$:

$$100 = \frac{200}{1 + 99e^{-\frac{x}{2}}}$$

3 Solve for x.

$$\begin{aligned} 1 + 99e^{-\frac{x}{2}} &= 2 \\ 99e^{-\frac{x}{2}} &= 1 \\ e^{-\frac{x}{2}} &= \frac{1}{99} \\ e^{\frac{x}{2}} &= 99 \end{aligned}$$

4 Use the definition of the logarithm.

$$\frac{x}{2} = \log_e(99)$$

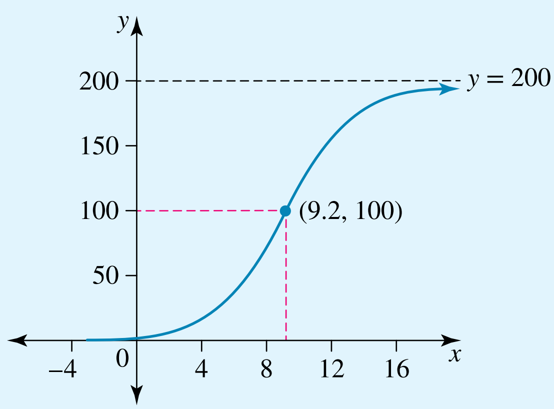
$$x = 2\log_e(99)$$

5 State the coordinates of the inflection point.

$(2\log_e(99), 100) \approx (9.2, 100)$

d 6 Sketch the graph.

d For $y = \frac{200}{1 + 99e^{-\frac{x}{2}}}$, $y \to 200$ as $x \to \infty$, and the inflection point is at $(2\log_e(99), 100) \approx (9.2, 100)$.



EXERCISE 11.6 The logistic equation

PRACTISE

1 WE12 A rumour is spreading in an elderly nursing home which contains 100 pensioners. The rumour spreads at a rate proportional to both the number who have heard the rumour and the number who have not yet heard the rumour. At first, 4 pensioners in the home have heard the rumour, and after 2 weeks, 20 of the pensioners have heard it.

a Let $N = N(t)$ be the number who have heard the rumour after t weeks. Write and solve the differential equation.

b Determine the number of pensioners in the home who have heard the rumour after 3 more weeks.

c How long is it before 50 pensioners in the home have heard the rumour?

2 An infection is spreading in another elderly nursing home which contains 80 pensioners. The infection spreads at a rate proportional to both the number who have the infection and the number who have not yet caught the infection. At first, only 2 pensioners in the home have the infection. After 3 days, 25% of the pensioners have the infection.

a Let $N = N(t)$ be the number who have the infection after t days. Write and solve the differential equation.

b Determine the number of pensioners in the home who have the infection after 4 more days.

c How long is it before 60% of the pensioners in the home have the infection?

3 WE13 The table shows the population of Sydney in millions of people.

Year	2001	2005	2009
Population (millions)	3.948	4.256	4.504

Assuming the population follows a logistic growth rate:

a determine the maximum population of Sydney

b determine the year in which the population of Sydney first exceeds 5 million.

4 The table shows the population of Brisbane in millions of people.

Year	2001	2003	2009
Population (millions)	1.609	1.735	2.004

Assuming the population follows a logistic growth rate:

a determine the year in which the population first exceeded 2 million

b estimate the population of Brisbane in 2020.

5 WE14 A logistic equation has the solution $y(x) = \dfrac{500}{1 + 9e^{-3x}}$.

a Using differentiation, show that it satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{3y(500 - y)}{500}$.

b Show that $\dfrac{d^2y}{dx^2} = \dfrac{9y(250 - y)(500 - y)}{125\,000}$.

c Hence, determine the coordinates of the point of inflection.

d Sketch the graph of y versus x.

6 The table shows the weights of females in kilograms at different ages in years. Assume a logistic growth rate.

Age	8	12	16
Weight (kg)	17	33	50

a Determine the weight in kilograms of a newborn baby girl.

b At what age do females grow most rapidly?

CONSOLIDATE

7 A logistic equation has the solution $y(x) = \dfrac{400}{1 + 199e^{-2x}}$.

a Using differentiation, show that it satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{y(400 - y)}{200}$.

b Show that $\dfrac{d^2y}{dx^2} = \dfrac{y(200 - y)(400 - y)}{20\,000}$.

c Hence, determine the coordinates of the point of inflection.

d Sketch the graph of y versus x.

8 A logistic equation has the solution $y(x) = \dfrac{600}{1 + 99e^{-\frac{x}{3}}}$.

a Using differentiation, show that it satisfies the differential equation $\dfrac{dy}{dt} = \dfrac{y(600 - y)}{1800}$.

b Show that $\dfrac{d^2y}{dx^2} = \dfrac{y(300 - y)(600 - y)}{1\,620\,000}$.

c Hence, determine the coordinates of the point of inflection.

d Sketch the graph of y versus x.

9 An area of the ocean initially contains 500 fish. The area can support a maximum of 4000 fish. The number of fish in the area grows continuously at a rate of 8% per year; that is, the constant of proportionality is 8%.

a Write the differential equation for the number of fish, $N(t)$, in the area at a time t years, assuming a logistic growth model.

b Solve the differential equation to determine the number of fish in the area after t years.

c Hence, determine the number of fish in the area after 5 years.

d After how many years will the number of fish reach 3000?

10 In a movie, people in a town are turned into zombies by a touch from an existing zombie. The town contains 360 people, and initially there is only one zombie. After 3 days, there are 60 zombies. The number of zombies in the town grows at a rate proportional to both the number of zombies and the number of people in the town who are not yet zombies.

a Write the differential equation for the number of zombies, N, after t days.

b After how many days will 75% of the people in the town become zombies?

11 A rumour is spreading in a school with 2000 students. The rumour spreads at a rate proportional to both the number who have heard the rumour and the number who have not yet heard it. Initially, only 2 students have heard the rumour, but after 3 days, 10% of the school has heard it.

a Write the differential equation for the number of students, N, in the school after t days who have heard the rumour, assuming a logistic growth model.

b After how many days have half the students heard the rumour?
c After how many days is the rumour spreading most rapidly?

12 A disease is spreading through an area with a population of 10 000. Initially, 4 people in the area have the disease. After 4 days, 50 people have the disease. The disease is spreading at a rate proportional to both the number of people who have the disease and the number of people who do not yet have the disease. After 2 weeks, how many people in the area have the disease?

13 The table below shows the number of people in billions in the world using the internet and the year in which that number was reached.

Year	2005	2010	2014
Internet users (billions)	1	2	3

Assume a logistic growth rate.

a Determine the ultimate number of people using the internet.
b In what year will the number of people using the internet first exceed 4 billion?
c In what year will the rate of increase in the number of internet users begin to slow down?
d According to this model, in what year were there only 1 million people using the internet?

14 a The table below shows the weights of males in kilograms at different ages in years.

Age	5	10	15
Weight (kg)	12.5	35	60

Assuming a logistic growth rate, determine:
i the weight in kilograms of a newborn baby boy
ii the weight in kilograms of a 24-year-old male.

b The table below shows the heights of males at different ages in years.

Age	2	6	15
Height (cm)	68	107	170

Assuming a logistic growth rate, determine:
i the height (length) in centimetres of a newborn baby boy
ii the ultimate height in centimetres of males
iii the age at which boys grow most rapidly.

15 The number of children who contract the common cold from a school nursery is found to be proportional to those who have the cold and those yet to get the cold. Initially, only 1 child from the nursery has the common cold. After 2 days, 5 children have the cold, and after a further 2 days, 15 children have the cold.

a How many children from the nursery will eventually get the cold?
b After how many days is the cold spreading most rapidly?
c How many children from the nursery have the cold after 6 days?
d Sketch the graph of the number of children with the cold against the time in days.

16 The table below shows the world population in billions and the year in which that number was reached.

Year	1987	1999	2011
Population (billions)	5	6	7

Assume a logistic growth rate.

a Estimate the world's maximum population.

b In what year did the world population first exceed:

  i 3 billion

  ii 4 billion?

c In what year will the world population first exceed 10 billion?

d In what year does the population growth slow down?

e Sketch the graph of the world population.

MASTER

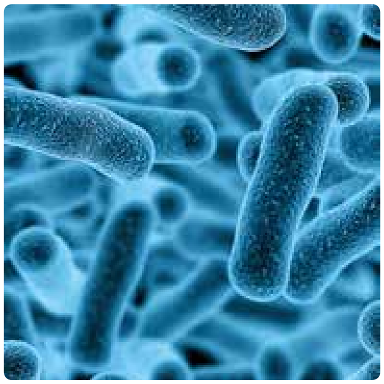
17 A biologist is studying the growth of two different strands of bacteria. He has two different experimental models, and both start at exactly the same time.

Type 1

The number of type 1 bacteria follows a logistic growth rate. Initially, there are 50 type 1 bacteria present, and it is known that the number of these bacteria cannot rise above 1000. The number of type 1 bacteria is increasing most rapidly before 10 hours and the rate slows down after 10 hours.

a Write and solve the differential equation for the number of type 1 bacteria present at t hours after the experiment starts.

Type 2



The number of type 2 bacteria follows the law of natural growth. Initially, there are 25 type 2 bacteria present, and after 18 hours there are 800 type 2 bacteria present.

b Write and solve the differential equation for the number of type 2 bacteria present at t hours after the experiment starts.

c After how many hours are there equal numbers of type 1 and type 2 bacteria present? Determine the actual number of each type.

d Sketch the graphs for both bacteria on the same set of axes.

18 Given the general logistic equation $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$, $N(0) = N_0$, where k is a positive constant and $P > N_0 > 0$:

a show, using integration, that $N(t) = \dfrac{PN_0}{N_0 + (P - N_0)e^{-kt}}$

b show that $\lim\limits_{t \to \infty} N(t) = P$

c show that $\dfrac{d^2N}{dt^2} = k^2N\left(1 - \dfrac{2N}{P}\right)\left(1 - \dfrac{N}{P}\right)$ and hence that there is a point of inflection at $\left(\dfrac{1}{k}\log_e\left(\dfrac{P}{N_0} - 1\right), \dfrac{P}{2}\right)$

d verify, using differentiation, that $N(t) = \dfrac{PN_0}{N_0 + (P - N_0)e^{-kt}}$ is a solution of $\dfrac{dN}{dt} = kN\left(1 - \dfrac{N}{P}\right)$, $N(0) = N_0$.

11.7 Euler's method

Introduction

The Swiss mathematician Leonhard Euler (1707–1783) is considered one of the greatest mathematicians of all time. He made significant discoveries in many areas of mathematics, including calculus, graph theory, fluid dynamics, optics and astronomy. It is his notation for functions that we still use today.

studyon

Units 3 & 4
AOS 3
Topic 3
Concept 7

Euler's method for first-order differential equations
Concept summary
Practice questions

Numerical solution of a differential equation

Many first-order differential equations can be solved by integration, obtaining the particular solution. A differential equation with a given initial condition is often called an 'initial value problem', abbreviated to IVP.

However, there are many types of first-order differential equations for which no solution can be found; that is, the integration cannot be done. In these cases a numerical approximation for the solution of the differential equation can be obtained instead. Euler's method is the simplest of many types of numerical approximations used for this purpose. It provides a solution in the form of a table of values without determining the particular solution of the function. An initial condition and a step size must be given to tabulate the solution.

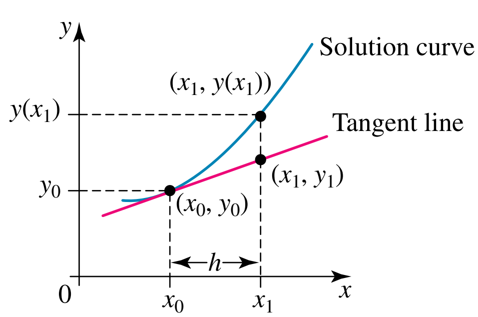
Euler's method is a first-order approximation method based on using the tangent to the curve at that point to estimate the next value. Consider the differential equation $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$. Assuming there is a solution of the form $y = F(x)$, which may not be known, we tabulate the solution for $x = x_0, x_1, x_2 \ldots$ up to $x = x_n$ by determining the y-values $y = y_0, y_1, y_2 \ldots y_n$, assuming that each x-value is equally spaced by h; that is, $x_1 = x_0 + h$, $x_2 = x_1 + h = x_0 + 2h \ldots$ and $x_n = x_{n-1} + h = x_0 + nh$.

The gradient of the tangent to the solution curve, $y = F(x)$, is $f(x)$, since $\frac{dy}{dx} = f(x)$.

Let m_T be the gradient of the tangent to the solution curve at (x_0, y_0).

For $m_T = f(x_0)$, the equation of the tangent to the solution curve is given by $y - y_0 = m_T(x - x_0) = f(x_0)(x - x_0)$.

When $x = x_1$, the next value of y is given by $y_1 = y_0 + f(x_0)(x_1 - x_0) = y_0 + hf(x_0)$.



This iterative process is repeated, with each coordinate being used to determine the next coordinate.

In general, to use Euler's method, iterate and use the result $y_{n+1} = y_n + hf(x_n)$. To obtain y_n, the value of y when $x = x_n = x_0 + nh$, repeat the procedure n times.

The result obtained by Euler's method can be compared to the exact solution.

WORKED EXAMPLE 15

a **Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} = \frac{1}{x^2 + 4}$, $y(0) = 0$, using $h = \frac{1}{2}$ to approximate $y(2)$.**

b **Solve the differential equation $\frac{dy}{dx} = \frac{1}{x^2 + 4}$, $y(0) = 0$ and determine $y(2)$. Determine the percentage error for the result obtained by Euler's method compared to the exact value.**

THINK	WRITE
a 1 State the initial values and the step size.	**a** $f(x) = \frac{1}{x^2 + 4}$, $x_0 = 0$, $y_0 = 0$, $h = \frac{1}{2}$ When $x = 2$, we require y_4 from x_4, so the procedure must be repeated 4 times.
2 Use Euler's method to determine the value of y_1.	$y_1 = y_0 + hf(x_0)$ $y_1 = 0 + \frac{1}{2}\left(\frac{1}{0 + 4}\right)$ $= \frac{1}{8}$ $= 0.125$
3 Use Euler's method to determine the value of y_2.	$y_2 = y_1 + hf(x_1)$, $x_1 = x_0 + h = \frac{1}{2} = 0.5$ $y_2 = \frac{1}{8} + \frac{1}{2}\left(\frac{1}{0.5^2 + 4}\right)$ $= \frac{33}{136}$ $= 0.242\,65$
4 Use Euler's method to determine the value of y_3.	$y_3 = y_2 + hf(x_2)$, $x_2 = x_1 + h = 1$ $y_3 = \frac{33}{136} + \frac{1}{2}\left(\frac{1}{1^2 + 4}\right)$ $= \frac{233}{680}$ $= 0.3426$
5 Use Euler's method to determine the value of y_4.	$y_4 = y_3 + hf(x_3)$, $x_3 = x_2 + h = \frac{3}{2} = 1.5$ $y_4 = \frac{233}{680} + \frac{1}{2}\left(\frac{1}{1.5^2 + 4}\right)$ $= \frac{1437}{3400}$ $= 0.4226$

6 Summarise the results. Note that exact fractions could be given in this particular case, but the answers are given to 4 decimal places.

x	0	0.5	1.0	1.5	2
y	0	0.125	0.2426	0.3426	0.4226

b 1 To solve the differential equation, integrate with respect to x.

b $\frac{dy}{dx} = \frac{1}{x^2 + 4}$

$y = \int \frac{1}{x^2 + 4}\, dx$

2 Perform the integration.

$y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

3 Determine the value of the constant of integration.

Since $y(0) = 0$, substitute $x = 0$ when $y = 0$:

$0 = \frac{1}{2}\tan^{-1}(0) + c$

$c = 0$

4 State the particular solution.

$y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

5 Determine the required value.

Substitute $x = 2$:

$y = \frac{1}{2}\tan^{-1}(1)$

$= \frac{1}{2} \times \frac{\pi}{4}$

$= \frac{\pi}{8}$

≈ 0.3927

6 Compare the exact result with the result obtained by Euler's method.

In this case Euler's method overestimates the exact result by

$\frac{0.4226 - 0.3927}{0.3927} \times 100 = 7.6\%.$

Using Euler's method to solve Type 2 differential equations

Type 2 first-order differential equations are of the form $\frac{dy}{dx} = f(y)$, $y(x)_0 = y_0$. Euler's method can also be used to approximate solutions for these equations, but in this case the iterative result $y_{n+1} = y_n + hf(y_n)$ is used.

WORKED EXAMPLE 16

a **Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} - e^{-y} = 0$, $y(1) = 0$, using $h = \frac{1}{4}$ to approximate $y(2)$.**

b **Solve the differential equation $\frac{dy}{dx} - e^{-y} = 0$, $y(1) = 0$ and determine $y(2)$. Determine the percentage error for the result obtained by Euler's method compared to the exact value.**

THINK	WRITE
a **1** Write the differential equation in standard form and state the initial values and the step size.	**a** $\frac{dy}{dx} = e^{-y},\ y(1) = 0$ $f(y) = e^{-y},\ x_0 = 1,\ y_0 = 0,\ h = \frac{1}{4}$ When $x = 2$, we require y_4 from y_3, so the procedure must be repeated 4 times.
2 Use Euler's method to determine the value of y_1.	$y_1 = y_0 + hf(y_0)$ $y_1 = 0 + \frac{1}{4} \times e^{-0}$ $= \frac{1}{4}$ $= 0.25$
3 Use Euler's method to determine the value of y_2.	$y_2 = y_1 + hf(y_1)$ $= 0.25 + \frac{1}{4} \times e^{-0.25}$ $= 0.4447$
4 Use Euler's method to determine the value of y_3.	$y_3 = y_2 + hf(y_2)$ $= 0.4447 + \frac{1}{4} \times e^{-0.4447}$ $= 0.6050$
5 Use Euler's method to determine the value of y_4.	$y_4 = y_3 + hf(y_3)$ $= 0.6050 + \frac{1}{4} \times e^{-0.6050}$ $= 0.7415$
6 Summarise the results. (Note that in the table they are given correct to 4 decimal places, but more decimal places were used in the working to avoid rounding errors.)	see table below

x	1	1.25	1.5	1.75	2
y	0	0.25	0.4447	0.6050	0.7415

THINK	WRITE
b **1** To solve the differential equation, first invert both sides, then integrate with respect to y.	**b** $\frac{dy}{dx} = e^{-y}$ $\frac{dx}{dy} = \frac{1}{e^{-y}}$ $\frac{dx}{dy} = e^{y}$ $x = \int e^{y}\, dy$
2 Perform the integration.	$x = e^{y} + c$
3 Determine the value of the constant of integration.	Since $y(1) = 0$, substitute $x = 1$ when $y = 0$: $1 = e^{0} + c$ $c = 0$
4 Rearrange to make y the subject and state the particular solution.	$x = e^{y}$ $y = \log_e(x),\ x > 0$
5 Determine the required value.	Substitute $x = 2$: $y = \log_e(2) \approx 0.6931$
6 Compare the exact result with the result obtained by Euler's method.	In this case Euler's method overestimates the exact result by $\frac{0.7415 - 0.6931}{0.6931} \times 100 = 7.0\%$.

Using Euler's method to solve Type 3 differential equations

Recall that first-order differential equations of the form $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ have been solved. Euler's method can also be applied to these equations, but in this case the iterative result $y_{n+1} = y_n + hf(x_n, y_i)$ is used.

WORKED EXAMPLE 17

a **Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 2$ using $h = \frac{1}{4}$ to approximate $y(1)$.**

b **Solve the differential equation $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 2$ and determine $y(1)$. Determine the percentage error for the result obtained by Euler's method compared to the exact value.**

THINK	WRITE
a **1** Write the differential equation in standard form and state the initial values and the step size.	**a** $\frac{dy}{dx} = -2xy^2$, $y(0) = 2$ $f(x, y) = -2xy^2$, $x_0 = 0$, $y_0 = 2$, $h = \frac{1}{4}$ When $x = 1$, we require y_4, so the procedure must be repeated 4 times.
2 Use Euler's method to determine the value of y_1.	$y_1 = y_0 + hf(x_0, y_0)$ $y_1 = 2 + \frac{1}{4} \times (-2 \times 0 \times 2^2)$ $= 2$
3 Use Euler's method to determine the value of y_2.	$y_2 = y_1 + hf(x_1, y_1)$, $x_1 = x_0 + h = \frac{1}{4}$ $y_2 = 2 + \frac{1}{4} \times \left(-2 \times \frac{1}{4} \times 2^2\right)$ $= \frac{3}{2}$ $= 1.5$
4 Use Euler's method to determine the value of y_3.	$y_3 = y_2 + hf(x_2, y_2)$, $x_2 = x_1 + h = \frac{1}{2}$ $y_3 = \frac{3}{2} + \frac{1}{4} \times \left(-2 \times \frac{1}{2} \times \left(\frac{3}{2}\right)^2\right)$ $= \frac{15}{16}$ $= 0.9375$
5 Use Euler's method to determine the value of y_4.	$y_4 = y_3 + hf(x_3, y_3)$, $x_3 = x_2 + h = \frac{3}{4}$ $y_4 = \frac{15}{16} + \frac{1}{4} \times \left(-2 \times \frac{3}{4} \times \left(\frac{15}{16}\right)^2\right)$ $= \frac{1245}{2048}$ $= 0.6079$

6 Summarise the results. (Note that they are given correct to 4 decimal places in the table, but exact answers were used when performing the calculations.)

x	0	0.25	0.5	0.75	1.0
y	2	2	1.5	0.9375	0.6079

b **1** To solve the differential equation, separate the variables, setting all x's on one side and all y's on the other side.

b $\frac{dy}{dx} = -2xy^2$

$\int 2x\, dx = \int \frac{-1}{y^2}\, dy$

2 Perform the integration.

$x^2 = \frac{1}{y} + c$

3 Determine the value of the constant of integration.

Since $y(0) = 2$, substitute $x = 0$ when $y = 2$:

$0 = \frac{1}{2} + c$

$c = -\frac{1}{2}$

4 Rearrange to make y the subject and state the particular solution.

$x^2 = \frac{1}{y} - \frac{1}{2}$

$\frac{1}{y} = x^2 + \frac{1}{2}$

$\frac{1}{y} = \frac{2x^2 + 1}{2}$

$y = \frac{2}{2x^2 + 1}$

5 Determine the required value.

Substitute $x = 1$:

$y = \frac{2}{3}$

$= 0.6667$

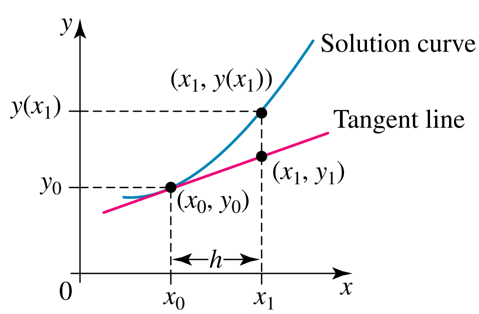
6 Compare the exact result with the result obtained by Euler's method.

In this case Euler's method underestimates the exact result by

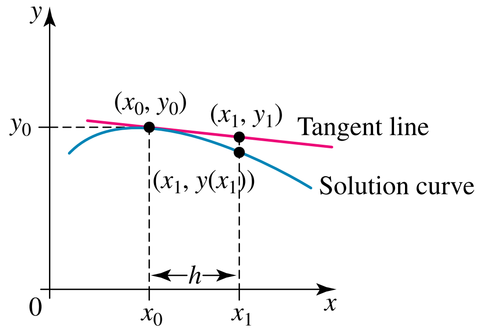
$\frac{0.6079 - 0.6667}{0.6667} \times 100 = -8.8\%$

Comparing Euler's method

Euler's method can overestimate or underestimate the exact solution of a differential equation. When a curve is concave up, the tangent line lies under the curve, so in these cases Euler's method will underestimate the exact solution.



When a curve is concave down, the tangent line lies above the curve, so in these cases Euler's method will overestimate the exact solution.



It is true that if the step size h is made smaller and the value of n larger, that is, the iterative process is repeated more times, then the percentage error can be reduced. However, there are occasions when Euler's method diverges and results are significantly wrong and incorrect.

EXERCISE 11.7 Euler's method

PRACTISE

1 WE15 **a** Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} = 3\sqrt{x}$, $y(4) = 1$, using $h = \frac{1}{4}$ to approximate $y(5)$.

b Solve the differential equation $\frac{dy}{dx} = 3\sqrt{x}$, $y(4) = 1$ and determine $y(5)$.

Determine the percentage error for the result obtained by Euler's method compared to the exact value.

2 Consider the differential equation $\frac{dy}{dx} = 4x^3$, $y(0) = k$. Use Euler's method with a step size of $h = \frac{1}{4}$ and the value of $y_4 = y(1) = 1$. Determine the value of k.

3 WE16 **a** Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} - \frac{2}{y} = 0$, $y(2) = 3$, using $h = \frac{1}{4}$ to approximate $y(3)$.

b Solve the differential equation $\frac{dy}{dx} - \frac{2}{y} = 0$, $y(2) = 3$ and determine $y(3)$.

Determine the percentage error for the result obtained by Euler's method compared to the exact value.

4 Use Euler's method with a step size of $h = 0.1$ to determine y_3 for the differential equation $\frac{dy}{dx} = \tan(y)$, $y(0.2) = 0.4$.

5 WE17 **a** Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} + 2x^3y^2 = 0$, $y(0) = 2$, using $h = \frac{1}{4}$ to approximate $y(1)$.

b Solve the differential equation $\frac{dy}{dx} + 2x^3y^2 = 0$, $y(0) = 2$ and determine $y(1)$.

Determine the percentage error for the result obtained by Euler's method compared to the exact value.

6 **a** Consider the differential equation $\frac{dy}{dx} - 2y\cos(x) = 0$, $y(0) = 1$. Use Euler's method with a step size of 0.1 to approximate y_3.

b Solve the differential equation $\frac{dy}{dx} - 2y\cos(x) = 0$, $y(0) = 1$ and determine $y(0.3)$. Determine the percentage error for the result obtained by Euler's method compared to the exact value.

CONSOLIDATE

7 Use Euler's method to determine the value of y_n indicated for each of the following initial value problems, using the given value of h. Give your answers correct to 4 decimal places, and compare the approximated answer to the exact answer.

a For $\frac{dy}{dx} = 2\cos\left(\frac{x}{2}\right)$, $y(0) = 2$, determine y_2 with $h = \frac{1}{2}$.

b For $\frac{dy}{dx} = \sin(3x)$, $y(0) = 3$, determine y_3 with $h = \frac{1}{3}$.

c For $\frac{dy}{dx} = 4e^{-2x}$, $y(0) = 2$, determine y_4 with $h = \frac{1}{4}$.

8 a Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} + 6x = 0$, $y(1) = 2$ up to $x = 2$ using:

i $h = \frac{1}{2}$ **ii** $h = \frac{1}{3}$ **iii** $h = \frac{1}{4}$ **iv** $h = \frac{1}{5}$.

b Determine the percentage error compared to the exact answer for each case in part **a**.

9 a Consider the differential equation $\frac{dy}{dx} = 6x^2$ with $x_0 = 1$ and $y_0 = k$. When Euler's method is used with a step size of $\frac{1}{3}$, $y_3 = 12$. Determine the value of k.

b Consider the differential equation $\frac{dy}{dx} + \frac{1}{x^2} = 0$ with $x_0 = 1$ and $y_0 = k$. When Euler's method is used with a step size of $\frac{1}{3}$, $y_3 = \frac{431}{1200}$. Determine the value of k.

c Consider the differential equation $\frac{dy}{dx} = \frac{2}{x}$ with $x_0 = 1$ and $y_0 = k$. When Euler's method is used with a step size of 0.25, $y_4 = 2$. Determine the value of k.

10 a Consider the differential equation $\frac{dy}{dx} = \log_e(3x + 2)$, $y(1) = 2$. Use Euler's method with a step size of $\frac{1}{3}$. Show that when $x = 2$, $y_3 = 2 + \frac{1}{3}\log_e(210)$.

b Consider the differential equation $\frac{dy}{dx} = \log_e(4x + 1)$, $y(3) = 5$. Use Euler's method with a step size of 0.25. Show that when $x = 4$, $y_4 = 5 + \frac{1}{4}\log_e(43\,680)$.

c i The solution to the initial value problem $\frac{dy}{dx} = \log_e(2x + 5)$, $y(2) = 4$ is approximated using Euler's method with a step size of 0.5. Show that when $x = 3$, $y_2 = 4 + \frac{1}{2}\log_e(90)$.

ii Differentiate $x\log_e(2x + 5)$ and hence solve the differential equation $\frac{dy}{dx} = \log_e(2x + 5)$, $y(2) = 4$ to determine the value of y when $x = 3$.

11 a For $\frac{dy}{dx} = \frac{y}{3}$, given $(x_0, y_0) = (0, 4)$:

i use Euler's method to determine the value of y_2 with $h = \frac{1}{2}$

ii use Euler's method to determine the value of y_3 with $h = \frac{1}{3}$

iii determine the percentage error compared to the exact answer for parts **i** and **ii**.

b For $\frac{dy}{dx} = \frac{y}{2}(5 - y)$, given $(x_0, y_0) = (0, 1)$:

i use Euler's method to determine the value of y_2 with $h = \frac{1}{2}$

ii use Euler's method to determine the value of y_3 with $h = \frac{1}{3}$

iii determine the percentage error compared to the exact answer for parts **i** and **ii**.

12 a Use Euler's method to tabulate the solutions to the differential equation $\frac{dy}{dx} + xy^2 = 0$, $y(1) = 2$ up to $x = 2$, giving your answers correct to 4 decimal places, with:

i $h = \frac{1}{3}$ **ii** $h = \frac{1}{4}$.

b Compare the approximations in parts **i** and **ii** to the exact answers and determine the percentage error in each case.

13 For each of the following equations, use Euler's method to determine (correct to 4 decimal places):

i y_3 with $h = \frac{1}{3}$ **ii** y_4 with $h = \frac{1}{4}$.

a $\frac{dy}{dx} = y\sqrt{x^2 + 5}$ if $(x_0, y_0) = (2, 5)$ **b** $\frac{dy}{dx} + \frac{e^{-2x}}{y} = 0$ if $(x_0, y_0) = (1, 5)$

14 a **i** Given the initial value problem $\frac{dy}{dx} = 4x - y + 2$, $y(0) = 2$, tabulate the solutions using Euler's method up to $x = 1$ with a step size of 0.25. Give your answers correct to 4 decimal places.

ii Given that $y = ax + b + ce^{kx}$ is a solution of the differential equation $\frac{dy}{dx} = 4x - y + 2$, $y(0) = 2$, determine the values of a, b, c and k. Hence, determine the value of y when $x = 1$.

b **i** Given the initial value problem $\frac{dy}{dx} = 5x + 2y + 1$, $y(0) = 1$, tabulate the solutions, using Euler's method with $h = \frac{1}{4}$ up to $x = 1$. Give your answers correct to 4 decimal places.

ii Given that $y = ax + b + ce^{kx}$ is a solution of the differential equation $\frac{dy}{dx} = 5x + 2y + 1$, $y(0) = 1$, determine the values of a, b, c and k. Hence, determine the value of y when $x = 1$.

15 a Given the initial value problem $\frac{dy}{dx} = x\sin\left(\frac{x}{2}\right)$, $y(0) = 2$:

i use Euler's method with $h = \frac{1}{3}$ to show that $y_3 = 2 + \frac{1}{9}\left[\sin\left(\frac{1}{6}\right) + 2\sin\left(\frac{1}{3}\right)\right]$

ii use Euler's method with $h = \frac{1}{4}$ to show that

$$y_4 = 2 + \frac{1}{16}\left[\sin\left(\frac{1}{8}\right) + 2\sin\left(\frac{1}{4}\right) + 3\sin\left(\frac{3}{8}\right)\right]$$

iii differentiate $x\cos\left(\frac{x}{2}\right)$ and hence solve the differential equation $\frac{dy}{dx} = x\sin\left(\frac{x}{2}\right)$, $y(0) = 2$ to determine the value of y when $x = 1$.

b Given the initial value problem $\frac{dy}{dx} = xe^{-3x}$, $y(0) = 4$:

i use Euler's method with $h = \frac{1}{3}$ to show that $y_3 = 4 + \frac{1}{9}(e^{-1} + 2e^{-2})$

ii use Euler's method with $h = \frac{1}{4}$ to show that $y_4 = 4 + \frac{1}{16}\left(e^{-\frac{3}{4}} + 2e^{-\frac{3}{2}} + 3e^{-\frac{9}{4}}\right)$

iii differentiate xe^{-3x} and hence solve the differential equation $\frac{dy}{dx} = xe^{-3x}$, $y(0) = 4$ to determine the value of y when $x = 1$.

16 Show that when the differential equation $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$ is approximated using Euler's method, for a small value of h, $y_n = y_0 + h\sum_{k=0}^{n-1} f(x_0 + kh)$.

MASTER

17 a Using a spreadsheet, tabulate the solutions to the initial value problem $\frac{dy}{dx} + 6x = 0$, $y(2) = 1$ for $y(3)$, giving answers correct to 5 decimal places.

Apply Euler's method with:

i $h = \frac{1}{10}$ ii $h = \frac{1}{20}$ iii $h = \frac{1}{50}$.

b Compared to the exact answer, determine the percentage error for each case.

18 a Using a spreadsheet, tabulate the solutions to the initial value problem $\frac{dy}{dx} = \frac{y}{3}$, $y(0) = 4$ for $y(1)$, giving answers correct to 5 decimal places. Apply Euler's method with:

i $h = \frac{1}{10}$ ii $h = \frac{1}{20}$ iii $h = \frac{1}{50}$.

b Compared to the exact answer, determine the percentage error for each case.

11.8 Slope fields

Introduction

study on

Units 3 & 4
AOS 3
Topic 3
Concept 6

Direction (slope) fields
Concept summary
Practice questions

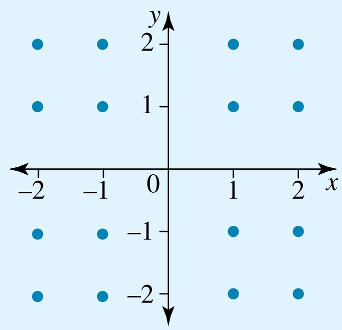
Slope fields, also known as direction fields, are a tool for graphically visualising the solutions to a first-order differential equation. A slope field is simply a graph on the Cartesian coordinate system showing short line segments that represent the slopes of the possible solutions at each point.

Type 1, $\frac{dy}{dx} = f(x)$

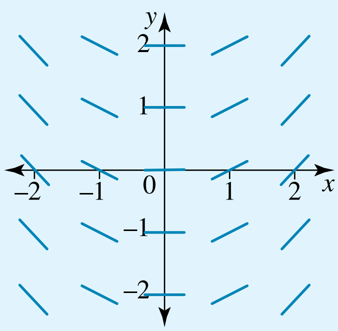
Type 1 first-order differential equations are of the form $\frac{dy}{dx} = f(x)$. In this section you will draw the slope fields for a simple differential equation of this type.

WORKED EXAMPLE 18

Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{x}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on the grid shown.

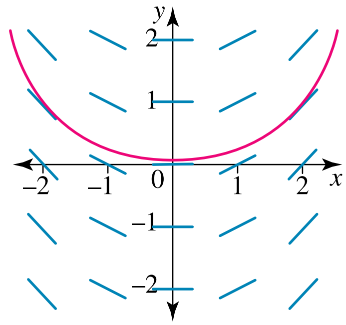


THINK	WRITE/DRAW
1 Make some observations about the slope.	When $x = 0$ (i.e. along the y-axis), the slope $\frac{dy}{dx}$ is zero. The y-values do not affect the slope.
2 Determine the slope at appropriate values of x.	When $x = 1$, the slope is $\frac{dy}{dx} = \frac{1}{2}$. A slope of $\frac{1}{2}$ makes an angle of $\tan^{-1}\left(\frac{1}{2}\right) \approx 27°$ with the positive x-axis.
3 Consider another x-value.	When $x = 2$, the slope is $\frac{dy}{dx} = 1$. A slope of 1 makes an angle of $\tan^{-1}(1) = 45°$ with the positive x-axis.
4 Consider the negative values of x.	When $x = -1$, the slope is $\frac{dy}{dx} = -\frac{1}{2}$. A slope of $-\frac{1}{2}$ makes an angle of $\tan^{-1}\left(-\frac{1}{2}\right) \approx 153°$ with the positive x-axis.
5 Consider the final x-value.	When $x = -2$, the slope is $\frac{dy}{dx} = -1$. A slope of -1 makes an angle of $\tan^{-1}(-1) = 135°$ with the positive x-axis.
6 Draw short line segments with the slopes found at each of the points.	The slope field in this case is symmetrical about the y-axis.



Note that we obtain these graphical solutions to a general solution of a first-order differential equation; that is, we do not necessarily require a particular condition.

In general, the solutions to this differential equation are a family of curves of the form $y = \frac{x^2}{4} + c$. However, if we joined some of the slopes shown in the answer to Worked example 18 and drew a particular curve passing through the origin, the result would represent the solution curve $y = \frac{x^2}{4}$, which is the particular solution to the differential equation $\frac{dy}{dx} = \frac{x}{2}$, $y(0) = 0$. The slope field simply represents tangents to the solution curve at the points.

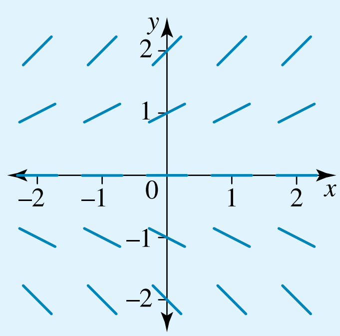


Type 2, $\frac{dy}{dx} = f(y)$

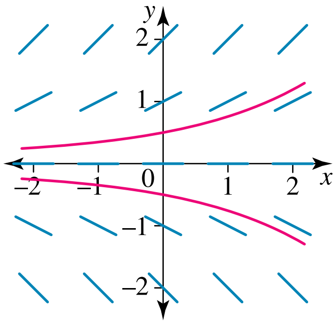
Type 2 first-order differential equations are of the form $\frac{dy}{dx} = f(y)$.

WORKED EXAMPLE 19

Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{y}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on a grid.

THINK	WRITE/DRAW
1 Make some observations about the slope.	When $y = 0$ (i.e. along the x-axis), the slope is zero. The x-values do not affect the slope.
2 Determine the slope at appropriate values of y.	When $y = 1$, the slope is $\frac{dy}{dx} = \frac{1}{2}$. A slope of $\frac{1}{2}$ makes an angle of $\tan^{-1}\left(\frac{1}{2}\right) \approx 27°$ with the positive x-axis.
3 Consider another y-value.	When $y = 2$, the slope is $\frac{dy}{dx} = 1$. A slope of 1 makes an angle of $\tan^{-1}(1) = 45°$ with the positive x-axis.
4 Consider the negative values of y.	When $y = -1$, the slope is $\frac{dy}{dx} = -\frac{1}{2}$. A slope of $-\frac{1}{2}$ makes an angle of $\tan^{-1}\left(-\frac{1}{2}\right) \approx 153°$ with the positive x-axis.
5 Consider the final y-value.	When $y = -2$, the slope is $\frac{dy}{dx} = -1$. A slope of -1 makes an angle of $\tan^{-1}(-1) = 135°$ with the positive x-axis.
6 Draw short line segments with the slope found at each of the points.	The slope field in this case is symmetrical about the x-axis. 

Note that, if we joined the slope field lines in Worked example 19 and drew curves, they would represent the curve $y = y_0 e^{\frac{x}{2}}$, which is the particular solution to the differential equation $\frac{dy}{dx} = \frac{y}{2}$, $y(0) = y_0$. The two curves shown at right have $y_0 > 0$ and $y_0 < 0$.



Type 3

Type 3 first-order differential equations are of the form $\frac{dy}{dx} = f(x, y)$.

WORKED EXAMPLE 20

Sketch the slope field for the differential equation $y\frac{dy}{dx} + x = 0$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on a grid.

THINK

1 Rearrange the differential equation to make $\frac{dy}{dx}$ the subject.

WRITE/DRAW

$$y\frac{dy}{dx} + x = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

2 We have to evaluate the slope at each of the 25 points on the grid. Rather than substituting point by point, draw up a table.

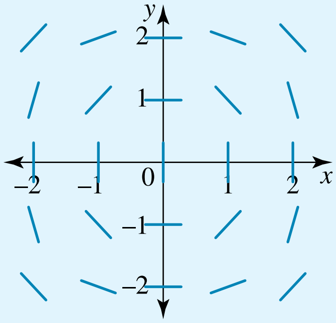
Substitute $x = 2$ and $y = 2$:

the slope $\frac{dy}{dx} = -\frac{2}{2} = -1$.

3 Complete the values. Note that an undefined slope is one that is parallel to the y-axis.

y \ x	−2	−1	0	1	2
−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
−1	−2	−1	0	1	2
0	Undef.	Undef.	Undef.	Undef.	Undef.
1	2	1	0	−1	−2
2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	−1

4 Draw the slope field by drawing short line segments with the slope found at each point.



In the answer for Worked example 20, if the line segments were joined, the field would look like a series of circles of varying radii centred at the origin. In fact, if $x^2 + y^2 = r^2$, using implicit differentiation gives $2x + 2y\frac{dy}{dx} = 0$ or $x + y\frac{dy}{dx} = 0$.

Interpreting a slope field

Because CAS calculators can draw slope fields, questions involving slope fields are often designed as multiple choice questions. In such questions we need to match a differential equation with the appropriate slope field. There are four common question types:

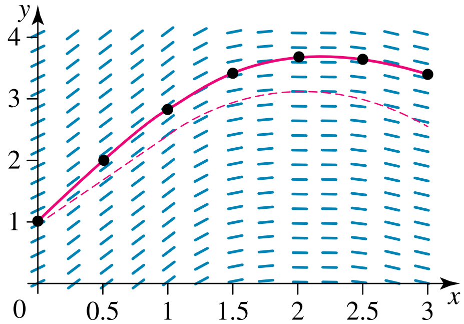
- Given a differential equation, choose the correct slope field.
- Given a solution of a differential equation, choose the correct slope field.
- Given a slope field, choose the correct differential equation.
- Given a slope field, choose the solution of the differential equation.

When analysing a slope field, the following approaches are useful.

- Determine the values where the slope is zero; that is, the values of x and y for which $\frac{dy}{dx} = 0$.
- Determine values for which the slope is parallel to the y-axis. In this case, the slope is infinite, and if the differential equation is of the form $\frac{dy}{dx} = \frac{g(x, y)}{h(x, y)}$, then the denominator $h(x, y)$ equals 0.
- Determine the values of the slopes along the x- and y-axes.
- Determine if the slopes are independent of x and therefore depend only on the y-value. In this case, the differential equation is of the form $\frac{dy}{dx} = f(y)$.
- Determine if the slopes are independent of y and therefore depend only on the x-value. In this case, the differential equation is of the form $\frac{dy}{dx} = f(x)$. (Differential equations of only one variable of the types $\frac{dy}{dx} = f(x)$ or $\frac{dy}{dx} = f(y)$ are called autonomous differential equations.)
- Determine where the slopes are positive and where the slopes are negative.
- The symmetry of the slope field and the slopes in each of the four quadrants can give information on the solution.
- Determine if there is no slope field for certain values. This could indicate a required domain.

Note that we are not solving the differential equation when we analyse a slope field. However, if we are given a solution, we can use implicit differentiation to understand the families of curves that are represented.

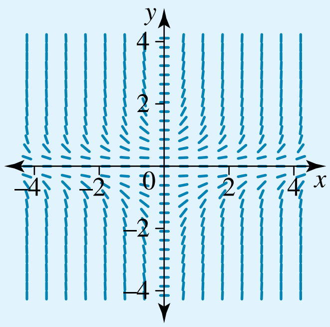
Also note that Euler's method is just a method of following along a certain slope field and obtaining a value.



WORKED EXAMPLE 21

The differential equation that best represents the slope field shown is:

A $\frac{dy}{dx} = \frac{y^2}{x}$ **B** $\frac{dy}{dx} = \frac{x^2}{y}$ **C** $\frac{dy}{dx} = yx^2$ **D** $\frac{dy}{dx} = y^2x$ **E** $\frac{dy}{dx} = x + y.$



THINK	WRITE
1 Consider the y-axis.	When $x = 0$ (i.e. along the y-axis), the slopes are all zero. Options A and E are incorrect.
2 Consider the first quadrant.	When $x = 1$ and $y = 1$, the slopes are 1. This satisfies B, C and D.
3 Consider the second quadrant.	When $x = -1$ and $y = 1$, the slopes are -1. Option C is incorrect.
4 Consider the third quadrant.	When $x = -1$ and $y = -1$, the slopes are 1. Options B and D are still valid.
5 Consider the fourth quadrant.	When $x = 1$ and $y = -1$, the slopes are 1. Option B is incorrect.
6 State the result.	Option D, $\frac{dy}{dx} = y^2x$, is the only differential equation that is represented by this slope field.

EXERCISE 11.8 Slope fields

PRACTISE

For questions 1–6, sketch the fields for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$.

1 WE18 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{x}$.

2 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

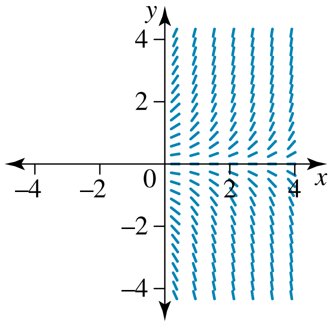
3 WE19 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{y}$.

4 Sketch the slope field for the differential equation $\frac{dy}{dx} = 2\sqrt{y}$.

5 WE20 Sketch the slope field for the differential equation $y\frac{dy}{dx} - x = 0$.

6 Sketch the slope field for the differential equation $\frac{dy}{dx} = xy$.

7 WE21 The differential equation that best represents the slope field shown is:



A $\frac{dy}{dx} = \sqrt{xy}$

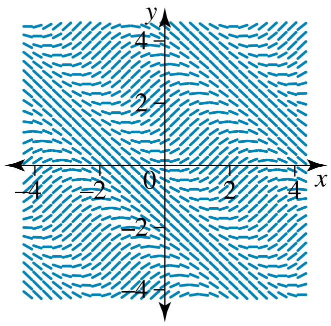
B $\frac{dy}{dx} = x\sqrt{y}$

C $\frac{dy}{dx} = y\sqrt{x}$

D $\frac{dy}{dx} = \frac{1}{\sqrt{xy}}$

E $\frac{dy}{dx} = \sqrt{x} + \sqrt{y}$

8 The differential equation that best represents the slope field shown is:



A $\frac{dy}{dx} = \cos(x + y)$

B $\frac{dy}{dx} = \cos(x - y)$

C $\frac{dy}{dx} = \sin(x) + \cos(y)$

D $\frac{dy}{dx} = \sin(x - y)$

E $\frac{dy}{dx} = \sin(x + y)$

CONSOLIDATE

For questions 9–13, sketch the fields for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$.

9 a Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{(x-1)^2}$.

b i Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{1}{x^2 + 1}$.

ii Solve the differential equation $\frac{dy}{dx} = \frac{1}{x^2 + 1}$, $y(0) = 0$ and sketch the solution curve on the diagram for part i.

10 a Sketch the slope field for the differential equation $\frac{dy}{dx} = y^2$.

b i Sketch the slope field for the differential equation $\frac{dy}{dx} = y^2 + 1$.

ii Solve the differential equation $\frac{dy}{dx} = y^2 + 1$, $y(0) = 0$ and sketch the solution curve on the diagram for part i.

11 Sketch the slope fields for each of the following differential equations.

a $\frac{dy}{dx} = \frac{-x}{y + 1}$

b $\frac{dy}{dx} = \frac{x - 2}{y}$

12 Sketch the slope fields for each of the following differential equations.

a $\frac{dy}{dx} = (y - 1)\cos\left(\frac{\pi x}{2}\right)$

b $\frac{dy}{dx} = y^2 \sin\left(\frac{\pi x}{2}\right)$

13 a Sketch the slope field for $\frac{dy}{dx} = x + y$.

b Given that $y = Ae^{kx} + Bx + C$ is a solution of the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 0$, determine the values of A, B, C and k.

c Show that the solution curve has a turning point at $(0, 0)$.

d Use Euler's method to approximate the solution to the differential equation $\frac{dy}{dx} = x + y$, $y(0) = 0$ up to $x = 1$ with a step size of $\frac{1}{2}$.

e On the slope field, sketch the particular solution to the differential equation $\frac{dy}{dx} = x + y$ that passes through the point (0, 0).

14 Match the four differential equations to the four slope fields shown below.

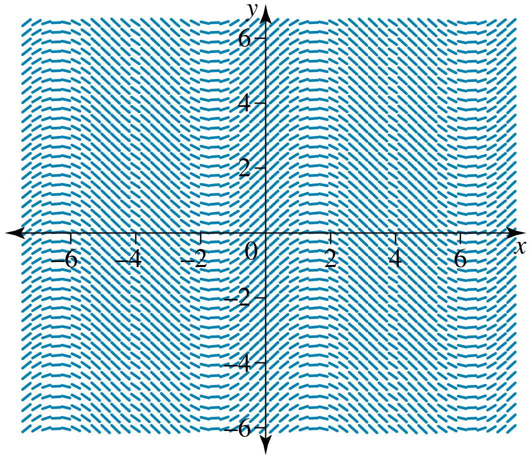
a $\frac{dy}{dx} + \sin(x) = 0$

b $\frac{dy}{dx} + \cos(x) = 0$

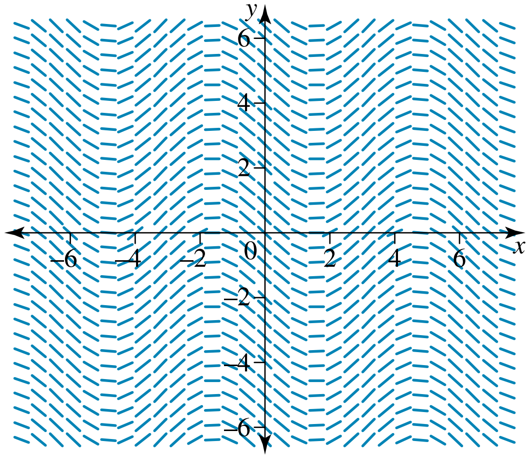
c $\frac{dy}{dx} - \sin(x) = 0$

d $\frac{dy}{dx} - \cos(x) = 0$

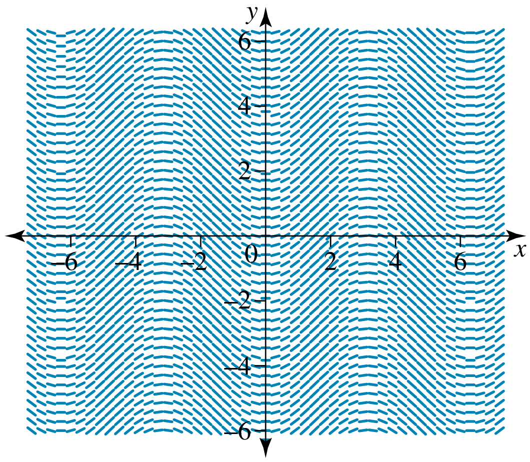
Slope 1



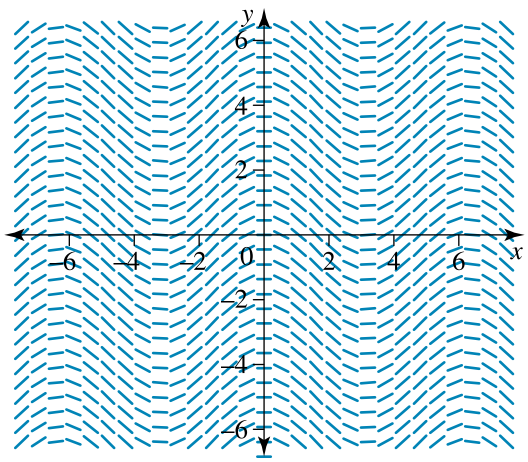
Slope 2



Slope 3



Slope 4



15 Match the four differential equations to the four slope fields shown on the next page.

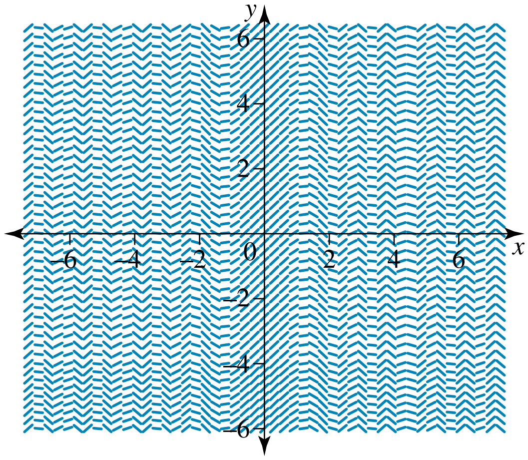
a $\frac{dy}{dx} = \sin(x^2)$

b $\frac{dy}{dx} = \cos(x^2)$

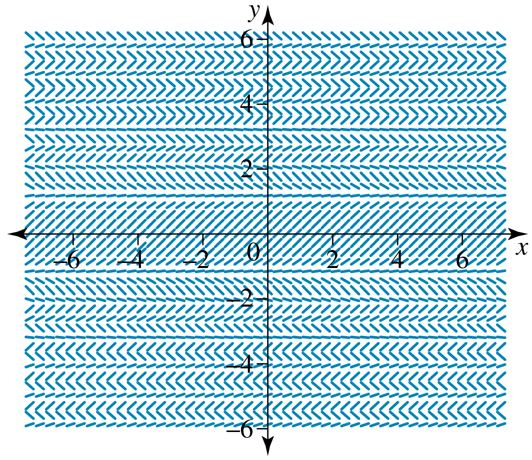
c $\frac{dy}{dx} = \sin(y^2)$

d $\frac{dy}{dx} = \cos(y^2)$

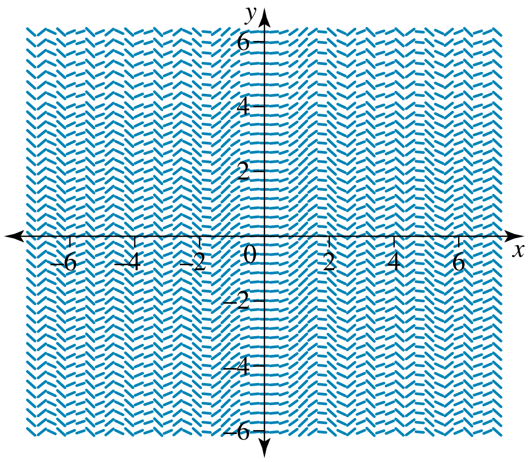
Slope 1



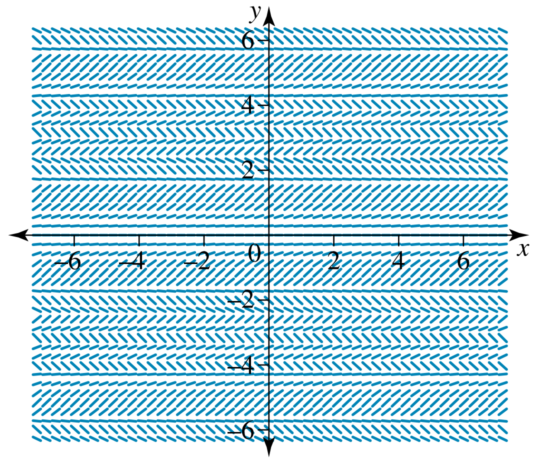
Slope 2



Slope 3



Slope 4



16 Match the four differential equations to the four slope fields shown below.

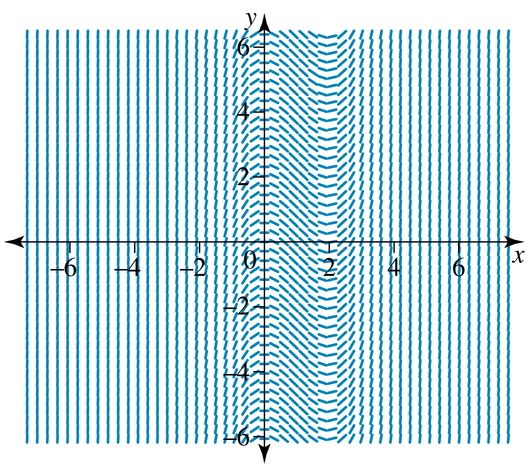
a $\dfrac{dy}{dx} = x(x - 2)$

b $\dfrac{dy}{dx} = y(y - 2)$

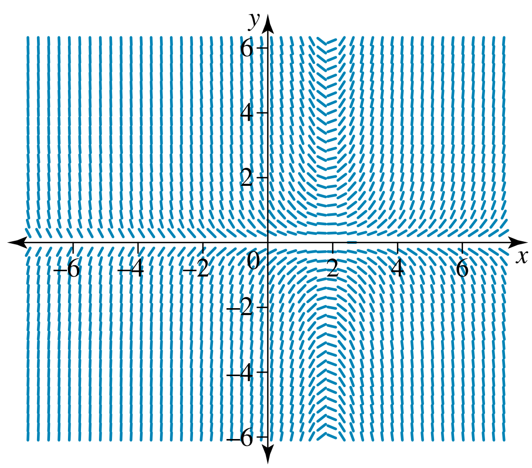
c $\dfrac{dy}{dx} = y(x - 2)$

d $\dfrac{dy}{dx} = x(y - 2)$

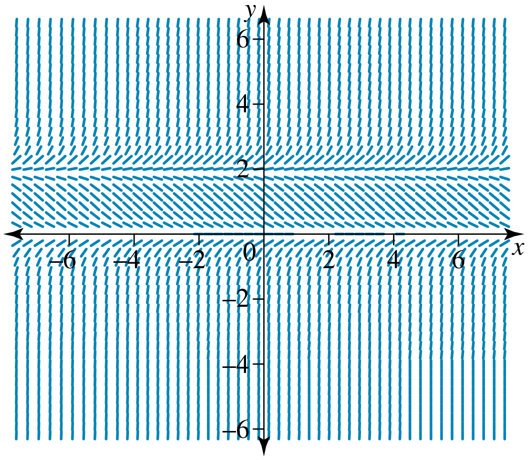
Slope 1



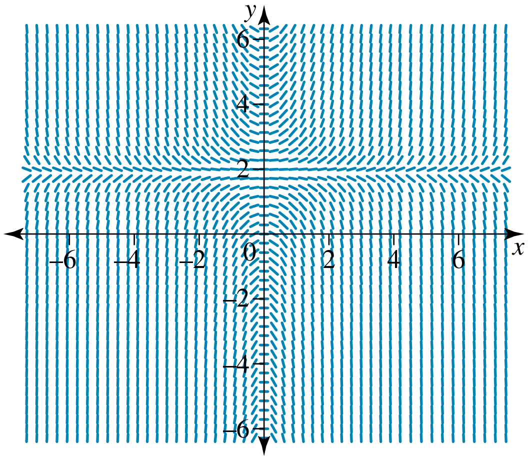
Slope 2



Slope 3



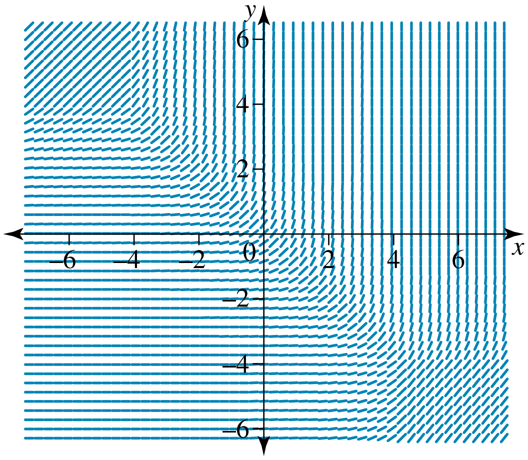
Slope 4



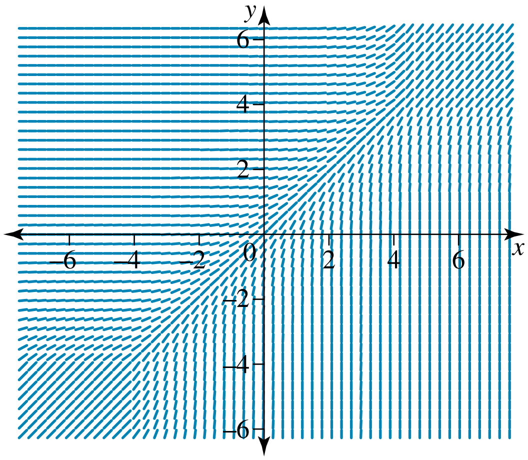
17 Match the four differential equations to the four slope fields shown below.

a $\frac{dy}{dx} = e^{x-y}$ **b** $\frac{dy}{dx} = e^{y-x}$ **c** $\frac{dy}{dx} = e^{x+y}$ **d** $\frac{dy}{dx} = e^{-(x+y)}$

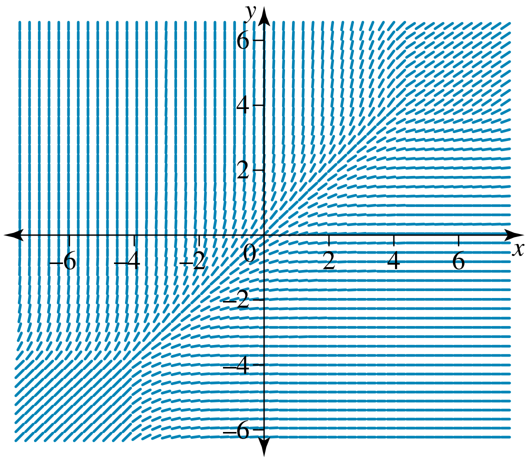
Slope 1



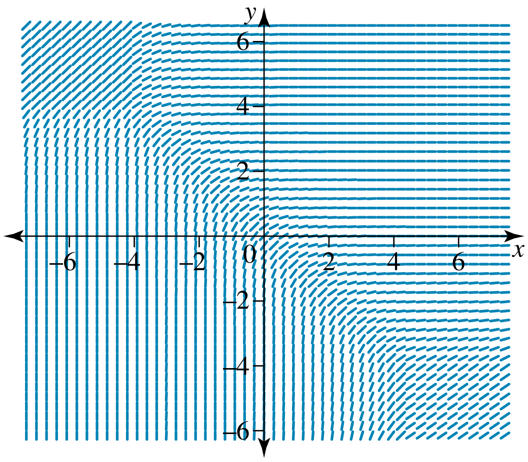
Slope 2



Slope 3



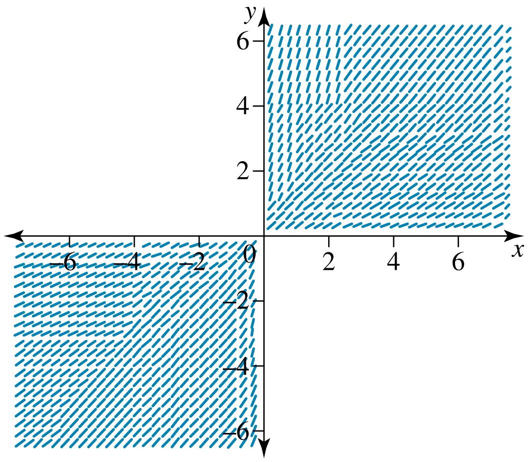
Slope 4



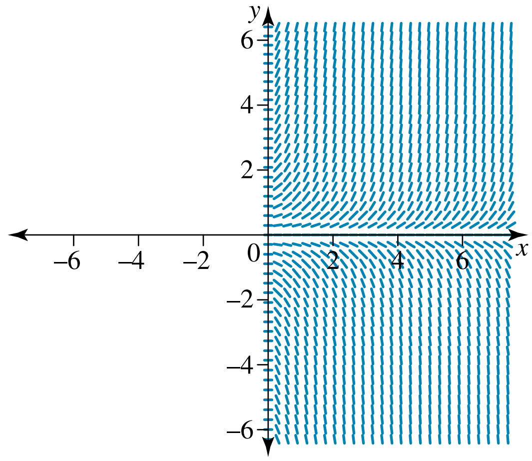
18 Match the four differential equations to the four slope fields shown below.

a $\dfrac{dy}{dx} = \sqrt{\dfrac{y}{x}}$ **b** $\dfrac{dy}{dx} = \sqrt{\dfrac{x}{y}}$ **c** $\dfrac{dy}{dx} = y\sqrt{x}$ **d** $\dfrac{dy}{dx} = x\sqrt{y}$

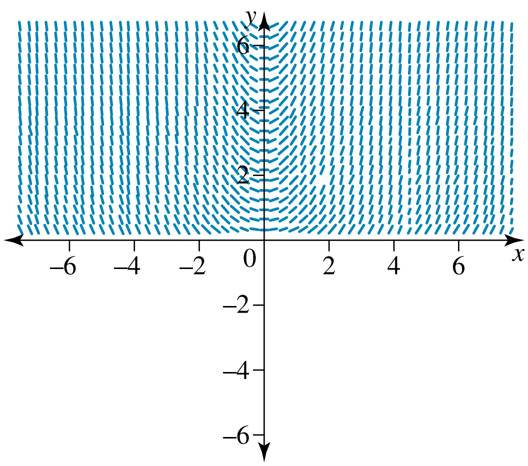
Slope 1



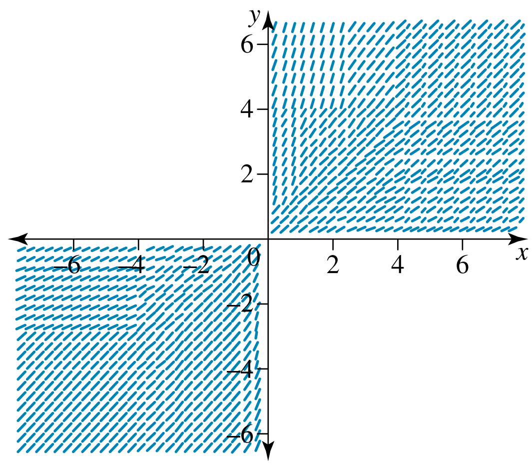
Slope 2



Slope 3



Slope 4



MASTER

19 Use CAS to draw the slope fields for:

a $\dfrac{dy}{dx} = \dfrac{1}{2}(x^2 + y^2)$ **b** $\dfrac{dy}{dx} = \dfrac{1}{2}(x^2 - y^2)$.

20 Use CAS to draw the slope fields for:

a $\dfrac{dy}{dx} = \sin(x^2y^2)$ **b** $\dfrac{dy}{dx} = \cos(x^2 + y^2)$.

eBook*plus* ONLINE ONLY

11.9 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

11 Answers

EXERCISE 11.2

1 a $N = 1.79 \times 10^6 e^{0.0277t}$
b 2.71 million
c 2042
2 137
3 58.31 mg
4 113.78 mg
5 12.8 years
6 65%
7 a 4146
b 11 275
8 942
9 a 24 057 541
b 2027
10 a 7.56 billion
b 2038
11 a 0.29 g
b 25.59 years
12 32.47 days
13 38.2 years
14 a 78.8%
b 24 110 years
15 11.76%
16 6.180 hours
17 a Check with your teacher.
b 2039
c 2043
d 2029
18 a $Q(t) = Q_0 e^{-k(t-t_0)}$
b 1.21×10^{-4}
c 10 356 years

EXERCISE 11.3

1 \$9048.37
2 \$630 000
3 25 million
4 140
5 a $\frac{dN}{dt} = \frac{N - 160}{4}$, $N(0) = 320$
b $N = 160\left(1 + e^{\frac{t}{4}}\right)$
c 1342
6 Check with your teacher.
7 a $P(h) = P_0 e^{kh}$, $k > 0$
b 50.94
8 a $D(t) = D_0 e^{-kt}$
b 2.08 hours
9 a Check with your teacher.
b 18.5%
10 a Check with your teacher.
b 205.31 newtons
11 a i Check with your teacher.
ii 35.48 ohms
b i 1000 units
ii 1.5 seconds
12 a $i(t) = i_0 e^{-\frac{Rt}{L}}$
b $Q(t) = Q_0 e^{-\frac{t}{RC}}$
13 a $\frac{dN}{dt} = \frac{N - 100}{20}$, $N(0) = 200$
b $N(t) = 100\left(1 + e^{\frac{t}{20}}\right)$
c 216
14 666 052
15 a 11 years
b 19
16 a $N(t) = \left(\frac{1}{2}kt + \sqrt{N_0}\right)^2$
b $N(t) = \sqrt{2kt + N_0^2}$
c $N(t) = \frac{4N_0}{(2 - kt\sqrt{N_0})^2}$
d $N(t) = \frac{N_0}{1 - ktN_0}$
17 Check with your teacher.
18 Check with your teacher.

EXERCISE 11.4

1 165.4 g
2 13 months
3 26.34 min
4 1.75 min
5 a 4.4 kg
b 3.06 kg
6 a 736
b 35.1 months
7 10.5 kg

8 a 5.43°C

b 26°C

9 5.27 min

10 5°C

11 108.37°C

12 a i $i(t) = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$

ii Check with your teacher.

b $v(t) = E\left(1 - e^{-\frac{t}{RC}}\right)$

13 Check with your teacher.

14 a Check with your teacher.

b 9:28 pm

15 a $t_1 = \frac{3}{2}, T_h = 95$

b $t_2 = \frac{3}{4}, T_c = 3$

c 1.12

16 a Check with your teacher.

b Monthly repayments: \$2146.52, total interest: \$215 164.59

c \$17 308.20

d 11.8 years

e 10.56%

EXERCISE 11.5

1 a $\frac{dQ}{dt} = 3 - \frac{3Q}{200 - t}, Q(0) = 20$

b Check with your teacher.

c $-\frac{7}{200000}$

d 1.15 kg/L

2 a $\frac{dQ}{dt} = 6b - \frac{3Q}{50 + 3t}$

b 2

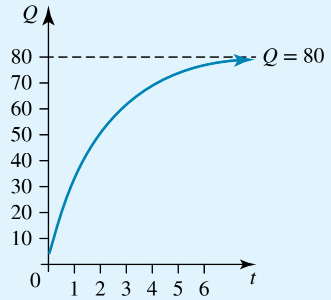
3 a $\frac{dQ}{dt} = \frac{3(80 - Q)}{20}, Q(0) = 5$

b $Q(t) = 5\left(16 - 15e^{-\frac{3t}{20}}\right)$

c 1.05 g/L

d Check with your teacher.

e



4 a $\frac{dQ}{dt} = \frac{4(75 - Q)}{15}, Q(0) = 0$

b $Q(t) = 75\left(1 - e^{-\frac{4t}{15}}\right), t \to \infty, Q \to 75$

c 6.04 min

5 a $\frac{dx}{dt} = k(2 - x)(6 - x), x(0) = 0$

b $x(t) = \frac{6(1 - e^{-4kt})}{3 - e^{-4kt}}$ where $k = \frac{1}{12}\log_e\left(\frac{5}{3}\right)$

c $\frac{16}{11}$ g or 1.4545 g

6 a $\frac{dx}{dt} = k(8 - x)^2, x(0) = 0$

b $x(t) = \frac{24t}{3t + 10}$

c 8 minutes (for 10 minutes total after the start of the reaction)

d 8 g

7 a i $\frac{dQ}{dt} = 12 - \frac{5Q}{50 - t}$

ii Check with your teacher.

b i $\frac{dQ}{dt} = 4 - \frac{2Q}{30 - t}, Q(0) = 0$

ii Check with your teacher.

8 a i $\frac{dQ}{dt} = 6 - \frac{Q}{20 + t}, Q(0) = 0$

ii Check with your teacher.

b i $\frac{dQ}{dt} = 8 - \frac{3Q}{20 - t}, Q(0) = 0$

ii Check with your teacher.

9 a i $\frac{dQ}{dt} = 20 - \frac{3Q}{64 + 2t}, Q(0) = 128$

ii Check with your teacher.

iii -2^{16}

iv 3.34 g/L

b 3

10 a i $30e^{-\frac{t}{120}}$

ii 11.04 kg

iii 83.18 min

b 1.659 g/L

11 a i 9.12 min

ii 34.66 min

b 5.5 min

12 a $\frac{dx}{dt} = k(4 - x)(8 - x), x(0) = 0$

b $x(t) = \frac{8(1 - e^{-4kt})}{2 - e^{-4kt}}$ where $k = \frac{1}{8}\log_e\left(\frac{7}{6}\right)$

c $\frac{52}{31}$ g or 1.677 g **d** 8 g

13 a 6.57 minutes

b $\frac{26}{9}$ g

c 4 g

14 a i Check with your teacher.

ii 3.61 g

b i Check with your teacher.

ii 3.85 g

iii 10 g

15 a i $\frac{dQ}{dt} = \frac{f}{V_0}(bV_0 - Q),\ Q(0) = 0$

ii Check with your teacher.

b i $\frac{dQ}{dt} = \frac{f}{V_0}(bV_0 - Q),\ Q(0) = q_0$

ii Check with your teacher.

16 Check with your teacher.

17 a $\frac{dQ}{dt} + \frac{Q}{100} = 4 + 2\sin\left(\frac{t}{6}\right),\ Q(0) = 2$

b $Q(t) = \frac{1}{2509}\left(1800\sin\left(\frac{t}{6}\right) - 30000\cos\left(\frac{t}{6}\right) - 968\,582e^{-\frac{t}{100}}\right)$

c 1.3 g/L

18 a $\frac{dQ}{dt} = 30e^{-\frac{t}{2}} - \frac{Q}{t+5},\ Q(0) = 0$

b $Q(t) = \frac{420}{t+5} - \frac{60(t+7)e^{-\frac{t}{2}}}{t+5}$

c 3.99 minutes, 0.82 g/L

d 0.02 g/L

e 53.54 g

EXERCISE 11.6

1 a $N(t) = \frac{100}{1 + 24e^{-0.8959t}}$

b 78

c 3.55 weeks

2 a $N(t) = \frac{80}{1 + 39e^{-0.855t}}$

b 72

c 4.76 days

3 a 5.236 million

b 2023

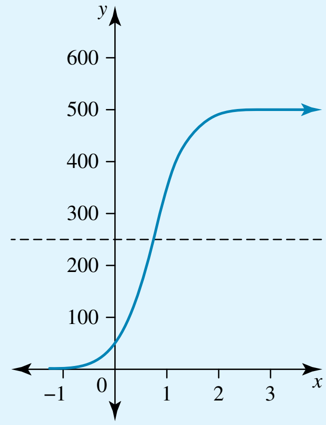
4 a 2008

b 2.197 million

5 a, b Check with your teacher.

c $\left(\frac{1}{3}\log_e(9), 250\right)$

d



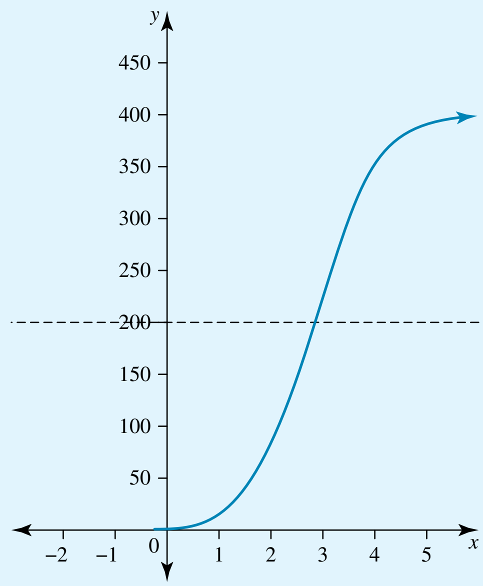
6 a 2.8 kg

b 12.5 years

7 a, b Check with your teacher.

c $\left(\frac{1}{2}\log_e(199), 200\right)$

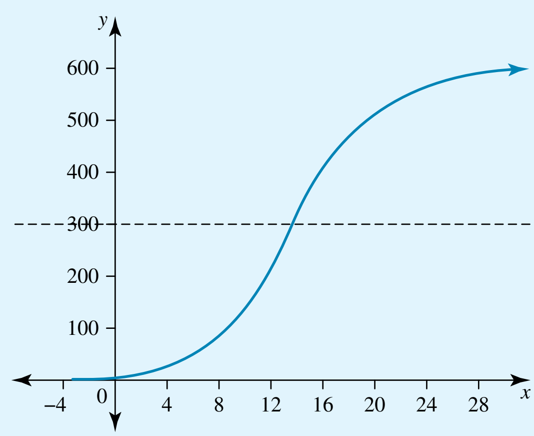
d



8 a, b Check with your teacher.

c $(3\log_e(99), 300)$

d



9 a $\frac{dN}{dt} = \frac{N(4000 - N)}{50\,000},\ N(0) = 500$

b $N(t) = \frac{4000}{1 + 7e^{-\frac{2t}{5}}}$

c 2054

d 7.61 years

10 a $\frac{dN}{dt} = kN(360 - N),\ N(0) = 1$

b 5 days (4.9 days)

11 a $\frac{dN}{dt} = kN(2000 - N),\ N(0) = 2$

b 5 days (4.4 days)

c 5 days (4.4 days)

12 7374

13 a 5.3 billion

b 2018

c 2012

d 1967

14 a i 3.31 kg

ii 74.965 kg

b i 51 cm

ii 190 cm

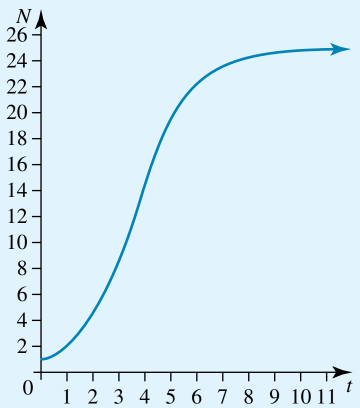
iii 5 years (4.8 years)

15 a 25

b 4 days (3.55 days)

c 22

d



16 a 12 billion

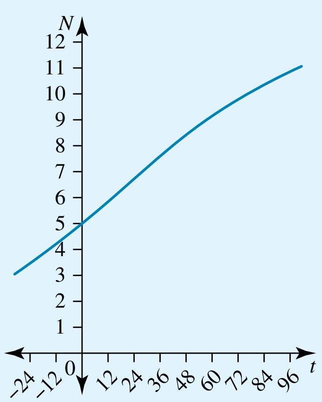
b i 1959

ii 1974

c 2056

d 1999

e

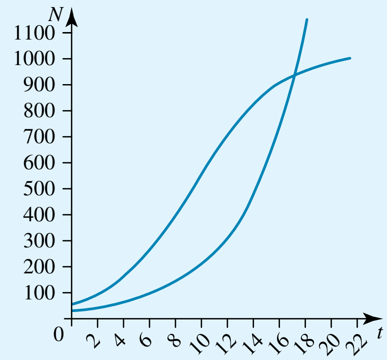


17 a $n1(t) = \frac{1000}{1 + 19e^{-0.2944t}}$

b $n2(t) = 25e^{0.19254t}$

c 18.8 h, 930 bacteria of each type

d



18 Check with your teacher.

EXERCISE 11.7

1 a

x	4	4.25	4.5	4.75	5
y	1	2.5	4.0462	5.6372	7.2717

b 7.3607 underestimates by 1.2%.

2 $\frac{7}{16}$

3 a

x	2	2.25	2.5	2.75	3
y	3	3.1667	3.3246	3.4750	3.6188

b 3.6056 overestimates by 0.4%.

4

x	0.2	0.3	0.4	0.5
y	0.4	0.4423	0.4896	0.5429

5 a

x	0	0.25	0.5	0.75	1
y	2	2	1.9688	1.7265	1.0977

b 1 overestimates by 9.8%.

6 a 1.7208

b 1.7208 underestimates by 4.7%.

7 a Approximated value $y_2 = 3.9689$; exact value of $y(1) = 3.9177$.

b Approximated value $y_3 = 3.5836$; exact value of $y(1) = 3.6633$.

c Approximated value $y_4 = 4.1975$; exact value of $y(1) = 3.7293$.

8 a i

x	1	$\frac{3}{2}$	2
y	2	-1	$-\frac{11}{2}$

ii

x	1	$\frac{4}{3}$	$\frac{5}{3}$	2
y	2	0	$-\frac{8}{3}$	-6

iii

x	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
y	2	$\frac{1}{2}$	$-\frac{11}{8}$	$-\frac{29}{8}$	$-\frac{25}{4}$

iv

x	1	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	2
y	2	$\frac{4}{5}$	$-\frac{16}{25}$	$-\frac{58}{25}$	$-\frac{106}{25}$	$-\frac{32}{5}$

b Exact value -7

i -21.4%

ii -14.3%

iii -10.7%

iv -8.6%

9 a $\frac{8}{9}$

b 1

c $\frac{101}{210}$

10 a, b Check with your teacher.

c i Check with your teacher.

ii $\frac{11}{2}\log_e(11) - \frac{9}{2}\log_e(9) + 3$

11 a i $\frac{49}{9}$

ii $\frac{4000}{729}$

iii Exact value 5.582 45; error for **i** is -2.5%, error for **ii** is -1.7%.

b i $\frac{7}{2}$

ii $\frac{7945}{2187}$

iii 3.7641; error for **i** is -7%, error for **ii** is -3.5%.

12 a i

x	1	1.3333	1.6667	2
y	2	0.6667	0.4691	0.3469

ii

x	1	1.25	1.5	1.75	2
y	2	1	0.6875	0.5103	0.3963

b 0.5; error for **i** is 31%, error for **ii** is 21%.

13 a i 44.8696

ii 54.4038

b i 4.9840

ii 4.9851

14 a i

x	0	0.25	0.5	0.75	1
y	2	2	2.25	2.6875	3.2656

ii $a = 4, b = -2, c = 4, k = -1; y(1) = 3.4715$

b i

x	0	0.25	0.5	0.75	1
y	1	1.75	3.1875	5.6563	9.6719

ii $a = -\frac{5}{2}, b = -\frac{7}{4}, c = \frac{11}{4}, k = 2; y(1) = 16.0699$

15 a i, ii Check with your teacher.

iii $2 - 4\cos(1) + 4\sin\left(\frac{1}{2}\right)$

b i, ii Check with your teacher.

iii $\frac{1}{9}(37 - 4e^{-3})$

16 Check with your teacher.

17 a Check with your teacher.

b i -2.14%

ii -1.07%

iii -0.42%

18 a Check with your teacher.

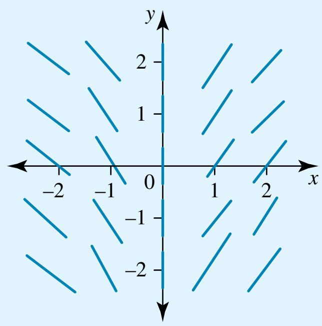
b i -0.54%

ii -0.27%

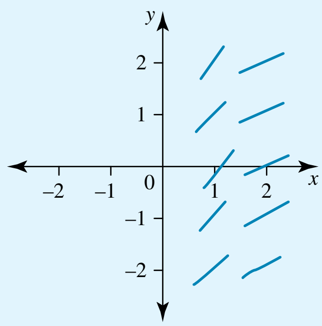
iii -0.11%

EXERCISE 11.8

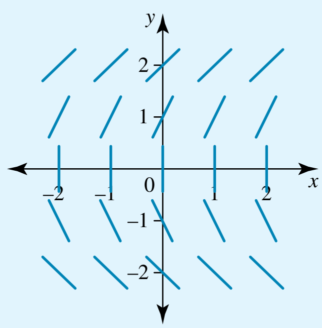
1



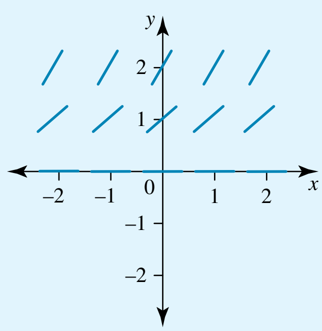
2



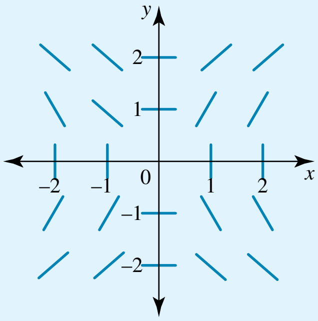
3



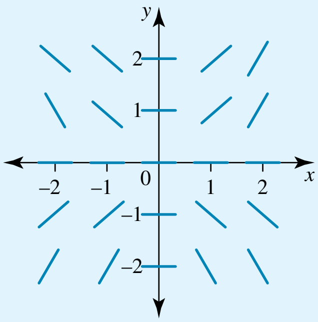
4



5



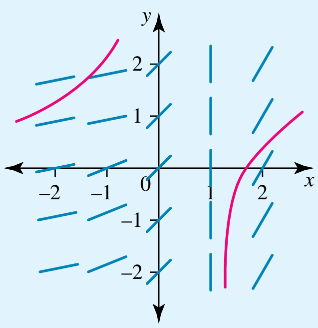
6



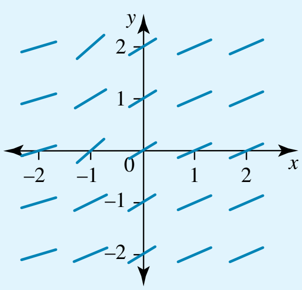
7 C

8 E

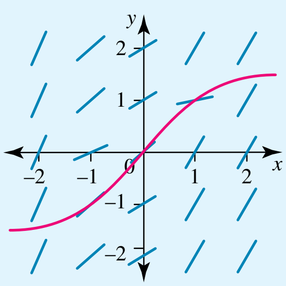
9 a



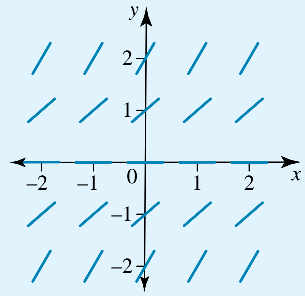
b **i**



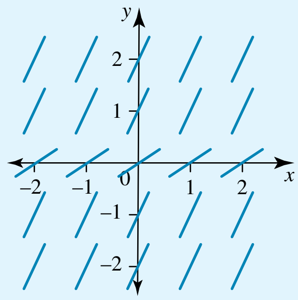
ii $y = \tan^{-1}(x)$



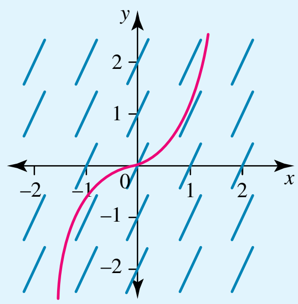
10 a



b **i**

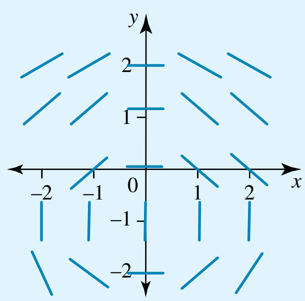


ii

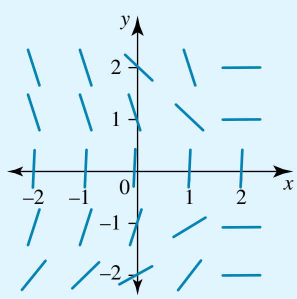


$y = \tan(x)$

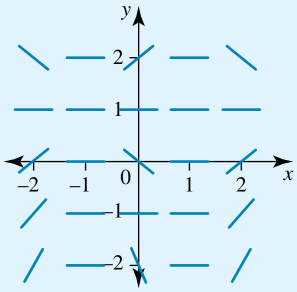
11 a



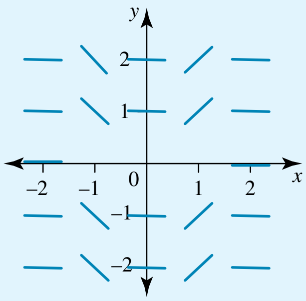
b



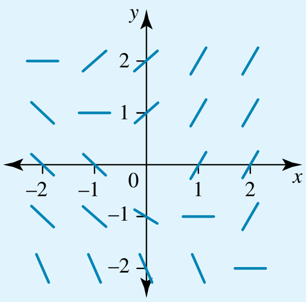
12 a



b



13 a



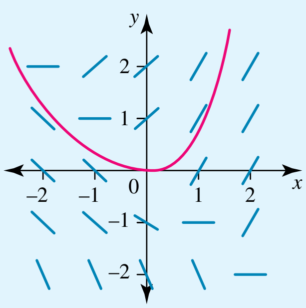
b $A = 2, B = -1, C = -1, k = 1$

c Check with your teacher.

d

x	0	0.5	1
y	0	0	0.25

e



14 a Slope 4 **b** Slope 2 **c** Slope 3 **d** Slope 1

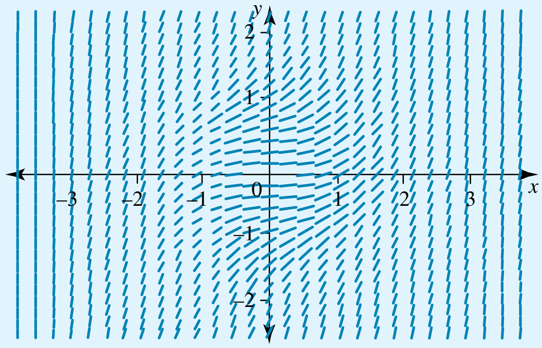
15 a Slope 3 **b** Slope 1 **c** Slope 4 **d** Slope 2

16 a Slope 1 **b** Slope 3 **c** Slope 2 **d** Slope 4

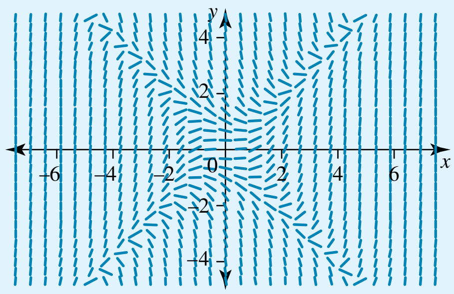
17 a Slope 2 **b** Slope 3 **c** Slope 1 **d** Slope 4

18 a Slope 1 **b** Slope 4 **c** Slope 2 **d** Slope 3

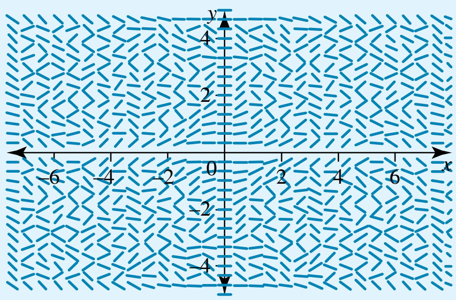
19 a



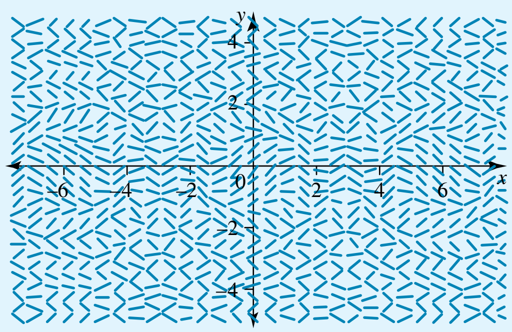
b



20 a



b



12

Variable forces

12.1 Kick off with CAS

Review of velocity and acceleration functions

Acceleration is the derivative of velocity, so $a(t) = v'(t)$, and given an acceleration function, the velocity function can be determined by the integral $v(t) = \int a(t)\,dt$ and the displacement function is $s(t) = \int v(t)\,dt$.

Using this information together with CAS, solve the following problems.

1 An object moves along a straight line with acceleration given by $a(t) = 1 + \sin(\pi t)$. When $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$ and $v(t)$.

2 An object is shot upwards from ground level with an initial velocity of 100 metres per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground.

3 For the following velocity functions, determine the net distance travelled and the total distance travelled in the specified time interval.

a $v = -9.8t + 49,\ 0 \le t \le 10$

b $v = 3(t - 3)(t - 1),\ 0 \le t \le 5$

c $v = \sin\left(\dfrac{\pi t}{3}\right) - t,\ 0 \le t \le 1$

4 A particle starts from rest at the origin and travels initially to the right with an acceleration of a m/s^2 where $a = 4t + 2$.

a Find equations to describe the velocity and position of the particle at time t.

b Find the velocity and displacement at time $t = 4$ seconds.

c Will the particle ever return to the origin?

5 A particle is travelling in a straight line with a velocity given by $v(t) = 50 - 10e^{-0.5t}$ m/s.

a What is the initial velocity?

b Calculate the velocity after 5 seconds.

c At what time will the particle's velocity be 45 m/s?

d Show that the particle's velocity is always positive.

e Calculate the total distance travelled by the particle in the first 5 seconds of motion.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

12.2 Forces that depend on time

Setting up the equation of motion

studyon

Units 3 & 4
AOS 3
Topic 5
Concept 1

Differentiation with displacement, velocity and acceleration
Concept summary
Practice questions

In this topic we consider the motion of an object of constant mass m moving in a straight line and subjected to a system of forces. In previous topics, the forces F have been constant, and since $F = ma$, the acceleration a was also constant, so the constant acceleration formulas could be used. However, if the forces are dependent upon time, t, velocity, v, or displacement, x, then the constant acceleration formulas *cannot* be used. In situations where the forces are not constant, a differential equation must be set up and solved.

Note: The notation used for acceleration in this topic is two dots above x, that is $\ddot{x}$.

The dot above a variable represents differentiation with respect to time.

If the resultant force, $F = F(t)$, depends upon time, t, then by Newton's Second Law of Motion, $m\ddot{x} = m\frac{dv}{dt} = F(t)$. Because the mass is constant, the acceleration, $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \ddot{x}$, is a function of time, t, where v is the velocity. Thus,

$$a = \ddot{x} = \frac{dv}{dt} = \frac{F(t)}{m}.$$

This equation is called the equation of motion. It can be solved by integrating both sides with respect to t to obtain the velocity as $v = \frac{1}{m}\int F(t)dt$. A constant of integration can be found using an initial condition for v and t. Because $v = \frac{dx}{dt}$, we can integrate again to express the displacement, x, in terms of t. This is another application of solving a second-order differential equation.

WORKED EXAMPLE 1

An object of mass 2 kilograms moves in a straight line and is acted upon by a force of $12t - 24$ newtons, where t is the time in seconds and $t \geq 0$. Initially the object is 5 metres to the right of the origin, and it comes to rest after 2 seconds. Find the distance travelled by the object in the first 2 seconds.

THINK	WRITE/DRAW
1 Use Newton's Second Law of Motion.	$m\ddot{x} = F(t)$ where $m = 2$ and $F(t) = 12t - 24$
2 Formulate the differential equation to be solved (the equation of motion).	$2\ddot{x} = 12t - 24$ $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 6t - 12$
3 Integrate both sides with respect to t.	$v = \int (6t - 12)dt$
4 Perform the integration using basic integration techniques, adding in the first constant of integration.	$v = 3t^2 - 12t + c_1$

5 Use the given initial condition to find the first constant of integration.

When $t = 2$, the object is at rest, so $v = 0$.
Substitute to find c_1:
$$0 = 12 - 24 + c_1$$
$$c_1 = 12$$

6 Substitute back for the constant of integration.

$$v = \frac{dx}{dt} = 3t^2 - 12t + 12$$

7 Integrate both sides again with respect to t.

$$x = \int (3t^2 - 12t + 12)dt$$

8 Perform the integration, adding in a second constant of integration.

$$x = t^3 - 6t^2 + 12t + c_2$$

9 Use the given initial condition to find the second constant of integration.

Initially when $t = 0$, $x = 5$.
Substitute to find c_2:
$$5 = 0 + c_2$$
$$c_2 = 5$$

10 Substitute back for the constant of integration.

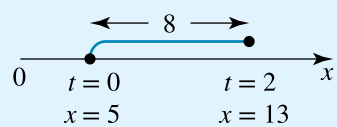
$$x = x(t) = t^3 - 6t^2 + 12t + 5$$

11 Find the displacement at the required time.

Substitute $t = 2$:
$$x(2) = 8 - 24 + 24 + 5$$
$$= 13$$

12 State the required result.

The object moves from $t = 0$, $x = 5$ to $t = 2$, $x = 13$. The distance travelled is 8 metres.



Integrals involving trigonometric functions

Recall the integrals involving basic trigonometric functions and exponential functions:

$$\int \sin(kx) = -\frac{1}{k}\cos(kx)$$

$$\int \cos(kx)dx = \frac{1}{k}\sin(kx)$$

$$\int e^{kx}dx = \frac{1}{k}e^{kx}.$$

WORKED EXAMPLE 2

A particle of mass 5 kilograms moves back and forth along the x-axis and is subjected to a force of $-20\sin(2t) - 20\cos(2t)$ newtons at time t seconds. If its initial velocity is 2 metres per second and it starts from a point 3 metres to the left of the origin, express the displacement x metres in terms of t.

THINK

1 Use Newton's Second Law of Motion.

WRITE

$m\ddot{x} = F(t)$ where
$m = 5$ and $F(t) = -20\sin(2t) - 20\cos(2t)$

2 Formulate the equation of motion to be solved.

$5\ddot{x} = -20\sin(2t) - 20\cos(2t)$

$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -4\sin(2t) - 4\cos(2t)$

3 Integrate both sides with respect to t.

$v = \int (-4\sin(2t) - 4\cos(2t))dt$

4 Perform the integration using the results $\int \sin(kx)dx = -\frac{1}{k}\cos(kx)$ and $\int \cos(kx)dx = \frac{1}{k}\sin(kx)$ with $k = 2$. Note that only one constant of integration is required.

$v = 2\cos(2t) - 2\sin(2t) + c_1$

5 Use the given initial condition to find the first constant of integration.

Initially, when $t = 0$, $v = 2$:
$2 = 2\cos(0) - 2\sin(0) + c_1$
$c_1 = 0$

6 Substitute back for the constant of integration.

$v = \frac{dx}{dt} = 2\cos(2t) - 2\sin(2t)$

7 Integrate both sides again with respect to t.

$x = \int (2\cos(2t) - 2\sin(2t))dt$

8 Perform the integration, adding in a second constant of integration.

$x = \sin(2t) + \cos(2t) + c_2$

9 Use the given initial condition to find the second constant of integration.

Initially, when $t = 0$, $x = -3$ (to the left):
$-3 = \sin(0) + \cos(0) + c_2$
$c_2 = -4$

10 Substitute back for the constant of integration. This is the required result.

$x = \sin(2t) + \cos(2t) - 4$

Horizontal rectilinear motion

When a driver of a car travelling at some initial speed applies the brakes, the braking force for a short period of time is a resistance force that opposes the direction of motion. The force is a function of the time during which the brakes are applied. This can be used to model the equation of motion during this time and determine, for example, the distance travelled while braking.

WORKED EXAMPLE 3

A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is 32 000 − 12 800t newtons, where t is the time in seconds after the driver applies the brakes. Find:

a the time after which the speed of the car has been reduced to 57.6 km/h

b the distance travelled in this time.

THINK	WRITE/DRAW
a 1 Use Newton's Second Law of Motion. The braking force opposes the direction of motion.	a $m\ddot{x} = -F(t)$ where $m = 1600$ and $F(t) = 32\,000 - 12\,800t$

2 Formulate the equation of motion to be solved.	$1600\ddot{x} = 12\,800t - 32\,000$ $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 8t - 20$
3 Integrate both sides with respect to t.	$v = \int (8t - 20)dt$
4 Perform the integration using the basic integration techniques, adding in the first constant of integration.	$v = 4t^2 - 20t + c_1$
5 Use the given initial conditions to find the first constant of integration. We need to use correct units.	The initial speed is 90 km/h. Convert km/h to m/s: $\frac{90 \times 1000}{60 \times 60} = 25$ m/s Initially, when $t = 0$, $v = 25$: $25 = 0 + c_1$ $c_1 = 25$
6 Substitute back for the constant of integration.	$v = 4t^2 - 20t + 25$
7 Determine the braking time.	The final speed is 57.6 km/h. Convert km/h to m/s: $\frac{57.6 \times 1000}{60 \times 60} = 16$ m/s Find t when $v = 16$.
8 The earlier time is the one required	$16 = 4t^2 - 20t + 25$ $4t^2 - 20t + 9 = 0$ $(2t - 1)(2t - 9) = 0$ $t = \frac{1}{2}, \frac{9}{2}$ $t = \frac{1}{2}$
9 State the required result.	After 0.5 seconds, the car's speed has been reduced from 90 to 57.6 km/h.
b 1 Express the velocity in terms of time, t.	**b** $v = \frac{dx}{dt} = 4t^2 - 20t + 25$
2 Integrate both sides again with respect to t.	$x = \int (4t^2 - 20t + 25)dt$
3 Perform the integration using basic integration techniques, adding in a second constant of integration.	$x = \frac{4t^3}{3} - 10t^2 + 25t + c_2$
4 Use the given initial condition to find the second constant of integration.	Since we want to find the distance travelled from first braking, assume that when $t = 0$, $x = 0$, so that $c_2 = 0$.

5 Express the displacement travelled in terms of time.

$$x = x(t) = \frac{4t^3}{3} - 10t^2 + 25t$$

6 Find the distance travelled, D metres, while the car is braking, as in these cases the distance travelled is the displacement.

$$D = x(t)$$

$$D = \frac{4 \times (0.5)^3}{3} - 10 \times 0.5^2 + 25 \times 0.5$$

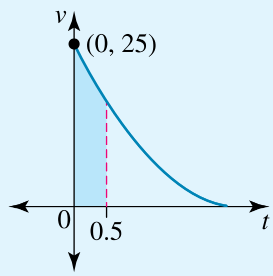
$$= 10\frac{1}{6}$$

7 State the final result.

The distance travelled while the car is braking is $10\frac{1}{6}$ metres.

8 An alternative method to find the distance travelled can be used.

Since the distance travelled is the area under the velocity–time graph, this distance is given by the definite integral.



$$D = \int_0^{\frac{1}{2}} (4t^2 - 20t + 25)dt$$

$$= \int_0^{\frac{1}{2}} (2t - 5)^2 dt$$

$$= \left[\frac{1}{6}(2t - 5)^3\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{6}(-4)^3 - \frac{1}{6}(-5)^3$$

$$= 10\frac{1}{6}$$

EXERCISE 12.2 Forces that depend on time

PRACTISE

1 WE1 An object of mass 3 kilograms moves in a straight line and is acted upon by a force of $6t - 18$ newtons, where t is the time in seconds and $t \geq 0$. Initially the object is 4 metres to the right of the origin, and it comes to rest after 3 seconds. Find the distance travelled by the object in the first 3 seconds.

2 A particle of mass 5 kilograms moves in a straight line and is acted upon by a force of $60t - 60$ newtons, where t is the time in seconds. Initially the particle is at the origin, and after 1 second the particle is 52 metres to the left of the origin. Find when the particle first comes to rest.

3 WE2 A particle of mass 8 kilograms moves back and forth along the x-axis and is subjected to a force of $-16\sin\left(\frac{t}{3}\right)$ newtons at time t seconds. If its initial velocity is 6 m/s and the particle starts at the origin, express the displacement, x metres, in terms of t.

4 A particle of mass 8 kilograms moves back and forth along the x-axis and is subjected to a force of $-16\cos\left(\frac{t}{2}\right)$ newtons at time t seconds. If initially the particle is at rest and starts from a point 8 metres from the origin, find the displacement of the particle after a time of $\frac{\pi}{2}$ seconds.

5 **WE3** A car of mass 1500 kg is moving along a straight road at a speed of 72 km/h when the driver brakes. The braking force is $180\,000t - 75\,000$ newtons, where t is the time in seconds after the driver applies the brakes. Find:

a the time after which the speed of the car has been reduced to 36 km/h

b the distance travelled in this time.

6 A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $\frac{320}{(t+2)^3}$ kN, where t is the time in seconds after the driver applies the brakes. Find:

a the time taken for the speed of the car to be reduced to 57.6 km/h

b the distance travelled in this time.

CONSOLIDATE

7 a A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $6 - 18t$ newtons, where t is the time in seconds and $t \geq 0$. Initially the body is at rest at the origin. Express x in terms of t.

b A particle of mass 500 grams is moving along the x-axis and at time t seconds is subject to a force of $4e^{2t} - 2$ newtons. Initially the body is at rest at the origin. Express x in terms of t.

8 a A particle of mass 5 kilograms moves in a straight line and is acted upon by a force of $30t - 40$ newtons, where t is the time in seconds. Initially the particle is 3 metres to the right of the origin and moving to the right with a speed of 2 m/s. Find its displacement after 2 seconds.

b A particle of mass 2 kilograms is moving along the x-axis and at time t seconds is subject to a force of $48t - 12$ newtons. If after 1 second its displacement is 4 metres and its initial velocity is 1 m/s, find its displacement at any time t seconds.

9 A bus of mass 6 tonnes moves in a straight line between two stops. The force acting on the bus as it moves between the two stops is given by $2000 - \frac{200t}{9}$ newtons, where t is the time in seconds after it leaves the first stop. Find the distance between the two stops and the time it takes to travel between them.

10 A particle of mass 2 kilograms moves in a straight line so that at time t seconds it is acted upon by a force of $8 - 8e^{-0.1t}$ newtons. If initially it is moving away from the origin with a velocity of 12 metres per second, find how far it has travelled in the first 5 seconds.

11 a A particle of mass 4 kilograms moves back and forth along the x-axis and is subjected to a force of $-144\cos(3t)$ newtons at time t seconds. If initially it is at rest 2 metres from the origin, find the furthest distance it reaches from the origin.

b A particle of mass 2 kilograms moves back and forth along the x-axis and is subjected to a force of $-6\sin\left(\frac{t}{2}\right)$ newtons at time t seconds. If its initial velocity is 6 m/s and the particle starts from a point 4 metres from the origin, find the greatest distance it reaches from the origin.

12 A particle of mass 800 kg moves in a straight line so that at time t seconds it is retarded by a force of $\frac{640}{(t+5)^3}$ kN. If the initial velocity of the particle is 16 m/s, find how far it has travelled in the first 5 seconds.

13 a A particle is moving in a straight line path and has a constant acceleration of a m/s^2. If its initial velocity is u m/s and it starts from the origin, show using calculus that its velocity, v m/s, and displacement, s metres, at any time, t seconds, are given by $v = u + at$ and $s = ut + \frac{1}{2}at^2$.

b A bullet of mass 15 g is fired vertically upwards with an initial speed of 49 m/s from a height of 2 m above the ground. It is subjected only to the gravitational force. Find:

i its height in metres above the ground at any time, t seconds

ii the greatest height above ground level that the bullet reaches.

14 a A car of mass 1500 kg moves in a straight line. When travelling at 60 km/h the driver applies the brakes. The braking force is $50t$ kN, where t is the time in seconds after the driver applies the brakes. Find the distance travelled until the car comes to rest.

b A car of mass m kg is travelling at a speed of U m/s along a level road when the driver applies the brakes. If the braking force is kt newtons, where t is the time in seconds after the driver applies the brakes and k is a positive constant, show that the car comes to rest after a time of $\sqrt{\frac{2mU}{k}}$ seconds and travels a distance of $\frac{2U}{3}\sqrt{\frac{2mU}{k}}$ metres.

15 a A particle of mass 3 kilograms is acted upon by a horizontal force of $29.4e^{-0.2t}$ newtons at a time t seconds. Its initial velocity is zero and it starts 20 metres to the right of the origin. Find the displacement at time t.

b A body of mass m kg is moving in a straight line path on a horizontal table by a force of be^{-kt} newtons, where b and k are positive constants. If its initial speed is U m/s, show that after a time t seconds its displacement is given by $x = Ut + \frac{b}{mk}\left[t + \frac{1}{k}(e^{-kt} - 1)\right]$ metres.

16 A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $\frac{59040\sqrt{2}}{\sqrt{(369t + 128)^3}}$ kN where t is the time in seconds after the driver applies the brakes. After a time T seconds, the speed of the car has been reduced to 57.6 km/h, and in this time the car has travelled D metres. Find the values of T and D for this situation.

MASTER

17 a A particle of mass 1 kg is acted upon by a time-varying force of $(4t^2 - 8t + 2)e^{-2t}$ newtons, where t is the time in seconds. Initially the particle is at rest at the origin. Using CAS, find the distance travelled over the first second of its motion.

b A particle of mass 3 kg is acted upon by a time varying force of $6\cos(2t)e^{\sin(2t)}$ newtons, where t is the time in seconds. Initially the particle is at rest at the origin. Find the distance travelled over the first 2 seconds of its motion.

18 a A particle of mass 2 kg is acted upon by a time-varying force of $2t\cos(t^2)$ newtons, where t is the time in seconds. Initially the particle is at rest at the origin. Find the distance travelled over the first 2 seconds of its motion.

b A particle of mass 2 kg is retarded by a time-varying force of $(24\cos(3t) + 10\sin(3t))e^{-2t}$ newtons, where t is the time in seconds. Initially the particle is at the origin, moving with a speed of 3 m/s. Using CAS, find the distance travelled over the first second of its motion.

12.3 Forces that depend on velocity

Setting up the equation of motion

studyon

Units 3 & 4
AOS 3
Topic 5
Concept 2

Antidifferentiation with displacement, velocity and acceleration
Concept summary
Practice questions

If the force $F = F(v)$ acting on a body of mass m depends upon the velocity, v, then by Newton's Second Law of Motion, $m\ddot{x} = F(v)$. Because $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt}.\frac{dv}{dx} = v\frac{dv}{dx}$ by the chain rule, there are two possible approaches.

Inverting $\ddot{x} = \frac{dv}{dt} = \frac{F(v)}{m}$ gives $\frac{dt}{dv} = \frac{m}{F(v)}$. Integrating both sides with respect to v gives $t = m\int \frac{1}{F(v)}\,dv$, since the mass is constant, with initial conditions on t and v. This gives us a relationship between v and t.

Alternatively, inverting $\ddot{x} = v\frac{dv}{dx} = \frac{F(v)}{m}$ gives $\frac{dx}{dv} = \frac{mv}{F(v)}$, and integrating both sides with respect to v gives $x = m\int \frac{v}{F(v)}\,dv$, with initial conditions on x and v. This gives us a relationship between v and x.

WORKED EXAMPLE 4

A body of mass 5 kg moving in a straight line is opposed by a force of $10(v + 3)$ newtons, where v is the velocity in m/s. Initially the body is moving at 3 m/s and is at the origin. Show that the displacement x at time t is given by $x = 3(1 - t - e^{-2t})$.

THINK	WRITE
1 Use Newton's Second Law of Motion. The force opposes the direction of motion.	$m\ddot{x} = -F(v)$ where $m = 5$ and $F(v) = 10(v + 3)$
2 Formulate the equation of motion to be solved.	$5\ddot{x} = -10(v + 3)$ $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -2(v + 3)$
3 Invert both sides or separate the variables.	$\frac{dt}{dv} = -\frac{1}{2(v + 3)}$ $-2\frac{dt}{dv} = \frac{1}{v + 3}$
4 Integrate both sides with respect to v.	$-2t = \int \frac{1}{v + 3}\,dv$
5 Perform the integration, placing the first constant of integration on one side of the equation.	$-2t = \log_e\left(\lvert v + 3\rvert\right) + c_1$
6 Use the given initial conditions to find the first constant of integration.	Initially, when $t = 0$, $v = 3$: $0 = \log_e(6) + c_1$ $c_1 = -\log_e(6)$
7 Substitute back for the first constant of integration.	$-2t = \log_e\left(\lvert v + 3\rvert\right) - \log_e(6)$
8 Use the laws of logarithms.	$-2t = \log_e\left(\frac{\lvert v + 3\rvert}{6}\right)$ The modulus signs are not needed.
9 Use the definition of logarithms to solve for v.	$\frac{v + 3}{6} = e^{-2t}$ $v + 3 = 6e^{-2t}$ $v = 6e^{-2t} - 3$
10 Use $v = \frac{dx}{dt}$.	$v = \frac{dx}{dt} = 6e^{-2t} - 3$
11 Integrate both sides with respect to t.	$x = \int (6e^{-2t} - 3)dt$
12 Perform the integration, placing the second constant of integration on one side of the equation.	$x = -3e^{-2t} - 3t + c_2$
13 Use the given initial conditions to find the second constant of integration.	Initially, when $t = 0$, $x = 0$ (it's at the origin): $0 = -3 + c_2$ $c_2 = 3$
14 Substitute back for the second constant of integration.	$x = -3e^{-2t} - 3t + 3$
15 Factorise and the result is shown.	$x = 3(1 - t - e^{-2t})$

Horizontal rectilinear motion

When a driver of a car travelling at some initial speed applies the brakes, the braking force for a short period of time is a resistance force that opposes the direction of motion and is a function of the speed as the brakes are applied. The equation of motion can be modelled during this time, and the distance travelled while braking can be determined.

WORKED EXAMPLE 5

A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $6400\sqrt{v}$ newtons where v is the speed in m/s after the driver applies the brakes. Find:

a the time taken for the speed of the car to be reduced to 57.6 km/h

b the distance travelled in this time.

THINK	WRITE
a 1 Use Newton's Second Law of Motion. The braking force opposes the direction of motion.	**a** $m\ddot{x} = -F(v)$ where $m = 1600$ and $F(v) = 6400\sqrt{v}$
2 Formulate the equation of motion to be solved.	$1600\ddot{x} = -6400\sqrt{v}$ $\ddot{x} = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = -4\sqrt{v}$
3 Invert both sides or separate the variables.	$\dfrac{dt}{dv} = -\dfrac{1}{4\sqrt{v}}$ $-4\dfrac{dt}{dv} = \dfrac{1}{\sqrt{v}} = v^{-\frac{1}{2}}$
4 Integrate both sides with respect to v.	$-4t = \int v^{-\frac{1}{2}}\,dv$
5 Perform the integration, placing the first constant of integration on one side of the equation.	$-4t + c_1 = 2v^{\frac{1}{2}}$ $= 2\sqrt{v}$
6 Use the given initial conditions to find the first constant of integration. We need to use correct units.	90 km/h = 25 m/s Initially, when $t = 0$, $v = 25$: $c_1 = 2\sqrt{25}$ $= 10$
7 Substitute back for the constant of integration.	$10 - 4t = 2\sqrt{v}$
8 Determine the braking time.	57.6 km/h = 16 m/s Find t when $v = 16$: $10 - 4t = 2\sqrt{16} = 8$ $4t = 2$ $t = \dfrac{1}{2}$
9 State the required result.	After 0.5 seconds, the car's speed has been reduced from 90 to 57.6 km/h.

b **1** Use the alternative form of the equation for the acceleration.

b $\ddot{x} = v\frac{dv}{dx} = -4\sqrt{v}$

$$\frac{dv}{dx} = -\frac{4\sqrt{v}}{v}$$
$$= -\frac{4}{\sqrt{v}}$$

2 Invert both sides or separate the variables.

$$-4\frac{dx}{dv} = v^{\frac{1}{2}}$$
$$-4\int dx = \int v^{\frac{1}{2}}dv$$

3 Perform the integration, placing the second constant of integration on one side of the equation.

$$-4x + c_2 = \frac{2}{3}v^{\frac{3}{2}}$$

4 Use the given initial condition to find the second constant of integration.

When $v = 25$, $x = 0$:
$$c_2 = \frac{2}{3}\sqrt{25^3}$$
$$= \frac{250}{3}$$

5 Substitute back for the second constant of integration.

$$\frac{250}{3} - 4x = \frac{2}{3}v^{\frac{3}{2}}$$

6 Find the distance travelled.

Solve for x when $v = 16$:
$$4x = \frac{250}{3} - \frac{2}{3}\left(\sqrt{16}\right)^3$$
$$4x = \frac{250 - 128}{3}$$
$$x = \frac{61}{6}$$

7 State the distance travelled while braking.

The distance travelled while braking is $10\frac{1}{6}$ metres.

General cases

When a body moves, the drag force is in general proportional to some power of the velocity. That is, the total air resistance and drag forces can be expressed as $F(v) = kv^n$; typical values of n are 1, 2, 3, 4, 5, $\frac{1}{2}$, $\frac{3}{2}$ and so on. Using these and generalising from the last example, some general expressions can be derived.

WORKED EXAMPLE 6

A car is moving along a straight road at a velocity of U m/s when the driver brakes. The braking force in newtons is proportional to the fourth power of the velocity, v, where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2}U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{9U}{14}$.

THINK	WRITE
1 Use Newton's Second Law of Motion. The braking force opposes the direction of motion.	Let the mass of the car be m and the proportionality constant be k. Then $F(v) = kv^4$ and $m\ddot{x} = -F(v) = -kv^4$ $\ddot{x} = -\lambda v^4$ where $\lambda = \frac{k}{m}$ is one single constant.
2 First obtain a relationship between v and t.	Use $\ddot{x} = \frac{dv}{dt} = -\lambda v^4$.
3 Invert both sides or separate the variables.	$\frac{dt}{dv} = -\frac{1}{\lambda v^4}$
4 Integrate both sides with respect to v.	$\int -\lambda dt = \int v^{-4} dv$
5 Perform the integration, placing the first constant of integration on one side of the equation.	$-\lambda t + c_1 = -\frac{1}{3}v^{-3}$
6 Use the given initial conditions to find the first constant of integration. We are using correct units.	Initially, when $t = 0$, $v = U$: $c_1 = -\frac{1}{3}U^{-3}$ $= -\frac{1}{3U^3}$
7 Substitute back for the constant of integration.	$-\lambda t - \frac{1}{3U^3} = -\frac{1}{3v^3}$ $\lambda t + \frac{1}{3U^3} = \frac{1}{3v^3}$
8 Obtain a relationship between the parameters.	When $t = T$, $v = \frac{1}{2}U$: $\lambda T + \frac{1}{3U^3} = \frac{1}{3\left(\frac{1}{2}U\right)^3}$ $= \frac{8}{3U^3}$
9 Simplify this relationship and express λ in terms of T and U.	$\lambda T = \frac{8}{3U^3} - \frac{1}{3U^3}$ $\lambda T = \frac{7}{3U^3}$ $\lambda = \frac{7}{3TU^3}$
10 Next, obtain a relationship between v and x.	Use $\ddot{x} = v\frac{dv}{dx} = -\lambda v^4$ so that $\frac{dv}{dx} = -\lambda v^3$
11 Invert both sides or separate the variables.	$\frac{dx}{dv} = -\frac{1}{\lambda v^3}$

12 Integrate both sides with respect to v.

$$\int -\lambda dx = \int v^{-3} dv$$

13 Perform the integration, placing the second constant of integration on one side of the equation.

$$-\lambda x + c_2 = -\frac{1}{2}v^{-2}$$

14 Use the given initial conditions to find the second constant of integration.

Initially (when $t = 0$) $x = 0$ and $v = U$:

$$c_2 = -\frac{1}{2}U^{-2} = -\frac{1}{2U^2}$$

15 Substitute back for the second constant of integration.

$$-\lambda x - \frac{1}{2U^2} = -\frac{1}{2v^2}$$

$$\lambda x + \frac{1}{2U^2} = \frac{1}{2v^2}$$

16 Obtain a relationship between the parameters.

When $x = D$, $v = \frac{1}{2}U$:

$$\lambda D + \frac{1}{2U^2} = \frac{1}{2\left(\frac{1}{2}U\right)^2} = \frac{2}{U^2}$$

17 Simplify this relationship and express λ in terms of D and U.

$$\lambda D = \frac{2}{U^2} - \frac{1}{2U^2}$$

$$\lambda D = \frac{3}{2U^2}$$

$$\lambda = \frac{3}{2DU^2}$$

18 Eliminate λ by equating the two expressions for λ.

$$\lambda = \frac{7}{3TU^3} = \frac{3}{2DU^2}$$

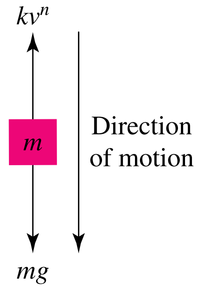
19 Simplify the resulting expression.

Hence $\frac{D}{T} = \frac{9U}{14}$ as required.

Vertical motion

When a body moves vertically, its weight force must be considered as part of its equation of motion.

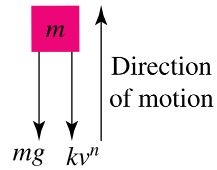
Downwards motion



Consider a body of mass m moving vertically downwards. The forces acting on the body are its weight force, which acts vertically downwards, and the force of air resistance, which opposes the direction of motion and acts vertically upwards.

The resistance force is proportional to some power of the body's velocity. Considering downwards as the positive direction, the body's equation of motion is given by $m\ddot{x} = mg - kv^n$. As the body falls, it reaches a so-called terminal or limiting velocity, v_T. This value can be obtained from $v_T = \lim_{t \to \infty} v(t)$, or as it is a constant speed when the acceleration is zero, $\ddot{x} = 0$. Thus, the terminal velocity satisfies $mg - k{v_T}^n = 0$.

Upwards motion



Consider a body of mass m moving vertically upwards. The forces acting on the body are its weight force, which acts vertically downwards, and the force of air resistance, which opposes the direction of motion and also acts vertically downwards.

The resistance force is proportional to some power of the body's velocity. Considering upwards as the positive direction, the body's equation of motion is given by $m\ddot{x} = -mg - kv^n = -(mg + kv^n)$.

Typical values of n are 1, 2, 3, 4, 5, $\frac{1}{2}$ and $\frac{3}{2}$.

WORKED EXAMPLE 7

A large brick of mass 5 kg is accidentally dropped from a high-rise construction site. As it falls vertically downwards it is retarded by a force of $0.01v^2$ newtons, where v m/s is the speed of the brick at a time t seconds after it was dropped. It has travelled a distance of x metres in this time.

a **Show that while the brick is falling, its equation of motion is given by $\ddot{x} = \frac{4900 - v^2}{500}$.**

b **Show that $x = 250 \log_e \left(\frac{4900}{4900 - v^2} \right)$.**

c **Find the magnitude of the momentum of the brick after it has fallen a distance of 100 m.**

d **Show that $t = \frac{25}{7} \log_e \left(\frac{70 + v}{70 - v} \right)$.**

e **Find the terminal velocity of the brick.**

f **Find the time taken for the brick to fall a distance of 100 m.**

THINK	WRITE
a 1 Use Newton's Second Law of Motion.	**a** $m = 5, k = 0.01$ $m\ddot{x} = mg - kv^2$
2 Formulate the equation of motion to be solved. Simplify and the required result is shown.	$5\ddot{x} = 5 \times 9.8 - 0.01v^2$ $5\ddot{x} = 49 - \frac{v^2}{100}$ $5\ddot{x} = \frac{4900 - v^2}{100}$ $\ddot{x} = \frac{4900 - v^2}{500}$

b **1** To obtain a relationship between v and x, use $\ddot{x} = v\dfrac{dv}{dx}$ and invert both sides.

b $\ddot{x} = v\dfrac{dv}{dx} = \dfrac{4900 - v^2}{500}$

$\dfrac{dx}{dv} = \dfrac{500v}{4900 - v^2}$

2 Integrate both sides with respect to v.

$x = 500\displaystyle\int \frac{v}{4900 - v^2}\,dv$

$= -\dfrac{500}{2}\displaystyle\int \frac{-2v}{4900 - v^2}\,dv$

3 Perform the integration, placing the first constant of integration on one side of the equation.

$x = -250\log_e(4900 - v^2) + c_1$

4 Use the given initial conditions to find the first constant of integration.

Since the brick was dropped when $t = 0$, $x = 0$ and $v = 0$:

$0 = -250\log_e(4900) + c_1$

$c_1 = 250\log_e(4900)$

5 Substitute back for the first constant of integration and take out common factors.

$x = -250\log_e(4900 - v^2) + 250\log_e(4900)$

$= 250[\log_e(4900) - \log_e(4900 - v^2)]$

6 Use log laws to show the required result.

$x = 250\log_e\left(\dfrac{4900}{4900 - v^2}\right)$

c **1** To find the momentum, we first need to find the speed.

c When $x = 100$, $v = ?$

$100 = 250\log_e\left(\dfrac{4900}{4900 - v^2}\right)$

2 Use the definition of the logarithm and transpose to make v the subject.

$e^{0.4} = \dfrac{4900}{4900 - v^2}$

$4900 - v^2 = 4900e^{-0.4}$

$v^2 = 4900(1 - e^{-0.4})$

3 Find the speed when the brick has fallen this required distance.

$v = 70\sqrt{(1 - e^{-0.4})}$

$v = 40.19$

4 Find the magnitude of the momentum.

$p = mv$

$= 5 \times 40.19$

$= 200.962$

The momentum of the brick is 200.96 kg m/s.

d **1** To obtain a relationship between v and t, use $\ddot{x} = \dfrac{dv}{dt}$ and invert both sides.

d $\dfrac{dv}{dt} = \dfrac{4900 - v^2}{500}$

$\dfrac{dt}{dv} = \dfrac{500}{4900 - v^2}$

2 Integrate both sides with respect to v.

$t = \displaystyle\int \frac{500}{4900 - v^2}\,dv$

3 To find this integral, use partial fractions. Express the integrand into its partial fractions decomposition.

$$\frac{500}{4900 - v^2} = \frac{A}{70 - v} + \frac{B}{70 + v}$$
$$= \frac{A(70 + v) + B(70 - v)}{(70 - v)(70 + v)}$$
$$= \frac{70(A + B) + v(A - B)}{4900 - v^2}$$

4 Find the values of the constants A and B.

Equate the coefficients:
$A - B = 0 \Rightarrow A = B$
$70(A + B) = 500$
So $A = B = \frac{500}{140} = \frac{25}{7}$.

5 Express the integrand in a form for which we can perform the integration.

$$t = \int \frac{500}{4900 - v^2} dv$$
$$= \frac{25}{7} \int \left(\frac{1}{70 + v} + \frac{1}{70 - v}\right) dv$$

6 Perform the integration, placing the second constant of integration on one side of the equation.

$$t = \frac{25}{7}\left(\log_e\left(|70 + v|\right) - \log_e\left(|70 - v|\right) + c_2\right)$$

7 Use the given initial condition to find the second constant of integration.

When $x = 0$, $t = 0$, $v = 0$:
$$0 = \frac{25}{7}(\log_e(70) - \log_e(70) + c_2)$$
$c_2 = 0$

8 Substitute back for the second constant of integration and use log laws again. The required result is shown.

$$t = \frac{25}{7}\left(\log_e\left(|70 + v|\right) - \log_e\left(|70 - v|\right)\right)$$
$$t = \frac{25}{7}\log_e\left(\left|\frac{70 + v}{70 - v}\right|\right)$$
But since $0 \le v < 70$, the modulus signs are not needed.
$$t = \frac{25}{7}\log_e\left(\frac{70 + v}{70 - v}\right)$$

e The terminal velocity can be found when the acceleration is zero.

e $\frac{4900 - v^2}{500} = 0$
$v^2 = 4900$
$v_T = \sqrt{4900}$
$v_T = 70$
The terminal velocity is 70 m/s.

f 1 Determine the time taken for the brick to fall the required distance.

f $t = ?$ when $v = 40.19$
$$t = \frac{25}{7}\log_e\left(\frac{70 + 40.19}{70 - 40.19}\right)$$
$$= 4.67$$

2 State the time to fall the required distance. | The time taken to fall is 4.67 s.

3 An alternative method to find the time is to numerically evaluate a definite integral. | $t = \int_0^{40.19} \frac{500}{4900 - v^2}\, dv = 4.67$

EXERCISE 12.3 Forces that depend on velocity

PRACTISE

1 **WE4** A body of mass 2 kg moving in a straight line is opposed by a force of $v - 8$ newtons, where v is the velocity in m/s. Initially the body is at rest at the origin. Show that the displacement at time t is given by $x = 8\left(1 + 2\left(e^{-\frac{t}{2}} - 1\right)\right)$.

2 A body of mass 6 kg moving in a straight line is acted upon by a resistance force of $2(v - 6)$ newtons, where v is the velocity in m/s. Initially the body is moving at 12 m/s and is at the origin. Show that the displacement at time t is given by $x = 6\left(t + 3\left(1 - e^{-\frac{t}{3}}\right)\right)$.

3 **WE5** A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is $320\sqrt{v^3}$ newtons, where v is the speed in m/s after the driver applies the brakes. Find:

a the time taken for the speed of the car to be reduced to 57.6 km/h

b the distance travelled in this time.

4 A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is $\frac{369}{100}v^3$ newtons, where v is the speed in m/s after the driver applies the brakes. After a time T seconds, the speed of the car has been reduced to 57.6 km/h and in this time the car has travelled a distance of D metres. Find the values of T and D for this situation.

5 **WE6** A car moves along a straight road at a speed of U m/s when the driver brakes. The resistance braking force in newtons is proportional to the cube of the velocity v, where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2}U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{2U}{3}$.

6 A car moves along a straight road at a speed of U m/s when the driver brakes. The resistance braking force in newtons is proportional to the square root of the velocity v cubed, where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2}U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{\sqrt{2}U}{2}$.

7 **WE7** A skydiver of mass 90 kg falls vertically from rest from a plane. While falling vertically downwards he is retarded by a force of $0.1v^2$ newtons, where v m/s is his speed at a time t seconds after falling a distance of x metres.

a Show that while the skydiver is falling, his equation of motion is given by $\ddot{x} = \frac{8820 - v^2}{900}$.

b Show that $x = 450 \log_e\left(\frac{8820}{8820 - v^2}\right)$.

c Find the magnitude of the momentum of the skydiver after he has fallen a distance of 150 m.

d Determine the terminal velocity of the skydiver.

e Find the time taken for the skydiver to fall a distance of 150 m.

8 A body of mass m kg falls vertically. While falling vertically downwards the particle is retarded by a force of kv^2 newtons, where k is a positive constant and v m/s is the speed at a time t seconds. Show that the terminal speed is given by $v_T = \sqrt{\frac{mg}{k}}$ and that $t = \frac{v_T}{2g} \log_e\left(\frac{v_T - v}{v_T + v}\right)$.

CONSOLIDATE

9 a A body of mass 2 kg moving in a straight line is acted upon by a resistance force of $4v$ newtons, where v is the velocity in m/s. Initially the body is moving at 1 m/s and is at the origin. Show that the displacement at time t is given by $x = \frac{1}{2}(1 - e^{-2t})$.

b A body of mass 3 kg moving in a straight line is acted upon by a resistance force of $9(v + 4)$ newtons, where v is the velocity in m/s. Initially the body is moving at 2 m/s and is at the origin. Show that the displacement at time t is given by $x = 2(1 - e^{-3t} - 2t)$.

10 A boat of mass 500 kg is sailing in a straight line at a speed of 57.6 km/h when the driver disengages the engine. The resistance force is $400\sqrt{v}$ newtons, where v m/s is the speed of the boat at a time t seconds. Find:

a the time taken for the speed of the boat to be reduced to 14.4 km/h

b the distance travelled in this time.

11 A sports car of mass 800 kg is moving along a level road at a speed of 57.6 km/h when the driver applies the brakes. The braking force is $80v^{\frac{3}{2}}$ newtons, where v m/s is the speed of the car at a time t seconds. After a time T s, the speed of the car is 14.4 km/h, and in this time it has travelled a distance of D m. Find the values of:

a T **b** D.

12 A body of mass m kg is moving in a straight line path on a horizontal surface and is acted upon by a resistive force that is proportional to its speed, the constant of proportionality being k. If its initial speed is U m/s, show that:

a its speed, v m/s, at a time t seconds satisfies $v = Ue^{-\frac{kt}{m}}$

b its displacement, x m, after a time t seconds is given by $x = \frac{mU}{k}\left(1 - e^{-\frac{kt}{m}}\right)$

c its speed, v m/s, after moving a distance x m is given by $v = U - \frac{kx}{m}$.

13 A block of mass m kg is moving in a straight line path on a smooth horizontal surface and is acted upon by a resistive force in newtons that is proportional to the square of its speed, the constant of proportionality being k. If its initial speed is U m/s, show that:

a its speed, v m/s, after moving a distance of x metres is given by $v = Ue^{-\frac{kt}{m}}$

b its speed, v m/s, at a time t seconds satisfies $v = \dfrac{mU}{m + kUt}$.

14 A train of mass m kg is moving in a straight line and is acted upon by a force of air resistance that is equal to $a + bv$ newtons, where a and b are positive constants and v m/s is its speed at any time t seconds. If its initial speed is U m/s and it travels a distance of D m before coming to rest in a time of T s, show that:

a $T = \dfrac{m}{b} \log_e\left(1 + \dfrac{bU}{a}\right)$

b $D = \dfrac{1}{b}\,(mU - aT)$.

15 A motor car of mass m kg is travelling with a speed of v m/s along a level section of road when the brakes are applied The resistance to the motion of the car is given by $a + bv^2$ newtons, where a and b are positive constants. Show that, with the engine disengaged, the brakes will bring the car from an initial speed of U m/s to rest in a time T s, and that the car will travel a distance of D m, where

$$T = \frac{m}{\sqrt{ab}} \tan^{-1}\left(U\sqrt{\frac{b}{a}}\right) \text{ and } D = \frac{m}{2b} \log_e\left(1 + \frac{bU^2}{a}\right).$$

16 A body of mass m kg falls from rest in a gravitational field and is subject to a force of air resistance in newtons that is proportional to its speed, the constant of proportionality being k.

a Show that the speed at any time, t seconds, is given by $v = \dfrac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$.

b Deduce that the limiting (or terminal speed) is given by $\dfrac{mg}{k}$.

c Show that when the speed is half the terminal speed, the distance of the body below the point of projection is given by $\dfrac{0.193m^2g}{k^2}$ m.

17 During a snow storm, a small block of ice of mass 10 grams falls from the sky. As it falls vertically downwards it is retarded by a force of $0.002v^2$ newtons, where v m/s is its speed at a time t seconds after falling a distance of x m.

a Show that while the ice block is falling, its equation of motion is given by $\ddot{x} = \dfrac{49 - v^2}{5}$.

b Show that $v = 7\sqrt{1 - e^{-0.4x}}$.

c Show that $v = \dfrac{7(1 - e^{-2.8t})}{1 + e^{-2.8t}}$ and hence find the terminal speed of the ice block.

d Show that $x = 5 \log_e\left(\dfrac{e^{1.4t} + e^{-1.4t}}{2}\right)$.

18 A ball of mass m kg is projected vertically upwards from ground level with an initial speed of U m/s. While travelling upwards or downwards it is subjected to a force of air resistance equal to kv^2 newtons, where k is a positive constant and v is its velocity in m/s.

a Show that the ball reaches a maximum height of $\frac{m}{2k}\log_e\left(1+\frac{kU^2}{mg}\right)$ m.

b Show that the time required for the ball to reach its maximum height is given by $\sqrt{\frac{m}{kg}}\tan^{-1}\left(U\sqrt{\frac{k}{mg}}\right)$ s.

c Show that the ball returns to its original point with a speed of $U\sqrt{\frac{mg}{mg+kU^2}}$ m/s.

MASTER

19 a A bullet of mass m kg is fired horizontally from a gun and experiences a force of air resistance in newtons that varies as the cube of its speed, the constant of proportionality being k. If the initial speed of the bullet is U m/s, show that after a time t seconds, if its displacement is x m, then $t=\frac{x}{U}+\frac{kx^2}{2m}$, assuming that the motion remains horizontal.

b A bullet of mass 10 grams is fired into a bullet-proof vest. While moving horizontally it is retarded by a force of $4(v^2+10000)$ newtons, where v is its velocity in m/s and t is the time in seconds after impact. The initial speed of the bullet is 400 m/s.

i Express the time, t, in terms of the velocity, v, and hence find how long it takes for the bullet to come to rest.

ii Find how far the bullet penetrates the bullet-proof vest before coming to rest.

20 A cyclist of mass m kg travelling horizontally at a speed U m/s reaches a level section of road and begins to freewheel. She observes that after travelling a distance of D metres in time T seconds along this section of road, her speed has fallen to V m/s, where $U > V > 0$.

a If during the freewheeling her retardation is proportional to:

i her speed, show that $\frac{D}{T}=\frac{U-V}{\log_e\left(\frac{U}{V}\right)}$

ii the square of her speed, show that $\frac{D}{T}=\frac{UV\log_e\left(\frac{U}{V}\right)}{U-V}$

iii the cube of her speed, show that $\frac{D}{T}=\frac{2UV}{U+V}$

iv the fourth power of her speed, show that $\frac{D}{T}=\frac{3UV(U+V)}{2(U^2+UV+V^2)}$

v the fifth power of her speed, show that $\frac{D}{T}=\frac{4UV(U^3-V^3)}{3(U^4-V^4)}$

vi the square root of her speed cubed, show that $\frac{D}{T} = \sqrt{UV}$

vii the nth power of her speed, show that $\frac{D}{T} = \frac{(U^{2-n} - V^{2-n})(1-n)}{(U^{1-n} - V^{1-n})(2-n)}$ where $n \in R \backslash \{1, 2\}$.

b If we assume a constant retardation, show that $\frac{D}{T} = \frac{U+V}{2}$.

12.4 Forces that depend on displacement

Setting up the equation of motion

If the force $F = F(x)$ depends upon the displacement x, then the acceleration is effectively a function of the displacement. Because $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dv}\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$ by the chain rule, then by Newton's Second Law of Motion, $\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{F(x)}{m}$. Integrating both sides with respect to x gives $\frac{1}{2}v^2 = \frac{1}{m}\int F(x)dx$ with initial conditions on x and v. This gives us a relationship between v and x.

WORKED EXAMPLE 8

A particle of mass 3 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{-(15-6x)}{x^3}$ newtons and the particle is at rest at a distance of 5 m from the origin. Find where else the particle comes to rest.

THINK	WRITE
1 Use Newton's Second Law of Motion.	$m\ddot{x} = F(x)$ where $m = 3$ and $F(x) = \frac{-(15-6x)}{x^3}$
2 Formulate the equation of motion to be solved.	$3\ddot{x} = \frac{-(15-6x)}{x^3}$ $\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{-(5-2x)}{x^3}$
3 Integrate both sides with respect to x.	$\frac{1}{2}v^2 = \int\left(\frac{-(5-2x)}{x^3}\right)dx$
4 Express the integrand in index notation.	$\frac{1}{2}v^2 = \int(-5x^{-3} + 2x^{-2})dx$
5 Perform the integration, placing the constant of integration on one side of the equation.	$\frac{1}{2}v^2 = \frac{5}{2}x^{-2} - 2x^{-1} + c$

6 Write the expression with positive indices.	$\frac{1}{2}v^2 = \frac{5}{2x^2} - \frac{2}{x} + c$
7 Use the given initial conditions to find the constant of integration.	When $v = 0$, $x = 5$: $0 = \frac{5}{2(5)^2} - \frac{2}{5} + c$ $c = \frac{2}{5} - \frac{1}{10} = \frac{3}{10}$
8 Substitute back for the constant of integration.	$\frac{1}{2}v^2 = \frac{5}{2x^2} - \frac{2}{x} + \frac{3}{10}$
9 Form a common denominator.	$\frac{1}{2}v^2 = \frac{25 - 20x + 3x^2}{10x^2}$
10 Factorise the quadratic in the numerator.	$v^2 = \frac{(3x-5)(x-5)}{5x^2}$
11 Express the velocity in terms of x.	$v = \frac{1}{\lvert x \rvert}\sqrt{\frac{(3x-5)(x-5)}{5}}$
12 Find the values of x when the particle comes to rest.	If $v = 0$, then $(3x-5)(x-5) = 0$, so $x = 5$ or $x = \frac{5}{3}$. As we were given $x = 5$ when $v = 0$, the required solution is $x = \frac{5}{3}$.
13 State the final result.	The particle comes to rest $\frac{5}{3}$ metres from the origin.

Relationships between time, displacement, velocity and acceleration

So far, we have used integration techniques to obtain relationships between time, t, displacement, x, velocity, v, and acceleration, a. However, we can also use differentiation to obtain relations by using

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dv}\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}.$$

WORKED EXAMPLE 9

A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{9 - 4x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = -4x$.

THINK	WRITE
1 Differentiate using the chain rule.	$v = \sqrt{9 - 4x^2} = (9 - 4x^2)^{\frac{1}{2}}$ $\frac{dv}{dx} = \frac{1}{2} \times -8x \times (9 - 4x^2)^{-\frac{1}{2}}$ $= \frac{-4x}{\sqrt{9 - 4x^2}}$

2 Use an expression for the acceleration.

$$a = v\frac{dv}{dx}$$
$$= \sqrt{9-4x^2} \times \frac{-4x}{\sqrt{9-4x^2}}$$
$$= -4x$$

3 Alternatively, square the velocity, halve it, and differentiate with respect to x.

$$v^2 = 9 - 4x^2$$
$$\frac{1}{2}v^2 = \frac{9}{2} - 2x^2$$
$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -4x$$

Expressing x in terms of t

In some cases it may be possible to rearrange and express the displacement x in terms of t by solving for v and using $v = \dfrac{dx}{dt}$.

WORKED EXAMPLE 10

A body of mass 5 kg moves in a straight line and is retarded by a force of $20x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 1 m from the origin and the initial velocity of the body is 2 m/s. Express x in terms of t where t is the time in seconds.

THINK	WRITE
1 Use Newton's Second Law of Motion. The braking force opposes the direction of motion.	$m\ddot{x} = F(x)$ where $m = 5$ and $F(x) = -20x$
2 Formulate the equation of motion to be solved.	$5\ddot{x} = -20x$ $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = -4x$
3 Integrate both sides with respect to x.	$\frac{1}{2}v^2 = \int -4x\,dx$
4 Perform the integration, placing the constant of integration on one side of the equation.	$\frac{1}{2}v^2 = -2x^2 + c_1$
5 Use the given initial conditions to find the first constant of integration.	Initially, when $t = 0$, $v = 2$ and $x = 1$: $2 = -2 + c_1$ $c_1 = 4$
6 Substitute back for the first constant of integration.	$\frac{1}{2}v^2 = -2x^2 + 4$
7 Rearrange and solve for v.	$v^2 = 8 - 4x^2$ $v^2 = 4(2 - x^2)$ $v = \pm 2\sqrt{2 - x^2}$

8 Initially, when $t = 0$, $x = 1$ and $v = 2$, $2 > 0$; therefore, we can take the positive root only.	$v = \frac{dx}{dt} = 2\sqrt{2 - x^2}, 0 \le x \le \sqrt{2}$
9 Invert both sides.	$\frac{dt}{dx} = \frac{1}{2\sqrt{2 - x^2}}$
10 Integrate both sides with respect to x.	$t = \frac{1}{2}\int \frac{1}{\sqrt{2 - x^2}}\,dx$
11 Perform the integration.	$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + c_2$
12 Use the given initial conditions to find the second constant of integration.	When $t = 0$, $v = 2$ and $x = 1$: $0 = \frac{1}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + c_2$ $c_2 = -\frac{\pi}{8}$
13 Substitute back for the second constant of integration.	$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{\pi}{8}$
14 Rearrange to make x the subject.	$\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) = 2t + \frac{\pi}{4}$ $\frac{x}{\sqrt{2}} = \sin\left(2t + \frac{\pi}{4}\right)$
15 Expand using compound-angle formula $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.	$\frac{x}{\sqrt{2}} = \sin(2t)\cos\left(\frac{\pi}{4}\right) + \cos(2t)\sin\left(\frac{\pi}{4}\right)$ $\frac{x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin(2t) + \frac{1}{\sqrt{2}}\cos(2t)$
16 State the final result.	$x = \sin(2t) + \cos(2t)$

EXERCISE 12.4 Forces that depend on displacement

PRACTISE

1 WE8 A particle of mass 2 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{3 - x}{x^3}$ newtons and the particle is at rest at a distance of 1 m from the origin. Find where else the particle comes to rest.

2 A particle of mass m kg moves so that its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{1 + bx}{x^3}$ newtons and the particle is at rest when $x = 1$ and also when $x = -\frac{1}{5}$. Find the value of the constant b.

3 WE9 A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{4 + 9x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 9x$.

4 A body is moving in a straight line. Its velocity, v m/s, is given by $v = x^4$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 4x^7$.

5 **WE10** A body of mass 8 kilograms moves in a straight line and is retarded by a force of $2x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 8 metres from the origin and at rest. Express x in terms of t where t is the time in seconds.

6 A body of mass 3 kg moves in a straight line and is retarded by a force of $\frac{x}{3}$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin and the initial velocity of the body is 6 m/s. Express x in terms of t where t is the time in seconds.

CONSOLIDATE

7 a A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $12x - 18$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 2 metres from the origin, and the initial velocity of the body is 2 metres per second. Express x in terms of t where t is the time in seconds.

b A body of mass 2 kilograms moves in a straight line and is acted upon by a force of $18x - 30$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 2 metres from the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.

8 a A body of mass 2 kilograms moves in a straight line and is acted upon by a force of $2x - 2$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.

b A body of mass 3 kilograms moves in a straight line and is retarded by a force of $12x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 8 metres per second. Express x in terms of t where t is the time in seconds.

9 a A particle of mass 2 kilograms moves in a straight line and is retarded by a force of $2e^{-2x}$ newtons, where x is its displacement in metres from a fixed origin. Initially the particle is at the origin, and its initial velocity is 1 metre per second. Express x in terms of t where t is the time in seconds.

b A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $6e^{4x}$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.

10 a A particle of mass 4 kg is subjected to a force of $\frac{28x - 24}{x^3}$ newtons, where x is its displacement in metres from a fixed origin. If the particle is at rest at a distance of 3 m from the origin, find where else the particle comes to rest.

b A particle of mass 2 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{20x - 16}{x^3}$ newtons, and the particle is at rest at a distance of 2 m from the origin. Find where else the particle comes to rest.

11 a A body of mass 16 kg moves in a straight line and is retarded by a force of $64x$ newtons, where x is its displacement in metres from a fixed origin. If the initial velocity of the body is 16 m/s, find where it comes to rest.

b A car of mass m kg is travelling at a speed of U m/s along a straight level road when the driver applies the brakes. The braking force is kx newtons, where x is the distance travelled in metres after the driver applies the brakes and k is a positive constant. Show that when the car comes to rest it has travelled a distance of $U\sqrt{\frac{m}{k}}$ metres.

12 a A body is moving in a straight line. Its velocity, v m/s, is given by $v = x^2$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 2x^3$.

b A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{16 - 25x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = -25x$.

c A body is moving in a straight line. Its velocity, v m/s, is given by $v = e^{2x} + e^{-2x}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 2(e^{4x} - e^{-4x})$

d A body is moving in a straight line. Its velocity, v m/s, at time t seconds is given by $v = 3 - 2e^{-2t}$. Show that its acceleration, a m/s^2, is given by $a = 2(v - 3)$.

13 a A body of mass 500 grams moves in a straight line and is retarded by a force of $8x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at rest 3 metres from the origin. Express x in terms of t where t is the time in seconds.

b A block of mass m kilograms moves back and forth along a straight line track. It is subjected to a force that opposes the motion and whose magnitude is proportional to its distance from the origin, O, the constant of proportionality being k. The particle starts from rest when its displacement from O is a metres. Show that if its speed is v metres per second at any time t seconds and displacement x metres, then $v^2 = \frac{k}{m}(a^2 - x^2)$ and $x = a\cos\left(\sqrt{\frac{k}{m}}\, t\right)$.

14 a A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = x^n$, where n is a constant. Show that its acceleration, a m/s^2, is given by $a = nx^{2n-1}$.

b A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = \sqrt{b - n^2x^2}$, where n and b are constants. Show that its acceleration, a m/s^2, is given by $a = -n^2x$.

c A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = e^{nx} + e^{-nx}$, where n is a constant. Show that its acceleration, a m/s^2, is given by $a = n(e^{2nx} - e^{-2nx})$.

d A body is moving in a straight line. Its velocity, v m/s, at time t seconds is given by $v = b - ne^{-nt}$, where n is a constant. Show that its acceleration, a m/s^2, is given by $a = -n(v - b)$.

15 a A block of mass m kg moves along a horizontal table top and is subjected to a resistance force of $\frac{a+bx}{x^3}$ newtons, where a and b are constants. If initially the block is at rest at a displacement of a metres, show that the block next comes to rest again when the displacement is $\frac{-a}{2b+1}$ m.

b A particle of mass m kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $ma(ax+b)$ newtons, where a and b are non-zero real constants. Initially the particle is $\frac{b}{a}$ m from the origin, moving with a speed of $2b$ m/s. Show that $x = \frac{b}{a}(2e^{at} - 1)$.

16 A particle of mass m kg moves in a straight line and is subjected to a force that opposes its motion. The magnitude of the opposing force is inversely proportional to the cube of the distance from a fixed point, O; the constant of proportionality is k. The particle starts from rest when its distance from O is a m.

a Show that its speed, v m/s, is given by $v = \frac{\sqrt{k(a^2 - x^2)}}{\sqrt{m}\,ax}$, where x m is its distance from O at a time t seconds.

b Obtain a relationship between t and x, and hence show that when x has the value of $\frac{\sqrt{7}a}{4}$ m, a time of $\frac{3a^2}{4}\sqrt{\frac{m}{k}}$ s has elapsed.

MASTER

17 a A particle of mass 4 kg moves in a straight line and is retarded by a force $\frac{32}{x^2}$ newtons, where x is its distance in metres from a fixed point, O. If the particle starts from rest when its distance from O is 4 m, find the time taken to travel to the origin. Give your answer correct to 2 decimal places.

b A particle of mass m kg moves in a straight line against a central force, that is one whose magnitude is inversely proportional to the square of the distance from a fixed point. O. The constant of proportionality is k. If the particle starts from rest when its distance from O is a metres, show that the speed, v m/s, at a distance x m from O satisfies $v = \sqrt{\frac{2k(a-x)}{max}}$.

18 A particle of mass 4 kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $32x(x^2 - 9)$ newtons and initially is 2 m from the origin, moving with a speed of 10 m/s.

a Express x in terms of t.

b As t approaches infinity, the particle approaches a fixed position. Find that position.

eBook*plus* ONLINE ONLY

12.5 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyON

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

12 Answers

EXERCISE 12.2

1 9 m

2 4 s

3 $18\sin\left(\frac{t}{3}\right)$

4 $4\sqrt{2}$ m

5 a 1 s b 25 m

6 a $\frac{1}{2}$ s b 10 m

7 a $x = t^2 - t^3$

b $x = 2(e^{2t} - 1) - 2t^2 - 4t$

8 a 1 m to the left of the origin

b $x = 4t^3 - 3t^2 + t + 2$

9 900 m, 90 s

10 67.388 m

11 a 6 m to the left of the origin

b 16 m

12 40 m

13 a Check with your teacher.

b i $49t - 4.9t^2 + 2$ ii 124.5 m

14 a 11.11 m

b Check with your teacher.

15 a $x = 49t + 245e^{-0.2t} - 225$

b Check with your teacher.

16 $T = \frac{1}{2}, D = \frac{400}{41}$

17 a 0.135 m b 1.669 m

18 a 0.492 m b 0.845 m

EXERCISE 12.3

1 Check with your teacher.

2 Check with your teacher.

3 a $\frac{1}{2}$ s b 10 m

4 $T = \frac{1}{2}, D = \frac{400}{41}$

5 Check with your teacher.

6 Check with your teacher.

7 a, b Check with your teacher.

c 4500.17 kg m/s

d $42\sqrt{5}$ m/s

e 5.69 s

8 Check with your teacher.

9 a $x = \frac{1}{2}(1 - e^{-2t})$

b $x = 2(1 - e^{-3t} - 2t)$

10 a 5 s b $46\frac{2}{3}$ m

11 a 5 b 40

12–18 Check with your teacher.

19 a Check with your teacher.

b i $t = \frac{1}{40000}\left(\tan^{-1}(4) - \tan^{-1}\left(\frac{v}{100}\right)\right)$

0.033 ms

ii 3.5 mm

20 Check with your teacher.

EXERCISE 12.4

1 3 m to the left of the origin

2 2

3 Check with your teacher.

4 Check with your teacher.

5 $8\cos\left(\frac{t}{2}\right)$

6 $18\sin\left(\frac{t}{3}\right)$

7 a $x = \frac{1}{2}(3 + e^{2t})$ b $x = \frac{1}{3}(5 + e^{3t})$

8 a $x = 1 - e^t$ b $x = 4\sin(2t)$

9 a $x = \log_e(t + 1)$ b $x = -\frac{1}{2}\log_e(1 - 2t)$

10 a $\frac{1}{2}$ m b $\frac{1}{2}$ m

11 a 8 m

b Check with your teacher.

12 Check with your teacher.

13 a $x = 3\cos(4t)$

b Check with your teacher.

14–16 Check with your teacher.

17 a 6.28 s

b Check with your teacher.

18 a $x = \frac{3(5 - e^{-12t})}{5 + e^{-12t}}$

b 3 m

13

Vector calculus

13.1 Kick off with CAS

Exploring parametric equations with CAS

Parametric equations arise when both the x- and y-coordinates depend upon a parameter t.

$x = x(t)$ (1)

$y = y(t)$ (2)

1 Using CAS technology, sketch the graphs of the following parametric equations.

a $x = 2\cos(t)$, $y = 2\sin(t)$

b $x = 3\cos(t)$, $y = 4\sin(t)$

c $x = 4\cos(2t)$, $y = 4\sin(2t)$

d $x = 5\cos(3t)$, $y = 8\sin(3t)$

e $x = 3\cos(2t)$, $y = 4\sin(t)$

f $x = 3\cos(4t)$, $y = 4\sin(2t)$

2 If $x = a\cos(nt)$ and $y = b\sin(mt)$, describe the parametric graph for the cases when:

a $m = n$ and $a = b$

b $m = n$ and $a \neq b$

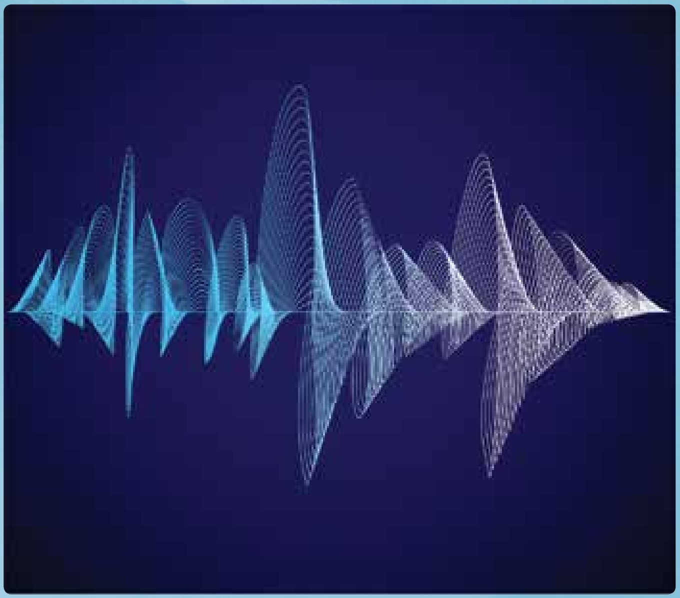
c $n = 2m$.

3 Comment on the effect of changing the values of a and b on the graphs of the above equations.

4 Using CAS technology, sketch the graphs of the following parametric equations.

a $x = 3\cos(2t)$, $y = 4\sin(6t)$

b $x = 4\cos(t) + \cos(4t)$, $y = 4\sin(t) - \sin(4t)$



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

13.2 Position vectors as functions of time

Parametric equations

studyon

Units 3 & 4
AOS 4
Topic 2
Concept 1

Position vectors as a function of time
Concept summary
Practice questions

The parametric equations of a particle as it moves in two dimensions are discussed in Topic 6. The path traced out by the particle is defined in terms of a third variable, t, which is called a parameter. In the two-dimensional case the path is described by two parametric equations, as both the x- and y-coordinates depend upon the parameter, t.

$$x = x(t) \quad (1)$$
$$y = y(t) \quad (2)$$

The position vector of the particle is given by $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the x and y directions. This is also called a vector function of the scalar real variable t, where t is called the parameter, and often represents time.

If we can eliminate the parameter from the two parametric equations and obtain an equation of the form $y = f(x)$, this is called an explicit relationship or the equation of the path. It may not be possible to obtain an explicit relationship, but often an implicit relationship of the form $f(x, y) = 0$ can be formed. The relationship between x and y is called the Cartesian equation of the path. Careful consideration must be given to the possible values of t, which then specify the domain (the x-values) and the range (the y-values) of the equation of the path.

Closest approach

Given the position vector of a particle, $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the x and y directions, it is possible to find the position or coordinates of the particle at a given value of t. It is also possible to find the value of t and the coordinates when the particle is closest to the origin.

WORKED EXAMPLE 1

A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (3t - 4)\underset{\sim}{i} + (4t - 3)\underset{\sim}{j}$ for $t \geq 0$.

a **Find the distance of the particle from the origin when $t = 2$.**

b **Find the distance of the particle from the origin at any time t.**

c **Find the closest distance of the particle from the origin.**

THINK	WRITE
a 1 Substitute the value of t.	**a** Substitute $t = 2$: $\underset{\sim}{r}(2) = (6 - 4)\underset{\sim}{i} + (8 - 3)\underset{\sim}{j}$ $= 2\underset{\sim}{i} + 5\underset{\sim}{j}$ at the point (2, 5)
2 Find the magnitude of the vector, which represents the distance from the origin.	$\lvert\underset{\sim}{r}(2)\rvert = \sqrt{2^2 + 5^2}$
3 State the distance from the origin at this time.	$\lvert\underset{\sim}{r}(2)\rvert = \sqrt{29}$
b 1 Find the magnitude of the vector at time t.	**b** $\underset{\sim}{r}(t) = (3t - 4)\underset{\sim}{i} + (4t - 3)\underset{\sim}{j}$ $\lvert\underset{\sim}{r}(t)\rvert = \sqrt{(3t - 4)^2 + (4t - 3)^2}$

2 Expand and simplify to state the distance in terms of t.	$\lvert \underset{\sim}{r}(t) \rvert = \sqrt{9t^2 - 24t + 16 + 16t^2 - 24t + 9}$ $= \sqrt{25t^2 - 48t + 25}$
c 1 For the closest or minimum distance, we need to use calculus.	**c** The minimum distance occurs when $\frac{d}{dt}\left(\lvert \underset{\sim}{r}(t) \rvert\right) = 0$.
2 Use the chain rule.	$\lvert \underset{\sim}{r}(t) \rvert = (25t^2 - 48t + 25)^{\frac{1}{2}}$ $\frac{d}{dt}\left(\lvert \underset{\sim}{r}(t) \rvert\right) = \frac{\frac{1}{2} \times (50t - 48)}{\sqrt{25t^2 - 48t + 25}} = 0$
3 Solve for the value of t.	$50t - 48 = 0$ $t = \frac{24}{25}$
4 Determine the position vector at this value.	Substitute $t = \frac{24}{25}$: $\underset{\sim}{r}\left(\frac{24}{25}\right) = \left(3 \times \frac{24}{25} - 4\right)\underset{\sim}{i} + \left(4 \times \frac{24}{25} - 3\right)\underset{\sim}{j}$ $= -\frac{28}{25}\underset{\sim}{i} + \frac{21}{25}\underset{\sim}{j}$
5 Determine the magnitude of the vector at this time, which represents the closest distance of the particle from the origin.	$\lvert \underset{\sim}{r}(t) \rvert_{\min} = \sqrt{\left(-\frac{28}{25}\right)^2 + \left(\frac{21}{25}\right)^2}$ $= \frac{1}{25}\sqrt{1225}$
6 State the final result.	$\lvert \underset{\sim}{r}(t) \rvert_{\min} = \frac{7}{5}$

Collision problems

There are a number of problems that can be formulated around the motion of two moving particles on different curves in two dimensions.

1. Do the particles collide? For two particles to collide, they must be at exactly the same coordinates at exactly the same time.
2. Do the paths of the particles cross without colliding? This will happen when they are at the same coordinates but at different times.
3. What is the distance between the particles at a particular time? To determine this, the magnitude of the difference between their respective position vectors must be found.

WORKED EXAMPLE 2

Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (3t - 8)\underset{\sim}{i} + (t^2 - 18t + 87)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (20 - t)\underset{\sim}{i} + (2t - 4)\underset{\sim}{j}$ for $t \geq 0$. Find:

a **when and where the particles collide**

b **where their paths cross**

c **the distance between the particles when $t = 10$.**

THINK	WRITE
a **1** Equate the $\underset{\sim}{i}$ components for each particle.	**a** $3t - 8 = 20 - t$
2 Solve this equation for t.	$4t = 28$ $t = 7$
3 Equate the $\underset{\sim}{j}$ components for each particle.	$t^2 - 18t + 87 = 2t - 4$
4 Solve this equation for t.	$t^2 - 20t + 91 = 0$ $(t-7)(t-13) = 0$ $t = 7, 13$
5 Evaluate the position vectors at the common time.	The common solution is when $t = 7$. $\underset{\sim}{r}_A(7) = 13\underset{\sim}{i} + 10\underset{\sim}{j}$ $\underset{\sim}{r}_B(7) = 13\underset{\sim}{i} + 10\underset{\sim}{j}$
6 State the result for when the particles collide.	The particles collide when $t = 7$ at the point (13, 10).
b **1** The particles paths cross at different values of t. Introduce a different parameter, s.	**b** Let $\underset{\sim}{r}_A(s) = (3s-8)\underset{\sim}{i} + (s^2 - 18s + 87)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (20-t)\underset{\sim}{i} + (2t-4)\underset{\sim}{j}$.
2 Equate the $\underset{\sim}{i}$ components for each particle.	$3s - 8 = 20 - t$
3 Solve this equation for t.	$t = 28 - 3s$
4 Equate the $\underset{\sim}{j}$ components for each particle.	$s^2 - 18s + 87 = 2t - 4$
5 Solve this equation for s.	Substitute $t = 28 - 3s$: $s^2 - 18s + 87 = 2(28 - 3s) - 4$ $s^2 - 18s + 87 = 56 - 6s - 4$ $s^2 - 12s + 35 = 0$ $(s-5)(s-7) = 0$ $s = 5, 7$
6 Evaluate the position vectors at the required time. Note that when $s = 7$, the particles collide.	$\underset{\sim}{r}_A(5) = (15-8)\underset{\sim}{i} + (25 - 90 + 87)\underset{\sim}{j}$ $= 7\underset{\sim}{i} + 22\underset{\sim}{j}$ When $s = 5$, $t = 28 - 15 = 13$ $\underset{\sim}{r}_B(13) = (20-13)\underset{\sim}{i} + (26-4)\underset{\sim}{j}$ $= 7\underset{\sim}{i} + 22\underset{\sim}{j}$
7 State the required result.	The paths cross at the point (7, 22).
c **1** Evaluate both the position vectors at the required time.	**c** Substitute $t = 10$: $\underset{\sim}{r}_A(10) = (30-8)\underset{\sim}{i} + (100 - 180 + 87)\underset{\sim}{j}$ $= 22\underset{\sim}{i} + 7\underset{\sim}{j}$ $\underset{\sim}{r}_B(10) = (20-10)\underset{\sim}{i} + (20-4)\underset{\sim}{j}$ $= 10\underset{\sim}{i} + 16\underset{\sim}{j}$
2 Find the difference between these two vectors.	$\underset{\sim}{r}_A(10) - \underset{\sim}{r}_B(10) = 22\underset{\sim}{i} + 7\underset{\sim}{j} - (10\underset{\sim}{i} + 16\underset{\sim}{j})$ $= 12\underset{\sim}{i} - 9\underset{\sim}{j}$
3 The distance between the particles is the magnitude of the difference between these vectors.	$\lvert\underset{\sim}{r}_A(10) - \underset{\sim}{r}_B(10)\rvert = \lvert 12\underset{\sim}{i} - 9\underset{\sim}{j}\rvert$ $= \sqrt{12^2 + (-9)^2}$ $= \sqrt{225}$
4 State the required distance.	$\lvert\underset{\sim}{r}_A(10) - \underset{\sim}{r}_B(10)\rvert = 15$

EXERCISE 13.2 Position vectors as functions of time

PRACTISE

1 **WE1** A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (t - 2)\underset{\sim}{i} + (3t - 1)\underset{\sim}{j}$ for $t \geq 0$.

a Find the distance of the particle from the origin when $t = 5$.

b Find the distance of the particle from the origin at any time t.

c Find the closest distance of the particle from the origin.

2 A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = \sqrt{t}\underset{\sim}{i} + (2t + 3)\underset{\sim}{j}$ for $t \geq 0$.

a Find the distance of the particle from the origin when $t = 4$.

b Find the value t when the distance of the particle from the origin is $15\sqrt{2}$.

3 **WE2** Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (2t + 6)\underset{\sim}{i} + (t^2 - 6t + 45)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (t + 11)\underset{\sim}{i} + (7t + 5)\underset{\sim}{j}$ for $t \geq 0$. Find:

a when and where the particles collide

b where their paths cross

c the distance between the particles when $t = 10$.

4 Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (-t^2 + 12t - 22)\underset{\sim}{i} + (19 - 3t)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (18 - 2t)\underset{\sim}{i} + (t + 3)\underset{\sim}{j}$ for $t \geq 0$. Find:

a when and where the particles collide

b where their paths cross

c the distance between the particles when $t = 5$.

CONSOLIDATE

5 a A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2t - 1)\underset{\sim}{i} + (t - 3)\underset{\sim}{j}$ for $t \geq 0$.

i Find the distance of the particle from the origin when $t = 4$.

ii Find the distance of the particle from the origin at any time t.

iii Find the closest distance of the particle from the origin.

b A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (4t - 3)\underset{\sim}{i} + (3t + 4)\underset{\sim}{j}$ for $t \geq 0$.

i Find the distance of the particle from the origin when $t = 2$.

ii Find the distance of the particle from the origin at any time t.

iii Find the closest distance of the particle from the origin.

6 a A boat moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2t - 3)\underset{\sim}{i} + 2\sqrt{t}\underset{\sim}{j}$ for $t \geq 0$, where distance is measured in kilometres.

i Find the distance of the boat from the origin when $t = 4$.

ii Find the closest distance of the boat from the origin.

iii Find the times when the boat is 3 kilometres from the origin.

b A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2t - 7)\underset{\sim}{i} + (2t + 2)\underset{\sim}{j}$ for $t \geq 0$.

i Find the closest distance of the particle from the origin.

ii Find the time when the particle is $9\sqrt{5}$ units from the origin.

7 a A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (at + b)\underset{\sim}{i} + (ct^2 + d)\underset{\sim}{j}$ for $t \geq 0$. If $\underset{\sim}{r}(2) = 5\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{r}(4) = 13\underset{\sim}{i} + 19\underset{\sim}{j}$, find the values of a, b, c and d.

b A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (at + b)\underset{\sim}{i} + (ct^2 + dt)\underset{\sim}{j}$ for $t \geq 0$. If $\underset{\sim}{r}(4) = 13\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{r}(6) = 17\underset{\sim}{i} + 18\underset{\sim}{j}$, find the values of a, b, c and d.

8 Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (3t - 43)\underset{\sim}{i} + (-t^2 + 26t - 160)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (17 - t)\underset{\sim}{i} + (2t - 25)\underset{\sim}{j}$ for $t \geq 0$. Find:

a when and where the particles collide
b where their paths cross
c the distance between the particles when $t = 12$.

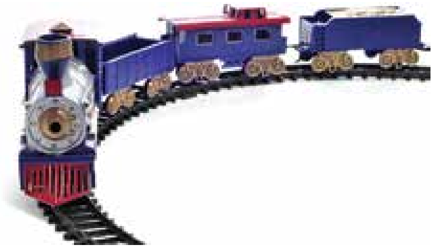
9 Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (t^2 - 6)\underset{\sim}{i} + (2t + 2)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (7t - 16)\underset{\sim}{i} + \frac{1}{5}(17t - t^2)\underset{\sim}{j}$ for $t \geq 0$. Find:

a when and where the particles collide
b the distance between the particles when $t = 10$.

10 Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (-t^2 + 12t + 53)\underset{\sim}{i} + (3t + 38)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = (2t + 29)\underset{\sim}{i} + (86 - t)\underset{\sim}{j}$ for $t \geq 0$. Find:

a when and where the particles collide
b where their paths cross
c the distance between the particles when $t = 20$.

11 A toy train moves so that its vector equation is given by $\underset{\sim}{r}(t) = (3 - 3\cos(t))\underset{\sim}{i} + (3 + 3\sin(t))\underset{\sim}{j}$ for $t \geq 0$.



a Find the position of the toy train at the times $t = 0$, π and 2π.
b Find and sketch the Cartesian equation of the path of the toy train, stating the domain and range.
c Find the distance of the toy train from the origin at any time t.
d Find the closest distance of the toy train from the origin.

12 A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2 + 4\cos(t))\underset{\sim}{i} + (4 + 4\sin(t))\underset{\sim}{j}$ for $t \geq 0$.

a Find the position of the particle at the times $t = 0$, π and 2π.
b Find and sketch the Cartesian equation of the path, stating the domain and range.
c Find the distance of the particle from the origin at any time t.
d Find the closest distance of the particle from the origin.

13 a A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = t\underset{\sim}{i} + \frac{1}{t}\underset{\sim}{j}$ for $t > 0$. Find the closest distance of the particle from the origin.
b A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = e^{-t}\underset{\sim}{i} + e^{t}\underset{\sim}{j}$ for $t \in R$. Find the closest distance of the particle from the origin.

14 a Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = a\cos(t)\underset{\sim}{i} + a\sin(t)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = a\cos^2(t)\underset{\sim}{i} + a\sin^2(t)\underset{\sim}{j}$ for $t \geq 0$, where $a \in R^+$. Determine if the particles collide or if their paths cross.

b Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (a + a\cos(t))\underset{\sim}{i} + (a + a\sin(t))\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t) = a\cos^2(t)\underset{\sim}{i} + a\sin^2(t)\underset{\sim}{j}$ for $t \geq 0$, where $a \in R^+$. Determine if the particles collide or if their paths cross.

MASTER

15 a A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (5t - 2)\underset{\sim}{i} + (12t - 2)\underset{\sim}{j}$ for $t \geq 0$. Find the closest distance of the particle from the origin.

b A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (at + b)\underset{\sim}{i} + (ct + d)\underset{\sim}{j}$ for $t \geq 0$. Find the closest distance of the particle from the origin.

16 a A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (4t - 3)\underset{\sim}{i} + (t^2 + 3)\underset{\sim}{j}$ for $t \geq 0$. Find the closest distance of the particle from the origin.

b A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2t - 1)\underset{\sim}{i} + (t^2 + 3t)\underset{\sim}{j}$ for $t \geq 0$. Find the closest distance of the particle from the origin.

13.3 Differentiation of vectors

Vector functions

studyon

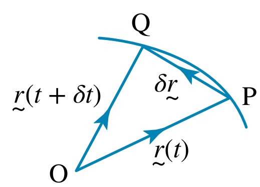
Units 3 & 4
AOS 4
Topic 2
Concept 2

Differentiation of vector functions
Concept summary
Practice questions

The function $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ is called a vector function. It represents the position vector of a particle at time t in two dimensions. As t changes, both the x- and y-coordinates change; thus, the particle is moving along a curve. The equation of the curve is called the Cartesian equation of the path.

The derivative of a vector function

Consider an origin, O, and let P be the position of the particle at time t, so that $\overrightarrow{OP} = \underset{\sim}{r}(t)$.



Suppose that Q is a neighbouring point close to P at time $t + \delta t$, so that $\overrightarrow{OQ} = \underset{\sim}{r}(t + \delta t)$.

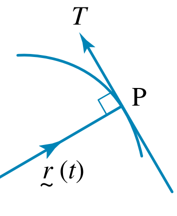
$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$\delta\underset{\sim}{r} = \underset{\sim}{r}(t + \delta t) - \underset{\sim}{r}(t)$$

Consider $\dfrac{\delta\underset{\sim}{r}}{\delta t} = \dfrac{\underset{\sim}{r}(t + \delta t) - \underset{\sim}{r}(t)}{\delta t}$, where $\delta t \neq 0$. Because $\delta\underset{\sim}{r}$ is a vector and δt is a scalar, $\dfrac{\delta\underset{\sim}{r}}{\delta t}$ is a vector parallel to $\delta\underset{\sim}{r}$ or the vector $\overrightarrow{PQ}$. As $\delta t \to 0$, provided the limit exists, we define $\dfrac{d\underset{\sim}{r}}{dt} = \lim\limits_{\delta t \to 0} \dfrac{\underset{\sim}{r}(t + \delta t) - \underset{\sim}{r}(t)}{\delta t}$

$$\frac{d\underset{\sim}{r}}{dt} = \lim_{\delta t \to 0} \frac{x(t + \delta t)\underset{\sim}{i} + y(t + \delta t)\underset{\sim}{j} - (x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j})}{\delta t}$$

$$\frac{d\underset{\sim}{r}}{dt} = \lim_{\delta t \to 0} \frac{x(t + \delta t) - x(t)}{\delta t}\underset{\sim}{i} + \lim_{\delta t \to 0} \frac{y(t + \delta t) - y(t)}{\delta t}\underset{\sim}{j}$$

$$\frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j}$$



The vector $\frac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}}$, where the dot indicates the derivative with respect to t, is a vector parallel to the tangent T to the curve at the point P.

A unit tangent vector at $t = a$ is denoted by $\hat{\underset{\sim}{s}} = \frac{\dot{\underset{\sim}{r}}(a)}{|\dot{\underset{\sim}{r}}(a)|}$.

We do not need to use first principles to find the derivatives of vectors.

Rules for differentiating vectors

In the following sections it is assumed that the derivatives exist for the functions given.

Derivative of a constant vector

If $\underset{\sim}{c}$ is a constant vector, that is a vector which does not change and is independent of t, then $\frac{d\underset{\sim}{c}}{dt} = \underset{\sim}{0}$. Note that $\frac{d\underset{\sim}{i}}{dt} = \frac{d\underset{\sim}{j}}{dt} = \frac{d\underset{\sim}{k}}{dt} = \underset{\sim}{0}$.

Derivative of a sum or difference of vectors

The sum or difference of two vectors can be differentiated as the sum or difference of the individual derivatives. That is,

$$\frac{d}{dt}(\underset{\sim}{a} + \underset{\sim}{b}) = \frac{d\underset{\sim}{a}}{dt} + \frac{d\underset{\sim}{b}}{dt} \text{ and } \frac{d}{dt}(\underset{\sim}{a} - \underset{\sim}{b}) = \frac{d\underset{\sim}{a}}{dt} - \frac{d\underset{\sim}{b}}{dt}.$$

Using these rules, if $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$, then $\frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j}$. Simply put, to differentiate a vector we merely differentiate each component using the rules for differentiation.

WORKED EXAMPLE 3

Find a unit tangent vector to $\underset{\sim}{r}(t) = e^{3t}\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$ at the point where $t = 0$.

THINK	WRITE
1 Differentiate the vector.	$\underset{\sim}{r}(t) = e^{3t}\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$ $\frac{d\underset{\sim}{r}}{dt} = \frac{d}{dt}(e^{3t})\underset{\sim}{i} + \frac{d}{dt}(\sin(2t))\underset{\sim}{j}$
2 State the derivative vector.	$\frac{d\underset{\sim}{r}}{dt} = 3e^{3t}\underset{\sim}{i} + 2\cos(2t)\underset{\sim}{j}$
3 Evaluate the derivative vector at the required value.	$\left.\frac{d\underset{\sim}{r}}{dt}\right\vert_{t=0} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$
4 Find the magnitude of the derivative vector.	$\left\lvert\frac{d\underset{\sim}{r}(0)}{dt}\right\rvert = \sqrt{3^2 + 2^2}$ $= \sqrt{13}$
5 State the required result, which is a unit vector.	$\hat{\underset{\sim}{r}} = \frac{1}{\sqrt{13}}(3\underset{\sim}{i} + 2\underset{\sim}{j})$

Velocity vector

Because $\underset{\sim}{r}(t)$ represents the position vector, $\underset{\sim}{v}(t) = \frac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}}(t)$ represents the velocity vector. Note the single dot above $\underset{\sim}{r}$ indicates the derivative with respect to time. Furthermore, if $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$, then $\dot{\underset{\sim}{r}}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j}$.

Speed

The speed of a moving particle is the magnitude of the velocity vector. The speed at time t is given by $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$. If the particle has a mass of m, then the magnitude of the momentum acting on the particle is given by $\underset{\sim}{p} = m|\dot{\underset{\sim}{r}}(t)|$.

Acceleration vector

Since $\underset{\sim}{v}(t) = \frac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}}(t)$ represents the velocity vector, differentiating again with respect to t gives the acceleration vector. The acceleration vector is given by $\underset{\sim}{a}(t) = \frac{d}{dt}(\dot{r}(t)) = \ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j}$. Note that the two dots above the variables indicate the second derivative with respect to time.

WORKED EXAMPLE 4

A particle spirals outwards so that its position vector is given by $\underset{\sim}{r}(t) = t\cos(t)\underset{\sim}{i} + t\sin(t)\underset{\sim}{j}$ for $t \geq 0$.

a **Find the velocity vector.**

b **Find the speed of the particle at time t and hence find the speed when $t = \frac{3\pi}{4}$.**

c **Find the acceleration vector.**

THINK	WRITE
a 1 State the parametric equations.	**a** $\underset{\sim}{r}(t) = t\cos(t)\underset{\sim}{i} + t\sin(t)\underset{\sim}{j}$ Then $x(t) = t\cos(t)$ and $y(t) = t\sin(t)$.
2 Differentiate x with respect to t. The dot notation is used for the derivative with respect to t. Use the product rule.	$\frac{dx}{dt} = \dot{x} = \frac{d}{dt}(t\cos(t))$ $= \cos(t)\frac{d}{dt}(t) + t\frac{d}{dt}(\cos(t))$ $= \cos(t) - t\sin(t)$
3 Differentiate y with respect to t. Use the product rule.	$\frac{dy}{dt} = \dot{y} = \frac{d}{dt}(t\sin(t))$ $= \sin(t)\frac{d}{dt}(t) + t\frac{d}{dt}(\sin(t))$ $= \sin(t) + t\cos(t)$
4 State the velocity vector.	$\dot{\underset{\sim}{r}}(t) = (\cos(t) - t\sin(t))\underset{\sim}{i} + (\sin(t) + t\cos(t))\underset{\sim}{j}$
b 1 Find the speed at time t. Substitute for the derivatives and expand.	**b** $\lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{\dot{x}^2 + \dot{y}^2}$ $\dot{x}^2 = (\cos(t) - t\sin(t))^2$ $= \cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t)$ $\dot{y}^2 = (\sin(t) + t\cos(t))^2$ $= \sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t)$

2 Simplify using trigonometry, since $\sin^2(t) + \cos^2(t) = 1$. $\quad \dot{x}^2 + \dot{y}^2 = 1 + t^2$

3 State the speed at time t. $\quad |\dot{\underset{\sim}{r}}(t)| = \sqrt{1 + t^2}$

4 Find the speed at the required value. $\quad$ Substitute $t = \frac{3\pi}{4}$:

$$\left|\dot{\underset{\sim}{r}}\left(\frac{3\pi}{4}\right)\right| = \sqrt{1 + \left(\frac{3\pi}{4}\right)^2} = \sqrt{\frac{16 + 9\pi^2}{16}}$$

5 State the speed in simplified form. $\quad \left|\dot{\underset{\sim}{r}}\left(\frac{3\pi}{4}\right)\right| = \frac{1}{4}\sqrt{16 + 9\pi^2}$

c 1 Determine the $\underset{\sim}{i}$ component of the acceleration vector.

c
$$\begin{aligned}\frac{d^2x}{dt^2} = \ddot{x} &= \frac{d}{dt}(\cos(t) - t\sin(t))\\ &= -\sin(t) - \frac{d}{dt}(t\sin(t))\\ &= -\sin(t) - \sin(t) - t\cos(t)\\ &= -2\sin(t) - t\cos(t)\end{aligned}$$

2 Determine the $\underset{\sim}{j}$ component of the acceleration vector.

$$\begin{aligned}\frac{d^2y}{dt^2} = \ddot{y} &= \frac{d}{dt}(\sin(t) + t\cos(t))\\ &= \cos(t) + \frac{d}{dt}(t\cos(t))\\ &= \cos(t) + \cos(t) - t\sin(t)\\ &= 2\cos(t) - t\sin(t)\end{aligned}$$

3 State the acceleration vector in terms of t.

$$\ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j}$$
$$\ddot{\underset{\sim}{r}}(t) = -(2\sin(t) + t\cos(t))\underset{\sim}{i} + (2\cos(t) - t\sin(t))\underset{\sim}{j}$$

Extension to three dimensions

If $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$ is the position vector of a particle moving in three dimensions, then the velocity vector is given by $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j} + \dot{z}(t)\underset{\sim}{k}$. The speed is given by $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and the acceleration vector is given by $\underset{\sim}{a}(t) = \frac{d\underset{\sim}{v}(t)}{dt} = \ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j} + \ddot{z}(t)\underset{\sim}{k}$. Note that alternative notations may be used.

WORKED EXAMPLE 5

A particle has a position vector given by $\underset{\sim}{r}(t) = t^3\underset{\sim}{i} + 6\sin(3t)\underset{\sim}{j} + 12e^{-\frac{t}{2}}\underset{\sim}{k}$ for $t \geq 0$. Find:

a the velocity vector 　　 **b the acceleration vector.**

THINK / **WRITE**

a 1 Differentiate the position vector.

a $\underset{\sim}{r}(t) = t^3\underset{\sim}{i} + 6\sin(3t)\underset{\sim}{j} + 12e^{-\frac{t}{2}}\underset{\sim}{k}$

$$\dot{\underset{\sim}{r}}(t) = \frac{d}{dt}(t^3)\underset{\sim}{i} + \frac{d}{dt}(6\sin(3t))\underset{\sim}{j} + \frac{d}{dt}\left(12e^{-\frac{t}{2}}\right)\underset{\sim}{k}$$

2 State the derivative or velocity vector. $\quad \dot{\underset{\sim}{r}}(t) = 3t^2\underset{\sim}{i} + 18\cos(3t)\underset{\sim}{j} - 6e^{-\frac{t}{2}}\underset{\sim}{k}$

b 1 Differentiate the velocity vector.

b $\ddot{\underset{\sim}{r}}(t) = \frac{d}{dt}(3t^2)\underset{\sim}{i} + \frac{d}{dt}(18\cos(3t))\underset{\sim}{j} - \frac{d}{dt}\left(6e^{-\frac{t}{2}}\right)\underset{\sim}{k}$

2 State the acceleration vector.

$\ddot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} - 54\sin(3t)\underset{\sim}{j} + 3e^{-\frac{t}{2}}\underset{\sim}{k}$

The gradient of the curve

Because we can find the Cartesian equation of the curve as either an explicit relationship, $y = f(x)$, or an implicit relationship, $f(x, y) = c$, we can find the gradient of the curve using either explicit or implicit differentiation. Alternatively, we can find the gradient of the curve using parametric differentiation, since $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$. Techniques such as these have been studied in earlier topics. It may be necessary to solve equations to find the maximum or minimum speeds of a particle, the maximum or minimum values of the acceleration, or the force acting on a particle, given its mass.

WORKED EXAMPLE 6

A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (3-2\cos(2t))\underset{\sim}{i} + (2 + 3\sin(2t))\underset{\sim}{j}$ for $0 \le t \le \pi$.

a **Find the coordinates where the gradient of the curve is parallel to the x-axis.**

b **Find and sketch the Cartesian equation of the path, stating its domain and range.**

c **Find the maximum and minimum values of the speed.**

THINK	WRITE/DRAW
a 1 State the parametric equations.	**a** $\underset{\sim}{r}(t) = (3 - 2\cos(2t))\underset{\sim}{i} + (2 + 3\sin(2t))\underset{\sim}{j}$ Then $x(t) = 3 - 2\cos(2t)$ and $y(t) = 2 + 3\sin(2t)$.
2 Differentiate x with respect to t. The dot notation is used for the derivative with respect to t.	$x(t) = 3 - 2\cos(2t)$ $\frac{dx}{dt} = \dot{x} = 4\sin(2t)$
3 Differentiate y with respect to t.	$y(t) = 2 + 3\sin(2t)$ $\frac{dy}{dt} = \dot{y} = 6\cos(2t)$
4 Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$
5 Substitute for the derivatives.	$\frac{dy}{dx} = \frac{6\cos(2t)}{4\sin(2t)}$ $= \frac{3}{2\tan(2t)}$
6 Find where the gradient is parallel to the x-axis or where the gradient is zero.	$\frac{dy}{dx} = 0 \Rightarrow \cos(2t) = 0$, but $\sin(2t) \neq 0$.

7 Solve for the values of t.

Since $0 \leq t \leq \pi$,

$2t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}$

8 Find the coordinates.

When $t = \frac{\pi}{4}$,

$x\left(\frac{\pi}{4}\right) = 3 - 2\cos\left(\frac{\pi}{2}\right) = 3$ and $y\left(\frac{\pi}{4}\right) = 2 + 3\sin\left(\frac{\pi}{2}\right) = 5.$

At (3, 5), the gradient is zero.

9 Find the other coordinate.

When $t = \frac{3\pi}{4}$,

$x\left(\frac{3\pi}{4}\right) = 3 - 2\cos\left(\frac{3\pi}{2}\right) = 3$ and

$y\left(\frac{3\pi}{4}\right) = 2 + 3\sin\left(\frac{3\pi}{2}\right) = -1.$

At (3, –1), the gradient is zero.

b 1 Use the parametric equations to eliminate the parameter.

b $x = 3 - 2\cos(2t) \Rightarrow \cos(2t) = \frac{x-3}{-2}$

$y = 2 + 3\sin(2t) \Rightarrow \sin(2t) = \frac{y-2}{3}$

2 State the Cartesian equation of the path.

Since $\cos^2(2t) + \sin^2(2t) = 1$,

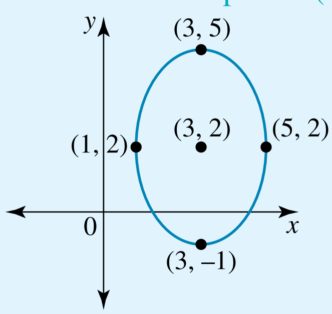
$\frac{(x-3)^2}{4} + \frac{(y-2)^2}{9} = 1.$

3 State the curve and its domain and range.

The curve is an ellipse with centre at (3, 2) and semi-major and minor axes of 2 and 3. The domain is 3 ± 2 or [1, 5] and the range is 2 ± 3 or [–1, 5].

4 Sketch the curve.

Note that at the points (3, 5) and (3, –1) the gradient is zero.



c 1 Find the speed at time t. Substitute for the derivatives.

c $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{(4\sin(2t))^2 + (6\cos(2t))^2}$

$= \sqrt{16\sin^2(2t) + 36\cos(2t)}$

2 Use $\sin^2(2t) = 1 - \cos^2(2t)$ to express the speed in terms of $\cos(2t)$ only.

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{16(1 - \cos^2(2t)) + 36\cos^2(2t)}$

$= \sqrt{16 + 20\cos^2(2t)}$

$= \sqrt{4(4 + 5\cos^2(2t))}$

3 State the speed at time t.	$\lvert \dot{\underset{\sim}{r}}(t) \rvert = 2\sqrt{4 + 5\cos^2(2t)}$
4 Determine when the maximum value of the speed will occur.	The maximum value of the speed occurs when $\cos^2(2t) = 1$; that is, $\lvert \dot{\underset{\sim}{r}}(t) \rvert_{max} = 2\sqrt{4 + 5}$
5 State the maximum value of the speed.	$\lvert \dot{\underset{\sim}{r}}(t) \rvert_{max} = 6$
6 Determine when the minimum value of the speed will occur.	The minimum value of the speed occurs when $\cos^2(2t) = 0$; that is, $\lvert \dot{\underset{\sim}{r}}(t) \rvert_{min} = 2\sqrt{4}$
7 State the minimum value of the speed.	$\lvert \dot{\underset{\sim}{r}}(t) \rvert_{min} = 4$

Applications of vector calculus

Given the vector equation of a moving particle, the motion can be analysed.

WORKED EXAMPLE 7

The position vector, $\underset{\sim}{r}(t)$, of a golf ball at a time t seconds is given by $\underset{\sim}{r}(t) = 15t\underset{\sim}{i} + (20t - 4.9t^2)\underset{\sim}{j}$ for $t \geq 0$, where the distance is in metres, $\underset{\sim}{i}$ is a unit vector horizontally forward and $\underset{\sim}{j}$ is a unit vector vertically upwards above ground level.

a **Find when the golf ball hits the ground.**

b **Find where the golf ball hits the ground.**

c **Determine the initial speed and angle of projection.**

d **Find the maximum height reached.**

e **Show that the golf ball travels in a parabolic path.**

THINK	WRITE
a **1** The time when the golf ball is at ground level is when the $\underset{\sim}{j}$ component is zero.	**a** $y(t) = 20t - 4.9t^2 = 0$ $t(20 - 4.9t) = 0$ $t = 0$ or $20 - 4.9t = 0$ $\frac{20}{4.9} = 4.08$ Since $t \geq 0$, ignore the initial time when $t = 0$.
2 State when the golf ball hits the ground.	The golf ball hits the ground after 4.08 seconds.
b **1** The distance R travelled when the golf ball hits the ground is the value of the $\underset{\sim}{i}$ component at this time.	**b** $R = x(4.08)$ $= 15 \times 4.08$
2 State where the golf ball hits the ground.	The golf ball hits the ground at a distance of 61.22 metres from the initial point.

c 1 Differentiate the position vector.

$\underset{\sim}{r}(t) = 15t\underset{\sim}{i} + (20t - 4.9t^2)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = 15\underset{\sim}{i} + (20 - 9.8t)\underset{\sim}{j}$

2 Find the initial velocity vector.

$\dot{\underset{\sim}{r}}(0) = 15\underset{\sim}{i} + 20\underset{\sim}{j}$

3 Find the initial speed.

The initial speed is the magnitude of the initial velocity vector.

$|\dot{\underset{\sim}{r}}(0)| = \sqrt{15^2 + 20^2} = 25$

4 Find the initial angle of projection.

The angle the initial velocity vector makes with the $\underset{\sim}{i}$ axis is $\cos^{-1}\left(\frac{15}{25}\right) = 53.13°$.

5 State the required results.

The golf ball is struck with an initial speed of 25 m/s at an angle of 53.13°.

d 1 The golf ball will rise until the vertical component of its velocity is zero.

$\dot{y}(t) = 20 - 9.8t = 0$

$t = \frac{20}{9.8} = 2.04$

2 The maximum height reached, H, is the value of the $\underset{\sim}{j}$ component at this time.

$H = y(2.04) = 20 \times 2.04 - 4.9 \times 2.04^2$

3 State the maximum height reached.

The golf ball reaches a maximum height of 20.41 metres.

e 1 Write the parametric equations.

$x = 15t \Rightarrow t = \frac{x}{15}$

$y = 20t - 4.9t^2$

2 Substitute the value of t into the equation for y.

$y = 20\left(\frac{x}{15}\right) - 4.9\left(\frac{x}{15}\right)^2$

3 Simplify and form common denominators.

$y = -\frac{x}{2250}(49x - 3000)$

4 State the result.

The parametric equation is of the form of a parabola, $y = ax(x - b)$ with $a < 0$. Therefore, the golf ball travels in a parabolic path.

EXERCISE 13.3 Differentiation of vectors

PRACTISE

1 **WE3** Find a unit tangent vector to $\underset{\sim}{r}(t) = (e^{2t} + e^{-2t})\underset{\sim}{i} + (e^{2t} - e^{-2t})\underset{\sim}{j}$ at the point where $t = 0$.

2 Find a unit tangent vector to $\underset{\sim}{r}(t) = \cos(2t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$ at the point where $t = \frac{\pi}{6}$.

3 **WE4** A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = te^{-2t}\underset{\sim}{i} + te^{2t}\underset{\sim}{j}$ for $t \geq 0$.

a Find the velocity vector.

b Find the speed of the particle at time t and hence find its speed when $t = \frac{1}{2}$.

c Find the acceleration vector.

4 A particle of mass 4 kg moves so that its position vector is given by $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \sin^2(t)\underset{\sim}{j}$ for $t \geq 0$, where t is measured in seconds and the

distance is in metres. Find the magnitude of the momentum acting on the particle at time $t = \frac{3\pi}{8}$.

5 WE5 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 2t^4\underset{\sim}{i} + 4\cos(2t)\underset{\sim}{j} + 6e^{-2t}\underset{\sim}{k}$ for $t \geq 0$. Find:

a the velocity vector

b the acceleration vector.

6 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 8\cos\left(\frac{\pi t}{4}\right)\underset{\sim}{i} + 8\sin\left(\frac{\pi t}{4}\right)\underset{\sim}{j} + 4e^{-2t}\underset{\sim}{k}$ for $t \geq 0$. Find the magnitude of the acceleration vector when $t = 1$.

7 WE6 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (4 + 3\cos(2t))\underset{\sim}{i} + (3 - 2\sin(2t))\underset{\sim}{j}$ for $0 \leq t \leq \pi$.

a Find the coordinates where the gradient of the curve is parallel to the x-axis.

b Find and sketch the Cartesian equation of the path, stating its domain and range.

c Find the maximum and minimum values of the speed.

8 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 3\sec(t)\underset{\sim}{i} + 2\tan(t)\underset{\sim}{j}$ for $0 \leq t \leq 2\pi$.

a Find and sketch the Cartesian equation of the path, stating its domain and range.

b Find the coordinates where the gradient is $\frac{4}{3}$.

9 WE7 The position vector $\underset{\sim}{r}(t)$ of a soccer ball at a time $t \geq 0$ seconds is given by $\underset{\sim}{r}(t) = 5t\underset{\sim}{i} + (12t - 4.9t^2)\underset{\sim}{j}$, where the distance is in metres, $\underset{\sim}{i}$ is a unit vector horizontally forward and $\underset{\sim}{j}$ is a unit vector vertically upwards above ground level.

a Find when the soccer ball hits the ground.

b Find where the soccer ball hits the ground.

c Determine the initial speed and angle of projection.

d Find the maximum height reached.

e Show that the soccer ball travels in a parabolic path.

10 A boy throws a tennis ball. The position vector, $\underset{\sim}{r}(t)$, of the tennis ball at a time $t \geq 0$ seconds is given by $\underset{\sim}{r}(t) = 24t\underset{\sim}{i} + (2 + 7t - 4.9t^2)\underset{\sim}{j}$, where the distance is in metres, $\underset{\sim}{i}$ is a unit vector horizontally forward and $\underset{\sim}{j}$ is a unit vector vertically upwards above ground level.

a How long before the tennis ball hits the ground?

b Find where the tennis ball hits the ground.

c Determine the initial speed and angle of projection.

d Find the maximum height reached.

e Show that the tennis ball travels in a parabolic path.

CONSOLIDATE

11 Find a unit tangent vector to each of the following at the point indicated.

a $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 4t^2\underset{\sim}{j}$, $t \geq 0$ at $t = 1$

b $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 8t^3\underset{\sim}{j}$, $t \geq 0$ at $t = 1$

c $\underset{\sim}{r}(t) = 3t^2\underset{\sim}{i} + (t^2 - 4t)\underset{\sim}{j}$, $t \geq 0$ at $t = 3$

d $\underset{\sim}{r}(t) = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j}$, $t \geq 0$ at $t = 2$

12 Find a unit tangent vector to each of the following at the point indicated.

a $\underset{\sim}{r}(t) = e^{-2t}\underset{\sim}{i} + e^{2t}\underset{\sim}{j}$, $t \geq 0$ at $t = 0$

b $\underset{\sim}{r}(t) = (t^2 + t)\underset{\sim}{i} + (t^2 - t)\underset{\sim}{j}$, $t \geq 0$ at $t = 1$

c $\underset{\sim}{r}(t) = \cos^2(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j},\ t \geq 0$ at $t = \frac{\pi}{3}$

d $\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j},\ t \geq 0$ at $t = \frac{\pi}{3}$

13 For each of the following position vectors, find the velocity vector and the acceleration vector.

a $\underset{\sim}{r}(t) = (t^2 + 9)\underset{\sim}{i} + \left(\frac{1}{1+t}\right)\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = \log_e(3t)\underset{\sim}{i} + (5t^2 + 4t)\underset{\sim}{j}$

c $\underset{\sim}{r}(t) = 8e^{-\frac{t}{2}}\underset{\sim}{i} + 4e^{2t}\underset{\sim}{j}$

d $\underset{\sim}{r}(t) = \log_e(3t + 4)\underset{\sim}{i} + 4\cos(3t)\underset{\sim}{j}$

14 For each of the following position vectors, find the velocity vector and the acceleration vector.

a $\underset{\sim}{r}(t) = e^{-2t}\underset{\sim}{i} + (t^4 - 2t^2)\underset{\sim}{j} + (5t^2 - 3)\underset{\sim}{k}$

b $\underset{\sim}{r}(t) = 3\cos(2t)\underset{\sim}{i} - 4\sin(2t)\underset{\sim}{j} + (12t - 5t^2)\underset{\sim}{k}$

c $\underset{\sim}{r}(t) = t^2\sin(2t)\underset{\sim}{i} + te^{-2t}\underset{\sim}{j} + 10t\underset{\sim}{k}$

d $\underset{\sim}{r}(t) = \cos^2(3t)\underset{\sim}{i} + t^3\cos(3t)\underset{\sim}{j} + (12 + t^2)\underset{\sim}{k}$

15 a A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 3\cos(2t)\underset{\sim}{i} + 3\sin(2t)\underset{\sim}{j},\ t \geq 0$.

i Find the Cartesian equation of the path.
ii Show that the speed is constant.
iii Show that the acceleration is directed inwards.
iv Show that the velocity vector is perpendicular to the acceleration vector.

b A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 4\cos(3t)\underset{\sim}{i} + 2\sqrt{2}\sin(3t)(\underset{\sim}{j} - \underset{\sim}{k}),\ t \geq 0$.

i Show that the speed is constant.
ii Show that the acceleration vector is perpendicular to the position vector.

16 a A particle of mass 3 kg moves so that its position vector is given by $\underset{\sim}{r}(t) = (2 + 4\sin(2t))\underset{\sim}{i} + (4\cos(2t) - 3)\underset{\sim}{j},\ t \geq 0$, where the distance is in metres.

i Find and sketch the Cartesian equation of the path.
ii Show that the velocity vector is perpendicular to the acceleration vector.
iii Find the magnitude of the momentum acting on the particle.
iv Find the magnitude of the resultant force acting on the body after $\frac{\pi}{8}$ seconds.
v Find the gradient of the curve when $t = \frac{\pi}{8}$.

b A particle of mass 2 kg moves so that its position vector is given by $\underset{\sim}{r}(t) = (5 + 2\cos(2t))\underset{\sim}{i} + (3 + 4\sin(2t))\underset{\sim}{j},\ 0 \leq t \leq 2\pi$, where the distance is in metres.

i Find and sketch the Cartesian equation of the path.
ii Find the magnitude of the momentum acting on the particle after $\frac{\pi}{12}$ seconds.
iii Find the magnitude of the resultant force acting on the body after $\frac{\pi}{12}$ seconds.
iv Find the maximum and minimum values of the speed.
v Find the values of t for which the gradient of the curve is $-2\sqrt{3}$.

17 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = a\cos(nt)\underset{\sim}{i} + a\sin(nt)\underset{\sim}{j}$ for $t \geq 0$, where a and n are positive constants.

a Find the Cartesian equation of the path.
b Show that the speed is constant.

c Show that the acceleration is directed inwards.
d Show that the velocity vector is perpendicular to the position vector.
e Describe the motion.

18 a A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 2\sec(t)\underset{\sim}{i} + 3\tan(t)\underset{\sim}{j}$ for $0 \le t \le 2\pi$.
i Find and sketch the Cartesian equation of the path, stating its domain and range.
ii Find the gradient at the point where $t = \frac{\pi}{4}$.

b A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 2\sec^2(t)\underset{\sim}{i} + 3\tan^2(t)\underset{\sim}{j}$ for $0 \le t \le 2\pi$.
i Find and sketch the Cartesian equation of the path, stating its domain and range.
ii Find the gradient at any point.
iii Find the minimum value of the speed.

19 a A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 3\,\text{cosec}(t)\underset{\sim}{i} + 4\cot(t)\underset{\sim}{j}$ for $0 \le t \le 2\pi$.
i Find and sketch the Cartesian equation of the path, stating its domain and range.
ii Find the gradient when $t = \frac{\pi}{3}$.
iii Find the minimum value of the speed.

b A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (1 + 2\,\text{cosec}(t))\underset{\sim}{i} + (4 - 3\cot(t))\underset{\sim}{j}$ for $0 \le t \le 2\pi$.
i Find and sketch the Cartesian equation of the path, stating its domain and range.
ii Find the values of t for which the gradient of the curve is 3.

20 A boy throws a tennis ball. The position vector $\underset{\sim}{r}(t)$ of the tennis ball at a time $t \ge 0$ seconds is given by $\underset{\sim}{r}(t) = 10t\underset{\sim}{i} + (10t - 4.9t^2)\underset{\sim}{j}$ where the distance is in metres, $\underset{\sim}{i}$ is a unit vector horizontally forward and $\underset{\sim}{j}$ is a unit vector vertically upwards above ground level.

a Find the time taken to reach the ground.
b Find the horizontal distance covered.
c Find the initial speed and angle of projection.
d Find the maximum height reached.
e Show that the tennis ball travels in a parabolic path.

MASTER

21 A javelin is thrown by an athlete. After the javelin is thrown, its position vector is given by $\underset{\sim}{r}(t) = 35t\underset{\sim}{i} + (1.8 + 9t - 4.9t^2)\underset{\sim}{j}$ at a time $t \ge 0$ seconds, where the distance is in metres, t is the time in seconds after the javelin is released, $\underset{\sim}{i}$ is a unit vector horizontally forward and $\underset{\sim}{j}$ is a unit vector vertically upwards above ground level.

a Find the initial height above the ground when the javelin was released.
b Find the time taken to reach the ground.
c Find the horizontal distance covered.

d Find the initial speed and angle of projection.
e Find the maximum height reached.
f Show that the javelin travels in a parabolic path.

22 A soccer ball is kicked off the ground. Its position vector is given by

$$\underset{\sim}{r}(t) = 23t\underset{\sim}{i} + 5t\underset{\sim}{j} + 4\sqrt{2}\sin\left(\frac{\pi t}{2}\right)\underset{\sim}{k},$$

where $\underset{\sim}{i}$ is a unit vector horizontally forward, $\underset{\sim}{j}$ is a unit vector to the right and $\underset{\sim}{k}$ is a unit vector vertically upwards above ground level. Measurements are in metres. Find:

a the time taken for the soccer ball to hit the ground
b the distance the soccer ball lands from the initial point
c the initial speed at which the soccer ball was kicked
d the maximum height above ground level reached by the soccer ball.

13.4 Special parametric curves

Plane curves

To sketch the graphs of curves defined by parametric equations, find important features such as the turning points and axis intercepts. Alternatively, use a calculator.

Finding areas using parametric forms

Areas bounded by curves can be found using the parametric equation of the curve and symmetrical properties. Consider the area bounded by a non-negative curve, which is a curve that lies entirely above the x-axis. The area bounded by the curve and the x-axis is given by $A = \int_a^b y\,dx$. However, we can substitute for $y = y(t)$ and $\frac{d}{dt}(x(t))$ to obtain an integral in terms of t for the area, $\int_{t=t_0}^{t=t_1} y(t)\frac{d}{dt}(x(t))dt$.

Finding the length of curves

Using symmetry, we can find the total length of the curve. The length of a curve between the values of t_0 and t_1 is given by $\int_{t_0}^{t_1} |\underset{\sim}{v}(t)|dt$. Sometimes the definite integrals obtained cannot be integrated; however, we can use a calculator to determine the numerical value of a definite integral.

WORKED EXAMPLE 8

A particle moves along a curve defined by the vector equation $\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$ for $t \in [0, 2\pi]$.

a **Find the gradient of the curve in terms of t.**

b **Find the values of t when the tangent to the curve is parallel to the x-axis, and hence find the turning points on the curve.**

c Sketch the graph of the curve defined by the parametric equations $x = 2\cos(t)$ and $y = \sin(2t)$ for $[0, 2\pi]$.

d Find the speed of the particle when $t = \frac{\pi}{4}$ and $t = \frac{\pi}{2}$.

e The area bounded by the curve and the x-axis can be expressed as $\int_{t=t_0}^{t=t_1} y(t)\frac{d}{dt}(x(t))dt$. Obtain a definite integral in terms of t for the area and show using calculus that the total area bounded by the curve and the x-axis is $\frac{16}{3}$ units^2.

f The total length of a curve between the values of t_0 and t_1 is given by $\int_{t_0}^{t_1} |\dot{\underset{\sim}{r}}(t)|dt$.

Set up a definite integral that gives the total length of the curve and, using technology, find the total length of the curve.

g Show that particle moves along the curve defined by the implicit equation $y^2 = \frac{x^2}{4}(4 - x^2)$.

h Hence, verify that the total area bounded by the curve and x-axis is given by $\frac{16}{3}$ units^2.

THINK	WRITE/DRAW
a 1 State the parametric equations.	**a** $x = 2\cos(t)$ and $y = \sin(2t)$
2 Differentiate x with respect to t.	$\frac{dx}{dt} = \dot{x} = -2\sin(t)$
3 Differentiate y with respect to t.	$\frac{dy}{dt} = \dot{y} = 2\cos(2t)$
4 Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$
5 Substitute for the derivatives and state the gradient in terms of t.	$\frac{dy}{dx} = -\frac{2\cos(2t)}{2\sin(t)} = -\frac{\cos(2t)}{\sin(t)}$
b 1 The tangent to the curve is parallel to the x-axis when the gradient is zero. Equate the numerator to zero but not the denominator.	**b** $\frac{dy}{dx} = 0 \Rightarrow \cos(2t) = 0$, but $\sin(t) \neq 0$.
2 Solve and find the values of t when the gradient is zero.	$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ since $t \in [0\,, 2\pi]$, $2t \in [0, 4\pi]$ $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

3 Find the coordinates of the turning points. Substitute $t = \frac{\pi}{4}$ into the position vector.

$$\underset{\sim}{r}\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right)\underset{\sim}{i} + \sin\left(\frac{\pi}{2}\right)\underset{\sim}{j}$$
$$= \sqrt{2}\underset{\sim}{i} + \underset{\sim}{j}$$

The coordinates are $(\sqrt{2}, 1)$.

4 Substitute $t = \frac{3\pi}{4}$ into the position vector.

$$\underset{\sim}{r}\left(\frac{3\pi}{4}\right) = 2\cos\left(\frac{3\pi}{4}\right)\underset{\sim}{i} + \sin\left(\frac{3\pi}{2}\right)\underset{\sim}{j}$$
$$= -\sqrt{2}\underset{\sim}{i} - \underset{\sim}{j}$$

The coordinates are $(-\sqrt{2}, -1)$.

5 Substitute $t = \frac{5\pi}{4}$ into the position vector.

$$\underset{\sim}{r}\left(\frac{5\pi}{4}\right) = 2\cos\left(\frac{5\pi}{4}\right)\underset{\sim}{i} + \sin\left(\frac{5\pi}{2}\right)\underset{\sim}{j}$$
$$= -\sqrt{2}\underset{\sim}{i} + \underset{\sim}{j}$$

The coordinates are $(-\sqrt{2}, 1)$.

6 Substitute $t = \frac{7\pi}{4}$ into the position vector.

$$\underset{\sim}{r}\left(\frac{7\pi}{4}\right) = 2\cos\left(\frac{7\pi}{4}\right)\underset{\sim}{i} + \sin\left(\frac{7\pi}{2}\right)\underset{\sim}{j}$$
$$= \sqrt{2}\underset{\sim}{i} - \underset{\sim}{j}$$

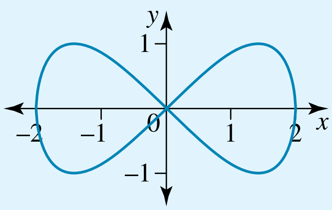
The coordinates are $(\sqrt{2}, -1)$.

7 State the coordinates of the turning points.

There are maximum turning points at $(\pm\sqrt{2}, 1)$ and minimum turning points at $(\pm\sqrt{2}, -1)$.

c 1 Use a calculator to sketch the graph of the parametric equations.

c



2 When the gradient of the denominator is zero, we have vertical asymptotes.

$$\sin(t) = 0$$
$$t = 0, \pi, 2\pi$$

3 State the coordinates on the graph where the tangent lines are vertical.

$$\underset{\sim}{r}(0) = 2\cos(0)\underset{\sim}{i} + \sin(0)\underset{\sim}{j}$$
$$= 2\underset{\sim}{i}$$

The coordinates are (2, 0).

$$\underset{\sim}{r}(\pi) = 2\cos(\pi)\underset{\sim}{i} + \sin(2\pi)\underset{\sim}{j}$$
$$= -2\underset{\sim}{i}$$

The coordinates are (–2, 0).

d 1 Find the velocity vector. Differentiate the position vector with respect to t.

d

$$\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$$
$$\dot{\underset{\sim}{r}}(t) = -2\sin(t)\underset{\sim}{i} + 2\cos(2t)\underset{\sim}{j}$$

2 Find the speed at time t by finding the magnitude of the velocity vector.

$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{\dot{x}^2 + \dot{y}^2}$$
$$= \sqrt{(-2\sin(t))^2 + (2\cos(2t))^2}$$
$$= \sqrt{4(\sin^2(t) + \cos^2(2t))}$$

3 Find the speed at $t = \frac{\pi}{4}$.

Substitute $t = \frac{\pi}{4}$:

$$\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right)\right| = \sqrt{4\left(\sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{2}\right)\right)}$$
$$= \sqrt{4\left(\frac{1}{2} + 0\right)}$$
$$= \sqrt{2}$$

4 Find the speed at $t = \frac{\pi}{2}$.

Substitute $t = \frac{\pi}{2}$:

$$\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{2}\right)\right| = \sqrt{4\left(\sin^2\left(\frac{\pi}{2}\right) + \cos^2(\pi)\right)}$$
$$= \sqrt{4(1 + (-1)^2)}$$
$$= \sqrt{8} = 2\sqrt{2}$$

e 1 Consider the movement of the particle.

e When $t = 0$,

$$\underset{\sim}{r}(0) = 2\cos(0)\underset{\sim}{i} + \sin(0)\underset{\sim}{j}$$
$$= 2\underset{\sim}{i}$$

The coordinates are (2, 0).

When $t = \frac{\pi}{2}$,

$$\underset{\sim}{r}(0) = 2\cos\left(\frac{\pi}{2}\right)\underset{\sim}{i} + \sin\left(2 \times \frac{\pi}{2}\right)\underset{\sim}{j}$$
$$= \underset{\sim}{0}$$

The coordinates are (0, 0).

2 Use the symmetry of the curve to state a definite integral that gives the total area bounded by the curve.

$$\int_{t=t_0}^{t=t_1} y(t)\frac{d}{dt}(x(t))\,dt = \int_{t=t_0}^{t=t_1} y(t)\dot{x}(t)\,dt$$

where $y = \sin(2t)$ and $\frac{dx}{dt} = \dot{x} = 2\sin(t)$

The total area A is 4 times the area from $t = \frac{\pi}{2}$ to $t = 0$.

3 Write the definite integral that gives the area.

$$A = 4\int_{\frac{\pi}{2}}^{0} \sin(2t) \times -2\sin(t)dt$$

4 Simplify the integrand using the double-angle formula $\sin(2t) = 2\sin(t)\cos(t)$.

$$A = -16\int_{\frac{\pi}{2}}^{0} \sin^2(t)\cos(t)dt$$

5 Perform the integration.

$$A = -\frac{16}{3}\left[\sin^3(t)\right]_{\frac{\pi}{2}}^{0}$$

6 Evaluate the area.

$$A = -\frac{16}{3}\left[\left(\sin^3(0) - \sin^3\left(\frac{\pi}{2}\right)\right)\right]$$

7 State the value of the required area.

$$A = \frac{16}{3}\text{ units}^2$$

f **1** The total length of a curve s between the values of t_0 and t_1 is given by $\int_{t_0}^{t_1} |\underset{\sim}{v}(t)| dt$.

f Since
$|\underset{\sim}{v}(t)| = |\dot{\underset{\sim}{r}}(t)| = 2\sqrt{(\sin^2(t) + \cos^2(2t)}$,
by symmetry,

$$S = 4\int_{\frac{\pi}{2}}^{0} 2\sqrt{(\sin^2(t) + \cos^2(2t))}\, dt$$

2 This integral cannot be evaluated by our techniques of calculus, so we resort to using a calculator to obtain a numerical answer.

Using a calculator, $s = 12.1944$ units.

g **1** Substitute the given parametric equations into the implicit equation. Consider the left-hand side. Substitute $y = \sin(2t)$.

g The implicit equation is $y^2 = \frac{x^2}{4}(4 - x^2)$.

LHS:

$$\begin{aligned} y^2 &= \sin^2(2t) \\ &= (2\sin(t)\cos(t))^2 \\ &= 4\sin^2(t)\cos^2(t) \end{aligned}$$

2 Consider the right-hand side. Substitute $x = \cos(t)$.

RHS:

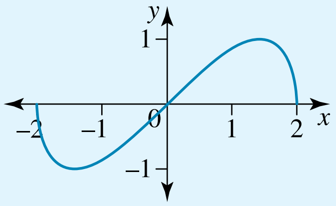
$$\begin{aligned} \frac{x^2}{4}(4 - x^2) &= \frac{(2\cos(t))^2}{4}(4 - 4\cos^2(t)) \\ &= \frac{4}{4}\cos^2(t)4(1 - \cos^2(t)) \\ &= 4\cos^2(t)\sin^2(t) \end{aligned}$$

h **1** The implicit equation is a relation, not a function.

h $y^2 = \frac{x^2}{4}(4 - x^2)$

$y = \pm\frac{x}{2}\sqrt{4 - x^2}$

Consider the graph of $y = \frac{x}{2}\sqrt{4 - x^2}$, which is a function.



2 The total area bounded by the parametric curves can be found by symmetry.

$$A = \int_a^b y\, dx$$

$$A = 4\int_0^2 \frac{x}{2}\sqrt{4 - x^2}dx$$

3 Use a non-linear substitution.

Let $u = 4 - x^2$.

$\frac{du}{dx} = -2x$ or $dx = -\frac{1}{2x}du$

4 Change the terminals to the new variable.

When $x = 0 \Rightarrow u = 4$, and when $x = 2 \Rightarrow u = 0$.

5 Substitute into the definite integral and use the properties of the definite integral. Note that the x terms cancel.

$$A = 4\int_4^0 \frac{x}{2} u^{\frac{1}{2}} \times -\frac{1}{2x}\, du$$

$$= -\int_4^0 u^{\frac{1}{2}}\, du$$

$$= \int_0^4 u^{\frac{1}{2}}\, du$$

6 Perform the integration.

$$A = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^4$$

7 Evaluate the definite integral.

$$A = \frac{2}{3}\left[4^{\frac{3}{2}} - 0\right]$$

8 State the final result, which agrees with the alternative method.

$$A = \frac{16}{3}$$

studyon

Units 3 & 4
AOS 4
Topic 2
Concept 4

Applications of vector calculus
Concept summary
Practice questions

Elegant curves

We can use calculators to sketch the graphs of parametric curves. The parametric form of some curves can appear to be quite complicated, but their graphs have interesting mathematical properties. For example, the types of curves in the example above are called Lissajous figures. Graphs of these types occur in electronics and appear on oscilloscopes. These graphs were first investigated by Nathaniel Bowditch in 1815, and were later explored in greater depth by Jules Lissajous in 1857.


Jules Lissajous

From the parametric equations we can often but not always determine an explicit or implicit relation for the equation of the curve. Sometimes, by using symmetry and the parametric form, we can obtain the area bounded by the curve, but sometimes it may not be possible to obtain an area from an implicitly defined curve, as it may not be possible to rearrange the implicit curve into one that has y as the subject. The arc length of a curve can sometimes be calculated using our techniques of calculus, but often, as above, we will need to use a calculator to find a numerical value for the definite integral obtained.

As seen in many examples, we will often need to use trigonometric addition theorems and double-angle formulas to simplify the results.

WORKED EXAMPLE 9

A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (4\cos(t) + \cos(4t))\underset{\sim}{i} + (4\sin(t) - \sin(4t))\underset{\sim}{j}$ for $0 \le t \le 2\pi$.

a **Find the gradient of the curve.**

b **Find the speed at time t.**

c **Find $\dot{\underset{\sim}{r}}(t) \cdot \underset{\sim}{r}(t)$**

d Find the maximum and minimum values of the acceleration.

e Sketch the path of the particle.

THINK	WRITE/DRAW
a 1 State the parametric equations.	**a** $\underset{\sim}{r}(t) = (4\cos(t) + \cos(4t))\underset{\sim}{i} + (4\sin(t) - \sin(4t))\underset{\sim}{j}$ Then $x(t) = 4\cos(t) + \cos(4t)$ and $y(t) = 4\sin(t) - \sin(4t)$
2 Differentiate x with respect to t.	$x = 4\cos(t) + \cos(4t)$ $\frac{dx}{dt} = \dot{x} = -4\sin(t) - 4\sin(4t)$
3 Differentiate y with respect to t.	$y = 4\sin(t) - \sin(4t)$ $\frac{dy}{dt} = \dot{y} = 4\cos(t) - 4\cos(4t)$
4 Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$
5 Substitute for the derivatives.	$\frac{dy}{dx} = \frac{4\cos(t) - 4\cos(4t)}{-4\sin(t) - 4\sin(4t)}$
6 Simplify by cancelling common factors and state the gradient in terms of t.	$\frac{dy}{dx} = \frac{\cos(4t) - \cos(t)}{\sin(4t) + \sin(t)}$
b 1 Find the speed at time t. Substitute for the derivatives and expand.	**b** $\lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{\dot{x}^2 + \dot{y}^2}$ $\dot{x}^2 = [-4\sin(t) - 4\sin(4t)]^2$ $= 16\sin^2(t) + 32\sin(t)\sin(4t) + 16\sin^2(4t)$ $\dot{y}^2 = [4(\cos(t) - 4\cos(4t))]^2$ $= 16\cos^2(t) - 32\cos(t)\cos(4t) + 16\cos^2(4t)$
2 Simplify using addition theorems and trigonometry. $\sin^2(A) + \cos^2(A) = 1$ and $\cos(A + B)$ $= \cos(A)\cos(B) - \sin(A)\sin(B)$	$\dot{x}^2 + \dot{y}^2$ $= 16(\sin^2(t) + \cos^2(t)) + 16(\sin^2(4t) + \cos^2(4t))$ $+ 32(\sin(t)\sin(4t) - \cos(t)\cos(4t))$ $= 32 - 32\cos(5t)$
3 Simplify using the double-angle formula $2\cos^2(A) = 1 - \cos(2A)$.	$\lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{32(1 - \cos(5t))}$ $= \sqrt{32 \times 2\cos^2\left(\frac{5t}{2}\right)}$
4 State the speed in simplified form at time t.	$\lvert\dot{\underset{\sim}{r}}(t)\rvert = 8\cos\left(\frac{5t}{2}\right)$
c 1 Find the dot product of the position and velocity vectors.	**c** $\dot{\underset{\sim}{r}}(t) \cdot \underset{\sim}{r}(t) = \dot{x}(t) \times x(t) + \dot{y}(t) \times y(t)$ $= -4(\sin(t) + \sin(4t)) \times (4\cos(t) + \cos(4t))$ $+ 4(\cos(t) - \cos(4t)) \times (4\sin(t) - \sin(4t))$
2 Expand the brackets.	$= -16\sin(t)\cos(t) - 4\sin(t)\cos(4t)$ $- 16\sin(4t)\cos(t) - 4\sin(4t)\cos(4t)$ $+ 16\sin(t)\cos(t) - 4\sin(4t)\cos(t)$ $- 16\sin(t)\cos(4t) + 4\sin(4t)\cos(4t)$

3 Cancel factors and group like terms.

$= -20(\sin(4t)\cos(t) + \cos(4t)\sin(t))$

4 Simplify using addition theorems and trigonometry.
$\sin(A + B)$
$= \sin(A)\cos(B) + \cos(A)\sin(B)$

$\dot{\underset{\sim}{r}}(t) \cdot \underset{\sim}{r}(t) = -20\sin(5t)$

d **1** Differentiate the velocity vector to find the acceleration vector.

d $\dot{\underset{\sim}{r}}(t) = -4(\sin(t) + \sin(4t))\underset{\sim}{i} + 4(\cos(t) - \cos(4t))\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}}(t) = -4(\cos(t) + 4\cos(4t))\underset{\sim}{i} + 4(-\sin(t) + 4\sin(4t))\underset{\sim}{j}$

2 Find the magnitude of the acceleration vector at time t. Substitute for the derivatives and expand.

$|\ddot{\underset{\sim}{r}}(t)| = \sqrt{\ddot{x}^2 + \ddot{y}^2}$

$\ddot{x}^2 = [-4(\cos(t) + 4\cos(4t))]^2$

$= 16(\cos^2(t) + 8\cos(t)\cos(4t) + 16\cos^2(4t))$

$\ddot{y}^2 = [4(-\sin(t) + 4\sin(4t))]^2$

$= 16(\sin^2(t) - 8\sin(t)\sin(4t) + 16\sin^2(4t))$

3 Simplify using addition theorems and trigonometry.
$\sin^2(A) + \cos^2(A) = 1$ and
$\cos(A + B)$
$= \cos(A)\cos(B) - \sin(A)\sin(B)$

$\ddot{x}^2 + \ddot{y}^2$

$= 16(\cos^2(t) + \sin^2(t)) + 16 \times 16(\cos^2(4t) + \sin^2(4t))$

$+ 16 \times 8(\cos(t)\cos(4t) - \sin(t)\sin(4t))$

$= 16(1 + 16 + 8\cos(5t))$

4 Simplify and state the magnitude of the acceleration in simplified form at time t.

$|\ddot{\underset{\sim}{r}}(t)| = \sqrt{16(17 + 8\cos(5t))}$

$= 4\sqrt{17 + 8\cos(5t)}$

5 Determine when the maximum value of the acceleration will occur.

The maximum value of the acceleration occurs when $\cos(5t) = 1$; that is

$|\ddot{\underset{\sim}{r}}(t)|_{max} = 4\sqrt{17 + 8}$.

6 State the maximum value of the acceleration.

$|\ddot{\underset{\sim}{r}}(t)|_{max} = 20$

7 Determine when the minimum value of the acceleration will occur.

The minimum value of the acceleration occurs when $\cos(5t) = -1$; that is

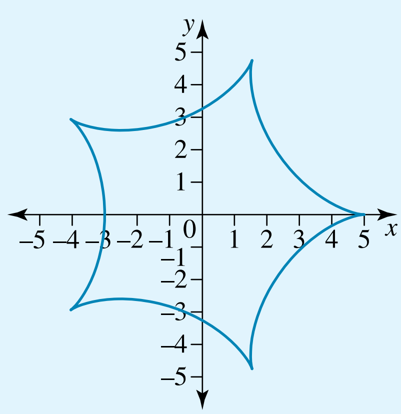
$|\ddot{\underset{\sim}{r}}(t)|_{min} = 4\sqrt{17 - 8}$.

8 State the minimum value of the acceleration.

$|\ddot{\underset{\sim}{r}}(t)|_{min} = 12$

e **1** Use a calculator in parametric mode to sketch the graph of the parametric equations.

e $x = 4\cos(t) + \cos(4t)$ and $y = 4\sin(t) - \sin(4t)$



EXERCISE 13.4 Special parametric curves

PRACTISE

1 **WE8** A particle moves along a curve defined by the vector equation $\underset{\sim}{r}(t) = 2\sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$ for $t \in [0, 2\pi]$.

a Find the gradient of the curve in terms of t.

b Find the value of t when the tangent to the curve is parallel to the x-axis, and hence find the turning point on the curve.

c Sketch the graph of the curve defined by the parametric equations $x = 2\sin(t)$ and $y = \cos(2t)$ for $[0, 2\pi]$.

d Find the speed of the particle when $t = \frac{\pi}{4}$ and $t = \pi$.

e The area bounded by the curve and the x-axis can be expressed as $\int_{t=t_0}^{t=t_1} y(t)\frac{d}{dt}(x(t))\,dt$. Obtain a definite integral in terms of t for the area and show using calculus that the total area bounded by the curve and the x-axis is $\frac{4\sqrt{2}}{3}$ units^2.

f The length of a curve between the values of t_0 and t_1 is given by $\int_{t_0}^{t_1} |\underset{\sim}{v}(t)|\,dt$.

Write a definite integral that gives the total length of the curve and, using technology, find the total length of the curve.

g Show that the particle moves along the parabola $y = 1 - \frac{x^2}{2}$.

h Hence verify that the total area bounded by the curve and x-axis is given by $\frac{4\sqrt{2}}{3}$ units^2.

2 A particle moves along a curve defined by the vector equation $\underset{\sim}{r}(t) = a\sin(nt)\underset{\sim}{i} + b\cos(mt)\underset{\sim}{j}$ for $t \geq 0$, where a, b, n and m are positive real numbers.

a Find the gradient of the curve in terms of t.

b Find the speed in terms of t.

c If $a = b$ and $m = n$, show that the particle moves in a circle.

d If $a \neq b$ and $m = n$, show that the particle moves along an ellipse.

e If $m = 2n$, show that the particle moves along a parabola.

3 **WE9** A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (2\cos(t) + \cos(2t))\underset{\sim}{i} + (2\sin(t) - \sin(2t))\underset{\sim}{j}$ for $0 \leq t \leq 2\pi$. This curve is called a deltoid.

a Find the gradient of the curve.

b Find the speed at time t.

c Find $\dot{\underset{\sim}{r}}(t) \cdot \underset{\sim}{r}(t)$.

d Find the maximum and minimum values of the acceleration.

e Sketch the path of the particle.

4 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (3\cos(t) - \cos(2t))\underset{\sim}{i} + (3\sin(t) + \sin(2t))\underset{\sim}{j}$ for $0 \leq t \leq 2\pi$. This curve is called a hypocycloid.

a Find the gradient of the curve at the point where $t = \frac{3\pi}{4}$.

b Find the maximum and minimum values of the speed.

c Find the maximum and minimum values of the acceleration.

d Sketch the path of the particle.

CONSOLIDATE

5 A ball rolls along a curve so that its position vector is given by $\underset{\sim}{r}(t) = (t - \sin(t))\underset{\sim}{i} + (1 + \cos(t))\underset{\sim}{j}$ for $t \in [0, 4\pi]$. The path of the curve is called a cycloid.

a Show that the gradient of the curve is given by $-\cot\left(\frac{t}{2}\right)$.

b Find the coordinates on the curve where the gradient is parallel to the x-axis.

c Sketch the path of the particle.

d Show that the speed at time t is given by $2\sin\left(\frac{t}{2}\right)$.

e Show using calculus that the total length of curve is 16 units.

f Show that the total area bounded by the cycloid and the x-axis is given by 2π units^2.

6 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (\cos(t) + t\sin(t))\underset{\sim}{i} + (\sin(t) - t\cos(t))\underset{\sim}{j}$ for $t \geq 0$. The path of the curve is called an involute of a circle.

a Show that the gradient of the curve is given by $\tan(t)$.

b Show that the speed at time t is given by t.

c Show that the magnitude of the acceleration at time t is given by $\sqrt{1 + t^2}$.

d Show using calculus that the length of the curve between $t = t_0$ and $t = t_1$ is given by $\frac{1}{2}(t_1^2 - t_0^2)$.

e Sketch the path of the particle when $a = 1$.

7 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = at\underset{\sim}{i} + \frac{a}{1 + t^2}\underset{\sim}{j}$ for $t \in R$, where a is a positive real constant. The path of the curve is called the witch of Agnesi. Maria Agnesi (1718–1799) was an Italian mathematician and philosopher. She is often considered to be the first woman to have achieved a reputation in mathematics.

a Show that the gradient of the curve is given by $\frac{-2t}{(1 + t^2)^2}$ and that the curve has a turning point at $(0, a)$.

b Show that $\frac{d^2y}{dx^2} = \frac{2(3t^2 - 1)}{a(1 + t^2)^2}$ and hence show that the points of inflection on the curve are given by $\left(\pm\frac{\sqrt{3}a}{3}, \frac{3a}{4}\right)$.

c Sketch the graph of the witch of Agnesi.

d Show that the area bounded by the curve, the coordinate axes and the point $x = a$ is given by $\frac{\pi a^2}{4}$.

e Show that the Cartesian equation of the curve is given by $y = \frac{a^3}{a^2 + x^2}$ and hence verify that the area bounded by the curve, the coordinate axes and the point $x = a$ is given by $\frac{\pi a^2}{4}$.

8 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = \dfrac{3at}{1+t^3}\underset{\sim}{i} + \dfrac{3at^2}{1+t^3}\underset{\sim}{j}$ for $t \in R$, where a is a positive real constant. This curve is called the folium of Descartes.

a Show that the Cartesian equation of the curve is given by $x^3 + y^3 = 3axy$.

b Find the gradient of the tangent to the curve in terms t.

c Find the gradient of the curve in terms of both x and y, and show that the curve has a turning point at $\left(a\sqrt[3]{2}, a\sqrt[3]{4}\right)$.

d If $a = 3$:

i sketch the graph of the folium of Descartes

ii find the position vector of the particle at the point where $t = 2$

iii find the speed of the particle at the point where $t = 2$

iv find the equation of the tangent to the curve at the point where $t = 2$.

9 A curve given by the Cartesian equation $y = \dfrac{abx}{x^2 + a^2}$, where a and b are positive constants, is called the serpentine curve.

a Show that $\dfrac{dy}{dx} = \dfrac{ab(a-x)(a+x)}{(x^2+a^2)^2}$ and that the graph of the serpentine curve has turning points at $\left(a, \dfrac{b}{2}\right)$ and $\left(-a, -\dfrac{b}{2}\right)$.

b Show that $\dfrac{d^2y}{dx^2} = \dfrac{2abx(x^2 - 3a^2)}{(x^2+a^2)^3}$ and hence show that the points of inflection on the serpentine curve are given by $\left(\pm\sqrt{3}a, \pm\dfrac{\sqrt{3}b}{4}\right)$.

c Show that the vector equation of the serpentine curve is given by $\underset{\sim}{r}(t) = a\cot(t)\underset{\sim}{i} + \dfrac{b}{2}\sin(2t)\underset{\sim}{j}$ for $t \geq 0$.

d Sketch the graph of the serpentine curve.

e Show that the area bounded by the serpentine curve $y = \dfrac{abx}{x^2+a^2}$, the x-axis, the origin and the point $x = a$ is given by $\dfrac{ab}{2}\log_e(2)$. Verify this result using another method.

10 A particle moves along the vector equation $\underset{\sim}{r}(t) = 8\sin^3(t)\underset{\sim}{i} + 8\cos^3(t)\underset{\sim}{j}$ for $t \in [0, 2\pi]$. This curve is called an astroid.

a Show that the Cartesian equation of the curve is given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$.

b Show that the gradient at t is given by $-\tan(t)$.

c Show that $\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$ and deduce that the curve has no turning points.

d Sketch the graph of the astroid.

e Show that the speed of the particle at a time t is given by $12\sin(2t)$.

f The total length of the astroid can be expressed as $4\displaystyle\int_0^{\frac{\pi}{2}} \left|\dot{\underset{\sim}{r}}(t)\right| dt$ Hence, show that the length of the astroid is 48 units.

g Show that the total area inside the astroid is given by $768\int_0^{\frac{\pi}{2}} \sin^4(t)\cos^2(t)dt$, and, using technology, show that this area is equal to 24π units^2.

Consider the general astroid with parametric equations $x = a\cos^3(t)$ and $y = a\sin^3(t)$, where a is a positive real constant.

h Let P $(a\cos^3(t), a\sin^3(t))$ be a general point on the astroid. Find, in terms of t, the equation of the tangent to the astroid at the point P.

i If the tangent to the astroid crosses the x-axis at Q and crosses the y-axis at R, show that distance between the points Q and R is always equal to a.

11 A curve given by the Cartesian equation $y^2(x^2 + y^2) = a^2x^2$, where a is a positive constant, is called the kappa curve.

a Show that $x^2 = \dfrac{y^4}{a^2 - y^2}$.

b Show that the curve can be represented by the parametric equations $x = a\cos(t)\cot(t)$ and $y = a\cos(t)$ for $t \in [0, 2\pi]$.

c Show that the gradient at t is given by $\dfrac{\sin^3(t)}{\cos(t)(\sin^2(t) + 1)}$ and hence deduce that the kappa curve has a vertical tangent at the origin, no turning points and horizontal asymptotes at $y = \pm a$.

d Sketch the graph of the kappa curve.

e The area bounded by the kappa curve, the origin, the y-axis and the line $y = \dfrac{a}{2}$ is rotated about the y-axis to form a solid of revolution. Use calculus and partial fractions to show that the volume obtained is given by $\pi a^3\left(\log_e(\sqrt{3}) - \dfrac{13}{24}\right)$.

12 The function $f: D \to R$, $f(x) = x\sqrt{1 - x^2}$ is called an eight curve.

a Find the coordinates where the curve crosses the x-axis and hence state the maximal domain, D, of the function.

b Show that $f'(x) = \dfrac{1 - 2x^2}{\sqrt{1 - x^2}}$ and that the graph of the function f has a maximum at $\left(\dfrac{\sqrt{2}}{2}, \dfrac{1}{2}\right)$ and a minimum at $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{1}{2}\right)$.

c Show that $f''(x) = \dfrac{2x^3 - 3x}{\sqrt{(1 - x^2)^3}}$ and hence show that the only point of inflection is at the origin.

d Discuss what happens to the gradient of the curve as x approaches ± 1.

e Consider the curve defined by the implicit relationship $y^2 = x^2 - x^4$. Using implicit differentiation, find $\dfrac{dy}{dx}$ and hence verify the result in **b**.

The position vector $\underset{\sim}{r}(t)$ of a particle moving on a curve is given by $\underset{\sim}{r}(t) = \sin(t)\underset{\sim}{i} + \dfrac{1}{2}\sin(2t)\underset{\sim}{j}$, for $t \in [0, 2\pi]$.

f Show that the particle moves on the the eight curve $y^2 = x^2 - x^4$.

g Sketch the graph of the eight curve $y^2 = x^2 - x^4$.

h Show that the total area inside the eight curve is given by $\dfrac{4}{3}$ units^2.

i If the eight curve is rotated about the x-axis, it forms a solid of revolution.
Show that the total volume formed is $\dfrac{4\pi}{15}$.

j Using technology, find the total length of the eight curve.

13 Consider the functions $f: D \to R$, $f(x) = x^{\frac{3}{2}}\sqrt{4-x}$ and $g: D \to R$, $g(x) = -\sqrt{x^3(4-x)}$.

a If the maximal domain of both functions f and g is $D = [a, b]$, state the values of a and b.

b Explain how the graph of g is obtained from the graph of f.

c If $y^2 = x^3(4-x)$:

i use implicit differentiation to obtain an expression for $\dfrac{dy}{dx}$ in terms of both x and y

ii show that $\dfrac{d^2y}{dx^2} = \dfrac{2(6-6x+x^2)}{\sqrt{x(4-x)^3}}$

iii find the exact coordinates of any turning points on the graph of f and use an appropriate test to verify its nature

iv show that the point of inflection on the graph of f is $(3-\sqrt{3}, 2.36)$

v discuss what happens to the gradient of both curves f and g as x approaches the value of b

vi sketch the graphs of f and g.

d The position vector $\underset{\sim}{r}(t)$ of a particle moving on a curve is given by
$\underset{\sim}{r}(t) = 4\sin^2\left(\dfrac{t}{2}\right)\underset{\sim}{i} + 16\cos\left(\dfrac{t}{2}\right)\sin^3\left(\dfrac{t}{2}\right)\underset{\sim}{j}$ for $t \geq 0$.

i Show that the particle moves on the curve $y^2 = x^3(4-x)$.

ii Find the first time the particle passes through the maximum turning point.

iii Find the speed of the particle at the first time it passes through the maximum turning point.

14 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (3\cos(t) - \cos(3t))\underset{\sim}{i} + (3\sin(t) - \sin(3t))\underset{\sim}{j}$ for $0 \leq t \leq 2\pi$. The graph of this curve is called a nephroid. The nephroid is a plane curve whose name means 'kidney-shaped'.

a Show that the speed of the particle at time t is given by $6\sin(t)$ and hence show that the maximum and minimum values of the speed are 6 and –6.

b Find the times when the gradient is zero and hence find the turning points on the graph of the nephroid.

c Sketch the path of the particle.

d Show that the nephroid can be expressed in the implicit form $(x^2 + y^2 - 4)^3 = 108y^2$.

e Show using calculus that the total length of the nephroid is 24 units.

f Use technology to show that the total area inside the nephroid is 12π units^2.

MASTER

15 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 2\cos(t)(1 + \cos(t))\underset{\sim}{i} + 2\sin(t)(1 + \cos(t))\underset{\sim}{j}$ for $0 \leq t \leq 2\pi$. The graph of this curve is called a cardioid. The cardioid is a plane curve that is heart-shaped.

a Find the distance of the particle from the origin at time t.

b Find the speed of the particle at time t.

c Find the times when the position vector is perpendicular to the velocity vector.
d Find the maximum and minimum values of the acceleration.
e Sketch the path of the particle.
f Show using calculus that the total length of the cardioid is 16 units.
g Use technology to show that the total area inside the cardioid is 6π units^2.
h Show that the cardioid can be expressed in the implicit form $(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$.

16 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (n\cos(t) + \cos(nt))\underset{\sim}{i} + (n\sin(t) - \sin(nt))\underset{\sim}{j}$, where $n > 1$.

a Show that the gradient of the curve at time t is given by $\dfrac{\cos(nt) - \cos(t)}{\sin(nt) + \sin(t)}$.
b Show that the speed at time t is given by $2n\cos\left(\left(\dfrac{n+1}{2}\right)t\right)$.
c Show that $\dot{\underset{\sim}{r}}(t) \cdot \underset{\sim}{r}(t) = -(n^2 + n)\sin((n+1)t)$.
d Show that the maximum and minimum values of the acceleration are given by $n(n+1)$ and $n(n-1)$ respectively.

13.5 Integration of vectors

The constant vector

studyon

Units 3 & 4
AOS 4
Topic 2
Concept 3

Antidifferentiation of vector functions
Concept summary
Practice questions

When integrating a function, always remember to include the constant of integration, which is a scalar. When integrating a vector function with respect to a scalar, the constant of integration is a vector. This follows since if $\underset{\sim}{c}$ is a constant vector, then $\dfrac{d}{dt}(\underset{\sim}{c}) = \underset{\sim}{0}$.

Rules for integrating vectors

When differentiating a vector, we differentiate its components, so when we integrate a vector, we integrate each component using the usual rules for finding antiderivatives. If $\underset{\sim}{q}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ is a vector function, then we define $\displaystyle\int \underset{\sim}{q}(t)\,dt = \int x(t)\,dt\,\underset{\sim}{i} + \int y(t)\,dt\,\underset{\sim}{j} + \underset{\sim}{c}$, where $\underset{\sim}{c}$ is a constant vector.

Note that in the two-dimensional case, $\underset{\sim}{c} = c_1\underset{\sim}{i} + c_2\underset{\sim}{j}$, where c_1 and c_2 are real numbers.

Velocity vector to position vector

Because differentiating the position vector with respect to time gives the velocity vector, if we integrate the velocity vector with respect to time, we will obtain the position vector. Thus, given the velocity vector $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j}$, we can obtain the position vector $\underset{\sim}{r}(t) = \displaystyle\int \dot{x}(t)\,dt\,\underset{\sim}{i} + \int \dot{y}(t)\,dt\,\underset{\sim}{j} = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + \underset{\sim}{c}$. Note that an initial condition must be given in order for us to be able to find the constant vector of integration.

WORKED EXAMPLE 10 The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 6t\underset{\sim}{j}$ for $t \geq 0$. If $\underset{\sim}{r}(1) = 3\underset{\sim}{i} + \underset{\sim}{j}$, find the position vector at time t.

THINK	WRITE
1 Integrate the velocity vector to obtain the position vector using the given rules.	$\dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 6t\underset{\sim}{j}$ $\underset{\sim}{r}(t) = \int 2dt\,\underset{\sim}{i} + \int 6tdt\,\underset{\sim}{j}$
2 Perform the integration. Do not forget to add in a constant vector.	$\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 3t^2\underset{\sim}{j} + \underset{\sim}{c}$
3 Substitute to find the value of the constant vector.	Substitute $t = 1$ and use the given condition. $\underset{\sim}{r}(1) = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j}$
4 Solve for the constant vector.	$\underset{\sim}{c} = (3\underset{\sim}{i} + \underset{\sim}{j}) - (2\underset{\sim}{i} + 3\underset{\sim}{j})$ $= \underset{\sim}{i} - 2\underset{\sim}{j}$
5 Substitute back for the constant vector.	$\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + 3t^2\underset{\sim}{j} + \underset{\sim}{c}$ $= (2t\underset{\sim}{i} + 3t^2\underset{\sim}{j}) + (\underset{\sim}{i} - 2\underset{\sim}{j})$
6 Simplify the position vector to give the final result.	$\underset{\sim}{r}(t) = (2t + 1)\underset{\sim}{i} + (3t^2 - 2)\underset{\sim}{j}$

Acceleration vector to position vector

Because differentiating the velocity vector with respect to time gives the acceleration vector, if we integrate the acceleration vector with respect to time, we will obtain the velocity vector. Given the acceleration vector

$\underset{\sim}{a}(t) = \dfrac{d\underset{\sim}{v}(t)}{dt} = \ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j}$, we can obtain the velocity vector

$\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \int \ddot{x}(t)\,dt\underset{\sim}{i} + \int \ddot{y}(t)\,dt\underset{\sim}{j} = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j} + \underset{\sim}{c}_1$, where $\underset{\sim}{c}_1$ is a constant vector.

By integrating again, as above, we can find the position vector.

Note that two sets of information must be given to find the two constant vectors of integration. This process is a generalisation of the techniques used in earlier topics when we integrated the acceleration to find the displacement.

WORKED EXAMPLE 11 The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i}$, where $t \geq 0$ is the time. Given that $\dot{\underset{\sim}{r}}(2) = 6\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{r}(2) = 4\underset{\sim}{i} - 2\underset{\sim}{j}$, find the position vector at time t.

THINK	WRITE
1 Integrate the acceleration vector to obtain the velocity vector using the given rules.	$\ddot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i}$ $\dot{\underset{\sim}{r}}(t) = \int 6tdt\,\underset{\sim}{i}$
2 Perform the integration. Do not forget to add in the first constant vector.	$\dot{\underset{\sim}{r}}(t) = 3t^2\underset{\sim}{i} + \underset{\sim}{c}_1$
3 Substitute to find the first constant vector.	Substitute $t = 2$ and use the first given condition. $\dot{\underset{\sim}{r}}(2) = 12\underset{\sim}{i} + \underset{\sim}{c}_1 = 6\underset{\sim}{i} - 3\underset{\sim}{j}$

4 Solve for the first constant vector. | $\underset{\sim}{c}_1 = (6\underset{\sim}{i} - 3\underset{\sim}{j}) - 12\underset{\sim}{i}$
$= -6\underset{\sim}{i} - 3\underset{\sim}{j}$

5 Substitute back for the first constant vector. | $\dot{\underset{\sim}{r}}(t) = 3t^2\underset{\sim}{i} + (-6\underset{\sim}{i} - 3\underset{\sim}{j})$

6 Simplify to give the velocity vector. | $\dot{\underset{\sim}{r}}(t) = (3t^2 - 6)\underset{\sim}{i} - 3\underset{\sim}{j}$

7 Integrate the velocity vector to obtain the position vector using the given rules. | $\underset{\sim}{r}(t) = \int (3t^2 - 6)dt\underset{\sim}{i} - \int 3dt\underset{\sim}{j}$

8 Perform the integration. Do not forget to add in a second constant vector. | $\underset{\sim}{r}(t) = (t^3 - 6t)\underset{\sim}{i} - 3t\underset{\sim}{j} + \underset{\sim}{c}_2$

9 Substitute to find the value of the second constant vector. | Substitute $t = 2$ and use the second given condition.
$\underset{\sim}{r}(2) = (8 - 12)\underset{\sim}{i} - 6\underset{\sim}{j} + \underset{\sim}{c}_2 = 4\underset{\sim}{i} - 2\underset{\sim}{j}$

10 Solve for the second constant vector. | $\underset{\sim}{c}_2 = (4\underset{\sim}{i} - 2\underset{\sim}{j}) - (-4\underset{\sim}{i} - 6\underset{\sim}{j})$
$= 8\underset{\sim}{i} + 4\underset{\sim}{j}$

11 Substitute back for the second constant vector. | $\underset{\sim}{r}(t) = (t^3 - 6t)\underset{\sim}{i} - 3t\,\underset{\sim}{j} + (8\underset{\sim}{i} + 4\underset{\sim}{j})$

12 Simplify the position vector to give the final result. | $\underset{\sim}{r}(t) = (t^3 - 6t + 8)\underset{\sim}{i} + (4 + 3t)\underset{\sim}{j}$

Finding the Cartesian equation

As previously, when we are given the position vector, we can determine the parametric equations of the path of the particle, and by eliminating the parameter, we can determine the Cartesian equation of the curve along which the particle moves.

WORKED EXAMPLE 12

The acceleration vector of a moving particle is given by $\cos\left(\frac{t}{2}\right)\underset{\sim}{i} - \sin\left(\frac{t}{2}\right)\underset{\sim}{j}$ for $0 \le t \le 4\pi$, where t is the time. The initial velocity is $2\underset{\sim}{j}$ and the initial position is $\underset{\sim}{i} - 3\underset{\sim}{j}$. Find the Cartesian equation of the path.

THINK | **WRITE**

1 State and integrate the acceleration vector to obtain the velocity vector using the given rules. | $\ddot{\underset{\sim}{r}}(t) = \cos\left(\frac{t}{2}\right)\underset{\sim}{i} - \sin\left(\frac{t}{2}\right)\underset{\sim}{j}$
$\dot{\underset{\sim}{r}}(t) = \int \cos\left(\frac{t}{2}\right)dt\underset{\sim}{i} - \int \sin\left(\frac{t}{2}\right)dt\underset{\sim}{j}$

2 Perform the integration. Do not forget to add in the first constant vector. | Since $\int \cos(kx)dx = \frac{1}{k}\sin(kx)$ and
$\int \sin(kx)dx = -\frac{1}{k}\cos(kx)$ with $k = \frac{1}{2}$,
$\dot{\underset{\sim}{r}}(t) = 2\sin\left(\frac{t}{2}\right)\underset{\sim}{i} + 2\cos\left(\frac{t}{2}\right)\underset{\sim}{j} + \underset{\sim}{c}_1$.

3 Substitute to solve for the first constant vector.	Initially means when $t = 0$. Substitute $t = 0$ and use the first given condition. $\dot{\underset{\sim}{r}}(0) = 2\sin(0)\underset{\sim}{i} + 2\cos(0)\underset{\sim}{j} + \underset{\sim}{c}_1 = 2\underset{\sim}{j}$ $2\underset{\sim}{j} + \underset{\sim}{c}_1 = 2\underset{\sim}{j}$ $\underset{\sim}{c}_1 = \underset{\sim}{0}$
4 Substitute back for the first constant vector and integrate the velocity vector to obtain the position vector, using the given rules.	$\dot{\underset{\sim}{r}}(t) = 2\sin\left(\frac{t}{2}\right)\underset{\sim}{i} + 2\cos\left(\frac{t}{2}\right)\underset{\sim}{j}$ $\dot{\underset{\sim}{r}}(t) = \int \cos\left(\frac{t}{2}\right)dt\underset{\sim}{i} - \int \sin\left(\frac{t}{2}\right)dt\underset{\sim}{j}$ $\underset{\sim}{r}(t) = \int 2\sin\left(\frac{t}{2}\right)dt\underset{\sim}{i} + \int 2\cos\left(\frac{t}{2}\right)dt\underset{\sim}{j}$
5 Perform the integration. Do not forget to add in a second constant vector.	$\underset{\sim}{r}(t) = -4\cos\left(\frac{t}{2}\right)\underset{\sim}{i} + 4\sin\left(\frac{t}{2}\right)\underset{\sim}{j} + \underset{\sim}{c}_2$
6 Substitute to solve for the second constant vector.	Initially means when $t = 0$. Substitute $t = 0$ and use the second given condition. $\underset{\sim}{r}(0) = -4\cos(0)\underset{\sim}{i} + 4\sin(0)\underset{\sim}{j} + \underset{\sim}{c}_2 = \underset{\sim}{i} - 3\underset{\sim}{j}$ $-4\underset{\sim}{i} + \underset{\sim}{c}_2 = \underset{\sim}{i} - 3\underset{\sim}{j}$ $\underset{\sim}{c}_2 = 5\underset{\sim}{i} - 3\underset{\sim}{j}$
7 Substitute back for the second constant vector and state the position vector.	$\underset{\sim}{r}(t) = -4\cos\left(\frac{t}{2}\right)\underset{\sim}{i} + 4\sin\left(\frac{t}{2}\right)\underset{\sim}{j} + (5\underset{\sim}{i} - 3\underset{\sim}{j})$ $\underset{\sim}{r}(t) = \left(5 - 4\cos\left(\frac{t}{2}\right)\right)\underset{\sim}{i} + \left(4\sin\left(\frac{t}{2}\right) - 3\right)\underset{\sim}{j}$
8 State the parametric equations.	$x = 5 - 4\cos\left(\frac{t}{2}\right),\ y = 4\sin\left(\frac{t}{2}\right) - 3$
9 Eliminate the parameter.	$\cos\left(\frac{t}{2}\right) = \frac{5 - x}{4},\ \sin\left(\frac{t}{2}\right) = \frac{y + 3}{4}$ Since $\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$, $\left(\frac{5 - x}{4}\right)^2 + \left(\frac{y + 3}{4}\right)^2 = 1$
10 State the Cartesian equation in the implicit form.	$(x - 5)^2 + (y + 3)^2 = 16$ This is a circle with centre at (5, –3) and radius 4.

EXERCISE 13.5 Integration of vectors

PRACTISE

1 WE10 The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = (4t - 4)\underset{\sim}{i} - 3\underset{\sim}{j}$ for $t \geq 0$. If $\underset{\sim}{r}(1) = 3\underset{\sim}{i} + \underset{\sim}{j}$, find the position vector at time t.

2 The initial position of a particle is given by $3\underset{\sim}{i} + \underset{\sim}{j}$. If the velocity vector of the particle is given by $\underset{\sim}{v}(t) = 6\sin(3t)\underset{\sim}{i} + 4e^{-2t}\underset{\sim}{j}$, find the position vector.

3 WE11 The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = -12t^2\underset{\sim}{j}$, where $t \geq 0$ is the time. Given that $\dot{\underset{\sim}{r}}(2) = -2\underset{\sim}{i} - 16\underset{\sim}{j}$ and $\underset{\sim}{r}(2) = \underset{\sim}{i} + 6\underset{\sim}{j}$, find the position vector at time t.

4 A particle is moving such that $\underset{\sim}{r}(1) = -2\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}(1) = 6\underset{\sim}{i} + 10\underset{\sim}{j}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = 6\underset{\sim}{i} + 2\underset{\sim}{j}$, where $t \geq 0$ is the time. Find the position vector at time t.

5 WE12 The acceleration vector of a moving particle is given by $-45\cos(3t)\underset{\sim}{i} + 45\sin(3t)\underset{\sim}{j}$, where $t \geq 0$ is the time. The initial velocity is $-15\underset{\sim}{j}$ and the initial position is $3\underset{\sim}{i} + 4\underset{\sim}{j}$. Find the Cartesian equation of the path.

6 A particle is moving such that its initial position is $2\underset{\sim}{i} - 2\underset{\sim}{j}$ and its initial velocity is $10\underset{\sim}{j}$. The acceleration vector of the particle is given by $12\cos(2t)\underset{\sim}{i} - 20\sin(2t)\underset{\sim}{j}$, where $0 \leq t \leq \pi$ and t is the time. Find the Cartesian equation of the path.

CONSOLIDATE

7 a The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = e^{-\frac{t}{3}}\underset{\sim}{i} + 4t^3\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the particle is at the origin, find the position vector.

b The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + 6\sin(2t)\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the particle is at the origin, find the position vector.

c The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = \frac{1}{\sqrt{16 - t^2}}\underset{\sim}{i} - \frac{t}{\sqrt{t^2 + 9}}\underset{\sim}{j}$, where $t \geq 0$ is the time. If $\underset{\sim}{r}(0) = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, find the position vector.

d The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = \frac{2}{2t + 1}\underset{\sim}{i} + \frac{72}{(3t + 2)^2}\underset{\sim}{j}$, where $t \geq 0$ is the time. If $\underset{\sim}{r}(0) = 5\underset{\sim}{i} + \underset{\sim}{j}$, find the position vector at time t.

8 a The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = 6\underset{\sim}{i}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underset{\sim}{r}}(0) = 4\underset{\sim}{j}$ and the initial position is $\underset{\sim}{r}(0) = 2\underset{\sim}{i} - 3\underset{\sim}{j}$, find the position vector at time t.

b The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = 4\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underset{\sim}{r}}(0) = 2\underset{\sim}{i}$ and the initial position is $\underset{\sim}{r}(0) = 4\underset{\sim}{i} + \underset{\sim}{j}$, find the position vector at time t.

c A moving particle starts at position $\underset{\sim}{r}(0) = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ with an initial velocity of zero. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = 8\underset{\sim}{i} + 6\underset{\sim}{j}$, where $t \geq 0$ is the time. Find the position vector at time t.

d A moving particle starts at position $\underset{\sim}{r}(0) = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ with an initial velocity $\dot{\underset{\sim}{r}}(0) = 8\underset{\sim}{j}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = 4\underset{\sim}{i} + 2\underset{\sim}{j}$, where $t \geq 0$ is the time. Find the position vector at time t.

9 a The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = (6t - 8)\underset{\sim}{i}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underset{\sim}{r}}(0) = 4\underset{\sim}{j}$ and the initial position is $\underset{\sim}{r}(0) = 5\underset{\sim}{i} - 3\underset{\sim}{j}$, find the position vector at time t.

b The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = (12t - 6)\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underset{\sim}{r}}(0) = 8\underset{\sim}{i}$ and the initial position is $\underset{\sim}{r}(0) = 7\underset{\sim}{i} - 4\underset{\sim}{j}$, find the position vector at time t.

c A particle is moving such that $\underset{\sim}{r}(1) = 9\underset{\sim}{i} - 15\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}(1) = 6\underset{\sim}{i} - 2\underset{\sim}{j}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + 2\underset{\sim}{j}$, where $t \geq 0$ is the time. Find the position vector at time t.

d A particle is moving such that $\underset{\sim}{r}(1) = 8\underset{\sim}{i} - 8\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}(1) = 2\underset{\sim}{i} - 10\underset{\sim}{j}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = -2\underset{\sim}{i} + 6\underset{\sim}{j}$, where $t \geq 0$ is the time. Find the position vector at time t.

10 a A particle is moving such that $\underset{\sim}{r}(1) = 2\underset{\sim}{i}$ and $\dot{\underset{\sim}{r}}(1) = 4\underset{\sim}{j}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = \frac{2}{t^3}\underset{\sim}{i} - \frac{4}{t^3}\underset{\sim}{j}$, where $t \geq 0$ is the time. Find the position vector at time t.

b A particle is moving such that $\underset{\sim}{r}(4) = 8\underset{\sim}{i} + \underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}(4) = \underset{\sim}{i} - 2\underset{\sim}{j}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = \frac{-1}{\sqrt{t^3}}\underset{\sim}{i} + 2\underset{\sim}{j}$, where $t \geq 0$ is the time. Find the position vector at time t.

c The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = \frac{-9}{(3t+1)^2}\underset{\sim}{i} + \frac{32}{(2t+1)^3}\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underset{\sim}{r}}(0) = 3\underset{\sim}{i} - 8\underset{\sim}{j}$ and the initial position is $\underset{\sim}{r}(0) = 4\underset{\sim}{i} + 3\underset{\sim}{j}$, find the position vector at time t.

d The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = \frac{-9}{(3t+1)^2}\underset{\sim}{i} - \frac{24}{(2t+1)^4}\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underset{\sim}{r}}(0) = 2\underset{\sim}{i} - \underset{\sim}{j}$ and the initial position is $\underset{\sim}{r}(0) = 6\underset{\sim}{i} + 8\underset{\sim}{j}$, find the position vector at time t.

11 a When a ball is thrown, its acceleration vector is given by $\ddot{\underset{\sim}{r}}(t) = -10\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $15\underset{\sim}{i} + 20\underset{\sim}{j}$ and the initial position is $2\underset{\sim}{j}$, find the Cartesian equation of the path.

b When a ball is thrown, its acceleration vector is given by $\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $5\underset{\sim}{i} + 10\underset{\sim}{j}$ and the initial position is $\underset{\sim}{j}$, find the Cartesian equation of the path.

12 a The acceleration vector of a moving particle is given by $\ddot{\underset{\sim}{r}}(t) = 4e^{-2t}\underset{\sim}{i} + 4e^{2t}\underset{\sim}{j}$, where $t \in R$. If $\dot{\underset{\sim}{r}}(0) = -2\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = 5\underset{\sim}{i} - 2\underset{\sim}{j}$, find the Cartesian equation of the path.

b The acceleration vector of a moving particle is given by $\ddot{\underset{\sim}{r}}(t) = -e^{-\frac{t}{2}}\underset{\sim}{i} + 2e^{\frac{t}{2}}\underset{\sim}{j}$, where $t \in R$. If $\dot{\underset{\sim}{r}}(0) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = -2\underset{\sim}{i} + 3\underset{\sim}{j}$, find the Cartesian equation of the path.

13 a A particle is moving such that $\underset{\sim}{r}(0) = \underset{\sim}{i} + 5\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}(0) = 4\underset{\sim}{j}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = 8\cos(2t)\underset{\sim}{i} - 8\sin(2t)\underset{\sim}{j}$, where t is the time and $0 \leq t \leq 2\pi$. Find the Cartesian equation of the path.

b The acceleration vector of a moving particle is given by $-\cos\left(\frac{t}{3}\right)\underset{\sim}{i} - \sin\left(\frac{t}{3}\right)\underset{\sim}{j}$, where $0 \leq t \leq 6\pi$. If $v(0) = 3\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = 2\underset{\sim}{i} + 5\underset{\sim}{j}$, find the Cartesian equation of the path.

14 a A particle is moving such that its initial position is $3\underset{\sim}{i} + 5\underset{\sim}{j}$ and its initial velocity $-6\underset{\sim}{j}$. The acceleration vector of the particle is given by $a(t) = 9\cos(3t)\underset{\sim}{i} + 18\sin(3t)\underset{\sim}{j}$, where t is the time and $0 \leq t \leq 2\pi$. Find the Cartesian equation of the path.

b The acceleration vector of a moving particle is given by $-3\cos\left(\frac{t}{2}\right)\underset{\sim}{i} + \sin\left(\frac{t}{2}\right)\underset{\sim}{j}$, where $0 \leq t \leq 4\pi$. If $v(0) = -2\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = 5\underset{\sim}{i} + 3\underset{\sim}{j}$, find the Cartesian equation of the path.

15 a The acceleration vector of a moving particle is given by $-2\cos(t)\underset{\sim}{i} - 8\cos(2t)\underset{\sim}{j}$, where $t \geq 0$. If initially the velocity vector is zero and the initial position is $2\underset{\sim}{i}$, find the Cartesian equation of the path, stating the domain and range.

b A moving particle is such that its initial position is $\underset{\sim}{i} + 3\underset{\sim}{j}$ and its initial velocity $4\underset{\sim}{i}$. The acceleration vector of the particle is given by $-8\sin(2t)\underset{\sim}{i} - 96\cos(4t)\underset{\sim}{j}$, where t is the time and $0 \le t \le \pi$. Find the Cartesian equation of the path, stating the domain and range.

16 a Particle A has an acceleration of $2\underset{\sim}{i} + 4\underset{\sim}{j}$, an initial velocity of $-6\underset{\sim}{i} + 5\underset{\sim}{j}$ and an initial position of $13\underset{\sim}{i} - 17\underset{\sim}{j}$. Particle B has an acceleration of $6\underset{\sim}{i} + 8\underset{\sim}{j}$, an initial velocity of $-8\underset{\sim}{i} - 20\underset{\sim}{j}$ and an initial position of $\underset{\sim}{i} + 40\underset{\sim}{j}$. Show that the two particles collide, and find the time and point of collision.

b Car A has an acceleration of $2\underset{\sim}{i} - 2\underset{\sim}{j}$ at time t, and after 1 second its velocity is $-5\underset{\sim}{i} + 6\underset{\sim}{j}$ and its position is $-3\underset{\sim}{i} + 2\underset{\sim}{j}$. Car B has an acceleration of $2\underset{\sim}{i} - 6\underset{\sim}{j}$ at time t, and after 1 second its velocity is $-2\underset{\sim}{i} + 6\underset{\sim}{j}$ and its position is $-15\underset{\sim}{i} + 34\underset{\sim}{j}$. Show that the two cars collide, and find the time and point of collision.

MASTER

17 a The acceleration vector of a particle is given by $-n^2 r\cos(nt)\underset{\sim}{i} - n^2 r\sin(nt)\underset{\sim}{j}$, at time t where a, b and r are all real constants. If $\underset{\sim}{v}(0) = nr\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = (a + r)\underset{\sim}{i} + b\underset{\sim}{j}$, find the Cartesian equation of the path.

b A particle is moving such that its initial position is $(h + a)\underset{\sim}{i} + k\underset{\sim}{j}$ and its initial velocity is $bn\underset{\sim}{j}$, where a, b, n, h and k are all real constants. The acceleration vector of the particle is given by $-n^2 a\cos(nt)\underset{\sim}{i} - n^2 b\sin(3t)\underset{\sim}{j}$, where t is the time. Find the Cartesian equation of the path.

18 a The acceleration vector of a moving particle is given by $(4\cos(2t) - 2\cos(t))\underset{\sim}{i} + (4\sin(2t) - 2\sin(t))\underset{\sim}{j}$, where $0 \le t \le 2\pi$, $\dot{\underset{\sim}{r}}(\pi) = -4\underset{\sim}{j}$ and $\underset{\sim}{r}(\pi) = -3\underset{\sim}{i}$. Find the position vector and sketch the equation of the path.

b A particle is moving such that $\underset{\sim}{r}\left(\frac{\pi}{2}\right) = \underset{\sim}{i} + 4\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}\left(\frac{\pi}{2}\right) = -4\underset{\sim}{i} - 4\underset{\sim}{j}$.

The acceleration vector of the particle is given by $-(16\cos(4t) + 4\cos(t))\underset{\sim}{i} + (16\sin(4t) - 4\sin(t))\underset{\sim}{j}$. Find the position vector and sketch the equation of the path.

c The acceleration vector of a moving particle is given by

$-8\cos(2t)\underset{\sim}{i} - 108\sin(6t)\underset{\sim}{j}$, where $0 \le t \le 2\pi$, $\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right) = -4\underset{\sim}{i}$ and $\underset{\sim}{r}\left(\frac{\pi}{4}\right) = -3\underset{\sim}{j}$.

Find the position vector and sketch the equation of the path.

13.6 Projectile motion

The motion of a particle when acted upon by gravity and air resistance is called projectile motion. The object or projectile considered can be a ball of any type, as considered in sport, or it can be a car or bullet; in fact, it can be any object which moves.

In our first modelling approach to projectile motion, certain assumptions are made. The first is to assume that the projectile, no matter how big, is treated as a point particle. Further assumptions are to ignore air resistance, assume that the Earth is flat, ignore the rotation of the Earth and ignore the variations in gravity due to height. For projectiles moving close to the Earth's surface at heights of no more than approximately two hundred metres, these assumptions are generally valid.

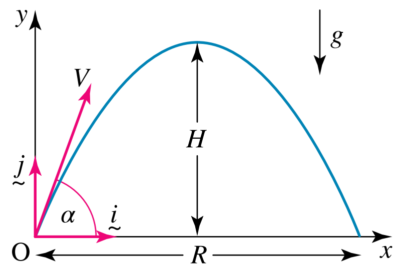
General theory of a projectile

study on

Units 3 & 4
AOS 4
Topic 2
Concept 5

Projectile motion
Concept summary
Practice questions

Consider a projectile fired in a vertical two-dimensional plane from the origin, O, with an initial speed of V m/s at an angle of α degrees to the horizontal. T is the time of flight, H is the maximum height reached by the projectile on its motion, and R is the range on the horizontal plane, that is, the horizontal distance travelled.



As the motion is in two dimensions, $x = x(t)$ is the horizontal displacement and $y = y(t)$ is the vertical displacement at time t seconds, where $0 \le t \le T$.

Taking $\underset{\sim}{i}$ as a unit vector of 1 metre in the x direction and $\underset{\sim}{j}$ as a unit vector of 1 metre in the positive upwards y direction,

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ is the position vector,

$\dot{\underset{\sim}{r}}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j}$ is the velocity vector and

$\ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j}$ is the acceleration vector.

The only force acting on the projectile is the weight force, which acts downwards, so that $\underset{\sim}{g} = -g\underset{\sim}{j}$, where $g = 9.8$ m/s^2.

Using vectors:

$\ddot{\underset{\sim}{r}}(t) = -g\underset{\sim}{j}$ and

$\dot{\underset{\sim}{r}}(t) = -gt\underset{\sim}{j} + \underset{\sim}{c}_1$ (integrating the acceleration vector to get the velocity vector).

But $\dot{\underset{\sim}{r}}(0) = V\cos(\alpha)\underset{\sim}{i} + V\sin(\alpha)\underset{\sim}{j}$ (the initial velocity vector).

Therefore, $\underset{\sim}{c}_1 = V\cos(\alpha)\underset{\sim}{i} + V\sin(\alpha)\underset{\sim}{j}$.

$\dot{\underset{\sim}{r}}(t) = V\cos(\alpha)\underset{\sim}{i} + (V\sin(\alpha) - gt)\underset{\sim}{j}$ (substituting for the first constant vector)

$\underset{\sim}{r}(t) = Vt\cos(\alpha)\underset{\sim}{i} + \left(Vt\sin(\alpha) - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{c}_2$ (integrating the velocity vector to get the position vector)

As the projectile is fired from the origin, O, $\underset{\sim}{r}(0) = \underset{\sim}{0}$, so $\underset{\sim}{c}_2 = \underset{\sim}{0}$; thus,

$\underset{\sim}{r}(t) = Vt\cos(\alpha)\underset{\sim}{i} + \left(Vt\sin(\alpha) - \frac{1}{2}gt^2\right)\underset{\sim}{j}$.

Note that if the projectile is not initially fired from the origin but from some initial height h above ground level, then $\underset{\sim}{c}_2 = h\underset{\sim}{j}$.

Therefore the parametric equations of the projectile are

$$x(t) = Vt\cos(\alpha) \text{ and } y(t) = Vt\sin(\alpha) - \frac{1}{2}gt^2 + h.$$

Time of flight

The time of flight is the time that the projectile takes to go up and come down again, or the time at which it returns to ground level and hits the ground. To find the time of flight, solve $y = 0$ for t.

$$Vt\sin(\alpha) - \frac{1}{2}gt^2 = 0$$

$$t\left(V\sin(\alpha) - \frac{1}{2}gt\right) = 0$$

The result $t = 0$ represents the time when the projectile was fired, so $T = \dfrac{2V\sin(\alpha)}{g}$ represents the time of flight.

The range

The range is the horizontal distance travelled in time T.

$$R = x(T) = VT\cos(\alpha)$$

Substitute $T = \dfrac{2V\sin(\alpha)}{g}$: $\quad R = V\cos(\alpha)\left(\dfrac{2V\sin(\alpha)}{g}\right)$

Expand: $\quad R = \dfrac{V^2\, 2\sin(\alpha)\cos(\alpha)}{g}$

Using the double-angle formula $\sin(2A) = 2\sin(A)\cos(A)$,

$$R = \frac{V^2\sin(2\alpha)}{g}$$

Note that for maximum range, $\sin(2\alpha) = 1$, so $2\alpha = 90°$ or $\alpha = 45°$. This applies only for a projectile fired from ground level.

Maximum height

The maximum height occurs when the particle is no longer rising. This occurs when the vertical component of the velocity is zero, that is, $\dot{y}(t) = 0$.

Solving $\dot{y} = V\sin(\alpha) - gt = 0$ gives $t = \dfrac{V\sin(\alpha)}{g} = \dfrac{T}{2}$, which is half the time of flight.

This applies only for a projectile fired from ground level.

Substituting for t into the y component gives

$$H = y\left(\frac{T}{2}\right) = V\sin(\alpha)\left(\frac{V\sin(\alpha)}{g}\right) - \frac{g}{2}\left(\frac{V\sin(\alpha)}{g}\right)^2$$

$$H = \frac{V^2\sin^2(\alpha)}{g} - \frac{V^2\sin^2(\alpha)}{2g}$$

$$H = \frac{V^2\sin^2(\alpha)}{2g}$$

WORKED EXAMPLE 13

A golf ball is hit off the ground at an angle of 53.13° with an initial speed of 25 m/s. Find:

a **the time of flight**

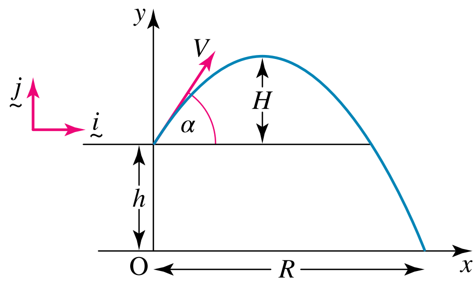
b **the range**

c **the maximum height reached.**

THINK	WRITE
a 1 State the value of the parameters.	**a** $V = 25$ and $\alpha = 53.13°$
2 Use the general results to find the time of flight.	$T = \dfrac{2V\sin(\alpha)}{g}$ $T = \dfrac{2 \times 25\sin(53.13°)}{9.8}$

3 State the time of flight.	$T = 4.08$ seconds.
b 1 Use the general results to find the range.	**b** $R = \dfrac{V^2\sin(2\alpha)}{g}$ $R = \dfrac{25^2\sin(2\times 53.13°)}{9.8}$
2 State the range.	$R = 61.22$ metres.
c 1 Use the general results to find the maximum height reached.	**c** $H = \dfrac{V^2\sin^2(\alpha)}{2g}$ $H = \dfrac{25^2\sin^2(53.13°)}{2\times 9.8}$
2 State the maximum height reached.	$H = 20.41$ metres.

Improving the range



The formulas for the time of flight $T = \dfrac{2V\sin(\alpha)}{g}$ and the range $R = \dfrac{V^2\sin(2\alpha)}{g}$ derived above are only true when the projectile is released from the origin, O. In many situations, the projectile is released from a height h above ground level.

In these situations we need to solve the appropriate parametric equations $x(t) = Vt\cos(\alpha)$ and $y(t) = h + Vt\sin(\alpha) - \frac{1}{2}gt^2$ to find the time of flight and the range. The result for the maximum height, $H = \dfrac{V^2\sin^2(\alpha)}{2g}$, is still valid, but it gives only the height above h.

It can also be shown that the path of the projectile is a parabola. An interesting question in these situations is how we determine the maximum range on the horizontal plane (ground level) for a given speed of projection.

WORKED EXAMPLE 14

A boy throws a ball from the top of a hill 2 metres above ground level with an initial speed of 10 m/s at an angle of 40° to the horizontal.

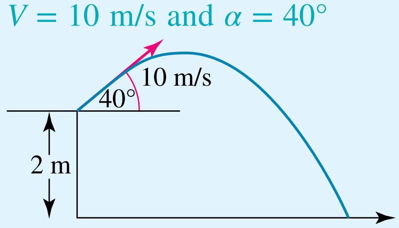
a Find the time of flight and the horizontal distance travelled.

b Find the maximum height reached.

c Find the speed and angle at which the ball lands.

d Show that the ball travels in a parabolic path.

THINK	WRITE/DRAW
a 1 State the value of the parameters.	**a** $V = 10$ m/s and $\alpha = 40°$



2 Start with the acceleration vector and the initial velocity vector to find the velocity vector at time t.	$\ddot{\underset{\sim}{r}}(t) = -9.8\underset{\sim}{j}$ $\dot{\underset{\sim}{r}}(0) = 10\cos(40°)\underset{\sim}{i} + 10\sin(40°)\underset{\sim}{j}$ $\dot{\underset{\sim}{r}}(t) = 10\cos(40°)\underset{\sim}{i} + (10\sin(40°) - 9.8t)\underset{\sim}{j}$
3 State the position vector at time t.	Since the initial position is 2 metres above the ground, $\underset{\sim}{r}(0) = 2\underset{\sim}{j}$, $r(t) = 10t\cos(40°)\underset{\sim}{i} + (2 + 10t\sin(40°) - 4.9t^2)\underset{\sim}{j}$
4 The ball hits the ground when the vertical component is zero.	Solve $y = 2 + 10t\sin(40°) - 4.9t^2 = 0$ for t. Solving this quadratic gives $t = -0.26, 1.57$. Since $t \geq 0$, the time of flight is 1.57 seconds. $x(1.57) = 10 \times 1.57\cos(40°)$ $= 12.04$ The horizontal distance travelled is 12.04 metres.
b 1 The ball will rise and reach maximum height when the vertical component of its velocity is zero.	**b** Solve $\dot{y} = 10\sin(40°) - 9.8t = 0$ for t. $t = \dfrac{10\sin(40°)}{9.8}$ $= 0.656$ Note that this is not half the time of flight.
2 State the maximum height reached.	$y(0.656) = 2 + 10 \times 0.656\sin(40^o) - 4.9 \times 0.656^2$ The maximum height reached is 4.11 metres.
c 1 Use the time when the ball lands to find its velocity vector at this time.	**c** $\dot{\underset{\sim}{r}}(1.57) = 10\cos(40°)\underset{\sim}{i} + (10\sin(40°) - 9.8 \times 1.57)\underset{\sim}{j}$ $= 7.66\underset{\sim}{i} - 8.96\underset{\sim}{j}$
2 Find the speed at which the ball lands.	$\lvert\underset{\sim}{r}(1.57)\rvert = \sqrt{\dot{x}^2 + \dot{y}^2}$ $= \sqrt{7.66^2 + (-8.96)^2}$ The ball lands with a speed of 11.8 m/s.
3 Determine the angle that the velocity vector makes with the ground, when the ball lands.	Let ψ be the angle with the ground when the ball lands. $\tan(\psi) = \dfrac{\dot{y}}{\dot{x}}$ $\psi = \tan^{-1}\left(\dfrac{8.96}{7.66}\right)$ $= 49.5°$
d 1 State the parametric equations and eliminate the parameter t.	**d** $x = 10t\cos(40°)$, $y = 2 + 10t\sin(40°) - 4.9t^2$ $t = \dfrac{x}{10\cos(40°)}$ $y = 2 + \dfrac{10x\sin(40°)}{10\cos(40°)} - 4.9\left(\dfrac{x}{10\cos(40°)}\right)^2$
2 The Cartesian equation is parabolic as it is of the form $y = ax^2 + bx + c$.	$y = 2 + x\tan(40°) - \dfrac{49x^2}{1000}\sec^2(40°)$

The equation of the path

It can be shown that the path of the projectile is a parabola. Transposing $x = Vt\cos(\alpha)$ for t gives $t = \dfrac{x}{V\cos(\alpha)}$. Substituting this into $y(t) = Vt\sin(\alpha) - \frac{1}{2}gt^2$ gives

$$y = V\sin(\alpha)\left(\frac{x}{V\cos(\alpha)}\right) - \frac{g}{2}\left(\frac{x}{V\cos(\alpha)}\right)^2.$$

Simplifying this gives

$$y = x\tan(\alpha) - \frac{gx^2}{2V^2\cos^2(\alpha)} \text{ or } y = x\tan(\alpha) - \frac{gx^2\sec^2(\alpha)}{2V^2}.$$

This is of the form $y = ax + bx^2$, so the path of the projectile is a parabola.

Finding the angle of projection

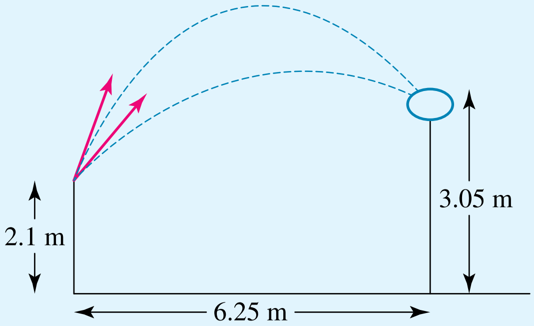
Because $\sec^2(\alpha) = 1 + \tan^2(\alpha)$, the Cartesian equation of a projectile can be written in the form $y = x\tan(\alpha) - \dfrac{gx^2}{2V^2}(1 + \tan^2\alpha)$. Alternatively, this can be rearranged into a quadratic in $\tan(\alpha)$ as $\dfrac{gx^2}{2V^2}\tan^2(\alpha) - x\tan(\alpha) + \left(y + \dfrac{gx^2}{2V^2}\right) = 0$.

This equation is useful for finding the angle of projection, α, if we are given the coordinates of a point (x, y) through which the particle passes and the initial speed of projection, V. Because this equation is a quadratic in $\tan(\alpha)$, it is probable that we can obtain two values for α.

WORKED EXAMPLE 15

A basketballer shoots for goal from the three-point line. She throws the ball with an initial speed of 15 m/s and the ball leaves her hands at a height of 2.1 metres above the ground. Find the possible angles of projection if she is to score a goal.

Data: Distance from goal to three-point line: 6.25 m
Height of ring: 3.05 m

THINK	WRITE/DRAW
1 State the value of the parameters. Drawing a diagram will help.	$V = 15$ but α is unknown and required to be found. 
2 State where the projectile will pass through.	The basketball must pass through the point where $x = 6.25$ and $y = 3.05 - 2.1 = 0.95$.

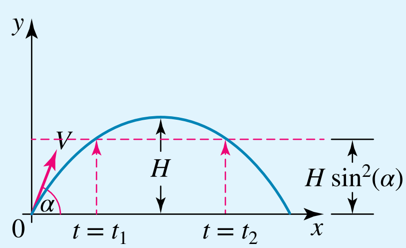
3 Substitute in the appropriate values to obtain a quadratic in tan(α).	Substitute $V = 15$, $x = 6.25$ and $y = 0.95$: $$\frac{gx^2}{2V^2}\tan(\alpha) - x\tan(\alpha) + \left(y + \frac{gx^2}{2V^2}\right) = 0$$ $$\frac{9.8 \times 6.25^2}{2 \times 15^2}\tan^2(\alpha) - 6.25\tan(\alpha) + \left(0.95 + \frac{9.8 \times 6.25^2}{2 \times 15^2}\right) = 0$$
4 Simplify the quadratic to be solved.	$$0.8507\tan^2(\alpha) - 6.25\tan(\alpha) + 1.8007 = 0$$
5 First find the discriminant of the quadratic equation.	$a = 0.8507$, $b = -6.25$, $c = 1.8007$ $$\Delta = b^2 - 4ac = (-6.25)^2 - 4 \times 0.8507 \times 1.8007 = 32.935$$ $$\sqrt{\Delta} = 5.739$$
6 Use the quadratic formula to solve for tan(α).	$$\tan(\alpha) = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{6.25 \pm 5.739}{2 \times 0.8507} = 7.0465,\ 0.3004$$
7 Find the values of α.	$\alpha = \tan^{-1}(7.0465), \tan^{-1}(0.3004)$
8 State the final result.	There are two possible angles of projection: $\alpha = 81.92°$ or $16.72°$.

Proofs involving projectile motion

Often projectile motion problems involve parameters rather than specific given values. In these types of problems, we are required to prove or show that a certain equation is valid. To do this, we can use the general equations and mathematically manipulate these to show the desired result. Note that we can only give results in terms of the given parameters.

WORKED EXAMPLE 16

A ball is projected at an angle α from a point on a horizontal plane. If the ball reaches a maximum height of H, show that the time interval between the instants when the ball is at heights of $H\sin^2(\alpha)$ is $2\cos(\alpha)\sqrt{\dfrac{2H}{g}}$.

THINK	WRITE/DRAW
1 Draw a diagram.	
2 Use the result for the maximum height.	$$H = \frac{V^2\sin^2(\alpha)}{2g}$$

3 Since the initial speed of projection is unknown, express V in terms of the given values.

$V^2 = \dfrac{2gH}{\sin^2(\alpha)}$

4 Use the given equation for y.

$y = Vt\sin(\alpha) - \frac{1}{2}gt^2$

5 We need to find the values of t at the given heights.

Let $y = H\sin^2(\alpha)$.

$$H\sin^2(\alpha) = Vt\sin(\alpha) - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 - Vt\sin(\alpha) + H\sin^2(\alpha) = 0$$

6 Find the discriminant of the quadratic.

$a = \frac{1}{2}g$, $b = -V\sin(\alpha)$, $c = H\sin^2(\alpha)$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= -V^2\sin^2(\alpha) - 4\times\frac{1}{2}g\times H\sin^2(\alpha)\\ &= -V^2\sin^2(\alpha) - 2gH\sin^2(\alpha)\end{aligned}$$

7 Use previous results to eliminate the unknown, V.

Substitute $-V^2\sin^2(\alpha) = 2gH$ into the discriminant:

$$\begin{aligned}\Delta &= 2gH - 2gH\sin^2(\alpha)\\ &= 2gH(1-\sin^2(\alpha))\\ &= 2gH\cos^2(\alpha)\end{aligned}$$

8 Use the quadratic formula to find the two times.

$t = \dfrac{V\sin(\alpha)\pm\sqrt{\Delta}}{g}$, so $t_1 = \dfrac{V\sin(\alpha)-\sqrt{\Delta}}{g}$ and $t_2 = \dfrac{V\sin(\alpha)+\sqrt{\Delta}}{g}$, where $t_2 > t_1$.

9 Find the time interval and simplify.

$$\begin{aligned}t_2 - t_1 &= \frac{2\sqrt{\Delta}}{g}\\ &= \frac{2\sqrt{2gH\cos^2(\alpha)}}{g}\\ &= \frac{2\cos(\alpha)}{g}\sqrt{2gH}\end{aligned}$$

10 State the required result.

$t_2 - t_1 = 2\cos(\alpha)\sqrt{\dfrac{2H}{g}}$

Incorporating air resistance and three-dimensional motion

In reality, projectiles may move in a three-dimensional framework rather than a two-dimensional plane. Also, with the effects of air resistance being included, the path is not necessarily a parabola.

WORKED EXAMPLE 17

A shot is thrown by a shot-put competitor on level ground. At a time t in seconds measured from the point of release of the shot, the position vector $\underset{\sim}{r}(t)$ of the shot is given by

$$\underset{\sim}{r}(t) = 7t\underset{\sim}{i} + \left(5t + 6\left(e^{-\frac{t}{2}} - 1\right)\right)\underset{\sim}{j} + (2 + 12t - 5t^2)\underset{\sim}{k}$$

where $\underset{\sim}{i}$ is a unit vector in the east direction, $\underset{\sim}{j}$ is a unit vector in the north direction and $\underset{\sim}{k}$ is a unit vector vertically up. The origin, O, of the coordinate system is at ground level and displacements are measured in metres. Let P be the point where the shot hits the ground.

a Find the time taken for the shot to hit the ground.

b How far from O does the shot hit the ground?

c Find the initial speed of projection.

d Find the speed and angle at which the shot hits the ground.

THINK	WRITE/DRAW
a 1 Determine when the shot hits the ground.	**a** The shot hits the ground when the $\underset{\sim}{k}$ component is zero, that is, when $2 + 12t - 5t^2 = 0$.
2 Solve for the values of t.	Solving using the quadratic formula gives $t = -0.1565$ or $t = 2.5565$. Since $t \geq 0$, the shot hits the ground at 2.56 seconds.
b 1 Find the position vector where the shot hits the ground.	**b** Substitute $t = 2.56$ into the position vector. $\underset{\sim}{r}(2.56) = 7 \times 2.56\underset{\sim}{i} + \left(5 \times 2.56 + 6\left(e^{-\frac{2.56}{2}} - 1\right)\right)\underset{\sim}{j} + (2 + 12 \times 2.56 - 5 \times 2.56^2)\underset{\sim}{k}$ $\underset{\sim}{r}(2.56) = 17.895\underset{\sim}{i} + 8.453\underset{\sim}{j}$
2 Determine the distance where the shot hits the ground.	$\lvert\underset{\sim}{r}(2.56)\rvert = \sqrt{17.895^2 + 8.453^2}$ $= 19.79$ metres
c 1 Determine the velocity vector.	**c** $\dot{\underset{\sim}{r}}(t) = 7\underset{\sim}{i} + \left(5 - 3e^{-\frac{t}{2}}\right)\underset{\sim}{j} + (12 - 10t)\underset{\sim}{k}$
2 Determine the initial velocity vector.	Substitute $t = 0$ into the velocity vector. $\dot{r}(0) = 7\underset{\sim}{i} + (5 - 3e^0)\underset{\sim}{j} + 12\underset{\sim}{k}$ $= 7\underset{\sim}{i} + 2\underset{\sim}{j} + 12\underset{\sim}{k}$
3 Find the initial speed of projection.	$\lvert\dot{\underset{\sim}{r}}(0)\rvert = \sqrt{7^2 + 2^2 + 12^2}$ $= \sqrt{197}$ The initial speed of projection is 14.04 m/s.

d 1 Determine the velocity vector when the shot hits the ground.

d Substitute $t = 2.56$ into the velocity vector.

$$\dot{\underset{\sim}{r}}(2.56) = 7\underset{\sim}{i} + \left(5 - 3e^{-\frac{2.56}{2}}\right)\underset{\sim}{j} + (12 - 10 \times 2.56)\underset{\sim}{k}$$

$$= 7\underset{\sim}{i} + 4.16\underset{\sim}{j} - 13.56\underset{\sim}{k}$$

2 Find the magnitude of the velocity vector.

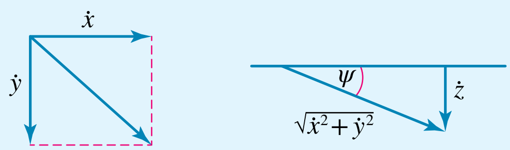
$$|\dot{\underset{\sim}{r}}(2.56)| = \sqrt{7^2 + 4.16^2 + (-13.56)^2}$$

$$= \sqrt{250.34}$$

3 Determine the angle at which the shot hits the ground.

The required angle, ψ, is the angle between the downwards component and the combined east and north components.

$$\tan(\psi) = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$



4 Find the angle.

$$\tan(\psi) = \frac{13.56}{\sqrt{7^2 + 4.16^2}}$$

$$\psi = \tan^{-1}(1.665)$$

5 State the final result.

The shot hits the ground with a speed of 15.82 m/s at an angle of 59°1′.

EXERCISE 13.6 Projectile motion

PRACTISE

1 WE13 A soccer ball is kicked off the ground at an angle of 67°23′ with an initial speed of 13 m/s. Find:

a the time of flight
b the range
c the maximum height reached.

2 A ball is thrown so that its time of flight is $\frac{10\sqrt{3}}{g}$ and the ratio of the range to the maximum height reached is $\frac{4\sqrt{3}}{3}$. Find the initial speed and angle of projection.

3 WE14 A cricketer smashes a cricket ball at an angle of 35° to the horizontal from a point 0.5 m above the ground with an initial speed of 30 m/s.

a Find the time of flight and the horizontal distance travelled.
b Find the maximum height reached.
c Find the speed and angle at which the ball lands.
d Show that the ball travels in a parabolic path.

4 A cricket outfielder attempts to throw the cricket ball back towards the stumps. He throws the ball from a height of 1.8 m above the ground with an initial speed of 20 m/s at an angle of 20° to the horizontal.

a Find the time of flight and the horizontal distance travelled.
b Find the maximum height reached.
c Find the speed and angle at which the ball lands.
d Show that the ball travels in a parabolic path.

5 WE15 A tennis player hits the ball 4 feet above the baseline of a tennis court with a speed of 100 feet per second. If the ball travels in a vertical plane towards the centre of the net and just grazes the top of the net, find the angles at which it could have been hit.

Data: Tennis court dimensions 78 × 27 feet
Net height at the centre 3 feet
Use $g = 32$ ft/s^2.

6 A gun is mounted on top of a cliff face 122.5 metres high and fires a shell at a speed of 49 m/s in order to hit a target which is located at a horizontal distance of 346.482 metres from the base of the cliff. Find:

a the angle of projection
b the time of flight
c the speed and angle at which the shell hits the target.

7 WE16 A particle is projected at a given angle α from a point on a horizontal plane. If the particle reaches a maximum height of H, show that the time interval between the instants when the particle is at heights of $\frac{H}{2}$ is $2\sqrt{\frac{H}{g}}$.

8 A particle is projected with an initial speed of $3g$ m/s. The particle reaches heights of g metres at times t_1 and t_2. Given that $t_2 - t_1 = 1$, show that the range of the particle is given by $\frac{9g\sqrt{3}}{2}$.

9 WE17 A football is kicked by a footballer. At a time t in seconds measured from the point of impact, the position vector $\underset{\sim}{r}(t)$ of the tip of the football is given by

$$\underset{\sim}{r}(t) = 60\left(1 - e^{-\frac{t}{2}}\right)\underset{\sim}{i} + 2t\underset{\sim}{j} + (1 + 12t - 4.9t^2)\underset{\sim}{k}$$

where $\underset{\sim}{i}$ is a unit vector horizontally forward, $\underset{\sim}{j}$ is a unit vector to the right and $\underset{\sim}{k}$ is a unit vector vertically up.

The origin, O, of the coordinate system is at ground level and all displacements are measured in metres.

a How far from the origin does the football hit the ground?
b Find the speed and angle at which the football hits the ground.

10 A javelin is thrown by an athlete on level ground. The time t is in seconds, measured from the release of the javelin. The position vector $\underset{\sim}{r}(t)$ of the tip of the javelin is given by $\underset{\sim}{r}(t) = 20t\underset{\sim}{i} + \left(2\pi t - 3\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j} + (1.8 + 14.4t - 5t^2)\underset{\sim}{k}$

where $\underset{\sim}{i}$ is a unit vector in the east direction, $\underset{\sim}{j}$ is a unit vector in the north direction and $\underset{\sim}{k}$ is a unit vector vertically up. The origin, O, of the coordinate system is at ground level, and all displacements are measured in metres.

a How long will it take for the javelin to strike the ground?
b Find how far from O the tip of the javelin hits the ground.
c Find the speed and angle at which the javelin's tip strikes the ground.

CONSOLIDATE

11 a A cricket ball is hit by a batsman off his toes with an initial speed of 20 m/s at an angle of 30° with the horizontal. Find:
 i the time of flight
 ii the range on the horizontal plane.
 iii the greatest height reached.
b A golf ball is hit at an angle of 20° and its range is 150 metres. Find:
 i the initial speed of projection
 ii the greatest height reached
 iii the time of flight.

12 a A rock is thrown horizontally from the top of a cliff face and strikes the ground 30 metres from the base of the cliff after 1 second.
 i Find the height of the cliff.
 ii Find the initial speed of projection.
 iii Find the speed and angle at which the rock strikes the ground.
 iv Show that the rock travels in a parabolic path.
b A motorcycle is driven at 150 km/h horizontally off the top of a cliff face and strikes the ground after 2 seconds. Find:
 i the height of the cliff
 ii how far from the base of the cliff the motorcycle strikes the ground
 iii the speed and angle at which it strikes the ground.

13 a An object is projected from the top of a building 100 metres high at an angle of 45° with a speed of 10 m/s. Find:
 i the time of flight to reach ground level
 ii how far from the edge of the building the object strikes the ground
 iii the greatest height the object reaches above the building
 iv the speed and angle at which the object strikes the ground.
b A stone is thrown with a speed of 15 m/s at an angle of 20° from a cliff face and strikes the ground after 3.053 seconds. Find:
 i the height of the cliff
 ii how far from the edge of the cliff the stone strikes the ground
 iii the greatest height the stone reaches above the ground
 iv the speed and angle at which the stone strikes the ground.

14 a A rugby player uses a place kick to kick the ball off the ground. To score a goal, the ball must pass through a point 40 metres out and 8 metres vertically above the point of release. If he kicks the ball at a speed of 30 m/s, find the possible angles of projection.

b A catapult throws a stone with a speed of 7 m/s from a point at the top of a cliff face 40 metres high to hit a ship which is at a horizontal distance of 20 metres from the base of the cliff. Find the possible angles of projection.

15 a A shell is fired with an initial speed of 147 m/s for maximum range. There is a target 2 km away on the same horizontal level.

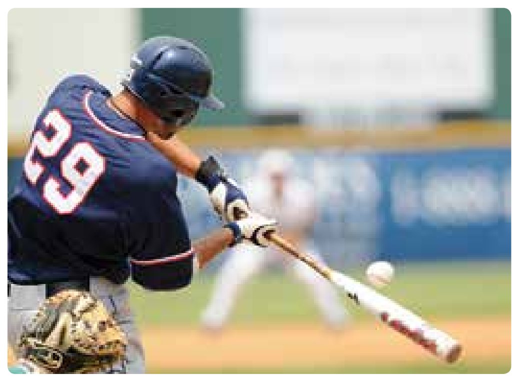
i How far above the target does the shell pass?
ii How far beyond the target does the shell strike the ground?
iii What is the minimum speed of projection to just reach the target?

b A projectile falls a metres short of its target when fired at an angle of α, and a metres beyond the target when fired at the same muzzle velocity but at an angle of β. If θ is the angle required for a direct hit on the target, show that $\sin(2\theta) = \frac{1}{2}(\sin(2\alpha) + \sin(2\beta))$.

16 a A gun is mounted on top of a cliff face 122.5 metres high and fires a shell at a speed of 49 m/s in order to hit a target that is located at a horizontal distance of 346.482 metres from the base of the cliff. Find:

i the angle of projection
ii the time of flight
iii the speed and angle at which the shell hits the target.

b A baseball is initially hit from a distance of 1.5 metres above the ground at an angle of 35°. The ball reaches a maximum height of 8 metres above the ground. Find:



i the initial speed of projection
ii the distance of the outfielder from the hitter if the outfielder just catches the ball at ground level
iii the speed and angle at which the baseball strikes the fielder's hands.

17 a A projectile is fired with an initial speed of $\sqrt{2ga}$ to hit a target at a horizontal distance of a from the point of projection and a vertical distance of $\frac{a}{2}$ above it. Show that there are two possible angles of projection, that the ratio of the two times to hit the target is $\sqrt{5}$, and that the ratio of the maximum heights reached in each case is $\frac{9}{5}$.

b A projectile is fired with an initial speed of $\sqrt{2ga}$ at an angle of α to hit a target at a horizontal distance of a from the point of projection and a vertical distance of b.

i Show that $a\tan^2(\alpha) - 4a\tan(\alpha) + (4b + a) = 0$.

ii Show that it is impossible to hit the target if $4b > 3a$.

iii Show that if $4b = 3a$, then $\alpha = \tan^{-1}(2)$.

18 **a** A particle is projected to just clear two walls. The walls are both of height 6 metres and are at distances of 5 metres and 10 metres from the point of projection. Show that if α is the angle of projection, then $\tan(\alpha) = \frac{9}{5}$.

b A particle is projected to just clear two walls. The walls are both h metres high and are at distances of a metres and b metres from the point of projection. Show that the angle of projection is $\tan^{-1}\left(\frac{h(a+b)}{ab}\right)$.

c A particle is projected to just clear two walls. The first wall is h_1 metres high and at a distance of a metres from the point of projection; the second wall is h_2 metres high and b metres from the point of projection, where $b > a > 0$. Show that the angle of projection is $\tan^{-1}\left(\frac{b^2h_1 - a^2h_2}{ab(b-a)}\right)$.

19 A soccer ball is kicked off the ground. Its position vector is given by $\underset{\sim}{r}(t) = 6t\underset{\sim}{i} + 28t\underset{\sim}{j} + \frac{12\sqrt{2}}{5}\sin\left(\frac{\pi t}{2}\right)\underset{\sim}{k}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the horizontal plane at right angles to each other, and $\underset{\sim}{k}$ is a unit vector in the vertical direction. Displacements are measured in metres, and t is the time in seconds after the ball is kicked.

a Find the initial speed at which the soccer ball is kicked.

b At what angle from the ground was the soccer ball kicked?

c Determine the maximum height reached by soccer ball.

d After the soccer ball has reached its maximum height and is on its downwards trajectory, a player jumps and heads the soccer ball when it is 2.4 metres above the ground.

i What time has elapsed from the instant that the soccer ball is kicked until the player heads the ball?

ii How far, measured along the ground, was the player from where the ball was kicked?

iii If the soccer ball has a mass of 430 grams, find the magnitude of the momentum in kg m/s of the ball at the instant when it strikes the player's head. Give your answer correct to 2 decimal places.

20 **a** An object is thrown horizontally with a speed of V m/s from the top of a cliff face h metres high and strikes the ground at a distance of R metres from the base of the cliff after a time of T seconds. Show that:

i $T = \sqrt{\frac{2h}{g}}$

ii $R = VT$

iii the speed at which the object hits the ground is $\sqrt{V^2 + 2gh}$

iv the angle at which the object strikes the ground is $\tan^{-1}\left(\frac{2h}{R}\right)$

v the object travels in a parabolic path, $y = h\left(1 - \frac{x^2}{R^2}\right)$.

b A stone is projected horizontally from the top of a cliff of height H metres with a speed of U m/s. At the same instant another object is fired from the base of the cliff with a speed of V m/s at an angle of α. Given that the two objects collide after a time of T seconds, show that

$$\alpha = \cos^{-1}\left(\frac{U}{V}\right) \text{ and } V^2 = U^2 + \frac{H^2}{T^2}.$$

MASTER

21 A projectile is fired at an angle of α with an initial speed of V. It reaches a maximum height of H and has a horizontal range of R. Show that:

a $R\tan(\alpha) = \frac{1}{2}gT^2$

b $\tan(\alpha) = \frac{4H}{R}$

c $T^2 = \frac{8H}{g}$

d $V = \sqrt{2g\left(H + \frac{R^2}{16H}\right)}$

e the equation of the path is $y = \frac{4Hx}{R^2}(R - x)$ for $0 \le x \le R$.

22 a When a projectile is fired at an angle of α with an initial speed of V from a cliff of height h for maximum range R, provided that $V < \sqrt{gR}$, show that:

i $\tan(\alpha) = \frac{V^2}{gR}$

ii $h = \frac{gR^2 - V^4}{2V^2g}$

iii $R = \frac{V}{g}\sqrt{V^2 + 2gH}$

iv $T = \frac{\sqrt{2(V^2 + gh)}}{g}$.

b A shot-putter can throw the shot with a release speed of 20 m/s. The shot leaves his hand 2 metres above the ground. Find the maximum range, the angle of projection and the time of flight.

eBook*plus* ONLINE ONLY

13.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

13 Answers

EXERCISE 13.2

1 a $\sqrt{205}$

b $\sqrt{10t^2 - 10t + 5}$

c $\frac{\sqrt{10}}{4}$

2 a $5\sqrt{2}$ **b** 9

3 a $t = 5$, (16, 40)

b (36, 180)

c $5\sqrt{5}$

4 a $t = 4$, (10, 7)

b (–2, 13)

c $\sqrt{41}$

5 a i $5\sqrt{2}$

ii $\sqrt{5t^2 - 10t + 10}$

iii $\sqrt{5}$

b i $5\sqrt{5}$

ii $5\sqrt{t^2 + 1}$

iii 5

6 a i $\sqrt{41}$

ii $\sqrt{5}$

iii 0, 2

b i $\frac{9\sqrt{2}}{2}$

ii 8

7 a $a = 4, b = -3, c = 1, d = 3$

b $a = 2, b = 5, c = 1, d = -3$

8 a $t = 15$, (2, 5)

b (8, –7)

c 15

9 a $t = 2$, (–2, 6); $t = 5$, (19, 12)

b $8\sqrt{26}$

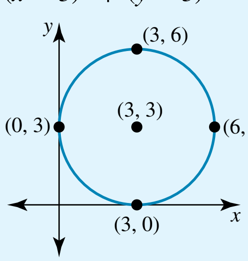
10 a $t = 12$, (53, 74)

b (89, 56)

c $80\sqrt{5}$

11 a $3\underset{\sim}{j}$, $6\underset{\sim}{i} + 3\underset{\sim}{j}$, $3\underset{\sim}{j}$

b $(x - 3)^2 + (y - 3)^2 = 9$; domain [0, 6], range [0, 6]

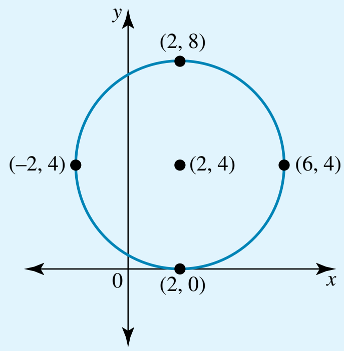


c $3\sqrt{3 + 2\sin(t) - 2\cos(t)}$

d $3\sqrt{2} - 3$

12 a $6\underset{\sim}{i} + 4\underset{\sim}{j}$, $-2\underset{\sim}{i} + 4\underset{\sim}{j}$, $6\underset{\sim}{i} + 4\underset{\sim}{j}$

b $(x - 2)^2 + (y - 4)^2 = 16$; domain [–2, 6], range [0, 8]



c $2\sqrt{9 + 4\cos(t) + 8\sin(t)}$

d $2\sqrt{5} - 4$

13 a $\sqrt{2}$

b $\sqrt{2}$

14 a The particles collide at $t = 0, 2\pi, (a, 0)$ and $t = \frac{\pi}{2}, (0, a)$

b The paths intersect at $(a, 0)$ and $(0, a)$

15 a $\frac{14}{13}$ **b** $\frac{ad - bc}{a^2 + c^2}$

16 a 3.397 **b** 0.844

EXERCISE 13.3

1 $\underset{\sim}{j}$

2 $\frac{1}{2}(-\sqrt{3}\,\underset{\sim}{i} + \underset{\sim}{j})$

3 a $e^{-2t}(1 - 2t)\underset{\sim}{i} + e^{2t}(1 + 2t)\underset{\sim}{j}$

b $2e$

c $4(t - 1)e^{-2t}\underset{\sim}{i} + 4(t + 1)e^{2t}\underset{\sim}{j}$

4 4 kg m/s

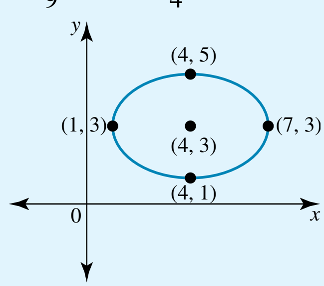
5 a $8t^3\underset{\sim}{i} - 8\sin(2t)\underset{\sim}{j} - 12e^{-2t}\underset{\sim}{k}$

b $24t^2\underset{\sim}{i} - 16\cos(2t)\underset{\sim}{j} + 24e^{-2t}\underset{\sim}{k}$

6 $\frac{1}{2}\sqrt{\pi^4 + 1024e^{-4}}$

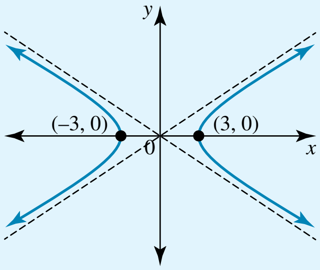
7 a (4, 1), (4, 5)

b $\frac{(x - 4)^2}{9} + \frac{(y - 3)^2}{4} = 1$; domain [1, 7], range [1, 5]



c $|\dot{\underset{\sim}{r}}(t)|_{max} = 4$, $|\dot{\underset{\sim}{r}}(t)|_{min} = 3$

8 a $\frac{x^2}{9} - \frac{y^2}{4} = 1$; domain $|x| \geq 3$, range R, asymptotes $y = \pm\frac{2x}{3}$



b $\left(2\sqrt{3}, \frac{2\sqrt{3}}{3}\right), \left(-2\sqrt{3}, -\frac{2\sqrt{3}}{3}\right)$

9 a 2.45 s **b** 12.25 m

c 13 m/s, 67.38° **d** 7.35 m

e $y = -\frac{x}{250}(49x - 600)$

10 a 1.67 s

b 40.14 m

c 25 m/s, 16.26°

d 4.5 m

e $y = 2 + \frac{7x}{24} - \frac{49x^2}{5760}$

11 a $\frac{1}{\sqrt{17}}(\underset{\sim}{i} + 4\underset{\sim}{j})$ **b** $\frac{1}{\sqrt{145}}(\underset{\sim}{i} + 12\underset{\sim}{j})$

c $\frac{1}{\sqrt{82}}(9\underset{\sim}{i} + \underset{\sim}{j})$ **d** $\frac{1}{\sqrt{34}}(3\underset{\sim}{i} + 5\underset{\sim}{j})$

12 a $\frac{1}{\sqrt{2}}(-\underset{\sim}{i} + \underset{\sim}{j})$ **b** $\frac{1}{\sqrt{10}}(3\underset{\sim}{i} + \underset{\sim}{j})$

c $-\frac{1}{\sqrt{5}}(\underset{\sim}{i} + 2\underset{\sim}{j})$ **d** $\frac{1}{\sqrt{5}}(\underset{\sim}{i} - 2\underset{\sim}{j})$

13 a $\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} - \frac{1}{(1+t)^2}\underset{\sim}{j}$, $\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} + \frac{2}{(1+t)^3}\underset{\sim}{j}$

b $\dot{\underset{\sim}{r}}(t) = \frac{1}{t}\underset{\sim}{i} + (10t + 4)\underset{\sim}{j}$, $\ddot{\underset{\sim}{r}}(t) = -\frac{1}{t^2}\underset{\sim}{i} + 10\underset{\sim}{j}$

c $\dot{\underset{\sim}{r}}(t) = -4e^{-\frac{t}{2}}\underset{\sim}{i} + 8e^{2t}\underset{\sim}{j}$, $\ddot{\underset{\sim}{r}}(t) = 2e^{-\frac{t}{2}}\underset{\sim}{i} + 16e^{2t}\underset{\sim}{j}$

d $\dot{\underset{\sim}{r}}(t) = \frac{3}{3t+4}\underset{\sim}{i} - 12\sin(3t)\underset{\sim}{j}$,

$\ddot{\underset{\sim}{r}}(t) = \frac{-9}{(3t+4)^2}\underset{\sim}{i} - 36\cos(3t)\underset{\sim}{j}$

14 a $\dot{\underset{\sim}{r}}(t) = -2e^{-2t}\underset{\sim}{i} + (4t^3 - 4t)\underset{\sim}{j} + 10t\underset{\sim}{k}$,

$\ddot{\underset{\sim}{r}}(t) = 4e^{-2t}\underset{\sim}{i} + (12t^2 - 4)\underset{\sim}{j} + 10t\underset{\sim}{k}$

b $\dot{\underset{\sim}{r}}(t) = -6\sin(2t)\underset{\sim}{i} - 8\cos(2t)\underset{\sim}{j} + (12 - 10t)\underset{\sim}{k}$,

$\ddot{\underset{\sim}{r}}(t) = -12\cos(2t)\underset{\sim}{i} + 16\sin(2t)\underset{\sim}{j} - 10\underset{\sim}{k}$

c $\dot{\underset{\sim}{r}}(t) = (2t\sin(2t) + 2t^2\cos(2t))\underset{\sim}{i}$
$+ e^{-2t}(1 - 2t)\underset{\sim}{j} + 10\underset{\sim}{k}$,

$\ddot{\underset{\sim}{r}}(t) = (8t\cos(2t) + (2 - 4t^2)\sin(2t))\underset{\sim}{i}$
$+ 4e^{-2t}(t - 1)\underset{\sim}{j}$

d $\dot{\underset{\sim}{r}}(t) = -3\sin(6t)\underset{\sim}{i} + 3t^2(\cos(3t)$
$- t\sin(3t))\underset{\sim}{j} + 2t\underset{\sim}{k}$,

$\ddot{\underset{\sim}{r}}(t) = -18\cos(6t)\underset{\sim}{i} + ((6t - 9t^3)\cos(3t)$
$- 18t^2\sin(3t))\underset{\sim}{j} + 2\underset{\sim}{k}$

15 a i Circle, $x^2 + y^2 = 9$

ii The speed is 6.

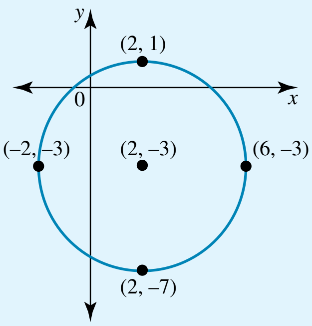
iii $\ddot{\underset{\sim}{r}}(t) = -4\underset{\sim}{r}(t)$

iv Check with your teacher.

b i The speed is 12.

ii Check with your teacher.

16 a i Circle, $(x - 2)^2 + (y + 3)^2 = 16$



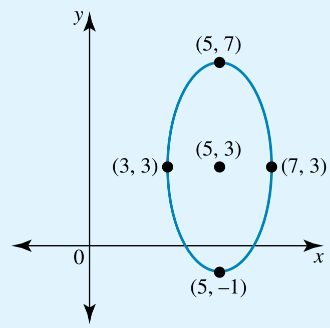
ii Check with your teacher.

iii 24 kg m/s

iv 48 N

v –1

b i Ellipse, $\frac{(x-5)^2}{4} + \frac{(y-3)^2}{16} = 1$



ii $4\sqrt{13}$ kg m/s

iii $8\sqrt{7}$ N

iv $|\dot{\underset{\sim}{r}}(t)|_{max} = 8$, $|\dot{\underset{\sim}{r}}(t)|_{min} = 4$

v $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

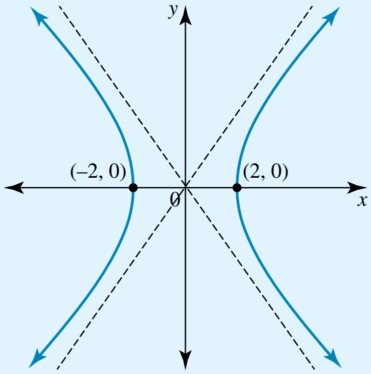
17 a Circle, $x^2 + y^2 = a^2$

b–d Check with your teacher.

e The motion is circular.

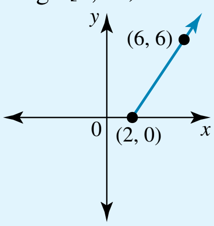
18 a i Hyperbola, $\frac{x^2}{4} - \frac{y^2}{9} = 1$; domain $|x| \geq 2$, range R,

asymptotes $y = \pm\frac{3x}{2}$



ii $\frac{3\sqrt{2}}{2}$

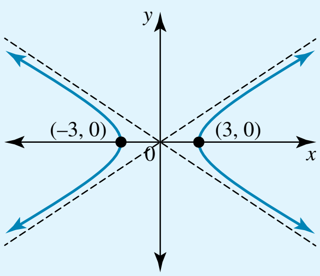
b i Straight line, $\frac{x}{2} - \frac{y}{3} = 1$; domain $[2, \infty)$, range $[0, \infty)$



ii $\frac{3}{2}$

iii 3

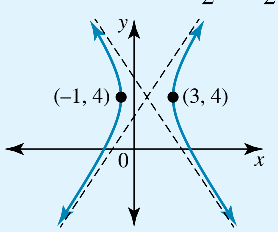
19 a i Hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$; domain $|x| \geq 3$, range R; asymptotes $y = \pm\frac{4x}{3}$



ii $\frac{8}{3}$

iii 4

b i Hyperbola, $\frac{(x-1)^2}{4} - \frac{(y-4)^2}{9} = 1$; domain $(-\infty, -1] \cup [3, \infty)$, range R, asymptotes $y = -\frac{3x}{2} + \frac{11}{2}$, $y = \frac{3x}{2} + \frac{5}{2}$



ii $\frac{2\pi}{3}, \frac{4\pi}{3}$

20 a 2.04 s

b 20.4 m

c 14.14 m/s, 45°

d 5.1 m

e $y = -\frac{x}{1000}(49x - 1000)$

21 a 1.8 m

b 2.019 s

c 70.65 m

d 36.14 m/s, 14.42°

e 5.93 m

f $y = 1.8 + \frac{9x}{35} - \frac{x^2}{250}$

22 a 2 s

b 47.1 m

c 25.16 m/s

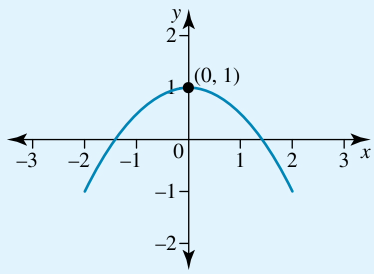
d 5.66 m

EXERCISE 13.4

1 a $-\frac{\sin(2t)}{\cos(t)}$

b $t = 0, \pi, 2\pi$; maximum at $(0, 1)$

c



d $\sqrt{6}, 2$

e Check with your teacher.

f 5.9158

g, h Check with your teacher.

2 a $-\frac{bm\sin(mt)}{an\cos(nt)}$

b $\sqrt{a^2n^2\cos^2(nt) + b^2m^2\sin^2(mt)}$

c–e Check with your teacher.

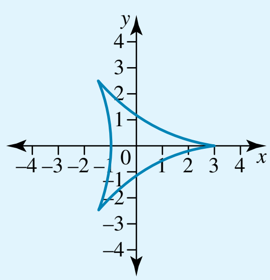
3 a $\frac{\cos(2t) - \cos(t)}{\sin(2t) - \sin(t)}$

b $4\cos\left(\frac{3t}{2}\right)$

c $-6\sin(3t)$

d $|\dot{\underset{\sim}{r}}(t)|_{max} = 6$, $|\dot{\underset{\sim}{r}}(t)|_{min} = 2$,

e

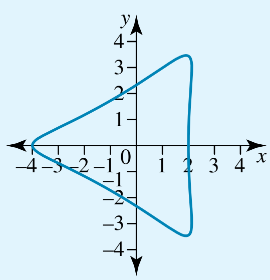


4 a $9 - 6\sqrt{2}$

b $|\dot{\underset{\sim}{r}}(t)|_{max} = 5$, $|\dot{\underset{\sim}{r}}(t)|_{min} = 1$

c $|\ddot{\underset{\sim}{r}}(t)|_{max} = 7$, $|\ddot{\underset{\sim}{r}}(t)|_{min} = 1$

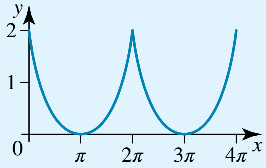
d



5 a Check with your teacher.

b $(\pi, 0),\ (3\pi, 0)$

c



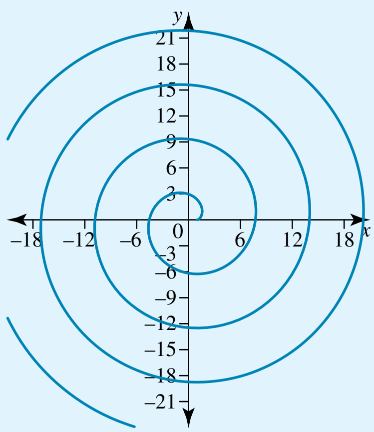
d Check with your teacher.

e Check with your teacher.

f Check with your teacher.

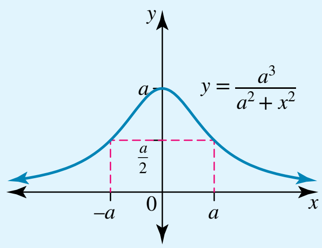
6 a–d Check with your teacher.

e



7 a, b, d, e Check with your teacher.

c

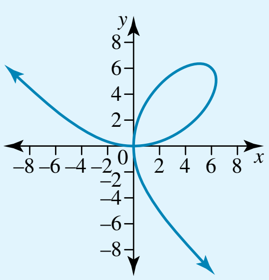


8 a Check with your teacher.

b $\dfrac{t(2 - t^3)}{2t^3 - 1}$

c $\dfrac{ay - x^2}{y^2 - ax}$

d i



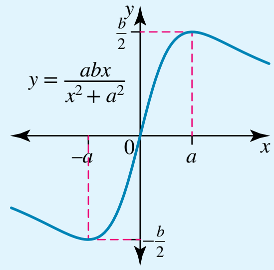
ii $2\underset{\sim}{i} + 4\underset{\sim}{j}$

iii $\dfrac{\sqrt{41}}{3}$

iv $5y - 4x - 12 = 0$

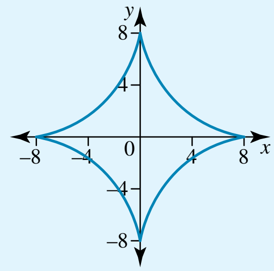
9 a, b, c, e Check with your teacher.

d



10 a, b, c, e, f, g, i Check with your teacher.

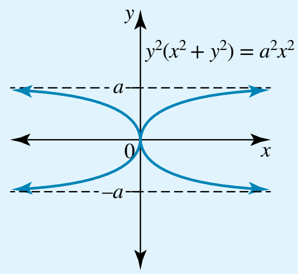
d



h $y = a^3\sin^3(t) - \tan(t)(x - a\cos^3(t))$

11 a, b, c, e Check with your teacher.

d

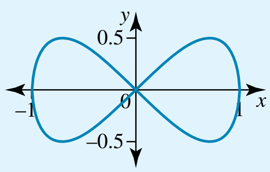


12 a (–1, 0), (0, 0), (1, 0), domain [–1, 1]

b, c, e, f, h, i Check with your teacher.

d The gradient approaches infinity, and there is a vertical tangent at $x = \pm 1$.

g



j 6.0972

13 a $a = 0, b = 4$

b Reflection in the x-axis

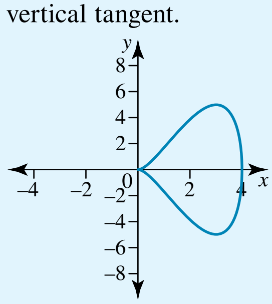
c i $\dfrac{2\sqrt{x}(3 - x)}{\sqrt{4 - x}}$

ii, iv Check with your teacher.

iii $(3, 3\sqrt{3})$

v The gradient becomes infinite, and there is a vertical tangent.

vi



d i Check with your teacher.

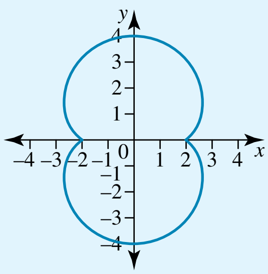
ii $\dfrac{2\pi}{3}$

iii $\sqrt{3}$

14 a Check with your teacher.

b $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$, (0, 4) and (0, –4)

c



d–f Check with your teacher.

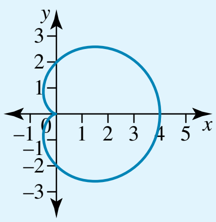
15 a $4\cos^2\left(\dfrac{t}{2}\right)$

b $4\cos\left(\dfrac{t}{2}\right)$

c $0, \pi, 2\pi$

d $|\dot{\underset{\sim}{r}}(t)|_{max} = 6, |\dot{\underset{\sim}{r}}(t)|_{min} = 2$

e



f–h Check with your teacher.

16 Check with your teacher.

EXERCISE 13.5

1 $(2t^2 - 4t + 5)\underset{\sim}{i} + (4 - 3t)\underset{\sim}{j}$

2 $(5 - 2\cos(3t))\underset{\sim}{i} + (3 - 2e^{-2t})\underset{\sim}{j}$

3 $(5 - 2t)\underset{\sim}{i} + (16t - t^4 - 10)\underset{\sim}{j}$

4 $(3t^2 - 5)\underset{\sim}{i} + (t^2 + 8t - 2)\underset{\sim}{j}$

5 $(x + 2)^2 + (y - 4)^2 = 25$; circle with centre (–2, 4), radius 5

6 $\dfrac{(x-5)^2}{9} + \dfrac{(y+2)^2}{25} = 1$; ellipse with centre (5, –2), semi-major and minor axes 3, 5

7 a $3\left(1 - e^{-\frac{t}{3}}\right)\underset{\sim}{i} + t^4\underset{\sim}{j}$

b $t^2\underset{\sim}{i} + 3(1 - \cos(2t))\underset{\sim}{j}$

c $\left(3 + \sin^{-1}\left(\dfrac{t}{4}\right)\right)\underset{\sim}{i} + \left(5 - \sqrt{t^2 + 9}\right)\underset{\sim}{j}$

d $(5 + \log_e(2t + 1))\underset{\sim}{i} + \left(4 - \dfrac{12}{(3t+2)^2}\right)\underset{\sim}{j}$

8 a $(3t^2 + 2)\underset{\sim}{i} + (4t - 3)\underset{\sim}{j}$

b $(2t + 4)\underset{\sim}{i} + (2t^2 + 1)\underset{\sim}{j}$

c $(4t^2 + 3)\underset{\sim}{i} + (3t^2 - 2)\underset{\sim}{j}$

d $(2t^2 + 3)\underset{\sim}{i} + (t^2 + 8t + 4)\underset{\sim}{j}$

9 a $(t^3 - 4t^2 + 5)\underset{\sim}{i} + (4t - 3)\underset{\sim}{j}$

b $(8t + 7)\underset{\sim}{i} + (2t^3 - 3t^2 - 4)\underset{\sim}{j}$

c $(t^2 + 4t + 4)\underset{\sim}{i} + (t^2 - 4t - 12)\underset{\sim}{j}$

d $(5 + 4t - t^2)\underset{\sim}{i} + (3t^2 - 16t + 5)\underset{\sim}{j}$

10 a $\left(t + \dfrac{1}{t}\right)\underset{\sim}{i} + 2\left(t - \dfrac{1}{t}\right)\underset{\sim}{j}$

b $4\sqrt{t}\underset{\sim}{i} + (t^2 - 10t + 25)\underset{\sim}{j}$

c $(\log_e(3t + 1) + 4)\underset{\sim}{i} + \left(\dfrac{4}{2t+1} - 1\right)\underset{\sim}{j}$

d $(\log_e(3t + 1) + 3t + 2)\underset{\sim}{i} + \left(4t - \dfrac{1}{(2t+1)^2}\right)\underset{\sim}{j}$

11 a $y = -\dfrac{x^2}{45} + \dfrac{4x}{3} + 2, x \geq 0$

b $y = -\dfrac{49x^2}{250} + 2x + 1, x \geq 0$

12 a $y = \dfrac{1}{x-4} + 3$

b $y = \dfrac{32}{2-x} - 5$

13 a $(x - 3)^2 + (y - 5)^2 = 4$

b $(x + 7)^2 + (y - 5)^2 = 81$

14 a $(x - 4)^2 + \dfrac{(y-5)^2}{4} = 1$

b $\dfrac{(x+7)^2}{144} + \dfrac{(y-3)^2}{16} = 1$

15 a $y = x^2 - 4$, [–2, 2], [–4, 0]

b $y = -3x(x - 2)$, [–1, 3], [–9, 3]

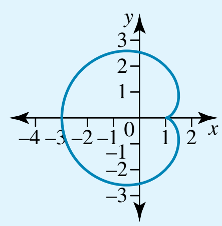
16 a $t = 3$, (4, 16)

b $t = 5$, (–7, 10)

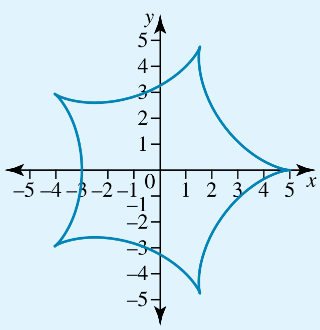
17 a $(x - a)^2 + (y - b)^2 = r^2$

b $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

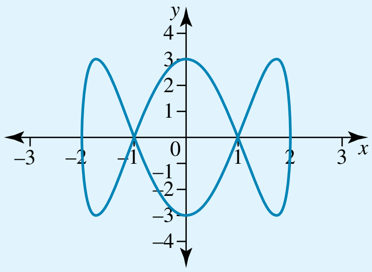
18 a $\underset{\sim}{r}(t) = (2\cos(t) - \cos(2t))\underset{\sim}{i} + (2\sin(t) - \sin(2t))\underset{\sim}{j}$



b $\underset{\sim}{r}(t) = (4\cos(t) + \cos(4t))\underset{\sim}{i} + (4\sin(t) - \sin(4t))\underset{\sim}{j}$



c $\underset{\sim}{r}(t) = 2\cos(2t)\underset{\sim}{i} + 3\sin(6t)\underset{\sim}{j}$



EXERCISE 13.6

1 a 2.45 s
b 12.25 m
c 7.35 m

2 10 m/s, 60°

3 a 3.54 s, 87 metres
b 15.61 metres
c 30.16 m/s, 35.43°
d Check with your teacher.

4 a 1.92 s, 34.7 metres
b 5.44 metres
c 20.87 m/s, 29.76°
d Check with your teacher.

5 86.4°, 2.1°

6 a 35.26°
b 8.662 seconds
c 69.31 m/s, 54.75°

7 Check with your teacher.

8 Check with your teacher.

9 a 2.53 s, 43.36 metres
b 15.47 m/s, 55.75°

10 a 3 s
b 62.06 metres
c 26.13 m/s, 36.65°

11 a i 2.04 s
ii 35.35 m
iii 5.10 m
b i 47.82 m/s
ii 13.65 m
iii 3.34 s

12 a i 4.9 m
ii 30 m/s
iii 31.56 m/s, 18.1°
iv Check with your teacher.
b i 19.6 m
ii 83.333 m
iii 46.05 m/s, 25.2°

13 a i 5.296 s
ii 37.45 m
iii 2.5 m
iv 45.39 m/s, 81°
b i 30 m
ii 43.03 m
iii 1.34 m
iv 28.51 m/s, 60.4°

14 a 76.4°, 25°
b 0°, 26.6°

15 a i 185.8 m
ii 205 m
iii 140 m/s
b Check with your teacher.

16 a i 35.3°
ii 8.662 s
iii 69.31 m/s, 54.8°
b i 19.68 m/s
ii 39.165 m
iii 20.41 m/s, 37.8°

17 Check with your teacher.

18 Check with your teacher.

19 a 29.13 m/s
b 10.5°
c 3.4 metres
d i 1.5 seconds
ii 42.95 metres
iii 12.42 kg m/s

20 Check with your teacher.

21 Check with your teacher.

22 a Check with your teacher.
b 42.77 m, 43.7°, 2.96 s

14

Statistical inference

14.1 Kick off with CAS

Mean and spread of sample distributions

If samples are taken from a large population, X, with a mean of μ and variance σ^2, then the sample mean, $\overline{X}$, is considered to be approximately normally distributed.

Use CAS to answer the following.

1 The age of truck drivers in Australia has a mean of 53 and a standard deviation of 10. If 20 truck drivers are randomly selected from the population, what is the probability that the sample mean of their ages is:

a less than 50

b greater than 55

c between 47 and 57?

2 The height of Australian women is normally distributed with a mean of 70 kg and a standard deviation of 8.5 kg. A lift is overloaded if the total weight exceeds 580 kg. If 8 women enter the lift, determine the probability that the lift is overloaded.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

14.2 Linear combinations of random variables

In this section we consider data about Melbourne temperatures in June 2014. The following table shows the minimum and maximum temperatures recorded for each day of the month.

studyon

Units 3 & 4
AOS 6
Topic 1
Concept 1

Mean and variance
Concept summary
Practice questions

Date	Day	Temperatures Min. (°C)	Temperatures Max. (°C)
1	Su	11.4	15.1
2	Mo	11.5	17.4
3	Tu	12.6	16.6
4	We	10.8	18.6
5	Th	11.9	17.1
6	Fr	8.5	16.6
7	Sa	10.5	17.5
8	Su	11.2	15.9
9	Mo	8.6	15.6
10	Tu	9.7	19.0
11	We	6.9	16.6
12	Th	9.2	16.3
13	Fr	12.2	17.0
14	Sa	10.3	15.8
15	Su	9.9	16.4
16	Mo	10.4	17.0

Date	Day	Temperatures Min. (°C)	Temperatures Max. (°C)
17	Tu	10.1	16.3
18	We	11.1	14.3
19	Th	8.0	12.3
20	Fr	9.3	15.5
21	Sa	11.3	15.7
22	Su	7.7	15.8
23	Mo	8.5	14.7
24	Tu	7.1	14.5
25	We	8.6	16.0
26	Th	11.2	16.8
27	Fr	11.0	14.7
28	Sa	9.6	13.0
29	Su	8.8	11.3
30	Mo	6.5	13.3

Source: Bureau of Meteorology
http://www.bom.gov.au/climate/dwo/201406/html/IDCJDW3050.201406.shtml

The following summary statistics can be calculated (correct to 2 decimal places) from the minimum temperatures:

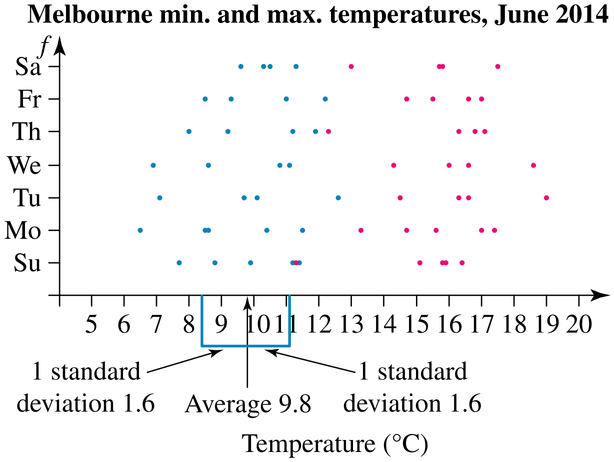
Mean: 9.81

Standard deviation: 1.60

Variance: 2.57

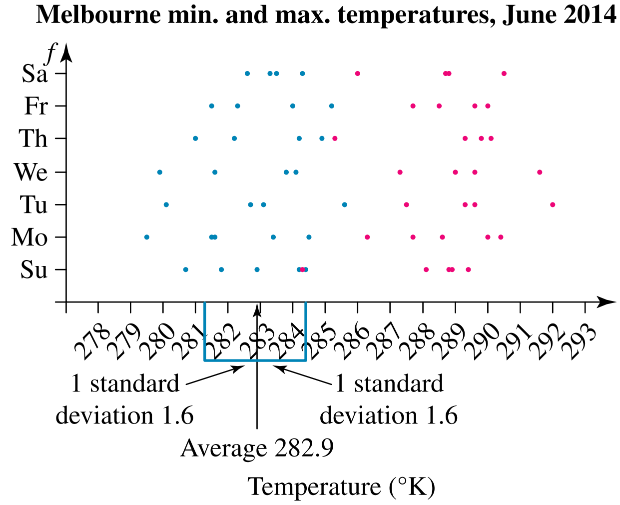
Let the name of this distribution be X. The mean of the distribution can also be called the expected value. (If you were going to Melbourne in June, what would you expect the minimum temperature to be?) The summary information about mean and variance can be written as $\mathrm{E}(X) = 9.81$ and $\mathrm{Var}(X) = 2.57$. Standard deviation has the same units as the mean, so it can also be shown on the distribution graph. As the variances of different distributions can be added together under certain circumstances, variance is also calculated for distributions.

A dot plot of the minimum (blue) and maximum (red) temperatures distribution would look like the following figure.



Change of origin

If, for some reason, we want to look at the minimum temperatures in degrees Kelvin instead of degrees Celsius, it is a simple matter to add 273.15 to each measurement. A dot plot of these temperatures is shown.



As you can see, the distribution has been moved up by 273.15 units. This repositioning is known as a change of origin. As all of the scores are increased by 273.15 units, the mean is also increased by 273.15. As the spread of the data is the same, the standard deviation and therefore the variance remain unchanged. This can be summarised as:

$$\begin{aligned} \mathrm{E}(X + 273.15) &= \mathrm{E}(X) + 273.15 \\ &= 9.81 + 273.15 \\ &= 282.96 \\ \mathrm{Var}(X + 273.15) &= \mathrm{Var}(X) \\ &= 1.60 \end{aligned}$$

In general, $\mathrm{E}(X + b) = \mathrm{E}(X) + b$ and $\mathrm{Var}(X + b) = \mathrm{Var}(X)$.

Change of scale

The temperature can be converted to a Fahrenheit scale. The formula for this conversion is $[°F] = \frac{5}{9}[°C] + 32$. This involves both a change in origin (adding 32) and a change in scale (multiplying by $\frac{5}{9}$). Let us explore what happens if the scale is changed first.

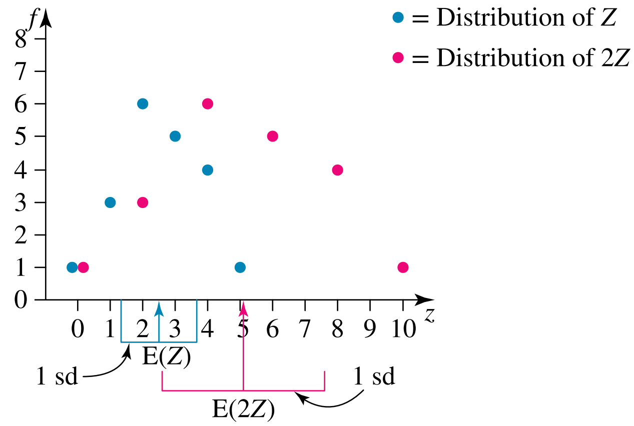
Consider the distribution shown in the table for a variable Z.

z	Frequency (f)	fz
0	1	0
1	3	3
2	6	12
3	5	15
4	4	16
5	1	5
Totals	**20**	**51**

For this distribution, E(Z) = 2.55, Var(Z) = 1.5475 and the standard deviation is 1.2439.

If for some reason, we decided to double the distribution, the values would become those in the following table.

$2Z$	Frequency (f)	$f(2Z)$
0	1	0
2	3	6
4	6	24
6	5	30
8	4	32
10	1	10
Totals	**20**	**102**



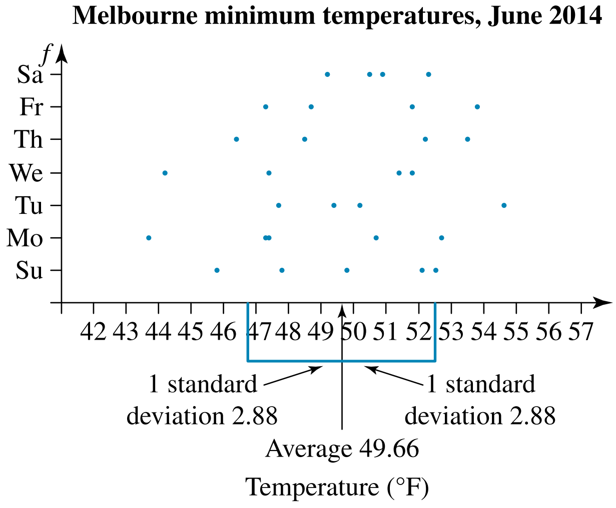
For this distribution, E($2Z$) = 5.1, Var($2Z$) = 6.19 and the standard deviation is 2.487 971. If you look at the graph of the distributions, it becomes clear why the measures of central tendency and spread have changed. When the Z values are doubled, both the mean and the standard deviation are doubled. This means that the variance has been quadrupled.

$$\begin{aligned} E(2Z) &= 2E(Z) \\ &= 2 \times 2.55 \\ &= 5.1 \\ Var(2Z) &= 2^2 Var(X) \\ &= 4 \times 1.5475 \\ &= 6.19 \end{aligned}$$

In summary, $E(aX) = aE(X)$ and $Var(aX) = a^2 Var(X)$.

Change of origin and scale

A dot plot of the Melbourne minimum temperatures in degrees Fahrenheit for June 2014 looks like this.



Notice that:

$$\begin{aligned} \mathrm{E}\left(\frac{5}{9}X + 32\right) &= \frac{5}{9}\mathrm{E}(X) + 32 \\ &= \frac{5}{9} \times 9.81 + 32 \\ &= 37.45 \\ \mathrm{Var}\left(\frac{5}{9}X + 32\right) &= \left(\frac{5}{9}\right)^2 \mathrm{Var}(X) \\ &= \frac{25}{81} \times 2.57 \\ &= 0.79 \end{aligned}$$

In general, $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$.

WORKED EXAMPLE 1

For a random variable X, the expected value of the distribution is 3 and the variance is 2.2.

a If 10 is added to each score in the distribution, what is the new expected value and variance?

b If each score in the distribution is doubled, what is the new expected value and variance?

THINK	WRITE
Write the information using correct notation.	$\mathrm{E}(X) = 3,\ \mathrm{Var}(X) = 2.2$
a Each score has 10 added to it. This will increase the expected value by 10 but will not change the variance.	**a** $\begin{aligned} \mathrm{E}(X + 10) &= \mathrm{E}(X) + 10 \\ &= 3 + 10 \\ &= 13 \\ \mathrm{Var}(X + 10) &= \mathrm{Var}(X) \\ &= 2.2 \end{aligned}$
b Each score is doubled. This will double the expected value and quadruple the variance.	**b** $\begin{aligned} \mathrm{E}(2X) &= 2\mathrm{E}(X) \\ &= 2 \times 3 \\ &= 6 \\ \mathrm{Var}(2X) &= 2^2\mathrm{Var}(X) \\ &= 4 \times 2.2 \\ &= 8.8 \end{aligned}$

studyon

Units 3 & 4

AOS 6

Topic 1

Concept 2

Distribution of linear combinations of random variables

Concept summary

Practice questions

Linear combinations of random variables

Say that we are concerned with the range of temperatures on a particular day rather than the actual temperatures. We have already called the distribution of minimum temperatures X, so we will call the distribution of maximum temperatures Y. The range for each day can be found by calculating $Y - X$.

The summary statistics are $\mathrm{E}(X) = 9.81$, $\mathrm{Var}(X) = 2.57$, $\mathrm{E}(Y) = 15.76$ and $\mathrm{Var}(Y) = 2.85$.

We can calculate the individual differences and then find the expected value and variance, as shown in the table on the next page.

June	Day	Temperature Min. (X) (°C)	Temperature Max. (Y) (°C)	Range ($Y - X$) (°C)
1	Su	11.4	15.1	3.7
2	Mo	11.5	17.4	5.9
3	Tu	12.6	16.6	4.0
4	We	10.8	18.6	7.8
5	Th	11.9	17.1	5.2
6	Fr	8.5	16.6	8.1
7	Sa	10.5	17.5	7.0
8	Su	11.2	15.9	4.7
9	Mo	8.6	15.6	7.0
10	Tu	9.7	19	9.3
11	We	6.9	16.6	9.7
12	Th	9.2	16.3	7.1
13	Fr	12.2	17	4.8
14	Sa	10.3	15.8	5.5
15	Su	9.9	16.4	6.5
16	Mo	10.4	17	6.6
17	Tu	10.1	16.3	6.2
18	We	11.1	14.3	3.2
19	Th	8	12.3	4.3
20	Fr	9.3	15.5	6.2
21	Sa	11.3	15.7	4.4
22	Su	7.7	15.8	8.1
23	Mo	8.5	14.7	6.2
24	Tu	7.1	14.5	7.4
25	We	8.6	16	7.4
26	Th	11.2	16.8	5.6
27	Fr	11	14.7	3.7
28	Sa	9.6	13	3.4
29	Su	8.8	11.3	2.5
30	Mo	6.5	13.3	6.8
	Average	**9.81**	**15.76**	**5.94**
	S. D.	**1.60**	**1.69**	**3.69**
	Variance	**2.57**	**2.85**	**13.63**

We are interested in using the original summary statistics to find the statistics for linear combinations of variables, rather than recalculating them.

Notice that:

$$\begin{aligned} E(Y - X) &= E(Y) - E(X) \\ &= 15.76 - 9.81 \\ &= 5.95 \end{aligned}$$

In general, $E(aX + bY) = aE(X) + bE(Y)$.

If X and Y are independent, it is possible to find the variance of the distribution using $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$.

In our example above, it is reasonable to assume that there is some correlation between the minimum and maximum temperatures for the day. Therefore, the variables are not independent and the relationship does not hold.

WORKED EXAMPLE 2

Ten patients in a hospital ward are taking vitamin C and vitamin D. Their doses of the two vitamins are shown in the table below. It is believed that the consumption of vitamin C and vitamin D by patients are independent random events. As vitamin C(X) is recorded in mg and vitamin D(Y) is recorded in μg, the formula $T = X + 0.001Y$ is used to find the total amount of vitamins taken.

If $E(X) = 3.9032$, $\text{Var}(X) = 9.898976$, $E(Y) = 121.71$ and $\text{Var}(Y) = 24499.7129$, what would be the expected value and variance of T?

***Note*: One microgram (μg) is equal to 10^{-6} g.**

Patient number	Vitamin C (mg), X	Vitamin D (μg), Y	Total vitamins, $X + 0.001Y$
1	0.909	94.5	1.0035
2	2.201	52.1	2.2531
3	8.945	73.4	9.0184
4	6.262	88	6.35
5	1.866	79.9	1.9459
6	0.697	60.4	0.7574
7	3.114	95.3	3.2093
8	9.835	19.4	9.8544
9	3.723	67.3	3.7903
10	1.48	586.8	2.0668

THINK	WRITE
1 Find the expected value of T using $E(aX + bY) = aE(X) + bE(Y)$.	$T = X + 0.001Y$ $E(X + 0.001Y) = E(X) + 0.001E(Y)$ $= 3.9032 + 0.001 \times 121.71$ $= 4.02491$

2 Find the variance of T using $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$.

$$\begin{aligned}\text{Var}(X + 0.001Y) &= \text{Var}(X) + 0.001^2\text{Var}(Y)\\ &= 9.898976 + 0.000001 \times 24499.7129\\ &= 9.923476\end{aligned}$$

Linear combinations of normally distributed random variables

If X and Y are independent normally distributed variables, then the distribution $aX + bY$ also has a normal distribution. The mean of this distribution is $\text{E}(aX + bY) = a\text{E}(X) + b\text{E}(Y)$ and the variance is $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$.

EXERCISE 14.2 Linear combinations of random variables

PRACTISE

1 WE1 For a random variable X, the expected value of the distribution is 15.3 and the variance is 1.8.

a If 10 is subtracted from each score in the distribution, what is the new expected value and variance?

b If each score in the distribution is doubled, what is the new expected value and variance?

2 For a random variable X, the expected value of the distribution is 13 and the variance is 3.2.

a If 4 is added to each score in the distribution, what is the new expected value and variance?

b If each score in the distribution is tripled, what is the new expected value and variance?

3 WE2 If $\text{E}(X) = 3.5$, $\text{Var}(X) = 1.1$, $\text{E}(Y) = 5.4$ and $\text{Var}(Y) = 2.44$, what would be the expected value and variance of $2X + 3Y$?

4 If $\text{E}(X) = 23.43$, $\text{Var}(X) = 5.89$, $\text{E}(Y) = 12.43$ and $\text{Var}(Y) = 9.7$, what would be the expected value and variance of $\frac{1}{2}X + Y$?

CONSOLIDATE

5 For a random variable X, the expected value of the distribution is 17.3 and the variance is 3.9.

a If 20 is added to each score in the distribution, what is the new expected value and variance?

b If each score in the distribution is halved, what is the new expected value and variance?

6 For a random variable X, the expected value of the distribution is 13 and the variance is 5.6.

a If 3 is added to each score in the distribution, what is the new expected value and variance?

b If each score in the distribution is multiplied by 1.5, what is the new expected value and variance?

7 $\text{E}(X) = 4.5$ and $\text{Var}(X) = 2.3$. If $Y = 2X + 3$, find $\text{E}(Y)$ and $\text{Var}(Y)$.

8 $\text{E}(Y) = 36.4$ and $\text{Var}(Y) = 4.2$. If $Y = -2X + 5$, find $\text{E}(X)$ and $\text{Var}(X)$.

9 $\text{E}(Y) = 56.3$ and $\text{Var}(Y) = 24.3$. If $Y = 3X + 11$, find $\text{E}(X)$ and $\text{Var}(X)$.

10 If $E(X) = 5.3$, $Var(X) = 0.1$, $E(Y) = 5.9$ and $Var(Y) = 2.4$, what would be the expected value and variance of $3X + 2Y$?

11 If $E(X) = 34.2$, $Var(X) = 1.1$, $E(Y) = 2.4$ and $Var(Y) = 0.3$, what would be the expected value and variance of $0.1X + 2Y$?

12 In a class of 30 students, each student has completed 2 tests. The results are recorded as percentages. If the distribution of results for the first test is called X and the distribution of results for the second test is called Y, the overall percentage is found by calculating $\frac{X + Y}{2}$. If $E(X) = 53.2$ and $E(Y) = 83$, what is the expected value for the overall percentage?

13 Using your results from question **12**, if $Var(X) = 104$ and $Var(Y) = 245$, what is the standard deviation of the overall percentage? You may assume that the variables are independent.

14 The notation $X \sim N(0, 9)$ means that X is normally distributed where $E(X) = 0$ and $Var(X) = 9$. If $Y \sim N(4, 1.5)$, describe the distribution $2X + 5Y$.

MASTER

15 The formula $y_i = \frac{x_i - \bar{x}}{s_x}$ can be used to change normally distributed $X\left(\bar{x}, s_x^{\,2}\right)$ into a standard normal distribution. Verify that the distribution Y has a mean of 0 and a variance of 1.

16 The expected value of a distribution can be written as $E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$.
Use this formula to prove that $E(aX + bY) = aE(X) + bE(Y)$.

14.3 Sample means

Estimating population parameters

Population parameters are often not known. For very large populations, or where data for the entire population is very difficult and/or expensive to obtain, samples can be used to estimate the population parameters. In this section, we are concerned with finding the mean of samples.

WORKED EXAMPLE 3

Dylan is interested in finding the average length of a television show on Netflix. He records the following data. All times are recorded in minutes. Estimate the mean program length.

Sample	Show 1	Show 2	Show 3	Show 4	Show 5	Show 6	Show 7	Show 8	Show 9	Show 10
1	60	55	60	60	60	5	55	30	30	30
2	60	30	30	60	35	130	35	60	30	35
3	85	30	50	55	60	60	55	25	55	30
4	30	30	30	30	30	30	30	30	30	30
5	30	30	30	30	30	25	25	25	25	25
6	60	60	60	60	60	60	60	30	30	60
7	30	60	60	60	90	60	35	35	35	30
8	55	55	60	35	35	55	55	55	50	55
9	85	60	60	30	30	60	60	60	60	60
10	30	30	30	45	45	45	45	45	45	45
11	70	70	130	35	35	70	70	35	35	70
12	60	105	60	120	60	60	60	60	60	105

THINK **WRITE**

1 Write the formula for the mean.

$$\bar{x} = \frac{\Sigma x_i}{n}$$

2 Find the mean for each sample.

$$\text{Sample 1: } \bar{x} = \frac{445}{10} = 44.5$$

$$\text{Sample 2: } \bar{x} = \frac{505}{10} = 50.5$$

$$\text{Sample 3: } \bar{x} = \frac{505}{10} = 50.5$$

$$\text{Sample 4: } \bar{x} = \frac{300}{10} = 30$$

$$\text{Sample 5: } \bar{x} = \frac{275}{10} = 27.5$$

$$\text{Sample 6: } \bar{x} = \frac{540}{10} = 54$$

$$\text{Sample 7: } \bar{x} = \frac{495}{10} = 49.5$$

$$\text{Sample 8: } \bar{x} = \frac{510}{10} = 51$$

Sample 9: $\bar{x} = \frac{565}{10}$

$= 56.5$

Sample 10: $\bar{x} = \frac{405}{10}$

$= 40.5$

Sample 11: $\bar{x} = \frac{620}{10}$

$= 62$

Sample 12: $\bar{x} = \frac{750}{10}$

$= 75$

3 Calculate the average, $\bar{x}$.

$$\bar{x} = \frac{44.5 + 50.5 + 50.5 + 30 + 27.5 + 54 + 49.5 + 51 + 56.5 + 40.5 + 62 + 75}{12}$$

$$= \frac{591.5}{12}$$

$$= 49.3$$

4 Answer the question.

An estimate of the population mean is 49.3 minutes.

studyon

Units 3 & 4

AOS 6

Topic 2

Concept 1

Mean and standard deviation of a sample

Concept summary

Practice questions

The distribution of sample means

For purposes of understanding, we will examine a very small population and an even smaller sample. Consider the following data points: 23, 42, 12, 21 and 11. The average of these points is $\mu = 21.8$ and the standard deviation is 11.16.

Consider all of the different samples of size 2 from this data set. The mean of a sample is used to give some indication of the likely population mean. The sample mean is given the symbol $\bar{x}$.

Data points	$\bar{x}$
23, 42	32.5
23, 12	17.5
23, 21	22
23, 11	17
42, 12	27
42, 21	31.5
42, 11	26.5
12, 21	16.5
12, 11	11.5
21, 11	16

As you can see, there is a lot of variety in the values of $\bar{x}$. This variability would be reduced by selecting larger samples. If you calculate the average of all of the sample $\bar{x}$ values, you will find it is equal to 21.8. Notice that this is the same as the population mean. Although the means of individual samples will vary, the mean of the sample means will be the same as the population mean. That is, $\mu_{\bar{x}} = \mu$.

The standard deviation of the sample means can be found using

$$\frac{\sigma}{\sqrt{n}} = \frac{11.16}{\sqrt{2}}$$

$$= 7.89$$

As the standard deviation is divided by the square root of the sample size, this measure of variability becomes smaller as the sample size increases: $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

Verifying the formulas

The original population X has $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. In this instance, we are concerned with the distribution of sample means taken from this population. Each sample has a mean, $\bar{x}_i$, and the distribution of these means can be referred to as $\bar{X}$.

We have observed that $\mathrm{E}(\bar{X}) = \mu$. As each sample comes from the original population, the expected value for the mean of each sample is the same as the population mean, μ. (In reality the actual value is $\bar{x}_i$, but we can't expect a random value; we expect it to be μ.).

$$\begin{aligned}\mathrm{E}(\bar{X}) &= \mathrm{E}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)\\ &= \frac{1}{n}\mathrm{E}(X_1 + X_2 + \ldots X_n)\\ &= \frac{1}{n}(n\mu)\\ &= \mu\end{aligned}$$

The variance of the distribution can be proven in a similar fashion. We assume that the random samples are mutually independent, and as each sample comes from the population, it will have the same variance as the population.

$$\begin{aligned}\mathrm{Var}(\bar{X}) &= \mathrm{Var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)\\ &= \frac{1}{n^2}\mathrm{Var}(X_1 + X_2 + \ldots + X_n)\\ &= \frac{1}{n^2}(n\sigma^2)\\ &= \frac{\sigma^2}{n}\end{aligned}$$

WORKED EXAMPLE 4

Five hundred students are studying a Mathematics course at Flinders University. On a recent exam, the mean score was 67.2 with a standard deviation of 17.

a **Samples of size 5 are selected and the means are found. Find the mean and standard deviation of the distribution of $\bar{X}$.**

b **If the sample size was increased to 30, what effect would this have on the mean and standard deviation of the distribution of sample means?**

THINK	WRITE
a 1 The mean of the distribution of sample means is the same as the population mean.	**a** $\mathrm{E}(\bar{X}) = \mu$ $= 67.2$
2 Write the formula for variance of the distribution of sample means.	$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$

3 Calculate the standard deviation.

$$\text{Var}(\overline{X}) = \frac{17^2}{5}$$
$$= 57.8$$
$$\text{sd}(\overline{X}) = 7.6$$

b 1 The mean is not dependant on sample size.

b $$\text{E}(\overline{X}) = \mu$$
$$= 67.2$$

2 Calculate the standard deviation using $n = 30$.

$$\text{Var}(\overline{X}) = \frac{\sigma^2}{n}$$
$$= \frac{17^2}{30}$$
$$= 9.6\dot{3}$$
$$\text{sd}(\overline{X}) = 3.1$$

3 Write your conclusions.

Increasing the sample size does not change the mean of the distribution, but it does reduce the standard deviation.

EXERCISE 14.3 Sample means

PRACTISE

1 WE3 Ronit decides to measure his mean travel time to school. He records the time, in minutes, every day for 7 weeks. Estimate his mean travel time.

Week	Monday	Tuesday	Wednesday	Thursday	Friday
1	92	43	41	39	35
2	118	81	46	51	38
3	62	48	46	41	49
4	82	48	42	43	41
5	78	51	42	41	38
6	63	62	41	43	44
7	55	41	46	41	32

2 Leesa wants to know the mean movie length for her favourite movie channel. She records the lengths of 8 movies every day for 1 week. Her results are recorded in minutes. Estimate the population mean movie length.

Day	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5	Movie 6	Movie 7	Movie 8
Monday	115	95	105	95	115	100	90	95
Tuesday	95	85	90	90	105	95	75	95
Wednesday	110	95	80	110	95	90	105	80
Thursday	95	100	90	95	105	100	90	85
Friday	105	95	90	100	105	100	90	105
Saturday	90	85	90	110	80	100	90	90
Sunday	105	100	100	95	90	90	90	110

3 WE4 Every year 500 students apply for a place at Monotreme University. The average enrolment test score is 600 with a standard deviation of $\sqrt{300}$.

a Samples of size 10 are selected and the means are found. Find the mean and standard deviation of the distribution of $\overline{X}$.

b If the sample size was increased to 20, what effect would this have on the mean and standard deviation of the distribution of sample means?

4 One thousand students are studying a statistics course at Echidna University. On a recent exam, the mean score was 90.5 with a standard deviation of 10.

a Samples of size 15 are selected and the means found. Find the mean and standard deviation of the distribution of $\overline{X}$.

b If the sample size was increased to 30, what effect would this have on the mean and standard deviation of the distribution of sample means?

CONSOLIDATE

5 Use the random number generator on your calculator to generate random numbers between 0 and 100. Perform the simulation 40 times and find the sample mean.

a Compare your results to those of your classmates. How close are they to the expected value of 50?

b Find the average of $\overline{X}$ for your class. How close is it to 50?

Questions 6–8 refer to the following data set — a simulation of the total obtained when a pair of dice were tossed. Eight people each tossed the dice 10 times.

Toss	Player 1	Player 2	Player 3	Player 4	Player 5	Player 6	Player 7	Player 8
Toss 1	7	6	10	7	11	2	10	8
Toss 2	8	9	8	6	6	11	4	10
Toss 3	2	4	3	10	6	8	8	6
Toss 4	8	5	6	5	11	7	6	2
Toss 5	10	12	6	8	10	8	3	4
Toss 6	4	10	9	5	3	6	5	5
Toss 7	7	6	5	6	9	10	4	2
Toss 8	11	8	9	8	9	9	6	10
Toss 9	4	7	10	10	7	4	12	8
Toss 10	7	8	3	8	10	4	7	11

6 Find an estimate of the population mean and compare it to the theoretical mean.

7 Repeat the experiment with your own simulation of dice tosses. Estimate the population mean.

8 Construct a dot plot for the distribution of sample means. Use the samples given here, the ones you simulated yourself and your classmates' simulations.

Questions 9–11 refer to a population with a mean of 73 and a standard deviation of 12.

9 If samples of size 20 are selected, find the mean and standard deviation of the distribution of sample means.

10 If the sample size is increased to 30, find the mean and standard deviation of the sample means.

11 What sample size would be needed to reduce the standard deviation of the distribution of sample means to less than 2?

Questions 12–14 refer to a population with a mean of 123 and a standard deviation of 43.

12 If samples of size 25 are selected, find the mean and standard deviation of the distribution of sample means.

13 If samples of size 40 are selected, find the mean and standard deviation of the distribution of sample means.

14 What sample size would be needed to reduce the standard deviation of the distribution of sample means to less than 5?

MASTER

15 Using Excel or similar spreadsheet software, record the song lengths from your music library. Select at least 15 random samples of 10 songs from your library and calculate the sample mean song length. Construct a dot plot of your sample means. What do you notice about the shape of the distribution?

16 We wish to create data points from a skewed population. Use Excel or similar spreadsheet software to create the data points.

- a Use a random number generator to find 70 numbers between 1 and 20. Use the generator to find another 30 numbers between 15 and 20. The distribution should now be skewed to the right. You should now have 100 numbers. Plot your distribution.
- b Now take samples from the distribution to see what the distribution of sample means might look like when the original population is skewed. Select random samples of size 10 from your distribution and calculate the sample means. Plot the distribution of sample means. What do you observe?

14.4 Confidence intervals

We have seen that different samples will have different means. Because a sample is used to tell us what the population parameters might be, we need to be able to say more than that the mean is approximately equal to the sample mean. Confidence intervals allow us to quantify the interval within which the population mean might lie.

Calculation of confidence intervals

We have learned that when the sample size is large, $\bar{x}$ is normally distributed with a mean of $\mu_{\bar{x}} = \mu$ and a standard deviation of $\frac{\sigma}{\sqrt{n}}$. As a sample is normally selected in order to estimate the population mean and standard deviation, the best estimates for

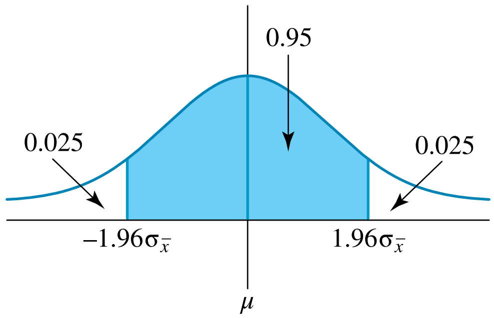
studyon

Units 3 & 4
AOS 6
Topic 3
Concept 1

Confidence intervals for means
Concept summary
Practice questions

μ and σ are the sample mean and standard deviation $\bar{x}$ and s respectively. We know that for normal distributions, $z = \dfrac{x - \mu}{\sigma}$. This means that to find the upper and lower values of z, $z = \dfrac{x \pm \bar{x}}{s/\sqrt{n}}$. Rearranging this gives us $x = \bar{x} \pm z\dfrac{s}{\sqrt{n}}$.

For a 95% confidence interval, 95% of the distribution is in the middle area of the distribution. This means that the tails combined contain 5% of the distribution (2.5% each). The z-score for this distribution is 1.96.



Returning to the sample data we used for investigating sample means, it is possible to calculate a 90% confidence interval (although it is not very accurate because of the small sample). This interval gives some indication of what the population mean might be.

Data points	$\bar{x}$	s	90% confidence interval	Contains population mean
23, 42	32.5	13.435	16.8–48.1	Yes
23, 12	17.5	7.778	8.5–26.5	Yes
23, 21	22	1.414	20.4–23.6	Yes
23, 11	17	8.485	7.1–26.9	Yes
42, 12	27	21.213	2.3–51.7	Yes
42, 21	31.5	14.849	14.2–48.8	Yes
42, 11	26.5	21.920	1–52	Yes
12, 21	16.5	6.364	9.1–23.9	Yes
12, 11	11.5	0.707	10.7–12.3	No
21, 11	16	7.071	7.8–24.2	Yes

As you can see, 90% of the possible confidence intervals contain the population mean. This is the meaning of the 90% confidence interval.

Note that the size of the confidence intervals varies between samples. The smaller the sample standard deviation is, the smaller the confidence interval for the sample will be.

WORKED EXAMPLE 5

After surveying the 20 people in your class, you find that they plan to spend an average of \$4.53 on their lunch today. The standard deviation for your sample was \$0.23. Estimate the average amount that your classmates will spend on lunch today. Find a 95% confidence interval for your estimate.

THINK	WRITE
1 There are 20 people in the class. This is the sample size. \$4.53 is spent on lunch. This is the sample mean. The sample standard deviation is \$0.23.	$n = 20$ $\bar{x} = 4.53$ $s = 0.23$
2 For a 95% confidence interval, $z = 1.96$.	$z = 1.96$

3 The confidence interval is $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$. Find $z\frac{s}{\sqrt{n}}$.

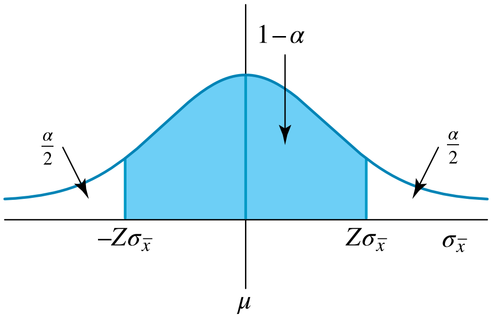
$z\frac{s}{\sqrt{n}} = 1.96 \times \frac{0.23}{\sqrt{20}}$
$= 0.1$

4 Identify the 95% confidence interval.

$\bar{x} - z\frac{s}{\sqrt{n}} = 4.53 - 0.1$
$= 4.43$
$\bar{x} + z\frac{s}{\sqrt{n}} = 4.53 + 0.1$
$= 4.63$

We can be 95% confident that, on average, between \$4.43 and \$4.63 will be spent on lunch by students from this class today.

More generally, we can talk about a $1 - \alpha$ confidence interval. In this case, the tails combined will have an area of α (or $\frac{\alpha}{2}$ in each tail). In this case, the z-score that has a tail area of $\frac{\alpha}{2}$ is used.



WORKED EXAMPLE 6

Arabella samples 102 people and finds, with a standard deviation of 0.8, that on average they eat 5.2 cups of vegetables per day. Estimate the average daily vegetable consumption. Find a 99% confidence interval for your estimate.

THINK

WRITE

1 There are 102 people in the sample. This is the sample size.
The sample mean is 5.2.
The sample standard deviation is 0.8.

$n = 102$
$\bar{x} = 5.2$
$s = 0.8$

2 For a 99% confidence interval, find the z-score.

99% confidence interval:
1% will be in the tails.
0.5% in each tail:
$z = 2.58$

3 The confidence interval is $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$. Find $z\frac{s}{\sqrt{n}}$.

$z\frac{s}{\sqrt{n}} = 2.58\frac{0.8}{\sqrt{102}}$
$= 0.2$

4 Identify the 99% confidence interval.

$$\bar{x} - z\frac{s}{\sqrt{n}} = 5.2 - 0.2$$
$$= 5$$
$$\bar{x} + z\frac{s}{\sqrt{n}} = 5.2 + 0.2$$
$$= 5.4$$

We can be 99% confident that, on average, a person consumes between 5 and 5.4 cups of vegetables.

WORKED EXAMPLE 7

In Worked example 5, the average amount spent on lunch was between \$4.43 and \$4.63. What sample size would be needed to reduce the interval to ±\$0.05 at the 95% level of confidence?

THINK	WRITE
1 The confidence interval formula is $\bar{x} \pm z\frac{s}{\sqrt{n}}$. This means that we need $z\frac{s}{\sqrt{n}} = 0.05$.	$z\frac{s}{\sqrt{n}} = 0.05$
2 Although the sample standard deviation will change with a larger sample, the current value is the best estimate that we have.	$s = 0.23$
3 For a 95% confidence interval, $z = 1.96$.	$z = 1.96$
4 Solve $z\frac{s}{\sqrt{n}} = 0.05$ for n.	$z\frac{s}{\sqrt{n}} = 0.05$ $1.96 \times \frac{0.23}{\sqrt{n}} = 0.05$ $1.96 \times \frac{0.23}{0.05} = \sqrt{n}$ $\sqrt{n} = 9.016$ $n = 81$ At least 81 people would need to be surveyed.

EXERCISE 14.4 Confidence intervals

PRACTISE

1 WE5 Among 75 Victorians who were surveyed, the average amount spent on holidays this year was \$2314 with a standard deviation of \$567. Find a 95% confidence interval for the average amount spent on holidays by Victorians.

2 After surveying 30 swimmers as they entered the local swimming complex, you found that the average distance that they intended to swim was 1.2 km. The sample standard deviation was 0.5 km. Estimate the average distance people were intending to swim that day. Find a 95% confidence interval for your estimate.

3 WE6 James samples 116 people and finds, with a standard deviation of \$537, that their average car value is \$23 456. Estimate, to the nearest dollar, the average car value for the population. Find a 99% confidence interval for your estimate.

4 Charles samples 95 people and finds, with a standard deviation of 5.8 g, that on average, they each consume 25.7 g of chocolate per day. Estimate the proportion of the population that eats chocolate daily. Find a 90% confidence interval for your estimate.

5 WE7 For question 1, what sample size would be needed to reduce the interval to ±\$100 at the 95% level of confidence?

6 For question 2, what sample size would be needed to reduce the interval to ±0.1 km at the 95% level of confidence?

CONSOLIDATE

7 You are auditing a bank. You take a sample of 50 cash deposits and find a mean of \$203.45 and a standard deviation of \$43.32. Estimate, with 95% confidence, the average cash deposit amount.

8 A sample of 40 AA batteries were tested to determine their average lifetime. It was found that they lasted an average of 2314 minutes with a standard deviation of 243 minutes. Estimate, with 95% confidence, the average battery life.

9 For her class assignment, Holly times her trips to school. She records the times for 30 days and finds the average trip time is 24.6 minutes with a standard deviation of 7.6 minutes. Estimate, with 95% confidence, the average time that Holly will take to travel to school over the year.

10 Repeat question 7 but find a 90% confidence interval.

11 Repeat question 8 but find a 99% confidence interval.

12 Repeat question 9 but find a 90% confidence interval.

13 For question 7, what sample size would be needed to reduce the interval to ±\$2 at the 95% level of confidence?

14 For question 8, what sample size would be needed to reduce the interval to ±50 minutes at the 95% level of confidence?

15 For question 9, what sample size would be needed to reduce the interval to ±2 minutes at the 95% level of confidence?

16 For question 9, what sample size would be needed to reduce the interval to ±2 minutes at the 90% level of confidence?

MASTER

17 Refer to the data about song tune length that you collected in Exercise 14.3 question 15. Construct a 95% confidence interval for your average song length prediction. Some people say that for a song to be popular, it must be less than 3.05 minutes long. How does this statement compare with your data?

18 Refer to the data set that you created for Exercise 14.3 question 16. Construct a 90% confidence interval for the population average.

14.5 Hypothesis testing

Calculating confidence intervals involves finding an interval in which the population mean is likely to lie. Different samples can have different means. If a sample mean is different to what might have been expected, we need to determine if the difference is large enough to state that the sample is from a different population, or if the difference is just due to chance.

studyon

Units 3 & 4

AOS 6

Topic 4

Concept 1

Hypothesis testing
Concept summary
Practice questions

Hypotheses

A hypothesis is a claim about a population that requires formal investigation before the claim can be determined. For example, if we are testing a particular drug, either the drug is no different to other treatments, or it performs differently in terms of treatment time. In this case we propose two hypotheses.

The first hypothesis, called the null hypothesis and indicated by the symbol H_0, assumes that nothing has happened. This is the statement that we accept unless we have sufficient evidence to claim otherwise. This should be the statement that 'does no harm'.

The alternative hypothesis, symbol H_1, is the statement that something is going on. It is possible to make the claim that the mean is greater than or less than the original value. This is called a one-tailed test. If the claim is that the mean is different to the original value, it is a two-tailed test.

- **The null hypothesis (H_0) makes the claim that there is no difference, $\mu = \mu_0$.**
- **The alternative hypothesis (H_1) makes the claim that there is a difference: $\mu > \mu_0$ and $\mu < \mu_0$ are one-tailed tests, and $\mu \neq \mu_0$ is a two-tailed test.**

In our example about the new drug, we would pose the following hypotheses:

H_0: The new drug has the same treatment time as the normal drug.

H_1: The new drug works within a different treatment time.

WORKED EXAMPLE 8

Riley has decided to participate in his schools Maths tutoring program. He decides to go for one hour per week. With reference to his exam results, state the null and alternative hypothesis.

THINK	WRITE
1 The null hypothesis states that there will be no effect.	H_0: The tutoring will have no effect on Riley's Maths results.
2 The alternative hypothesis states that there will be an effect. (In this case Riley is hoping that there is improvement in his results.)	H_1: The tutoring will change Riley's Maths results.

Type I and Type II errors

Before analysis, we assume that the null hypothesis is true. After analysis, if there is sufficient evidence, we may choose to reject the null hypotheses and believe that the alternative hypothesis is true. Note that we never prove the null hypothesis; we just accept that it is true unless we have evidence to say otherwise.

In the decision making process, two types of errors are possible.

- **A type I error is rejecting the null hypothesis when it is actually true.**
- **A type II error is accepting the null hypotheses when it is actually false.**

It is not possible to reduce the chances of making both types of errors. Reducing the chances of making a type I error increases the likelihood of making a type II error and vice versa.

WORKED EXAMPLE 9

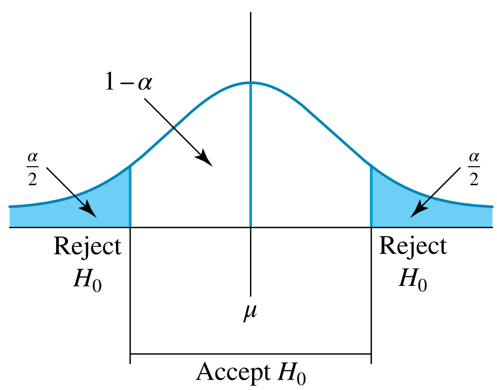
Suppose you are testing a new medication. You are going to conclude that the drug has no effect unless there is sufficient evidence to say otherwise.

a **If a type I error was made, what conclusions were reached?**

b **If a type II error was made, what conclusions were reached?**

THINK	WRITE
Create the null and alternative hypotheses.	H_0: The new drug has the same treatment time as the normal drug. H_1: The new drug has a different treatment time.
a A type I error means that the null hypothesis was rejected when it was actually true.	**a** The null hypothesis was rejected. The new drug works differently to the normal drug.
b A type II error means that the null hypothesis is accepted when it is actually false.	**b** The null hypothesis was accepted. There is insufficient evidence to support the claim that the new drug works differently.

The level of significance predicts the likelihood of making a type I error. Often tests are conducted at a 5% level of significance, meaning that 5% of the time the null hypothesis is rejected when it is actually true. If that level of significance is not acceptable, it is also possible to test at, say, a 1% level of significance. Reducing the level of significance means that the null hypothesis is less likely to be falsely rejected, but it increases the chances



of accepting the null hypothesis when it is false. In general, we talk about the α level of significance.

Hypothesis testing

Once we have created hypotheses and decided on the level of significance, it is necessary to test the hypotheses and draw any conclusions.

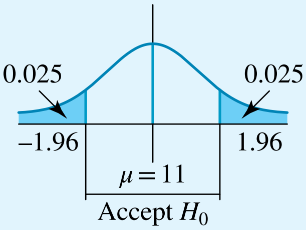
We are going to assume that we know the standard deviation of the population that we are testing. In practice, this information is often not available and there are other statistical tests that may be used, but the overall process is the same as this one.

The test statistic $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ can be used.

WORKED EXAMPLE 10

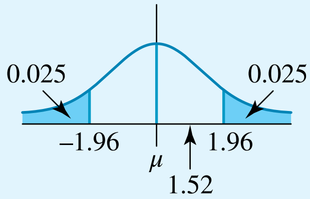
Bi More supermarkets' sales records show that the average monthly expenditure per person on a certain product was \$11 with a standard deviation of \$1.80.

The company recently ran a promotion campaign and would like to know if sales have changed. They sampled 30 families and found their average expenditure to be \$11.50. Has the promotion campaign changed sales figures? Test this at the 5% level of significance.

THINK	WRITE
1 Create the null and alternative hypotheses. Assume that the promotion campaign has not changed the average sales unless there is sufficient evidence otherwise.	H_0: $\mu = 11$; the promotion has not changed sales. H_1: $\mu \neq 11$; the promotion has changed sales.
2 We are testing at the 5% level of significance and are concerned with means not equal to 11.	
3 Find the z-score that corresponds with the 5% level of significance. Values greater than that score will be rejected. *Note:* If $\Pr(Z > z) = 0.025$, then, using invnorm, $z = 1.96$. If $\Pr(Z < z) = 0.025$, then, using invnorm, $z = -1.96$.	At the 5% level of significance, $z = \pm 1.96$. Reject values if $z > 1.96$ or $z < -1.96$.
4 Calculate the test statistic for our sample.	$z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ $= \dfrac{11.5 - 11}{1.8 / \sqrt{30}}$ $= 1.52$

5 The test statistic is $z = 1.52$. We would reject it if $z > 1.96$.

The test statistic does not fall in the rejection region.



6 Draw your conclusions.

Accept H_0.
There is insufficient evidence to support the claim that the promotion has changed sales.

P-value

The *P*-value is a probability used to test the hypothesis. It is the likelihood that a value at least as extreme as the sample statistic occurs, given that the null hypothesis is true. The *P*-value is another way that the hypothesis can be tested.

WORKED EXAMPLE 11

Repeat Worked example 10 using the *P*-value to test the hypothesis.

THINK / **WRITE**

1 State the null and alternative hypotheses.

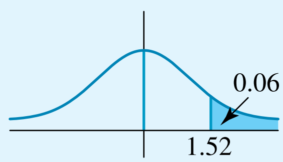
H_0: $\mu = 11$; the promotion has not changed sales.
H_1: $\mu \neq 11$; the promotion has changed sales.

2 Convert 11.5 to a *z*-score.

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{11.5 - 11}{1.8 / \sqrt{30}} = 1.52$$

3 Calculate the *P*-value.

$P(Z > 1.52) = 0.06$



4 If the tail area is less than 0.025, then we can reject H_0. As this tail area is greater than 0.025, we accept the null hypothesis.

Accept H_0.
There is insufficient evidence to support the claim that the promotion has changed sales.

EXERCISE 14.5 Hypothesis testing

PRACTISE

1 WE8 Des has decided to increase his mountain bike training by adding an extra three 1-hour sessions to his weekly training schedule. He is hoping to improve his best race time. With reference to his race times, state the null and alternative hypotheses.

2 Gabby attempts to improve her cross country time by adding an extra training session per week. With reference to race times, state the null and alternative hypotheses.

3 WE9 Ezy Pet Food believes that it has a contaminated batch of pet food. The company decides to test some of the tins. It will assume that the batch is contaminated unless there is evidence to the contrary.

a If a type I error is made, what conclusions are reached?
b If a type II error is made, what conclusions are reached?

4 The probability of low birth weight in Australia is about 6%. Out of 200 babies born at Bundaberg during a period when aerial spraying of sugar cane was common, 18 had low birth weight. The local paper claimed that this proved that aerial spraying was dangerous because the rate of birth problems was 150% of the national average. Aerial spraying will only be stopped if there is sufficient evidence to do so.

a If a type I error was made, what conclusions were reached?
b If a type II error was made, what conclusions were reached?

5 WE10 Bi More supermarkets' sales records show that the average monthly expenditure per person on a certain product was \$11 with a standard deviation of \$1.80. The company recently ran a promotion campaign and would like to know if sales have changed. They sampled 30 families and found their average expenditure to be \$11.80. Has the promotion campaign made a difference? Test this at the 5% level of significance.

6 Bi More supermarkets' sales records show that the average monthly expenditure per person on a certain product was \$11 with a standard deviation of \$1. The company recently ran a promotion campaign and would like to know if sales have changed. They sampled 25 families and found that their average expenditure had increased to \$11.50. Has the promotion campaign made a difference? Test this at the 5% level of significance.

7 WE11 Repeat question 5 using the P-value to test the hypothesis.

8 Repeat question 6 using the P-value to test the hypothesis.

CONSOLIDATE

Questions 9–13 refer to the following information.

A manufacturer of a particular car tyre uses a machine to study the wear and tear characteristics of tyres. The average tyre tread is believed to last for 17 000 km with a standard deviation of 1500 km. A change in manufacturing is being considered. A sample of 25 tyres created with the new process was found to have a mean life of 17 400 km. Does the new process change the average mileage for the tyres?

9 Create the null and alternative hypotheses.

10 Test the claim at a 5% level of significance.

11 If $\bar{x} = 17\,852$ and a 1% level of significance is used, what is the decision?

12 If $\bar{x} = 17\,852$, what is the smallest α at which H_0 is rejected (using $n = 25$)?

13 If $\bar{x} = 18\,000$, what is the P-value (using $n = 25$)?

Questions **14–18** refer to the following information.

The drying time for a particular type of paint is known to be normally distributed with $\mu = 75$ minutes and $\sigma = 9.4$ minutes. A new additive has been developed that is supposed to change the drying time. After testing 100 samples, the observed average drying time is $\bar{x} = 71.2$. Does the new additive change the drying time?

14 Create the null and alternative hypotheses.

15 The new additive is expensive. Test the claim at the 1% level of significance.

16 If $\bar{x} = 72.9$, what is the conclusion using $\alpha = 0.01$?

17 What is the P-value if $\bar{x} = 72.9$?

18 What is the α for the test procedure that rejects H_0 when $Z < -2.88$ or $Z > 2.88$?

MASTER

19 A random sample is drawn from a normal population $\sim N(\mu, \sigma^2)$. The mean of this sample is μ_1. When a statistician tests the hypothesis that the mean is μ using this sample, she calculates a P-value ≥ 0.01. Show that μ_1 is in the 99% confidence interval.

20 An expensive drug has been shown to improve recovery time from a disease. Without the drug, the recovery time is normally distributed with $\mu = 7.3$ days. Because the drug is expensive, it was tested at a 1% level of significance. A sample of 20 patients was found to have an average recovery time of 6.9 days and the null hypothesis was rejected. Even though the drug has been shown to improve recovery time, what might prevent it from being introduced?

eBook*plus* ONLINE ONLY

14.6 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

14 Answers

EXERCISE 14.2

1 a $E(X-10)=5.3$, $Var(X-10)=1.8$

b $E(2X)=30.6$, $Var(2X)=7.2$

2 a $E(X+4)=17$, $Var(X+4)=3.2$

b $E(3X)=39$, $Var(3X)=28.8$

3 $E(2X+3Y)=23.2$, $Var(2X+3Y)=26.36$

4 $E\left(\frac{1}{2}X+Y\right)=24.145$, $Var\left(\frac{1}{2}X+Y\right)=11.1725$

5 a $E(X+20)=37.3$, $Var(X+20)=3.9$

b $E(2X)=8.65$, $Var(2X)=0.975$

6 a $E(X+3)=16$, $Var(X+3)=5.6$

b $E(1.5X)=19.5$, $Var(1.5X)=12.6$

7 $E(Y)=12$, $Var(Y)=9.2$

8 $E(X)=-15.7$, $Var(X)=1.05$

9 $E(X)=15.1$, $Var(X)=2.7$

10 $E(3X+2Y)=27.7$, $Var(3X+2Y)=10.5$

11 $E(0.1X+2Y)=8.22$, $Var(0.1X+2Y)=1.211$

12 $E\left(\frac{X+Y}{2}\right)=68.1$

13 $SD\left(\frac{X+Y}{2}\right)=9.34$

14 $2X+5Y \sim N(20, 73.5)$

15 Use $Y=\frac{1}{s_x}X-\frac{\bar{x}}{s_x}$ and then check with your teacher.

16 Check with your teacher.

EXERCISE 14.3

1 51.5 minutes

2 95.9 minutes

3 a $\mu_{\bar{x}}=600$, $\sigma_{\bar{x}}=5.48$ **b** $\mu_{\bar{x}}=600$, $\sigma_{\bar{x}}=3.87$

4 a $\mu_{\bar{x}}=90.5$, $\sigma_{\bar{x}}=2.58$ **b** $\mu_{\bar{x}}=90.5$, $\sigma_{\bar{x}}=1.83$

5 Check with your teacher.

6 $E(\bar{X})=7.09$, true mean 7

7 Check with your teacher.

8 Check with your teacher.

9 $E(\bar{X})=73$, $SD(\bar{X})=2.68$

10 $E(\bar{X})=73$, $SD(\bar{X})=2.19$

11 36

12 $E(\bar{X})=123$, $SD(\bar{X})=8.6$

13 $E(\bar{X})=123$, $SD(\bar{X})=6.8$

14 74

15 The distribution should be symmetrical.

16 Check with your teacher, but the distribution should be symmetrical.

EXERCISE 14.4

1 \$2186–\$2442

2 1–1.4 km

3 \$23 327–\$23 585

4 24.7–26.7 g

5 124

6 96

7 \$191.44–\$215.46

8 2239–2389 minutes

9 21.9–27.3 minutes

10 \$193.40–\$213.50

11 2215–2413 minutes

12 22.3–26.9 minutes

13 1803

14 91

15 56

16 39

17 Check with your teacher.

18 Check with your teacher.

EXERCISE 14.5

1 H_0: the extra training sessions do not change his race time.

H_1: the extra training sessions improve his race time.

2 H_0: the extra training sessions do not change her race time.

H_1: the extra training sessions improve her race time.

3 a The pet food is safe.

b The pet food is contaminated.

4 a Aerial spraying is not safe.

b There is insufficient evidence to conclude that aerial spraying is safe.

5 The promotion has changed sales.

6 The promotion has changed sales.

7 The promotion has changed sales. $P=0.0075$

8 The promotion has changed sales. $P=0.0062$

9 H_0: $\mu=17000$; the new process does not change the mileage.

H_1: $\mu \neq 17000$; the new process changes the mileage.

10 Accept H_0. There is insufficient evidence to claim that the process changes mileage.

11 Reject H_0. The new process changes the mileage.

12 $\alpha=0.0046$

13 $P = 0.000\,429$

14 H_0: $\mu = 75$; the new additive does not change the drying time.

H_1: $\mu \neq 75$; the new additive changes the drying time.

15 Reject H_0. The new additive changes the drying time.

16 Accept H_0. There is insufficient evidence to claim that the additive changes drying time.

17 $P = 0.0129$

18 $\alpha = 0.004$

19 Check with your teacher.

20 The improvement is only 0.4 days. As the drug is expensive, it probably is not worth the money.

Index

W

Z