1 Combinatorics

Topic	1	Combinatorics
Subtopic	1.2	Counting techniques

online only

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Question 1 (1 mark)

$A = \{3, 4, \pi, 7\}, B = \{1, 4, 5, 9\}, C = \{n: n = 3r \text{ for } 1 \leq r \leq 4, r \in Z\}$. $(A \cap B) \cup C$ equals

A. ϕ
B. $\{4\}$
C. $\{3, 6, 9\}$
D. $\{3, 4, 6, 9\}$
E. $\{3, 4, 6, 9, 12\}$

Question 2 (1 mark)

Michael is buying a sound system for his music studio. He has a choice of 6 pairs of speakers, 4 amplifiers, 3 DJ controllers and 2 turntables. Calculate the number of different systems possible if he chooses one of each type of component.

☐ systems

Question 3 (1 mark)

Janelle must choose 1 first semester unit from 3 units of geography and 2 units of mathematics and 1 second semester unit from 2 units of English, 3 units of science and 3 units of IT. Determine how many different 2-units courses are possible.

☐ courses

Question 4 (1 mark)

If $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{3, 4, 5, 6, 7\}$, find $A \cap B$.

Question 5 (1 mark)

If $\varepsilon = \{a, b, c, d, e, f, g, h, i\}$ and $P = \{b, d, f\}$, find P'.

Question 6 (2 marks)

If $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $P = \{3, 4, 5, 6, 7\}$ and $Q = \{1, 3\}$, find $P' \cap Q'$.

Question 7 (3 marks)

M is the set of prime numbers less than 15. P is the set of even numbers less than 20. Find $M \cup P$ and $M \cap P$.

Question 8 (1 mark)

If $A = \{1, 3, 5\}$ and $B = \{1, 3, 4, 5, 6, 7\}$, $A \cup B$ equals:

A. A

B. B

C. $\{1, 3, 5\}$

D. $\{4, 6, 7\}$

E. $(A \cap B)'$

Question 9 (1 mark)

$\xi = \{a, b, c, d, e, f, g, h, i\}$ and $P = \{b, d, f\}$, therefore $\left(P' \cap \xi\right)$ equals:

A. $\{b, d, f\}$

B. $\{a, b, c, d, e, f, g, h, i\}$

C. $\left(P \cup \xi\right)$

D. $\{a, b, b, c, d, d, e, f, f, g, h, i\}$

E. $\{a, c, e, g, h, i\}$

Topic	1	Combinatorics
Subtopic	1.3	Factorials and permutations

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Question 1 (1 mark)

If ${}^{2n}P_n = 1680$ then n is equal to

A. 3
B. 4
C. 5
D. 6
E. 7

Question 2 (2 marks)

A child has wooden letters spelling out their name GRACE on their wardrobe. Determine how many names can be created by rearranging these wooden letters in any combination that is more than 2 letters long.

☐ names

Question 3 (1 mark)

Determine how many ways the Mathematics, Science, Physics and Chemistry prizes can be awarded from 25 students if no student can win more than one prize.

☐ ways

Question 4 (1 mark)

The number of arrangements of the letters in the word 'BREAK' where the vowels are together is:

A. 24
B. 12
C. 48
D. 36
E. 40

Question 5 (1 mark)

$\dfrac{n!}{(n-3)!}$ simplifies to:

A. $(n-1)(n-2)$
B. $n^2 - n$
C. $n^3 - 3n^2 + 2n$
D. $(n-1)(n-2)(n-3)$
E. $n^2 - 3x + 3$

Question 6 (1 mark)

Calculate the number of ways a family consisting of a mother, father, two daughters and three sons can be seated at a circular table if the daughters are to sit together.

Question 7 (1 mark)

Five students have made an appointment to see their teacher. The number of ways the appointments can be arranged if Sam, a difficult student, must go first is:

A. 120
B. 60
C. 24
D. 16
E. 30

Question 8 (1 mark)

The number of arrangements of the letters in the word 'FINDER' where the vowels are together is:

A. 720
B. 360
C. 240
D. 120
E. 320

Topic	1	Combinatorics
Subtopic	1.4	Permutations with restrictions

online only

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Question 1 (1 mark)

Determine the total number of arrangements of the letters of the word FACTORIAL, which starts and ends with A .

No. of arrangements = ☐

Question 2 (1 mark)

The number of ten–digit mobile phone numbers having at least one of their digits repeated is:

A. 36 288
B. 3 628 800
C. 99 963 712
D. 36 288 000
E. 9 996 371 200

Question 3 (2 marks)

Determine how many ways the letters of the word CALCULUS can be arranged in a row

a. If there are no restrictions. **(1 mark)**

☐ ways

b. If the two C's are separated. **(1 mark)**

☐ ways

Topic 1 | Subtopic 1.4 Permutations with restrictions

Question 4 (1 mark)
Emma has a special collection of bottles. Eight are red, five are blue and seven are green. She likes to arrange them in threes on a shelf so that one of each colour is displayed with green in the middle.
If the bottles are all different in shape and size, how many arrangements can she make?

Question 5 (1 mark)
The number of different three-digit numbers that can be formed using $\{4, 6, 8, 1, 2\}$ if each digit can be used only once is:
A. 24
B. 120
C. 15
D. 60
E. 40

Question 6 (1 mark)
The number of ten–digit mobile phone numbers having at least one of their digits repeated is:
A. 3 628 800
B. 99 963 712
C. 36 288 000
D. 9 996 371 200
E. 362 880

Question 7 (2 marks)
Determine the probability that in a randomly chosen arrangement of the word PERMUTATION, the letters T are together.

Topic	1	Combinatorics
Subtopic	1.5	Combinations

online only

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Question 1 (1 mark)

Four marbles are to be selected from a bag containing six green and three purple marbles. Determine how many different ways this can be done if there are to be two purple marbles in the selection.

A. ${}^{6}C_{2} \times {}^{3}C_{2}$
B. ${}^{9}C_{2} \times {}^{7}C_{2}$
C. ${}^{6}C_{2} + {}^{3}C_{2}$
D. ${}^{9}C_{2} + {}^{7}C_{2}$
E. $\dfrac{{}^{9}C_{2} \times {}^{7}C_{2}}{21}$

Question 2 (1 mark)

At Marina's café, sandwiches can be made with cos lettuce, carrot, avocado, capsicum, tomato and red onion. Determine how many different sandwiches are possible.

☐ sandwiches

Question 3 (2 marks)

A rugby union squad has 12 forwards and 10 backs in training. A team consists of 8 forwards and 7 backs. Determine how many different teams can be chosen from the squad.

☐ teams

Question 4 (3 marks)

In a basket there are 6 red balls and 14 blue balls. How likely is it that 5 balls chosen at random should all be red?

Question 5 (4 marks)

A review panel is to be established. It is to consist of the Principal and four other teachers selected from the 7 male and 9 female staff members.

a. How many different panels can be formed? **(1 mark)**

b. How many panels can be formed that consist of at least 3 female teachers? **(1 mark)**

c. What is the probability that only 1 male teacher is selected to be on the panel? **(2 marks)**

Question 6 (3 marks)

Determine the value of n if ${}^nC_2 = 10$.

Topic	1	Combinatorics
Subtopic	1.6	Applications of permutations and combinations

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Question 1 (1 mark)

To win a racing quinella you have to select the first two runners across the finishing line in any order. At \$0.50 a bet, determine how much it would cost to cover all quinella possibilities in a 24-horse race.

It would cost \$☐

Question 2 (1 mark)

Determine how many 4-digit numbers that are divisible by 10 can be formed from the numbers $3, 5, 7, 8, 9, 0$ such that no number repeats.

☐ 4-digit numbers

Question 3 (1 mark)

The official version of Oz Lotto requires you to select seven numbers from a total of 45 numbers. A standard game consists of 7 randomly chosen numbers and costs \$1.45. Determine how much it would cost you to ensure that you won first prize.

It would cost \$☐

Question 4 (2 marks)

A domino set contains all number pairs from double zero to double six with each number pair occurring exactly once. How many dominos are there in the set?

Question 5 (1 mark)

How many 4-digit numbers that are divisible by 10 can be formed from the numbers 3, 5, 7, 8, 9, 0 such that no number repeats?

Question 6 (2 marks)

A poker hand (five cards) is dealt from a standard pack of 52 cards. What is the probability of being dealt at least two aces? Give your answer correct to two decimal places.

Topic	1	Combinatorics
Subtopic	1.7	Pascal's triangle and the pigeon-hole principle

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Question 1 (1 mark)

On one evening in Australia, 1.4 million people watched the national news. We can be certain that at least x people from the same state/territory watched the news. Determine the maximum value of x.

A. 139 999
B. 174 999
C. 175 000
D. 200 000
E. 233 333

Question 2 (1 mark)

There are 21 – 30 people swimming at the local pool. The pool has 10 lanes. Show that there is at least 1 lane with 3 or more people in it.

Question 3 (2 marks)

In an all-boys school of 393 students, every students must wear the school uniform comprising of the following options: short-sleeve shirt, long-sleeve shirt, jumper, shorts and pants. A students must wear 1 type of shirt, shorts or pants and has the option of wearing the jumper.
Show that there are at least 50 students wearing the exact same uniform.

Question 4 (1 mark)

Using a copy of Pascal's triangle, find the number of ways six objects can be selected three at a time.

Question 5 (1 mark)

The number of ways that 7 objects can be chosen three at a time is:

A. 20
B. 21
C. 7
D. 35
E. 70

Question 6 (1 mark)

The middle number in the ninth row of Pascal's triangle is:

A. 56
B. 70
C. 126
D. 84
E. 252

Question 7 (6 marks)

The first six terms in a particular row of Pascal's triangle are $1, 25, 300, 2300, 12650, 53130$. Determine the first six terms in the row after this.

Question 8 (1 mark)
In Pascal's triangle, what do you notice about the horizontal sums?
Is there a pattern?

Question 9 (2 marks)
A dance concert is split into 3 age groups, $< 10, 10 - 16$ and > 16. If there are 46 dancers performing in the concert, show that at least 1 age group must contain 16 or more dancers.

Question 10 (2 marks)
There are $21 - 30$ people swimming at the local pool. The pool has 10 lanes. Show that there is at least 1 lane with 3 or more people in it.

Topic	1	Combinatorics
Subtopic	1.8	Review

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Question 1 (2 marks)

Determine how many ways the letters of the word DIFFERENTIAL can be arranged in a row

a. if there are no restrictions. **(1 mark)**

☐ combinations

b. if the F's are separated. **(1 mark)**

☐ combinations

Question 2 (1 mark)

Old MacDonald had a farm E – I – E – I – O.

The number of different arrangements of the letters E – I – E – I – O in a straight line is

A. $5!$

B. 5

C. $\frac{5!}{2!}$

D. $\frac{5!}{2!\,2!}$

E. $5!\ 2!\ 2!$

Question 3 (2 marks)

A netball team of 7 players is to be chosen from a squad of 11 players. Suppose and member can play any position. Determine how many ways this can be done

a. if each player is chosen to play a particular position. **(1 mark)**

No. of ways = ☐

b. If players have no particular position. **(1 mark)**

No. of ways = ☐

Question 4 (3 marks)

Determine how many ways five men and five women can be arranged in a row if

a. there are no restrictions. **(1 mark)**

No. of ways = ☐

b. the men and women occupy alternate positions. **(1 mark)**

No. of ways = ☐

c. all the men are next to each other. **(1 mark)**

No. of ways = ☐

Question 5 (1 mark)

In the expansion of $(ax + b)^4$ two of the terms are $-540x^3$ and $-1500x$. Determine the possible values of a and b.

Question 6 (2 marks)
Find the probability of selecting one banana, one apple and one grapefruit when selecting three pieces of fruit from nine bananas, four apples and three grapefruit.

Question 7 (2 marks)
A shelf holds 8 French books and 10 German books. If 6 French books and 7 German books are selected from the shelf, in how many different ways can they be arranged in a row. Assume each book is unique.

Question 8 (2 marks)
In how many different ways can 3 men and 4 women be arranged in a circle if:

a. the women must sit together. **(1 mark)**

b. men and women must sit alternatively. **(1 mark)**

Question 9 (4 marks)

A committee of 5 goats and 2 wolves is to be formed from 9 goats and 6 wolves. Find the number of different committees if:

a. the committee contains the thinnest goat and the lone wolf. **(2 marks)**

b. the committee does not contain the black goat or the toothless wolf. **(2 marks)**

Question 10 (8 marks)

In the expansion of $(a+b)^n$, the fifth term equals the sixth term. **(5 marks)**

a. Find a relationship between a and b.

b. Use the relationship you have found between a and b in part (a) to show that the second last term in the expansion of $(a+b)^n$ is $\dfrac{n}{5(n-5)}b^n$. **(3 marks)**

Answers and marking guide

1.2 Counting techniques

Question 1

$$A \cup B = \{4\}$$
$$C = \{3,6,9,12\}$$
$$(A \cap B) \cup C = \{3,4,6,9,12\}$$

Question 2

The multiplication principle can be used to determine the number of different systems.

Speakers	Amplifiers	DJ controllers	Turntables
6	4	3	2

$6 \times 4 \times 3 \times 2 = 144$ ways

There are 144 different system. **[1 mark]**

Question 3

Total number of 2-unit courses:

Semester 1	Semester 2
5	8

$5 \times 8 = 40$

There are 40 different 2-unit courses. **[1 mark]**

Question 4

The numbers that overlap are 3, 4, 5 and 6, so $A \cap B = \{3, 4, 5, 6\}$. **[1 mark]**

Question 5

P' is everything within the universal set that is not in P.

$P' = \{a, c, e, g, h, i\}$ **[1 mark]**

Question 6

$$P' = \{1, 2, 8, 9, 10, 11, 12\}$$
$$Q' = \{2, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$ **[1 mark]**
$$P' \cap Q' = \{2, 8, 9, 10, 11, 12\}$$ **[1 mark]**

Question 7

$$M = \{2, 3, 5, 7, 11, 13\}$$
$$P = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$$ **[1 mark]**
$$M \cup P = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 18\}$$ **[1 mark]**
$$M \cap P = \{2\}$$ **[1 mark]**

Question 8

$A \cup B$ combines everything in A with everything in B, excluding duplications.

Question 9

$P' = \{a, c, e, g, h, i\}$

The intersection of P' with the universal set is simply P.

1.3 Factorials and permutations

Question 1

$$\begin{aligned} {}^{2n}P_n &= \frac{(2n)!}{(2n-n)!} \\ &= \frac{(2n)!}{n!} \end{aligned}$$

Test each value until you find the expression which

When $n = 3$:

$$\begin{aligned} {}^{2n}P_n &= {}^{6}P_3 \\ &= \frac{6!}{3!} \\ &= 6 \times 5 \times 4 \\ &= 120 \end{aligned}$$

When $n = 4$:

$$\begin{aligned} {}^{2n}P_n &= {}^{8}P_4 \\ &= \frac{8!}{4!} \\ &= 8 \times 7 \times 6 \times 5 \\ &= 1680 \end{aligned}$$

Therefore, the answer is $n = 4$

Question 2

There are 5 letters in the name Grace.

The number of names of length 3 that can be formed by rearranging these letters is

$${}^{5}P_3 = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

The number of names of length 4 that can be formed by rearranging these letters is

$${}^{5}P_4 = \frac{5!}{1!} = 5 \times 4 \times 3 \times 2 = 120$$

The number of names of length 5 that can be formed by rearranging these letters is

$${}^{5}P_5 = \frac{5!}{0!} = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Award 1 mark for correctly calculating the number of names of length 3, 4 and 5.

In total, the number of names that are 3 or more letters long that can be formed from the letters in the name Grace is $60 + 120 + 120 = 300$. **[1 mark]**

Question 3

$$\begin{aligned} {}^{25}P_4 &= \frac{25!}{21!} \\ &= 25 \times 24 \times 23 \times 22 \\ &= 303\,600 \end{aligned}$$ **[1 mark]**

Question 4

Count the EA as one unit.

Therefore, there are 4! number of ways to arrange the letters.

The EA can also be AE.

$$\begin{aligned} & 4! \times 2 \\ = \; & 4 \times 3 \times 2 \times 1 \times 2 \\ = \; & 48 \end{aligned}$$

Question 5

$$\frac{n!}{(n-3)!} = \frac{n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times ... \times 1}{(n-3) \times (n-4) \times ... \times 1} \quad \text{cancelling down common factors gives}$$
$$= \frac{n \times (n-1) \times (n-2)}{1}$$
$$= n(n-1)(n-2)$$
$$= \left(n^2 - n\right)(n-2)$$
$$= n^3 - 2n^2 - n^2 + 2n$$
$$= n^3 - 3n^2 + 2n$$

Question 6

The two daughters can be considered as 1 object, giving a total of 6 objects needing to be placed in a circle.
The number of distinct ways n objects can be arranged in a circle = $(n-1)!$
In this case $n = 6$.

$(n-1)! = (6-1)!$

$= 5!$

$= 120$ ways **[1 mark]**

For each arrangement the daughters can sit in 2 different ways; daughter 1/daughter 2 or daughter 2/daughter 1

$120 \times 2 = 240$

There are 240 possible arrangements. **[1 mark]**

Question 7

Sam is not counted in calculations because we know he must be first. The other appointments can then be arranged in any order.

$4! = 4 \times 3 \times 2 \times 1$

$= 24$

Question 8

Count the IE as one unit.

$\therefore$ there are $5!$ ways to arrange the letters.

The IE can also be EI.

$5! \times 2 = 5 \times 4 \times 3 \times 2 \times 1 \times 2$

$= 240$

1.4 Permutations with restrictions

Question 1

The 2 A's have set positions, first and last. This leaves 7 letters in the middle that can be arranged in any order.

The number of arrangements is therefore $7! = 5040$. **[1 mark]**

Question 2

The number of ten-digit mobile phone numbers that can be formed using the digits of $0, 1, 2, ..., 9$ is 10^{10}

The number of ten-digit mobile phone numbers that have none of their digits repeated is

${}^{10}P_{10} = 10! = 3\,628\,800$.

Hence, the required number $10^{10} - 3\,628\,800 = 9\,996\,371\,200$.

Question 3

a. There are 8 letters in the word CALCULUS, but there are 2 C's, 2 L's and 2 U's. With no restrictions, there are $\frac{8!}{2!\,2!\,2!} = 5040$ ways of arranging the letters. **[1 mark]**

b. Consider the 2 C's together as one character. There are $\frac{7!}{2!\,2!} = 1260$ combinations that have consecutive C's.
Therefore, the number of combinations in which the C's are separated is $5040 - 1260 = 3780$. **[1 mark]**

Question 4

The colours can be arranged in 2 ways: RGB or BGR. **[1 mark]**

$$2 \times {}^8C_1 \times {}^7C_1 \times {}^5C_1 = 2 \times 8 \times 7 \times 5 \text{ [1 mark]}$$
$$= 560$$

Question 5

$${}^5P_3 = \frac{5!}{(5-3)} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

Question 6

The number of ten–digit mobile phone numbers that can be formed using the digits of $0, 1, 2, \ldots, 9$ is 10^{10}
The number of ten–digit mobile phone numbers that have none of their digits repeated is
${}^{10}P_{10} = 10! = 3628800$.
Hence the required number $= 10^{10} - 3628800 = 9\,996\,371\,200$. **[1 mark]**

Question 7

The word PERMUTATION contains 11 letters of which there are 2 Ts.

$$\frac{11!}{2!} = 19\,958\,400$$

There are 19 958 400 arrangements. **[1 mark]**
For the letters T to be together, we treat these two letters as one object. This creates 10 groups: (TT), P, E, R, M, U, A, I, O, N.
The 10 groups can be arranged in 10! ways.

$$\begin{aligned} P(\text{That all the Ts are together}) &= \frac{\text{number of arrangemens with the Ts together}}{\text{total number of arrangements}} \\ &= \frac{10!}{\left(\frac{11!}{2!}\right)} \\ &= \frac{10! \times 2!}{11!} \\ &= \frac{2}{11} \text{ [1 mark]} \end{aligned}$$

1.5 Combinations

Question 1

Two of the marbles are to be purple, so we have to choose 2 out of the three purple: ${}^3C_2$
The other two marbles will be green, so we have to choose 2 out of the six green: ${}^6C_2$
Total number of ways $= {}^6C_2 \times {}^3C_2$

Question 2

There are 6 ingredients and customers can choose 1, 2, 3, 4, 5 or 6 of these ingredients.

$$\begin{aligned} \text{The number of sandwiches possible} &= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6 \\ &= 63 \text{ [1 mark]} \end{aligned}$$

Question 3

From 12 forwards choose 8 and from 10 backs choose 7.

$$
\begin{aligned}
{}^{12}C_8\,{}^{10}C_7 &= \frac{12!}{(12-8)!\,8!} \times \frac{10!}{(10-7)!\,7!} \\
&= \frac{12!}{4!\,8!} \times \frac{10!}{3!\,7!} \textbf{ [1 mark]} \\
&= \frac{12 \times 11 \times 10 \times 9 \times 8!}{8! \times 4 \times 3 \times 2 \times 1} \times \frac{10 \times 9 \times 8 \times 7!}{7! \times 3 \times 2 \times 1} \\
&= 495 \times 120 \\
&= 59\,400 \textbf{ [1 mark]}
\end{aligned}
$$

Question 4

Number of ways of choosing 5 from 6 is ${}^{6}C_5 = 6$ **[1 mark]**

Number of ways of choosing 5 from 20 is ${}^{20}C_5 = 15\,504$ **[1 mark]**

Likelihood (probability) $= \dfrac{6}{15\,504} = \dfrac{1}{2584}$ **[1 mark]**

Question 5

a.

$$
\begin{aligned}
{}^{16}C_4 &= \frac{16!}{4! \times 12!} \\
&= \frac{16 \times 15 \times 14 \times 13 \times \cancel{12!}}{4 \times \cancel{12!}} \\
&= \frac{43680}{24} \\
&= 1820 \textbf{ [1 mark]}
\end{aligned}
$$

b. 3 females (and 1 male) or 4 females

$$
\begin{aligned}
{}^{9}C_3 \times {}^{7}C_1 \times {}^{9}C_4 &= \frac{9!}{3! \times 6!} \times 7 + \frac{9!}{4! \times 5} \textbf{ [1 mark]} \\
&= 588 + 126 \\
&= 714
\end{aligned}
$$

c. Calculations from parts (a) and (b)

$$
\begin{aligned}
\text{Pr (only 1 male)} &= \frac{\text{n (3 females + 1 male)}}{\text{total number of panels}} \textbf{ [1 mark]} \\
&= \frac{588}{1820} \\
&= 0.3231 \textbf{ [1 mark]}
\end{aligned}
$$

Question 6

$$
\begin{aligned}
{}^{n}C_2 &= 10 \\
\frac{n!}{2!\,(n-2)!} &= 10 \textbf{ [1 mark]} \\
\frac{n!}{(n-2)!} &= 10 \times 21 \\
\frac{n \times (n-1) \times (n-2) \times (n-3) \ldots \times 1}{(n-2) \times (n-3) \times \ldots \times 1} &= 20 \\
n\,(n-1) &= 20 \\
n^2 - n - 20 &= 0 \textbf{ [1 mark]} \\
(n-5)\,(n+4) &= 20 \\
n &= 5 \text{ or } -4
\end{aligned}
$$

Only the positive solution makes sense in this situation, so $n = 5$ **[1 mark]**

1.6 Applications of permutations and combinations

Question 1

Total number of quinellas = ${}^{24}C_2 = 276$
At $0.50 a bet it would cost you $138.
($\$0.50 \times 276 = \$138$). **[1 mark]**

Question 2

If a number is divisible by 10, its units place should contain a 0: _ _ _0
After 0 is placed in the units column, the tens place can be filled with any of the other 5 digits.
Selecting one digit out 5 digits can be done in ${}^{5}C_1 = 5$ ways. After filling the tens place, we are left with 4 digits. Selecting 1 digits can be done in ${}^{4}C_1 = 4$ ways.
After filling the hundreds place, the thousands place can be filled in ${}^{3}C_1 = 3$ ways. **[1 mark]**
Therefore, the total combinations possible = $5 \times 4 \times 3 = 60$.

Question 3

There are ${}^{45}C_7 = 45\,379\,620$ possible combination. At $1.45 per game, it would cost you $45\,379\,620 \times \$1.45 = \$65\,800\,449$ to ensure you won first prize. **[1 mark]**

Question 4

Preface: There are 7 possible numbers that can appear on one half of a tile; $\{0, 1, 2, 3, 4, 5, 6\}$
There are ${}^{7}C_2$ ways to select a domino in which the ends have different numbers of spots and ${}^{7}C_1$ ways to select a domino in which the two ends have the same number of spots.
Hence,

$$\begin{aligned}\text{The number of dominoes in a set} &= {}^{7}C_2 + {}^{7}C_1 \ \textbf{[1 mark]}\\ &= 21 + 7\\ &= 28 \ \textbf{[1 mark]}\end{aligned}$$

Question 5

If a number is divisible by 10, its units place should contain a 0. _ _ _ 0
After 0 is placed in the units column, the tens place can be filled with any of the other 5 digits.
Selecting one digit out of 5 digits can be done in ${}^{5}C_1 = 5$ ways.
After filling the tens place, we are left with 4 digits. Selecting 1 digit out of 4 digits can be done in ${}^{4}C_1 = 4$ ways.
After filling the hundreds place, the thousands place can be filled in ${}^{3}C_1 = 3$ ways.
Therefore, the total combinations possible = $5 \times 4 \times 3 = 60$ **[1 mark]**

Question 6

Order is not important, so this is a selection problem. There are three mutually exclusive possibilities:
2 aces and three non-aces
3 aces and two non-aces
4 aces and one non-ace.

$$\Pr(2\text{ aces, } 3\text{ non} - \text{aces}) + \Pr(3\text{ aces, } 2\text{ non} - \text{aces}) + \Pr(4\text{ aces, } 1\text{ non} - \text{aces}) = \frac{{}^{4}C_2 \times {}^{48}C_3}{{}^{32}C_5} + \frac{{}^{4}C_3 \times {}^{48}C_2}{{}^{32}C_5} + \frac{{}^{4}C_4 \times {}^{48}C_1}{{}^{32}C_5} \ \textbf{[1 mark]}$$

$$= 0.04 \ \textbf{[1 mark]}$$

1.7 Pascal's triangle and the pigeon-hole principle

Question 1

There are 8 states/territories. $n = 8$

1 400 000 people watched the news.

$1\,400\,000 = 8 \times 174\ 999 + 8 > 8 \times 174\ 999 + 1$, $k = 174\,999$

Therefore, at least $k + 1 = 175\,000$ people from the same state watched the news.

Question 2

There are 10 lanes in the pool. $n = 10$

There are at least 21 people in the pool.

$21 = 10 \times 2 + 1 = nk + 1$, $k = 2$

Therefore, there is at least 1 lane with $k + 1 = 3$ people in it. **[1 mark]**

Question 3

Uniform options:

short-sleeve + shorts

short-sleeve + pants

long-sleeve + shorts

long-sleeve + pants

short-sleeve + shorts + jumber

short-sleeve + pants + jumber

long-sleeve + shorts + jumber

long-sleeve + pants + jumber

There are 8 uniform options. $n = 8$ **[1 mark]**

There are 393 students.

$$\begin{aligned} 393 &= 8 \times 49 + 1 \\ &= nk + 1\ (k = 39) \end{aligned}$$

Therefore, by the pigeonhole principal, there must be at least $k + 1 = 50$ students wearing the same uniform. **[1 mark]**

Question 4

Identify the relevant row in Pascal's triangle. Six objects, so seventh row.

1 6 15 20 15 6 1

Three at a time, so the fourth element.

The answer is 20. **[1 mark]**

Question 5

Identify the relevant row in Pascal's triangle. 7 objects, so eighth row.

1 7 21 35 35 21 7 1

Three at a time, so the fourth element.

The answer is 35. **[1 mark]**

Question 6

Ninth row is 1 8 28 56 70 56 28 8 1

The middle number is 70. **[1 mark]**

Question 7

Term 1 = 1

$$\begin{aligned}\text{Term 2} &= 1 + 25\\ &= 26\\ \text{Term 3} &= 25 + 300\\ &= 325\\ \text{Term 4} &= 300 + 2300\\ &= 2600\\ \text{Term 5} &= 2300 + 12650\\ &= 14950\\ \text{Term 6} &= 12650 + 53130\\ &= 65780\end{aligned}$$

Award **1 mark** for all terms correct.

Question 8

The horizontal sums double each time. **[1 mark]**

Question 9

There are 3 age groups. $n = 3$

There are 46 dancers. $46 = 3 \times 15 + 1 = nk + 1,\ \ k = 15$ **[1 mark]**

Therefore, there is at least 1 age group with 16 or more dancers. **[1 mark]**

Question 10

There are 10 lanes in the pool. $n = 10$

There are at least 21 people in the pool.

$21 = 10 \times 2 + 1 = nk + 1,\ \ k = 2$ **[1 mark]**

Therefore, there is at least 1 lane with $k + 1 = 3$ people in it. **[1 mark]**

1.8 Review

Question 1

a. The word DIFFERENTIAL has 12 letters, with 2 I's, 2 F's and 2 E's.

With no restrictions there are $\dfrac{12!}{2!\,2!\,2!} = 59\,875\,200$ possible combinations. **[1 mark]**

b. Consider the 2 F's together as one character. There are $\dfrac{11!}{2!\,2!} = 9\,979\,200$ combinations that have consecutive F's.

Therefore, the number of combinations in which F's are separated is

$59\,875\,200 - 9\,979\,200 = 49\,896\,000$. **[1 mark]**

Question 2

There are 5 letters of which there are two doubles. $\dfrac{5!}{2!\,2!} = 30$

Question 3

a. To play in a particular position means order is important.

This is a permutation.

${}^{11}P_7 = 1\,663\,200$ **[1 mark]**

b. If position is not important, this is a combination.

${}^{11}C_7 = 330$ **[1 mark]**

Question 4

a. In a group of 10 people with no restrictions, the number of arrangements is $10! = 3\,628\,800$ **[1 mark]**

b. Alternating MW or WM, there are $2 \times 5! \times 5! = 28\,800$ possible arrangements. **[1 mark]**

c. The number of arrangements within the group of men is $5! = 120$. The 5 women and the group of men can be thought of as an arrangement of 6 groups. Therefore, the number of possible arrangements with all men standing next to each other is $6! \times 5! = 86\,400$ **[1 mark]**

Question 5

$(ax+b)^4 = a^4x^4 + 4a^3bx^3 + 6a^2b^2x^2 + 4ab^3x + b^4$

x^3: $\quad -540 = 4a^3b \quad [1] \quad a^3b = -135$

x: $\quad -1500 = 4ab^3 \quad [2] \quad ab^3 = -375$ **[1 mark]**

$[2] \quad a = -\dfrac{375}{b^3}$ into $[1]: \left(-\dfrac{375}{b^3}\right)^3 \times b = -135$ **[1 mark]**

$b^8 = 390\,625$

$b = \pm 5,\ a = \pm\dfrac{375}{125} = \pm 3$

$a = -3,\ b = 5$ or $a = 3,\ b = -5$ **[1 mark]**

Question 6

Using combinations, there are 16 pieces of fruit, so there are $\binom{16}{3}$ ways of selecting 3 pieces of fruit.

There are $\binom{9}{1}\binom{4}{1}\binom{3}{1}$ ways of selecting one of each fruit. **[1 mark]**

$$P(1,1,1) = \frac{\binom{9}{1}\binom{4}{1}\binom{3}{1}}{\binom{16}{3}} = 0.1929$$ **[1 mark]**

Question 7

We select the books, making 13 books in all. We then arrange the 13 books.

Select in $\binom{8}{6}\binom{10}{7}$ ways. **[1mark]**

Now arrange. Total number of arrangements is $\binom{8}{6}\binom{10}{7} \times 13!$ **[1mark]**

Question 8

a. Treat the women as a single unit. There are 4 units to arrange (women plus 3 men) in 3! ways.
Then we need to arrange the women within the unit of 4 women in 4! ways.
Arrangements equal $3!\,4! = 144$ways. **[1 mark]**

b. Seat the women in 3! ways spaced around the table (remember $(n-1)!$ Now seat the men in 3! ways in the empty seats.
Arrangements equal $3!\,3! = 36$ ways. **[1 mark]**

Question 9

a. If the thinnest goat and the lone wolf are on the committee we still need 4 goats out of the 8 still available and 1 wolf out of the remaining 5. **[1 mark]**
Number of committees equals $\binom{8}{5}\binom{5}{1} = 350$. **[1 mark]**

b. We need 5 goats out of 8 goats and 2 wolves out of 5 wolves. **[1 mark]**

Number of committees equals $\binom{8}{5}\binom{5}{2} = 560$. **[1 mark]**

Question 10

a. $t_5 = \binom{n}{4} a^{(n-4)} b^4$; $t_6 = \binom{n}{5} a^{(n-5)} b^5$ **[2 marks]**

$\binom{n}{4} a^{(n-4)} b^4 = \binom{n}{5} a^{(n-5)} b^5$ **[2 marks]**

$a = \dfrac{b}{5(n-5)}$ **[1 mark]**

b. $t_{n-1} = \binom{n}{1} a^1 b^{(n-1)}$ **[1 mark]**

$= n \times \dfrac{b}{5(n-5)} \times b^{(n-1)}$ **[1 mark]**

$= \dfrac{n}{5(n-5)} b^n$ **[1mark]**

2 Sequences and series

Topic	2	Sequences and series
Subtopic	2.2	Describing sequences

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (1 mark)

Specify a rule for the sequence 2, 5, 10, 17, 26 ...

$t_n = \square, n \in N$

Question 2 (1 mark)

Determine the next three terms in the sequence $a^2, -a, 1, ...$

$\square$, $\square$ and $\square$

Question 3 (1 mark)

Write down the first five terms in the sequence $t_n = \dfrac{(n+1)^3}{n}$.

$t_1 = \square$

$t_2 = \square$

$t_3 = \square$

$t_4 = \square$

$t_5 = \square$

Question 4 (1 mark)

For a particular sequence $t_1 = 6, t_n = 2n - 1, n > 1$.

List the first six terms of the sequence.

Question 5 (1 mark)

Specify a rule for the sequence $2, 5, 10, 17, 26, \ldots$

Topic	2	Sequences and series
Subtopic	2.3	Arithmetic sequences

online only

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Question 1 (2 marks)

Determine the common difference of the arithmetic sequence given by

$t_n = (n+1)^2 - n(n+3) + 4,\ n \in N$

$d = \square$

Question 2 (1 mark)

In an arithmetic sequence, $t_{16} = 5t_3$ and the common difference is 4. Determine the value of the first term.

$a = \square$

Question 3 (2 marks)

$p,\ q,\ r,\ s$ are four consecutive terms in an arithmetic sequence. Show that $qr - ps = 2(r-s)^2$.

Question 4 (1 mark)

Is the sequence given by $t_n = 2n + 5, n \geq 1$ arithmetic?

Question 5 (2 marks)

What is the common difference of the arithmetic sequence given by

$t_n = (n+1)^2 - n(n+3) + 4, n \in N$?

Question 6 (2 marks)

In an arithmetic sequence, $t_7 = 7$ and $t_{20} = 33$, find the first term and the common difference.

Question 7 (1 mark)

In an arithmetic sequence, $t_{16} = 5t_3$ and the common difference is 4. Find the first term.

Question 8 (2 marks)

Insert five terms between $(3q - 7p)$ and $(-3q - p)$ so that all seven terms form an arithmetic sequence.

Question 9 (1 mark)

Wine bottles are stacked on their side in rows, one row on top of another. There are a bottles on the bottom row and one less bottle on each subsequent row. If there are 71 bottles on the sixth row, then a equals:

A. 64
B. 65
C. 66
D. 76
E. 77

Question 10 (1 mark)
A team of runners sets off. Igor was the first runner and ran 1000 metres. Each other runner in the team ran 500 metres further than the previous runner. Natasha was the final runner, and she ran 6500 metres. The total number of runners in the team was:

A. 12
B. 11
C. 10
D. 9
E. 8

Question 11 (1 mark)
The first three terms of an arithmetic sequence are $-a$, $-3a$, $-5a$, ...
Find an expression for t_{n+1}, the $(n + 1)$th term of this sequence.

A. $-2an$
B. $(2n + 1)$
C. $-(2n + 1)$
D. $(2n + 1) + a$
E. $-(2n + 1)a$

Question 12 (1 mark)
A toy raceway consists of a number of small pieces of raceway that join together. The shortest piece is 250 mm and each of the other pieces increase in length by 30 mm. The total length of the complete toy raceway is 3.85 metres. The number of small pieces is:

A. 4
B. 10
C. 25
D. 26
E. 77

Topic	2	Sequences and series
Subtopic	2.4	Arithmetic series

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (2 mark)

Calculate the sum of all natural numbers between 1 and 100 that are divisible by three.

The sum of all natural numbers between 1 and 100 that are divisible by three is ☐

Question 2 (2 marks)

Calculate the sum of the first 15 terms of the sequence $5, 9, 13, 17, \ldots$

$S_{15} =$ ☐

Question 3 (2 marks)

Consider the sequence $t_n = pn + q$. Calculate the sum of the first 24 terms in the sequence by first showing that the sequence is arithmetic.

$S_{24} =$ ☐

Question 4 (3 marks)

The sum of three consecutive terms of an arithmetic series is 36 and the sum of the next three terms is 45. If the first term is a, find the sum of the first 12 terms in terms of a.

Question 5 (2 marks)

p, q, r, s are four terms of an arithmetic series. Show that $qr - ps = 2(r - s)^2$.

Question 6 (1 mark)

The first three terms of an arithmetic sequence are $10, 13, 16, \ldots$

S_n, the sum of the first n terms of this sequence is:

A. $\frac{n}{2}(3n + 23)$

B. $\frac{n}{2}(n + 23)$

C. $\frac{n}{2}(3n + 17)$

D. $\frac{n}{2}(n + 17)$

E. $\frac{n}{2}(3n + 7)$

Topic	**2**	**Sequences and series**
Subtopic	**2.5**	**Geometric sequences**

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Question 1 (1 mark)

For the geometric sequence $m,\ -\frac{n}{m},\ \frac{n^2}{m^3},\ ...$ the common ration and the tenth term are

A. $-\frac{n}{m}, -\frac{n^9}{m^8}$

B. $-\frac{n}{m}, \frac{n^9}{m^8}$

C. $-\frac{n}{m^2},\ -\frac{n^9}{m^{17}}$

D. $-\frac{n}{m^2}, \frac{n^9}{m^{17}}$

E. $-\frac{n^2}{m^2}, -\frac{n^{18}}{m^{17}}$

Question 2 (1 mark)

Insert one term between 2 and 8 to form a geometric sequence.

The missing term is ☐ or ☐

Question 3 (2 marks)

Determine the number which forms a geometric sequence when added to each of the numbers $11,\ 17,\ 25$.

The number to be added is ☐.

Question 4 (1 mark)

Find the next three terms in the sequence $a^2, -a, 1, ...$

Question 5 (1 mark)
Specify the rule for the sequence $\frac{m}{n}, \ -1, \ \frac{n}{m}, \ldots$

Question 6 (2 marks)
The planet Angloopa has a population of 6.6 billion nargs in the year 2094 AP. If the population of nargs increases at a rate of 10% per Angloopa year (approx. 1.75 Earth years), during which year, AP, will the population have tripled?

Question 7 (1 mark)
The common ratio of the geometric sequence $2, \ -6, \ 18, \ldots$ is:
A. $\frac{1}{3}$
B. $-\frac{1}{3}$
C. 3
D. -3
E. -3.333

Question 8 (1 mark)
The first term of a geometric sequence is p and the common ratio is also p. If the nth term is n, then p, expressed in terms of n, is:
A. n^n
B. $\sqrt[n]{n-1}$
C. $\sqrt[(n-1)]{n-1}$
D. $\sqrt[(n-1)]{n}$
E. $\sqrt[n]{n}$

Question 9 (1 mark)

Two geometric sequences S_1 and S_2 are such that $a_1 = pa_2$ and $r_1 = qr_2$. The ratio of the seventh term of S_1 to the seventh term of S_2 is:

A. $\frac{p}{q}$

B. pq^6

C. $\frac{p}{q^6}$

D. $\frac{q}{p^6}$

E. $(pq)^6$

Question 10 (1 mark)

The length of a fish in a pond increases by 5% per month until its length has doubled. The number of months the length takes to double is closest to:

A. 8

B. 10

C. 14

D. 15

E. 20

Question 11 (1 mark)

A Snurk living on the planet Onguardo decreases its body weight by 10% every Onguardo year (approximately 5.2 Earth years) ultimately reaching its ideal weight. Initially its weight was 5 krongs (1 krong = 0.425 kg). Find the Snurk's ideal weight.

Topic	2	Sequences and series
Subtopic	2.6	Geometric series

online only

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Question 1 (1 mark)

Calculate the sum of the first 10 terms of the geometric series $-2, \ +4, \ -8, \ +15, \ -32, ...$

$S_{10} = \square$

Question 2 (1 mark)

The sum of the first 12 terms of a geometric sequence is 12 285. If the first term is 3, determine the common ratio.

$r = \square$

Question 3 (3 marks)

For a particular geometric sequence $S_{2n} = 5S_n$. Show that $r = \sqrt[n]{4}$.

Question 4 (1 mark)

Find the sum of n terms of the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

Question 5 (1 mark)

Find the sum of the first 10 terms of the geometric series $-2, +4, -8, +16, -32, ...$

Question 6 (2 marks)
The first term of a geometric series is 4 and the common ratio is 2. Find the sum of the second 10 terms, i.e. t_{11} to t_{20}.

Question 7 (1 mark)
The sum of the first 12 terms of a geometric sequence is 12 285. If the first term is 3, find the common ratio.

Question 8 (1 mark)
Find S_∞ for the geometric series 4, 2, 1, ...

Question 9 (2 marks)
The nth term of a geometric series is $6(0.25)^n$. Find the sum to infinity.

Question 10 (1 mark)
Explain why some geometric series have a sum to infinity and some do not.

Question 11 (3 marks)

For a particular geometric series where $S_\infty = mS_p$, show that $r = \sqrt[p]{\dfrac{m-1}{m}}$, where r is the common ratio.

Question 12 (1 mark)

The first four terms of a geometric sequence are $3, -9, 27, -81$.

The sum of the first 10 terms of this sequence is:

A. −44 286
B. 44 286
C. −59 049
D. 597 049
E. 177 147

Topic	2	Sequences and series
Subtopic	2.7	Applications of sequences and series

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Question 1 (2 marks)

Using compound interest, calculate the future value of $1000 invested for 10 years 8% per annum.

Value = $ ☐ to the nearest cent

Question 2 (2 marks)

The planet Angloopa has a population of 6.6 billion nargs in the year 2094 AP. If the population of nargs increases at a rate of 10% per Angloopa year (approx. 1.75 Earth years), determine during which year, AP, the population will have tripled.

During year ☐

Question 3 (3 marks)

The number of koalas K_n on a plantation at the start of the nth year is given by $K_{n+1} = 0.98K_n - 5$ and at the start of the first year there were 200 koalas. (*Note*: round all decimal answers down to the nearest integer when stating answers.)

a. Determine the number of koalas on the plantation at the start of the fourth year. **(1 mark)**

There are ☐ koalas at the start of the fourth year.

b. The number of koalas at the start of the nth year is given by the formula $K_n = 450 \times 0.98^{n-1} - 250$. Use this formula to determine the year in which the number of koalas falls below 100.

During the ☐th year **(2 marks)**

Topic	2	Sequences and series
Subtopic	2.8	Review

online only

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Question 1 (1 mark)

A pendulum starts its swing through an angle of 60° from one extreme to the other.
Each oscillation of the pendulum swings out an angle which is 60% of the preceding oscillation.
Determine the angle that the pendulum swings through in its 10th oscillation.
Angle = $\square$°, to two decimal places

Question 2 (2 marks)

Two friends, Thanh and Maria, start work at Blogg Brothers Emporium. Initially they are paid the same annual salary of $25 000.
Thanh's salary increases at the end of each year at the flat rate of $1250 per annum while Maria's salary increases at the end of each year at 4% per annum.
Determine who earns the greater salary at the end of the sixth year.

Question 3 (2 marks)

The nth term of a geometric series is $6(0.25)^n$. Calculate the sum to infinity.
$S_\infty = \square$

Question 4 (2 marks)

A ball drops from a height of 10 metres. After bouncing, it reaches 70% of its original height.
Assuming a geometric sequence, determine on which bounce the ball passes a rebound height of 1 metre for the last time.
On the $\square$th bounce.

Question 5 (3 marks)

The sum of three consecutive terms of an arithmetic series is 36 and the sum of the next three terms is 45. If the first term is a, calculate the sum of the first 12 terms in terms of a.

$S_{12} = \square$

Question 6 (3 marks)

Explain why the sequence $a, a^2, a^3, a^4, a^5, \ldots$ is not arithmetic.

Find the nth term of the sequence.

Calculate $t_{n+1} - t_n$.

Question 7 (1 mark)

\$20 000 is invested at 7% compound interest for 5 years. What is the total value of the investment at the end of 5 years?

Question 8 (8 marks)

Answer the following.

a. For a geometric series, the sum to infinity is four times the sum of the first two terms. Find the common ratio. **(2 marks)**

b. Show that for any geometric series, if $S_{2n} = 3S_n$, then $r = \sqrt[n]{2}$. **(3 marks)**

c. Show that for any geometric series, if $S_{2n} = qS_n,\ q > 0,\ $ then $r = \sqrt[n]{(q-1)}$. **(3 marks)**

Answers and marking guide

2.2 Describing sequences

Question 1

Each term is one greater than a square. That is, $2 = 1^2 + 1$, $5 = 2^2 + 1$, and so on.

$t_n = n^2 + 1,\ n \in N$ **[1 mark]**

Question 2

Divide each term by $-a$

$a^2, -a, 1, -\frac{1}{a}, \frac{1}{a^2}, -\frac{1}{a^3}, \ldots$ **[1 mark]**

Question 3

Substitute $n = 1$, $n = 2$ and so on.

$t_1 = \frac{(1+1)^3}{1} = 2^3 = 8$

$\therefore 8, \frac{27}{2}, \frac{64}{3}, \frac{125}{4}, \frac{216}{5}, \ldots$ **[1 mark]**

Question 4

$t_1 = 6$

$t_2 = 2 \times 2 - 1 = 3$

$t_3 = 2 \times 3 - 1 = 5$

$t_4 = 2 \times 4 - 1 = 7$

$t_5 = 2 \times 5 - 1 = 9$

$t_6 = 2 \times 6 - 1 = 11$ **[1 mark]**

Question 5

Each term is one greater than a square. That is, $2 = 1^2 + 1$, $5 = 2^2 + 1$, etc.

$t_n = n^2 + 1,\ n \geq 1$ **[1 mark]**

2.3 Arithmetic sequences

Question 1

$t_n = (n+1)^2 - n(n+3) + 4$

$= n^2 + 2n + 1 - n^2 - 3n + 4$

$= -n + 5$ **[1 mark]**

$t_1 = 4,\ t_2 = 3 \Rightarrow d = -1$ **[1 mark]**

Question 2

$t_{16} = 5t_3$

$a + 15d = 5a + 10d$

$4a = 5d$

Given $d = 4 \Rightarrow 4a = 20$

$\therefore a = 5$ **[1 mark]**

Question 3

$qr - ps = 2(r - s)^2$

$\text{LHS} = (p + d)\ (p + 2d) - p\ (p + 3d)$

$= 2d^2$ **[1 mark]**

$\text{RHS} = 2(r - s)^2$

$= 2(-d)^2$

$= 2d^2$

$\therefore \text{LHS} = \text{RHS}$ **[1 mark]**

Question 4

$t_{n+1} - t_n = 2\,(n + 1) + 5 - (2n + 5) = 2 \Rightarrow$ arithmetic, $\quad d = 2$ **[1 mark]**

Question 5

$t_n = (n + 1)^2 - n\,(n + 3) + 4$

$= n^2 + 2n + 1 - n^2 - 3n + 4$

$= -n + 5$ **[1 mark]**

$t_1 = 4,\ t_2 = 3 \Rightarrow d = -1$ **[1 mark]**

Question 6

$a + 6d = 7$

$a + 19d = 33$ **[1 mark]**

Solving simultaneously.

$a = -5,\ d = 2$ **[1 mark]**

Question 7

$t_{16} = 5t_3$

$a + 15d = 5a + 10d$

$4a = 5d$

Given $d = 4 \Rightarrow 4a = 20$

$\therefore a = 5$ **[1 mark]**

Question 8

$a = 3q - 7p$

$a + 6d = -3q - p$

$d = -q + p$ **[1 mark]**

Sequence is:

$(3q - 7p)\,, (2q - 6p)\,, (q - 5p)\,, (-4p)\,, (-q - 3p)\,, (-2q - 2p)\,, (-3q - p)$ **[1 mark]**

Question 9

$t_n = a + (n - 1)d$

$t_6 = a + (6 - 1) \times -1$

$= 71$

$a = 76$

Question 10

$6500 = 1000 + 500(n - 1)$

$5500 = 500(n - 1)$

$n - 1 = 11$

$\therefore n = 12$

Question 11

$t_{n+1} = -a + (n + 1 - 1) \times -2a$

$= -(2n + 1)a$

Question 12

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$3850 = \frac{n}{2}[2 \times 250 + (n-1) \times 30]$$
$$7700 = 470n + 30n^2$$
$$3n^2 + 47n - 770 = 0$$
$$(3n + 77)(n - 10) = 0$$
$$n = 10$$

2.4 Arithmetic series

Question 1

Arithmetic series, $a = 3,\ d = 3,\ l = 99$. **[1 mark]**

To find n:

$l = a + (n-1)\,d \Rightarrow n = 33$ **[1 mark]**

$$S_{33} = \frac{33}{2}[2 \times 3 + (33 - 1) \times 3]$$
$= 1683$ **[1 mark]**

Question 2

Arithmetic, $a = 5,\ d = 4$ **[1 mark]**

$$S_{15} = \frac{15}{2}[2 \times 5 + (15 - 1) \times 4]$$
$= 495$ **[1 mark]**

Question 3

$$t_n = pn + q$$
$$t_{n+1} - t_n = p(n + 1) + q - (pn + q) = p$$

Arithmetic $d = p$

$$a = t_1$$
$= p + q$ **[1 mark]**

$$S_{24} = \frac{24}{2}[2\,(p + q) + 23p]$$
$= 12(25p + 2q)$ **[1 mark]**

Question 4

Three consecutive terms are $(p - d),\ p, (p + d)$

$(p - d) + p + (p + d) = 36 \Rightarrow p = 12$ **[1 mark]**

Next three terms are $(12 + 2d), (12 + 3d), (12 + 4d)$

$(12 + 2d) + (12 + 3d) + (12 + 4d) = 45 \Rightarrow d = 1$ **[1 mark]**

$$S_{12} = \frac{12}{2}[2 \times a + (12 - 1) \times 1]$$
$= 12a + 66$ **[1 mark]**

Question 5

$qr - ps = 2(r - s)^2$

$\text{LHS} = (p + d)\,(p + 2d) - p\,(p + 3d)$

$= 2d^2$ **[1 mark]**

$$\begin{aligned}\text{RHS} &= 2(r-s)^2 \\ &= 2(-d)^2 \\ &= 2d^2 \text{ [1 mark]}\end{aligned}$$

$\therefore$ LHS = RHS

Question 6

$$\begin{aligned}S_n &= \frac{n}{2}[2a+(n-1)d] \\ &= \frac{n}{2}[2\times 10+(n-1)\times 3] \\ &= \frac{n}{2}(3n+17)\end{aligned}$$

2.5 Geometric sequences

Question 1

$$\begin{aligned}r &= -\frac{n}{m} \div m \\ &= -\frac{n}{m^2}\end{aligned}$$

$$\begin{aligned}t_{10} &= ar^9 \\ &= m\left(-\frac{n}{m^2}\right)^9 \\ &= -\frac{n^9}{m^{17}}\end{aligned}$$

Question 2

Let sequence be $2,\ m,\ 8$

$\frac{8}{m} = \frac{m}{2} \Rightarrow m^2 = 16 \Rightarrow m \pm 4.$

Either 4 or -4 can be inserted. **[1 mark]**

Question 3

$(11+b), (17+b), (25+b)$ form a geometric sequence.

$$\begin{aligned}\frac{t_2}{t_1} &= \frac{t_3}{t_2} \\ \frac{17+b}{11+b} &= \frac{25+b}{17+b} \\ (17+b)^2 &= (11+b)(25+b) \text{ [1 mark]} \\ 34b+289 &= 36b+275 \\ b &= 7 \text{ [1 mark]}\end{aligned}$$

Question 4

Divide each term by $-a$.

$a^2, -a, 1, -\frac{1}{a}, \frac{1}{a^2}, -\frac{1}{a^3}, \ldots$ **[1 mark]**

Question 5

Multiply by $-\frac{n}{m}$ to get the next term; $\frac{m}{n}, -1, \frac{n}{m}, -\frac{n^2}{m^2}, \frac{n^3}{m^3}, \ldots$ **[1 mark]**

Question 6

$$3 \times 6.6 = 6.6(1 + 0.1)^n \text{ } \textbf{[1 mark]}$$
$$1.1^n = 3$$
$$n = 11.53$$

Year is 2094 + 11.53 = 2105th year. **[1 mark]**

Question 7

$$r = \frac{-6}{2} = \frac{18}{-6} = -3$$

Question 8

$$t_n = p \times p^{(n-1)}$$
$$= p^n$$
$$= n$$
$$\Rightarrow p = \sqrt[n]{n}$$

Question 9

$$\frac{a_1(r_1)^6}{a_2(r_2)^6} = \frac{pa_2(qr_2)^6}{a_2(r_2)^6}$$
$$= pq^6$$

Question 10

$$t_n = 2a$$
$$= a(1.05)^n$$
$$(1.05)^n = 2$$
$$n = \frac{\log_e(2)}{\log_e(1.05)}$$
$$= 14.21 \text{ months}$$

Question 11

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{5}{1-0.1}$$
$$= 4.5 \text{ krongs } \textbf{[1 mark]}$$

2.6 Geometric series

Question 1

$a = -2,\ r = -2$

$$S_{10} = \frac{-2\left(1-(-2)^{10}\right)}{1-(-2)}$$
$$= 682 \text{ } \textbf{[1 mark]}$$

Question 2

$$S_{12} = \frac{3 \times \left(1 - r^{12}\right)}{1 - r}$$
$$= 12\,285$$

$$\frac{\left(1 - r^{12}\right)}{1 - r} = 4095$$

Solving, $r = 2$ **[1 mark]**

Question 3

$$S_{2n} = 5S_n$$
$$\frac{a\left(r^n - 1\right)}{r - 1} = \frac{5a\left(r^n - 1\right)}{r - 1} \text{ [1 mark]}$$
$$\frac{a\left(r^n - 1\right)\left(r^n - 1\right)}{r - 1} = \frac{5a\left(r^n - 1\right)}{r - 1} \text{ [1 mark]}$$
$$r^n + 1 = 5$$
$$r = \sqrt[n]{4} \text{ [1 mark]}$$

Question 4

$a = 1,\ r = \frac{1}{2}$

$$S_n = \frac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}}$$
$$= 2 - \left(\frac{1}{2}\right)^{n-1} \text{ [1 mark]}$$

Question 5

$a = -2,\ r = -2$

$$S_{10} = \frac{-2\left(1 - (-2)^{10}\right)}{1 - (-2)}$$
$$= 682 \text{ [1 mark]}$$

Question 6

$$S_{20} - S_{10} = \frac{4\left(2^{20} - 1\right)}{2 - 1} - \frac{4\left(20^{10} - 1\right)}{2 - 1} \text{ [1 mark]}$$
$$= 4\,190\,208 \text{ [1 mark]}$$

Question 7

$$S_{12} = \frac{3 \times \left(1 - r^{12}\right)}{1 - r}$$
$$= 12\,285$$

$$\frac{\left(1 - r^{12}\right)}{1 - r} = 4095$$

Solving, $r = 2$ **[1 mark]**

Question 8

$$a = 4, \ r = \frac{1}{2}$$
$$S_\infty = \frac{a}{1-r}$$
$$= \frac{4}{1-\frac{1}{2}}$$
$$= 8$$ **[1 mark]**

Question 9

$$t_n = 6(0.25)^n$$
$$a = t_1 = 6(0.25)^1 = 1.5, \ r = 0.25$$ **[1 mark]**
$$S_\infty = \frac{a}{1-r}$$
$$= \frac{1.5}{1-0.25}$$
$$= 2$$ **[1 mark]**

Question 10

For a sum to infinity to exist each subsequent term must be numerically smaller than the previous term. This only happens if $|r| < 1$. **[1 mark]**

Question 11

$$S_\infty = mS_p$$
$$\frac{a}{1-r} = m\frac{a(1-r^p)}{1-r}$$ **[1 mark]**
$$1 = m(1-r^p)$$
$$r^p = \frac{m-1}{m}$$ **[1 mark]**
$$r = \sqrt[p]{\frac{m-1}{m}}$$ **[1 mark]**

Question 12

$$S_{10} = \frac{a(1-r^n)}{1-r}$$
$$= \frac{3\left(1-(-3)^{10}\right)}{1-(-3)}$$
$$= -44\,286$$

2.7 Applications of sequences and series

Question 1

Value $= \$1000 \times 1.08^{10}$ **[1 mark]**

$= \$2158.92$ **[1 mark]**

Question 2

$$3 \times 6.6 = 6.6(1+0.1)^n$$
$$1.1^n = 3$$
$$n = 11.53$$ **[1 mark]**

Year is $2094 + 11.53 = 2105$th year **[1 mark]**

Question 3

a. $K_{n+1} = 0.98K_n - 5,\ K_1 = 200$

$K_2 = 0.98K_1 - 5$
$K_2 = 0.98 \times 200 - 5$
$K_2 = 191$

There are 191 koalas at the start of the second year. **[1 mark]**

$K_3 = 0.98K_2 - 5$
$K_3 = 0.98 \times 191 - 5$
$K_3 = 182.18$

There are 182 koalas at the start of the third year. **[1 mark]**

$K_4 = 0.98K_3 - 5$
$K_4 = 0.98 \times 182.18 - 5$
$K_4 = 173.54$

There are 173 koalas at the start of the fourth year. **[1 mark]**

b. $K_n = 450 \times 0.98^{n-1} - 250$

To determine when the number of koalas reaches 100, solve the following equation for n using your CAS calculator.

$100 = 450 \times 0.98^{n-1} - 250$

The solution is $n = 13.4396$ **[1 mark]**

$K_{13} = 450 \times (0.98)^{12} - 250 = 103.12$

$K_{14} = 450 \times (0.98)^{13} - 250 = 96.06$

The number of koalas will fall below 100 during the 13th year. **[1 mark]**

2.8 Review

Question 1

$\text{Angle} = 60° \times (0.6)^9$
$= 0.60°$ **[1 mark]**

Question 2

Thanh : $\$25\,000 + 6 \times \$1250 = \$32\,500$ **[1 mark]**

Maria : $\$25\,000(10.4)^6 = \$31\,632.98$ **[1 mark]**

Thanh earns a greater salary at the end of the sixth year.

Question 3

$t_n = 6(0.25)^n$

$a = t_1 = 6(0.25)^1 = 1.5,\ r = 0.25$ **[1 mark]**

$$S_\infty = \frac{a}{1-r} = \frac{1.5}{1-0.25} = 2$$ **[1 mark]**

Question 4

$a = 7,\ r = 0.7$ [1 mark]

t_n height ≥ 1

$1 \leq 7 \times (0.7)^n$

$\Rightarrow n = 5.456$

That is, on the 6th bounce **[1 mark]**

Question 5

Three consecutive terms are $(p-d),\ p,\ (p+d)$

$(p-d)+p+(p+d)=36 \Rightarrow p=12$ **[1 mark]**

Next three terms are $(12+2d),\ (12+3d),\ (12+4d)$

$(12+2d)+(12+3d)+(12+4d)=45 \Rightarrow d=1$ **[1 mark]**

$$S_{12}=\frac{12}{2}\left[2\times a+(12-1)\times 1\right]$$
$$=12a+66$$ **[1 mark]**

Question 6

In this instance, each term is multiplied by a constant term to find the next term. For an arithmetic sequence, a constant is added. **[1 mark]**

$t_n=a^n$ **[1 mark]**

$t_{n+1}-t_n=a^{n+1}-a^n=a^n(a-1)$ **[1 mark]**

Question 7

Value $=\$20\,000(1.07)^5$

$=\$28\,051.03$ **[1 mark]**

Question 8

a. $\dfrac{a}{1-r}=4\dfrac{a\left(1-r^2\right)}{1-r}$ **[1 mark]**

$$1-r^2=\frac{1}{4}$$
$$r^2=\frac{3}{4}$$
$$r=\pm\frac{\sqrt{3}}{2}$$ **[1 mark]**

b. $\dfrac{a\left(1-r^{2n}\right)}{1-r}=\dfrac{3a\left(1-r^n\right)}{1-r}$ **[1 mark]**

$$1-r^{2n}=3\left(1-r^n\right)$$
$$\left(1-r^n\right)\left(1+r^n\right)=3\left(1-r^n\right)$$ **[1 mark]**

$$r=\sqrt[n]{2}$$ **[1 mark]**

c. $\dfrac{a\left(1-r^{2n}\right)}{1-r}=\dfrac{qa\left(1-r^n\right)}{1-r}$ **[1 mark]**

$$1-r^{2n}=q\left(1-r^n\right)$$
$$\left(1-r^n\right)\left(1+r^n\right)=q\left(1-r^n\right)$$ **[1 mark]**

$$r=\sqrt[n]{(q-1)}$$ **[1 mark]**

3 Logic and algorithms

Topic	3	Logic and algorithms
Subtopic	3.2	Statements (propositions), connectives and truth tables

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Question 1 (1 mark)

Complete the truth table for the compound statement $p \wedge \neg p$.

Question 2 (1 mark)

p and q, or p and r can be written as

A. $(p \wedge q) \wedge (p \wedge r)$

B. $(p \wedge q) \vee (p \wedge r)$

C. $(p \vee q) \wedge (p \vee r)$

D. $(p \vee q) \vee (p \vee r)$

E. $(p \wedge q) \neg (p \wedge r)$

Question 3 (2 marks)

Determine the missing connectives between p, q and r in the last column of the following truth table.

p	q	r	p?q?r
T	T	T	T
T	T	F	T
T	F	T	T
F	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T
F	F	F	F

p ☐ q ☐ r

Question 4 (1 mark)
Is the following sentence a statement?
'When the engine fails, a yacht can always revert to sail power.'
If not, what can be added to create a statement?
Is the statement created true?

Question 5 (1 mark)
Construct the truth table for $p \wedge q$.

Question 6 (1 mark)
Construct the truth table for $p \vee \neg q$.

Question 7 (2 marks)
Use truth tables to establish if $(\neg p \Rightarrow q) \Leftrightarrow (p \Rightarrow \neg q)$.

Topic 3 | Subtopic 3.2 Statements (propositions), connectives and truth tables

Question 8 (1 mark)
Which of the following is a negation of 'All plongs are umple'?
A. No plongs are umple.
B. All plongs are not umple.
C. No plongs are not umple.
D. Some plongs are umple.
E. Some plongs are not umple.

Question 9 (1 mark)
Prove $(p \vee \neg p)$ to be a tautology.

Question 10 (2 marks)
Use a truth table to determine if $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$.

Question 11 (2 marks)
Prove by counter example that the statement 'the cube root of x^3 is $\pm x$' is false.

Topic 3 | Subtopic 3.2 Statements (propositions), connectives and truth tables

Question 12 (1 mark)
This is the truth table for:

T	T	T
T	F	F
F	T	F
F	F	F

A. $p \vee q$
B. $p \wedge q'$
C. $p' \wedge q'$
D. $p' \wedge q$
E. $p \wedge q$

Question 13 (1 mark)
This is the truth table for:

T	T	T
T	F	T
F	T	F
F	F	T

A. $p \vee q$
B. $p \wedge q'$
C. $p' \wedge q'$
D. $p \vee \neg q$
E. $\neg p \vee q$

Question 14 (1 mark)
Where the universal set is the set of real numbers, R, a tautology is true for:
A. $x \in R^+ \cup \{0\}$
B. $x \in R^+$
C. $x \in R^- \cup \{0\}$
D. $x \in R^-$
E. $x \in R$

Question 15 (1 mark)
$(P \vee Q) \wedge \neg Q$ is equivalent to:
A. $P \wedge \neg Q$
B. $P \vee \neg Q$
C. 1
D. 0
E. P

Question 16 (2 marks)
Determine by using truth tables if $(p \Rightarrow q) \Leftrightarrow (q \Rightarrow \neg p)$.

Topic	3	Logic and algorithms
Subtopic	3.3	Valid and invalid arguments

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Question 1 (2 marks)

Examine the validity of the following argument.

The supplement of an obtuse angle is an acute angle.

Angle A is not obtuse.

Hence, the supplement of angle A is not acute.

The argument is ☐.

Question 2 (2 marks)

Use a truth table to determine whether $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$.

The statements are ☐.

Question 3 (2 marks)

Use tautology to verify the validity or otherwise of the following *modus tollens* argument.

$\neg p \rightarrow \neg q$

$\underline{q \qquad\quad}$

p

Question 4 (2 marks)
Use tautology to verify the validity or otherwise of the following *modus tollens* argument:

$\neg p \Rightarrow \neg q$

q

p

Question 5 (2 marks)
Determine if the following statement is a tautology: 'If it rained yesterday, today is Tuesday'.

Topic	**3**	**Logic and algorithms**
Subtopic	**3.4**	**Boolean algebra and digital logic**

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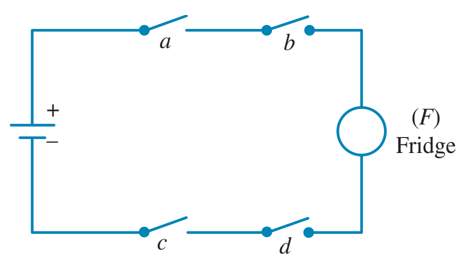
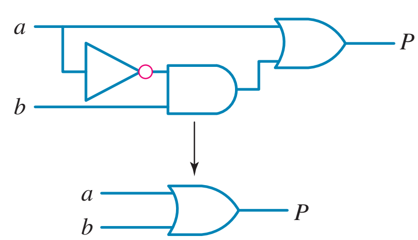
Question 1 (2 marks)
Using the laws of Boolean algebra, show that $A + A \cdot B = A$.

Question 2 (2 marks)
Determine the Boolean equations for this electrical circuit.



$F = \square$

Question 3 (3 marks)
Use truth tables to show that the second logic circuit is a simplified version of the first logic circuit.

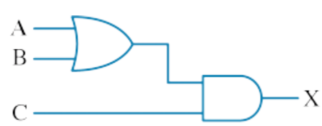


Question 4 (1 mark)
In the language of logic gates, explain NAND.

Question 5 (4 marks)
Draw the logic circuit for the Boolean equation $P = a + a \cdot b$, and construct the truth table.

Question 6 (3 marks)
What is a tautology? Give one example using set theory and one example using Boolean algebra.

Question 7 (1 mark)
In Boolean algebra, this logic gate system is:



A. $(A \cdot B) \cdot C$
B. $(A + B) + C$
C. $(A \cdot B) + C$
D. $A \cdot (B + C)$
E. $(A + B) \cdot C$

Topic	3	Logic and algorithms
Subtopic	3.5	Sets and Boolean algebra

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Question 1 (3 marks)

Determine using Boolean algebra if $(p \wedge q) \vee \neg q$ and $p \vee \neg q$ are equivalent.

They are ☐.

Question 2 (1 mark)

If $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{3, 5, 7, 8\}$, $(B \cap C) \cup B'$ equals

A. $\{2, 4, 6, 8\}$
B. $\varnothing$
C. $\{2, 3, 4, 5, 6, 7, 8\}$
D. $\{6\}$
E. $\{2, 3, 4, 5, 6, 7, 8, 8\}$

Question 3 (4 marks)

If $A = \{p, q, r\}$, $B = \{p, q, r, s\}$, $I = \{p, q, r, s, t, u, v\}$, verify De Morgan's Law $(A + B)' = A' \cdot B'$.

Question 4 (2 marks)

Write out the Distributive Laws in both set and Boolean notation.

Question 5 (1 mark)
Write out deMorgan's Laws.

Question 6 (3 marks)
Determine using Boolean algebra if $(P \wedge Q) \vee \neg Q$ and $P \vee \neg Q$ are equivalent.

Question 7 (4 marks)
If $A = \{p, q, r\}, B = \{p, q, r, s\}, 1 = \{p, q, r, s, t, u, v\}$, prove deMorgan's Law: $(A + B)' = A' \cdot B'$.

Question 8 (1 mark)
Written in set notation, $A \cup (B \cap C)$ equals:
A. $(A \cup B) \cap (A \cup C)$
B. $(A \cap B) \cup (A \cap C)$
C. $(A \cup B) \cap (A \cap C)$
D. $(A \cdot B) + (A \cdot C)$
E. $(A + B) \cdot (A + C)$

Question 9 (1 mark)
$(A \cdot B)'$ equals:
A. $A' \cdot B'$
B. $(A' \cdot B)'$
C. $(A \cdot B)'$
D. $A' + B'$
E. $(A + B)'$

Topic	3	Logic and algorithms
Subtopic	3.6	Algorithms and pseudocode

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Question 1 (2 marks)

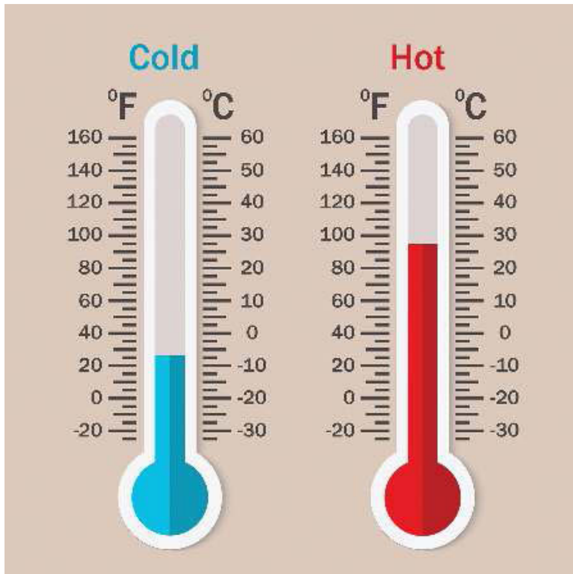
Write a pseudocode to determine the volume of a cylinder.

Question 2 (5 marks)

The following equation shows how to convert from degrees Fahrenheit to degrees Celsius:

$C = \frac{5}{9}(F - 32)$.

a. Write pseudocode to convert a temperature entered as Fahrenheit to its Celsius equivalent. **(2 marks)**



b. Write pseudocode to tabulate temperatures in Celsius to their Fahrenheit equivalence for values of Celsius from 0 to 40 in steps of 2 degrees. **(3 marks)**

Question 3 (4 marks)

Write a pseudocode to solve a quadratic equation, where the coefficient of the quadratic term is non-zero.

Topic	3	Logic and algorithms
Subtopic	3.7	Review

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Question 1 (1 mark)

Consider the following valid argument.

If John plays for us on Saturday, then we will win.

If we win on Saturday, then we will come in first place on the ladder.

If we come in first place on the ladder, then we play our first final at home.

Therefore, if John plays for us on Saturday, then we play our first final at home.

This is an example of

A. *modus ponens*

B. disjunctive syllogism

C. hypothetical syllogism

D. *modus tollens*

E. constructive dilemma

Question 2 (1 mark)

Premise: All argans are flimps. Premise: All flimps are doinks.

The conclusion is

A. all doinks are flimps.

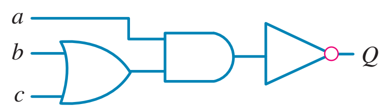
B. all doinks are argans.

C. all argans are doinks.

D. all flimps are argans.

E. doinks are a subset of argans.

Question 3 (1 mark)
Consider the following logic circuit
The Boolean equivalent to this circuit is:



A. $[a \cdot (b + c)]'$
B. $a' + (b' \cdot c')$
C. $(a' + b') \cdot (a' + c')$
D. Both **A** and **B** are equivalent to the circuit
E. **A**, **B** and **C** are all equivalent to the circuit

Question 4 (2 marks)
Use truth tables to determine whether $(p \rightarrow q) \leftrightarrow (q \rightarrow \neg p)$.
The are ☐.

Question 5 (8 marks)

$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

$A = \{1, 2, 3, 4, 5\}$

$B = \{4, 5, 6, 7, 8, 9\}$

$C = \{5, 8, 9, 10, 11\}$

By considering the elements in the various sets below, verify each of the following statements.

a. $(A \cup B)' = A' \cap B'$ **(2 marks)**

b. $(A \cap B)' = A' \cup B'$ **(2 marks)**

c. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ **(2 marks)**

d. $(A \cap B \cap C)' = A' \cup B' \cup C'$ **(2 marks)**

Answers and marking guide

3.2 Statements (propositions), connectives and truth tables

Question 1

p	$\neg p$	$p \vee \neg p$
T	F	F
F	T	F

Award **1 mark** for the correct truth table.

Question 2

'and' is written as $\wedge$.
'or' is written as $\vee$.

Question 3

The last row is false, which implies at least one of p, q and r is required.
The second last row is true; r is required but not with the other two, which are false. This implies $\vee$.
Through a process of elimination, the answer is $p \wedge q \vee r$.
Award **1 mark** for a process of elimination.
Award **1 mark** for the correct answer.

Question 4

No, this sentence is not a statement.
To make this sentence a statement, 'It is true that…' needs to be added to the beginning of the sentence.
'It is true that when the engine fails, a yacht can always revert to sail power.'
This statement is true. **[1 mark]**

Question 5

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

[1 mark]

Question 6

p	q	$p \vee \neg q$
T	T	T
T	F	T
F	T	F
F	F	T

[1 mark]

Question 7

p	q	$\neg p$	$\neg p \Rightarrow q$	$\neg q$	$p \Rightarrow \neg q$
T	T	F	T	F	F
T	F	F	T	T	T
F	T	T	T	F	T
F	F	T	F	T	T

[1 mark]

They are not equivalent. **[1 mark]**

Question 8
A negation is merely the opposite of the original statement.
If P = 'All plongs are umple', then $\sim P$ = 'Some plongs are not umple'.
Another example:
If P = 'All animals are equal', then $\sim P$ = 'Some animals are not equal'.

Question 9

p	$\neg p$	$(p \vee \neg p)$
T	T	T
T	F	T
F	T	T
F	T	T

Award **1 mark** for the correct truth table.

Question 10

p	q	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$	$\neg p$	$\neg q$
T	T	T	T	F	F
T	F	F	F	F	T
F	T	T	T	T	F
F	F	T	T	T	T

Award **1 mark** for the correct truth table.
$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$ is equivalent. **[1 mark]**

Question 11
Consider a single case, let $x = 5$.
The cube root of 5^3 is $+5 \Rightarrow$ the cube root of x^3 is $+x$. **[1 mark]**
The original statement is false. **[1 mark]**

Question 12
$p \wedge q$, as p and q are true simultaneously.

Question 13
$p \vee \neg q$
The last row gives $\neg p$ or $\neg q$, and the first row gives p or q. Therefore, it must be option D or option E. Now just use trial and error.

Question 14
Here the universal set is R. A tautology is always true, hence it is true for all values of the universal set.

Question 15

$$\begin{aligned}(P \vee Q) \wedge \neg Q &= (P \wedge \neg Q) \vee (Q \wedge \neg Q)\\ &= P \wedge \neg Q \vee 0 \qquad [Q \wedge \neg Q = 0]\\ &= P \wedge \neg Q\end{aligned}$$

Question 16

p	q	$p \Rightarrow q$	$\neg p$	$q \Rightarrow \neg p$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Conclusion: They are not equivalent.

Award **1 mark** for constructing the correct truth table.
Award **1 mark** for concluding that the statements are not equivalent.

3.3 Valid and invalid arguments

Question 1

p = Angle A is an obtuse angle.
q = The supplement of angle A is an acute angle.
Set up truth tables for both premises.
$(p \to q) \leftrightarrow (\neg p \to \neg q)$

p	q	$(p \to q)$	$\neg p \to \neg q$	$\neg p$	$\neg q$
T	T	T	T	F	F
T	F	F	F	F	T
F	T	T	T	T	F
F	F	T	T	T	T

Award **1 mark** for the correct truth table.

Question 2

p	q	$(p \to q)$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Award **1 mark** for the correct truth table.
$(p \to q) \leftrightarrow (\neg q \to \neg p)$ is equivalent. **[1 mark]**

Question 3

$\neg p$	$\neg q$	$\neg p \to \neg q$	p	q	$q \to p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Award **1 mark** for the correct truth table.
The *modus tollens* arugument is valid. **[1 mark]**

Question 4

$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$	$q \Rightarrow p$	p	q
T	T	T	T	F	F
T	F	F	F	F	T
F	T	T	T	T	F
F	F	T	T	T	T

Award **1 mark** for the correct truth table.
The *modus tollens* argument is not valid. **[1 mark]**

Question 5

p = If it rained yesterday
q = Today is Tuesday

Prove $(p \wedge q)$ to be a tautology.

p	q	$(p \wedge q)$
T	T	T
T	F	F
F	T	T
F	F	T

Award **1 mark** for the correct truth table.
False, $(p \wedge q)$ is not a tautology. **[1 mark]**

3.4 Boolean algebra and digital logic

Question 1

$$
\begin{aligned}
A + A \cdot B &= (A + A) \cdot (A + B) \\
&= A \cdot (A + B) \\
&= (A + 0) \cdot (A + B) \quad \textbf{[1 mark]} \\
&= A + (0 \cdot B) \\
&= A + 0 \\
&= A \quad \textbf{[1 mark]}
\end{aligned}
$$

Question 2

$F = (a \cdot b) + (c \cdot d)$

Award **1 mark** for each of the two branches.

Question 3

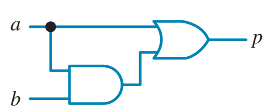
a	b	$a' \cdot b$	$a + a' \cdot b$	$a + b$
0	0	0	0	0
0	1	1	1	1
1	0	0	1	1
1	1	0	1	1

Award **1 mark** for each of the last three columns correct in the truth table.

Question 4

Essentially this gate means 'not and', so it does not allow both to occur simultaneously, every other combination is acceptable. **[1 mark]**

Question 5



Award **1 mark** for each of the two gates.

a	b	$a \cdot b$	$a + a \cdot b$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Award **1 mark** for each of the last two columns correct in the truth table.

Question 6

A tautology is something that is always true, no matter what values are used. For example, 'If today is Friday, tomorrow is Saturday'.

Award **1 mark** for a correct explanation.

$A \cap A = A$ **[1 mark]**

$A + A = A$ **[1 mark]**

Question 7

A and B enter an OR gate, which is represented by +.

(A + B) then enters an AND gate, which is represented by ·, with C, hence the result.

3.5 Sets and Boolean algebra

Question 1

$(p \wedge q) \vee \neg q$

$= (P \cdot Q) + Q'$

$= (P + Q') \cdot (Q + Q')$ **[1 mark]**

$= (P + Q') \cdot I$ **[1 mark]**

$= P + Q'$

$= p \vee \neg q$

$\therefore$ They are equivalent. **[1 mark]**

Question 2

$$\begin{aligned}(B \cap C) \cup B' &= (B \cup B') \cap (C \cup B') \\ &= C \cup B' \\ &= \{3, 5, 7, 8\} \cup \{2, 4, 6, 8\} \\ &= \{2, 3, 4, 5, 6, 7, 8\}\end{aligned}$$

Question 3

$A = \{p, q, r\}, B = \{p, q, r, s\}, \mathrm{I} = \{p, q, r, s, t, u, v\}$

$A + B = \{p, q, r, s\}$

$A + B = B$ **[1 mark]**

$(A + B)' = \{t, u, v\}$ .... (1) **[1 mark]**

$A' = \{s, t, u, v\}$

$B' = \{t, u, v\}$ **[1 mark]**

$A' \cdot B' = \{t, u, v\}$ .... (2) **[1 mark]**

$\therefore (1) = (2)$

$\therefore (A + B)' = A' \cdot B'$

Question 4

Set notation:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **[1 mark]**

Boolean notation:

$A + (B \cdot C) = (A + B) \cdot (A + C)$

$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$ **[1 mark]**

Question 5

$(A + B)' = A' \cdot B'$

$(A \cdot B)' = A' + B'$ **[1 mark]**

Question 6

$(P \wedge Q) \vee \neg Q = (P \vee \neg Q) \wedge (Q \vee \neg Q)$ **[1 mark]**

$= P \vee \neg Q \quad (Q \vee \neg Q = 1)$ **[1 mark]**

$\therefore$ They are equivalent. **[1 mark]**

Question 7

$A = \{p, q, r\}, B\{p, q, r, s\}, 1 = \{p, q, r, s, t, u, v\}$

$A + B = \{p, q, r, s\}$

$= B$ **[1 mark]**

$(A + B)' = \{t, u, v\}$ .... (1)

$A' = \{s, t, u, v\}$

$B' = \{t, u, v\}$ **[1 mark]**

$A' \cdot B' = \{t, u, v\}$ .... (2) **[1 mark]**

$\therefore (1) = (2)$

$\therefore (A + B)' = A' \cdot B'$

Question 8

Using distributive law, $A \cup (B \cap C)$ is equivalent to $(A \cup B) \cap (A \cup C)$

Question 9

Using de Morgan's Law, $(A \cdot B)'$ equals $A' + B'$

3.6 Algorithms and pseudocode

Question 1

```
begin
    numeric radius, height, volume
    display 'Enter the radius'
    input radius
    display 'Enter the height'
    input height
    volume = pi* r**2 * h
    display 'The volume of the cylinder is', volume [2 marks]
end
```

Question 2

a.
```
begin
    numeric c, f
    display 'Enter the temperature in Fahrenheit'
    input f
    c = 5/9*(f – 32)
    display 'The equivalent temperature in Centigrade is',
    c
    end [2 marks]
```

b.
```
begin
    numeric c, f
    disp 'Centigrade Fahrenheit'
    for (c = 0, 40, 2)
    f = 32 + 9*c/5 [3 marks]
    disp c, f
    endfor
    end
```

Question 3

```
begin
    numeric  a, b, c, delta, x1, x2
    display 'Enter the values of a, b and c'
    input a, b, c
    delta = b* b − 4* a* c
    If (delta > 0) then
        x1 = (−b + sqrt(delta))/(2* a)
        x2 = (−b − sqrt(delta))/(2* a)
        disp 'There are two solutions' x1, 'and', x2
    elseif (delta == 0) then
        x1 = −b/(2* a)
        disp 'There is one solution' x1
    else
        disp 'There are no real solutions'
    endif
end
```
[4 marks]

3.7 Review

Question 1

Let p — John plays for us on Saturday.
q — we win on Saturday.
r — we came in first place on the ladder.
s — we play our first final at home.

$p \rightarrow q$
$q \rightarrow r$
$r \rightarrow s$
$\overline{p \rightarrow s}$

Question 2

arugans ⊆ flimps
flimps ⊆ doinks
→ arugans ⊆ doinks
→ all argans are doinks

Question 3

Working from left to right:
b and c connect to an OR gate. Boolean: $b + c$
$b + c$ and a connect to an AND gate. Boolean: $a.(b + c)$
The output of this connects to a NOT gate. Boolean
$[a \cdot (b + c)]'$
A is a Boolean equivalent to the logic circuit.
$[a \cdot (b + c)]' = a' + (b' \cdot c')$, so B is also a Boolean equivalent.

$$\begin{aligned}[a \cdot (b + c)]' &= [(a \cdot b) + (a \cdot c)]' \\ &= (a' + b') \cdot (a' + c')\end{aligned}$$

So C is also a Boolean equivalent.

Question 4

p	q	$p \to q$	$\neg p$	$q \to \neg p$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Conclusion: They are not equivalent.
Award **1 mark** for constructing the correct truth table.
Award **1 mark** for concluding that the statements are not equivalent.

Question 5

a. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$(A \cup B)' = \{10, 11, 12, 13, 14, 15\}$ **[1 mark]**

$A' = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

$B' = \{1, 2, 3, 10, 11, 12, 13, 14, 15\}$

$A' \cap B' = \{10, 11, 12, 13, 14, 15\}$ **[1 mark]**

$(A \cup B)' = A' \cap B'$

b. $(\text{A} \cap \text{B})' = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ **[1 mark]**

$\text{A}' \cup \text{B}' = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ **[1 mark]**

$(\text{A} \cap \text{B})' = \text{A}' \cup \text{B}'$

c. $\text{A} \cup \text{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$(\text{A} \cup \text{B}) \cap \text{C} = \{5, 8, 9\}$ **[1 mark]**

$\text{A} \cap \text{C} = \{5\}, \text{B} \cap \text{C} = \{5, 8, 9\}$

$(\text{A} \cap \text{C}) \cup (\text{B} \cap \text{C}) = \{5, 8, 9\}$ **[1 mark]**

$(\text{A} \cup \text{B}) \cap \text{C} = (\text{A} \cap \text{C}) \cup (\text{B} \cap \text{C})$

d. $\text{A} \cap \text{B} \cap \text{C} = \{5\}$

$(\text{A} \cap \text{B} \cap \text{C})' = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ **[1 mark]**

$\text{A}' \cup \text{B}' \cup \text{C}' = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ **[1 mark]**

$(\text{A} \cap \text{B} \cap \text{C})' = \text{A}' \cup \text{B}' \cup \text{C}'$

4 Proof and number

Topic	**4**	**Proof and number**
Subtopic	**4.2**	**Number systems and mathematical statements**

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Question 1 (1 mark)

The recurring decimal $0.\dot{2}0\dot{9}$ expressed as a fraction in the form $\frac{a}{b}$ is

A. $\frac{209}{1009}$

B. $\frac{209}{999}$

C. $\frac{209}{990}$

D. $\frac{209}{909}$

E. $\frac{29}{101}$

Question 2 (1 mark)

Determine if the proposition $\forall x \in R, \forall y \in (x, \infty), x < y$ is true or false.

A. True

B. False

Question 3 (2 marks)

Consider the statement '$\forall x \in N, x(x+1)$ is even'. Decide if this is a true statement and justify your decision.

Topic 4 | Subtopic 4.2 Number systems and mathematical statements

Question 4 (1 mark)
Within the set of real numbers, the interval greater than 6 but less than or equal to 11 is written as:
A. $[6, 11]$
B. $[6, 11)$
C. $(6, 11]$
D. $(6, 11)$
E. $6 < x < 11, x \in R$

Question 5 (1 mark)
If $a \in N$ and $b \in R^- \cup \{0\}$, show that $a > b$.

Question 6 (1 mark)
Is $b\sqrt{a^2}$ a surd?

Question 7 (1 mark)
The statement 'x is greater than or equal to a but less than b' is written as:
A. (a, b)
B. $(a, b]$
C. $[a, b)$
D. $[b, a)$
E. $[a, b]$

Topic	4	Proof and number
Subtopic	4.3	Direct and indirect methods of proof

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Question 1 (2 marks)
Prove by counter example that the statement 'the cube root of $x^3$is $\pm x$' is false.

Question 2 (1 mark)
The contrapositive of the conjecture 'If p, then q' is
A. 'If not p, then not q'.
B. 'If not q, then p'.
C. 'If not q, then not p'.
D. 'If q, then not p'.
E. 'If q, then p'.

Question 3 (3 marks)
If you were trying to prove that $M \rightarrow N$ is true by contradiction, state what steps you would take. Explain the process.

Question 4 (2 marks)
Prove that $\forall p > 0,\ p \geq 2 - \frac{1}{p}$.

Question 5 (4 marks)
Consider the statement $\forall x \in N,\ x(x-1)$ is odd. Is this a true statement? Show proof.

Topic	4	Proof and number
Subtopic	4.4	Proofs with rational and irrational numbers

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Question 1 (1 mark)

Consider four consecutive numbers, $a_1,\ a_2,\ a_3,\ a_4$. Then $a_1 + a_2 + a_3 + a_4$ equals

A. $a_3 \times a_4 - a_1 \times a_2$
B. $a_3 \times a_4 + a_1 \times a_2$
C. $a_1 \times a_2 - a_3 \times a_4$
D. $a_1 \times a_2 + a_3 \times a_4$
E. $a_2 \times a_4 - a_1 \times a_3$

Question 2 (1 mark)

If a is a factor of b and b is a factor of c, then

A. $\frac{a}{b}$ is a factor of $\frac{b}{c}$
B. ab is a factor of $\frac{c}{b}$
C. ab is a factor of c
D. a is a factor of c
E. a^3 is a factor of bc

Question 3 (3 marks)

Prove that for all $x, y \in R, (x+1)^2 + (y-1)^2 \geq 2x - 2y - 2xy + 2$

Topic 4 | Subtopic 4.4 Proofs with rational and irrational numbers

Question 4 (3 marks)

Prove that for all $x, y \in R$, $(x+1)^2 + (y-1)^2 \geq -4x + 2y + 2xy - 4$.

Question 5 (3 marks)

Show that if p and q are multiples of a, then $p + q$ is a multiple of a.

Question 6 (1 mark)

If proof by contradiction was to be used to prove $\sqrt{pq} \leq \frac{p+q}{2}$, $p, q \in N$, the opening line would be:

A. $\sqrt{pq} > \frac{p+q}{2}$, $p, q \in N$

B. $\sqrt{pq} < \frac{p+q}{2}$, $p, q \in N$

C. $\sqrt{pq} \leq \frac{p+q}{2}$, $p, q \in N$

D. $\sqrt{pq} \geq \frac{p+q}{2}$, $p, q \in N$

E. $\sqrt{pq} = \frac{p+q}{2}$, $p, q \in N$

Topic	4	Proof and number
Subtopic	4.5	Proof by mathematical induction

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Question 1 (1 mark)

The first step in the proof by induction to prove $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$, is to prove the statement is true for

A. $n > 1$
B. $k > 1$
C. $n = 1$
D. $n = k$
E. $n = k + 1$

Question 2 (1 mark)

Consider the proposition $P(n)$ that

$1^2 + 2^2 + 3^2 + \ldots.. + n^2 = \frac{1}{6}n(n+1)(2n+1)$

If $n = k + 1$, the proposition would become

A. $1^2 + 2^2 + \ldots.. + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+1)$
B. $(k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$
C. $1^2 + 2^2 + \ldots.. + \frac{1}{6}(k+1)^2 = k(k+1)(k+2)(2k+3)$
D. $1^2 + 2^2 + \ldots.. + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$
E. $1^2 + 2^2 + \ldots.. + k^2 + (k+1)^2 = \frac{k}{6}(k+1)(2k+2)$

Question 3 (5 marks)

Assuming the triangle inequality $|a+b| \le |a| + |b|$ for $a,\ b \in R$.
Use mathematical induction to prove that
$|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|, x_1,\ x_2,\ \ldots,\ x_n \in R$.

Question 4 (1 mark)

Consider the proposition $P(n)$ that 4 is a factor of $5n - 1$ for all natural numbers n.

To prove this by mathematical induction, the first step would be to

A. assume $P(k)$ holds

B. consider $P(k+1)$

C. test if $P(1)$ holds

D. express $(5^{k+1} - 1)$ as $(5^{k+1} - 5 + 4)$

E. assume $P(1)$ holds

Question 5 (1 mark)

In using mathematical induction to prove De Moivre's theorem that if $z = r\text{cis}\theta$ then $z^n = r^n\text{cis}n\theta$ for any natural number n, the proof proceeds as follows:

Step 1$n = 1$ is true.

Step 2 Assume $n = k$ is true and consider z^{k+1}.

Which of the following would show the necessary explanation for the next part of Step 2 of the proof?

A. $z^{k+1} = z^k z$

$= r^k\text{cis } k\theta r \text{ cis}\theta$

$= r^{k+1}\text{cis } k\theta$

B. $z^{k+1} = zz^k$

$= r\,\text{cis}\theta r^k\text{cis } k\theta$ by assumption

$= r^{k+1}\text{cis }(k+1)\,\theta$

$n = k + 1$ is true if $n = k$ is true

C. $z^{k+1} = zz^k$

$= r\,\text{cis}\theta r^k\text{cis } k\theta$ by assumption

$= r^{k+1}\text{cis } k\theta^2$so $n = k + 1$ is true

D. $z^{k+1} = zz^k$

$= r \text{ cis } \theta r^k \text{ cis } \theta^k$ by assumption

$= r^{n+1}\text{cis}\theta^{k+1}$so $n = k + 1$ is true

E. $z^{k+1} = zz^k$

$= r\,\text{cis}\theta r^k\text{cis } k\theta$

$= r^{k+1}\text{cis }(k+1)\,\theta$

Topic	4	Proof and number
Subtopic	4.6	Proof of divisibility using induction

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Question 1 (4 marks)
Use proof by induction to prove that $12 \mid \left(3^{(2n+1)} - 3\right)$, $n \in N$.

Question 2 (4 marks)
Use proof by induction to prove that $n^4 - n^2$ is divisible by 2, $n \in N$

Question 3 (4 marks)
Prove by induction that $2 \mid \left(5^{(n-3)} + 1\right)$, where n is a natural number greater than 3.

Question 4 (4 marks)
Use proof by induction to prove that $12| \left(3^{(2n+1)} - 3\right)$

Question 5 (4 marks)
Use proof by induction to prove that $9^n - 6$ is divisible by 3.

Topic	4	Proof and number
Subtopic	4.7	Review

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Question 1 (1 mark)

$\frac{22}{7}, 3.14, 3.146, \sqrt{2\frac{1}{4}}, \sqrt[3]{9}, \sqrt{\left(5^2 + 12^2\right)}$

Within this list of numbers, the irrational number is

A. $\frac{22}{7}$

B. 3.146

C. $\sqrt[3]{9}$

D. $\sqrt{\left(5^2 + 12^2\right)}$

E. $\sqrt{2\frac{1}{4}}$

Question 2 (1 mark)

If $p = q, p, q \in N$, then $p + q$ is even.

The contrapositive is:

A. If $p \neq q, p, q \in N$, then $p + q$ is odd.

B. If $p = q, p, q \in N$, then $p + q$ is odd.

C. If $p + q$ is odd, $p, q \in N$, then $p \neq q$

D. If $p + q$ is odd, $p, q \in N$, then $p = q$

E. If $p = q$, $p, q \in N$, then $p - q$ is even.

Question 3 (5 marks)

Prove by induction that $\sum_{i=1}^{n} 5^i = \frac{5}{4}\left(5^n - 1\right)$.

Question 4 (4 marks)

Consider the following statement: $\exists x \in N, \; x(x-1)$ is odd. Is this a true statement? Show proof.

Question 5 (3 marks)

Prove that for $a, b \in Z$, if $4|\left(a^2+b^2\right)$, then a and b are not both odd.

Question 6 (1 mark)

If $l \in Z^-$ and $l = r^4$, prove that r does not exist.

Question 7 (4 marks)

Answer the following

a. Identify what is wrong with the following proof. **(2 marks)**

Proposition: $17 = 9$

Let x and y be two non $-$ zero integers where $8x = 9y$.

$8x = 9y$

$8x^2 = 9xy$

$8\left(x^2+y^2\right) = 9\left(xy-y^2\right)$

$8(x+y)(x-y) = 9y(x-y)$

$8x + 8y = 9y$

Substitude $8x = 9y$

$9y + 8y = 9y$

$17y = 9y$

$17 = 9$

b. Prove, by any means you choose, that if $p, q \in N$, $p^2 - q^2 \neq 1$. **(2 marks)**

Answers and marking guide

4.2 Number systems and mathematical statements

Question 1

The denominator consists of 9s, one for each digit in the repeated decimal.

Question 2

$\forall x,\ \forall y$ means for all possible values of x and all possible values of y the statement is true. In this instance, y is selected from the set (x, ∞). This means that y is guaranteed to be larger than x. The statement is true. **[1 mark]**

Question 3

The statement $\forall$ natural numbers, x , $x\,(x+1)$ is even is true. **[1 mark]**
If x is even, then the statement will be true (an even number multiplied by an odd number is even). If x is odd, then $x+1$ is even and the statement is still true. **[1 mark]**

Question 4

Square brackets for equality; round brackets for non-equality.

Question 5

$a \in N \Rightarrow a > 0, \qquad b \in R^{-} \cup \{0\} \Rightarrow b \le 0.$ Hence $a > b$. **[1 mark]**

Question 6

No, because $b\sqrt{a^2} = ab$ which does not contain a radical (root) sign. **[1 mark]**

Question 7

'Equality' requires square brackets, whereas 'greater than' or 'less than' requires round brackets.

4.3 Direct and indirect methods of proof

Question 1

Consider a single case, let $x = 5$.
The cube root of 5^3 is $+5 \rightarrow -5$ is not the cube root of 5^3. **[1 mark]**
The original statement is false. **[1 mark]**

Question 2

The contrapositive of the conjecture 'If p, then q' is 'If not q, then not p'.

Question 3

Assume the proposition to be proved is false, i.e. $M \rightarrow N$ is false. **[1 mark]**
The next step is to show that this assumption leads to some mathematical contradiction, for example $0 = 1$ **[1 mark]**
Once you have reached a contradiction we can conclude that our assumption was false, and hence the proposition $M \rightarrow N$ is true. **[1 mark]**

Question 4

$p \ge 2 - \dfrac{1}{p}$

$p^2 - 2p + 1 \ge 0$ **[1 mark]**

$(p-1)^2 \ge 0$ True $\forall p$ **[1 mark]**

Question 5

x and $(x-1)$ are consecutive numbers, hence one is odd and the other even. **[1 mark]**
The product of an odd number and an even number is even. **[1 mark]**

If either is zero, the product is zero, taken as even. **[1 mark]**
Thus the statement is untrue. $\forall x \in N,\ x(x-1)$ is even. **[1 mark]**

4.4 Proofs with rational and irrational numbers

Question 1
Let the numbers be $m,\ m+1,\ m+2,\ m+3$
$a_3 \times a_4 - a_1 \times a_2$
$(m+2)(m+3) - m(m+1) = 4m + 6$
$m + (m+1) + (m+2) + (m+3) = 4m + 6$

Question 2
If a is a factor of b then $b = na$
If b is a factor of c then $c = mb = mna$
$\rightarrow a$ is a factor of c

Question 3
$$[(x+1)+(y-1)]^2 \geq 0 \text{ [1 mark]}$$
$$(x+1)^2 + (y-1)^2 + 2(x+1)(y-1)$$
$$(x+1)^2 + (y-1)^2 \geq -2(x+1)(y-1) \text{ [1 mark]}$$
$$\geq -2(xy - x + y - 1)$$
$$\geq 2x - 2y - 2xy + 2 \text{ [1 mark]}$$

Question 4
$$-4x + 2y + 2xy - 4 = -4(x+1) + 2y(x+1) \text{ [1 mark]}$$
$$= 2(x+1)(y+2) \text{ [1 mark]}$$
$$(x+1)^2 + (y-1)^2 \geq 2(x+1)(y+1)$$
$$(x+1)^2 + (y-1)^2 - 2(x+1)(y-2) \geq 0$$
$$\left\{(x+1)+(y-1)^2\right\} \geq 0 \quad \text{which is true for all } x,\ y \in R \text{ [1 mark]}$$

Question 5
$p = ma$ **[1 mark]**
$q = na$ **[1 mark]**
$p + q = ma + na = (m+n)a$ **[1 mark]**

Question 6
$\sqrt{pq} > \dfrac{p+q}{2}$, $p,\ q \in N$. Since this is 'proof by contradiction' it would seem obvious that one should start with a contradictory statement.

4.5 Proof by mathematical induction

Question 1
The first step in the proof by induction is to show that the statement is true for $n = 1$

Question 2
$$1^2 + 2^2 + 3^2 \ldots\ldots k^2 + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$$

Question 3
We are trying to prove that $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$,
$x_1,\ x_2,\ \ldots,\ x_n \in R$
Test that it holds for $n = 1$:
$|x_1| \leq |x_1|$. This is true. **[1 mark]**
Assume that the statement is true for $n = k$:

$|x_1 + x_2 + \ldots + x_k| \le |x_1| + |x_2| + \ldots + |x_k|$ **[1 mark]**
Consider the case when $n = k + 1$:
$|x_1 + x_2 + \ldots + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}|$
$\le |(x_1 + x_2 + \ldots + x_k)| + |x_{k+1}|$
(By the triangle inequality) **[1 mark]**
At this point we can use the assumption:
$|x_1 + x_2 + \ldots + x_k| \le |x_1| + |x_2| + \ldots + |x_k|$
$\le |x_1| + |x_2| + \ldots + |x_k| + |x_{k+1}|$ **[1 mark]**
We have shown that the statement is true for $n = 1$, and that if it is true for $n = k$, it must also be true for $n = k + 1$. By induction, the statement is true for all $n \in N$. **[1 mark]**

Question 4
The first step in any proof by mathematical induction is to establish that the proposition holds for an initial value, $n = 1$ in this case.

Question 5
$z^{k+1} = zz^k$
$= r\,\text{cis}\theta r^k \text{cis}\,k\theta$ by assumption
$= r^{k+1}\text{cis}\,(k+1)\,\theta$
$n = k + 1$ is true if $n = k$ is true

4.6 Proof of divisibility using induction

Question 1
When $n = 1$, $3^{(2n+1)} - 3$
$= 3^{(2+1)} - 3$
$= 24$
$\therefore 3^{(2n+1)} - 3$ is divisible by 12 when $n = 1$. **[1 mark]**
Assume that $3^{(2n+1)} - 3$ is divisible by 12 when $n = k$.
When $n = k$, $3^{(2n+1)} - 3$
$= 3^{(2k+1)} - 3$
$= 12m$ **[1 mark]**
When $n = k + 1$, $3^{(2n+1)} - 3$
$= 3^{(2k+2+1)} - 3$
$= 9 \times 3^{(2k+1)} - 3$
$= 9\left(3^{(2k+1)} - 3\right) + 9 \times 3 - 3$
$= 9\left(3^{(2k+1)} - 3\right) + 24\text{v}$ **[1 mark]**
$= 9\,(12m) + 2 \times 12$
$= 12\,(9m + 2)$ **[1 mark]**
By induction $3^{(2n+1)} - 3$ is divisible by 12.
for all $n \in N$

Question 2
If $n = 2, n^4 - n^2 = 16 - 4 = 12$, which is divisible by 2. **[1 mark]**
Assume that $n^4 - n^2$ is divisible by 2 when $n = k$.
When $n = k$, $n^4 - n^2$
$= k^4 - k^2$
$= 2m$ **[1 mark]**

When $n = k + 1, n^4 - n^2$
$= (k+1)^4 - (k+1)^2$
$= k^4 + 4k^3 + 6k^2 + 4k + 1 - k^2 - 2k - 1$
$= k^4 - k^2 + 4k^3 + 6k^2 + 2k$ **[1 mark]**
$= 2m + 4k^3 + 6k^2 + 2k$
$= 2m + 2\left(2k^3 + 3k^2 + k\right)$
$= 2\left(m + 2k^3 + 3k^2 + k\right)$ **[1 mark]**
By induction, $n^4 - n^2$ is divisible by 2 for all $n \in N$.

Question 3
When $n = 4,\ 5^{(n-3)} + 1$
$= 5^{(4-3)} + 1$
$= 6$
$\therefore 5^{(4-3)} + 1$ is divisible by 2 when $n = 4$. **[1 mark]**
Assume that $5^{(4-3)} + 1$ is divisible by 2 when $n = k$.
When $n = k,\ 5^{(n-3)} + 1$ **[1 mark]**
$= 5^{(k-3)} + 1$
$= 2m$
When $n = k + 1,\ 5^{(n-3)} + 1$
$= 5^{(k+1-3)} + 1$
$= 5^{(k-2)} + 1$
$= 5\left(5^{(k-3)} + 1\right) - 4$
$= 5(2m) - 4$
$= 2(5m - 2)$ **[1 mark]**
By induction $5^{(n-3)} + 1$ is divisible by 2.
For all natural numbers $n > 3$.

Question 4
When $n \quad = 1,\ 3^{(2n+1)} - 3$
$= 3^{(2n+1)} - 3$
$= 24$
$\therefore 3^{(2n+1)} - 3$ is divisible by 12 when $n = 1$. **[1 mark]**
Assume that $3^{(2n+1)} - 3$ is divisible by 12 when $n = k$.
When $n = k,\ 3^{(2n+1)} - 3$
$= 3^{(2k+1)} - 3$
$= 12m$ **[1 mark]**
When $n = k + 1,\ 3^{(2n+1)} - 3$
$= 3^{(2k+2+1)} - 3$
$= 9 \times 3^{(2k+1)} - 3$
$= 9\left(3^{(2k+1)} - 3\right) + 9 \times 3 - 3$
$= 9\left(3^{(2k+1)} - 3\right) + 24$ **[1 mark]**
$= 9(12m) + 2 \times 12$
$= 12(9m + 2)$ **[1 mark]**
By induction, $3^{(2n+1)} - 3$ is divisible by 12.

Question 5

When $n = 1,\ 9^n - 6$
$= 9 - 9$
$= 3$

$\therefore 9^n - 6$ is divisible by 3 when $n = 1$. **[1 mark]**

Assume that $9^n - 6$ is divisible by 3 when $n = k$.

When $n = k,\ 9^n - 6$
$= 9^k - 6$
$= 3m$ **[1 mark]**

When $n = k + 1,\ 9^n - 6$
$= 9^{k+1} - 6$
$= 9\left(9^k - 6\right) + 48$
$= 9m + 48$
$= 9m + 3 \times 16$
$= 3(3m + 16)$ **[1 mark]**

By induction, $9^n - 6$ is divisible by 3.

4.7 Review

Question 1

$\sqrt[3]{9}$ is irrational. The answer lies between 2 and 3

Question 2

If $p + q$ is odd, $p, q \in N$, then $p \neq q$. The contrapositive of a true statement is also true.

Question 3

We are trying to prove that $\sum_{i=1}^{n} 5^i = \frac{5}{4}(5^n - 1)$ for all $n \in N$.

When $n = 1$:

$$\sum_{i=1}^{1} 5^i = \frac{5}{4}\left(5^1 - 1\right)$$
$$5 = \frac{5}{4}(5 - 1)$$
$$5 = \frac{5}{4} \times 4$$
$$5 = 5$$

The statement is true for $n = 1$ **[1 mark]**

Assume that it is true for $n = k$.

$\sum_{i=1}^{k} 5^i = \frac{5}{4}\left(5^k - 1\right)$ **[1 mark]**

Consider the case when $n = k + 1$:

$$\sum_{i=1}^{k+1} 5^i = \sum_{i=1}^{k} 5^i + 5^{k+1}$$

$= \frac{5}{4}\left(5^k - 1\right) + 5^{k+1}$ (Using the inductive hypothesis) **[1 mark]**

$$= \frac{5}{4}\left(5^k - 1\right) + 5 \times 5^k$$
$$= \frac{5}{4}\left(5^k - 1\right) + \frac{5}{4}\left(4 \times 5^k\right)$$

$= \frac{5}{4}\left(5^k - 1 + 4 \times 5^k\right)$

$= \frac{5}{4}\left(5 \times 5^k - 1\right)$

$= \frac{5}{4}\left(5^{k+1} - 1\right)$ **[1 mark]**

We have shown that the statement is true for $n = 1$, and that if it is true for $n = k$, it must also be true for $n = k + 1$. By induction, the statement is true for all $n \in N$. **[1 mark]**

Question 4

x and $(x - 1)$ are consecutive numbers, hence one is odd and the other even. **[1 mark]**
he product of an odd number and an even number is even. **[1 mark]**
If either is zero, the product is zero, taken as even. **[1 mark]**
Thus the statement is untrue. $\forall x \in N, x(x-1)$ is even. **[1 mark]**

Question 5

R.T.P: if $a, b \in Z$ and $4 \mid \left(a^2 + b^2\right)$ then a and b are not both odd
Prove the contrapositive: If a and b are both odd, then 4 does not divide $a^2 + b^2$. **[1 mark]**
Assume a and b are both odd.
$a = 2m + 1, b = 2n + 1, m, n \in Z$

$a^2 + b^2 = (2m+1)^2 + (2n+1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$

$= 4\left(m^2 + m + n^2 + n\right) + 2$ **[1 mark]**

Therefore 4 does not divide $a^2 + b^2$.
Therefore if $4 \mid \left(a^2 + b^2\right)$ then a and b are not both odd. **[1 mark]**

Question 6

$l = r^4 \Rightarrow r = \pm\sqrt[4]{l}$, but $l < 0$ and an even root of a negative number does not exist. **[1 mark]**

Question 7

a. Error:
$8\left(x^2 - y^2\right) = 9\left(xy - y^2\right)$ This line has an error. Expanded we get
$8x^2 - 8y^2 = 9xy - 9y^2$. We have subtracted different quantities from each side. **[2 Marks]**

b. $p^2 - q^2 = (p - q)(p + q)$
If $(p - q) = 1$ then $(p = q) \neq 1$
Hence $(p - q)(p + q) \neq 1$ **[1 mark]**
If $(p + q) = 1$ then $(p - q) \neq 1$
Hence $(p - q)(p + q) \neq 1$ **[1 mark]**

5 Matrices

Topic	5	Matrices
Subtopic	5.2	Addition, subtraction and scalar multiplication of matrices

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Question 1 (4 marks)

Given the matrices $A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -1 \\ 2 & 3 \end{bmatrix}$, determine the matrix X if $X = 2B - 3A + 2I$.

$X = \square$

Question 2 (1 mark)

If $\begin{bmatrix} 7 & 5 \\ a & -2 \end{bmatrix} - \begin{bmatrix} -2 & b \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ -3 & -9 \end{bmatrix}$, then the values of a and b are

A. $a = 0$ and $b = 1$.
B. $a = 0$ and $b = -1$.
C. $a = -6$ and $b = -1$.
D. $a = -6$ and $b = 1$.
E. $a = 6$ and $b = 1$.

Question 3 (2 marks)

Given the matrices $A = \begin{bmatrix} a & 3 \\ x & -6 \end{bmatrix}$, $B = \begin{bmatrix} -2 & b \\ 4 & y \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, determine the values of a, b, x, and y given that $2A - 3B = 0$.

$a = \square$, $b = \square$, $x = \square$, and $y = \square$

Question 4 (4 marks)

Given matrices $A = \begin{bmatrix} 0 & 1 \\ -4 & -3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ 6 & 0 \\ 3 & 6 \end{bmatrix}$, find matrices $2A$ and $-3B$ and hence find matrix C such that $-2A + C = -3B$.

$C = \square$

Question 5 (1 mark)

In the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 5 & -3 & 4 \\ -5 & -4 & 0 \end{bmatrix}$, element a_{32} is:

A. 4
B. −4
C. −3
D. −2
E. 0

Question 6 (1 mark)

If $A = \begin{bmatrix} 2 & -3 \\ -5 & 5 \end{bmatrix}$, then $-2A$ equals:

A. $\begin{bmatrix} 4 & -6 \\ -10 & 10 \end{bmatrix}$

B. $\begin{bmatrix} -4 & -6 \\ -10 & -10 \end{bmatrix}$

C. $\begin{bmatrix} -4 & 6 \\ 10 & -10 \end{bmatrix}$

D. $\begin{bmatrix} 10 & -10 \\ -4 & 6 \end{bmatrix}$

E. $\begin{bmatrix} -10 & 10 \\ -4 & -6 \end{bmatrix}$

Topic	5	Matrices
Subtopic	5.3	Matrix multiplication

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Question 1 (1 mark)

If A is a (2×3) matrix and B is a (3×1) matrix, then the product AB

A. is a (2×1) matrix.

B. is a (3×3) matrix.

C. is a (2×3) matrix.

D. does not exist.

E. is not defined.

Question 2 (1 mark)

The set of matrices that can be multiplied is

A. $\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$

B. $[4 \;\; 2]$ and $[5 \;\; 3]$

C. $[3 \;\; 1 \;\; 2]$ and $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 4 & 2 \\ 7 & 3 \end{bmatrix}$ and $\begin{bmatrix} 9 & 1 & 5 \\ 3 & 2 & 4 \\ 8 & 1 & 3 \end{bmatrix}$

E. $[7]$ and $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$

Question 3 (1 mark)

$\begin{bmatrix} 1 & 4 \\ 6 & 3 \end{bmatrix} \times \begin{bmatrix} 8 & 1 & 3 \\ 0 & 9 & 4 \end{bmatrix}$ is equal to

A. $\begin{bmatrix} 8 & 19 \\ 48 & 37 \end{bmatrix}$

B. $\begin{bmatrix} 8 & 37 & 19 \\ 48 & 33 & 30 \end{bmatrix}$

C. $\begin{bmatrix} 48 & 37 \\ 30 & 33 \end{bmatrix}$

D. $\begin{bmatrix} 8 & 48 \\ 37 & 33 \\ 19 & 30 \end{bmatrix}$

E. $\begin{bmatrix} 8 & 28 & 15 \\ 48 & 42 & 34 \end{bmatrix}$

Question 4 (6 mark)

Using $A = \begin{bmatrix} 5 & 2 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$, verify the Distributive Law: $A(B + C) = AB + AC$.

Question 5 (4 marks)

Given matrices $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, verify that $A^2I = IA^2 = A^2$.

Topic	5	Matrices
Subtopic	5.4	Determinants and inverses

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Question 1 (1 mark)

Which of these matrices is singular?

A. $\begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 4 & -1 \\ 8 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

E. $\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}$

Question 2 (1 mark)

The inverse of the matrix $\begin{bmatrix} 4 & 7 \\ 2 & 9 \end{bmatrix}$ is:

A. $\begin{bmatrix} 9 & -7 \\ -1 & 2 \end{bmatrix}$

B. $\begin{bmatrix} \frac{9}{22} & \frac{-7}{22} \\ \frac{-1}{22} & \frac{2}{22} \end{bmatrix}$

C. 22

D. $\begin{bmatrix} \frac{4}{22} & \frac{7}{22} \\ \frac{2}{22} & \frac{9}{22} \end{bmatrix}$

E. $\begin{bmatrix} \frac{4}{22} & -\frac{7}{22} \\ -\frac{2}{22} & \frac{9}{22} \end{bmatrix}$

Question 3 (3 marks)

If $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, determine the value ($s$) of k for which the determinant of the matrix $A - kI = 0$.

$k = \square$ or $\square$

Question 4 (4 marks)

If $M^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$, find M and verify that $MM^{-1} = M^{-1}M = I$.

Question 5 (3 marks)

Find the multiplicative inverse of the matrix $\begin{bmatrix} 6 & 2 \\ -3 & 2 \end{bmatrix}$.

Question 6 (4 marks)

$M^{-1} = \frac{1}{11}\begin{bmatrix} -2 & -5 \\ 3 & 2 \end{bmatrix}$ Find M and verify that $MM^{-1} = M^{-1}M = I$.

Question 7 (1 mark)

The inverse of the matrix $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

A. $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

B. $= \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$.

C. $= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.

D. does not exist.

E. $= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Topic	5	Matrices
Subtopic	5.5	Matrix equations

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Question 1 (1 mark)

Express this system of simultaneous equations in matrix form and solve for x and y.

$2x + 5y = -3$

$7x - 2y = 9$

$x = \square$

$y = \square$

Question 2 (1 mark)

Given matrices $A = \begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 7 \\ -8 & -2 \end{bmatrix}$, matrix X, such that $AX = B$, is

A. $\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

B. $\begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 20 & -30 \\ 10 & -20 \end{bmatrix}$

D. $\frac{1}{10}\begin{bmatrix} 20 & -30 \\ 10 & -20 \end{bmatrix}$

E. $\frac{1}{5}\begin{bmatrix} -7 & -7 \\ -18 & 7 \end{bmatrix}$

Question 3 (1 mark)

Which of the following system of equations has a unique solution?

System I:	System II:	System III:
$2x - y = -3$	$y = 5 - 2x$	$7x - 5y = 1$
$5x + y = 4$	$4x + 2y = 5$	$2x = 6 + y$

A. System I only
B. Systems I and II only
C. Systems I and III only
D. All systems
E. None of the systems

Question 4 (3 marks)

Two shops, A and B, are supplied with boxes of different flavours of corn chips – plain (P), chilli (C) and salty (S) – according to the following table.

	P	C	S
A	30	30	20
B	20	15	20

The costs of the boxes are plain \$10, chilli \$12 and salty \$12.

a. Write down the costs in a 3×1 matrix. **(1 mark)**

b. Use matrix multiplication to find the total costs for each shop. **(2 marks)**

Question 5 (8 marks)

Kylie bought 3 Mars Bars and 1 Snickers for \$9. Jarrad bought 2 Mars Bars and 3 Snickers for \$13.

a. Express these situations in two simultaneous equations. **(1 mark)**

b. Write the simultaneous equations in matrix form. **(1 mark)**

c. Calculate the determinant of the coefficient matrix. **(2 marks)**

d. Use the determinant to find the inverse of the coefficient matrix. **(1 mark)**

e. Use the inverse of the coefficient matrix to solve for both variables. **(3 marks)**

Question 6 (4 marks)

Express the following system of simultaneous equations in matrix form and solve for x and y.

$3x - 2y = 7$

$5x + y = 3$

Question 7 (1 mark)

The matrix equation $\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ represents which of the following pairs of simultaneous equations?

A. $3x - 2y = 4$
$2x - y = 5$

B. $2x + y = 5$
$3x - 2y = 4$

C. $3x + 2y = 4$
$2x + y = 5$

D. $3x - 2y = 1$
$2x + y = 5$

E. $2x + y = 5$
$3x + 2y = 4$

Topic	5	Matrices
Subtopic	5.6	Review

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Question 1 (4 marks)

Express the following system of simultaneous equations in matrix form and solve for x and y.

$3x - 2y = 7$

$5x + y = 3$

$x = \square, y = \square$

Question 2 (1 mark)

The inverse of the matrix $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ is

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

C. $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

D. not defined

E. none of these

Question 3 (6 marks)

Using $A = \begin{bmatrix} 5 & 2 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$, verify the Associative Law:

$A(B + C) = AB + AC$.

Question 4 (4 marks)

Given matrices $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, verify that $A^2I = IA^2 = A^2$.

Question 5 (4 marks)

If $M^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$, calculate M, and verify that $MM^{-1} = M^{-1}M = I$.

Question 6 (1 mark)

If $B = \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$, then the matrix B^2 is equal to:

A. $\begin{bmatrix} 16 & 4 \\ 1 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 16 & -4 \\ 1 & -1 \end{bmatrix}$

C. $\begin{bmatrix} 14 & -6 \\ 3 & -1 \end{bmatrix}$

D. $\begin{bmatrix} 8 & -4 \\ 2 & -2 \end{bmatrix}$

E. $\begin{bmatrix} 14 & 3 \\ -6 & -1 \end{bmatrix}$

Question 7 (1 mark)

If $A^2 + 9I = O$ where O represents the null (zero) matrix and I represents the identity matrix, then the matrix A^{-1} is equal to:

A. $\frac{1}{9}A$

B. $-\frac{1}{9}A$

C. $\frac{1}{3}I$

D. $-\frac{1}{3}I$

E. $9A$

Question 8 (6 mark)

Given the matrices $A = \begin{bmatrix} 3 & 7 \\ 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 19 & -24 \\ 33 & 36 \end{bmatrix}$ and $C = \begin{bmatrix} 15 \\ -16 \end{bmatrix}$:

a. Find the inverse of the matrix A. **(2 marks)**

b. Find the matrix X if $AX = C$. **(2 marks)**

c. Find a matrix D such that $DA = B$. **(2 marks)**

Question 9 (10 marks)

Given the matrices

$A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

a. Express the determinant of the matrix $A - kI$ in the form $pk^2 + qk + r$. **(2 marks)**

b. Evaluate the matrix $pA^2 + qA + rI$. **(2 marks)**

c. Find the values of k for which the determinant of the matrix $A - kI$ is equal to zero. **(2 marks)**

d. Find the inverse of the matrix P. **(2 marks)**

e. Determine the matrix $P^{-1}AP$. **(2 marks)**

Question 10 (5 marks)

Given the matrices $D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$:

a. The matrix D is called a diagonal matrix as it has non-zeros on the leading diagonal and zeros elsewhere. Find the matrices D^2, D^3 and D^{-1}. **(3 marks)**

b. If $F = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ where e and f are non-zero real numbers, find the matrix F^2 and hence write down the matrices F^6 and F^{-1}. **(2 marks)**

Question 11 (4 marks)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The transpose of the matrix A is given by $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Find the matrix AA^T and show that $\det\left(AA^T\right) = \det(A)\det\left(A^T\right)$.

Answers and marking guide

5.2 Addition, subtraction and scalar multiplication of matrices

Question 1

$2B = \begin{bmatrix} -14 & -2 \\ 4 & 6 \end{bmatrix}$ **[1 mark]**

$-3A = \begin{bmatrix} -6 & -12 \\ -15 & -24 \end{bmatrix}$ **[1 mark]**

$2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **[1 mark]**

$$\begin{aligned} X &= 2B - 3A + 2I \\ &= \begin{bmatrix} -14 - 6 + 2 & -2 - 12 + 0 \\ 4 - 15 + 0 & 6 - 24 + 2 \end{bmatrix} \\ &= \begin{bmatrix} -18 & -14 \\ -11 & -16 \end{bmatrix} \end{aligned}$$ **[1 mark]**

Question 2

$$\begin{bmatrix} 7 & 5 \\ a & -2 \end{bmatrix} - \begin{bmatrix} -2 & b \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ -3 & -9 \end{bmatrix}$$

$5 - b = 6$

$\therefore b = -1$

$a - 3 = -3$

$\therefore a = 0$

Question 3

$$A = \begin{bmatrix} a & 3 \\ x & -6 \end{bmatrix}, B = \begin{bmatrix} -2 & b \\ 4 & y \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2A - 3B = O$$

$$\begin{bmatrix} 2a & 6 \\ 2x & -12 \end{bmatrix} - \begin{bmatrix} -6 & 3b \\ 12 & 3y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2a + 6 & 6 - 3b \\ 2x - 12 & -12 - 3y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore

$2a + 6 = 0$

$2a = -6$

$a = -3$

$6 - 3b = 0$

$3b = 6$

$b = 2$

$2x - 12 = 0$

$2x = 12$

$x = 6$

$-12 - 3y = 0$

$3y = -12$

$y = -4$

Award **2 marks** for the correct values of a, b, x and y.

Question 4

$$2A = \begin{bmatrix} 0 & 2 \\ -8 & -6 \\ 4 & 2 \end{bmatrix} \textbf{[1 mark]}$$

$$-3B = \begin{bmatrix} -6 & 12 \\ -18 & 0 \\ -9 & -18 \end{bmatrix} \textbf{[1 mark]}$$

$-2A + C = -3B$

$$C = -3B + 2A \textbf{ [1 mark]}$$

$$= \begin{bmatrix} -6 & 12 \\ -18 & 0 \\ -9 & -18 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -8 & -6 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 14 \\ -26 & -6 \\ -5 & -16 \end{bmatrix} \textbf{[1 mark]}$$

Question 5

a_{32} is the element in row 3 and column 2.

Question 6

$$A = \begin{bmatrix} 2 & -3 \\ -5 & 5 \end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} 2 \times 2 & -3 \times 2 \\ -5 \times 2 & 5 \times 2 \end{bmatrix}$$

$$\Rightarrow -2A = \begin{bmatrix} -4 & 6 \\ 10 & -10 \end{bmatrix}$$

5.3 Matrix multiplication

Question 1

$(2 \times 3) \times (3 \times 1)$

Multiplication is defined, since the number of columns in $A(3)$ is equal to the number of rows in $B(3)$, and the resultant matrix is of the order (2×1).

Question 2

To multiply matrices, the number of columns of the first matrix must be the same as the number of rows of the second matrix. Option C contains 1×3 matrix followed by a 3×1 matrix. The number of columns of the first matrix, 3, is the same as the number of the second matrix, 3, and therefore the matrices can be multiplied.

Question 3

$$\begin{bmatrix} 1 & 4 \\ 6 & 3 \end{bmatrix} \times \begin{bmatrix} 8 & 1 & 3 \\ 0 & 9 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 8 + 4 \times 0 & 1 \times 1 + 4 \times 9 & 1 \times 3 + 4 \times 4 \\ 6 \times 8 + 3 \times 0 & 6 \times 1 + 3 \times 9 & 6 \times 3 + 3 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 37 & 19 \\ 48 & 33 & 30 \end{bmatrix}$$

Question 4

$$A(B+C)=\begin{bmatrix}5 & 2\\0 & 4\end{bmatrix}\begin{bmatrix}2-1 & 4+0\\-1+0 & 3+4\end{bmatrix}$$

$$=\begin{bmatrix}5 & 2\\0 & 4\end{bmatrix}\begin{bmatrix}1 & 4\\-1 & 7\end{bmatrix}$$ **[1 mark]**

$$=\begin{bmatrix}3 & 34\\-4 & 28\end{bmatrix}$$ **[1 mark]**

$$AB=\begin{bmatrix}5 & 2\\0 & 4\end{bmatrix}\begin{bmatrix}2 & 4\\-1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}8 & 26\\-4 & 12\end{bmatrix}$$ **[1 mark]**

$$AC=\begin{bmatrix}5 & 2\\0 & 4\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}-5 & 8\\0 & 16\end{bmatrix}$$ **[1 mark]**

$$AB+AC=\begin{bmatrix}8 & 26\\-4 & 12\end{bmatrix}+\begin{bmatrix}-5 & 8\\0 & 16\end{bmatrix}$$

$$=\begin{bmatrix}3 & 34\\-4 & 28\end{bmatrix}$$ **[1 mark]**

$$\therefore A(B+C)=AB+AC=\begin{bmatrix}3 & 34\\-4 & 28\end{bmatrix}$$

so the Distributive Law holds **[1 mark]**

Question 5

$$A^2=\begin{bmatrix}1 & -2\\-3 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\-3 & 4\end{bmatrix}$$

$$=\begin{bmatrix}7 & -10\\-15 & 22\end{bmatrix}$$ **[1 mark]**

$$A^2I=\begin{bmatrix}7 & -10\\-15 & 22\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}7 & -10\\-15 & 22\end{bmatrix}$$ **[1 mark]**

$$IA^2=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}7 & -10\\-15 & 22\end{bmatrix}$$

$$=\begin{bmatrix}7 & -10\\-15 & 22\end{bmatrix}$$ **[1 mark]**

$\therefore A^2I=IA^2=A^2$ **[1 mark]**

5.4 Determinants and inverses

Question 1

$\Delta = 2\times 3-1\times 6$

$\quad = 0$

$\therefore$ the matrix is singular.

Question 2

$$\begin{bmatrix} 4 & 7 \\ 2 & 9 \end{bmatrix}^{-1} = \frac{1}{(4 \times 9 - 7 \times 2)} \times \begin{bmatrix} 9 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{22} \times \begin{bmatrix} 9 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{22} \times 9 & \frac{1}{22} \times -7 \\ \frac{1}{22} \times -2 & \frac{1}{22} \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{22} & \frac{-7}{22} \\ \frac{-1}{11} & \frac{2}{11} \end{bmatrix}$$

Question 3

$$A - kI = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \textbf{ [1 mark]}$$

$$= \begin{bmatrix} 1-k & 2 \\ 5 & 4-k \end{bmatrix}$$

$$\det(A - kI) = (1-k)(4-k) - 2 \times 5$$

$$0 = 4 - 5k + k^2 - 10$$

$$0 = k^2 - 5k - 6$$

$$0 = (k-6)(k+1) \textbf{ [1 mark]}$$

$$\therefore k = 6, -1 \textbf{ [1 mark]}$$

Question 4

$$\text{Inverse of } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$M^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \textbf{ [1 mark]}$$

$$\therefore M = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \textbf{ [1 mark]}$$

$$MM^{-1} = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \textbf{ [1 mark]}$$

$$M^{-1}M = \frac{1}{5} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \textbf{ [1 mark]}$$

$$\therefore MM^{-1} = M^{-1}M = I$$

Question 5

$$\begin{bmatrix} 6 & 2 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{6 \times 2 - 2 \times 3} \times \begin{bmatrix} 2 & -2 \\ 3 & 6 \end{bmatrix} \textbf{ [1 mark]}$$

$$= \frac{1}{18} \times \begin{bmatrix} 2 & -2 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{18} & \frac{-2}{18} \\ \frac{3}{18} & \frac{6}{18} \end{bmatrix} \textbf{ [1 mark]}$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{-1}{9} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \textbf{ [1 mark]}$$

Question 6

Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$M^{-1} = \frac{1}{11} \begin{bmatrix} -2 & -5 \\ 3 & 2 \end{bmatrix}$

$M = \begin{bmatrix} 2 & 5 \\ -3 & -2 \end{bmatrix}$ **[1 mark]**

$$MM^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -2 \end{bmatrix} \frac{1}{11} \begin{bmatrix} -2 & -5 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 2 & 5 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -2 & -5 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \textbf{ [1 mark]}$$

$$M^{-1}M = \frac{1}{11} \begin{bmatrix} -2 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -2 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \textbf{ [1 mark]}$$

$\therefore MM^{-1} = M^{-1}M = I$ **[1 mark]**

Question 7

$$\det = 2 \times 1 - 0 \times 1$$
$$= 2$$

$$\therefore \text{Inverse} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

5.5 Matrix equations

Question 1

$2x + 5y = -3$

$7x - 2y = 9$

$$\begin{bmatrix} 2 & 5 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$ **[1 mark]**

$$A = \begin{bmatrix} 2 & 5 \\ 7 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{(2 \times -2) - (5 \times 7)} \begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}$$ **[1 mark]**

$$AX = B$$

$$X = A^{-1}B$$

$$X = \frac{1}{-39} \begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$X = -\frac{1}{39} \begin{bmatrix} -39 \\ 39 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ **[1 mark]**

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$x = 1,\ y = -1$ **[1 mark]**

Question 2

$$A = \begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (3 \times -4) - (2 \times -1) \\ &= -12 + 2 \\ &= -10 \end{aligned}$$

$$A^{-1} = -\frac{1}{10} \begin{bmatrix} -4 & 1 \\ -2 & 3 \end{bmatrix}$$

$$AX = B$$

$$\begin{aligned} X &= A^{-1}B \\ &= -\frac{1}{10} \begin{bmatrix} -4 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -7 & 7 \\ -8 & -2 \end{bmatrix} \\ &= -\frac{1}{10} \begin{bmatrix} 20 & -30 \\ -10 & -20 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

Question 3

If the solution is unique, $\Delta \neq 0$.

System I:

$$2x - y = -3$$
$$5x + y = 4$$

In matrix form, $\begin{bmatrix} 2 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$\begin{aligned} \Delta &= (2 \times 1) - (-1 \times 5) \\ &= 7 \\ &\neq 0 \end{aligned}$$

$\therefore$ system I has a unique solution.

System II:

$$\begin{aligned} y &= 5 - 2x \\ 2x + y &= 5 \\ 4x + 2y &= 5 \end{aligned}$$

In matrix form, $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$$\begin{aligned} \Delta &= (2 \times 2) - (4 \times 1) \\ &= 0 \\ &= 0 \end{aligned}$$

$\therefore$ system II has no unique solution.

System III:

$$\begin{aligned} 7x - 5y &= 1 \\ 2x &= 6 + y \\ 2x - y &= 6 \end{aligned}$$

In matrix form, $\begin{bmatrix} 7 & -5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

$$\begin{aligned} \Delta &= (7 \times -1) - (2 \times -5) \\ &= 3 \\ &\neq 0 \end{aligned}$$

$\therefore$ system III has a unique solutions.

$\therefore$ systems I and III have unique solution

Question 4

a. Cost matrix: $C = \begin{bmatrix} 10 \\ 12 \\ 12 \end{bmatrix}$ **[1 mark]**

b. $\begin{matrix} A \\ B \end{matrix} \begin{bmatrix} 30 & 30 & 20 \\ 20 & 15 & 20 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 12 \end{bmatrix}$ **[1 mark]**

$$\begin{aligned} \text{Shop A} &= 30 \times 10 + 30 \times 12 + 20 \times 12 \\ &= 900 \\ \text{Shop B} &= 20 \times 10 + 15 \times 12 + 20 \times 12 \\ &= 620 \end{aligned}$$

$\therefore$ the cost for shop A is \$900, and the cost for shop B is \$620. **[1 mark]**

Question 5

a. $3m + s = 9$

$2m + 3s = 13$ **[1 mark]**

b. $\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} m \\ s \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \end{bmatrix}$ **[1 mark]**

c. $\det\begin{bmatrix}3 & 1\\2 & 3\end{bmatrix} = 3\times 3 - 1\times 2$ **[1 mark]**

$= 9 - 2$

$= 7$ **[1 mark]**

d. $\begin{bmatrix}3 & 1\\2 & 3\end{bmatrix}^{-1} = \frac{1}{7}\times\begin{bmatrix}3 & -1\\-2 & 3\end{bmatrix}$

$= \begin{bmatrix}\frac{3}{7} & \frac{-1}{7}\\ \frac{-2}{7} & \frac{3}{7}\end{bmatrix}$ **[1 mark]**

e. $\begin{bmatrix}3 & 1\\2 & 3\end{bmatrix}\times\begin{bmatrix}m\\s\end{bmatrix} = \begin{bmatrix}9\\13\end{bmatrix}$

$\begin{bmatrix}m\\s\end{bmatrix} = \begin{bmatrix}3 & 1\\2 & 3\end{bmatrix}^{-1}\times\begin{bmatrix}9\\13\end{bmatrix}$ **[1 mark]**

$= \begin{bmatrix}\frac{3}{7} & \frac{-1}{7}\\ \frac{-2}{7} & \frac{3}{7}\end{bmatrix}\times\begin{bmatrix}9\\13\end{bmatrix}$

$= \begin{bmatrix}\frac{3}{7}\times 9 & + & \frac{-1}{7}\times 13\\ \frac{-2}{7}\times 9 & + & \frac{3}{7}\times 13\end{bmatrix}$ **[1 mark]**

$= \begin{bmatrix}\frac{27}{7} - & \frac{13}{7}\\ \frac{39}{7} - & \frac{18}{7}\end{bmatrix}$

$= \begin{bmatrix}\frac{14}{7}\\ \frac{21}{7}\end{bmatrix}$

$= \begin{bmatrix}2\\3\end{bmatrix}$ **[1 mark]**

Therefore, Mars Bars cost \$2 each and Snickers cost \$3.

Question 6

$3x - 2y = 7$

$5x + y = 3$

$\begin{bmatrix}3 & -2\\5 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}7\\3\end{bmatrix}$ **[1 mark]**

$A = \begin{bmatrix}3 & -2\\5 & 1\end{bmatrix}, X = \begin{bmatrix}x\\y\end{bmatrix}$ and $B = \begin{bmatrix}7\\3\end{bmatrix}$

$A^{-1} = \frac{1}{(3\times 1) - (-2\times 5)}\begin{bmatrix}1 & 2\\-5 & 3\end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix}1 & 2\\-5 & 3\end{bmatrix}$ **[1 mark]**

$AX = B$

$X = A^{-1}B$

$= \frac{1}{13}\begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 13 \\ -26 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **[1 mark]**

$x = 1, y = -2$ **[1 mark]**

Question 7

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2\times & x + 1 \times y \\ 3\times & x + (-2) \times y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$\therefore 2x + y = 5$

$\therefore 3x - 2y = 4$

5.6 Review

Question 1

$3x - 2y = 7$

$5x + y = 3$

$\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ **[1 mark]**

$\text{A} = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}, \text{X} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and B} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$\text{A}^{-1} = \frac{1}{(3 \times 1) - (-2 \times 5)}\begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$ **[1 mark]**

$\text{AX} = \text{B}$

$\text{X} = \text{A}^{-1}\text{B}$

$= \frac{1}{13}\begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 13 \\ -26 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **[1 mark]**

$x = 1, y = -2$ **[1 mark]**

Question 2

$$\begin{aligned}\det &= 2\times 1 - 0\times 1\\ &= 2\end{aligned}$$

$$\therefore \text{Inverse} = \frac{1}{2}\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}$$

Question 3

$$\text{A (B + C)} = \begin{bmatrix}5 & 2\\ 0 & 4\end{bmatrix}\begin{bmatrix}2-1 & 4+0\\ -1+0 & 3+4\end{bmatrix}$$

$$= \begin{bmatrix}5 & 2\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & 4\\ -1 & 7\end{bmatrix}$$ **[1 mark]**

$$= \begin{bmatrix}3 & 34\\ -4 & 28\end{bmatrix}$$ **[1 mark]**

$$\text{AB} = \begin{bmatrix}5 & 2\\ 0 & 4\end{bmatrix}\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}8 & 26\\ -4 & 12\end{bmatrix}$$ **[1 mark]**

$$\text{AC} = \begin{bmatrix}5 & 2\\ 0 & 4\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 4\end{bmatrix}$$

$$= \begin{bmatrix}-5 & 8\\ 0 & 16\end{bmatrix}$$ **[1 mark]**

$$\text{AB + AC} = \begin{bmatrix}8 & 26\\ -4 & 12\end{bmatrix} + \begin{bmatrix}-5 & 8\\ 0 & 16\end{bmatrix}$$

$$= \begin{bmatrix}3 & 34\\ -4 & 28\end{bmatrix}$$ **[1 mark]**

$$\therefore \text{A (B + C)} = \text{AB + AC} = \begin{bmatrix}3 & 34\\ -4 & 28\end{bmatrix}$$

So the Associative Law holds. **[1 mark]**

Question 4

$$\text{A}^2 = \begin{bmatrix}1 & -2\\ -3 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\ -3 & 4\end{bmatrix}$$

$$= \begin{bmatrix}7 & -10\\ -15 & 22\end{bmatrix}$$ **[1 mark]**

$$\text{A}^2\text{I} = \begin{bmatrix}7 & -10\\ -15 & 22\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}7 & -10\\ -15 & 22\end{bmatrix}$$ **[1 mark]**

$$\text{IA}^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}7 & -10\\ -15 & 22\end{bmatrix}$$

$$= \begin{bmatrix}7 & -10\\ -15 & 22\end{bmatrix}$$ **[1 mark]**

$\therefore A^2I = IA^2 = A^2$ **[1 mark]**

Question 5

Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$M^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \quad \textbf{[1 mark]}$$

$$\begin{aligned}
\text{MM}^{-1} &= \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}\frac{1}{5}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{I} \ \textbf{[1 mark]}
\end{aligned}$$

$$\begin{aligned}
\text{M}^{-1}\text{M} &= \frac{1}{5}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \ \textbf{[1 mark]}
\end{aligned}$$

$\therefore MM^{(-1)} = M^{(-1)}M = I$ **[1 mark]**

Question 6

$$\begin{aligned}
B^2 &= \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 4 \times 4 + (-2) \times 1 & 4 \times (-2) + (-2) \times (-1) \\ 1 \times 4 + (-1) \times 1 & 1 \times (-2) + (-1) \times (-1) \end{bmatrix} \\
&= \begin{bmatrix} 14 & -6 \\ 3 & -1 \end{bmatrix}
\end{aligned}$$

Question 7

$$\begin{aligned}
A^2 + 9I &= O \\
A^2 &= -9I \\
AA &= -9I \\
AAA^{-1} &= -9IA^{-1} \\
A &= -9A^{-1} \\
-\frac{1}{9}A &= A^{-1}
\end{aligned}$$

Question 8

a. $A = \begin{bmatrix} 3 & 7 \\ 5 & -2 \end{bmatrix}$ $\quad \det(A) = -6 - 35 = -41$ **[1 mark]**

$$A^{-1} = -\frac{1}{41}\begin{bmatrix} -2 & -7 \\ -5 & 3 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 2 & 7 \\ 5 & -3 \end{bmatrix} \quad \textbf{[1 mark]}$$

b. $AX = C$

$A^{-1}AX = A^{-1}C$ **[1 mark]**

$IX = A^{-1}C$

$X = A^{-1}C = \frac{1}{41}\begin{bmatrix} 2 & 7 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 15 \\ -16 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} -82 \\ 123 \end{bmatrix}$

$X = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **[1mark]**

c. $DA = B$

$DAA^{-1} = BA^{-1}$ **[1 mark]**

$DI = BA^{-1}$

$D = BA^{-1} = \begin{bmatrix} 19 & -24 \\ 33 & 36 \end{bmatrix}\frac{1}{41}\begin{bmatrix} 2 & 7 \\ 5 & -3 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} -82 & 205 \\ 246 & 123 \end{bmatrix}$

$D = \begin{bmatrix} -2 & 5 \\ 6 & 3 \end{bmatrix}$ **[1 mark]**

Question 9

a. $A - kI = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-k & -3 \\ -1 & 2-k \end{bmatrix}$ **[1 mark]**

$\det(A - kI) = (4-k)(2-k) - 3 = 8 - 6k + k^2 - 3$

$\det(A - kI) = k^2 - 6k + 5$

$p = 1, \quad q = -6, \ r = 5$ **[1 mark]**

b. $pA^2 + qA + rI = A^2 - 6A + 5I$

$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} - 6\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **[1 mark]**

$= \begin{bmatrix} 19 & -18 \\ -6 & 7 \end{bmatrix} - \begin{bmatrix} 24 & -18 \\ -6 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **[1 mark]**

c. $\det(A - kI) = k^2 - 6k + 5 = (k-5)(k-1) = 0$ **[1 mark]**

$k = 5, -1$ **[1 mark]**

d. $P = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ $\det(P) = 3 + 1 = 4$ **[1 mark]**

$P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ **[1 mark]**

e. $P^{-1}AP$

$= \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{4}\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}\right)\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ **[1 mark]**

$= \frac{1}{4}\begin{bmatrix} 5 & -5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 20 & 0 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ **[1 mark]**

Question 10

a. $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 1 \end{bmatrix}$ **[1 mark]**

$D^3 = D^2 D = \begin{bmatrix} 25 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 125 & 0 \\ 0 & 1 \end{bmatrix}$ **[1 mark]**

$\det(D) = 5, \quad D^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}$ **[1 mark]**

b. $F = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \quad F^2 = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} e^2 & 0 \\ 0 & f^2 \end{bmatrix}$ **[1 mark]**

$F^6 = \begin{bmatrix} e^6 & 0 \\ 0 & f^6 \end{bmatrix} \quad F^{-1} = \begin{bmatrix} \frac{1}{e} & 0 \\ 0 & \frac{1}{f} \end{bmatrix}$ **[1 mark]**

Question 11

$AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$ **[1 mark]**

Now $\det(A) = \det\left(A^T\right) = ad - bc$ **[1 mark]**

$$\begin{aligned} \det\left(AA^T\right) &= \left(a^2 + b^2\right)\left(c^2 + d^2\right) - (ac + bd)^2 \\ &= \left(a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2\right) - \left(a^2c^2 + 2abcd + b^2d^2\right) \quad \textbf{[1 mark]} \\ &= a^2d^2 - 2abcd + b^2c^2 \\ &= (ad - bc)^2 = \det(A)\det\left(A^T\right) \quad \textbf{[1 mark]} \end{aligned}$$

6 Graph Theory

Topic	6	Graph theory
Subtopic	6.2	Introduction to graph theory

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Question 1 (2 marks)

If two simple graphs are isomorphic, determine if the complement of each graph also isomorphic. Justify your answer.

Question 2 (3 marks)

A graph G is self-complementary if G is isomorphic to the complement of G. Show that if G is self-complementary, then the number of vertices is equal to $4k$ or $4k + 1$, where k is an integer.

Question 3 (1 mark)

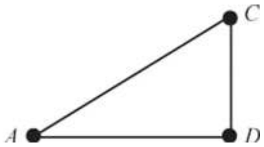
Consider the following adjacency matrix for a graph G.

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The sequence that gives the possible degrees of each vertex in G is

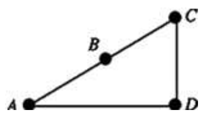
A. (2, 2, 3, 3, 4)
B. (0, 1, 1, 2, 0)
C. (0, 0, 1, 1, 2)
D. (2, 3, 3, 3, 3)
E. (2, 3, 4)

Question 4 (1 mark)

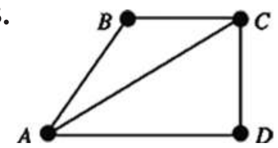


The graph above is a subgraph of which one of the following graphs?

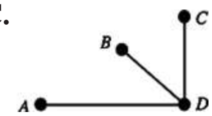
A.



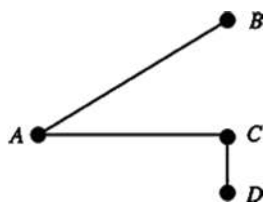
B.



C.



D.

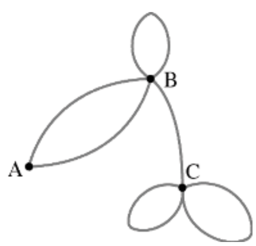


E. None of the above

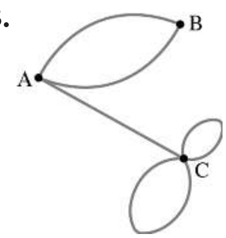
Question 5 (1 mark)

The adjacency matrix $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ represents which of the following networks?

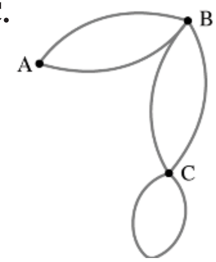
A.



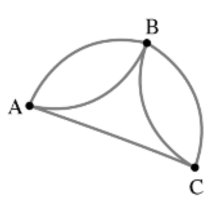
B.



C.

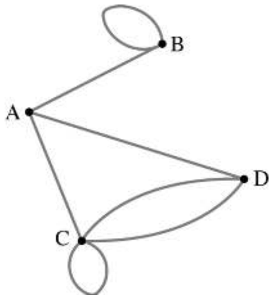


D.



E. None of the above

Question 6 (1 mark)



Construct the adjacency matrix for the network above.

Question 7 (1 mark)

Which of the adjacency matrices describes a connected graph?

A. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}$

E. All of the above

Question 8 (1 mark)

Which of the following statements best describes a simple graph?

A. A graph in which each vertex is connected directly to every other vertex
B. A graph with no loops or multiple edges between vertices
C. A graph with loops and multiple edges
D. A graph that can be drawn without any edges crossing each other
E. A graph in which pairs of vertices are connected by at least two edges

Topic	6	Graph theory
Subtopic	6.3	Planar graphs and Euler's formula

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Question 1 (1 mark)

If G is a 3-regular planar graph with 24 vertices, determine how many faces there will be in the planar drawing of G.

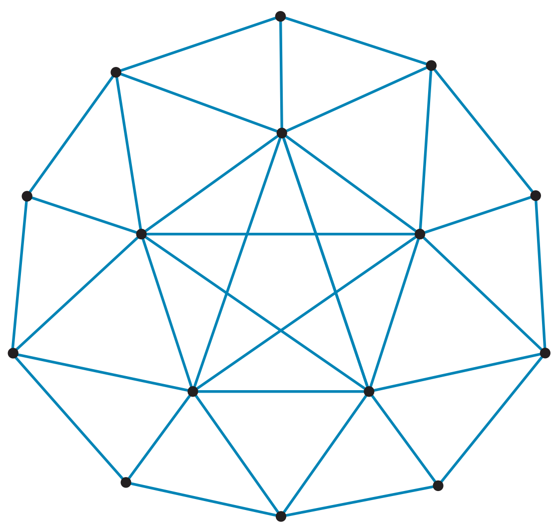
$f = \square$

Question 2 (2 marks)

A graph has 6 vertices and 10 edges. Explain whether it is planar, non-planar or either. Justify your answer.

Question 3 (2 marks)

State whether the following graph is planar. Justify your answer.



Question 4 (1 mark)
The given adjacency matrix represents a planar graph with 3 vertices. The number of faces (regions) on this planar graph is

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

A. 4
B. 5
C. 6
D. 7
E. 8

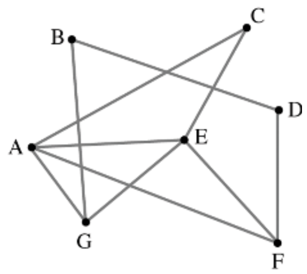
Question 5 (1 mark)
A connected planar graph has 5 vertices and 7 edges. How many faces does it have?
A. 2
B. 3
C. 4
D. 5
E. 6

Question 6 (1 mark)
A connected planar graph has 14 edges and 7 faces. How many vertices does it have?
A. 3
B. 5
C. 7
D. 9
E. 10

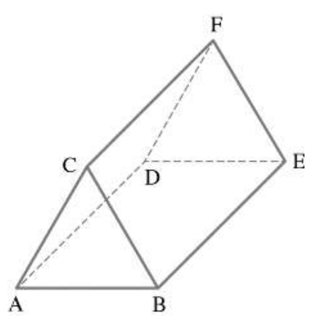
Topic 6 | Subtopic 6.3 Planar graphs and Euler's formula

Question 7 (1 mark)

Redraw the following graph to show that it is planar.



Question 8 (1 mark)



Represent this triangular prism as a planar graph.

Topic	6	Graph theory
Subtopic	6.4	Eulerian and Hamiltonian graphs

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Question 1 (1 mark)
State what values of n will result in K_n being an Eulerian graph.

Question 2 (4 marks)
Hamiltonian cycles of a graph are only considered distinct from one another if they use a different set of edges. That is, the order the edges are traversed is not relevant. It can be seen then that K_3 only has 1 Hamiltonian cycle.

a. Determine the number of distinct Hamiltonian cycles in: **(2 marks)**

i. K_4 **(1 mark)**

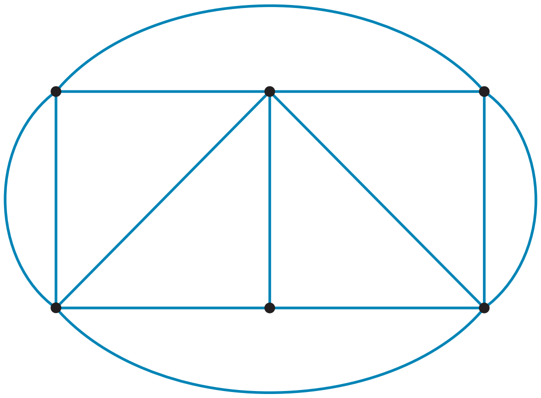
☐ distinct Hamiltonian cycles

ii. K_5 **(1 mark)**

☐ distinct Hamiltonian cycles

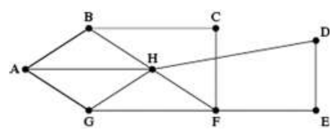
b. Determine a rule that connects the number of Hamiltonian cycles with n, the number of vertices in the complete graph K_n. **(2 marks)**

Question 3 (1 mark)
An Eulerian trail for the graph will be possible if only one edge is removed. The number of different ways in which this could this be done is



A. 3
B. 4
C. 5
D. 6
E. 7

Question 4 (1 mark)
The sequence of vertices that represent a trail in the network shown is



A. BHFGHBA
B. HFEDHFG
C. HBAHGFH
D. AHGAHBC
E. All of the above

Question 5 (1 mark)

Which of the following statements is incorrect?

A. A path uses vertices only once.
B. A trail uses edges only once.
C. A cycle finishes on its starting edge.
D. A circuit finishes on its starting vertex.
E. A simple cycle is a cycle in a graph with no-repeated vertices.

Question 6 (1 mark)

Which of the following statements best describes a Hamiltonian path?

A. A path that begins and ends at the same vertex and passes through each vertex exactly once.
B. A path that begins and ends at the same vertex and passes through each edge exactly once.
C. A path that begins and ends at different vertices and passes through each vertex exactly once.
D. A path that begins and ends at different vertices and passes through each edge exactly once.
E. A path that begins and ends at the same vertex and passes through each vertex at least once.

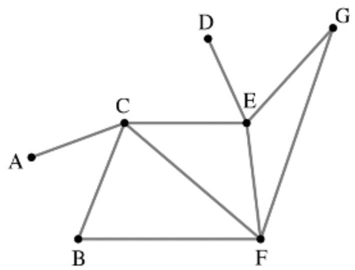
Question 7 (1 mark)

How many edges need to be removed to ensure that there is no Hamiltonian cycle starting at X?

A. 4
B. 5
C. 6
D. 7
E. 8

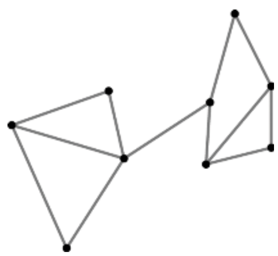
Question 8 (1 mark)

Identify the semi-Hamiltonian cycle that starts at vertex A in the following graph.



Question 9 (2 marks)

Explain why the following graph is semi-Hamiltonian, but not Hamiltonian.



Question 10 (1 mark)

Draw an Eulerian graph with 4 vertices and 5 edges.

Topic	6	Graph theory
Subtopic	6.5	Weighted graphs and trees

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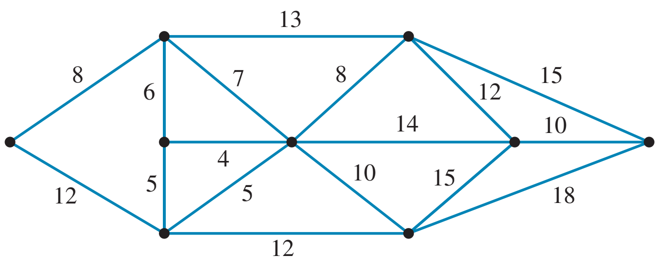
Question 1 (1 mark)

The following showing the distances (in km) between different camping sites in a national park. Determine the minimum length of cable required to connect all the camps to the same network.



The minimum length of cable required is ☐km

Question 2 (2 marks)

Let G be a connected graph.

a. Describe what can be said about an edge of G that appears in every spanning tree. **(1 mark)**

b. Describe what can be said about an edge of G that appears in no spanning tree. **(1 mark)**

Question 3 (4 marks)

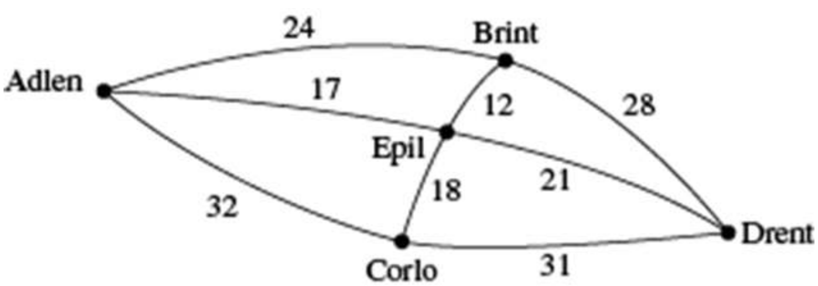
Consider the complete graph K_n where the vertices are labelled 1 through to n.

a. Determine the value of the minimum spanning tree if the weight of each edge is the difference of the value of the vertices it connects. **(2 marks)**

b. Determine the value of the minimum spanning tree if the weight of each edge is the sum of the value of the vertices it connects. **(2 marks)**

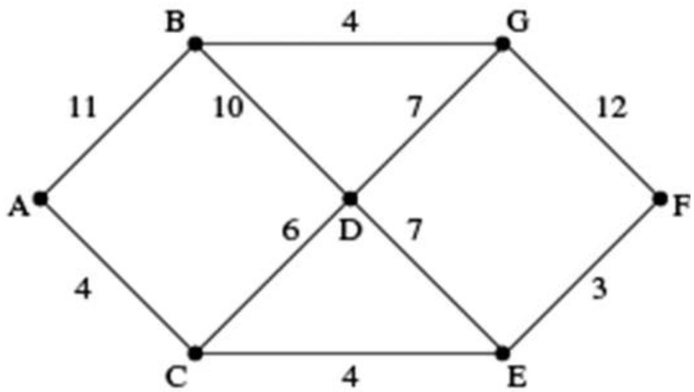
Question 4 (1 mark)

The weighted network shows the distance, in kilometres, between a series of towns, Adlen, Brint, Corlo, Epil and Drent. Which of the following distances is not possible to achieve in travelling from Adlen to Drent?



A. 90 km
B. 66 km
C. 52 km
D. 40 km
E. None of the above

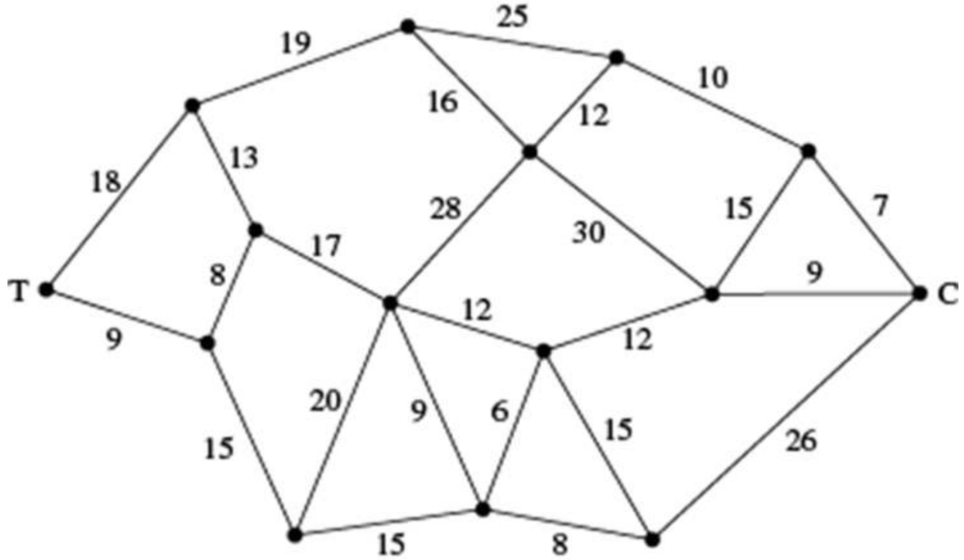
Question 5 (1 mark)
A journey is taken from point A to point G where the numbers on edges indicate distances in kilometres. What is the maximum possible distance that can be travelled if the journey must pass through each vertex only once?



A. 15 km
B. 23 km
C. 46 km
D. 52 km
E. 43 km

Question 6 (1 mark)

The network diagram shows the distances, in kilometres, between a major regional town, T, and the nearest capital city, C. The area in between is home to many smaller towns. On a particular day a locum doctor leaves the regional town T and must visit each small town and the capital city once only, and return home to town T. Calculate the minimum distance he must travel to achieve this task.



A. 195 km
B. 215 km
C. 236 km
D. 245 km
E. 180 km

Topic	6	Graph theory
Subtopic	6.6	Bipartite graphs and the Hungarian algorithm

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Question 1 (4 marks)

Consider the complete bipartite graph $K_{3,4}$.

a. Draw the graph of $K_{3,4}$. **(1 mark)**

b. Determine if the graph has a Eulerian cycle. Justify your answer. **(1 mark)**

c. Determine if the graph has a Hamiltonian cycle. Justify your answer. **(2 marks)**

Question 2 (2 marks)

Answer the following

a. State which complete bipartite graphs are also Eulerian graphs. **(1 mark)**

b. State which complete bipartite graphs also semi-Eulerian Graphs. **(1 mark)**

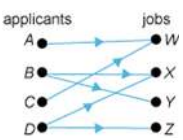
Topic 6 | Subtopic 6.6 Bipartite graphs and the Hungarian algorithm

Question 3 (3 marks)
Four swimmers are to compete in a medley relay. During training each swimmer has swum each leg of the relay and their times have been recorded in the following table.

	Leg 1	Leg 2	Leg 3	Leg 4
Mary	2	4	3	5
Jenny	3	5	3	4
Pauline	2	3	4	2
Jacinta	2	4	2	3

Determine the optimal allocation and the minimum time to complete the race.

Question 4 (1 mark)



Which of the following statements is true about the bipartite graph above?

A. Person *A* must do job *W*.
B. Person *B* must do job *Z* .
C. Person *C* cannot do job *W*.
D. Person *D* can do jobs *W*, *X* or *Y*.
E. Job *X* can only be done by person *D*.

Question 5 (1 mark)
Four friends are discussing the countries they would like to visit over the winter.
Georgie would like to visit New Zealand or Indonesia.
Tim wants to visit Japan. Sarah would like to visit Thailand or Japan.
Karan would like to go to any of the countries already suggested.
Draw a graph to show the travel preferences of the four friends.

Question 6 (2 marks)

A company has 4 vacant positions and 4 applicants in total.
They believe that Naomi would be a good fit for jobs 1 and 2; Aaron would work well in jobs 2 and 4; Julian only suits job 2 and Maria would be outstanding in jobs 1, 3 or 4.
Draw the graph representing the situation and determine which person gets each job.

Question 7 (1 mark)

The time in minutes for a team of people to complete a series of tasks is given in the following table.

	T1	T2	T3	T4	T5
P1	12	9	11	8	15
P2	12	15	13	12	6
P3	9	11	8	11	9
P4	8	12	9	13	14
P5	12	15	13	15	13

Determine the allocation of tasks to each individual to ensure all tasks are completed in the minimum time possible. What is the minimum time possible?

A. 38 mins
B. 39 mins
C. 42 mins
D. 43 mins
E. 45 mins

Question 8 (1 mark)

Four workers need to be allocated four tasks. The time required for each worker for each task is summarised in the table below.

	Task 1	Task 2	Task 3	Task 4
Worker 1	30	70	19	32
Worker 2	45	62	21	44
Worker 3	38	75	26	37
Worker 4	26	73	24	39

Use the Hungarian algorithm to minimise the time required to complete the tasks by allocating one job to each worker.

Topic	**6**	**Graph theory**
Subtopic	**6.7**	**Review**

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Question 1 (2 marks)

Three houses need to be connected to three utilities: gas water and electricity. Determine if it is possible to connect each house to each utility without the lines intersecting. Justify your answer.

Question 2 (1 mark)

A complete graph with 7 vertices will have a total number of edges of

A. 7
B. 8
C. 14
D. 21
E. 28

Question 3 (1 mark)

If G is a simple connected 3-regular planar graph where every face is bounded by exactly 3 edges, then the number of edges in G is

A. 4
B. 5
C. 6
D. 7
E. 8

Question 4 (5 marks)

A team of four — Barnie, Ruth, Shelley and Carlos have been selected to play four holes of golf for a social event for their workplace. Each member of the team will play one of the holes. Their scores for their previous rounds of golf for these four holes have been summarised in the following table.

	Hole 1	**Hole 2**	**Hole 3**	**Hole 4**
Barnie	5	7	5	9
Ruth	6	10	10	7
Shelley	7	5	3	8
Carlos	7	8	8	9

a. Perform a row reduction on the matrix formed from this table. **(2 marks)**

b. Perform a column reduction on the matrix from part **a**. **(1 mark)**

c. Apply the Hungarian algorithm if necessary. **(1 mark)**

d. State the optimal team for this event. **(1 mark)**

Question 5 (5 marks)

A cruise ship takes passengers around Tasmania between the seven locations marked on the map.

The sailing distances between locations are indicated in the table.

	Hobart	Bruny I.	Maria I.	Flinders I.	Devonport	Robbins I.	King I.
Hobart	–	65 km	145 km	595 km	625 km	–	–
Bruny I.	65 km	–	130 km	–	–	715 km	–
Maria I.	145 km	130 km	–	450 km	–	–	–
Flinders I.	595 km	–	450 km	–	330 km	405 km	465 km
Devonport	625 km	–	–	330 km	–	265 km	395 km
Robbins I.	–	715 km	–	405 km	265 km	–	120 km
King I.	–	–	–	465 km	395 km	120 km	–

a. Draw a weighted graph to represent all possible ways of travelling to the locations. **(2 marks)**

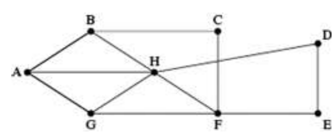
b. Determine the shortest route from Hobart to Robbins Island. **(1 mark)**

c. Determine the shortest way of travelling from Hobart to visit each location only once. **(1 mark)**

d. Determine the shortest way of sailing from King Island, visiting each location once and returning to King Island. **(1 mark)**

Question 6 (1 mark)

The sequence of vertices that represent a trail in the network shown is



A. BHFGHBA
B. HFEDHFG
C. HBAHGFH
D. AHGAHBC
E. GHFEDHBA

Question 7 (1 mark)

Which of the following statements is incorrect?

A. A path uses vertices only once.
B. A trail uses edges only once.
C. A cycle finishes on its starting edge.
D. A circuit finishes on its starting vertex.
E. A closed trail begins and ends at the same vertex.

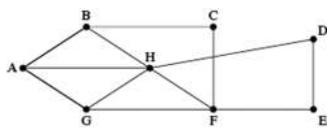
Question 8 (1 mark)

Which of the following statements best describes a Hamiltonian path?

A. A path that begins and ends at the same vertex and passes through each vertex exactly once.
B. A path that begins and ends at the same vertex and passes through each edge exactly once.
C. A path that begins and ends at different vertices and passes through each vertex exactly once.
D. A path that begins and ends at different vertices and passes through each edge exactly once.
E. A path that uses every edge of a graph exactly one.

Question 9 (1 mark)

The sequence of vertices that represent a trail in the network shown is



A. BHFGHBA
B. HFEDHFG
C. HBAHGFH
D. AHGAHBC
E. EFCBAHDEFC

Question 10 (1 mark)

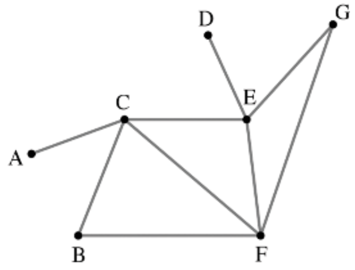
How many edges need to be removed to ensure that there is no Hamiltonian cycle starting at X?

A. 4
B. 5
C. 6
D. 7
E. 8

***Source:** Jacaranda (John Wiley & Sons Australia, Ltd), Practice Question*

Question 11 (1 mark)

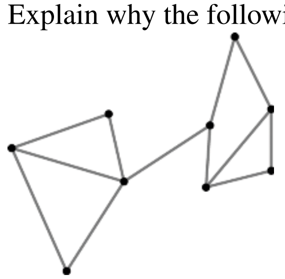
Identify the semi-Hamiltonian cycle that starts at vertex A in the following graph.



***Source:** Jacaranda (John Wiley & Sons Australia, Ltd), Practice Question*

Question 12 (2 marks)

Explain why the following graph is semi-Hamiltonian, but not Hamiltonian.



***Source:** Jacaranda (John Wiley & Sons Australia, Ltd), Practice Question*

Question 13 (1 mark)

Draw an Eulerian graph with 4 vertices and 5 edges.

Answers and marking guide

6.2 Introduction to graph theory

Question 1

Since the two graphs are isomorphic, there is a one-to-one correspondence between the two graphs. This correspondence between the two graphs preserves not only adjacency between vertices but also non-adjacency. **[1 mark]**
Therefore, the non-adjacency between each vertex in both graphs is also isomorphic. Hence, the complement of the two graphs is isomorphic. **[1 mark]**

Question 2

If G is self-complementary with n vertices, then G and its complement must have the same number of edges. The sum of their numbers of edges must be the number of edges of the complete graph.

Hence, edges in G must be $\frac{1}{2} \times$ the number of edges in K_n

Edges in G must be $\frac{1}{2} \times \frac{n(n-1)}{2} = \frac{n(n-1)}{4}$ **[1 mark]**

Since the number of edges in G must be a whole positive number, $n(n-1)$ must be divisible by 4.
If n is odd then $(n-1)$ is even or n is even then $(n-1)$ is odd. This means that n and $(n-1)$ cannot both be divisible by 2. Thus, either n or $(n-1)$ is divisible by four. **[1 mark]**
Giving:
$n = 4k$ or $n - 1 = 4k \rightarrow n = 4k + 1$ **[1 mark]**

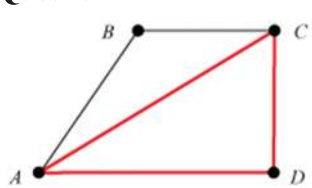
Question 3

The number of edges can be determined from the adjacency matrix by adding up the values in the upper triangle of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

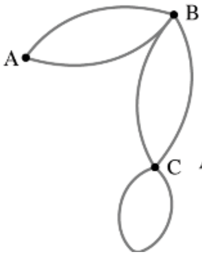
The number of edges is equal to $1 + 1 + 2 + 1 + 1 + 1 = 7$
Therefore the sum of the degrees is equal to $2 \times 7 = 14$
So only A and B are possible options.
Looking at the adjacency matric, the vertex represented by the top row has 4 edges coming from it, so its degree must be 4. This is only present in option A.

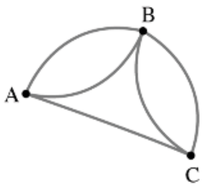
Question 4

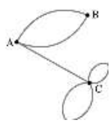


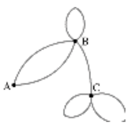
Question 5

The bottom right entry of the matrix states that there are 2 edges connecting vertex C with vertex C. This rules out the following networks:





The middle entry in the matrix states that there is an edge connecting vertex B with vertex B. This rules out the following network: 

 is the correct network for the adjacency matrix.

Question 6

The adjacency matrix details how many edges connect vertices. The columns and rows represent the vertices A to D.
The number of edges between vertices A and A is 0, place a zero in the top left corner. The number of edges between vertices A and B is 1, place a 1 in the 1st row 2nd column. Continue for the rest of the vertices.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$ **[1 mark]**

Question 7

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ describes a connected graph

A is connected to B and C, and C is connected to D.
All other adjacency matrices contain a vertex that has no connections to any other vertices.

Question 8

A simple graph has no loops or multiple edges between vertices.

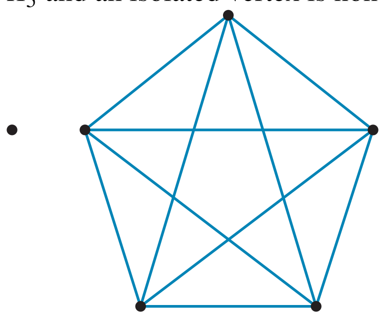
6.3 Planar graphs and Euler's formula

Question 1

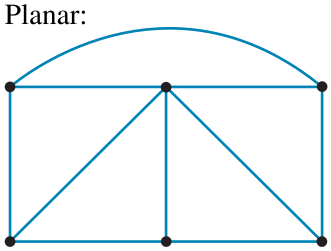
A 3-regular graph will have degree 3 for every vertex.
With 24 vertices this means that $\sum \deg(V) = 24 \times 3 = 72$.
We know that $\sum \deg(V) = 2n(E) = 72$
So the number of edges will be equal to $72 \div 2 = 36$
Using Euler's formula $v - e + f = 2$ gives
$24 - 36 + f = 2$
$f = 14$ **[1 mark]**

Question 2

It is possible to produce a non-planar and planar graph using 6 vertices and 10 edges. Non-planar:
K_5 and an isolated vertex is non-planar



Planar:



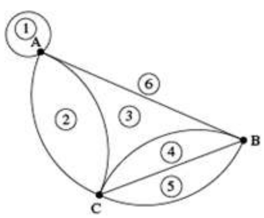
Award 1 mark for an example of a non-planar graph and 1 mark for an example of a planar graph.

Question 3

It can be seen that K_5 is a subgraph contained in the middle of the graph. **[1 mark]**
Since K_5 is a subgraph the graph is non-planar. **[1 mark]**

Question 4

A planar graph representing the above adjacency matrix is



Number each of the separate regions to find the total number of faces (regions). Make sure you include the outside or infinite region. There are 6 regions.

Question 5

$v - e + f = 2$

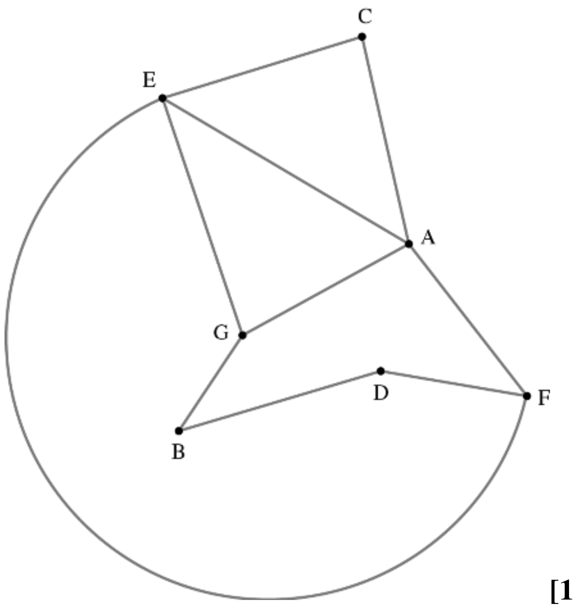
$5 - 7 + f = 2$

$f = 4$

Question 6

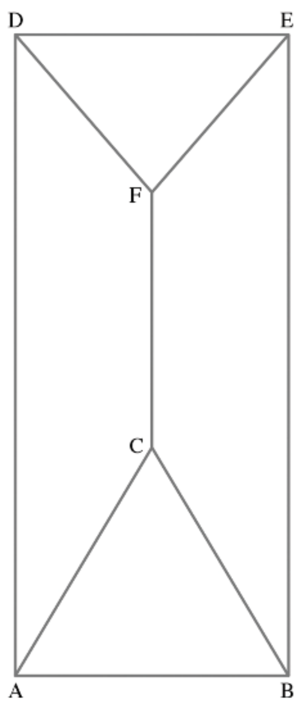
$$
\begin{aligned}
v - e + f &= 2 \\
v - 14 + 7 &= 2 \\
v &= 9
\end{aligned}
$$

Question 7



[1 mark]

Question 8



[1 mark]

6.4 Eulerian and Hamiltonian graphs

Question 1

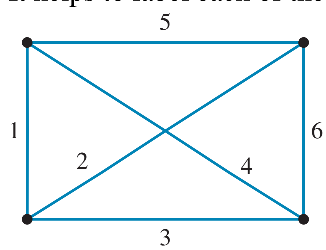
A graph will be a Eulerian graph if the degrees of all vertices are even. A complete graph K_n is also an $(n-1)$-regular graph. It follows that all r-regular graphs will be Eulerian graphs if r is even. Therefore, if $r = n - 1$, then $n = r + 1$.

So if r is even, then n must be odd.

Thus, complete graphs K_n will be Eulerian graphs when n is an odd positive integer. **[1 mark]**

Question 2

a. i. It helps to label each of the edges before starting.



Starting from the bottom left vertex, there are three possible paths (1, 2, 3).

Depending on which path is taken, there will be two possible paths from each of the initial choices:

$1 \to 5$ and $1 \to 4$
$2 \to 5$ and $2 \to 6$
$3 \to 4$ and $3 \to 6$

Finally for each choice, there will only be one path left to return to the original vertex.

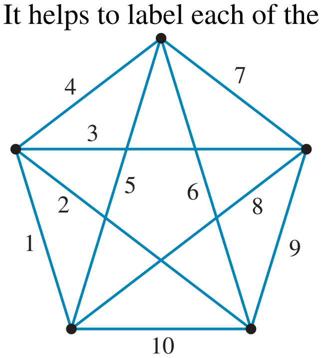
$1 \to 5 \to 6 \to 3$ and $1 \to 4 \to 6 \to 2$
$2 \to 5 \to 4 \to 3$ and $2 \to 6 \to 4 \to 1$
$3 \to 4 \to 5 \to 2$ and $3 \to 6 \to 5 \to 1$

We can then see that a number of these are repeats in reverse order.

$1 \to 5 \to 6 \to 3$ and $1 \to 4 \to 6 \to 2$
$2 \to 5 \to 4 \to 3$ and $2 \to 6 \to 4 \to 1$
$3 \to 4 \to 5 \to 2$ and $3 \to 6 \to 5 \to 1$

These are considered the same Hamiltonian cycle, so there is only 3 Hamiltonian cycles for K_4 **[1 mark]**

ii. It helps to label each of the edges before starting.



Starting from the left vertex, there are three possible paths (1, 2, 3, 4).

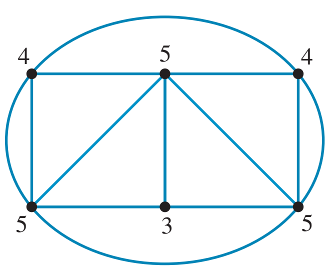
Depending on which path is taken, there will be three possible paths from each of the initial choices:

$1 \to 5$, $1 \to 8$ and $1 \to 10$
$2 \to 6$, $2 \to 9$ and $2 \to 10$
$3 \to 7$, $3 \to 8$ and $3 \to 9$
$4 \to 5$, $4 \to 6$ and $4 \to 7$

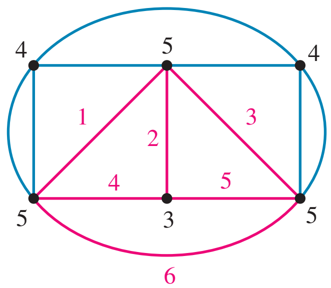
Following on, there will be two choices that remain for each path:
$1 \to 5 \to 6$, $1 \to 8 \to 7$ and $1 \to 10 \to 8$
$1 \to 5 \to 7$, $1 \to 8 \to 9$ and $1 \to 10 \to 9$
$2 \to 6 \to 5$, $2 \to 9 \to 7$ and $2 \to 10 \to 5$
$2 \to 6 \to 7$, $2 \to 9 \to 8$ and $2 \to 10 \to 8$
$3 \to 7 \to 5$, $3 \to 8 \to 5$ and $3 \to 9 \to 10$
$3 \to 7 \to 6$, $3 \to 8 \to 10$ and $3 \to 9 \to 6$
$4 \to 5 \to 8$, $4 \to 6 \to 9$ and $4 \to 7 \to 8$
$4 \to 5 \to 10$, $4 \to 6 \to 10$ and $4 \to 7 \to 9$
Finally, there is one route back to the original vertex for each possible path so far.
$1 \to 5 \to 6 \to 9 \to 3$, $1 \to 8 \to 7 \to 6 \to 10$ and $1 \to 10 \to 8 \to 7 \to 3$
$1 \to 5 \to 7 \to 9 \to 2$, $1 \to 8 \to 9 \to 6 \to 4$ and $1 \to 10 \to 9 \to 7 \to 4$
$2 \to 6 \to 5 \to 8 \to 3$, $2 \to 9 \to 7 \to 5 \to 1$ and $2 \to 10 \to 5 \to 7 \to 3$
$2 \to 6 \to 7 \to 8 \to 1$, $2 \to 9 \to 8 \to 5 \to 4$ and $2 \to 10 \to 8 \to 7 \to 4$
$3 \to 7 \to 5 \to 10 \to 2$, $3 \to 8 \to 5 \to 6 \to 2$ and $3 \to 9 \to 10 \to 5 \to 4$
$3 \to 7 \to 6 \to 10 \to 1$, $3 \to 8 \to 10 \to 6 \to 4$ and $3 \to 9 \to 6 \to 5 \to 1$
$4 \to 5 \to 8 \to 9 \to 2$, $4 \to 6 \to 9 \to 8 \to 1$ and $4 \to 7 \to 8 \to 10 \to 2$
$4 \to 5 \to 10 \to 9 \to 3$, $4 \to 6 \to 10 \to 8 \to 3$ and $4 \to 7 \to 9 \to 10 \to 1$
Removing the repeats (paths appearing in reverse order) leaves 12 Hamiltonian cycles.
$1 \to 5 \to 6 \to 9 \to 3$, $1 \to 8 \to 7 \to 6 \to 10$ and $1 \to 10 \to 8 \to 7 \to 3$
$1 \to 5 \to 7 \to 9 \to 2$, $1 \to 8 \to 9 \to 6 \to 4$ and $1 \to 10 \to 9 \to 7 \to 4$
$2 \to 6 \to 5 \to 8 \to 3$, $2 \to 10 \to 5 \to 7 \to 3$ and $2 \to 9 \to 8 \to 5 \to 4$
$2 \to 10 \to 8 \to 7 \to 4$, $3 \to 9 \to 10 \to 5 \to 4$ and $3 \to 8 \to 10 \to 6 \to 4$ **[1 mark]**

b. Using the approach taken in parts a. when given a K_n graph, there will be $(n-1)$ edges that can be taken from the first vertex. After making a choice, the number of edges available to traverse will be one fewer than at the previous vertex until there is only one remain path to the original vertex. So that total number of cycles created will be equal to:
$(n-1) \times (n-2) \times (n-3) \times \ \ldots \ \times 1 = (n-1)!$ **[1 mark]**
Half of these cycles end up being the reverse of another cycle and are considered the same cycle (as they traverse the exact same edges).
Therefore, the total number of Hamiltonian cycles of a complete graph K_n is equal to $\frac{(n-1)!}{2}$ **[1 mark]**

Question 3
An Eulerian trail for the graph will be possible if there are only two odd vertices. Removing an edge will decrease the value of two vertices by 1. So we can only remove edges that will leave the graph with only two odd vertices.
The following graph shows the degrees of the vertices in red.



There are 4 odd vertices, so we can remove any edge that joins two odd vertices. There are 6 edges that could be removed, which are indicated in pink below.

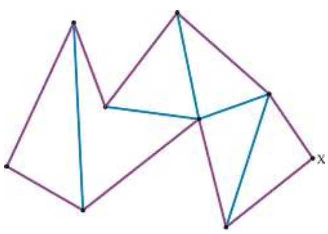


Question 4
The only sequence of vertices that does not pass over the same edge twice is HBAHGFH

Question 5
A cycle finishes on its starting vertex, not its starting edge and it can only use vertices once.

Question 6
A Hamiltonian path is a path that begins and ends at different vertices and passes through each vertex exactly once.

Question 7
The only Hamiltonian cycle starting at X is the path shown below in purple.

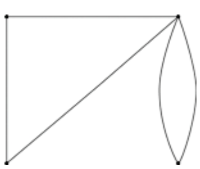


There are 5 unnecessary edges that can be removed, then 1 more edge will need to be removed to break the Hamiltonian cycle. The total number of edges that need to be removed is 6.

Question 8
The semi-Hamiltonian path starting at vertex A is
A – C – B – F – G – E – D **[1 mark]**

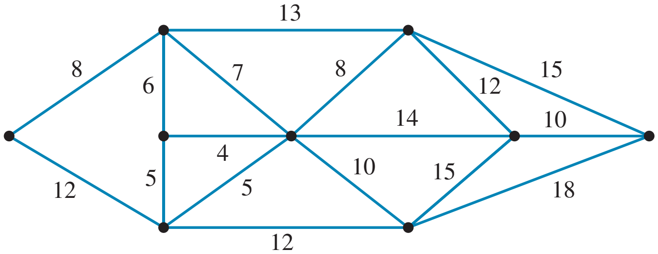
Question 9
This graph contains a bridge, which means that it cannot contain a Hamiltonian cycle as it can only be traversed once. **[1 mark]**. However it can contain a semi-Hamiltonian cycle, as the starting vertex could be on one side of the bridge and the end vertex could be on the other side of the bridge. **[1 mark]**

Question 10

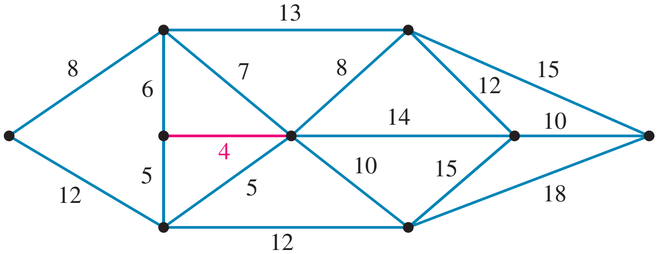

[1 mark]

6.5 Weighted graphs and trees

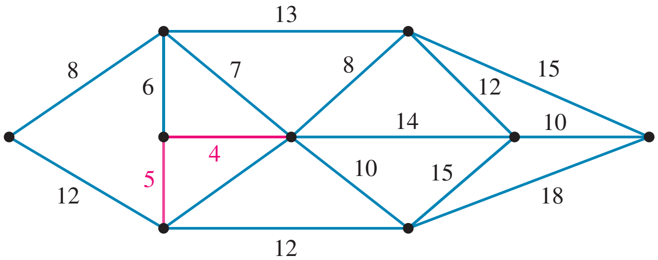
Question 1



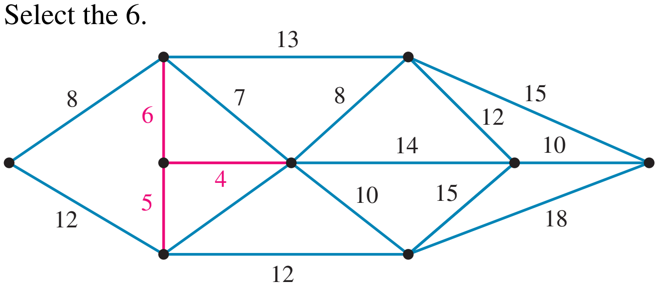
Using Prim's algorithm, the sides selected in order are: Select the 4



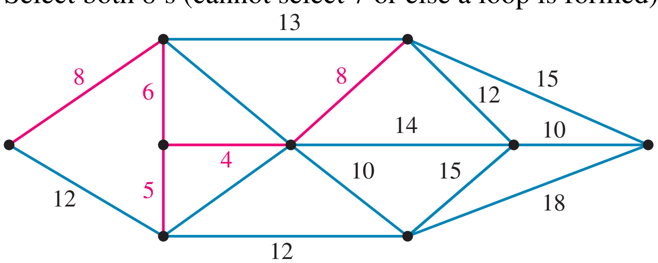
Then select only one of the 5,s (otherwise a loop is formed).



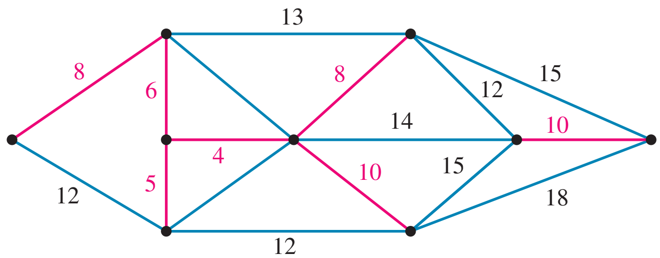
Select the 6.



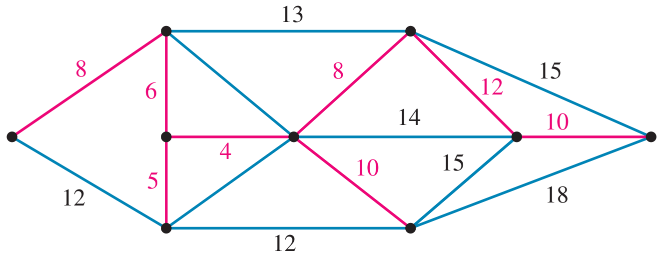
Select both 8's (cannot select 7 or else a loop is formed).



Select both 10's.



Select 12 to finish the tree.



The minimum value is therefore $4 + 5 + 6 + 8 + 8 + 10 + 10 + 12 = 63$ km **[1 mark]**

Question 2

a. Every edge in a tree is a bridge, since removing it would disconnect the graph. **[1 mark]**

b. If an edge never appears in a spanning tree, then it is a loop, as a loop will never be found in any tree as it is a cycle. **[1 mark]**

Question 3

a. In this situation the edges are weighted according to the difference between the numbers of the vertices they connect.
The defining feature of a complete graph is the fact that every vertex is joined to every other vertex.
Therefore the minimum spanning tree will start at vertex 1 and connect to vertex 2, then connect to vertex 3 to vertex 4 to vertex 5 and so on until it reaches vertex n **[1 mark]**
Each of these edges has the minimum weight of 1, since they connect consecutively numbered vertices.
There are $(n - 1)$ edges required to connect all vertices in a spanning tree. The value of the minimum spanning tree is therefore $(n - 1) \times 1 = (n - 1)$. **[1 mark]**

b. In this situation the edges are weighted according to the sum of the numbers of the vertices they connect.
Since a spanning tree must connect to all vertices in a graph and each vertex in a complete graph is connected to every other vertex via an edge, the minimum spanning tree will be the tree which connect every vertex to the vertex with the lowest number (vertex 1). This will minimise the weights of all edges. **[1 mark]**
The minimum spanning tree will therefore contain edges of weights $1 + 2 = 3$, $1 + 3 = 4$, $1 + 4 = 5$, and so on, up to $(1 + n)$. The value of the minimum spanning tree will therefore be $3 + 4 + 5 + 6 + ... + n + (n + 1)$. **[1 mark]**

Question 4

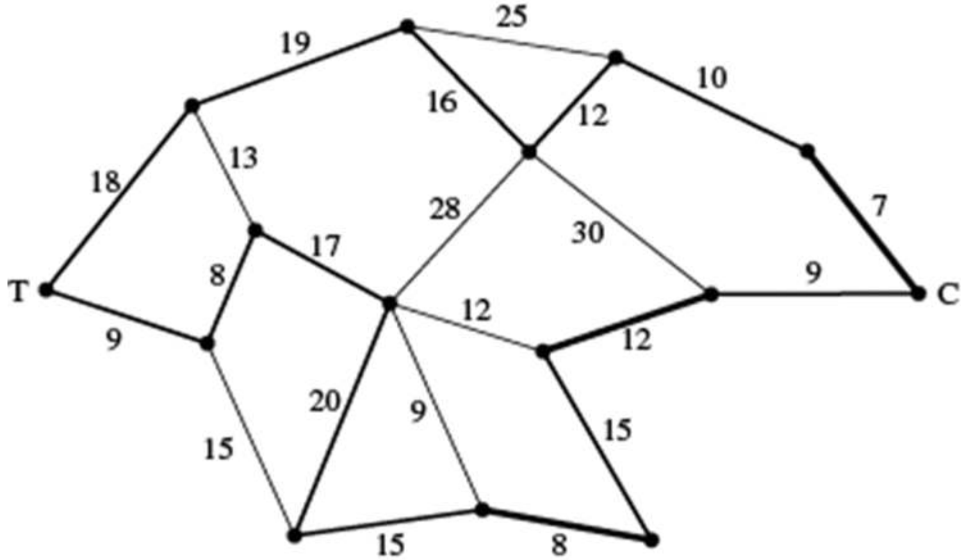
57 km is travelled using the route AEBD or ABED
66 km is travelled using the route AECD
52 km is travelled using the route ABD
The distance of 40 km cannot be achieved by any valid route.

Question 5

Starting the journey at A, travelling to B (11), then to D (10), onto C (6) , then E (4), followed by F (3) and finishing at G (12).
This route takes in the three largest distances in the network.
Adding the values gives a total of 46 km.

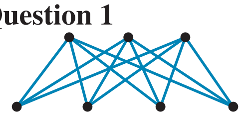
Question 6

The network below shows the route needed. The total distance of this route is
$18 + 19 + 16 + 12 + 10 + 7 + 9 + 12 + 15 + 8 + 15 + 20 + 17 + 8 + 9 = 195$ km



6.6 Bipartite graphs and the Hungarian algorithm

Question 1

a.  **[1 mark]**

b. Since all the vertices are odd, the graph will not have a Eulerian cycle. **[1 mark]**

c. Start by establishing the two groups: group A has the 4 vertices and group B has the 3 vertices. Any Hamiltonian path will start in group A and zig-zag through the graph and thus also finish in group A. **[1 mark]**
It is not possible to therefore complete the cycle as group A vertices are only connected to group B vertices, so it is impossible to return directly to the group A vertex from which the Hamiltonian path started. The graph will therefore not have a Hamiltonian cycle. **[1 mark]**

Question 2

a. A Eulerian graph must satisfy the condition that every vertex has an even degree. That means the two groups in the bipartite graph must have an even number of vertices. Therefore: $K_{m,n}$, where m, n are even positive integers will be Eulerian graphs. **[1 mark]**

b. A semi-Eulerian graph must satisfy the condition that every vertex has an even degree with two odd vertices. That means that one group can only have two vertices (both of which will be odd) and the other group can have any odd number of vertices. Therefore: $K_{m,n}$, where $m = 2$ and n is an odd positive integer will be semi-Eulerian graphs. **[1 mark]**

Question 3

$$\begin{bmatrix} 2 & 4 & 3 & 5 \\ 3 & 5 & 3 & 4 \\ 2 & 3 & 4 & 2 \\ 2 & 4 & 2 & 3 \end{bmatrix}$$

Conduct a row reduction

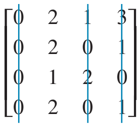
Row 1, the smallest number is 2, so subtract 2 from all other numbers in the row.

Row 2, the smallest number is 3, so subtract 3 from all other numbers in the row.

Row 3, the smallest number is 2, so subtract 2 from all other numbers in the row.
Row 4, the smallest number is 2, so subtract 2 from all other numbers in the row.

$$\begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

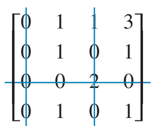
The minimum number of lines needed to cover all zeroes is 3, so all 4 tasks cannot be allocated.



Conduct a column reduction
Column 1, the smallest number is 0, so do nothing.
Column 2, the smallest number is 1, so subtract 1 from all other numbers in the column.
Column 3, the smallest number is 0, so do nothing.
Column 4, the smallest number is 0, so do nothing.

$$\begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The minimum number of lines needed to cover all zeroes is 3, so all 4 tasks cannot be allocated.



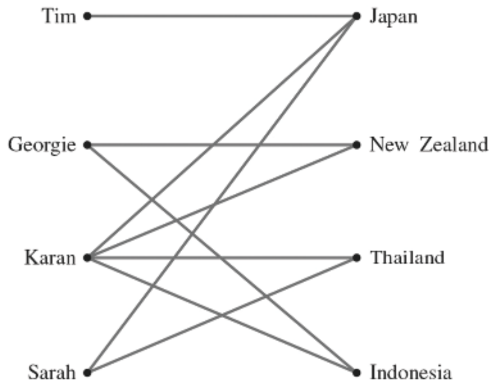
Use the Hungarian algorithm.
The smallest uncovered number is 1, so subtract that from all uncovered numbers and add 1 to the two numbers with two lines going through them.

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ **[1 mark]**

$M \rightarrow L1$ or $L2$
$Je \rightarrow L1, L2, L3$ or $L4$
$P \rightarrow L2$ or $L4$
$Ja \rightarrow L1, L2, L3$ or $L4$
So one possible allocation would be
$M \rightarrow L1$
$Je \rightarrow L3$
$P \rightarrow L2$ **[1 mark]**
$Ja \rightarrow L4$
The minimum time is therefore: $2 + 3 + 3 + 3 = 11$ minutes **[1 mark]**

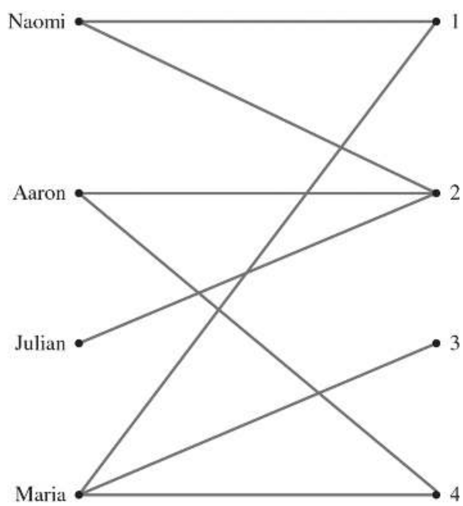
Question 4
The only job that person A can do is job W.

Question 5



[1 mark]

Question 6



[1 mark]

The only applicant able to do job 3 is Maria.
Julian can only do job 2.
Naomi must do job 1 and Aaron must do job 4. **[1 mark]**

Question 7

Using the Hungarian algorithm.

1. Subtracting the lowest value in each row from all values in that row gives:

	T1	T2	T3	T4	T5
P1	4	0	3	0	7
P2	6	8	7	6	0
P3	1	2	0	3	1
P4	0	3	1	5	6
P5	0	2	1	3	1

2. Subtracting the lowest value in each column form all other values in the column.

	T1	T2	T3	T4	T5
P1	4	0	3	0	7
P2	6	8	7	6	0
P3	1	2	0	3	1
P4	0	3	1	5	6
P5	0	2	1	3	1

3. Cover the zeros with the minimum number of lines.

	T1	T2	T3	T4	T5
P1	4	0	3	0	7
P2	6	8	7	6	0
P3	1	2	0	3	1
P4	0	3	1	5	6
P5	0	2	1	3	1

4. Covering the zeros would take a minimum of 4 lines (we need 5 as there are 5 rows). Therefore subtract the lowest value from all uncovered values and add the lowest value to any places where two covering lines intersect.

	T1	T2	T3	T4	T5
P1	5	0	3	0	8
P2	6	7	6	5	0
P3	2	2	0	3	2
P4	0	2	0	4	6
P5	0	1	0	2	1

5. Cover the zeros with the minimum number of lines.

	T1	T2	T3	T4	T5
P1	5	0	3	0	8
P2	6	7	6	5	0
P3	2	2	0	3	2
P4	0	2	0	4	6
P5	0	1	0	2	1

6. Covering the zeros would take a minimum of 4 lines (we need 5, as there are 5 rows).
Therefore, subtract the lowest value from all uncovered values and add the lowest value to any places where two covering lines intersect.

	T1	T2	T3	T4	T5
P1	6	0	4	0	9
P2	6	6	6	4	0
P3	2	1	0	2	2
P4	0	1	0	3	6
P5	0	0	0	1	1

7. Covering the zeros would take a minimum of 5 lines; therefore, allocations can now occur.

	T1	T2	T3	T4	T5
P1	6	0	4	0	9
P2	6	6	6	4	0
P3	2	1	0	2	2
P4	0	1	0	3	6
P5	0	0	0	1	1

Allocation of tasks to people for minimum time is:
P1 to T4, P2 to T5, P3 to T3, P4 to T1 and P5 to T2 For these allocations, the times are
$8 + 6 + 8 + 8 + 15 = 45$ minutes

Question 8

Subtract the row minimum from each row.

	Task 1	**Task 2**	**Task 3**	**Task 4**
Worker 1	11	51	0	13
Worker 2	24	41	0	23
Worker 3	12	49	0	11
Worker 4	2	49	0	15

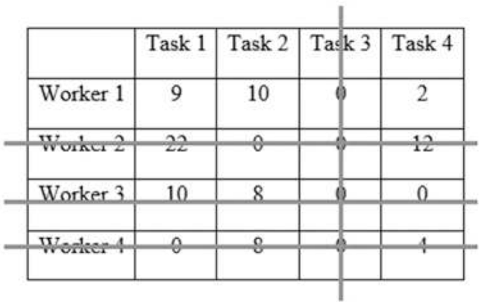
[1 mark]

Subtract the column minimum from each column.

	Task 1	**Task 2**	**Task 3**	**Task 4**
Worker 1	9	10	0	13
Worker 2	22	0	0	12
Worker 3	10	8	0	0
Worker 4	0	8	0	4

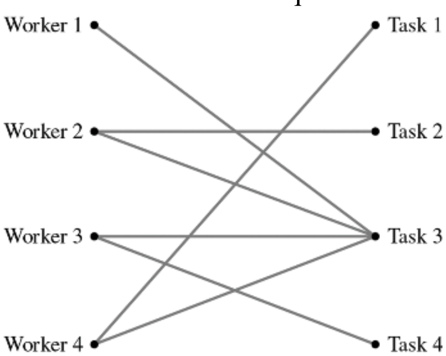
[1 mark]

Cover all of the 0s with the minimum number of lines.



	Task 1	Task 2	Task 3	Task 4
Worker 1	9	10	0	2
Worker 2	22	0	0	12
Worker 3	10	8	0	0
Worker 4	0	8	0	4

The number of lines is equal to the number of tasks. Draw the bipartite graph.



[1 mark]

Worker 1 can only do task 3.

Task 2 can only be done by worker 2.

Task 1 can only be done by worker 4.

Task 4 can only be done by worker 3.

The minimum time for all 4 tasks to be completed is $19 + 62 + 37 + 26 = 144$ minutes. **[1 mark]**

6.7 Review

Question 1

The three house must each be joined to the three utilities. Thus we can consider the problem to be made up of 6 vertices splits into two groups of 3 (the houses and the utilities). The graph of this situation would be the graph of $K_{3,3}$ which is non-planar. **[1 mark]**

Therefore it is impossible to connect each house to each utility without crossing lines. **[1 mark]**

Question 2

Edges $= \dfrac{n(n-1)}{2}$, where $n =$ the number of vertices

Edges $= \dfrac{7(7-1)}{2} = 21$

Question 3

A 3-regular graph will have $\dfrac{3n}{2}$ edges, where n is the number of vertices.

Given the graph is planar we have:

$$v - e + f = 2$$
$$n - \frac{3n}{2} + f = 2$$
$$f = 2 + \frac{n}{2}$$

Since f must be an integer, n must be even.

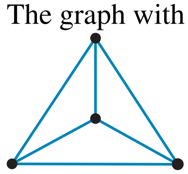
if $n = 2$, then there 3 faces with 3 edges. In this graph each face is bounded by two faces only, so A is not correct.

if $n = 4$, then there are only 4 faces with 6 edges.

Since each face is bounded by exactly three edges, the total number of boundaries required is 12.

As each edge is a boundary on two faces, 6 edges can provide 12 boundaries.

The graph with $n = 4$ is:



This graph satisfies all the properties of G, so the answer is 6, as G has 6 edges.

Question 4

a. $\begin{bmatrix} 5 & 7 & 5 & 9 \\ 6 & 10 & 10 & 7 \\ 7 & 5 & 3 & 8 \\ 7 & 8 & 8 & 9 \end{bmatrix}$

Conduct a row reduction.

Row 1, the smallest number is 5, so subtract 5 from all other numbers in the row.

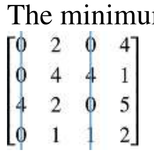
Row 2, the smallest number is 6, so subtract 6 from all other numbers in the row.

Row 3, the smallest number is 3, so subtract 3 from all other numbers in the row.

Row 4, the smallest number is 7, so subtract 7 from all other numbers in the row.

$\begin{bmatrix} 0 & 2 & 0 & 4 \\ 0 & 4 & 4 & 1 \\ 4 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ **[1 mark]**

The minimum number of lines needed to cover all zeroes is 2, so all 4 tasks cannot be allocated.



$\begin{bmatrix} 0 & 2 & 0 & 4 \\ 0 & 4 & 4 & 1 \\ 4 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

b. Conduct a column reduction.

Column 1, the smallest number is 0, so do nothing.

Column 2, the smallest number is 1, so subtract 1 from all other numbers in the column.

Column 3, the smallest number is 0, so do nothing.

Column 4, the smallest number is 1, so subtract 1 from all other numbers in the column.

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 3 & 4 & 0 \\ 4 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The minimum number of lines needed to cover all zeros is 4, so all 4 tasks can be allocated. **[1 mark]**

c. Hungarian algorithm is not necessary. **[1 mark]**

d. $B \rightarrow H1$

$R \rightarrow H4$

$S \rightarrow H3$

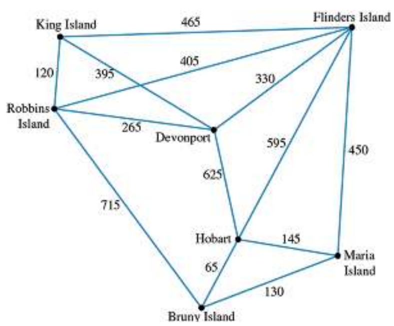
$C \rightarrow H2$

The minimum time is therefore: **[1 mark]**

$5 + 7 + 3 + 8 = 23$

Question 5

a.



b. obart–Bruny–Robbins: $715 + 65 = 780$ km **[1 mark]**

c. Hobart–Bruny–Robbins–King–Devonport–Flinders–Maria:

$715 + 65 + 120 + 395 + 330 + 450 = 2075$ km **[1 mark]**

d. King–Devonport–Flinders–Maria–Hobart–Bruny– Robbins–King:

$395 + 330 + 450 + 145 + 65 + 715 + 120 = 2220$ km **[1 mark]**

Question 6

The only sequence of vertices that does not pass over the same edge twice is HBAHGFH.

Question 7

A cycle finishes on its starting vertex, not its starting edge and it can only use vertices once.

Question 8

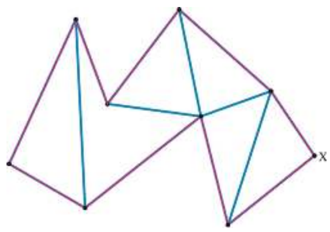
A Hamiltonian path is a path that begins and ends at different vertices and passes through each vertex exactly once.

Question 9

The only sequence of vertices that does not pass over the same edge twice is HBAHGFH.

Question 10

The only Hamiltonian cycle starting at X is the path shown below in purple.



There are 5 unnecessary edges that can be removed, then 1 more edge will need to be removed to break the Hamiltonian cycle. The total number of edges that need to be removed is 6.

Question 11

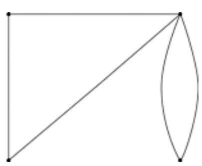
The semi-Hamiltonian path starting at vertex A is

A – C – B – F – G – E – D **[1 mark]**

Question 12

This graph contains a bridge, which means that it cannot contain a Hamiltonian cycle as it can only be traversed once. **[1 mark]**. However it can contain a semi-Hamiltonian cycle, as the starting vertex could be on one side of the bridge and the end vertex could be on the other side of the bridge. **[1 mark]**

Question 13



[1 mark]

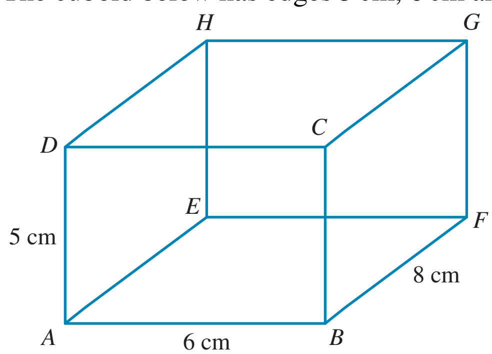
7 Trigonometric ratios and applications

Topic	7	Trigonometric ratios and applications
Subtopic	7.2	Review of trigonometry

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au.

Question 1 (2 marks)

The cuboid below has edges 5 cm, 6 cm and 8 cm.



Determine the size of the angle between the diagonal AG and the edge AD. Give your answer in the form $a° - \tan^{-1}\left(\frac{b}{c}\right)$.

The size of the angle is $a° - \tan^{-1}\left(\frac{b}{c}\right)$ with

$a = \square°$ and $\frac{b}{c} = \square$

Question 2 (1 mark)

From the top of an office building 100 metres high, the angle of depression to a second building, 80 metres away, is 16°. The height of the lower building is

A. 68 m
B. 71 m
C. 75 m
D. 77 m
E. 79 m

Question 3 (2 marks)

The gallant knight Sir George, whose eye level is 1.8 metres from the ground, is standing on top of a small mound, 3 metres high. From this vantage point he notices that the angle of elevation of the tall tower, at the top of which the fair damsel Charlotte is being held captive, is 2. Using his expert judgement he calculates the distance to the foot of the tower as 3 kilometres. Sir George requires a rope twice the height of the tower to aid him to rescue the damsel. Determine what length of rope, rounded to the nearest metre, he should purchase from the conveniently located rope shop.

The length of rope is ☐ m, to the nearest metre.

Question 4 (1 mark)

In a right-angled triangle ABC, AB = 5 cm, BC = 12 cm and AC = 13 cm. Find the size of angle ACB in degrees correct to 2 decimal places.

Question 5 (2 marks)

A cuboid has edges 5 cm, 6 cm and 8 cm. Find the size of the angle between the space diagonal and the shortest edge, in degrees correct to 2 decimal places.

Question 6 (2 marks)

A rectangle is 10 cm long and 5 cm wide. Find the angle between the diagonal and the long side, in degrees correct to 2 decimal places.

Question 7 (2 marks)

The gradient sign beside a road up a hill says 1:5. This means that for every 5 metres horizontally the road rises 1 metre vertically. What angle is the road to the horizontal, in degrees correct to 2 decimal places?

Question 8 (1 mark)
From a yacht, the angle of elevation of the top of a cliff is $a°$.
From the top of the cliff, the angle of depression of the yacht is:
A. $(\alpha - 90)°$
B. $(90 + \alpha)°$
C. $\alpha°$
D. $(90 - \alpha)°$
E. $(180 - \alpha)°$

Question 9 (2 marks)
From ground level the angle of elevation of the top of a tree 120 metres away over flat ground is 23.7°. How high is the tree, in metres to 2 decimal places?

Question 10 (3 marks)
From the top of an office building 100 metres high, the angle of depression to a second building, 80 metres away, is 16°. What is the height of the lower building, in metres to 2 decimal places?

Question 11 (3 marks)
For a horizontal distance of 1 kilometre from its base a road rises 100 metres vertically, that is, a gradient of 1:10.
For the next kilometre horizontally the road has a gradient of 1:8 until it reaches its top.
From a point at the top of the road, what is the angle of depression to a point at the bottom of the road? Write your answer in degrees correct to 3 decimal places.

Question 12 (2 marks)
To walk in the direction $p°$ west of south is equivalent to walking on what bearing?
If I walk for k kilometres in this direction, how far west of my starting point do I finish up?

Question 13 (4 marks)
An orienteering course is as follows:
From S go 7 km to P on a bearing of 050°. Then go 5 km to Q on a bearing of 120°.
How far is S from Q and what is its bearing?

Question 14 (3 marks)
A yacht sails on a bearing of 045° for 10 km to point P, then sails on a bearing of 250° for 15 km to point Q. What is the bearing of the yacht's starting point from the point Q?

Question 15 (3 marks)
Two ships leave port. One sails on a bearing of 325° at 20 km/h, the other on a bearing of 100° at 15 km/h. How far apart are the ships after 4 hours?

Question 16 (1 mark)
The angle of elevation of the top of a cliff from A is 30°. One kilometre due east of A lies B. From B, the angle of elevation of the cliff is 25°. How high is the cliff?

Question 17 (1 mark)
What is the angle of elevation from a point on the ground 29 m away to the top of a 20 m high building?
A. 34.59°
B. 43.60°
C. 46.40°
D. 55.41°
E. 61.93°

Question 18 (1 mark)

From the top of a 3.2 km mountain, a car 6 km away can be seen.
What is the angle of depression from the mountain to the car?

A. 28.07°
B. 32.23°
C. 45°
D. 57.77°
E. 61.93°

Question 19 (1 mark)

The bearing of ship X from a harbour control tower, T, is 025°. Ship Y is due south of the control tower. The bearing of ship X from ship Y is 015°. The size of angle TXY is:

A. 5°
B. 10°
C. 15°
D. 20°
E. 25°

Question 20 (2 marks)

The points A, B and C are the vertices of a triangular yachting course. If the three legs of the course form an equilateral triangle and the bearing of B from A is 050°, find the possible bearings of C from A.

Question 21 (3 marks)

Two ships leave the same point. One sails on a bearing of 125° at 20 km/h, the other on a bearing of 030° at 10 km/h. How far apart are the ships after 2 hours?

Topic	7	Trigonometric ratios and applications
Subtopic	7.3	The sine rule

online only

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Question 1 (2 marks)

In triangle ABC, $AB = 6$ cm, $AC = 8$ cm and angle $ACB = 28°$. Calculate the size of angle ABC correct to 2 decimal places.

$\angle ABC = \square°$

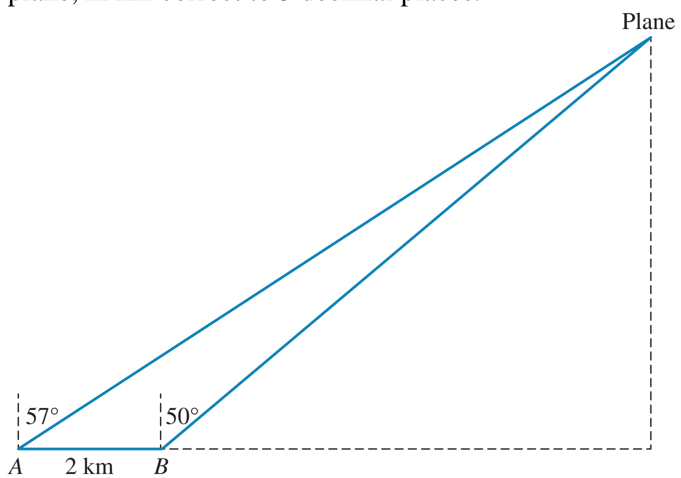
Question 2 (2 marks)

In triangle PQR, PQ is 15 mm and angle RPQ is 40°. if RQ is 12 mm, determine the size of angle PRQ correct to 2 decimal places.

$\angle PQR = \square°$

Question 3 (3 marks)

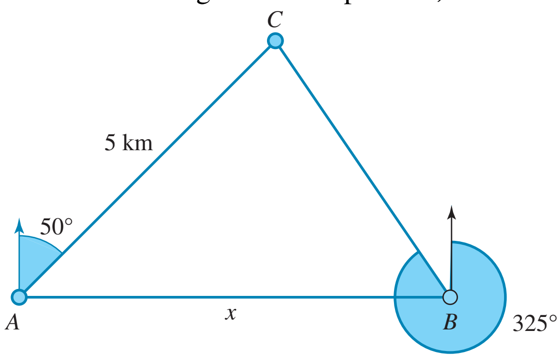
From point A on a straight road a plane is at an angle of 57° to the vertical. From point B, 2 km away, looking in the same direction, the plane is at an angle of 50° to the vertical. Calculate the altitude of the plane, in km correct to 3 decimal places.



The altitude of the plane is $\square$ km

Question 4 (1 mark)

Consider the diagram of campsites A, B and C.



Campsite A is due west of campsite B. Campsite C is 5 km away on a bearing of 50° T from campsite A. Campsite C is on a bearing of 325° from campsite B

What is the distance between campsite A and campsite B?

A. 4.11 km
B. 6.08 km
C. 7.78 km
D. 8.71 km
E. 9.07 km

Question 5 (2 marks)

In triangle PQR, $PR = 5$ cm, $PQ = 7$ cm and angle $RPQ = 30°$. Find the length of QR.

Question 6 (2 marks)

In triangle ACB, ∠CAB = 55° and ∠ABC = 80°. The length of side AB is 40 m. Find the length of AC.

Topic	7	Trigonometric ratios and applications
Subtopic	7.4	The cosine rule

online only

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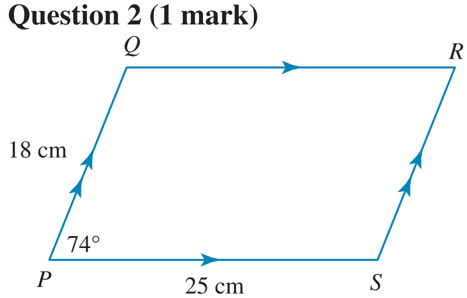
Question 1 (2 marks)

Tri looks at a door. One side of the door is 6 metres away while the other side of the door is 7.5 metres away. The angle between his sight lines to either side of the door is 17°. Calculate how wide the door is. Give your answer rounded to the nearest centimetre.

The width of the door is ☐ m

Question 2 (1 mark)



In parallelogram $PQRS$, $\angle QPS = 74°$.

In this parallelogram, $PQ = 18$ cm and $PS = 25$ cm.

The length of the shorter diagonal is closest to

A. 26.5 cm

B. 30.1 cm

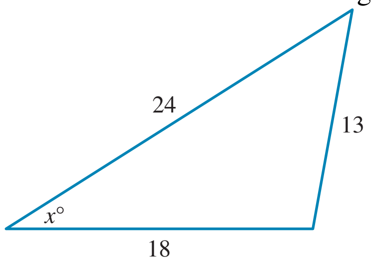
C. 30.8 cm

D. 26.3 cm

E. 39.9 cm

Question 3 (2 marks)

Determine the value of x in degrees, rounded to 2 decimal places.

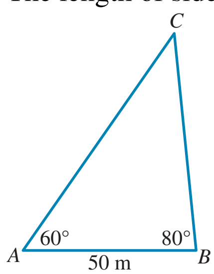


$x = \square°$

Question 4 (1 mark)

In triangle ACB, $\angle CAB = 60°$ and $\angle ABC = 80°$.

The length of side $AB = 50$ m.



Find the length of BC.

A. 57 m
B. 67 m
C. 77 m
D. 81 m
E. 100 m

Question 5 (3 marks)

The sides of a triangle are in the ratio of 8:14:7. What is the size of the largest angle?

Topic	7	Trigonometric ratios and applications
Subtopic	7.5	Arc length, sectors and segments

online only

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Question 1 (2 marks)

A segment of a circle, radius 10 cm, has an angle of 30°. Determine the area of the segment, in square centimetres correct to 3 decimal places.

The area is ☐ units^2

Question 2 (2 marks)

The area of a sector of a circle is 500 square centimetres. The central angle is 40°. Calculate the radius, in centimetres correct to 2 decimal places.

The radius is ☐ cm

Question 3 (1 mark)

For a particular circle, the area of a sector and the arc length of the same sector are numerically the same. Determine the radius of the circle.

The radius is ☐ units

Question 4 (1 mark)

A sector of a circle, radius r cm, has an angle of $\theta°$. The area of the sector is:

A. $\frac{1}{2}\theta r^2$

B. $\frac{1}{2}(\theta r)^2$

C. $\frac{1}{2}\theta^2 r$

D. $\frac{1}{2}\left(\frac{\theta}{180}\right)^\circ r^2$

E. $\frac{1}{2}\left(\frac{180}{\theta}\right)^\circ r^2$

Question 5 (1 mark)

An arc of a circle, radius r cm, has an angle of $\theta°$. The length of the arc is:

A. $\theta° r$

B. $(\theta r)^2$

C. (θr)

D. $\left(\frac{\theta}{180}\right)^\circ r^2$

E $\left(\frac{180}{\theta}\right)^\circ r^2$

Topic	7	Trigonometric ratios and applications
Subtopic	7.6	Review

online only

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Question 1 (4 marks)

An orienteering course is as follows:

From S go 7 km to P on a bearing of 050°. Then go 5 km to Q on a bearing of 120°. Calculate how far S is from Q, in km to 2 decimal places, and determine the true bearing of S from Q in degrees to 2 decimal places.

$SQ =$ ☐ km

The bearing is ☐° T

Question 2 (4 marks)

A yacht sails on a bearing of 045° for 10 km to point P, then sails on a bearing of 250° for 15 km to point Q. Calculate the true bearing of the yacht's starting point from the point Q, in degrees to 2 decimal places.

The bearing is ☐° T

Question 3 (1 mark)

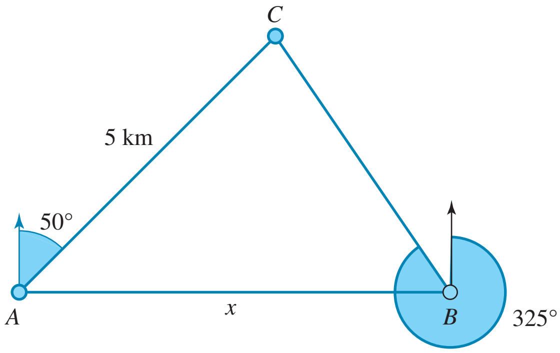
From a point, A, west of a cliff, the angle of elevation of the top of a cliff from A is 30°. One kilometre due east of A lies B. From B, the angle of elevation of the cliff is 25°. The height of the cliff is

A. 208 m
B. 258 m
C. 299 m
D. 320 m
E. 323 m

Question 4 (1 mark)

Consider the diagram of campsites A, B and C.

Campsite A is due west of campsite B. Campsite C is 5 km away on a bearing of 50° T from campsite A. Campsite C is on a bearing of 325° from campsite B.

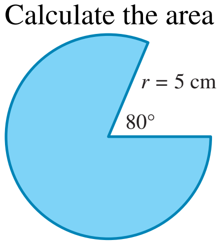


The distance between campsite A and campsite B.

A. 4.11 km
B. 6.08 km
C. 7.78 km
D. 8.71 km
E. 9.07 km

Question 5 (1 mark)

Calculate the area of the shape in the diagram to the nearest square centimetre.



The area is ☐ cm^2

Question 6 (1 mark)

There is a particular circle for which the numerical value of the area of a sector is six times the numerical value of the arc of that sector. Find the radius of this particular circle.

Question 7

Answer the following.

a. Show that the perimeter of a sector of a circle of radius r cm and central angle $\theta°$ is $r(\theta + 2)$. **(2 marks)**

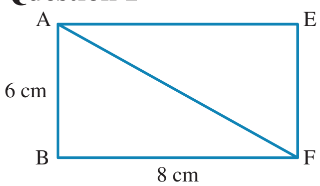
b. Show that the perimeter of a segment of a circle of radius r cm and central angle $\theta°$ is $r\left(\theta + 2\sin\frac{\theta}{2}\right)$. **(3 marks)**

c. solve $r(\theta + 2) = r\left(\theta + 2\sin\frac{\theta}{2}\right)$. **(2 marks)**

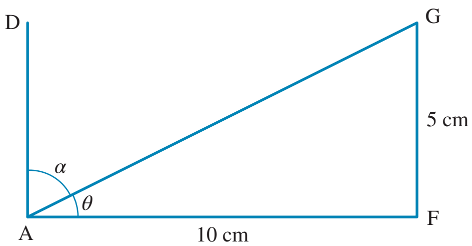
Answers and marking guide

7.2 Review of trigonometry

Question 1



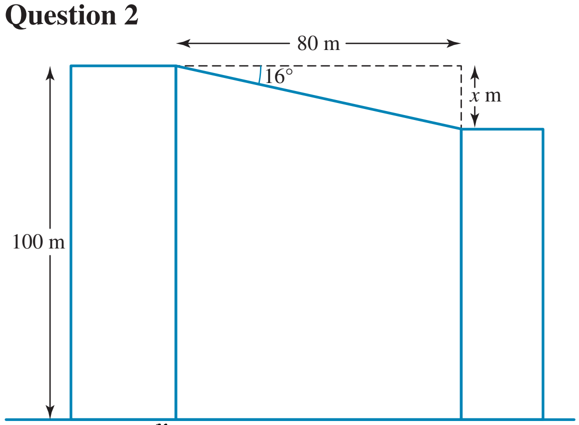
$$\begin{aligned} AF &= \sqrt{6^2 + 8^2}\text{ cm} \\ &= \sqrt{100}\text{ cm} \\ &= 10\text{ cm } \textbf{[1 mark]} \end{aligned}$$



$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{5}{10}\right) \\ &= \tan^{-1}\left(\frac{1}{2}\right) \end{aligned}$$

$$\alpha = 90° - \tan^{-1}\left(\frac{1}{2}\right) \quad \textbf{[1 mark]}$$

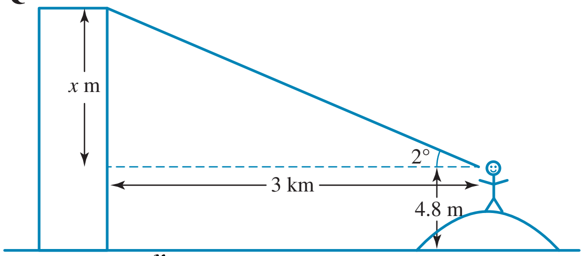
Question 2



$$\begin{aligned} \tan(16°) &= \frac{x}{80} \\ x &= 80\tan(16°) \\ &\approx 23\text{ m} \end{aligned}$$

$$\begin{aligned} \text{Height of building 2} &= 100 - x\text{ m} \\ &\approx 100 - 23\text{ m} \\ &\approx 77\text{ m} \end{aligned}$$

Question 3



$\tan(2°) = \frac{x}{3\text{ km}}$

$x = 3\tan(2°)\text{ km}$

$= 0.1047\text{ km}$

$= 104.7\text{ m}$ **[1 mark]**

The height of the tower is $104.76\text{ m} + 3\text{ m} + 1.8\text{ m} = 109.56\text{ m}$

The length of rope required is two times the height of the tower.

Length of rope $= 2 \times 109.56\text{ m}$

$= 219.12\text{ m}$

$\approx 219\text{ m}$ **[1 mark]**

Question 4

$\sin(\angle ACB) = \frac{5}{13}$

$\angle ACB = 22.62°$ **[1 mark]**

Question 5

Space diagonal $= \sqrt{(5^2 + 6^2 + 8^2)}$

$= 5\sqrt{5}$

$= 11.18\text{ cm}$ **[1 mark]**

$\cos(\theta) = \frac{5}{5\sqrt{5}}$

$\theta = 63.43°$ **[1 mark]**

Question 6

$\cos(\theta) = \frac{10}{\sqrt{5^2 + 10^2}}$ **[1 mark]**

$\theta = \cos^{-1}\left(\frac{10}{\sqrt{5^2 + 10^2}}\right)$

$= 26.57°$ **[1 mark]**

Question 7

$\tan(\theta) = \frac{1}{5}$

$= 0.2$ **[1 mark]**

$\theta = \tan^{-1}(0.2)$

$= 11.31°$ **[1 mark]**

Question 8

$\alpha°$

Question 9

$$\tan(23.7)° = \frac{\text{height of tree}}{\text{ground distance}}$$
$$= \frac{h}{120} \text{ [1 mark]}$$
$$h = 120 \times \tan(23.7)°$$
$$= 52.68 \text{ meters [1 mark]}$$

Question 10

Difference in height is found by:

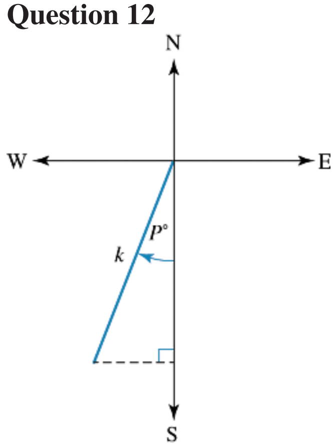
$$\tan(16)° = \frac{\text{height difference}}{\text{ground distance}}$$
$$= \frac{h}{80} \text{ [1 mark]}$$
$$h = 80 \times \tan(16)°$$
$$= 22.94 \text{ meters [1 mark]}$$
$$\text{Height of lower building} = (100 - 22.94) \text{ metres}$$
$$= 77.06 \text{ meters [1 mark]}$$

Question 11

Angle of depression (θ) is found by:

$$\tan(\theta)° = \frac{\text{height difference (m)}}{\text{ground distance (m)}}$$
$$= \frac{h_1 + h_2}{d_1 + d_2} \text{ [1 mark]}$$
$$h = \frac{100 + \frac{1}{8} \times 1000}{1000 + 1000} \text{ [1 mark]}$$
$$= \frac{225}{2000}$$
$$\theta = 6.419° \text{ [1 mark]}$$

Question 12



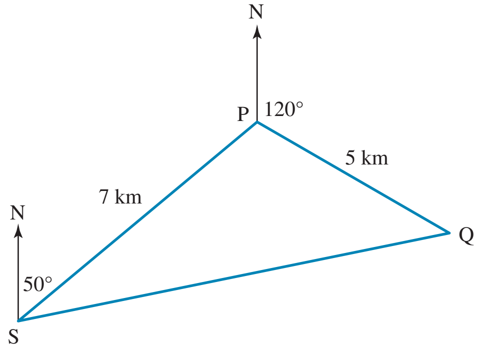
Bearings are given as three digit number, clockwise from north.

$(180 + p)°$ **[1 mark]**

Right-angled triangle:

$k \sin(p)°$ **[1 mark]**

Question 13



$\angle SPQ = 50° + 60°$

$= 110°$ **[1 mark]**

Cosine rule:

$SQ = \sqrt{(7^2 + 5^2 - 2 \times 7 \times 5 \times \cos(110°))}$

$= 9.90$ km **[1 mark]**

Sine rule:

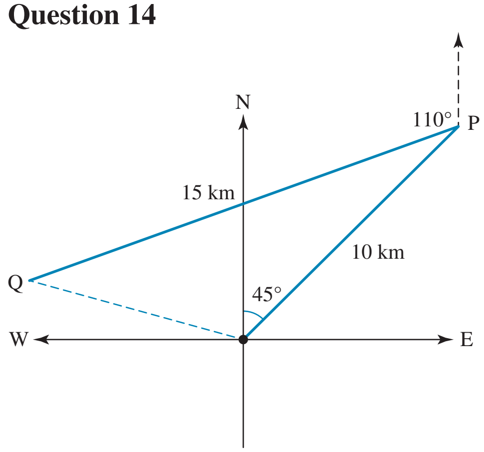
$$\frac{\sin(\angle PQS)}{7} = \frac{\sin(\angle SPQ)}{9.90}$$

$\angle SPQ = 41.64°$ **[1 mark]**

Bearing $= (360 - (60 + 41.64))°$

$= 258.36°$ **[1 mark]**

Question 14



Method: find ∠QOP, then subtract 45° to give ∠QOW.
The bearing of O from Q is (∠QOW + 90)°.

$\angle QOP = (180 - 110 - 45)°$

$= 35°$ **[1 mark]**

In ΔQOP

$OQ^2 = OP^2 + OP^2 - 2 \times OP \times OQ \times \cos(QOP)$

$= 100 + 225 - 300\cos(35°)$

$OQ = 8.902$ **[1 mark]**

$$\frac{\sin(QOP)}{OQ} = \frac{\sin(QPO)}{OQ}$$

$$\sin(QOP) = \frac{QP \sin(QPO)}{OQ}$$
$$= 0.9665$$
$$\angle QOP = 75.12°$$ **[1 mark]**

$$\angle QOW = (75.12 - 45)°$$
$$= 30.12°$$

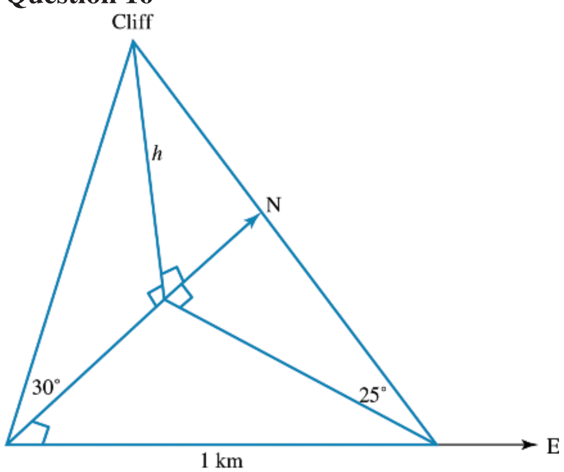
∴ Bearing of O from Q is $(\angle QOW + 90)° = 120.12°$ **[1 mark]**

Question 15

Distances travelled are 4×20 km and 4×15 km **[1 mark]**
Angle between is $(35 + 110)°$ **[1 mark]**

$$d^2 = 80^2 + 60^2 - 2 \times 80 \times 60 \times \cos(145°)$$
$$= 17\,863$$
$$d = 133.7 \text{ km}$$ **[1 mark]**

Question 16



Cliff is h metres high.
In the right-angled horizontal triangle:

$(h\tan(60°))^2 + 1000^2 = (h\tan(65°))^2$ **[1 mark]**

$$h^2 = \frac{1000\,000}{\left[(\tan(65°))^2 - (\tan(60°))^2\right]}$$
$$h^2 = 790.8 \text{ m}$$ **[1 mark]**

Question 17

$$\tan(\theta) = \frac{20}{29}$$
$$\theta = 34.59°$$

Question 18

$$\tan(\theta) = \frac{3.2}{6}$$
$$\theta = 28.07°$$

Question 19

ΔTXY:
$\angle TYX = 15°$

$$\angle XTY = 65° + 90° = 155° \Rightarrow \angle TXY = 10°$$

Question 20

Angle BAC is 60°. The possible bearings are
$050° + 60°$ or $050° - 60°$. **[1 mark]**
Bearings are 290° and 110°. **[1 mark]**

Question 21

Distances travelled are 2×20 km and 2×10 km. **[1 mark]**
Angle between the ships is $(125 - 30)°$ **[1 mark]**

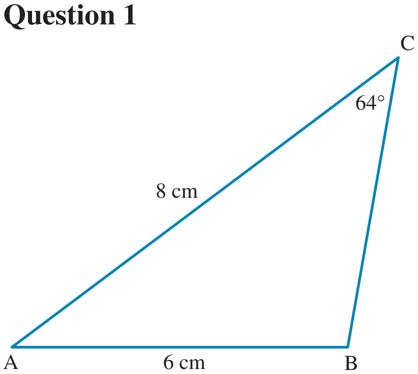
$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$d^2 = 40^2 + 20^2 - 2 \times 40 \times 20 \times \cos(95°) = 2139.45$$

$d = 46.25$ km **[1 mark]**

7.3 The sine rule

Question 1



$$\frac{\sin(ABC)}{AC} = \frac{\sin(ACB)}{AB}$$

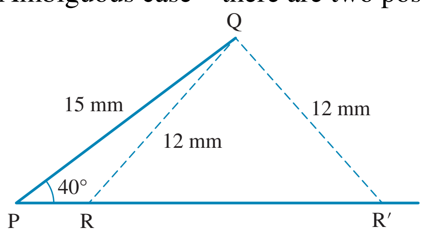
$$\sin(ABC) = \frac{AC\sin(ACB)}{AB}$$ **[1 mark]**

$$= \frac{8 \times \sin(28°)}{6}$$

$$\angle ABC = \sin^{-1}\left(\frac{8 \times \sin(28°)}{6}\right) = 38.75°$$ **[1 mark]**

Question 2

Ambiguous case – there are two possible answers



$$\frac{\sin(\text{PRQ})}{\text{PQ}} = \frac{\sin(\text{QPR})}{\text{QR}}$$

$$\sin(\text{PRQ}) = \frac{\text{PQ}\sin(\text{QPR})}{\text{QR}} \quad \textbf{[1 mark]}$$

$$= \frac{15 \times \sin(40°)}{12}$$

$$\angle\text{PRQ} = \sin^{-1}\left(\frac{15 \times \sin(40°)}{12}\right)$$

$$= 53.46° \text{ or } (180 - 53.46)°$$

$$= 126.54° \quad \textbf{[1 mark]}$$

Question 3

$\angle\text{PAB} = 33°, \angle\text{PBA} = 140° \Rightarrow \angle\text{APB} = 7°$ **[1 mark]**

$$\frac{\text{PB}}{\sin(\text{PAB})} = \frac{\text{AB}}{\sin(\text{APB})}$$

$$\text{PB} = \frac{\text{AB}\sin(\text{PAB})}{\sin(\text{APB})}$$

$$= 8.938 \quad \textbf{[1 mark]}$$

In $\Delta\text{BPL}, \angle\text{PBL} = 40°$

$$\sin(40°) = \frac{\text{PL}}{\text{PB}}$$

$$\text{PL} = 8.938\sin(40°)$$

$$= 5.745\text{ km} \quad \textbf{[1 mark]}$$

Question 4

$\angle CAB = 40°$ (complementary angles), $\angle ABC = 55°$ (exterior and complementary angles),
$\angle ACB = 85°$ (angles in a triangle),

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 85°} = \frac{5}{\sin 55°}$$

$$x = \frac{5 \times \sin 85°}{\sin 55°}$$

$$= 6.08\text{ km}$$

Question 5

Use the cosine rule. **[1 mark]**

$$QR = \sqrt{(5^2 + 7^2 - 2 \times 5 \times 7 \times \cos(30°))}$$

$$= 3.658\text{ cm} \quad \textbf{[1 mark]}$$

Question 6

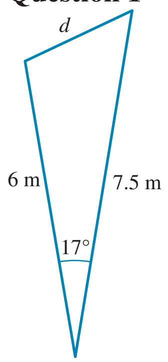
$\angle\text{ACB} = 45°$ **[1 mark]**

$$\frac{40}{\sin(45°)} = \frac{\text{AC}}{\sin(80°)}$$

$$\text{AC} = 55.71 \quad \textbf{[1 mark]}$$

7.4 The cosine rule

Question 1



$d^2 = 6^2 + 7.5^2 - 2 \times 6 \times 7.5 \times \cos(17°)$ **[1 mark]**

$= 36 + 56.25 - 86.067$

$d = 2.49$ m **[1 mark]**

Question 2

Use the cosine rule

$\text{PR} = \sqrt{(18^2 + 25^2 - 2 \times 18 \times 25 \times \cos(74°))}$

$= 26.28$ cm

Question 3

$\cos(x) = \dfrac{18^2 + 24^2 - 13^2}{2 \times 18 \times 24}$ **[1 mark]**

$x = 32.21°$ **[1 mark]**

Question 4

Use the sine rule.

$\angle\text{ACB} = 40°$

$\dfrac{\text{BC}}{\sin(60°)} = \dfrac{50}{\sin(40°)}$ **[1 mark]**

$\text{BC} = \dfrac{50\sin(60°)}{\sin(40°)}$

$= 67.36$ cm **[1 mark]**

Question 5

Largest angle is opposite the longest side. **[1 mark]**

$\cos(x) = \dfrac{8^2 + 7^2 - 14^2}{2 \times 8 \times 7}$ **[1 mark]**

$x = 137.8°$ **[1 mark]**

7.5 Arc length, sectors and segments

Question 1

$A = \dfrac{1}{2}r^2(\theta - \sin(\theta))$

$= \dfrac{1}{2}10^2\left(\left(\dfrac{\pi}{6}\right)^c - \sin\left(\dfrac{\pi}{6}\right)^c\right)$ **[1 mark]**

$= 50(0.5236 - 0.5)$

$= 1.180$ **[1 mark]**

Question 2

$A = \frac{1}{2}r^2\theta$

$r = \sqrt{\frac{2A}{\theta}}$ (positive value) **[1 mark]**

$r = \sqrt{\frac{2 \times 500}{\left(\frac{40\pi}{180}\right)}}$

$= 37.85\text{ cm}$ **[1 mark]**

Question 3

$\frac{1}{2}r^2\theta A = r\theta$ **[1 mark]**

$\frac{1}{2}r = 1$

$r = 2$

Therefore, the radius is 2 units. **[1 mark]**

Question 4

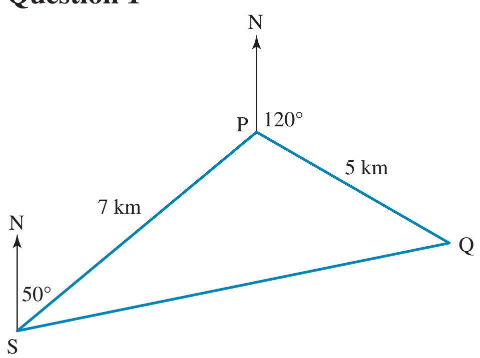
$\frac{1}{2}\theta r^2$

Question 5

(θr)

7.6 Review

Question 1



$\angle SPQ = 50° + 60°$

$= 110°$ **[1 mark]**

Cosine rule:

$SQ = \sqrt{(7^2 + 5^2 - 2 \times 7 \times 5 \times \cos(110°))}$

$= 9.90\text{ km}$ **[1 mark]**

Sine rule:

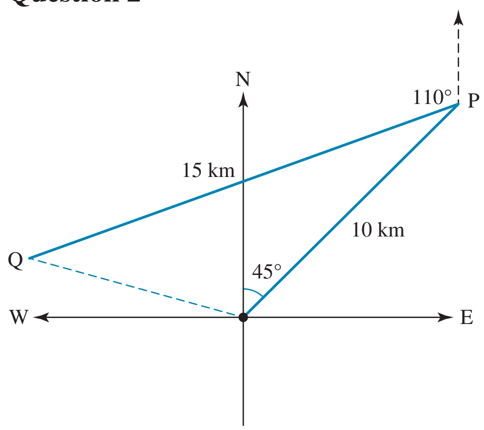
$\frac{\sin(\angle PQS)}{7} = \frac{\sin(\angle SPQ)}{9.90}$

$\angle PQS = 41.64°$ **[1 mark]**

$\text{Bearing} = (360 - (60 + 41.64))°$

$= 258.36°\text{T}$ **[1 mark]**

Question 2



Method: find ∠QOP, then subtract 45° to give ∠QON then subtract ∠QON from 90 degrees to give ∠QOW.
The bearing of O from Q is (∠QOW + 90)°.

$\angle QPO = (180 - 110 - 45)°$

$= 35°$ **[1 mark]**

In ΔQOP:

$OQ^2 = OP^2 + QP^2 - 2 \times OP \times QP \times \cos(QPO)$

$= 100 + 225 - 300\cos(35°)$

$OQ = 8.902$ **[1 mark]**

$$\frac{\sin(QOP)}{QP} = \frac{\sin(QPO)}{OQ}$$

$$\sin(QOP) = \frac{QP\sin(QPO)}{OQ}$$

$= 0.9664$

$\angle QOP = 180 - 75.11 = 104.89°$ **[1 mark]**

$\angle QOP = (104.89 - 45)°$

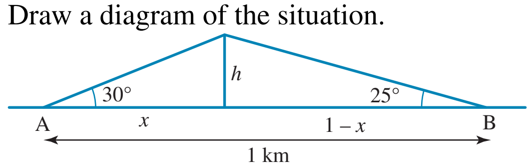
$= 59.89°$

$\angle QOW = 90 - 59.89 = 30.11°$

$\therefore$ Bearing of O from Q is $(\angle QOW + 90)° = 120.11°$T **[1 mark]**

Question 3

Draw a diagram of the situation.



$$\tan(30^o) = \frac{h}{x}$$

$$\tan(25^o) = \frac{h}{1 - x}$$

$$\frac{\tan(30^o)}{\tan(25^o)} = \left(\frac{h}{x}\right) \times \left(\frac{1 - x}{h}\right)$$

$$1.23813 = \frac{1 - x}{x}$$

$$1.23813x = 1 - x$$

$$2.23813x = 1$$

$$x = \frac{1}{2.23813}$$

$$= 0.4468\text{ km}$$

Substituting this value back into the first equation:

$$\tan(30°) = \frac{h}{x}$$

$$\frac{\sqrt{3}}{3} = \frac{h}{0.4468}$$

$$h = \frac{0.4468\sqrt{3}}{3}$$

$$= 0.258\text{ km}$$

$$= 258\text{ m}$$

The cliff is 258 metres high.

Question 4

$\angle CAB = 40°$ (complementary angles)

$\angle ABC = 55°$ (exterior and complementary angles)

$\angle ACB = 85°$ (angles in a triangle)

$$\frac{c}{\sin(C)} = \frac{b}{\sin(B)}$$

$$\frac{x}{\sin(85°)} = \frac{5}{\sin(55°)}$$

$$x = \frac{5\sin(85°)}{\sin(55°)}$$

$$= 6.08\text{ km}$$

Question 5

This is a major sector.

Therefore, the angle subtended by the major sector is $(360 - 80)° = 280°$

This is equivalent to $280 \times \frac{\pi}{180} = \frac{28\pi}{18} = 4.8869^c$

Therefore, the area of the major sector is:

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2} \times 5^2 \times 4.8869^c$$

$$= 61.08625$$

$\approx 61\text{ cm}^2$ **[1 mark]**

Question 6

$$\frac{1}{2}r^2\theta = 6r\theta$$

$r = 12$ **[1 mark]**

Question 7

a. A sector is bounded by two radii and one arc. **[1 mark]**

Perimeter $= r + r + r\theta$

$= r(\theta + 2)$ **[1 mark]**

b. A segment is bounded by a chord and an arc. **[1 mark]**

$$\text{Chord length} = 2 \times r\sin\left(\frac{\theta}{2}\right)$$

$$= 2r\sin\left(\frac{\theta}{2}\right)$$ **[1 mark]**

$$\text{Perimeter} = 2r\sin\left(\frac{\theta}{2}\right) + r\theta$$

$$= r\left(\theta + 2\sin\left(\frac{\theta}{2}\right)\right)$$ **[1 mark]**

c. z $r(\theta + 2) = r\left(\theta + 2\sin\left(\frac{\theta}{2}\right)\right)$

$$r\theta + 2r = r\theta + 2r\sin\left(\frac{\theta}{2}\right)$$

$$\sin\left(\frac{\theta}{2}\right) =$$ **[1 mark]**

$$\theta = 180°$$ **[1 mark]**

8 Trigonometric identities

Topic	8	Trigonometric identities
Subtopic	8.2	Pythagorean identities

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Question 1 (1 mark)

The expression $\dfrac{1+\tan^2(x)}{\tan^2(x)}$ can be simplified to

A. $\sec^2(x)$

B. $\dfrac{\sin^2(x)}{\cos^2(x)}$

C. $\operatorname{cosec}^2(x)$

D. $\cot^2(x)$

E. 2

Question 2 (1 mark)

Determine which of the following statements is **false**.

A. $\sin\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right)$

B. $\cos^3(\pi)+\sin^3(\pi)=1$

C. $\cos^2(\pi)+\sin^2(\pi)=1$

D. $\cos(\pi)+\sin(\pi)=-1$

E. $\cos\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)=1$

Question 3 (1 mark)

Determine which of the following statements is **false**.

A. $2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right)$

B. $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$

C. $\tan\left(\frac{3\pi}{2}\right) = \dfrac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)}$

D. $\dfrac{1}{\tan\left(\frac{3\pi}{2}\right)} = \dfrac{\cos\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{2}\right)}$

E. $\sin\left(\frac{\pi}{6}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\left(\frac{\pi}{3}\right)\right)}$

Question 4 (1 mark)

Which of the following statements is true?

A. $\left(2\sin\left(\frac{x}{2}\right)\right)^2 + \left(2\cos\left(\frac{x}{2}\right)\right)^2 = 1$

B. $\left(2\sin\left(\frac{x^2}{2}\right)\right)^2 + \left(2\cos\left(\frac{x^2}{2}\right)\right)^2 = 1$

C. $\left(2\sin^2\left(\frac{x}{2}\right)\right)^2 + \left(2\cos^2\left(\frac{x}{2}\right)\right)^2 = 1$

D. $\left(\sin\left(\frac{x}{2}\right)\right)^2 + \left(\cos\left(\frac{x}{2}\right)\right)^2 = 1$

E. $\sqrt{\sin^2\left(\frac{x^2}{2}\right)} + \sqrt{\cos^2\left(\frac{x^2}{2}\right)} = 1$

Topic 8 Subtopic 8.2 Pythagorean identities

Question 5 (1 mark)

$\frac{\sin(A)}{1+\cos(A)} + \frac{1+\cos(A)}{\sin(A)}$ is equal to

A. $2\sin(A)$
B. $2\cos(A)$
C. 2
D. $\frac{2}{\sin(A)}$
E. $\frac{2}{\cos(A)}$

Question 6 (2 marks)

Prove that $\sin^4(x) - \cos^4(x) = (\sin(x) - \cos(x))(\sin(x) + \cos(x))$

Question 7 (2 marks)

Prove that $\frac{1-\sin(x)}{1+\sin(x)} = \frac{1}{(\sec(x)+\tan(x))^2}$

Question 8 (2 marks)

Prove that $\frac{1-\cos(x)}{\sin(x)} + \frac{\sin(x)}{1+\cos(x)} = \frac{2}{\operatorname{cosec}(x)+\cot(x)}$.

Question 9 (2 marks)

Simplify $\frac{\cos(2x)-1}{\cos(2x)+1}$.

Topic	8	Trigonometric identities
Subtopic	8.3	Compound angle formulas

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Question 1 (1 mark)

When expanded, $\cos(a-b)+\cos(a+b)$ equals

A. 0
B. $2\cos(a)\cos(b)$
C. $-2\cos(a)\cos(b)$
D. $2\sin(a)\sin(b)$
E. $-2\sin(a)\sin(b)$

Question 2 (1 mark)

Determine which of the following is **not** a correct trigonometric identity.

A. $\cos\left(x+\frac{\pi}{4}\right)-\cos\left(x-\frac{\pi}{4}\right)=\sqrt{2}\sin(x)$
B. $\cos\left(x+\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}\right)=\sqrt{2}\cos(x)$
C. $\sin\left(x+\frac{\pi}{4}\right)-\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}\cos(x)$
D. $\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}\sin(x)$
E. $\tan\left(x+\frac{\pi}{4}\right)-\tan\left(x-\frac{\pi}{4}\right)=\frac{2\left(\tan^2(x)+1\right)}{1-\tan^2(x)}$

Question 3 (2 marks)

Simplify $\cos(x+y)\cos(x-y)+\sin(x+y)\sin(x-y)$.

Question 4 (1 mark)
Expand $\tan(3x + 2y)$.

Question 5 (2 marks)
Simplify $\sin(A + B) + \sin(A - B)$.

Question 6 (1 mark)
Use the compound-angle formula to prove that $\cos(2A) = \cos^2(A) - \sin^2(A)$.

Question 7 (1 mark)
If $\cos(2\theta) = \dfrac{1 - a}{b}$, find $\cos(\theta)$.

Question 8 (1 mark)
When expanded, $\cos(a - b) + \cos(a + b)$ equals:
A. 0
B. $2\cos(a)\cos(b)$
C. $-2\cos(a)\cos(b)$
D. $2\sin(a)\sin(b)$
E. $-2\sin(a)\sin(b)$

Topic	8	Trigonometric identities
Subtopic	8.4	Double and half angle formulas

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Question 1 (3 marks)

Simplify $\cos(4\theta)$.

Question 2 (2 marks)

Answer the following

a. Simplify $\sin(2\theta)\cos(\theta) - \sin(\theta)\cos(2\theta)$. **(1 mark)**

b. Simplify $\sin(6\theta)\cos(2\theta) - \sin(2\theta)\cos(6\theta)$. **(1 mark)**

Question 3 (1 mark)

Determine which of the following is **not** equal to $\cot\left(\frac{8\pi}{3}\right)$.

A. $\tan\left(-\frac{\pi}{6}\right)$

B. $\dfrac{\cos\left(\frac{8\pi}{3}\right)}{\sin\left(\frac{8\pi}{3}\right)}$

C. $\dfrac{1}{\tan\left(\frac{8\pi}{3}\right)}$

D. $\dfrac{1-\tan^2\left(\frac{2\pi}{3}\right)}{2\tan\left(\frac{2\pi}{3}\right)}$

E. $\dfrac{1-\tan^2\left(\frac{2\pi}{6}\right)}{2\tan\left(\frac{2\pi}{6}\right)}$

Question 4 (1 mark)

Which of the following is **not** a correct trigonometric identity?

A. $\sin(2x) = \dfrac{2\tan(x)}{1+\tan^2(x)}$

B. $\cos(2x) = \dfrac{1-\tan^2(x)}{1+\tan^2(x)}$

C. $\dfrac{1-\cos(2x)}{\sin(2x)} = \tan(x)$

D. $\dfrac{\sin(2x)}{1+\cos(2x)} = \cot(x)$

E. $\dfrac{\sin(3x)}{\sin(x)} - \dfrac{\cos(3x)}{\cos(x)} = 2$

Question 5 (3 marks)

Show that the multiple angle formula $\cos(3x) = 4\cos^3(x) - 3\cos(x)$ holds for $x = \dfrac{\pi}{3}$.

Topic	8	Trigonometric identities
Subtopic	8.5	Converting *a* cos(*x*) + *b* sin(*x*) to *r* cos ($x \pm \alpha$) or r sin ($x \pm \alpha$)

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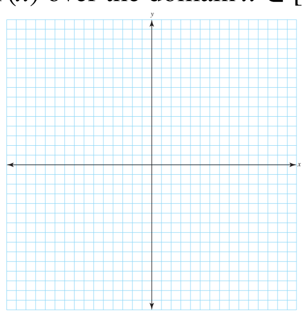
Question 1 (3 marks)

Solve the trigonometric equation $\sqrt{3}\cos(x) + \sin(x) = 1$ for $0 \le x \le 2\pi$ by converting the LHS of the equation into the form $r\cos(x - a)$.

Question 2 (1 mark)

Sketch the graph of $y = \cos(x) + \sqrt{3}\sin(x)$ over the domain $x \in [0, 2\pi]$.



Question 3 (1 mark)

$\cos(x) - \sin(x)$ can be rewritten as:

A. $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

B. $\sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$

C. $\sqrt{2}\sin\left(x + \frac{5\pi}{4}\right)$

D. $\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$

E. $\sqrt{2}\sin\left(x + \frac{\pi}{2}\right)$

Question 4 (1 mark)

$4\cos\left(\theta + \frac{\pi}{6}\right)$ is equal to:

A. $2\sqrt{3}\cos(\theta) + 2\sin(\theta)$

B. $2\sqrt{3}\cos(\theta) - 2\sin(\theta)$

C. $2\sqrt{3}\sin(\theta) - 2\cos(\theta)$

D. $2\sqrt{3}\sin(\theta) + 2\cos(\theta)$

E. $2\sqrt{3}\cos(\theta) + 2\cos(\theta)$

Topic	8	Trigonometric identities
Subtopic	8.6	Review

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (3 marks)

Show that $\dfrac{\sin(2\theta)\cos(\theta) - \sin(\theta)\cos(2\theta)}{\sin(6\theta)\cos(2\theta) - \sin(2\theta)\cos(6\theta)} = \dfrac{1}{4}\sec(\theta)\sec(2\theta)$.

Question 2 (2 marks)

If $\dfrac{1}{4}\sec(\theta)\sec(2\theta) = -\dfrac{1}{4}$, calculate θ, $0 \le \theta \le 2\pi$.

Question 3 (1 mark)

Determine which of the following statements is **false**.

A. $2\sin\left(\dfrac{\pi}{4}\right)\cos\left(\dfrac{\pi}{4}\right) = 1$

B. $\cos^2\left(\dfrac{\pi}{4}\right) - \sin^2\left(\dfrac{\pi}{4}\right) = 0$

C. $\tan\left(\dfrac{\pi}{2}\right) - \tan\left(\dfrac{5\pi}{2}\right) = 0$

D. $\tan\left(\dfrac{\pi}{3}\right) = \dfrac{2\tan\left(\frac{\pi}{6}\right)}{1 - \tan^2\left(\frac{\pi}{6}\right)}$

E. $\cos\left(\dfrac{\pi}{6}\right) = \sqrt{\dfrac{1}{2}\left(1 + \cos\left(\dfrac{\pi}{3}\right)\right)}$

Question 4 (2 marks)

If $\cot(2x) = \frac{4\sqrt{2}}{7}$, and $\frac{\pi}{2} \leq x \leq \pi$, determine an exact value for $\sin(2x)$.

$\sin(2x) = \square$

Question 5 (2 marks)

If $\cos(2\theta) = \frac{1-a}{b}$, determine $\cos(\theta)$ in terms of a and b.

$\cos(\theta) = \square$

Question 6 (1 mark)

Which of the following is not equal to $\cot\left(\frac{8\pi}{3}\right)$?

A. $\tan\left(-\frac{\pi}{6}\right)$

B. $\frac{\cos\left(\frac{8\pi}{3}\right)}{\sin\left(\frac{8\pi}{3}\right)}$

C. $\frac{1}{\tan\left(\frac{8\pi}{3}\right)}$

D. $\frac{1-\tan^2\left(\frac{2\pi}{3}\right)}{2\tan\left(\frac{2\pi}{3}\right)}$

E. $\frac{1-\tan^2\left(\frac{8\pi}{6}\right)}{2\tan\left(\frac{8\pi}{6}\right)}$

Question 7 (1 mark)

Which of the following is **not** true?

A. $(\sec(x) + \tan(x))(\sec(x) - \tan(x)) = 1$

B. $(\operatorname{cosec}(x) + \cot(x))(\operatorname{cosec}(x) - \cot(x)) = 1$

C. $\dfrac{1 + \sin(x)}{\cos(x)} = \sec(x) + \tan(x)$

D. $\dfrac{1 - \cos(x)}{\sin(x)} = \operatorname{cosec}(x) - \cot(x)$

E. $\dfrac{\sin(x)}{\operatorname{cosec}(x)} + \dfrac{\cos(x)}{\sec(x)} = 2$

Question 8 (1 mark)

Which of the following is **not** a correct trigonometric identity?

A. $\sec^2(x) + \operatorname{cosec}^2(x) = \sec^2(x)\operatorname{cosec}^2(x)$

B. $\left(\sec^2(x) - 1\right)\left(\operatorname{cosec}^2(x) - 1\right) = 1$

C. $\cot(x) + \tan(x) = \sec(x)\operatorname{cosec}(x)$

D. $\dfrac{1}{1 - \sin(x)} - \dfrac{1}{1 + \sin(x)} = 2\operatorname{cosec}^2(x)$

E. $\sec^2(x) - \operatorname{cosec}^2(x) = \tan^2(x) - \cot^2(x)$

Question 9 (3 marks)

Prove that $\dfrac{\cos(3\theta)}{\cos(\theta)} = 1 - 4\sin^2(\theta)$.

Question 10 (4 marks)

Factorise $2\sin(x)\sin(3x) - \sin(8x)$.

Question 11 (3 marks)
Prove the identity $\sin 4x + \sin 2x = 6\sin x \cos x - 8\sin^3 x \cos x$.

Question 12 (2 marks)
Write $\cos(3x) - \cos(5x)$ as a product.

Question 13 (1 mark)
Answer the following.
a. Simplify $\sin(2\theta)\cos(\theta) - \sin(\theta)\cos(2\theta)$

Question 14 (1 mark)
Which of the following statements is **false**?
A. $\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)$
B. $\cos^3(\pi) + \sin^3(\pi) = 1$
C. $\cos^2(\pi) + \sin^2(\pi) = 1$
D. $\cos(\pi) + \sin(\pi) = -1$
E. $\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1$

Topic 8 Subtopic 8.6 Review

Question 15 (1 mark)

Answer the following

a. Simplify $\sin(2\theta)\cos(\theta) - \sin(\theta)\cos(2\theta)$

b. Simplify $\sin(6\theta)\cos(2\theta) - \sin(2\theta)\cos(6\theta)$

c. Show that $\dfrac{\sin(2\theta)\cos(\theta) - \sin(\theta)\cos(2\theta)}{\sin(6\theta)\cos(2\theta) - \sin(2\theta)\cos(6\theta)} = \dfrac{1}{4}\sec(\theta)\sec(2\theta)$

d. If $\dfrac{1}{4}\sec(\theta)\sec(2\theta) = -\dfrac{1}{4}$, find $\theta, 0 \le \theta \le 2\pi$.

Answers and marking guide

8.2 Pythagorean identities

Question 1

$$\frac{1+\tan^2(x)}{\tan^2(x)}$$
$$=\frac{1}{\tan^2(x)}+\frac{\tan^2(x)}{\tan^2(x)}$$
$$=\cot^2(x)+1=\operatorname{cosec}^2(x)$$

Question 2

$\cos^3(\pi)+\sin^3(\pi)=1$ is false since $\cos^3(\pi)+\sin^3(\pi)=(-1)^3+0^3=-1$

Question 3

$\tan\left(\frac{3\pi}{2}\right)=\frac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)}$ is false since $\cos\left(\frac{3\pi}{2}\right)=0$

Question 4

For statements A, B and C the right-hand side is equal to 4, and E is false; therefore D is true.

Question 5

$$\frac{\sin(A)}{1+\cos(A)}+\frac{1+\cos(A)}{\sin(A)}$$
$$=\frac{\sin^2(A)+(1+\cos(A))^2}{(1+\cos(A))\sin(A)}$$
$$=\frac{\sin^2(A)+1+2\cos(A)+\cos^2(A)}{\sin(A)(1+\cos(A))}$$
$$=\frac{2+2\cos(A)}{\sin(A)(1+\cos(A))}=\frac{2(1+\cos(A))}{\sin(A)(1+\cos(A))}$$
$$=\frac{2}{\sin(A)}$$

Question 6

LHS:

$\sin^4 x-\cos^4 x=\left(\sin^2 x-\cos^2 x\right)\left(\sin^2 x+\cos^2 x\right)$

$=\left(\sin^2 x-\cos^2 x\right)$ **[1 mark]**

RHS:

$(\sin x-\cos x)(\sin x+\cos x)=\left(\sin^2 x-\cos^2 x\right)$ **[1 mark]**

∴ LHS = RHS

Question 7

LHS:

$$\frac{1-\sin(x)}{1+\sin(x)}\times\frac{1+\sin(x)}{1+\sin(x)}=\frac{1-\sin^2(x)}{(1+\sin(x))^2}$$
$$=\frac{\cos^2(x)}{(1+\sin(x))^2}$$ **[1 mark]**

RHS:

$$\frac{1}{(\sec(x)+\tan(x))^2}=\frac{1}{\left(\frac{1}{\cos(x)}+\frac{\sin(x)}{\cos(x)}\right)^2}$$
$$=\frac{\cos^2(x)}{(1+\sin(x))^2}$$ **[1 mark]**

$\therefore$ LHS = RHS

Question 8

LHS:

$$\frac{1-\cos(x)}{\sin(x)}+\frac{\sin(x)}{1+\cos(x)}=\frac{\left(1-\cos^2(x)+\sin^2(x)\right)}{\sin x\,(1+\cos(x))}$$
$$=\frac{\sin^2(x)+\sin^2(x)}{\sin(x)\,(1+\cos(x))}$$
$$=\frac{2\sin^2(x)}{\sin(x)\,(1+\cos(x))}$$
$$=\frac{2\sin(x)}{(1+\cos(x))}\qquad\textbf{[1 mark]}$$

RHS:

$$\frac{2}{\operatorname{cosec}(x)+\cot(x)}=\frac{2}{\frac{1}{\sin(x)}+\frac{\cos(x)}{\sin(x)}}$$
$$=\frac{2\sin(x)}{1+\cos(x)}\qquad\textbf{[1 mark]}$$

$\therefore$ LHS = RHS

Question 9

$$\frac{\cos(2x)-1}{\cos(2x)+1}=\frac{\left(1-2\sin^2(x)\right)-1}{\left(2\cos^2(x)-1\right)+1}\qquad\textbf{[1 mark]}$$
$$=\frac{-2\sin^2(x)}{2\cos^2(x)}$$
$$=-\tan^2(x)\qquad\textbf{[1 mark]}$$

8.3 Compound angle formulas

Question 1

$\cos(a-b)+\cos(a+b)$

$=\cos(a)\cos(b)+\sin(a)\sin(b)+\cos(a)\cos(b)-\sin(a)\sin(b)$

$=2\cos(a)\cos(b)$

Question 2

B, C, D, and E can all be shown to be correct trigonometric identities.

Question 3

$\cos(x+y)\ \cos(x-y)+\sin(x+y)\ \sin(x-y)$

$=\cos\left[(x+y)-(x-y)\right]$ **[1 mark]**

$=\cos(2y)$ **[1 mark]**

Question 4

$$\tan(3x+2y)=\frac{\tan(3x)+\tan(2y)}{1-\tan(3x)\tan(2y)}\qquad\textbf{[1 mark]}$$

Question 5

$\sin(A+B)+\sin(A-B)=\sin(A)\cos(B)+\cos(A)\sin(B)+\sin(A)\cos(B)-\cos(A)\sin(B)$ **[1 mark]**

$=2\sin(A)\cos(B)$ **[1 mark]**

Question 6

$\cos(2A)=\cos(A+A)$

$=\cos(A)\cos(A)-\sin(A)\sin(A)$

$=\cos^2(A)-\sin^2(A)$ **[1 mark]**

Question 7

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

$$2\cos^2(\theta) = 1 + \left(\frac{1-a}{b}\right) \qquad \textbf{[1 mark]}$$

$$\cos(\theta) = \frac{1}{2}\left(1 + \left(\frac{1-a}{b}\right)\right)$$

$$= \left(\frac{1-a+b}{2b}\right) \qquad \textbf{[1 mark]}$$

Question 8

$$\cos(a-b) + \cos(a+b)$$

$$= \cos(a)\cos(b) + \sin(a)\sin(a) + \cos(a)\cos(b) - \sin(a)\sin(a)$$

$$= 2\cos(a)\cos(b)$$

8.4 Double and half angle formulas

Question 1

$$\cos(4\theta) = \cos(2(2\theta))$$

$$= 2\cos^2(2\theta) - 1 \qquad \textbf{[1 mark]}$$

$$2\cos^2(2\theta) - 1 = 2\left(2\cos^2(\theta) - 1\right)^2 - 1$$

$$= 2\left(4\cos^4(\theta) - 4\cos^2(\theta) + 1\right) - 1 \quad \textbf{[1 mark]}$$

$$= 8\cos^4(\theta) - 8\cos^2(\theta) + 1 \qquad \textbf{[1 mark]}$$

Question 2

a. $\sin(2\theta)\cos(\theta) - \sin(\theta)\cos(2\theta) = \sin(2\theta - \theta)$

$= \sin(\theta)$ **[1 mark]**

b. $\sin(6\theta)\cos(2\theta) - \sin(2\theta)\cos(6\theta) = \sin(6\theta - 2\theta)$

$= \sin(4\theta)$ **[1 mark]**

Question 3

$$\frac{1 - \tan^2\left(\frac{2\pi}{3}\right)}{2\tan\left(\frac{2\pi}{3}\right)} = \cot\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{3}$$

$$\neq \cot\left(\frac{8\pi}{3}\right)$$

Question 4

D. $$\frac{\sin(2x)}{1+\cos(2x)} = \frac{2\sin(x)\cos(x)}{1 + \left(2\cos^2(x) - 1\right)}$$

$$= \frac{2\sin(x)\cos(x)}{2\cos^2(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x)$$

A, B, C and E can be shown to be true identities.

$$\sin(2x) = 2\sin(x)\cos(x)$$

A. $$= \frac{2\sin(x)\cos^2(x)}{\cos(x)} = \frac{2\tan(x)}{\sec^2(x)} = \frac{2\tan(x)}{1+\tan^2(x)}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

B. $$= \frac{\frac{1}{\cos^2(x)}\left(\cos^2(x) - \sin^2(x)\right)}{\frac{1}{\cos^2(x)}} = \frac{1-\tan^2(x)}{\sec^2(x)} = \frac{1-\tan^2(x)}{1+\tan^2(x)}$$

C. $\frac{1-\cos(2x)}{\sin(2x)} = \frac{1-\left(1-2\sin^2(x)\right)}{2\sin(x)\cos(x)}$

$= \frac{2\sin^2(x)}{2\sin(x)\cos(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x)$

E. $\frac{\sin(3x)}{\sin(x)} - \frac{\cos(3x)}{\cos(x)} = \frac{\sin(3x)\cos(x) - \sin(x)\cos(3x)}{\sin(x)\cos(x)}$

$= \frac{\sin(3x-x)}{\frac{1}{2}\sin(2x)} = 2$

Question 5

$\cos\left(3\left(\frac{\pi}{3}\right)\right) = \cos(\pi)$

$= 1$ **[1 mark]**

$\cos\left(3\left(\frac{\pi}{3}\right)\right) = 4\cos^3\left(\frac{\pi}{3}\right) - 3\cos\left(\frac{\pi}{3}\right)$ **[1 mark]**

$= 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)$

$= 4\left(\frac{1}{8}\right) - \frac{3}{2}$

$= \frac{1}{2} - \frac{3}{2}$

$= -1$ **[1 mark]**

8.5 Converting $a\cos(X) + b\sin(X)$ to $r\cos(X \pm \alpha)$ or $r\sin(X \pm \alpha)$

Question 1

Let $\sqrt{3}\cos(x) + \sin(x) = r\cos(x-\alpha)$

$= r(\cos(x)\cos(\alpha) + \sin(x)\sin(\alpha))$

$= r\cos(x)\cos(\alpha) + \sin(x)\sin(\alpha)$

$[1] : r\cos(\alpha) = \sqrt{3}$

$: r\sin(\alpha) = 1$

$\frac{[2]}{[1]} : \frac{r\sin(\alpha)}{r\cos(\alpha)} = \frac{1}{\sqrt{3}}$

$\tan(\alpha) = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$[1]\ r\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$

$r \times \frac{\sqrt{3}}{2} = \sqrt{3} \therefore r = 2$

$\therefore \sqrt{3}\cos(x) + \sin(x) = 2\cos\left(x - \frac{\pi}{6}\right)$ **[1 mark]**

Now solving for x:

$2\cos\left(x - \frac{\pi}{6}\right) = 1$

$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

Over the domain $0 \leq x - \frac{\pi}{6} \leq 2\pi$

$\therefore$ Domain is now $\frac{\pi}{6} \leq x \leq \frac{13\pi}{6}$ **[1 mark]**

Reference angle is $\frac{\pi}{3}$ and cos is postive in first and fourth quadrants

$x - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{2}, \frac{11\pi}{6}$ **[1 mark]**

Question 2

$$\begin{aligned}\cos(x) + \sqrt{3}\sin(x) &= r\cos(x - \alpha)\\ &= r(\cos(x)\cos(\alpha) + \sin(x)\sin(\alpha))\\ &= r\cos(x)\cos(\alpha) + r\sin(x)\sin(\alpha)\end{aligned}$$

[1]:$r\cos(\alpha) = 1$

:$r\sin(\alpha) = \sqrt{3}$

$\frac{[2]}{[1]}:\frac{r\sin(\alpha)}{r\cos(\alpha)} = \sqrt{3}$

$\tan(\alpha) = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

[1] $r\cos\left(\frac{\pi}{3}\right) = 1$

$r \times \frac{1}{2} = 1 \therefore r = 2$

$\therefore \cos(x) + \sqrt{3}\sin(x) = 2\cos\left(x - \frac{\pi}{3}\right)$ **[1 mark]**

This graph has a maximum of 2 and a minimum of -2. It has a period of 2π and a starting point of $(0, 1)$ and an end point of $(2\pi, 1)$.

Its x-intercepts are at:

$2\cos\left(x - \frac{\pi}{3}\right) = 0$

$\cos\left(x - \frac{\pi}{3}\right) = 0$

Over the domain $0 \leq x - \frac{\pi}{3} \leq 2\pi$

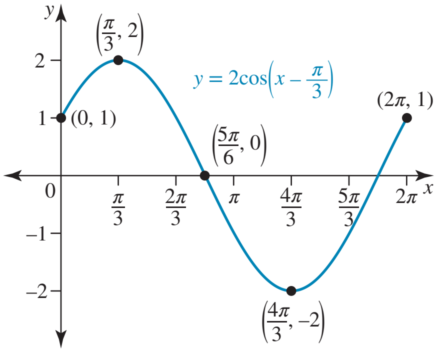
$\therefore \frac{\pi}{3} \leq x \leq \frac{7\pi}{3}$

Reference angle is $\frac{\pi}{2}$

$x - \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{5\pi}{6}, \frac{11\pi}{6}$ **[1 mark]**

The graph is shown below:



Award 1 mark for correct graph shape
Award 1 mark for the correct endpoints, axis intercepts and stationary points

Question 3

Using a CAS calculator to collect the expression $\cos(x) - \sin(x)$

$$\cos(x) - \sin(x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$$

Recall that $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$

Therefore,

$$\begin{aligned}\cos(x) - \sin(x) &= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)\\ &= \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)\end{aligned}$$

Question 4

$$\begin{aligned}4\cos\left(\theta + \frac{\pi}{6}\right) &= 4\left(\cos(\theta)\cos\left(\frac{\pi}{6}\right) - \sin(\theta)\sin\left(\frac{\pi}{6}\right)\right)\\ &= 4\left(\frac{\sqrt{3}}{2}\cos(\theta) - \frac{1}{2}\sin(\theta)\right)\\ &= 2\left(\sqrt{3}\cos(\theta) - \sin(\theta)\right)\\ &= 2\sqrt{3}\cos(\theta) - 2\sin(\theta)\end{aligned}$$

8.6 Review

Question 1

$$\begin{aligned}\frac{\sin(2\theta)\cos(\theta) - \sin(\theta)\cos(2\theta)}{\sin(6\theta)\cos(2\theta) - \sin(2\theta)\cos(6\theta)} &= \frac{\sin(\theta)}{\sin(4\theta)} && \textbf{[1 mark]}\\ &= \frac{\sin(\theta)}{2\sin(2\theta)\cos(2\theta)}\\ &= \frac{\sin(\theta)}{4\sin(\theta)\cos(\theta)\cos(2\theta)} && \textbf{[1 mark]}\\ &= \frac{1}{4\cos(\theta)\cos(2\theta)}\\ &= \frac{1}{4}\sec(\theta)\sec(2\theta) && \textbf{[1 mark]}\end{aligned}$$

Question 2

$$\frac{1}{4}\sec(\theta)\sec(2\theta) = -\frac{1}{4}$$
$$\frac{1}{\cos(\theta)\cos(2\theta)} = -1$$
$$\cos(\theta)\left(2\cos^2(\theta) - 1\right) = -1 \qquad \textbf{[1 mark]}$$
$$2\cos^3(\theta) - \cos(\theta) + 1 = 0$$
$$(\cos(\theta) + 1)\left(2\cos^2(\theta) - 2\cos(\theta) + 1\right) = 0$$
$$\cos(\theta) + 1 = 0 \Rightarrow \theta = \frac{3\pi}{2}$$
$$2\cos^2(\theta) - 2\cos(\theta) + 1 = 0, \text{ no solutions} \qquad \textbf{[1 mark]}$$

Question 3

C is false since $\tan\left(\frac{\pi}{2}\right) - \tan\left(\frac{3\pi}{2}\right)$ is undefined.

Question 4

$$\begin{aligned}\operatorname{cosec}^2(2x) &= 1 + \cot^2(2x)\\ &= 1 + \frac{32}{49}\\ &= \frac{81}{49} \qquad \textbf{[1 mark]}\end{aligned}$$

$$\operatorname{cosec}(2x) = \pm\frac{9}{7}$$
$$\sin(2x) = \pm\frac{7}{9}$$
$$\frac{\pi}{2} \le x \le \pi \Rightarrow \pi \le 2x \le 2\pi$$

Hence $\sin(2x)$ is negative $\Rightarrow \sin(2x) = -\frac{7}{9}$ **[1 mark]**

Question 5

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$
$$2\cos^2(\theta) = 1 + \left(\frac{1-a}{b}\right) \qquad \textbf{[1 mark]}$$
$$\cos^2(\theta) = \frac{1}{2}\left(1 + \left(\frac{1-a}{b}\right)\right)$$
$$\cos(\theta) = \sqrt{\frac{1-a+b}{2b}} \qquad \textbf{[1 mark]}$$

Question 6

$$\frac{1 - \tan^2\left(\frac{2\pi}{3}\right)}{2\tan\left(\frac{2\pi}{3}\right)} = \cot\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{3}$$
$$\ne \cot\left(\frac{8\pi}{3}\right)$$

Question 7

Options A, B, C and D can be shown to be true identities.

A. $\sec^2(x) - \tan^2(x) = 1 \Rightarrow (\sec(x) + \tan(x))(\sec(x) - \tan(x)) = 1$

B. $\operatorname{cosec}^2(x) - \cot^2(x) = 1 \Rightarrow (\operatorname{cosec}(x) + \cot(x))(\operatorname{cosec}(x) - \cot(x)) = 1$

C. $\dfrac{1 + \sin(x)}{\cos(x)} = \dfrac{1}{\cos(x)} + \dfrac{\sin(x)}{\cos(x)} = \sec(x) + \tan(x)$

D. $\dfrac{1-\cos(x)}{\sin(x)} = \dfrac{1}{\sin(x)} - \dfrac{\cos(x)}{\sin(x)} = \operatorname{cosec}(x) - \cot(x)$

E. $\dfrac{\sin(x)}{\operatorname{cosec}(x)} + \dfrac{\cos(x)}{\sec(x)} = \dfrac{\sin(x)}{\frac{1}{\sin(x)}} + \dfrac{\cos(x)}{\frac{1}{\cos(x)}} = \sin^2(x) + \cos^2(x) = 1$

Question 8

Options **A**, **B**, **C** and **E** can be shown to be true identities.

A. $$\begin{aligned}\sec^2(x) + \operatorname{cosec}^2(x) &= \frac{1}{\cos^2(x)} + \frac{1}{\sin^2(x)}\\ &= \frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)\cos^2(x)}\\ &= \frac{1}{\sin^2(x)\cos^2(x)}\\ &= \sec^2(x)\operatorname{cosec}^2(x)\end{aligned}$$

B. $\left(\sec^2(x)-1\right)\left(\operatorname{cosec}^2(x)-1\right) = \tan^2(x)\cot^2(x) = 1$

C. $$\begin{aligned}\cot(x) + \tan(x) &= \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\sin(x)\cos(x)}\\ &= \frac{1}{\cos(x)\sin(x)}\\ &= \frac{1}{\cos(x)}\frac{1}{\sin(x)}\\ &= \sec(x)\operatorname{cosec}(x)\end{aligned}$$

D. $$\begin{aligned}\frac{1}{1-\sin(x)} - \frac{1}{1+\sin(x)} &= \frac{(1+\sin(x))+(1-\sin(x))}{(1-\sin(x))(1+\sin(x))}\\ &= \frac{2}{1-\sin^2(x)}\\ &= \frac{2}{\cos^2(x)}\\ &= 2\sec^2(x)\end{aligned}$$

E. $\sec^2(x) - \operatorname{cosec}^2(x) = \tan^2(x) - \cot^2(x)$

$\Rightarrow \sec^2(x) - \tan^2(x) = \operatorname{cosec}^2(x) - \cot^2(x) = 1$

Question 9

$$\begin{aligned}\frac{\cos(3\theta)}{\cos(\theta)} &= \frac{\cos(2\theta+\theta)}{\cos(\theta)}\\ &= \frac{\cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)}{\cos(\theta)} && \textbf{[1 mark]}\\ &= \frac{\left(1-2\sin^2(\theta)\right)\cos(\theta) - 2\sin(\theta)\cos(\theta)\sin(\theta)}{\cos(\theta)} && \textbf{[1 mark]}\\ &= \frac{\cos(\theta)\left(1-2\sin^2(\theta)-2\sin^2(\theta)\right)}{\cos(\theta)}\\ &= 1-4\sin^2(\theta) && \textbf{[1 mark]}\end{aligned}$$

$\therefore \dfrac{\cos(3\theta)}{\cos(\theta)} = 1 - 4\sin^2(\theta)$

Question 10

$$\begin{aligned}2\sin(x)\sin(3x)-\sin(8x) &= \cos(4x)+\cos(2x)-2\sin(4x)\cos(4x) \quad \textbf{[1 mark]}\\ &= \cos(4x)(1+\cos(2x)-4\sin(2x)\cos(2x)) \quad \textbf{[1 mark]}\\ &= \cos(4x)\left(2\cos^2(x)-8\sin(x)\cos(x)\cos(2x)\right) \quad \textbf{[1 mark]}\\ &= 2\cos(x)\cos(4x)(\cos(x)-4\sin(x)\cos(2x)) \quad \textbf{[1 mark]}\end{aligned}$$

Question 11

LHS:

$$\begin{aligned}\sin(4x)+\sin(2x) &= 2\sin(3x)\cos(x)\\ &= 2\sin(2x+x)\cos(x)\\ &= 2\cos(x)(\sin(2x)\cos(x)+\cos 2x\sin x)\\ &= 2\cos(x)\left(2\sin(x)\cos^2(x)+\left(1-2\sin^2(x)\right)\sin(x)\right) \quad \textbf{[1 mark]}\\ &= 2\sin(x)\cos(x)\left(2\cos^2(x)+\left(1-2\sin^2(x)\right)\right)\\ &= 2\sin(x)\cos(x)\left(2\left(1-\sin^2(x)\right)+\left(1-2\sin^2(x)\right)\right)\\ &= 2\sin(x)\cos(x)\left(3-4\sin^2(x)\right) \quad \textbf{[1 mark]}\end{aligned}$$

RHS:

$$6\sin x\cos x-8\sin^3 x\cos x = 2\sin x\cos x\left(3-4\sin^2 x\right)$$

$\therefore$ RHS = LHS

Question 12

$$\begin{aligned}\cos(3x)-\cos(5x) &= -2\sin\left(\frac{3x+5x}{2}\right)\sin\left(\frac{3x-5x}{2}\right) \quad \textbf{[1 mark]}\\ &= -2\sin(4x)\sin(-x)\\ &= 2\sin(4x)\sin(x) \quad \textbf{[1 mark]}\end{aligned}$$

Question 13

a. $$\begin{aligned}\sin(2\theta)\cos(\theta)-\sin(\theta)\cos(2\theta) &= \sin(2\theta-\theta)\\ &= \sin(\theta) \quad \textbf{[1 mark]}\end{aligned}$$

Question 14

Option B is false since $\cos^3(\pi)+\sin^3(\pi)=(-1)^3+0^3=-1$, therefore A, C, D, and E, are true.

Question 15

a. $$\begin{aligned}\sin(2\theta)\cos(\theta)-\sin(\theta)\cos(2\theta) &= \sin(2\theta-\theta)\\ &= \sin(\theta) \quad \textbf{[1 mark]}\end{aligned}$$

b. $$\begin{aligned}\sin(6\theta)\cos(2\theta)-\sin(2\theta)\cos(6\theta) &= \sin(6\theta-2\theta)\\ &= \sin(4\theta) \quad \textbf{[1 mark]}\end{aligned}$$

c. $$\begin{aligned}\frac{\sin(2\theta)\cos(\theta)-\sin(\theta)\cos(2\theta)}{\sin(6\theta)\cos(2\theta)-\sin(2\theta)\cos(6\theta)} &= \frac{\sin(\theta)}{\sin(4\theta)}\\ &= \frac{\sin(\theta)}{2\sin(2\theta)\cos(2\theta)} \quad \textbf{[1 mark]}\\ &= \frac{\sin(\theta)}{4\sin(\theta)\cos(\theta)\cos(2\theta)} \quad \textbf{[1 mark]}\\ &= \frac{1}{4\cos(\theta)\cos(2\theta)}\\ &= \frac{1}{4}\sec(\theta)\sec(2\theta) \quad \textbf{[1 mark]}\end{aligned}$$

d.

$$\begin{aligned}
\frac{1}{4}\sec(\theta)\sec(2\theta) &= -\frac{1}{4}\\
\frac{1}{\cos(\theta)\cos(2\theta)} &= -1\\
\cos(\theta)\left(2\cos^2(\theta-1)\right) &= -1 \qquad \textbf{[1 mark]}\\
2\cos^3(\theta)-\cos(\theta)+1 &= 0\\
(\cos(\theta)+1)\left(2\cos^2(\theta)-2\cos(\theta)+1\right) &= 0 \qquad \textbf{[1 mark]}\\
\cos(\theta)+1 = 0 \Rightarrow \theta &= \frac{3\pi}{2}\\
2\cos^2(\theta)-2\cos(\theta)+1 = 0,&\ \text{no solutions} \qquad \textbf{[1 mark]}
\end{aligned}$$

9 Vectors in the plane

Topic	9	Vectors in the plane
Subtopic	9.2	Vectors and scalars

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Question 1 (1 mark)

OABC is a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

The sum of the diagonal vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$ is

A. $2\underset{\sim}{a} + 2\underset{\sim}{c}$

B. $2\underset{\sim}{a} - 2\underset{\sim}{c}$

C. $2\underset{\sim}{a}$

D. $2\underset{\sim}{c}$

E. $\underset{\sim}{a} - \underset{\sim}{c}$

Question 2 (2 marks)

Clementine, an intrepid hiker, walks 10 km east through rugged terrain, then 6 km due north. After a suitable break, she then walks a further 3 km west and 3 km south, then camps for the night. Determine how far she is from her starting point as an exact value, and determine the bearing of her camp from her starting point, in degrees rounded to 2 decimal places.

The distance from the starting point is ☐ and the bearing is ☐ °T

Question 3 (2 marks)

A river current flows parallel to the river banks at 5 km/h. A boat heads directly across the river from one river bank to the other side at 12 km/h. Determine the actual velocity and direction of the boat, in degrees to 2 decimal places.

The actual velocity is ☐ km/h

The direction is ☐°

Question 4 (1 mark)

What is the difference between a scalar and a vector?

Question 5 (1 mark)

What does the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ mean?

Topic	9	Vectors in the plane
Subtopic	9.3	Position vectors in the plane

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Question 1 (1 mark)

The magnitude of the vector $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ is equal to

A. 7
B. 5
C. $\sqrt{7}$
D. $\sqrt{5}$
E. 25

Question 2 (1 mark)

The vector with a magnitude of $20\sqrt{5}$ is

A. $2\underset{\sim}{i} - 15\underset{\sim}{j}$
B. $7\underset{\sim}{i} + 5\underset{\sim}{j}$
C. $20\underset{\sim}{i} + 40\underset{\sim}{j}$
D. $6\underset{\sim}{i} + 5\underset{\sim}{j}$
E. $80\underset{\sim}{i} + 5\underset{\sim}{j}$

Question 3 (1 mark)

If $a = i + 3j$ and $\underset{\sim}{b} = -i - 2j$, determine $|\underset{\sim}{a} - \underset{\sim}{b}|$.

$|\underset{\sim}{a} - \underset{\sim}{b}| = \square$

Question 4 (1 mark)

What is the vector from the origin to the point (p, q)?

Question 5 (1 mark)

$\underset{\sim}{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, find $|\underset{\sim}{a}|$.

Question 6 (1 mark)

Find the magnitude of the position vector $\begin{bmatrix} -7 \\ 1 \end{bmatrix}$.

Question 7 (1 mark)

The vectors $\underset{\sim}{a} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ are added giving $\underset{\sim}{a} + \underset{\sim}{b}$.

Find $|\underset{\sim}{a} + \underset{\sim}{b}|$.

Question 8 (1 mark)

Find the angle that the position vector $\begin{bmatrix} p \\ \sqrt{3}p \end{bmatrix}$ makes with the positive direction of the x-axis.

Question 9 (1 mark)

What is a unit vector?

Question 10 (1 mark)

Find the angle that the vector $2i - 5j$ makes with the positive direction of the x-axis.

Question 11 (1 mark)
If $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -5\underset{\sim}{i} - 6\underset{\sim}{j}$, find $2\underset{\sim}{a} - 3\underset{\sim}{b}$

Question 12 (2 marks)
If $\underset{\sim}{a} = i + 3j$ and $\underset{\sim}{b} = -i - 2j$, find the position vector for M, the midpoint of $\underset{\sim}{a} - \underset{\sim}{b}$.

Question 13 (1 mark)
A boat travels on a bearing of 210° for 100 km Its position in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ will be

A. $-50\sqrt{3}\underset{\sim}{i} - 50\underset{\sim}{j}$
B. $50\underset{\sim}{i} - 50\sqrt{3}\underset{\sim}{j}$
C. $-50\underset{\sim}{i} - 50\underset{\sim}{j}$
D. $-50\underset{\sim}{i} - 50\sqrt{3}\underset{\sim}{j}$
E. $50\underset{\sim}{i} - 50\underset{\sim}{j}$

Question 14 (4 marks)
Redbeard, the evil pirate, sails his ship, the Lady Kirsty, from her home port of Tortegula. Redbeard sails at 15 knots on a vector of $20i + 30j$ for 8 hours when he anchors. If i represents east and j represents north, how far east and north of Tortegula is he anchored? (One knot is a speed of one nautical mile per hour.)

Topic 9 Subtopic 9.3 Position vectors in the plane

Question 15 (1 mark)
A ship leaves a point $O$and travels a distance of 30 km on a bearing N60°W while a car leaves the same point O and travels 20 km due south. If $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors of magnitude 1 km in the directions of east and north respectively, then the position vector of the car from the ship is given by

A. $15\sqrt{3}\underset{\sim}{i} - 35\underset{\sim}{j}$

B. $-15\sqrt{3}\underset{\sim}{i} + 35\underset{\sim}{j}$

C. $-15\sqrt{3}\underset{\sim}{i} - 5\underset{\sim}{j}$

D. $15\underset{\sim}{i} + \left(-15\sqrt{3} + 20\right)\underset{\sim}{j}$

E. $-15\underset{\sim}{i} - \left(15\sqrt{3} + 20\right)\underset{\sim}{j}$

Question 16 (1 mark)
A student walks a distance of 100 m on a bearing south 30° east to a lift. The lift travels vertically up a distance of 10 m. If $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors of magnitude 1 m in the directions of east, north and vertically upwards respectively, then the position vector of the student relative to her initial position is given by

A. $50\underset{\sim}{i} - 50\sqrt{3}\underset{\sim}{j}$

B. $50\underset{\sim}{i} - 50\sqrt{3}\underset{\sim}{j} + 10\underset{\sim}{k}$

C. $50\sqrt{3}\underset{\sim}{i} - 50\underset{\sim}{j}$

D. $-50\underset{\sim}{i} + 50\sqrt{3}\underset{\sim}{j} + 10\underset{\sim}{k}$

E. $50\sqrt{3}\underset{\sim}{i} - 50\underset{\sim}{j} + 10\underset{\sim}{k}$

Topic	9	Vectors in the plane
Subtopic	9.4	Scalar multiplication of vectors

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Question 1 (1 mark)

Consider the following relationships between vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$.

$\underset{\sim}{p} = 2\underset{\sim}{q} - \underset{\sim}{r}$ and $\underset{\sim}{r} = \underset{\sim}{q} + \underset{\sim}{p}$

Out of the following, the statement which is true is

A. $\underset{\sim}{p} = \dfrac{\underset{\sim}{q}}{2}$

B. $\underset{\sim}{p} = 2\underset{\sim}{q}$

C. $\underset{\sim}{q} = 0,\ \underset{\sim}{p} = \underset{\sim}{r}$

D. $\underset{\sim}{q} = 0, \underset{\sim}{p} = -\underset{\sim}{r}$

E. $\underset{\sim}{p} = \underset{\sim}{q} + \underset{\sim}{r}$

Question 2 (1 mark)

$a = \left(\dfrac{3}{7}\underset{\sim}{i} - \dfrac{8}{5}\underset{\sim}{j}\right)$, if $k\underset{\sim}{a} = 15\underset{\sim}{i} + m\underset{\sim}{j}$, the values of k and m are

A. $k = 35$
$m = 56$

B. $k = 35$
$m = -56$

C. $k = 7$
$m = 56$

D. $k = 7$
$m = -56$

E. $k = -7$
$m = 35$

Topic 9 | Subtopic 9.4 Scalar multiplication of vectors

Question 3 (2 marks)

$\underset{\sim}{r} = x\underset{\sim}{i} + 4\underset{\sim}{j}$ and $s = -5\underset{\sim}{i} + 6\underset{\sim}{j}$. If $3\underset{\sim}{r} + \underset{\sim}{s} = 2\underset{\sim}{i} + 18\underset{\sim}{j}$, determine the value of x.

$x = \square$

Question 4 (3 marks)

A cyclist is riding up a hill at 22km/h in the southwest direction. Once he gets to the top of the hill he turns around and rides back down at twice the speed.

If the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ represent the directions east and north respectively, the vector which represents his journey back down the hill is:

Topic	9	Vectors in the plane
Subtopic	9.5	The scalar (dot) product

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Question 1 (1 mark)

If $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -\underset{\sim}{i} - 2\underset{\sim}{j}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b}$, the scalar product, is equal to

A. 7
B. −7
C. 5
D. $-5\sqrt{2}$
E. $\sqrt{7}$

Question 2 (1 mark)

If $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -\underset{\sim}{i} - 2\underset{\sim}{j}$, determine the value of θ, the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in degrees correct to 1 decimal place.

$\theta = \square^\circ$

Question 3 (2 marks)

$\underset{\sim}{r} = a\underset{\sim}{i} + (a-1)\underset{\sim}{j}$, $\underset{\sim}{s} = 3a\underset{\sim}{i} + (a+1)\underset{\sim}{j}$. If r and s are perpendicular, determine the value(s) of a.

$a = \square$

Question 4 (1 mark)

If $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = -4\underset{\sim}{i} + t\underset{\sim}{j} - 2\underset{\sim}{k}$, then

A. if $t = -2$, the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$ and if $t = 10$, the vector $\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$.

B. if $t = 2$, the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$ and if $t = -10$, the vector $\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$.

C. if $t = 2$, the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$ and if $t = 10$, the vector $\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$.

D. if $t = 10$, the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$ and if $t = 2$, the vector $\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$.

E. if $t = -10$, the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$ and if $t = 2$, the vector $\underset{\sim}{a}$ is perpendicular to the vector $\underset{\sim}{b}$.

Question 5 (2 marks)

If $\underset{\sim}{a} = -i + 3j$ and $\underset{\sim}{b} = -i + 2j$, find θ, the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Topic	9	Vectors in the plane
Subtopic	9.6	Projections of vectors — scalar and vector resolutes

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Question 1 (1 mark)

If $\underset{\sim}{a} = -i + 3j$ and $\underset{\sim}{a} = -i + 3j$, the scalar resolute of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is equal to

A. $\dfrac{-14}{5}$

B. $\dfrac{14}{5}$

C. $\dfrac{14}{\sqrt{29}}$

D. $\dfrac{-14}{\sqrt{29}}$

E. $\dfrac{14}{\sqrt{5}}$

Question 2 (1 mark)

If $\underset{\sim}{u} = 6\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{v} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$, the vector resolute of$\underset{\sim}{u}$ parallel to $\underset{\sim}{v}$ is equal to

A. $\dfrac{2}{5}\underset{\sim}{v}$

B. $-\dfrac{2}{5}\underset{\sim}{v}$

C. $-6\underset{\sim}{i} + 8\underset{\sim}{j}$

D. $6\underset{\sim}{i} + 8\underset{\sim}{j}$

E. $6\underset{\sim}{i} - 8\underset{\sim}{j}$

Topic 9 | Subtopic 9.6 Projections of vectors — scalar and vector resolutes

Question 3 (3 marks)

If $\underset{\sim}{r} = \sqrt{5}\left(14\underset{\sim}{i} + 18\underset{\sim}{j}\right)$ and $\underset{\sim}{s} = 2\underset{\sim}{i} + \underset{\sim}{j}$, calculate the vector resolute of $\underset{\sim}{r}$perpendicular to $\underset{\sim}{s}$.
the vector resolute of $\underset{\sim}{r}$ perpendicular to $\underset{\sim}{s}$ is ☐

Question 4 (3 marks)

If $\underset{\sim}{u} = 8\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 9\underset{\sim}{j}$, calculate the vector resolute of $\underset{\sim}{u}$ parallel to $\underset{\sim}{v}$.

Topic	9	Vectors in the plane
Subtopic	9.7	Review

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Question 1 (1 mark)
Out of the following, the vector which is a unit vector is

A. $\underset{\sim}{i} + \underset{\sim}{j}$

B. $\sqrt{2}\left(2\underset{\sim}{i} - \underset{\sim}{j}\right)$

C. $\frac{1}{11}\left(-3\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

D. $\frac{1}{\sqrt{21}}\left(4\underset{\sim}{i} + \sqrt{5}\underset{\sim}{j}\right)$

E. $\frac{1}{2}\left(\underset{\sim}{i} - \underset{\sim}{j}\right)$

Question 2 (2 marks)
An airship is headed east at a speed of 50 km/h. A wind blows from the southwest at 10 km/h. Determine the velocity of the airship relative to the ground, rounded to one decimal place.
The velocity is ☐ km/h

Question 3 (1 mark)
Calculate the magnitude of the position vector $\begin{bmatrix} -7 \\ 1 \end{bmatrix}$.

The magnitude is ☐

Question 4 (2 marks)

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -\dfrac{\sqrt{2}|\underset{\sim}{a}||\underset{\sim}{b}|}{2}$ Determine the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

The angle is $\square°$

Question 5 (3 marks)

OAB is a triangle, with O the origin. $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, M is the midpoint of $\overrightarrow{OB}$, N is the midpoint of $\overrightarrow{OA}$ and Q is the midpoint of $\overrightarrow{AB}$. Show the ONQM is a parallelogram.

Question 6 (1 mark)

ABC is a triangle with *P* the midpoint of *AC* and *Q* the midpoint of *CB*. Which of the following statements is false?

A. $\overrightarrow{AP} = \overrightarrow{PC}$ and $\overrightarrow{CQ} = \overrightarrow{QB}$

B. $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB}$

C. $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$

D. The length of $\overrightarrow{PQ}$ is one-half the length of $\overrightarrow{AB}$.

E. $\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{AP} + \overrightarrow{CQ}\right)$

Question 7 (1 mark)

OABC is a square with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

Which of the following statements is false?

A. $\overrightarrow{CB} = \underset{\sim}{a}$

B. $\overrightarrow{AB} = \underset{\sim}{c}$

C. $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$

D. $\overrightarrow{AC} = \underset{\sim}{a} - \underset{\sim}{c}$

E. $|\underset{\sim}{a}| = |\underset{\sim}{c}|$

Question 8 (3 marks)

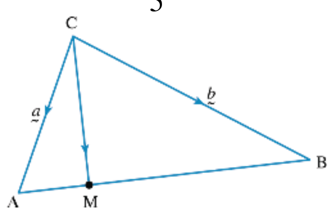
Use vectors to show that the points O(0, 0), A(2, 1) and B(0, 5) form a right-angled triangle.

Question 9 (3 marks)

A circle cuts the x-axis at $A(-a,\ 0)$, $B(a,\ 0)$ and the y-axis at C(0, a) and D(0, $-a$). Prove that angle ACB is a right angle by using the dot product.

Question 10 (2 marks)

In triangle ABC, AB is divided in the ratio AM:MB = 1:4. If vector $\overrightarrow{CA} = \underset{\sim}{a}$ and $\overrightarrow{CB} = \underset{\sim}{b}$, show that $\overrightarrow{CM} = \frac{1}{5}(4\underset{\sim}{a} + \underset{\sim}{b})$.



Question 11 (1 mark)

In the Cartesian plane, a vector perpendicular to the line $-2x + 3y + 6 = 0$ is:

A. $-3x - 2y + 3 = 0$
B. $-3x + 2y + 3 = 0$
C. $-2x - 3y + 3 = 0$
D. $-2x + 3y + 3 = 0$
E. $\frac{1}{3}x - \frac{1}{2}y + 1 = 0$

Question 12 (1 mark)

ABC is a triangle with P the midpoint of AC and Q the midpoint of CB. Which of the following statements is true?

A. $\overrightarrow{PC} + \overrightarrow{CQ} = \overrightarrow{QP}$
B. $\overrightarrow{PA} + \overrightarrow{AB} + \overrightarrow{BQ} = \overrightarrow{QP}$
C. $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{BA}$
D. Area of ΔPCQ $= \frac{1}{2}$ Area of ΔACB
E. $\frac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{CB}\right) = \overrightarrow{PQ}$

Question 13 (1 mark)

If $\overrightarrow{PQ} = -\overrightarrow{PR}$ and $|\overrightarrow{RQ}| = 2$, then which of the following is false?

A. P, Q and R are collinear.
B. $|\overrightarrow{PQ}| = 1$
C. $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 0$
D. $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RP}$
E. $\overrightarrow{PQ} \cdot \overrightarrow{PR} = -1$

Question 14 (1 mark)
The adjacent sides of a rhombus are $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Which one of the following is true?
A. $\underset{\sim}{b} \cdot \underset{\sim}{a} = 0$
B. $\underset{\sim}{b} \cdot \underset{\sim}{a} = 1$
C. $\left(\underset{\sim}{a} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{b}\right) = 0$
D. $\left(\underset{\sim}{a} - \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{b}\right) = 1$
E. $2\underset{\sim}{a} + 2\underset{\sim}{b} = 0$

Question 15 (1 mark)
The point *M* cuts the line segment *PQ* in the ratio of 3:5, with *M* being closer to *P*. If the position vectors of *P* and *Q* are $\underset{\sim}{p}$ and $\underset{\sim}{q}$ respectively, then the position vector of *M* is
A. $\frac{1}{8}(3\underset{\sim}{p} + 5\underset{\sim}{q})$
B. $\frac{1}{8}(3\underset{\sim}{p} - 5\underset{\sim}{q})$
C. $\frac{1}{8}(5\underset{\sim}{p} + 3\underset{\sim}{q})$
D. $\frac{1}{8}(5\underset{\sim}{q} - 3\underset{\sim}{p})$
E. $\frac{1}{8}(5\underset{\sim}{p} - 3\underset{\sim}{q})$

Question 16 (1 mark)
The sides of a vector triangle are $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$. The angle between $\underset{\sim}{b}$ and $\underset{\sim}{c}$ is 60°. Which one of the following is true?
A. $|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2$
B. $|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 - |\underset{\sim}{b}||\underset{\sim}{c}|$
C. $|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{b}||\underset{\sim}{c}|$
D. $|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 - 2|\underset{\sim}{b}||\underset{\sim}{c}|$
E. $|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + 2|\underset{\sim}{b}||\underset{\sim}{c}|$

Question 17 (1 mark)

A vector of magnitude $3\sqrt{2}$ units is at an angle of 30° to the vertical. Its vertical component will be

A. $\frac{3\sqrt{2}}{2}\underset{\sim}{j}$

B. $\sqrt{6}\underset{\sim}{j}$

C. $\frac{3\sqrt{6}}{2}\underset{\sim}{j}$

D. $\sqrt{2}\underset{\sim}{j}$

E. $6\sqrt{2}\underset{\sim}{j}$

Question 18 (1 mark)

A vector of magnitude $5\sqrt{3}$ units is at an angle of 30° to the horizontal. Its vertical component will be

A. $\frac{5\sqrt{3}}{2}\underset{\sim}{j}$

B. $\frac{15}{2}\underset{\sim}{j}$

C. $5\sqrt{3}\underset{\sim}{j}$

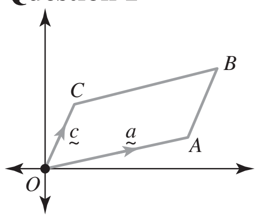
D. $150\sqrt{3}\underset{\sim}{j}$

E. $\frac{5\sqrt{2}}{3}\underset{\sim}{j}$

Answers and marking guide

9.2 Vectors and scalars

Question 1



$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{AC} = -\underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{OB} + \overrightarrow{AC} = 2\underset{\sim}{c}$

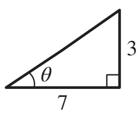
Question 2

Vector $= (10i + 6j) + (-3i - 3j)$

$= 7i + 3j$

Distance $= \sqrt{7^2 + 3^2}$

$= \sqrt{49 + 9} = \sqrt{58}$ **[1 mark]**



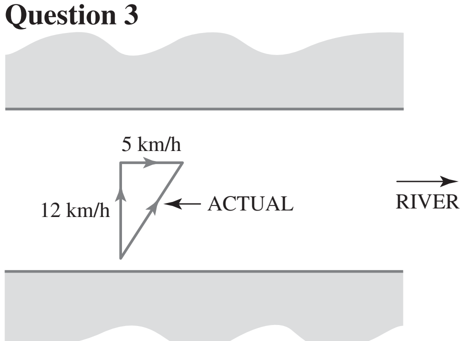
$\tan(\theta) = \frac{3}{7}$

$\theta = 23.20°$

$90 - 23.20 = 66.80$

Hence, bearing = 066.80°T **[1 mark]**

Question 3



The actual velocity is $\sqrt{(5^2 + 12^2)} = 13$ km/h. **[1 mark]**

The actual direction is $\cos^{-1}\left(\frac{12}{13}\right) = 22.62°$ to the original river bank. **[1 mark]**

Question 4

Both have magnitude, but only a vector has direction. **[1 mark]**

Question 5

Move a units horizontally and b units vertically. **[1 mark]**

9.3 Position vectors in the plane

Question 1

$|a| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Question 2

$|20\underset{\sim}{i} + 40\underset{\sim}{j}| = \sqrt{20^2 + 40^2} = \sqrt{2000} = 20\sqrt{5}$

Question 3

$\underset{\sim}{a} - \underset{\sim}{b} = \left(\underset{\sim}{i} + 3\underset{\sim}{j}\right) - \left(-\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

$= 2\underset{\sim}{i} + 5\underset{\sim}{j}$

$|\underset{\sim}{a} - \underset{\sim}{b}| = \sqrt{2^2 + 5^2}$

$= \sqrt{29}$ **[1 mark]**

Question 4

Across $+p$ units, then $+q$ vertically. The vector is $\begin{bmatrix} p \\ q \end{bmatrix}$. **[1 mark]**

Question 5

$|\underset{\sim}{a}| = \sqrt{2^2 + 3^2}$

$= \sqrt{13}$ **[1 mark]**

Question 6

Length $= \sqrt{(-7)^2 + 1^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$ **[1 mark]**

Question 7

$\underset{\sim}{a} + \underset{\sim}{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **[1 mark]**

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}$ **[1 mark]**

Question 8

$\tan(\theta) = \dfrac{\sqrt{3}p}{p}$

$= \sqrt{3}$

$\Rightarrow \theta = 60°$ **[1 mark]**

Question 9

A unit vector is a vector whose magnitude is one unit. **[1 mark]**

Question 10

$\tan(\theta) = \dfrac{-5}{2}$

$= -2.5$

$\Rightarrow \theta = 68.20°$

Note: This angle is in the 4th quadrant **[1 mark]**

Question 11

$2\underset{\sim}{a} - 3\underset{\sim}{b} = 2(2\underset{\sim}{i} - 3\underset{\sim}{j}) - 3(-5\underset{\sim}{i} - 6\underset{\sim}{j})$

$= 19\underset{\sim}{i} + 12\underset{\sim}{j}$ **[1 mark]**

Question 12

$\underset{\sim}{a} - \underset{\sim}{b} = (i + 3j) - (-i - 2j)$

$= 2i + 5j$

$\frac{\underset{\sim}{a} - \underset{\sim}{b}}{2} = i + \frac{5}{2}j$ **[1 mark]**

$\overrightarrow{OM} = \begin{bmatrix} 1 \\ \frac{5}{2} \end{bmatrix}$ **[1 mark]**

Question 13

$x: -100\cos(60)\underset{\sim}{i} = -50\underset{\sim}{k}$

$y: -100\sin(60)\underset{\sim}{i} = -50\sqrt{3}\underset{\sim}{j}$

Position: $-50\underset{\sim}{i} - 50\sqrt{3}\underset{\sim}{j}$

Question 14

15 knots for 10 hours = 150 nautical miles **[1 mark]**

θ is the angle to the east at which he sails.

$\tan(\theta) = \frac{30}{20}$

$\theta = 56.31°$ **[1 mark]**

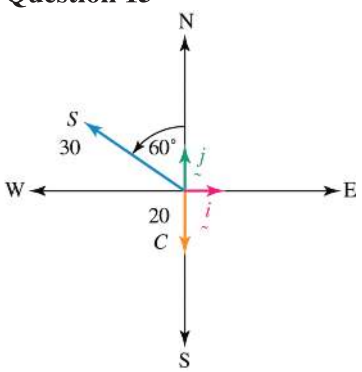
Distance east = $150\cos(56.31°)$

= 83.21 nautical miles **[1 mark]**

Distance north = $150\sin(56.31°)$

= 124.81 nautical miles **[1 mark]**

Question 15

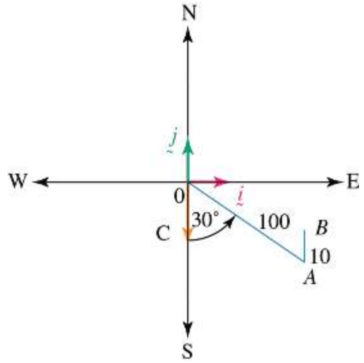


$\overrightarrow{OC} = -20\underset{\sim}{j}$

$\overrightarrow{OS} = -30\sin(60°)\underset{\sim}{i} + 30\cos(60°)\underset{\sim}{j} = -15\sqrt{3}\underset{\sim}{i} + 15\underset{\sim}{j}$

We require $\overrightarrow{SC} = \overrightarrow{OC} - \overrightarrow{OS} = 15\sqrt{3}\underset{\sim}{i} - 35\underset{\sim}{j}$

Question 16



$\overrightarrow{OA} = 100\sin(30°)\underset{\sim}{i} - 100\cos(30°)\underset{\sim}{j} = 50\underset{\sim}{i} - 50\sqrt{3}\underset{\sim}{j}$

$\overrightarrow{AB} = 10\underset{\sim}{k}$

So $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = 50\underset{\sim}{i} - 50\sqrt{3}\underset{\sim}{j} + 10\underset{\sim}{k}$

9.4 Scalar multiplication of vectors

Question 1

$\underset{\sim}{p} = 2\underset{\sim}{q} - \underset{\sim}{r}$

$\underset{\sim}{r} = \underset{\sim}{q} + \underset{\sim}{p}$

$\underset{\sim}{p} = 2\underset{\sim}{q} - \left(\underset{\sim}{q} + \underset{\sim}{p}\right)$

$\underset{\sim}{p} = \underset{\sim}{q} - \underset{\sim}{p}$

$\underset{\sim}{p} = \dfrac{\underset{\sim}{q}}{2}$

Question 2

$k\underset{\sim}{a} = \left(\dfrac{3k}{7}\underset{\sim}{i} - \dfrac{8k}{5}\underset{\sim}{j}\right)$

$\dfrac{3k}{7} = 15$

$3k = 105$

$k = 35$

$k\underset{\sim}{a} = 35\underset{\sim}{a}$

$= \dfrac{105}{7}\underset{\sim}{i} - \dfrac{280}{5}\underset{\sim}{j}$

$= 15\underset{\sim}{i} - 56\underset{\sim}{j}$

$m = -56$

Question 3

$3\underset{\sim}{r} = 3x\underset{\sim}{i} + 12\underset{\sim}{j}$

$3\underset{\sim}{r} + \underset{\sim}{s} = 3x\underset{\sim}{i} + 12\underset{\sim}{j} - 5\underset{\sim}{i} + 6\underset{\sim}{j}$

$= (3x - 5)\,\underset{\sim}{i} + 18\underset{\sim}{j}$ **[1 mark]**

$(3x - 5)\,\underset{\sim}{i} + 18\underset{\sim}{j} = 2\underset{\sim}{i} + 18\underset{\sim}{j}$

$3x - 5 = 2$

$3x = 7$

$x = \dfrac{7}{3}$ **[1 mark]**

Question 4

Southwest direction: $k\left(-\underset{\sim}{i}-\underset{\sim}{j}\right)$. **[1 mark]**

Speed is 22 km/h, so magnitude = 22,

$$k\sqrt{(-1)^2+(-1)^2}=22$$
$$k\sqrt{2}=22$$
$$k=\frac{22}{\sqrt{2}}$$
$$k=11\sqrt{2}\ \textbf{[1 mark]}$$

Uphill Journey: $11\sqrt{2}\left(\underset{\sim}{i}-\underset{\sim}{j}\right)$

Downhill Journey (double speed, opposite direction) = $22\sqrt{2}\left(-\underset{\sim}{i}-\underset{\sim}{j}\right)$ **[1 mark]**

9.5 The scalar (dot) product

Question 1

$$\begin{aligned}\underset{\sim}{a}\cdot\underset{\sim}{b}&=1\times-1+3\times-2\\&=-7\end{aligned}$$

Question 2

$$\begin{aligned}\cos(\theta)&=\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\\&=\frac{1\times-1+3\times-2}{\sqrt{10}\sqrt{5}}\\&=-\frac{7}{5\sqrt{2}}\\\theta&=171.9^\circ\end{aligned}$$

Question 3

$\underset{\sim}{r}\cdot\underset{\sim}{s}=0=3a^2+\left(a^2-1\right)$

$\Rightarrow 4a^2=1\Rightarrow a=\pm\frac{1}{2}$

Question 4

If $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$, $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$.

Now $\underset{\sim}{a}\cdot\underset{\sim}{b}=-8-t-2=0\Rightarrow t=-10$

If $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$, $\underset{\sim}{b}=\lambda\underset{\sim}{a}$, $\underset{\sim}{b}=-2\underset{\sim}{a}\Rightarrow t=2$

Question 5

$$\begin{aligned}\cos(\theta)&=\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\\&=\frac{-1\times-1+3\times2}{\sqrt{10}\sqrt{5}}\\&=\frac{7}{5\sqrt{2}}\ \textbf{[1 mark]}\\\theta&=8.13^\circ\ \textbf{[1 mark]}\end{aligned}$$

9.6 Projections of vectors — scalar and vector resolutes

Question 1

The scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{a}|}$

$$\frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{-20+6}{\sqrt{16+9}} = \frac{-14}{5}$$

Question 2

The vector resolute of $\underset{\sim}{u}$ on $\underset{\sim}{v} = \left(\hat{\underset{\sim}{v}} \cdot \underset{\sim}{u}\right)\hat{\underset{\sim}{v}}$

$$\begin{aligned}\left(\hat{\underset{\sim}{v}} \cdot \underset{\sim}{u}\right)\hat{\underset{\sim}{v}} &= \left(\left(\frac{-3\underset{\sim}{i}+4\underset{\sim}{j}}{\sqrt{(-3)^2+4^2}}\right)\cdot\left(6\underset{\sim}{i}+7\underset{\sim}{j}\right)\right)\frac{\left(-3\underset{\sim}{i}+4\underset{\sim}{j}\right)}{\sqrt{(-3)^2+4^2}}\\
&= \left(\left(\frac{-3\underset{\sim}{i}+4\underset{\sim}{j}}{\sqrt{25}}\right)\cdot\left(6\underset{\sim}{i}+7\underset{\sim}{j}\right)\right)\frac{\left(-3\underset{\sim}{i}+4\underset{\sim}{j}\right)}{\sqrt{25}}\\
&= \left(\frac{-18+28}{5}\right)\left(\frac{-3\underset{\sim}{i}+4\underset{\sim}{j}}{5}\right)\\
&= \frac{2}{5}\left(-3\underset{\sim}{i}+4\underset{\sim}{j}\right)\\
&= \frac{2}{5}\underset{\sim}{v}\end{aligned}$$

Question 3

The vector resolute of $\underset{\sim}{r}$ perpendicular to $\underset{\sim}{s} = \underset{\sim}{r} - \left(\hat{\underset{\sim}{s}} \cdot \underset{\sim}{r}\right)\hat{\underset{\sim}{s}}$

$$\begin{aligned}\underset{\sim}{r} - \left(\hat{\underset{\sim}{s}} \cdot \underset{\sim}{r}\right)\hat{\underset{\sim}{s}} &= \sqrt{5}\left(14\underset{\sim}{i}+18\underset{\sim}{j}\right) - \left(\left(\frac{2\underset{\sim}{i}+\underset{\sim}{j}}{\sqrt{5}}\right)\cdot\left(\sqrt{5}\left(14\underset{\sim}{i}+18\underset{\sim}{j}\right)\right)\right)\left(\frac{2\underset{\sim}{i}+\underset{\sim}{j}}{\sqrt{5}}\right)\\
&= \sqrt{5}\left(14\underset{\sim}{i}+18\underset{\sim}{j}\right) - (28+18)\left(\frac{2\underset{\sim}{i}+\underset{\sim}{j}}{\sqrt{5}}\right)\\
&= \sqrt{5}\left(14\underset{\sim}{i}+18\underset{\sim}{j}\right) - \left(\frac{92\underset{\sim}{i}+46\underset{\sim}{j}}{\sqrt{5}}\right)\\
&= \left(\frac{70\underset{\sim}{i}+90\underset{\sim}{j}}{\sqrt{5}}\right) - \left(\frac{92\underset{\sim}{i}+46\underset{\sim}{j}}{\sqrt{5}}\right)\\
&= \frac{-22}{\sqrt{5}}\underset{\sim}{i} + \frac{44}{\sqrt{5}}\underset{\sim}{j}\end{aligned}$$

Question 4

$$\left(\hat{\underset{\sim}{v}} \cdot \underset{\sim}{u}\right)\hat{\underset{\sim}{v}} = \left(\left(\frac{-\underset{\sim}{i}+9\underset{\sim}{j}}{\sqrt{(-1)^2+9^2}}\right)\cdot\left(8\underset{\sim}{i}+3\underset{\sim}{j}\right)\right)\left(-\underset{\sim}{i}+9\underset{\sim}{j}\right) \textbf{ [1 mark]}$$

$$= \left(\left(\frac{-\underset{\sim}{i}+9\underset{\sim}{j}}{\sqrt{82}}\right)\cdot\left(8\underset{\sim}{i}-3\underset{\sim}{j}\right)\right)\left(-\underset{\sim}{i}+9\underset{\sim}{j}\right) \textbf{ [1 mark]}$$

$$= \left(\frac{1}{\sqrt{82}}(-8-27)\right)\left(-\underset{\sim}{i}+9\underset{\sim}{j}\right)$$

$$= \frac{-35}{\sqrt{82}}\left(-\underset{\sim}{i}+9\underset{\sim}{j}\right) \textbf{ [1 mark]}$$

9.7 Review

Question 1

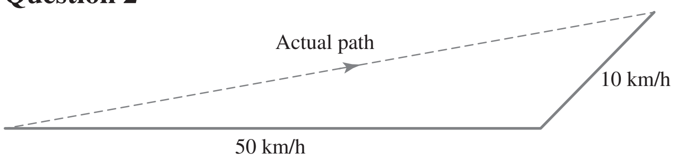
$$\left|\frac{1}{\sqrt{21}}\left(4\underset{\sim}{i}+\sqrt{5}\underset{\sim}{j}\right)\right| = \frac{1}{\sqrt{21}}\left|4\underset{\sim}{i}+\sqrt{5}\underset{\sim}{j}\right|$$

$$\frac{1}{\sqrt{21}}\sqrt{4^2+\sqrt{5}^2} = \frac{1}{\sqrt{21}}\sqrt{21}$$

$$= 1$$

$$\therefore \frac{1}{\sqrt{21}}\left(4\underset{\sim}{i}+\sqrt{5}\underset{\sim}{j}\right) \text{ is a unit vector.}$$

Question 2



Cosine Rule **[1 mark]**

$$v = \sqrt{\left(50^2 + 10^2 - 2 \times 50 \times 10 \times \cos(135)^\circ\right)}$$

$= 57.5$ km/h **[1 mark]**

Question 3

$$\text{Length} = \sqrt{(-7)^2 + 1^2}$$

$$= \sqrt{50}$$

$= 5\sqrt{2}$ **[1 mark]**

Question 4

$$\cos(\theta) = -\frac{\sqrt{2}}{2}$$

$= -\frac{1}{\sqrt{2}}$ **[1 mark]**

$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

$$= 135^\circ$$

$\therefore$ The angle between the vectors is 135°. **[1 mark]**

Question 5

$\overrightarrow{NQ} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$ **[1 mark]**

$$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$$

$= \frac{1}{2}\underset{\sim}{b} = \overrightarrow{OM}$ **[1 mark]**

One pair of opposite parallel and equal sides is sufficient to prove that ONQM is a parallelogram. **[1 mark]**

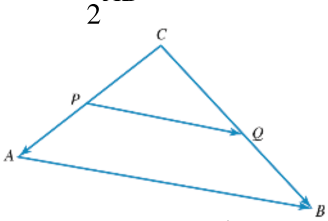
Question 6

A, B, C and D are true; E is false.

Since P and Q are the mid-points of the sides $\overrightarrow{AC}$ and $\overrightarrow{CB}$ respectively, then

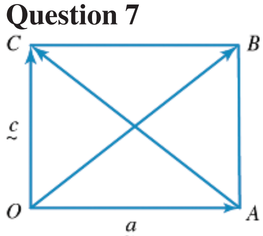
$\overrightarrow{AP} = \overrightarrow{PC} = \frac{1}{2}\overrightarrow{AC}$ and $\overrightarrow{CQ} = \overrightarrow{QB} = \frac{1}{2}\overrightarrow{CB}$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PC} + \overrightarrow{CQ} \\ &= \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} \\ &= \frac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{CB}\right) \\ &= \frac{1}{2}\overrightarrow{AB}\end{aligned}$$



Notice that $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB}$ is one vector statement; however; this states that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$ and that the length of $\overrightarrow{PQ}$ is half that of $\overrightarrow{AB}$

Question 7



A. $\underset{\sim}{a} = \overrightarrow{OA} = \overrightarrow{CB}$ is true.
B. $\underset{\sim}{c} = \overrightarrow{OC} = \overrightarrow{AB}$ is true.
C. $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \underset{\sim}{a} + \underset{\sim}{c}$ is true.
D. $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OA} = \underset{\sim}{c} - \underset{\sim}{a}$, so D is false.
E. $|\underset{\sim}{a}| = |\underset{\sim}{c}|$. Since it is a square, the magnitudes of the vectors are equal.

Question 8

$\overrightarrow{OA} = \begin{bmatrix}2\\1\end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix}-2\\4\end{bmatrix}$, $\overrightarrow{BO} = \begin{bmatrix}0\\-5\end{bmatrix}$ **[1 mark]**

$|\overrightarrow{OA}| = \sqrt{5}$, $|\overrightarrow{AB}| = \sqrt{20}$, $|\overrightarrow{BO}| = \sqrt{25}$ **[1 mark]**

$\left(\sqrt{5}\right)^2 + \left(\sqrt{20}\right)^2 = \left(\sqrt{25}\right)^2$ **[1 mark]**

Question 9

$\overrightarrow{AC} = ai + aj$ **[1 mark]**

$\overrightarrow{BC} = -ai + aj$ **[1 mark]**

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BC} &= -a^2 + a^2 \\ &= 0\end{aligned}$$

$\Rightarrow$ perpendicularity **[1 mark]**

Question 10

$\overrightarrow{AB} = -\underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{AM} = \frac{1}{5}(-\underset{\sim}{a} + \underset{\sim}{b})$ **[1 mark]**

$\overrightarrow{CM} = \overrightarrow{CA} + \overrightarrow{AM}$

$= \underset{\sim}{a} + \frac{1}{5}(-\underset{\sim}{a} + \underset{\sim}{b})$

$= \frac{1}{5}(4\underset{\sim}{a} + \underset{\sim}{b})$ **[1 mark]**

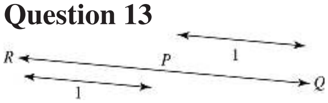
Question 11

For $ax + by + c = 0$, the perpendicular vector is $bx - ay + d = 0$
For $-2x + 3y + 6 = 0$, the perpendicular is $3x + 2y + d = 0$

Question 12

The length of PQ is half the length of AB, but the direction of travel must be the same; that is, P → Q and A → B.

Question 13



P, Q and R are collinear.
$\overrightarrow{PQ} \cdot \overrightarrow{PR} = |\overrightarrow{PQ}||\overrightarrow{PR}| \cos(180°) = -1$
Since $|\overrightarrow{PQ}| = |\overrightarrow{PR}| = 1$.
All of A, B, D and E are true.
If $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 0 \Rightarrow \overrightarrow{PQ}$ is perpendicular to $\overrightarrow{QR}$, this is false.

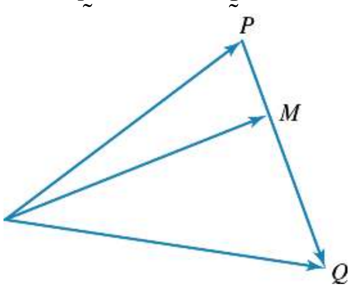
Question 14

The diagonals of a rhombus are perpendicular. They are $\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$ and $\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$.
Perpendicularity implies the dot product equals zero.

Question 15

$\overrightarrow{PM} = \frac{3}{8}\overrightarrow{PQ}$

$\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$



$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{8}\overrightarrow{PQ}$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{8}\left(\overrightarrow{PO} + \overrightarrow{OQ}\right)$$

$$\overrightarrow{OM} = \overrightarrow{OP} + \frac{3}{8}\left(\overrightarrow{OQ} - \overrightarrow{OP}\right)$$

$$\overrightarrow{OM} = \frac{1}{8}\left(5\overrightarrow{OP} + 3\overrightarrow{OQ}\right)$$

$$\overrightarrow{OM} = \frac{1}{8}(5\underset{\sim}{p} + 3\underset{\sim}{q})$$

Question 16

$$|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 - 2|\underset{\sim}{b}||\underset{\sim}{a}|\cos(60°)$$

$$|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 - 2|\underset{\sim}{b}||\underset{\sim}{a}| \times \frac{1}{2}$$

$$|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 - |\underset{\sim}{b}||\underset{\sim}{a}|$$

Question 17

Vertical component:

$$3\sqrt{2}\sin(60)\underset{\sim}{j} = \frac{3\sqrt{6}}{2}\underset{\sim}{j}$$

Question 18

Vertical component:

$$5\sqrt{3}\sin(30)\underset{\sim}{j} = \frac{5\sqrt{3}}{2}\underset{\sim}{j}$$

10 Complex numbers

Topic	10	Complex numbers
Subtopic	10.2	Introduction to complex numbers

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Question 1 (2 marks)

Answer the following

a. Simplify each of the following. **(1 mark)**

$\sqrt{-4} = \square$

$\sqrt{-9} = \square$

$\sqrt{-4} \times \sqrt{-9} = \square$

$\sqrt{-4 \times 9} = \square$

b. Consider the following: $-1 = i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1} = \sqrt{1} = 1$

Determine where the logic breaks down. **(1 mark)**

Question 2 (3 marks)

Answer the following

a. If n is a natural number evaluate i^{4n}. **(1 mark)**

$i^{4n} = \square$

b. If n is a natural number evaluate i^{4n+3}. **(1 mark)**

$i^{4n+3} = \square$

c. If n is an even natural number show that $(-1)^{\frac{n}{2}} = i^n$. **(1 mark)**

Question 3 (2 marks)

Let $f(n) = i + 2i^2 + 3i^3 + ... + ni^n$. Determine:

a. $\text{Re}(f(10)) = \square$ **(1 mark)**

b. $\text{Im}(f(11)) = \square$ **(1 mark)**

Question 4 (1 mark)

If $u = 4a + 3b + 12i$, $v = -1 + (3a - 2b)\,i$ and $u = v$, where a and b are real numbers, then

A. $a = 2$ and b $= 3$
B. $a = 2$ and b $= -3$
C. $a = 5$ and b $= -7$
D. $a = 3$ and b $= -4$
E. $a = 4$ and b $= 0$

Question 5 (2 marks)

If $(x - 3) + (y + 5)\,i = (3x - y) - 2xi$, find x and y.

Topic 10 | Subtopic 10.2 Introduction to complex numbers

Question 6 (1 mark)
The imaginary component of $z = -4 - i$ is:
A. -4
B. -1
C. $-i$
D. i
E. 4

Question 7 (1 mark)
If $z_1 = a + bi$ and $z_2 = p + qi$, then $z_1 = z_2$ if:
A. $a^2 + b^2 = p^2 + q^2$
B. $a + b = p + q$
C. $ab = pq$
D. $a = b,\ p = q$
E. $a = p,\ b = q$

Topic	10	Complex numbers
Subtopic	10.3	Basic operations on complex numbers

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Question 1 (5 marks)

Let $u = 3 - 4i$ and $v = 4 + 5i$. Evaluate each of the following.

a. $u + v = \square$ **(1 mark)**

b. $2u - 3v = \square$ **(1 mark)**

c. $uv = \square$ **(1 mark)**

d. $\text{Re}\left(u^2 - 2uv + v^2\right) = \square$ **(1 mark)**

e. $\text{Im}\left(u^2 + 2uv + v^2\right) = \square$ **(1 mark)**

Question 2 (3 marks)

Let $u = 3 - 4i$ and $v = 4 + 5i$. Determine the values of the real numbers x and y if:

a. $xu + yv = 17 - 2i$ **(2 marks)**

$x = \square$, $y = \square$

b. $xu^2 + yv^2 + 5 + 88i = 0$ **(1 mark)**

$x = \square$, $y = \square$

Question 3 (4 marks)

Let $u = 2 - 3i$ and $v = 3 + 4i$. Determine the values of the real numbers x and y if:

a. $(x + yi)\,u = v$ **(2 marks)**

$x = \square$, $y = \square$

b. $(x + yi)\,v = u$ **(2 marks)**

$x = \square$, $y = \square$

Question 4 (1 mark)

If $z = a + bi$ and $w = c + di$ where $a,\ b,\ c$ and d are non-zero real numbers, then which one of the following is a real number?

A. $zw - \bar{z}\bar{w}$

B. $z\overline{w} - \bar{z}w$

C. $\dfrac{z}{\bar{z}} \times w\bar{w}$

D. $\dfrac{z}{\bar{z}} + w\bar{w}$

E. $zw + \bar{z}\bar{w}$

Question 5 (2 marks)

If $z_1 = 2 - 3i$, $z_2 = -4 - i$ and $z_3 = 3 + 2i$, find:

a. $2z_1 + 3z_2$ **(1 mark)**

b. $-z_1 - 5z_3$ **(1 mark)**

Question 6 (2 marks)

Simplify:

a. $-i(3-2i)$ **(1 mark)**

b. $(2-3i)^2$ **(1 mark)**

Topic	10	Complex numbers
Subtopic	10.4	Complex conjugates and division of complex numbers

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Question 1 (5 marks)

Let $u = 3 - 4i$ and $v = 4 + 5i$, evaluate each of the following.

a. $\frac{1}{u} = \square$ **(1 mark)**

b. $\frac{1}{v} = \square$ **(1 mark)**

c. $\frac{u}{v} = \square$ **(1 mark)**

d. $\frac{v}{u} = \square$ **(1 mark)**

e. $\frac{1}{uv} = \square$ **(1 mark)**

Question 2 (1 mark)

Let $u = a + bi$ and $v = c + di$. Simplify $(u + v)^{-1}$.

$(u + v)^{-1} = \square$

Question 3 (1 mark)

Determine the value of the complex number z in Cartesian form if $\frac{z-i}{z+i} = -2i$.

$s = \square$

Question 4 (1 mark)

If $z = 2 - 4i$ and $w = 1 + 3i$, find $\frac{z}{w}$.

A. $2 + \frac{4}{3}i$
B. $-1 - i$
C. $1 - i$
D. $2 - \frac{4}{3}i$
E. $-1 + i$

Question 5 (3 marks)

If $z = a + ib$ and $w = c + id$ show that $\bar{z} \times \bar{w} = \overline{z \times w}$.

Question 6 (1 mark)

If $u = \frac{1}{a} - \frac{1}{b}i$, where a and b are non-zero real numbers, then $a^2b^2u^2$ is equal to

A. $\left(b^2 + a^2\right) - 2a^2b^2i$
B. $\left(b^2 - a^2\right) - 2abi$
C. $\left(b^2 + a^2\right) - 2abi$
D. $b^2 + a^2$
E. $b^2 - a^2$

Question 7 (1 mark)

If $u = \frac{1}{a} + \frac{1}{b}i$ and $v = a - bi$, where a and b are non-zero real numbers, then $\frac{u}{v}$ in Cartesian form is equal to

A. 0

B. $\frac{2}{a^2 + b^2}$

C. $-\frac{i}{ab}$

D. $\frac{i}{ab}$

E. $\frac{abi}{(a^2 + b^2)}$

Topic	**10**	**Complex numbers**
Subtopic	**10.5**	**The complex plane (the Argand plane)**

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Question 1 (4 marks)

Let $u = 3 - 4i$. Plot $\overline{u}$, iu, i^2u on an Argand diagram and comment on their relative positions.

Question 2 (1 mark)

A triangle is formed when the complex number $u = a + bi$ and $\overline{u}$ are plotted on an Argand diagram with the origin O. Determine the area of this triangle.

Area = ☐

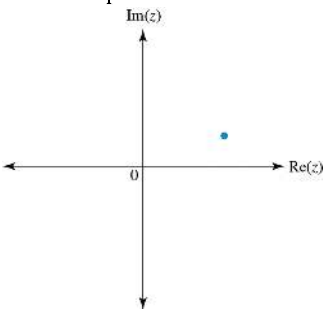
Question 3 (1 mark)

A triangle is formed when the complex number $u = a + bi$ are plotted on an Argand diagram with the origin O. Determine the area of this triangle.

Area = ☐

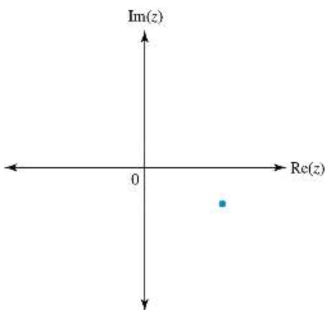
Question 4 (1 mark)

The complex number $a + bi$, where a and b are real constants, is represented in the following diagram.

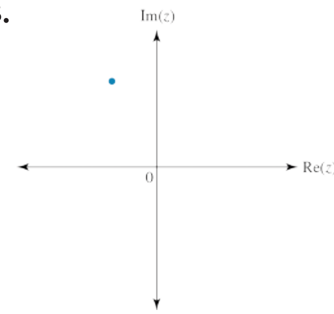


All axes below have the same scale as in the diagram above. The complex number $-i^2(a - bi)$ could be represented by

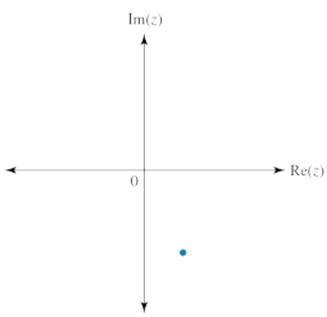
A.



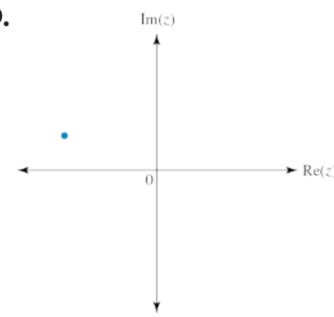
B.



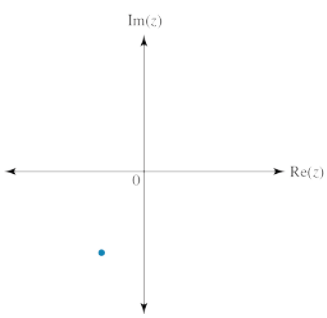
C.



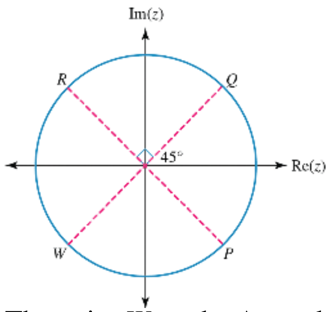
D.



E.



Question 5 (1 mark)



The point W on the Argand diagram represents the complex number w. The points P, Q, R and W all lie on a circle and are all equally spaced around the circle as shown above.
The complex number $i^2\overline{w}$ is best represented by the point

A. W
B. P
C. Q
D. R
E. None of the above

Question 6 (3 marks)
If $z = 3 + 2i$, on an Argand diagram plot $2z$, iz, $\overline{z}$.

Question 7 (2 marks)
The points P and Q represent the complex numbers z and w in the Argand plane; O is the origin. What can you say about the position of P and Q if
$|z| = |w|$?

Topic	10	Complex numbers
Subtopic	10.6	Complex numbers in polar form

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Question 1 (2 marks)

Answer the following

a. Let $u = \sqrt{3} + i$. Express u, $\overline{u}$, $\frac{1}{u}$ and $-u$ in polar form. **(1 mark)**

b. Let $w = 5\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$. Express w in Cartesian form. **(1 mark)**

$w = \square$

Question 2 (1 mark)

Let $u = 1 - i$ and $v = -1 - i$. Express u, v, $u + v$ and $\frac{u}{v}$ in polar form.

Question 3 (3 marks)

Let $u = a - \sqrt{6}i$. Determine the value of the real number a if:

a. $|u| = 9$ **(1 mark)**

$a = \square$

b. $\text{Arg}(u) = -\frac{\pi}{3}$ **(1 mark)**

$a = \square$

c. $u^2 = 8\text{cis}\left(-\frac{2\pi}{3}\right)$ **(1 mark)**

$a = \square$

Question 4 (2 marks)

Convert $(1,\ 1)$ to polar form.

Question 5 (2 marks)

Convert $5\ \text{cis}\,(150°)$ to Cartesian form.

Question 6 (2 marks)

Find the distance parallel to the x-axis between $2\ \text{cis}\left(\frac{\pi}{6}\right)$ and $8\ \text{cis}\left(\frac{\pi}{6}\right)$.

Question 7 (3 marks)

Find the distance between $\text{A} = 2\ \text{cis}\,(45°)$ and $\text{B} = 4\ \text{cis}\,(60°)$.

Topic	10	Complex numbers
Subtopic	10.7	Basic operations on complex numbers in polar form

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Question 1 (4 marks)

Answer the following

a. Let $u = -1 - \sqrt{3}i$. Express u^{12} in Cartesian form. **(2 marks)**

$u^{12} = \square$

b. Let $w = -1 - i$. Express $\frac{1}{w^5}$ in Cartesian form. **(2 marks)**

$\frac{1}{w^5} = \square$

Question 2 (4 marks)

Simplify $\frac{(1-i)^{10}}{(\sqrt{3}-i)^6}$, stating your answer in Cartesian form.

$\frac{(1-i)^{10}}{(\sqrt{3}-i)^6} = \square$

Question 3 (6 marks)

If $u = r\text{cis}\left(\frac{\pi}{3}\right)$ and $v = 4\text{cis}(\theta)$, determine the values of r and θ if:

a. $uv = -12$ **(2 marks)**

$r = \square$, $\theta = \square$

b. $\frac{u}{v} = -12i$ **(2 marks)**

$r = \square,\ \theta = \square$

c. $u^2v^2 = 32$ **(2 marks)**

$r = \square,\ \theta = \square$

Question 4 (1 mark)

Let $u = 5\,\text{cis}\left(\frac{\pi}{4}\right)$ and $v = a\,\text{cis}\,(b)$, where a and b are real constants.

If $\frac{u}{v} = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$, a and b respectively are equal to:

A. $\frac{5}{2}$ and $\frac{\pi}{2}$

B. 10 and $\frac{\pi}{2}$

C. 10 and $\frac{2\pi}{2}$

D. 10 and $-\frac{11\pi}{12}$

E. $\frac{5}{2}$ rm $\frac{2\pi}{5}$

Question 5 (1 mark)

Let $u = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ and $v = a\,\text{cis}(b)$, where a and b are real constants.

If $uv = 3\,\text{cis}\left(\frac{3\pi}{6}\right)$, a and b respectively are equal to:

A. $\frac{2}{3}$ and $\frac{4\pi}{2}$

B. 1.5 and $\frac{\pi}{2}$

C. 1.5 and $\frac{14\pi}{15}$

D. 1.5 and $\frac{14\pi}{15}$

E. $\frac{2}{3}$ and $\frac{4\pi}{15}$

Topic 10 Subtopic 10.7 Basic operations on complex numbers in polar form

Question 6 (1 mark)

If $z = -1 - i$, then $\dfrac{1}{\bar{z}^7}$ is equal to:

A. $-\dfrac{1}{16}(1 + i)$

B. $\dfrac{1}{16}(-1 + i)$

C. $\dfrac{1}{16}(1 - i)$

D. $8(1 - i)$

E. $\dfrac{1}{8}(-1 + i)$

Topic	10	Complex numbers
Subtopic	10.8	Solving quadratic equations with complex roots

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Question 1 (2 marks)

Solve for z in each of the following. Write your answer in the form $a \pm bi$.

a. $z^2 + 12z + 52 = 0$ **(1 mark)**

$z = \square$

b. $2z^2 - 5z + 6 = 0$ **(1 mark)**

$z = \square$

Question 2 (2 marks)

Answer the following

a. Form a quadratic with integer coefficients that has $2 - \sqrt{3}i$ as one of its roots. **(1 mark)**

Quadratic $= \square$

b. The quadratie $z^2 + bz + 41 = 0$ has $3 + \sqrt{c}i$ as one of its roots, determine the values of the real numbers b and c. **(1 mark)**

$b = \square$, $c = \square$

Question 3 (3 marks)

Determine the real numbers a and b if $(a + bi)^2 = 7 - 24i$.

$a = \square$, $b = \square$

or

$a = \square$, $b = \square$

Question 4 (1 mark)
If $z^2 + a = 0$, where a is a real positive number, then the most correct solution(s) for z is
A. $i\sqrt{a}$
B. $\sqrt{a}$
C. $\pm i\sqrt{a}$
D. $\pm\sqrt{a}$
E. $\pm ai$

Question 5 (1 mark)
Factorise $a^2 + 1, a \in C$.

Question 6 (2 marks)
Factorise $z(-z + 1) - 4, z \in C$.

Question 7 (1 mark)
If the quadratic expression $ax^2 + bx + c$ cannot be factorised over R, it means that:
A. $c < 0$.
B. $b^2 - 4ac > 0$.
C. $b^2 - 4ac < 0$.
D. $-\frac{c}{a} > 0$.
E. $\frac{ac}{b} \notin Z$.

Question 8 (1 mark)
The quadratic expression $p^2x^2 + q^3x + 2p$ does not have factors over R. Find a relationship between p and q.

Question 9 (3 marks)
If $x^2 + x + b$ does not factorise over R, factorize $x^2 + x + b$ over C. Explain why $b \neq -4$.

Question 10 (1 mark)
Solve for $x \in C,\ x^2 + 6 = 0$.

Question 11 (2 marks)
Solve for $x \in C,\ x^2 + 6x + 10 = 0$.

Question 12 (2 marks)
By substituting $a = z^2$, solve $z^4 - z^2 - 6 = 0,\ z \in C$.

Question 13 (1 mark)
$a^2 = -b$ and $a, b \in C$. Which one of the following is true?

A. $a = \pm\sqrt{b}$
B. No solution
C. $a = -\sqrt{b}, \sqrt{bi}$
D. $a = \pm\sqrt{bi}$
E. $a = \pm\sqrt{bi}$

Question 14 (1 mark)
$z = i$ is a solution to the equation $z^2 + zi + 2 = 0,\ z \in C$. Find the other solution.

Topic	10	Complex numbers
Subtopic	10.9	Lines, rays, circles, ellipses and regions in the complex plane

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Question 1 (4 marks)

Sketch and describe each of the following sets, clearly indicating which boundaries are included.

a. $\{z: |z-2| = |z-4|\}$

(2 marks)

b. $\{z: |z+4i| = |z-4|\}$

(2 marks)

Question 2 (2 marks)

Determine the Cartesian equation and sketch the graph defined by $\left\{z\text{:}\operatorname{Arg}(z-2)=\dfrac{\pi}{6}\right\}$.

Question 3 (2 marks)

Describe and sketch the region defined by $\{z\text{:}1 \le |z| \le 2\}$.

Topic	10	Complex numbers
Subtopic	10.10	Review

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Question 1 (1 mark)

Simplify the expression $\frac{2i}{1+i} - \frac{3}{2-i}$, giving your answer in the form $a + bi$.

$\frac{2i}{1+i} - \frac{3}{2-i} = \square$

Question 2 (1 mark)

If $f(i) = \frac{1 + i + i^2 + \ldots + i^{11}}{4}$, then the statement below that is true is

A. $f(i) = 2 + i$

B. $\text{Re}[f(i)] = 5$

C. $\text{Im}[f(i)] - \frac{1}{4}$

D. $f(i) = 0$

E. $\text{Re}[f(i)] = \frac{1}{4}$

Question 3 (1 mark)

If $z = 8 - 7i$ and $w = 3 + 4i$, then $3z - 2w$ is equal to

A. $30 - 13i$

B. $30 - 29i$

C. $18 - 29i$

D. 18

E. $18 + 29i$

Question 4 (1 mark)

If $z = -1 - \sqrt{3}i$ and $w = 2 + 2i$, then $\dfrac{w^4}{z^3}$ is equal to

A. $-4 + 4i$
B. $2\sqrt{3}$
C. $-4i$
D. -8
E. $\dfrac{1}{2i}$

Question 5 (4 marks)

Answer the following

a. Evaluate the complex number z in Cartesian form if $\dfrac{z+1}{z-1} = \dfrac{1}{2}(3 + i)$.

$z = \square$ **(1 mark)**

b. Determine the real numbers a and b if $(a + bi)^2 = 5 - 12i$.

$a = \square$, $b = \square$

or

$a = \square$, $b = \square$ **(3 marks)**

Question 6 (2 marks)

On an Argand diagram, plot the points corresponding to A $= -3 - 2i$, B $= -1 + 3i$, C $= 4 + 5i$.
If A,Band C represent three vertices of a parallelogram, find the complex number to represent D, the fourth vertex of the parallelogram.

Question 7 (1 mark)

For any complex number $z = x + yi$, where both x and y are real, then the complex number $w = i^3z$ is found by

A. reflecting z in the real axis.
B. reflecting z in the imaginary axis.
C. rotating z through 90° in a clockwise direction.
D. rotating z through 90° in an anti-clockwise direction.
E. rotating 180°.

Question 8 (1 mark)

If $z = a + bi$ and $w = c + di$ where a, b, c and d are non-zero real numbers, then $\text{Re}(w^2) + \text{Im}(z^2)$ is equal to:

A. $c^2 + b^2$
B. $c^2 - d^2 + 2ab$
C. $c^2 + d^2 - 2ab$
D. $c^2d^2 + (ab)^2$
E. $c^2 - b^2$

Question 9 (1 mark)

$\dfrac{\sqrt{r} - \sqrt{p}}{\sqrt{r} + \sqrt{p}}$ equals:

A. $1 - \dfrac{\sqrt{p}}{\sqrt{r} + \sqrt{p}}$
B. $1 - \dfrac{\sqrt{p}}{\sqrt{r} - \sqrt{p}}$
C. $\dfrac{r + p - 2\sqrt{pr}}{r - p}$
D. $\dfrac{r + p - 2\sqrt{pr}}{r + p}$
E. $\dfrac{r + p + 2\sqrt{pr}}{r - p}$

Question 10 (1 mark)

If p is a real constant and the imaginary part of $\dfrac{p + 3i}{4 + pi}$ is equal to zero, then

A. $p = 0$ only
B. $p = 4$ only
C. $p = -3$
D. $p = \pm 2\sqrt{3}$
E. $p = \pm 4$

Question 11 (1 mark)

If $z = x + yi$ where x and y are two non-zero real numbers, then which one of the following is not a real number?

A. $z\bar{z}$

B. $z + \bar{z}$

C. $z - \bar{z}$

D. $\frac{1}{z} + \frac{1}{\bar{z}}$

E. $\frac{1}{z} \times \frac{1}{\bar{z}}$

Question 12 (2 marks)

Solve the equation $iz^2 - 2z + \sqrt{3}i = 0$ for z.

Answers and marking guide

10.2 Introduction to complex numbers

Question 1

a. $\sqrt{-4}$

$= \sqrt{4i^2}$

$= 2i$

$\sqrt{-9}$

$= \sqrt{9i^2}$

$= 3i$

$\sqrt{-4} \times \sqrt{-9}$

$= 2i \times 3i$

$= 6i^2$

$= -6$

$\sqrt{-4 \times -9}$

$= \sqrt{36}$

$= 6$ **[1 mark]**

b. $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

Only for $a \geq 0,\ b \geq 0$ **[1 mark]**

Question 2

a. i^{4n}

$= \left(i^4\right)^n$

$= (1)^n$

$= 1$ **[1 mark]**

b. i^{4n+3}

$= \left(i^4\right)^n \times i^3$

$= 1 \times i^2 \times i$

$= -i$ **[1 mark]**

c. $(-1)^{\frac{n}{2}}$

$= \left(\sqrt{-1}\right)^n$

$= i^n$ **[1 mark]**

Question 3

a. $\text{Re}\,(f(10))$

$= 2i^2 + 4i^4 + 6i^6 + 8i^8 + 10i^{10}$

$= -2 + 4 - 6 + 8 - 10$

$= -6$ **[1 mark]**

b. $\text{Im}\,(f(11))$

$= \text{Im}\left(i + 3i^3 + 5i^5 + 7i^7 + 9i^9 + 11i^{11}\right)$

$= \text{Im}\,(i - 3i + 5i - 7i + 9i - 11i)$

$= \text{Im}\,(-6i)$

$= -6$ **[1 mark]**

Question 4

Equating real parts (1) $\quad 4a + 3b = -1$

Equating imaginary parts (2) $\quad 3a - 2b = 12$

$2 \times (1) \qquad 8a + 6b = -2$
$3 \times (2) \qquad 9a - 6b = 36$
Adding gives $17a = 34 \Rightarrow a = 2$ and $b = -3$

Question 5
$(x - 3) + (y + 5)\,i = (3x - y) - 2xi$
$x - 3 = 3x - y$
$y + 5 = -2x \Rightarrow y = -2x - 5$ **[1 mark]**
Substitute $y = -2x - 5$

$$\begin{aligned} x - 3 &= 3x - (-2x - 5) \\ -8 &= 4x \\ x &= -2 \\ y &= -1 \end{aligned}$$ **[1 mark]**

Question 6
The imaginary component is the coefficient of i. That is, -1.

Question 7
Complex numbers are equal if their real components are equal and their imaginary components are equal.

10.3 Basic operations on complex numbers

Question 1
a. $u = 3 - 4i,\ v = 4 + 5i$
$$\begin{aligned} u + v &= (3 - 4i) + (4 + 5i) \\ &= 3 + 4 + i\,(-4 + 5) \\ &= 7 + i \end{aligned}$$ **[1 mark]**

b. $u = 3 - 4i,\ v = 4 + 5i$
$$\begin{aligned} 2u - 3v &= 2\,(3 - 4i) - 3\,(4 + 5i) \\ &= 6 - 8i - 12 - 15i \\ &= (6 - 12) + i\,(-8 - 15) \\ &= -6 - 23i \end{aligned}$$ **[1 mark]**

c. $u = 3 - 4i,\ v = 4 + 5i$
$$\begin{aligned} u - v &= (3 - 4i)\,(4 + 5i) \\ &= 12 - 16i + 15i - 20i^2 \\ &= 12 - 16i + 15i + 20 \\ &= (12 + 20) + i\,(15 - 16) \\ &= 32 - i \end{aligned}$$ **[1 mark]**

d. $u = 3 - 4i,\ v = 4 + 5i$
$\text{Re}\left(u^2 - 2uv + v^2\right)$
$= \text{Re}\left((u - v)^2\right)$
$= \text{Re}(\left(3 - 4i - (4 + 5i)\right)^2)$
$= \text{Re}(\left((3 - 4) - i\,(4 + 5)\right)^2)$
$= \text{Re}\left((-1 - 9i)^2\right)$
$= \text{Re}\left(1 + 18i + 81i^2\right)$
$= \text{Re}\,(1 - 81 + 18i)$
$= -80$ **[1 mark]**

e. $u = 3 - 4i,\ v = 4 + 5i$
$\text{Im}\ \left(u^2 + 2uv + v^2\right)$

$= \text{Im}\left((u+v)^2\right)$
$= \text{Im}\left((7+i)^2\right)$
$= \text{Im}\left(49+14i+i^2\right)$
$= 14$

Question 2

a. $xu + yv = 17 - 2i$
$x(3-4i) + y(4+5i) = 17 - 2i$
$3x + 4y + i(5y - 4x) = 17 - 2i$
Re: [1] $3x + 4y = 17$
Im: [2] $5y - 4x = -2$ **[1 mark]**
$[1] \times 4 \quad 12x + 16y = 68$
$[2] \times 3 \quad 15y - 12x = -6$
$31y = 62$
$y = 2$
$3x = 17 - 4y$
$= 17 - 8 = 9$
$x = 3$ **[1 mark]**

b. $u = 3 - 4i \qquad v = 4 + 5i$
$u^2 = 9 + 16i^2 - 24i \qquad v^2 = 16 + 40i + 25i^2$
$= -7 - 24i \qquad = -9 + 40i$
$xu^2 + yv^2 + 5 + 88i = 0$
$x(-7 - 24i) + y(-9 + 40i) = -5 - 88i$
$-7x - 9y + i(40y - 24x) = -5 - 88i$
Re: [1] $-7x - 9y = -5$
Im: [2] $40y - 24x = -88$ **[1 mark]**
$-[1] \; 7x + 9y = 5$
$[2] \times \frac{1}{8} \; 5y - 3x = -11$
$-[1] \times 3 \; 21x + 27y = 15$
$[2] \times \frac{7}{8} \; 35y - 21x = -77$
$62y = -62$
$y = -1 \quad x = 2$ **[1 mark]**

Question 3

a. $(x + yi)u = v$
$(x + yi)(2 - 3i) = (3 + 4i)$
$2x + 2yi - 3xi - 3yi^2 = 3 + 4i$
$(2x + 3y) + (-3x + 2y) = 3 + 4i$
Re: [1] $2x + 3y = 3$
Im: [2] $-3x + 2y = 4$ **[1 mark]**
$[1] \times 3 \quad 6x + 9y = 9$
$[2] \times 2 \quad -6x + 4y = 8$

$$13y = 17$$
$$y = \frac{17}{13}$$
$$2x = 3 - 3y$$
$$2x = 3 - \frac{51}{13}$$
$$2x = \frac{39 - 51}{13}$$
$$2x = -\frac{12}{13}$$
$$x = -\frac{6}{13}$$ **[1 mark]**

b. $(x + yi)\, v = u$

$(x + yi)\,(3 + 4i) = 2 - 3i$

$3x + 3yi + 4xi + 4yi^2 = 2 - 3i$

$(3x - 4y) + i\,(4x + 3y) = 2 - 3i$

Re: [1] $3x - 4y = 2$

Im: [2] $4x + 3y = -3$ **[1 mark]**

[1] × 4 $12x - 16y = 8$

[2] × 3 $12x + 9y = -9$

$$25y = -17$$
$$y = \frac{-17}{25}$$
$$3x = 2 + 4y$$
$$3x = 2 - \frac{68}{25}$$
$$3x = \frac{50 - 68}{25}$$
$$3x = \frac{-18}{25}$$
$$x = \frac{-6}{25}$$ **[1 mark]**

Quesrion 4

$zw + \bar{z}\bar{w} = (a + bi)(c + di) + (a - bi)(c - di) = ac + adi + bci - bd + ac - adi - bci - bd = 2ac - 2bd$

Question 5

a. $2z_1 + 3z_2 = 2\,(2 - 3i) + 3\,(-4 - i)$

$= -8 - 9i$ **[1 mark]**

b. $-z_1 - 5z_3 = -\,(2 - 3i) - 5\,(3 + 2i)$

$= -17 - 7i$ **[1 mark]**

Question 6

a. $-i\,(3 - 2i) = -3i + 2i^2$

$= -3i - 2$ **[1 mark]**

b. $(2 - 3i)^2 = 4 - 12i + 9i^2$

$= 4 - 12i - 9$

$= -5 - 12i$ **[1 mark]**

10.4 Complex conjugates and division of complex numbers

Question 1

a. $u = 3 - 4i, v = 4 + 5i$

$$\begin{aligned}\frac{1}{u} &= \frac{1}{3-4i}\\ &= \frac{1}{3-4i} \times \frac{3+4i}{3+4i}\\ &= \frac{3+4i}{9-16i^2}\\ &= \frac{3}{25} + \frac{4}{25}i\end{aligned}$$

[1 mark]

b. $u = 3 - 4i, v = 4 + 5i$

$$\begin{aligned}\frac{1}{v} &= \frac{1}{4+5i}\\ &= \frac{1}{4+5i} \times \frac{4-5i}{4-5i}\\ &= \frac{4-5i}{16-25i^2}\\ &= \frac{4}{41} - \frac{5}{41}i\end{aligned}$$

[1 mark]

c. $u = 3 - 4i, v = 4 + 5i$

$$\begin{aligned}\frac{u}{v} &= \frac{3-4i}{4+5i}\\ &= \frac{(3-4i)\times(4-5i)}{(4+5i)(4-5i)}\\ &= \frac{12-16i-15i+20i^2}{16-25i^2}\\ &= \frac{(12-20)+i(-15-16)}{41}\\ &= \frac{-8}{41} - \frac{31}{41}i\end{aligned}$$

[1 mark]

d. $u = 3 - 4i, v = 4 + 5i$

$$\begin{aligned}\frac{v}{u} &= \frac{4+5i}{3-4i}\\ &= \frac{4+5i}{3-4i} \times \frac{3+4i}{3+4i}\\ &= \frac{12+15i+16i+20i^2}{9-16i^2}\\ &= \frac{(12-20)+i(15+16)}{25}\\ &= \frac{-8}{25} + \frac{31}{25}i\end{aligned}$$

[1 mark]

e. $u = 3 - 4i, v = 4 + 5i$

$$\begin{aligned}\frac{1}{uv} &= \frac{1}{(3-4i)(4+5i)}\\ &= \frac{1}{12 - 16i + 15i - 20i^2}\\ &= \frac{1}{12 + 20 + i(15 - 16)}\\ &= \frac{1}{32 - i}\\ &= \frac{1}{32 - i} \times \frac{32 + i}{32 + i}\\ &= \frac{32 + i}{1024 - i^2}\\ &= \frac{32}{1025} + \frac{1}{1025}i\end{aligned}$$

[1 mark]

Question 2

$u = a + bi \qquad v = c + di$

$$\begin{aligned}(u+v)^{-1} &= \frac{1}{u+v}\\ &= \frac{1}{(a+bi)+(c+di)}\\ &= \frac{1}{a+c+i(b+d)} \times \frac{a+c-(b+d)i}{a+c-(b+d)i}\end{aligned}$$

[1 mark]

$$\begin{aligned}&= \frac{a+c-(b+d)i}{(a+c)^2-(b+d)^2i^2}\\ &= \frac{a+c-(b+d)i}{(a+c)^2+(b+d)^2}\end{aligned}$$

[1 mark]

Question 3

$$\begin{aligned}\frac{z-i}{z+i} &= -2i\\ (z-i) &= -2i(z+i)\\ z - i &= -2iz - 2i^2\\ z - i &= -2iz + 2\\ z(1+2i) &= 2 + i\\ z &= \frac{2+i}{1+2i} \times \frac{1-2i}{1-2i}\end{aligned}$$

[1 mark]

$$\begin{aligned}&= \frac{2 + i - 4i - 2i^2}{1 - 4i^2}\\ &= \frac{4-3i}{5}\\ &= \frac{4}{5} - \frac{3}{5}i\end{aligned}$$

[1 mark]

Question 4

$$\begin{aligned}\frac{z}{w} &= \frac{2-4i}{1+3i} \times \frac{1-3i}{1-3i}\\ &= \frac{-10-10i}{10}\\ &= -1 - i\end{aligned}$$

Question 5

$z = a + ib \qquad \bar{z} = a - ibw = c + idw^{-} = c - id$

$\dfrac{\overline{z} \times \overline{w}}{z \times w} = \dfrac{ac - bd - i(cd + ad)}{ac - bd + i(cd + ad)}$ **[1 mark]**

$= ac - bd - i(cd + ad)$

$= \overline{z} \times \overline{w}$ **[1 mark]**

Question 6

$a^2b^2u^2 = a^2b^2\left(\dfrac{1}{a} - \dfrac{1}{b}i\right)^2$

$a^2b^2u^2 = a^2b^2\left(\dfrac{1}{a^2} - \dfrac{2i}{ab} + \dfrac{i^2}{b^2}\right)$

$a^2b^2u^2 = b^2 - 2abi + a^2i^2$

$a^2b^2u^2 = \left(b^2 - a^2\right) - 2abi$

Question 7

$\dfrac{u}{v} = \dfrac{\frac{1}{a} + \frac{1}{b}i}{a - bi} \times \dfrac{a + bi}{a + bi}$

$\dfrac{u}{v} = \dfrac{1 + i^2 + \left(\frac{a}{b} + \frac{b}{a}\right)i}{a^2 - b^2i^2}$

$\dfrac{u}{v} = \dfrac{\left(\frac{a^2+b^2}{ab}\right)i}{a^2 + b^2} = \dfrac{i}{ab}$

10.5 The complex plane (the Argand plane)

Question 1

$u = 3 - 4i$

$\overline{u} = 3 + 4i$ reflection in the real axis

$iu = i(3 - 4i)$

$= 3i - 4i^2$ a rotation by 90° anti − clockwise

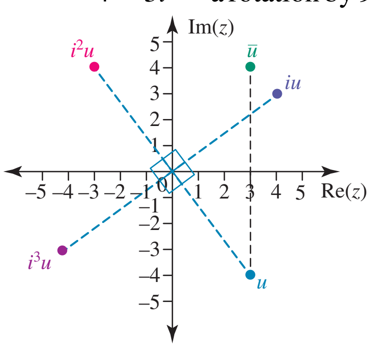
$= 4 + 3i$

$i^2u = -(3 - 4i)$

$= -3 + 4i$ a rotation by 180° anti − clockwise

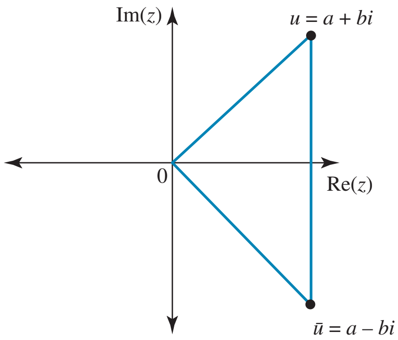
$i^3u = -i(3 - 4i)$

$= -4 - 3i$ a rotation by 90° clockwise



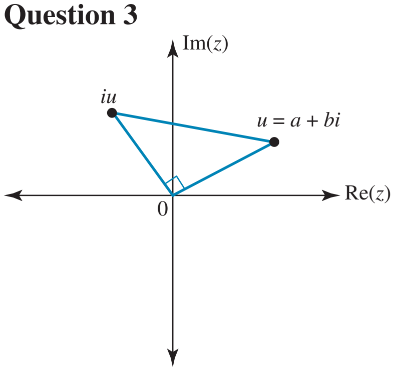
Award **1 mark** for each correctly plotted point and description of how it differs geometrically from u.

Question 2



$$\text{Area} = \frac{1}{2} \times |a| \times |2b| \quad \textbf{[1 mark]}$$
$$= |ab| \quad \textbf{[1 mark]}$$

Question 3



$$|u| = \sqrt{a^2 + b^2} = |iu| \quad \textbf{[1 mark]}$$
$$\text{Area} = \frac{1}{2} \times \left(\sqrt{a^2 + b^2}\right)^2$$
$$= \frac{1}{2}\left(a^2 + b^2\right) \quad \textbf{[1 mark]}$$

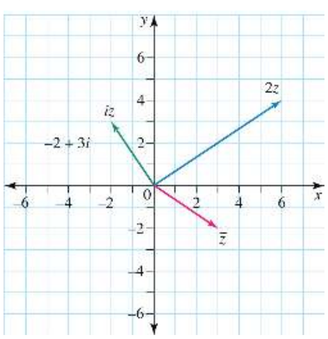
Question 4

$-i^2 (a - bi) = -(-1)(a - bi) = a - bi$

Question 5

$\overline{w}$ is the refection of w in the real axis, which is the point R.
$i^2\overline{w}$ is a rotation by 180° anti-clockwise; this takes us to the point P.

Question 6



Award **1 mark** for each correctly plotted point.

Question 7

P and Q lie on the circumference of a circle, with centre at the origin and radius $|z| = |w|$. **[1 mark]**

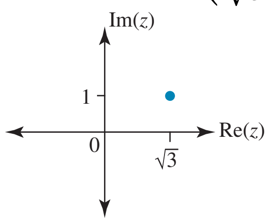
10.6 Complex numbers in polar form

Question 1

a. $u = \sqrt{3} + i$

$|u| = \sqrt{3+1} = 2$

$\text{Arg}(u) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$



$$u = \sqrt{3} + i = 2\,\text{cis}\left(\frac{\pi}{6}\right)$$

$$\overline{u} = \sqrt{3} - i = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{1}{u} = \frac{1}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$$

$$= \frac{\sqrt{3}-i}{3-i^2}$$

$$= \frac{1}{4}\left(\sqrt{3}-i\right) = \frac{1}{2}\,\text{cis}\left(-\frac{\pi}{6}\right)$$

$$-u = -\sqrt{3} - i = 2\,\text{cis}\left(-\frac{5\pi}{6}\right)$$ **[1 mark]**

b. $w = 5\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

$$= 5\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$= 5\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= 5\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= -5 - 5i$$ **[1 mark]**

Question 2

$$u = 1 - i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

$$v = -1 - i = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$$

$$u + v = -2i = 2\,\text{cis}\left(-\frac{\pi}{2}\right)$$

$$\frac{u}{v}=\frac{1-i}{-1-i}\times\frac{-1+i}{-1+i}=\frac{\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)}{\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)}$$

$$=\frac{-1+i+i-i^2}{1-i^2}$$

$$=\frac{2i}{2}$$

$$=i=\text{cis}\left(-\frac{\pi}{4}+\frac{3\pi}{4}\right)$$

$$=\text{cis}\left(\frac{\pi}{2}\right)$$ **[1 mark]**

Question 3

a $u=a-\sqrt{6}i$

$|u|=\sqrt{a^2+6}=9$

$a^2+6=81$

$a^2=75=25\times 5$

$a=\pm 5\sqrt{5}$ **[1 mark]**

b $u=a-\sqrt{6}i$

$\text{Arg}(u)=-\frac{\pi}{3}$

$\tan\left(-\frac{\pi}{3}\right)=\frac{-\sqrt{6}}{a}$

$-\sqrt{3}=\frac{-\sqrt{6}}{a}$

$a=\frac{\sqrt{6}}{\sqrt{3}}$

$a=\sqrt{2}$ **[1 mark]**

c $u=a-\sqrt{6}i$

$u^2=8\,\text{cis}\left(-\frac{2\pi}{3}\right)$

$=\left(a-\sqrt{6}i\right)^2$

$=a^2-2a\sqrt{6}i+6i^2$

$=\left(a^2-6\right)-2a\sqrt{6}i$

$|u|^2=a^2+6=8$

$a^2=2, a>0$

$u=\sqrt{2}-\sqrt{6}i$

$\text{Arg}\left(u^2\right)=-\frac{2\pi}{3}$

So $a=\sqrt{2}$ **[1 mark]**

Question 4

$\tan(\theta)=\frac{1}{1}$

$\theta=\frac{\pi}{4}$ **[1 mark]**

[(1, 1) is in the first quadrant]

$r = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$ **[1 mark]**

$\therefore (1, 1) = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

Question 5

$x = r\cos(\theta), \quad y = r\sin(\theta)$

$x = 5\cos(150°), \quad y = 5\sin(150°)$ **[1 mark]**

$x = \frac{5\sqrt{3}}{2}, \quad y = \frac{5}{2}$ $\quad 5\,\text{cis}(150°) = \frac{-5\sqrt{3}}{2} + \frac{5}{2}i$ **[1 mark]**

Question 6

Actual distance apart is 6 units. **[1 mark]**

Distance parallel to the x-axis is $6\cos\left(\frac{\pi}{6}\right) = 3\sqrt{3}$ **[1 mark]**

Question 7

$\text{OA} = 2, \quad \text{OB} = 4,$

$\angle\text{AOB} = (60 - 45)°$

$= 15°$ **[1 mark]**

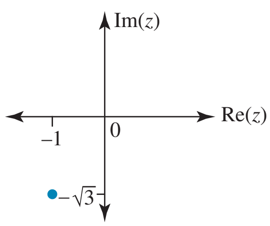
cosine rule $\text{AB} = \sqrt{2^2 + 4^2 - 2 \times 2 \times 4 \times \cos(15°)}$ **[1 mark]**

$= 2.132$ **[1 mark]**

10.7 Basic operations on complex numbers in polar form

Question 1

a.



$u = -1 - \sqrt{3}i$

$|u| = 2$

$\text{Arg}(u) = \tan^{-1}\left(\sqrt{3}\right) - \pi$

$= \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

$u = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ **[1 mark]**

$u^{12} = \left(2\,\text{cis}\left(-\frac{2\pi}{3}\right)\right)^{12}$

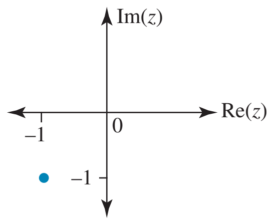
$= 2^{12}\text{cis}\left(12 \times -\frac{2\pi}{3}\right)$

$= 2^{12}\text{cis}(-8\pi)$

$= 2^{12}\text{cis}(0)$

$= 4096$ **[1 mark]**

b.



$$w = -1 - i$$
$$= \sqrt{2}\,\text{cis}\left(\tan^{-1}(1) - \pi\right)$$
$$= \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right) \quad \textbf{[1 mark]}$$

$$\frac{1}{w^5} = w^{-5} = \left(\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right)^{-5}$$
$$= \left(\sqrt{2}\right)^{-5}\text{cis}\left(5 \times \frac{3\pi}{4}\right)$$
$$= 2^{\frac{-5}{2}}\,\text{cis}\left(\frac{15\pi}{4}\right)$$
$$= \frac{1}{\sqrt{32}}\,\text{cis}\left(\frac{15\pi}{4} - 4\pi\right)$$
$$= \frac{1}{\sqrt{32}}\,\text{cis}\left(-\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{32}}\left(\cos\left(-\frac{\pi}{4}\right) - i\,\sin\left(\frac{\pi}{4}\right)\right)$$
$$= \frac{1}{8} - \frac{1}{8}i \quad \textbf{[1 mark]}$$

Question 2

$$\frac{(1-i)^{10}}{\left(\sqrt{3}-i\right)^6}$$
$$= \frac{\left(\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^{10}}{\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^6} \quad \textbf{[1 mark]}$$
$$= \frac{\left(\sqrt{2}\right)^{10}\text{cis}\left(-\frac{10\pi}{4}\right)}{(2)^6\,\text{cis}(-\pi)} \quad \textbf{[1 mark]}$$
$$= \frac{\left(\sqrt{2}\right)^{10}}{2^6}\,\text{cis}\left(-\frac{5\pi}{2} + \pi\right)$$
$$= \frac{2^5}{2^6}\,\text{cis}\left(-\frac{3\pi}{2}\right)$$
$$= \frac{1}{2}\,\text{cis}\left(-\frac{3\pi}{2} + 2\pi\right)i \quad \textbf{[1 mark]}$$
$$= \frac{1}{2}\,\text{cis}\left(\frac{\pi}{2}\right)$$
$$= \frac{1}{2}i \quad \textbf{[1 mark]}$$

Question 3

a. $u = r\,\text{cis}\left(\frac{\pi}{3}\right) \qquad v = 4\,\text{cis}\,(\theta)$

$$uv = r\,\text{cis}\left(\frac{\pi}{3}\right) \times 4\,\text{cis}\,(\theta)$$
$$= 4r\,\text{cis}\left(\frac{\pi}{3} + \theta\right) = -12 \quad \textbf{[1 mark]}$$
$$= 12\,\text{cis}\,(\pi)$$
$$r = 3 \qquad \theta = \frac{2\pi}{3} \qquad \textbf{[1 mark]}$$

b. $u = r\,\text{cis}\left(\frac{\pi}{3}\right) \qquad v = 4\,\text{cis}\,(\theta)$

$$\frac{u}{v} = \frac{r\,\text{cis}\left(\frac{\pi}{3}\right)}{4\,\text{cis}\,(\theta)}$$
$$= \frac{r}{4}\,\text{cis}\left(\frac{\pi}{3} - \theta\right) = -12i \quad \textbf{[1 mark]}$$
$$= 12\,\text{cis}\left(-\frac{\pi}{2}\right)$$
$$\frac{r}{4} = 12 \qquad \frac{\pi}{3} - \theta = -\frac{\pi}{2}$$
$$r = 48 \qquad \theta = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6} \quad \textbf{[1 mark]}$$

c. $u = r\,\text{cis}\left(\frac{\pi}{3}\right) \qquad v = 4\,\text{cis}\,(\theta)$

$$u^2v^2 = (uv)^2 = 16r^2\,\text{cis}\left(\frac{2\pi}{3} + 2\theta\right) = 32$$
$$= 32\,\text{cis}\,(0)$$
$$16r^2 = 32$$
$$r^2 = 2$$
$$r = \sqrt{2} \quad \textbf{[1 mark]}$$
$$2\theta + \frac{2\pi}{3} = 0$$
$$\theta = -\frac{\pi}{3} \quad \textbf{[1 mark]}$$

Question 4

$$\frac{v}{u} = \frac{a}{5}\text{cis}\left(b - \frac{\pi}{4}\right) = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$$
$$\therefore \frac{a}{5} = 2$$
$$a = 10$$
$$b - \frac{\pi}{4} = \frac{5\pi}{6}$$
$$b = \frac{5\pi}{6} + \frac{\pi}{4} = \frac{13\pi}{6}$$
$$b = \frac{11\pi}{12}$$

Question 5

$$uv = 2a\,\text{cis}\left(b - \frac{\pi}{3}\right) = 3\,\text{cis}\left(\frac{3\pi}{6}\right)$$
$$\therefore 2a = 3$$
$$a = \frac{3}{2}$$

$b + \frac{\pi}{3} = \frac{3\pi}{5}$

$b = \frac{3\pi}{5} + \frac{\pi}{3} = \frac{4\pi}{15}$

Question 6

$z = -1 - i \Rightarrow \bar{z} = -1 + i = 2cis(3\pi 4)1z^{-}7 = 1(2cis(3\pi 4))7 = (2) - 7cis(-21\pi 4)1z^{-}7 = 182cis(3\pi 4) =$
$182(cos\,(3\pi 4) + isin\,(3\pi 4)) = 182(-22 + 22)1z^{-}7 = 116(-1 + i)$

10.8 Solving quadratic equations with complex roots

Question 1

a. $z^2 + 12z + 52 = 0$

$z^2 + 12z + 36 = -52 + 36$

$(z + 6)^2 = -16$

$z + 6 = \pm 4i$

$z = -6 \pm 4i$ **[1 mark]**

b. $2z^2 - 5z + 6 = 0$

$a = 2 \quad b = -5 \quad c = 6$

$\Delta = b^2 - 4ac$

$= 25 - 48$

$= -23 = 23i^2$

$z = \frac{-b \pm \sqrt{\Delta}}{2a}$

$= \frac{5 \pm \sqrt{23}i}{4}$ **[1 mark]**

Question 2

a. $\alpha = 2 - \sqrt{3}i \qquad \bar{\alpha} = 2 + \sqrt{3}i$

$\alpha + \bar{\alpha} = 4 \qquad \alpha\bar{\alpha} = 4 - 3i^2 = 7$

$z^2 - 4z + 7$ **[1 mark]**

b. $\alpha = 3 + \sqrt{c}i \qquad \bar{\alpha} = 3 - \sqrt{c}i$

$\alpha + \bar{\alpha} = 6 \qquad \alpha\bar{\alpha} = 9 - ci^2 = 9 + c$

So $b = -6$

$41 = 9 + c$

$c = 32$ **[1 mark]**

Question 3

$(a + bi)^2 = a^2 + 2abi + b^2i^2$

$= a^2 - b^2 + 2abi = 7 - 24i$

Re: [1] $a^2 - b^2 = 7$

Im: [2] $2ab = -24 \Rightarrow b = \frac{-12}{a}$ into [1] **[1 mark]**

$a^2 - \left(\frac{-12}{a}\right)^2 = 7$

$a^2 - \frac{144}{a^2} = 7$ **[1 mark]**

$a^4 - 7a^2 - 144 = 0, a \in R$

$a = \pm 4 \Rightarrow b = \mp 3$

$4 - 3i \quad or \quad -4 + 3i$ **[1 mark]**

Question 4

$$z^2 + a = z^2 - i^2\left(\sqrt{a}\right)^2 = \left(z + \sqrt{a}i\right)\left(z + \sqrt{a}i\right) = 0$$

$$z = \pm i\sqrt{a} \quad \text{for} \quad a > 0$$

Question 5

$$a^2 + 1 = a^2 - i^2$$
$$= (a - i)(a + i) \quad \textbf{[1 mark]}$$

Question 6

$$z(-z + 1) - 4 = -z^2 + z - 4$$
$$= -\left(z^2 - z + 4\right)$$
$$= -\left[\left(z - \frac{1}{2}\right)^2 + \frac{15}{4}\right] \quad \textbf{[1 mark]}$$
$$= -\left[\left(z - \frac{1}{2}\right)^2 - \frac{15}{4}i^2\right]$$
$$= -\left(z - \frac{1 - \sqrt{15}i}{2}\right)\left(z - \frac{1 + \sqrt{15}i}{2}\right) \quad \textbf{[1 mark]}$$

Question 7

The expression does not factorise over R if the discriminant $b^2 - 4ac < 0$.

Question 8

$$\Delta < 0 \Rightarrow b^2 - 4ac < 0$$
$$\left(q^3\right)^2 - 8p^2 \times p < 0$$
$$q^6 - 8p^3 < 0$$
$$q^2 < 2p \quad \textbf{[1 mark]}$$

Question 9

$$x^2 + x + b = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + b$$
$$= \left(x + \frac{1}{2}\right)^2 - \left(b - \frac{1}{4}\right)i^2$$
$$= \left[\left(x + \frac{1}{2}\right) - \left(\frac{\sqrt{(4b - 1)}}{2}\right)i\right]\left[\left(x + \frac{1}{2}\right) + \left(\frac{\sqrt{(4b - 1)}}{2}\right)i\right] \quad \textbf{[1 mark]}$$

$$-\frac{1}{4} + b > 0 \Rightarrow b > \frac{1}{4}$$
$$\therefore b \neq -4 \quad \textbf{[1 mark]}$$

Question 10

$$x^2 + 6 = 0$$
$$x^2 = -6$$
$$x^2 = 6i^2$$
$$x = \pm\sqrt{6}i \quad \textbf{[1 mark]}$$

Question 11

$$x^2 + 6x + 10 = 0$$

$$(x+3)^2 - 9 + 10 = 0 \quad \textbf{[1 mark]}$$

$$(x+3)^2 = -1 = i^2$$

$$x = -3 \pm i \quad \textbf{[1 mark]}$$

Question 12

$$z^4 - z^2 - 6 = 0$$

$$a^2 - a - 6 = 0$$

$$(a-3)(a+2) = 0 \quad \textbf{[1 mark]}$$

$$a = 3, -2 \Rightarrow z^2 = 3, -2$$

$$z = \pm\sqrt{3}, \pm\sqrt{2}i \quad \textbf{[1 mark]}$$

Question 13

$$a^2 = -b$$

$$= bi^2$$

$$a = \pm\sqrt{b}i$$

Question 14

Using long or synthetic division, the other solution is $z = -2i$. **[1 mark]**

10.9 Lines, rays, circles, ellipses and regions in the complex plane

Question 1

a. $(z: |z-2| = |z-4|)$

Let $z = x + iy$

$$|(x-2) + iy| = |(x-4) + iy|$$

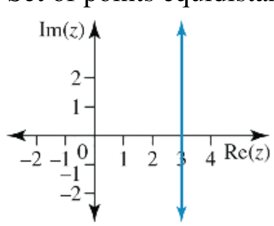
$$\sqrt{(x-2)^2 + y^2} = \sqrt{(x-4)^2 + y^2}$$

$$x^2 - 4x + 4 + y^2 = x^2 - 8x + 16 + y^2$$

$$4x = 12$$

$$x = 3 \ \textbf{[1 mark]}$$

Set of points equidistant from (0, –4) to (4, 0)



Award **1 mark** for correctly sketching the line Re $(z) = 3$

b. $(z: |z+4i| = |z-4|)$

Let $z = x + iy$

$$|x + (y+4)i| = |(x-4) + iy|$$

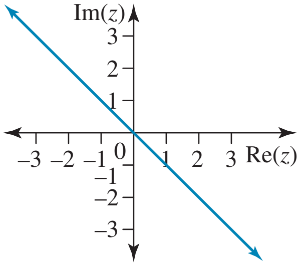
$$\sqrt{x^2 + (y+4)^2} = \sqrt{(x-4)^2 + y^2}$$

$$x^2 + y^2 + 8y + 16 = x^2 - 8x + 16 + y^2$$

$$8x + 8y = 0$$

$$y = -x \ \textbf{[1 mark]}$$

Set of points equidistant from 0, –4 and 4, 0.



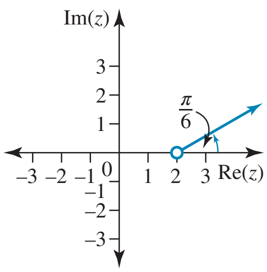
Award **1 mark** for correctly sketching the line $\text{Im}(z) = -\text{Re}(z)$.

Question 2

$$\left(z{:}\text{Arg}(z-2) = \frac{\pi}{6}\right)$$

Ray starts from the point (2, 0) making an angle of $\frac{\pi}{6}$ with the positive real axis

$$\text{Arg}(z-2) = \frac{\pi}{6}$$
$$\text{Arg}(x+yi-2) = \frac{\pi}{6}$$
$$\text{Arg}((x-2)+yi) = \frac{\pi}{6}$$
$$\tan^{-1}\left(\frac{y}{x-2}\right) = \frac{\pi}{6}, \text{ for } x > 2$$
$$\frac{y}{x-2} = \tan^{-1}\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \text{ for } x > 2$$
$$\sqrt{3}y = (x-2), \text{ for } x > 2$$ **[1 mark]**



Question 3

$1 \le |z| \le 2$

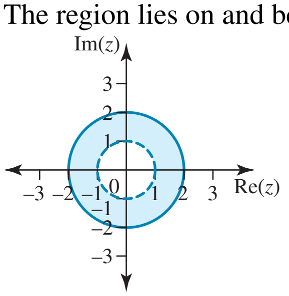
$1 \le \sqrt{x^2+y^2} \le 2$

$1 \le x^2 + y^2 \le 4$

Boundary lines are two circles, centre (0, 0), with radius 1 and 2 units.

The boundary lines are included.

The region lies on and between the two circles, centred at (0, 0) of radius 1 and 2 units. **[1 mark]**



Award **1 mark** for correctly sketching the region as shown above.

10.10 Review

Question 1

$$\frac{2i}{1+i} - \frac{3}{2-i}$$

$$\frac{4i+2-3-3i}{(1+i)(2-i)}$$ **[1 mark]**

$$= \frac{-1+i}{2-i+2i+1}$$

$$= \frac{-1+i}{3+i} \times \frac{3-i}{3-i}$$

$$= \frac{-3+i+3i+1}{9+1}$$ **[1 mark]**

$$= \frac{-2+4i}{10}$$

$$= \frac{-1}{5} + \frac{2}{5}i$$ **[1 mark]**

Question 2

$$\frac{1+i+i^2+\cdots+i^{11}}{1-i}$$

$$= \frac{1+i+i^2+i^2\times i+i^2\times i^2+i^4\times i+i^4\times i^2+\cdots+i^4\times i^4\times i^2\times i}{1-i}$$

$$= \frac{1+i+-1-i+1+i+-1-i+1+i+-1-i}{1-i}$$

$$= \frac{0}{1-i}$$

$$= 0$$

Question 3

$$\begin{aligned} 3z-2w &= 3(8-7i)-2(3+4i) \\ &= 24-21i-6-8i \\ &= 18-29i \end{aligned}$$

Question 4

$$z = -1-\sqrt{3}i$$

$$\begin{aligned} |z| &= \sqrt{(-1)^2+\left(-\sqrt{3}\right)^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\tan(\theta) = \frac{-\sqrt{3}}{-1} \text{ (third quadrant)}$$
$$= \sqrt{3}$$
$$\theta = -\pi + \tan^{-1}\left(\sqrt{3}\right)$$
$$= -\pi + \frac{\pi}{3}$$
$$= \frac{-2\pi}{3}$$
$$z = 2\operatorname{cis}\left(\frac{-2\pi}{3}\right)$$
$$w = 2 + 2i$$
$$|w| = \sqrt{2^2 + 2^2}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$
$$\tan(\theta) = \frac{2}{2}$$
$$= 1$$
$$\theta = \tan^{-1}(1)$$
$$= \frac{\pi}{4}$$
$$w = 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$
$$\frac{w^4}{z^3} = \frac{\left(2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^4}{\left(2\operatorname{cis}\left(-\frac{2\pi}{3}\right)\right)^3}$$
$$= \frac{64\operatorname{cis}(\pi)}{8\operatorname{cis}(-2\pi)}$$
$$= 8\operatorname{cis}(3\pi)$$
$$= 8\operatorname{cis}(\pi)$$
$$= 8\cos(\pi) + \sin(\pi)\, i$$
$$= -8 + 0i$$
$$= -8$$

Question 5

a. $\dfrac{z+1}{z-1} = \dfrac{1}{2}(3+i)$

$$2(z+1) = (3+i)(z-1)$$
$$2z + 2 = 3z + zi - 3 - i$$
$$5 + i = z(1+i)$$
$$z = \frac{5+i}{1+i} \times \frac{1-i}{1-i} \qquad \textbf{[1 mark]}$$
$$z = \frac{5 + i - 5i - i^2}{1 - i^2}$$
$$z = 3 - 2i \qquad \textbf{[1 mark]}$$

b. $(a+bi)^2 = a^2 + 2abi + b^2 i^2 \qquad a, b \in R$

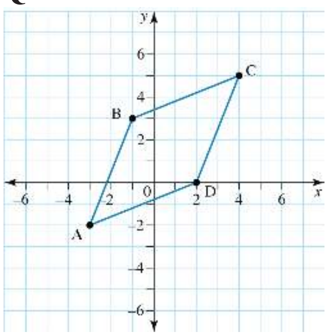
$$= a^2 - b^2 + 2abi = 5 - 12i \qquad \textbf{[1 mark]}$$

Re: [1] $\quad a^2 - b^2 = 5$

Im: [2] $\quad 2ab = -12 \Rightarrow b = -\dfrac{6}{a}$ into [1] **[1 mark]**

$$a^2 - \left(-\frac{6}{a}\right)^2 = 5$$
$$a^2 - \frac{36}{a^2} = 5$$
$$a^4 - 5a^2 - 36 = 0, \quad a \in R$$
$$a = \pm 3 \Rightarrow b = \mp 2$$
$3 - 2i$ *or* $-3 + 2i$ **[1 mark]**

Question 6



Award 1 mark for correct plot.
$D = 2 + 0i$ **[1 mark]**

Question 7

i is a rotation of 90° anti-clockwise, so i^3 is a rotation of 270° anti-clockwise or 90° clockwise.

Question 8

$$z^2 = a^2 - b^2 + 2abi$$
$$w^2 = c^2 - d^2 + 2cdi$$
$$\text{Re}\left(w^2\right) = c^2 - d^2$$
$$\text{Im}\left(z^2\right) = 2abi$$

Question 9

$$\frac{\sqrt{r} - \sqrt{p}}{\sqrt{r} + \sqrt{p}} = \frac{\sqrt{r} - \sqrt{p}}{\sqrt{r} + \sqrt{p}} \times \frac{\sqrt{r} - \sqrt{p}}{\sqrt{r} - \sqrt{p}}$$
$$= \frac{r + p - 2\sqrt{pr}}{r - p}$$

Question 10

$$\frac{p + 3i}{4 + pi} = \frac{p + 3i}{4 + pi} \times \frac{4 - pi}{4 - pi}$$
$$\frac{p + 3i}{4 + pi} = \frac{4p + 12i - p^2 i - 3pi^2}{16 - p^2 i^2}$$
$$\frac{p + 3i}{4 + pi} = \frac{7p + \left(12 - p^2\right) i}{16 + p^2}$$
$$\text{If Im}\left(\frac{p + 3i}{4 + pi}\right) = 0 \Rightarrow \text{Im}\left(\frac{7p + \left(12 - p^2\right) i}{16 + p^2}\right) = 0$$
$$\Rightarrow 12 - p^2 = 0 \Rightarrow p = \pm 2\sqrt{3}$$

Question 11

$$(x + yi) - (x - yi) = x - x + yi + yi$$
$$z - \bar{z} = 2yi$$

Question 12

$iz^2 - 2z + \sqrt{3}i = 0$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times i \times \sqrt{3}i}}{2i} \quad \textbf{[1 mark]}$$

$$= \frac{2 \pm 2\sqrt{\left(1 + \sqrt{3}\right)}}{2i}$$

$$= -\left[1 \pm \sqrt{\left(1 + \sqrt{3}\right)}\right] i \quad \textbf{[1 mark]}$$

11 Transformations

Topic	11	Transformations
Subtopic	11.2	Translations

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Question 1 (1 mark)

The coordinates of the image of $(-3, 2)$ under the transformation given by $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ are

A. $(-2, 1)$
B. $(2, -1)$
C. $(-6, 7)$
D. $(6, -7)$
E. $(6, 7)$

Question 2 (1 mark)

The image equation when the line with equation $y = x - 5$ is transformed by the translation matrix $T = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is

A. $y' = 3x' - 1$
B. $y' = -3x' - 1$
C. $y' = x' - 9$
D. $y' = x' - 7$
E. $y' = x' - 1$

Question 3 (3 marks)

Determine the image equation when the equation $y = x^2 + 1$ is transformed by the translation matrix $T = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$y = \square$

Question 4 (1 mark)

If the graph of $y = x^3$, is translated 3 units in the positive direction of the x-axis and then translated 3 units down in the negative direction of the y-axis, it becomes the graph of

A. $y = 3x^3 - 3$
B. $y = (x - 3)^3 + 3$
C. $y = (x - 3)^3 - 3$
D. $y = 3 - (x - 3)^3$
E. $y = 3 - (x + 3)^3$

Question 5 (1 mark)

The point $(a,\ b)$ is translated under $T_{p,\,q}$ giving the image point $(-2a,\ 3b)$. Which one of the following is true?

A. $p = 3a,\ q = 4b$
B. $p = -3a,\ q = 4b$
C. $p = 3a,\ q = -4b$
D. $p = -3a,\ q = -4b$
E. $p = -3,\ q = -4$

Question 6 (1 mark)

The point $(a,\ b)$ is translated under $T_{p,\,q}$. Find the coordinates of the image point.

Question 7 (2 marks)

$y = f(x)$ is translated under $T_{p,\,q}$. Find the new equation.

Question 8 (2 marks)
The image of $y = x^2$ under a translation is $y = (x - 3)^2 - 7$. Find the translation.

Question 9 (1 mark)
If the graph of $y = x^2$, is translated 2 units to the left parallel to the x-axis and then translated 3 units up, parallel to the y-axis, it becomes the graph of

A. $y = (x + 2)^2 + 3$
B. $y = (x - 2)^2 + 3$
C. $y = (x + 2)^2 - 3$
D. $y = (x - 3)^2 + 2$
E. $y = (x + 3)^2 - 2$

Question 10 (1 mark)
The point $(-a,\ b)$ is translated under $T_{-p,\,-q}$.
The coordinates of the image point are:

A. $(a + p,\ b + q)$
B. $(a + p,\ -b - q)$
C. $(a + p,\ b - q)$
D. $(-a - p,\ b - q)$
E. $(a - p,\ b - q)$

Topic	11	Transformations
Subtopic	11.3	Reflections and rotations

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Question 1 (1 mark)

The coordinates of the point $(-5, -2)$ after a reflection in the y-axis are

A. $(-2, 5)$
B. $(5, -2)$
C. $(-5, 2)$
D. $(5, 2)$
E. $(2, 5)$

Question 2 (1 mark)

An anticlockwise rotation about the origin of 60° is best represented by

A. $\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$

B. $\frac{1}{2}\begin{bmatrix} 2 & \sqrt{3} \\ -\sqrt{3} & 2 \end{bmatrix}$

C. $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

D. $\frac{1}{2}\begin{bmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$

E. $\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

Question 3 (3 marks)

Determine the equation of the image of the line $2y - 3x = 1$ under a rotation 90° about the origin in a clockwise direction.

The equation is ☐

Question 4 (1 mark)

If the graph of $y = (x - 2)^2 + 2$, is reflected in the x-axis and then in the y-axis, it becomes the graph of

A. $y = -(x - 2)^2 - 2$
B. $y = -(x - 2)^2 + 2$
C. $y = (x - 2)^2 - 2$
D. $y = -(x + 2)^2 - 2$
E. $y = (x + 2)^2 - 2$

Question 5 (2 marks)

The point (a, b) is reflected in the y-axis. The image point in then reflected in the x-axis. What are the coordinates of the final point?

Question 6 (3 marks)

$y = x^3 - 3x + 1$ is reflected in the y-axis. What is the new equation?

Question 7 (3 marks)

$y = x^2 + x + 1$ is reflected in the x-axis. What is the new equation?

Question 8 (1 mark)

The graph of the function $g: (-\infty, \quad 0) \to R$ where $g(x) = x^2 + x$ is reflected in the y-axis. The equation of the new graph is:

A. $g'(x) = x^2 - x, \quad x \in (-\infty, \quad 0)$
B. $g'(x) = x^2 - x, \quad x \in [0, \quad \infty]$
C. $g'(x) = -x^2 - x, \quad x \in (-\infty, \quad 0)$
D. $g'(x) = -x^2 - x, \quad x \in [0, \quad \infty]$
E. $g'(x) = x^2 + x, \quad x \in [0, \quad \infty]$

Question 9 (1 mark)

Consider the graph of $y = x^3 - x^2$. Which of the following statements is true?

A. When reflected in the x-axis, the graph becomes $y = x^3 + x^2$.
B. When reflected in the x-axis, or the y-axis, the graph becomes $y = -x^3 + x^2$.
C. When reflected in the line, the graph becomes $y = x^3 - x^2$.
D. When reflected in the line, the graph becomes $y = -x^3 + x^2$.
E. When reflected in both the x-axis and y-axis, the graph becomes $y = x^3 + x^2$.

Question 10 (1 mark)

The image point of point A after reflection in the line $y = -x$ is $(-m, -n)$. The coordinates of A are:

A. $(-m, -n)$
B. $(-n, -m)$
C. $(-n, m)$
D. $(n, -m)$
E. (n, m)

Question 11 (2 marks)

The parabola $y = x^2 - 5x + 6$ is reflected in the x-axis. What is the new equation?

Question 12 (2 marks)

The parabola $y = 2x^2 - 3x - 7$ is reflected in the y-axis. What is the new equation?

Topic	11	Transformations
Subtopic	11.4	Dilations

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Question 1 (3 marks)

A transformation T is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

a. Determine the image of point A $(-3, 1)$. **(1 mark)**

The coordinates of the image are ☐

b. Describe the transformation represented by T. **(2 marks)**

Question 2 (3 marks)

Determine the image equation when the hyperbola $y = \dfrac{1}{x-2}$ is dilated by a factor of 3 from the x-axis.

$y =$ ☐

Question 3 (1 mark)

The image of the point $(-6, 2)$ after a dilation of factor 2 from the x-axis is

A. $(-12, 2)$
B. $(-6, 4)$
C. $(-3, 2)$
D. $(-3, 1)$
E. $(-6, 2)$

Question 4 (1 mark)

The graph $y = x^2$ is transformed to the graph of $y = 4x^2$ by

A. a dilation parallel to the x-axis by a scale factor of 4.
B. a dilation parallel to the y-axis by a scale factor of 4.
C. a dilation parallel to the x-axis by a scale factor of 2.
D. a dilation parallel to the y-axis by a scale factor of 2.
E. a dilation parallel to the y-axis by a scale factor of $\frac{1}{2}$.

Question 5 (1 mark)

The graph $y = x^3$ is transformed to the graph of $y = 8x^3$ by

A. a dilation parallel to the x-axis by a scale factor of 8.
B. a dilation parallel to the y-axis by a scale factor of 8.
C. a dilation parallel to the x-axis by a scale factor of 2.
D. a dilation parallel to the y-axis by a scale factor of 2.
E. a dilation parallel to the y-axis by a scale factor of $\frac{1}{2}$.

Question 6 (2 marks)

The point (a, b) is dilated by a factor of p parallel to the x-axis. What are the coordinates of the image point?

Question 7 (3 marks)

$y = x^2 + x + 1$ is dilated by a factor of 9 parallel to the y-axis. Find the new equation.

Question 8 (2 marks)

The image of $y = x^2$ under a dilation is $y = \frac{x^2}{16}$. Find the dilation, giving your answer as:

a. a dilation parallel to the x-axis **(1 mark)**

b. a dilation parallel to the y-axis **(1 mark)**

☐

Question 9 (1 mark)

The graph of $y = f(3x)$ is dilated by a factor of 5 from the x-axis and by a factor of 6 from the y-axis. The new equation is:

A. $y = 30f(3x)$

B. $y = \frac{1}{30}f(3x)$

C. $y = \frac{1}{5}f(2x)$

D. $y = \frac{1}{5}f\left(\frac{x}{2}\right)$

E. $y = 5f\left(\frac{x}{2}\right)$

Question 10 (1 mark)

The graph $y = x^2$ is transformed to the graph of $y = \frac{x^2}{4}$ by

A. a dilation from the x-axis by a scale factor of 4.

B. a dilation from the y-axis by a scale factor of 4.

C. a dilation from the x-axis by a scale factor of 2.

D. a dilation from the y-axis by a scale factor of 2.

E. a dilation from the y-axis by a scale factor of $\frac{1}{2}$.

Topic	11	Transformations
Subtopic	11.5	Combinations of transformations

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Question 1 (1 mark)

The transformation $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is

A. a translation of 2 units in both the x- and y-directions.
B. a dilation of factor 2 parallel to the y-axis.
C. a dilation of factor 2 from the y-axis.
D. a dilation of factor 2 parallel to both the x- and y-axis.
E. a dilation of factor 2 from the x-axis.

Question 2 (3 marks)

Determine the image of the point (−1, 3) transformed by the following matrices in order:

Dilation $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, reflection $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, translation $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$

The coordinates of the image are ☐

Question 3 (3 marks)

A rectangle ABCD with vertices A (0, 0), B (0, 2), C (3, 2) and D (3, 0) is transformed under the transformation matrix $T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Describe the effect of the transformation on the rectangle's position and area.

Question 4 (4 marks)

The point (a, b) is dilated by a factor of p parallel to the x-axis and by a factor of q parallel to the y-axis. The image point is reflected in the y-axis and then translated by $T_{m,n}$. What are the coordinates of the final image point?

Question 5 (4 marks)

$y = x^5$ is dilated by a factor of p parallel to the x-axis and by a factor of q parallel to the y-axis. The image point is reflected in the y-axis and then translated by $T_{m,n}$. What is the final equation?

Question 6 (1 mark)

A reflection in the line $y = x$ followed by a dilation of 2 from the from the x-axis is represented by the matrix:

A. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$

Question 7 (3 marks)

The point $(a,\ b)$ is translated under $T_{m,\,n}$ and then reflected in the x-axis. Write these transformation in matrix form and hence find the final image point.

Question 8 (3 marks)

The point $(a,\ b)$ is dilated by a factor of p parallel to the x-axis, then rotated by $45°$ anticlockwise around the origin. Write these transformations in matrix form and hence find the final image point.

Question 9 (1 mark)

The graph of $y = x^2$ is transformed to the graph of $y = 4 - (x - 2)^2$ by

A. a reflection in the y-axis, a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

B. a reflection in the x-axis, a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

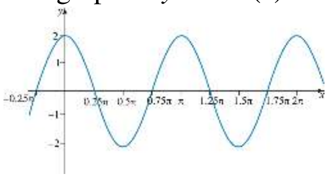
C. a reflection in the y-axis, a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

D. a reflection in the x-axis, a translation of 2 units in the negative direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

E. a reflection in the y-axis, a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.

Question 10 (1 mark)

The graph of $y = \cos(x)$ is transformed in to the graph below.



Which of the following statements describes the correct transformations?

A. A dilation by a factor of 2 parallel to the y-axis and a dilation by a factor of 2 parallel to the x-axis.

B. A dilation by a factor of 2 parallel to the y-axis and a dilation by a factor of $\frac{1}{2}$ parallel to the x-axis.

C. A dilation by a factor of 2 away from the y-axis and a dilation by a factor of 2 away from the x-axis.

D. A dilation by a factor of $\frac{1}{2}$ away from the y-axis and a dilation by a factor of $\frac{1}{2}$ away from the x-axis.

E. A dilation by a factor of $\frac{1}{2}$ parallel to the y-axis and a dilation by a factor of $\frac{1}{2}$ away from the x-axis.

Topic	11	Transformations
Subtopic	11.6	Review

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Question 1 (1 mark)

The image of the point $(-a, b)$ after using the matrix equation for translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ is

A. $(-(5+a), -(b-4))$
B. $(-(5+a), (b+4))$
C. $((5+a), -(b-4))$
D. $((5+a), (b-4))$
E. $((5+a), (b+4))$

Question 2 (1 mark)

The matrix equation for the translation that maps the circle $(x-p)^2 + (y+q)^2 = 1$ onto the circle $x^2 + y^2 = 1$ is

A. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -p \\ q \end{bmatrix}$

B. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p \\ -q \end{bmatrix}$

C. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} p \\ -q \end{bmatrix}$

D. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -p \\ -q \end{bmatrix}$

E. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -p \\ -q \end{bmatrix}$

Question 3 (1 mark)

The single transformation matrix T that describes a reflection in the line $y = x$ followed by a dilation of factor 2 from both the x- and y-axes is

A. $T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

B. $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

C. $T = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

D. $T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

E. $T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Question 4 (1 mark)

The point $P(a, b)$ comes under a reflection in the y-axis followed by a dilation of factor 3 from the y-axis. The coordinates of the image point $P'(x, y)$ are

A. $(-a, -3b)$

B. $(a, 3b)$

C. $(a, -3b)$

D. $(-3a, b)$

E. $(-3b, a)$

Question 5 (1 mark)

The equation of the image of $y = x^2$ under a double transformation of a reflection in the line $y = -x$ followed by a clockwise rotation of 90° about the origin is

A. $y = \sqrt{x}$

B. $y = -\sqrt{x}$

C. $y = -x^2$

D. $y = x^2$

E. none of these

Question 6 (1 mark)

The transformation $T : R2 \rightarrow R2$, which maps the curve with equation $y = \log\ e(x)$ to the curve with equation $y = \log\ e(2x - 4) + 3$, could have rule

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Question 7 (1 mark)

The point $(2, -3)$ is mapped to the point $(-3, 4)$ by which of the following transformation matrices?

A. $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & -\frac{1}{2} \\ -1 & 0 \end{bmatrix}$

Question 8 (4 marks)

Answer the following

a. The graph of $y = f(x)$ is translated h units to the left and k units upwards. What is the new equation? **(1 mark)**

b. The new equation is dilated by a factor of a parallel to the x-axis and a factor of b parallel to the y-axis. What is the new equation? **(1 mark)**

$y = \square$

c. Had the dilations in part b been completed first, *before* the translations in part a, what would be the final equation? Are the final equations the same? **(1 mark)**

$y = \square$

d. Take the equation $7y = -\frac{3}{4}(x+3)^3 + 6$ Which transformations and in which order would be required to result in a final equation $x = y^3$ **(1 mark)**

Question 9 (1 mark)

The transformation $T: R^2 \to R^2$ with rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ maps the curve with equation $y = x^2$ to the curve with equation:

A. $y = 4 - \dfrac{3(x+1)^2}{4}$

B. $y = 4 + \dfrac{3(x+1)^2}{4}$

C. $y = -4 + \dfrac{3(x+1)^2}{4}$

D. $y = 4 + \dfrac{3(x-1)^2}{4}$

E. $y = -4 + \dfrac{3(x-1)^2}{4}$

Question 10 (1 mark)

The transformation $T:R^2 \to R^2$, that maps the curve with equation $y = x^3$ to the curve with equation $y = (4x - 6)^3 + 1$ could, have which of the following rules.

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -6 \\ -1 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -6 \\ -1 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

Question 11 (1 mark)

A transformation $T : R^2 \rightarrow R^2$ that maps the curve with equation $y = \cos(x)$ onto the curve with equation $y = 2\cos(3x)$ is given by:

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Question 12 (2 marks)

Compare the graph of the equation $y =$ to the graph of the equation $y = 6\{$  $-4\} - 7$.

Which transformations have taken place?

Question 13 (8 marks)

Answer the following

a. The graph of $y = f(x)$ is translated h units to the left and k units upwards. What is the new equation? **(1 mark)**

b. The new equation is dilated by a factor of a parallel to the x-axis and a factor of b parallel to the y-axis. What is the new equation? **(1 mark)**

a. Had the dilations in part **b**. been completed first, *before* the translations in part **a.**, what would be the final equation? Are the final equations the same? **(2 marks)**

b. Take the equation $7y = -\frac{3}{4}(x+3)^3 + 6$. Which transformations and in which order would be required to result in a final equation $x = y^3$? **(4 marks)**

Answers and marking guide

11.2 Translations

Question 1

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

$$(x', y') = (x + a, \ y + b)$$

$$(x', y') = (-3 + 5, 2 - 3)$$

$$= (2, -1)$$

Question 2

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

The image equation for the two coordinates is:

$$x' = x + 3$$
$$y' = y - 1$$
$$x = x' - 3$$
$$y = y' + 1$$
$$y = x - 5$$
$$y' + 1 = (x' - 3) - 5$$
$$y' = x' - 9$$

Question 3

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$x' = x + 2$

$y' = y - 1$ **[1 mark]**

$x = x' - 2$

$y = y' + 1$

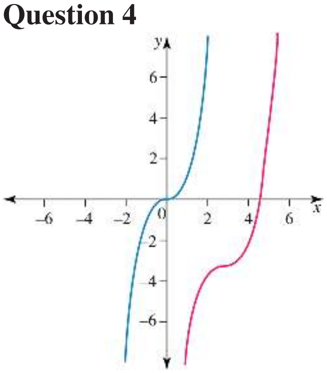
$y = x^2 + 1$

$y' + 1 = (x' - 2)^2 + 1$ **[1 mark]**

$y' = (x' - 2)^2$

$\therefore$ the image equations is $y = (x - 2)^2$. **[1 mark]**

Question 4



Question 5

$(a, \quad b) \rightarrow (-2a, -3b) \Rightarrow x' = x - 3a, \quad y' = y - 4b$

Question 6

$x' = x + p,\ y' = y + q$

$\therefore$ The image point is $(a + p,\ b + q)$. **[1 mark]**

Question 7

$x' = x + p, \quad y' + q$

$x = x' - p, \quad y = y' - q$ **[1 mark]**

$y' - q = f(x' - p)$

$y - q = f(x - p)$

$y = f(x - p) + q$ **[1 mark]**

Question 8

$y = (x - 3)^2 - 7$

$y + 7 = (x - 3)^2$ **[1 mark]**

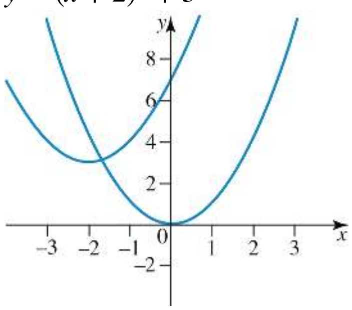
$y - k = (x - h)^2$

This is a translation of $+h$ horizontally and $+k$ vertically.

i.e., $T_{7, -3}$ **[1 mark]**

Question 9

$y = (x + 2)^2 + 3$



Question 10

The coordinates of the image point are:

$x' = x - p,\ y' = y - q$

The image point is $(-a - p,\ b - q)$.

11.3 Reflections and rotations

Question 1

The coordinates of the point (–5, –2) after a reflection in the y-axis are (5, –2)

Question 2

Anticlockwise rotation is represented by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Here the angle is 60°, giving $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

Question 3

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad [1\text{ mark}]$$

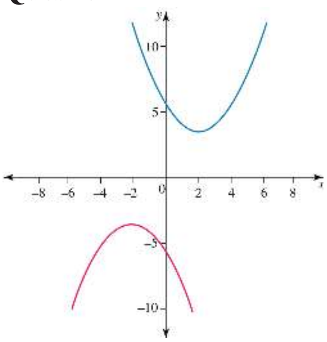
$$= \begin{bmatrix} y \\ -x \end{bmatrix}$$

$x = -y', y = x'$ [1 mark]

$2y - 3x = 1$

$2x' + 3y' = 1 \Rightarrow 2x + 3y = 1$ [1 mark]

Question 4



Question 5

$x' = -x,\ y' = y$

Image is $(-a,\ b)$ **[1 mark]**

$x'' = x',\ y'' = -y'$

Final point $(-a,\ -b)$ **[1 mark]**

Question 6

$x' = -x, \quad y' = y$

$x = -x', \quad y = y'$ **[1 mark]**

$y' = -[x']^3 + 3(x') + 1$ **[1 mark]**

$y = -x^3 + 3x + 1$ **[1 mark]**

Question 7

$x' = x, \quad y' = -y$

$x = x', \quad y = -y'$ **[1 mark]**

$-y' = [x']^2 + (x') + 1$ **[1 mark]**

$y = -x^2 - x - 1$ **[1 mark]**

Question 8

Substitute $(-x)$ for x.

$g(-x) = (-x)^2 + (-x)$

$\quad = x^2 - x,\ [0,\ \infty]$

Question 9

If $y = x^3 - x^2$, when reflected in the x-axis,
the graph becomes $y = x^3 + x^2$, and if $y = -x^3 - x^2$,
is reflected in the y-axis, the graph becomes $y = x^3 + x^2$.

Question 10

On reflection, y becomes $-x$ and x becomes $-y$

Question 11

$y' = -y$; negative sign is placed in front of the equation. **[1 mark]**

$$y' = -\left(x^2 - 5x + 6\right)$$
$$= -x^2 + 5x - 6$$ **[1 mark]**

Question 12

y':x is changed to negative x. **[1 mark]**

$$y' = 2(-x)^2 - 3(-x) - 7$$
$$= 2x^2 + 3x - 7$$ **[1 mark]**

11.4 Dilations

Question 1

a.
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \times -3 + 0 \times 1 \\ 0 \times -3 + 3 \times 1 \end{bmatrix}$$
$$= \begin{bmatrix} -12 \\ 3 \end{bmatrix}$$
$\therefore (-3, 1) \to (-12, 3)$ **[1 mark]**

b. A dilation of factor 4 from the y-axis (parallel to the x-axis) [1 mark] and a dilation of factor 3 from the x-axis (parallel to the y-axis). [1 mark]

Question 2

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ **[1 mark]**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ 3y \end{bmatrix}$$
$$x' = x$$
$$y' = 3y$$
$\therefore y = \dfrac{y'}{3}$ **[1 mark]**
$$y = \frac{1}{x-2}$$
$$\frac{y'}{3} = \frac{1}{x'-2}$$
$$y' = \frac{3}{x'-2}$$
$\therefore$ the image equation is $y = \dfrac{3}{x-2}$. **[1 mark]**

Question 3

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times -6 + 0 \times 2 \\ 0 \times -6 + 2 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$
$\therefore (-6, 2) \to (-6, 4)$

Question 4

a dilation parallel to the y-axis by a scale factor of 4.

Question 5

a dilation parallel to the y-axis by a scale factor of 8.

Question 6

$x' = pa, \quad y' = b$ **[1 mark]**

The image point is $(pa, \ b)$. **[1 mark]**

Question 7

$x' = x, \quad y' = 9y$

$x = x', \quad y = \dfrac{y'}{9}$ **[1 mark]**

$\dfrac{y'}{9} = (x')^2 + x' + 1$ **[1 mark]**

$y = 9(x^2 + x + 1)$ **[1 mark]**

Question 8

a. $y = \dfrac{x^2}{16}$

$y = \left(\dfrac{x}{4}\right)^2$. This is a dilation of 4 parallel to the $x-$ axis. **[1 mark]**

b. $y = \dfrac{x^2}{16}$

$\dfrac{y}{\frac{1}{16}} = x^2$. This is a dilation of $\dfrac{1}{16}$ parallel to the $y-$ axis. **[1 mark]**

Remember, in equations, dilations divide.

Question 9

$y = f(3x)$ becomes $\dfrac{Y}{5} = f\left(\dfrac{3x}{6}\right) \Rightarrow y = 5f\left(\dfrac{x}{2}\right)$

Question 10

a dilation from the y-axis by a scale factor of 2.

$y = \left(\dfrac{x}{2}\right)^2 = \dfrac{x^2}{4}$

11.5 Combinations of transformations

Question 1

The matrix D_{k1k2} is a dilation parallel to both the x- and y-axes (k_1 and k_2 are the dilation factors).

Question 2

Let $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

First transformation:

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **[1 mark]**

Second transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{[1 mark]}$$

Third transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 8 \end{bmatrix} \quad \textbf{[1 mark]}$$

$\therefore (-1, 3) \rightarrow (-2, 3)$

Question 3

$$\text{A}: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{B}: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\text{C}: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

$$\text{A}: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$\therefore$ A′ $(0, 0)$, B′ $(0, -2)$, C′ $(-3, -2)$, D′ $(-3, 0)$ **[1 mark]**

$\therefore$ Original rectangle is reflected in both the axes. **[1 mark]**

$\therefore$ The area is unchanged. **[1 mark]**

Question 4

$(a, b) \rightarrow (pa, b)$ **[1 mark]**

$(pa, b) \rightarrow (pa, qb)$ **[1 mark]**

$(pa, qb) \rightarrow (-pa, qb)$ **[1 mark]**

$(-pa, qb) \rightarrow (-pa + m, qb + n)$ **[1 mark]**

$\therefore$ The final image point is $(-pa + m,\ qb + n)$.

Question 5

$$y = x^5 \rightarrow y = \left(\frac{x}{p}\right)^5 \quad \textbf{[1 mark]}$$

$$y = \left(\frac{x}{p}\right)^5 \rightarrow \frac{y}{q} = \left(\frac{x}{p}\right)^5 \Rightarrow y = q\left(\frac{x}{p}\right)^5 \quad \textbf{[1 mark]}$$

$$y = q\left(\frac{x}{p}\right)^5 \rightarrow y = q\left(\frac{(-x)}{p}\right)^5 \Rightarrow y = -q\left(\frac{x}{p}\right)^5 \quad \textbf{[1 mark]}$$

$$y = -q\left(\frac{x}{p}\right)^5 \to y - n = -q\left(\frac{(x-m)}{p}\right)^5 + y = -q\left(\frac{(x-m)}{p}\right)^5 + n \quad \textbf{[1 mark]}$$

$$\therefore \text{The final equation } y = -q\left(\frac{(x-m)}{p}\right)^5 + n. \quad \textbf{[1 mark]}$$

Question 6

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Question 7

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} a+m \\ b+n \end{bmatrix} \quad \textbf{[1 mark]}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a+m \\ b+n \end{bmatrix} = \begin{bmatrix} a+m \\ -(b+n) \end{bmatrix} \quad \textbf{[1 mark]}$$

$\therefore$ The final image point is $[a+m, \ -(b+n)]$. **[1 mark]**

Question 8

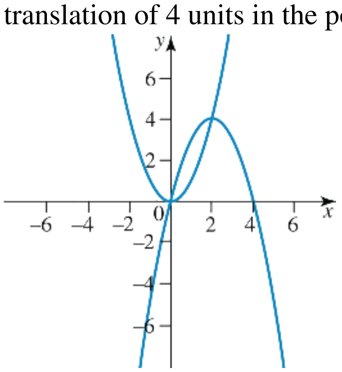
$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} pa \\ b \end{bmatrix} \quad \textbf{[1 mark]}$$

$$\begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}\begin{bmatrix} pa \\ b \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} pa-b \\ pa+b \end{bmatrix} \quad \textbf{[1 mark]}$$

$\therefore$ The final image point is $\dfrac{\sqrt{2}}{2}(pa-b)$, $\dfrac{\sqrt{2}}{2}(pa+b)$. **[1 mark]**

Question 9

A reflection in the x-axis, a translation of 2 units in the positive direction of the x-axis, followed by a translation of 4 units in the positive direction of the y-axis.



Question 10

The graph is $y = 2\cos(2x)$ which has an amplitude of 2 and a period of $T = \dfrac{2\pi}{2} = \pi$

11.6 Review

Question 1

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -a-5 \\ b+4 \end{bmatrix} = \begin{bmatrix} -(a+5) \\ b+4 \end{bmatrix}$$

Question 2

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p \\ -q \end{bmatrix}$$

$(x-p)^2 + (y+q)^2 = 1$

$= (x-p+p)^2 + (y+q-q)^2 = 1$

$= x^2 + y^2 = 1$

Question 3

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Question 4

Reflection in the y-axis changes the sign of the x coordinate.

Point is now $(-a, b)$.

Dilation from the y-axis multiplies the x coordinate by the given factor, 3. Point is now $(-3a, b)$.

Question 5

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

$x' = -x, \quad y' = y$

$y = x'$

$y' = (-x')^2 = (x')^2$

Dropping dashes

$y = x^2$

Question 6

Dilation of factor 0.5 from the y-axis

Horizontal translation of 2

Vertical translation of 3

Question 7

$(2, -3)$ is reflected in the line $y = x$ the point then becomes $(-3, 2)$, by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$(-3, 2)$ is dilated parallel to the y-axis by a scale factor of 2, so that it becomes $(-3, 4)$, and the matrix is $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ therefore $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Question 8

a.
$y - k = f(x-h)$

$y = f(x-h) + k$ **[1 mark]**

b.
$\dfrac{y}{b} = f\left(\dfrac{x}{a} - h\right) + k$

$y = bf\left(\dfrac{x - ah}{a}\right) + bk$ **[1 mark]**

$\dfrac{y}{b} = f\left(\dfrac{x}{a}\right)$ **[1 mark]**

c.
$\dfrac{y-k}{b} = f\left(\dfrac{x-h}{a}\right)$

$y = bf\left(\dfrac{x-h}{a}\right) + k$ **[1 mark]**

d. $7y = -\frac{3}{4}(x+3)^3 + 6 \Rightarrow \dfrac{y}{\left(\frac{3}{28}\right)} = -(x+3)^3 + 8$

Step 1: a dilation of factor $\frac{28}{3}$ parallel to the y-axis. **[1 mark]**

$\Rightarrow y = -(x+3)^3 + 8$

Step 2: a translation of 3 units to the right and 8 units downwards. **[1 mark]**

$\Rightarrow y = -x^3$

Step 3: a reflection in the x-axis. **[1 mark]**

$\Rightarrow y = x^3$

Step 4: a reflection in the line $y = x$. **[1 mark]**

$\Rightarrow x = y^3$

Question 9

$x' = 2x - 1 \Rightarrow x = \frac{x'+1}{2}$

$y' = -3y + 4 \Rightarrow y = \frac{4-y'}{3}$

$\frac{4-y'}{3} = \left(\frac{x'+1}{2}\right)^2$

$y' = 4 - \frac{3(x'+1)^2}{4}$

$y = 4 - \frac{3(x+1)^2}{4}$

Question 10

$y = x^3$ into $y = (4x-6)^3 + 1$ or $y - 1 = (4x-6)^3$, then $y = y' - 1$ and $x = 4x' - 6$ becomes

$x' = \frac{x}{4} + \frac{3}{2} = 0.25x + 1.5$ and $y' = y + 1$ in matrix form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$

Question 11

$x' = \frac{x}{3}$

$y' = 2y$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{x}{3} \\ 2y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Question 12

There has been a horizontal translation of +4 and a vertical translation of −7. **[1 mark]**

There has been a dilation factor 6, parallel to the y-axis. **[1 mark]**

Question 13

a. $y - k = f(x - h)$

$y = f(x - h) + k$ **[1 mark]**

b. $\frac{y}{b} = f\left(\frac{x}{a} - h\right) + k$

$y = bf\left(\frac{x - ah}{a}\right) + bk$ **[1 mark]**

c. $\dfrac{y}{b} = f\left(\dfrac{x}{a}\right)$ **[1 mark]**

$\dfrac{y-k}{b} = f\left(\dfrac{x-h}{a}\right)$

$y = bf\left(\dfrac{x-h}{a}\right) + k$ **[1 mark]**

d. $7y = -\dfrac{3}{4}(x+3)^3 + 6 \Rightarrow \dfrac{y}{\left(\dfrac{3}{28}\right)} = -(x+3)^3 + 8$

Step 1: a dilation of factor $\dfrac{28}{3}$ parallel to the y-axis. **[1 mark]**

$\Rightarrow y = -(x+3)^3 + 8$

Step 2: a translation of 3 units to the right and 8 units downwards. **[1 mark]**

$\Rightarrow y = -x^3$

Step 3: a reflection in the x-axis. **[1 mark]**

$\Rightarrow y = x^3$

Step 4: a reflection in the line $y = x$. **[1 mark]**

$\Rightarrow x = y^3$

12 Functions, relations and graphs

Topic	12	Functions, relations and graphs
Subtopic	12.2	The absolute value function

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Question 1 (5 marks)

Given the function $f(x) = 4 - |3 - 2x|$

a. Sketch the graph of $y = f(x)$ stating the range.

(3 marks)

b. Determine $\{x: f(x) < 3\}$.

$x \in \square$

(2 marks)

Question 2 (2 marks)

Solve $|5 - 2x| \geqslant x$.

Answer = ☐

Question 3 (3 marks)

Sketch the graph of $y = |x^2 + 2x - 3|$ and solve $|3 - 2x - x^2| < 5$.

Question 4 (2 marks)
If $|x| = -|x| + 1$, show that $x = \pm\frac{1}{2}$.

Question 5 (1 mark)
Which one of the following is not true about the function
$f: R \to R, f(x) = |ax + 3a|$?
A. The graph of f is continuous for $x \in R$.
B. The graph of f' is continuous for $x \in R$.
C. $f(x) \geq 0$ for all values of x.
D. $f'(x) =$ a for all $x > 0$.
E. None of the above.

Question 6 (4 marks)
The graph of $y = |x - 1|$ is translated by 2 units to the left and then reflected in the x axis. This reflected graph is now subject to a dilation of 3 units parallel to the y axis and 2 units parallel to the x axis. Find the final equation.

Topic	12	Functions, relations and graphs
Subtopic	12.3	Partial fractions

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Question 1 (1 mark)

The partial fraction decomposition of $\frac{x}{(x+2)^3}$ can be expressed A, B, C, where, and are non-zero real numbers, as

A. $\frac{A}{(x+2)} + \frac{B}{(x+2)^3}$

B. $\frac{A}{(x+2)^2} + \frac{B}{(x+2)^3}$

C. $\frac{Ax}{(x+2)^2} + \frac{B}{(x+2)^3}$

D. $\frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$

E. $\frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{Cx}{(x+2)^3}$

Question 2 (1 mark)

The partial fractions decomposition of $\frac{a}{a^2 - x^2}$ is given by

A. $\frac{a}{a-x} + \frac{a}{a+x}$

B. $\frac{1}{a-x} + \frac{1}{a+x}$

C. $\frac{1}{a-x} - \frac{1}{a+x}$

D. $\frac{1}{2(a-x)} + \frac{1}{2(a+x)}$

E. $\frac{1}{2(a+x)} - \frac{1}{2(a-x)}$

Question 3 (1 mark)

The partial fractions decomposition of $\dfrac{x}{x^2\left(x^2+a^2\right)}$ is given by

A. $\dfrac{A}{x}+\dfrac{B}{x^2+a^2}$

B. $\dfrac{A}{x}+\dfrac{Bx}{x^2+a^2}$

C. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+a^2}$

D. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx}{x^2+a^2}$

E. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^2+a^2}$

Topic	12	Functions, relations and graphs
Subtopic	12.4	Reciprocal graphs

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Question 1 (3 marks)

Sketch the graph of $y = \dfrac{1}{x^2 + 2x - 3}$.

Question 2 (4 marks)

Sketch the graphs of each of the following

a. $y = \dfrac{1}{x^2 + 9}$

(2 marks)

b. $y = \dfrac{1}{x^2 + 6x + 9}$

(2 marks)

Question 3 (3 marks)

Given the function $y = \dfrac{1}{x^2 + kx + 16}$ determine the values of k, for which the function has:

a. two straight line vertical asymptotes

$k = \square$ **(1 mark)**

b. one straight line vertical asymptote

$k = \square$ **(1 mark)**

c. no straight line vertical asymptotes.

$k = \square$ **(1 mark)**

Question 4 (2 marks)

Find the equations of the vertical asymptotes of $\dfrac{1}{f(x)}$, where $f(x) = x^2 - 3x - 4$.

Question 5 (2 marks)

The graph of $\dfrac{1}{f(x)}$ has vertical asymptotes of $x = 2,\ x = 0,\ x = -1$. Give a possible equation for $f(x)$.

Question 6 (1 mark)

Comment on the reciprocal graph of $f(x) = x^2 + 1$.

Topic	**12**	**Functions, relations and graphs**
Subtopic	**12.5**	**The reciprocal trigonometric functions**

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Question 1 (1 mark)

Sketch the graph of the function $f: [-3\pi, 3\pi] \rightarrow R,\ f(x) = 2\cot\left(\frac{x}{2}\right)$.

Question 2 (1 mark)

Sketch the graph of the function $f: (-3\pi, 3\pi) \rightarrow R,\ f(x) = 3\ \sec\left(\frac{x}{2}\right)$.

Question 3 (1 mark)

Sketch the graph of the function $f\colon [-3\pi, 3\pi] \to R,\ f(x) = 2\,\text{cosec}\left(\frac{x}{2}\right)$.

Question 4 (1 mark)

The graph of $y = 3\sec\frac{x}{2}$ has a range of:

A. R

B. $\left(-\infty, \frac{-1}{3}\right] \cup \left[\frac{-1}{3}, \infty\right)$

C. $(-\infty, -3] \cup [3, \infty)$

D. $(-\infty, -2] \cup [2, \infty)$

E. $\left(-\infty, \frac{-1}{2}\right] \cup \left[\frac{-1}{2}, \infty\right)$

Question 5 (1 mark)

The graph of $y = \frac{1}{2}\sec(2\pi x)$ has a period of:

A. 1
B. $\frac{1}{2}$
C. π
D. $\frac{1}{2\pi}$
E. $\frac{1}{\pi}$

Question 6 (2 marks)

State the domain and range of $y = 2\sec(2x) + 2$.

Question 7 (1 mark)

The graph of $y = \cot\frac{3x}{2}$:

A. crosses the x-axis at $x = \frac{2\pi}{3}$ and has a vertical asymptote at $x = \frac{\pi}{2}$.
B. crosses the x-axis at $x = \pi$ and has a vertical asymptote at $x = \frac{2\pi}{3}$.
C. does not cross the x-axis and has a vertical asymptote at $x = \frac{2\pi}{3}$.
D. does not cross the x-axis and has a vertical asymptote at $x = \frac{\pi}{2}$.
E. none of the above.

Question 8 (2 marks)

If $\sin(x) = 2$, find the value of $\text{cosec}(x)$.

Question 9 (2 marks)

If $\cot(x) = p$, find the value of $\sec(x)$.

Question 10 (3 marks)

If $\operatorname{cosec}(x) = \sqrt{2},\ \pi \leq x \leq \frac{3\pi}{2}$, find $\sec(x)$.

Question 11 (2 marks)

The graphs of $y = \operatorname{cosec}(bx)$ and $y = \sec(bx)$ intersect at P.
If $0 \leq bx \leq 2\pi$, what are the coordinates of P?

Question 12 (3 marks)

Find the vertical asymptotes of the graph of $y = \sec\left(\frac{\pi}{a}\right)x,\ -a \leq x \leq a$.

Topic	12	Functions, relations and graphs
Subtopic	12.6	Inverse trigonometric functions

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Question 1 (1 mark)

Consider the function, f, defined by $f(x) = \dfrac{1}{\cos^{-1}\left(\dfrac{3-4x}{5}\right)}$. The maximal domain of f is

A. $x \in R\backslash\left\{\dfrac{3}{4}\right\}$

B. $\left(-\dfrac{1}{2}, 2\right)$

C. $\left[-\dfrac{1}{2}, 2\right]$

D. $\left[-\dfrac{1}{2}, 2\right)$

E. $\left(-\dfrac{1}{2}, 2\right]$

Question 2 (6 marks)

Sketch the graph of the following, stating coordinates of any axial intercepts and the coordinates of any endpoints.

a. $y = \sin^{-1}\left(\frac{3x - 2}{4}\right)$

(3 marks)

b. $y = \cos^{-1}\left(\frac{3-2x}{6}\right)$

(3 marks)

Question 3 (3 marks)

Sketch the graph of $y = \frac{4}{\pi}\tan^{-1}(3x)$ stating the equations of any asymptotes.

Topic	12	Functions, relations and graphs
Subtopic	12.7	Circles and ellipses

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Question 1 (4 marks)

Answer the following

a. Determine the equation of the circle which has a domain of $[1, 7]$ and range of $[-2, 4]$.
Express the equation in the form $(x-h)^2 + (y-k)^2 = r^2$.
Equation: ☐ **(2 marks)**

b. Determine the equation of the ellipse which has a domain of $[-4, 6]$ and range of $[-2, 6]$.
Express your equation in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
Equation: ☐ **(2 marks)**

Question 2 (3 marks)

Show that $x^2 - 4x + y^2 + 6y - 12 = 0$ represents a circle, sketch its graph, stating the coordinates of the centre and the domain and range.

Question 3 (3 marks)

Show that $49x^2 + 294x + 25y^2 - 200y - 384 = 0$ represents an ellipse, sketch its graph, stating the coordinates of the centre and the domain and range.

Question 4 (1 mark)

The circle $(x-2)^2 + (y+3)^2 = 25$ can also be represented as

A. $x^2 + 4x + y^2 - 6y + 12 = 0$
B. $x^2 - 4x + y^2 + 6y = 38$
C. $x^2 - 4x + y^2 + 6y - 12 = 0$
D. $x^2 + 4x + y^2 - 6y = 38$
E. $x^2 + 4x - y^2 + 6y + 12 = 0$

Question 5 (1 mark)

The graph of a circle touches the x-axis at $x = -3$ and touches the y-axis at $y = 3$. Its equation is given by

A. $x^2 + 6x + y^2 - 6y = 9$
B. $x^2 + 6x + y^2 - 6y + 9 = 0$
C. $x^2 - 6x + y^2 + 6y + 9 = 0$
D. $x^2 - 6x + y^2 + 6y = 9$
E. $x^2 + y^2 + 3x - 3y = 9$

Question 6 (2 marks)

Find the coordinates of the centre and the radius of the circle $x^2 + y^2 - 4x + 6y + 9 = 0$.

Question 7 (3 marks)

Find the coordinates of the point(s) of intersection of the circle $(x-3)^2 + (y+4)^2 = 16$ and the x-axis. Comment on your answer.

Question 8 (3 marks)

Find the equation of the locus point which is always $\sqrt{2}p$ units from the point $(p,\ p)$.

Question 9 (1 mark)

The equation $p\left(x^2 + 2x\right) + y^2 = 1$ will represent an ellipse if

A. $p > 0$
B. $p > 1$
C. $-1 < p < 0$
D. $p = 1$
E. $p = \pm 1$

Question 10 (1 mark)

Find the equation of an ellipse whose horizontal semi-axis is 1, vertical semi-axis is 2 and whose centre is $(3, -5)$.

Question 11 (2 marks)

Find the exact values of the axial intercepts on the ellipse $\frac{x^2}{400} + \frac{(y+5)^2}{900} = 1$.

Question 12 (2 marks)

For a particular ellipse, the semi-major axis is 3 and the semi-minor axis is 5. The centre of the ellipse is the point $(-2, -1)$. State the equation of the ellipse.

Question 13 (2 marks)

Find the coordinates of the vertices of the ellipse $\frac{(x+4)^2}{4} + \frac{(y-1)^2}{9} = 1$.

Question 14 (1 mark)

An ellipse has a domain of $(-3, 11)$ and a range of $(-7, 1)$. Its equation is given by

A. $\dfrac{(x+4)^2}{49}+\dfrac{(y-3)^2}{16}=1$

B. $\dfrac{(x-7)^2}{49}+\dfrac{(y+4)^2}{16}=1$

C. $\dfrac{(x-4)^2}{49}+\dfrac{(y+3)^2}{16}=1$

D. $\dfrac{(x+3)^2}{16}+\dfrac{(y-4)^2}{49}=1$

E. $\dfrac{(x+3)^2}{49}+\dfrac{(y-4)^2}{16}=1$

Question 15 (1 mark)

The ellipse with the equation $\dfrac{(x-2)^2}{4}+\dfrac{2(y+3)^2}{5}=2$ has a maximum y-value of

A. $3+\sqrt{5}$

B. $5-\sqrt{3}$

C. $-3+\sqrt{5}$

D. $3-\sqrt{5}$

E. $5+\sqrt{3}$

Question 16 (1 mark)

The coordinates of the centre of the circle $x^2+4x+y^2=36$ are

A. $(0,-2)$

B. $(-2,2)$

C. $(2,0)$

D. $(-2,0)$

E. $(2,2)$

Question 17 (1 mark)
In equation $x^2 + px + y^2 = -4$ a circle will not be formed when p is equal to:
A. 4
B. -5
C. 6
D. 5
E. -6

Question 18 (1 mark)
The ellipse with the equation $\frac{(x-2)^2}{4} + \frac{2(y+3)^2}{5} = 2$ has a maximum y-value of
A. $3 + \sqrt{5}$
B. $5 - \sqrt{3}$
C. $-3 + \sqrt{5}$
D. $3 - \sqrt{5}$
E. $5 + \sqrt{3}$

Question 19 (1 mark)
An ellipse has a domain of $(-3, 11)$ and a range of $(-7, 1)$ Its equation is given by
A. $\frac{(x+4)^2}{49} + \frac{(y-3)^2}{16} = 1$
B. $\frac{(x-7)^2}{49} + \frac{(y+4)^2}{16} = 1$
C. $\frac{(x-4)^2}{49} + \frac{(y+3)^2}{16} = 1$
D. $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{49} = 1$
E. $\frac{(x+3)^2}{49} + \frac{(y-4)^2}{16} = 1$

Question 20 (1 mark)

An ellipse has a horizontal semi-axis length of 4 and a vertical semi-axis length of 3.
Given that the centre of the ellipse has coordinates $(-1, 3)$, the ellipse is:

A. $\dfrac{(x+1)^2}{16} - \dfrac{(y-3)^2}{9} = 1$

B. $\dfrac{(x+1)^2}{16} + \dfrac{(y-3)^2}{9} = 1$

C. $\dfrac{(x-1)^2}{16} - \dfrac{(y+3)^2}{9} = 1$

D. $\dfrac{(x-1)^2}{16} + \dfrac{(y+3)^2}{9} = 1$

E. $\dfrac{(x+1)^2}{9} + \dfrac{(y-3)^2}{16} = 1$

Topic	12	Functions, relations and graphs
Subtopic	12.8	Hyperbolas

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Question 1 (3 marks)

Determine the equation of the hyperbola that has asymptotes given by $y = \pm\frac{x}{2}$ and vertex at the points $(\pm 4,\ 0)$.

Express your equation in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation: ☐

Question 2 (4 marks)

A hyperbola has the equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, asymptotes given by $y = \frac{3x}{2}$ and $y = -\frac{3x}{2} - 6$ and a vertex at the point $(0,\ -3)$.

Determine the values of h, k, a and b.

$h =$ ☐, $k =$ ☐, $a =$ ☐, $b =$ ☐

Question 3 (4 marks)

Show that $9x^2 - 54x - 16y^2 - 64y - 127 = 0$ represents a hyperbola, sketch its graph, stating the coordinates of the centre, the domain and range and the equations of any asymptotes.

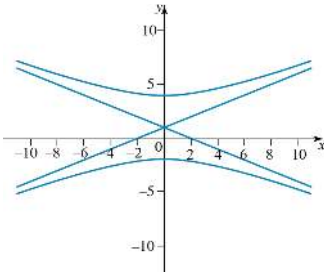
Question 4 (1 mark)

The asymptotes on the graph of $y = -\frac{3}{2(x-1)} + 4$ are:

A. $x = 1,\ x = 4$
B. $x = -1,\ x = 4$
C. $x = -1,\ y = 4$
D. $x = 1,\ y = 4$
E. $x = \frac{1}{2},\ y = 4$

Question 5 (1 mark)

The equation for the hyperbola shown is



A. $\frac{(y+1)^2}{9} - x^2 = 1$

B. $\frac{(y-1)^2}{9} - x^2 = 1$

C. $\frac{(y-1)^2}{9} - \frac{x^2}{36} = 1$

D. $\frac{x^2}{36} - \frac{(y-1)^2}{9} = 1$

E. $\frac{(y-1)^2}{9} - \frac{x^2}{6} = 1$

Question 6 (3 marks)

Find the equations of the asymptotes of the hyperbola $\frac{(x+5)^2}{4} - \frac{(y-3)^2}{9} = 1$.

Question 7 (4 marks)

The equations of the asymptotes of a hyperbola are $y = \pm 2x$ and the x-axial intercepts are $\left(\pm 2\sqrt{2}, 0\right)$. Find the equation of the hyperbola.

Question 8 (1 mark)

The hyperbola $\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}=1$ has asymptotes given by:

A. $y=\pm\frac{bq(x-p)}{a}$

B. $y=\frac{b(x-p)}{a}\pm q$

C. $y=-\frac{b(x-p)}{a}\pm q$

D. $y=\pm\frac{b(x-p)}{a}+q$

E. $y=\pm\frac{b(x-p)}{a}-q$

Question 9 (1 mark)

A hyperbola has asymptotes given by $y=2x$ and $y=-2x-8$. Which one of the following could be its equation?

A. $\frac{(x+2)^2}{4}-\frac{(y+4)^2}{8}=1$

B. $\frac{(x+2)^2}{4}-\frac{(y-4)^2}{8}=1$

C. $\frac{(x+2)^2}{4}-\frac{(y+4)^2}{16}=1$

D. $\frac{(x-2)^2}{4}-\frac{(y-4)^2}{16}=1$

E. $\frac{(x-2)^2}{2}-\frac{(y-4)^2}{4}=1$

Topic	12	Functions, relations and graphs
Subtopic	12.9	Polar coordinates, equations and graphs

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Question 1 (4 marks)

Answer the following

a. Convert each of the following to polar coordinates, expressing θ in radians.

i. $\left(-1, \sqrt{3}\right) = \square$ **(1 mark)**

ii. $(-\sqrt{3}, -1) = \square$ **(1 mark)**

b. Convert each of the following to Cartesian coordinates.

i. $\left[4\sqrt{2}, -\frac{3\pi}{4}\right] = \square$ **(1 mark)**

ii. $\left[8, \frac{2\pi}{3}\right] = \square$ **(1 mark)**

Question 2 (7 marks)

Convert each of the following Cartesian equations into polar equations.

a. $x^2 + y^2 = 49$

$r = \square$, circle **(2 marks)**

b. $5x - 12y = 13$

$r = \square$, line **(2 marks)**

c. $\frac{x^2}{49} + \frac{y^2}{25} = 1$

$r^2 = \square$, ellipse **(3 marks)**

Question 3 (7 marks)

Convert each of the following polar equations into Cartesian equations and sketch the graphs.

a. $r = 7\sin(2\theta)$

(2 marks)

b. $\tan(\theta) + 1 = 0$

(2 marks)

c. $r = \dfrac{3}{1 + \cos(\theta)}$

(3 marks)

Topic 12 | Subtopic 12.9 Polar coordinates, equations and graphs

Question 4 (2 marks)
Convert $(1, 1)$ to polar coordinates.

Question 5 (2 marks)
Convert $[5, 150°]$ to Cartesian coordinates.

Question 6 (3 marks)
The line from the origin to the point (x, y) makes an angle of $\theta°$with the x-axis. If $\tan(\theta°) < 0$, what conclusion can be made about the value of y?

Question 7 (2 marks)
Find the distance parallel to the x-axis between A $[2, 30°]$ and B $[8, 30°]$.

Question 8 (3 marks)
Find the distance between A $[2, 45°]$ and B $[4, 60°]$.

Question 9 (2 marks)
Rewrite $y = x^2$ as a polar equation.

Question 10 (2 marks)
Rewrite $r = 2$ as a Cartesian equation.

Question 11 (3 marks)
Rewrite $y = \sqrt{3}x$ as a polar equation.

Question 12 (3 marks)
Rewrite $(x-3)^2 + (y+4)^2 = 25$ as a polar equation.

Question 13 (4 marks)
Rewrite $r = \dfrac{1}{1+\cos(\theta)}$ as a Cartesian equation.

Question 14 (1 mark)
Written as a Cartesian equation, $r = a$ is:

A. $x + y = a^2$
B. $x + y = a$
C. $x^2 + y^2 = a$
D. $\sqrt{x^2 + y^2} = a^2$
E. $\sqrt{x^2 + y^2} = a$

Topic	12	Functions, relations and graphs
Subtopic	12.10	Parametric equations

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Question 1 (4 marks)

Given the parametric equations $x = \cos(t)$ and $y = \cos(2t)$ for $0 \le t \le 2\pi$, determine the Cartesian equation and sketch the graph, stating the domain and range.

Question 2 (4 marks)

Given the parametric equations $x = 4\ \cos(t)$ and $y = 3\ \sin(t)$ for $0 \le t \le 2\pi$, determine the Cartesian equation and sketch the graph, stating the domain and range.

Question 3 (4 marks)

Given the parametric equations $x = 3\ \sec(t)$ and $y = 4\ \tan(t)$ for $0 \le t \le 2\pi$, determine the Cartesian equation and sketch the graph, stating the domain and range.

Question 4 (1 mark)

Find the Cartesian equation with the parametric form $x = t,\ y = 2t^3 - 1$.

Question 5 (3 marks)

Find the Cartesian equation with the parametric form $x = 2\cos(\theta)\,,\ y = 5\sin(\theta)$.

Question 6 (3 marks)

Find a suitable parametric form for $\left(\frac{x}{4}\right)^4 - \left(\frac{y}{5}\right)^2 = 1$.

Question 7 (2 marks)

Find the Cartesian equation with the parametric form $x = 2t,\ y = \left(1 - t^2\right)^2$.

Question 8 (2 marks)

Find the Cartesian equation with the parametric form $x = \sqrt{(t+2)},\ y = t^3$.

Question 9 (3 marks)

Find the Cartesian equation with the parametric form $x = 2\cos(\theta),\ y = 2\sin(\theta)$. Describe the shape of the curve.

Question 10 (3 marks)

Find the Cartesian equation with the parametric form $x = 3\cos(\theta),\ y = 2\sin(\theta)$. Describe the shape of the curve.

Question 11 (3 marks)

Find the Cartesian equation with the parametric form $x = 3\sec(\theta), y = 5\tan(\theta)$. Describe the shape of the curve.

Question 12 (3 marks)

Find parametric forms for the equation of a circle with centre $(2, -1)$ and radius 3.

Question 13 (3 marks)

Find parametric forms for the equation of an ellipse with centre $(-3, -2)$, semi-major axis 4 and semi-minor axis 2.

Question 14 (3 marks)

Find the parametric forms for the equation of a circle with centre $(-2, 5)$ and radius 4.

Topic	12	Functions, relations and graphs
Subtopic	12.11	Review

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Question 1 (4 marks)

Express $r = \dfrac{1}{1 + \cos(\theta)}$ in Cartesian form.

Question 2 (2 marks)

Determine the Cartesian equation with the parametric form $x = \sqrt{t + 2},\ y = t^3$.

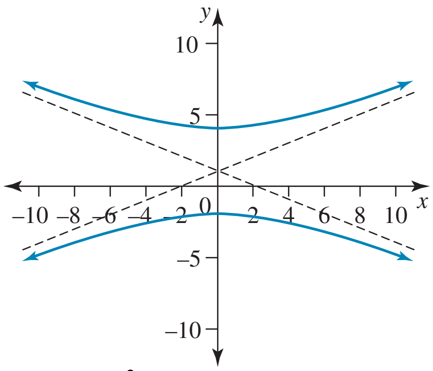
Question 3 (1 mark)

The equation $p\left(x^2 + 2x\right) + y^2 = 1$ will represent an ellipse if

A. $p > 0$
B. $p > 1$
C. $-1 < p < 0$
D. $p = 1$
E. $p = \pm 1$

Question 4 (1 mark)

The equation for the hyperbola shown is



A. $\frac{(y+1)^2}{9} - x^2 = 1$

B. $\frac{(y-1)^2}{9} - x^2 = 1$

C. $\frac{(y-1)^2}{9} - \frac{x^2}{36} = 1$

D. $\frac{x^2}{36} - \frac{(y-1)^2}{9} = 1$

E. $\frac{(y-1)^2}{9} - \frac{x^2}{6} = 1$

Question 5 (4 marks)

a. Show that the graph defined in polar form $r = 2$ and the graph of the parametric equations $x = 2\cos(t)$, $y = 2\sin(t)$ both give the circle $x^2 + y^2 = 4$. **(2 marks)**

b. Show that the graph defined in polar form $r = \frac{2}{1-\cos(\theta)}$ and the graph of the parametric equations $x = \cos(2t)$, $y = 2\sqrt{2}\cos(t)$ both give the parabola $y^2 = 4(x+1)$. **(2 marks)**

Question 6 (1 mark)

The graph of $y = |3 - x|$, is reflected in the x-axis and then in the y-axis, it becomes the graph of

A. $y = |3 - x|$

B. $y = |3 + x|$

C. $y = -|3 - x|$

D. $y = -|3 + x|$

E. $y = 3 - |x|$

Question 7 (1 mark)

What is an asymptote?

Question 8 (1 mark)

Explain why it is possible to have a horizontal asymptote.

Question 9 (10 marks)

a. Answer the following.

i. Rewrite $r = \dfrac{a}{b - p\cos(\theta)}$ as a Cartesian equation. **(4 marks)**

ii. What shape is the curve if $b = p$? **(1 mark)**

b. Answer the following.

i. Rewrite $1 = \dfrac{a\sin(\theta)}{b - p\cos(\theta)}$ as a Cartesian equation. **(4 marks)**

ii. What shape is the curve if $b = p$? **(1 mark)**

Answers and marking guide

12.2 The absolute value function

Question 1

a. $y = f(x) = 4 - |3 - 2x|$

Crosses the x-axis when

$$\begin{aligned} 4 &= 3 - 2x \text{ and } -4 = 3 - 2x \\ 2x &= -1 \qquad\qquad 2x = 7 \\ x &= -\frac{1}{2} \qquad\qquad x = \frac{7}{2} \end{aligned}$$

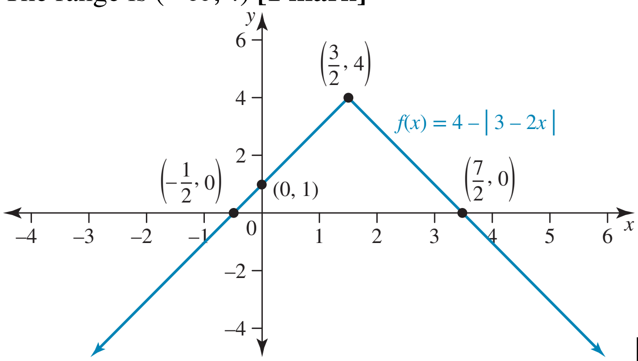
$\left(-\frac{1}{2}, 0\right)$ $\left(\frac{7}{2}, 0\right)$ **[1 mark]**

Crosses the y-axis when $x = 0$: $y = 4 - |3| = 1$

$(0, 1)$

The vertex is at $\left(\frac{3}{2}, 4\right)$

The range is $(-\infty, 4)$ **[1 mark]**



[1 mark]

b. $\{x : f(x) < 3\}$

$$\begin{aligned} &4 - |3 - 2x| < 3 \\ &|3 - 2x| > 1 \\ &3 - 2x < -1 \text{ or } 3 - 2x > 1 \\ &\quad -2x < -4 \qquad -2x > -2 \\ &\qquad x > 2 \qquad\qquad x < 1 \end{aligned}$$ **[1 mark]**

So $x \in (-\infty, 1) \cup (2, \infty)$ **[1 mark]**

Question 2

$$\begin{aligned} &|5 - 2x| \geq x \\ &5 - 2x \geq x \text{ and } 2x - 5 \geq x \\ &\quad 5 \geq 3x \qquad\qquad x \geq 5 \\ &\quad x \leq \frac{5}{3} \end{aligned}$$ **[1 mark]**

So $\left(-\infty, \frac{5}{3}\right] \cup [5, \infty)$ **[1 mark]**

Question 3

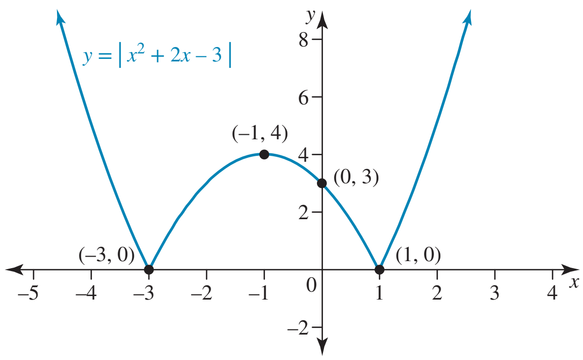
Consider $\begin{aligned} y &= x^2 + 2x - 3 \\ &= (x + 3)(x - 1) \end{aligned}$

Crosses the x-axis at $x = -3, 1$ $(-3, 0)$ $(1, 0)$

Crosses the y-axis at $y = -3$ $(0, -3)$

$$\begin{aligned} y &= x^2 + 2x - 3 \\ &= x^2 + 2x + 1 - 3 - 1 \\ &= (x+1)^2 - 4 \end{aligned}$$

The minimum turning point is at $(-1, -4)$
The graph of $y = |x^2 + 2x - 3|$
crosses the x-axis at $(-3, 0)$ $(-1, 0)$ and the y-axis at $(0, 3)$ and has a maximum point at $(-1, 4)$. **[1 mark]**



[1 mark]

$$\text{Solve} \quad \begin{aligned} x^2 + 2x - 3 &= 5 \\ x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \\ x &= -4, 2 \end{aligned}$$

So $|x^2 + 2x - 3| < 5$

Gives $\begin{aligned} -4 \le x \le 2 \\ x \in [-4, 2] \end{aligned}$ **[1 mark]**

Question 4

$|x| = |x| + 1$

$2|x| = 1$ **[1 mark]**

$|x| = \dfrac{1}{2}$

$\Rightarrow x = \pm\dfrac{1}{2}$ **[1 mark]**

Question 5

The function has a sharp point at $x = -3$. The gradient function is not continuous at $x = -3$. A sharp point is called a *cusp*.

Question 6

$y = |x - 1|$

$y = |(x + 2) - 1|$

$\quad = |x + 1|$ **[1 mark]**

$y = -|x + 1|$ **[1 mark]**

$\dfrac{y}{3} = -\left|\dfrac{x}{2} + 1\right|$ **[1 mark]**

$y = -\dfrac{3}{2}|x + 2|$ **[1 mark]**

12.3 Partial fractions

Question 1

Since we have a repeated factor

$$\frac{x}{(x+2)^3} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

The correct answer is **D**.

Question 2

$$\frac{a}{a^2 - x^2} = \frac{A}{a-x} + \frac{B}{a+x}$$

$$= \frac{A(a+x) + B(a-x)}{(a-x)(a+x)}$$

$$= \frac{x(A-B) + a(A+B)}{a^2 - x^2}$$

(1) $\quad A - B = 0 \quad \Rightarrow A = B$

(2) $\quad a(A+B) = a \quad \Rightarrow A = B = \frac{1}{2}$

$$\frac{a}{a^2 - x^2} = \frac{1}{2(a-x)} + \frac{1}{2(a+x)}$$

The correct answer is **D**.

Question 3

$$\frac{x}{x^2\left(x^2+a^2\right)} = \frac{1}{x\left(x^2+a^2\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+a^2},\ x \neq 0$$

The correct answer is **B**.

12.4 Reciprocal graphs

Question 1

$$y = \frac{1}{x^2+2x-3} = \frac{1}{x^2+2x+1-4}$$

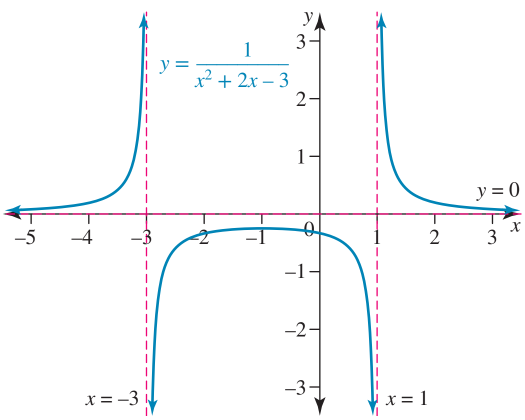
$$= \frac{1}{(x+1)^2-4} = \frac{1}{(x+3)(x-1)}$$ **[1 mark]**

Does not cross x or y-axis

Vertical asymptotes $x = -3, x = 1$

Horizontal asymptote $y = 0$

Maximum turning point at $\left(-1, -\frac{1}{4}\right)$ **[1 mark]**



[1 mark]

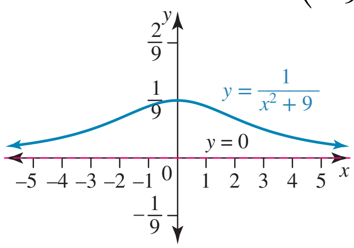
Question 2

a. $y = \dfrac{1}{x^2 + 9}$

No vertical asymptote

$y = 0$ horizontal asymptote

Maximum turning point $\left(0, \dfrac{1}{9}\right)$ **[1 mark]**



[1 mark]

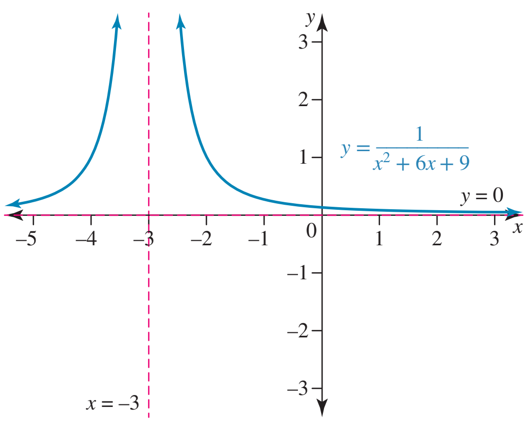
b. $y = \dfrac{1}{x^2 + 6x + 9}$

$= \dfrac{1}{(x+3)^2}$

$x = -3$ is a vertical asymptote

$y = 0$ is a horizontal asymptote

Crosses y-axis $x = 0$ $y = \dfrac{1}{9}$ $\left(0, \dfrac{1}{9}\right)$ **[1 mark]**



[1 mark]

Question 3

a. $y = \dfrac{1}{x^2 + kx + 16}$

$\Delta = b^2 - 4ac$

$a = 1, b = k, c = 16$

$\Delta = k^2 - 4 \times 16$

$= k^2 - 64$

Two straight line vertical asymptotes

$\Delta > 0$

$k^2 > 64$

$|k| > 8$

$(-\infty, -8) \cup (8, \infty)$ **[1 mark]**

b. $y = \dfrac{1}{x^2 + kx + 16}$

$\Delta = b^2 - 4ac$

$a = 1, b = k, c = 16$

$\Delta = k^2 - 4 \times 16$

$= k^2 - 64$

One straight line vertical asymptote

$\Delta = 0$

$k^2 = 64$ **[1 mark]**

$k = \pm 8$

c. $y = \dfrac{1}{x^2 + kx + 16}$

$\Delta = b^2 - 4ac$

$a = 1, b = k, c = 16$

$\Delta = k^2 - 4 \times 16$

$= k^2 - 64$

No straight line vertical asymptotes

$\Delta < 0$

$k^2 < 64$

$|k| < 8$

$(-8, 8)$ **[1 mark]**

Question 4

$x^2 - 3x - 4 = (x - 4)(x + 1)$ **[1 mark]**

$\dfrac{1}{f(x)} = \dfrac{1}{(x - 4)(x + 1)}$

$\therefore$ Vertical asymptotes are $x = 4, x = -1$ **[1 mark]**

Question 5

$\dfrac{1}{f(x)} = \dfrac{1}{kx(x - 2)(x + 1)}$ **[1 mark]**

$f(x) = kx(x - 2)(x + 1)$ **[1 mark]**

Question 6

The graph has no vertical asymptotes. It has the x-axis as a horizontal asymptote. **[1 mark]**

12.5 The reciprocal trigonometric functions

Question 1

$f: [-3\pi, 3\pi] \to R,\ f(x) = 2\cot\left(\dfrac{x}{2}\right)$

$y = 2\cot\left(\dfrac{x}{2}\right) = \dfrac{2\cos\left(\dfrac{x}{2}\right)}{\sin\left(\dfrac{x}{2}\right)}$

Vertical asymptote $\sin\left(\dfrac{x}{2}\right) = 0$

$\dfrac{x}{2} = -\pi, 0, \pi$

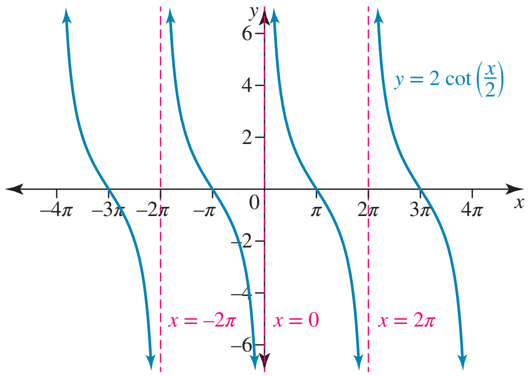
$x = -2\pi, 0, 2\pi$ **[1 mark]**

Crosses x-axis $y = 0 \ \cos\left(\frac{x}{2}\right) = 0$

$\frac{x}{2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$x = \pm\pi, \ \pm3\pi$

$(-3\pi, 0), (-\pi, 0), (\pi, 0), (3\pi, 0)$ **[1 mark]**



[1 mark]

Question 2

$f: (-3\pi, 3\pi) \rightarrow R, \ f(x) = 3\sec\left(\frac{x}{2}\right)$

$y = 3\sec\left(\frac{x}{2}\right) = \frac{3}{\cos\left(\frac{x}{2}\right)}$

Does not cross the x-axis.

Crosses the y-axis when $x = 0$:

$y = \frac{3}{\cos(0)}$
$= 3$

y-intercept at $(0, 3)$.

Vertical intercepts occur when $\cos\left(\frac{x}{2}\right) = 0$:

$\cos\left(\frac{x}{2}\right) = 0$

$\frac{x}{2} = -\frac{\pi}{2}, \ \frac{\pi}{2}$ **[1 mark]**

$x = -\pi, \ \pi$

Turning points occur when $\cos\left(\frac{x}{2}\right) = -1$ or $\cos\left(\frac{x}{2}\right) = 1$:

$\cos\left(\frac{x}{2}\right) = -1$

$\frac{x}{2} = -\pi, \ \pi$

$x = -2\pi, \ 2\pi$

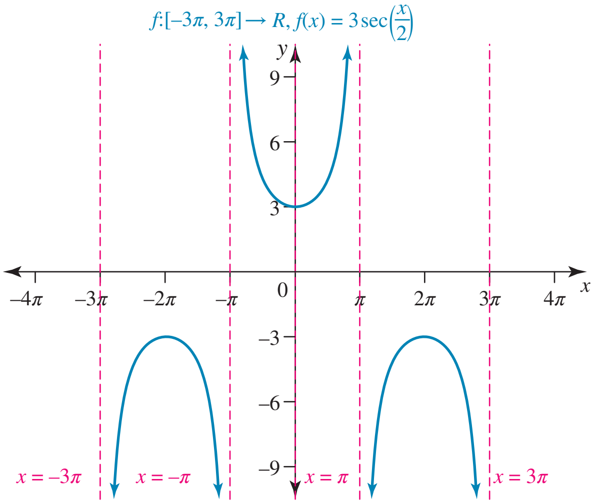
$\cos\left(\frac{x}{2}\right) = 1$

$\frac{x}{2} = 0$

$x = 0$

Turning points:

$(-2\pi, -3), (0, 3), (2\pi, -3)$ **[1 mark]**



[1 mark]

Question 3

f: $[-3\pi, 3\pi] \rightarrow \text{R}, \ \ \text{f}(x) = 2\,\text{cosec}\left(\dfrac{x}{2}\right)$

$y = 2\,\text{cosec}\left(\dfrac{x}{2}\right) = \dfrac{2}{\sin\left(\dfrac{x}{2}\right)}$

Does not cross x-axis or y-axis

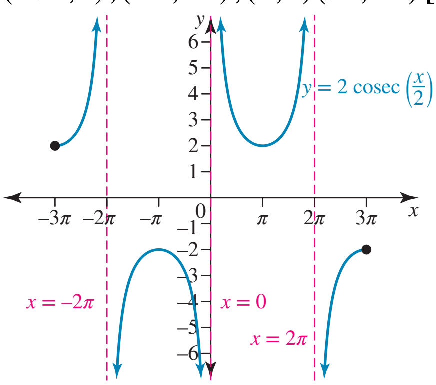
Vertical asymptotes $\sin\left(\dfrac{x}{2}\right) = 0$

$\dfrac{x}{2} = -\pi, 0, \pi$

$x = -2\pi, 0, 2\pi$ **[1 mark]**]

Turning points

$(-3\pi, 2), (-\pi, -2), (\pi, 2)\,(3\pi, -2)$ **[1 mark]**



[1 mark]

Question 4

$y = 3\sec\dfrac{x}{2} = \dfrac{3}{\cos\frac{x}{2}}$

Range: $(-\infty, -3] \cup [3, \infty)$

Question 5

$y = \dfrac{1}{2}\sec 2\pi x = \dfrac{1}{2\cos 2\pi x}$

Period $T = \dfrac{2\pi}{2\pi} = 1$

Question 6

$y = 2\sec 2x + 2 = \dfrac{2}{\cos 2x} + 2$

Vertical asymptote when:

$\cos 2x = 0$

$\Rightarrow 2x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{(2n+1)\pi}{2}$

$\Rightarrow x = \dfrac{(2n+1)\pi}{2}$

Domain: $x \in R \backslash \left\{\dfrac{(2n+1)\pi}{4}\right\}, \ n \in Z$ **[1 mark]**

Range: $(-\infty, 0] \cup (4, -\infty]$ **[1 mark]**

Question 7

$y = \cot \dfrac{3x}{2} = \dfrac{\cos \frac{3x}{2}}{\sin \frac{3x}{2}}$

Crosses the x-axis when:

$\cos \dfrac{3x}{2} = 0$

$\dfrac{3x}{2} = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \Rightarrow x = \dfrac{\pi}{2}, \pi$

Vertical asymptote when:

$\sin \dfrac{3x}{2} = 0, \pi \Rightarrow \dfrac{3x}{2} = \pi \Rightarrow x = \dfrac{2\pi}{3}$

Question 8

$\sec(x) = 2$

$\cos(x) = \dfrac{1}{2}$

$\sin(x) = \dfrac{\pm\sqrt{3}}{2}$ **[1 mark]**

$\therefore \operatorname{cosec}(x) = \dfrac{\pm 2}{\sqrt{3}}$

$= \dfrac{\pm 2\sqrt{3}}{3}$ **[1 mark]**

Question 9

$\cot(x) = p \Rightarrow \tan(x) = \dfrac{1}{p}$

$\sec^2(x) = 1 + \tan^2(x)$

$= 1\left(\dfrac{1}{p}\right)^2$

$= 1 + \dfrac{1}{p^2}$ **[1 mark]**

$\sec(x) = \pm\sqrt{\left(1 + \dfrac{1}{p^2}\right)}$

$= \pm\dfrac{1}{p}\sqrt{(1 + p^2)}$ **[1 mark]**

Question 10

x lies in the third quadrant where $\sec(x)$ is negative. **[1 mark]**

$$\operatorname{cosec}(a) = \sqrt{2} \Rightarrow \sin(a) = \frac{1}{\sqrt{2}} \Rightarrow a = \frac{\pi}{4} \quad \textbf{[1 mark]}$$

$$\begin{aligned} x &= (\pi + a) \\ &= \frac{5\pi}{4} \end{aligned}$$

$$\sec\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}} \textbf{ [1 mark]}$$

Question 11

$y = \operatorname{cosec}(bx)$

$y = \sec(bx)$

$$\frac{1}{\sin(bx)} = \frac{1}{\cos(bx)} \Rightarrow \tan(bx) = 1, \quad \sin(bx) \neq 0, \ \cos(bx) \neq 0$$

$$\tan(bx) = 1 \Rightarrow bx = \frac{\pi}{4}, \frac{5\pi}{4} \textbf{ [1 mark]}$$

$$y = \frac{\pi}{4b}, \frac{5\pi}{4b}$$

$y = \sec(bx)$

$y = \sqrt{2}, -\sqrt{2}$

Points are $\left(\frac{\pi}{4b}, \sqrt{2}\right), \left(\frac{5\pi}{4b}, -\sqrt{2}\right)$ **[1 mark]**

Question 12

Period of $y = \sec\left(\frac{\pi}{a}\right)x$ is $2\pi \div \frac{\pi}{a} = 2a$ **[1 mark]**

Asymptotes where $\cos\left(\frac{\pi}{a}\right)x = 0$ **[1 mark]**

i.e. where $\left(\frac{\pi}{a}\right)x = \pm\frac{\pi}{2}$

$x = \pm\frac{1}{2a}$ **[1 mark]**

12.6 Inverse trigonometric functions

Question 1

Domain of $\cos^{-1}\left(\frac{3-4x}{5}\right)$ is given by:

$$\left|\frac{3-4x}{5}\right| \leq 1$$

$$\begin{aligned} -1 \leq \frac{3-4x}{5} &\leq 1 \\ -5 \leq 3 - 4x &\leq 5 \\ -8 \leq -4x &\leq 2 \\ 8 \geq 4x &\geq -2 \\ 2 \geq x &\geq -\frac{1}{2} \\ -\frac{1}{2} \leq x &\leq 2 \end{aligned}$$

Domain of $\cos^{-1}\left(\frac{3-4x}{5}\right)$ is $\left[-\frac{1}{2}, 2\right]$.

$f(x) = \dfrac{1}{\cos^{-1}\left(\dfrac{3-4x}{5}\right)}$ will be undefined when $\cos^{-1}\left(\dfrac{3-4x}{5}\right) = 0$:

$$\begin{aligned}\cos^{-1}\left(\frac{3-4x}{5}\right) &= 0\\ \frac{3-4x}{5} &= 1\\ 3-4x &= 5\\ -4x &= 2\\ x &= -\frac{1}{2}\end{aligned}$$

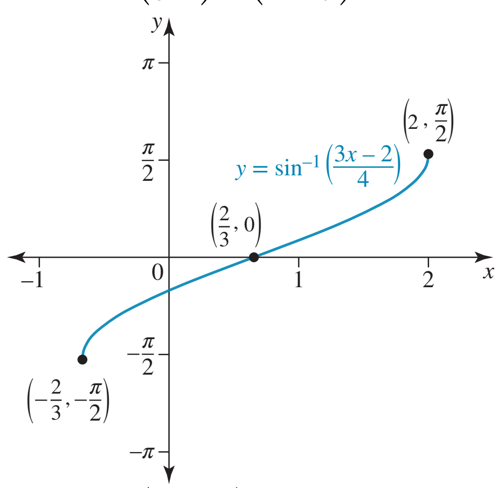
So the maximal domain of $f(x) = \dfrac{1}{\cos^{-1}\left(\dfrac{3-4x}{5}\right)}$ is $\left(-\dfrac{1}{2}, 2\right]$.

The correct answer is **E**.

Question 2

a. Endpoints $\left(-\dfrac{2}{3}, 2\right)$, $\left(2, \dfrac{\pi}{2}\right)$

Intercepts $\left(\dfrac{2}{3}, 0\right)$, $\left(0, -\dfrac{\pi}{6}\right)$



b. $y = \cos^{-1}\left(\dfrac{3-2x}{6}\right)$

Domain: $\left|\dfrac{3-2x}{6}\right| \le 1$

$-6 \le 3x - 2x \le 6$

$-9 \le -2x \le 3$

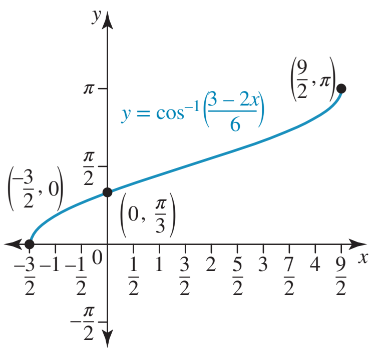
$-3 \le 2x \le 9$

$-\dfrac{3}{2} \le x \le \dfrac{9}{2}$ domain $\left[-\dfrac{3}{2}, \dfrac{9}{2}\right]$ **[1 mark]**

Endpoints $\left(-\dfrac{3}{2}, 0\right)$, $\left(\dfrac{9}{2}, \pi\right)$

Crosses y-axis $x = 0$, $y = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

$\left(0, \dfrac{\pi}{3}\right)$ **[1 mark]**



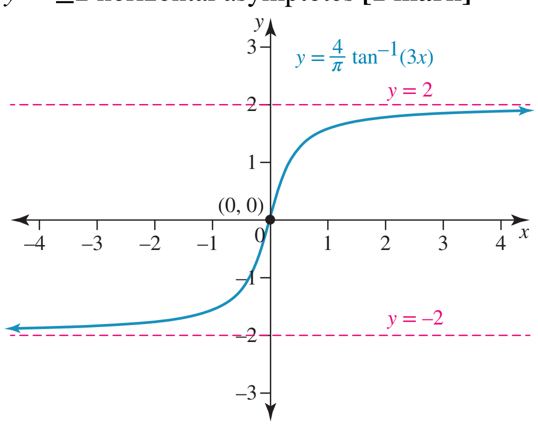
Question 3

$y = \frac{4}{\pi}\tan^{-1}(3x)$

Crosses x and y-axis at origin $(0, 0)$ **[1 mark]**

$$\lim_{x \to \infty}\left(\frac{4}{\pi}\tan^{-1}(3x)\right) = \frac{4}{\pi} \times \frac{\pi}{2} = 2$$

$y = \pm 2$ horizontal asymptotes **[1 mark]**



[1 mark]

12.7 Circles and ellipses

Question 1

a. Circle: domain $[1, 7]$

Distance between is $7 - 1 = 6$ so radius is 3

Centre at $x = \frac{7+1}{2} = 4$

Range $[-2, 4]$

Distance between is $4 - 2 = 6$ so radius is 3 **[1 mark]**

Centre at $y = \frac{4-2}{2} = 1$

Circle $(x-h)^2 + (y-k)^2 = r^2$

$(x-4)^2 + (y-1)^2 = 9$ **[1 mark]**

b. Ellipse: domain $[-4, 6]$

Distance between is $10 - (-4) = 14 = 2a$ so $a = 7$

Midpoint $\frac{10-4}{2} = 3 = h$

Range $[-2, 6]$

Distance between $6 - (-2) = 8 = 2b$, $b = 4$

Midpoint $\frac{6-2}{2} = 2 = k$

Ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ **[1 mark]**

$\frac{(x-3)^2}{49} + \frac{(y-2)^2}{16} = 1$ **[1 mark]**

Question 2

$x^2 - 4x + y^2 + 6y - 12 = 0$

$x^2 - 4x + 4 + y^2 + 6y + 9 = 12 + 4 + 9$

$(x-2)^2 + (y+3)^2 = 25$ circle **[1 mark]**

Centre $(-2, 3)$ radius 5

Vertices $(7, -3)$ $(-3, -3)$

$(2, 2)$ $(2, -8)$

Domain $[-3, 7]$

Range $[-8, 2]$ **[1 mark]**

Crosses x-axis $y = 0$

$x^2 - 4x = 12$

$x^2 - 4x + 4 = 16$

$(x-2)^2 = 16$

$x = 2 \pm 4$

$x = -2, 6$

Crosses y-axis $x = 0$

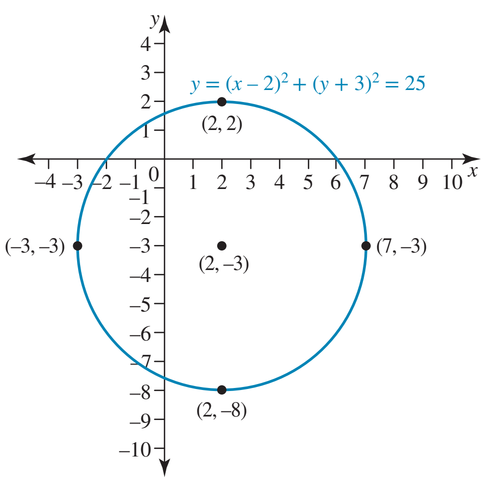
$y^2 + 6y = 12$

$y^2 + 6y + 9 = 12 + 9$

$(y+3)^2 = 21$

$y = -3 \pm \sqrt{21}$

$= -7.58,\ 1.58$



Question 3

$49x^2 + 294x + 25y^2 - 200y - 384 = 0$

$49\left(x^2 + 6x\right) + 25\left(y^2 - 8y\right) = 384$

$49\left(x^2 + 6x + 9\right) + 25\left(y^2 - 8y + 16\right) = 384 + 49 \times 9 + 25 \times 16$

$49(x+3)^2 + 25(y-4)^2 = 1225$

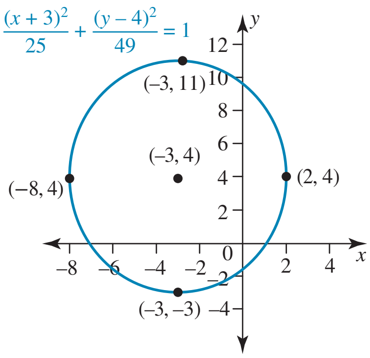
$\frac{(x+3)^2}{25} + \frac{(y-4)^2}{49} = 1$ ellipse **[1 mark]**

Centre $(-3, 4)$
Vertices $(-8, 4)$ $(2, 4)$ $(-3, -3)$ $(-3, 11)$
Domain $[-8, 2]$
Range $[-3, 11]$ **[1 mark]**
Crosses x-axis $y = 0$

$$49\left(x^2 + 6x + 9\right) = 384 + 49 \times 9$$
$$49(x + 3)^2 = 825$$
$$x = -3 \pm \frac{5\sqrt{33}}{7}$$
$$= -7.1,\ 1.1$$

Crosses y-axis $y = 0$

$$25\left(y^2 - 8y + 16\right) = 384 + 25 \times 16$$
$$25(y - 4)^2 = 784$$
$$y = 4 \pm \sqrt{\frac{784}{25}}$$
$$= 9.6,\ -1.6$$



Question 4

$$(x - 2)^2 + (y + 3)^2 = 25$$
$$x^2 - 4x + 4 + y^2 + 6y + 9 = 25$$
$$x^2 - 4x + y^2 - 12 = 0$$

Question 5

The graph of the circle has its centre at $C(-3, 3)$ with a radius of 3. Its equation is given by

$$(x + 3)^2 + (y - 3)^2 = 3^2$$
$$x^2 + 6x + 9 + y^2 - 6y + 9 = 9$$
$$x^2 + 6x + y^2 - 6y + 9 = 0$$

Question 6

Centre: $\left(-\frac{(-4)}{2}, \frac{-6}{2}\right) = (2,\ -3)$ **[1 mark]**

Radius: $\sqrt{2^2 + (-3)^2 - 9} = 2$ **[1 mark]**

Question 7

x-axis, $y = 0$.

$(x - 3)^2 + (0 + 4)^2 = 16$ **[1 mark]**

$$(x - 3)^2 = 0$$
$$x = 3,\ x = 3$$

$\therefore$ there is only one point, $(3, 0)$. **[1 mark]**

The double answer indicates that the x-axis is a tangent to the circle at $(3, 0)$. **[1 mark]**

Question 8

Circle centre (p, p) radius $\sqrt{2}p$. **[1 mark]**

$$(x-p)^2+(y-p)^2=\left(\sqrt{2}p\right)^2$$ **[1 mark]**

$x^2+y^2-2px-2py=0$ **[1 mark]**

Question 9

We require $p>1$, for the equation to represent an ellipse.

Question 10

$$\frac{(x-3)^2}{1}+\frac{(y+5)^2}{4}=1$$

$4(x-3)^2+(y+5)^2=4$ **[1 mark]**

Question 11

$$\frac{x^2}{400}+\frac{(y+5)^2}{900}=1$$

x-axis, $y=0$:

$$\frac{x^2}{400}+\frac{(0+5)^2}{900}=1$$

$$x=\pm\frac{10\sqrt{35}}{3}$$ **[1 mark]**

y-axis, $x=0$:

$$\frac{0}{400}+\frac{(y+5)^2}{900}=1$$

$y=25,-35$ **[1 mark]**

Question 12

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-(-2))^2}{3^2}+\frac{(y-(-1))^2}{5^2}=1$$ **[1 mark]**

$$\frac{(x+2)^2}{9}+\frac{(y+1)^2}{25}=1$$ **[1 mark]**

Question 13

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Horizontal:

$(h\pm a, k)=(-4\pm 2, 1)$

$=(-6, 1), (-2, 1)$ **[1 mark]**

Vertical:

$(h, k\pm b)=(-4, 1\pm 3)$

$=(-4, 4), (-4, -2)$ **[1 mark]**

Question 14

Given that the domain is $(-3, 11)$, the x-value at the midpoint of this interval is $h=4$. This is the x-coordinate of the centre of the ellipse. Also, $a=7$ and the range is $(-7, 1)$. The y-value at the midpoint of this interval is $k=-3$. This is the y-coordinate of the centre of the ellipse. Also, $b=4$, and the standard equation of the ellipse is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

So, $\dfrac{(x-4)^2}{49}+\dfrac{(y+3)^2}{16}=1.$

Question 15

$$\frac{(x-2)^2}{8}+\frac{(y+3)^2}{5}=1$$

Maximum when $x-2=0$:

$$(y+3)^2=5$$
$$y+3=\sqrt{5}$$
$$y=-3+\sqrt{5}$$

Question 16

$$x^2+4x+y^2=36$$
$$x^2+4x+4+y^2=36+4$$
$$(x+2)^2+y^2=40$$

$\therefore$ Centre at $(-2, 0)$

Question 17

$$x^2+px+y^2=-4$$
$$x^2+px+\left(\frac{p}{2}\right)^2+y^2=\left(\frac{p}{2}\right)^2-4$$

$\left(x+\frac{p}{2}\right)^2+y^2=\frac{p^2}{4}-4$; Which forms a circle if

$$\frac{p^2}{4}-4>0$$
$$p^2>16$$

$p>4$ or $p<-4$

Question 18

$$\frac{(x-2)^2}{8}+\frac{(y+3)^2}{5}=1$$

Maximum when $x-2=0$:

$$(y+3)^2=5$$
$$y+3=\sqrt{5}$$
$$y=-3+\sqrt{5}$$

Question 19

Given that the domain is $(-3, 11)$, the x-value at the midpoint of this interval is $h=4$. This is the x-coordinate of the centre of the ellipse. Also, $a=7$ and the range is $(-7, 1)$. The y-value at the midpoint of this interval is $k=-3$. This is the y-coordinate of the centre of the ellipse. Also, $b=4$, and the standard equation of the ellipse is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

So, $\frac{(x-4)^2}{49}+\frac{(y+3)^2}{16}=1.$

Question 20

$$\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}=1$$
$$\frac{(x+1)^2}{16}+\frac{(y-3)^2}{9}=1$$

12.8 Hyperbolas

Question 1

Asymptote $y = \pm\frac{x}{2}$

Intersect at the origin $(0, 0)$

So $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **[1 mark]**

Asymptote $y = \pm\frac{bx}{a} = \pm\frac{x}{2}$

So $\frac{b}{a} = \frac{1}{2}$ **[1 mark]**

Vertex $(\pm 4, 0)$ is $\pm(a, 0)$

So $a = 4 \quad b = 2$

$\frac{x^2}{16} - \frac{y^2}{4} = 1$ **[1 mark]**

Question 2

Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Asymptote $y = \frac{3x}{2}, \quad y = -\frac{3x}{2} - 6$

These intersect $\frac{3x}{2} = -\frac{3x}{2} - 6$

$3x = -6$

$x = -2 \Rightarrow y = -3$

So centre $(-2, 3)$

$h = -2, k = -3$ **[2 marks]**

$\frac{y-k}{b} = \pm\frac{(x-h)}{a}$

$y = \pm\frac{b}{a}(x-h) + k$

$y = -3 \pm \frac{b}{a}(x+2)$

So $b = 3 \quad a = 2$ since the asymptote $y = \frac{3x}{2}$ passes through the origin

$\frac{(x+2)^2}{4} - \frac{(y+3)^2}{9} = 1$ **[2 marks]**

Question 3

$9x^2 - 54x - 16y^2 - 64y - 127 = 0$

$9\left(x^2 - 6x\right) - 16\left(y^2 + 4y\right) = 127$

$9\left(x^2 - 6x + 9\right) - 16\left(y^2 + 4y + 4\right) = 127 + 9 \times 9 - 16 \times 4$

$9(x-3)^2 - 16(y+2)^2 = 144$

$\frac{(x-3)^2}{16} - \frac{(y+2)^2}{9} = 1$ **[1 mark]**

Hyperbola centre $(3, -2)$

Domain $(-\infty, -1] \cup [7, \infty)$, range R **[1 mark]**

Asymptotes $\frac{y+2}{3} = \pm\frac{x-3}{4}$

$y + 2 = \pm\frac{3}{4}(x-3)$

$y = \frac{1}{4}(3x - 17), \; y = \frac{1}{4}(1 - 3x)$ **[1 mark]**

Crosses x-axis $y = 0$

$9\left(x^2 - 6x + 9\right) = 127 + 81$

$9(x-3)^2 = 208$

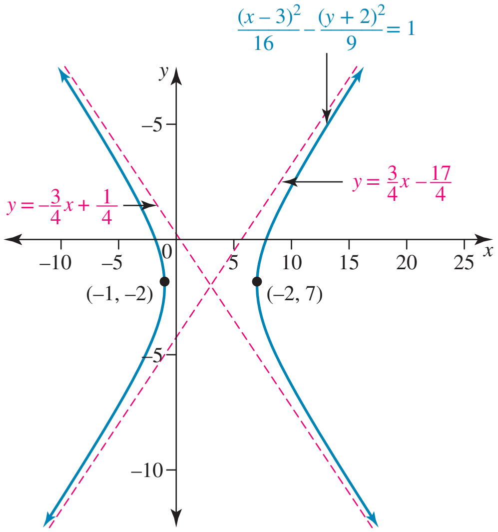
$x = 3 \pm \dfrac{\sqrt{208}}{3}$

$x = -1.81, 7.81$

Crosses y-axis $x = 0$

$-16(y+2)^2 = 144$

Doesn't cross y-axis



Question 4

Vertical asymptote, $x - 1 = 0 \Rightarrow x = 1$

Horizontal asymptote, $\dfrac{-3}{2\,(x-1)} = 0 \Rightarrow y = 4$

Question 5

Hyperbola centre at $(0, 1)$:$h = 1, k = 0$; distance from centre to graph along y-axis is 3. So $a = 3$. The general solution $\dfrac{(y-h)^2}{a^2} - \dfrac{(x-k)^2}{b^2} = 1$ becomes $\dfrac{(y-1)^2}{9} - \dfrac{x^2}{b^2} = 1$. To find b, the asymptotes are $\dfrac{y-1}{3} = \pm\dfrac{x}{b}$ or $y - 1 = \pm\dfrac{3x}{b}$. When $x = 2, y = 0$, so $b = 6$. The equation is $\dfrac{(y-1)^2}{9} - \dfrac{x^2}{36} = 1$.

Question 6

$$y - 3 = \pm\sqrt{\frac{9}{4}}\,(x+5) \quad \textbf{[1 mark]}$$

$$y = 3 \pm \frac{3}{2}\,(x+5)$$

$2y - 3x - 21 = 0$ **[1 mark]**

$2y + 3x + 9 = 0$ **[1 mark]**

Question 7

Centre $(0, 0)$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ **[1 mark]**

x-axis, $y = 0$

$\Rightarrow x = \pm a \Rightarrow a$

$= 2\sqrt{2}$ **[1 mark]**

Assymptotes are $y = \pm\frac{b}{a}x$

$$y = \pm 2x$$
$$\Rightarrow b^2 = 4a^2$$
$$= 32 \text{ [1 mark]}$$
$$\frac{x^2}{8} - \frac{y^2}{32} = 1 \text{ [1 mark]}$$

Question 8

$$\frac{(y-q)}{b} = \frac{\pm(x-p)}{a}$$
$$y = \pm\frac{b(x-p)}{a} + q$$

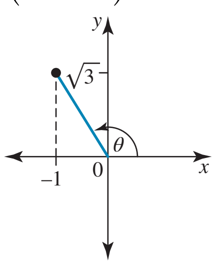
Question 9

The asymptotes $y = 2x$ and $y = -2x - 8$ intersect at $2x = -2x - 8$ or when $4x = -8$ at $x = -2$ and $y = -4$. This is the centre of the hyperbola, so its equation is $\frac{(x+2)^2}{a^2} - \frac{(y+4)^2}{b^2} = 1$. Now the asymptotes are $\frac{y+4}{b} = \pm\left(\frac{x+2}{a}\right)$ or $y = \pm\frac{b}{a}(x+2) - 4$. Since the gradient of the asymptotes is 2, $\frac{b}{a} = 2$ and $\frac{b^2}{a^2} = 4$. Option **C** is the only one which satisfies this.

12.9 Polar coordinates, equations and graphs

Question 1

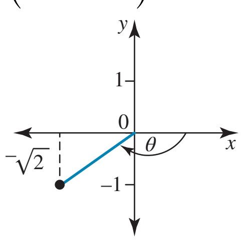
a. i. $\left(-1, \sqrt{3}\right)$



$$x = -1,\ y = \sqrt{3}$$
$$r^2 = x^2 + y^2$$
$$r^2 = 1 + 3$$
$$r = 2$$
$$\theta = \tan^{-1}\left(-\sqrt{3}\right) + \pi$$
$$= -\frac{\pi}{3} + \pi$$
$$= \frac{2\pi}{3}$$

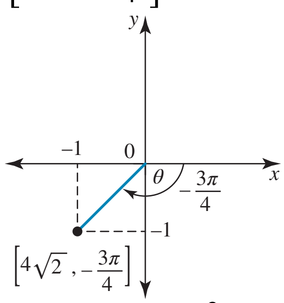
$\left[2, \frac{2\pi}{3}\right]$ **[1 mark]**

ii. $\left(-\sqrt{3}, -1\right)$



$$x = -\sqrt{3},\ y = 1$$
$$r^2 = x^2 + y^2$$
$$= 3 + 1$$
$$r = 2$$
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \pi$$
$$= \frac{\pi}{6} - \pi$$
$$= -\frac{5\pi}{6}$$
$\left[2, -\frac{5\pi}{6}\right]$ **[1 mark]**

b. i. $\left[4\sqrt{2}, -\frac{3\pi}{4}\right]$



$$r = 4\sqrt{2},\ \theta = -\frac{3\pi}{4}$$
$$x = r\cos(\theta)$$
$$= 4\sqrt{2}\cos\left(-\frac{3\pi}{4}\right)$$
$$= 4\sqrt{2}\cos\left(\frac{3\pi}{4}\right)$$
$$= 4\sqrt{2} \times \left(-\frac{\sqrt{2}}{2}\right)$$
$$= -4$$
$$y = r\sin(\theta)$$
$$= 4\sqrt{2}\sin\left(-\frac{3\pi}{4}\right)$$
$$= 4\sqrt{2}\sin\left(\frac{3\pi}{4}\right)$$
$$= -4\sqrt{2} \times \frac{\sqrt{2}}{2}$$
$$= -4$$
$(-4, -4)$ **[1 mark]**

ii. $\left[8, \frac{2\pi}{3}\right]$

$$r = 8, \quad \theta = \frac{2\pi}{3}$$

$$x = r\cos(\theta)$$
$$= 8\cos\left(\frac{2\pi}{3}\right)$$
$$= 8 \times \left(-\frac{1}{2}\right)$$
$$= -4$$

$\left(-4,\ 4\sqrt{3}\right)$ **[1 mark]**

Question 2

a. $x^2 + y^2 = 49$

$x = r\cos(\theta)\quad y = r\sin(\theta)$

$r^2\cos^2(\theta) + r^2\sin^2(\theta) = 49$ **[1 mark]**

$r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) = 49$

$r^2 = 49$

$r = 7$ circle **[1 mark]**

b. $5x - 12y = 13$

$5r\cos(\theta) - 12r\sin(\theta) = 13$ **[1 mark]**

$r(5\cos(\theta) - 12\sin(\theta)) = 13$

$r = \dfrac{13}{5\cos(\theta) - 12\sin(\theta)}$ line **[1 mark]**

c. $\dfrac{x^2}{49} + \dfrac{y^2}{25} = 1$

$\dfrac{r^2\cos^2(\theta)}{49} + \dfrac{r^2\sin^2(\theta)}{25} = 1$ **[1 mark]**

$\dfrac{25r^2\cos^2(\theta) + 49r^2\sin^2(\theta)}{1225} = 1$

$r^2\left(25\cos^2(\theta) + 49\sin^2(\theta)\right) = 1225$ **[1 mark]**

$r^2 = \dfrac{1225}{25\cos^2(\theta) + 49\sin^2(\theta)}$ ellipse **[1 mark]**

Question 3

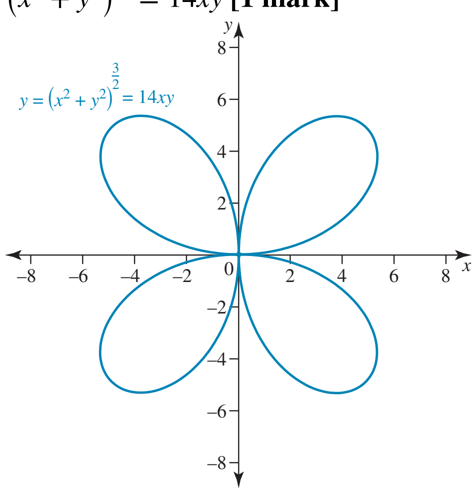
a. $r = 7\sin(2\theta)$

$r = 14\sin(\theta)\cos(\theta)$

$r^3 = 14r^2\sin(\theta)\cos(\theta)$ **[1 mark]**

$r^3 = 14xy$

$\left(x^2 + y^2\right)^{\frac{3}{2}} = 14xy$ **[1 mark]**



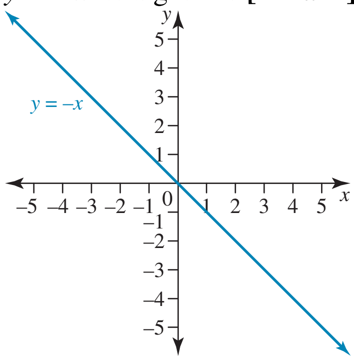
b. $\tan(\theta) + 1 = 0$

$\tan(\theta) = -1$

$\tan(\theta) = -1$

$\frac{y}{x} = -1$

$y = -x$ straight line **[1 mark]**



[1 mark]

c. $r = \frac{3}{1 + \cos(\theta)}$

$r + r\cos(\theta) = 3$ **[1 mark]**

$r + x = 3$

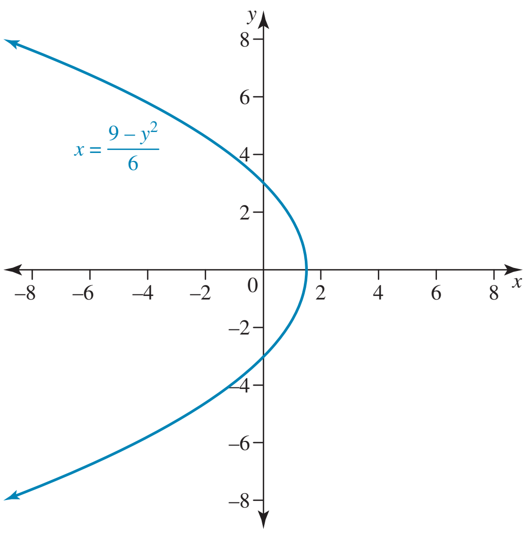
$r = 3 - x$

$r^2 = (3 - x)^2$

$x^2 + y^2 = 9 - 6x + x^2$

$y^2 = 9 - 6x$

$x = \frac{9 - y^2}{6}$ sideways parabola **[1 mark]**



[1 mark]

Question 4

$\tan(\theta) = \frac{1}{1}$

$\theta = 45°$ **[1 mark]**

[(1, 1) is in the first quadrant]

$r = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$ **[1 mark]**

$\therefore (1, 1) = [\sqrt{2}, 45°]$

Question 5

$x = r\cos(\theta), y = r\sin(\theta)$

$x = 5\cos(150°), y = 5\sin(150°)$ **[1 mark]**

$x = -\frac{5\sqrt{3}}{2}, y = \frac{5}{2}$ **[1 mark]**

Question 6

$\tan(\theta) < 0 \Rightarrow \theta$ lies in the 2nd/4th quadrants. **[1 mark]**

$x > 0 \Rightarrow \theta$ lies in the 1st/4th quadrants. **[1 mark]**

y lies in the 4th quadrant $\Rightarrow y$ is negative. **[1 mark]**

Question 7

Actual distance apart is 6 units. **[1 mark]**

Distance parallel to the x-axis is $6\cos(30°) = 3\sqrt{3}$ **[1 mark]**

Question 8

$$\begin{aligned} \text{OA} &= 2,\ \text{OB} = 4, \\ \angle\text{AOB} &= (60 - 45)° \\ &= 15° \text{ [1 mark]} \\ \text{cosine rule AB} &= \sqrt{2^2 + 4^2 - 2 \times 2 \times 4 \times \cos(15°)} \text{ [1 mark]} \\ &= 2.132 \text{ [1 mark]} \end{aligned}$$

Question 9

$$\begin{aligned} r\sin(\theta) &= r^2\cos^2(\theta) \text{ [1 mark]} \\ r &= \frac{\sin(\theta)}{\cos^2(\theta)} \\ &= \tan(\theta)\sec(\theta) \text{ [1 mark]} \end{aligned}$$

Question 10

$$\begin{aligned} \sqrt{x^2 + y^2} &= 2 \text{ [1 mark]} \\ x^2 + y^2 &= 4 \text{ [1 mark]} \end{aligned}$$

Question 11

$$\begin{aligned} r\sin(\theta) &= \sqrt{3}r\cos(\theta) \text{ [1 mark]} \\ \tan(\theta) &= \sqrt{3} \text{ [1 mark]} \\ \theta &= 60° \text{ [1 mark]} \end{aligned}$$

Question 12

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ (r\cos(\theta) - 3)^2 + (r\sin(\theta) + 4)^2 &= 25 \text{ [1 mark]} \\ r^2 - 6r\cos(\theta) + 8r\sin(\theta) &= 0 \text{ [1 mark]} \\ r &= 6\cos(\theta) - 8\sin(\theta) \text{ [1 mark]} \end{aligned}$$

Question 13

$$\begin{aligned} r &= \frac{1}{1 + \cos(\theta)} \\ r + r\cos(\theta) &= 1 \\ r &= 1 - r\cos(\theta) \text{ [1 mark]} \\ \sqrt{x^2 + y^2} &= 1 - x \text{ [1 mark]} \\ x^2 + y^2 &= (1 - x)^2 \\ &= 1 - 2x + x^2 \text{ [1 mark]} \\ y^2 &= 1 - 2x \text{ [1 mark]} \end{aligned}$$

Question 14

$r = \sqrt{x^2 + y^2}$

$\quad = a$

12.10 Parametric equations

Question 1

$x = \cos(t)$

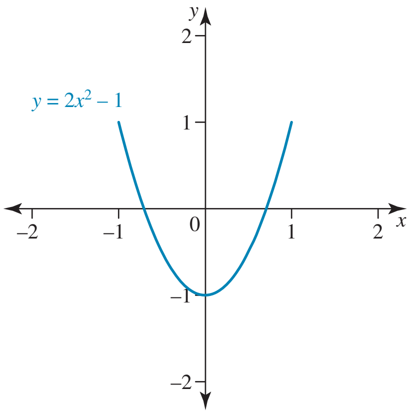
$y = \cos(2t) = 2\cos^2(t) - 1, \quad 0 \le t \le 2\pi$

$\quad = 2x^2 - 1$ part of a parabola **[1 mark]**

So $-1 \le x \le 1,\ -1 \le y \le 1$

Domain $[-1, 1]$ **[1 mark]**

Range $[-1, 1]$ **[1 mark]**



Question 2

$x = 4\cos(t) \quad \cos(t) = \dfrac{x}{4}$

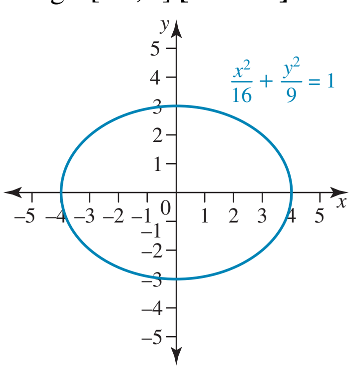
$y = 3\sin(t) \quad \sin(t) = \dfrac{y}{3}$

$\cos^2(t) + \sin^2(t) = 1$ **[1 mark]**

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ ellipse

Domain $[-4, 4]$ **[1 mark]**

Range $[-3, 3]$ **[1 mark]**



Question 3

$x = 3\sec(t) \qquad \sec(t) = \dfrac{x}{3}$

$y = 4\tan(t) \qquad \tan(t) = \dfrac{y}{4}$

$\tan^2(t) + 1 = \sec^2(t)$

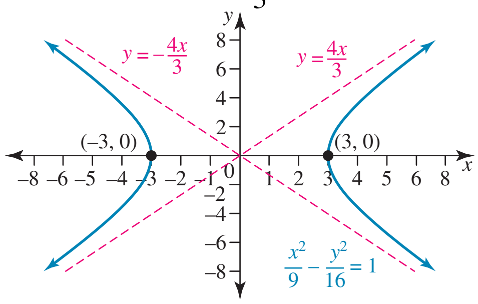
$\sec^2(t) - \tan^2(t) = 1$

$\frac{x^2}{9} - \frac{y^2}{16} = 1$ hyperbole **[1 mark]**

Crosses x -axis at $(\pm 3, 0)$

doesn't cross y-axis

asymptotes at $y = \pm\frac{4x}{3}$ **[1 mark]**



Domain $(-\infty, -3] \cup [3, \infty)$

Range R **[1 mark]**

Question 4

Substituting $x = t$ into $y = 2t^3 - 1$

$y = 2x^3 - 1$ **[1 mark]**

Question 5

$\cos(\theta) = \frac{x}{2}$, $\sin(\theta) = \frac{y}{5}$ **[1 mark]**

Using identity $\cos^2(\theta) + \sin^2(\theta) = 1$ **[1 mark]**

$$\cos^2(\theta) + \sin^2(\theta) = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2$$

$= 1$ **[1 mark]**

$\frac{x^2}{4} + \frac{y^2}{25} = 1$ **[1 mark]**

Question 6

This hyperbola is similar to a circle where $x = r\cos(\theta)$, $y = r\sin(\theta)$ are used. The trigonometric identity best suited here is $\sec^2(\theta) - \tan^2(\theta) = 1$ **[1 mark]**

Substituting $x = 4\sec(\theta)$, $y = 5\tan(\theta)$ **[1 mark]**

gives $\sec^2(\theta) - \tan^2(\theta) = 1$

Hences the parametric form is: $x = 4\sec(\theta)$, $y = 5\tan(\theta)$ **[1 mark]**

Question 7

$x = 2t$

$t = \frac{x}{2}$ **[1 mark]**

$y = (1 - t^2)^2$

$y = \left(1 - \left(\frac{x}{2}\right)^2\right)^2$ **[1 mark]**

Question 8

$x = \sqrt{(t + 2)}$

$x^2 = t + 2$

$t = x^2 - 2$ **[1 mark]**

$y = (x^2 - 2)^3$ **[1 mark]**

Question 9

$$\cos(\theta) = \frac{\pi}{2},\ \sin(\theta) = \frac{y}{2}\ \textbf{[1 mark]}$$

$$\cos^2(\theta) + \sin^2(\theta) = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2$$

$$= 1\ \textbf{[1 mark]}$$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$x^2 + y^2 = 4$$

∴ Circle, centre the origin, radius 2. **[1 mark]**

Question 10

$$\cos(\theta) = \frac{x}{3},\ \sin(\theta) = \frac{y}{2}\ \textbf{[1 mark]}$$

$$\cos^2(\theta) + \sin^2(\theta) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2$$

$$= 1\ \textbf{[1 mark]}$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

∴ Ellipse, centre the origin, semi-major axis 3, semi-major axis 2. **[1 mark]**

Question 11

$$\sec(\theta) = \frac{x}{3},\ \tan(\theta) = \frac{y}{5}\ \textbf{[1 mark]}$$

$$\sec^2(\theta) - \tan^2(\theta) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2$$

$$= 1$$

$$\frac{x^2}{9} - \frac{y^2}{25} = 1\ \textbf{[1 mark]}$$

∴ Hyperbola, centre the origin, asymptotes $y = \pm\frac{5x}{3}$. **[1 mark]**

Question 12

$(x - 2)^2 + (y + 1)^2 = 9$ **[1 mark]**

$x - 2 = 3\cos(\theta), y + 1 = 3\sin(\theta)$ **[1 mark]**

$x = 3\cos(\theta) + 2,\ y = 3\sin(\theta) - 1$ **[1 mark]**

Question 13

$\left(\frac{x + 3}{4}\right)^2 + \left(\frac{y + 2}{2}\right)^2 = 1$ **[1 mark]**

$\frac{x + 3}{4} = \cos(\theta),\ \frac{y + 2}{2} = \sin(\theta)$ **[1 mark]**

$x = 4\cos(\theta) - 3,\ y = 2\sin(\theta) - 2$ **[1 mark]**

Question 14

$(x - (-2))^2 + (y - 5)^2 = 16$

$(x + 2)^2 + (y - 5)^2 = 16$ **[1 mark]**

$x + 2 = 4\cos(\theta),\ y - 5 = 4\sin(\theta)$ **[1 mark]**

$x = 4\cos(\theta) - 2,\ y = 4\sin(\theta) + 5$ **[1 mark]**

12.11 Review

Question 1

$$r = \frac{1}{1 + \cos(\theta)}$$

$$r + r\cos(\theta) = 1$$

$$r = 1 - r\cos(\theta) \quad \textbf{[1 mark]}$$

$$\sqrt{x^2 + y^2} = 1 - x \quad \textbf{[1 mark]}$$

$$x^2 + y^2 = (1 - x)^2$$

$$= 1 - 2x + x^2 \quad \textbf{[1 mark]}$$

$$y^2 = 1 - 2x \quad \textbf{[1 mark]}$$

Question 2

$$x = \sqrt{(t + 2)}$$

$$x^2 = t + 2$$

$$t = x^2 - 2 \quad \textbf{[1 mark]}$$

$$y = \left(x^2 - 2\right)^3 \quad \textbf{[1 mark]}$$

Question 3

We require $p > 0$, for the equation to represent an ellipse.

The correct answer is **A**.

Question 4

Hyperbola centre at $(0, 1) : h = 1, k = 0$; distance from centre to graph along y-axis is $a = 3$. The general solution $\frac{(y - h)^2}{a^2} - \frac{(x - k)^2}{b^2} = 1$ becomes $\frac{(y - 1)^2}{9} - \frac{x^2}{b^2} = 1$. To find b, the asymptotes are $\frac{y - 1}{3} = \pm\frac{x}{b}$ or $y - 1 = \pm\frac{3x}{b}$. When $x = 2, y = 0,$ so $b = 6$. The equation is $\frac{(y - 1)^2}{9} - \frac{x^2}{36} = 1$.

The correct answer is **C**.

Question 5

a. $r = 2$

$$\sqrt{x^2 + y^2} = 2$$

$$x^2 + y^2 = 4 \quad \textbf{[1 mark]}$$

$$x = 2\cos(t), \quad y = 2\sin(t)$$

$$\frac{x}{2} = \cos(t) \quad \frac{y}{2} = \sin(t)$$

$$\cos^2(t) + \sin^2(t) = 1$$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$x^2 + y^2 = 4 \quad \textbf{[1 mark]}$$

b. $r = \frac{2}{1 - \cos(\theta)}$

$$r(1 - \cos(\theta)) = 2$$

$$r - r\cos(\theta) = 2$$

$$r - x = 2$$

$$r = 2 + x$$

$$\sqrt{x^2 + y^2} = 2 + x$$

$$x^2 + y^2 = 4 + 4x + x^2$$

$$y^2 = 4x + 4$$

$$y^2 = 4(x + 1) \quad \textbf{[1 mark]}$$

$x = \cos(2t), \quad y = 2\sqrt{2}\cos(t)$

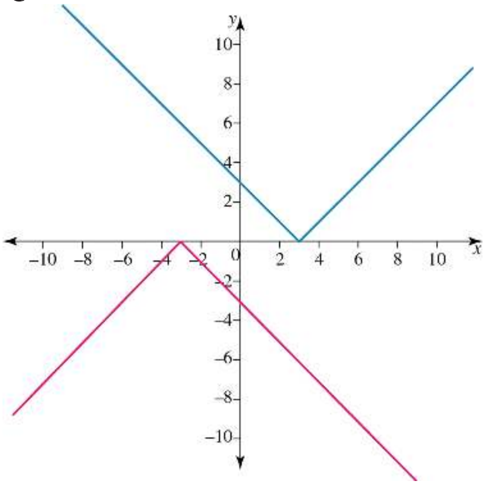
$x = 2\cos^2(t) - 1 \quad \frac{y}{2\sqrt{2}} = \cos(t)$

$x = 2\left(\frac{y}{2\sqrt{2}}\right)^2 - 1$

$x + 1 = \frac{y^2}{4}$

$y^2 = 4(x + 1)$ **[1 mark]**

Question 6



Question 7

An asymptote is the shape a curve approaches, that is gets closer to, as x gets closer and closer to critical values. For example, as x gets closer and closer to 2 for the curve $y = \frac{1}{x-2}$, then the curve gets closer and closer to the shape of the vertical line $x = 2$. The vertical line $x = 2$ is an asymptote. **[1 mark]**

Question 8

An asymptote is the shape a curve approaches as x gets closer and closer to critical values. For example, as x gets larger and approaches infinity for the curve $y = 3 - \frac{1}{x+5}$, then the curve gets closer and closer to the shape of the horizontal line $y = 3$. The horizontal line $y = 3$ is an asymptote. **[1 mark]**

Question 9

a. i. $r = \frac{a}{b - p\cos(\theta)}$

$br - pr\cos(\theta) = a$

$br = a + pr\cos(\theta)$ **[1 mark]**

$b\sqrt{x^2 + y^2} = a + px$ **[1 mark]**

$b^2x^2 + b^2y^2 = (a + px)^2$

$= a^2 + 2apx + p^2x^2$ **[1 mark]**

$y^2 = \frac{1}{b^2}\left[a^2 + 2apx + \left(p^2 - b^2\right)x^2\right]$ **[1 mark]**

ii. If $b = p$

$y^2 = \frac{1}{b^2}\left(a^2 + 2bx\right)$, which is a reflected parabola. **[1 mark]**

b. **i.**
$$1 = \frac{a\sin(\theta)}{b - p\cos(\theta)}$$
$$b - p\cos(\theta) = a\sin(\theta)$$
$$br = ar\sin(\theta) + pr\cos(\theta)$$ **[1 mark]**
$$b\sqrt{x^2 + y^2} = ay + px$$ **[1 mark]**
$$b^2x^2 + b^2y^2 = (ay + px)^2$$
$$= a^2y^2 + 2apxy + p^2x^2$$ **[1 mark]**
$$\left(b^2 - a^2\right)y^2 = \left[2apxy + \left(p^2 - b^2\right)x^2\right]$$ **[1 mark]**

ii. If $b = p$,
$$\left(b^2 - a^2\right)y^2 = 2apxy$$
$y = \dfrac{2ap}{\left(b^2 - a^2\right)}x$, which is a straight line with gradient $\dfrac{2ap}{\left(b^2 - a^2\right)}$. **[1 mark]**

13 Simulation, sampling and sampling distributions

Topic	13	Simulation, sampling and sampling distributions
Subtopic	13.2	Random experiments, events and event spaces

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Question 1 (2 marks)

The three hearts picture cards are drawn at random. List the elements of the event space and list the elements of K, the event of a king being drawn first or a jack being drawn last.

Question 2 (2 marks)

Answer the following

a. A card is drawn from a standard pack of 52 cards. List the elements of Q, the event of a red picture card or an even spade. **(1 mark)**

b. The event in Q could be described as (red card AND picture card) OR (even card AND spade). List the elements of R, the event of (red card OR picture card) AND (even card OR spade). **(1 mark)**

Question 3 (2 marks)

An urn contains four balls numbered 1 to 4. A ball is withdrawn and its number noted. A second ball is then drawn out and its number noted (without replacement of the first ball). List the elements of the event space, and list the elements of R, the event of an odd number on both balls.

Question 4 (1 mark)
Three dice are rolled. List the event space for all three dice showing the same number upwards.

Question 5 (1 mark)
A die is biased. Does the concept of 'equally likely' apply here? Explain your answer.

Question 6 (1 mark)
In how many ways can a five or a Jack be selected in one pick from a standard pack of playing cards?

Question 7 (1 mark)
Kirsty has designed an experiment to get some idea of how many families with three children have either three boys or three girls. Kirsty will toss three coins one thousand times in her experiment. If heads represents girls and tails represents boys, approximately how many families out of 1000 should she expect to find have three girls?

Question 8 (1 mark)
What is the long run proportion for choosing a heart from a standard pack of playing cards?

Topic	**13**	**Simulation, sampling and sampling distributions**
Subtopic	**13.3**	**Simulation**

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Question 1 (4 marks)

The data below show the number of bullseyes scored by 40 dart players after 55 throws each.

1	4	0	3	4	2	1	5	4	2
3	0	4	5	2	1	2	3	2	1
0	2	1	4	3	5	3	2	4	4
0	2	1	0	3	5	4	2	3	1

a. Explain how a calculator can be used to obtain the range of numbers given in the table. **(1 mark)**

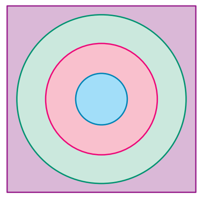
b. Calculate the proportion of players that scored at least 3 bullseyes. **(1 mark)**

c. Conduct 20 trials and obtain another possible value for the proportion of players who scored at least 3 bullseyes. **(1 mark)**

d. Comment on your results. **(1 mark)**

Question 2 (4 marks)

Use a spreadsheet to simulate the following situation. A target consists of 3 concentric circles and darts are randomly thrown at the target. Let the smallest circle have a radius of cm, the next r_2 cm and the largest r_3 cm. They sit on a square board $y \times y$ cm $(y \geq 2r_3)$. Assume that all darts hit the target or the square board outside the target.



Start with some numeric examples, say $r_1 = 5$, $r_2 = 10$, $r_3 = 15$ and $y = 32$.

If you 'hit' a circle, you score points: the inner circle, 5 points; the middle circle, 3 points; the outer circle, 1 point; and you score no points for hitting the board outside the target.

a. Perform enough simulations so that you can predict the 'expected' number of points per throw.

b. Experiment with different values of the radii and board size, and tabulate your results.

c. Experiment with different scoring systems and tabulate your results.

d. Calculate the 'theoretical' probabilities and expected scores for your different values of radii and board size.

Hints on setting up simulation:

1. Generate two random numbers which represent x- and y-coordinates with the target at $(0, 0)$.

2. For each random pair, calculate the distance from the origin and compare it to r_1, $r_2$and r_3.

3. Allocate points appropriately.

Question 3 (4 marks)

A football league has 8 teams. Each team plays all the other teams once. Thus there are 28 games played in all.

a. Simulate a full season's play, assuming that each team has a 50 : 50 chance of winning each game. **(1 mark)**

b. Modify the probabilities so that they are unequal (*hint:* sum of probabilities = 4) and simulate a full season's play. Determine whether the better teams reached the top of the ladder. Discuss your results with other students.
Hint: If Team 1 has a probability of winning of 0.7 and Team 2 has a probability of winning of 0.6, then when they play against each other, the probability of Team 1 winning is $\frac{0.7}{0.7 + 0.6}$. **(3 marks)**

Question 4 (1 mark)
A six-sided die is rolled and a coin is tossed. What proportion of outcomes would you expect to have a number divisible by three and a head?

A. $\frac{1}{8}$

B. $\frac{1}{6}$

C. $\frac{1}{4}$

D. $\frac{1}{3}$

E. $\frac{1}{2}$

Topic	13	Simulation, sampling and sampling distributions
Subtopic	13.4	Discrete random variables

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Question 1 (2 marks)

Casey decides to apply for a job selling mobile phones. She receives a base salary of \$200 per month and \$15 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, x, being sold per month. Determine the expected salary that Casey would receive each month.

x	50	100	150	200	250
$\Pr(X = x)$	0.48	0.32	0.1	0.06	0.04

Expected salary = \$ ☐

Question 2 (2 marks)

The table below represents the probability distribution of the number of accidents per week in a factory.

x	1	2	3	4	5	6	7	8	9
$\Pr(X = x)$	0.02	0.22	0.16	0.14	0.07	0.13	0.03	0.05	0.18

Given that $\mu = 4.36$ and $\sigma = 2.105$ evaluate $\Pr \mu - 2\sigma \leq X \leq \mu + 2\sigma$.

$\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)=$ ☐

Question 3 (3 marks)

The probability distribution of X is given by the formula:

$\Pr(X = x) = \dfrac{x^2}{54}$ where $x = 2, 3, 4, 5$

Determine:

a. the probability distribution of X as a table **(1 mark)**

b. the expected value of X, correct to 4 decimal places

$E(X) =$ ☐ **(1 mark)**

c. the standard deviation of X, correct to 4 decimal places.

$\sigma = \square$ **(1 mark)**

Question 4 (1 mark)

The mean of a population is μ and the standard deviation is σ. The probability of success is p. Samples of size n are taken. The standard deviation of the distribution of sample means is:

A. $\sigma_{\bar{x}} = \dfrac{p(1-p)}{\sqrt{n}}$

B. $\sigma_{\bar{x}} = \sqrt{\dfrac{p}{n}}$

C. $\sigma_{\bar{x}} = \sqrt{\dfrac{\sigma}{n}}$

D. $\sigma_{\bar{x}} = \dfrac{\sqrt{\sigma}}{n(n-1)}$

E. $\sigma_{\bar{x}} = \dfrac{1}{n}\sqrt{p(1-p)}$

Question 5 (2 marks)

The mean of a population is μ and the standard deviation is σ. The standard deviation of the distribution of sample means is 0.25σ. What is the size of each sample used?

Question 6 (1 mark)

In a normally distributed population of the heights of Fyongs, the mean height is p kpls and the standard deviation is r kpls. It is known that a height of q kpls corresponds to s standard deviations below the mean. The relationship between $p,\ q,\ r$ and s is:

A. $s = \dfrac{q-p}{r}$

B. $s = \dfrac{p-q}{r}$

C. $q = \dfrac{p-s}{r}$

D. $q = \dfrac{s-p}{r}$

E. $s = p - qr$

Topic	13	Simulation, sampling and sampling distributions
Subtopic	13.5	Sampling

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Question 1 (1 mark)

A survey is given to 50 students randomly selected from the Year 12 students at your school. The population is

A. the 50 selected student
B. all year 12 students at your school
C. all students at your school
D. all Year 12 students at in the state
E. all people who live within the school catchment zone

Question 2 (1 mark)

A standard deviation is known as a parameter if it is calculated from the

A. mean
B. sample
C. population
D. statistic
E. variance

Question 3 (1 mark)

The different between $\bar{x}$ and μ is that

A. $\bar{x}$ is a parameter and μ is a statistic
B. $\bar{x}$ is calculated from a population and μ is calculated from a sample
C. $\bar{x}$ is calculated from a sample and μ is calculated from a population
D. $\bar{x}$ is the average value and μ describes the spread of the data
E. μ is the average value and $\bar{x}$ describes the spread of the data

Question 4 (1 mark)

A survey at a football match found that of the 40 000 people attending, roughly 20% were female.
A small sample of 80 of the attendees was taken at half-time. How many of these would you expect to be female?

A. 400
B. 16
C. 1788
D. 1789
E. 64

Question 5 (1 mark)

The mean of a population is μ and the standard deviation is σ.
The standard deviation of the distribution of sample means is 0.125σ.
What is the size of each sample used?

A. kh 64
B. 8
C. $5\sqrt{5}$
D. 156
E. 12.5

Topic	13	Simulation, sampling and sampling distributions
Subtopic	13.6	Sampling distribution of sample means

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Question 1 (3 marks)

It was stated that to accurately determine the estimate of the population mean from a set of sample means, the size of each sample must be the same. This is not strictly true. Consider the following example of the sample means $\bar{x}$ medical cases requiring hospitalisation at a group of medical clinics. (Note that the hospitals have used different-sized samples.)

Clinic	$\bar{x}$	Sample size
Abbotsford	3	7
Brunswick	5	13
Carlton	3	10
Dandenong	5	15
Eltham	4	10
Frankston	2	8
Geelong	8	21
Hawthorn	3	11
Inner Melbourne	2	15
N. Melbourne	3	13
S. Melbourne	5	17
E. Melbourne	3	7
W. Melbourne	5	14
St Kilda	3	8

a. Convert the sample means of each hospital to the equivalent sample mean of a sample of size 1. (Give exact answers)

Abbortsford: $\bar{x}_1 = \square$

Brunswick: $\bar{x}_1 = \square$

Carlton: $\bar{x}_1 = \square$

Dandenong: $\bar{x}_1 = \square$

Eltham: $\bar{x}_1 = \square$

Frankston: $\bar{x}_1 = \square$

Geelong: $\bar{x}_1 = \square$

Hawthorn: $\bar{x}_1 = \square$

Inner Melbourne: $\bar{x}_1 = \square$

N. Melbourne: $\bar{x}_1 = \square$

S. Melbourne: $\bar{x}_1 = \square$

E. Melbourne: $\bar{x}_1 = \square$

W. Melbourne $\bar{x}_1 = \square$

St Kilda: $\bar{x}_1 = \square$ **(1 mark)**

b. Calculate an estimate of the population mean if the population is 10 000 cases.
An estimate is ☐, to the nearest whole number. **(1 mark)**

c. State whether a dotplot would be an appropriate way to display the spread of the sample proportions. **(1 mark)**

Question 2 (3 marks)

Consider a population with $N = 600$, $\mu = 180$ and $\sigma^2 = 121$and samples of size 60 are taken from it.

a. Calculate the mean of the distribution of $\bar{x}$.
$\mu_{\bar{x}} =$ ☐ **(1 mark)**

b. Calculate the standard deviation of the distribution of $\bar{x}$.
$\sigma_{\bar{x}} =$ ☐ to 2 decimal places. **(1 mark)**

c. Graph the distribution of $\bar{x}$.

(1 mark)

Question 3 (1 mark)

A population of N items has a mean of M and a variance of V.

P samples, each of size q, are drawn from the population.

The mean and standard deviation of the distribution of sample means are respectively

A. $M, \sqrt{V}$

B. $\frac{M}{q}, \sqrt{\frac{V}{p}}$

C. $M, \sqrt{\frac{V}{q}}$

D. $\frac{M}{p}, \frac{V}{\sqrt{q}}$

E. $M, \frac{V}{\sqrt{p}}$

Question 4 (1 mark)

A survey at a football match found that of the 60 000 people attending, roughly 30% were female.

A small sample of 60 of the attendees was taken at half-time. How many of these would you expect to be female?

☐ would be expected to be females.

Question 5 (1 mark)

Of the students at Oldenivy University, 60% study English. 20% of those who study English, also study Latin. If students can only study Latin if they study English, what proportion of students at Oldenivy University study Latin?

That proportion is ☐ %

Question 6 (1 mark)

The mean of a population is M and the standard deviation is t. The probability of success is p. Samples of size n are taken.

The standard deviation, σ_x, of the distribution of sample means is:

A. $\sigma_x = \dfrac{p(1-p)}{\sqrt{n}}$

B. $\sigma_x = \sqrt{\dfrac{p}{n}}$

C. $\sigma_x = \dfrac{t}{\sqrt{n}}$

D. $\sigma_x = \dfrac{\sqrt{t}}{n(n-1)}$

E. $\sigma_x = \dfrac{1}{2}\sqrt{p(1-p)}$

Topic	13	Simulation, sampling and sampling distributions
Subtopic	13.7	Review

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Question 1 (1 mark)

Which of the following statements best describes the population, as used in statistics?

A. The population is the total number of people living in a country.
B. The population is the total number of people who will participate in a research study.
C. The population consists of people only.
D. The population is the group about whom conclusions are drawn on the basis of a random sample of that group.
E. The population is a subgroup selected from a larger group of research interest.

Question 2 (2 marks)

Approximately 40% of all Trogfins in the world's population of 1 million Trogfins are tartan with a variance of 250 000. A random sample of 1600 Trogfins is taken. Determine the mean and variance of this sample.

$\bar{x} = \square$

Sample variance $= \square$

Question 3 (2 marks)

Using a calculator simulate the outcomes of a game whereby a coin and a six-sided die are tossed simultaneously.

From your simulated results, calculate the probability of tossing a head and rolling a six. Compare your result to the theoretical result.

Question 4 (3 marks)

Igor is playing a game of Pingo which involves throwing 3 darts at a dart board while blindfolded. The dartboard comprises three sections marked by circles. The inner circle, in black is known as the bull. There is a middle yellow section between the bull and the outer ring which is red. All three sections are of equal area.

Assuming that all three darts hit the target, simulate the throwing of three darts 20 times, and from your results calculate the probability that one dart lands in each of the three coloured areas. Compare this to the theoretical probability of this occurrence.

Question 5 (4 marks)

The Specialist Mathematics students at Perfect High School perform very well on exams, with a mean score of 91% and a standard deviation of 3.5%.

a. Samples of the percentages gained by 35 students are taken. Calculate the mean and standard deviation of the distribution of sample means.

$\mu_{\bar{x}} = \square\%$

$\sigma_{\bar{x}} = \square\%$ to 4 decimal places. **(2 marks)**

b. If the sample sizes were decreased to 15, state the effect this would this have on your previous answers. **(2 marks)**

Question 6 (4 marks)

An estimation of a population mean can be obtained by taking the average of the mean of samples taken from the population. Comment on this statement, pointing out any difficulties.

Answers and marking guide

13.2 Random experiments, events and event spaces

Question 1

$\xi = \{KQJ, KJQ, QKJ, QJK, JKQ, JQK\}$ **[1 mark]**
$K = \{KQJ, KJQ, QKJ\}$ **[1 mark]**

Question 2

a. Red picture cards {2S, 4S, 6S, 8S , 10S}
Even spades {2S, 4S, 6S, 8S , 10S}
Q = {JH, QH, KH, JD, QD, KD, 2S,
4S, 6S, 8S, 10S} **[1 mark]**

b. Red card OR Picture card:
{A, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH,
[QH, KH, AD, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D,
[10D, JD, QD, KDJS, QS, KS, JC, QC, KC} **[1mark]**
Even card OR Spade:
{2H, 4H, 6H, 8H, 10H,2 D, 4D, 6D, 8D, 10D,
[AS, 2S, 3S, 4S, 5S, S, 7S, 8S, 9S, 10S, JS, QS, KS,
[2C, 4C, 6C, 8C, 10C}
$R = \{2H, 4H, 6H, 8H, 10H, 2D, 4D, 6D,$
$18D, 10D, JS, QS, KS\}$
$R = \{2H, 4H, 6H, 8H, 10H, 2D, 4D, 6D, 8D, 10D, JS, QS, KS\}$

Question 3

$\xi = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4),$
$(3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}$ **[1 mark]**
$R = \{(1,3), (3,1)\}$ **[1 mark]**

Question 4

$\{(1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)\}$ **[1 mark]**

Question 5

If the die is biased it means that one number is more likely to come up than any other. The outcomes are therefore not ‘equally likely’ to occur. **[1 mark]**

Question 6

There are 4 fives and 4 Jacks so there are 8 ways of choosing either a five or a Jack.
$4 + 4 = 8$

Question 7

Kirsty will find that three heads will appear roughly 125 times.
Three tails will give a similar result. **[1 mark]**
$\left(\frac{1}{8} \text{ of } 1000 = 125\right)$

Question 8

As the number of trials increases, the proportion of hearts should get closer to one-quarter. **[1 mark]**

13.3 Simulation

Question 1

a. Generate 20 numbers from 0 to 5 using randInt function on the calculator, that is, randInt $(0, 5, 20)$. **[1 mark]**

b. $\Pr(\text{at least } 3) = \dfrac{19}{40}$

$= 0.475$ **[1 mark]**

c. Sampe response shown: Using a calculator, these numbers were generated 53223513423110433001

Number	Frequency
0	3
1	4
2	3
3	6
4	2
5	2

$\Pr(\text{at least } 3) = \dfrac{10}{20}$

$= \dfrac{1}{2}$ **[1 mark]**

Note: Answers will vary depending on the result of the simulation. The important thing is that the calculation based on the results is correct.

d. Results compare well, however for greater accuracy more numbers need to be generated. **[1 mark]**

Question 2

Results will vary depending on the parameters chosen. If you use the suggested starting values, $r_1 = 5$, $r_2 = 10$, $r_3 = 15$ and $y = 32$ with scores for inner circle, 5; the middle circle 3, the outer circle 1, and outside the circle, 0. The total square has an area of $32 \times 32 = 1024$.

Region	Area	Probability	Points
Inner circle	$\pi \times 5^2 = 78.54$	$\frac{78.54}{1024} = 0.077$	5
Middle circle	$\pi \times 10^2 - \pi \times 5^2 = 235.62$	$\frac{235.62}{1024} = 0.23$	10
Outer circle	$\pi \times 15^2 - \pi \times 10^2 = 392.7$	$\frac{392.7}{1024} = 0.383$	15
Outside circles	$1024 - \pi \times 15^2 = 317.14$	$\frac{317.14}{1024} = 0.31$	0

The theoretical expected number of points per throw is

$0.077 \times 5 + 0.23 \times 10 + 0.383 \times 15 + 0.31 \times 0 = 8.43$

This question is marked holistically. Award 1 mark for each step (parts **a–d**) so long as the response correctly uses the results of the selected parameters.

Question 3

a. Results will vary depending on the outcome of the simulation. Students should explain their simulation methodology in their response.

Award 1 mark for a simulation that has been performed correctly

b.

Team Number	Probability
1	0.9
2	0.8
3	0.7
4	0.6
5	0.4
6	0.3
7	0.2
8	0.1

This will result in a theoretical probability of winning each match as given below (winning team in left hand column):

	Team 1	Team 2	Team 3	Team 4	Team 5	Team 6	Team 7	Team 8
Team 1	–	$\frac{0.9}{0.9+0.8}$ $= 0.53$	$\frac{0.9}{0.9+0.7}$ $= 0.56$	$\frac{0.9}{0.9+0.6}$ $= 0.6$	$\frac{0.9}{0.9+0.4}$ $= 0.69$	$\frac{0.9}{0.9+0.3}$ $= 0.75$	$\frac{0.9}{0.9+0.2}$ $= 0.82$	$\frac{0.9}{0.9+0.1}$ $= 0.9$
Team 2	$1 - 0.53$ $= 0.47$	–	$\frac{0.8}{0.8+0.7}$ $= 0.53$	$\frac{0.8}{0.8+0.6}$ $= 0.57$	$\frac{0.8}{0.8+0.4}$ $= 0.67$	$\frac{0.8}{0.8+0.3}$ $= 0.73$	$\frac{0.8}{0.8+0.2}$ $= 0.8$	$\frac{0.8}{0.8+0.1}$ $= 0.89$
Team 3	$1 - 0.56$ $= 0.44$	$1 - 0.53 =$ 0.47	–	$\frac{0.7}{0.7+0.6}$ $= 0.54$	$\frac{0.7}{0.7+0.4}$ $= 0.64$	$\frac{0.7}{0.7+0.3}$ $= 0.7$	$\frac{0.7}{0.7+0.2}$ $= 0.78$	$\frac{0.7}{0.7+0.1}$ $= 0.88$
Team 4	$1 - 0.6$ $= 0.4$	$1 - 0.57 =$ 0.43	$1 - 0.54 =$ 0.46	–	$\frac{0.6}{0.6+0.4}$ $= 0.6$	$\frac{0.6}{0.6+0.3}$ $= 0.67$	$\frac{0.6}{0.6+0.2}$ $= 0.75$	$\frac{0.6}{0.6+0.1}$ $= 0.86$
Team 5	$1 - 0.69$ $= 0.31$	$1 - 0.67 =$ 0.33	$1 - 0.64 =$ 0.36	$1 - 0.6 =$ 0.4	–	$\frac{0.4}{0.4+0.3}$ $= 0.57$	$\frac{0.4}{0.4+0.2}$ $= 0.67$	$\frac{0.4}{0.4+0.1}$ $= 0.8$
Team 6	$1 - 0.75$ $= 0.25$	$1 - 0.73 =$ 0.27	$1 - 0.7 =$ 0.3	$1 - 0.67 =$ 0.33	$1 - 0.57 =$ 0.43	–	$\frac{0.3}{0.3+0.2}$ $= 0.6$	$\frac{0.3}{0.3+0.1}$ $= 0.75$
Team 7	$1 - 0.82$ $= 0.18$	$1 - 0.8 =$ 0.2	$1 - 0.78 =$ 0.22	$1 - 0.75 =$ 0.25	$1 - 0.67 =$ 0.33	$1 - 0.6 =$ 0.4	–	$\frac{0.2}{0.2+0.1}$ $= 0.67$
Team 8	$1 - 0.9$ $= 0.1$	$1 - 0.89 =$ 0.11	$1 - 0.88 =$ 0.12	$1 - 0.86 =$ 0.14	$1 - 0.8 =$ 0.2	$1 - 0.75 =$ 0.25	$1 - 0.67 =$ 0.33	–

The games in each round would need to be simulated using these probabilities to decide who the ultimate winner would be. **[1 mark]**

Award 1 mark for selecting a valid set of probabilities.

Award 1 mark for correctly determining the probability of the outcomes of each individual game in the season.

Award 1 mark for a correctly performed simulation based on the probabilities calculated for each match.

Question 4

3 and 6 are divisible by 3. The only possible outcomes which are acceptable are $\{3, \text{H}\}$ and $\{6, \text{H}\}$.

There are 12 possible outcomes, so the answer is $\frac{2}{12}$ or $\frac{1}{6}$.

13.4 Discrete random variables

Question 1

The expected number of phones that Casey sells in a month is:

$$\text{E}(X) = 50 \times 0.48 + 100 \times 0.32 + 150 \times 0.1 + 200 \times 0.06 + 250 \times 0.04$$

$$= 93 \quad \textbf{[1 mark]}$$

$$\text{Expected salary} = \$200 + 93 \times \$15$$

$$= \$1595 \quad \textbf{[1 mark]}$$

Question 2

$\mu - 2\sigma \leq X \leq \mu + 2\sigma$

$= 4.36 - 2 \times 2.105 \leq X \leq 4.36 + 2 \times 2.105$ **[1 mark]**

$= 1 \leq X \leq 8$

$\Pr(1 \leq X \leq 8)$

$= 0.95$ **[1 mark]**

Question 3

a.

x	2	3	4	5
$\text{pr}(X = x)$	$\frac{2}{27}$	$\frac{1}{6}$	$\frac{8}{27}$	$\frac{25}{54}$

b. $\text{E}(X) = 2 \times \frac{2}{27} + 3 \times \frac{1}{6} + 4 \times \frac{8}{27} + 5 \times \frac{25}{54} = 4.1484$ **[1 mark]**

c. $\text{Var}(X) = \text{E}\left(X^2\right) - [\text{E}(X)]^2$

$= \left(2^2 \times \frac{2}{7} + \ldots\ldots\ldots\ldots\ldots\ldots + 5^2 \times \frac{25}{54}\right) - 4.1481\ldots^2$

$= 18.111\ldots - 4.1481\ldots^2 = 0.9039\ldots$

$\sigma = \sqrt{0.9039\ldots} = 0.9509$ **[1 mark]**

Question 4

The standard deviation of the mean is $\sigma_{\bar{x}} = \sqrt{\frac{\sigma}{n}}$

Question 5

$0.25\sigma = \frac{\sigma}{\sqrt{n}}$ **[1 mark]**

$\sqrt{n} = \frac{1}{0.25}$

$n = 16$ **[1 mark]**

Question 6

$s = \frac{q - p}{r}$

13.5 Sampling

Question 1

The population is the Year 12 students at the school as the sample is specifically targeting this group.

Question 2

A parameter is a characteristic of the population, not of a sample.

Question 3

$\bar{x}$ is a sample statistic and is calculated from a sample. μ is a parameter and is calculated from a population.

Question 4

The proportion in the sample should be the same as the proportion in the population.
20% of 80 is 16.

Question 5

$$0.125\sigma = \frac{\sigma}{\sqrt{n}}$$
$$\sqrt{n} = \frac{1}{0.125}$$
$$n = 64$$

13.6 Sampling distribution of sample means

Question 1

a. Convert each sample mean to the equivalent sample mean for a sample of size 1 by dividing the sample mean by the sample size.

Abbotsford: $\bar{x}_1 = \frac{3}{7} \approx 0.43$ **[1 mark]**

Abbortsford: $\bar{x}_1 = \frac{3}{7} \approx 0.43$

Brunswick: $\bar{x}_1 = \frac{5}{13} \approx 0.38$

Carlton: $\bar{x}_1 = \frac{3}{10} = 0.3$

Dandenong: $\bar{x}_1 = \frac{5}{15} = \frac{1}{3} \approx 0.33$

Eltham: $\bar{x}_1 = \frac{4}{10} = 0.4$

Frankston: $\bar{x}_1 = \frac{2}{8} = 0.25$

Geelong: $\bar{x}_1 = \frac{8}{21} \approx 0.38$

Hawthorn: $\bar{x}_1 = \frac{3}{11} \approx 0.27$

Inner Melbourne: $\bar{x}_1 = \frac{2}{15} \approx 0.13$

N. Melbourne: $\bar{x}_1 = \frac{3}{13} \approx 0.23$

S. Melbourne: $\bar{x}_1 = \frac{5}{17} \approx 0.29$

E. Melbourne: $\bar{x}_1 = \frac{3}{7} \approx 0.43$

W. Melbourne $\bar{x}_1 = \frac{5}{14} \approx 0.36$

St Kilda: $\bar{x}_1 = \frac{3}{8} = 0.375$

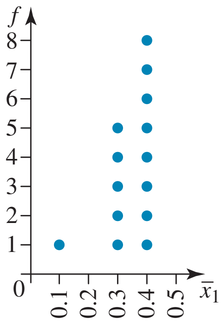
b. $\sum \bar{x}_1 = \frac{3}{7} + \frac{5}{13} + ... + \frac{3}{8}$

$= 4.569134...$

Average $\bar{x}_1 = \frac{4.569134...}{14} = 0.326366...$

An estimate of the population mean if the population is 10000 is 0.326366... × 10000 ≈ 3264 **[1 mark]**

c. Yes. A dotplot is very effective, rounding to one decimal place. **[1 mark]**

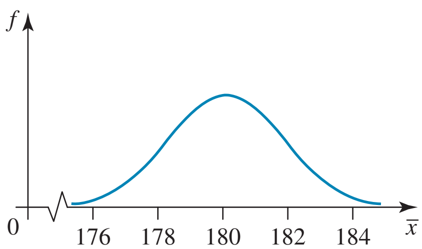


Question 2

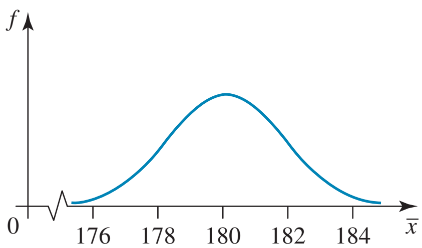
a. $\mu_{\bar{x}} = \mu = 180$ **[1 mark]**

b. $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{11}{\sqrt{60}} = 1.42$

Note: the sample size is used here NOT the population size **[1 mark]**



c.



Question 3

The mean of the distribution of sample means is equal to the mean of the population which is M he standard deviation of the distribution of sample means is equal to the standard deviation of the population divided by the square root of the number of items in each sample (not the number of samples). In this case there are q items in each sample. Mean = M

Standard deviation = $\sqrt{\frac{V}{q}}$

Question 4

Assume the same proportion in the sample as in the population. Proportion in population is 30%

30% × 60 = 18 females. **[1 mark]**

Question 5

The proportion is 20% of 60% = 20% × 60%

= 12% **[1 mark]**

Question 6

$$\sigma_x = \frac{\sigma}{\sqrt{n}}$$
$$= \frac{t}{\sqrt{n}}$$

13.7 Review

Question 1

Population is the totality of everything under discussion. This means that the population is the group about whom conclusions are drawn on the basis of a random sample of that group.

Question 2

$\overline{x} = 40\% \text{of} 1600 = 640$ **[1 mark]**

Sample variance $\frac{\sigma^2}{n} = \frac{250000}{1600} = 156.25$ **[1 mark]**

Question 3

Create a list of 100 random integers between 1 and 2 and a list of 100&nb sp;random integers between 1 and 6 to represent the outcomes of the coin toss and the dice roll. **[1 mark]** Analyse the data to determine the number of times a head was tossed and a 6 was rolled simultaneously. **[1 mark]**

The following screen shows how this can be computed using the TI Nspire CAS calculator:

Note: define t 1 *as the coin toss (1 = Heads) and define* t 2 *as the dice roll (numbered* $1 - 6$.

A head and a 6 occurs when $t2 - t1 = 5$.

Simulated probability is $\frac{7}{100} = 0.07$

Theoretic probability is $\frac{1}{12} \approx 0.0833$

Question 4

Each section has the same area so we can assume that there is the same probability of landing on each.

Create 3 lists of 20 random integers between 1 and 3 to simulate the result of each dart throw. **[1 mark]**

Analyse the data to determine how many times the three darts landed in the three coloured sections. **[1 mark]** Then compare the simulated probability to the

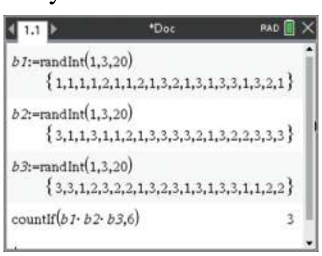
theoretical probability. **[1 mark]**

Then compare the simulated probability to the theoretical probability. **[1 mark]**

The following screen shows how this can be computed using the TI Nspire CAS calculator:

Note: define b1 as the first dart, b2 as the second dart and b3 as the third dart.

They land in each of the 3 different sections when $b1 \times b2 \times b3 = 6$.



Simulated probability is $\frac{3}{20} = 0.15$

Theoretic probability:

The first dart can land in any area, then the second dart must land in either of the other 2 areas, and the third dart must land in the 1 remaining area.

$$\frac{3 \times 2 \times 1}{3 \times 3 \times 3} = \frac{6}{27} = 0.\dot{2}$$

Question 5

a. $\mu_{\bar{x}} = 91\%$

$\sigma_{\bar{x}} = 0.5916\%$

b. If n is reduced, the mean of the sample means is unchanged. **[1 mark]**

The standard deviation of the sample means will increase as we are dividing by a smaller number.

$\mu_{\bar{x}} = \mu = 91\%$

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{3.5}{\sqrt{15}} = 0.9037\%$ **[1 mark]**

Question 6

The samples whose means are to be averaged must be of equal size, otherwise a weighting would have to be given to each. **[1 mark]**

The larger number of samples, the greater the accuracy; that is, taking the average of the means of 100 samples gives a closer measure of the mean of the population than taking the average of the means of two samples. **[1 mark]**

The larger the size of the samples, the greater the accuracy. **[1 mark]**

The result can only ever be an estimate; the actual mean of the population can only be calculated using all of the population data. **[1 mark]**