2 Complex numbers

Topic	2	Complex numbers
Subtopic	2.2	Complex numbers in Cartesian form

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Source: *VCE 2020 Specialist Mathematics Exam 2, Section A, Q5; © VCAA*

Question 1 (1 mark)

Given the complex number $z = a + bi$, where $a \in R \setminus \{0\}$ and $b \in R$, $\dfrac{4z\bar{z}}{(z+\bar{z})^2}$ is equivalent to

A. $1 + \left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$
B. $4\,[\text{Re}(z) \times \text{Im}(z)]$
C. $4\left([\text{Re}(z)]^2 \times [\text{Im}(z)]^2\right)$
D. $4\left[1 + (\text{Re}(z) + \text{Im}(z))^2\right]$
E. $\dfrac{2 \times \text{Im}(z)}{[\text{Re}(z)]^2}$

Source: *VCE 2019 Specialist Mathematics Exam 2, Section A, Q4; © VCAA*

Question 2 (1 mark)

The expression $i^{1!} + i^{2!} + i^{3!} + \ldots + i^{100!}$ is equal to

A. 0
B. 96
C. $95 + i$
D. $94 + 2i$
E. $98 + 2i$

Question 3 (1 mark)

$\sum_{n=1}^{100} ni^n$ is equal to

A. $50 - 50i$
B. 50
C. $-50i$
D. 100
E. 5050

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.6; © VCAA*

Question 4 (1 mark)

Given that $i^n = p$ and $i^2 = -1$, then i^{2n+3} in terms of p is equal to

A. $p^2 - i$
B. $p^2 + i$
C. $-p^2$
D. $-ip^2$
E. ip^2

Question 5 (1 mark)

If $z = x + yi$ where x and y are two non-zero real numbers, then which one of the following is not a real number?

A. $z\bar{z}$
B. $z + \bar{z}$
C. $z - \bar{z}$
D. $\frac{1}{z} + \frac{1}{\bar{z}}$
E. $\frac{1}{z} \times \frac{1}{\bar{z}}$

Question 6 (1 mark)

If $u = 4a + 3b + 12i$, $v = -1 + (3a - 2b)\,i$, and $u = v$, where a and b are real numbers, then

A. $a = 2$ and $b = 3$

B. $a = 2$ and $b = -3$

C. $a = 5$ and $b = -7$

D. $a = 3$ and $b = -4$

E. $a = 4$ and $b = 0$

Question 7 (1 mark)

If $z = -x - yi$, where x and y are non-zero real numbers, which one of the following is a real number?

A. z^{-1}

B. $\frac{1}{\bar{z}}$

C. $\frac{1}{z - \bar{z}}$

D. $\frac{1}{z} - \frac{1}{\bar{z}}$

E. $\frac{1}{z} + \frac{1}{\bar{z}}$

Question 8 (1 mark)

If $(x + yi)^3$ is a real number, find a relationship between x and y.

Question 9 (1 mark)

For any complex number $z = x + yi$, where both x and y are real, then the complex number $w = i^3 z$ is found by

A. reflecting z in the real axis.

B. reflecting z in the imaginary axis.

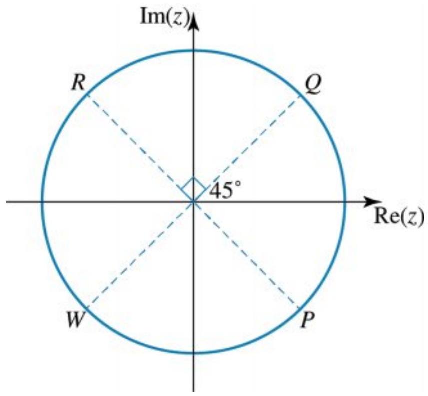
C. rotating z through 90° in a clockwise direction.

D. rotating z through 90° in an anti-clockwise direction.

E. rotating 180°.

Topic 2 | Subtopic 2.2 Complex numbers in Cartesian form

Question 10 (1 mark)



The point W on the Argand diagram represents the complex number w. The points P, Q, R and S all lie on a circle and are all equally spaced around the circle as shown above.

The complex number $i^2\overline{w}$ is best represented by the point

A. W
B. P
C. Q
D. R
E. None of the above

Question 11 (1 mark)

If $z = a + bi$ and $w = c + di$ where a, b, c and d are non-zero real numbers, then which one of the following is a real number?

A. $zw - \overline{z}\,\overline{w}$
B. $z\overline{w} - \overline{z}w$
C. $\frac{z}{\overline{z}} \times \frac{w}{\overline{w}}$
D. $\frac{z}{\overline{z}} + \frac{w}{\overline{w}}$
E. $zw + \overline{z}\,\overline{w}$

Question 12 (1 mark)

If $z = a + bi$ and $w = c + di$ where a, b, c and d are non-zero real numbers, then $\text{Re}(w^2) + \text{Im}(z^2)$ is equal to:

A. $c^2 + b^2$
B. $c^2 - d^2 + 2ab$
C. $c^2 + d^2 - 2ab$
D. $c^2d^2 + (ab)^2$
E. $c^2 - b^2$

Question 13 (1 mark)

If $z = a + bi$ and $w = c + di$, where a, b, c and d are non-zero real numbers, then $\text{Re}(z\bar{z}) + \text{Im}(w\overline{w})$ is equal to

A. $\left(a^2 - b^2\right) - \left(c^2 + b^2\right)$
B. $\left(c^2 - d^2\right) + \left(a^2 - b^2\right)$
C. $a^2 + c^2$
D. $a^2 - b^2$
E. $a^2 - c^2$

Question 14 (1 mark)

If $u = \frac{1}{a} - \frac{1}{b}i$, where a and b are non-zero real numbers, then $a^2b^2u^2$ is equal to

A. $\left(b^2 + a^2\right) - 2a^2b^2i$
B. $\left(b^2 - a^2\right) - 2abi$
C. $\left(b^2 + a^2\right) - 2abi$
D. $b^2 + a^2$
E. $b^2 - a^2$

Question 15 (1 mark)

If $u = a + bi$ and $v = \frac{1}{a} - \frac{1}{b}i$ where a and b are non-zero real numbers, then uv in Cartesian form is equal to

A. 2

B. $2 + \left(\frac{b^2 + a^2}{ab}\right)i$

C. $2 + \left(\frac{b^2 - a^2}{ab}\right)i$

D. $\left(\frac{b^2 - a^2}{ab}\right)i$

E. $\left(\frac{b^2 + a^2}{ab}\right)i$

Question 16 (1 mark)

If $z = 2 - 4i$ and $w = 1 + 3i$, find $\frac{z}{w}$.

A. $2 + \frac{4}{3}i$

B. $-1 - i$

C. $1 - i$

D. $2 - \frac{4}{3}i$

E. $-1 + i$

Question 17 (1 mark)

If $u = \frac{1}{a} + \frac{1}{b}i$ and $v = a - bi$, where a and b are non-zero real numbers, then $\frac{u}{v}$ in Cartesian form is equal to

A. 0

B. $\frac{2}{a^2 + b^2}$

C. $-\frac{i}{ab}$

D. $\frac{i}{ab}$

E. $\frac{abi}{(a^2 + b^2)}$

Topic	2	Complex numbers
Subtopic	2.3	Complex numbers in polar form

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Source: *VCE 2021 Specialist Mathematics Exam 2, Section A, Q.4; © VCAA*

Question 1 (1 mark)

For $z \in C$, if $\text{Im}(z) > 0$, then $\text{Arg}\left(\dfrac{z\bar{z}}{z - \bar{z}}\right)$ is

A. $-\dfrac{\pi}{2}$

B. 0

C. $\dfrac{\pi}{4}$

D. $\dfrac{\pi}{2}$

E. π

Source: *VCE 2021 Specialist Mathematics Exam 2, Section A, Q.6; © VCAA*

Question 2 (1 mark)

If $z \in C$, $z \neq 0$ and $z^2 \in R$, then the possible values of $\arg(z)$ are

A. $\dfrac{k\pi}{2}, k \in Z$

B. $k\pi, k \in Z$

C. $\dfrac{(2k+1)\pi}{2}, k \in Z$

D. $\dfrac{(4k+1)\pi}{2}, k \in Z$

E. $\dfrac{(4k-1)\pi}{2}, k \in Z$

Source: *VCE 2019 Specialist Mathematics Exam 1, Q.7; © VCAA*

Question 3 (5 marks)

a. Show that $3 - \sqrt{3}i = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$. **(1 mark)**

b. Find $\left(3 - \sqrt{3}i\right)^3$, expressing your answer in the form $x + iy$, where $x, y \in R$. **(2 marks)**

c. Find the integer values of n for which $\left(3 - \sqrt{3}i\right)^n$ is real. **(1 mark)**

d. Find the integer values of n for which $\left(3 - \sqrt{3}i\right)^n = ai$, where a is a real number. **(1 mark)**

Source: *VCE 2019, Specialist Mathematics 2, Section A, Q.6; © VCAA*

Question 4 (1 mark)

Let $z, w \in C$ where $\text{Arg}(z) = \frac{\pi}{2}$ and $\text{Arg}(w) = \frac{\pi}{4}$.

The value of $\text{Arg}\left(\frac{z^5}{w^4}\right)$ is

A. $-\frac{\pi}{2}$

B. $\frac{\pi}{2}$

C. π

D. $\frac{5\pi}{2}$

E. $\frac{7\pi}{2}$

Topic 2 | Subtopic 2.3 Complex numbers in polar form

Source: *VCE 2016, Specialist Mathematics 2, Section A, Q.5; © VCAA*

Question 5 (1 mark)

If $\text{Arg}(-1+ai)=-\frac{2\pi}{3}$, then the real number a is

A. $-\sqrt{3}$

B. $-\frac{\sqrt{3}}{2}$

C. $-\frac{1}{\sqrt{3}}$

D. $\frac{1}{\sqrt{3}}$

E. $\sqrt{3}$

Source: *VCE 2016, Specialist Mathematics 1, Q.6; © VCAA*

Question 6 (3 marks)

Write $\frac{\left(1-\sqrt{3}i\right)^4}{1+\sqrt{3}i}$ in the form $a+bi$, where a and b are real constants.

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.5; © VCAA*

Question 7 (1 mark)

Given $z=\frac{1+i\sqrt{3}}{1+i}$, the modulus and argument of the complex number z^5 are respectively

A. $2\sqrt{2}$ and $\frac{5\pi}{6}$

B. $4\sqrt{2}$ and $\frac{5\pi}{12}$

C. $4\sqrt{2}$ and $\frac{7\pi}{12}$

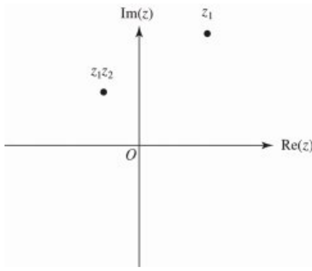
D. $2\sqrt{2}$ and $\frac{5\pi}{12}$

E. $4\sqrt{2}$ and $-\frac{\pi}{12}$

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.9; © VCAA*

Question 8 (1 mark)

Let $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, where z_1 and $z_1 z_2$ are shown in the Argand diagram below; θ_1 and θ_2 are acute angles.



A statement that is **necessarily** true is

A. $r_2 > 1$

B. $\theta_1 < \theta_2$

C. $\left|\dfrac{z_1}{z_2}\right| > r_1$

D. $\theta_1 = \theta_2$

E. $r_1 > 1$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.5; © VCAA*

Question 9 (1 mark)

If the complex number z has modulus $2\sqrt{2}$ and argument $\dfrac{3\pi}{4}$ then z^2 is equal to

A. $-8i$

B. $4i$

C. $-2\sqrt{2}i$

D. $2\sqrt{2}i$

E. $-4i$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.8; © VCAA*

Question 10 (1 mark)

The principal argument of $\frac{-3\sqrt{2} - i\sqrt{6}}{2 + 2i}$ is

A. $\frac{-13\pi}{12}$

B. $\frac{7\pi}{12}$

C. $\frac{11\pi}{12}$

D. $\frac{13\pi}{12}$

E. $\frac{-11\pi}{12}$

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.7; © VCAA*

Question 11 (1 mark)

If $z = r\,\text{cis}(\theta)$, then $\frac{z^2}{\bar{z}}$ is equivalent to

A. $r^3\,\text{cis}(3\theta)$

B. $r^3\,\text{cis}(-\theta)$

C. $2\,\text{cis}(3\theta)$

D. $r^3\,\text{cis}(\theta)$

E. $r\,\text{cis}(3\theta)$

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.6; © VCAA*

Question 12 (1 mark)

Let $z = a + bi$, $z \in C$.

If the principal argument of z^3 is in the second quadrant, then the complete set of values for Arg(z) is

A. $\left(\frac{\pi}{2}, \pi\right)$

B. $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

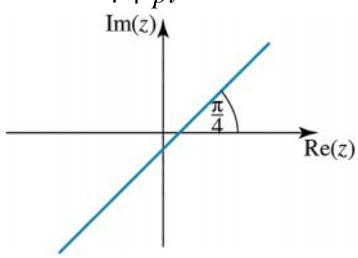
C. $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

D. $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

E. $\left(\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

Question 13 (1 mark)

If p is a real constant and the imaginary part of $\frac{p + 3i}{4 + pi}$ is equal to zero, then



A. $p = 0$ only

B. $p = 4$ only

C. $p = -3$ only

D. $p = \pm 2\sqrt{3}$

E. $p = \pm 4$

Question 14 (1 mark)

Which of the following could not represent the complex number $-2-2\sqrt{3}i$?

A. $4\,\text{cis}(-120°)$

B. $4\,\text{cis}\left(\frac{4\pi}{3}\right)$

C. $4\,\text{cis}\left(-\frac{2\pi}{3}\right)$

D. $-4\,\text{cis}(60°)$

E. $-4\,\text{cis}\left(\frac{2\pi}{3}\right)$

Question 15 (1 mark)

If $z=-2a-ai$, where a is a real positive number, then which of the following represents $\text{Arg}(z)$?

A. $\tan^{-1}(2)$

B. $-\tan^{-1}(2)$

C. $\tan^{-1}\left(\frac{1}{2}\right)$

D. $-\tan^{-1}\left(\frac{1}{2}\right)-\frac{\pi}{2}$

E. $\tan^{-1}\left(\frac{1}{2}\right)-\pi$

Question 16 (1 mark)

Let $u = 5\,\text{cis}\left(\frac{\pi}{4}\right)$ and $v = a\,\text{cis}(b)$, where a and b are real constants.

If $\frac{v}{u} = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$, a and b respectively are equal to

A. $\frac{5}{2}$ and $\frac{\pi}{2}$

B. 10 and $\frac{\pi}{2}$

C. 10 and $\frac{2\pi}{5}$

D. 10 and $-\frac{11\pi}{12}$

E. 2.5 and $\frac{13\pi}{12}$

Question 17 (1 mark)

Let $u = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ and $v = a\,\text{cis}(b)$, where a and b are real constants.

If $uv = 3\text{cis}\left(\frac{3\pi}{5}\right)$, a and b respectively are equal to

A. $\frac{2}{3}$ and $\frac{4\pi}{15}$

B. 1.5 and $\frac{\pi}{2}$

C. 1.5 and $\frac{14\pi}{15}$

D. 6 and $-\frac{4\pi}{15}$

E. 1.5 and $\frac{4\pi}{15}$

Question 18 (1 mark)

If $u = 3\,\text{cis}(\theta)$, $v = r\,\text{cis}\left(\frac{2\pi}{3}\right)$ and $uv = -6i$, then

A. $r = 3, \theta = -\frac{7\pi}{6}$

B. $r = 3, \theta = -\frac{\pi}{6}$

C. $r = 2, \theta = \frac{5\pi}{6}$

D. $r = -2, \theta = \frac{5\pi}{6}$

E. $r = -2, \theta = -\frac{2\pi}{6}$

Question 19 (1 mark)

If $u = r\,\text{cis}\,(\theta)\,,\, v = 2\,\text{cis}\left(-\frac{5\pi}{3}\right)$ and $\frac{u}{v} = -6$, then

A. $r = -4, \theta = -\frac{\pi}{3}$

B. $r = -4, \theta = \frac{5\pi}{3}$

C. $r = 12, \theta = \frac{5\pi}{3}$

D. $r = 12, \theta = -\frac{\pi}{3}$

E. $r = 12, \theta = -\frac{2\pi}{3}$

Question 20 (1 mark)

If $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, then $\text{Arg}(z^3)$ is equal to

A. 0

B. 2π

C. π

D. -2π

E. $-\frac{8\pi^3}{27}$

Topic	2	Complex numbers
Subtopic	2.4	Solving polynomial equations over *C*

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Source: *VCE 2021 Specialist Mathematics Exam 1, Q.8; © VCAA*

Question 1 (4 marks)

a. Solve $z^2 + 2z + 2 = 0$ for z, where $z \in C$. **(1 mark)**

b. Solve $z^2 + 2\bar{z} + 2 = 0$ for z, where $z \in C$. **(3 marks)**

Source: *VCE 2020 Specialist Mathematics Exam 2, Section A, Q6; © VCAA*

Question 2 (1 mark)

For the complex polynomial $P(z) = z^3 + az^2 + bz + c$ with real coefficients a, b and c, $P(-2) = 0$ and $P(3i) = 0$.

The values of a, b and c are respectively

A. $-2, 9, -18$
B. $3, 4, 12$
C. $2, 9, 18$
D. $-3, -4, 12$
E. $2, -9, -18$

Source: *VCE 2017 Specialist Mathematics Exam 2, Section A, Q3; © VCAA*

Question 3 (1 mark)

The number of distinct roots of the equation $\left(z^4 - 1\right)\left(z^2 + 3iz - 2\right) = 0$, where $z \in C$, is

A. 2
B. 3
C. 4
D. 5
E. 6

Source: *VCE 2019, Specialist Mathematics 1, Q.7; © VCAA*

Question 4 (5 marks)

Answer the following.

a. Show that $3 - \sqrt{3}i = 2\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{6}\right)$. **(1 mark)**

b. Find $\left(3 - \sqrt{3}i\right)^3$, expressing your answer in the form $x + yi$, where $x, y \in R$. **(2 marks)**

c. Find the integer values of n for which $\left(3 - \sqrt{3}i\right)^n$ is real. **(1 mark)**

d. Find the integer values of n for which $\left(3 - \sqrt{3}i\right)^n = ai$, where a is a real number. **(1 mark)**

Source: *VCE 2017, Specialist Mathematics 2, Section A, Q.4*; © VCAA

Question 5 (1 mark)

The solutions to $z^n = 1 + i$, $n \in Z^+$ are given by

A. $2^{\frac{1}{2n}} \text{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in R$

B. $2^{\frac{1}{n}} \text{cis}\left(\frac{\pi}{4n} + 2\pi k\right), k \in Z$

C. $2^{\frac{1}{2n}} \text{cis}\left(\frac{\pi}{4} + \frac{2\pi k}{n}\right), k \in R$

D. $2^{\frac{1}{n}} \text{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in Z$

E. $2^{\frac{1}{2n}} \text{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in R$

Source: *VCE 2017, Specialist Mathematics 1, Q.3; © VCAA*

Question 6 (3 marks)

Let $z^3 + az^2 + 6z + a = 0$, $z \in C$, where a is a real constant.

Given that $z = 1 - i$ is a solution to the equation, find all other solutions.

Source: *VCE 2016, Specialist Mathematics 2, Section A, Q.4*; © VCAA

Question 7 (1 mark)

One of the roots of $z^3 + bz^2 + cz = 0$ is $3 - 2i$, where b and c are real numbers.

The values of b and c respectively are

A. $6, 13$

B. $3, -2$

C. $-3, 2$

D. $2, 3$

E. $-6, 13$

Source: *VCE 2015, Specialist Mathematics 1, Q.4; © VCAA*

Question 8 (4 marks)

a. Find all solutions of $z^3 = 8i$, $z \in C$ in cartesian form. **(3 marks)**

b. Find all solutions of $(z - 2i)^3 = 8i$, $z \in C$ in cartesian form. **(1 mark)**

Source: *VCE 2014, Specialist Mathematics 1, Q.3; © VCAA*

Question 9 (5 marks)

Let f be a function of a complex variable, defined by the rule $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$.

a. Given that $z = i$ is a solution of $f(z) = 0$, write down a quadratic factor of $f(z)$. **(2 marks)**

b. Given that the other quadratic factor of $f(z)$ has the form $z^2 + bz + c$ find all solutions of $z^4 - 4z^3 + 7z^2 - 4z + 6 = 0$ in cartesian form. **(3 marks)**

Source: *VCE 2013, Specialist Mathematics 1, Q.8; © VCAA*

Question 10 (4 marks)

Find all solutions of $z^4 - 2z^2 + 4 = 0$, $z \in c$ in cartesian form.

Topic 2 Subtopic 2.4 Solving polynomial equations over C

Question 11 (1 mark)

If $z^2 + a = 0$, where a is a real positive number, then the most correct solution(s) for z is

A. $i\sqrt{a}$
B. $\sqrt{a}$
C. $\pm i\sqrt{a}$
D. $\pm\sqrt{a}$
E. $\pm ai$

Question 12 (1 mark)

If $z^2 = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$, then z equals

A. $1 - \sqrt{3}i$ or $-1 + \sqrt{3}i$
B. $\sqrt{3} + i$ or $-\sqrt{3} - i$
C. $-1 - \sqrt{3}i$ or $-1 + \sqrt{3}i$
D. $\sqrt{3} - i$ or $-\sqrt{3} + i$
E. $1 - \sqrt{3}i$ or $1 + \sqrt{3}i$

Question 13 (1 mark)

A linear factor of $P(z) = z^3 - 3z^2 + 6z - 18$ is

A. 3
B. $\sqrt{6}i$
C. $z + \sqrt{6}i$
D. $z - \sqrt{6}$
E. $z^2 + 6$

Question 14 (1 mark)

$P(z)$ is a polynomial in z of degree 5 with real coefficients. Which of the following statements must be false?

A. $P(z) = 0$ has five real roots.
B. $P(z) = 0$ has one real root and two pairs of complex conjugate roots.
C. $P(z) = 0$ has three real roots and one pair of complex conjugate roots.
D. $P(z) = 0$ has one repeated real root and three non-real roots.
E. $P(z) = 0$ has three repeated real roots and two non-real roots.

Question 15 (1 mark)

The linear factors of $z^3 - 3iz^2 + 3z - 9i$ are

A. $\left(z^2 + 3\right)(z - 3i)$
B. $\left(z^2 - 3\right)(z + 3i)$
C. $(z - 3)\left(z^2 - 3i\right)$
D. $(z + 3i)(z - 3i)\left(z - \sqrt{3}i\right)$
E. $\left(z + \sqrt{3}i\right)\left(z - \sqrt{3}i\right)(z - 3i)$

Question 16 (1 mark)

$P(z)$ is a polynomial in z of degree 6 with real coefficients. Which one of the following statements must be false?

A. $P(z) = 0$ has no real roots.
B. $P(z) = 0$ has three real roots and three non-real roots.
C. $P(z) = 0$ has one (repeated) real root and two non-real roots.
D. $P(z) = 0$ has four real roots and two non-real roots.
E. $P(z) = 0$ has six real roots.

Topic 2 | Subtopic 2.4 Solving polynomial equations over C

Question 17 (1 mark)
If $z^4 + 16 = 0$ then z is equal to
A. $\sqrt{2}(1 \pm i)$
B. $-\sqrt{2}(1 \pm i)$
C. $\sqrt{2}(\pm 1 + i)$
D. $\sqrt{2}(\pm 1 + i)$ and $\sqrt{2}(1 \pm i)$
E. $\sqrt{2}(1 \pm i)$ and $-\sqrt{2}(1 \pm i)$

Question 18 (1 mark)
Given that $z - 1 + \sqrt{3}i$ is a factor of $P(z) = z^3 + z^2 - 2z + a$ where $a \in R$ the value of a is
A. -12
B. 4
C. -1
D. 3
E. 12

Question 19 (1 mark)
The conjugate pair of factors $P(z) = z^3 + 2z^2 + 5z + 10$ are
A. $\left(z - 2\sqrt{5}i\right)$ and $\left(z + 2\sqrt{5}i\right)$
B. $\left(z - \sqrt{5}i\right)$ and $\left(z + \sqrt{5}i\right)$
C. $\left(z + 2 - \sqrt{5}i\right)$ and $\left(z + 2 + \sqrt{5}i\right)$
D. $\left(z - 2 - \sqrt{5}i\right)$ and $\left(z - 2 + \sqrt{5}i\right)$
E. $\left(z - \frac{\sqrt{5}}{2}i\right)$ and $\left(z + \frac{\sqrt{5}}{2}i\right)$

Question 20 (1 mark)
If $1 - i$ and $\pm\sqrt{2}i$ are solutions to the equation $P(z) = z^4 + bz^3 + cz^2 + dz + 4 = 0$ where b, c and d are real, then
A. $b = -2$, $c = 4$ and $d = -4$
B. $b = -2$, $c = -4$ and $d = 4$
C. $b = -2$, $c = -4$ and $d = -4$
D. $b = 2$, $c = 4$ and $d = 4$
E. $b = 2$, $c = -4$ and $d = 4$

Topic	2	Complex numbers
Subtopic	2.5	Subsets of the complex plane

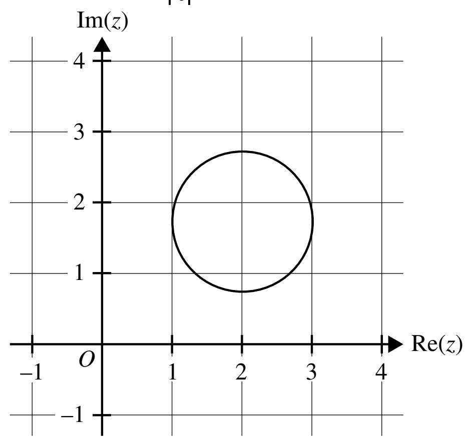
To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2021 Specialist Mathematics Exam 2, Section A, Q.5; © VCAA*

Question 1 (1 mark)

The graph of the circle given by $\left|z - 2 - \sqrt{3}\right| i = 1$, where $z \in C$ is shown.

For points on this circle, the maximum value of $|z|$ is



A. $\sqrt{3} + 1$
B. 3
C. $\sqrt{3}$
D. $\sqrt{7} + 1$
E. 8

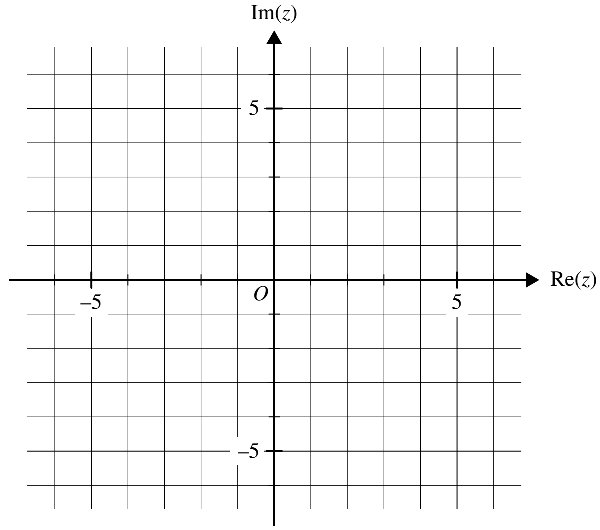
Source: *VCE 2020 Specialist Mathematics Exam 2, Section B, Q2; © VCAA*

Question 2 (11 marks)

Two complex numbers, u and v, are defined as $u = -2 - i$ and $v = -4 - 3i$.

a. Express the relation $|z - u| = |z - v|$ in the cartesian form $y = mx + c$. where $m, c \in R$. **(3 marks)**

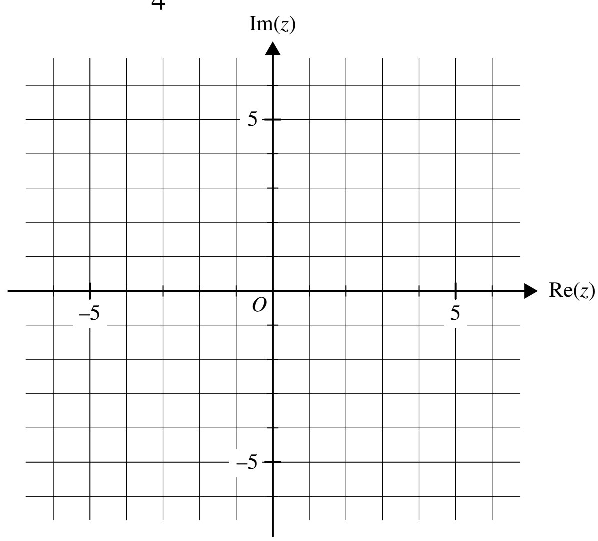
b. Plot the points that represent u and v and the relation $|z - u| = |z - v|$ on the Argand diagram below. **(2 marks)**



c. State a geometrical interpretation of the graph of $|z - u| = |z - v|$ in the points that represent u and v. **(1 mark)**

d. Answer the following.

i. Sketch the ray given by $\text{Arg}(z - u) = \dfrac{\pi}{4}$ on the Argand diagram in part **b.** **(1 mark)**



ii. Write down the function that describes the ray $\text{Arg}(z - u) = \dfrac{\pi}{4}$, giving the rule in cartesian form. **(1 mark)**

$y = \square$ for $x\,\square$

e. The points representing u and v and $-5i$ lie on the circle given by $|z - zc| = r$,where z_c is the centre of the circle and r is the radius.
Find z_c in the form $a + ib$, where $a,b \in R$, and find the radius r. **(3 marks)**

Source: *VCE 2019 Specialist Mathematics Exam 2, Section A, Q5; © VCAA*

Question 3 (1 mark)

Let $z = x + yi$, where $x, y \in R$. The rays $\text{Arg}(z - 2) = \dfrac{\pi}{4}$ and $\text{Arg}(z - (5 + i)) = \dfrac{5\pi}{6}$, where $z \in C$, intersect on the complex plane at a point $(a,\ b)$.

The value of b is

A. $-\sqrt{3}$
B. $2 - \sqrt{3}$
C. 0
D. $\sqrt{3}$
E. $2 + \sqrt{3}$

Source: *VCE 2017, Specialist Mathematics 2, Section A, Q.5; © VCAA*

Question 4 (1 mark)

On an Argand diagram, a point that lies on the path defined by $|z - 2 + i| = |z - 4|$ is

A. $\left(3, -\dfrac{1}{2}\right)$

B. $\left(-3, -\dfrac{1}{2}\right)$

C. $\left(-3, \dfrac{3}{2}\right)$

D. $\left(3, \dfrac{1}{2}\right)$

E. $\left(3, -\dfrac{3}{2}\right)$

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.3; © VCAA*

Question 5 (1 mark)

If both a and c are non-zero real numbers, the relation $a^2x^2 + (1 - a^2)y^2 = c^2$ **cannot** represent

A. a circle.

B. an ellipse.

C. a hyperbola.

D. a single straight line.

E. a pair of straight lines.

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.6; © VCAA*

Question 6 (1 mark)

Which one of the following relations has a graph that passes through the point $1 + 2i$ in the complex plane?

A. $z\bar{z} = \sqrt{5}$

B. $\text{Arg}(z) = \dfrac{\pi}{3}$

C. $|z - 1| = |z - 2i|$

D. $\text{Re}(z) = 2\text{Im}(z)$

E. $z + \bar{z} = 2$

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.8; © VCAA*

Question 7 (1 mark)

A relation that does **not** represent a circle in the complex plane is

A. $z\bar{z} = 4$

B. $|z + 3i| = 2|z - i|$

C. $|z - i| = |z + 2|$

D. $|z - 1 + i| = 4$

E. $|z| + 2|\bar{z}| = 4$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.9; © VCAA*

Question 8 (1 mark)

The circle $|z - 3 - 2i| = 2$ is intersected exactly twice by the line given by

A. $|z - i| = |z + 1|$

B. $|z - 3 - 2i| = |z - 5|$

C. $|z - 3 - 2i| = |z - 10i|$

D. $\text{Im}(z) = 0$

E. $\text{Re}(z) = 5$

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.5; © VCAA*

Question 9 (1 mark)

The region in the complex plane that is **outside** the circle of radius b centred at the origin is given by the set of points z, where $z \in C$, such that

A. $|z| < b$

B. $|z| > b$

C. $|z| > b^2$

D. $|z| = b$

E. $|z| < b^2$

Question 10 (1 mark)

If the ray represented by $\left\{z: \arg(z) = \frac{2\pi}{3}\right\}$ is translated 3 units downwards parallel to the imaginary axis, then it will be described by

A. $\left\{z: \arg(z + 3i) = \frac{2\pi}{3}\right\}$

B. $\left\{z: \arg(z - 3i) = \frac{2\pi}{3}\right\}$

C. $\left\{z: \arg(z + 3) = \frac{2\pi}{3}\right\}$

D. $\left\{z: \arg(z - 3) = \frac{2\pi}{3}\right\}$

E. $\left\{z: \arg(z) = \frac{2\pi}{3} - 3\right\}$

Question 11 (1 mark)

Find the equation of the perpendicular bisector of the line which joins the points $2 + 3i$ and $1 - i$.

Question 12 (1 mark)

If the ray represented by $\left\{z: \arg(z) = \frac{5\pi}{6}\right\}$ is translated 2 units upwards parallel to the imaginary axis and 3 units to the right parallel to the real axis, then it will be described by

A. $\left\{z: \arg(z - 3 - 2i) = \frac{5\pi}{6}\right\}$

B. $\left\{z: \arg(z + 3 + 2i) = \frac{5\pi}{6}\right\}$

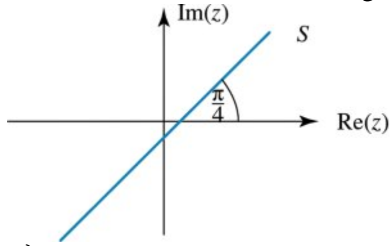
C. $\left\{z: \arg(z + 3 - 2i) = \frac{5\pi}{6}\right\}$

D. $\left\{z: \arg(z + 2 - 3i) = \frac{5\pi}{6}\right\}$

E. $\left\{z: \arg(z - 2 - 3i) = \frac{5\pi}{6}\right\}$

Question 13 (1 mark)

Which of the following does **not** describe the line S shown in the Argand diagram?



A. $\left\{z:\text{Arg}(z) = \frac{\pi}{4}\right\} \cup \left\{z:\text{Arg}(z) = -\frac{3\pi}{4}\right\}$

B. $\{z:\text{Re}(z) = \text{Im}(z)\}$

C. $\{z: |z - 2| = |z - 2i|\}$

D. $\{z: |z + 3| = |z + 3i|\}$

E. $\{z: |z + 1 - i| = |z - 1 + i|\}$

Question 14 (1 mark)

The set of points which are equidistant from the two complex numbers $3i$ and -3 is given by

A. $\{z: |z + 3| = |z - 3i|\}$

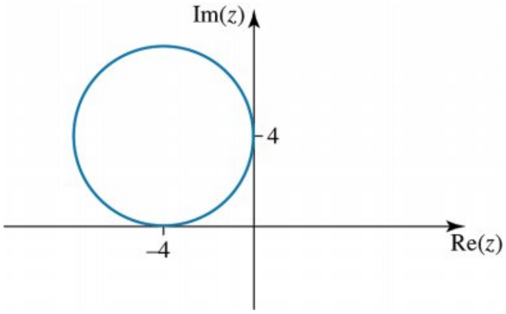
B. $\{z: |z + 3| = |z + 3i|\}$

C. $\{z: |z - 3| = |z - 3i|\}$

D. $\{z: |z - 3i| - |z + 3| = 3\}$

E. $\{z: |z - 3i| + |z + 3| = 3\}$

Question 15 (1 mark)
The diagram shows a circle in the complex plane. The circle is specified by



A. $\{z: (z+4+4i)\left(\overline{z}-4+4i\right)=16\}$
B. $\{z: (z+4-4i)\left(\overline{z}+4+4i\right)=16\}$
C. $\{z: (z-4-4i)\left(\overline{z}-4+4i\right)=16\}$
D. $\{z: |z-4+4i|=16\}$
E. $\{z: |z+4-4i|=16\}$

Question 16 (1 mark)
The set of points in the complex plane described by $\{z: |z-3|+|z+3|=10\}$ best represents
A. a straight line.
B. a circle.
C. an ellipse.
D. a branch of a hyperbola.
E. a null set.

Question 17 (1 mark)
If $z \in C$, which one of the following represents a ellipse on an Argand diagram?
A. $z\bar{z} = 6$
B. $|z - 2| + |z + 2| = 6$
C. $(z - 4 + 2i)\left(\bar{z} - 4 - 2i\right) = 7$
D. $4|z - 1 + i| = 6$
E. $|z - 3| = 4$

Question 18 (1 mark)
The Cartesian equation representing $\text{Im}(z + 2i) = 3[\text{Re}(z - 2)]^2 - 1$ is given by
A. $y = 3x^2 - 12x - 9$
B. $y = 3\left(x^2 - 6x + 4\right)$
C. $y = 3x^2 + 12x - 9$
D. $y = 3x^2 - 12x + 9$
E. $y = 3(x - 2)^2 + 1$

Question 19 (1 mark)
The Cartesian equation representing $\text{Im}(z - i) = 2[\text{Re}(z + 1)]^3$ is given by
A. $y = 2x^3 - 6x^2 + 6x + 3$
B. $y = 2x^3 + 6x^2 + 6x + 3$
C. $y = 2x^3 + 6x^2 - 6x + 3$
D. $y = 2\left(x^3 + 3x^2 + 3x + 3\right)$
E. $y = 2(x + 1)^3 + 3$

Question 20 (1 mark)

The Cartesian equation representing $[\text{Im}(z)]^2\,[\text{Re}(z)]^2 = 8$ is given by

A. $y = 4\sqrt{\dfrac{2}{x^3}}$

B. $y = \sqrt[3]{\dfrac{8}{8}}$

C. $y = \sqrt{\dfrac{2}{x^3}}$

D. $y = \sqrt{\dfrac{x^3}{8}}$

E. $y = 2\sqrt{\dfrac{2}{x^3}}$

Question 21 (1 mark)

The set of points in the complex plane described by $\left\{z\colon \text{Im}\left(\dfrac{z - ai}{z - b}\right) = 0\right\}$ where a and b are real and $ab \neq 0$, represents

A. a straight line

B. a circle.

C. an ellipse.

D. a branch of a hyperbola.

E. a null set.

Question 22 (1 mark)

The set of points in the complex plane described by $\{z\colon |z + 3| - |z - 3| = 2\}$ represents

A. a straight line.

B. a circle.

C. an ellipse.

D. a branch of a hyperbola.

E. a null set.

Topic	2	Complex numbers
Subtopic	2.6	Roots of complex numbers

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Source: *VCE 2020 Specialist Mathematics Exam 1, Q.3; © VCAA*

Question 1 (3 marks)

Find the cube roots of $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. Express your answer in polar form using principal values of the argument.

Source: *VCE 2017 Specialist Mathematics Exam 2, Section A, Q4; © VCAA*

Question 2 (1 mark)

The solution to $z^n = 1 + i, n \in Z^+$ are given by

A. $2^{\frac{1}{2n}} \text{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in R$

B. $2^{\frac{1}{n}} \text{cis}\left(\frac{\pi}{4n} + 2\pi k\right), k \in Z$

C. $2^{\frac{1}{2n}} \text{cis}\left(\frac{\pi}{4} + \frac{2\pi k}{n}\right), k \in R$

D. $2^{\frac{1}{n}} \text{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in Z$

E. $2^{\frac{1}{2n}} \text{cis}\left(\frac{\pi}{4n} + \frac{2\pi k}{n}\right), k \in Z$

Source: *VCE 2015 Specialist Mathematics Exam 1, Q4; © VCAA*

Question 3 (4 marks)

Answer the following.

a. Find all solutions of $z^3 = 8i,\ z \in C$ in cartesian form. **(3 marks)**

b. Find all solutions of $(z - 2i)^3 = 8i, z \in C$ in cartesian form. **(1 mark)**

Source: *VCE 2020, Specialist Mathematics 2, Section A, Q.6; © VCAA*

Question 4 (1 mark)

For the complex polynomial $P(z) = z^3 + az^2 + bz + c$ with real coefficients $a,\ b$ and c, $P(-2) = 0$ and $P(3i) = 0$.

The values of $a,\ b$ and c are respectively

A. $-2,\ 9, -18$

B. $3,\ 4,\ 12$

C. $2,\ 9,\ 18$

D. $-3, -4,\ 12$

E. $2, -9, -18$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.7; © VCAA*

Question 5 (1 mark)

The sum of the roots of $z^3 - 5z^2 + 11z - 7 = 0$, where $z \in c$, is

A. $1 + 2\sqrt{3}i$

B. $5i$

C. $4 - 2\sqrt{3}i$

D. $2\sqrt{3}i$

E. 5

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.8; © VCAA*

Question 6 (1 mark)

The principal arguments of the solutions to the equation $z^2 = 1 + i$ are

A. $\frac{\pi}{8}$ and $\frac{9\pi}{8}$

B. $-\frac{\pi}{8}$ and $\frac{7\pi}{8}$

C. $-\frac{7\pi}{8}$ and $\frac{\pi}{8}$

D. $\frac{7\pi}{8}$ and $\frac{15\pi}{8}$

E. $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$

Question 7 (1 mark)

The solutions to $z^4 = 36$ in Cartesian form are

A. $6, 6i, -6$ and $-6i$

B. $\sqrt{6}$ and $\sqrt{6i}$

C. $\sqrt{6}, \sqrt{6i} - \sqrt{6}$ and $-\sqrt{6i}$

D. $\sqrt{36}, \sqrt{36i}, -\sqrt{36}$ and $-\sqrt{36i}$

E. $\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}i, -\frac{\sqrt{6}}{4}$ and $-\frac{\sqrt{6}}{4}i$

Question 8 (1 mark)

The solutions to $z^8 = 16$ in Cartesian form are

A. $\sqrt{2}, -\sqrt{2}, -\sqrt{2}i, -\sqrt{2}\,i, 1 - i, -1 - i, 1 + i, -1 + i$

B. $\sqrt{8}, -\sqrt{8}, -\sqrt{8}\,i, -\sqrt{8}\,i, 1 - 2i, -1 - 2i, 1 - 2i, -1 + 2i$

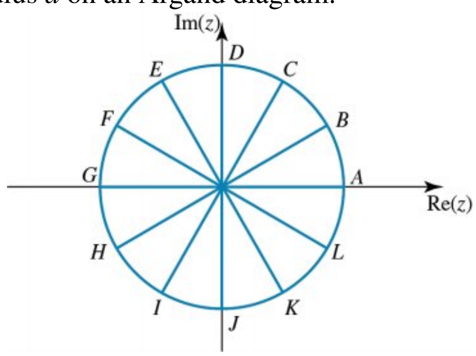
C. $\sqrt{2}, -\sqrt{2}, -\sqrt{2}i, -\sqrt{2}\,i, 1 - \sqrt{2i}, -1 - \sqrt{2i}, 1 - \sqrt{2i}, -1 + \sqrt{2i}$

D. $2, -2, -2i, -2i, 2 - i, -2 - i, 2 - i, -2 + i$

E. $\sqrt{2}, -\sqrt{2}, -\sqrt{2}\,i, -\sqrt{2}\,i, 1 - 2i, -1 - 2i, 1 - 2i, -1 + 2i$

Question 9 (1 mark)
The solutions of $z^4 = 256$ are
A. ± 16
B. $\pm 16i$
C. ± 16 and $\pm 16i$
D. $\pm 4i$
E. ± 4 and $\pm 4i$

Question 10 (1 mark)
The diagram shows a circle of radius a on an Argand diagram.



The roots of the equation $z^3 + a^3 i = 0$, where a is a positive real number, are given by
A. D only
B. J, G and B.
C. G, C and K.
D. A, E and I.
E. D, H and L.

Question 11 (1 mark)

Solve $z^3 = -27i$ in Cartesian form.

Question 12 (1 mark)

If $z^2 = 1 - 4\sqrt{3}i$, then z is equal to

A. $2 - \sqrt{3}i$ only

B. $2 + \sqrt{3}i$ only

C. $2 \pm \sqrt{3}i$

D. $2 - \sqrt{3}i$ and $-2 + \sqrt{3}i$

E. $\sqrt{3} - 2i$ and $\sqrt{3} + 2i$

Topic	2	Complex numbers
Subtopic	2.7	Review

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Source: *VCE 2021 Specialist Mathematics Exam 2, Section B, Q.2; © VCAA*

Question 1 (9 marks)

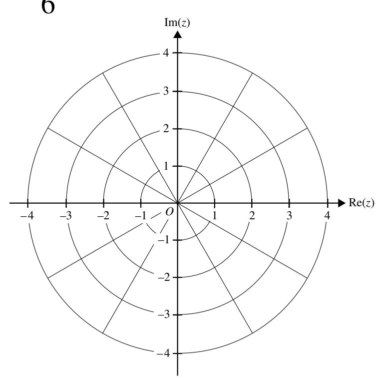
The polynomial $p(z) = z^3 + \alpha z^2 + \beta z + \gamma$, where $z \in C$ and $\alpha,\ \beta,\ \gamma \in R$, can also be written as $p(z) = (z - z_1)(z - z_2)(z - z_3)$, where $z_1 \in R$ and $z_2, z_3 \in C$.

a. i. State the relationship between z_2 and z_3. **(1 mark)**

ii. Determine the values of α, β and γ, given that $p(2) = -13$, $|z_2 + z_3| = 0$ and $|z_2 - z_3| = 6$. **(3 marks)**

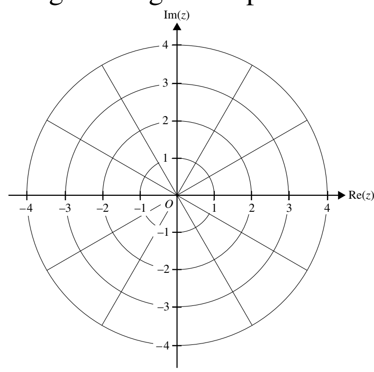
Consider the point $z_4 = \sqrt{3} + i$.

b. Sketch the ray given by $\text{Arg}(z - z_4) = \dfrac{5\pi}{6}$ on the Argand diagram below. **(2 marks)**



c. The ray $\text{Arg}(z - z_4) = \frac{5\pi}{6}$ intersects the circle $|z - 3i| = 1$, dividing it into a major and a minor segment.

i. Sketch the circle $|z - 3i| = 1$ on the Argand diagram in part **b**. **(1 mark)**



ii. Find the area of the minor segment. **(2 marks)**

Source: *VCE 2018 Specialist Mathematics Exam 2, Section B, Q2; © VCAA*

Question 2 (10 marks)

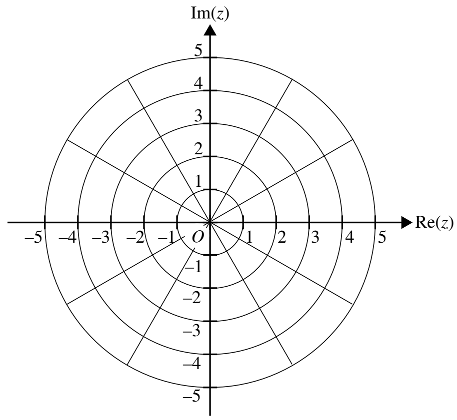
a. State the center in the form $(x, \ y)$, where $x, y \in R$, and the radius of the circle given by $|z - (1 + 2i)| = 2$, where $z \in C$. **(1 mark)**

Center = ☐

Radius = ☐

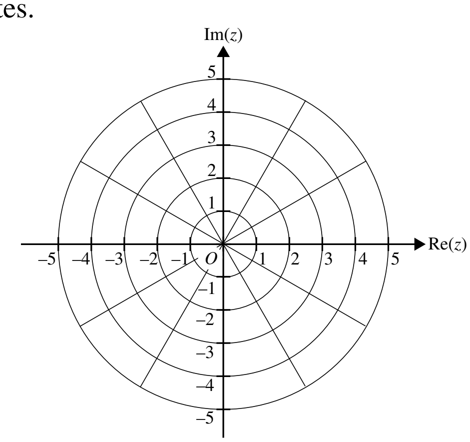
b. By expressing the circle given by $|z + 1| = \sqrt{2}\,|z - i|$ in cartesian form, show that this circle has the same centre and radius as the circle given by $|z - (1 + 2i)| = 2$. **(2 marks)**

c. Graph the circle given by $|z+1| = \sqrt{2}\,|z-i|$ on the Argand diagram below, labelling the intercepts with the vertical axis. **(2 marks)**



The line given by $|z-1| = |z-3|$ intersects the circle given by $|z+1| = \sqrt{2}\,|z-i|$ in two places.

d. Draw the line given by $|z-1| = |z-3|$ on the Argand diagram in part **c**. Label the points of intersection with their coordinates. **(2 marks)**



e. Find the area of the minor segment enclosed by an arc of the circle given by $|z+1| = \sqrt{2}\,|z-i|$ and part of the line given by $|z-1| = |z-3|$. **(3 marks)**

Source: *VCE 2016 Specialist Mathematics Exam 2, Section B, Q2; © VCAA*

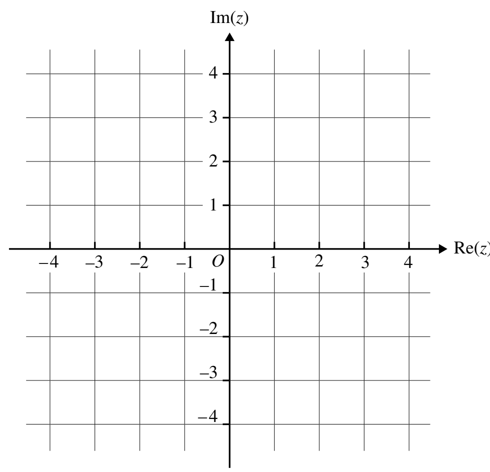
Question 3 (11 marks)

A line in the complex plane is given by $|z-1| = |z+2-3i|,\ z \in C$.

a. Find the equation of this line in the form $y = mx + c$. **(2 marks)**

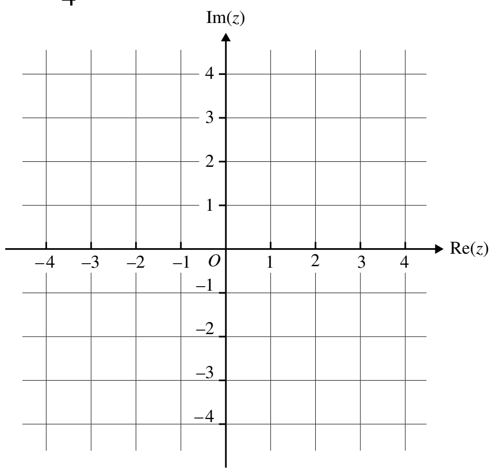
b. Find the points of intersection of the line $|z-1| = |z+2-3i|$ with the circle $|z-1| = 3$. **(2 marks)**

c. Sketch both the line $|z-1| = |z+2-3i|$ and the circle $|z-1| = 3$ on the Argand diagram below. **(2 marks)**



d. The line $|z-1| = |z+2-3i|$ cuts the circle $|z-1| = 3$ into two segments. Find the area of the major segment. **(2 marks)**

e. Sketch the ray given by $\text{Arg}(z) = -\frac{3\pi}{4}$ on the Argand diagram in part **c**. **(1 mark)**



f. Write down the range of values of α, $\alpha \in R$, for which a ray with equation $\text{Arg}(z) = \alpha\pi$ intersects the line $|z-1| = |z+2-3i|$. **(2 marks)**

Source: *VCE 2015 Specialist Mathematics Exam 2, Section A, Q5; © VCAA*

Question 4 (1 mark)

Given $z = \frac{1 + i\sqrt{3}}{1 + i}$, the modulus and argument of the complex number z^5 are respectively

A. $2\sqrt{2}$ and $\frac{5\pi}{6}$

B. $4\sqrt{2}$ and $\frac{5\pi}{12}$

C. $4\sqrt{2}$ and $\frac{7\pi}{12}$

D. $2\sqrt{2}$ and $\frac{5\pi}{12}$

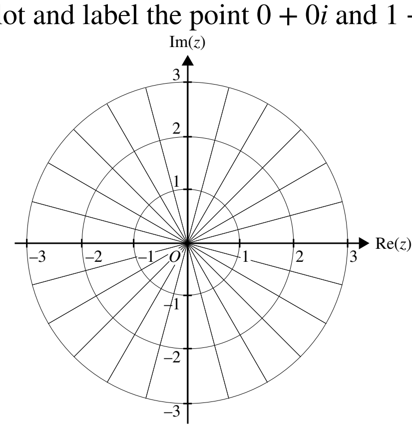
E. $4\sqrt{2}$ and $-\frac{\pi}{12}$

Source: *VCE 2015 Specialist Mathematics Exam 2,* Section 2, Q2; *© VCAA*

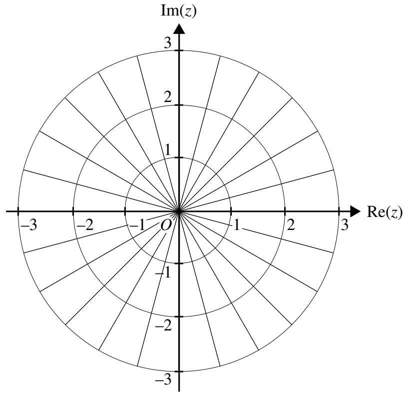
Question 5 (12 marks)

Answer the following

a. i. On the Argand diagram below, plot and label the point $0 + 0i$ and $1 + i\sqrt{3}$. **(2 marks)**



ii. On the same Argand diagram above, sketch the line $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$ and the circle $|z - 2| = 1$. **(2 marks)**



iii. Use the fact that the line $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$ passes through the point $z = 2$, or otherwise, to find the equation of this line in cartesian form. **(1 mark)**

$y = \square$

iv. Find the points of intersection of the line and the circle, expressing your answer in the form $a + ib$. **(3 marks)**

b. i. Consider the equation $z^2 - 4\cos(\alpha)\,z + 4 = 0$, where α is a real constant and $0 < \alpha < \dfrac{\pi}{2}$.
Find the roots z_1 and z_2 of this equation, in terms of α, expressing your answers in polar form. **(3 marks)**

ii. Find the values of α for which $\left|\text{Arg}\left(\frac{z_1}{z_2}\right)\right| = \frac{5\pi}{6}$. **(1 mark)**

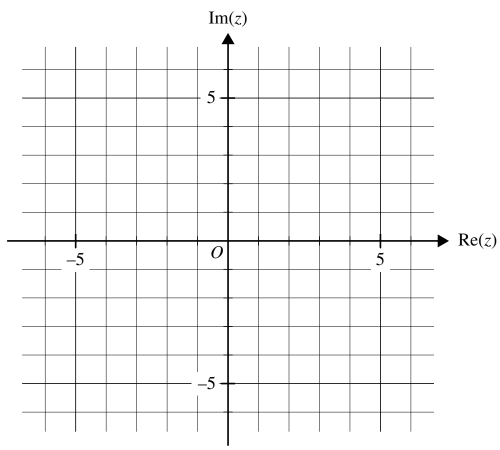
Source: *VCE 2020, Specialist Mathematics 2, Section B, Q.2; © VCAA*

Question 6 (11 marks)

Two complex numbers, u and v, are defined as $u = -2 - i$ and $v = -4 - 3i$.

a. Express the relation $|z - u| = |z - v|$ in the cartesian form $y = mx + c$, where $m, c \in R$. **(3 marks)**

b. Plot the points that represent u and v and the relation $|z - u| = |z - v|$ on the Argand diagram below. **(2 marks)**



c. State a geometrical interpretation of the graph of $|z - u| = |z - v|$ in relation to the points that represent u and v. **(1 mark)**

d. **i.** Sketch the ray given by $\text{Arg}(z-u) = \frac{\pi}{4}$ on the Argand diagram in **part b**. **(1 mark)**

ii. Write down the function that describes the ray $\text{Arg}(z-u) = \frac{\pi}{4}$, giving the rule in cartesian form. **(1 mark)**

e. The points representing u and v and $-5i$ on the circle given by $|z - z_c| = r$. where z_c is the centre of the circle and r is the radius. Find z_c in the form $a + ib$, where $a, b \in R$, and find the radius r. **(3 marks)**

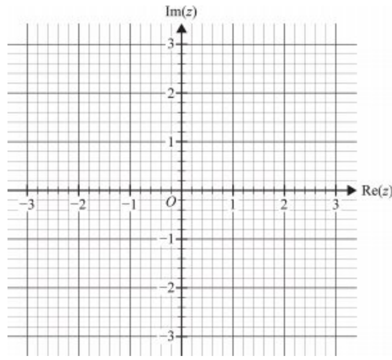
Source: *VCE 2019, Specialist Mathematics 2, Section B, Q.2; © VCAA*

Question 7 (9 marks)

Answer the following.

a. **i.** Show that the solutions of $2z^2 + 4z + 5 = 0$, where $z \in C$, are $z = -1 \pm \frac{\sqrt{6}}{2}i$. **(1 mark)**

ii. Plot the solutions of $2z^2 + 4z + 5 = 0$ on the Argand diagram below. **(1 mark)**



b. Let $|z + m| = n$, where $m, n \in R$ represent the circle of minimum radius that passes through the solutions of $2z^2 + 4z + 5 = 0$.

i. Find m and n. **(2 marks)**

ii. Find the cartesian equation of the circle $|z + m| = n$. **(1 mark)**

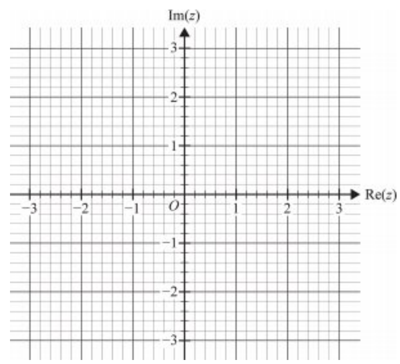
iii. Sketch the circle on the Argand diagram in **part a.ii**. Intercepts with the coordinate axes do not need to be calculated or labelled. **(1 mark)**

c. Find all values of d, where $d \in R$, for which the solutions of $2z^2 + 4z + d = 0$ satisfy the relation $|z + m| \leq n$. **(2 marks)**

d. All complex solutions of $az^2 + bz + c = 0$ have non-zero real and imaginary parts.
Let $|z + p| = q$ represent the circle of minimum radius in the complex plane that passes through these solutions, where $a, b, c, p, q \in R$.

i. Find p and q in terms of a, b and c. **(2 marks)**

ii. Plot the solutions of $2z^2 + 4z + 5 = 0$ on the Argand diagram below. **(1 mark)**



Source: *VCE 2018, Specialist Mathematics 2, Section A, Q.6; © VCAA*

Question 8 (1 mark)

The complex numbers z, iz and $z + iz$, where $z \in C \setminus \{0\}$, are plotted in the Argand plane, forming the vertices of a triangle.
The area of this triangle is given by

A. $|z|$

B. $|z| + |z|^2$

C. $\dfrac{|z|^2}{2}$

D. $|z|^2$

E. $\dfrac{\sqrt{3}\,|z|^2}{2}$

Source: *VCE 2018, Specialist Mathematics 2, Section A, Q.5; © VCAA*

Question 9 (1 mark)

Let $z = a + bi$, where $a, b \in R \setminus \{0\}$.

If $z + \frac{1}{z} \in R$, which one of the following must be **true**?

A. $\text{Arg}(z) = \frac{\pi}{4}$

B. $a = -b$

C. $a = b$

D. $|z| = 1$

E. $z^2 = 1$

Source: *VCE 2018, Specialist Mathematics 1, Q.2; © VCAA*

Question 10 (4 marks)

Answer the following.

a. Show that $1 + i = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$. **(1 mark)**

b. Evaluate $\frac{\left(\sqrt{3} - i\right)^{10}}{(1 + i)^{12}}$, giving your answer in the form $a + bi$ where $a, b \in R$. **(3 marks)**

Source: *VCE 2016, Specialist Mathematics 2, Section A, Q.6; © VCAA*

Question 11 (1 mark)

The points corresponding to the four complex numbers given by

$z_1 = 2\text{cis}\left(\frac{\pi}{3}\right), z_2 = \text{cis}\left(\frac{3\pi}{4}\right), z_3 = 2\text{cis}\left(-\frac{2\pi}{3}\right), z_4 = \text{cis}\left(-\frac{\pi}{4}\right)$ are the vertices of a parallelogram in the complex plane.

Which one of the following statements is **not** true?

A. The acute angle between the diagonals of the parallelogram is $\frac{5\pi}{12}$

B. The diagonals of the parallelogram have lengths 2 and 4

C. $z_1 z_2 z_3 z_4 = 0$

D. $z_1 + z_2 + z_3 + z_4 = 0$

E. $1 \le |z| \le 2$ for all four of z_1, z_2, z_3, z_4

Source: *VCE 2016, Specialist Mathematics 2, Section B, Q.2; © VCAA*

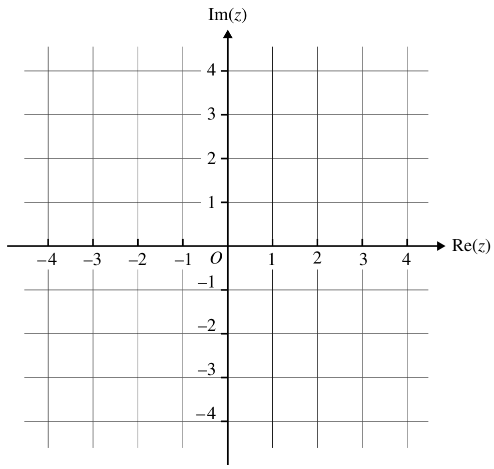
Question 12 (11 marks)

A line in the complex plane is given by $|z - 1| = |z + 2 - 3i|,\ z \in C$.

a. Find the equation of this line in the form $y = mx + c$. **(2 marks)**

b. Find the points of intersection of the line $|z - 1| = |z + 2 - 3i|$ with the circle $|z - 1| = 3$. **(2 mark)**

c. Sketch both the line $|z-1| = |z+2-3i|$ and the circle $|z-1| = 3$ on the Argand diagram below. **(2 marks)**



d. The line $|z-1| = |z+2-3i|$ cuts the circle $|z-1| = 3$ into two segments. Find the area of the major segment. **(2 marks)**

e. Sketch the ray given by $\text{Arg}(z) = -\frac{3\pi}{4}$ on the Argand diagram in **part c**. **(1 mark)**

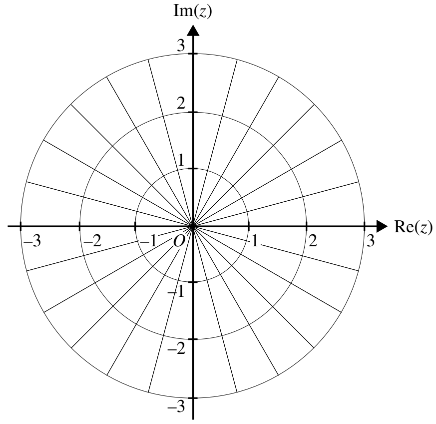
f. Write down the range of values of α, $\alpha \in R$, for which a ray with equation $\text{Arg}(z) = \alpha\pi$ intersects the line $|z-1| = |z+2-3i|$. **(2 marks)**

Source: *VCE 2015, Specialist Mathematics 2, Section 2, Q.2; © VCAA*

Question 13 (12 marks)

Answer the following.

a. i. On the Argand diagram below, plot and label the points $0 + 0i$ and $1 + i\sqrt{3}$. **(2 marks)**



ii. On the same Argand diagram above, sketch the line $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$ and the circle $|z - 2| = 1$. **(2 marks)**

iii. Use the fact that the line $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$ passes through the point $z = 2$, or otherwise, to find the equation of this line in cartesian form. **(1 mark)**

iv. Find the points of intersection of the line and the circle, expressing your answers in the form $a + ib$. **(3 marks)**

b. i. Consider the equation $z^2 - 4\cos(\alpha)z + 4 = 0$, where α is a real constant and $0 < \alpha < \frac{\pi}{2}$.
Find the roots z_1 and z_2 of this equation, in terms of α, expressing your answers in polar form. **(3 marks)**

ii. Find the value of α for which $\left|\text{Arg}\left(\frac{z_1}{z_2}\right)\right| = \frac{5\pi}{6}$. **(1 mark)**

Source: *VCE 2014, Specialist Mathematics 2, Section 2, Q.2; © VCAA*

Question 14 (13 marks)

Consider the complex number $z_1 = \sqrt{3} - 3i$.

a. i. Express z_1 in polar form. **(2 marks)**

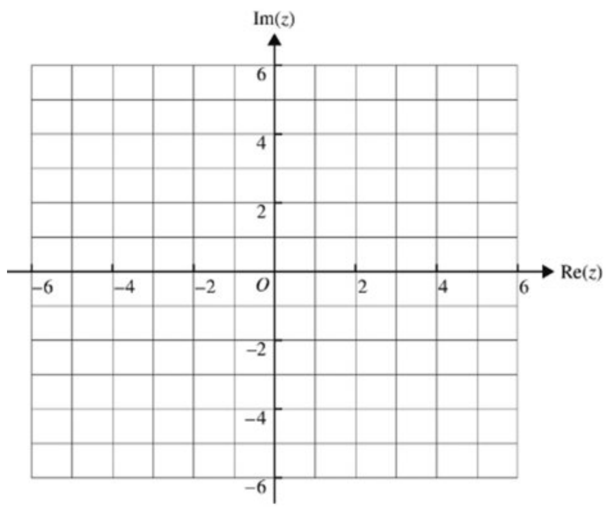
ii. Find $\text{Arg}(z_1^4)$. **(1 mark)**

iii. Given that $z_1 = \sqrt{3} - 3i$ is one root of the equation $z^3 + 24\sqrt{3} = 0$, find the other two roots, expressing your answers in cartesian form. **(2 marks)**

b. i. Find the value of $(z_1 + 2i)\left(\bar{z}_1 - 2i\right)$, where $z_1 = \sqrt{3} - 3i$. **(1 mark)**

ii. Show that the relation $(z + 2i)\left(\bar{z} - 2i\right) = 4$ can be expressed in cartesian form as $x^2 + (y + 2)^2 = 4$. **(2 marks)**

iii. Sketch $\left\{z: (z + 2i)\left(\bar{z} - 2i\right) = 4\right\}$ on the axes below. **(2 marks)**



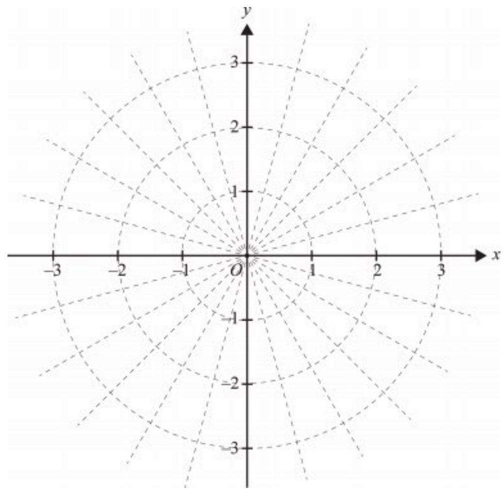
c. The line joining the points corresponding to $k - 2i$ and $-(2 + k)i$, where $k < 0$, is tangent to the curve given by $\left\{z: (z + 2i)\left(\bar{z} - 2i\right) = 4\right\}$.
Find the value of k. **(3 marks)**

Source: *VCE 2013, Specialist Mathematics, Exam 2, Section 2, Q.2; © VCAA*

Question 15 (15 marks)

Answer the following.

a. On the Argand diagram below, sketch $\{z: z\bar{z} = 4,\ z \in C\}$ **and** sketch $\{z: |z + \bar{z}| = |z - \bar{z}|, z \in C\}$. **(3 marks)**



b. Find all elements of $\{z: z\bar{z} = 4,\ z \in C\} \cap \{z: |z + \bar{z}| = |z - \bar{z}|,\ z \in C\}$, expressing your answer(s) in the form $a + ib$. **(3 marks)**

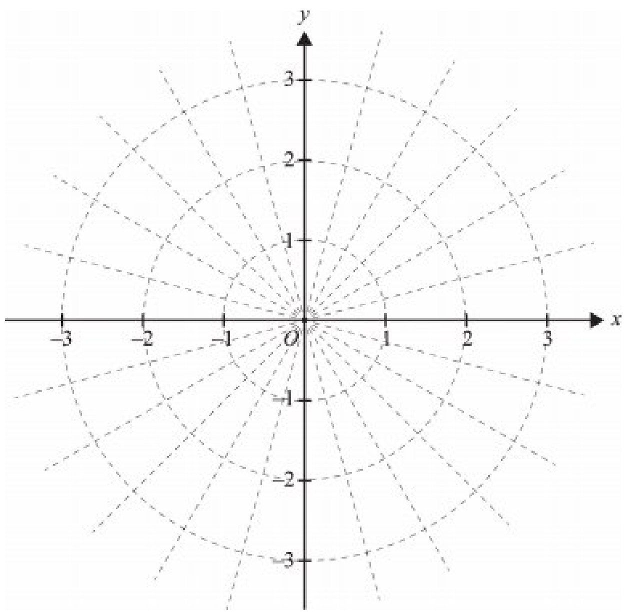
c. One of the roots of the equation $z^2 + 16 = 0$ is $z = \sqrt{2} + i\sqrt{2}$.
Write down the other roots in cartesian form.
Plot and label all of these roots on the Argand diagram provided in **part a**. **(2 marks)**

d. Express $z^4 + 16$ as the product of four linear factors in terms of z. **(1 mark)**

e. On the Argand diagram provided in **part a.**, shade the region defined by
$\{z: |z| \leq 2, z \in C\} \cap \left\{z: \text{Re}(z) \leq \sqrt{2}, z \in C\right\}$ **(1 mark)**

f. Find the area of the shaded region in **part e**. **(2 marks)**

g. On the Argand diagram below, sketch $\{z: z\bar{z} = 4, z \in C\}$ **and** sketch
$\left\{z: |z + \bar{z}| = |z - \bar{z}|, z \in C\right\}$. **(3 marks)**



Question 16 (1 mark)

If $z = -1 - i$, then $\frac{1}{z^7}$ is equal to

A. $-\frac{1}{16}(1 + i)$

B. $\frac{1}{16}(-1 + i)$

C. $\frac{1}{16}(1 - i)$

D. $8(1 - i)$

E. $-8(1 + i)$

Question 17 (1 mark)

If $P(z) = z^3 + az^2 + bz - 12$ and $P(2 - \alpha i) = 0$ and $P(2) = 0$, where a, b and α are all real numbers, and $\alpha \neq 0$, then

A. $a = -6$ and $b = -14$

B. $a = -2$ and $b = -14$

C. $\alpha = -2$

D. $\alpha = 2$

E. $\alpha = \pm\sqrt{2}$

Answers and marking guide

2.2 Complex numbers in Cartesian form

Question 1

$z = a + bi,\ a = \text{Re}(z) \in R \setminus \{0\},\ b = \text{Im}(z) \in R$

$\bar{z} = a - bi,\ \bar{z} + z = 2a,\ \bar{z}z = a^2 - b^2 i^2 = a^2 + b^2$

$$\frac{4z\bar{z}}{(z+\bar{z})^2} = \frac{4(a^2+b^2)}{4a^2}$$
$$= 1 + \frac{b^2}{a^2}$$
$$= 1 + \left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)^2$$

The correct answer is **A**.

Question 2

$i^{1!} = i,\ i^{2!} = -1,\ i^{3!} = -1,\ i^{4!} = i^{5!} = ... = i^{100!} = 1$

sum $= i - 2 + 97 = 95 + i$

$i^{1!} + i^{2!} + i^{3!} + ... + i^{100!} = 95 + i$

The correct answer is **C**.

Question 3

$$\sum_{n=1}^{4} ni^n = 1 \times i + 2 \times i^2 + 3 \times i^3 + 4 \times i^4$$
$$= i - 2 - 3i + 4 = 2 - 2i$$
$$\sum_{n=5}^{9} ni^n = 5 \times i^5 + 6 \times i^6 + 7 \times i^7 + 8 \times i^8$$
$$= 5i - 6 - 7i + 8 = 2 - 2i$$
$$\sum_{n=1}^{100} ni^n = 25\,(2 - i) = 50 - 50i$$

Alternatively, use a CAS calculator and evaluate:

$$\sum_{n=1}^{100} ni^n = 50 - 50i$$

The correct answer is **A**.

Question 4

$i^n = p$

$i^2 = -1$

$$i^{2n+3} = i^{2n} \times i^3$$
$$= (i^n)^2 \times i^2 \times i$$
$$= -ip^2$$

Question 5

$(x + yi) - (x - yi) = x - x + yi + yi$

$z - \bar{z} = 2yi$

Question 6

Equating real parts (1) $4a + 3b = -1$

Equating imaginary parts (2) $3a - 2b = 12$

$2 \times (1)\ \ 8a + 6b = -2$

$3 \times (2)\ \ 9a - 6b = 36$

Adding gives $17a = 34 \ \Rightarrow\ a = 2$ and $b = -3$

Question 7

$$\frac{1}{z}+\frac{1}{\bar{z}}=\frac{\bar{z}+z}{z\bar{z}}$$
$$=\frac{-x+iy-x-iy}{(-x+iy)(-x-iy)}$$
$$=\frac{-2x}{x^2+y^2},\quad \text{a real number}$$

Question 8

$$(x+yi)^3=x^3+3x^2yi-3xy^2-y^3i$$
$$3x^2yi-y^3i=0$$
$$y=\pm\sqrt{3}x\quad (y=0\,\text{discarded})\quad \textbf{[1 mark]}$$

Question 9

i is a rotation of 90° anti-clockwise, so i^3 is a rotation of 270° anti-clockwise or 90°clockwise.

Question 10

$\overline{w}$ is the refection of w in the real axis, which is the point R.
$i^2\overline{w}$ is a rotation by 180° anti-clockwise; this takes us to the point P.

Question 11

$$zw+\overline{zw}=(a+bi)(c+di)+(a-bi)(c-di)$$
$$=ac+adi+bci-bd+ac-adi-bci-bd$$
$$=2ac-2bd$$

Question 12

$$z^2=a^2-b^2+2abi$$
$$w^2=c^2-d^2+2cdi$$
$$\text{Re}\left(w^2\right)=c^2-d^2$$
$$\text{Im}\left(z^2\right)=2abi$$

Question 13

$$z\bar{z}=a^2-b^2$$
$$\text{Re}\left(z\bar{z}\right)=a^2-b^2$$
$$w\overline{w}=c^2-d^2$$
$$\text{Im}\left(w\overline{w}\right)=0i$$

Question 14

$$a^2b^2u^2=a^2b^2\left(\frac{1}{a}-\frac{1}{b}i\right)^2$$
$$a^2b^2u^2=a^2b^2\left(\frac{1}{a^2}-\frac{2i}{ab}+\frac{i^2}{b^2}\right)$$
$$a^2b^2u^2=b^2-2abi+a^2i^2$$
$$a^2b^2u^2=\left(b^2-a^2\right)-2abi$$

Question 15

$uv = (a + bi)\left(\frac{1}{a} - \frac{1}{b}i\right)$

$uv = 1 + \frac{b}{a}i - \frac{a}{b}i - i^2$

$uv = 2 + \left(\frac{b}{a} - \frac{a}{b}\right)i$

$uv = 2 + \left(\frac{b^2 - a^2}{ab}\right)i$

Question 16

$\frac{z}{w} = \frac{2 - 4i}{1 + 3i} \times \frac{1 - 3i}{1 - 3i}$

$= \frac{-10 - 10i}{10}$

$= -1 - i$

Question 17

$\frac{u}{v} = \frac{\frac{1}{a} + \frac{1}{b}i}{a - bi} \times \frac{a + bi}{a + bi}$

$\frac{u}{v} = \frac{1 + i^2 + \left(\frac{a}{b} + \frac{b}{a}\right)i}{a^2 - b^2 i^2}$

$\frac{u}{v} = \frac{\left(\frac{a^2 + b^2}{ab}\right)i}{a^2 + b^2} = \frac{i}{ab}$

2.3 Complex numbers in polar form

Question 1

$z = a + bi, b = im(z) > 0, z = a - bi$

$\frac{\bar{z}\,z}{z - \bar{z}} = \frac{a^2 - b^2 i^2}{2bi} = \frac{a^2 + b^2}{2bi} \times \frac{i}{i} = \frac{-\left(a^2 + b^2\right)}{2b}i$

$\text{Re}\left(\frac{\bar{z}\,z}{z - \bar{z}}\right) = 0, \text{Im}\left(\frac{\bar{z}\,z}{z - \bar{z}}\right) = \frac{-\left(a^2 + b^2\right)}{2b} < 0$

$\text{Arg}\left(\frac{\bar{z}\,z}{z - \bar{z}}\right) = -\frac{\pi}{2}$

The correct answer is **A**.

Question 2

$z = r\,\text{cis}(\theta),\ z^2 = r^2\text{cis}(2\theta)$

$\text{Im}(z^2) = 0 \Rightarrow \sin(2\theta) = 0$

$\arg(z) = \theta = \frac{\text{k}\pi}{2},\ k \in Z$

The correct answer is **A.**

Question 3

a. $z = 3 - \sqrt{3}i$

$|z| = \sqrt{(3)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{12} = 2\sqrt{3}$

$\text{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right) = -\frac{\pi}{6}$

$z = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$

Award **1 mark** for correctly converting between the two forms.

***VCAA Examination Report* note:**

Students were required to show that $3 - \sqrt{3}i = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$ and students generally did this quite well. Some particular errors were noted. Some students wrote such things as $\tan\left(\frac{\sqrt{3}}{3}\right)$ or $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6} = -\frac{\pi}{6}$.

b. $z^3 = \left(3 - \sqrt{3}i\right)^3 = \left(2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^3$

$z^3 = \left(2\sqrt{3}\right)^3 \text{cis}\left(-\frac{3\pi}{6}\right) = 24\sqrt{3}\,\text{cis}\left(-\frac{\pi}{2}\right)$

$z^3 = 24\sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 0 - 24\sqrt{3}i$

Award **1 mark** for using De Moivre's Theorem.

Award **1 mark** for the correct answer.

***VCAA Examination Report* note:**

The efficient method was to use de Moivre's theorem although some students attempted to expand $\left(3 - \sqrt{3}i\right)^3$. Students who chose the latter approach generally did not score as well.

c. $\text{Im}\left(\left(3 - \sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \sin\left(-\frac{n\pi}{6}\right) = 0$

$\sin\left(-\frac{n\pi}{6}\right) = 0,\ \frac{n\pi}{6} = k\pi$

$n = 6k,\ k \in Z$ **[1 mark]**

n is an integer multiple of 6.

***VCAA Examination Report* note:**

There were several ways to answer this question. Some students realised that if n was a positive or negative multiple of 6 then $\left(3 - \sqrt{3}i\right)^n$ was real, but were unable to express this mathematically. Some students did not indicate that k was a member of Z, the set of integers.

d. $\left(3 - \sqrt{3}i\right)^n = ai,\ a \in R$

$\text{Re}\left(\left(3 - \sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \cos\left(-\frac{n\pi}{6}\right) = 0$

$\cos\left(\frac{n\pi}{6}\right) = 0,\ \frac{n\pi}{6} = \frac{\pi}{2} + k\pi$

$n = 6k + 3,\ k \in Z$ **[1 mark]**

n is an odd integer multiple of 3.

***VCAA Examination Report* note:**

This question was answered poorly. There were a number of equivalent correct answers but many students were unable to find a general solution.

Question 4

$\text{Arg}(z) = \frac{\pi}{2}$, $\text{Arg}(w) = \frac{\pi}{4}$

$$\arg\left(\frac{z^5}{w^4}\right) = 5 \times \frac{\pi}{2} - 4 \times \frac{\pi}{4} = \frac{3\pi}{2}$$

$$\text{Arg}\left(\frac{z^5}{w^4}\right) = \frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$$

Question 5

$\text{Arg}(-1 + ai) = -\frac{2\pi}{3}$,

The complex number is in the third quadrant.

$$\tan^{-1}(-a) = -\frac{2\pi}{3} + \pi = \frac{\pi}{3}$$

$$-a = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\Rightarrow a = -\sqrt{3}$$

Question 6

$$\begin{aligned}\frac{\left(1 - \sqrt{3}i\right)^4}{1 + \sqrt{3}i} &= \frac{\left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^4}{2\,\text{cis}\left(\frac{\pi}{3}\right)} \\ &= \frac{2^4\,\text{cis}\left(-\frac{4\pi}{3}\right)}{2\,\text{cis}\left(\frac{\pi}{3}\right)} \\ &= 2^3\,\text{cis}\left(-\frac{4\pi}{3} - \frac{\pi}{3}\right) \\ &= 8\,\text{cis}\left(-\frac{5\pi}{3}\right) \\ &= 8\,\text{cis}\left(\frac{\pi}{3}\right) \\ &= 8\cos\left(\frac{\pi}{3}\right) + 8i\sin\left(\frac{\pi}{3}\right) \\ &= 8 \times \frac{1}{2} + 8i \times \frac{\sqrt{3}}{2} \\ &= 4 + 4\sqrt{3}i\end{aligned}$$

Award **1 mark** for expressing both complex numbers in rectangular form.
Award **1 mark** for correctly applying DeMoivre's theorem and division of complex numbers in polar form.
Award **1 mark** for the final correct result.

***VCAA Assessment Report* note:**

This question was well answered overall. Those students who used polar form tended to have greater success than those who tried to solve the equation in Cartesian form and often made algebraic or arithmetical errors. It was common for the incorrect argument to be used, usually due to the incorrect quadrant but sometimes due to not knowing exact values. A sketch may have been helpful. Many sign errors were seen. An elegant solution used the fact that the numerator turns out to be four times the square of the denominator.

Question 7

$z = \frac{1 + i\sqrt{3}}{1 + i} = \frac{2\,\text{cis}\left(\frac{\pi}{3}\right)}{\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)} = \sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)$

$\text{Arg}\left(z^5\right) = \frac{5\pi}{12}$

$\left|z^5\right| = 4\sqrt{2}$

This was solved using CAS.

Question 8

Require $r_2 < r_1$, since $z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$. **A, B, D** and **E** are incorrect.

Question 9

$z = 2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

$z^2 = \left(2\sqrt{2}\right)^2\,\text{cis}\left(2 \times \frac{3\pi}{4}\right)$

$z^2 = 8\,\text{cis}\left(\frac{3\pi}{2}\right)$

$z^2 = -8i$

Question 10

$\text{Arg}\left(\frac{-3\sqrt{2} - i\sqrt{6}}{2 + 2i}\right) = \frac{11\pi}{12}$ by CAS

Question 11

$z = r\,\text{cis}(\theta)\frac{z^2}{\bar{z}} = \frac{r^2\,\text{cis}(2\theta)}{r\,\text{cis}(-\theta)} = r\,\text{cis}(2\theta - (-\theta)) = r\,\text{cis}(3\theta)$

Question 12

$z = a + bi,\ z \in C$

$\frac{\pi}{2} < \text{Arg}\left(z^3\right) < \pi$ then $\frac{\pi}{6} < \text{Arg}(z) < \frac{\pi}{3}$

also if $-\frac{\pi}{2} < \text{Arg}(z) < -\frac{\pi}{3}$ then $-\frac{3\pi}{2} < \text{Arg}\left(z^3\right) < -\pi \Leftrightarrow \frac{\pi}{2} < \text{Arg}\left(z^3\right) < \pi$

so $\text{Arg}(z) \in \left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

***VCAA Assessment Report* note:**

All students were awarded marks for this question.

Question 13

$\frac{p + 3i}{4 + pi} = \frac{p + 3i}{4 + pi} \times \frac{4 - pi}{4 - pi}$

$\frac{p + 3i}{4 + pi} = \frac{4p + 12i - p^2 i - 3pi^2}{16 - p^2 i^2}$

$\frac{p + 3i}{4 + pi} = \frac{7p + \left(12 - p^2\right)i}{16 + p^2}$

If $\text{Im}\left(\frac{p + 3i}{4 + pi}\right) = 0 \Rightarrow \text{Im}\left(\frac{7p + \left(12 - p^2\right)i}{16 + p^2}\right) = 0$

$\Rightarrow 12 - p^2 = 0 \Rightarrow p \pm 2\sqrt{3}$

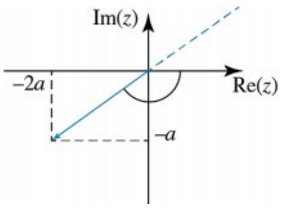
Question 14

$$4\,\text{cis}(-120°) = 4\,\text{cis}\left(\frac{4\pi}{3}\right) = 4\,\text{cis}\left(-\frac{2\pi}{3}\right) = -4\,\text{cis}(60°) = -2 - 2\sqrt{3}i$$

$$-4\,\text{cis}\left(\frac{2\pi}{3}\right) = -4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2 - 2\sqrt{3}i$$

Question 15

z is a complex number in the 3rd quadrant.



$$\text{Arg}(z) = \tan^{-1}\left(\frac{1}{2}\right) - \pi$$

Question 16

$$\frac{v}{u} = \frac{a}{5}\,\text{cis}\left(b - \frac{\pi}{4}\right) = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$$

$$\therefore \frac{a}{5} = 2$$

$$a = 10$$

$$b - \frac{\pi}{4} = \frac{5\pi}{5}$$

$$b = \frac{5\pi}{6} + \frac{\pi}{4} = \frac{13\pi}{12}$$

$$b = -\frac{11\pi}{12}$$

Question 17

$$uv = 2a\,\text{cis}\left(b + \frac{\pi}{3}\right) = 3\,\text{cis}\left(\frac{3\pi}{5}\right)$$

$$\therefore 2a = 3$$

$$a = \frac{3}{2}$$

$$b + \frac{\pi}{3} = \frac{3\pi}{5}$$

$$b = \frac{3\pi}{5} - \frac{\pi}{3} = \frac{4\pi}{15}$$

Question 18

$$uv = (3\,\text{cis}(\theta))\left(r\,\text{cis}\left(\frac{2\pi}{3}\right)\right) = 3r\,\text{cis}\left(\frac{2\pi}{3} + \theta\right) = -6i = 6\,\text{cis}\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow 3r = 6 \Rightarrow r = 2 \text{ and } \frac{2\pi}{3} + \theta = -\frac{\pi}{2}, \theta = -\frac{\pi}{2} - \frac{2\pi}{3} = -\frac{7\pi}{6}$$

However, we can add 2π to the value of θ.

$$\theta = -\frac{7\pi}{6} + 2\pi = \frac{5\pi}{6}$$

Question 19

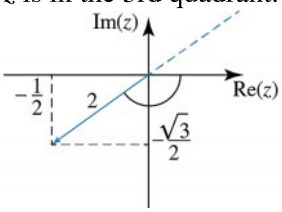
$$\frac{u}{v} = \frac{r\,\text{cis}(\theta)}{2\,\text{cis}\left(-\frac{5\pi}{3}\right)} = \frac{r}{2}\,\text{cis}\left(\theta + \frac{5\pi}{3}\right) = -6 = 6\,\text{cis}(\pi)$$

$$\frac{r}{2} = 6 \Rightarrow r = 12 \text{ and } \theta + \frac{5\pi}{3} = \pi, \theta = \pi - \frac{5\pi}{3} = -\frac{2\pi}{3}$$

Question 20

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

z is in the 3rd quadrant.



$$\text{Arg}(z) = -\pi + \tan^{-1}\left(\sqrt{3}\right) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$$

$\text{Arg}\left(z^3\right) = 3 \times -\frac{2\pi}{3} = -2\pi$ but $-\pi < \text{Arg}(z) \le \pi$, so that

$$\text{Arg}\left(z^3\right) = -2\pi + 2\pi = 0$$

2.4 Solving polynomial equations over *C*

Question 1

a. $z^2 + 2z + 2 = 0$

$z^2 + 2z + 1 = -1 = i^2$

$(z+1)^2 = i^2$

$z + 1 = \pm i$

$z = -1 \pm i$ **[1 mark]**

b. $z^2 + 2\bar{z} + 2 = 0$

Let $z = a + bi, a, b \in R, z \in C$

$(a+bi)^2 + 2(a - bi) + 2 = 0$

$a^2 + 2abi + b^2i^2 + 2a - 2bi + 2 = 0$

$\left(a^2 + 2a + 2 - b^2\right) + i(2ab - 2b) = 0$

Re: (1) $a^2 + 2a + 2 - b^2 = 0$

Im: (2) $2ab - 2b = 2b(a-1) = 0$

(2) $\Rightarrow a = 1 \; or \; b = 0$

If $b = 0$ then $a^2 + 2a + 2 = 0 \Rightarrow a = 1 \pm i$, but $a \in R$

So $a = 1, b^2 = 1 + 2 + 2 = 5, b = \pm\sqrt{5}$

$z = 1 \pm \sqrt{5i}$

Award **1 mark** for setting-up using Cartesian form.

Award **1 mark** for equating real and imaginary parts.

Award **1 mark** for the final correct values of z

Question 2

$P(z) = z^3 + az^2 + bz + c$

$P(-2) = 0,\ P(3i) = 0$

$z = -2,\ z = \pm 3i$

$P(z) = (z+2)(z^2 = 9)$

$P(z) = z^3 + 2z^2 + 9z + 18$

$a = 2,\ b = 9,\ c = 18$

The correct answer is **C**.

Question 3
Solve the equation to determine the number of distinct roots.
$(z^4 - 1)(z^2 + 3iz - 2) = 0$
$(z + i)^2 (z - i)(z + 2i)(z - 1)(z + 1) = 0$
$z = -i,\ i,\ -2i,\ -1,\ 1$
There are 5 distinct roots
To solve this equation using CAS, complete the entry line as:
cSolve $((z^4 - 1)(z^2 + 3iz - 2) = 0,\ z)$, remembering to use the correct symbol for i, rather than the variable i.
The correct answer is **D**.

Question 4
a. $z = 3 - \sqrt{3}i$

$$|z| = \sqrt{(3)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{12} = 2\sqrt{3}$$

$$\text{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

$$z = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$$ **[1 mark]**

***VCAA Examination Report* note:**

Students were required to show that $3 - \sqrt{3}i = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$ and students generally did this quite well. Some particular errors were noted. Some students wrote such things as

$\tan\left(\frac{\sqrt{3}}{3}\right)$ or $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6} = -\frac{\pi}{6}$.

b. $z^3 = \left(3 - \sqrt{3}i\right)^3 = \left(2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^3$

$$z^3 = \left(2\sqrt{3}\right)^3 \text{cis}\left(-\frac{3\pi}{6}\right) = 24\sqrt{3}\,\text{cis}\left(-\frac{\pi}{2}\right)$$

$$z^3 = 24\sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 0 - 24\sqrt{3}i$$

Award **1 mark** for using De Moivre's Theorem.
Award **1 mark** for the correct answer.
***VCAA Examination Report* note:**
The efficient method was to use de Moivre's theorem although some students attempted to expand $\left(3 - \sqrt{3}i\right)^3$. Students who chose the latter approach generally did not score as well.

c. $\text{Im}\left(\left(3 - \sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \sin\left(-\frac{n\pi}{6}\right) = 0$

$$\sin\left(-\frac{n\pi}{6}\right) = 0,\ \frac{n\pi}{6} = k\pi$$

$n = 6k,\ k \in Z$
n is an integer multiple of 6. [**1 mark**]
***VCAA Examination Report* note:**
There were several ways to answer this question. Some students realised that if n was a positive or negative multiple of 6 then $\left(3 - \sqrt{3}i\right)^n$ was real, but were unable to express this mathematically. Some students did not indicate that k was a member of Z, the set of integers.

d. $\left(3-\sqrt{3}i\right)^n = ai,\ a \in R$

$\text{Re}\left(\left(3-\sqrt{3}i\right)^n\right) = \left(2\sqrt{3}\right)^n \cos\left(-\frac{n\pi}{6}\right) = 0$

$\cos\left(\frac{n\pi}{6}\right) = 0,\ \frac{n\pi}{6} = \frac{\pi}{2} + k\pi$

$n = 3 + 6k,\ k \in Z$

n is an odd integer multiple of 3. [**1 mark**]

***VCAA Examination Report* note:**

This question was answered poorly. There were a number of equivalent correct answers but many students were unable to find a general solution.

Question 5

$z^n = 1 + i,\ n \in Z^+$

$$z^n = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$$
$$= 2^{\frac{1}{2}}\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$$
$$z = 2^{\frac{1}{2n}}\text{cis}\left(\frac{\pi}{4n} + \frac{2k\pi}{n}\right),\ k \in Z$$

Question 6

$z^3 + az^2 + 6z + a = 0$,where $a \in R$

Given that $z = 1 - i$, by the conjugate root theorem then $\bar{z} = 1 + i$ is also a root.

Now $z + \bar{z} = 2,\ z\bar{z} = 1 - i^2$, so that $z^2 - 2z + 2$ is a factor

$z^3 + az^2 + 6z + a = 0$

$\left(z^2 - 2z + 2\right)(z + k) = 0$, expanding

$z^3 + (k-2)z^2 + (2-2k)z + 2k = 0$

Equating coefficients

$\Rightarrow 6 = 2 - 2k$

$\Rightarrow k = -2.$

$z^3 + az^2 + 6z - a = 0$

$\left(z^2 - 2z + 2\right)(z - 2) = 0$

$\therefore$ The roots are $z = 2,\ \ z = 1 + i,\ z = 1 - i$

Award **1 mark** for using the conjugate root theorem.

Award **1 mark** for the correct quadratic.

Award **1 mark** for all correct roots.

***VCAA Examination Report* note:**

Students generally performed well on this question, with most students able to obtain at least two marks. Typical errors included:

- giving a second solution as $-1 - i$
- correctly giving $1 + i$ as a second solution then multiplying this by the given solution to get 2 and stating 2 as the third solution, which was a correct answer but incorrect reasoning
- not being able to correctly determine G. Students could correctly find $a = -4$ but were unable to get the third solution.

A small number of students expressed the real solution in terms of a. Some students quoted the answer as factors rather than solutions. Those who attempted to use polar form were unsuccessful.

Question 7

$\alpha = 3 - 2i,\ \overline{\alpha} = 3 + 2i$

$\alpha + \overline{\alpha} = 6,\ \alpha\overline{\alpha} = 9 - 4i^2 = 13$

$z^2 - (\text{sum of roots})\,z + \text{product of roots} = 0$

$z^2 - 6z + 13$ is one factor.

$z(z^2 - 6z + 13) = z^3 - 6z^2 + 13z = 0$

$\Rightarrow b = -6,\ c = 13$

Question 8

a. $z^3 = 8i, z \in C$

$$z^3 = 8\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)$$

$$z = 8^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$$

$$k = 0, z = 2\text{cis}\left(\frac{\pi}{6} = \sqrt{3} + i\right)$$ **[1 mark]**

$$k = 1, z = 2\text{cis}\left(\frac{5\pi}{6}\right) = -\sqrt{3} + i$$ **[1 mark]**

$$k = 1, z = 2\text{cis}\left(-\frac{\pi}{2}\right) = -2i$$ **[1 mark]**

VCAA Assessment Report **note:**

This question was quite well answered by students who used polar form, but not by the small number of students who tried to solve the equation in cartesian form. The most common errors included finding the incorrect polar form for $8i$ or finding the correct polar form for $8i$ but making errors in finding the other two solutions. Some students who found the correct solutions in polar form either left them in polar form or converted them to cartesian form with arithmetical errors. Many students assumed that the Conjugate Root Theorem applied. Others tried to use the formula for perfect cubes. Some gave factors rather than solutions, and a number of students gave only one solution for this cubic.

b. $(z - 2i)^3 = 8i,\ z \in C$

$z - 2i = \sqrt{3} + i,\ -\sqrt{3} + i,\ -2i$

$z = \sqrt{3} + 3i,\ -\sqrt{3} + 3i,\ 0$ **[1 mark]**

VCAA Assessment Report **note:**

Students were expected to recognise that the solutions to Question 4a. needed to be translated two units to the right, and so add $2i$. Several students subtracted $2i$ from the answers in part **a.**, and a small number tried to solve the equation without using their answer to part **a**.

Question 9

a. $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$

By the conjugate root theorem,

$f(i) = f(-i) = 0,$

So $(z + i)(z - i) = z^2 - i^2 = z^2 + 1$ is a quadratic factor.

Award **1 mark** for stating the conjugate root theorem.

Award **1 mark** for the correct length.

VCAA Assessment Report **notes:**

Most students identified the need to use the conjugate root theorem but some then gave $z^2 - 1$ as their answer. Confusion between solutions and factors was often evident. Several students gave the conjugate root or factor and then did no further work. Some students seemed not to realise that they had completed what was required and found the second quadratic factor. Some quoted $z = \pm 1$ as solutions rather than $z = \pm i$

b. $z^4 - 4z^3 + 7z^2 - 4z + 6 = 0$

$\left(z^2+1\right)\left(z^2-4z+6\right)=0$

$\left(z^2+1\right)\left(z^2-4z+4+2\right)=0$

$\left(z^2+1\right)\left((z-2)^2+2\right)=0$

$\left(z^2-i^2\right)\left((z-2)^2-2i^2\right)=0$

$(z+i)(z-i)\left(z-2+\sqrt{2}i\right)\left(z-2-\sqrt{2}i\right)=0$

$z=\pm i,\ 2\pm\sqrt{2}i$

Award **1 mark** for division and finding the other quadratic factor.
Award **1 mark** for solving the other quadratic by completing the square or using formulae.
Award **1 mark** for all three correct solutions.

***VCAA Assessment Report* notes:**

Students made many sign errors and other algebraic errors in finding the second quadratic factor. Some students, having found this factor, gave $-2\pm\sqrt{2}i,\ 2\pm 2i$ or $2\pm\sqrt{10}i$ as the solution. Some confusion between solutions and factors was evident.

Question 10

$z^4-2z^2+4=0,\ z\in C$

$z^4-2z^2+1=-4+1=-3$

$\left(z^2-1\right)^2=3i^2$

$z^2-1=\pm\sqrt{3}i$

$z^2=1\pm\sqrt{3}i$

Let $z=a+bi, a, b\in R$

$z^2=a^2+2abi+b^2i^2=a^2-b^2+2abi$

$\text{Re}(1)\Rightarrow a^2-b^2=1$

$\text{Im}(2)\Rightarrow 2ab=\pm\sqrt{3}$

$b=\pm\dfrac{\sqrt{3}}{2a}$ into (1)

$a^2-\dfrac{3}{4a^2}=1$

$4a^4-4a^2-3=0$

$\left(2a^2+1\right)\left(2a^2-3\right)=0\quad a\in R$

$a=\pm\sqrt{\dfrac{3}{2}}=\pm\dfrac{\sqrt{6}}{2}\Rightarrow b=\pm\dfrac{\sqrt{3}}{2a}=\pm\dfrac{\sqrt{3}}{\sqrt{6}}=\pm\dfrac{1}{\sqrt{2}}=\pm\dfrac{\sqrt{2}}{2}$

Altogether there are four solutions:

$z=\dfrac{\sqrt{6}}{2}+\dfrac{\sqrt{2}}{2}i,\dfrac{\sqrt{6}}{2}-\dfrac{\sqrt{2}}{2}i,-\dfrac{\sqrt{6}}{2}+\dfrac{\sqrt{2}}{2}i,-\dfrac{\sqrt{6}}{2}-\dfrac{\sqrt{2}}{2}i$

$z=\dfrac{\sqrt{6}}{2}\pm\dfrac{\sqrt{2}}{2}i,-\dfrac{\sqrt{6}}{2}\pm\dfrac{\sqrt{2}}{2}i$

Award **1 mark** for attempting to solve the quartic.
Award **1 mark** for the correct method.
Award **2 marks** for all four correct final answers.

***VCAA Examination Report* note:**
There were many good attempts at this question, using either completing the square or the quadratic formula, so many students were able to find that $z^2 = 1 \pm i\sqrt{3}$. The most frequently occurring answer from that point was $z = \pm\sqrt{1 \pm i\sqrt{3}}$, which is not of the form $z = x \pm iy$. Those who used polar form often achieved correct answers, although a few forgot to change back to Cartesian form. Some students made errors in the sine and cosine of standard angles. A few students solved the question successfully by other methods and did not need to use polar form. Solving $(x + iy)^2 = 1 \pm i\sqrt{3}$ was a viable approach but most who used this approach struggled with the algebra. There were some interesting attempts including treating the original expression as a perfect square as well as working that involved the factor theorem.

Question 11

$z^2 + a = z^2 - i^2\left(\sqrt{a}\right)^2 = \left(z + \sqrt{a}i\right)\left(z - \sqrt{a}i\right) = 0$

$z = \pm i\sqrt{a}\quad \text{for}\quad a > 0$

Question 12

$$z^2 = 4\text{cis}\left(-\frac{\pi}{3}\right),\ r = 2$$
$$2\theta = -\frac{\pi}{3}, -\frac{\pi}{3} + 2\pi$$
$$\theta = -\frac{\pi}{6}, \frac{5\pi}{6}$$

$2\text{cis}\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$

$2\text{cis}\left(\frac{5\pi}{6}\right) = -\sqrt{3} + i$

Question 13

$z^2(z-3) + 6(z-3) = (z-3)\left(z^2+6\right)$

$= (z-3)\left(z + \sqrt{6}i\right)\left(z - \sqrt{6}i\right)$

Question 14

$P(z) = 0$ has one repeated real root and three non-real roots.

Question 15

By grouping,

$z^3 - 3iz^2 + 3z - 9i = z^2(z - 3i) + 3(z - 3i) = \left(z^2 + 3\right)(z - 3i)$

$z^3 - 3iz^2 + 3z - 9i = \left(z + \sqrt{3}i\right)\left(z - \sqrt{3}i\right)(z - 3i)$

Question 16

A polynomial of degree 6 must have an even number of non-real roots; therefore, B is a false statement.

Question 17

$z^4 + 16 = 0$

$$z^4 = -16 = 16\text{cis}(\pi + 2k\pi)$$

$$z = 16^{\frac{1}{4}}\text{cis}\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right) = 4\text{cis}\left(\frac{(2k+1)\pi}{4}\right)$$

$$k = z = 4\text{cis}\left(\frac{\pi}{4}\right) = 4\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}(1+i)$$

$$k = z = 4\text{cis}\left(\frac{3\pi}{4}\right) = 4\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}(-1+i) = -\sqrt{2}(1-i)$$

$$k = z = 4\text{cis}\left(-\frac{\pi}{4}\right) = 4\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}(1-i)$$

$$k = z = 4\text{cis}\left(-\frac{3\pi}{4}\right) = 4\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}(-1-i) = -\sqrt{2}(1+i)$$

Question 18

Answer E

$$\left(z - 1 + \sqrt{3}i\right)\left(z - 1 - \sqrt{3}i\right)$$
$$= z^2 - 2z + 4$$

$P(z) = \left(z^2 - 2z + 4\right)(z + k)$

$P(z) = z^3 + kz^2 - 2z^2 - 2kz + 4z + 4k$

$P(z) = z^3 + (k-2)z^2 + (4-2k)z + 4k$

$4 - 2k = -2$

$k = 3$

$\therefore a = 12$

Question 19

Answer B

$P(-2) = (-2)^3 + 2(-2)^2 + 5(-2) + 10$

$= 0$

$P(z) = (z+2)\left(z^2+5\right)$

$= (z+2)\left(z+\sqrt{5}i\right)\left(z-\sqrt{5}i\right)$

Question 20

Now $u + \overline{u} = 2$ and $u\overline{u} = 1 - i^2 = 2$

Let $v = \sqrt{2}i$ by the conjugate root theorem. $\overline{v} = \sqrt{2}i$ is also a root.

Now $v + \overline{v} = 0\,\text{M}$ and $v + \overline{v} = -2i^2 = 2$

$(z-u)\left(z-\overline{u}\right)(z-v)\left(z-\overline{v}\right) = \left(z^2 - \left(u+\overline{u}\right) + u\overline{u}\right)\left(z^2 - \left(v+\overline{v}\right) + v\overline{v}\right)$

$\left(z^2 - 2z + 2\right)\left(z^2 + 2\right) = \left(z^4 - 2z^3 + 4z^2 - 4z + 4\right)$

Coefficient of z^3: $\Rightarrow b = -2$

Coefficient of z^2: $\Rightarrow c = 4$

Coefficient of z: $\Rightarrow d = -4$

2.5 Subsets of the complex plane

Question 1

The circle is centred around the point z = $2 + \sqrt{3}$i. $|z|$ represents the distance of a complex number from the origin. The maximum distance from the origin will be the distance of the centre of the circle from the origin, plus the radius of the circle.

$$|z|_{max} = \left|2 + \sqrt{3}i\right| + 1$$
$$= \sqrt{2^2 + \left(\sqrt{3}\right)^2} + 1$$
$$= \sqrt{4 + 3} + 1$$
$$= \sqrt{7} + 1$$

The correct answer is **D**.

Question 2

a. $u = -2 - i, v = -4 - 3i$

$|z - u| = |z - v|, z = x + yi$

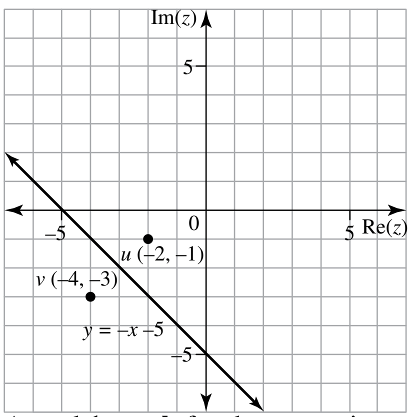
$$|(x + 2) + i(y + 1)| = |(x + 4) + i(y + 3)|$$
$$(x + 2)^2 + (y + 1)^2 = (x + 4)^2 + (y + 3)^2$$
$$x^2 + 4x + 4 + y^2 + 2y + 1 = x^2 + 8x + 16 + y^2 + 6y + 9$$
$$4x + 4y + 20 = 0$$
$$y = -x - 5, m = -1, c = -5$$

Award **1 mark** for expressing in modulus form.

Award **1 mark** for simplifying.

Award **1 mark** for the correct line.

b.



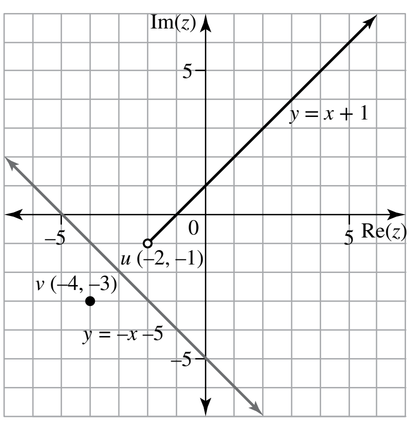
Award **1 mark** for the two points.

Award **1 mark** for the correctly graphed line.

c. The line is the perpendicular bisector of the line joining u and v, or the set points equidistant from both u and v.

Award **1 mark** for the correct geometrical interpretation.

d. i.



Award **1 mark** for correctly sketching the ray on the diagram in part **b**.

Note this diagram must show an open circle at u.

ii. $\text{Arg}(z-u)=\dfrac{\pi}{4}$

$\text{Arg}((x+2)+i(y+1))=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{y+1}{x+2}\right)=\dfrac{\pi}{4}\text{for } x>-2$

$\dfrac{y+1}{x+2}=1$

$y=x+1 \text{ for } x>-2$

A ray with open circle at $u\,(-2,-1)$

Award **1 mark** for the correct line.

e. $z_c=a+bi$

$|z-z_c|=r,\ (x-a)^2+(y-b)^2=r^2$

$(-2,\ -1)\ (1)\ (-2-a)^2+(-1-b)^2=r^2$

$(-4,\ -3)\ (2)\ (-4-a)^2+(-3-b)^2=r^2$

$(0,\ -5)\ (3)\ a^2+(-5-b)^2=r^2$

Solving using CAS:

$a=-\dfrac{5}{3}, b=-\dfrac{10}{3}, r=\dfrac{5\sqrt{2}}{3}$

$z_c=-\dfrac{5}{3}-\dfrac{10}{3}i=-\dfrac{5}{3}(1+2i)$

Award **1 mark** for correctly setting up three equations.

Award **1 mark** for the correct value of a and b.

Award **1 mark** for the correct value of r.

Question 3

$\text{Arg}(z-2)=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{y}{x-2}\right)=\dfrac{\pi}{4}, x>2$

$\dfrac{y}{x-2}=\tan\left(\dfrac{\pi}{4}\right)=1$

$(1)\ y=x-2, x>2$

$\text{Arg}(z-(5+i))=\dfrac{5\pi}{6}$

$\tan^{-1}\left(\dfrac{y-1}{x-5}\right)=\dfrac{5\pi}{6}, x<5$

$\dfrac{y-1}{x-5}=\tan\left(\dfrac{5\pi}{6}\right)$

$(2)\ y=\dfrac{\sqrt{3}\,(5-x)}{3}+1, x<5$

$(1)\ x=y+2, \text{into } (2)\ y=\dfrac{\sqrt{3}\,(5-x)}{3}+1$

$y=\dfrac{\sqrt{3}\,(5-y-2)}{3}+1=\dfrac{(3-y)}{\sqrt{3}}+1$

$\sqrt{3}\,(y-1)=3-y$

$y\left(\sqrt{3}+1\right)=3+\sqrt{3}$

$y=\dfrac{3+\sqrt{3}}{\sqrt{3}+1}\times\dfrac{\sqrt{3}-1}{\sqrt{3}-1}=\dfrac{3-3+2\sqrt{3}}{3-1}=\sqrt{3}$

The correct answer is **D**.

Question 4

$|z-2+i|=|z-4|,\ z=x+yi$

$|x-2+(y+1)i|=|x-4+yi|$

$\sqrt{(x-2)^2+(y+1)^2}=\sqrt{(x-4)^2+y^2}$

$x^2-4x+4+y^2+2y+1=x^2-8x+16+y^2$

$4x+2y=11$ satisfied by $x=3, y=-\dfrac{1}{2}$.

Alternatively, considering the points between $z=2-i$ and $z=4$ the point $z=3-\dfrac{i}{2}$ lies on the midpoint, that is the perpendicular bisector.

Question 5

$a^2x^2+\left(1-a^2\right)y^2=c^2 \Rightarrow \dfrac{x^2}{\frac{c^2}{a^2}}+\dfrac{y^2}{\frac{c^2}{1-a^2}}=1$

Therefore a circle, an ellipse, a hyperbola and a pair of lines are all possible, $c=0, a=\dfrac{1}{2}y=\pm$....

A single straight line is not possible.

Question 6

$z=1+2i,\ \bar{z}=1-2i$

$z\cdot\bar{z}=1-4i^2=5,\ \text{Arg}(z)=\tan^{-1}(2)$

$z+\bar{z}=2$

Question 7

$|z-i|=|z+2|$

$|x+(y-1)|=|((x+2)+yi)|$

$\sqrt{x^2+(y-1)^2}=\sqrt{(x+2)^2+y^2}$

$x^2+y^2-2y+1=x^2+4x+4+y^2$

$4x+2y+3=0$ is a line; all the rest are circles.

Question 8

$$|z-3-2i|=2$$
$$|x-3+(y-2)i|=2$$
$$\sqrt{(x-3)^2+(y-2)^2}=2$$
$$x^2-6x+9+y^2-4y+4=4$$
$$x^2-6x+y^2-4y+13=4$$
$$|z-3-2i|=|z-5|$$
$$|x-3+(y-2)i|=|x-5+yi|$$
$$\sqrt{(x-3)^2+(y-2)^2}=\sqrt{(x-5)^2+y^2}$$
$$x^2-6x+y^2-4y+13=x^2-10x+25+y^2$$
$$4x-4y=12$$
$$x-y=3$$
$$y=x-3$$

The line $y=x-3$ intersects $x^2-6x+y^2-4y+13=4$ at $(3,0)$ and $(5,2)$.

Question 9

$|z|>b$ is the set of points **outside** a circle of radius b, centred at the origin.

Question 10

$\left\{z: \arg(z+3i) = \frac{2\pi}{3}\right\}$

Question 11

Let $x + yi$ be a point on the perpendicular bisector.

$(x-2)^2 + (y-3)^2 = (x-1)^2 + (y+1)^2$

$-4x + 4 - 6y + 9 = -2x + 1 + 2y + 1$

$-2x - 8y + 11 = 0$ **[1 mark]**

Question 12

$\left\{z: \arg(z+3-2i) = \frac{5\pi}{6}\right\}$

Question 13

All of B, C, D and E are correct; they all give $y = x$.

A is incorrect as it does not include the origin.

Question 14

The set of points which are equidistant from the two complex numbers u and v is given by

$\{z: |z-u| = |z-v|\}$

$\{z: |z+3| = |z-3i|\}$

Let $z = x + yi$

$|x + yi + 3| = |x + yi - 3i|$

$|(x+3) + yi| = |x + (y-3)i|$

$\sqrt{(x+3)^2 + y^2} = \sqrt{x^2 + (y-3)^2}$

$x^2 + 6x + 9 + y^2 = x^2 + y^2 - 6y + 9$

$6x = -6y$

$y = -x$

Question 15

Let $z = x + yi$ $\bar{z} = x - yi$ $c = a + bi$ $\bar{c} = a - bi$

$Now\ (z-c)(\bar{z}-\bar{c}) = z\bar{z} - c\bar{z} - z\bar{c} + c\bar{c}$

$= (x+yi)(x-yi) - (a+bi)(x-yi) - (x+yi)(a-bi) + (a+bi)(a-bi)$

$= (x^2 - y^2i^2) - (ax + bxi - ayi - byi^2) - (ax + ayi - bxi - byi^2) + (a^2 - b^2i^2)$

$= x^2 + y^2 - 2ax + 2byi^2 + (a^2 + b^2) = (x^2 - 2ax + a^2) + (y^2 - 2by + b^2)$

$= (x-a)^2 + (y-b)^2$

So $(z-c)(\bar{z}-\bar{c}) = r^2$ represents $(x-a)^2 + (y-b)^2 = r^2$, which is a circle with centre $c = a + bi$ and radius r.

In our case, $c = -4 + 4i$, $\bar{c} = -4 - 4i$ and the radius is 4.

$\{z: (z+4-4i)(\bar{z}+4+4i) = 16\}$ specifies the circle.

A circle with centre $c = a + bi$ and radius r is also represented as $\{z: |z-c| = r\}$, so E has the wrong radius and is therefore incorrect.

Question 16

$\{z: |z-3| + |z+3| = 10\}$

Let $z = x + yi$

$|(x-3) + yi| + |(x+3) + yi| = 10$

$\sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10$

$$So\sqrt{(x-3)^2+y^2} = 10-\sqrt{(x+3)^2+y^2} \text{ squaring both sides gives}$$

$$(x-3)^2+y^2 = 100+(x+3)^2+y^2-20\sqrt{(x+3)^2+y^2}$$

$$x^2-6x+9+y^2 = 100+x^2+6x+9+y^2-20\sqrt{(x+3)^2+y^2}$$

$$20\sqrt{(x+3)^2+y^2} = 12x+100 \text{ for } x > -\frac{25}{3}$$

$$5\sqrt{(x+3)^2+y^2} = 3x+25 \text{ squaring both sides again gives}$$

$$25\left(\left(x^2+6x+9\right)+y^2\right) = 9x^2+150x+625$$

$$25x^2+150x+225+25y^2 = 9x^2+150x+625$$

$$16x^2+25y^2 = 400$$

$$\frac{x^2}{25}+\frac{y^2}{16} = 1$$

This is an ellipse.

Question 17

$|z-2|+|z+2| = 6$ represents an ellipse. All the others represent circles.

Question 18

$y+2 = 3(x-2)^2-1$

$y = 3x^2-12x+9$

Question 19

$y-1 = 2(x+1)^3$

$y = 2x^3+6x^2+6x+3$

Question 20

$y^2x^3 = 8$

$$y = \sqrt{\frac{8}{x^3}}$$

$$= 2\sqrt{\frac{2}{x^3}}$$

Question 21

$$\left\{z\text{: Im}\left(\frac{z-ai}{z-b}\right) = 0\right\}$$

Let $z = x+yi$

$$\text{Im}\left(\frac{(x+yi)-ai}{(x+yi)-b}\right) = 0 \Rightarrow \text{Im}\left(\frac{x+(y-a)\,i}{(x-b)+yi}\right) = 0$$

$$\text{Im}\left(\frac{x+(y-a)\,i}{(x-b)+yi}\times\left(\frac{(x-b)-yi}{(x-b)-yi}\right)\right) = 0$$

$$\Rightarrow (x-b)\,(y-a)-xy = 0$$

$$xy-by-ax+ab-xy = 0$$

$$ax+by = ab \text{ since } ab \neq 0$$

This represents a straight line.

Question 22

$\{z: |z+3| - |z-3| = 2\}$

Let $z = x + yi$

$|(x+3) + yi| - |(x-3) + yi| = 2$

$\sqrt{(x+3)^2 + y^2} - \sqrt{(x-3)^2 + y^2} = 2$

So $\sqrt{(x+3)^2 + y^2} = 2 + \sqrt{(x-3)^2 + y^2}$

Squaring both sides gives

$$(x+3)^2 + y^2 = 4 + (x-3)^2 + y^2 + 4\sqrt{(x-3)^2 + y^2}$$

$$x^2 + 6x + 9 + y^2 = 4 + x^2 - 6x + 9 + y^2 + 4\sqrt{(x-3)^2 + y^2}$$

$$4\sqrt{(x-3)^2 + y^2} = 12x - 4 \text{ for } x > \frac{1}{3}$$

$$\sqrt{(x-3)^2 + y^2} = 3x - 1$$

Squaring both sides again gives

$$\left(x^2 + 6x + 9\right) + y^2 = 9x^2 - 6x + 1$$

$$8x^2 - y^2 = 8$$

$$x^2 - \frac{y^2}{8} = 1$$

this is a branch of a hyperbola, since $x > \frac{1}{3}$

2.6 Roots of complex numbers

Question 1

$$z^3 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \text{cis}\left(-\frac{\pi}{4} + 2k\pi\right) = \text{cis}\left(\frac{\pi(8k-1)}{4}\right)$$

$$z = \text{cis}\left(\frac{\pi(8k-1)}{12}\right)$$

$$k = 0,\ \ z = \text{cis}\left(-\frac{\pi}{12}\right)$$

$$k = 1,\ \ z = \text{cis}\left(\frac{7\pi}{12}\right)$$

$$k = -1,\ \ z = \text{cis}\left(\frac{-9\pi}{12}\right) = \text{cis}\left(-\frac{3\pi}{4}\right)$$

Award **1 mark** for using *ks* with DeMoivre's Theorem.
Award **2 marks** for the correct three values.

Question 2

$z^n = 1 + i, \text{n} \in Z^+$

$$z^n = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$$

$$= 2^{\frac{1}{2}}\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$$

$$z = 2^{\frac{1}{2n}}\text{cis}\left(\frac{\pi}{4n} + \frac{2k\pi}{n}\right),\ k \in Z$$

The correct answer is **E**.

Question 3

a. $z^3 = 8i, z \in C$

$$z^3 = 8\operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)$$

$$z = 8^{\frac{1}{3}}\operatorname{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$$

$k = 0, z = 2\operatorname{cis}\left(\frac{\pi}{6}\right) = \sqrt{3} + i$ **[1 mark]**

$k = 1, z = 2\operatorname{cis}\left(\frac{5\pi}{6}\right) = -\sqrt{3} + i$ **[1 mark]**

$k = -1, z = 2\operatorname{cis}\left(-\frac{\pi}{2}\right) = -2i$ **[1 mark]**

b. $(z - 2i)^3 = 8i, z \in C$

$z - 2i = \sqrt{3} + i, -\sqrt{3} + i, -2i$

$z = \sqrt{3} + 3i, -\sqrt{3} + 3i, 0$ **[1 mark]**

Question 4

$P(z) = z^3 + az^2 + bz + c$

$P(-2) = 0,\ P(3i) = 0$

$z = -2,\ z = \pm 3i$

$P(z) = (z + 2)\left(z^2 + 9\right)$

$P(z) = z^3 + 2z^2 + 9z + 18$

$a = 2,\ b = 9,\ c = 18$

Question 5

$z^3 - 5z^2 + 11z - 7 = 0$

$z_1 = 2 + \sqrt{3}i,\ z_2 = 2 - \sqrt{3}i,\ z_3 = 1$

Sum of the roots:

$z_1 + z_2 + z_3 = \left(2 + \sqrt{3}i\right) + \left(2 - \sqrt{3}i\right) + 1$

$= 5$

Question 6

$$z^2 = 1 + i = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4} + 2k\pi\right)$$

$$z = \sqrt{\sqrt{2}}\operatorname{cis}\left(\frac{\pi}{8} + k\pi\right)$$

Let $k = 0,\ z = \sqrt{\sqrt{2}}\operatorname{cis}\left(\frac{\pi}{8}\right)$

Let $k = -1,\ z = \sqrt{\sqrt{2}}\operatorname{cis}\left(-\frac{7\pi}{8}\right)$

Principal arguments are $-\frac{7\pi}{8}$ and $\frac{\pi}{8}$.

Question 7

$z^4 = 36\operatorname{cis}(2k\pi)$

$$z = \sqrt{6}\operatorname{cis}\left(\frac{k\pi}{2}\right)$$

$k = 0 \rightarrow z = \sqrt{6}$

$k = 1$

$z = \sqrt{6}\,\text{cis}\left(\frac{\pi}{2}\right)$

$\quad = \sqrt{6}i$

$k = 2,\ z = \sqrt{6}\,\text{cis}\,(\pi) = -\sqrt{6}$

$k = 3,\ z = \sqrt{6}\,\text{cis}\left(\frac{3\pi}{2}\right) = -\sqrt{6}$

Question 8

$z^8 = 16\,\text{cis}\,(2k\pi)$

$z = \sqrt{2}\,\text{cis}\left(\frac{k\pi}{8}\right)$

$k = 0,\ z = \sqrt{2}$

$k = 1, z = \sqrt{2}\,\text{cis}\left(\frac{\pi}{8}\right)$ etc.

or use a calculator to give the eight solutions that occur as conjugate pairs:

$z = \sqrt{2}, -\sqrt{2}, -\sqrt{2}\,i, -\sqrt{2}\,i, 1+i, -1-i, 1-i, -1+i$

Question 9

$z^4 = 256 = 256\,\text{cis}\ (0 + 2k\pi)$

$z = 256^{\frac{1}{4}}\,\text{cis}\left(\frac{2k\pi}{4}\right) = 4\,\text{cis}\left(\frac{k\pi}{2}\right)$

$k = 0 \quad z = 4\,\text{cis}(0) = 4$

$k = 1 \quad z = 4\,\text{cis}\left(\frac{\pi}{2}\right) = 4i$

$k = 2 \quad z = 4\,\text{cis}(\pi) = -4$

$k = -1 \quad z = 4\,\text{cis}\left(-\frac{\pi}{2}\right) = -4i$

Question 10

$z^3 + a^3 i = 0 \Rightarrow z^3 = -a^3 i = a^3 i^3$ since $i^2 = -1$

One solution is $z = ai$ the point D on the diagram above; however, there must be three solutions, and they must be equally spaced around the circle. Hence, the solutions are$D$, H and L.

Question 11

$z^3 = -27i$

$\quad = 27\,\text{cis}\left(\frac{3\pi}{2} + 2k\pi\right)$

$z = 27^{\frac{1}{3}}\text{cis}\left(\frac{\frac{3\pi}{2}}{3} + \frac{2k\pi}{3}\right)$

$\quad = 3i, \frac{3}{2}\left(\sqrt{3} - i\right), \frac{3}{2}\left(-\sqrt{3} - i\right)$ **[1 mark]**

Question 12

Let $z = x + yi$ so that $z^2 = z^2 - y^2 + 2xyi$

Now $z^2 = 1 - 4\sqrt{3}i$, so equating real and imaginary parts

$\text{Re}(1)\,x^2 - y^2 = 1$

$\text{Im}(2)\,2xy = -4\sqrt{3}$ so $y = \frac{-2\sqrt{3}}{x}$

Substituting gives

$$x^2 - \frac{12}{x^2} = 1$$
$$x^4 - 12 = x^2$$
$$x^4 - x^2 - 12 = 0$$
$$\left(x^2 - 4\right)\left(x^2 + 3\right) = 0$$

$x = \pm 2$ or $x = \pm\sqrt{3}i$ but $x \in R$ so $x = \pm 2$ only

So if $x = 2$, then and if $x = -2$, then $y = \sqrt{3}$

$z = \pm\left(2 - \sqrt{3}i\right)\, y = -\sqrt{3}$

2.7 Review

Question 1

a. i. $p(z) = z^3 + \alpha z^2 + \beta z + \gamma,\ \ z \in C, \alpha, \beta, \gamma \in R,$

By the conjugate root theorem, and since the coefficient are real, z_2 and z_3 are conjugate pairs. **[1 mark]**

ii. Let $z_2 = a + bi$, $z_3 = a - bi$

$z_2 + z_3 = 2a,\ \ |z_2 + z_3| = 0,$

$\Rightarrow\ a = 0$

$z_2 - z_3 = 2bi, |z_2 - z_3| = |2bi| = 2\,|bi| = 2\,|b| = 6$

$\Rightarrow\ b = \pm 3$

So $(z - 3i)(z + 3i) = \left(z^2 + 9\right)$ is a factor

$p(z) = z^3 + \alpha z^2 + \beta z + \gamma = (z + c)\left(z^2 + 9\right)$

$= z^3 + cz^2 + 9z + 9c$

$\alpha = c, \beta = 9, \gamma = 9c$

$p(2) = -13$

$-13 = 8 + 4\alpha + 2\beta + \gamma$

$-13 = 8 + 4c + 18 + 9c = 26 + 13c$

$13c = -39,\ \ c = -3$

$\alpha = -3,\ \beta = 9, \gamma = -27$

$p(z) = z^3 - 3z^2 + 9z - 27 = (z - 3)\left(z^2 + 9\right)$

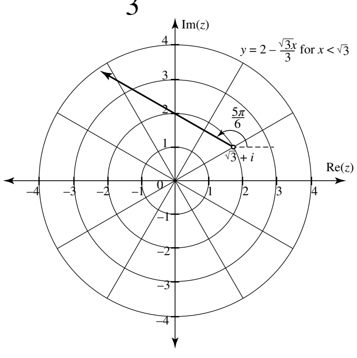
Award **1 mark** for setting up cubic coeffcents.

Award **1 mark** for the correct values of a and b.

Award **1 mark** for the correct values of α, β and γ.

b. $z_4 = \sqrt{3} + i$, the ray $\text{Arg}(z - z_4) = \dfrac{5\pi}{6}$ is the line

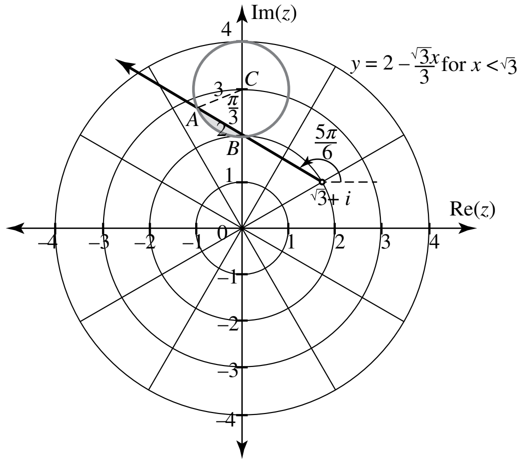
$y = 2 - \dfrac{\sqrt{3}x}{3}$ for $x < \sqrt{3}$.



Award **1 mark** for the correct ray.

Award **1 mark** for including open circle at z_4.

c. i. The circle $|z - 3i| = 1$ is $x^2 + (y-3)^2 = 1$
with centre at $C(0, 3)$ and radius $r = 1$. **[1 mark]**



ii. The ray and the circle intersect at the points $A\left(-\frac{\sqrt{3}}{2}, \frac{5}{2}\right)$, $B(0, 2)$, and triangle ABC is equilateral, so $\angle BCA = \frac{\pi}{3}$, the area of the minor segment is

$$\begin{aligned}\frac{1}{2}r^2(\theta - \sin(\theta)) &= \frac{1}{2}\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)\\ &= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\\ &= \frac{1}{12}\left(2\pi - 3\sqrt{3}\right)\end{aligned}$$

Award **1 mark** for correctly identifying the segment.
Award **1 mark** for correctly calculating the area of the segment.

Question 2

a. $|z - (1 + 2i)| = 2, z = x + yi$
$(x-1)^2 + (y-2)^2 = 4$
Circle center $(1, 2)$ radius 2 **[1 mark]**

***VCAA Examination Report* notes:**

Some students gave only one of the two required parts of the answer. An incorrect radius of $\sqrt{2}$ was occasionally given. Students were not asked to find the expression of the circle at this point but a number did so.

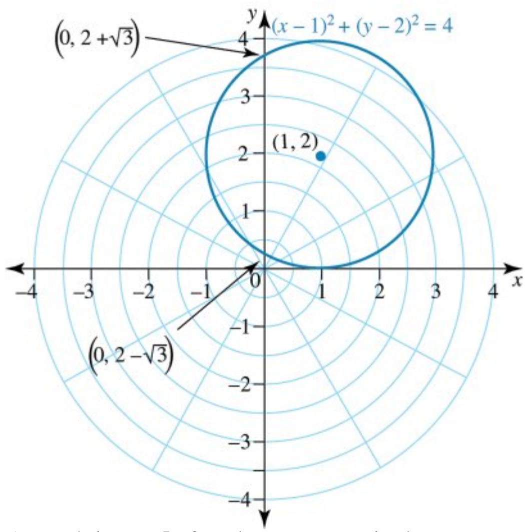
b. $|z + 1| = \sqrt{2}|z - i|$

$$\begin{aligned}|(x+1) + iy| &= \sqrt{2}|x + (y-1)i|\\ \sqrt{(x+1)^2 + y^2} &= \sqrt{2}\sqrt{x^2 + (y-1)^2}\\ x^2 + 2x + 1 + y^2 &= 2\left[x^2 + y^2 - 2y + 1\right]\\ x^2 - 2x + y^2 - 4y + 1 &= 0\\ x^2 - 2x + 1 + y^2 - 4y + 4 &= 4\\ (x-1)^2 + (y-2)^2 &= 4\end{aligned}$$

Circle centre $(1, 2)$ radius 2
Award **1 mark** for correctly setting up an equation involving x and y.
Award **1 mark** for correctly manipulating this equation to give the same cartesian equation found in part **a**.

***VCAA Examination Report* notes:**
Most students were able to correctly find an expression that did not involve i. In a 'show that' question such as this, students are expected to explicitly show that the given relation leads to the required conclusion. The working shown above is an example of a suitable response.

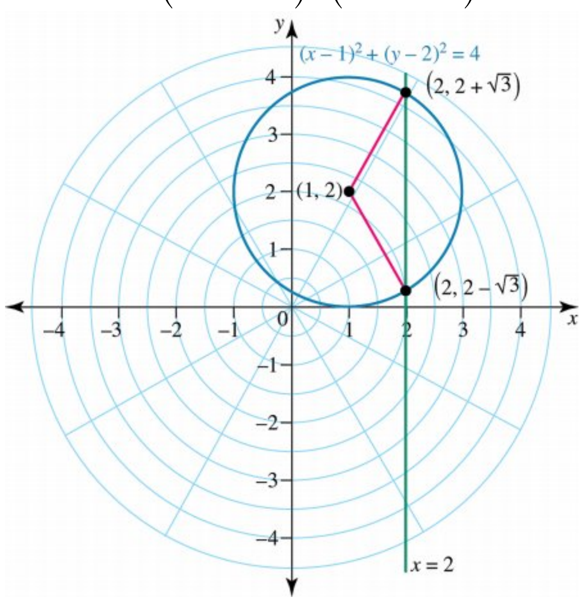
c.



Award **1 mark** for the correct circle.
Award **1 mark** for correct intercepts.

***VCAA Examination Report* note:**
The circle was generally drawn correctly. Students did not always supply the coordinates of the y-intercepts as required by the question. Some coordinates were incorrectly given imaginary numbers.

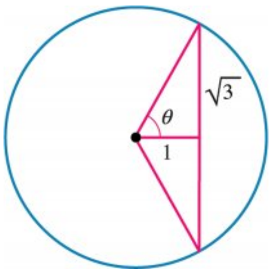
d. $|z-1| = |z-3|$ is the line $x = 2$, solving
$x = 2$ and $(x-1)^2 + (y-2)^2 = 4$ gives
$y = 2 \pm \sqrt{3}, \left(2, 2+\sqrt{3}\right), \left(2, 2-\sqrt{3}\right)$



Award **1 mark** for drawing a correct graph.
Award **1 mark** for correctly labelling the points of intersection.

***VCAA Examination Report* notes:**
While the vertical line was usually sketched correctly, coordinates of the points of intersection with the circle were not always shown. Coordinates were sometimes presented as decimal approximations.

e.



$\tan(\theta) = \frac{\sqrt{3}}{1}$

$\theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

The angle subtended is $2\theta = \frac{2\pi}{3}$.

$A = \frac{1}{2}r^2(\theta - \sin(\theta))$

$A = \frac{1}{2} \times 4\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$A = \frac{1}{3}\left(4\pi - 3\sqrt{3}\right)$

Award **1 mark** for identifying the required area.
Award **1 mark** for correctly calculating the area.

***VCAA Examination Report* notes:**
Students who used standard formulas to find the segment area were generally more successful than those who took a definite integral approach. Many students did not start the problem with a correct sector angle.

Question 3

a. Let $z = x + yi$

$$|(x-1) + iy| = |(x+2) + (y-3)i|$$
$$\sqrt{(x-1)^2 + y^2} = \sqrt{(x+2)^2 + (y-3)^2}$$
$$x^2 - 2x + 1 + y^2 = x^2 + 4x + 4 + y^2 - 6y + 9$$
$$6x - 6y + 12 = 0$$
$$y = x + 2$$

Award **1 mark** for find the linear equation.
Award **1 mark** for correct linear equation.

***VCAA Assessment Report* note:**
The majority of correct answers resulted from substituting $z = x + yi$ into the expression provided. Very few students used a perpendicular bisector approach at this stage. The most common error was a negative gradient.

b. The circle $|z - 1| = 3$ has centre $(1, 0)$ and radius 3.

$$|(x-1) + iy| = 3$$
$$\sqrt{(x-1)^2 + y^2} = 3$$
$$(x-1)^2 + y^2 = 9$$

For points of intersection, solve

(1) $y = x + 2$ and (2) $(x-1)^2 + y^2 = 9$

$$(x-1)^2 + (x+2)^2 = 9$$
$$x^2 - 2x + 1 + x^2 + 4x + 4 = 9$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0 \Rightarrow x = 1, -2$$
$$(1, 3), (-2, 0)$$

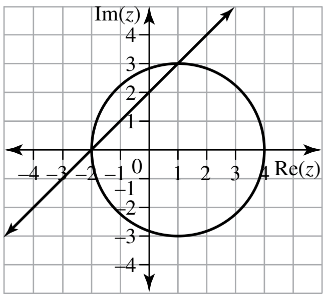
Award **1 mark** for solving the line and circle.

Award **1 mark** for final correct coordinates of the two points.

***VCAA Assessment Report* note:**

Most students were able to find the cartesian expression for the circle and the line and then apply substitution or use technology to find the required points

c.



Award **1 mark** for the correct circle and centre axial intercepts.

Award **1 mark** for correct line and intersection points.

***VCAA Assessment Report* note:**

This question was generally well answered. The circle was sketched correctly in cases. However, the line was not always placed with sufficient accuracy. Some students who were unable to find the correct equation in Question 2a. were able to use a perpendicular bisector approach to draw a correct line.

d. The area of the major segment is the area of the right-angled isosceles triangle with two side of 3 units, plus the area of the sector, $\frac{1}{2}r^2\theta$, where $r = 3$ and $\theta = \frac{3\pi}{2}$.

$$A = \frac{1}{2} \times 3 \times 3 + \frac{1}{2} \times 3^2 \times \frac{3\pi}{2}$$
$$= \frac{18 + 27\pi}{4}$$

Award **1 mark** for the section area.

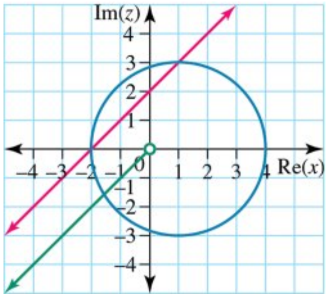
Award **1 mark** for the correct area.

***VCAA Assessment Report* note:**

Students found this question more difficult than previous parts of Question 2. A correct answer was most easily found by adding a right-angled triangle to three-questions of a circle. Subtracting the minor segment area from the circle area was a common approach. Some students correctly used a segment area formula and a larger number set up elaborate definite integrals to find the area, occasionally successfully, but this was not an efficient approach. Sign and factorisation errors meant that some students moved from a correct approach and evaluation to an incorrect final answer.

e. The ray $\text{Arg}(z) = -\frac{3\pi}{4}$ is the line $y = x$ for $x < 0$ and has an open circle at the origin.

Since the origin is not included and the ray makes an angle of 135° measured clockwise from the positive real axis.



Award **1 mark** for the correct ray on the diagram in part **c**.

***VCAA Assessment Report* note:**

Many students were not able to sketch the required ray. Some students sketched a line but did not restrict their ray appropriately, either including or extending past the origin.

f. Since $-\pi < \text{Arg}(z) \leq \pi$, the ray $\text{Arg}(z) = \alpha\pi$ is parallel to the line $|z - 1| = |z + 2 - 3i|$ when $\text{Arg}(z) = -\dfrac{3\pi}{4}$ or $\text{Arg}(z) = \dfrac{\pi}{4}$ and therefore does not intersect the line,

$\alpha \in \left(-1, -\dfrac{3}{4}\right) \cup \left(-\dfrac{1}{4}, 1\right]$.

Award **1 mark** for each correct value.

***VCAA Assessment Report* note:**

Students found this question demanding, with few students giving a fully correct answer. Many students did not respond to this question. Common incorrect answers contained multiples of π or included the endpoint $\alpha = -1$. Some students did not note that the principal value of the argument was used in the question. Of the correct answers, a variety of correct notations were presented.

Question 4

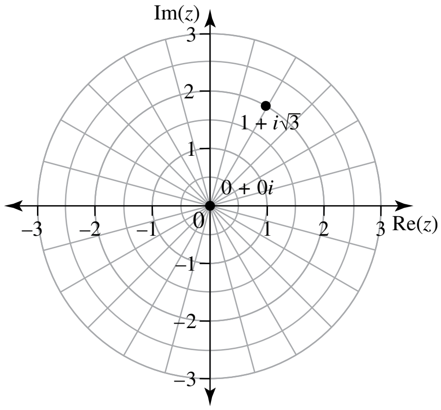
$$
\begin{aligned}
z &= \frac{1 + i\sqrt{3}}{1 + i} \\
&= \frac{2\,\text{cis}\left(\frac{\pi}{3}\right)}{\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)} \\
&= \frac{2}{\sqrt{2}}\,\text{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)
\end{aligned}
$$

$$
\begin{aligned}
z^5 &= \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)\right)^5 \\
&= \left(\sqrt{2}\right)^5\,\text{cis}\left(\frac{5\pi}{12}\right) \\
&= 4\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)
\end{aligned}
$$

The correct answer is **B**.

Question 5

a. **i.**



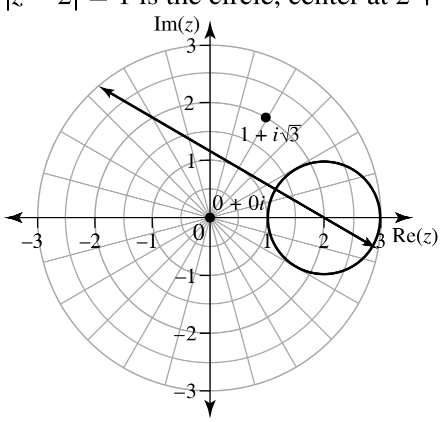
Award **1 mark** for each correctly positioned point.

***VCAA Assessment Report* note:**

This question was answered quite well, but many students could not accurately position $1 + i\sqrt{3}$, not realising it lay on the circle of radius 2. A number of students did not fully label both points.

ii. $|z - (1 + i\sqrt{3})| = |z|$ is the line equidistant or the perpendicular bisector from the points $0 + 0i$ and $1 + i\sqrt{3}$.

$|z - 2| = 1$ is the circle, center at $2 + 0i$ and radius 1.



Award **1 mark** for the line and **1 mark** for the circle on the diagram in **a**. **i**.

***VCAA Assessment report* note:**

Most students graphed the circle correctly, although some circles were poorly drawn.

A common error was to draw a straight line with a positive gradient. A number of students terminated their line at (2, 0). Few students seemed to realise that the required line was the perpendicular bisector of the line interval joining (0, 0) and $\left(1, \sqrt{3}\right)$.

iii. $|z - (1 + i\sqrt{3})| = |z|$

Let $z = x + yi$

$$\left|(x-1) + \left(y - \sqrt{3}\right)i\right| = |x + yi|$$

$$\sqrt{(x-1)^2 + \left(y - \sqrt{3}\right)^2} = \sqrt{x^2 + y^2}$$

$$x^2 - 2x + 1 + y^2 - 2\sqrt{3}y + 3 = x^2 + y^2$$

$$x + \sqrt{3}y = 2$$ **[1 mark]**

***VCAA Assessment Report* note:**

This question was generally well answered. The most common error was the gradient given as positive.

iv. (1) $\left|z-\left(1+i\sqrt{3}\right)\right|=|z| \Rightarrow x+\sqrt{3}y=2$

(2) $|z-2|=1 \Rightarrow (x-2)^2+y^2=1$

Substitute (1)$x=2-\sqrt{3}y$ into (2)

$\left(-\sqrt{3}y\right)^2+y^2=4y^2=1$

$y^2=\frac{1}{4} \Rightarrow y=\pm\frac{1}{2}$

$y=\frac{1}{2} \Rightarrow x=2-\frac{\sqrt{3}}{2}$

$y=-\frac{1}{2} \Rightarrow x=2+\frac{\sqrt{3}}{2}$

$\frac{1}{2}\left(4-\sqrt{3}\right)+\frac{1}{2}i$ and $\frac{1}{2}\left(4+\sqrt{3}\right)-\frac{1}{2}i$

Award **1 mark** for substitution.

Award **1 mark** for each correct point of intersection.

***VCAA Assessment Report* note:**

The question was reasonably well answered. Common errors were answers given in the wrong form and sign errors. Most students attempted to solve the equations of the line and circle simultaneously.

b i. $z^2-4\cos(\alpha)z+4=0,\ \alpha \in R,\ 0<\alpha<\frac{\pi}{2}$

$\Delta=(4\cos(\alpha))^2-4\times 4=16\cos^2(\alpha)-16$

$=16(\cos^2(\alpha)-1)=-16\sin^2(\alpha)$

$=16i^2\sin^2(\alpha)$

$$z=\frac{4\cos(\alpha)\pm\sqrt{16i^2\sin^2(\alpha)}}{2}$$

$z=2\cos(\alpha)\pm 2i\,|\sin(\alpha)|$ since $0<\alpha<\frac{\pi}{2}$

$z_1=2(\cos(\alpha)+i\sin(\alpha))=2\,\text{cis}(\alpha)$

$z_2=2(\cos(\alpha)-i\sin(\alpha))=2\,\text{cis}(-\alpha)$

Award **1 mark** for using the quadratic formula.

Award **1 mark** for simplifications.

Award **1 mark** for both correct roots.

***VCAA Assessment Report* note:**

Most students attempted to apply the quadratic formula or complete the square, but few managed to find the values of z in polar form. Dealing with the discriminant proved to be a problem for many. A number of students left answers in cartesian form, and some erroneously converted the correct cartesian form answer to $2\sqrt{2}\,\text{cis}(\alpha)$ and $2\sqrt{2}\,\text{cis}(-\alpha)$.

ii. $\frac{z_1}{z_2}=\frac{2\,\text{cis}(\alpha)}{2\,\text{cis}(-\alpha)}=\text{cis}(2\alpha)$

$\text{Arg}\left(\frac{z_1}{z_2}\right)=2\alpha=\frac{5\pi}{6}$

$\alpha=\frac{5\pi}{12}$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students did not attempt this question. Common errors were unsimplified expressions involving z, $-\frac{5\pi}{12}$ and $\frac{5\pi}{6}$.

Question 6

a. $u = -2 - i, v = -4 - 3i$

$|z - u| = |z - v|, z = x + yi$

$$|(x + 2) + i(y + 1)| = |(x + 4) + i(y + 3)|$$

$$(x + 2)^2 + (y + 1)^2 = (x + 4)^2 + (y + 3)^2$$

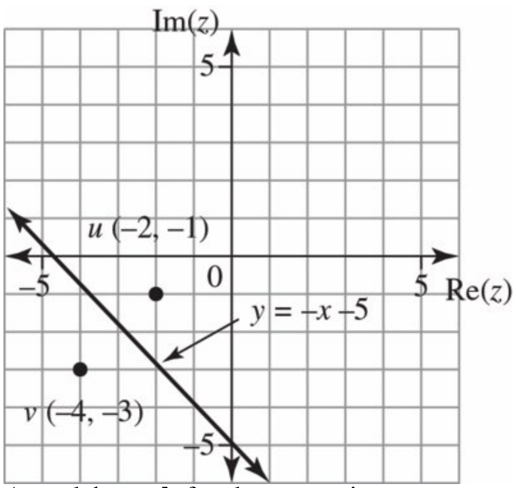
$$x^2 + 4x + 4 + y^2 + 2y + 1 = x^2 + 8x + 16 + y^2 + 6y + 9$$

$$4x + 4y + 20 = 0$$

$$y = -x - 5, m = -1, c = -5$$

Award **1 mark** for expressing in modulus form.
Award **1 mark** for simplifying.
Award **1 mark** for the correct line.

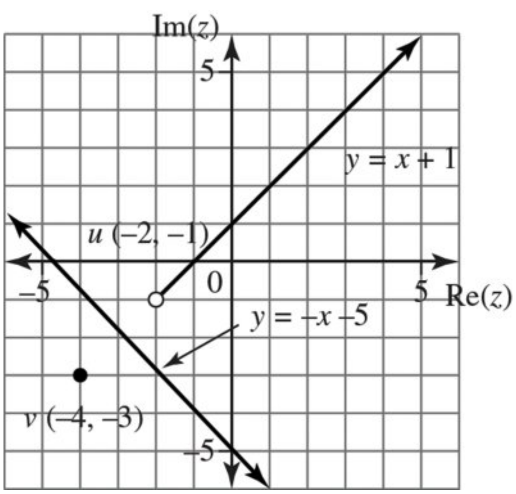
b.



Award **1 mark** for the two points.
Award **1 mark** for the correctly graphed line.

c. The line is the perpendicular bisector of the line joining u and v, or the set of points equidistant from both u and v.
Award **1 mark** for the correct geometrical interpretation.

d i.



Award **1 mark** for correctly sketching the ray on the diagram in part **b**.
Note this diagram must show an open circle at u.

ii. $\text{Arg}(z-u)=\dfrac{\pi}{4}$

$\text{Arg}((x+2)+i(y+1)=\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{y+1}{x+2}\right)=\dfrac{\pi}{4}$ for $x>-2$

$\dfrac{y+1}{x+2}=1$

$y=x+1$ for $x>-2$

A ray with open circle at $u\,(-2,-1)$

Award **1 mark** for the correct line.

e. $z_c=a+bi$

$|z-z_c|=r,(x-a)^2+(y-b)^2=r^2$

$(-2,-1)$ (1) $(-2-a)^2+(-1-b)^2=r^2$

$(-4,-3)$ (2) $(-4-a)^2+(-3-b)^2=r^2$

$(0,-5)$ (3) $h^2+(-5-b)^2=r^2$

Solving using CAS:

$a=-\dfrac{5}{3},\ b=-\dfrac{10}{3}\ r=\dfrac{5\sqrt{2}}{3}$

$z_c=-\dfrac{5}{3}-\dfrac{10}{3}i=-\dfrac{5}{3}(1+2i)$

Award **1 mark** for correctly setting up three equations.

Award **1 mark** for the correct values of a and b.

Award **1 mark** for the correct value of r.

Question 7

a. i. $2z^2+4z+5=0$

Completing the square

$2\left(z^2+2z+1\right)=-5+2=-3=3i^2$

$(z+1)^2=\dfrac{3i^2}{2}$

$z+1=\pm\sqrt{\dfrac{3i^2}{2}}=\pm\dfrac{\sqrt{6}i}{2}$

$z=-1\pm\dfrac{\sqrt{6}i}{2}$

Quadratic formula

$a=2,\ b=4,\ c=5$

$\Delta=b^2-4ac=16-40=-24=24i^2=4\times 6i^2$

$z=\dfrac{-b\pm\sqrt{\Delta}}{2a}=\dfrac{-4\pm 2\sqrt{6}i}{4}$

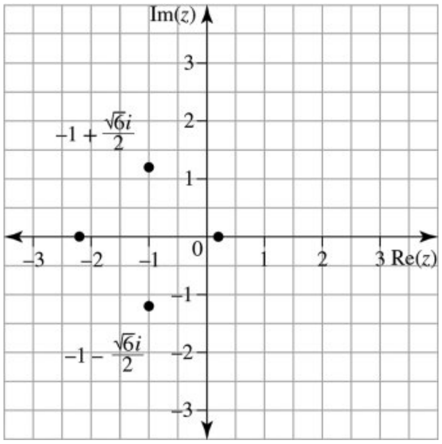
$z=-1\pm\dfrac{\sqrt{6}i}{2}$

Award **1 mark** for the correct method and proof.

***VCAA Examination Report* note:**

Most students were able to show that the quadratic equation had the given solutions either by the quadratic formula as above or by completing the square. In a ‘show that’ question such as this, students are expected to explicitly show that the given information leads to the required conclusion rather than ‘verify’ that the given values of z are solutions of the equation. The working shown above is an example of a suitable response.

ii.



Award **1 mark** for the correct points.

***VCAA Examination Report* note:**

Most students correctly plotted points representing the two solutions. In some cases insufficient care was taken to plot points accurately relative to the supplied gridlines and scale.

b. i. $m = 1, n = \frac{\sqrt{6}}{2}$

Award **1 mark** each for the correct values of n and m.

***VCAA Examination Report* note:**

$m = -1$ was a common incorrect response.

ii. $|z+m| = n,\ |z+1| = \frac{\sqrt{6}}{2},\ z = x + yi$

$|(x+1)+yi| = \frac{\sqrt{6}}{2}$

$\sqrt{(x+1)^2+y^2} = \frac{\sqrt{6}}{2}$

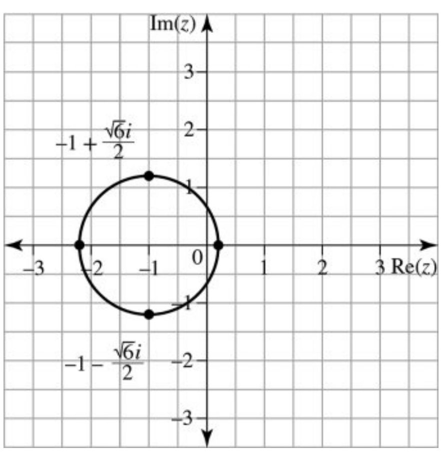
$(x+1)^2 + y^2 = \frac{3}{2}$,circle with centre $(-1, 0)$ and radius $\frac{\sqrt{6}}{2}$

Award **1 mark** for the correct equation of the circle.

***VCAA Examination Report* note:**

Incorrect responses included sign errors and algebraic errors resulting from unnecessary attempts to isolate y.

iii.



c. $p(z) = 2z^2 + 4z + d = 0,\ z = -1 \pm \dfrac{i\sqrt{2(d-2)}}{2}$ for complex solutions,

So, $d - 2 > 0$ but for solutions of $p(z)$ to lie in $|z + m| \le n$ so $2 < d \le 5$.

However, $-1 \le d \le 2$ when the solutions are real and lie on the real axis, inside the circle, so $-1 \le d \le 5$.

Alternatively

$$\left|\frac{\sqrt{2}(d-2)}{2}\right| = \left|\frac{(d-2)}{\sqrt{2}}\right| \le \sqrt{\frac{3}{2}}$$

$$|d-2| \le 3,\ -1 \le d \le 5$$

Award **1 mark** for using the correct method.

Award **1 mark** for the correct value of d.

***VCAA Examination Report* note:**

Many students did not attempt this question. Of those who did, most abandoned a potentially correct approach before they reached a conclusion. Those who reached a conclusion generally got there via the quadratic formula but unfortunately most of these students found only one end point of the interval.

d. i. $az^2 + bz + c = 0,\ z = \dfrac{-b \pm i\sqrt{-\Delta}}{2a}$ since we want complex solutions, so $\Delta = b^2 - 4ac < 0$

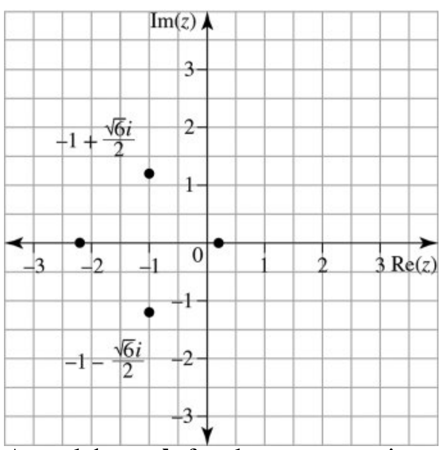
$p = \dfrac{b}{2a},\ \ q = \dfrac{\sqrt{4ac - b^2}}{2|a|}$ since $q > 0$ and q represents the radius.

Award **1 mark** each for the correct values of p and q.

***VCAA Examination Report* note:**

Many students did not attempt this question. Of those who attempted it, most made errors relating to signs; giving the negative of the correct p value or not accounting for the sign of a.

ii.



Award **1 mark** for the correct points.

***VCAA Examination Report* note:**

Most students correctly plotted points representing the two solutions. In some cases insufficient care was taken to plot points accurately relative to the supplied gridlines and scale.

Question 8

$z = a + bi, A(a, b)$

$iz = -b + ai,\ B(-b, a)$

$z + iz = (a - b) + (a + b)i,\ C(a - b, a + b)$

$CA \cdot CB = 0 \quad \angle ACB = 90°$

$d(AC) = \sqrt{a^2 + b^2}, d(BC) = \sqrt{(-b)^2 + a^2}$

$A_\Delta = \dfrac{1}{2} d(AC)\, d(BC) = \dfrac{1}{2}\left(a^2 + b^2\right) = \dfrac{|z|^2}{2}$

Question 9

$z = a + bi, a, b \in R \setminus \{0\}$

$$z + \frac{1}{z} = a + bi + \frac{1}{a + bi} \times \frac{a - bi}{a - bi}$$
$$= a + bi + \frac{a - bi}{a^2 + b^2}$$
$$= \left(a + \frac{a}{a^2 + b^2}\right) + i\left(b - \frac{b}{a^2 + b^2}\right)$$
$$\text{Im}\left(z + \frac{1}{z}\right) = 0 \Rightarrow b - \frac{b}{a^2 + b^2} = 0$$
$$|z|^2 = a^2 + b^2 = 1, |z| = 1$$

Question 10

a. $u = 1 + i$

$|u| = \sqrt{1 + 1} = \sqrt{2}$

$\text{Arg}(u) = \tan^{-1}(1) = \frac{\pi}{4}$

$u = 1 + i = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ **[1 mark]**

VCAA Examination Report **note:**

This question was answered well by most students. A common incorrect response was to write $\tan\left(\frac{1}{1}\right) = \frac{\pi}{4}$ rather than $\arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$.

b. $v = \sqrt{3} - i$

$|v| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ $\text{Arg}(v) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}, v = \sqrt{3} - i = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$ **[1 mark]**

$$\frac{\left(\sqrt{3} - i\right)^{10}}{(1 + i)^{12}} = \frac{\left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^{10}}{\left(\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\right)^{12}} = \frac{2^{10}\text{cis}\left(-\frac{5\pi}{3}\right)}{2^6\text{cis}(3\pi)}$$ **[1 mark]**

$$= 2^4\text{cis}\left(-\frac{5\pi}{3} - 3\pi + 4\pi\right)$$
$$= 16\text{cis}\left(-\frac{2\pi}{3}\right)$$
$$= 16\cos\left(\frac{2\pi}{3}\right) - 16i\sin\left(\frac{2\pi}{3}\right)$$
$$= 16 \times -\frac{1}{2} - 16 \times \frac{\sqrt{3}}{2}$$

$= -8 - 8\sqrt{3}i$ **[1 mark]**

VCAA Examination Report **note:**

Most students realised that they needed to use polar form and de Moirvre's theorem. Quite a few students were not able to write $\sqrt{3} - i$ in polar form correctly with arguments of $\frac{\pi}{6}, \frac{5\pi}{6}$ and $\frac{\pi}{3}$ being given frequently.Students are reminded that a diagram placing the complex number in the correct quadrant can

be helpful in avoiding errors. Of those students who obtained the result $16\,\text{cis}\left(-\frac{2\pi}{3}\right)$, some neglected to write the final answer in the required form or made errors in their attempt.
A small number of students attempted to expand brackets. This approach was rarely successful.

Question 11

$z_1 = 2\,\text{cis}\left(\frac{\pi}{3}\right), z_2 = \text{cis}\left(\frac{3\pi}{4}\right), z_3 = 2\,\text{cis}\left(-\frac{2\pi}{3}\right), z_4 = \text{cis}\left(-\frac{\pi}{4}\right)$

$$z_1z_2z_3z_4 = 4\,\text{cis}\left(\frac{\pi}{3} + \frac{3\pi}{4} - \frac{2\pi}{3} - \frac{\pi}{4}\right)$$
$$= 4\,\text{cis}\left(\frac{\pi}{6}\right) = 2\sqrt{3} + 2i \neq 0$$

All other options are true.

Question 12

a. Let $z = x + iy$

$$|(x-1) + iy| = |(x+2) + (y-3)\,i|$$
$$\sqrt{(x-1)^2 + y^2} = \sqrt{(x+2)^2 + (y-3)^2}$$
$$x^2 - 2x + 1 + y^2 = x^2 + 4x + 4 + y^2 - 6y + 9$$
$$6x - 6y + 12 = 0$$
$$y = x + 2$$

Award **1 mark** for finding the linear equation.
Award **1 mark** for the correct linear equation.
***VCAA Assessment Report* note:**
The majority of correct answers resulted from substituting $z = x + yi$ into the expression provided. Very few students used a perpendicular bisector approach at this stage. The most common error was a negative gradient.

b. The circle $|z-1| = 3$ has centre $(1, 0)$ and radius 3.

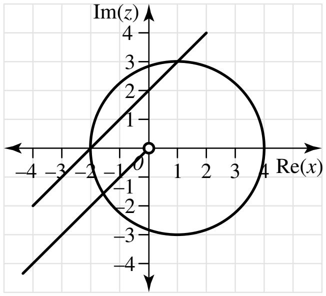
$$|(x-1) + iy| = 3$$
$$\sqrt{(x-1)^2 + y^2} = 3$$
$$(x-1)^2 + y^2 = 9$$

For points of intersection, solve
(1) $y = x + 2$ and (2) $(x-1)^2 + y^2 = 9$

$$(x-1)^2 + (x+2)^2 = 9$$
$$x^2 - 2x + 1 + x^2 + 4x + 4 = 9$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0 \Rightarrow x = 1, -2$$

$(1, 3)$, $(-2, 0)$

Award **1 mark** for solving the line and circle.
Award **1 mark** for final correct coordinates of the two points.
***VCAA Assessment Report* note:**
Most students were able to find cartesian expressions for the circle and the line and then apply substitution or use technology to find the required points.

c. 

Award **1 mark** for the correct circle and centre axial intercepts.
Award **1 mark** for the correct line and intersection points.
VCAA Assessment Report **note:**
This question was generally well answered. The circle was sketched correctly in almost all cases. However, the line was not always placed with sufficient accuracy. Some students who were unable to find the correct equation in part a. were able to use a perpendicular bisector approach to draw a correct line.

d. The area of the major segment is the area of the rightangled isosceles triangle with two sides of 3 units, plus the area of the sector, $\frac{1}{2}r^2\theta$ where $r = 3$ and $\theta = \frac{3\pi}{2}$.

$$A = \frac{1}{2} \times 3 \times 3 + \frac{1}{2} \times 3^2 \times \frac{3\pi}{2}$$
$$= \frac{18 + 27\pi}{4}$$

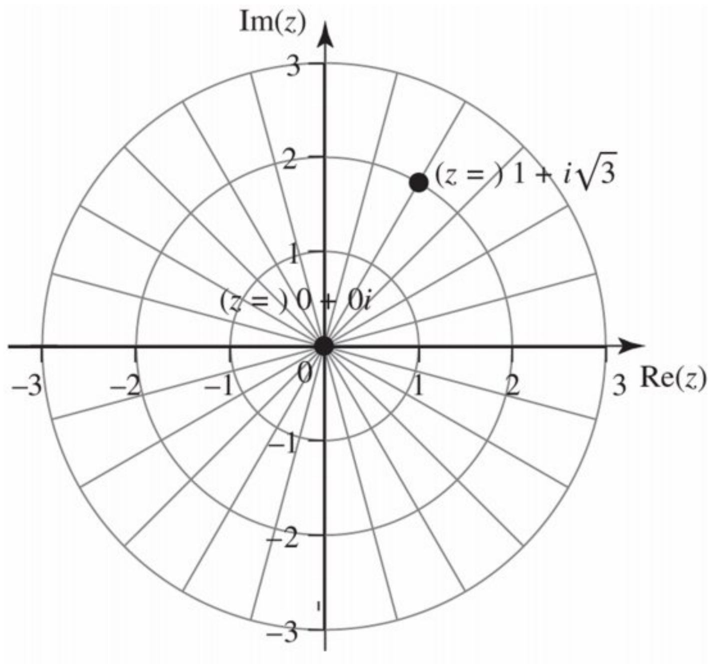
Award **1 mark** for the sector area.
Award **1 mark** for the final correct area.
VCAA Assessment Report **note:**
Students found this question more difficult than previous parts. A correct answer was most easily found by adding a right-angled triangle to three-quarters of a circle. Subtracting the minor segment area from the circle area was a common approach. Some students correctly used a segment area formula and a larger number set up elaborate definite integrals to find the area, occasionally successfully, but this was not an efficient approach. Sign and factorisation errors meant that some students moved from a correct approach and evaluation to an incorrect final answer.

e. The ray $\text{Arg}(z) = -\frac{3\pi}{4}$ is the line $y = x$ for $x < 0$ and has an open circle at the origin, since the origin is not included and makes an angle of 135° measured clockwise from the positive real axis.
Award **1 mark** for the correct ray on the diagram in part **c**.
VCAA Assessment Report **note:**
Many students were not able to sketch the required ray. Some students sketched a line but did not restrict their ray appropriately, either including or extending past the origin.

f. Since $-\pi < \text{Arg}(z) \le \pi$, the ray $\text{Arg}(z) = \alpha\pi$ is parallel to the line $|z - 1| = |z + 2 - 3i|$.
When $\text{Arg}(z) = -\frac{3\pi}{4}$ or $\text{Arg}(z) = \frac{\pi}{4}$ and therefore does not intersect the line, $\alpha \in \left(-1, -\frac{3}{4}\right) \cup \left(\frac{1}{4}, 1\right]$.
Award **1 mark** for each correct value.
VCAA Assessment Report **note:**
Students found this question demanding, with few students giving a fully correct answer. Many students did not respond to this question. Common incorrect answers contained multiples of π or included the endpoint $\alpha = -1$ Some students did not note that the principal value of the argument was used in the question. Of the correct answers, a variety of correct notations were presented.

Question 13

a. i



Award **1 mark** for each correctly positioned point.

VCAA Assessment Report note:

This question was answered quite well, but many students could not accurately position $1 + i\sqrt{3}$, not realising it lay on the circle of radius 2 A number of students did not fully label both points.

ii. $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$ is the line equidistant or the perpendicular bisector from the points $0 + 0i$ and $1 + i\sqrt{3}$.

$|z - 2| = 1$ is the circle, centre at $2 + 0i$ and radius 1.

Award **1 mark** for the line and **1 mark** for the circle on the diagram in a. i.

***VCAA Assessment Report* note:**

Most students graphed the circle correctly, although some circles were poorly drawn.
A common error was to draw a straight line with a positive gradient. A number of students terminated their line at $(2, 0)$. Few students seemed to realise that the required line was the perpendicular bisector of the line interval joining $(0, 0)$ and $\left(1, \sqrt{3}\right)$.

iii. $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z|$

Let $z = x + yi$

$$\left|(x-1) + \left(y - \sqrt{3}\right)i\right| = |x + yi|$$

$$\sqrt{(x-1)^2 + \left(y - \sqrt{3}\right)^2} = \sqrt{x^2 + y^2}$$

$$x^2 - 2x + 1 + y^2 - 2\sqrt{3}y + 3 = x^2 + y^2$$

$$x + \sqrt{3}y = 2$$ **[1 mark]**

***VCAA Assessment Report* note:**

This question was generally well answered. The most common error was the gradient given as positive.

iv. (1) $\left|z - \left(1 + i\sqrt{3}\right)\right| = |z| \Rightarrow x + \sqrt{3}y = 2$

(2) $|z - 2| = 1 \Rightarrow (x-2)^2 + y^2 = 1$

Substitute (1) $x = 2 - \sqrt{3}y$ into (2)

$\left(-\sqrt{3}y\right)^2 + y^2 = 4y^2 = 1$

$y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

$y = \frac{1}{2} \Rightarrow x = 2 - \frac{\sqrt{3}}{2}$

$y = -\frac{1}{2} \Rightarrow x = 2 + \frac{\sqrt{3}}{2}$

$\frac{1}{2}\left(4 - \sqrt{3}\right) + \frac{1}{2}i$ and $\frac{1}{2}\left(4 + \sqrt{3}\right) - \frac{1}{2}i$

Award **1 mark** for substitution.

Award **1 mark** for each correct point of intersection.

***VCAA Assessment Report* note:**

This question was reasonably well answered. Common errors were answers given in the wrong form and sign errors. Most students attempted to solve the equations of the line and circle simultaneously.

b. i. $z^2 - 4\cos(\alpha)z + 4 = 0, \alpha \in R, 0 < \alpha < \frac{\pi}{2}$

$\Delta = (4\cos(\alpha))^2 - 4 \times 4 = 16\cos^2(\alpha) - 16$

$= 16(\cos^2(\alpha) - 1) = -16\sin^2(\alpha)$

$= 16i^2\sin^2(\alpha)$

$z = \frac{4\cos(\alpha) \pm \sqrt{16i^2\sin^2(\alpha)}}{2}$

$z = 2\cos(\alpha) \pm 2i\,|\sin(\alpha)|$ since $0 < \alpha < \frac{\pi}{2}$

$z_1 = 2(\cos(\alpha) + i\sin(\alpha)) = 2\,\text{cis}(\alpha)$

$z_2 = 2(\cos(\alpha) - i\sin(\alpha) = 2\,\text{cis}(-\alpha)$

Award **1 mark** for using the quadratic formula.

Award **1 mark** for simplifications.

Award **1 mark** for both correct roots.

VCAA Assessment Report note:

Most students attempted to apply the quadratic formula or complete the square, but few managed to find the values of z in polar form. Dealing with the discriminant proved to be a problem for many. A number of students left answers in cartesian form, and some erroneously converted correct cartesian form answers to $2\sqrt{2}\text{cis}(\alpha)$ and $2\sqrt{2}\text{cis}(-\alpha)$.

ii. $\frac{z_1}{z_2} = \frac{2\,\text{cis}(\alpha)}{2\,\text{cis}(-\alpha)} = \text{cis}(2\alpha)$

$\text{Arg}\left(\frac{z_1}{z_2}\right) = 2\alpha = \frac{5\pi}{6}$

$\alpha = \frac{5\pi}{12}$ **[1 mark]**

***VCAA Assessment Report* note:**

Many students did not attempt this question. Common errors were unsimplified expressions involving z, $-\frac{5\pi}{12}$ and $\frac{5\pi}{6}$.

Question 14

a. i. $z_1 = \sqrt{3} - 3i$

$|z_1| = \sqrt{\left(\sqrt{3}\right)^2 + (-3)^2}$

$= \sqrt{12}$

$= 2\sqrt{3}$

$\text{Arg}(z_1) = \tan^{-1}\left(\frac{-3}{\sqrt{3}}\right)$

$= -\frac{\pi}{3}$

$z_1 = 2\sqrt{3}\text{cis}\left(-\frac{\pi}{3}\right)$

Award **1 mark** for the correct magnitude.

Award **1 mark** for the correct argument (angle).

ii. $\text{Arg}\left(z_1^4\right) = -4 \times \frac{\pi}{3} + 2\pi$

$\text{Arg}\left(z_1^4\right) = \frac{2\pi}{3}$

Award **1 mark** for the correct argument (angle).

iii. $z_1 = \sqrt{3} - 3i,\ \bar{z}_1 = \sqrt{3} + 3i$

$z_1\bar{z}_1 = 3 - 9i^2 = 12,\ z_1 + \bar{z}_1 = 2\sqrt{3}$

So $z^2 - 2\sqrt{3}z + 12$ is a factor.

$z^3 + 24\sqrt{3} = 0$

$\left(z^2 - 2\sqrt{3}z + 12\right)\left(z + 2\sqrt{3}\right) = 0$

$z = -2\sqrt{3},\ \sqrt{3} \pm 3i$

Award **1 mark** for the correct factors.

Award **1 mark** for the other two roots.

b. i. $(z_1 + 2i)(z_1 - 2i) = z_1\bar{z}_1 + 2iz_1 - 2i\bar{z}_1 - 4i^2$

$= 12 + 2i\left(\sqrt{3} + 3i\right) - 2i\left(\sqrt{3} - 3i\right) + 4$

$= 12 + 6\sqrt{3}i + 6i^2 - 6\sqrt{3}i + 6i^2 + 4$

$= 12 - 6 - 6 + 4$

$= 4$

Award **1 mark** for the correct value.

ii. $(z + 2i)\left(\bar{z} - 2i\right) = 4$

$z\bar{z} + 2i\bar{z} - 2iz - 4i^2 = 4$

$(x + yi)(x - yi) + 2i(x - yi) - 2i(x + yi) + 4 = 4$

$x^2 + xyi - xyi - i^2y^2 + 2ix - 2yi^2 - 2ix - 2yi^2 + 4 = 4$

$x^2 + y^2 + 4y + 4 = 4$

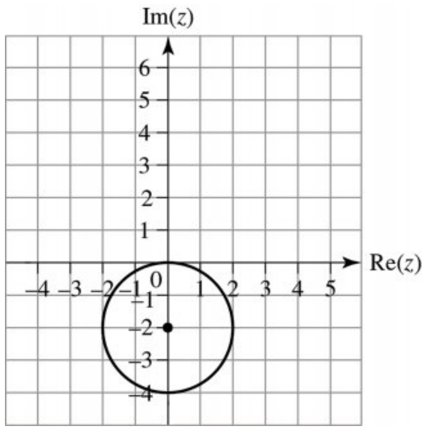
$x^2 + (y + 2)^2 = 4$

Award **1 mark** for correct expansions.

Award **1 mark** for showing the correct relation.

iii. $x^2 + (y + 2)^2 = 4$

Circle with centre $(0, -2)$ and radius 2



Award **1 mark** for the correct radius.
Award **1 mark** for the correct centre.

c. $P(k, -2),\ Q(0, -(2+k)),\ k < 0$

$m_{PQ} = \dfrac{-(2+k)+2}{0-k} = 1$

$y + 2 = 1(x - k)$

The line touches the circle $x^2 + (y+2)^2 = 4$

$x^2 + (x-k)^2 = 4$

$2x^2 - 2xk + \left(k^2 - 4\right) = 0$ has only one solution.

$\Delta = 0$

$(-2k)^2 - 4 \times 2 \times \left(k^2 - 4\right) = 0$

$4k^2 - 8k^2 + 32 = 0$

$4k^2 = 32$

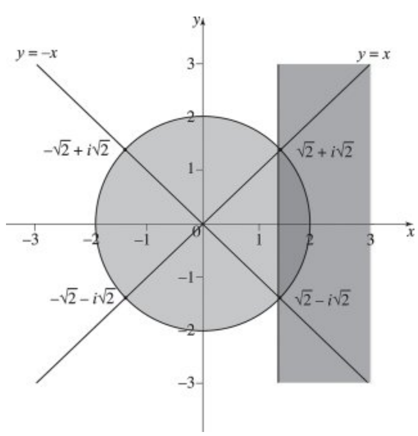
$k^2 = 8,\ k = \pm 2\sqrt{2}$

$k = -2\sqrt{2}$ since $k < 0$

Award **1 mark** for the correct gradient.
Award **1 mark** for using the discriminant.
Award **1 mark** for the correct value of k.

Question 15

a.



$z = x + yi,\ \bar{z} = x - yi$

$z\bar{z} = x^2 - y^2 i^2 = x^2 + y^2 = 4$

A circle with centre at the origin and radius 2

$|z + \bar{z}| = |z - \bar{z}| \Rightarrow |2x| = |2yi| \Rightarrow |x| = |y|$

$y = \pm x$, which implies two straight lines through the origin.

Award **1 mark** for the correct Cartesian equation of the line.

Award **1 mark** for the correct Cartesian equation of the circle.
Award **1 mark** for correct drawing, lines and circle.
***VCAA Assessment Report* note:**
The majority of students did not draw the line $y = -x$. Most got the circle and the line $y = x$. Some students drew more than one circle, and straight lines were sometimes drawn roughly.

b. Substitute $y = \pm x$ into $x^2 + y^2 = 4$.
$2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2} \Rightarrow y = \pm\sqrt{2}$
Altogether there are four answers.
$\sqrt{2} + i\sqrt{2},\ \sqrt{2} - i\sqrt{2}, -\sqrt{2} + i\sqrt{2}, -\sqrt{2} - i\sqrt{2}$
Award **1 mark** for substituting.
Award **2 marks** for all four correct answers.
***VCAA Assessment Report* note:**
This question was only moderately well done. Most students found only two solutions for z, $z = \sqrt{2} + i\sqrt{2}$ and $z = -\sqrt{2} - i\sqrt{2}$.

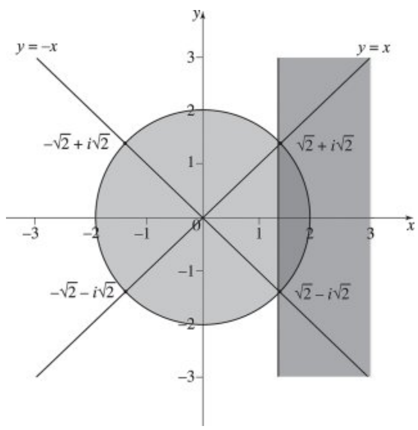
c. $z^4 + 16 = 0$
One root: $z = \sqrt{2} + i\sqrt{2}$
The other three roots are $z = \sqrt{2} - i\sqrt{2},\ z = -\sqrt{2} + i\sqrt{2},\ z = -\sqrt{2} - i\sqrt{2}$
Award 1 mark for the correct answers.
Award **1 mark** for correct labelling in part a.
***VCAA Assessment Report* note:**
Major errors involved not plotting the roots, not labelling the plotted roots, and plotting the roots on the incorrect circle and in other locations.

d. $z^4 + 16 = \left(z + \sqrt{2} + i\sqrt{2}\right)\left(z + \sqrt{2} - i\sqrt{2}\right)\left(z - \sqrt{2} + i\sqrt{2}\right)\left(z - \sqrt{2} - i\sqrt{2}\right)$
Award **1 mark** for the correct linear factors.
***VCAA Assessment Report* note:**
A number of students confused factors and roots. Some wrote down the correct factors but not as a product as required.

e. $z = x + iy$
$\text{Re}(z) \geq \sqrt{2} \Rightarrow x \geq \sqrt{2}$
Award 1 mark for correct shading in part a.
***VCAA Assessment Report* note:**
Frequent errors included segments shaded in other quadrants, the major segment shaded, shading of an annulus, and inaccurate borders and shading of the defined region.

f. Area of segment $= \frac{1}{2}r^2(\theta - \sin(\theta))$
$$A = \frac{1}{2} \times 2^2\left(\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right) = 2\left(\frac{\pi}{2} - 1\right)$$
$A = \pi - 2$
Award **1 mark** for using the correct formulae.
Award **1 mark** for the correct area.
***VCAA Assessment Report* note:**
This question was done reasonably well by students who shaded the correct region in **part e.** Some students gave a decimal approximation. A frequent error was $A = \frac{1}{2}(\pi - 2)$, where only half the segment was sketched, or where integration was used without doubling the integral.

g.



$z = x + yi$, $\bar{z} = x - yi$

$z\bar{z} = x^2 - y^2i^2 = x^2 + y^2 = 4$

A circle with centre at the origin and radius 2

$|z + \bar{z}| = |z - \bar{z}| \Rightarrow |2x| = |2yi| \Rightarrow |x| = |y|$

$y = \pm x$, which implies two straight lines through the origin.

Award **1 mark** for the correct Cartesian equation of the line.

Award **1 mark** for the correct Cartesian equation of the circle.

Award **1 mark** for correct drawing, lines and circle.

***VCAA Assessment Report* note:**

The majority of students did not draw the line $y = -x$. Most got the circle and the line $y = x$. Some students drew more than one circle, and straight lines were sometimes drawn roughly.

Question 16

$z = -1 - i \Rightarrow \bar{z} = -1 + i = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

$\frac{1}{\bar{z}^7} = \frac{1}{\left(\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right)^7} = \left(\sqrt{2}\right)^{-7}\text{cis}\left(-\frac{21\pi}{4}\right)$

$\frac{1}{\bar{z}^7} = \frac{1}{8\sqrt{2}}\text{cis}\left(\frac{3\pi}{4}\right) = \frac{1}{8\sqrt{2}}\left(\cos\left(\frac{3\pi}{4}\right) + i\,\sin\left(\frac{3\pi}{4}\right)\right) = \frac{1}{8\sqrt{2}}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$

$\frac{1}{\bar{z}^7} = \frac{1}{16}(-1 + i)$

Question 17

Let $u = 2 - \alpha i$ by the conjugate root theorem. $\bar{u} = 2 + \alpha i$ is also a root.

Now $u + \bar{u} = 4$ and $u\bar{u} = 4 - \alpha^2 i^2 = 1 + \alpha^2$.

$(z-2)(z-u)\left(z-\bar{u}\right) = (z-2)\left(z^2 - \left(u+\bar{u}\right) + u + \bar{u}\right)$

$= (z-2)\left(z^2 - 4z + \left(4 + \alpha^2\right)\right) = z^3 + az^2 + bz - 12$

$-2\left(4 + \alpha^2\right) = -12 \Rightarrow 4 + \alpha^2 = 6\alpha^2 = 2\alpha = \pm\sqrt{2}$

Coefficient of z^2: $a = -2 - 4 \Rightarrow a = -6$

Coefficient of z: $b = 8 + 4 + \alpha^2 \Rightarrow b = 14$

4 Vector equations of lines and planes

Topic	4	Vector equations of lines and planes
Subtopic	4.2	Linear equations and inequations

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Question 1 (1 mark)

Which of the following statements imply that $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{0}$?

A. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular
B. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel
C. $|\underset{\sim}{a}| = |\underset{\sim}{b}|$
D. Neither $\underset{\sim}{a}$ or $\underset{\sim}{b}$ are the zero vector
E. $|\underset{\sim}{a}| < |\underset{\sim}{b}| < 1$

Question 2 (3 marks)

Consider the points $P(3, 0, 0)$, $Q(0, -2, 0)$ and $R(0, 0, 1)$.
Determine the area of the triangle PQR using a vector method.

Question 3 (2 marks)

For any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, show that $|\underset{\sim}{a} \times \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2$.

Topic	4	Vector equations of lines and planes
Subtopic	4.3	Lines in three dimensions

online only

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Question 1 (1 mark)

Which of the following is a vector equation for the line passing through $A(-3, 0, 5)$ and $B(2, 5, -7)$?

A. $\underset{\sim}{r}(t) = \left(-3\underset{\sim}{i} + 5\underset{\sim}{k}\right) + t\left(5\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}\right)$

B. $\underset{\sim}{r}(t) = \left(-3\underset{\sim}{i} + 5\underset{\sim}{k}\right) + t\left(-5\underset{\sim}{i} - 7\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

C. $\underset{\sim}{r}(t) = \left(-8\underset{\sim}{i} + 5\underset{\sim}{j} - 7\underset{\sim}{k}\right) + t\left(5\underset{\sim}{i} - 7\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

D. $\underset{\sim}{r}(t) = \left(2\underset{\sim}{i} - 7\underset{\sim}{j} + 17\underset{\sim}{k}\right) + t\left(-5\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}\right)$

E. $\underset{\sim}{r}(t) = \left(2\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}\right) + t\left(-5\underset{\sim}{i} - 5\underset{\sim}{j} - 12\underset{\sim}{k}\right)$

Question 2 (3 marks)

Determine the vector equation of the line given by the Cartesian equation $\frac{x-2}{a} = \frac{y+1}{-2} = \frac{z-7}{-4}$ and which passes through the point $(8, -5, -1)$.

Question 3 (5 marks)

One line passes through the two points $(5, 4, -1)$ and $(9, 14, -7)$. A second line passes through the two points $(5, 3, 10)$ and $(6, 5, 14)$. Show that the two lines intersect and calculate the point of intersection and angle between the two lines.

Topic	4	Vector equations of lines and planes
Subtopic	4.4	Planes

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Question 1 (3 marks)

Determine the equation of the plane which contains the points $A(1, 3, -4)$, $B(2, 0, -1)$ and $C(0, -3, 4)$.

Question 2 (2 marks)

Determine the equation of the plane which is perpendicular to $\underset{\sim}{n} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and which contains the point $P_0(1, 3, -1)$.

Question 3 (2 marks)

The parametric equation $x = t + 3, y = 5 - t$ and $z = 2t + 1$ defines a line. The line intersects the plane $3x - 2y + z = 7$ at a point. Determine the coordinates of this point of intersection.

Topic	4	Vector equations of lines and planes
Subtopic	4.5	Review

online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (1 mark)

Consider the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and the plane $a_2x + b_2y + c_2z = d$, the acute angle between the angle and the plane is given by

A. $\sin^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

B. $\cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

C. $\sin^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

D. $\cos^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

E. $\sin^{-1}\left(\frac{|a_1a_2| + |b_1b_2| + |c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

Question 2 (1 mark)

Consider the two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ the acute angle between the lines is given by

A. $\sin^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

B. $\cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

C. $\sin^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

D. $\cos^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

E. $\cos^{-1}\left(\frac{|a_1a_2| + |b_1b_2| + |c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

Question 3 (1 mark)

In three dimensions the equation $y = x$ represents

A. a line parallel to the *xy* plane.

B. a line parallel to the *z*-axis.

C. a plane parallel to the *xy* plane.

D. a plane parallel to the *z*-axis.

E. a plane passing through the origin and perpendicular to the *z*-axis.

Question 4 (1 mark)

Consider the line $\frac{x}{2} = -y = z$ and the plane $x + y - z = 0$ then the line and the plane

A. are parallel and do not intersect.

B. are skew and do not intersect.

C. are perpendicular and intersect only at the origin.

D. intersect at two points $(2, -1, 1)$ and the origin.

E. intersect at an infinite number of points as the line lies in the plane.

Question 5 (3 marks)

A triangle is formed by the points $A(-3, 5, 6)$, $B(-2, 7, 9)$ and $C(2, 1, 7)$. Classify the triangle as either scalene, isosceles or equilateral and prove the triangle is right-angled.

Answers and marking guide

4.2 Linear equations and inequations

Question 1

$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta) = 0$

$\sin(\theta) = 0$

$\therefore \theta = 0$

$|\underset{\sim}{a}| = 0$

$|\underset{\sim}{b}| = 0$

When the vector product is zero, the angle between the vectors is 0, or at least one of the vectors is a zero vector.

The correct answer is **B**.

Question 2

$P(3, 0, 0)$, $Q(0, -2, 0)$, $R(0, 0, 1)$

$\overrightarrow{OP} = 3\underset{\sim}{i}$

$\overrightarrow{OQ} = -2\underset{\sim}{j}$

$\overrightarrow{OR} = \underset{\sim}{k}$ **[1 mark]**

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -3\underset{\sim}{i} - 2\underset{\sim}{j}$ $\quad \overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = 2\underset{\sim}{j} + \underset{\sim}{k}$

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3 & -2 & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -2 & 0 \\ 2 & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -3 & -2 \\ 0 & 2 \end{vmatrix}$$

$-2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$ **[1 mark]**

$$\left|\overrightarrow{PQ} \times \overrightarrow{QR}\right| = \sqrt{(-2)^2 + 3^2 + (-6)^2} = \sqrt{49} = 7$$

$$\text{Area of triangle} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{QR}\right| = \frac{7}{2}$$ **[1 mark]**

Question 3

$|\underset{\sim}{a} \times \underset{\sim}{b}|^2 = ||\underset{\sim}{a}||\underset{\sim}{b}|\sin(\theta) \cdot \hat{\underset{\sim}{n}}|^2$ $\qquad$ since $|\underset{\sim}{n}| = 1$

$= |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 \sin^2(\theta)$ **[1 mark]**

$\text{RHS} = |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2$

$= |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - (|\underset{\sim}{a}|\,|b|\cos(\theta))^2$

$= |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 - \left(|\underset{\sim}{a}|^2 |b|^2 \cos^2(\theta)\right)$

$= |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 \left(1 - \cos^2(\theta)\right)$

$= |\underset{\sim}{a}|^2 |\underset{\sim}{b}|^2 \sin^2(\theta)$

$=$ LHS shown **[1 mark]**

4.3 Lines in three dimensions

Question 1

$\underset{\sim}{a} = -3\underset{\sim}{i} + 5\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + 5\underset{\sim}{j} - 7\underset{\sim}{k}$

$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a}$

$= 5\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{a} + k\underset{\sim}{b}$

$= \left(-3\underset{\sim}{i} + 5\underset{\sim}{k}\right) + k\left(5\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}\right)$

The correct answer is **A**.

Question 2

$\frac{x-2}{a} = \frac{y+1}{-2} = \frac{z-7}{-4} = t$

$x = 2 + at,\ y = -2t - 1,\ z = 7 - 4t$ **[1 mark]**

$y = -5 = -2t - 1$

$2t = 4$

$t = 2$

$x = 8 = 2 + 2a$ **[1 mark]**

$2a = 6$

$a = 3$

$\underset{\sim}{r} = \left(2\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k}\right) + t\left(3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}\right)$ **[1 mark]**

Question 3

Line 1: $A(5, 4, -1)$ $B(9, 14, -7)$

$\underset{\sim}{v}_1 = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 4\underset{\sim}{i} + 10\underset{\sim}{j} - 6\underset{\sim}{k}$

$t = \frac{x-5}{4} = \frac{y-4}{10} = \frac{z+1}{-6}$

$x = 5 + 4t,\ y = 4 + 10t,\ z = -1 - 6t$ **[1 mark]**

Line 2: $P(5, 3, 10)$ $Q(6, 5, 14)$

$\underset{\sim}{v}_2 = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$s = \frac{x-5}{1} = \frac{y-3}{2} = \frac{z-10}{4}$

$x = 5 + s,\ y = 3 + 2s,\ z = 10 + 4s$ **[1 mark]**

(1) $5 + 4t = 5 + s \quad \Rightarrow s = 4t$ into (2)

(2) $4 + 10t = 3 + 2s \quad 4 + 10t = 3 + 8t$

(3) $-1 - 6t = 10 + 4s \quad 2t = -1$

$t = -\frac{1}{2} \Rightarrow s = -2$

Check in (3) LHS $= -1 - 6t = 2$

RHS $= 10 + 4s = 2$ **[1 mark]**

$t = -\frac{1}{2} \Rightarrow x = 3,\ y = -1,\ z = 2$

$s = -2 \Rightarrow x = 3,\ y = -1,\ z = 2$

Point of intersection $(3, -1, 2)$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = 4 + 20 - 24 = 0$ **[1 mark]**

Lines intersect at 90° **[1 mark]**

4.4 Planes

Question 1

$A(1, 3, -4)$, $B(2, 0, -1)$ and $C(0, -3, 4)$

$\overrightarrow{AB} = \underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$, $\overrightarrow{AC} = -\underset{\sim}{i} - 6\underset{\sim}{j} + 8\underset{\sim}{k}$

$\overrightarrow{AB} = \underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$, $\overrightarrow{AC} = -\underset{\sim}{i} - 6\underset{\sim}{j} + 8\underset{\sim}{k}$ **[1 mark]**

$$\underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -3 & 3 \\ -1 & -6 & 8 \end{vmatrix}$$

[1 mark]

$\underset{\sim}{n} = -6\underset{\sim}{i} - 11\underset{\sim}{j} - 9\underset{\sim}{k}$

$0 = -6(x-1) - 11(y-3) - 9(z+4)$

$0 = -6x + 6 - 11y + 33 - 9z - 36$

$6x + 11y + 9z = 3$ **[1 mark]**

Question 2

$\underset{\sim}{n} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k} \quad P_0 = (1, 3, -1)$

$2(x-1) + 3(y-3) - (z+1) = 0$ [**1 mark**]

$2x - 2 + 3y - 9 - z - 1 = 0$

$2x + 3y - z = 12$ [**1 mark**]

Question 3

$x = t + 3,\ y = 5 - t$ and $z = 2t + 1$

$7 = 3x - 2y + z$

$7 = 3(t+3) - 2(5-t) + 1(2t+1)$

$7 = 7t$

$t = 1$ [**1 mark**]

$x = 1 + 3 = 4$

$y = 5 - 1 = 4$

$z = 2 \times 1 + 1 = 3$

Point$(4, 4, 3)$ [**1 mark**]

4.5 Review

Question 1

Direction of the line: $\underset{\sim}{v}_1 = a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k}$

Normal to the plane: $\underset{\sim}{n} = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$

The acute angle between these two vectors is:

$$\frac{\pi}{2} - \cos^{-1}\left(\frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{n}|}{|\underset{\sim}{v}_1||\underset{\sim}{n}|}\right) = \sin^{-1}\left(\frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{n}|}{|\underset{\sim}{v}_1||\underset{\sim}{n}|}\right)$$

$$= \sin^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

The correct answer is **C**.

Question 2

Direction of the first line: $\underset{\sim}{v}_1 = a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k}$ direction of the second line: $\underset{\sim}{v}_2 = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$.

The acute angle between these two vectors is:

$$\cos^{-1}\left(\frac{|\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2|}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|}\right) = \cos^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

The correct answer is **D**.

Question 3

In two dimensions it is a line in the xy plane, in three dimensions it is plane parallel to the z-axis.

The correct answer is **D**.

Question 4

Line: $\frac{x}{2} = -y = z,\ y = -\frac{x}{2}$.

Substitute into the plane $x + y - z = 0$. $x - \frac{x}{2} - \frac{x}{2} = 0 = 0$.

Consistent. There is an infinite number of solutions, as the line lies in the plane.

The correct answer is **E**.

Question 5

$$\begin{aligned}
\overrightarrow{AB} &= \left(-2\underset{\sim}{i} + 7\underset{\sim}{j} + 9\underset{\sim}{k}\right) - \left(-3\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}\right) \\
&= \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k} \\
\left|\overrightarrow{AB}\right| &= \sqrt{14} \\
\overrightarrow{BC} &= \left(2\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}\right) - \left(-2\underset{\sim}{i} + 7\underset{\sim}{j} + 9\underset{\sim}{k}\right) \\
&= 4\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k} \\
\left|\overrightarrow{BC}\right| &= \sqrt{56} \\
\overrightarrow{CA} &= \left(-3\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}\right) - \left(-2\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}\right) \\
&= -5\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k} \\
\left|\overrightarrow{CA}\right| &= \sqrt{42}\ \text{[1 mark]} \\
\overrightarrow{AB} \cdot \overrightarrow{BC} &= \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(4\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}\right) \\
&= 4 + -24 + -6 \\
&= -14 \\
\overrightarrow{BC} \cdot \overrightarrow{CA} &= \left(4\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \left(-5\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right) \\
&= -20 + -24 + 2 \\
&= -42 \\
\overrightarrow{AB} \cdot \overrightarrow{CA} &= \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(-5\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right) \\
&= -5 + 8 - 3 \\
&= 0 \qquad \text{[1 mark]} \\
&\therefore \overrightarrow{AB} \cdot \overrightarrow{CA} \text{are} \perp
\end{aligned}$$

Triangle ABC is a right-angled scalene triangle. **[1 mark]**

5 Differential Calculus

Topic	5	Differential Calculus
Subtopic	5.2	Review of differentiation techniques

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Question 1 (2 marks)

Calculate the derivative of $y = \log_e\left(\sqrt{2x^2 + 9}\right)$.

Question 2 (2 marks)

If $y = \log_e(\tan(3x) + \sec(3x))$, show that $\frac{dy}{dx} = 3\sec(3x)$.

Question 3 (2 marks)

If n and k are both constants, verify the following.

$$\frac{d}{dx}[x^n \cos(kx)] = x^{n-1}(n\cos(kx) - kx\sin(kx))$$

Question 4 (1 mark)

If $y = \dfrac{4x-3}{3x+4}$ then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{4}{3}$

B. $\dfrac{25}{(3x+4)^2}$

C. $\dfrac{-7}{(3x+4)^2}$

D. $\dfrac{7}{(3x+4)^2}$

E. $\dfrac{12x+9}{(3x+4)^2}$

Question 5 (1 mark)

If $y = \dfrac{\log_e(3x)}{3x}$ then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{1}{x^2} - \dfrac{\log_e(3x)}{3x^2}$

B. $\dfrac{1}{3x^2}\left(1 - \log_e(3x)\right)$

C. $\dfrac{1}{3x^2}\left(\log_e(3x) - 3\right)$

D. $\dfrac{7}{(3x+4)^2}$

E. $\dfrac{12x+9}{(3x+4)^2}$

Topic	5	Differential Calculus
Subtopic	5.3	Applications of differentiation

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Question 1 (4 marks)

The tangent to the curve $y = \frac{1}{x}$ at the point where $x = a$ and $a > 0$ crosses the x-axis at B and crosses the y-axis at C if O is the origin, determine the area of the triangle OBC.

Question 2 (4 marks)

Given the function $f: R \to R, f(x) = x^3 e^{-2x}$:

a. determine the equations of the tangents to the curve $y = x^3 e^{-2x}$ at the points where $x = 1$ and $x = 0$ **(2 marks)**

Tangent at the point where $x = 1$: $y = \square$

Tangent at the point where $x = 0$: $y = \square$

b. show that the tangents above are the only two tangents to the curve that pass through the origin. **(2 marks)**

Question 3 (3 marks)

The population number, $P(t)$, of ants in a certain area is given by

$$P(t) = \frac{520}{0.3 + e^{-0.15t}}$$

where $t \geq 0$ is the time in months.

a. Calculate the initial population of the ants.

The initial population was ☐ ants **(1 mark)**

b. Determine the rate at which the ant population is increasing with respect to time and evaluate this rate after 10 months, correct to 1 decimal place.

After 10 months the population is increasing at a rate of ☐ ants per month **(1 mark)**

c. Over the first 10 months determine the average rate at which the ant population is increasing, correct to 1 decimal place. **(1 mark)**

☐ ants per month

Topic	5	Differential Calculus
Subtopic	5.4	Implicit and parametric differentiation

online only

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Source: *VCE 2019 Specialist Mathematics Exam 1, Q10; © VCAA.*

Question 1 (5 marks)

Find $\frac{dy}{dx}$ at the point $\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}}\right)$ for the curve defined by the relation $\sin\left(x^2\right) + \cos\left(y^2\right) = \frac{3\sqrt{2}}{\pi}xy$.

Give your answer in the form $\frac{\pi - a\sqrt{b}}{\sqrt{a}\left(\pi + \sqrt{b}\right)}$, where $a, b \in Z^+$.

Source: *VCE 2017 Specialist Mathematics Exam 1, Q1; © VCAA.*

Question 2 (3 marks)

Find the equation of the tangent to the curve given by $3xy^2 + 2y = x$ at the point $(1, -1)$.

Source: *VCE 2016 Specialist Mathematics Exam 2, Section A, Q7; © VCAA.*

Question 3 (1 mark)

Given that $x = \sin(t) - \cos(t)$ and $y = \frac{1}{2}\sin(2t)$ then $\frac{dy}{dx}$ in term of t is

A. $\cos(t) - \sin(t)$
B. $\cos(t) + \sin(t)$
C. $\sec(t) + \operatorname{cosec}(t)$
D. $\sec(t) + \operatorname{cosec}(t)$
E. $\frac{\cos(2t)}{\cos(t) - \sin(t)}$

Source: *VCE 2018, Specialist Mathematics 1, Q.3; © VCAA*

Question 4 (4 marks)

Find the gradient of the curve $2x^2 \sin(y) + xy = \frac{\pi^2}{18}$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Give your answer in the form $\frac{a}{\pi\sqrt{b} + c}$, where a, b and c are integers.

Source: *VCE 2017, Specialist Mathematics 2, Section A, Q.6; © VCAA*

Question 5 (1 mark)

Given that $\frac{dy}{dx} = e^x \arctan(y)$, the value of $\frac{d^2y}{dx^2}$ at the point (0, 1) is

A. $\frac{1}{2}$

B. $\frac{3\pi}{8}$

C. $-\frac{1}{2}$

D. $\frac{\pi}{4}$

E. $-\frac{\pi}{8}$

Source: *VCE 2016, Specialist Mathematics 1, Q.3; © VCAA*

Question 6 (4 marks)

Find the equation of the line perpendicular to the graph of $\cos(y) + y\sin(x) = x^2$ at $\left(0, -\frac{\pi}{2}\right)$.

Source: *VCE 2014, Specialist Mathematics 1, Q.4; © VCAA*

Question 7 (3 marks)

Find the gradient of the normal to the curve defined by $y = -3e^{3x}e^{y}$ at the point $(1, -3)$.

Question 8 (3 marks)

Find the gradient of the curve with equation $e^{x}e^{2y} + e^{4y^2} = 2e^{4}$ at the point (2, 1).

Question 9 (1 mark)

If $2x^2 + 2xy + y^2 = 0$, then an expression for the gradient is given by

A. $4x + 2y$

B. $4x + 2y + y^2$

C. $\dfrac{2x - y}{x - y}$

D. $4x + 2$

E. $\dfrac{-2x - y}{x + y}$

Question 10 (1 mark)

If $yx^2 + xy^2 = 10$, $\dfrac{dy}{dx}$, is equal to

A. $\dfrac{-2xy - y^2}{(x^2 + 2xy)}$

B. $2xy + y^2$

C. $\dfrac{-2xy + y^2}{(x^2 + 2xy)}$

D. $2x + 1$

E. $\dfrac{-2x - y}{x + y}$

Question 11 (1 mark)

The gradient of the normal to the curve $x^2 + y^2 = 25$ at the point in the third quadrant where $x = -3$ is equal to

A. -6

B. $\frac{3}{4}$

C. $-\frac{3}{4}$

D. $\frac{4}{3}$

E. $-\frac{4}{3}$

Question 12 (1 mark)

Consider the curve $x^2 - xy - 6y^2 = 0$. Which one of the following is false?

A. The gradient of tangent is given by $\frac{2x - y}{x + 12y}$

B. At the point $(-2, 1)$, the gradient of the normal is 2.

C. At the point $(3, 1)$, the gradient of the normal is $\frac{1}{3}$.

D. At the point $(4, -2)$, the gradient of the normal is $-\frac{1}{2}$.

E. At the point $(-6, -2)$, the gradient of the normal is 3.

Topic	5	Differential Calculus
Subtopic	5.5	Second derivatives

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Source: *VCE 2017 Specialist Mathematics Exam 2, Section A, Q6; © VCAA.*

Question 1 (1 mark)

Given that $\frac{dy}{dx} = e^x \arctan(y)$, the value of $\frac{d^2y}{dx^2}$ at the point $(0, 1)$ is

A. $\frac{1}{2}$

B. $\frac{3\pi}{8}$

C. $-\frac{1}{2}$

D. $\frac{\pi}{4}$

E. $-\frac{8\pi}{8}$

Question 2 (4 marks)

Use mathematical induction to show that $\frac{d}{dx}(x^n) = nx^{n-1}, \forall n \in N$.

Question 3 (4 marks)

If a, b, and n are constants, show that $\frac{d^2}{dx^2}\left[\left(ax^2+b\right)^n\right] = 2an\left(ax^2+b\right)^{n-2}\left(a(2n-1)x^2+b\right)$.

Topic 5 Subtopic 5.5 Second derivatives

Source: *VCE 2017, Specialist Mathematics 2, Section A, Q.8; © VCAA*

Question 4 (1 mark)

Let $f(x) = x^3 - mx^2 + 4$, where $m, x \in R$.

The **gradient** of f will always be strictly increasing for

A. $x \geq 0$

B. $x \geq \frac{m}{3}$

C. $x \leq \frac{m}{3}$

D. $x \geq \frac{2m}{3}$

E. $x \leq \frac{2m}{3}$

Question 5 (1 mark)

If $y = \frac{2x\sqrt{x}}{x^2 - 1}$ for $x > 0$, then $\frac{d^2y}{dx^2}$ is equal to

A. $\frac{-\left(x^{\frac{7}{2}} + 3\sqrt{x}\right)}{\left(x^2 - 1\right)^4}$

B. $\frac{-\left(x^{\frac{5}{2}} + 3\sqrt{x}\right)}{\left(x^2 - 1\right)^2}$

C. $\frac{3x + 24x^2 + 2}{\sqrt{x}\left(x^2 - 1\right)^3}$

D. $\frac{3x^4 + 26x^2 + 3}{2\sqrt{x}\left(x^2 - 1\right)^3}$

E. $\frac{2x^3 + 6x^2 + x}{\left(x^2 - 1\right)^4}$

Question 6 (1 mark)

The second derivative of $f(x) = x^3 + x^{\frac{5}{2}}$ is equal to

A. $3x^2 + \frac{5x^{\frac{3}{2}}}{2}$

B. $6x + \frac{15\sqrt{x}}{4}$

C. $3x^2 + \frac{5x^{\frac{1}{2}}}{2}$

D. $6x + \frac{5x^{\frac{1}{2}}}{4}$

E. $6x + \frac{15\sqrt{x}}{2}$

Question 7 (1 mark)

If $f(x) = e^{x^2}$, then $f''(1)$ is equal to

A. $4xe^{x^2} + 2e^{x^2}$

B. $6e^{x^2}$

C. $4e$

D. $6e$

E. $2e$

Question 8 (1 mark)

If $f(x) = \tan^{-1}\left(\frac{bx}{a}\right)$, where a and b are positive real constants, then $f''(1)$ is equal to

A. $\frac{1}{a^2 + b^2}$

B. $\frac{ab}{a^2 + b^2}$

C. $\frac{-2ab}{\left(a^2 + b^2\right)^2}$

D. $\frac{2ab}{\left(a^2 + b^2\right)^2}$

E. $\frac{-2ab^3}{\left(a^2 + b^2\right)^2}$

Question 9 (1 mark)

If $y = \sin^{-1}(bx)$, where b is a real non-zero number, then $\frac{d^2y}{dx^2}$ is equal to

A. $\frac{-bx}{(1 - b^2x^2)}$

B. $\frac{-bx}{\sqrt{(1 - b^2x^2)^3}}$

C. $\frac{bx}{\sqrt{(1 - b^2x^2)^3}}$

D. $\frac{-b^3x}{\sqrt{(1 - b^2x^2)^3}}$

E. $\frac{b^3x}{\sqrt{(1 - b^2x^2)^3}}$

Topic	5	Differential Calculus
Subtopic	5.6	Derivatives of inverse trigonometric functions

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Source: *VCE 2020 Specialist Mathematics Exam 1, Q6a; © VCAA.*

Question 1 (1 mark)

Let $f(x) = \arctan(3x - 6) + \pi$.

Show that $f'(x) = \dfrac{3}{9x^2 - 36x + 37}$.

Source: *VCE 2017 Specialist Mathematics Exam 1, Q10a; © VCAA.*

Question 2 (1 mark)

Show that $\dfrac{d}{dx}\left(x \arccos\left(\dfrac{x}{a}\right)\right) = \arccos\left(\dfrac{x}{a}\right) - \dfrac{x}{\sqrt{a^2 - x^2}}$, where $a > 0$.

Source: *VCE 2015 Specialist Mathematics Exam 2, Section 1, Q2; © VCAA.*

Question 3 (1 mark)

The range of the function with rule $f(x) = (2 - x)\arcsin\left(\dfrac{x}{2} - 1\right)$ is

A. $[-\pi, 0]$

B. $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

C. $\left[-\dfrac{(2-x)\pi}{2}, \dfrac{(2-x)\pi}{2}\right]$

D. $[0, 4]$

E. $[0, \pi]$

Source: *VCE 2017, Specialist Mathematics 1, Q.6; © VCAA*

Question 4 (3 marks)

Let $f(x) = \dfrac{1}{\arcsin(x)}$.

Find $f'(x)$ and state the largest set of values of x for which $f'(x)$ is defined.

Question 5 (1 mark)

If $y = \tan^{-1}\left(\dfrac{5}{3x}\right)$ and $x \neq 0$, then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{5}{25 + 9x^2}$

B. $\dfrac{-15}{25 + 9x^2}$

C. $\dfrac{-15}{x^2\left(25 + 9x^2\right)}$

D. $\dfrac{15x^2}{25 + 9x^2}$

E. $\dfrac{25 + 9x^2}{5}$

Question 6 (1 mark)

If $y = \cos^{-1}\left(\dfrac{4}{5x}\right)$ and $x > 0$, then $\dfrac{dy}{dx}$ is equal to

A. $\dfrac{4}{\sqrt{16 - 25x^2}}$

B. $\dfrac{5}{\sqrt{16 - 25x^2}}$

C. $\dfrac{\sqrt{16 - 25x^2}}{5}$

D. $\dfrac{4}{x\sqrt{25x^2 - 16}}$

E. $\dfrac{-5}{x\sqrt{16 - 25x^2}}$

Question 7 (1 mark)

If $y = \tan^{-1}\left(\sqrt{3x}\right)$, then $\frac{dy}{dx}$ equals

A. $\frac{1}{1 + 3x}$

B. $\frac{\sqrt{3}}{1 + 3x}$

C. $\frac{\sqrt{3}}{1 + 3x^2}$

D. $\frac{1}{2\sqrt{3x}(1 + 3x)}$

E. $\frac{\sqrt{3}}{2\sqrt{x}(1 + 3x)}$

Topic	5	Differential Calculus
Subtopic	5.7	Related rates

online only

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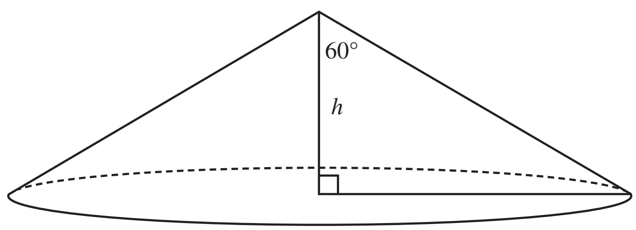
Source: *VCE 2019, Specialist Mathematics 2, Section A, Q.10; © VCAA*

Question 1 (1 mark)



Sand falls from a chute to form a pile in the shape of a right circular cone with semi-vertex angle 60°. Sand is added to the pile at a rate of 1.5 m^3 per minute.

The rate at which the height h metres of the pile is increasing, in metres per minute, when the height of the pile is 0.5 m, correct to two decimal places, is

A. 0.21
B. 0.31
C. 0.64
D. 3.82
E. 3.53

Source: *VCE 2016, Specialist Mathematics 1, Q.4; © VCAA*

Question 2 (4 marks)

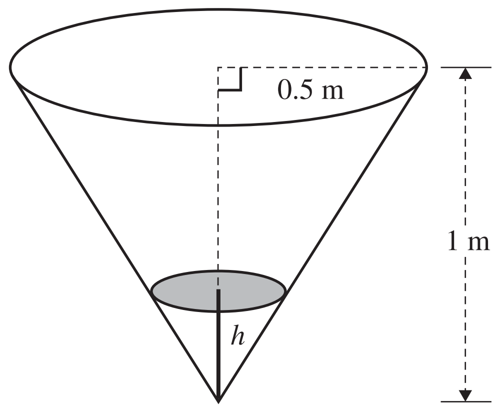
Chemicals are added to a container so that a particular crystal will grow in the shape of a cube. The side length of the crystal, x millimetres, t days after the chemicals were added to the container, is given by $x = \arctan(t)$.

Find the rate at which the surface area, A square millimetres, of the crystal is growing one day after the chemicals were added. Give your answer in square millimetres per day.

Source: *VCE 2014 Specialist Mathematics Exam 2, Section 2, Q4 a, b; © VCAA.*

Question 3 (5 marks)

At a water fun park, a conical tank of radius 0.5 m and height 1 m is filling with water. At the same time, some water flows out from the vertex, wetting those underneath. When the tank eventually fills, it tips over and the water falls out, drenching all those underneath. The tank then returns to its original position and begins to refill.



Water flows in at a constant rate of $0.02\pi \text{m}^3/\text{min}$ and flows out at a variable rate of $0.01\pi\sqrt{h}\,\text{m}^3/\text{min}$, where h metres is the depth of the water at any instant.

a. Show that the volume, V cubic metres, of water in the cone when it is filled to a depth of h metres is given by $V = \frac{\pi}{12}h^3$. **(1 mark)**

b. Find the rate, in m/min, at which the depth of the water in the tank is increasing when the depth is 0.25 m **(4 marks)**

Topic 5 Subtopic 5.7 Related rates

Source: *VCE 2016, Specialist Mathematics 2, Section A, Q.7; © VCAA*

Question 4 (1 mark)

Given that $x = \sin(t) - \cos(t)$ and $y = \frac{1}{2}\sin(2t)$, then $\frac{dy}{dx}$ in terms of t is

A. $\cos(t) - \sin(t)$
B. $\cos(t) + \sin(t)$
C. $\sec(t) + \text{cosec}(t)$
D. $\sec(t) - \text{cosec}(t)$
E. $\frac{\cos(2t)}{\cos(t) - \sin(t)}$

Question 5 (1 mark)

An oil slick is in the shape of a circle. Its surface area is increasing at a rate of $20\,\text{m}^2/\text{s}$.
Let r metre be the radius of the oil slick and time t seconds.
The rate of increase of r, In m/s, is given by

A. $\frac{10}{\pi r}$
B. $\frac{20}{\pi r}$
C. $\frac{10}{\pi r^2}$
D. $\frac{1}{20\pi r}$
E. $20\pi r$

Question 6 (1 mark)

The sides of an equilateral triangle are increasing at a rate of 4 cm/ min. The rate at which the area is increasing in cm^2/ min when the sides are all $\sqrt{3}$ cm is

A. $4\sqrt{3}$
B. $2\sqrt{3}$
C. $\sqrt{3}$
D. $\frac{\sqrt{3}}{2}$
E. 6

Question 7 (1 mark)

A regular hexagon has each side length being L cm. The area A of the hexagon is given by $\frac{3\sqrt{3}}{2}L^2$. If each of the six sides of the hexagon increases at a rate of $2\sqrt{3}$ cm/s, then the rate in cm^2/s at which the area of the hexagon is increasing, when the side lengths are $\sqrt{3}$ cm, is given by

A. $18\sqrt{3}$
B. 18
C. $12\sqrt{3}$
D. $9\sqrt{3}$
E. 27

Question 8 (1 mark)

A puddle is in the shape of a circle. Its surface area, s cm^2, is increasing at a rate of 10 cm^2/s. Let r metres be the radius of the puddle at time t seconds.
The rate of increase of r, in cm/s, is given by

A. $\frac{5}{\pi r}$
B. $\frac{20}{\pi r}$
C. $\frac{10}{\pi r^2}$
D. $\frac{1}{20\pi r}$
E. 20π

Topic	5	Differential Calculus
Subtopic	5.8	Review

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Source: *VCE 2021 Specialist Mathematics Exam 1, Q.5; © VCAA.*

Question 1 (3 marks)

Find the gradient of the curve with equation $e^x e^{2y} + e^{4y^2} = 2e^4$ at the point $(2, 1)$.

Source: *VCE 2016 Specialist Mathematics Exam 1, Q3; © VCAA.*

Question 2 (4 marks)

Find the equation of the line perpendicular to the graph of $\cos(y) + y\sin(x) = x^2$ at $\left(0, -\frac{\pi}{2}\right)$.

Source: *VCE 2015 Specialist Mathematics Exam 1, Q9; © VCAA.*

Question 3 (6 marks)

Consider the curve represented by $x^2 - xy + \frac{3}{2}y^2 = 9$.

a. Find the gradient of the curve at any point (x, y). **(2 marks)**

b. Find the equation of the tangent to the curve at the point $(3, 0)$ and find the equation of the tangent to the curve at the point $\left(0, \sqrt{6}\right)$. **(2 marks)**

c. Find the acute angle between the tangent to the curve at the point $(3, 0)$ and the tangent to the curve at the point $\left(0, \sqrt{6}\right)$. **(2 marks)**

Give your answer in the form $k\pi$, where k k is a real constant.

Source: *VCE 2013 Specialist Mathematics Exam 1, Q6; © VCAA.*

Question 4 (4 marks)

Find the value of c, where $c \in R$, such that the curve defined by $y^2 + \dfrac{3e^{(x-1)}}{x-2} = c$ has a gradient of 2 where $x = 1$.

Question 5 (5 marks)

Use mathematical induction to prove that $\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}, \forall n \in N.$

Source: *VCE 2020, Specialist Mathematics 1, Q.7; © VCAA*

Question 6 (5 marks)

Consider the function defined by

$$f(x) = \begin{cases} mx + n, & x < 1 \\ \dfrac{4}{1+x^2}, & x \geq 1 \end{cases}$$

where m and n are real numbers.

a. Given that $f(x)$ and $f'(x)$ are continuous over R, show $m = -2$ and $n = 4$. **(2 marks)**

b. Find the area enclosed by the graph of the function, the x-axis and the lines $x = 0$ and $x = \sqrt{3}$. **(3 marks)**

Answers and marking guide

5.2 Review of differentiation techniques

Question 1

$$\begin{aligned} y &= \log_e\left(\sqrt{2x^2+9}\right) \\ &= \log_e\left(2x^2+9\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\log_e\left(2x^2+9\right) \qquad \textbf{[1 mark]} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\times\frac{4x}{2x^2+9} \\ &= \frac{2x}{2x^2+9} \qquad \textbf{[1 mark]} \end{aligned}$$

Question 2

$y = \log_e(\tan(3x) + \sec(3x))$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dx}(\tan(3x)+\sec(3x))}{\tan(3x)+\sec(3x)} \\ &= \frac{3\sec^2(3x)+3\sec(3x)\tan(3x)}{\tan(3x)+\sec(3x)} \qquad \textbf{[1 mark]} \\ &= \frac{3\sec(3x)(\sec(3x)+\tan(3x))}{\tan(3x)+\sec(3x)} \\ &= 3\sec(3x) \qquad \textbf{[1 mark]} \end{aligned}$$

Question 3

$y = x^n\cos(kx)$

Let $v = \cos(kx)$

$\frac{dv}{dx} = -k\sin(kx)$

Let $u = x^n$

$\frac{du}{dx} = nx^{n-1}$

$$\begin{aligned} y &= uv \\ \frac{dy}{dx} &= -kx^{n-1}\sin(kx) + nx^{n-1}\cos(kx) \\ &= x^{n-1}(n\cos(nx) - kx\sin(nx)) \end{aligned}$$

Award **1 mark** for correctly identifying u and v and their derivatives.

Award **1 mark** for correctly applying the product rule to obtain the result.

Question 4

$$y = \frac{4x-3}{3x+4}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{4(3x+4)-3(4x-3)}{(3x+4)^2} = \frac{(12x+16)-(12x-9)}{(3x+4)^2} \\ &= \frac{25}{(3x+4)^2} \end{aligned}$$

Question 5

$$y = \frac{\log_e(3x)}{3x}$$

let $u = \log_e(3x)$ and $v = 3x$

$$\frac{du}{dx} = \frac{1}{x} \text{ and } \frac{dv}{dx} = 3$$

$$\frac{dy}{dx} = \frac{\frac{1}{x} \times 3x - 3\log_e(3x)}{\left(3x^2\right)} = \frac{3\left(1 - \log_e(3x)\right)}{9x^2}$$

$$= \frac{1}{3x^2}\left(1 - \log_e(3x)\right)$$

5.3 Applications of differentiation

Question 1

$y = \frac{1}{x}$ when $x = a,\ a > 0 : \text{P}\left(a, \frac{1}{a}\right)$ $y = 0$ **[1 mark]**

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$m_T = -\frac{1}{a^2}$$

$$T : y - \frac{1}{a} = \frac{1}{a^2}(x - a)$$

Crosses x-axis when $y = 0$ at point B. **[1 mark]**

$$-\frac{1}{a} = \frac{1}{a^2}(x - a)$$

$$x - a = a$$

$$x = 2a$$

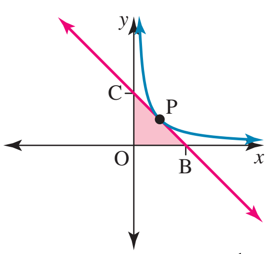
B $(2a, 0)$

Crosses y-axis when $x = 0$ at point C.

$y - \frac{1}{a} = -\frac{1}{a^2}(-a)$ **[1 mark]**

$$y - \frac{1}{a} = \frac{1}{a}$$

$$\text{C}\left(0, \frac{2}{a}\right)$$



Area of Δ OBC $= \frac{1}{2} \times 2a \times \frac{2}{a}$ **[1 mark]**

$= 2$

Question 2

a. $f(x) = x^3 e^{-2x}$

$$f'(x) = 3x^2 e^{-2x} - 2x^3 e^{-2x}$$

$$= x^2 e^{-2x}(3 - 2x)$$

At $x = 1$
$f(1) = e^{-2}$
$f'(1) = e^{-2}$
$T: y - e^{-2} = e^{-2}(x - 1)$
$y = xe^{-2}$ **[1 mark]**
At $x = 0$
$f(0) = 0$
$f(0) = 0$
$T: y = 0$

b. At $x = a$ **[1 mark]**
$P\left(a, a^3e^{-2a}\right)$
$f'(a) = e^{-2a}\left(3a^2 - 2a^3\right)$
Tangent at $x = a$
$y - a^3e^{-2a} = e^{-2a}\left(3a^2 - 2a^3\right)(x - a)$ **[1 mark]**
If the tangent passes through the origin $x = 0, \quad y = 0$
$-a^3e^{-2a} = e^{-2a}\left(3a^2 - 2a^3\right) \times -a$
$-a^3 = -a\left(3a^2 - 2a^3\right)$
$a^2 = 3a^2 - 2a^3$
$2a^3 - 2a^2 = 0$
$2a^2(a - 1) = 0$
So $a = 0$ or $a = 1$
Are the only tangents which pass through the origin. **[1 mark]**

Question 3

a. $P(t) = \dfrac{520}{0.3 + e^{-0.15t}}$

$P(0) = \dfrac{520}{1.3} = 400$ **[1 mark]**

b. $P(t) = \dfrac{520}{0.3 + e^{-0.15t}}$

$P(t) = 520\left(0.3 + e^{-0.15t}\right)^{-1}$

$\dfrac{dP}{dt} = 0.15e^{-0.15t} \times 520\left(0.3 + e^{-0.15t}\right)^{-2}$

$= \dfrac{78e^{-0.15t}}{\left(0.3 + e^{-0.15t}\right)^2}$

$\left.\dfrac{dP}{dt}\right|_{t=10} = \dfrac{78e^{-0.15\times10}}{\left(0.3 + e^{-0.15\times10t}\right)^2}$ **[1 mark]**

$= 63.6$

c. $P(t) = \dfrac{520}{0.3 + e^{-0.15t}}$

$P(10) = 994.016$

$\dfrac{P(10) - P(0)}{10 - 0} = \dfrac{994 - 400}{10}$

$= 59.4$ **[1 mark]**

5.4 Implicit and parametric differentiation

Question 1

$\sin\left(x^2\right) + cos\left(y^2\right) = \frac{3\sqrt{2}}{\pi}xy$ using implicit differentiation

$$\frac{d}{dx}\left(\sin\left(x^2\right)\right) + \frac{d}{dx}\left(\cos\left(y^2\right)\right) = \frac{3\sqrt{2}}{\pi}\frac{d}{dx}(xy)$$

$$2x\cos\left(x^2\right) - 2y\sin\left(y^2\right)\frac{dy}{dx} = \frac{3\sqrt{2}}{\pi}\left(y + x\frac{dy}{dx}\right)$$

$$2x\cos\left(x^2\right) - \frac{3\sqrt{2}y}{\pi} = \frac{dy}{dx}\left(\frac{3\sqrt{2}x}{\pi} + 2y\sin\left(y^2\right)\right)$$

Now at the point $\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}},\right) x = \frac{\sqrt{\pi}}{\sqrt{6}}, y = \frac{\sqrt{\pi}}{\sqrt{3}}$

$$2\frac{\sqrt{\pi}}{\sqrt{6}}cos\left(\frac{\pi}{6}\right) - \frac{3\sqrt{2}}{\pi} \times \frac{\sqrt{\pi}}{\sqrt{3}} = \frac{dy}{dx}\left(\frac{3\sqrt{2}}{\pi} \times \frac{\sqrt{\pi}}{\sqrt{6}} + 2\frac{\sqrt{\pi}}{\sqrt{3}}\sin\left(\frac{\pi}{3}\right)\right)$$

$$2\frac{\sqrt{\pi}}{\sqrt{6}} \times \frac{\sqrt{3}}{2} - \frac{3\sqrt{2}}{\pi} \times \frac{\sqrt{\pi}}{\sqrt{3}} = \frac{dy}{dx}\left(\frac{3\sqrt{2}}{\pi} \times \frac{\sqrt{\pi}}{\sqrt{6}} + 2\frac{\sqrt{\pi}}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\right)$$

$$\frac{\sqrt{\pi}}{\sqrt{2}} - \frac{\sqrt{6}}{\pi} = \frac{dy}{dx}\left[\sqrt{\pi} + \frac{\sqrt{3}}{\sqrt{\pi}}\right]$$

$$\frac{\pi - \sqrt{12}}{\sqrt{2\pi}} = \frac{dy}{dx}\left(\frac{\pi + \sqrt{3}}{\sqrt{\pi}}\right)$$

$$\frac{dy}{dx} = \frac{\pi - 2\sqrt{3}}{\sqrt{2}\left(\pi + \sqrt{3}\right)} a = 2, b = 3$$

Award **1 mark** for using implicit differentiation and product rule.
Award **1 mark** for correctly rearranging.
Award **1 mark** for correctly substituting the x and y values.
Award **1 mark** for transforming.
Award **1 mark** for the final correct values of a and b.

Question 2

$3xy^2 + 2y = x$ using implicit differentiation, with product rule in the first term

$$3y^2 + 6xy\frac{dy}{dx} + 2\frac{dy}{dx} = 1$$

$$\frac{dy}{dx}(6xy + 2) = 1 - 3y^2$$

$\frac{dy}{dx} = \frac{1 - 3y^2}{6xy + 2}$ at the point $(1, , -1)$

$$\frac{dy}{dx}\Big|_{(1\,,\,-1)} = \frac{1 - 3}{-6 + 2}$$

$$= \frac{1}{2}$$

$$T: y + 1 = \frac{1}{2}(x - 1)$$

$$2y + 2 = x - 1$$

$$2y - x + 3 = 0$$

Award **1 mark** for using implicit differentiation with the product rule.
Award **1 mark** for the correct gradient.
Award **1 mark** for the correct tangent line.

Question 3

$x\sin(t) - \cos(t) \quad y = \frac{1}{2}\sin(2t)$

$\frac{dx}{dt} = \dot{x} = \cos(t) + \sin(t) \quad \frac{dy}{dt} = \dot{y} = \cos(2t)$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\cos(2)}{\cos(t) + \sin(t)}$$

$$= \frac{\cos(2t)}{\cos(t) + \sin(t)} \times \frac{\cos(t) - \sin(t)}{\cos(t) - \sin(t)}$$

$$= \frac{\cos(2t)(\cos(t) - \sin(t))}{\cos^2(t) - \sin^2(t)}$$

$$= \frac{\cos(2t)(\cos(t) - \sin(t))}{\cos(2t)}$$

$$= \cos(t) - \sin(t)$$

The correct answer is **E**.

Question 4

$2x^2\sin(y) + xy = \frac{\pi^2}{18}$

Using implicit differentiation, with product rule in the first and second terms:

$$4x\sin(y) + 2x^2\cos(y)\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$4x\sin(y) + y = -\frac{dy}{dx}(2x^2\cos(y) + x)$$

Award **1 mark** for using implicit differentiation.
Award **1 mark** for correctly using the product rule.

At the point $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

$$4\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + \frac{\pi}{6} = -\frac{dy}{dx}\left(2\left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{6}\right) + \frac{\pi}{6}\right)$$ **[1 mark]**

$$\frac{\pi}{3} + \frac{\pi}{6} = -\frac{dy}{dx}\left(\frac{\pi^2\sqrt{3}}{36} + \frac{\pi}{6}\right)$$

$$\frac{\pi}{2} = -\frac{dy}{dx}\left(\frac{\pi}{36}\left(\pi\sqrt{3} + 6\right)\right)$$

$$\frac{dy}{dx} = \frac{-18}{\pi\sqrt{3} + 6}$$

$a = -18,\ b = 3,\ c = 6$ **[1 mark]**

***VCAA Examination Report* note:**

Most students knew to use implicit differentiation in this problem and were successful in their application of the chain and product rules. Many students attempted to find an expression for $\frac{dy}{dx}$ in terms of x and y This was not necessary, with a more effective approach being to substitute $x = \frac{\pi}{6}$ and $y = \frac{\pi}{6}$ immediately following the implicit differentiation. Some students had difficulty with arithmetic.

Question 5

$\frac{dy}{dx} = e^x \tan^{-1}(y)$ at $(0, 1)$ when $x = 0,\ y = 1,$

$\frac{dy}{dx} = \tan^{-1}(1) = \frac{\pi}{4}$, using implicit differentiation and the product rule:

$$\frac{d^2y}{dx^2} = e^x \tan^{-1}(y) + \frac{e^x}{1+y^2}\frac{dy}{dx}$$

when $x = 0$ $y = 1$

$$\frac{d^2y}{dx^2} = \tan^{-1}(1) + \frac{1}{2}\frac{dy}{dx}$$
$$= \frac{\pi}{4} + \frac{\pi}{8}$$
$$= \frac{3\pi}{8}$$

Question 6

$\cos(y) + y\sin(x) = x^2, \left(0, -\frac{\pi}{2}\right)$

Using implicit differentiation, with the product rule on second term:

$$-\sin(y)\frac{dy}{dx} + \sin(x)\frac{dy}{dx} + y\cos(x) = 2x$$
$$\frac{dy}{dx}[\sin(x) - \sin(y)] = 2x - y\cos(x)$$
$$\frac{dy}{dx} = \frac{2x - y\cos(x)}{\sin(x) - \sin(y)}$$

At$x = 0,\ y = -\frac{\pi}{2}$:

$$m_T = \frac{0 + \frac{\pi}{2}\cos(0)}{\sin(0) - \sin\left(-\frac{\pi}{2}\right)}$$
$$= \frac{\pi}{2}$$
$$\Rightarrow m_N = -\frac{2}{\pi}$$

Find the equation of N:

$$y + \frac{\pi}{2} = -\frac{2}{\pi}(x - 0)$$
$$y = -\frac{2x}{\pi} - \frac{\pi}{2}$$

Award **1 mark** for using implicit differentiation and applying the product rule.
Award **1 mark** for finding the correct gradient at the indicated point.
Award **1 mark** for the gradient of the normal.
Award **1 mark** for the correct equation of the normal.

VCAA Assessment Report **note:**

Students generally dealt well with the implicit differentiation, with most realising the need for the chain rule and the product rule. Several students made a sign error when substituting the given point; others found the gradient of the perpendicular line and did not continue. Others found the gradient and/or equation of the tangent, while some thought that the gradient of the normal was equal to the reciprocal rather than the negative reciprocal of the gradient of the tangent. Others used the negative of the gradient of the tangent.

Question 7

$y = -3e^{3x}e^{y}$

$y + 3e^{3x}e^{y} = 0$

$$\frac{dy}{dx} + 9e^{3x}e^{y} + 3e^{3x}e^{y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(1 + 3e^{3x}e^{y}\right) = -9e^{3x}e^{y}$$

$$\frac{dy}{dx} = \frac{-9e^{3x}e^{y}}{1 + 3e^{3x}e^{y}}$$

$$\frac{dy}{dx} = \frac{-9e^{3x+y}}{1 + 3e^{3x}e^{y}}$$

$$m_T = \left.\frac{dy}{dx}\right|_{(1,-3)} = \frac{9e^{3-3}}{1 + 3e^{3-3}} = -\frac{9}{4}$$

$$m_N = \frac{4}{9}$$

Award **1 mark** for using implicit differentiation and the product rule.
Award **1 mark** for evaluating the gradient of the tangent.
Award **1 mark** for the correct gradient of the normal.

***VCAA Assessment Report* notes:**

Many students answered this question well. Most recognised the need for implicit differentiation and attempted to use the product rule. The most common differentiation errors were $\frac{d}{dy}(e^{y}) = ye^{y}$ and $\frac{d}{dy}(y) = 0$, occasionally $= 1$. Some students rearranged the equation prior to attempting to find the derivative. On most occasions this either led to complications or was an incomplete attempt. A number of students were unable to take the $\frac{dy}{dx}$ terms to one side of the equation or made algebraic errors in doing so. Many students did not substitute in the given values. Some who did substitute in the given values made numerical errors or were unable to simplify $e^{-3}e^{3}$. Several students correctly found the gradient of the tangent and then did no further work. Some students found the equation of the normal, which was not required.

Question 8

$e^{x}e^{2y} + e^{4y^2} = 2e^{4}$

$$\frac{d}{dx}\left(e^{x}e^{2y}\right) + \frac{d}{dx}\left(e^{4y^2}\right) = \frac{d}{dx}\left(2e^{4}\right)$$ using the product rule on the first term

$$e^{2y}\frac{d}{dx}\left(e^{x}\right) + e^{x}\frac{d}{dx}\left(e^{2y}\right) + \frac{d}{dy}\left(e^{4y^2}\right)\frac{dy}{dx} = 0$$

$$e^{x}e^{2y} + 2e^{x}e^{2y}\frac{dy}{dx} + 8ye^{4y^2}\frac{dy}{dx} = 0$$

substitute $x = 1,\ y = 2$

$$e^{2}e^{2} + 2e^{2}e^{2}\frac{dy}{dx} + 8e^{4}\frac{dy}{dx} = 0$$

$$e^{4} = -10e^{4}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{10}$$

Award **1 mark** for applying the product rule.
Award **1 mark** for correctly using implicit differentiation.
Award **1 mark** for the final correct gradient.

Question 9

$$0 = 4x + 2y + \frac{dy}{dx}(2x + 2y)$$

$$\frac{dy}{dx}(2x + 2y) = -4x - 2y$$

$$\frac{dy}{dx} = \frac{-4x - 2y}{(2x + 2y)}$$

$$= \frac{-2x - y}{x + y}$$

Question 10

$$0 = 2xy + x^2\frac{dy}{dx} + y^2 + \frac{dy}{dx}2xy$$

$$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{(x^2 + 2xy)}$$

Question 11

$x^2 + y^2 = 25$

Taking $\frac{d}{dx}$ of each term $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Now when $x = -3$, $9 + y^2 = 25 \Rightarrow y^2 = 16 \quad y = \pm 4$

In the third quadrant, $x = -3$ and $y = -4$ Point $(-3, -4)$

$$\left.\frac{dy}{dx}\right|_{(-3,-4)} = -\left(\frac{-3}{-4}\right) = -\frac{3}{4} = m_T$$

$$m_N = \frac{4}{3}$$

Question 12

Using implicit differentiation $x^2 - xy - 6y^2 = 0$.

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) - \frac{d}{dx}(6y^2) = 0$$

Using the product rule in the middle term

$$\frac{d}{dx}(x^2) - x\frac{d}{dx}(y) - y\frac{d}{dx}(x) - \frac{d}{dx}(6y^2) = 0$$

$$2x - x\frac{dy}{dx} - y - 12y\frac{dy}{dx} = 0$$

$$2x - y = \frac{dy}{dx}(12y + x)$$

$$\frac{dy}{dx} = \frac{2x - y}{x + 12y}$$

A is true

B $\left.\frac{dy}{dx}\right|_{(-2,\ 1)} = \frac{-4 - 1}{12 - 2} = -\frac{1}{2} = m_T \Rightarrow m_N = 2$ is true.

C $\left.\frac{dy}{dx}\right|_{(3,\ 1)} = \frac{6 - 1}{3 + 12} = \frac{1}{3} = m_T$ is true.

D $\left.\frac{dy}{dx}\right|_{(4,\,-2)} = \frac{8+2}{4-24} = -\frac{1}{2} = m_T$ is true.

E $\left.\frac{dy}{dx}\right|_{(-6,\,-2)} = \frac{-12+2}{-6-24} = 3 = m_T \quad m_N = -\frac{1}{3}$ So **E** is false.

5.5 Second derivatives

Question 1

$\frac{dy}{dx} = e^x \tan^{-1}(y)$ at $(0, 1)$ when $x = 0$, $y = 1$,

$\frac{dy}{dx} = \tan^{-1}(1) = \frac{\pi}{4}$, using implicit differentiation and the product rule:

$$\frac{d^2y}{dx^2} = e^x \tan^{-1}(y) + \frac{e^x}{1+y^2}\frac{dy}{dx}$$

When $x = 0,\ y = 1$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \tan^{-1}(1) + \frac{1}{2}\frac{dy}{dx}\\ &= \frac{\pi}{4} + \frac{\pi}{8}\\ &= \frac{3\pi}{8}\end{aligned}$$

The correct answer is **B**.

Question 2

To show $\frac{d}{dx}(x^n) = nx^{n-1}$,

the first base step is to prove it is true for the base case, that is when $n = 1$

$\frac{d}{dx}\left(x^1\right) = 1 = 1x^0 = 1$ so it is true when $n = 1$.

We now assume it is true when $n = k$, that is assume

$\frac{d}{dx}\left(x^k\right) = kx^{k-1}$ the kth derivative is true,

we now consider the derivative of x^{k+1}

$\frac{d}{dx}\left(x^{k+1}\right) = \frac{d}{dx}\left(x^k \times x\right)$ using the product rule

$\frac{d}{dx}\left(x^{k+1}\right) = x\frac{d}{dx}\left(x^k\right) + x^k\frac{d}{dx}(x)$

$\frac{d}{dx}\left(x^{k+1}\right) = x \times kx^{k-1} + x^k \times 1 = kx^k + x^k$

$\frac{d}{dx}\left(x^{k+1}\right) = (k+1)\,x^k$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

Award **1 mark** for base case when $n = 1$.

Award **1 mark** for assumption case.

Award **1 mark** for simplifying using the product rule.

Award **1 mark** for the final conclusion.

Question 3

Let

$$y = \left(ax^2 + b\right)^n$$

$$\frac{dy}{dx} = n \times 2ax\left(ax^2 + b\right)^{n-1}$$

$$= 2an\left[x \times \left(ax^2 + b\right)^{n-1}\right]$$ **[1 mark]**

$$\frac{d^2y}{dx^2} = 2an\left[\left(ax^2 + b\right)^{n-1} + x\frac{d}{dx}\left(ax^2 + b\right)^{n-1}\right]$$ **[1 mark]**

$$= 2an\left[\left(ax^2 + b\right)^{n-1} + 2ax^2(n-1)\left(ax^2 + b\right)^{n-2}\right]$$

$$= 2an\left(ax^2 + b\right)^{n-2}\left[\left(ax^2 + b\right) + 2ax^2(n-1)\right]$$ **[1 mark]**

$$= 2an\left(ax^2 + b\right)^{n-2}\left(ax^2(1 + 2(n-1)) + b\right)$$

$$= 2an\left(ax^2 + b\right)^{n-2}\left(a(2n-1)x^2 + b\right)$$

$$\frac{d^2}{dx^2}\left[\left(ax^2 + b\right)^n\right] = 2an\left(ax^2 + b\right)^{n-2}\left(a(2n-1)x^2 + b\right)$$ **[1 mark]**

Question 4

$f(x) = x^3 - mx^2 + 4$

$f'(x) = 3x^2 - 2mx$

$f''(x) = 6x - 2m$

$f''(x) > 0 \Rightarrow x \geq \frac{m}{3}$

Question 5

$$\frac{dy}{dx} = \frac{-\left(x^{\frac{5}{2}} + 3\sqrt{x}\right)}{\left(x^2 - 1\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{3x^4 + 26x^2 + 3}{2\sqrt{x}\left(x^2 - 1\right)^3}$$

Question 6

$$f'(x) = 3x^2 + \frac{5x^{\frac{3}{2}}}{2}$$

$$f''(x) = 6x + \frac{15\sqrt{x}}{4}$$

Question 7

$f'(x) = 2xe^{x^2}$

$f''(x) = 4x^2e^x + 2e^x$

$= 2e^{x^2}(1 + 2x)$

$f''(1) = 4e + 2e$

$= 6e$

Question 8

$$f(x) = \tan^{-1}\left(\frac{bx}{a}\right)$$

$$f'(x) = \frac{ab}{a^2 + b^2x^2} = ab\left(a^2 + b^2x^2\right)^{-1}$$

$$f''(x) = -ab(2b^2x)(a^2 + b^2x^2)^{-2}$$

$$f''(x) = \frac{-2ab^3x}{\left(a^2 + b^2x^2\right)^2}$$

$$f''(1) = \frac{-2ab^3}{(a^2 + b^2)^2}$$

Question 9

$$y = \sin^{-1}(bx)$$

$$\frac{dy}{dx} = \frac{b}{\sqrt{1 - b^2x^2}} = b(1 - b^2x^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(-2b^2x)(1 - b^2x^2)^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{b^3x}{\sqrt{(1 - b^2x^2)^3}}$$

5.6 Derivatives of inverse trigonometric functions

Question 1

$$f(x) = \arctan(3x - 6) + \pi$$

$$f(x) = y = \tan^{-1}(u) + \pi, u = 3x - 6$$

$$\frac{dy}{du} = \frac{1}{1 + u^2}, \frac{du}{dx} = 3$$

$$f'(x) = \frac{3}{1 + (3x - 6)^2}$$

$$f'(x) = \frac{3}{9x^2 - 36x + 37}$$

Award **1 mark** for the correct use of the chain rule.

Question 2

Using the product rule

$$\frac{d}{dx}\left(x \arccos\left(\frac{x}{a}\right)\right), a > 0$$

$$= \frac{d}{dx}(x) \arccos\left(\frac{x}{a}\right) + x\frac{d}{dx}\left(\arccos\left(\frac{x}{a}\right)\right)$$

$$= \arccos\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}}$$

Award 1 mark for the correct use of the product rule.

Question 3

$$y = (2 - x)\sin^{-1}\left(\frac{x}{2} - 1\right)$$

Domain $\left|\frac{x}{2} - 1\right| \leq 1 \Rightarrow -1 \leq \frac{x}{2} - 1 \leq 1$

$0 \le \frac{x}{2} \le 2 \Rightarrow x \in [0, 4]$
$f(0) = -\pi, f(4) = -\pi, f(2) = 0$
Range $[-\pi, 0]$
The correct answer is **A**.

Question 4

$$f(x) = \frac{1}{\arcsin(x)} = \left(\sin^{-1}(x)\right)^{-1}$$

Let $f(u) = u^{-1}$, $\quad u(x) = \sin^{-1}(x)$ using the chain rule

$$f'(u) = -u^{-2}, \quad u'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}\left(\sin^{-1}(x)\right)^2} \quad \text{for } -1 < x < 1$$

Award **1 mark** for using the chain rule.
Award **1 mark** for the correct derivative.
Award **1 mark** for the correct interval for the gradient.

***VCAA Examination Report* note:**

This question was not answered well. Some students confused the inverse function with the reciprocal function. The most common incorrect derivatives were $\sqrt{1-x^2}$ and $\frac{\log_e\left(\sin^{-1} x\right)}{\sqrt{1-x^2}}$, while some had $\frac{d(\arcsin x)}{dx} = \log_e(\arcsin x)$. Common errors for the domain included R, $R \setminus \{-1, 0, 1\}$, $[-1, 1]$, $(-1, 1)$ and $[-1, 1] \setminus \{0\}$ Many students did not exclude zero. The incorrect answer $(-\infty, 0) \cup (0, \infty)$ was also relatively common.

Question 5

Let $y = \tan^{-1}\left(\frac{5}{3x}\right)$

$y = \tan^{-1}(u)$, where $u = \frac{5}{3x} = \frac{5}{3}x^{-1}$ (using the chain rule)

$$\frac{dy}{du} = \frac{1}{1+u^2} \qquad \frac{du}{dx} = -\frac{5}{3}x^{-2} = \frac{-5}{3x^2}$$

$$\frac{dy}{dx} = \frac{-5}{3x^2}\frac{1}{\left(1+\left(\frac{5}{3x}\right)^2\right)} = -\frac{5}{3x^2}\left(\frac{1}{1+\frac{25}{9x^2}}\right) = -\frac{5}{3x^2}\left(\frac{1}{\frac{9x^2+25}{9x^2}}\right)$$

$$\frac{dy}{dx} = \frac{-5 \times 9x^2}{3x^2(9x^2+25)} \quad \text{since } x \neq 0$$

$$\frac{dy}{dx} = \frac{-15}{25+9x^2}$$

Question 6

Let $y = \cos^{-1}\left(\frac{4}{5x}\right)$

$y = \cos^{-1}(u)$, where $u = \frac{4}{5x} = \frac{4}{5}x^{-1}$ (using the chain rule)

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \qquad \frac{du}{dx} = -\frac{4}{5}x^{-2} = \frac{-4}{5x^2}$$

$$\frac{dy}{dx} = \frac{4}{5x^2\sqrt{1-\left(\frac{4}{5x}\right)^2}} = \frac{4}{5x^2\sqrt{1-\frac{16}{25x^2}}}$$

$$\frac{dy}{dx} = \frac{4}{5x^2\sqrt{\frac{25x^2-16}{25x^2}}} = \frac{4\times 5x}{5x^2\sqrt{25x^2-16}} \quad \text{since } x>0$$

$$\frac{dy}{dx} = \frac{4}{x\sqrt{25x^2-16}}$$

Question 7

$$y = \tan^{-1}\left(\sqrt{3}x\right)$$

$$\tan(y) = \sqrt{3}x$$

$$\sec^2 y\,dy = \sqrt{3}dx$$

$$\frac{dy}{dx} = \frac{\sqrt{3}}{\sec^2(y)}$$

$$= \frac{\sqrt{3}}{1+\tan^2(y)}$$

$$= \frac{\sqrt{3}}{1+3x^2}$$ **[1 mark]**

5.7 Related rates

Question 1

Given $\frac{dV}{dt} = 1.5\,\text{m}^3/\text{min}$

$$\tan(60°) = \frac{r}{h} = \sqrt{3}, \quad r = \sqrt{3}h \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\sqrt{3}h\right)^2 h = \pi h^3$$

$$\frac{dV}{dh} = 3\pi h^2$$

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{1.5}{3\pi h^2}, \quad \left.\frac{dh}{dt}\right|_{h-0.5} = \frac{1.5}{3\pi(0.5)^2} = 0.64\,\text{m/min}$$

Question 2

$x = \tan^{-1}(t),\; A(x) = 6x^2$

$A(t) = 6\left(\tan^{-1}(t)\right)^2$

$$\frac{dA}{dt} = \frac{6\times 2\tan^{-1}(t)}{1+t^2}$$

When $t = 1$:

$$\frac{dA}{dt} = \frac{12\tan^{-1}(1)}{2}$$

$$= 6\times\frac{\pi}{4}$$

$$= \frac{3\pi}{2}\,\text{mm}^2/\text{day}$$

Award **1 mark** for correctly expressing A in terms of t.
Award **1 mark** for the correct rate.
Award **1 mark** for evaluating the rate.
Award **1 mark** for the final correct result.

***VCAA Assessment Report* note:**

Students had mixed success with this question. Quite a few students made errors with the formula for the surface area of a cube, including, $A = x^2$, $2x^2$, $4x^2$ or more commonly x^3. There was some confusion with t and x, which resulted in a denominator $(1 + x^2)$ rather than $(1 + t^2)$. Some found the correct derivative using the chain rule but then did not continue by substituting for t and/or x. Some students did not use the chain rule. A few wrote $t = \tan(x)$ and some progressed successfully from there. Other methods were seen, including some students converting completely to the variable x.

Some students wrote $\arctan(t)$ as $\tan^{-1}(t)$ and then converted this to $\dfrac{1}{\tan(t)}$

Question 3

a.
$$\begin{aligned}\frac{r}{h} &= \frac{0.5}{1}\\ &= \frac{1}{2}\\ r &= \frac{h}{2}\\ V &= \frac{1}{3}\pi r^2 h\\ &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h\\ &= \frac{\pi}{12}h^3 \qquad \textbf{[1 mark]}\end{aligned}$$

b.
$$\begin{aligned}\frac{dV}{dt} &= 0.02\pi - 0.01\pi\sqrt{h}\\ &= \frac{\pi}{100}\left(2-\sqrt{h}\right) \qquad \textbf{[1 mark]}\\ \frac{dV}{dh} &= \frac{\pi h^2}{4} \qquad \textbf{[1 mark]}\\ \frac{dh}{dt} &= \frac{dh}{dV}\cdot\frac{dV}{dt}\\ &= \frac{4}{\pi h^2}\times\frac{\pi}{100}\left(2-\sqrt{h}\right)\\ &= \frac{2-\sqrt{h}}{25h^2} \qquad \textbf{[1 mark]}\\ \left.\frac{dh}{dt}\right|_{h=0.25} &= \frac{2-\sqrt{0.25}}{25\,(0.25)^2}\\ &= \frac{1.5}{\left(\frac{25}{16}\right)}\\ &= \frac{24}{25}\\ &= 0.96\,\text{m/min} \qquad \textbf{[1 mark]}\end{aligned}$$

Question 4

$x = \sin(t) - \cos(t) \qquad y = \frac{1}{2}\sin(2t)$

$\frac{dx}{dt} = \dot{x} = \cos(t) + \sin(t) \quad \frac{dy}{dt} = \dot{y} = \cos(2t)$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\cos(2t)}{\cos(t) + \sin(t)}$$

$$= \frac{\cos(2t)}{\cos(t) + \sin(t)} \times \frac{\cos(t) - \sin(t)}{\cos(t) - \sin(t)}$$

$$= \frac{\cos(2t)(\cos(t) - \sin(t))}{\cos^2(t) - \sin^2(t)}$$

$$= \frac{\cos(2t)(\cos(t) - \sin(t))}{\cos(2t)}$$

$$= \cos(t) - \sin(t)$$

Question 5

$\frac{dS}{dt} = 20$

$\frac{ds}{dr} = 2\pi r$

$\frac{dr}{dt} = \frac{dr}{ds} \times \frac{ds}{dt}$

$= \frac{1}{2\pi r} \times 20$

$= \frac{10}{\pi r}$

Question 6

$A = \frac{1}{2}\sin(60°)L^2 \quad A = \frac{\sqrt{3}L^2}{4} \quad \frac{dA}{dL} = \frac{\sqrt{3}L}{2}$ given $\frac{dL}{dt} = 4\,\text{cm/min}$

$\frac{dA}{dt} = \frac{dA}{dL} \cdot \frac{dL}{dt} = \frac{\sqrt{3}L}{2} \times 4 = 2\sqrt{3}L$

$\left.\frac{dA}{dt}\right|_{L=\sqrt{3}} = 6\,\text{cm}^2/\text{min}$

Question 7

$A = \frac{3\sqrt{3}}{2}L^2 \quad \frac{dA}{dL} = 3\sqrt{3}L$ gives $\frac{dL}{\text{d}t} = 2\sqrt{3}\,\text{cm/s}$

$\frac{\text{d}A}{\text{d}t} = \frac{\text{d}A}{\text{d}L} \cdot \frac{\text{d}L}{\text{d}t} = 3\sqrt{3}L \times 2\sqrt{3} = 18L$

$\left.\frac{\text{d}A}{\text{d}t}\right|_{L=\sqrt{3}} = 18\sqrt{3}\,\text{cm}^2/\text{s}$

Question 8

$$\frac{ds}{dt} = 10$$

$$\frac{ds}{dr} = 2\pi r$$

$$\frac{dr}{dt} = \frac{dr}{ds} \times \frac{ds}{dt}$$
$$= \frac{1}{2\pi r} \times 10$$
$$= \frac{5}{\pi r} \quad \textbf{[1 mark]}$$

5.8 Review

Question 1

$$e^x e^{2y} + e^{4y^2} = 2e^4$$

$$\frac{d}{dx}\left(e^x e^{2y}\right) + \frac{d}{dx}\left(e^{4y^2}\right) = \frac{d}{dx}\left(2e^4\right)$$ using the product rule on the first term

$$e^{2y}\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(e^{2y}\right) + \frac{d}{dy}\left(e^{4y^2}\right)\frac{dy}{dx} = 0$$

$$e^x e^{2y} + 2e^x e^{2y}\frac{d}{dx} + 8ye^{4y^2}\frac{dy}{dx} = 0$$

Substitute $x = 1,\ y = 2$

$$e^2e^2 + 2e^2e^2\frac{dy}{dx} + 8e^4\frac{dy}{dx} = 0$$

$$e^4 = -10e^4\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{10}$$

Award **1 mark** for applying the product rule.
Award **1 mark** for correctly using implicit differentiation.
Award **1 mark** for the final correct gradient.

Question 2

$$\cos(y) + y\ \sin(x) = x^2, \left(0, -\frac{\pi}{2}\right)$$

Using implicit differentiation with the product rule on second term:

$$-\sin(y)\frac{dy}{dx} + \sin(x)\frac{dy}{dx} + y\ \cos(x) = 2x$$

$$\frac{dy}{dx}\left[\sin(x) - \sin(y)\right] = 2x - y\ \cos(x)$$

$$\frac{dy}{dx} = \frac{2x - y\ \cos(x)}{\sin(x) - \sin(y)}$$

At $x = 0$, $y = -\frac{\pi}{2}$:

$$m_T = \frac{0 + \frac{\pi}{2}\cos(0)}{\sin(0) - \sin\left(-\frac{\pi}{2}\right)}$$
$$= \frac{\pi}{2}$$

$$\Rightarrow\ m_N = -\frac{2}{\pi}$$

Find the equation of N:

$$y+\frac{\pi}{2}=-\frac{2}{\pi}(x-0)$$
$$y=-\frac{2x}{\pi}-\frac{\pi}{2}$$

Award **1 mark** for using implicit differentiation and applying the product rule.
Award **1 mark** for finding the correct gradient at the indicated point.
Award **1 mark** for gradient of the normal.
Award **1 mark** for the equation of the normal.

Question 3

a. $x^2-xy+\frac{3}{2}y^2=9$ implicit differentiation

$$2x-y-x\frac{dy}{dx}+3y\frac{dy}{dx}=0$$
$$2x-y=(x-3y)\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{2x-y}{x-3y}$$

Award **1 mark** for using implicit differentiation.
Award **1 mark** for the correct gradient.

b. $A:(3,0)\ m_A=\left.\frac{dy}{dx}\right|_{(3,\,0)}=2\ \ T_A: y-0=2(x-3)$

$T_A: y=2x-6$

$B: \left(0,\ \sqrt{6}\right)\ m_B=\left.\frac{dy}{dx}\right|_{\left(0,\ \sqrt{6}\right)}=\frac{1}{3}\ \ T_B: y-\sqrt{6}=\frac{1}{3}(x-0)$

$T_B: y=\frac{x}{3}+\sqrt{6}$

Award **1 mark** for the correct tangent at $(3,0)$.
Award **1 mark** for the correct tangent at $(0,\sqrt{6})$.

c. Let $m_B=\tan(\theta_1)=\frac{1}{3}\ \Rightarrow \theta_1=\tan^{-1}\left(\frac{1}{3}\right)$ and

$m_A=\tan(\theta_2)=2\ \Rightarrow\ \theta_2=\tan^{-1}(2)$

$\theta=\theta_2-\theta_1$

$$\tan(\theta)=\tan(\theta_2-\theta_1)$$
$$=\frac{\tan(\theta_2)-\tan(\theta_1)}{1+\tan(\theta_2)\tan(\theta_1)}=\frac{2-\frac{1}{3}}{1+2\times\frac{1}{3}}=\frac{\frac{5}{3}}{\frac{5}{3}}=1$$
$$\theta=\tan^{-1}(1)$$
$$\theta=\frac{\pi}{4}$$

Award **1 mark** for using the tan angle formula.
Award **1 mark** for the final correct angle.

Question 4

$$y^2+\frac{3e^{x-1}}{x-2}=c$$

Using implicit differentiation and the quotient rule.

$$2y\frac{dy}{dx}+\frac{3e^{(x-1)}(x-2)-3e^{(x-1)}}{(x-2)^2}=0$$
$$2y\frac{dy}{dx}+\frac{3e^{(x-1)}(x-3)}{(x-2)^2}=0$$

When $x = 1, \dfrac{dy}{dx} = 2$

$4y + \dfrac{3e^0 \times -2}{(-1)^2} = 0 \;\Rightarrow\; 4y - 6 = 0 \;\Rightarrow\; y = \dfrac{3}{2}$

$c = \dfrac{9}{4} - 3$

$c = -\dfrac{3}{4}$

Award **1 mark** for using implicit differentiation.
Award **1 mark** for using the quotient rule.
Award **1 mark** for substituting.
Award **1 mark** for using the final correct value c.

Question 5

To show $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$

The first base step is to prove it is true for the base case, that is when $n = 1$,

$$\text{LHS} = \frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{d}{dx}\left((ax+b)^{-1}\right)$$
$$= -a\,(ax+b)^{-2} = \frac{-a}{(ax+b)^2}$$

$\text{RHS} = \dfrac{(-1)^1\, 1!\, a^1}{(ax+b)^2} = \dfrac{-a}{(ax+b)^2}$ so is true when $n = 1$.

We now assume it is true when $n = k$, that is assume

$\dfrac{d^k}{dx^k}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^k\, k!\, a^k}{(ax+b)^{k+1}}$ for the k th derivative is true,

we now consider the $(k+1)$ th derivative, that is

$\dfrac{d^{k+1}}{dx^{k+1}}\left(\dfrac{1}{ax+b}\right) = \dfrac{d}{dx}\left(\dfrac{d^k}{dx^k}\left(\dfrac{1}{ax+b}\right)\right)$ using the assumption

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = \frac{d}{dx}\left(\frac{(-1)^k\, k!\, a^k}{(ax+b)^{k+1}}\right)$$

Taking the constant factor outside the derivative simplifying

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = (-1)^k\, k!\, a^k \frac{d}{dx}\left((ax+b)^{-k-1}\right)$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = (-1)^k\, k!\, a^k\,(-k-1) \times a\left((ax+b)^{-k-2}\right)$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = -(-1)^k\,(k+1)\,k!\,a^{k+1}\left(\frac{1}{(ax+b)^{k+2}}\right)$$

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{ax+b}\right) = \frac{(-1)^{k+1}\,(k+1)\,!\,a^{k+1}}{(ax+b)^{k+2}}$$

Which is what we want to show, this completes the proof by induction, so it is true for all $n \in N$.

Award **1 mark** for base case when $n = 1$.
Award **1 mark** for assumption case.
Award **2 marks** for simplifying.
Award **1 mark** for the final conclusion.

Question 6

a. $f(x) \quad = \begin{cases} mx + n, \; x < 1 \\ \dfrac{4}{1+x^2}, \; x \geq 1 \end{cases}$

$f'(x) \quad = \begin{cases} m, & x < 1 \\ \dfrac{-8x}{(1+x^2)^2}, & x \geq 1 \end{cases}$

Since $f(x)$ is continuous, $(1) f(1) = m + n = 2$.

Since $f'(x)$ is continuous, $(2) f'(1) = m = -\dfrac{8}{4} = -2 \Rightarrow n = 4$.

Award **1 mark** for equating components.

Award **1 mark** for solving for m and n.

b. $A = \displaystyle\int_0^1 (4 - 2x)dx + \int_1^{\sqrt{3}} \frac{4}{1+x^2}dx$

$A = [4x - x^2]_0^1 + [4\tan^{-1}(x)]_1^{\sqrt{3}}$

$A = (4 - 1) - 0 + 4\left(\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\right)$

$A = 3 + 4\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$

$A = 3 + \dfrac{\pi}{3}$

Award **1 mark** for the correct definite integral.

Award **1 mark** for the correct integral.

Award **1 mark** for the final correct area.

6 Functions and graphs

Topic	6	Functions and graphs
Subtopic	6.2	Sketching graphs of cubics and quartics

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Source: *VCE 2021 Specialist Mathematics Exam 2, Section A, Q.9; © VCAA*

Question 1 (1 mark)

Which one of the following derivatives corresponds to a graph of f that has no points of inflection?

A. $f'(x) = 2(x-2)^2 + 5$

B. $f'(x) = 2(x-3)^3 + 5$

C. $f'(x) = \frac{5}{2}(x-3)^2$

D. $f'(x) = \frac{1}{2}(x-3)^3 - 5$

E. $f'(x) = (x-3)^3 - 12x$

Source: *VCE 2017 Specialist Mathematics Exam 2, Section A, Q10; © VCAA*

Question 2 (1 mark)

A function f, its derivative f' and its second derivative f'' are defined for $x \in R$ with the following properties.

$f(a) = 1, f(-a) = -1$

$f(b) = -1, f(-b) = 1$

and

$f''(x) = \frac{(x+a)^2(x-b)}{g(x)}$, where $g(x) < 0$

The coordinates of any points of inflection of $|f(x)|$ are

A. $(-a, 1)$ and $(b, 1)$

B. $(b, -1)$

C. $(-a, -1)$ and $(b, -1)$

D. $(-a, 1)$

E. $(b, 1)$

Topic 6 | Subtopic 6.2 Sketching graphs of cubics and quartics

Source: *VCE 2017 Specialist Mathematics Exam 2, Section A, Q8; © VCAA*

Question 3 (1 mark)

Let $f(x) = x^3 - mx^2 + 4$, where $m,\ x \in R$.

The **gradient** of f will always be strictly increasing for

A. $x \geq 0$

B. $x \geq \frac{m}{3}$

C. $x \leq \frac{m}{3}$

D. $x \geq \frac{2m}{3}$

E. $x \leq \frac{2m}{3}$

Topic	6	Functions and graphs
Subtopic	6.3	Sketching graphs of rational functions

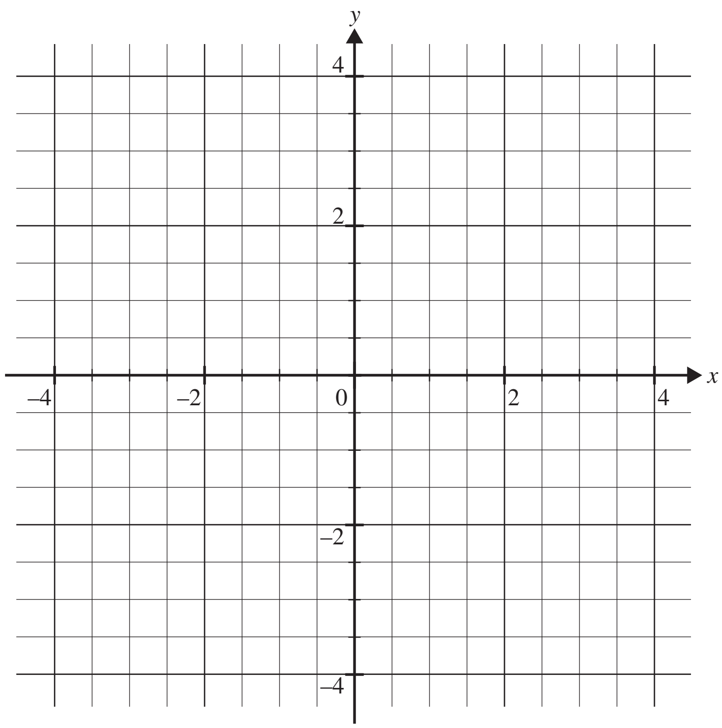
online only

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Source: *VCE 2018 Specialist Mathematics Exam 1, Q5; © VCAA*

Question 1 (4 marks)

Sketch the graph of $f(x) = \dfrac{x+1}{x^2-4}$ on the axes provided below, labelling any asymptotes with their equations and any intercepts with their coordinates.



Source: *VCE 2016 Specialist Mathematics Exam 2, Section A, Q3; © VCAA*

Question 2 (1 mark)

The straight-line asymptote(s) of the graph of the function with rule $f(x) = \dfrac{x^3 - ax}{x^2}$, where a is a non-zero real constant, is given by

A. $x = 0$ only.
B. $x = 0$ and $y = 0$ only.
C. $x = 0$ and $y = x$ only.
D. $x = 0,\ x = \sqrt{a}$ and $x = -\sqrt{a}$ only.
E. $x = 0$ and $y = a$ only.

Source: VCE 2016 Specialist Mathematics Exam 2, *Section B, Q1 a, b, c; © VCAA*

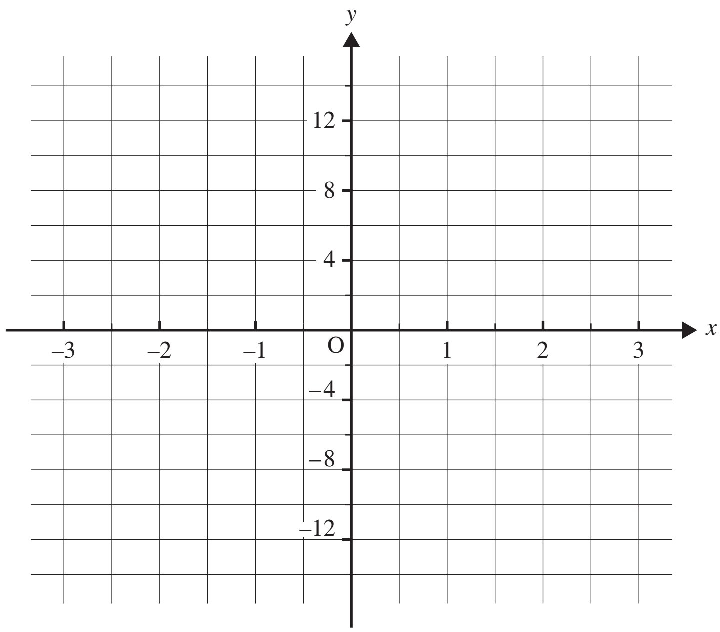
Question 3 (4 marks)

a. Find the stationary point of the graph of $f(x) = \dfrac{4 + x^2 + x^3}{x}$, $x \in R \setminus \{0\}$. Express your answer in coordinate form, giving values correct to two decimal places. **(1 mark)**

The stationary point is ☐.

b. Find the point of inflection of the graph given in part **a**. Express your answer in coordinate form, giving values correct to two decimal places. **(2 marks)**

c. Sketch the graph of $f(x) = \dfrac{4 + x^2 + x^3}{x}$ for $x \in [-3, 3]$ on the axes below, labelling the turning point and the point of inflection with their coordinates, correct to two decimal places. **(1 mark)**



Source: *VCE 2020, Specialist Mathematics 2, Section A, Q.1; © VCAA*

Question 4 (1 mark)

The y-intercept of the graph of $y = f(x)$, where $f(x) = \dfrac{(x-a)(x+3)}{(x-2)}$, is also a stationary point when a equals

A. -2

B. $-\dfrac{6}{5}$

C. 0

D. $\dfrac{6}{5}$

E. 2

Topic 6 | Subtopic 6.3 Sketching graphs of rational functions

Source: *VCE 2019, Specialist Mathematics 2, Section A, Q.2; © VCAA*

Question 5 (1 mark)

The asymptote(s) of the graph of $f(x) = \dfrac{x^2 + 1}{2x - 8}$ has equation(s)

A. $x = 4$
B. $x = 4$ and $y = \dfrac{x}{2}$
C. $x = 4$ and $y = \dfrac{x}{2} + 2$
D. $x = 8$ and $y = \dfrac{x}{2}$
E. $x = 8$ and $y = 2x + 2$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.3; © VCAA*

Question 6 (1 mark)

The features of the graph of the function with rule $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - x - 6}$ include

A. asymptotes at $x = 1$ and $x = -2$
B. asymptotes at $x = 3$ and $x = -2$
C. asymptotes at $x = 1$ and a point of discontinuity at $x = 3$
D. asymptotes at $x = -2$ and a point of discontinuity at $x = 3$
E. asymptotes at $x = 3$ and a point of discontinuity at $x = -2$

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.3; © VCAA*

Question 7 (1 mark)

The graph of $y = \dfrac{1}{ax^2 + bx + c}$ has asymptotes at $x = -5$, $x = 3$ and $y = 0$.

Given that the graph has one stationary point with a y-coordinate of $-\dfrac{1}{8}$, it follows that

A. $a = 1,\ b = 2,\ c = -15$
B. $a = \dfrac{1}{2},\ b = -1,\ c = -\dfrac{15}{2}$
C. $a = -\dfrac{1}{2},\ b = -1,\ c = 15$
D. $a = -1,\ b = -2,\ c = -15$
E. $a = \dfrac{1}{2},\ b = 1,\ c = -\dfrac{15}{2}$

Question 8 (1 mark)

The graph of $y = 3 - \frac{2}{x^2} + 2x$ has asymptotes at

A. $x = 0$ and $y = 3$

B. $x = 0$ and $y = 2x$

C. $x = 2$ and $y = 2x + 3$

D. $x = 0$ and $y = 2x + 3$

E. $x = 0$ and $y = 3 - 2x$

Question 9 (1 mark)

The graph of $y = \frac{x^3 + 2}{3x^2}$ has

A. $x = 0$ as its only straight line asymptote.

B. $x = 0$ and $y = \frac{x}{3}$ as its only straight line asymptotes.

C. no straight line asymptotes.

D. $y = 0$ as its only straight line asymptote.

E. $x = 0$ and $y = x^3 + 2$ as its only straight line asymptotes.

Question 10 (1 mark)

$2y + 5x = \frac{3}{2x^2} - 1$ has straight line asymptotes at

A. $x = 0$ and $y = \frac{-(5x + 1)}{2}$

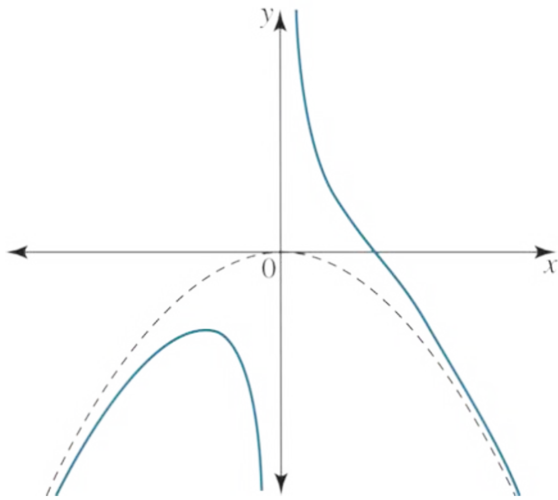
B. $x = 0$ and $y = -5x + 1$

C. $x = 0$ and $y = \frac{3}{4x^2}$

D. $x = 0$ and $y = \frac{3}{2x^2} - 1$

E. $x = 0$ and $y = \frac{5x}{2} - 1$

Question 11 (1 mark)



A possible equation for the graph of the curve shown is

A. $y = \dfrac{ax^2 + b}{x}$, $a > 0$ and $b > 0$

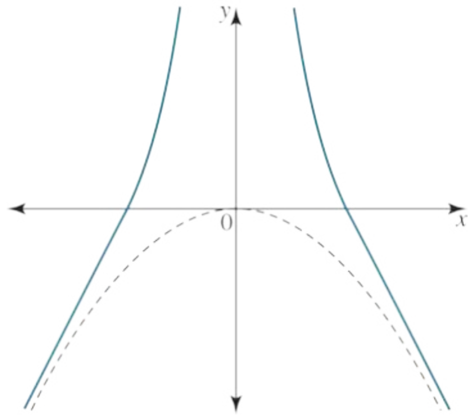
B. $y = \dfrac{ax^3 + b}{x}$, $a < 0$ and $b < 0$

C. $y = \dfrac{ax^3 + b}{x}$, $a < 0$ and $b > 0$

D. $y = \dfrac{ax^3 + b}{x^2}$, $a > 0$ and $b > 0$

E. $y = \dfrac{ax^3 + b}{x^2}$, $a < 0$ and $b < 0$

Question 12 (1 mark)



A possible equation for the graph of the curve shown is

A. $y = \dfrac{ax^2 + b}{x}$, $a > 0$ and $b > 0$

B. $y = \dfrac{ax^4 + b}{x^2}$, $a < 0$ and $b < 0$

C. $y = \dfrac{ax^4 + b}{x^2}$, $a < 0$ and $b > 0$

D. $y = \dfrac{ax^4 + b}{x^2}$, $a > 0$ and $b > 0$

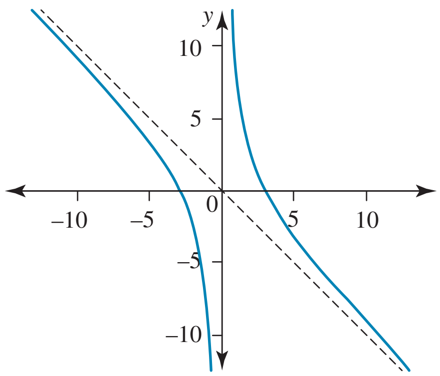
E. $y = \dfrac{ax^4 + b}{x^2}$, $a > 0$ and $b < 0$

Question 13 (4 marks)

Sketch the graph of $f: R \rightarrow, f(x) = \dfrac{x^2 + 4}{2x}$.

Give the equations of any asymptotes and the coordinates of any axial intercepts and stationary points.

Question 14 (1 mark)



A possible equation for the graph of the curve shown is

A. $y = \dfrac{ax^2 + b}{x}$, $a > 0$ and $b > 0$

B. $y = \dfrac{ax + b}{x}$, $a < 0$ and $b < 0$

C. $y = \dfrac{ax^2 + b}{x}$, $a < 0$ and $b > 0$

D. $y = \dfrac{ax^3 + b}{x^2}$, $a > 0$ and $b > 0$

E. $y = \dfrac{ax^3 + b}{x^2}$, $a < 0$ and $b < 0$

Topic	6	Functions and graphs
Subtopic	6.4	Sketching graphs of product and quotient functions

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Source: *VCE 2021 Specialist Mathematics Exam 2, Section A, Q.3; © VCAA*

Question 1 (1 mark)

The coordinates of the local maxima of the graph of $y = \dfrac{1}{(\cos(ax)+1)^2+3}$, where $a \in R \setminus \{0\}$, are

A. $\left(\dfrac{2\pi k}{a}, \dfrac{1}{7}\right)$, $k \in Z$

B. $\left(\dfrac{2\pi k}{a}, \dfrac{1}{3}\right)$, $k \in Z$

C. $\left(\dfrac{(1+2k)\pi}{2a}, \dfrac{1}{4}\right)$, $k \in Z$

D. $\left(\dfrac{\pi(1+2k)}{a}, \dfrac{1}{4}\right)$, $k \in Z$

E. $\left(\dfrac{\pi(1+2k)}{a}, \dfrac{1}{3}\right)$, $k \in Z$

Source: *VCE 2020 Specialist Mathematics Exam 2, Section B, Q3; © VCAA*

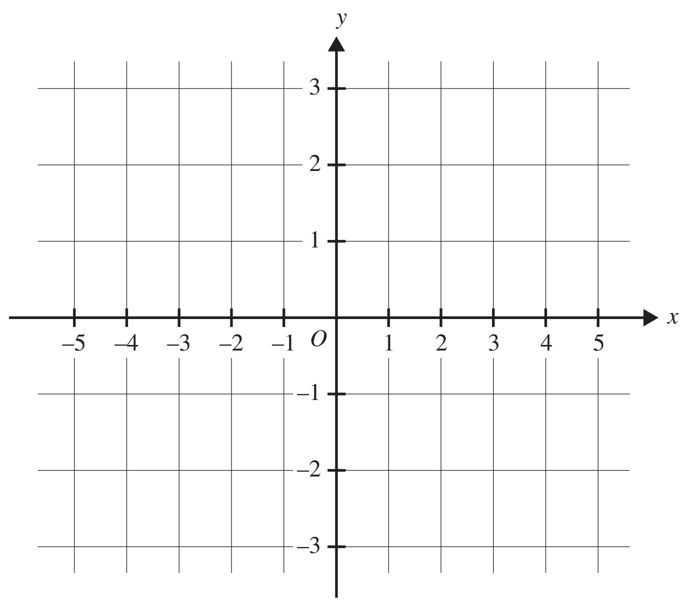
Question 2 (10 marks)

Let $f(x) = x^2e^{-x}$

a. Find an expression for $f'(x)$ and state the coordinates of the stationary points of $f(x)$ **(2 marks)**

b. State the equation(s) of any asymptotes of $f(x)$ **(1 mark)**

c. Sketch the graph of $y = f(x)$ on the axes provided below, labelling the local maximum stationary point and all points of inflection with their coordinates, correct to two decimal places. **(3 marks)**



d. Write down an expression for $g''(x)$ **(1 mark)**

e. i. Find the non-zero values of x for which $g''(x) = 0$. **(1 mark)**

ii. Complete the following table by stating the values(s) of n for which the graph of $g(x)$ has the given number of points of inflection. **(2 marks)**

Number of points of inflection	Value(s) of n (where $n \in Z$)
0	
1	
2	
3	

Source: *VCE 2019 Specialist Mathematics Exam 2, Section A, Q1; © VCAA*

Question 3 (1 mark)

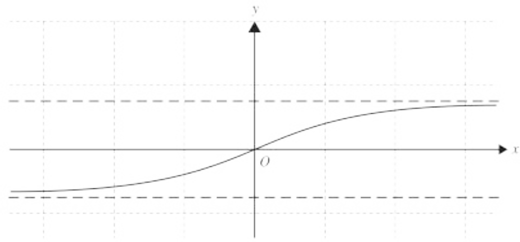
The graph of $f(x) = \dfrac{e^x}{x-1}$ does **not** have a

A. horizontal asymptote.
B. vertical asymptote.
C. local minimum.
D. vertical axis intercept.
E. point of inflection.

Source: *VCE 2018, Specialist Mathematics 2, Section A, Q.1; © VCAA*

Question 4 (1 mark)

Part of the graph of $y = \frac{1}{2}\tan^{-1}(x)$ is shown below.



The equations of its asymptotes are

A. $y = \pm\frac{1}{2}$

B. $y = \pm\frac{3}{4}$

C. $y = \pm 1$

D. $y = \pm\frac{\pi}{2}$

E. $y = \pm\frac{\pi}{4}$

Source: *VCE 2018, Specialist Mathematics 2, Section A, Q.2; © VCAA*

Question 5 (1 mark)

Consider the function f with rule $f(x) = \dfrac{1}{\sqrt{\sin^{-1}(cx+d)}}$, where $c, d \in R$ and $c > 0$.

The domain of f is

A. $x > -\dfrac{d}{c}$

B. $-\dfrac{d}{c} < x \le \dfrac{1-d}{c}$

C. $\dfrac{-1-d}{c} \le x \le \dfrac{1-d}{c}$

D. $x \in R \setminus \left\{-\dfrac{d}{c}\right\}$

E. $x \in R$

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.2; © VCAA*

Question 6 (1 mark)

The range of the function with rule is $f(x) = (2-x)\arcsin\left(\dfrac{x}{2} - 1\right)$ is

A. $[-\pi, 0]$

B. $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

C. $\left[-\dfrac{(2-x)\pi}{2}, \dfrac{(2-x)\pi}{2}\right]$

D. $[0, 4]$

E. $[0, \pi]$

Topic 6 | Subtopic 6.4 Sketching graphs of product and quotient functions

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.4; © VCAA*

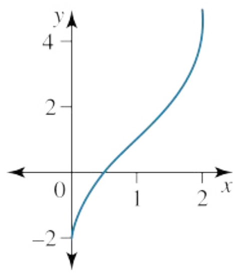
Question 7 (1 mark)

The domain of $\arcsin(2x - 1)$ is

A. $[-1,\ 1]$
B. $[-1,\ 0]$
C. $[0,\ 1]$
D. $[0,\ 1]$
E. $\left[0,\ \frac{1}{2}\right]$

Question 8 (1 mark)

The graph shown below could have the equation



A. $y = 2\sin^{-1}(x - 1) + 1$
B. $y = 2\sin^{-1}(x + 1) - 1$
C. $y = \sin^{-1}(x + 1) - 1$
D. $y = 2\sin^{-1}(x + 1) + 1$
E. $y = \sin^{-1}(x + 1) + 1$

Question 9 (1 mark)

Show that the range of the function $y = b\cos^{-1}(\theta - a) + c$ is $c \le y \le b\pi + c$

Question 10 (1 mark)

Find the domain of the function $y = p\sin^{-1}(q(rx - s)) - t$

Question 11 (1 mark)

The maximal domain of the function $f(x) = \sin^{-1}(2x + 3) - \frac{\pi}{4}$ will be

A. $[-4, -2]$

B. $[-1, -1]$

C. $[-2, -1]$

D. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

E. $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Question 12 (1 mark)

The maximal domain of the function $f(x) = \cos^{-1}(3x - 7) + \frac{2\pi}{3}$ will be

A. $[6, 8]$

B. $[0, \pi]$

C. $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$

D. $\left[2, \frac{8}{3}\right]$

E. $[-1, 1]$

Question 13 (1 mark)

The range of the function $f(x) = 2\sin^{-1}(2x - 1) + \frac{\pi}{4}$ will be

A. $[0, 1]$

B. $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

C. $\left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

D. $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$

E. $[0, 2]$

Topic 6 Subtopic 6.4 Sketching graphs of product and quotient functions

Question 14 (1 mark)

The implied domain of the function with the rule $f(x) = (a+b)\sin^{-1}(cx)$ is

A. $(-c,\ c)$

B. $\left[-\frac{1}{c},\ \frac{1}{c}\right]$

C. $[a-b,\ a+b]$

D. $\left[a-\frac{b\pi}{2},\ a+\frac{b\pi}{2}\right]$

E. $[a-bc,\ a+bc]$

Question 15 (1 mark)

The graph of $y = \cos^{-1}(x)$ is dilated by a scale factor of 2 units parallel to the x-axis and then translated 2 units away from the y-axis. It becomes the graph of

A. $y = 2\cos^{-1}\left(\frac{x}{2}\right)$

B. $y = \frac{1}{2}\cos^{-1}\left(\frac{x}{2}\right)$

C. $y = 2\cos^{-1}(2x)$

D. $y = \frac{1}{2}\cos^{-1}(2x)$

E. $y = \frac{1}{2\cos\left(\frac{x}{2}\right)}$

Question 16 (1 mark)

The graph of $y = 3\tan^{-1}\left(\frac{x}{3}\right) + \frac{\pi}{2}$ has asymptotes at

A. $x = \pm 3$

B. $x = \pm\frac{3\pi}{2}$

C. $y = \pm\frac{\pi}{2}$

D. $y = -\pi$ and $y = 2\pi$

E. $y \pm \frac{3\pi}{2}$

Topic	6	Functions and graphs
Subtopic	6.5	Review

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Source: *VCE 2021, Specialist Mathematics 2, Section B, Q.1; © VCAA*

Question 1 (10 marks)

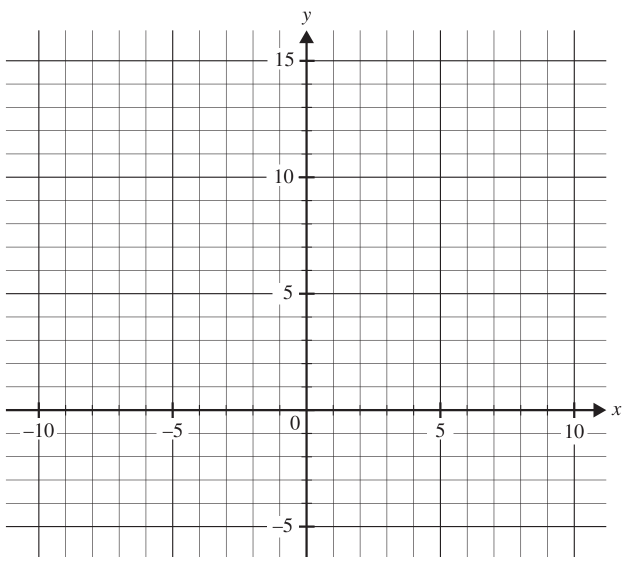
Let $f(x) = \dfrac{(2x-3)(x+5)}{(x-1)(x+2)}$.

a. Express $f(x)$ in the form $A + \dfrac{Bx + C}{(x-1)(x+2)}$ where A, B and C are real constants. **(1 mark)**

$f(x) = \square$

b. State the equations of the asymptotes of the graph of f. **(2 marks)**

c. Sketch the graph of f on the set of axes below. Label the asymptotes with their equations, and label the maximum turning point and the point of inflection with their coordinates, correct to two decimal places. Label the intercepts with the coordinate axes. **(3 marks)**



d. Let $g_k(x) = \dfrac{(2x-3)(x+5)}{(x-k)(x+2)}$, where k is a real constant.

i. For what values of k will the graph of g_k have two asymptotes? **(2 marks)**

ii. Given that the graph of g_k has more than two asymptotes, for what values of k will the graph of g_k have no stationary points? **(2 marks)**

Source: *VCE 2020, Specialist Mathematics 1, Q.6; © VCAA*

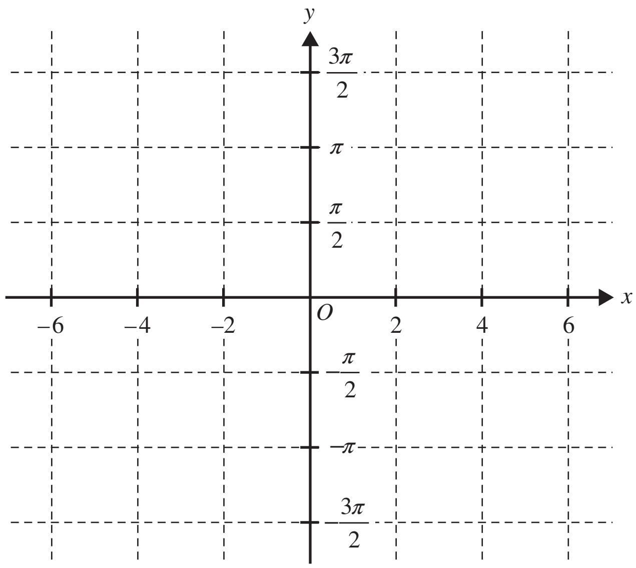
Question 2 (5 marks)

Let $f(x) = \arctan(3x-6) + \pi$.

a. Show that $f'(x) = \dfrac{3}{9x^2 - 36x + 37}$. **(1 mark)**

b. Hence, show that the graph of f has a point of inflection at $x = 2$. **(2 marks)**

c. Sketch the graph of $y = f(x)$ on the axes provided below. Label any asymptotes with their equations and the point of inflection with its coordinates. **(2 marks)**



Source: *VCE 2019 Specialist Mathematics Exam 2, Section A, Q2; © VCAA*

Question 3 (1 mark)

The asymptote(s) of the graph of $f(x) = \dfrac{x^2 + 1}{2x - 8}$ has equation(s)

A. $x = 4$

B. $x = 4$ and $y = \dfrac{x}{2}$

C. $x = 4$ and $y = \dfrac{x}{2} + 2$

D. $x = 8$ and $y = \dfrac{x}{2}$

E. $x = 8$ and $y = 2x + 2$

Source: *VCE 2017 Specialist Mathematics Exam 2, Section B, Q1 a,b; © VCAA*

Question 4 (8 marks)

Let $f: D \rightarrow R, f(x) = \dfrac{x}{1+x^3}$, where D is the maximal domain of f.

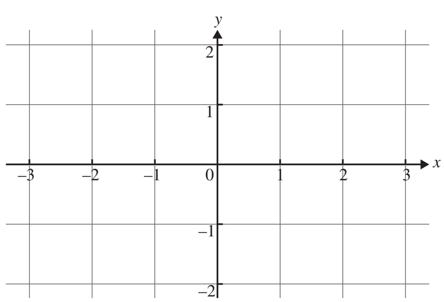
a. Answer the following.

i. Find the equations of any asymptotes of the graph of f. **(1 mark)**

ii. Find $f'(x)$ and state the coordinates of any stationary points of the graph of f, correct to two decimal places. **(2 marks)**

iii. Find the coordinates of any point of inflection of the graph of f correct to two decimal places. **(2 marks)**

b. Sketch the graph of $f(x) = \dfrac{x}{1+x^3}$ form $x = -3$ to $x = 3$ on the axes provided below, marking all stationary points, points of inflection and intercepts with axes, labelling them with their coordinates. Show any asymptotes and label them with their equations. **(3 marks)**



Source: *VCE 2014 Specialist Mathematics Exam 2, Section A, Q3; © VCAA*

Question 5 (1 mark)

The features of the graph of the function with rule $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - x - 6}$ include

A. asymptotes at $x = 1$ and $x = -2$
B. asymptotes at $x = 3$ and $x = -2$
C. an asymptote at $x = 1$ and a point of discontinuity at $x = 3$
D. an asymptote at $x = -2$ and a point of discontinuity at $x = 3$
E. an asymptote at $x = 3$ and a point of discontinuity at $x = -2$

Source: *VCE 2020, Specialist Mathematics 2, Section A, Q.2; © VCAA*

Question 6 (1 mark)

A function f has the rule $f(x) = \left|b\cos^{-1}(x) - a\right|$, where $a > 0\, b > 0$ and $a < \dfrac{b\pi}{2}$.

The range of f is

A. $[-a, b\pi - a]$
B. $[0, b\pi - a]$
C. $[a, b\pi - a]$
D. $[0, b\pi + a]$
E. $[a - b\pi, a]$

Source: *VCE 2017, Specialist Mathematics 2, Section A, Q.10; © VCAA*

Question 7 (1 mark)

A function f, its derivative f' and its second derivative f'' are defined for $x \in R$ with the following properties.

$f(a) = 1, f(-a) = -1$
$f(b) = -1, f(-b) = 1$
and $f''(x) = \dfrac{(x+a)^2(x-b)}{g(x)}$, where $g(x) < 0$

The coordinates of any points of inflection of $|f(x)|$ are

A. $(-a, 1)$ and $(b, 1)$
B. $(b, -1)$
C. $(-a, -1)$ and $(b, -1)$
D. $(-a, 1)$
E. $(b, 1)$

Source: *VCE 2020 Specialist Mathematics Exam 1, Q.6; © VCAA*

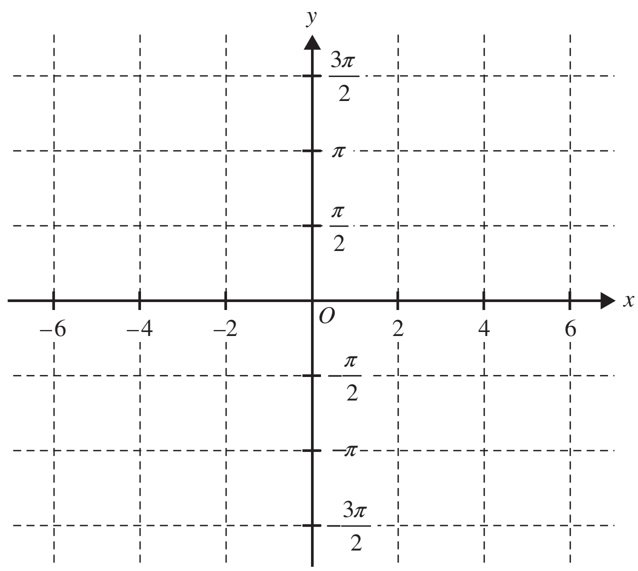
Question 8 (5 marks)

Let $f(x) = \arctan(3x - 6) + \pi$.

a. Show that $f'(x) = \dfrac{3}{9x^2 - 36x + 37}$. **(1 mark)**

b. Hence, show that the graph of f has a point of inflection at $x = 2$. **(2 marks)**

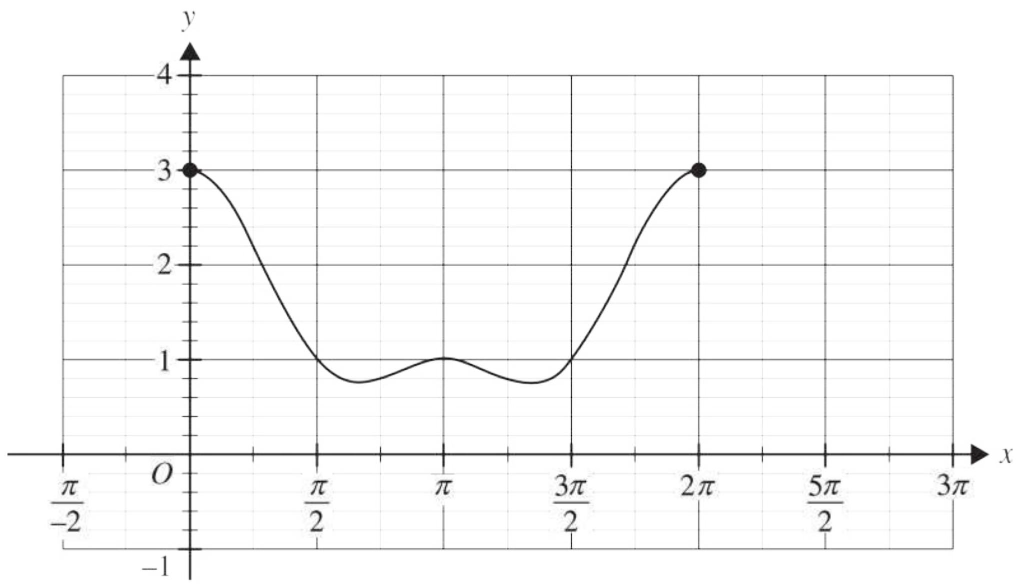
c. Sketch the graph of $y = f(x)$ on the axes provided below. Label any asymptotes with their equations and the point of inflection with it's coordinates. **(2 marks)**



Source: *VCE 2019, Specialist Mathematics 1, Q.5; © VCAA*

Question 9 (6 marks)

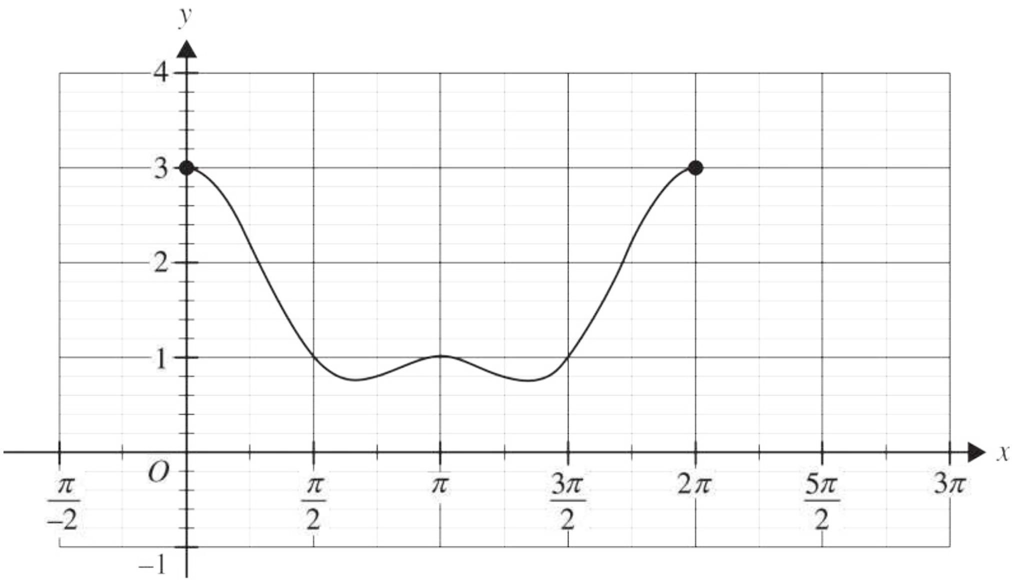
The graph of $f(x) = \cos^2(x) + \cos(x) + 1$ over the domain $0 \le x \le 2\pi$ is shown below.



a. i. Find $f'(x)$ **(1 mark)**

ii. Hence, find the coordinates of the turning points of the graph in the interval $(0, 2\pi)$. **(2 marks)**

b. Sketch the graph of $y = \dfrac{1}{f(x)}$ on the set of axes above. Clearly label the turning points and endpoints of this graph with their coordinates. **(3 marks)**



Source: *VCE 2016, Specialist Mathematics 2, Section B, Q.1; © VCAA*

Question 10 (8 marks)

Answer the following.

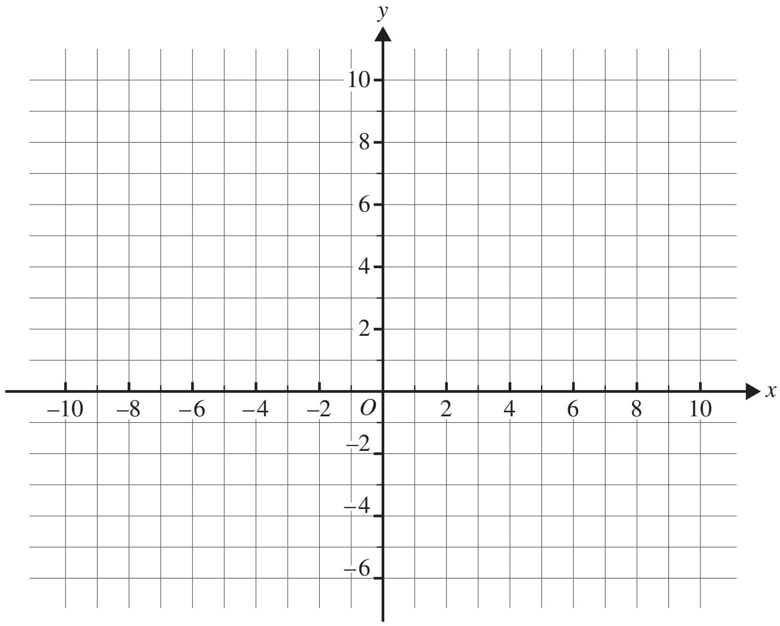
a. Find the stationary point of the graph of $f(x) = \dfrac{4 + x^2 + x^3}{x}$, $x \in R \setminus \{0\}$ Express your answer in coordinate form, giving values correct to two decimal places. **(1 mark)**

Stationary Point = ☐

b. Find the point of inflection of the graph given in **part a**. Express your answer in coordinate form, giving values correct to two decimal places. **(1 mark)**

Point of inflexion = ☐

c. Sketch the graph of $f(x) = \dfrac{4 + x^2 + x^3}{x}$ for $x \in [-3, 3]$ on the axes below, labelling the turning point and the point of inflection with their coordinates, correct to two decimal places. **(3 marks)**



d. A glass is to be modelled by rotating the curve that is the part of the graph where $x \in [-3, -0.5]$ about the y-axis, to form a solid of revolution.

i. Write down a definite integral, in terms of x, which gives the length of the curve to be rotated. **(1 mark)**

ii. Find the length of this curve, correct to two decimal places. **(1 mark)**

e. The volume of the solid formed is given by $V = a\int_c^b x^2 dy$.

Find the values of a, b, and c. Do **not** attempt to evaluate this integral. **(1 mark)**

Source: *VCE 2015, Specialist Mathematics 2, Section 2, Q.1; © VCAA*

Question 11 (12 marks)

Consider $y = \sqrt{2 - \sin^2(x)}$.

a. Use the relation $y^2 = 2 - \sin^2(x)$ to find $\frac{dy}{dx}$ in terms of x and y. **(1 mark)**

$\frac{dy}{dx} = \square$

b. i. Write down the values of y where $x = 0$ and where $x = \frac{\pi}{2}$. **(1 mark)**

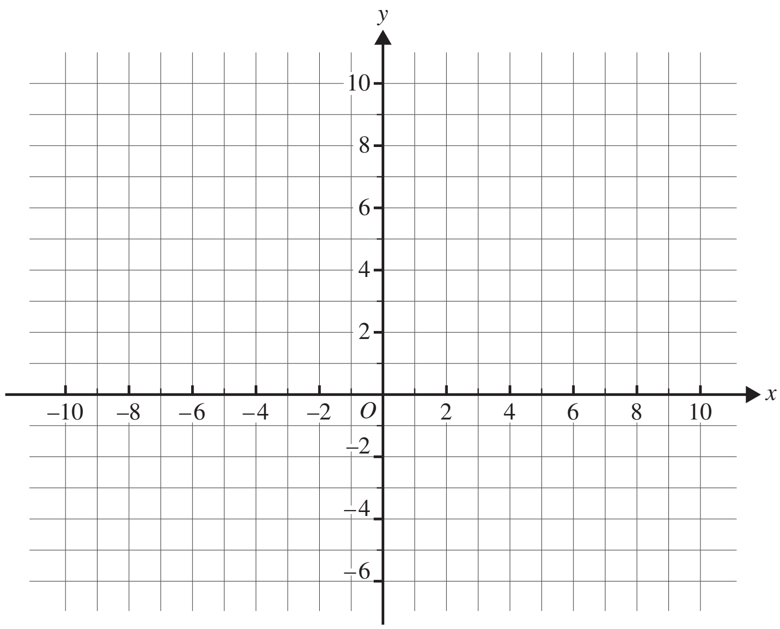
ii. Write down the values of $\frac{dy}{dx}$ where $x = 0$ and where $x = \frac{\pi}{2}$. **(1 mark)**

c. Now consider the function f with rule

$f(x) = \sqrt{2 - \sin^2(x)}$ for $0 \leq x \leq \frac{\pi}{2}$.

Find the rule for the inverse function f^{-1}, and state the domain and range of f^{-1}. **(3 marks)**

d. Sketch and label the graphs of f and f^{-1} on the axes below. **(2 marks)**



e. The graphs of f and f^{-1} intersect at the point $P(a, a)$.
Find $\boldsymbol{a}$, correct to three decimal places. **(1 mark)**

f. The region bounded by the graph of f, the coordinate axes and the line $x = 1$ is rotated about the x-axis to form a solid of revolution.

i. Write down a definite integral in terms of x that gives the volume of this solid of revolution. **(2 marks)**

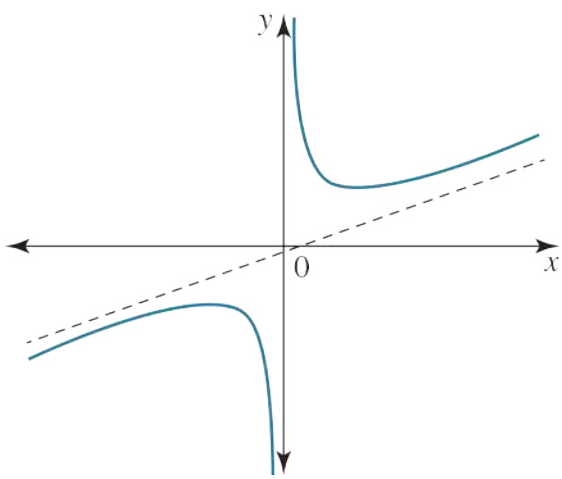
ii. Find the volume of this solid, correct to one decimal place. **(1 mark)**

Question 12 (1 mark)

The number of real solutions to $x^4 + x^3 = \text{cosec}^2(x) - \cot^2(x)$ is

A. 0
B. 1
C. 2
D. 3
E. 4

Question 13 (1 mark)



A possible equation for the graph of the curve shown is

A. $y = \dfrac{ax^2 + b}{x}$, $a > 0$ and $b > 0$

B. $y = \dfrac{ax^2 + b}{x}$, $a < 0$ and $b < 0$

C. $y = \dfrac{ax^2 + b}{x}$, $a < 0$ and $b > 0$

D. $y = \dfrac{ax^3 + b}{x^2}$, $a > 0$ and $b > 0$

E. $y = \dfrac{ax^3 + b}{x^2}$, $a < 0$ and $b < 0$

Question 14 (1 mark)

The graph of $y = \dfrac{-3x^2 + 4}{6x}$ has

A. no straight line asymptotes.

B. $y = -\dfrac{x}{2}$ as its only straight line asymptote.

C. $x = 0$ as its only straight line asymptote.

D. $y = \dfrac{x}{2}$, $x = 0$ as its only straight line asymptotes.

E. $y = -\dfrac{x}{2}$, $x = 0$ as its only straight line asymptotes.

Question 15 (1 mark)

The reciprocal of $y = 5x + 2$ will

A. intersect with $y = 5x + 2$ when $x = 1$ and $x = -1$ and have an asymptote at $x = 0$.

B. intersect with $y = 5x + 2$ when $x = \dfrac{1}{5}$ and $x = \dfrac{3}{5}$ and have an asymptote at $x = -\dfrac{2}{5}$

C. intersect with $y = 5x + 2$ when $x = -\dfrac{1}{5}$ and $x = -\dfrac{3}{5}$ and have an asymptote at $x = -\dfrac{2}{5}$

D. intersect with $y = 5x + 2$ when $x = \dfrac{1}{5}$ and $x = -\dfrac{3}{5}$ and have an asymptote at $x = \dfrac{2}{5}$

E. intersect with $y = 5x + 2$ when $x = \dfrac{1}{5}$ and $x = -\dfrac{3}{5}$ and have an asymptote at $x = 0$

Question 16 (1 mark)

The graph of the function with the rule: $f(x) = \dfrac{1}{(x+2)(x-3)}$ over its maximal domain has

A. asymptotes $x = 2$ and $x = -3$ and a turning point and $x = \dfrac{1}{2}$.

B. asymptotes $x = -2$ and $x = 3$ and a turning point and $x = -\dfrac{1}{2}$.

C. asymptotes $x = -2$ and $x = -3$ and a turning point and $x = \dfrac{1}{2}$.

D. asymptotes $x = -2$ and $x = 3$ and a turning point and $x = \dfrac{1}{2}$.

E. asymptotes $x = 2$ and $x = 3$ and a turning point and $x = -\dfrac{1}{2}$.

Question 17 (1 mark)

Consider the graph of the function $y = f(x)$, where $3y - 2x = 6$ The graph of $\frac{1}{f(x)}$ wil

A. intersect with $y = f(x)$ when $x = \frac{3}{2}$ and $x = -\frac{9}{2}$ and have an asymptote at $x = -3$.

B. intersect with $y = f(x)$ when $x = -\frac{3}{2}$ and $x = \frac{9}{2}$ and have an asymptote at $x = 3$.

C. intersect with $y = f(x)$ when $x = -\frac{3}{2}$ and $x = -\frac{9}{2}$ and have an asymptote at $x = 3$.

D. intersect with $y = f(x)$ when $x = \frac{3}{2}$ and $x = \frac{9}{2}$ and have an asymptote at $x = -3$.

E. intersect with $y = f(x)$ when $x = -\frac{3}{2}$ and $x = -\frac{9}{2}$ and have an asymptote at $x = -3$.

Question 18 (1 mark)

The graph of the function $y = f(x)$ has a maximum turning point at $(2, -4)$ The graph of the function $y = \frac{1}{f(x)}$ will have

A. a maximum turning point at $(2, 4)$.

B. a maximum turning point at $\left(2, -\frac{1}{4}\right)$.

C. a maximum turning point at $(2, 4)$.

D. a minimum turning point at $\left(2, -\frac{1}{4}\right)$.

E. a minimum turning point at $\left(\frac{1}{2}, -\frac{1}{4}\right)$.

Question 19 (1 mark)

The graph of the function $y = f(x)$ crosses the x-axis at $x = 3$ and crosses the y-axis at $y = 4$. The graph of the function $y = \dfrac{1}{f(x)}$ will

A. cross the y-axis at $y = -4$ and cross the x-axis at $x = -3$.

B. cross the y-axis at $y = 3$ and cross the x-axis at $x = 4$.

C. cross the y-axis at $y = \dfrac{1}{3}$ and cross the x-axis at $x = \dfrac{1}{4}$.

D. have a vertical asymptote at $x = 3$ and cross the y-axis $y = \dfrac{1}{4}$.

E. have a vertical asymptote at $x = \dfrac{1}{3}$ and cross the y-axis $y = \dfrac{1}{4}$.

Question 20 (1 mark)

The graph of $f(x) = \dfrac{1}{x^2 + mx + n}$ where m and n are real constants, has no vertical asymptotes if

A. $m^2 < 4n$
B. $m^2 > 4n$
C. $m^2 = -4n$
D. $m^2 < -4n$
E. $m^2 > -4n$

Question 21 (1 mark)

The graph of $y = \dfrac{1}{nx^2 + mx - 1}$, where m and n are real constants, has no vertical asymptotes if

A. $m^2 < 4n$
B. $m^2 > 4n$
C. $m^2 = -4n$
D. $m^2 < -4n$
E. $m^2 > -4n$

Answers and marking guide

6.2 Sketching graphs of cubics and quartics

Question 1

$f'(x) = 2(x-3)^3 + 5$

$f''(x) = 6(x-3)^2 \geq 0$

$f(x) = \frac{1}{2}(x-3)^4 + 5x$

$f(x)$ is a quartic, and has a minimum point, has no inflection points, the second derivative does not change sign.

(All cubics have an inflection point)

Question 2

$f(a) = 1, \;\; f(-a) = -1 \;\; f(b) = -1, \;\; f(-b) = 1$

$f''(x) = \frac{(x+a)^2(x-b)}{g(x)}, \;\; g(x) < 0, \;\; f''(x) = 0$

$\Rightarrow x = -a, \; x = b, \; f(b) = -1 \;\; |f(b) = 1|$

$x = -a$ is a turning point, $(b, 1)$ is an inflection point.

Question 3

$f(x) = x^3 - mx^2 + 4$

$f'(x) = 3x^2 - 2mx$

$f''(x) = 6x - 2m$

$f'(x) \geq 0 \Rightarrow x \geq \frac{m}{3}$

6.3 Sketching graphs of rational functions

Question 1

$f(x) = \frac{x+1}{x^2-4}$

Vertical asymptotes at $x = \pm 2$

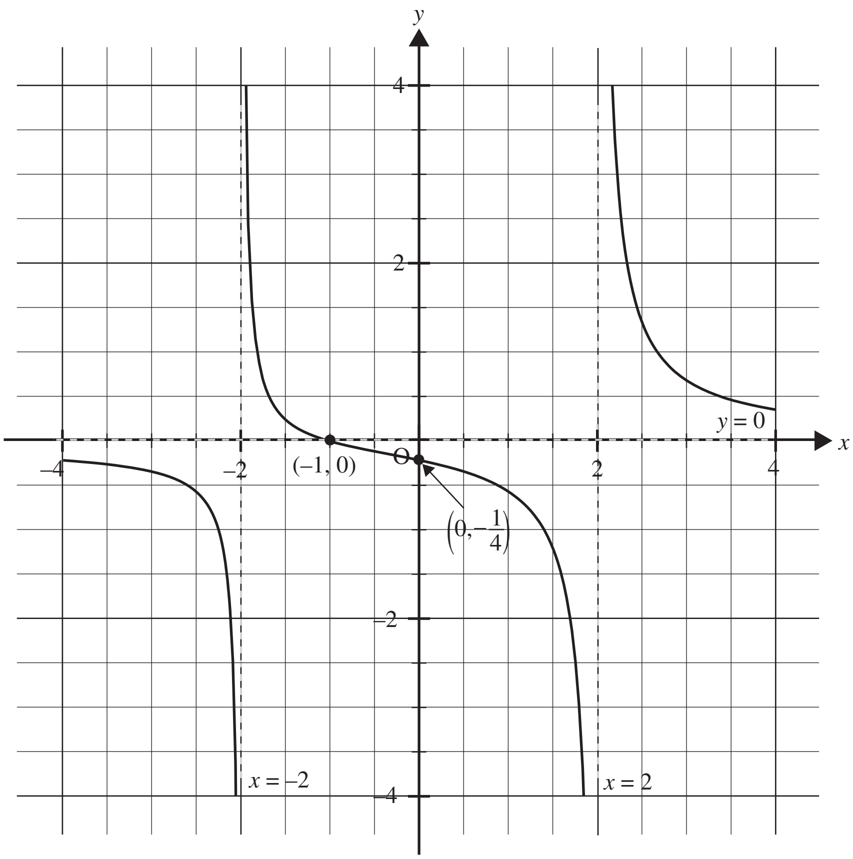
Horizontal asymptote at $y = 0$

Crosses the x-axis (horizontal asymptote) at $x = -1, (-1, 0)$

Crosses the y-axis at $y = -\frac{1}{4} \left(0, -\frac{1}{4}\right)$

$x > 2, f(x) > 0$

$x < -2, f(x) < 0$



Award **1 mark** for correct vertical asymptotes.
Award **1 mark** for correct horizontal asymptotes.
Award **1 mark** for correct axial intercepts.
Award **1 mark** for correct shape and horizontal asymptote intercept.

***VCAA Examination Report* note:**

Most students realised that $x = -2$ and $x = 2$ were vertical asymptotes, although the horizontal asymptote $y = 0$ was often not stated. Students who found the axis intercepts were not always able to position them correctly on the axes. Some students showed a stationary point of inflection on their graph or were missing the outer branches.

Question 2

$f(x) = \dfrac{x^3 - ax}{x^2} = x - \dfrac{a}{x}$

$x = 0$ is a vertical asymptote

$y = x$ is an oblique asymptote

Question 3

a. $f(x) = \dfrac{4 + x^2 + x^3}{x}, \ x \in R\backslash\{0\}$

$f(x) = \dfrac{2x^3 + x^2 - 4}{x^2}$

For stationary points, solving $f'(x) = 0$ gives $x = 1.1134, \ f(1.1134) = 5.946 \ (1.11, 5.95)$

Award **1 mark** for the correct coordinates of a stationary point.

***VCAA Examination Report* note:**

This question was answered very well. A small number of students gave the coordinates for the point of inflection rather than the stationary point.

b. $f''(x) = \dfrac{2\left(x^3+4\right)}{x^3}$

For inflection points, $f''(x) = 0$:

$x = \sqrt[3]{-4} = -1.587,\ f(-1.587) = -1.587$

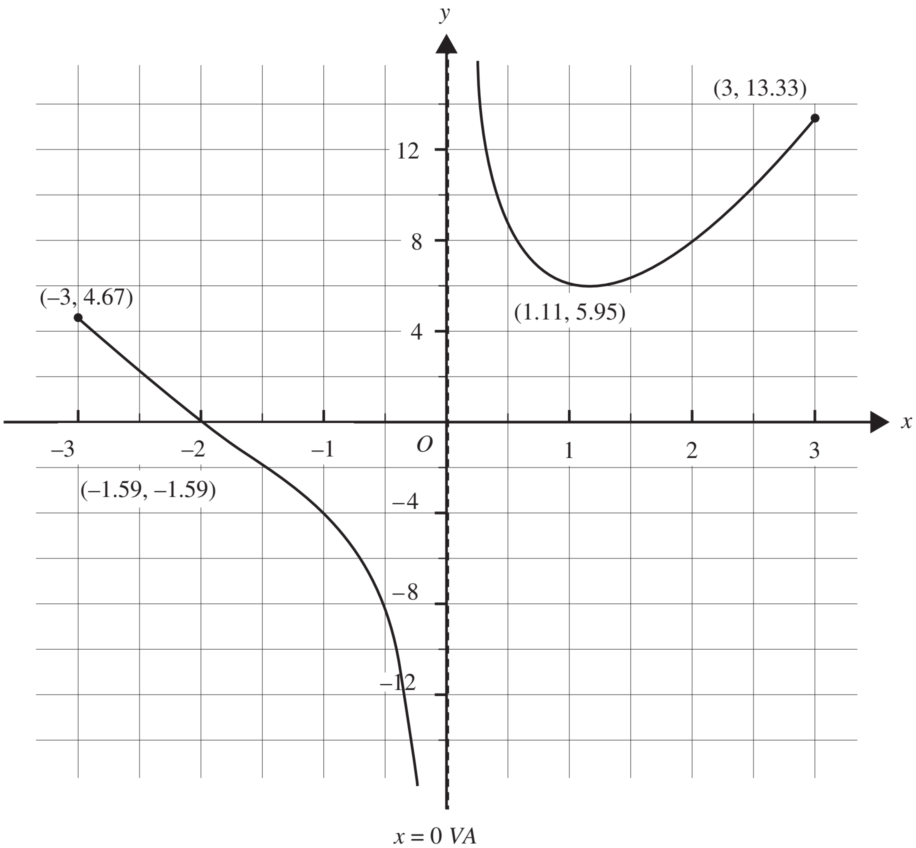
$(-1.59, -1.59)$

Award **1 mark** for setting the second derivative to zero. Award **1 mark** for the correct coordinates of the inflection point.

***VCAA Assessment Report* note:**

This question was generally answered well. The most common error resulted from substituting a rounded x value, yielding an incorrect y value of -1.58 Some students left off the negative sign of the y value.

c.



Award **1 mark** for the correct graph on the domain.

Award **1 mark** for the stationary point and inflection point.

Award **1 mark** for the endpoints and vertical asymptote.

Question 4

$f(x) = \dfrac{(x-a)(x+3)}{(x-2)}$

$f'(x) = \dfrac{x^2 - 4x + 5a - 6}{(x-2)^2}$

$f'(0) = 0 \Rightarrow 5a - 6 = 0,\ a = \dfrac{6}{5}$

Question 5

$f(x) = \dfrac{x^2+1}{2x-8} = \dfrac{17}{2(x-4)} + \dfrac{x}{2} + 2$

$x = 4$ is a vertical asymptote.

$y = \dfrac{x}{2} + 2$ is an oblique asymptote.

Question 6

$$\begin{aligned}y &= \frac{x^2-4x+3}{x^2-x-6}\\ &= \frac{(x-3)(x-1)}{(x-3)(x+2)},\ x \neq 3\\ &= \frac{x-1}{x+2}\\ &= \frac{x+2-3}{x+2}\\ &= 1-\frac{3}{x+2}\end{aligned}$$

The graph has a vertical asymptote at $x = -2$, a horizontal asymptote at $y = 1$ and a point of discontinuity at $x = 3$

Question 7

Since the asymptotes are at $x = -5$ and $x = 3$, then $y = \dfrac{1}{ax^2+bx+c} = \dfrac{1}{a(x+5)(x-3)}$

The stationary point has an x-coordinate halfway between $x = -5$ and $x = 3$ at $x = -1$.

$$\left(-1, -\frac{1}{8}\right) \Rightarrow -\frac{1}{8} = \frac{1}{a(4\times -4)} = -\frac{1}{16a} \Rightarrow a = \frac{1}{2}$$

$$y = \frac{1}{\frac{1}{2}(x+5)(x-3)} = \frac{1}{\frac{1}{2}\left(x^2+2x-15\right)} = \frac{1}{\frac{1}{2}x^2+x-\frac{15}{2}} \Rightarrow a = \frac{1}{2},\ b = 1,\ c = -\frac{15}{2}$$

Question 8

Asymptotes at $x = 0$ and $y = 2x + 3$

Question 9

$$\begin{aligned}\frac{x^3+2}{3x^2} &= \frac{x^3}{3x^2}+\frac{2}{3x^2}\\ &= \frac{x}{3}+\frac{2}{3x^2}\end{aligned}$$

Asymptotes at $x = 0$ and $y = \dfrac{x}{3}$

Question 10

$$\begin{aligned}2y &= \frac{3}{2x^2}-5x-1\\ y &= \frac{3}{4x^2}-\frac{(5x+1)}{2}\end{aligned}$$

Asymptotes at $x = 0$ and $y = \dfrac{-(5x+1)}{2}$

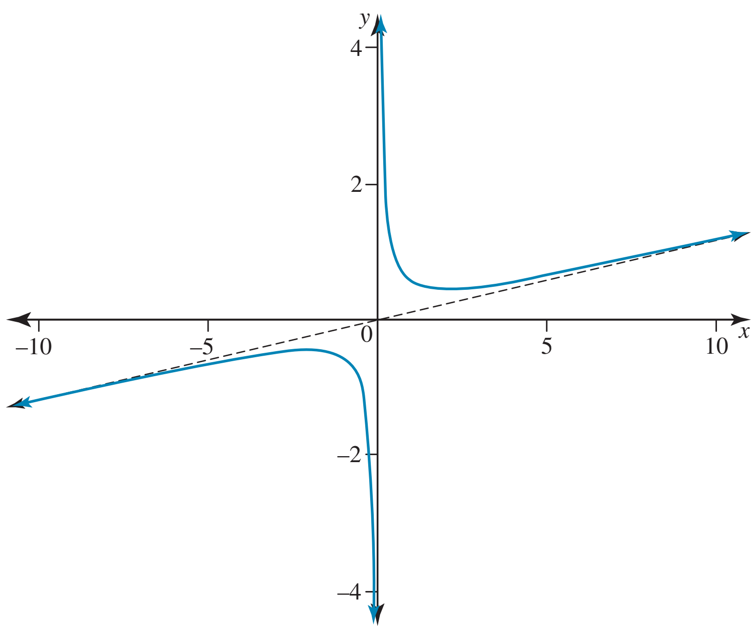
Question 11

$y = \dfrac{ax^3+b}{x} = ax^2 + \dfrac{b}{x}$. The parabola $y = ax^2$ with $a < 0$ is an asymptote. The graph crosses the x-axis at one point, so that there is one real positive solution for $ax^3 + b = 0$, which is $x = \sqrt[3]{-\dfrac{b}{a}}$. Since $a < 0$, we require $b > 0$. Also $\dfrac{dy}{dx} = 2ax - \dfrac{b}{x^2}$. For a single turning point, we require $\dfrac{dy}{dx} = 0$, so that $x = \sqrt[3]{\dfrac{b}{2a}}$.
The value of x is negative for the single turning point, so that $a < 0$ and $b > 0$.

Question 12

$y = \frac{ax^4 + b}{x^2} = ax^2 + \frac{b}{x^2}$. The parabola $y = ax^2$ with $a < 0$ is an asymptote. The graph crosses the x-axis at one point, so that there are two real solutions for $ax^4 + b = 0$, which is $x^2 = \pm\sqrt{-\frac{b}{a}}$. Since $a < 0$, we require $b > 0$. Also $\frac{dy}{dx} = 2ax - \frac{2b}{x^3}$. For a single turning point, we require $\frac{dy}{dx} \neq 0$ so that $x^4 = \frac{b}{a}$ or $x^2 = \pm\sqrt{\frac{b}{a}}$. The value of x is negative for the single turning point, so that $a < 0$ and $b > 0$.

Question 13



Award **1 mark** for sketching a graph with the correct shape.
Award **1 mark** for identifying the asymptotes: $x = 0,\ y = \frac{x}{8}$.
Award **1 mark** for identifying that there are no intercepts.
Award **1 mark** for identifying that the stationary points are turning points $\left(\pm 2, \pm\frac{1}{2}\right)$.

Question 14

$y = \frac{ax^2 + b}{x} = ax + \frac{b}{x}$. The line $y = ax$ with $a < 0$ is an asymptote. The graph crosses the x-axis at two distinct points, so there is a real solution for $ax^2 + b = 0$, which is $x = \pm\sqrt{\frac{b}{a}}$. Since $a < 0$, we require $b > 0$. Also $\frac{dy}{dx} = a - \frac{b}{x^2}$ for turning points. This graph has no turning points, so we require no solutions for $\frac{dy}{dx} = 0$, so $x = \pm\sqrt{\frac{b}{a}}$ has no solution. Since $a < 0$, we require $b > 0$.

6.4 Sketching graphs of product and quotient functions

Question 1

For local maximum $y = \dfrac{1}{(\cos(ax)+1)^2+3}$, $a \in R \setminus \{0\}$

$(\cos(ax)+1)^2 + 3$ has local minimum of 3

When $\cos(ax) = -1$

$ax = \cos^{-1}(-1) + 2k\pi$

$ax = \pi + 2k\pi = \pi(1+2k)$

Local max $\left(\dfrac{\pi(1+2k)}{a}, \dfrac{1}{3}\right)$, $k \in Z$

The correct answer is **E**.

Question 2

a. $f(x) = x^2 e^{-x}$

$f'(x) = \left(2x - x^2\right) e^{-x}$

Solving $f'(x) = x(2-x)e^{-x} = 0$ for stationary points:

$x = 0, 2$ since $e^{-x} \neq 0$

Absolute minimum $(0, 0)$

Local maximum $\left(2, 4e^{-2}\right) \approx (2, 0.54)$

Award **1 mark** for the correct derivative.

Award **1 mark** for the correct derivative.

b. $y = 0$ is a horizontal asymptote.

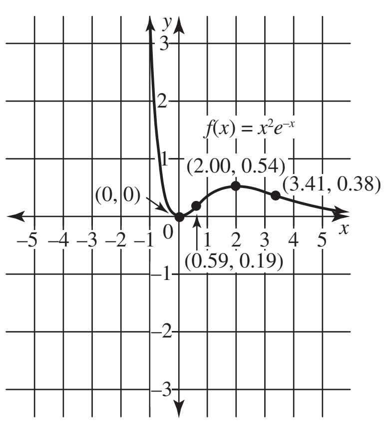
Award **1 mark** for the correct equation.

c. $f''(x) = \left(x^2 - 4x + 2\right) e^{-x}$

Solving $f''(x) = \left(x^2 - 4x + 2\right) e^{-x} = 0$ for inflection points:

$x = 2 \pm \sqrt{2} \approx 0.59, 3.41$ since $e^{-x} \neq 0$

Inflection points $(0.59, 0.19), (3.41, 0.38)$



Award **1 mark** for the correct second derivative and inflection points.

Award **2 marks** for the correct graph and labelled diagram.

d. $g(x) = x^n e^{-x}$

$g''(x) = x^{n-2}\left(x^2 - 2nx + n(n-1)\right) e^{-x}$

Award **1 mark** for the correct expression.

e. i. $g''(x) = 0 \;\Rightarrow\; x = n \pm \sqrt{n}$

Award **1 mark** for the correct values.

ii.

Number of points ofinflection	Value(s) of n (where $n \in Z$)
0	$n \leq 0$
1	1
2	$2,\ 4,\ 6\ldots\ n$ even $n = 2k,\ k \in Z^+$
3	$3,\ 5,\ 7\ldots\ n$ odd $n = 2k+1,\ k \in Z^+$

When $n$is odd, the point at the origin is a horizontal point of inflection, but when n is even, the point at the origin is an absolute minimum turning point (not an inflection point).
Award **1 mark** for the correct table.
Award **1 mark** for the correct explanation.

Question 3

$f(x) = \dfrac{e^x}{x-1}$
$x = 1$ is a vertical asymptote.
$y = 0$ is a horizontal asymptote.
The graph has a minimum turning point at $x = 2$.
The graph does cross the y-axis, $f(0) = -1$.
The graph does not have a point of inflection, $f''(x) \neq 0$
The correct answer is **E**.

Question 4

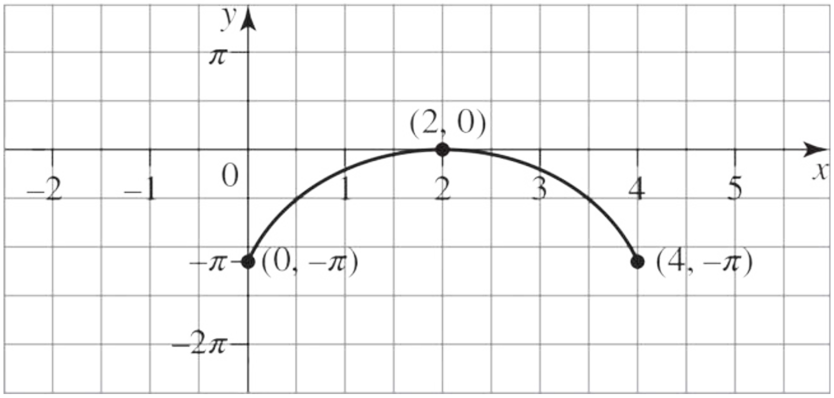
$y = \dfrac{1}{2}\tan^{-1}(x)$
range $\left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$
Horizontal asymptotes at $y = \pm\dfrac{\pi}{4}$

Question 5

$f(x) = \dfrac{1}{\sqrt{\sin^{-1}(cx+d)}}, c, d \in R, c > 0$
For the maximal domain, we require $\sin^{-1}(cx+d) > 0$:
$0 < cx + d \leq 1$
$-d < cx \leq 1 - d$
$-\dfrac{d}{c} < x \leq \dfrac{1-d}{c}$

Question 6

$y = (2-x)\sin^{-1}\left(\dfrac{x}{2} - 1\right)$
Domain $\left|\dfrac{x}{2} - 1\right| \leq 1 \quad \Rightarrow \ -1 \leq \dfrac{x}{2} - 1 \leq 1$
$0 \leq \dfrac{x}{2} \leq 2 \quad \Rightarrow \quad x \in [0, 4]$
$f(0) = -\pi,\ f(4) = -\pi,\ f(2) = 0$
Range $[-\pi, 0]$



Question 7

$y = \sin^{-1}(2x - 1)$

$|2x - 1| \le 1$

$-1 \le 2x - 1 \le 1$

$0 \le 2x \le 2$

$0 \le 2x \le 1$

Question 8

$y = 2\sin^{-1}(x - 1) + 1$ is the only possible graph, as it has the correct domain and range.

The domain of $y = 2\sin^{-1}(x - 1) + 1$ is $|x - 1| \le 1 \Rightarrow -1 \le x - 1 \le 1$ or $(0, 2)$ and the range is $(-\pi + 1, \pi + 1) \approx (-2.14, 4.14)$.

Question 9

a gives a horizontal translation which does not affect the range.

So consider $y = b\cos^{-1}(\theta) + c$

The effect of b is to dilate in the y direction.

The effect of c is a translation in the y direction.

The range is $c \le y \le b\pi + c$ **[1 mark]**

Question 10

The dilation factor in the y direction, p, does not affect the domain and can be ignored.

The vertical translation, $-t$, does not affect the domain and can be ignored.

Consider $q(rx - s) = qr\left(x - \frac{s}{r}\right)$

Domain of $y = \sin^{-1}(x)$ is $[1,\ 1]$

Domain of $y = p\sin^{-1}(q(r - s) - t)$ is $[-1,\ 1]$

$$= \left[-\frac{1}{qr} + \frac{s}{r}, \frac{1}{qr} + \frac{s}{r}\right]$$ **[1 mark]**

Question 11

Domain:

$-1 \le 2x + 3 \le 1$

$-4 \le 2x \le -2$

$-2 \le 2x \le -1$

Domain $= [-2, -1]$

Question 12

Domain:

$-1 \leq 3x - 7 \leq 1$

$6 \leq 3x \leq 8$

$2 \leq x \leq \frac{8}{3}$

Domain $= \left[2, \frac{8}{3}\right]$

Question 13

Domain:

$-1 \leq 2x - 1 \leq 1$

$0 \leq 2x \leq 2$

$0 \leq x \leq 1$

Domain $= [0, 1]$

$f(0) = 2\sin^{-1}(2(0) - 1) + \frac{\pi}{4} = -\frac{3\pi}{4}$

$f(1) = 2\sin^{-1}(2(1) - 1) + \frac{\pi}{4} = \frac{5\pi}{4}$

Range $= \left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

Question 14

For the implied or maximal domain, we require

$|cx| \leq 1,\ -1 \leq cx \leq 1$ or $-\frac{1}{c} \leq x \leq \frac{1}{c} = \left[-\frac{1}{c}, \frac{1}{c}\right]$

Question 15

As $y = \cos^{-1}(x)$ is dilated by a scale factor of 2 units parallel to the x-axis,replace x with $\frac{x}{2}$. It becomes

$y = \cos^{-1}\left(\frac{x}{2}\right)$.

Dilated 2 units parallel to the y-axis, it becomes the graph of $y = 2\cos^{-1}\left(\frac{x}{2}\right)$.

Question 16

$y = \tan^{-1}(x)$ has horizontal asymptotes at $y = \pm\frac{\pi}{2}$,

$y = 3\tan^{-1}\left(\frac{x}{3}\right) + \frac{\pi}{2}$ has horizontal asymptotes at $y = \pm\frac{3\pi}{2} + \frac{\pi}{2}$ or $y = -\pi$ and $y = 2\pi$,

6.5 Review

Question 1

a. $f(x) = \frac{(2x-3)(x+5)}{(x-1)(x+2)}$

$f(x) = 2 + \frac{5x - 11}{(x-1)(x+2)}$ **[1 mark]**

b. Vertical asymptotes $x = 1,\ \ x = -2$ **[1 mark]**

Horizontal asymptote $y = 2$ **[1 mark]**

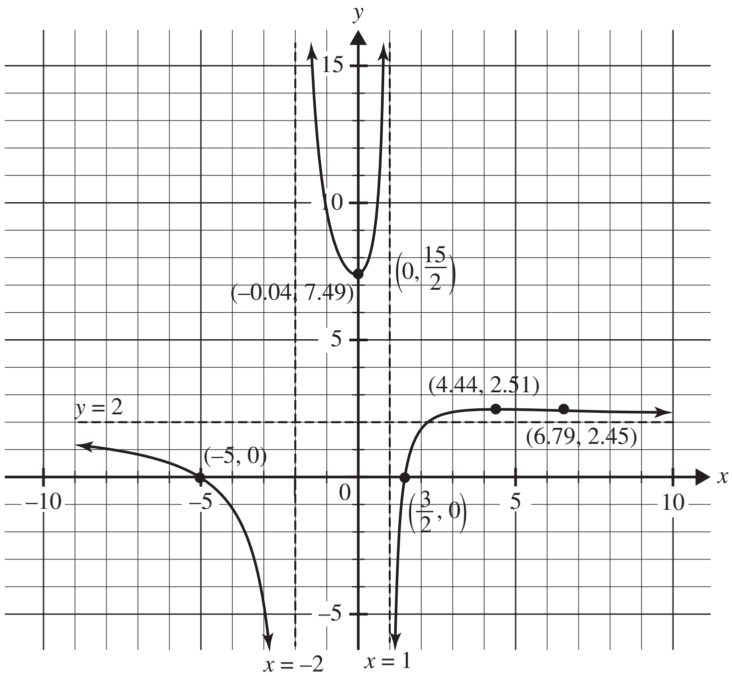
c. The graph crosses x-axis $\left(\frac{3}{2}, 0\right)$, $(-5, 0)$

The graph crosses y-axis $\left(0, \frac{15}{2}\right)$

The graph crosses HA at $\left(\frac{11}{5}, 2\right)$

The turning points are $(-0.04, 7.49)$, $(4.44, 2.51)$

The inflection point is $(6.79, 2.45)$



Award **1 mark** for the correct graph shapes.
Award **1 mark** for the correct axial intercepts.
Award **1 mark** for the correct turning points and inflection point.

d. i. $g_k(x) = \dfrac{(2x-3)(x+5)}{(x-k)(x+2)}$ $k \in R$

There are two asymptotes: one horizontal and one vertical.

$k = -5, \quad g_k(x) = \dfrac{2x-3}{x+2}$ $k \in R$

$k = \dfrac{3}{2} \quad g_k(x) = \dfrac{2(x+5)}{x+2}$

$k = -2 \quad g_k(x) = \dfrac{(2x-3)(x+5)}{(x+2)^2}$

Award **2 marks** for all 3 correct values of k.

ii. For more than two asymptotes but no stationary points, solving the discriminant for k for values less than zero when $\dfrac{dg_k(x)}{dx} = 0$ gives $k < -5$ or $k > \dfrac{3}{2}$

Award **1 mark** for each correct value of k.

Question 2

a. $f(x) = \arctan(3x - 6) + \pi$

$f(x) = y - \tan^{-1}(u) + \pi,\ u = 3x - 6$

$$\frac{dy}{du} = \frac{1}{1+u^2},\ \frac{du}{dx} = 3$$

$$f'(x) = \frac{3}{1+(3x-6)^2}$$

$$f'(x) = \frac{3}{9x^2 - 36x + 37}$$

Award **1 mark** for the correct derivative.

b. $f'(x) = 3(9x^2 - 36x + 37)^{-1}$

$f''(x) = -3(18x - 36)(9x^2 - 36x + 37)^{-2}$

$$f''(x) = \frac{-3 \times 18(x-2)}{(9x^2 - 36x + 37)^2} = 0 \text{ for point of inflection}$$

$f''(1) > 0,\ f''(3) < 0$: change of sign, so $x = 2$ is a point of inflection.

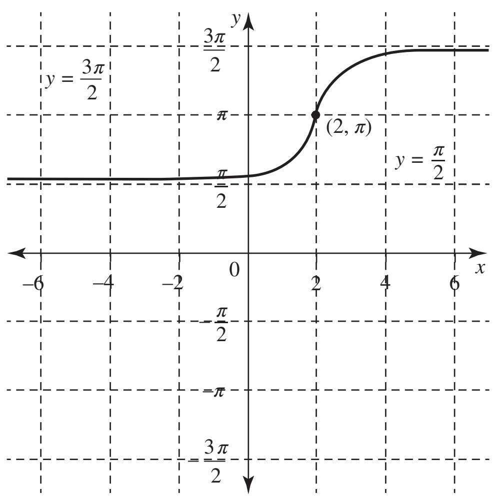
Award **1 mark** for the correct second derivative.

Award **1 mark** for the correct value of the point of inflection.

c. $f(2) = \tan^{-1}(0) + \pi = \pi \rightarrow (2, \pi)$

The range of $y = \tan^{-1}(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) + \pi,\ y = \frac{3\pi}{2}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes.



Award **1 mark** for the correct graph shape.

Award **1 mark** for the correct asymptotes.

Question 3

$$f(x) = \frac{x^2+1}{2x-8} = \frac{17}{2(x-4)} + \frac{x}{2} + 2$$

$x = 4$ is a vertical asymptote.

$y = \frac{x}{2} + 2$ is an oblique asymptote.

Question 4

a. i. $f(x) = \frac{x}{1+x^3}$, $x = -1$ is a vertical asymptote and $y = 0$ is the horizontal asymptote

Award **1 mark** for both correct asymptotes.

***VCAA Examination Report* note:**
The majority of students stated the vertical asymptote but significantly fewer stated the horizontal asymptote. Various incorrect attempts at partial fraction forms were made.

ii. $f'(x) = \dfrac{1-2x^3}{\left(1+x^3\right)^2}$

Stationary points at $f'(x) = 0$

$x = \dfrac{1}{\sqrt[3]{2}} \approx 0.7937,\ f(0.7937) = 0.529$

$(0.79\ ,\ 0.53)$ is a local maximum.

Award **1 mark** for the correct derivative.

Award **1 mark** for the correct coordinate of the stationary point.

***VCAA Examination Report* note:**
This question was generally answered well. Some students did not give the coordinates of the stationary point in the required form.

iii. $f''(x) = \dfrac{6x^2\left(x^3-2\right)}{\left(1+x^3\right)^2} f''(0.79) < 0$

Inflection points at $f''(x) = 0$

$x = \sqrt[3]{2} \approx 1.2599,\ f(1.26) = 0.42$

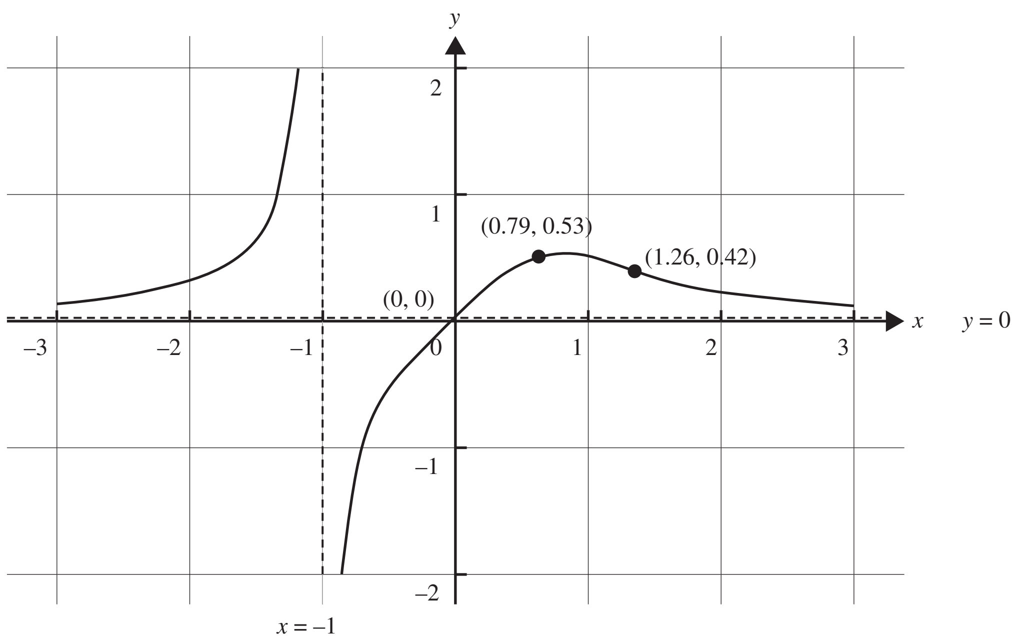
$(1.26, 0.42)$ is a point of inflection

Award **1 mark** for setting the second derivative to zero.

Award **1 mark** for the correct coordinate of inflection point.

***VCAA Examination Report* note:**
The majority of students provided the correct inflection point. A common error was to erroneously include the point (0, 0), which is another point where $f''(x) = 0$, but it is not a point of inflection as there is no change of concavity; $f''(x)$ does not change sign.

b.



Award **1 mark** for the correct graph on the domain with the correct intercept.

Award **1 mark** for the stationary point and the inflection point.

Award **1 mark** for the correct asymptote and its equation.

***VCAA Examination Report* note:**
Graphing was generally completed to a reasonable standard. In some cases the shape of the graph was poor and the required points were not marked clearly or were not placed in the correct position.

Question 5

$$y = \frac{x^2 - 4x + 3}{x^2 - x - 6}$$
$$= \frac{(x-3)(x-1)}{(x-3)(x+2)}, x \neq 3$$
$$= \frac{x-1}{x+2}$$
$$= \frac{x+2-3}{x+2}$$
$$= 1 - \frac{3}{x+2}$$

The graph has a vertical asymptote at $x = -2$, a horizontal asymptote at $y = 1$ and a point of discountinutity at $x = 3$.
The correct answer is **D**.

Question 6

$f(x) = \left|b\cos^{-1}(x) - a\right|, a > 0, b > 0, a < \dfrac{b\pi}{2}$

$0 \le \cos^{-1}(x) \le \pi,\ 0 \le b\cos^{-1}(x) \le b\pi,\ -a \le b\cos^{-1}(x) - a \le b\pi - a$

$0 \le \left|b\cos^{-1}(x) - a\right| \le b\pi - a$

range $[0,\ b\pi - a]$

Question 7

$f(a) = 1,\quad f(-a) = -1,\quad f(b) = -1,\quad f(-b) = 1$

$f''(x) = \dfrac{(x+a)^2(x-b)}{g(x)},\quad g(x) < 0,\quad f''(x) = 0 \Rightarrow x = -a,\ x = b,\ f(b) = -1\ |f(b) = 1|$

$x = -a$ is a turning point, $(b, 1)$ is an inflection point.

Question 8

a. $f(x) = \arctan(3x - 6) + \pi$

$f(x) = y = \tan^{-1}(u) + \pi, u = 3x - 6$

$\dfrac{dy}{du} = \dfrac{1}{1+u^2}, \dfrac{du}{dx} = 3$

$f'(x) = \dfrac{3}{1 + (3x-6)^2}$

$f'(x) = \dfrac{3}{9x^2 - 36x + 37}$

Award **1 mark** for the correct derivative.

b. $f'(x) = 3\left(9x^2 - 36x + 37\right)^{-1}$

$f''(x) = -3(18x - 36)\left(9x^2 - 36x + 37\right)^{-2}$

$f''(x) = \dfrac{-3 \times 18(x-2)}{\left(9x^2 - 36x + 37\right)^2} = 0$ for point of inflection

$f''(1) > 0, f''(3) < 0$: change of sign, so $x = 2$ is a point of inflection.

Award **1 mark** for the correct second derivative.

Award **1 mark** for the correct value of the point of inflection.

c. $f(2) = \tan^{-1}(0) + \pi = \pi \rightarrow (2, \pi)$

The range of $y = \tan^{-1}(x)$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

So $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) + \pi, y = \dfrac{3\pi}{2}$ and $y = \dfrac{\pi}{2}$ are horizontal asymptotes.

Award **1 mark** for the correct graph shape.
Award **1 mark** for the correct asymptotes.

Question 9

a. i. $f(x) = \cos^2(x) + \cos(x) + 1,\ 0 \le x \le 2\pi$

$f'(x) = -2\sin(x)\cos(x) - \sin(x),\ 0 < x < 2\pi$ **[1 mark]**

***VCAA Examination Report* note:**

This question was well done. A small number of students had difficulty finding the derivative and some students who correctly differentiated attempted to factorise their answer with mixed success. Some students used a double angle formula to write the answer in an alternative form. This was not always done correctly nor was it helpful for the next part of the question.

ii. $f'(x) = -\sin(x)(2\cos(x) + 1) = 0$ for turning points

$\sin(x) = 0, \quad \cos(x) = -\frac{1}{2}$

$x = 0, \pi, 2\pi, \quad x = \pi - \frac{\pi}{3}, \pi, +\frac{\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}$

$f\left(\frac{2\pi}{3}\right) = f\left(\frac{4\pi}{3}\right) = \left(-\frac{1}{2}\right)^2 - \frac{1}{2} + 1 = \frac{3}{4}$

$\left(\frac{2\pi}{3}, \frac{3}{4}\right), \left(\frac{4\pi}{3}, \frac{3}{4}\right), (\pi, 1)$

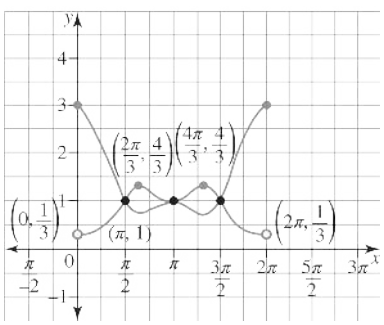
Award **1 mark** for correctly setting to zero and solving.
Award **1 mark** for all 3 correct coordinates.

***VCAA Examination Report* note:**

This question was generally well done. A common mistake was to include the endpoints at $x = 0$ and $x = 2\pi$ even though the question specifically asked students to find the coordinates of the turning points in the interval $(0, 2\pi)$.

b. The graph of $y = \frac{1}{f(x)}$ has turning points at

$\left(\frac{2\pi}{3}, \frac{4}{3}\right), \left(\frac{4\pi}{3}, \frac{4}{3}\right), (\pi, 1)$ the endpoints are $\left(0, \frac{1}{3}\right), \left(2\pi, \frac{1}{3}\right)$ not included.



Award **1 mark** for the correct graph shape.
Award **1 mark** for correct endpoints.
Award **1 mark** for correct turning points.

***VCAA Examination Report* note:**

Students' graph-sketching abilities were reasonable, with most drawing a single, smooth curve with the correct shape. Common errors included neglecting to label the turning point at $(\pi, 1)$, their graph not passing through the intersection points $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{4\pi}{3}, \frac{4}{3}\right)$, and poor estimation of the location of the heights $\frac{1}{3}$ and $\frac{4}{3}$ with respect to the given scale. Some students drew their graphs with an open circle at the endpoints.

Question 10

a. $f(x) = \dfrac{4 + x^2 + x^3}{x}, \quad x \in R \setminus \{0\}$

$f'(x) = \dfrac{2x^3 + x^2 - 4}{x^2}$

For stationary points, solving $f'(x) = 0$ gives $x = 1.1134,\ f(1.1134) = 5.946$

Award **1 mark** for the correct coordinates of the stationary point.

***VCAA Assessment Report* note:**

This question was answered very well. A small number of students gave the coordinates for the point of inflection rather than the stationary point.

b. $f''(x) = \dfrac{2(x^3 + 4)}{x^3}$

For inflection points, $f''(x) = 0$:

$x = \sqrt[3]{-4} = -1.587,\ f(-1.587) = -1.587$

$(-1.59, -1.59)$

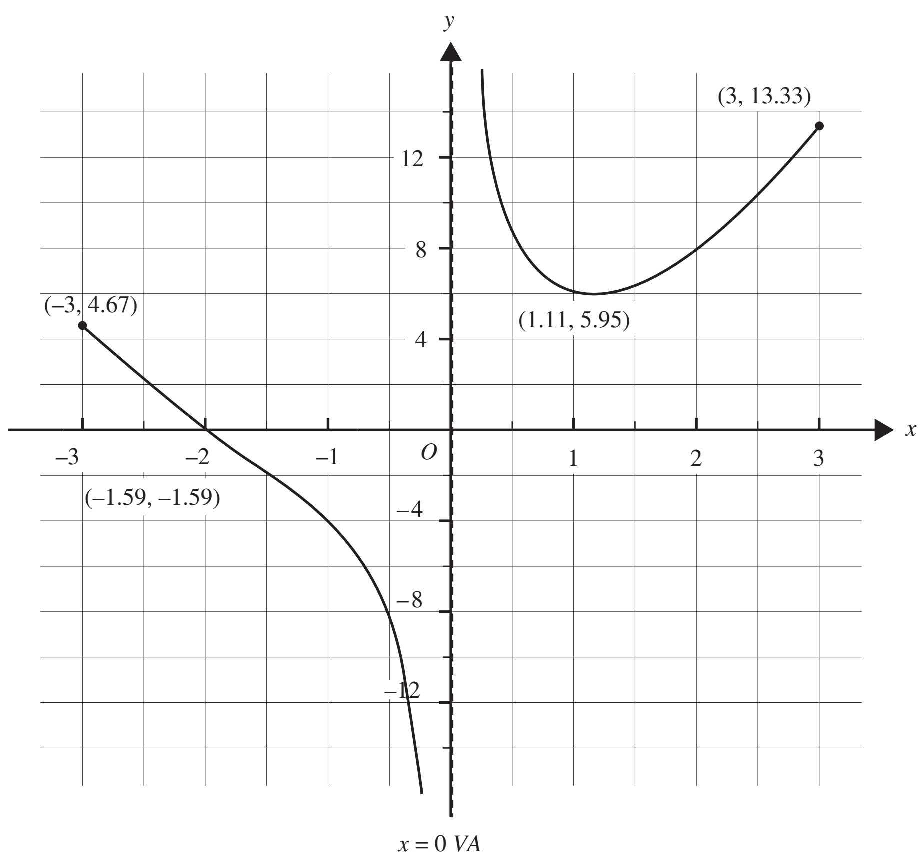
Award **1 mark** for setting the second derivative to zero.

Award **1 mark** for the correct coordinates of the inflexion point.

***VCAA Assessment Report* note:**

This question was generally answered well. The most common error resulted from substituting a rounded x value, yielding an incorrect y value of -1.58. Some students left off the negative sign of the y value.

c.



Award **1 mark** for the correct graph on the domain.

Award **1 mark** for the stationary point and inflexion point.

Award **1 mark** for the endpoints and vertical asymptote.

***VCAA Assessment Report* note:**

Students missed out on marks for ignoring the domain of the function or a lack of accuracy in the placement of the endpoints. Students are advised to use their technology as a tool to support the sketching of an accurate graph rather than simply copying a roughly correct shape from a screen. Students generally followed the instruction to label particular points but these points were not always plotted with appropriate accuracy. Careful attention to the axes scale is required.

d. i. $s = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$

$$s = \int_{-3}^{-0.5} \sqrt{1 + \left(\frac{2x^3 + x^2 - 4}{x^2}\right)^2}\,dx$$

Award **1 mark** for the equivalent definite integral representing arc length.

***VCAA Assessment Report* note:**

This question was answered fairly well. A variety of equivalent correct forms were presented. A common error was an integrand containing the square of $f(x)$ rather than the square of $f'(x)$. Other errors included incorrect terminals, sign errors within the integrand and expressions that appeared to represent the volume of a solid of revolution.

ii. $s = 13.18$

Award **1 mark** for the correct value.

***VCAA Assessment Report* note:**

The majority of students who answered **part d. i.** correctly were able to answer **part d. ii.** correctly. A number of students who wrote an incorrect expression for **part d. i.** went on to evaluate a correct expression and gained the mark for **part d. ii.**

e. $a = \pi$

$$b = f(-3) = \frac{14}{3}$$

$$c = f\left(-\frac{1}{2}\right) = -\frac{33}{4}$$

Award **1 mark** for all values a, b, and c correct.

***VCAA Assessment Report* note:**

Part e. was not answered well. A number of students incorrectly gave decimal approximations for the value of b. Students must note and follow the general instructions given at the start of Section B. Some students interchanged the values of b and c, but this would only be correct if they wrote $a = -\pi$.

Question 11

a. $y^2 = 2 - \sin^2(x)$

By implicit differentiation:

$$2y\frac{dy}{dx} = -2\sin(x)(\cos(x)) = -\sin(2x)$$

$$\frac{dy}{dx} = \frac{-\sin(2x)}{2y}$$ **[1 mark]**

***VCAA Assessment Report* note:**

This question was answered reasonably well. The main error was some answers were given only in terms of x. Some students moved from a correct answer involving x and y to an incorrect answer involving only x. Omission of the negative sign occurred occasionally.

b. i. $x = 0,\ y = \sqrt{2 - \sin^2(0)} = \sqrt{2}$

$x = \frac{\pi}{2},\ y = \sqrt{2 - \sin^2\left(\frac{\pi}{2}\right)} = 1$ **[1 mark]**

***VCAA Assessment Report* note:**

This question was well answered. Some students wrote 2 instead of $\sqrt{2}$ and others included $\pm$ alternatives in their answers.

ii. $x = 0, \ \dfrac{dy}{dx} = \dfrac{-\sin(0)}{\sqrt{2}} = 0$

$x = \dfrac{\pi}{2}, \ \dfrac{dy}{dx} = \dfrac{-\sin(\pi)}{1} = 0$ **[1 mark]**

***VCAA Assessment Report* note:**

This question was answered quite well. A few students had answers other than zero, while some students did not make it clear that both answers were zero.

c. $f\text{:} \quad y = \sqrt{2 - \sin^2(x)}$

$f^{-1} \quad x = \sqrt{2 - \sin^2(y)}$

$x^2 = 2 - \sin^2(y)$

$\sin^2(y) = 2 - x^2$

$\sin(y) = \pm\sqrt{2 - x^2}$ (take the positive)

$f^{-1}(x) = y = \sin^{-1}\left(\sqrt{2 - x^2}\right)$

$\operatorname{dom} f^{-1} = \operatorname{ran} f = [1, \sqrt{2}]$

$\operatorname{dom} f = \operatorname{ran} f^{-1} = \left[0, \dfrac{\pi}{2}\right]$

Award **1 mark** for the inverse function.

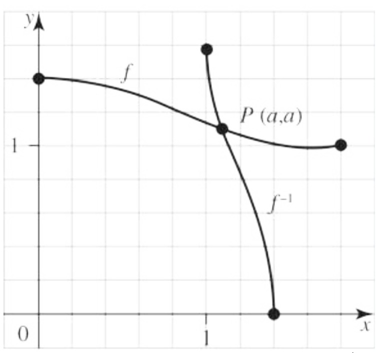
Award **1 mark** for the domain of the inverse function.

Award **1 mark** for the range of the inverse function.

***VCAA Assessment Report* note:**

This question was answered fairly well. The main errors were the incorrect domain and/or range, or the omission of one or both. A small number of students gave the inverse relation by including $\pm$ in front of the square root.

Most students knew to interchange x and y as a first step.



d. f passes through $\left(0, \sqrt{2}\right), \left(\dfrac{\pi}{2}, 1\right)$.

$f^{-1}\text{:} \left[1, \sqrt{2}\right] \rightarrow R, f^{-1}(x) = \sin^{-1}\left(\sqrt{2 - x^2}\right)$

f^{-1} passes through $\left(\sqrt{2}, 0\right), \left(1, \dfrac{\pi}{2}\right)$.

Award **1 mark** for the correct graph in terms of shape and **1 mark** for the scale.

***VCAA Assessment Report* note:**

This question was answered moderately well. Many students did not accurately transfer the graphs from a CAS screen to the axes provided. Of those who managed to draw the graphs correctly, a significant number did not label them. Incorrect location of endpoints and incorrect concavity were common.

e. Solving $f^{-1}(x) = f(x)$ or $f(x) = x$ or graphically

$x = a = 1.099$ **[1 mark]**

***VCAA Assessment Report* note:**

A significant number of students did not give their answer correct to three decimal places.

f. i. $V = \pi \int_0^1 \left(2 - \sin^2(x)\right) dx$

Award **1 mark** for the upper and lower bound, and **1 mark** for the definite integral.

***VCAA Assessment Report* note:**

This question was answered reasonably well. Common errors included not squaring $f(x)$, incorrect terminals and the occasional omission of π and/or dx.

ii. $V = 5.4\,\text{unit}^3$

[1 mark]

***VCAA Assessment Report* note:**

This question was answered very well by students who set up the integral correctly. A number of students who set up the integral correctly did not include the π in their calculations and obtained 1.7.

Question 12

$x^4 + x^3 = 1$

$x^4 + x^3 - 1 = 0$

Then use calculator functions to solve.

$\therefore$ there are two real solutions

Question 13

$y = \dfrac{ax^2 + b}{x} = ax + \dfrac{b}{x}$. The line $y = ax$ with $a > 0$ is an asymptote. The graph does not cross the x-axis at two distinct points, so there is no real solution for $ax^2 + b = 0$, which is $x = \pm\sqrt{-\dfrac{b}{a}}$. Since $a > 0$, we also require $b > 0$. Also $\dfrac{dy}{dx} = a - \dfrac{b}{x^2}$. For turning points we require $\dfrac{dy}{dx} = 0$, so that $x = \pm\sqrt{\dfrac{b}{a}}$. So $a > 0$ and $b > 0$.

Question 14

$$y = \frac{-3x^2 + 4}{6x}$$

$$= -\frac{3x^2}{6x} + \frac{4}{6x}$$

$$y = -\frac{1}{2}x + \frac{2}{3x}$$

$\therefore$ Asymptotes are: $x = 0,\ y = -\dfrac{x}{2}$ **[1 mark]**

Question 15

$y = \dfrac{1}{5x + 2}$ and $y = 5x + 2$ intersect

when $\dfrac{1}{5x + 2} = 5x + 2$

Solving for x gives $x = -\dfrac{1}{5}$ and $x = -\dfrac{3}{5}$.

Asymptote at $5x + 2 = 0 \Rightarrow x = -\dfrac{2}{5}$.

Question 16

Turning point at $\frac{-b}{2a}$

$(x+2)(x-3) = x^2 - x - 6$

$\therefore \frac{-b}{2a} = \frac{1}{2}$

Asymptotes at $x + 2 = 0 \Rightarrow x = -2$

and $x - 3 = 0 \Rightarrow x = 3$.

Question 17

$3y - 2x = 6$

$\Rightarrow y = \frac{6+2x}{3}$

The graphs of $y = f(x)$ and $\frac{1}{f(x)}$ will intersect when $\frac{6+2x}{3} = \frac{3}{6+2x}$.

Solving for x gives $x = -\frac{3}{2}$ and $x = -\frac{9}{2}$.

Asymptotes at $3(0) - 2x = 6 \Rightarrow x = -3$

Question 18

The reciprocal graph has the same x-value for the turning point, but the y-value is the reciprocal. Also, the maximum becomes a minimum.

Question 19

The reciprocal graph has a vertical asymptote where the original crosses the x-axis; the y-value is the reciprocal.

Question 20

Need $x^2 + mx + n \neq 0$

$b^2 - 4ac < 0$

$m^2 - 4n < 0$

$m^2 < 4n$

Question 21

Need $nx^2 + mx - 1 > 0$

$b^2 - 4ac < 0$

$m^2 + 4n < 0$

$m^2 < -4n$

7 Integral calculus

Topic	7	Integral calculus
Subtopic	7.2	Areas under and between curves

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Question 1 (1 mark)

Calculate the area bounded by the curve $y = x^2 - 2x - 24$, the x axis $x = 2$ and the lines and $x = 8$.

Area = $\square$ units^2

Question 2 (2 marks)

If a is a positive constant, calculate the area bounded by the curve $y = x^3 - a^2x$ and the x-axis.

Question 3 (2 marks)

Consider the graphs of $y = \dfrac{190}{x^2} - 5$ and $y = -32\cos\left(\dfrac{x}{5}\right)$ for $x \geq 0$.

a. Determine the coordinates of the first two points of intersection, giving your answers rounded to 4 decimal places.

($\square$, $\square$) ; ($\square$, $\square$) **(1 mark)**

b. Calculate the area between the curves and these first two points of intersection, giving your answer correct to 4 decimal places.
Area = ☐ units^2 **(1 mark)**

Source: *VCE 2014, Specialist Mathematics 1, Q.7; © VCAA*

Question 4 (5 marks)

Consider $f(x) = 3x \arctan(2x)$.

a. Write down the range of f. **(1 mark)**

b. Show that $f'(x) = 3\arctan(2x) + \dfrac{6x}{1 + 4x^2}$. **(1 mark)**

c. Hence evaluate the area enclosed by the graph of $g(x) = \arctan(2x)$, the x-axis $x = \dfrac{1}{2}$ and the lines and $x = \dfrac{\sqrt{3}}{2}$. **(3 marks)**

Source: *VCE 2009, Specialist Mathematics 1, Q.8; © VCAA*

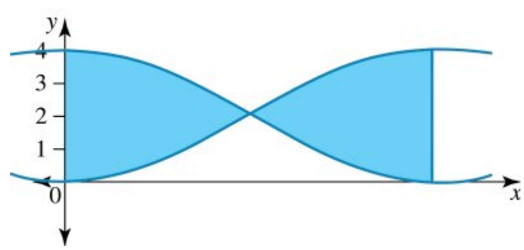
Question 5 (4 marks)

Answer the following.

a. Show that $f(x) = \dfrac{2+x^2}{4-x^2}$ can be written in the form $f(x) = -1 + \dfrac{6}{4-x^2}$. **(1 mark)**

b. Find the exact area enclosed by the graph of $f(x) = \dfrac{2+x^2}{4-x^2}$, the x-axis, and the lines $x = -1$ and $x = 1$. **(3 marks)**

Question 6 (1 mark)



The graphs of $y = 4\sin^2(2x)$ and $y = 4\cos^2(2x)$ are shown for $0 \le x \le \dfrac{\pi}{4}$. Given that the graphs intersect at $x = \dfrac{\pi}{8}$, the shaded area has a magnitude in square units of

A. 0.5
B. 1
C. 2
D. 4
E. 8

Topic	**7**	**Integral calculus**
Subtopic	**7.3**	**Linear substitutions**

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Source: *VCE 2020, Specialist Mathematics Exam 1, Q.2; © VCAA.*

Question 1 (4 marks)

Evaluate $\int_{-1}^{0} \frac{1+x}{\sqrt{1-x}}dx$. Give your answer in the form $a\sqrt{b}+c$, where $a, b, c \in R$.

Source: *VCE 2019, Specialist Mathematics Exam 2, Section A, Q8; © VCAA.*

Question 2 (1 mark)

With a suitable substitution, $\int_{1}^{5} (2x-1)\sqrt{2x+1}\,dx$ can be expressed as

A. $\frac{1}{2}\int_{1}^{5}\left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du$

B. $2\int_{3}^{11}\left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du$

C. $2\int_{1}^{5}\left(u^{\frac{3}{2}}+2u^{\frac{1}{2}}\right)du$

D. $2\int_{3}^{11}\left(u^{\frac{3}{2}}+2u^{\frac{1}{2}}\right)du$

E. $\frac{1}{2}\int_{3}^{11}\left(u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right)du$

Source: *VCE 2017, Specialist Mathematics Exam 2, Section A, Q7; © VCAA.*

Question 3 (1 mark)

With a suitable substitution, $\int_1^2 x^2\sqrt{2-x}\,dx$ can be expressed as

A. $-\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

B. $\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

C. $\int_0^1 \left(-4u^{\frac{1}{2}} + 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

D. $-\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

E. $\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

Source: *VCE 2020, Specialist Mathematics 1, Q.2; © VCAA*

Question 4 (4 marks)

Evaluate $\int_{-1}^0 \frac{1+x}{\sqrt{1-x}}dx$. Give your answer in the form $a\sqrt{b}+c$, where $a, b, c \in R$.

Source: *VCE 2019, Specialist Mathematics 2, Section A, Q.8; © VCAA*

Question 5 (1 mark)

With a suitable substitution, $\int_1^5 (2x-1)\sqrt{2x+1}\,dx$ can be expressed as

A. $\frac{1}{2}\int_1^5 \left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right) du$

B. $2\int_3^{11} \left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right) du$

C. $2\int_1^5 \left(u^{\frac{3}{2}}+2u^{\frac{1}{2}}\right) du$

D. $2\int_3^{11} \left(u^{\frac{3}{2}}+2u^{\frac{1}{2}}\right) du$

E. $\frac{1}{2}\int_3^{11} \left(u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right) du$

Source: *VCE 2017, Specialist Mathematics 2, Section A, Q.7; © VCAA*

Question 6 (1 mark)

With a suitable substitution, $\int_1^2 x^2\sqrt{2-x}dx$ can be expressed as

A. $-\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

B. $\int_1^2 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$

C. $\int_0^1 \left(-4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

D. $-\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

E. $\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} - u^{\frac{5}{2}}\right) du$

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.10; © VCAA*

Question 7 (1 mark)

Using a suitable substitution, the definite integral $\int_0^1 \left(x^2\sqrt{3x+1}\right) dx$ is equivalent to

A. $\frac{1}{9}\int_0^1 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

B. $\frac{1}{27}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

C. $\frac{1}{9}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

D. $\frac{1}{27}\int_0^1 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

E. $\frac{1}{3}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

Question 8 (2 marks)

Find an antiderivative of $ax\sqrt{(cx+e)}$.

Question 9 (1 mark)

With a suitable substitution, $\int 2x\sqrt{x-2}dx$ can be written as

A. $2\int\left(u+2\sqrt{u}\right)du$

B. $\int\left(2u^{\frac{5}{2}}+\sqrt{u}\right)du$

C. $\int\left(2+\sqrt{u}\right)du$

D. $\int\left(2u^{\frac{3}{2}}+4\sqrt{u}\right)du$

E. $\int\left(2u\sqrt{u-2}\right)du$

Question 10 (1 mark)

With a suitable substitution, $\int \frac{3x}{x+2}dx$ can be written as

A. $\int \frac{3}{u}du$

B. $\int\left(3-\frac{6}{u}\right)du$

C. $\frac{1}{2}\int \frac{3}{u}du$

D. $\int \frac{3u}{u+2}du$

E. $\int \frac{3u+6}{u}du$

Question 11 (1 mark)

If a and b are positive real constants, then $\int_0^{\frac{1}{a}} \frac{x}{ax+b}\,dx$ is equal to

A. $\frac{b}{a}\log_e\left(\frac{b}{b+1}\right)$

B. $\frac{b}{a^2}\log_e\left(\frac{b}{b+1}\right)$

C. $\frac{1}{a}\left(b\log_e\left(\frac{b}{b+1}\right)+1\right)$

D. $\frac{b}{a^2}\left(\log_e\left(\frac{b}{b+1}\right)+1\right)$

E. $\frac{1}{a^2}\left(b\log_e\left(\frac{b}{b+1}\right)+1\right)$

Topic	7	Integral calculus
Subtopic	7.4	Non-linear substitutions

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Source: *VCE 2018, Specialist Mathematics Exam 2, Section A, Q8; © VCAA.*

Question 1 (1 mark)

Using a suitable substitution, $\int_0^{\frac{\pi}{6}} \tan^2(x)\sec^2(x)dx$ can be expressed as

A. $\int_0^{\frac{1}{\sqrt{3}}} \left(u^4 + u^2\right) du$

B. $\int_0^{\frac{2}{\sqrt{3}}} \left(u^4 + u^2\right) du$

C. $\int_0^{\frac{1}{\sqrt{3}}} u\, du$

D. $\int_0^{\frac{\pi}{6}} u^2\, du$

E. $\int_0^{\frac{1}{\sqrt{3}}} u^2\, du$

Source: *VCE 2016, Specialist Mathematics Exam 2, Section A, Q8; © VCAA.*

Question 2 (1 mark)

Using a suitable substitution, $\int_a^b \left(x^3 e^{2x^4}\right) dx$, where a and b are real constants, can be written as

A. $\int_a^b \left(e^{2u}\right) du$

B. $\int_{a^4}^{b^4} \left(e^{2u}\right) du$

C. $\frac{1}{8}\int_a^b \left(e^{u}\right) du$

D. $\frac{1}{4}\int_a^{4b^4} \left(e^{2u}\right) du$

E. $\frac{1}{8}\int_{8a^3}^{8b^3} \left(e^{u}\right) du$

Source: *VCE 2013 Specialist Mathematics Exam 2, Section A, Q9; © VCAA.*

Question 3 (1 mark)

The definite integral, $\int_{e^3}^{e^4} \frac{1}{x\log_e(x)} dx$ can be written in the form $\int_a^b \frac{1}{u} du$ where

A. $u = \log_e(x), a = \log_e(3), b = \log_e(4)$

B. $u = \log_e(x), a = 3, b = 4$

C. $u = \log_e(x), a = e^3, b = e^4$

D. $u = \frac{1}{x}, a = e^{-3}, b = e^{-4}$

E. $u = \frac{1}{x}, a = e^3, b = e^4$

Source: *VCE 2020, Specialist Mathematics 2, Section A, Q.11; © VCAA*

Question 4 (1 mark)

With a suitable substitution, $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(x)}{\sec^2(x) - 3\tan(x) + 1} dx$ can be expressed as

A. $\int_{1}^{\frac{1}{\sqrt{3}}} \left(\frac{1}{u-1} - \frac{1}{u-2}\right) du$

B. $\int_{1}^{\sqrt{3}} \left(\frac{1}{3(u-3)} - \frac{1}{3u}\right) du$

C. $\int_{1}^{\sqrt{3}} \left(\frac{1}{u-2} - \frac{1}{u-1}\right) du$

D. $\int_{1}^{\sqrt{3}} \left(\frac{1}{u-1} - \frac{1}{u-2}\right) du$

E. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{3(u-1)} - \frac{1}{3(u+2)}\right) du$

Topic 7 › Subtopic 7.4 Non-linear substitutions

Source: *VCE 2018, Specialist Mathematics 2, Section A, Q.8; © VCAA*

Question 5 (1 mark)

Using a suitable substitution, $\int_0^{\frac{\pi}{6}} \tan^2(x)\sec^2(x)\,dx$ can be expressed as

A. $\int_0^{\frac{1}{\sqrt{3}}} \left(u^4 + u^2\right) du$

B. $\int_1^{\frac{2}{\sqrt{3}}} \left(u^4 + u^2\right) du$

C. $\int_0^{\frac{1}{\sqrt{3}}} u\,du$

D. $\int_0^{\frac{\pi}{6}} u^2 du$

E. $\int_0^{\frac{1}{\sqrt{3}}} u^2 du$

Source: *VCE 2016, Specialist Mathematics 2, Section A, Q.8; © VCAA*

Question 6 (1 mark)

Using a suitable substitution, $\int_a^b \left(x^3 e^{2x^4}\right) dx$, where a and b are real constants, can be written as

A. $\int_a^b \left(e^{2u}\right) du$

B. $\int_{a^4}^{b^4} \left(e^{2u}\right) du$

C. $\frac{1}{8}\int_a^b \left(e^{u}\right) du$

D. $\frac{1}{4}\int_{a^4}^{b^4} \left(e^{2u}\right) du$

E. $\frac{1}{8}\int_{8a^3}^{8b^3} \left(e^{u}\right) du$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.13; © VCAA*

Question 7 (1 mark)

Using the substitution $u = \sqrt{x+1}$ then $\int_0^2 \frac{dx}{(x+2)\sqrt{x+1}}$ can be expressed as

A. $\int_1^{\sqrt{3}} \frac{1}{\sqrt{u}\left(u^2+1\right)} du$

B. $\int_0^2 \frac{2}{u^2+1} du$

C. $\int_1^3 \frac{1}{\sqrt{u}\,(u+1)} du$

D. $\frac{1}{4}\int_0^2 \frac{1}{u^2\left(u^2+1\right)} du$

E. $2\int_1^{\sqrt{3}} \frac{1}{u^2+1} du$

Question 8 (1 mark)

An antiderivative of $\frac{e^{-3x}}{1-e^{-3x}}$ is equal to

A. $\log_e\left|e^{-3x}-1\right|+c$

B. $3\log_e\left|e^{-3x}-1\right|+c$

C. $-3\log_e\left|e^{-3x}-1\right|+c$

D. $\frac{1}{3}\log_e\left|e^{-3x}-1\right|+c$

E. $-\frac{1}{3}\log_e\left|e^{-3x}-1\right|+c$

Question 9 (1 mark)

An antiderivative of $\frac{x^3}{4x^2+9}$ is equal to

A. $\frac{x^2}{8}\log_e\left(4x^2+9\right)+c$

B. $\frac{x^3}{8}\tan^{-1}\left(\frac{2x}{3}\right)+c$

C. $\frac{x^2}{8}\log_e\left(4x^2+9\right)+\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+c$

D. $-\frac{1}{32}\left(9\log_e\left(4x^2+9\right)-4x^2\right)+c$

E. $\frac{1}{32}\left(9\log_e\left(4x^2+9\right)-4x^2\right)+c$

Question 10 (1 mark)

An antiderivative of $\tan(3x)$ is equal to

A. $\frac{1}{3}\log_e(\sin(3x))+c$

B. $-\frac{1}{3}\log_e(\sin(3x))+c$

C. $\frac{1}{3}\log_e(\cos(3x))+c$

D. $-\frac{1}{3}\log_e(\cos(3x))+c$

E. $-3\log_e(\cos(3x))+c$

Question 11 (1 mark)

Using a suitable substitution, $\int \frac{x^3}{\sqrt{3x^2+4}}dx$ is equal to

A. $\frac{1}{18}\int \frac{u-4}{\sqrt{u}}du$

B. $2\int \frac{u-4}{\sqrt{u}}du$

C. $\frac{1}{2}\int \frac{u}{\sqrt{3u+4}}du$

D. $2\int \frac{u}{\sqrt{3u+4}}du$

E. $\frac{1}{9}\int \frac{u^2-4}{\sqrt{u}}du$

Topic	7	Integral calculus
Subtopic	7.5	Integrals of powers of trigonometric functions

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Question 1 (1 mark)

The exact value of $\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{3x}{2}\right) dx$ is

A. $\frac{\pi}{3}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{12}$

D. $\frac{2\pi}{3}$

E. 0

Question 2 (1 mark)

State which of the following is an anti-derivative of $\sec^2(3x)\tan^2(3x)$.

A. $3\tan^3(3x) + c$

B. $3\sec^3(3x) + c$

C. $\frac{1}{3}\tan^3(3x) + c$

D. $\frac{1}{9}\tan^3(3x) + c$

E. $\frac{1}{3}\sec^3(3x) + c$

Topic 7 Subtopic 7.5 Integrals of powers of trigonometric functions

Question 3 (2 marks)
Evaluate the following in terms of p.

$$\int_0^p \left(\sin^2(5x) - \cos^2(5x)\right) dx = \square$$

Source: *VCE 2009, Specialist Mathematics 2, Section 1, Q.10; © VCAA*

Question 4 (1 mark)
Let $f: [-\pi, 2\pi] \to R$, where $f(x) = \sin^3(x)$.
Using the substitution $u = \cos(x)$, the area bounded by the graph of f and the x-axis could be found by evaluating

A. $-\int_{-\pi}^{2\pi} \left(1 - u^2\right) du$

B. $3\int_{-1}^{1} \left(1 - u^2\right) du$

C. $-3\int_{0}^{\pi} \left(1 - u^2\right) du$

D. $3\int_{1}^{-1} \left(1 - u^2\right) du$

E. $-\int_{-1}^{1} \left(1 - u^2\right) du$

Topic	7	Integral calculus
Subtopic	7.6	Integrals involving inverse trigonometric functions

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Source: *VCE 2021, Specialist Mathematics Exam 1, Q.2; © VCAA*

Question 1 (3 marks)

Evaluate $\int_0^1 \frac{2x+1}{x^2+1}dx$.

Question 2 (1 mark)

$\int \frac{dx}{2+5x^2}$ is equal to

A. $\tan^{-1}\left(\frac{\sqrt{5}x}{\sqrt{2}}\right)+c$

B. $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{10}x}{2}\right)+c$

C. $\frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{\sqrt{10}x}{2}\right)+c$

D. $\frac{\sqrt{10}}{10}\tan^{-1}\left(\frac{\sqrt{10}x}{2}\right)+c$

E. $\frac{1}{10x}\log_e\left(2+5x^2\right)+c$

Topic 7 | Subtopic 7.6 Integrals involving inverse trigonometric functions

Question 3 (2 marks)

Determine an anti-derivative of $\dfrac{\frac{a}{b}}{\sqrt{(b^2 - a^2x^2)}}$.

Question 4 (2 marks)

Find an antiderivative of $\dfrac{1}{\sqrt{(1 - 3x^2)}}$.

Question 5 (2 marks)

Find an antiderivative of $\dfrac{\cos(x)}{\sqrt{\left(1 - \sin^2(x)\right)}}$.

Question 6 (1 mark)

$\int \frac{dx}{\sqrt{(2-7x^2)}}$ is equal to

A. $\frac{\sqrt{7}}{7}\sin^{-1}\left(\frac{\sqrt{14}x}{2}\right)+c$

B. $\frac{\sqrt{7}}{7}\cos^{-1}\left(\frac{\sqrt{14}x}{2}\right)+c$

C. $\frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{\sqrt{14}x}{2}\right)+c$

D. $\sin^{-1}\left(\frac{\sqrt{7}x}{\sqrt{2}}\right)+c$

E. $-\sqrt{7}\cos^{-1}\left(\frac{\sqrt{7}x}{\sqrt{2}}\right)+c$

Topic	7	Integral calculus
Subtopic	7.7	Integrals involving partial fractions

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Source: *VCE 2018 Specialist Mathematics Exam 2, Section A, Q3; © VCAA*

Question 1 (1 mark)

Which one of the following, where A, B, C and D are non-zero real numbers, is the partial fraction form for the expression $\dfrac{2x^2+3x+1}{(2x+1)^3\left(x^2-1\right)}$?

1. $\dfrac{A}{2x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$

2. $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{(2x+1)^3}+\dfrac{Dx}{x^2-1}$

3. $\dfrac{A}{2x+1}+\dfrac{Bx+C}{x^2-1}$

4. $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{x-1}$

5. $\dfrac{A}{2x+1}+\dfrac{Bx+C}{(2x+1)^2}+\dfrac{D}{x-1}$

Source: *VCE 2017 Specialist Mathematics Exam 1, Q2; © VCAA*

Question 2 (4 marks)

Find $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x\left(1+x^2\right)}dx$ expressing your answer in the form $\log_e\left(\sqrt{\dfrac{a}{b}}\right)$ where a and b are positive integers.

Question 3 (1 mark)

If a is a positive real constant, then $\int \frac{a}{x(x-a)}\,dx$ for $x > a$ is equal to

A. $\log_e\left(\frac{a-x}{x}\right)$

B. $\log_e\left(\frac{x-a}{x}\right)$

C. $-\log_e\left(\frac{x-a}{x}\right)$

D. $-\log_e\left(\frac{a-x}{x}\right)$

E. $-\log_e(x(x-a))$

Source: *VCE 2020, Specialist Mathematics 2, Section A, Q.7; © VCAA*

Question 4 (1 mark)

For non-zero real constants a and b, where $b < 0$, the expression $\frac{1}{ax\left(x^2+b\right)}$ in partial fraction form with linear denominators, where A, B and C are real constants, is

A. $\frac{A}{ax} + \frac{Bx+C}{x^2+b}$

B. $\frac{A}{ax} + \frac{B}{x+\sqrt{b}} + \frac{C}{ax-\sqrt{b}}$

C. $\frac{A}{x} + \frac{B}{ax+\sqrt{|b|}} + \frac{C}{ax-\sqrt{|b|}}$

D. $\frac{A}{x} + \frac{B}{x+\sqrt{|b|}} + \frac{C}{x-\sqrt{|b|}}$

E. $\frac{A}{ax} + \frac{B}{\left(x+\sqrt{b}\right)^2} + \frac{C}{x+\sqrt{b}}$

Source: *VCE 2018, Specialist Mathematics 2, Section A, Q.3; © VCAA*

Question 5 (1 mark)

Which one of the following, where A, B, C and D are non-zero real numbers, is the partial fraction form for the expression $\dfrac{2x^2+3x+1}{(2x+1)^3\left(x^2-1\right)}$?

A. $\dfrac{A}{2x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$

B. $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{(2x+1)^3}+\dfrac{Dx}{x^2-1}$

C. $\dfrac{A}{2x+1}+\dfrac{Bx+c}{x^2-1}$

D. $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{x-1}$

E. $\dfrac{A}{2x+1}+\dfrac{Bx+c}{(2x+1)^2}+\dfrac{D}{x-1}$

Source: *VCE 2013, Specialist Mathematics, Exam 1, Q.2; © VCAA*

Question 6 (4 marks)

Evaluate $\displaystyle\int_0^1 \frac{x-5}{x^2-5x+6}dx$.

Source: *VCE 2011, Specialist Mathematics 1, Q.1; © VCAA*

Question 7 (3 marks)

Find an antiderivative of $\dfrac{1+x}{9-x^2}, x \in R\backslash\{-3,3\}$.

Question 8 (1 mark)

When $\frac{4x-6}{x^2+6x+8}$ is expressed in the form $\frac{A}{(x+4)}+\frac{B}{(x+2)}$, the values of A and B respectively are

A. 8 and –4
B. –4 and 8
C. –7 and 11
D. –6 and 10
E. 11 and –7

Question 9 (1 mark)

when $\frac{4}{(x-3)}-\frac{2}{(x-2)}$ is expressed in the form $\frac{Ax+B}{(x-3)(x-2)}$, the values of A and B respectively are

A. 4 and –2
B. 2 and –2
C. –2 and 4
D. 2 and 2
E. 3 and 2

Question 10 (1 mark)

$\frac{2x+1}{3x^2+4x+1}$ can be expressed as a partial fraction in the form $\frac{A}{(ax+c)}+\frac{B}{(dx+e)}$ as

A. $\frac{2}{3x+1}+\frac{1}{x+2}$
B. $\frac{1}{6x+1}+\frac{1}{2x+1}$
C. $\frac{1}{3x+1}+\frac{2}{x+1}$
D. $\frac{1}{3x+2}+\frac{1}{2x+1}$
E. $\frac{1}{6x+2}+\frac{1}{2x+2}$

Question 11 (1 mark)

The algebraic fraction $\frac{10}{4x^2 - 25}$ can be written in partial fractions form as

A. $\frac{5}{2x+5} + \frac{5}{2x-5}$

B. $\frac{2}{2x+5} + \frac{2}{2x-5}$

C. $\frac{1}{2x+5} + \frac{1}{2x-5}$

D. $\frac{2}{2x-5} - \frac{2}{2x+5}$

E. $\frac{1}{2x-5} - \frac{1}{2x+5}$

Question 12 (1 mark)

The algebraic fraction $\frac{1}{abx^2 + (pb + qa)\,x + pq}$, where a, b, p and q are real numbers, can be written in partial fractions form as

A. $\frac{1}{aq - pb}\left(\frac{q}{bx+q} - \frac{p}{ax+q}\right)$ provided that $aq - bp \neq 0$

B. $\frac{1}{aq - pb}\left(\frac{q}{bx+q} + \frac{p}{ax+q}\right)$ provided that $aq - bp \neq 0$

C. $\frac{1}{aq - pb}\left(\frac{b}{bx+q} - \frac{a}{ax+p}\right)$ provided that $aq - bp \neq 0$

D. $\frac{1}{aq - pb}\left(\frac{a}{ax+p} - \frac{b}{bx+q}\right)$ provided that $aq - bp \neq 0$

E. $\frac{a}{ax+p} - \frac{b}{bx+q}$

Question 13 (1 mark)

When $\frac{x+3}{x^2+8x+16}$ is expressed in the form $\frac{A}{(x+4)}+\frac{B}{(x+4)^2}$, the values of A and B respectively are

A. 1 and 3
B. −1 and 3
C. 1 and 1
D. 0 and 8
E. 1 and −1

Question 14 (1 mark)

When $\frac{1}{(x-3)}+\frac{1}{(x-3)^2}$ is expressed in the form $\frac{Ax+B}{(x-3)^2}$, the values of A and B respectively are

A. 6 and −3
B. 1 and 6
C. 1 and 3
D. 3 and 2
E. 6 and 1

Question 15 (1 mark)

$\frac{2-3x}{x^2-4x+4}$ can be expressed as a partial fraction in the form $\frac{A}{(ax+c)^2}+\frac{B}{(ax+c)}$ as

A. $\frac{-4}{(x-2)^2}-\frac{3}{(x-2)}$
B. $\frac{-3}{(x-2)^2}-\frac{4}{(x-2)}$
C. $\frac{-2}{(x-2)^2}-\frac{3}{(x-2)}$
D. $\frac{-3}{(x-2)^2}-\frac{2}{(x-2)}$
E. $\frac{3}{(x-2)^2}-\frac{4}{(x-2)}$

Question 16 (1 mark)

The algebraic fraction $\frac{2x}{(2x+5)^2}$ can be written in partial fractions form as

A. $\frac{1}{2x+5} - \frac{5}{(2x+5)^2}$

B. $\frac{5}{2x+5} - \frac{1}{(2x+5)^2}$

C. $\frac{2}{2x+5} - \frac{10}{(2x+5)^2}$

D. $\frac{1}{2x+5} + \frac{5}{(2x+5)^2}$

E. $\frac{5}{2x+5} + \frac{1}{(2x+5)^2}$

Question 17 (1 mark)

The algebraic fraction $\frac{ax+b}{(px+q)^2}$, where a, b, p and q are real numbers, provided that $p \neq 0$, can be written in partial fractions form as

A. $\frac{1}{p}\left(\frac{a}{px+q} + \frac{aq-bp}{(px+q)^2}\right)$

B. $\frac{1}{p}\left(\frac{a}{px+q} - \frac{aq-bp}{(px+q)^2}\right)$

C. $\frac{1}{p}\left(\frac{aq-bp}{px+q} - \frac{1}{(px+q)^2}\right)$

D. $\frac{1}{p}\left(\frac{1}{(px+q)^2} - \frac{aq-bp}{px+q}\right)$

E. $\frac{1}{p}\left(\frac{1}{(px+q)^2} + \frac{aq-bp}{px+q}\right)$

Question 18 (1 mark)

When $\dfrac{x-2}{\left(x^2+2\right)(x-1)}$ is expressed in the form $\dfrac{Ax+B}{\left(x^2+2\right)}+\dfrac{C}{(x-1)}$, the values of A, B and C respectively are

A. $-\frac{1}{3}, -\frac{4}{3}$ and $-\frac{1}{3}$

B. $\frac{1}{3}, \frac{4}{3}$ and $-\frac{1}{3}$

C. $\frac{2}{3}, \frac{1}{3}$ and $-\frac{1}{3}$

D. $\frac{2}{3}, -\frac{1}{3}$ and $\frac{1}{3}$

E. $2, -2$ and -1

Question 19 (1 mark)

$\dfrac{5-20x}{\left(x^2+4\right)(x-1)}$ can be expressed as a partial fraction in the form $\dfrac{Ax+B}{\left(x^2+4\right)}+\dfrac{C}{(x-1)}$ as

A. $\dfrac{5x-20}{\left(x^2+4\right)}-\dfrac{4}{(x-1)}$

B. $\dfrac{17-3x}{\left(x^2+4\right)}+\dfrac{3}{(x-1)}$

C. $\dfrac{3x-17}{\left(x^2+4\right)}-\dfrac{3}{(x-1)}$

D. $\dfrac{4x-20}{\left(x^2+4\right)}-\dfrac{5}{(x-1)}$

E. $\dfrac{20-4x}{\left(x^2+4\right)}-\dfrac{5}{(x-1)}$

Topic 7 | Subtopic 7.7 Integrals involving partial fractions

Question 20 (1 mark)

The algebraic fraction $\dfrac{7x^2 - 6x + 4}{3x\left(x^2 + 4\right)}$ can be written in partial fractions form, as

A. $\dfrac{1}{x} + \dfrac{2x - 2}{x^2 + 4}$

B. $\dfrac{1}{3x} + \dfrac{2x - 2}{x^2 + 4}$

C. $\dfrac{1}{3x} + \dfrac{2x + 2}{x^2 + 4}$

D. $\dfrac{1}{3x} - \dfrac{2x + 2}{x^2 + 4}$

E. $\dfrac{2}{3x} - \dfrac{x + 1}{x^2 + 4}$

Question 21 (1 mark)

The algebraic fraction $\dfrac{1}{\left(x^2 + a^2\right)(x + b)^2}$, where a and b are non-zero real numbers, can be written in partial fractions form, where A, B, C and D are real numbers, as

A. $\dfrac{A}{x + b} + \dfrac{B}{(x + b)^2} + \dfrac{Cx + D}{(x + a)^2}$

B. $\dfrac{A}{x + b} + \dfrac{B}{(x + b)^2} + \dfrac{C}{x + a} + \dfrac{D}{(x + a)^2}$

C. $\dfrac{A}{x + b} + \dfrac{Bx + C}{(x + b)^2} + \dfrac{D}{x^2 + a^2}$

D. $\dfrac{A}{x + b} + \dfrac{B}{(x + b)^2} + \dfrac{Cx + D}{x^2 + a^2}$

E. $\dfrac{Ax + B}{x + b} + \dfrac{C}{(x + b)^2} + \dfrac{D}{x^2 + a^2}$

Question 22 (2 marks)

Find an antiderivative of $\dfrac{2}{x(4-x)}$.

Question 23 (1 mark)

If a is a positive real constant, then $\int \dfrac{a}{x(x-a)}dx$ for $x > a$ is equal to

A. $\log_e\left(\dfrac{a-x}{x}\right)$

B. $\log_e\left(\dfrac{x-a}{x}\right)$

C. $-\log_e\left(\dfrac{x-a}{x}\right)$

D. $-\log_e\left(\dfrac{a-x}{x}\right)$

E. $-\log_e(x(x-a))$

Question 24 (1 mark)

If a and b are positive real constants, then $\int \dfrac{a^2x^2+b^2}{a^2x^2-b^2}dx$ is equal to

A. $x+\dfrac{2b}{a}\tan^{-1}\left(\dfrac{bx}{a}\right)$

B. $x-b\log_a\left(\left|\dfrac{ax+b}{ax-b}\right|\right)$

C. $x+\dfrac{b}{a}\log_e\left(\left|\dfrac{ax-b}{ax+b}\right|\right)$

D. $x+\dfrac{b}{a}\log_a\left(\left|\dfrac{ax+b}{ax-b}\right|\right)$

E. $x+\dfrac{b}{a}\log_a\left(\left|a^2x^2-b^2\right|\right)$

Topic	7	Integral calculus
Subtopic	7.8	Review

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Source: *VCE 2020, Specialist Mathematics Exam 2, Section A, Q11; © VCAA.*

Question 1 (1 mark)

With a suitable substitution, $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(x)}{\sec^2(x) - 3\tan(x) + 1} dx$ can be expressed as

A. $\int_{1}^{\frac{1}{\sqrt{3}}} \left(\frac{1}{u-1} - \frac{1}{u-2} \right) du$

B. $\int_{1}^{\sqrt{3}} \left(\frac{1}{3(u-3)} - \frac{1}{3u} \right) du$

C. $\int_{1}^{\sqrt{3}} \left(\frac{1}{u-2} - \frac{1}{u-1} \right) du$

D. $\int_{1}^{\sqrt{3}} \left(\frac{1}{u-1} - \frac{1}{u-2} \right) du$

E. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{3(u-1)} - \frac{1}{3(u+2)} \right) du$

Source: *VCE 2015, Specialist Mathematics Exam 2, Section A, Q1a; © VCAA.*

Question 2 (2 marks)

Show that $\int \tan(2x)dx = \frac{1}{2}\log_e|\sec(2x)| + c$.

Source: *VCE 2014, Specialist Mathematics Exam 1, Q7; © VCAA.*

Question 3 (5 marks)

Consider $f(x) = 3x\arctan(2x)$.

a. Write down the range of f.

Range = ☐ **(1 mark)**

b. Show that $f'(x) = 3\arctan(2x) + \frac{6x}{1 + 4x^2}$. **(1 mark)**

c. Hence evaluate the area enclosed by the graph of $g(x) = \arctan(2x)$, the x-axis and the lines $x = \frac{1}{2}$ and $x = \frac{\sqrt{3}}{2}$. **(3 marks)**

Question 4 (1 mark)

If a and b are positive real constants, then $\int \frac{a^2x^2 + b^2}{a^2x^2 - b^2} dx$ is equal to

A. $x + \frac{2b}{a}\tan^{-1}\left(\frac{bx}{a}\right)$

B. $x - b\log_e\left(\left|\frac{ax+b}{ax-b}\right|\right)$

C. $x + \frac{b}{a}\log_e\left(\left|\frac{ax-b}{ax+b}\right|\right)$

D. $x + \frac{b}{a}\log_e\left(\left|\frac{ax+b}{ax-b}\right|\right)$

E. $x + \frac{b}{a}\log_e\left(\left|a^2x^2 - b^2\right|\right)$

Question 5 (1 mark)

Evaluate the following in terms of a and b.

$$\int_a^b \left(\text{cosec}^2(3x)e^{3cot(3x)}\right) dx = \square$$

Source: *VCE 2021, Specialist Mathematics 1, Q.2; © VCAA*

Question 6 (3 marks)

Evaluate $\int_0^1 \frac{2x+1}{x^2+1} dx$.

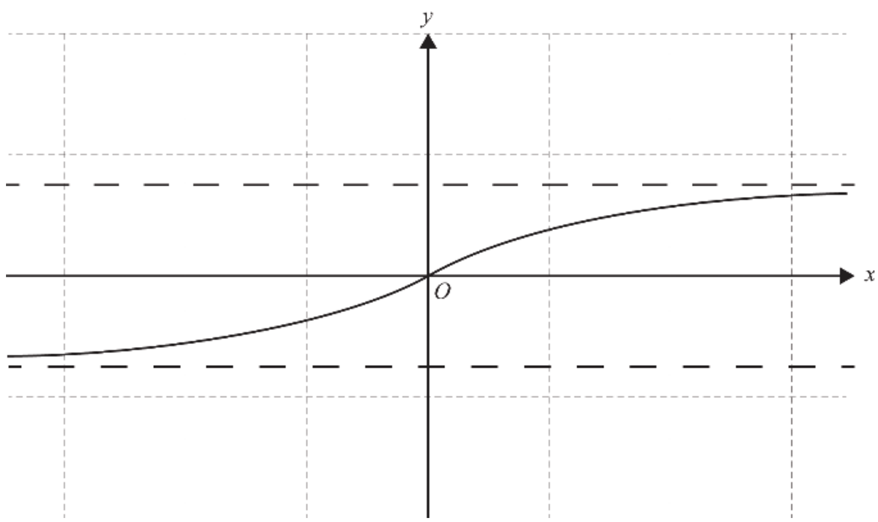
Source: *VCE 2015, Specialist Mathematics 1, Q.8; © VCAA*

Question 7 (7 marks)

Answer the following.

a. Show that $\int \tan(2x)\,dx = \frac{1}{2}\log_e|\sec(2x)| + c$. **(2 marks)**

b. The graph of $f(x) = \frac{1}{2}\arctan(x)$ is shown below. **(2 marks)**



i. Write down the equations of the asymptotes. **(1 mark)**

ii. On the axes above, sketch the graph of f^{-1}, labelling any asymptotes with their equations. **(1 mark)**

c. Find $f\left(\sqrt{3}\right)$. **(1 mark)**

d. Find the area enclosed by the graph of f, the x-axis and the line $x = \sqrt{3}$. **(2 marks)**

Source: *VCE 2011, Specialist Mathematics 1, Q.3a; © VCAA*

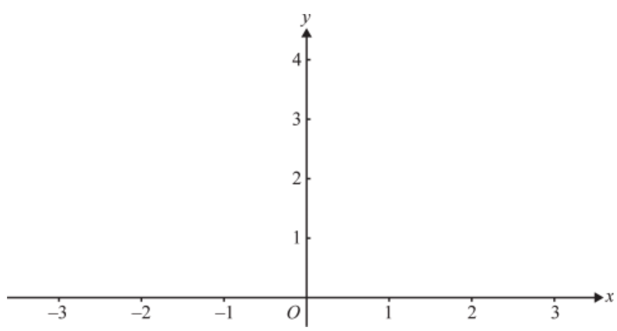
Question 8 (7 marks)

Answer the following.

a. Show that $f(x) = \dfrac{2x^2 + 3}{x^2 + 1}$ can be written in the form $f(x) = 2 + \dfrac{1}{x^2 + 1}$. **(1 mark)**

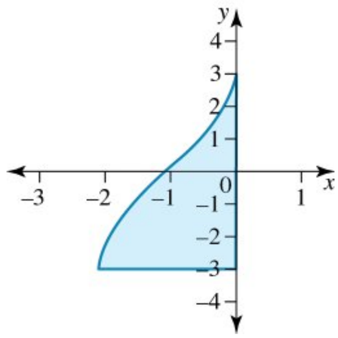
b. Sketch the graph of the relation $y = \dfrac{2x^2 + 3}{x^2 + 1}$ on the axes below.

Label any asymptotes with their equations and label any intercepts with the axes, writing them as coordinates. **(3 marks)**



c.Find the area enclosed by the graph of the relation $y = \frac{2x^2 + 3}{x^2 + 1}$, the x-axis, and the lines $x = -1$ and $x = 1$. **(3 marks)**

Question 9 (1 mark)
The graph of $y = 2\sin^{-1}(x + 1)$ is shown. The domain of the graph is $[-2, 0]$ and the range is $[-\pi, \pi]$. The shaded area is equal to



A. $\int_{-2}^{0} 2\sin^{-1}(x+1)\,dx$

B. $\int_{-\pi}^{\pi} 2\sin^{-1}(x+1)\,dx$

C. $\left|\int_{-\pi}^{0} 2\sin^{-1}(x+1)\,dx\right| + \int_{0}^{\pi} 2\sin^{-1}(x+1)\,dx$

D. $\int_{-\pi}^{\pi} \left(\sin\left(\frac{x}{2}\right) - 1\right) dx$

E. $\int_{-\pi}^{\pi} \left(1 - \sin\left(\frac{x}{2}\right)\right) dx$

Answer

7.2 Areas under and between curves

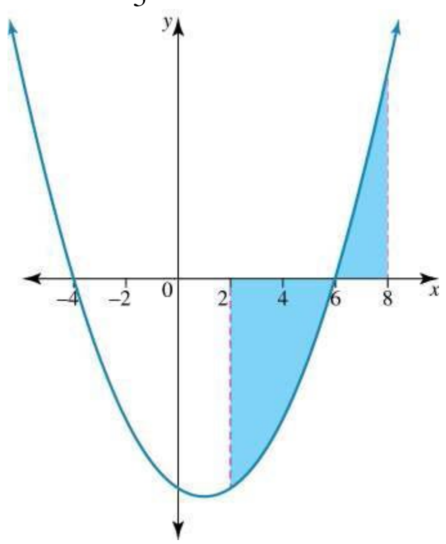
Question 1

$$y = x^2 - 2x - 24$$
$$= (x - 6)(x + 4)$$

$$A_1 = \int_2^6 \left(x^2 - 2x - 24\right)\, dx$$
$$= \left[\frac{1}{3}x^3 - x^2 - 24x\right]_2^6$$
$$= \left[\left(\frac{1}{3} \times 6 \times 36 - 36 - 24 \times 6\right) - \left(\frac{1}{3} \times 8 - 4 - 48\right)\right]$$
$$= -58\frac{2}{3}$$

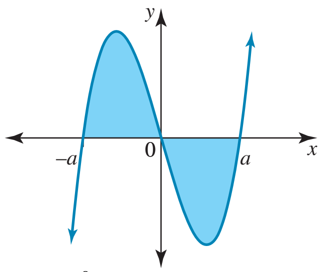
$$A_2 = \int_6^8 \left(x^2 - 2x - 24\right)\, dx$$
$$= \left[\frac{1}{3}x^3 - x^2 - 24x\right]_6^8$$
$$= \left[\left(\frac{1}{3} \times 8^3 - 8^2 - 24 \times 8\right) - \left(\frac{1}{3} \times 6^3 - 6^2 - 24 \times 6\right)\right]$$
$$= 22\frac{2}{3}$$

$$\text{Area} = |A_1| + A_2$$
$$= \left|-58\frac{2}{3}\right| + 22\frac{2}{3}$$
$$= 81\frac{1}{3}$$ **[1 mark]**



Question 2

$$y = x^3 - a^2x$$
$$= x\left(x^2 - a^2\right)$$
$$= x(x + a)(x - a)$$



$$A_1 = \int_{-a}^{0} \left(x^3 - a^2 x\right)\, dx$$
$$= \left[\frac{1}{4}x^4 - \frac{1}{2}a^2x^2\right]_{-a}^{0}$$
$$= (0) - \left[\frac{1}{4}a^4 - \frac{1}{2}a^2a^2\right]$$
$$= \frac{1}{4}a^4 \text{ [1 mark]}$$

So required area $= \frac{1}{2}a^4$ [**1 mark**]

Question 3

a. $y_1 = \frac{190}{x^2} - 5$

$y_2 = -32\cos\left(\frac{x}{5}\right)$

$y_1 = y_2$

$x = 7.5882,\ 24.2955$

Points of intersection $(7.5882, -1.7003)$
$(24.2955, -4.6781)$ [**1 mark**]

b. $A = \int_{7.5882}^{24.2955} \left(-32\cos\left(\frac{x}{5}\right) - \frac{190}{x^2} + 5\right) dx$

$= 384.3732$ [**1 mark**]

Question 4

a. $f(x) = 3x\tan^{-1}(2x)$, range $f = [0, \infty]$

Award 1 mark for the correct range.

***VCAA Assessment Report* note:**

This question was answered poorly by most students. Few realised that x and the arctan function are both positive for the same values, negative for the same values and zero for the same values. Incorrect responses included R, $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ and $\left(-\frac{3\pi x}{2}, \frac{3\pi x}{2}\right)$. Many students seemed to use the product of the range of each of the 'parts', some ignored one part and others found the product of the range of one part and the variable x. Some ignored the presence of one of the two functions involved.

b. $f'(x) = 3\tan^{-1}(2x) + 3x \times \frac{2}{1+4x^2}$, using the product rule

$= 3\tan^{-1}(2x) + \frac{6x}{1+4x^2}$

Award 1 mark for correct differentiation using the product rule.

***VCAA Assessment Report* note:**

This question was well answered. Most students were able to obtain the given result. There were, however, some unconvincing arguments, often because insufficient steps were shown. A few students used $\tan^{-1}(2x)$ and then confused inverses with reciprocals.

c. $\frac{d}{dx}\left[3x\tan^{-1}(2x)\right] = 3\tan^{-1}(2x) + \frac{6x}{1+4x^2}$

$$\int\left(3\tan^{-1}(2x) + \frac{6x}{1+4x^2}\right)dx = 3x\tan^{-1}(2x)$$

$$\int 3\tan^{-1}(2x)\,dx + \int\frac{6x}{1+4x^2}dx = 3x\tan^{-1}(2x)$$

$$\int 3\tan^{-1}(2x)\,dx = 3x\tan^{-1}(2x) - \int\frac{6x}{1+4x^2}dx$$

$$3\int\tan^{-1}(2x)\,dx = 3x\tan^{-1}(2x) - \frac{3}{4}\log_e\left(1+4x^2\right)$$

$$\int\tan^{-1}(2x)\,dx = x\tan^{-1}(2x) - \frac{1}{4}\log_e\left(1+4x^2\right)$$

$$A = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\tan^{-1}(2x)\,dx$$

$$= \left[x\tan^{-1}(2x) - \frac{1}{4}\log_e\left(1+4x^2\right)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \left(\frac{\sqrt{3}}{2}\tan^{-1}\left(\sqrt{3}\right) - \frac{1}{4}\log_e(4)\right) - \left(\frac{1}{2}\tan^{-1}(1) - \frac{1}{4}\log_e(2)\right)$$

$$= \left(\frac{\sqrt{3}}{2}\times\frac{\pi}{3} - \frac{1}{4}\log_e(4)\right) - \left(\frac{1}{2}\times\frac{\pi}{4} - \frac{1}{4}\log_e(2)\right)$$

$$= \pi\left(\frac{\sqrt{3}}{6} - \frac{1}{8}\right) + \frac{1}{4}\log_e\left(\frac{1}{2}\right)$$

$$= \frac{\left(4\sqrt{3}-3\right)\pi}{24} - \frac{1}{4}\log_e(2)$$

Award 1 mark for the correct definite integral for the area.
Award 1 mark for deducing the correct antiderivative.
Award 1 mark for the correct final area.

***VCAA Assessment Report* notes:**

Most students who used the result from **part b.** made good attempts at this question. Some ignored the word ‘hence’. Most attempted to apply this method but many made algebraic errors. When attempting to integrate $\frac{6x}{1+4x^2}$, some gave $3\arctan(2x)$ or similar, others made the correct substitution but made errors in either changing the terminals or with the arithmetic.

Question 5

a. $f(x)=\dfrac{2+x^2}{4-x^2}$

$$f(x)=\frac{-\left(-x^2-2\right)}{4+x^2}=\frac{-\left(4-x^2\right)+6}{4+x^2}$$

$$f(x)=-1+\frac{6}{4-x^2}$$ [**1 mark**]

***VCAA Assessment Report* note:**

This was a straight forward 'show that' question, which was done well by most students; however, some divided the numerator into the denominator.

b. The area $A=\int_a^b f(x)\,dx=\int_{-1}^{1}\left(-1+\dfrac{6}{4-x^2}\right)dx$

By partial fractions:

$$\frac{6}{4-x^2}=\frac{B}{2-x}+\frac{C}{2+x}=\frac{B(2+x)+C(2-x)}{(2-x)(2+x)}=\frac{x(B-C)+2(B+C)}{4-x^2}$$

$$\Rightarrow B-C=0 \text{ and } 2(B+C)=6\Rightarrow B+C=\frac{3}{2}$$

$$A=\int_{-1}^{1}\left(-1+\frac{3}{2}\left(\frac{1}{2-x}+\frac{1}{2+x}\right)\right)dx$$

$$A=\left[-x-\frac{3}{2}\log_e(2-x)+\frac{3}{2}\log_e(2+x)\right]_{-1}^{1}=\left[-x+\frac{3}{2}\log_e\left(\frac{2+x}{2-x}\right)\right]_{-1}^{1}$$

$$A=\left(-1+\frac{3}{2}\log_e(3)\right)-\left(1+\frac{3}{2}\log_e\left(\frac{1}{3}\right)\right)=\frac{3}{2}\log_e(9)-2$$

$$A=3\log_e(3)-2$$

$$=\log_e(27)-2$$ [**1 mark**]

***VCAA Assessment Report* note:**

A number of equally alternative answers were accepted. Most students realised the answer from **part a.** should be used and attempted partial fractions. Some wrote incorrect formulae for the area with an incorrect negative sign, or had the terminals reversed, or omitted the −1 term. Some did not use partial fractions and claimed incorrectly that $\int \dfrac{6}{4-x^2}dx=6\log_e\left(4-x^2\right)$, while others who found the correct partial fractions made sign errors when integrating. The evaluation of the terms, including the log, was poor, with numerous errors occurring.

Question 6

$$A=2\int_0^{\frac{\pi}{8}}\left(4\cos^2(2x)-4\sin^2(2x)\right)\,dx \text{ by symmetry}$$

$$A=8\int_0^{\frac{\pi}{8}}\cos(4x)\,dx=8\left[\frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{8}}=2\left(\sin\left(\frac{\pi}{2}\right)-\sin(0)\right)=2$$

7.3 Linear substitutions

Question 1

Let $u = 1 - x,\ x = 1 - u,\ x + 1 = 2 - u,\ \dfrac{du}{dx} = -1.$

Terminals $x = 0,\ u = 1,\ x = -1,\ u = 2$

$$I = \int_{-1}^{0} \frac{1+x}{\sqrt{1-x}}dx$$

$$= -\int_{2}^{1} \frac{2-u}{\sqrt{u}}du$$

$$= \int_{1}^{2} (2-u)u^{-\frac{1}{2}}du$$

$$= \int_{1}^{2} \left(2u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \left[4u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{1}^{2}$$

$$= \left(4\sqrt{2} - \frac{2}{3} \times 2\sqrt{2}\right) - \left(4 - \frac{2}{3}\right)$$

$$= \frac{8\sqrt{2}}{3} - \frac{10}{3}$$

Award 1 mark for the substitution.
Award 1 mark for changing terminals.
Award 1 mark for the correct definite integral.
Award 1 mark for the final correct answer.

Question 2

$$\int_{1}^{5} (2x-1)\sqrt{2x+1}dx$$

Let $u = 2x + 1,\quad 2x = u - 1,\quad 2x - 1 = u - 2,\quad \dfrac{du}{dx} = 2$

Terminals, when $x = 5, u = 11$ and when $x = 1, u = 3$

$$\int_{1}^{5} (2x-1)\sqrt{2x-1}\,dx = \frac{1}{2}\int_{3}^{11} (u-2)\sqrt{u}\,du$$

$$= \frac{1}{2}\int_{3}^{11} \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$$

The correct answer is **E**.

Question 3

$$I = \int_1^2 x^2 \sqrt{2-x}\, dx$$

$$u = 2 - x \; \frac{du}{dx} = -1$$

$$x = 2 - u,\; x^2 = (2-u)^2 = 4 - 4u + u^2$$

$$x = 2 \Rightarrow u = 0,\; x = 1 \Rightarrow u = 1$$

$$I = -\int_1^0 \left(4 - 4u + u^2\right) u^{\frac{1}{2}} du$$

$$I = -\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$$

The correct answer is **D**.

Question 4

Let $u = 1 - x, x = 1 - u, x + 1 = 2 - u, \dfrac{du}{dx} = -1.$

Terminals $x = 0, u = 1, x = -1, u = 2$

$$I = \int_{-1}^0 \frac{1+x}{\sqrt{1-x}} dx$$

$$= \int_2^1 \frac{2-u}{\sqrt{u}} - du$$

$$= \int_1^2 (2-u)\, u^{-\frac{1}{2}} du$$

$$= \int_1^2 \left(2u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \left[4u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^2$$

$$= \left(4\sqrt{2} - \frac{2}{3} \times 2\sqrt{2}\right) - \left(4 - \frac{2}{3}\right)$$

$$= \frac{8\sqrt{2}}{3} - \frac{10}{3}$$

$$I = \frac{2}{3}\left(4\sqrt{2} - 5\right)$$

Award 1 mark for the substitution.
Award 1 mark for changing terminals.
Award 1 mark for the correct definite integral.
Award 1 mark for the final correct answer.

Question 5

$$\int_0^5 (2x-1)\sqrt{2x+1}\,dx$$

Let $u = 2x+1, 2x = u-1, 2x-1 = u-2, \dfrac{du}{dx} = 2$

Terminals, when $x = 5, u = 11$ and when $x = 1, u = 3$

$$\int_0^5 (2x-1)\sqrt{2x+1}\,dx = \frac{1}{2}\int_3^{11} (u-2)\sqrt{u}\,du$$

$$= \frac{1}{2}\int_3^{11} \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$$

Question 6

$$I = \int_1^2 x^2\sqrt{2-x}\,dx$$

$$u = 2 - x\frac{du}{dx} = -1$$

$$x = 2-u,\ x^2 = (2-u)^2 = 4 - 4u + u^2$$

$$x = 2 \Rightarrow u = 0,\ x = 1 \Rightarrow u = 1$$

$$I = -\int_1^0 \left(4 - 4u + u^2\right) u^{\frac{1}{2}} du$$

$$I = -\int_1^0 \left(4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du$$

Question 7

$$I = \int_0^1 \left(x^2\sqrt{3x+1}\right) dx$$

Let $u = 3x+1,\ \dfrac{du}{dx} = 3,\ \dfrac{dx}{du} = \dfrac{1}{3}$

$$x = \frac{1}{3}(u-1),\ x^2 = \frac{1}{9}(u-1)^2$$

Terminals, $x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 4$

$$I = \int_1^4 \frac{1}{9}(u-1)^2\sqrt{u}\frac{1}{3}du$$

$$= \frac{1}{27}\int_1^4 \left(u^2 - 2u + 1\right) u^{\frac{1}{2}} du$$

$$I = \frac{1}{27}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

Question 8

$$cx + e = u \quad \Rightarrow x = \frac{u - e}{c}$$

$$c.dx = du \quad \Rightarrow dx = \frac{du}{c}$$

$$\int ax\sqrt{(cx + e)}\,dx = \int a\left(\frac{u - e}{c}\right)u^{\frac{1}{2}}\frac{du}{c} \text{ [1 mark]}$$

$$= \left(\frac{a}{c^2}u^{\frac{3}{2}} - \frac{a}{c^2}eu^{\frac{1}{2}}\right)du$$

$$= \frac{2a}{5c^2}u^{\frac{5}{2}} - \frac{2ae}{3c^2}u^{\frac{3}{2}}$$

$$= \frac{2a}{5c^2}(cx + e)^{\frac{5}{2}} - \frac{2ae}{3c^2}(cx + e)^{\frac{3}{2}} \text{ [1 mark]}$$

Question 9

Let $u = x - 2$

$$\frac{du}{dx} = 1 \text{ and } 2x = 2u + 4$$

$$\int 2x\sqrt{x - 2}dx = \int (2u + 4)\sqrt{u}du$$

$$\int \left(2u\sqrt{u} + 4\sqrt{u}\right)du = \int \left(2u^{\frac{3}{2}} + 4\sqrt{u}\right)du$$

Question 10

Let $u = x + 2$

$$\frac{du}{dx} = 1 \text{ and } 3x = 3u - 6$$

$$\int \frac{3x}{x + 2}dx = \int \frac{3u - 6}{u}du$$

$$= \int \left(3 - \frac{6}{u}\right)du$$

Question 11

$$\int_0^{\frac{1}{a}} \frac{x}{ax + b}dx \text{ let } u = ax + b \;\; \frac{du}{dx} = a$$

$$x = \frac{1}{a}(u - b) \text{ when } x = \frac{1}{a},\ u = 1 + b \text{ and when } x = 0, u = b$$

$$\int_0^{\frac{1}{a}} \frac{x}{ax + b}dx = \frac{1}{a^2}\int_b^{1+b}\left(\frac{u - b}{u}\right)dx = \frac{1}{a^2}\int_b^{1+b}\left(1 - \frac{b}{u}\right)dx$$

$$= \frac{1}{a^2}\left[u - b\log_e(u)\right]_b^{1+b} = \frac{1}{a^2}\left(1 + b - b\log_e(1 + b)\right) - \left(b - b\log_e(b)\right)$$

$$= \frac{1}{a^2}\left(1 + b\log_e\left(\frac{b}{b + 1}\right)\right)$$

7.4 Non-linear substitutions

Question 1

$$\int_0^{\frac{\pi}{6}} \tan^2(x)\sec^2(x)\,dx$$

Let $u = \tan(x)$.

$$\frac{du}{dx} = \sec^2(x)$$

Terminals: when $x = \frac{\pi}{6}$, $u = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ and when $x = 0, u = \tan(0) = 0$.

$$\int_0^{\frac{\pi}{6}} \tan^2(x)\ \sec^2(x)\,dx = \int_0^{\frac{1}{\sqrt{3}}} u^2 du$$

The correct answer is **E**.

Question 2

$$I = \int_a^b \left(x^3 e^{2x^4}\right) dx$$

Let $u = x^4$: $\frac{du}{dx} = 4x^3 \Rightarrow \frac{dx}{du} = \frac{1}{4x^3}$.

Terminals: $x = a \Rightarrow u = a^4$, $x = b \Rightarrow u = b^4$

$$I = \int_{a^4}^{b^4} x^3 e^{2u} \frac{1}{4x^3} du = \frac{1}{4}\int_{a^4}^{b^4} \left(e^{2u}\right) du$$

The correct answer is **D**.

Question 3

$$\int_{e^3}^{e^4} \frac{1}{x\log_e(x)} dx \text{ Let } u = \log_e(x), \frac{du}{dx} = \frac{1}{x}$$

Terminals $x = e^4$, $u = \log_e\left(e^4\right) = 4$ and $x = e^3$,
$u = \log_e\left(e^3\right) = 3$

$$\int_{e^3}^{e^4} \frac{1}{x\log_e(x)} dx = \int_3^4 \frac{1}{u} du = \int_a^b \frac{1}{u} du, \text{ so } a = 3 \text{ and } b = 4.$$

The correct answer is B.

Question 4

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(x)}{\sec^2(x) - 3\tan(x) + 1} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(x)}{1 + \tan^2(x) - 3\tan(x) + 1} dx \text{ using } 1 + \tan^2(x) = \sec^2(x)$$

Let $u = \tan(x)$.

$$\frac{du}{dx} = \sec^2(x)$$

Terminals $x = \frac{\pi}{3}, u = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}, x = \frac{\pi}{4}, u = \tan\left(\frac{\pi}{4}\right) = 1$

$$I = \int_1^{\sqrt{3}} \frac{1}{1 + u^2 - 3u + 1} du$$

$$= \int_1^{\sqrt{3}} \frac{1}{u^2 - 3u + 2} du$$

$$\frac{1}{u^2 - 3u + 2} = \frac{1}{(u-1)(u-2)}$$

$$= \frac{A}{u-1} + \frac{B}{u-2}$$

$$= \frac{A(u-2) + B(u-1)}{(u-1)(u-2)}$$

$$= \frac{u(A+B) - 2A - B}{u^2 - 3u + 2}$$

$A + B = 0, 2A + B = -1 \Rightarrow A = -1, B = 1$

$$I = \int_1^{\sqrt{3}} \left(\frac{1}{u-2} - \frac{1}{u-1}\right) du$$

Question 5

$$\int_0^{\frac{\pi}{6}} \tan^2(x) \sec^2(x)\, dx$$

Let $u = \tan(x)$.

$$\frac{du}{dx} = \sec^2(x)$$

Terminals: when $x = \frac{\pi}{6}, u = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

and when $x = 0, u = \tan(0) = 0$.

$$\int_0^{\frac{\pi}{6}} \tan^2(x) \sec^2(x)\, dx = \int_0^{\frac{1}{\sqrt{3}}} u^2 du$$

Question 6

$$I = \int_a^b \left(x^3 e^{2x^4}\right) dx$$

Let $u = x^4$: $\frac{du}{dx} = 4x^3 \Rightarrow \frac{dx}{du} = \frac{1}{4x^3}$.

Terminals: $x = a \Rightarrow u = a^4, x = b \Rightarrow u = b^4$

$$I = \int_{a^4}^{b^4} x^3 e^{2u} \frac{1}{4x^3} du = \frac{1}{4} \int_{a^4}^{b^4} \left(e^{2u}\right) du$$

Question 7

$$\int_0^2 \frac{dx}{(x+2)\sqrt{x+1}}$$

Let $u = \sqrt{x+1}$.

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$$
$$= \frac{1}{2u}$$

$x + 1 = u^2, (x+2) = u^2 + 1$

Terminals $x = 2, u = \sqrt{3}$ and $x = 0, u = 1$

$$\int_0^2 \frac{dx}{(x+2)\sqrt{x+1}} = \int_0^{\sqrt{3}} \frac{1}{\left(u^2+1\right)u}\frac{dx}{du}du$$
$$= \int_0^{\sqrt{3}} \frac{2u}{\left(u^2+1\right)u}du$$
$$= 2\int_0^{\sqrt{3}} \frac{1}{u^2+1}du$$

Question 8

Let $u = 1 - e^{-3x}, \dfrac{du}{dx} = 3e^{-3x}$

$$\int \frac{e^{-3x}}{1-e^{-3x}}dx$$
$$= \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\log_e|u| + c$$
$$= \frac{1}{3}\log_e\left|e^{-3x} - 1\right| + c$$

Question 9

$$\int \frac{x^3}{4x^2+9}dx = \int \frac{x \times x^2}{4x^2+9}dx$$

Let $u = 4x^2 + 9, \dfrac{du}{dx} = 8x$ and $x = \dfrac{1}{4}(u-9)$

$$= \frac{1}{8}\int \frac{\frac{1}{4}(u-9)}{u}du = \frac{1}{32}\int \frac{(u-9)}{u}du = \frac{1}{32}\int \left(1 - \frac{9}{u}\right)du$$
$$= \frac{1}{32}\left(u - 9\log_e(u)\right) + C$$
$$= \frac{1}{32}\left(4x^2+9\right) - \frac{9}{32}\log_e\left(4x^2+9\right) + C$$
$$= -\frac{1}{32}\left(9\log_e\left(4x^2+9\right) - 4x^2\right) + c \text{ let } c = C + \frac{9}{32}$$

Question 10

Let $u = \cos(3x)$, $\dfrac{du}{dx} = -3\sin(3x)$

$$\int \tan(3x)\,dx = \int \frac{\sin(3x)}{\cos(3x)}dx$$

$$= -\frac{1}{3}\int \frac{1}{u}du = -\frac{1}{3}\log_e(u) + c$$

$$= -\frac{1}{3}\log_e(\cos(3x)) + c$$

Question 11

$$\int \frac{x^3}{\sqrt{3x^2+4}}dx = \int \frac{x \times x^2}{\sqrt{3x^2+4}}dx$$

Let $u = 3x^2 + 4$, $\dfrac{du}{dx} = 6x$ and $x^2 = \dfrac{1}{3}(u-4)$

$$= \frac{1}{6}\int \frac{\frac{1}{3}(u-4)}{\sqrt{u}}du$$

$$= \frac{1}{18}\int \frac{(u-4)}{\sqrt{u}}du$$

7.5 Integrals of powers of trigonometric functions

Question 1

$$\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{3x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}}(1 + \cos(3x))dx$$

$$= \frac{1}{2}\left[x + \frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{3}} = \frac{1}{2}\left(\frac{\pi}{3} + \frac{1}{3}\sin(\pi) - \left(0 + \frac{1}{3}\sin(0)\right)\right)$$

$$= \frac{\pi}{6}$$

The correct answer is **B**.

Question 2

Let $u = \tan(3x)$, $\dfrac{du}{dx} = 3\sec^2(3x)$

$$\int \sec^2(3x)\tan^2(3x)\,dx$$

$$= \frac{1}{3}\int u^2du = \frac{1}{9}u^2 + c$$

$$= \frac{1}{9}\tan^3(3x) + c$$

The correct answer is **D**.

Question 3

$$\int_0^p \left(\sin^2(5x) - \cos^2(5x)\right) dx = -\int_0^p \cos(10x)\ dx\ [\mathbf{1\ mark}]$$

$$= \left[-\frac{1}{10}\sin(10x)\right]_0^p$$

$$= -\frac{1}{10}\sin(10p)\ \ [\mathbf{1\ mark}]$$

Question 4

$$A = 3\int_0^{\pi} \sin^3(x)\, dx \text{ by symmetry}$$

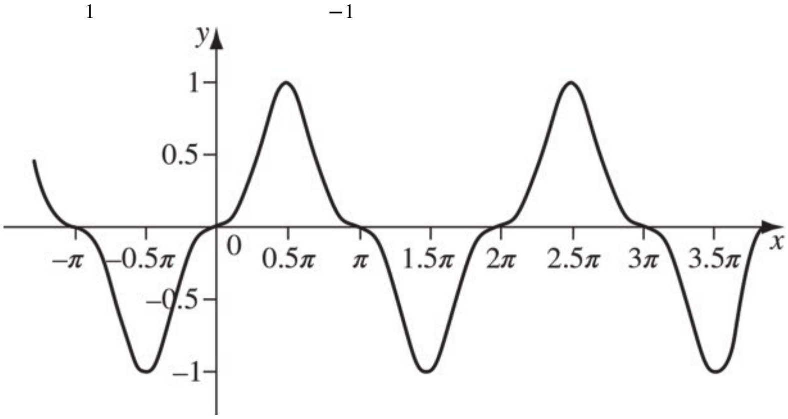
$$A = 3\int_0^{\pi} \sin(x)\sin^2(x)\, dx = 3\int_0^{\pi} \sin(x)\left(1 - \cos^2(x)\right) dx$$

Let $u = \cos(x)$ and $\frac{du}{dx} = -\sin(x)$

Terminals when $x = 0$ and $u = \cos(0) = 1$

and when $x = \pi$ and $u = \cos(\pi) = -1$

$$A = 3\int_1^{-1} -\left(1 - u^2\right) du = 3\int_{-1}^{1} \left(1 - u^2\right) du$$



***VCAA Assessment Report* note:**

This question was poorly done; only 23% of students chose the correct answer, which was the lowest percentage correct for a multiple choice question on this exam. 52% incorrectly chose E.

7.6 Integrals involving inverse trigonometric functions

Question 1

$$\int_0^1 \frac{2x+1}{x^2+1}dx = \int_0^1 \frac{2x}{x^2+1}dx + \int_0^1 \frac{1}{x^2+1}dx$$

$$= \left[\log_e\left(x^2+1\right) + \tan^{-1}(x)\right]_0^1$$

$$= \left(\log_e(2) + \tan^{-1}(1)\right) - \left(\log_e(1) + \tan^{-1}(0)\right)$$

$$= \log_e(2) + \frac{\pi}{4}$$

Award 1 mark for two integrals.
Award 1 mark for either correct integrals and evaluating.
Award 1 mark for final correct answer.

Question 2

Let $u = \sqrt{5}x, \ \dfrac{du}{dx} = \sqrt{5}$

$$\int \frac{dx}{2+5x^2}$$

$$= \frac{1}{\sqrt{5}}\int \frac{du}{2+u^2} = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{5}x}{\sqrt{2}}\right) + c$$

$$= \frac{\sqrt{10}}{10}\tan^{-1}\left(\frac{\sqrt{10}x}{2}\right) + c$$

The correct answer is **D**.

Question 3

$$\int \frac{\frac{a}{b}}{\sqrt{(b^2 - a^2x^2)}}dx = \int \frac{\frac{a}{b}}{\sqrt{a^2\left(\left(\frac{b^2}{a^2}\right) - x^2\right)}}dx \text{ [1 mark]}$$

$$= \int \frac{\frac{1}{b}}{\sqrt{\left(\frac{b^2}{a^2}\right) - x^2}}dx$$

$$= \frac{1}{b}\sin^{-1}\left(\frac{ax}{b}\right) \text{ [1 mark]}$$

Question 4

$$\frac{1}{\sqrt{(1-3x^2)}} = \frac{1}{\sqrt{\left(1 - \left(\sqrt{3}x\right)^2\right)}} \text{ [1 mark]}$$

$$\int\left(\frac{1}{\sqrt{\left(1 - \left(\sqrt{3}x\right)^2\right)}}\right) = \frac{1}{\sqrt{3}}\sin^{-1}\left(\sqrt{3}x\right) \text{ [1 mark]}$$

Question 5

$$\int \frac{\cos(x)}{\sqrt{\left(1 - \sin^2(x)\right)}}dx = \int \frac{\cos(x)}{\sqrt{\cos^2(x)}}dx \text{ [1 mark]}$$

$$= \int \frac{\cos(x)}{\cos(x)}dx$$

$$= \int 1dx$$

$$= x \text{ [1 mark]}$$

Question 6

Let $u = \sqrt{7}x, \dfrac{du}{dx} = \sqrt{7}$

$$\int \frac{dx}{\sqrt{(2 - 7x^2)}}$$

$$= \frac{1}{\sqrt{7}} \int \frac{du}{\sqrt{2 - u^2}} = \frac{1}{\sqrt{7}} \sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$

$$= \frac{\sqrt{7}}{7} \sin^{-1}\left(\frac{\sqrt{7}x}{\sqrt{2}}\right) + c$$

$$= \frac{\sqrt{7}}{7} \sin^{-1}\left(\frac{\sqrt{14}x}{2}\right) + c$$

7.7 Integrals involving partial fractions

Question 1

$$\frac{2x^2 + 3x + 1}{(2x + 1)^3 \left(x^2 - 1\right)} = \frac{(2x + 1)(x + 1)}{(2x + 1)^3 (x + 1)(x - 1)}$$

$$= \frac{1}{(2x + 1)^2 (x - 1)} \quad x \neq \pm 1, -\frac{1}{2}$$

$$= \frac{A}{2x + 1} + \frac{B}{(2x + 1)^2} + \frac{C}{x - 1}$$

The correct answer is **D**.

Question 2

Using partial fractions

$$\frac{1}{x\left(1 + x^2\right)} = \frac{A}{x} + \frac{Bx + C}{1 + x^2}$$

$$= \frac{A\left(1 + x^2\right) + x(Bx + C)}{x\left(1 + x^2\right)}$$

$$= \frac{x^2 (A + B) + Cx + A}{x\left(1 + x^2\right)}$$

$A + B = 0,\ A = 1,\ C = 0,\ B = -1$

$$\int_1^{\sqrt{3}} \frac{1}{x\left(1 + x^2\right)} dx = \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{1 + x^2}\right) dx$$

$$= \left[\log_e |x| - \frac{1}{2}\log_e \left(1 + x^2\right)\right]_1^{\sqrt{3}}$$

$$= \left[\frac{1}{2}\log_e \left(\frac{x^2}{1 + x^2}\right)\right]_1^{\sqrt{3}}$$

$$= \left(\frac{1}{2}\log_e\left(\frac{3}{4}\right) - \frac{1}{2}\log_e\left(\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\log_e\left(\frac{3}{2}\right)$$

$$= \log_e\left(\sqrt{\frac{3}{2}}\right)$$

Award 1 mark for the correct partial fractions decomposition.
Award 1 mark for the solving for the coefficients.
Award 1 mark for the correct integration.
Award 1 mark for the correct final values of a and b.

Question 3

By partial fractions,

$$\frac{a}{x(x-a)} = \frac{A}{x} + \frac{B}{x-a} = \frac{A(x-a)+Bx}{x(x-a)} = \frac{x(A+B)-Aa}{x(x-a)}$$

(1) $A + B = 0$ (2) $-Aa = a \Rightarrow A = -1 \quad B = 1$

$$\int \frac{a}{x(x-a)}dx = \int\left(\frac{1}{x-a} - \frac{1}{x}\right)dx$$

$$= \log_e(x-a) - \log_e(x)$$

$$= \log_e\left(\frac{x-a}{x}\right) + c \quad \text{since } x > a > 0$$

The correct answer is **B**.

Question 4

$a \in R\backslash\{0\}$, $b \in R$, $b < 0$

$$\frac{1}{ax\left(x^2+b\right)} = \frac{1}{ax\left(x^2-(-b)\right)} \frac{1}{ax\left(x^2-\sqrt{|b|^2}\right)}$$

$$= \frac{A}{x} + \frac{B}{x+\sqrt{|b|}} + \frac{C}{x-\sqrt{|b|}}$$

Question 5

$$\frac{2x^2+3x+1}{(2x+1)^3\left(x^2-1\right)} = \frac{(2x+1)(x+1)}{(2x+1)^3(x+1)(x-1)}$$

$$= \frac{1}{(2x+1)^2(x-1)} \quad x \neq \pm, -\frac{1}{2}$$

$$= \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-1}$$

Question 6

By partial fractions

$$\frac{x-5}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)}$$

$x - 5 = A \quad (x-2) + B(x-3)$

let $x = 2 \quad -3 = -B \quad \Rightarrow B = 3$

let $x = 3 \quad -2 = A \quad \Rightarrow A = -2$

$$\int_0^1 \frac{x-5}{x^2-5x+6}dx = \int_0^1 \left(\frac{3}{x-2} - \frac{2}{x-3}\right) dx$$

$$= \left[3\log_e|x-2| - 2\log_e|x-3|\right]_0^1$$

$$= \left(3\log_e|-1| - 2\log_e|-2|\right) - \left(3\log_e|-2| - 2\log_e|-3|\right)$$

$$= -2\log_e(2) - 3\log_e(2) + 2\log_e(3)$$

$$= \log_e\left(\frac{9}{4\times 8}\right)$$

$$= \log_e\left(\frac{9}{32}\right)$$

Award 1 mark for using partial fractions.
Award 1 mark for finding constants.
Award 1 mark for integration.
Award 1 mark for final correct answer.

***VCAA Assessment Report* note:**

Most students identified the need to use partial fractions. However, quite a few students were unable to successfully factorise the quadratic in the denominator, often giving $x^2 - 5x + 6 = (x-6)(x+1)$. Most students who had the correct factors managed to obtain the correct partial fractions. The most common error was the lack of modulus signs leading to the logarithms of negative numbers. A number of students were unable to apply the log laws and simplified incorrectly.

For instance, the following 'simplification' occurred often: $2\log_e(3) - 5\log_e(2) = \frac{2}{5}\log_e\left(\frac{3}{2}\right)$.

A significant number of students increased the work required either by first writing the given fraction as $\frac{x-5}{x^2-5x+6} = \frac{x}{x^2-5x+6} - \frac{5}{x^2-5x+6}$ and then using partial fractions on each term, or by $\frac{x-5}{x^2-5x+6} = \frac{1}{2}\left(\frac{2x-5}{x^2-5x+6}\right) - \frac{1}{2}\left(\frac{5}{x^2-5x+6}\right)$ using and then using substitution and partial fractions. Few students who used an unnecessarily complicated approach were successful.

Question 7

By partial fractions

$$\frac{x+1}{9-x^2} = \frac{A}{3-x} + \frac{B}{3+x}$$

$$= \frac{A(3+x) + B(3-x)}{(3-x)(3+x)}$$

$$= \frac{x(A-B) + 3(A+B)}{9-x^2}$$

$\Rightarrow A - B = 1$ and $3(A+B) = 1$ [**1 mark**]

$A = \frac{2}{3},\ B = -\frac{1}{3}$ [**1 mark**]

$$\int \left(\frac{1+x}{9-x^2}\right) dx = \frac{1}{3}\int \left(\frac{2}{3-x} - \frac{1}{3+x}\right) dx$$

$$= \frac{1}{3}\left[-2\log_e|3-x| - \log_e|3+x|\right]$$

$$= -\frac{1}{3}\log_e\left((3-x)^2|3+x|\right) + c$$ [**1 mark**]

VCAA Assessment Report **note:**
This question was quite well done. Most students recognized that partial fractions were required. Some changed the entire fraction into partial fractions immediately, while others split the original fraction first, using substitution on part of it. The latter method led to the correct answer but was inefficient as partial fractions were still required. Typical errors seen included using denominators of $9 - x$ and $9 + x$ in the partial fractions, integrating $\frac{1}{3-x}$ to give $|3-x|$ (missing the negative sign) and omitting modulus signs. Some students first changed the denominator to $x^2 - 9$ and many of these subsequently made sign errors. A few students justified removing modulus signs due to $x \in R \backslash \{-3, 3\}$. There was often an incorrect attempt at simplification of the two terms in the answer a the end (and missing out on the final mark); for example $-\frac{2}{3}\ln|3-x| - \frac{1}{3}\ln|3+x| = -\left(\frac{2}{3}\ln|3-x| - \frac{1}{3}\ln|3+x|\right)$. A small number of students produced answers involving arctan (x).

Question 8

$A(x+2) + B(x+4) = 4x - 6$

$(A+B)x + 2A + 4B = 4x - 6$

$A + B = 4$

$2A + 4B = -6$

$2A + 2B = 8$

$2B = -14$

$B = -7$

$A - 7 = 4$

$A = 11$

Question 9

$4(x-2) - 2(x-3)$

$= 4x - 8 - 2x + 6$

$= 2x - 2$

$A = 2$ and $B = -2$

Question 10

$$\frac{A}{(3x+1)} + \frac{B}{(x+1)}$$

$A(x+1) + B(3x+1) = 2x + 1$

$Ax + A + 3Bx + B = 2x + 1$

$(A+3B)x + A + B = 2x + 1$

$A + 3B = 2$ and $A + B = 1$

$A = B = \frac{1}{2}$

$$\frac{1}{2(3x+1)} + \frac{1}{2(x+1)}$$

$$\frac{1}{6x+2} + \frac{1}{2x+2}$$

Question 11

$$\frac{10}{4x^2 - 25} = \frac{A}{2x+5} + \frac{B}{2x-5}$$

$$= \frac{A(2x-5) + B(2x+5)}{4x^2 - 25}$$

$$= \frac{2x(A+B) + 5(B-A)}{4x^2 - 25}$$

(1) $A + B = 0$ (2) $5(B - A) = 10 \Rightarrow B - A = 2$

Add $2B = 2, B = 1$ and $A = -1$.

$$\frac{10}{4x^2 - 25} = \frac{1}{2x - 5} - \frac{1}{2x + 5}$$

Question 12

$$\frac{1}{abx^2 + (pb + qa)x + pq} = \frac{1}{(ax + p)(bx + q)} = \frac{A}{ax + p} + \frac{B}{bx + q}$$

$$= \frac{A(bx + q) + B(ax + p)}{(ax + p)(bx + q)} = \frac{x(Ab + Ba) + qA + Bp}{(ax + p)(bx + q)}$$

(1) $Aq + Bp = 1$ (2) $Ab + Ba = 0$ to eliminate A

$b \times (1)$ $Abq + Bbp = b$ $q \times (2)$ $Aqb + Baq = 0$ subtracting gives

$(aq - bp)B = -b \Rightarrow B = \dfrac{-b}{aq - bp}$

(1) $Aq + Bp = 1$ (2) $Ab + Ba = 0$ to eliminate B

$a \times (1)$ $Aaq + Bap = a$ $p \times (2)$ $Abp + Bap = 0$ subtracting gives

$(aq - bp)A = a \Rightarrow A = \dfrac{a}{aq - bp}$

$$\frac{1}{abx^2 + (pb + qa)x + pq} = \frac{1}{aq - pb}\left(\frac{a}{ax + p} - \frac{b}{bx + q}\right) \text{ provided that } aq - bp \neq 0$$

Question 13

$A(x + 4) + B = x + 3$

$Ax + 4A + B = x + 3$

$A = 1 \Rightarrow B = -1$

Question 14

$1(x - 3) + 6 = Ax + B$

$qx - 3 + 6 = Ax + B$

$x + 3 = Ax + B$

$A = 1$ and $B = 3$

Question 15

$$\frac{A}{(x - 2)^2} - \frac{B}{(x - 2)}$$

$A + B(x - 2) = 2 - 3x$

$A + Bx - 2B = 2 - 3x$

$A - 2B = 2$ and $Bx = -3x$

$A = -4$ and $B = -3$

$$\frac{-4}{(x - 2)^2} - \frac{3}{(x - 2)}$$

Question 16

$$\frac{2x}{(2x + 5)^2} = \frac{A}{2x + 5} + \frac{B}{(2x + 5)^2} = \frac{A(2x + 5) + B}{(2x + 5)^2}$$

$$= \frac{2Ax + 5A + B}{(2x + 5)^2}$$

$2A = 2 \Rightarrow A = 1$

$B + 5A = 0 \Rightarrow B = -5$

$$\frac{2x}{(2x + 5)^2} = \frac{1}{2x + 5} - \frac{5}{(2x + 5)^2}$$

Question 17

$$\frac{ax+b}{(px+q)^2}=\frac{A}{px+q}+\frac{B}{(px+q)^2}=\frac{A(px+q)+B}{(px+q)^2}=\frac{pAx+B+Aq}{(px+q)^2}$$

$$pA=a\Rightarrow A=\frac{a}{p}$$

$$B+Aq=b\Rightarrow B=b-Aq=b-\frac{aq}{p}=\frac{bp-aq}{p}=-\frac{aq-bp}{p}$$

$$\frac{ax+b}{(px+q)^2}=\frac{1}{p}\left(\frac{a}{px+q}-\frac{aq-bp}{(px+q)^2}\right)\quad p\neq 0$$

Question 18

$$(Ax+B)(x-1)+C\left(x^2+2\right)=x-2$$

$$Ax^2-Ax+Bx-B+Cx^2+2C=x-2$$

$$A+C=0, B-A=1, 2C-B-2$$

$$B+C=1$$

$$2C-B=-2$$

$$3C=-1$$

$$C=-\frac{1}{3}, B=\frac{4}{3}, A=\frac{1}{3}$$

Question 19

$$(Ax+B)(x-1)+C\left(x^2+4\right)=5-20x$$

$$Ax^2-Ax+Bx-B+Cx^2+4C=5-20x$$

$$A+C=0;\ B-A=-20;\ 4C-B=5$$

$$B+C=-20;\ 4C-B=5;$$

$$5C=-15$$

$$C=-3;\ B=-17;\ A=3$$

$$\frac{3x-17}{\left(x^2+4\right)}-\frac{3}{(x-1)}$$

Question 20

$$\frac{7x^2-6x+4}{3x\left(x^2+4\right)}=\frac{1}{3}\left(\frac{A}{x}+\frac{Bx+C}{x^2+4}\right)=\frac{1}{3}\left(\frac{A\left(x^2+4\right)+x(Bx+C)}{x\left(x^2+4\right)}\right)=\frac{x^2(A+B)+Cx+4A}{3x\left(x^2+4\right)}$$

$$A+B=7\quad C=-6\quad 4A=A\quad\Rightarrow A=1\quad B=6$$

$$\frac{7x^2-6x+4}{3x\left(x^2+4\right)}=\frac{1}{3}\left(\frac{1}{x}+\frac{6x-6}{x^2+4}\right)=\frac{1}{3x}+\frac{2x-2}{x^2+4}$$

Question 21

We have repeated and non-linear factors:

$$\frac{1}{\left(x^2+a^2\right)(x+b)^2}=\frac{A}{x+b}+\frac{B}{(x+b)^2}+\frac{Cx+D}{x^2+a^2}$$

Question 22

$$\frac{2}{x(4-x)}=\frac{1}{2}\left[\frac{1}{x}+\frac{1}{4-x}\right]\quad \textbf{[1 mark]}$$

$$\frac{1}{2}\int\left[\frac{1}{x}+\frac{1}{4-x}\right]dx=\frac{1}{2}\left[\ln|x|-\ln|(4-x)|\right]$$

$$=\frac{1}{2}\ln\left|\frac{x}{4-x}\right|\quad \textbf{[1 mark]}$$

Question 23

By partial fractions, $\dfrac{a}{x(x-a)} = \dfrac{A}{x} + \dfrac{B}{x-a} = \dfrac{A(x-a)+Bx}{x(x-a)} = \dfrac{x(A+B)-Aa}{x(x-a)}$

(1) $A+B=0$ (2) $-Aa=a \Rightarrow A=-1 \quad B=1$

$$\int \frac{a}{x(x-a)}dx = \int \left(\frac{1}{x-a} - \frac{1}{x}\right) dx$$

$= \log_e(x-a) - \log_e(x)$

$= \log_e\left(\dfrac{x-a}{x}\right) + c$ since $x > a > 0$

Question 24

$$\int \frac{a^2x^2+b^2}{a^2x^2-b^2}dx = \int \frac{a^2x^2-b^2+2b^2}{a^2x^2-b^2}dx = \int \left(1+\frac{2b^2}{a^2x^2-b^2}\right)dx$$

By partial fractions, $\dfrac{2b^2}{a^2x^2-b^2} = \dfrac{A}{ax+b} + \dfrac{B}{ax-b} = \dfrac{A(ax-b)+B(ax+b)}{(ax+b)(ax-b)} = \dfrac{ax(A+B)+b(B-A)}{x(x-a)}$

(1) $B-A=2b$ (2) $A+B=0$ add $\Rightarrow B=b \; A=-b$

$$\int \left(1+\frac{2b^2}{a^2x^2-b^2}\right)dx = \int \left(1+\frac{b}{ax-b} - \frac{b}{ax-b}\right)dx$$

$= x + \dfrac{b}{a}\log_e(|ax-b|) - \dfrac{b}{a}\log_e(|ax+b|)$

$= x + \dfrac{b}{a}\log_e\left(\left|\dfrac{ax-b}{ax+b}\right|\right)$

7.8 Review

Question 1

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(x)}{\sec^2(x) - 3\tan(x) + 1}dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(x)}{1+\tan^2(x) - 3\tan(x) + 1}dx$$

using $1+\tan^2(x) = \sec^2(x)$

Let $u = \tan(x)$.

$\dfrac{du}{dx} = \sec^2(x)$

Terminals

$x = \dfrac{\pi}{3},\ u = \tan\left(\dfrac{\pi}{3}\right) = \sqrt{3},\ x = \dfrac{\pi}{4},\ u = \tan\left(\dfrac{\pi}{4}\right) = 1$

$$I = \int_1^{\sqrt{3}} \frac{1}{1 + u^2 - 3u + 1} du$$

$$= \int_1^{\sqrt{3}} \frac{1}{u^2 - 3u + 2} du$$

$$\frac{1}{u^2 - 3u + 2} = \frac{1}{(u-1)(u-2)}$$

$$= \frac{A}{u-1} + \frac{B}{u-2}$$

$$= \frac{A(u-2) + B(u-1)}{(u-1)(u-2)}$$

$$= \frac{u(A+B) - 2A - B}{u^2 - 3u + 2}$$

$A + B = 0,\ 2A + B = -1 \Rightarrow A = -1,\ B = 1$

$$I = \int_1^{\sqrt{3}} \left(\frac{1}{u-2} - \frac{1}{u-1} \right) du$$

The correct answer is **C**.

Question 2

$$\int \tan(2x)\, dx = \int \frac{\sin(2x)}{\cos(2x)} dx$$

Let $u = \cos(2x)$, $\dfrac{du}{dx} = -2\sin(2x)$

$$\int \tan(2x)\, dx = -\frac{1}{2} \int \frac{1}{u} du = -\frac{1}{2} \log_e(|u|) + c$$

$$= -\frac{1}{2} \log_e(|\cos(2x)|) + c$$

$$= \frac{1}{2} \log_e\left(\frac{1}{|\cos(2x)|} \right) + c$$

$$= \frac{1}{2} \log_e(|\sec(2x)|) + c$$

Award 1 mark for the correct substitution.
Award 1 mark for the correct result.

Question 3

a. $f(x) = 3x \tan^{-1}(2x)$, range $f = [0, \infty]$

Award 1 mark for the correct range.

b. $f'(x) = 3\tan^{-1}(2x) + 3x \times \dfrac{2}{1 + 4x^2}$, using the product rule

$$= 3\tan^{-1}(2x) + \frac{6x}{1 + 4x^2} \quad \textbf{[1 mark]}$$

c. $\dfrac{d}{dx}\left[3x \tan^{-1}(2x)\right] = 3\tan^{-1}(2x) + \dfrac{6x}{1 + 4x^2}$

$$\int\left(3\tan^{-1}(2x)+\frac{6x}{1+4x^2}\right)dx=3x\tan^{-1}(2x)$$

$$\int 3\tan^{-1}(2x)\,dx+\int\frac{6x}{1+4x^2}\,dx=3x\tan^{-1}(2x)$$

$$\int 3\tan^{-1}(2x)\,dx=3x\tan^{-1}(2x)-\int\frac{6x}{1+4x^2}\,dx$$

$$3\int\tan^{-1}(2x)\,dx=3x\tan^{-1}(2x)-\frac{3}{4}\log_e\left(1+4x^2\right)$$

$$\int\tan^{-1}(2x)\,dx=x\tan^{-1}(2x)-\frac{1}{4}\log_e\left(1+4x^2\right)$$

$$A=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\tan^{-1}(2x)\,dx$$

$$=\left[x\tan^{-1}(2x)-\frac{1}{4}\log_e\left(1+4x^2\right)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$=\left(\frac{\sqrt{3}}{2}\tan^{-1}\left(\sqrt{3}\right)-\frac{1}{4}\log_e(4)\right)-\left(\frac{1}{2}\tan^{-1}(1)-\frac{1}{4}\log_e(2)\right)$$

$$=\left(\frac{\sqrt{3}}{2}\times\frac{\pi}{3}-\frac{1}{4}\log_e(4)\right)-\left(\frac{1}{2}\times\frac{\pi}{4}-\frac{1}{4}\log_e(2)\right)$$

$$=\pi\left(\frac{\sqrt{3}}{6}-\frac{1}{8}\right)+\frac{1}{4}\log_e\left(\frac{1}{2}\right)$$

$$=\frac{\left(4\sqrt{3}-3\right)\pi}{24}-\frac{1}{4}\log_e(2)$$

Award 1 mark for the correct definite integral for the area.
Award 1 mark for deducing the correct antiderivative.
Award 1 mark for the correct final area.

Question 4

$$\int\frac{a^2x^2+b^2}{a^2x^2-b^2}dx=\int\frac{a^2x^2-b^2+2b^2}{a^2x^2-b^2}dx=\int\left(1+\frac{2b^2}{a^2x^2-b^2}\right)dx$$

By partial fractions,

$$\frac{2b^2}{a^2x^2-b^2}=\frac{A}{ax+b}+\frac{B}{ax-b}$$

$$=\frac{A(ax-b)+B(ax+b)}{(ax+b)(ax-b)}=\frac{ax(A+B)+b(B-A)}{(ax+b)(ax-b)}$$

(1) $B - A = 2b$ (2) $A + B = 0$ add $\Rightarrow B = b \;\; A = -b$

$$\int \left(1 + \frac{2b^2}{a^2x^2 - b^2}\right) dx = \int \left(1 + \frac{b}{ax - b} - \frac{b}{ax + b}\right) dx$$

$$= x + \frac{b}{a}\log_e(|ax - b|) - \frac{b}{a}\log_e(|ax + b|)$$

$$= x + \frac{b}{a}\log_e\left(\left|\frac{ax - b}{ax + b}\right|\right)$$

The correct answer is **C**.

Question 5

$$\int_a^b \left(\operatorname{cosec}^2(3x)\, e^{3\cot(3x)}\right) dx = \left[-\frac{1}{9}e^{3\cot(3x)}\right]_a^b$$

$$= -\frac{1}{9}\left(e^{3\cot(3b)} - e^{3\cot(3a)}\right) \quad \textbf{[1 mark]}$$

Question 6

$$\int_0^1 \frac{2x + 1}{x^2 + 1} dx = \int_0^1 \frac{2x}{x^2 + 1} dx + \int_0^1 \frac{1}{x^2 + 1} dx$$

$$= \left[\log_e\left(x^2 + 1\right) + \tan^{-1}(x)\right]_0^1$$

$$= \left(\log_e(2) + \tan^{-1}(1)\right) - \left(\log_e(1) + \tan^{-1}(0)\right)$$

$$= \log_e(2) + \frac{\pi}{4}$$

Award 1 mark for two integrals.
Award 1 mark for either correct integrals and evaluating.
Award 1 mark for final correct answer.

Question 7

a. $\displaystyle\int \tan(2x)\, dx = \int \frac{\sin(2x)}{\cos(2x)} dx$

Let $u = \cos(2x)$, $\quad \dfrac{du}{dx} = -2\sin(2x)$

$$\int \tan(2x)\, dx = -\frac{1}{2}\int \frac{1}{u} du = -\frac{1}{2}\log_e(|u|) + c$$

$$= -\frac{1}{2}\log_e(|\cos(2x)|) + c = \frac{1}{2}\log_e\left(\frac{1}{|\cos(2x)|}\right) + c$$

$$= \frac{1}{2}\log_e(|\sec(2x)|) + c$$

Award 1 mark for the correct substitution.
Award 1 mark for the correct result.

VCAA Assessment Report **note:**

This question was answered reasonably well. There were many instances of poor choices of substitution, such as $u = \sin(2x)$, $u = \tan(2x)$, $u = \sec(2x)$ or $u = \cos(x)$ (after the use of double-angle formulas) rather than $u = \cos(2x)$. These attempts led to a more complicated solution and were rarely successful. Some students who used the correct substitution then made sign or arithmetical errors or did not use a modulus sign at the integration stage (although it often appeared at the end). Some attempted to use the tan double-angle formula, but this was rarely successful. A few students used differentiation as their method, but only a small number could correctly obtain the derivative of the right-hand scale. There were some unconvincing arguments, often due to insufficient steps shown.

b. i. $y = \pm\frac{\pi}{4}$ are the asymptotes of $f(x) = \arctan(x)$. **[1 mark]**

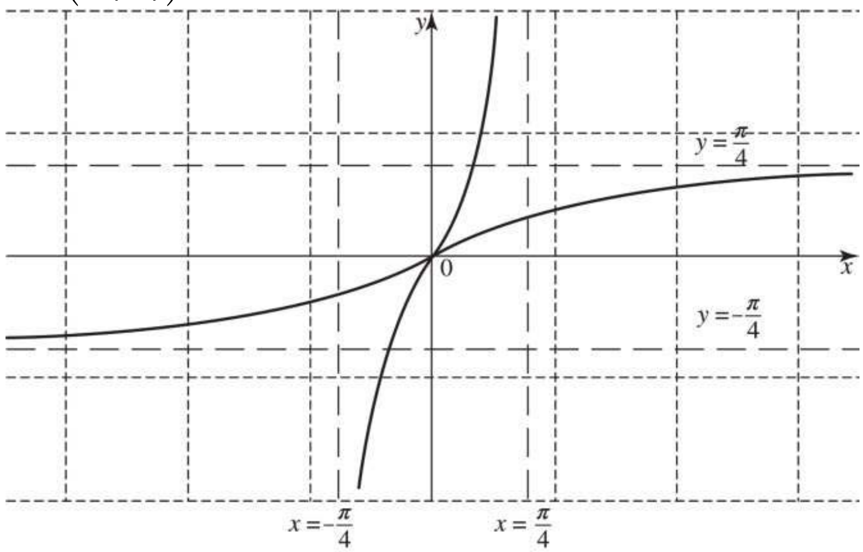
***VCAA Assessment Report* note:**

Most students answered this question correctly. Some gave $y = \pm\frac{\pi}{2}$ or $y = \pm\pi$. Others gave $\pm\frac{\pi}{4}$ rather than equations.

ii. $f(x) = y = \frac{1}{2}\tan^{-1}(x) \qquad f^{-1}: x = \frac{1}{2}\tan^{-1}(y)$

$2x = \tan^{-1}(y) \Rightarrow y = \tan(2x)$

$f^{-1}: \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to R,\ f^{-1}(x) = \tan(2x)$



[1 mark]

***VCAA Assessment Report* note:**

Students were expected to be able to reflect the given graph in the line $y = x$, or find the equation $y = \tan(2x)$ and sketch that directly. Typical errors included poor attempts at the shape of the inverse (sometimes graphed as $y = -\tan(2x)$), poor positioning of the vertical asymptotes, and either not labelling or incorrect labelling of the vertical asymptotes – for example, $y = \pm\frac{\pi}{4}$ – and drawing the graph beyond its domain. Asymptotic behaviour was lacking in some of the attempts, with curves sometimes moving away from the asymptotes.

c. $f\left(\sqrt{3}\right) = \frac{1}{2}\tan^{-1}\left(\sqrt{3}\right) = \frac{1}{2} \times \frac{\pi}{3} = \frac{\pi}{6}$ **[1 mark]**

***VCAA Assessment Report* note:**

This question was well answered. The main errors seen were $\frac{\pi}{3}$ and $\frac{\pi}{12}$, with some students not knowing the exact values.

d. $A_1 = \int_0^{\sqrt{3}} \frac{1}{2}\tan^{-1}(x)\,dx, \quad A_2 = \int_0^{\frac{\pi}{6}} \tan(2x)\,dx \quad \text{and} \quad A_1 + A_2 = \sqrt{3} \times \frac{\pi}{6}$

$$A_2 = \int_0^{\frac{\pi}{6}} \tan(2x)\,dx = \left[\frac{1}{2}\log\ e(|\sec(2x)|)\right]_0^{\frac{\pi}{6}}$$

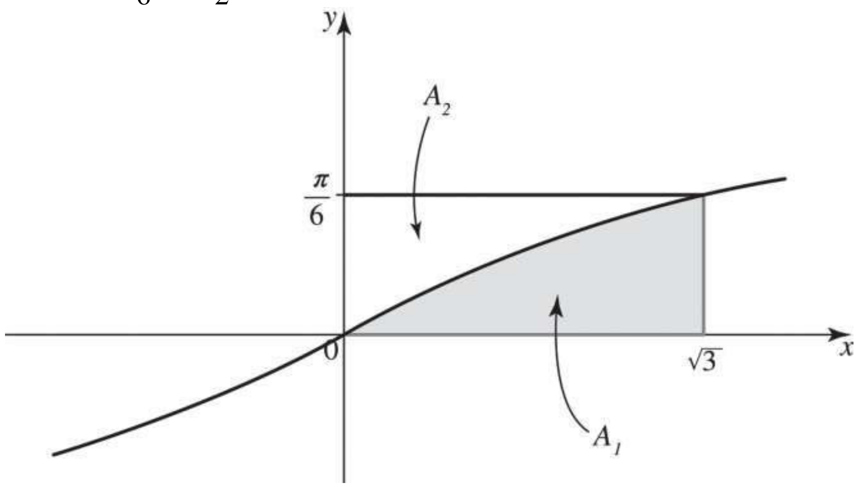
$$= \frac{1}{2}\left[\left(\log_e\left(\sec\left(\frac{\pi}{3}\right)\right)\right) - \left(\log_e(\sec(0))\right)\right]$$

$$= \frac{1}{2}\left[\left(\log_e\left(\frac{1}{\cos\left(\frac{\pi}{3}\right)}\right)\right) - \left(\log_e\left(\frac{1}{\cos(0)}\right)\right)\right]$$

$$= \frac{1}{2}\left[\log_e(2) - \log_e(1)\right]$$

$$= \frac{1}{2}\log_e(2)$$

So $A_1 = \frac{\sqrt{3}\pi}{6} - \frac{1}{2}\log_e(2)$



Award 1 mark for the correct integral and area of the square.
Award 1 mark for the correct area.

VCAA Assessment Report **note:**

Only a small proportion of students answered this question correctly. Some gave the correct expression for the area in terms of arctan, but no progress or poor attempts at integration (usually where the supposed antiderivative was actually the derivative) was made. Many gave an incorrect (incomplete) expression for the area in terms of the inverse $\left(\textit{omitting } \frac{\sqrt{3}\pi}{6}\right)$ or incorrect terminals. Some students tried to integrate tan(2y) rather than using the information contained in part a. but were usually unsuccessful. Several of those who used the correct antiderivative of tan(2y) using part a., made subsequent substitution errors. A number of students found the wrong area and obtained $\frac{1}{2}\log_e 2$. Using a diagram would have been helpful for many students.

Question 8

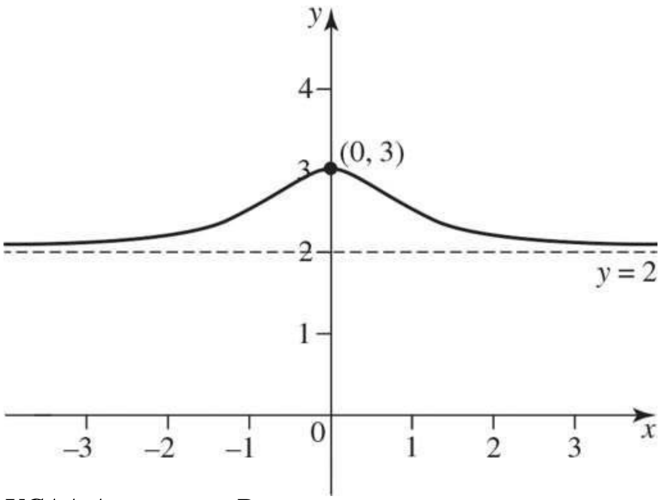
a. By long division $\frac{2x^2+3}{x^2+1} = \frac{2x^2+2+1}{x^2+1}$

$$= \frac{2\left(x^2+1\right)+1}{x^2+1}$$

$$= 2 + \frac{1}{x^2+1}$$ [**1 mark**]

***VCAA Assessment Report* note:**
This question was generally very well done. However, as often happens in a 'show that' question, some students were unable to do the relevant algebra yet somehow still managed to give the result stated.

b.



***VCAA Assessment Report* note:**
This question was well done by many students, but there were some strange graphs. A large number of students seemed to interpret the denominator as $x + 1$ or as $(x + 1)(x - 1)$ and consequently had one or two vertical asymptotes. A few students drew the graph of $y = \dfrac{3}{x^2 + 1}$. Among those who had the correct idea, typical errors included omitting the label of the intercept or the asymptote, showing the maximum point as a cusp rather than as a turning point and not showing asymptotic behaviour.

c. $A = \displaystyle\int_{-1}^{1} \frac{2x^2 + 3}{x^2 + 1} dx$

$A = 2\displaystyle\int_{0}^{1} \frac{2x^2 + 3}{x^2 + 1} dx$ by symmetry

$A = 2\displaystyle\int_{0}^{1} \left(2 + \frac{1}{x^2 + 1}\right) dx$ from a. [**1 mark**]

$A = 2\left[2x + \tan^{-1}(x)\right]_0^1$ [**1 mark**]

$A = 2\left[\left(2 + \tan^{-1}(1)\right) - \left(0 + \tan^{-1}(0)\right)\right]$

$A = 2\left[2 + \dfrac{\pi}{4}\right]$

$= 4 + \dfrac{\pi}{2} = \dfrac{\pi + 8}{2}$ units^2 [**1 mark**]

***VCAA Assessment Report* note:**

Most students handled this question very well, realising that they needed to express the fraction in two parts. Some students used symmetry successfully. Common errors included integrating the fraction to give $\ln\left(x^2 + 1\right)$, sign problems (often caused by lack of brackets) in the evaluation to give either 4 or $\dfrac{\pi}{2}$ as the answer and the incorrect evaluation of $\tan^{-1}(-1)$ as $\dfrac{3\pi}{4}$ (ignoring the quadrant). Some found the area between the graph and the asymptote.

Question 9

It is the area bounded by the inverse function with the y-axis, from $y = -\pi$ to $y = \pi$.

f: $y = 2\sin^{-1}(x+1)$

f^{-1}: $x = 2\sin^{-1}(y+1) \Rightarrow \dfrac{x}{2} = \sin^{-1}(y+1) \quad y + 1 = \sin\left(\dfrac{x}{2}\right)$

f^{-1}: $y = \sin\left(\dfrac{x}{2}\right) - 1$

So the required area is $\displaystyle\int_{-\pi}^{\pi} \left(\sin\left(\frac{x}{2}\right) - 1\right) dx$.

8 Differential equations

Topic	8	Differential equations
Subtopic	8.2	Verifying solutions to differential equations

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Source: *VCE 2015, Specialist Mathematics Exam 2, Section A, Q14; © VCAA*

Question 1 (1 mark)

A differential equation that has $y = x\sin(x)$ as a solution is

A. $\dfrac{d^2y}{dx^2} + y = 0$

B. $x\dfrac{d^2y}{dx^2} + y = 0$

C. $\dfrac{d^2y}{dx^2} + y = -\sin(x)$

D. $\dfrac{d^2y}{dx^2} + y = -2\cos(x)$

E. $\dfrac{d^2y}{dx^2} + y = 2\cos(x)$

Question 2 (3 marks)

Find the value of m where $m \in C$ if $y = e^{mx}$ satisfies $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 0$.

Topic 8 | Subtopic 8.2 Verifying solutions to differential equations

Question 3 (4 marks)

Verify that $y = \tan^{-1}(2x)$ is a solution of the differential equation $\left(1 + 4x^2\right)\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} = 0$.

Question 4 (1 mark)

Which one of the following differential equations is satisfied by $y = 3e^{-2x}$?

A. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = -6e^{-2x}$

B. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = -6e^{-2x}$

C. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 2y = -6e^{-2x}$

D. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 2y = -6e^{-2x}$

E. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 2y = -6e^{-2x}$

Question 5 (1 mark)

Which of the following is not a solution of the differential equation $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$?

A. $y = 3e^{-2x}$

B. $y = 5e^{-2x}$

C. $\dfrac{d^2x}{dt^2} + 27x = 0$

D. $y = (4 + 2x)\, e^{-2x}$

E. $y = x^2e^{-2x}$

Topic	8	Differential equations
Subtopic	8.3	Solving Type 1 differential equations, $\frac{dy}{dx} = f(x)$

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Question 1 (2 marks)

Determine the general solution to the differential equation $e^{2x}\frac{dy}{dx} + 6 = 2e^{4x}$.

Question 2 (2 marks)

Determine the general solution to the differential equation $\sqrt{x^2+9}\frac{dy}{dx} - x = 0$.

Question 3 (3 marks)

Solve the following differential equation and state the maximal domain for which the solution is valid.

$\operatorname{cosec}(3x)\frac{dy}{dx} + 9\cos^2(3x) = 0,\ \ y(0) = 0$

Question 4 (1 mark)

The solution to $\frac{dy}{dx} = 2x\sqrt{x^2 + 8}$, when $y = 1$ when $x = 1$, will be

A. $y = \frac{2}{3}\left(x^2 + 8\right)^{\frac{3}{2}} - 17$

B. $y = \frac{3}{2}\left(x^2 + 8\right)^{\frac{2}{3}} - 9$

C. $y = 2\left(x^2 + 8\right)^{\frac{1}{2}} - 17$

D. $y = 3\left(x^2 + 8\right)^{\frac{3}{2}} - 9$

E. $y = 2x\left(x^2 + 8\right)^{\frac{3}{2}} - 17$

Question 5 (1 mark)

The solution, in terms of u, to $\frac{dy}{dx} = x^2\sqrt{4 - x^3}$, can be evaluated using

A. $\int \sqrt{u}\, du$

B. $-\frac{1}{3}\int \sqrt{u}\, du$

C. $-3\int \sqrt{u}\, du$

D. $-\int \frac{u}{3}\sqrt{u}\, du$

E. $-\int \frac{u^2}{3}\sqrt{u}\, du$

Question 6 (1 mark)

The general solution to $\frac{dy}{dx} = 4x\sqrt{2 - x^2}$ will be

A. $-\frac{2(2 - x^2)^{\frac{3}{2}}}{3} + c$

B. $-\frac{2x(2 - x^2)^{\frac{3}{2}}}{3} + c$

C. $-\frac{8(1 - x^2)^{\frac{3}{2}}}{3} + c$

D. $-\frac{3(2 - x^2)^{\frac{3}{2}}}{4} + c$

E. $-\frac{4(2 - x^2)^{\frac{3}{2}}}{3} + c$

Question 7 (1 mark)

The gradient of a curve is given by $\frac{1}{x - 2}$. The equation of the curve which passes through the origin is given by

A. $y = \log_e(x - 2)$

B. $y = \log_e(2 - x)$

C. $y = \log_e\left(1 - \frac{x}{2}\right)$

D. $y = -\log_e|x - 2|$

E. $y = -\log_e(2 - x) + \log_e(2)$

Question 8 (1 mark)

Given that $\frac{dy}{dx} - \frac{1}{(2x+3)^2} = 0$ and $y(-1) = 1$, then

A. $y = \frac{3x+4}{2x+3}$

B. $y = \frac{1}{2(2x+3)} - \frac{1}{2}$

C. $y = 2 - \frac{1}{(2x+3)}$

D. $y = \frac{-1}{2(2x+3)} - \frac{9}{10}$

E. $y = \frac{-1}{2x+3} - \frac{4}{5}$

Topic	**8**	**Differential equations**
Subtopic	**8.4**	**Solving Type 2 differential equations, $\frac{dy}{dx} = f(y)$**

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Source: *VCE 2015, Specialist Mathematics Exam 2, Section 1, Q12; © VCAA*

Question 1 (1 mark)

Given $\frac{dy}{dx} = 1 - \frac{y}{3}$ and $y = 4$ when $x = 2$, then

A. $y = e^{\frac{-(x-2)}{3}} - 3$

B. $y = e^{\frac{-(x-2)}{3}} + 3$

C. $y = 4e^{\frac{-(x-2)}{3}}$

D. $y = e^{\frac{4(y-x-2)}{3}}$

E. $y = e^{\frac{(x-2)}{3}} + 3$

Question 2 (4 marks)

Solve the following differential equation and state the maximal domain for which the solution is valid.

$\frac{dy}{dx} + 6\operatorname{cosec}\left(\frac{y}{2}\right) = 0, \ y\left(\frac{1}{3}\right) = 0$

Question 3 (1 mark)

Given that a, b and c are non-zero real constants, solve the differential equation $\frac{dy}{dx} + ay = by^2$, $y(0) = c$.

$y = \square$

Source: *VCE 2019, Specialist Mathematics 1, Q.1; © VCAA*

Question 4 (4 marks)

Solve the differential equation $\frac{dy}{dx} = \frac{2ye^{2x}}{1 + e^{2x}}$ given that $y(0) = \pi$.

Question 5 (1 mark)

The general solution to $\frac{dy}{dx} = 3y$ will be

A. $y = \sqrt[3]{3(x - c)}$
B. $y = \log_e(3x) + c$
C. $y = Ae^{3x}$
D. $y = \frac{x^3}{2} + c$
E. $y = \frac{3x^2}{2} + c$

Topic 8 | Subtopic 8.4 Solving Type 2 differential equations

Question 6 (1 mark)

The general solution to $\frac{dy}{dx} = \frac{y}{4}$ will be

A. $y = \log_e(4x) + C$

B. $y = Ae^{4x}$

C. $y = 4\log_e x + C$

D. $y = Ae^{\frac{x}{4}}$

E. $y = \frac{x^2}{2} + c$

Question 7 (1 mark)

The solution of $\left(\frac{dy}{dx}\right) = \frac{-\sqrt{(9-y^2)}}{3}$, $y(3) = 0$, is given by

A. $y = 3\sin\left(\frac{x}{3}\right)$

B. $y = 3\cos\left(\frac{x}{3}\right)$

C. $x = \frac{\left(9-y^2\right)^{\frac{3}{2}}}{3y}$

D. $x = 3\sin\left(\frac{y}{3}\right)$

E. $x = 3\cos\left(\frac{y}{3}\right)$

Question 8 (1 mark)

The general solution to $\frac{dy}{dx} = \sqrt{2 - y^2}$ will be

A. $y = \sin\left(\frac{x - c}{\sqrt{2}}\right)$

B. $y = \sin\left(\sqrt{2}\,(x - c)\right)$

C. $y = \sqrt{2}\sin(x - c)$

D. $y = \frac{\sin(x - c)}{\sqrt{2}}$

E. $y = \sin\left(\frac{\sqrt{2}}{x - c}\right)$

Question 9 (1 mark)

The solution of $\frac{dy}{dx} + 4y = 0, y(0) = 3$ is given by

A. $y = 4e^{-3x}$

B. $y = 4e^{3x}$

C. $y = 3e^{-4x}$

D. $y = 3e^{4x}$

E. $y = 3e^{-\frac{x}{4}}$

Question 10 (1 mark)

The solution of $6\frac{dy}{dx} - 4y^2 = 9,\ y(0) = \frac{3}{2}$ is given by

A. $y = \frac{3}{2}\tan\left(x + \frac{\pi}{4}\right)$

B. $y = \frac{3}{2}\tan\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

C. $y = \frac{3}{2}\tan\left(\frac{\pi}{2} + \frac{3\pi}{4}\right)$

D. $y = \frac{3}{2}\tan\left(x + \frac{\pi}{24}\right)$

E. $y = \frac{3}{2}\tan^{-1}(x)$

Topic	8	Differential equations
Subtopic	8.5	Solving Type 3 differential equations, $\frac{dy}{dx} = f(x)g(y)$

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Source: *VCE 2019 Specialist Mathematics Exam 1, Q.1; © VCAA*

Question 1 (4 marks)

Solve the differential equation $\frac{dy}{dx} = \frac{2ye^{2x}}{1 + e^{2x}}$ given that $y(0) = \pi$.

Source: *VCE 2018 Specialist Mathematics Exam 2, Section A, Q9; © VCAA*

Question 2 (1 mark)

A solution to the differential equation $\frac{dy}{dx} = \frac{2}{\sin(x+y) - \sin(x-y)}$ can be obtained from

A. $\int 1dx = \int 2\sin(y)\,dy$

B. $\int \cos(y)\,dy = \int \operatorname{cosec}(x)\,dx$

C. $\int \cos(x)\,dx = \int \operatorname{cosec}(y)\,dy$

D. $\int \sec(x)\,dx = \int \sin(y)\,dy$

E. $\int \sec(x)\,dx = \int \operatorname{cosec}(y)\,dy$

Source: *VCE 2017 Specialist Mathematics Exam 1, 8b; © VCAA*

Question 3 (2 marks)

Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$ with the condition $y(-1) = 1$. Express your answer in the form $ay^3 + by + cx^2 + d = 0$ where a, b, c and d are integers.

Topic	8	Differential equations
Subtopic	8.6	Solving Type 4 differential equations, $\frac{d^2y}{dx^2} = f(x)$

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Question 1 (1 mark)

Solve the differential equation

$\frac{d^2y}{dx^2} + 24x^2 = 0, y(1) = 2, y(2) = 3$

$y = \square$

Question 2 (2 marks)

Solve the differential equation

$e^x \frac{d^2y}{dx^2} + 4e^{-2x} = 5, x = 0, \frac{dy}{dx} = 0, y = 0$

Question 3 (2 marks)

a. If a and b are positive real constants, show that $\frac{d}{dx}\left[\frac{x}{\sqrt{a + bx^2}}\right] = \frac{a}{\sqrt{(a + bx^2)^3}}$. (1 mark)

b. Hence, find the general solution to $\frac{d^2y}{dx^2} + \frac{1}{\sqrt{(a + bx^2)^3}} = 0$. (1 mark)

$y = \square$

Question 4 (1 mark)

Given that $\frac{d^2y}{dx^2} = 4\sin^2\left(\frac{x}{4}\right)$ and $y'(0) = 0,\ y(0) = 0$, then

A. $y = \frac{1}{48}\sin^4\left(\frac{x}{4}\right)$

B. $y = x^2 + 8\cos\left(\frac{x}{2}\right) - 8$

C. $y = x^2 - 8\cos\left(\frac{x}{2}\right) + 8$

D. $y = x^2 - 2\cos\left(\frac{x}{2}\right) + 2$

E. $y = x^2 + 2\cos\left(\frac{x}{2}\right) - 2$

Question 5 (1 mark)

Given that $\frac{d^2y}{dx^2} = \frac{1}{(4x-3)^2}$ and $y'(1) = 0,\ y(1) = 0$, then

A. $y = \frac{1}{4}(x-1) + \frac{1}{16}\log_e(4x-3)$

B. $y = \frac{1}{4}(x-1) - \frac{1}{16}\log_e(4x-3)$

C. $y = x - 1 - \frac{1}{4}\log_e(4x-3)$

D. $y = 1 - x + \frac{1}{4}\log_e(4x-3)$

E. $y = \frac{1}{16}\log_e(4x-3)$

Topic	8	Differential equations
Subtopic	8.7	Review

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Question 1 (3 marks)

Show that $y = \sin\left(x^2\right)$ satisfies the differential equation $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4x^3y = 0$.

Questionn 2 (5 marks)

Given that $y = 4e^{-2x}\sin(3x)$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 0$, determine the values of a and b.

Question 3 (3 marks)

Find the particular solution to the following differential equation, stating the maximal domain over which the solution is valid.

$(5x + 3)^2\dfrac{dy}{dx} + 4 = 0, y(-1) = 2$

Question 4 (1 mark)

$y = \tan^{-1}\left(\frac{x}{3}\right)$ satisfies which of the following differential equations?

A. $(x^2 + 9)\frac{dy}{dx} - 3 = 0$

B. $(x^2 + 9)\frac{dy}{dx} + 3 = 0$

C. $(x^2 - 3)\frac{dy}{dx} - 9 = 0$

D. $(9 - x^2)\frac{dy}{dx} + 3 = 0$

E. $(9 - x^2)\frac{dy}{dx} + 3x^2 = 0$

Question 5 (3 marks)

Let $y = ct\sin(2t)$. Find the value of c given that $\frac{d^2y}{dt^2} + 4y = 8\cos(2t)$.

Source: *VCE 2016, Specialist Mathematics 1, Q.10; © VCAA*

Question 6 (5 marks)

Solve the differential equation $\sqrt{2 - x^2}\frac{dy}{dx} = \frac{1}{2 - y}$, given that $y(1) = 0$. Express y as a function of x.

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.12; © VCAA*

Question 7 (1 mark)

If $\frac{dy}{dx} = \sqrt{(2x^6 + 1)}$ and $y = 5$ when $x = 1$, then the value of y when $x = 4$ is given by

A. $\int_1^4 \left(\sqrt{(2x^6+1)} + 5\right) dx$

B. $\int_1^4 \sqrt{(2x^6+1)}dx$

C. $\int_1^4 \sqrt{(2x^6+1)}dx + 5$

D. $\int_1^4 \sqrt{(2x^6+1)}dx - 5$

E. $\int_1^4 \left(\sqrt{(2x^6+1)} - 5\right) dx$

Question 8 (1 mark)

$y = -e^{-x}(\sin(x))$ is a solution to $\frac{d^2y}{dx^2} + k\frac{dy}{dx} = 2e^{-x}\sin(x)$ where $k \in R$.

Find the value of k.

A. 1
B. 2
C. −1
D. −2
E. 0
F. e^x

Question 9 (1 mark)

The value of $f(x)$ correct to 1 decimal place at $x = 2$ when $f'(x) = \sin(x^2)$, given $f(0) = 4$, will be equal to

A. 2.7
B. −0.1
C. 4.8
D. 6.0
E. 4.9

Question 10 (1 mark)

The value of $f(x)$ correct to 2 decimal place at $x = 1$ when $f'(x) = e^{2x}$, given $f(0) = 2$, will be equal to

A. 5.10
B. 5.21
C. 5.18
D. 5.09
E. 5.19

Answer

8.2 Verifying solutions to differential equations

Question 1

$$y = x\sin(x)$$
$$\frac{dy}{dx} = x\cos(x) + \sin(x)$$
$$\frac{d^2y}{dx^2} = \cos(x) - x\sin(x) + \cos(x)$$
$$\frac{d^2y}{dx^2} = 2\cos(x) - x\sin(x)$$
$$\frac{d^2y}{dx^2} + x\sin(x) = 2\cos(x)$$
$$\frac{d^2y}{dx^2} + y = 2\cos(x)$$

The correct answer is **E**.

Question 2

$$y = e^{mx}$$
$$\frac{dy}{dx} = me^{mx}$$
$$\frac{d^2y}{dx^2} = m^2e^{mx} \text{ [1 mark]}$$
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$
$$m^2e^{mx} + 4me^{mx} + 13e^{mx} = 0$$
$$e^{mx}\left(m^2 + 4m + 13\right) = 0$$
$$e^{mx} \neq 0 \text{ [1 mark]}$$
$$m^2 + 4m + 13 = 0$$
$$m^2 + 4m + 4 = -13 + 4$$
$$(m+2)^2 = -9$$
$$(m+2)^2 = 9i^2$$
$$m + 2 = \pm 3i$$
$$m = -2 \pm 3i \text{ [1 mark]}$$

Question 3

$$y = \tan^{-1}(2x)$$
$$\frac{dy}{dx} = \frac{2}{1+4x^2} \text{ [1 mark]}$$
$$= 2\left(1+4x^2\right)^{-1}$$
$$\frac{d^2y}{dx^2} = -1 \times 2 \times 8x\left(1+4x^2\right)^{-2}$$
$$= \frac{-16x}{\left(1+4x^2\right)^2} \text{ [1 mark]}$$

$$\begin{aligned}\text{LHS} &= \left(1+4x^2\right)\frac{d^2y}{dx^2}+8x\frac{dy}{dx}\\ &= \left(1+4x^2\right)\frac{-16x}{\left(1+4x^2\right)^2}+8x\times\frac{2}{1+4x^2}\ \text{[1 mark]}\\ &= \frac{-16x}{1+4x^2}+\frac{16x}{1+4x^2}\\ &= 0\\ &= \text{RHS}\ \text{[1 mark]}\end{aligned}$$

Question 4

$y = 3e^{-2x}$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-2y = 12e^{-2x}+2\left(-6e^{-2x}\right)-2\left(3e^{-2x}\right)$$

$$12e^{-2x}-12e^{-2x}-6e^{-2x}=-6e^{-2x}$$

Question 5

For any value A

If $y = Ae^{-2x}$, $\frac{dy}{dx} = -2Ae^{-2x}$ and $\frac{d^2y}{dx^2} = 4Ae^{-2x}$

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y = e^{-2x}(4A-8A+4A)=0$$

If $y = Axe^{-2x}$, $\frac{dy}{dx} = -2Axe^{-2x}+Ae^{-2x}=Ae^{-2x}(1-2x)$

$$\frac{d^2y}{dx^2} = -2Ae^{-2x}-2Ae^{-2x}(1-2x)=Ae^{-2x}(4x-4)$$

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y = Ae^{-2x}(4x-4+4(1-2x)+4x)=0$$

All of **A**, **B**, **C** and **D** (by a linear combination) satisfy the differential equation; **E** does not.

8.3 Solving Type 1 differential equations, $\frac{dy}{dx}=f(x)$

Question 1

$$\begin{aligned}e^{2x}\frac{dy}{dx}+6 &= 2e^{4x}\\ e^{2x}\frac{dy}{dx} &= 2e^{4x}-6\\ \frac{dy}{dx} &= \frac{2e^{4x}-6}{e^{2x}}\ \text{[1 mark]}\\ y &= \int\left(2e^{2x}-6e^{-2x}\right)dx\\ y &= e^{2x}+3e^{-2x}+c\ \text{[1 mark]}\end{aligned}$$

Question 2

$$\begin{aligned}\sqrt{x^2+9}\frac{dy}{dx}-x &= 0\\ \sqrt{x^2+9}\frac{dy}{dx} &= x\end{aligned}$$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+9}}$ [1 mark]

$y = \int \frac{x}{\sqrt{x^2+9}}$

$y = \sqrt{x^2+9} + c$ [1 mark]

Question 3

$\text{cosec}\,(3x)\,\frac{dy}{dx} + 9\cos^2(3x) = 0, \quad y(0) = 0$

$\text{cosec}\,(3x)\,\frac{dy}{dx} = -9\cos^2(3x)$

$\frac{dy}{dx} = -9\sin(3x)\,\cos^2(2x)$

$y = \int -9\sin(3x)\,\cos^2(3x)\,dx$ [1 mark]

Let $u = \cos(3x) \qquad \frac{du}{dx} = -3\sin(3x)$

$y = 3\int u^2 du$

$= u^3 + c$

$y = \cos^3(3x) + c$ [1 mark]

When $x = 0, \quad y = 0$

$0 = \cos^3(0) + c$

$c = -1$

$y = \cos^3(3x) - 1, \quad x \in R \backslash \frac{n\pi}{3}, n \in Z$ [1 mark]

Question 4

Let $u = x^2 + 8$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int 2x\sqrt{x^2+8}\,dx = \int 2x\sqrt{u}\frac{du}{2x}$

$= \int \sqrt{u}\,du$

$y = \frac{2}{3}\left(x^2+8\right)^{\frac{3}{2}} + c$

$y = 1$ when $x = 1$:

$1 = \frac{2}{3}(1+8)^{\frac{3}{2}} + c$

$1 = \frac{2}{3}(27) + c$

$c = -17$

$y = \frac{2}{3}\left(x^2+8\right)^{\frac{3}{2}} - 17$

Question 5

Let $u = 4 - x^3$

$\frac{du}{dx} = -3x^2$

$dx = -\frac{du}{3x^2}$

$$\int x^2\sqrt{4-x^2}\,dx = -\int x^2\sqrt{u}\frac{du}{3x^2}$$
$$= -\frac{1}{3}\int\sqrt{u}\,du$$

Question 6

Let $u = 2 - x^2$

$$\frac{du}{dx} = -2x$$
$$dx = -\frac{du}{2x}$$
$$\int 4x\sqrt{2-x^2}dx = -\int 4x\sqrt{u}\frac{du}{2x}$$
$$2\int\sqrt{u}du = -\frac{4u^{\frac{3}{2}}}{3} + c$$
$$= -\frac{4\left(2-x^2\right)^{\frac{3}{2}}}{3} + c$$

Question 7

$$\frac{dy}{dx} = \frac{1}{x-2} = \frac{-1}{2-x} \text{ since at } (0,0), x < 2.$$
$$y = \int\frac{-1}{2-x}dx = \log_e(2-x) + c$$

To find c, use $x = 0$ when $y = 0$.

$$0 = \log_e(2) + c \Rightarrow c = -\log_e(2)$$
$$y = \log_e(2-x) - \log_e(2) = \log_e\left(\frac{2-x}{2}\right) = \log_e\left(1-\frac{x}{2}\right)$$

Question 8

$$\frac{dy}{dx} - \frac{1}{(2x+3)^2} = 0 \text{ and } y(-1) = 1$$
$$\frac{dy}{dx} = \frac{1}{(2x+3)^2}$$
$$y = \int\frac{1}{(2x+3)^2}dx = \frac{-1}{2(2x+3)} + c$$

To find c, use $x = -1$ when $y = 1$.

$$1 = -\frac{1}{2} + c \Rightarrow c = \frac{3}{2}$$
$$y = \frac{-1}{2(2x+3)} + \frac{3}{2} = \frac{-1+3(2x+3)}{2(2x+3)} = \frac{6x+8}{2(2x+3)}$$
$$y = \frac{3x+4}{2x+3}$$

8.4 Solving Type 2 differential equations, $\frac{dy}{dx} = f(y)$

Question 1

$$\frac{dy}{dx} = 1 - \frac{y}{3},\quad y(2) = 4$$
$$\frac{dy}{dx} = \frac{3-y}{3} \Rightarrow \frac{dy}{dx} = \frac{3}{3-y}$$

$$x = 3\int \frac{1}{3-y}dy = -3\log_e(|3-y|) + c$$
$$2 = -3\log_e(|1|) + c, \quad c = 2$$
$$x = -3\log_e(|3-y|) + 2$$
$$\frac{2-x}{3} = \log_e(|3-y|) \quad \text{since } x = 2 \text{ when } y = 4$$
$$y - 3 = e^{\frac{2-x}{3}}$$
$$y = 3 + e^{\frac{2-x}{3}} = 3 + e^{\frac{-(x-2)}{3}}$$

Or use CAS to solve.

The correct answer is **B**.

Question 2

$$\frac{dy}{dx} + 6\operatorname{cosec}\left(\frac{y}{2}\right) = 0, \qquad y\left(\frac{1}{3}\right) = 0$$
$$\frac{dy}{dx} = -6\operatorname{cosec}\left(\frac{y}{2}\right)$$
$$\frac{dy}{dx} = \frac{-1}{-6\operatorname{cosec}\left(\frac{y}{2}\right)}$$
$$x = \frac{-1}{6}\int \sin\left(\frac{y}{2}\right)dy$$
$$x = \frac{1}{3}\cos\left(\frac{y}{2}\right) + c \ \text{[1 mark]}$$

When $y = 0, \quad x = \frac{1}{3}$

$$\frac{1}{3} = \frac{1}{3}\cos(0) + c$$
$$c = 0 \ \text{[1 mark]}$$
$$x = \frac{1}{3}\cos\left(\frac{y}{2}\right)$$
$$3x = \cos\left(\frac{y}{2}\right)$$
$$\frac{y}{2} = \cos^{-1}(3x)$$
$$y = 2\cos^{-1}(3x) \ \text{[1 mark]}$$

Require $|3x| \le 1$

$$|x| \le \frac{1}{3} \ \text{[1 mark]}$$

Question 3

$$\frac{dy}{dx} + ay = by^2, \qquad y(0) = c$$
$$\frac{dy}{dx} = by^2 - ay$$
$$\frac{dy}{dx} = \frac{1}{by^2 - ay}$$
$$= \frac{1}{y(by-a)}$$
$$x = \int \frac{1}{y(by-a)}dy \ \text{[1 mark]}$$

Partial fractions

$$\frac{1}{y(by-a)} = \frac{A}{y} + \frac{B}{by-a}$$

$$= \frac{A(by-a)+By}{y(by-a)}$$

$$= \frac{y(\text{A}b+B)-aA}{y(by-a)}$$

$$aA = -1 \Rightarrow A = \frac{-1}{a}$$

$$Ab + B = 0 \quad \Rightarrow B = -Ab = \frac{b}{a}$$

$$x = \frac{1}{a}\int\left(\frac{b}{by-a} - \frac{1}{y}\right)dy \text{ [1 mark]}$$

$$ax = \log_e\left|\frac{by-a}{y}\right| + K \text{ [1 mark]}$$

When $x = 0,\ y = c$

$$0 = \log_e\left|\frac{bc-a}{c}\right| + K$$

$$K = -\log_e\left|\frac{bc-a}{c}\right|$$

$$ax = \log_e\left|\frac{c(by-a)}{(bc-a)y}\right|$$

$$e^{ax} = \frac{c(by-a)}{y(bc-a)} \text{ [1 mark]}$$

Note: The absolute value can be removed since LHS = RHS when the given condition is substituted in.

$$\text{y}(bc-a)e^{ax} = c(by-a) = bcy - ac$$

$$-ac = y(bc-a)e^{ax} - bcy$$

$$ac = y\left(bc + (a-bc)e^{ax}\right)$$

$$y = \frac{ac}{bc + (a-bc)e^{ax}} \quad \text{[1 mark]}$$

Question 4

$\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}$ separating the variables

$$\int \frac{1}{y}dy = \int \frac{2e^{2x}}{1+e^{2x}}dx$$

$\log_e(|y|) = \log_e\left(1+e^{2x}\right) + c$ to find c use $x = 0,\ y = \pi$

$$\log_e(\pi) = \log_e(2) + c,\ c = \log_e\left(\frac{\pi}{2}\right)$$

$\log_e(y) = \log_e\left(1+e^{2x}\right) + \log_e\left(\frac{\pi}{2}\right)$ since $y > 0$ modulus is not needed

$$\log_e(y) = \log_e\left(\frac{\pi}{2}\left(1+e^{2x}\right)\right)$$

$$y = \frac{\pi}{2}\left(1+e^{2x}\right)$$

Award 1 mark for separating variables.

Award 1 mark for correct integration.

Award 1 mark for finding constant of integration.

Award 1 mark for correct rule as $y = \ldots$

***VCAA Examination Report* note:**

Most students recognised that they needed to separate and integrate in order to solve the differential equation although not all were then able to obtain the correct equation. Common errors were $\int 2y dy = \int \frac{e^{2x}}{1+e^{2x}} dx$ and $\int 2ye^{2x} dx = \int \frac{1}{1+e^{2x}} dx$. Students who failed to recognise that $\frac{d}{dx}\left(1+e^{2x}\right) = 2e^{2x}$ did not score highly. Some students spent time using a substitution to determine $\int \frac{2e^{2x}}{1+e^{2x}} dx$, which was not necessary. Some students who managed to correctly find the value of the constant of integration did not use log or index laws correctly, presenting incorrect solutions such as $y = e^{2x} + 1 + \frac{\pi}{2}$.

An alternative approach was to solve $\int_x^y \frac{1}{t} dt = \int_0^x \frac{2e^{2x}}{1+e^{2x}} dt$. Note that a different variable of integration must be used. Students using this method typically retained x and y as the variables of integration and thus did not obtain full marks.

Question 5

$$x = \int \frac{1}{3y} dy$$
$$= \log_e(3y) + C$$
$$y = Ae^{3x}$$

Question 6

$$x = \int \frac{4}{y} dy$$
$$= 4\log_e(y) + C$$
$$y = Ae^{\frac{x}{4}}$$

Question 7

$$\left(\frac{dx}{dy}\right) = \frac{3}{-\sqrt{(9-y^2)}}$$
$$\left(\frac{dx}{dy}\right) = \frac{-1}{\sqrt{\left(1-\left(\frac{y}{3}\right)^2\right)}}$$
$$x = 3\cos^{-1}\left(\frac{y}{3}\right) + c;\ \ y = 3, x = 0 \Rightarrow c = 0$$
$$x = 3\cos^{-1}\left(\frac{y}{3}\right) \Rightarrow y = 3\cos\left(\frac{x}{3}\right) \quad \text{[1 mark]}$$

Question 8

$$x = \int \frac{1}{\sqrt{2-y^2}} dy$$
$$x = \sin^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$$
$$y = \sqrt{2}\sin(x-c)$$

Question 9

Rewriting as $\frac{dy}{dx} = -4y$ and then inverting both sides gives

$$\frac{dx}{dy} = -\frac{1}{4y}$$

Integrating w.r.t. y gives

$x = -\frac{1}{4}\int \frac{1}{y}dy = -\frac{1}{4}\ln y + C$

$x = -\frac{1}{4}\ln y + C$

First find C and then rearrange. Now $y = 3$ when $x = 0$ $0 = -\frac{1}{4}\ln 3 + C$ so that $C = \frac{1}{4}\ln 3$ (do not evaluate)

Substituting gives:

$x = -\frac{1}{4}\ln y + \frac{1}{4}\ln 3$

$x = \frac{1}{4}\left[\ln 3 - \ln y\right]$

$x = \frac{1}{4}\ln\left(\frac{3}{y}\right)$

using log laws.

$4x = \ln\left(\frac{3}{y}\right)$ or $\frac{3}{y} = e^{4x}$

Inverting again gives:

$\frac{y}{2} = e^{-4x}$

So $y = 3e^{-4x}$

Question 10

Rewriting as $6\frac{dy}{dx} = 9 + 4y^2$ and then inverting both sides gives

$\frac{dx}{dy} = \frac{6}{4y^2 + 9}$

Integrating again w.r.t. y gives

$x = \int \frac{6}{4y^2 + 9}dy = \tan^{-1}\left(\frac{2y}{3}\right) + C$

Now $y = \frac{3}{2}$ when $x = 0$ so that

$0 = \tan^{-1}(1) + C \Rightarrow C = -\frac{\pi}{4}$

Substituting gives

$x = \tan^{-1}\left(\frac{2y}{3}\right) - \frac{\pi}{4}$ $x + \frac{\pi}{4} = \tan^{-1}\left(\frac{2y}{3}\right)$

$\tan\left(x + \frac{\pi}{4}\right) = \left(\frac{2y}{3}\right)$

$y = \frac{3}{2}\tan\left(x + \frac{\pi}{4}\right)$

8.5 Solving Type 3 differential equations, $\frac{dy}{dx} = f(x)g(y)$

Question 1

$\frac{dy}{dx} = \frac{2ye^{2x}}{1 + e^{2x}}$ Separating the variables:

$\log_e(|y|) = \log_e\left(1 + e^{2x}\right) + c$ To find c use $x = 0,\ y = \pi$

$\log_e(\pi) = \log_e(2) + c,\ \ c = \log_e\left(\frac{\pi}{2}\right)$

$\log_e(y) = \log_e\left(1 + e^{2x}\right) + \log_e\left(\frac{\pi}{2}\right)$

Since $y > 0$
modulus is not needed

$$\log_e(y) = \log_e\left(\frac{\pi}{2}\left(1 + e^{2x}\right)\right)$$

$$y = \frac{\pi}{2}\left(1 + e^{2x}\right)$$

Award 1 mark for separating variables.
Award 1 mark for correct integration.
Award 1 mark for finding constant of integration.
Award 1 mark for correct rule as $y = \ldots$

Question 2

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\sin(x + y) - \sin(x - y) = 2\cos(x)\sin(y)$

$$\frac{dy}{dx} = \frac{2}{\sin(x + y) - \sin(x - y)}$$

$$\frac{dy}{dx} = \frac{2}{2\cos(x)\sin(y)}$$

$$\int \sin(y)\,dy = \int \frac{1}{\cos(x)}dx = \int \sec(x)\,dx$$

The correct answer is **D**.

Question 3

$$\frac{dy}{dx} = \frac{-x}{1 + y^2}, y(-1) = 1$$

$$\int \left(1 + y^2\right) dy = \int -x\,dx$$

$$y + \frac{1}{3}y^3 = -\frac{1}{2}x^2 + c$$

When $x = -1,\ y = 1$

$$1 + \frac{1}{3} = -\frac{1}{2} + c \Rightarrow c = \frac{11}{6}$$

$$y + \frac{1}{3}y^3 = -\frac{1}{2}x^2 + \frac{11}{6}$$

$2y^3 + 6y + 3x^2 - 11 = 0,\quad a = 2,\ b = 6,\ c = 3,\ d = -11$

Award 1 mark for the correct separation of the variables.
Award 1 mark for the correct values.

8.6 Solving Type 4 differential equations, $\frac{d^2y}{dx^2} = f(x)$

Question 1

$$\frac{d^2y}{dx^2} + 24x^2 = 0,\quad y(1) = 2,\ y(2) = 3$$

$$\frac{d^2y}{dx^2} = -24x^2$$

$$\frac{dy}{dx} = \int -24x^2 dx$$

$$\frac{dy}{dx} = -8x^3 + c_1$$

$$y = \int \left(-8x^3 + c_1\right) dx$$

$$y = -2x^4 + c_1 x + c_2$$

When $y = 2,\ x = 1$

(1) $2 = -2 + c_1 + c_2 \Rightarrow c_1 + c_2 = 4$

When $y = 3,\ x = 2$

(2) $3 = -32 + 2c_1 + c_2 \Rightarrow 2c_1 + c_2 = 35$

$c_1 = 31,\ c_2 = -27$

$y = -2x^4 + 31x - 27$ [1 mark]

Question 2

$$e^x \frac{d^2y}{dx^2} + 4e^{-2x} = 5, \quad x = 0, \frac{dy}{dx} = 0, y = 0$$

$$e^x \frac{d^2y}{dx^2} = 5 - 4e^{-2x}$$

$$\frac{dy}{dx} = \int \left(5e^{-x} - 4e^{-3x}\right) dx$$

$$= 5e^{-x} + \frac{4}{3}e^{-3x} + c1$$

$$\frac{dy}{dx} = 0,\ x = 0$$

$$0 = -5 + \frac{4}{3} + c_1$$

$$c_1 = \frac{11}{3}$$

$$\frac{dy}{dx} = -5e^{-x} + \frac{4}{3}e^{-3x} + \frac{11}{3}$$ [1 mark]

$$y = \int \left(-5e^{-x} + \frac{4}{3}e^{-3x} + \frac{11}{3}\right) dx$$

$$y = 5e^{-x} - \frac{4}{9}e^{-3x} + \frac{11x}{3} + c_2$$

$$y = 0, \qquad x = 0$$

$$0 = 5 - \frac{4}{9} + c_2$$

$$c_2 = \frac{-41}{9}$$

$$y = 5e^{-x} - \frac{4}{9}e^{-3x} + \frac{11x}{3} - \frac{41}{9}$$ [1 mark]

Question 3

a. $\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{a + bx^2}}\right)$

Using the quotient rule,

$$= \frac{1\sqrt{a + bx^2} - 2bx \times \frac{1}{2}\left(a + bx^2\right)^{-\frac{1}{2}} \times x}{a + bx^2}$$

$$= \frac{1}{a + bx^2}\left(\sqrt{a + bx^2} - \frac{bx^2}{\sqrt{a + bx^2}}\right)$$

$$= \frac{1}{a + bx^2}\left(\frac{a + bx^2 - bx^2}{\sqrt{a + bx^2}}\right)$$

$$= \frac{a}{\left(\sqrt{a + bx^2}\right)^3}$$ [1mark]

b. $\frac{d^2y}{dx^2} + \frac{1}{\sqrt{(a+bx^2)^3}} = 0$

$$\frac{d^2y}{dx^2} = -\frac{1}{\sqrt{(a+bx^2)^3}}$$

$$\frac{dy}{dx} = -\int \frac{1}{\sqrt{(a+bx^2)^3}}dx$$

$$= \frac{-x}{a\sqrt{a+bx^2}} + c_1$$

$$y = \frac{-1}{a}\int\left(\frac{x}{\sqrt{a+bx^2}} + c_1\right)dx$$

$$= \frac{-1}{a} \times \frac{1}{2b} \times 2\sqrt{a+bx^2} + c_1x + c_2$$

$$= c_2 + c_1x - \frac{1}{ab}\sqrt{a+bx^2}$$ [1 mark]

Question 4

$\frac{d^2y}{dx^2} = 4\sin^2\left(\frac{x}{4}\right)$ and $y'(0) = 0,\ y(0) = 0$

$\frac{dy}{dx} = \int 4\sin^2\left(\frac{x}{4}\right)dx = 2\int\left(1 - \cos\left(\frac{x}{2}\right)\right)dx$

$\frac{dy}{dx} = 2\left(x - 2\sin\left(\frac{x}{2}\right)\right) + C_1$

To find the value of C_1, $y'(0) = 0$ means $x = 0$ when $\frac{dy}{dx} = 0$.

Substituting gives $0 = 2(0 - \sin(0)) + C_1$ so that $C_1 = 0$.

Now, $\frac{dy}{dx} = 2\left(x - 2\sin\left(\frac{x}{2}\right)\right)$.

Integrating again w.r.t. x gives $y = \int\left(2x - 4\sin\left(\frac{x}{2}\right)\right)dx = x^2 + 8\cos\left(\frac{x}{2}\right) + c_2$.

To find the value of C_2, $y(0) = 0$ means $x = 0$ when $y = 0$.

Substituting gives $0 = 0 + 8\cos(0) + C_2$ so that $C_2 = -8$.

$y = x^2 + 8\cos\left(\frac{x}{2}\right) - 8$

Question 5

$\frac{d^2y}{dx^2} = \frac{1}{(4x-3)^2}$ and $y'(1) = 0, y(1) = 0$

$\frac{dy}{dx} = \int\frac{1}{(4x-3)^2}dx$

$\frac{dy}{dx} = -\frac{1}{4(4x-3)} + C_1$

To find the value of C_1, $y'(1) = 0$ p;means $x = 1$ when $\frac{dy}{dx} = 0$

Substituting gives $0 = -\frac{1}{4} + C_1$ so that $C_1 = \frac{1}{4}$.

Now, $\frac{dy}{dx} = \frac{1}{4} - \frac{1}{4(4x-3)}$.

Integrating again w.r.t. x gives

$$y = \int \left(\frac{1}{4} - \frac{1}{4(4x-3)}\right) dx = \frac{x}{4} - \frac{1}{16}\log_e(4x-3) + C_2$$

To find the value of C_2, $y(1) = 0$ means $x = 1$ when $y = 0$.

Substituting gives $0 = \frac{1}{4} - \frac{1}{16}\log_e(1) + C_2$ so that $C_2 = -\frac{1}{4}$.

$$y = \frac{1}{4}(x-1) - \frac{1}{16}\log_e(4x-3)$$

8.7 Review

Question 1

$$y = \sin\left(x^2\right)$$

$$\frac{dy}{dx} = 2x\cos\left(x^2\right) \text{ [1 mark]}$$

$$\frac{d^2y}{dx^2} = 2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right) \text{ [1 mark]}$$

$$\begin{aligned} x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y &= x\left(2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right)\right) - 2x\cos\left(x^2\right) + 4x^3\sin\left(x^2\right) \\ &= 2x\cos\left(x^2\right) - 4x^3\sin\left(x^2\right) - 2x\cos\left(x^2\right) + 4x^3\sin\left(x^2\right) \\ &= 0 \\ &= \text{RHS [1 mark]} \end{aligned}$$

Question 2

$$\begin{aligned} y &= 4e^{-2x}\sin(3x) \\ \frac{dy}{dx} &= -8e^{-2x}\sin(3x) + 12e^{-2x}\cos(3x) \text{ [1 mark]} \\ &= 4e^{-2x}(3\cos(3x) - 2\sin(3x)) \\ \frac{d^2y}{dx^2} &= -8e^{-2x}(3\cos(3x) - 2\sin(3x)) + 4e^{-2x}(-6\sin(3x) - 6\cos(3x)) \\ &= -4e^{-2x}(5\sin(3x) + 12\cos(3x)) \text{ [1 mark]} \end{aligned}$$

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

$0 = 4e^{-2x}(-5\sin(3x) - 12\cos(3x) + a(3\cos(3x) - 2\sin(3x)) + b\sin(3x))$

$0 = 4e^{-2x}((-5 - 2a + b)\sin(3x) + (-12 + 3a)\cos(3x))$ [1 mark]

$-12 + 3a = 0$

$\Rightarrow a = 4$ [1 mark]

$-5 - 2a + b = 0$

$\Rightarrow b = 2a + 5$

$b = 13$ [1 mark]

Question 3

$$(5x+3)^2\frac{dy}{dx} + 4 = 0, \quad y(-1) = 2$$

$$\begin{aligned} (5x+3)^2\frac{dy}{dx} &= -4 \\ \frac{dy}{dx} &= \frac{-4}{(5x+3)^2} \\ y &= \int \frac{-4}{(5x+3)^2} dx \\ &= \frac{4}{5(5x+3)} + c \text{ [1 mark]} \end{aligned}$$

When $x = -1$, $y = 2$

$2 = \dfrac{2}{-5} + c$

$c = \dfrac{12}{5}$ [1 mark]

$y = \dfrac{4}{5(5x+3)} + \dfrac{12}{5}$

$y = \dfrac{4 + 12(5x+3)}{5(5x+3)}$

$y = \dfrac{60x + 40}{5(5x+3)}$

$y = \dfrac{12x+8}{5x+3}, x \neq \dfrac{-3}{5}$ [1 mark]

Question 4

$y = \tan^{-1}\left(\dfrac{x}{3}\right)$

$\dfrac{dy}{dx} = \dfrac{3}{x^2+9}$

$\left(x^2+9\right)\dfrac{dy}{dx} = 3$

Therefore, $y = \tan^{-1}\left(\dfrac{x}{3}\right)$ is the solution to the differential equation $\left(x^2+9\right)\dfrac{dy}{dx} - 3 = 0$

The correct answer is **A**.

Question 5

$y = c\,t\sin(2t)$

$\dfrac{dy}{dx} = c\left[\sin(2t) + 2t\cos(2t)\right]$ [1 mark]

$\dfrac{d^2y}{dx^2} = c\left[2\cos(2t) + 2\cos(2t) - 4t\sin(2t)\right]$ [1 mark]

$= c\left[4\cos(2t) - 4t\sin(2t)\right]$

$\dfrac{d^2y}{dx^2} + 4y = 4c\cos(2t) - 4ct\sin(2t) + 4ct\sin(2t)$

$= 4c\cos(2t)$

$4c\cos(2t) = 8\cos(2t)$

$c = 2$ [1 mark]

Question 6

$\sqrt{2-x^2}\dfrac{dy}{dx} = \dfrac{1}{2-y}$

Separate the variables:

$\displaystyle\int (2-y)\,dy = \int \frac{1}{\sqrt{2-x^2}}dx$

$2y - \dfrac{y^2}{2} = \sin^{-1}\left(\dfrac{x}{\sqrt{2}}\right) + c$

When $y = 0, x = 1$:$0 = \sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) + c \Rightarrow c = -\dfrac{\pi}{4}$

$$4y - y^2 = 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{\pi}{2}$$

$$y^2 - 4y + 4 = (y-2)^2 = 4 + \frac{\pi}{2} - 2\sin^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$$

$$y - 2 = \pm\sqrt{4 + \frac{\pi}{2} - 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)}$$

Since $y = 1$ when $x = 0$, take the negative result:

$$y = 2 - \sqrt{4 + \frac{\pi}{2} - 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)}$$

Award 1 mark for separating the variables.
Award 1 mark for correct integration on both sides.
Award 1 mark for evaluating the constant of integration.
Award 1 mark for completing the square.
Award 1 mark for the final correct expression.

***VCAA Assessment Report* note:**

Students found this question challenging. Most students were able to separate the variables correctly, but there were then many errors in the subsequent integration. Errors were seen in the integration of the polynomial part but far more in the integration of the term involving the reciprocal of the square root, despite the formula being on the formula sheet. It was common for logarithms to be seen. Some students were unable to proceed by completing the square or otherwise having integrated. A large number of students, when confronted with a square equals a constant, gave only the positive root. Many gave both roots but did not realise that only the negative root satisfied the initial conditions. Several students omitted the constant of integration; others made mistakes when attempting to evaluate the constant. A number of students interpreted $y(1) = 0$ as $x = 0$ when $y = 1$, while some attempted to solve for x rather than y.

Question 7

$$\frac{dy}{dx} = \sqrt{2x^6 + 1} \Rightarrow y = \int_0^x \sqrt{2t^6 + 1}\, dt + c$$

When $y = 5, x = 1$

$$\Rightarrow 5 = \int_0^1 \sqrt{2t^6 + 1}dt + c$$

$$c = 5 - \int_0^1 \sqrt{2t^6 + 1}dt$$

$$y = \int_0^x \sqrt{2t^6 + 1}\, dt + 5 - \int_0^1 \sqrt{2t^6 + 1}\, dt$$

When x = 4,

$$y = \int_0^4 \sqrt{2t^6 + 1}\, dt - \int_0^1 \sqrt{2t^6 + 1}\, dt + 5$$

$$y = \int_0^4 \sqrt{2t^6 + 1}\, dt + \int_0^1 \sqrt{2t^6 + 1}\, dt + 5$$

$$y = \int_1^4 \sqrt{2t^6 + 1}\, dt + 5$$

Question 8

$$\frac{dy}{dx} = -(\cos(x) - \sin(x))\, e^{-x}$$

$$\frac{d^2y}{dx^2} = 2\cos(x)\, e^{-x}$$

$$2\cos(x)\, e^{-x} - k(\cos(x) - \sin(x))\, e^{-x}$$

$$= 2\sin(x)\, e^{-x}$$

$$-k = \frac{2\sin(x)\, e^{-x} - 2\cos(x)\, e^{-x}}{(\cos(x) - \sin(x))\, e^{-x}}$$

$$= -2$$

$$\therefore k = 2$$

Question 9

$$f(x) = \int_0^2 \sin\left(x^2\right) dx + 4$$

$$\approx 4.8$$

Question 10

$$f(x) = \int_0^1 e^{2x} dx + 2$$

$$\approx 5.19$$

9 Further integration techniques and applications

Topic	9	Further integration techniques and applications
Subtopic	9.2	Integration by parts

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To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at **www.jacplus.com.au**.

Question 1 (2 marks)

Find $\int x^3 \log_e(2x)\, dx$.

Question 2 (4 marks)

In determining the Fourier series to represent the function $f(x) = x$ in the interval $-\pi \le x \le \pi$, the Fourier coefficients are given by $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\ dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\ dx$, where n is a positive integer.

Show that $a_n = 0$ and $b_n = -\frac{2(-1)^n}{n}$.

Question 3 (3 marks)

Let $J_n(x) = \int x^n e^{kx} dx$.

a. Use integration by parts to show that $J_n(x) = \frac{1}{k}x^n e^{kx} - \frac{n}{k}J_{n-1}(x)$. **(1 mark)**

b. Hence, evaluate $\int_0^1 x^3 e^{2x} dx$. **(2 marks)**

Topic	**9**	**Further integration techniques and applications**
Subtopic	**9.3**	**Integration by recognition and graphs of anti-derivatives**

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Question 1 (4 marks)

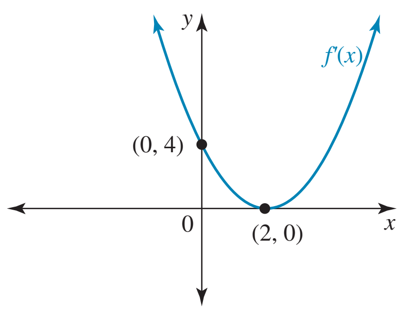
Determine $\frac{d}{dx}\left[\cos^{-1}\left(\frac{4}{x^2}\right)\right]$ and hence evaluate $\int_2^{2\sqrt{2}} \frac{1}{x\sqrt{x^4-16}}dx$.

Question 2 (4 marks)

Find $\frac{d}{dx}\left[\log_e(\tan(x)+\sec(x))\right]$ and hence determine the area enclosed between the curve $y=\sec(x)$, the coordinate axes and the line $x=\frac{\pi}{4}$.

Question 3 (2 marks)

Consider the following graph of a gradient function, $f'(x)$.



Sketch the graph of the original function, $f(x)$.

Topic	9	Further integration techniques and applications
Subtopic	9.4	Solids of revolution

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Source: *VCE 2020, Specialist Mathematics Exam 1, Q.8; © VCAA*

Question 1 (5 marks)

Find the volume, V of the solid of revolution formed when the graph of $y = 2\sqrt{\dfrac{x^2+x+1}{(x+1)\left(x^2+1\right)}}$ is rotated about the x-axis over the interval $[0, \sqrt{3}]$.
Give your answer in the form $V = 2\pi\left(\log_e(a) + b\right)$, where $a,\ b,\ \in R$.

Source: *VCE 2019, Specialist Mathematics Exam 1, Q.8; © VCAA*

Question 2 (4 marks)

Find the volume of the solid of revolution formed when the graph of $y = \sqrt{\dfrac{1+2x}{1+x^2}}$ is rotated about the x-axis over the interval $[0, 1]$.

Source: *VCE 2015, Specialist Mathematics Exam 1, Q5; © VCAA*

Question 3 (3 marks)

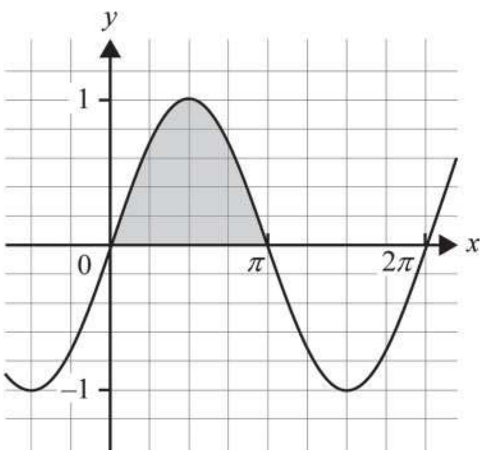
Find the volume generated when the region bounded by the graph of $y = 2x^2 - 3$, the line $y = 5$ and the y-axis is rotated about the y-axis.

Source: *VCE 2021, Specialist Mathematics 1, Q.4; © VCAA*

Question 4 (4 marks)

Answer the following.

a. The shaded region in the diagram below is bounded by the graph of $y = \sin(x)$ and the x-axis between the first two non-negative x-intercepts of the curve, that is, the interval $[0, \pi]$. The shaded region is rotated about the x-axis to form a solid of revolution. **(3 marks)**



Find the volume, V_s, of the solid formed.

b. Now consider the function $y = \sin(kx)$, where k is a positive real constant. The region bounded by the graph of the function and the x-axis between the first two non-negative x-intercepts of the graph is rotated about the x-axis to form a solid of revolution.

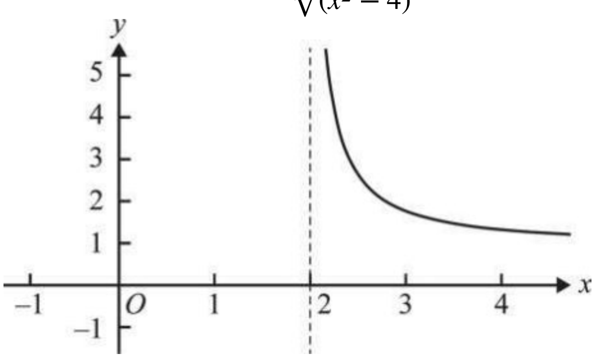
Find the volume of this solid in terms of V_s. **(1 mark)**

Source: *VCE 2014, Specialist Mathematics 1, Q.6; © VCAA*

Question 5 (4 marks)

a. Verify that $\dfrac{a}{a-4} = 1 + \dfrac{4}{a-4}$. **(1 mark)**

Part of the graph of $y = \dfrac{x}{\sqrt{(x^2-4)}}$ is shown below.



b. The region enclosed by the graph of $y = \dfrac{x}{\sqrt{(x^2-4)}}$ and the lines $y = 0$, $x = 3$ and $x = 4$ is rotated about the x-axis.

Find the volume of the resulting solid of revolution. **(4 marks)**

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.10; © VCAA*

Question 6 (1 mark)

The region bounded by the lines $x = 0, y = 3$ and the graph of $y = x^{\frac{4}{3}}$ where $x \geq 0$ is rotated about the y-axis to form a solid of revolution. The volume of this solid is

A. $\frac{81\pi 3^{\frac{2}{3}}}{11}$

B. $\frac{12\pi 3^{\frac{3}{4}}}{7}$

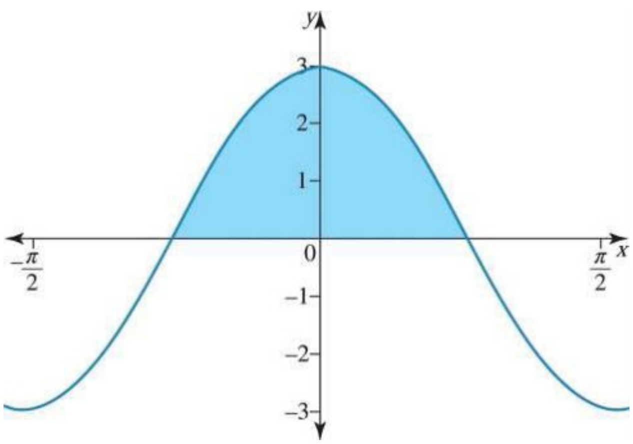
C. $\frac{27\pi 3^{\frac{1}{3}}}{7}$

D. $\frac{18\pi 3^{\frac{1}{2}}}{5}$

E. $\frac{6\pi 3^{\frac{1}{2}}}{5}$

Question 7 (1 mark)

The shaded region is the area bounded by the x-axis and the graph of $y = 3\cos(2x)$. This region is rotated about the x-axis to form a solid of revolution. The volume of the solid, in cubic units, is



A. 44.4132

B. $\frac{9\pi^2}{2}$

C. $18\pi^2$

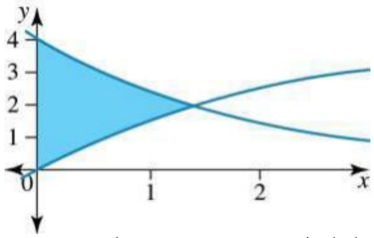
D. 3

E. $\frac{9\pi^2}{4}$

Question 8 (1 mark)

The diagram shows the graphs of $y = 4e^{-\frac{x}{2}}$ and $y = 3\sin\left(\frac{x}{2}\right)$.

If the curves intersect at the point (b, c) and the shaded area is rotated about the x-axis to form a volume of revolution, the volume is given by



A. $\pi \int_0^b \left(16e^{-x} - 9\sin^2\left(\frac{x}{2}\right)\right) dx$

B. $\pi \int_0^b \left(9\sin^2\left(\frac{x}{2}\right) - 16e^{-x}\right) dx$

C. $\pi \int_0^b \left(4e^{-\frac{x}{2}} - 3\sin\left(\frac{x}{2}\right)\right)^2 dx$

D. $\pi \int_0^b \left(3\sin\left(\frac{x}{2}\right) - 4e^{-\frac{x}{2}}\right)^2 dx$

E. $\pi \int_0^b \left(16e^{-\frac{x^2}{4}} - 9\sin^2\left(\frac{x}{2}\right)\right) dx$

Topic	9	Further integration techniques and applications
Subtopic	9.5	Volumes

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Source: *VCE 2018, Specialist Mathematics Exam 1, Q9bc; © VCAA*

Question 1 (3 marks)

Answer the following.

a. Find the x-coordinates of the points of intersection of the curve $x^2 - 2y^2 = 1$ and the line $y = x - 1$. **(1 mark)**

$x = \square$, $x = \square$

b. Find the volume of the solid of revolution formed when the region bounded by the curve and the line is rotated about the x-axis. **(2 marks)**

Source: *VCE 2013, Specialist Mathematics Exam 2, Section 1, Q10; © VCAA*

Question 2 (1 mark)

The region bounded by the lines $x = 0$, $y = 3$ and the graph of $y = x^{\frac{4}{3}}$ where $x \geq 0$ is rotated about the y-axis to form a solid of revolution. The volume of this solid is

A. $\dfrac{81\pi\, 3^{\frac{2}{3}}}{11}$

B. $\dfrac{12\pi\, 3^{\frac{1}{4}}}{7}$

C. $\dfrac{27\pi\, 3^{\frac{1}{3}}}{7}$

D. $\dfrac{18\pi\, 3^{\frac{1}{2}}}{5}$

E. $\dfrac{6\pi\, 3^{\frac{1}{2}}}{5}$

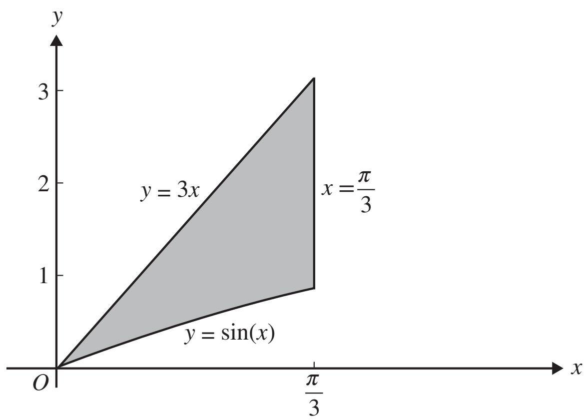
Source: *VCE 2013, Specialist Mathematics Exam 1, Q9; © VCAA*

Question 3 (4 marks)

The shaded region below is enclosed by the graph of $y = \sin(x)$ and the lines $y = 3x$ and $x = \frac{\pi}{3}$.

This region is rotated about the x-axis.

Find the volume of the resulting solid of revolution.



Topic	9	Further integration techniques and applications
Subtopic	9.6	Arc length and surface area

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Source: *VCE 2016, Specialist Mathematics Exam 1, Q7; © VCAA*

Question 1 (4 marks)

Find the arc length of the curve $y = \frac{1}{3}\left(x^2 + 2\right)^{\frac{3}{2}}$ from $x = 0$ to $x = 2$.

Question 2 (3 marks)

Find the surface area obtained by rotating the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$, about the x-axis.

Question 3 (8 marks)

a. Prove that the circumference of a circle of radius r is $2\pi r$. **(4 marks)**

b. Prove that the total surface area of a sphere of radius r is $4\pi r^2$. **(4 marks)**

Topic	9	Further integration techniques and applications
Subtopic	9.7	Water flow

online only

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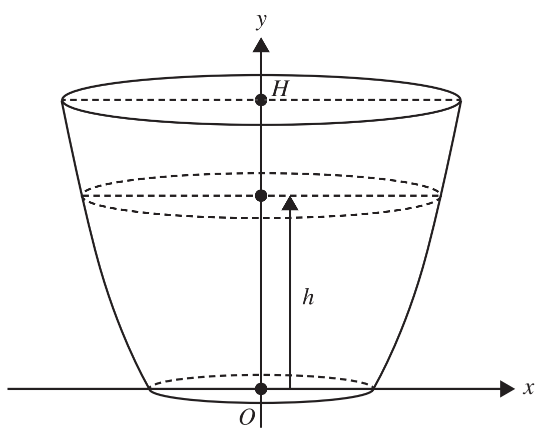
Source: *VCE 2021, Specialist Mathematics Exam 2, Section B, Q.3; © VCAA*

Question 1 (10 marks)

A thin-walled vessel is produced by rotating the graph of $y = x^3 - 8$ about the y-axis for $0 \leq y \leq H$.
All lengths are measured in centimetres.



a. i. Write down a definite integral in terms of y and H for the volume of the vessel in cubic centimetres. **(1 mark)**

ii. Hence, find an expression for the volume of the vessel in terms of H. **(1 mark)**

$V(H) = \square$

Water is poured into the vessel. However, due to a crack in the base, water leaks out at a rate proportional to the square root of the depth h of water in the vessel, that is $\frac{dv}{dt} = -4\sqrt{h}$, where V is the volume of water remaining in the vessel, in cubic centimetres, after t minutes.

b. **i.** Show that $\frac{dh}{dt} = \frac{-4\sqrt{h}}{\pi(h+8)^{\frac{2}{3}}}$. **(2 marks)**

ii. Find the maximum rate, in centimetres per minute, at which the depth of water in the vessel decreases, correct to two decimal places, and find the corresponding depth in centimetres. **(2 marks)**

Depth = ☐ cm, rate = ☐ cm/min

iii. Let $H = 50$ for a particular vessel. The vessel is initially full and water continues to leak out at a rate of $4\sqrt{h}\,\text{cm}^3\,\text{min}^{-1}$. Find the maximum rate at which water can be added, in cubic centimetres per minute, without the vessel overflowing. **(1 mark)**

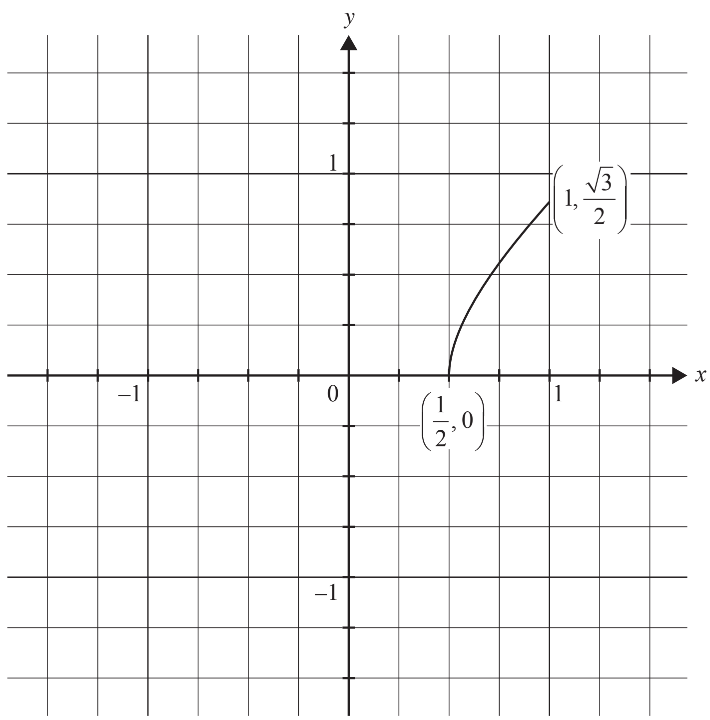
☐ cm^3/min

c. The vessel is initially full where $H = 50$ and water leaks out at a rate of $4\sqrt{h}\,\text{cm}^3\,\text{min}^{-1}$. When the depth of the water drops to 25 cm, extra water is poured in at a rate of $40\sqrt{2}\,\text{cm}^3\,\text{min}^{-1}$. Find how long it takes for the vessel to refill completely from a depth of 25 cm. Give your answer in minutes, correct to one decimal place. **(3 marks)**

Source: *VCE 2018, Specialist Mathematics Exam 2, Section B, Q3a-d,f; © VCAA*

Question 2 (11 marks)

Part of the graph of $y = \frac{1}{2}\sqrt{4x^2 - 1}$ is shown below.



The curve shown is rotated about the y-axis to form a volume of revolution that is to model a fountain, where length units are in metres.

a. Show that the volume, V cubic metres, of water in the fountain when it is filled to a depth of h metres is given by $V = \frac{\pi}{4}\left(\frac{4}{3}h^3 + h\right)$. **(2 marks)**

b. Find the depth h when the fountain is filled to half its volume. Give your answer in metres, correct to two decimal places. **(2 marks)**

The fountain is initially empty. A vertical jet of water in the centre fills the fountain at a rate of 0.04 cubic metres per second and, at the same time, water flows out from the bottom of the fountain at a rate of $0.05\sqrt{h}$ cubic metres per second when the depth is h metres.

c. i. Show that $\dfrac{dh}{dt} = \dfrac{4 - 5\sqrt{h}}{25\pi\left(4h^2 + 1\right)}$. **(2 marks)**

ii. Find the rate, in metres per second, correct to four decimal places, at which the depth is increasing when the depth is 0.25 m. **(1 mark)**

☐ m/s

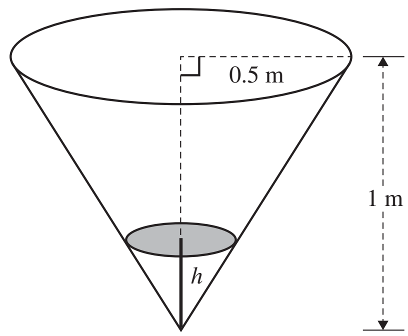
d. Express the time taken for the depth to reach 0.25 m as a definite integral and evaluate this integral correct to the nearest tenth of a second. **(2 marks)**

e. How far from the top of the fountain does the water level ultimately stabilise? Give your answer in metres, correct to two decimal places. **(2 marks)**

Source: *VCE 2014, Specialist Mathematics Exam 2, Section 2, Q4; © VCAA*

Question 3 (12 marks)

At a water fun park, a conical tank of radius 0.5 m and height 1 m is filling with water. At the same time, some water flows out from the vertex wetting those underneath. When the tank eventually fills, it tips over and the water falls out, drenching all those underneath. The tank then returns to its original position and begins to refill.



Water flows in at a constant rate of 0.02π m^3/min and flows out at a variable rate of $0.01\pi\sqrt{h}$ m^3/min, where h metres is the depth of the water at any instant.

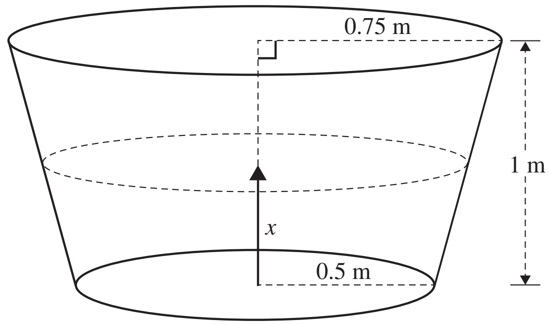
a. Show that the volume, V cubic metres, of water in the cone when it is filled to a depth of h metres is given by $V = \frac{\pi}{12}h^3$. **(1 mark)**

b. Find the rate, in m/min at which the depth of the water in the tank is increasing when the depth is 0.25 m. **(4 marks)**

The tank is empty at time $t = 0$ minutes.

c. By using an appropriate definite integral, find the time it takes for the tank to fill. Give your answer in minutes, correct to one decimal place. **(2 marks)**

Another water tank, shown below, has the shape of a large bucket (part of a cone) with the dimensions given. Water fills the tank at a rate of 0.05π m^3/min, but no water leaks out.



When filled to a depth of x metres, the volume of water, V cubic metres, in the tank is given by $V = \frac{\pi}{48}\left(x^3 + 6x^2 + 12x\right)$.

d. Given that the tank is initially empty, find the depth, x metres, as a function of time t. **(5 marks)**

Topic	9	Further integration techniques and applications
Subtopic	9.8	Review

online only

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Source: *VCE 2019, Specialist Mathematics Exam 2, Section B, Q1e; © VCAA*

Question 1 (2 marks)

The portion of the curve given by $y = \sqrt{x^2 - 2x}$ for $x \in [2, 4]$ is rotated about the y-axis to form a solid of revolution. Write down, but do not evaluate, a definite integral in terms of t that gives the volume of the solid formed. **(2 marks)**

Source: *VCE 2014, Specialist Mathematics Exam 2, Section 2, Q1; © VCAA*

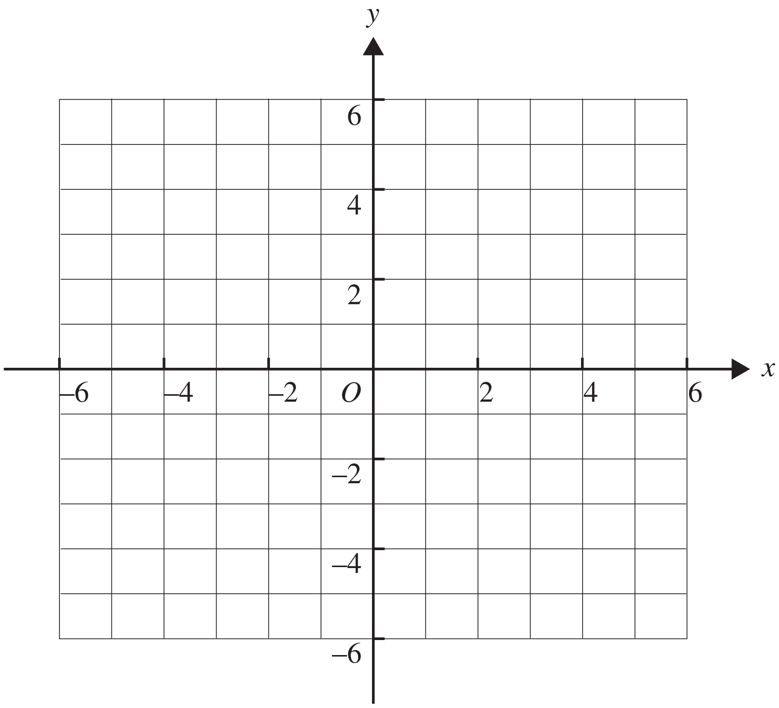
Question 2 (11 marks)

Consider the function f with rule $f(x) = \dfrac{9}{(x+2)(x-4)}$ over its maximal domain.

a. Find the coordinates of the stationary point(s). **(3 marks)**

b. State the equation of all asymptotes of the graph of f. **(2 marks)**

c. Sketch the graph of f for $x \in [-6, 6]$ on the axes below, showing asymptotes, the values of the coordinates of any intercepts with the axes, and the stationary point(s). You may wish to upload an image using the image upload tool. **(3 marks)**



The region bounded by the coordinate axes, the graph of f and the line $x = 3$, is rotated about the x-axis to form a solid of revolution.

d. i. Write down a definite integral in terms of x that gives the volume of this solid of revolution. **(2 marks)**

ii. Find the volume of this solid, correct to two decimal places. **(1 mark)**

Volume = ☐ units^3

Question 3 (3 marks)

A vase is formed, when part of the curve $\frac{x^2}{16} - \frac{65y^2}{4096} = 1$ for $4 \leq x \leq 9$, $y \geq 0$, is rotated about the y-axis to form a volume of revolution. The x and y coordinates are measured in centimetres. The vase has a small crack in the base and the water leaks out at a rate proportional to the square root of the remaining height of the water. Initially the vase was full and after 10 minutes the height of the water in the vase is 9 cm. Determine how much more time will pass before the vase is empty, in minutes correct to one decimal place.

Question 4 (6 marks)

If n is a positive integer, and a is a positive real number then:

a. Show that $\int \cos^n(ax)\,dx = \frac{1}{an}\sin(ax)\cos^{n-1}(ax) + \frac{n-1}{n}\int \cos^{n-2}(ax)\,dx$. **(3 marks)**

b. Hence, determine $\int \cos^5(3x)\,dx$. **(3 marks)**

Question 5 (4 marks)

A soup bowl has a circular base and top, with radii of 4 and 9 cm respectively. Its height is 16 cm. The origin O is at the centre of the base of the bowl. The bowl is formed when the curve $y = ax^2 + b$ is rotated about the y-axis. Initially the bowl is filled with soup, but the soup leaks out at a rate equal to $2\sqrt{h}\ \text{cm}^3/\text{min}$, where h cm is the height of the soup remaining in the bowl after t minutes.
Set up the differential equation for h and t and determine how long before the bowl is empty, in minutes correct to one decimal place.

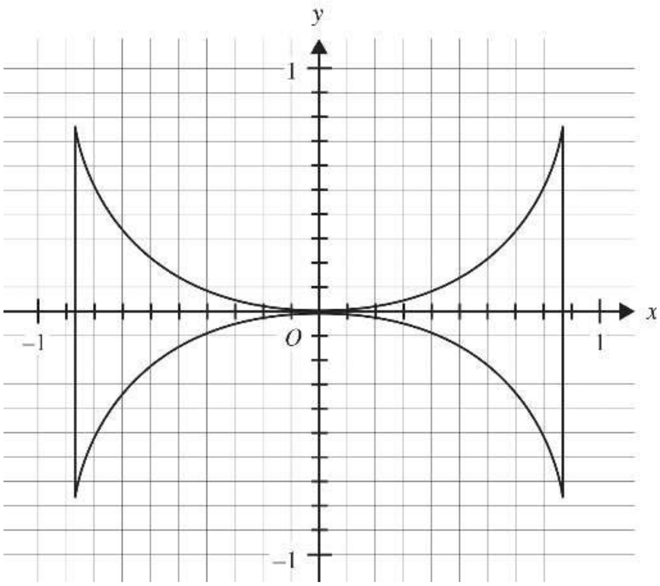
Source: *VCE 2015, Specialist Mathematics 2, Section 2, Q.3; © VCAA*

Question 6 (11 marks)

A manufacturer of bow ties wishes to design an advertising logo, represented below, where the upper boundary curve in the first and second quadrants is given by the parametric relations

$x = \sin(t)$, $y = \frac{1}{2}\sin(t)\tan(t)$, for $t \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

The logo is symmetrical about the x-axis.



a. Find an expression for $\frac{dy}{dx}$ in terms of t. **(2 marks)**

b. Find the slope of the upper boundary curve where $t = \frac{\pi}{6}$. Give your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are positive integers. **(2 marks)**

c. i. Verify that the cartesian equation of the upper boundary curve is $y = \dfrac{x^2}{2\sqrt{1-x^2}}$ **(1 mark)**

ii. State the domain for x of the upper boundary curve. **(1 mark)**

d. Show that $\dfrac{d}{dx}(\arcsin(x)) = \dfrac{2x^2}{\sqrt{1-x^2}} + \dfrac{d}{dx}\left(x\sqrt{1-x^2}\right)$ by simplifying the right-hand side of this equation. **(2 marks)**

e. Hence write down an antiderivative in terms of x, to be evaluated between two appropriate terminals, and find the area of the advertising logo. **(3 marks)**

Answers and marking guide

9.2 Integration by parts

Question 1

$$\int x^3 \log_e(2x)\, dx$$

$u = \log_e(2x) \quad \frac{dv}{dx} = x^3$

$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{1}{4}x^4$

$$\int x^3 \log_e(2x)\, dx = \frac{1}{4}x^4 \log_e(2x) - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\, dx$$ **[1 mark]**

$$= \frac{1}{4}x^4 \log_e(2x) - \frac{1}{4}\int x^3 dx$$

$$= \frac{1}{4}x^4 \log_e(2x) - \frac{1}{16}x^4 + c$$

$$= \frac{1}{16}x^4 \left(4\log_e(2x) - 1\right) + c$$ **[1 mark]**

Question 2

$n \in Z$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\, dx$$

$u = x \quad \frac{dv}{dx} = \cos(nx)$

$\frac{du}{dx} = 1 \quad v = \frac{1}{n}\sin(nx)$

$$a_n = \frac{1}{\pi}\left[\left[\frac{x}{n}\sin(nx)\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} \sin(nx)\, dx\right]$$ **[1 mark]**

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\cos(n\pi) + \frac{\pi}{n}\sin(-n\pi) - \frac{1}{n^2}\cos(-n\pi)\right]$$

Now $\sin(n\pi) = 0 \quad \sin(-n\pi) = 0$

$\cos(n\pi) = \cos(-n\pi) \quad = (-1)^n$

So $a_n = 0$ **[1 mark]**

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\, dx$$

$u = x \quad \frac{dv}{dx} = \sin(nx)$

$\frac{du}{dx} = 1 \quad v = \frac{-1}{n}\cos(nx)$

$$b_n = \frac{1}{\pi}\left[\left[-\frac{x}{n}\cos(nx)\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right]$$ **[1 mark]**

$$= \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - \frac{\pi}{n}\cos(-n\pi) - \frac{1}{n^2}\sin(-n\pi)\right]$$

$$= -\frac{2}{n}\cos(n\pi)$$

$$= -\frac{2}{n}(-1)^n$$ **[1 mark]**

Question 3

a. $J_n = \int x^n e^{kx}\,dx$

$u = x^n \qquad \frac{dv}{dx} = e^{kx}$

$\frac{du}{dx} = nx^{n-1} \quad v = \frac{1}{k}e^{kx}$

$J_n = \frac{1}{k}x^n e^{kx} - \frac{n}{k}\int x^{n-1}e^{kx}\,dx$

$J_n = \frac{1}{k}x^n e^{kx} - \frac{n}{k}J_{n-1}$ **[1 mark]**

b. $J_0 = \int e^{kx}\,dx = \frac{1}{k}e^{kx}$

$J_1 = \int xe^{kx}dx = \frac{1}{k}xe^{kx} - \frac{1}{k}J_0$

$= \frac{1}{k}xe^{kx} - \frac{1}{k^2}e^{kx}$

$J_2 = \int x^2e^{kx}dx = \frac{1}{k}x^2e^{kx} - \frac{2}{k}J_1$

$= \frac{1}{k}x^2e^{kx} - \frac{2}{k}\left(\frac{1}{k}xe^{kx} - \frac{1}{k^2}e^{kx}\right)$

$= \frac{1}{k}e^{kx}\left(x^2 - \frac{2x}{k} + \frac{2}{k^2}\right)$

$J_3 = \int x^3e^{kx}\,dx = \frac{1}{k}x^3e^{kx} - \frac{3}{k}J_2$

$= \frac{1}{k}x^3e^{kx} - \frac{3}{k}\left(\frac{1}{k}e^{kx}\left(x^2 - \frac{2x}{k} + \frac{2}{k^2}\right)\right)$

$= \frac{1}{k}e^{kx}\left(x^3 - \frac{3x^2}{k} + \frac{6x}{k^2} - \frac{6}{k^3}\right)$ **[1 mark]**

$$\text{So} \int_0^1 x^3 e^{2x}\,dx \quad k = 2 \qquad n = 3$$

$$= \left[\frac{1}{2}e^{2x}\left(x^3 - \frac{3x^2}{2} + \frac{6x}{4} - \frac{6}{8}\right)\right]_0^1$$

$$= \frac{1}{2}e^2\left(1 - \frac{3}{2} + \frac{6}{4} - \frac{3}{4}\right) - \frac{1}{2}e^0\left(0 - \frac{3}{4}\right)$$

$$= \frac{1}{8}\left(e^2 + 3\right)$$ **[1 mark]**

9.3 Integration by recognition and graphs of anti-derivatives

Question 1

$$y = \cos^{-1}\left(\frac{4}{x^2}\right)$$

$$= \cos^{-1}(u) \qquad u = \frac{4}{x^2} = 4x^{-2}$$

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \quad \frac{du}{dx} = -8x^{-3} = -\frac{8}{x^3}$$

$$\frac{dy}{dx} = \frac{8}{x^3}\sqrt{\frac{1}{1-\frac{16}{x^4}}}$$

$$= \frac{8}{x^3}\sqrt{\frac{x^4}{x^4-16}}$$

$$= \frac{8}{x^3} \times \frac{x^2}{\sqrt{x^4-16}}$$ **[1 mark]**

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{4}{x^2}\right)\right) = \frac{8}{x\sqrt{x^4-16}}, \qquad |x| > 2$$ **[1 mark]**

$$\int_2^{2\sqrt{2}} \frac{1}{x\sqrt{x^4-16}}\,dx = \frac{1}{8}\left[\cos^{-1}\left(\frac{4}{x^2}\right)\right]_2^{2\sqrt{2}}$$

$$= \frac{1}{8}\left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(1)\right)$$ **[1 mark]**

$$= \frac{1}{8}\left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi}{24}$$ **[1 mark]**

Question 2

$$\frac{d}{dx}(\sec(x)) = \tan(x)\sec(x), \quad \frac{d}{dx}(\tan(x))$$

$$= \sec^2(x)$$ **[1 mark]**

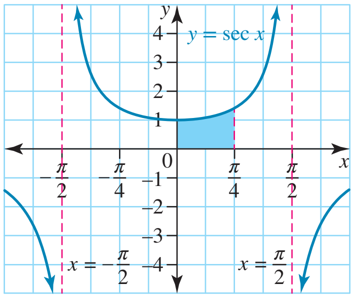
If $y = \log_e(\tan(\mathrm{x}) + \sec(\mathrm{x}))$

$$\frac{dy}{dx} = \frac{\sec^2(x) + \tan(x)\sec(x)}{\tan(x) + \sec(x)}$$

$$= \frac{\sec(x)(\tan(x) + \sec(x))}{\tan(x) + \sec(x)}$$

$$= \sec(x)$$ **[1 mark]**

so $\int \sec(x)\,dx = \log_e(\tan(x) + \sec(x)) + c$



$$A = \int_0^{\frac{\pi}{4}} \sec(x)\ dx$$

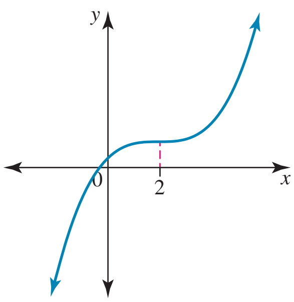
$= \left[\log_e(\tan(x) + \sec(x))\right]_0^{\frac{\pi}{4}}$ **[1 mark]**

$= \log_e\left(\tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right)\right) - \log_e(\tan(0) + \sec(0))$

$= \log_e\left(1 + \sqrt{2}\right)$ **[1 mark]**

Question 3

A stationary point of inflection at $x = 2$.



Award **1 mark** for correct shape of graph and **1 mark** for the stationary point of inflection at $x = 2$.

9.4 Solids of revolution

Question 1

$$y = 2\sqrt{\frac{x^2 + x + 1}{(x+1)\left(x^2+1\right)}}$$

$$V = \pi\int_a^b y^2\,dx = 4\pi\int_0^{\sqrt{3}} \frac{x^2 + x + 1}{(x+1)\left(x^2+1\right)}\,dx$$

By partial fractions,

$$\frac{A}{x+1} + \frac{Bx + C}{x^2+1} = \frac{A\left(x^2+1\right) + (x+1)(Bx+C)}{(x+1)\left(x^2+1\right)}$$

$$= \frac{x^2(A+B) + x(B+C) + \mathrm{A} + \mathrm{C}}{(x+1)\left(x^2+1\right)}$$

$$= \frac{x^2 + x + 1}{(x+1)\left(x^2+1\right)}$$

Equating coefficients:
$x^2 : A + B = 1,\ x^1 : B + C = 1,\ x^0 : A + C = 1$
$\Rightarrow A = B = C = \frac{1}{2}$

$$v = 2\pi \int_0^{\sqrt{3}} \left(\frac{1}{x+1} + \frac{x+1}{x^2+1} \right) dx$$

$$v = 2\pi \int_0^{\sqrt{3}} \left(\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1} \right) dx$$

$$v = 2\pi \left[\log_e (x+1) + \frac{1}{2} \log_e \left(x^2+1 \right) + \tan^{-1}(x) \right]_0^{\sqrt{3}}$$

$$v = 2\pi \left[\log_e \left(1+\sqrt{3} \right) + \frac{1}{2} \log_e (4) + \tan^{-1}\left(\sqrt{3} \right) - 0 \right]$$

$$v = 2\pi \left(\log_e \left(2+2\sqrt{3} \right) + \frac{\pi}{3} \right) \text{ units}^3,\ a = 2+2\sqrt{3},\ b = \frac{\pi}{3}$$

Award **1 mark** for the correct define integral for the volume.
Award **1 mark** for using partial fractions.
Award **2 marks** for correct integration.
Award **1 mark** for the final correct volume.

Question 2

$$y = \sqrt{\frac{1+2x}{1+x^2}},\ V = \pi \int_0^b y^2\, dx$$

$$v = \pi \int_0^1 \left(\frac{1+2x}{1+x^2} \right) dx$$

$$v = \pi \left[\int_0^1 \frac{1}{1+x^2}\, dx + \int_0^1 \frac{2x}{1+x^2}\, dx \right]$$

$$v = \pi \left[\tan^{-1}(x) + \log_e \left(1+x^2 \right) \right]_0^1$$

$$v = \pi \left[\tan^{-1}(1) + \log_e (2) - \tan^{-1}(0) - \log_e (1) \right]$$

$$v = \pi \left(\frac{\pi}{4} + \log_e (2) \right) \text{ units}^3$$

Award **1 mark** for the correct definite integral.
Award **1 mark** each for the correct integration of each term.
Award **2 marks** for the correct final volume.
VCAA Examination Report note:
Most students were able to write down the correct integral to find the volume of solid of revolution. Some students did not recognise the way in which the integrand split naturally and had difficulty proceeding further with the question. Some attempted solutions using partial fractions were seen. Many students who were able to successfully split the integrand used a substitution method to integrate $\int_0^1 \frac{2x}{1+x^2}\, dx$. This was unnecessary and resulted in a loss of marks if not done correctly.

Question 3

$y = 2x^2 - 3$

$x^2 = \frac{1}{2}(y+3)$

$V_y = \pi \int_a^b x^2 \, dy$

$= \pi \int_{-3}^{5} \frac{1}{2}(y+3) \, dy$ **[1 mark]**

$= \frac{\pi}{2}\left[\frac{1}{2}y^2 + 3y\right]_{-3}^{5}$ **[1 mark]**

$= \frac{\pi}{2}\left[\left(\frac{25}{2} + 15\right) - \left(\frac{9}{2} - 9\right)\right]$

$= 16\pi$ units^3 **[1 mark]**

Question 4

a. $V_s = \pi \int_0^{\pi} \sin^2(x) \, dx$

$V_s = \frac{\pi}{2}\int_0^{\pi}(1 - \cos(2x)) \, dx$

$V_s = \frac{\pi}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi}$

$V_s = \frac{\pi}{2}\left[\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right]$

$V_s = \frac{\pi^2}{2}$

Award **1 mark** for the correct definite integral for the volume.
Award **1 mark** for the correct antiderivative and evaluating.
Award **1 mark** for the final correct volume.

b. The volume is dilated by a factor of $\frac{1}{k}$ parallel to the x-axis, so $V = \frac{1}{k}V_s$.

$V = \pi \int_0^{\frac{\pi}{k}} \sin^2(kx) \, dx$ alternatively

$V = \frac{\pi}{2}\int_0^{\frac{\pi}{k}}(1 - \cos(2kx)) \, dx$

$V = \frac{\pi}{2}\left[x - \frac{1}{2k}\sin(2kx)\right]_0^{\frac{\pi}{k}}$

$V = \frac{\pi}{2}\left[\left(\frac{\pi}{k} - \frac{1}{2k}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right]$

$V = \frac{\pi^2}{2k} = \frac{1}{k}V_s$ **[1 mark]**

$\frac{1}{k}$

Question 5

a. $\frac{a}{a-4}=\frac{a-4+4}{a-4}$

$=\frac{a-4}{a-4}+\frac{4}{a-4}$

$=1+\frac{4}{a-4}$

Award 1 mark for the correct final verification.

***VCAA Assessment Report* notes:**

This question was answered well, but many students did not know how a verification or proof should be set out. Some arguments were not convincing, and some eventually showed that $a = a$ or similar.

b. $y=\frac{x}{\sqrt{(x^2-4)}}$

$V=\pi\int_3^4 \frac{x^2}{x^2-4}dx$

$=\pi\int_3^4\left(1+\frac{4}{x^2-4}\right)dx$

$\frac{4}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$

$=\frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$

$=\frac{x(A+B)+2(A-B)}{x^2-4}$

(1) $A=B=0$ (2) $A-B=2$

(1) + (2) $\Rightarrow 2A=2$ $\Rightarrow A=1, B=-1$

$V=\pi\int_3^4\left(1+\frac{1}{x-2}-\frac{1}{x+2}\right)dx$

$V=\pi\left[x+\log_e\left(\left|\frac{x-2}{x+2}\right|\right)\right]_3^4$

$V=\pi\left[\left(4+\log_e\left(\frac{2}{6}\right)\right)-\left(3+\log_e\left(\frac{1}{5}\right)\right)\right]$

$V=\pi\left(1+\log_e\frac{5}{3}\right)$

Award **1 mark** for the correct definite integral for the volume.

Award **1 mark** for using partial fractions.

Award **1 mark** for correct integration.

Award **1 mark** for the correct final volume.

***VCAA Assessment Report* notes:**

Many students did not use the result from **part a**. Those who used the result from **part a**. generally answered this question well. Those who did not answer this question well commonly used partial fractions, incorrectly attempting $\frac{x^2}{x^2-4}=\frac{A}{x-2}+\frac{B}{x-2}$. Students are reminded that it is often necessary or beneficial to use the results from earlier parts of a question in the latter parts. Many students performed the division in this part, missing the prompt given. Several did not use partial fractions at all, giving the log of the denominator as their answer. A number of arithmetic and simplification errors were seen. Most students remembered to include π in their integral, but some did not square the expression for y.

Question 6

Volume around y-axis:

$$y = x^{\frac{4}{3}} \Rightarrow y^3 = x^4 \Rightarrow x^2 = y^{\frac{3}{2}},\ a = 0,\ b = 3$$

$$\begin{aligned} V &= \pi \int_0^3 y^{\frac{3}{2}} dy \\ &= \pi \left[\frac{2}{5} y^{\frac{5}{2}}\right]_0^3 \\ &= \frac{2\pi}{5}\left(3^{\frac{5}{2}} - 0\right) \\ &= \frac{2\pi}{5} \times 3^2 \times 3^{\frac{1}{2}} \\ &= \frac{18\pi 3^{\frac{1}{2}}}{5} \end{aligned}$$

Question 7

$y = 0 \qquad 3\cos(2x) = 0 \qquad \Rightarrow \qquad 2x = \pm\frac{\pi}{2} \qquad x = \pm\frac{\pi}{4}$

$$V_y = \pi \int_a^b y^2 dx = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 9\cos^2(3x)\, dx = 18\pi \int_0^{\frac{\pi}{4}} \cos^2(3x)\, dx \text{ by symmetry}$$

$$V = 9\pi \int_0^{\frac{\pi}{4}} (1 - \cos(4x))\, dx = 9\pi \left[x - \frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{4}}$$

$$V = 9\pi\left(\left(\frac{\pi}{4} - \frac{1}{4}\sin(\pi)\right) - \left(0 - \frac{1}{4}\sin(0)\right)\right)$$

$$V = \frac{9\pi^2}{4}$$

Question 8

Let $y_2 = 3\sin\left(\frac{x}{2}\right)$ (lower) and $y_1 = 4e^{-\frac{x}{2}}$ (upper). The volume required is $V_x = \pi \int_a^b \left(y_1^2 - y_2^2\right) dx$, where y_1 and y_2 are the outer (upper) and inner (lower) radii respectively,

$$\pi \int_0^b \left(16e^{-x} - 9\sin^2\left(\frac{x}{2}\right)\right) dx$$

9.5 Volumes

Question 1

a. $x^2 - 2(x - 1)^2 = 1$

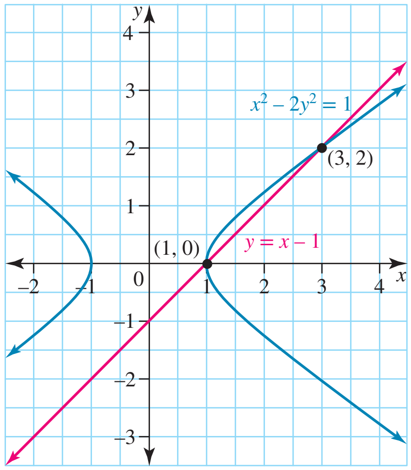
$x^2 - 2\left(x^2 - 2x + 1\right) = 1$

$x^2 - 4x + 3 = 0$

$x = 1,\ x = 3$ **[1 mark]**

***VCAA Examination Report* note:**

Students needed to substitute $y = x - 1$ into the equation $x^2 - 2y^2 = 1$ and solve the resulting quadratic equation for x. A number of students gave the coordinates of the points of intersection and in some cases did not do this correctly.

b. 

$a = 1,\ b = 3,\ y_2^2 = \dfrac{x^2 - 1}{2},\ y_1 = x - 1$

$V = \pi \displaystyle\int_1^3 \left(\frac{x^2 - 1}{2} - (x - 1)^2\right) dx$ **[1 mark]**

$V = \dfrac{\pi}{2} \displaystyle\int_1^3 \left(-x^2 + 4x - 3\right)\, dx$

$V = \dfrac{\pi}{2}\left[-\dfrac{x^3}{3} + 2x^2 - 3x\right]_1^3$

$V = \dfrac{\pi}{2}\left[(-9 + 18 - 9) - \left(-\dfrac{1}{3} + 2 - 3\right)\right]$

$V = \dfrac{2\pi}{3}$ units^3 **[1 mark]**

***VCAA Examination Report* note:**

Students found this question challenging. Some students did not apply the formula for the volume of a solid of revolution correctly. Many students made algebraic or arithmetic errors. A small number of students realised that the volume required could be found by finding the volume obtained by rotating the region bounded by the hyperbola, the x-axis and the lines $x = 1$ and $x = 3$ about the x-axis and then subtracting the volume of an appropriate cone.

Question 2

Volume around y-axis: $V_y = \pi \displaystyle\int_a^{b^2} dy$

$y = x^{\frac{4}{3}} \Rightarrow y^3 = x^4 \Rightarrow x^2 = y^{\frac{3}{2}},\ a = 0,\ b = 3$

$V = \pi \displaystyle\int_0^3 y^{\frac{3}{2}}\, dy$

$= \pi \left[\dfrac{2}{5} y^{\frac{5}{2}}\right]_0^3$

$= \dfrac{2\pi}{5}\left(3^{\frac{5}{2}} - 0\right)$

$$= \frac{2\pi}{5} \times 3^2 \times 3^{\frac{1}{2}}$$

$$= \frac{18\pi 3^{\frac{1}{2}}}{5}$$

The correct answer is **D**.

Question 3

$$V = \pi \int_a^b \left(y_2^2 - y_1^2\right) dx$$

$$a = 0,\ b = \frac{\pi}{3},\ y_1 = \sin(x),\ y_2 = 3x$$

$$V = \pi \int_0^{\frac{\pi}{3}} \left(9x^2 - \sin^2(x)\right) dx$$

$$V = \pi \int_0^{\frac{\pi}{3}} \left(9x^2 - \frac{1}{2}(1 - \cos(2x))\right) dx$$

$$V = \pi \int_0^{\frac{\pi}{3}} \left(9x^2 - \frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$$

$$V = \pi \left[3x^3 - \frac{x}{2} + \frac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{3}}$$

$$V = \pi \left[3 \times \frac{\pi^3}{27} - \frac{\pi}{6} + \frac{1}{4}\sin\left(\frac{2\pi}{3}\right)\right]$$

$$V = \frac{\pi^4}{9} - \frac{\pi^2}{6} + \frac{\sqrt{3}\pi}{8}$$

Award **1 mark** for the correct volume.
Award **1 mark** for using the double angle formula.
Award **1 mark** for the correct integration.
Award **1 mark** for the correct final volume.

VCAA Assessment Report **note:**

This question was well done by some students and poorly by others. Common errors included π being omitted from the outset, an incorrect expression for the volume - using$(3x - \sin x)^2$ as the integrand was common -the double angle formula not being used or being used incorrectly (for example, a sign error), occasional mistakes with the integration step (for example, a sign error), algebraic simplification errors at the end and finding the area rather than the volume. Several students did not distribute a negative through the brackets correctly. Too many students were unable to evaluate $\sin\left(\frac{2\pi}{3}\right)$ and many errors were made when expanding brackets, especially sign errors. Also, $(3x)^2 = 6x^2$ was often seen. A few students treated the shape to be rotated as a triangle. Some students arrived at the correct answer but then made errors when attempting to use a common denominator (which was unnecessary).

9.6 Arc length and surface area

Question 1

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \frac{1}{3}\left(x^2 + 2\right)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{3} \times 2x \times \frac{3}{2}\left(x^2 + 2\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x\sqrt{x^2 + 2}$$

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + x^2\left(x^2 + 2\right) \\ &= 1 + 2x^2 + x^4 \\ &= \left(x^2 + 1\right)^2 \end{aligned}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x^2 + 1$$

$$s = \int_0^2 \left(x^2 + 1\right) dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^2 = \left(\frac{8}{3} + 2\right) - 0$$

$$s = \frac{14}{3}$$

Award **1 mark** for the correct gradient.
Award **1 mark** for substituting into the arc length formula.
Award **1 mark** for the correct definite integral giving the arc length.
Award **1 mark** for the final correct arc length.

***VCAA Assessment Report* note:**

Some students responded very well to this question but others had some difficulty. A number made an error in the formula, despite it being on the formula sheet. Most students found the derivative correctly, but some made errors leading to an impossible integral. A common error was to give the derivative as $\frac{x}{\sqrt{x^2 + 2}}$.

A large proportion of those who found the correct derivative and substituted correctly into the formula were then unable to recognise the perfect square inside the square root. Some of the functions that were used in attempts to substitute could not lead to a correct answer. A small number of students took the square root of individual terms. Some students did not use dx.

Question 2

$$f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

$$S = 2\pi \int_0^1 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx \text{ [1 mark]}$$

$$= 2\pi \int_0^1 \sqrt{x}\sqrt{\frac{4x + 1}{4x}}\, dx$$

$$= \pi \int_0^1 \sqrt{4x + 1}\, dx$$

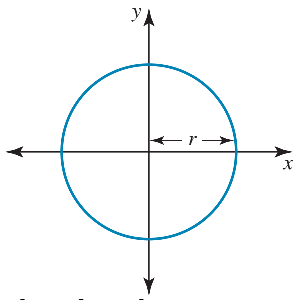
$$= \pi \left[\frac{1}{4} \times \frac{2}{3} (4x + 1)^{\frac{3}{2}} \right]_0^1$$ **[1 mark]**

$$= \frac{\pi}{6} \left[5^{\frac{3}{2}} - 1 \right]$$

$$= \frac{\pi}{6} \left(5\sqrt{5} - 1 \right)$$ **[1 mark]**

Question 3

a.



$$x^2 + y^2 = r^2$$

$$y = \sqrt{r^2 - x^2} \qquad \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$$ **[1 mark]**

$$S = 4 \int_0^r \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \; dx$$

$$1 + \left(\frac{dy}{dx} \right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$ **[1 mark]**

$$s = 4 \int_0^r \frac{r}{\sqrt{r^2 - x^2}} dx$$

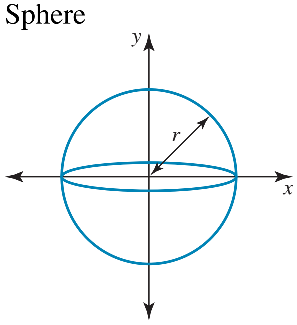
$$S = 4r \left[\sin^{-1} \left(\frac{x}{r} \right) \right]_0^r$$ **[1 mark]**

$$= 4r \left[\sin^{-1}(1) - \sin^{-1}(0) \right]$$

$$= 4r \times \frac{\pi}{2}$$

$C = 2\pi r$ **[1 mark]**

b. Sphere



$$y = \sqrt{r^2 - x^2}$$

$$x^2 + y^2 = r^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$\frac{dy}{dx} = -\frac{x}{y}$ **[1 mark]**

$$S = 2 \times 2\pi \int_0^r y\sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= 4\pi \int_0^r y\sqrt{\frac{x^2 + y^2}{y^2}}\, dx \text{ [1 mark]}$$

$$= 4\pi \int_0^r r\, dx$$

$$= 4\pi r \int_0^r 1\, dx$$

$$= 4\pi r\, [x]_0^r \text{ [1 mark]}$$

$$= 4\pi r\, (r - 0)$$

$S = 4\pi r^2$ **[1 mark]**

9.7 Water flow

Question 1

a. i. $y = x^3 - 8, \quad x^3 = y + 8$

$$V = \pi \int_0^H (y + 8)^{\frac{2}{3}}\, dy \text{ [1 mark]}$$

ii. $y = x^3 - 8, \quad x^3 = y + 8$

$$V(H) = \frac{3\pi}{5}\left[(H + 8)^{\frac{5}{3}} - 32\right] \text{ [1 mark]}$$

b. i. $\frac{dV}{dh} = \pi(h + 8)^{\frac{2}{3}}, \quad \frac{dV}{dt} = -4\sqrt{h}$ **[1 mark]**

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{-4\sqrt{h}}{\pi(h + 8)^{\frac{2}{3}}} \text{ [1 mark]}$$

ii. $\frac{d^2h}{dt^2} = 0 \quad \Rightarrow \quad h = 24, \ \left.\frac{dh}{dt}\right|_{h=24} = -0.62$ **[1 mark]**

Decreases at 0.62cm/min **[1 mark]**

iii. Maximum rate when $h = 50$

$\frac{dV}{dt} = 4\sqrt{50} = 20\sqrt{2}\ \ \text{cm}^3/\text{min}$ **[1 mark]**

c. $\frac{dh}{dt} = \frac{40\sqrt{2} - 4\sqrt{h}}{\pi(h + 8)^{\frac{2}{3}}}$ **[1 mark]**

$$\frac{dt}{dh} = \frac{\pi(h + 8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)} \text{ [1 mark]}$$

$$t = \int_{25}^{50} \frac{\pi(h + 8)^{\frac{2}{3}}}{4\left(10\sqrt{2} - \sqrt{h}\right)}\, dh = 31.4 \text{ [1 mark]}$$

Question 2

a. $y = \frac{1}{2}\sqrt{4x^2 - 1}$

$V = \pi \int_a^{b^2} dy$ **[1 mark]**

$4y^2 = 4x^2 - 1 \Rightarrow x^2 = \frac{1}{4}\left(1 + 4y^2\right)$ **[1 mark]**

$V = \frac{\pi}{4}\int_0^h \left(1 + 4y^2\right)\, dy$

$V = \frac{\pi}{4}\left[y + \frac{4y^3}{3}\right]_0^h = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$ **[1 mark]**

***VCAA Examination Report* note:**

Approximately half of the students were able to either set up an appropriate definite integral or find an anti–derivative and attempt to evaluate the constant of integration. Of these, many did not explicitly show that the first part of their response yielded the required volume.

b. When the fountain is full, $h = \frac{\sqrt{3}}{2}$

so $V\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{4}$. **[1 mark]**

When the fountain is half-full, $V = \frac{\pi\sqrt{3}}{8}$

Solving $V = \frac{\pi\sqrt{3}}{8} = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$

gives $h = 0.59\, m$ **[1 mark]**

***VCAA Examination Report* note:**

While the approach above was the most common, other correct approaches were used. A common error was to fail to halve the volume.

c. i. Inflow $0.04\text{m}^3/\text{s}$, outflow $0.05\sqrt{\text{h}}\,\text{m}^3/\text{s}$

$\frac{dV}{dt} = \text{inflow} - \text{outflow} = 0.04 - 0.05\sqrt{\text{h}}$

$= \frac{1}{100}\left(4 - 5\sqrt{h}\right)$ **[1 mark]**

since $V = \frac{\pi}{4}\left(h + \frac{4h^3}{3}\right)$

$\frac{dV}{dh} = \frac{\pi}{4}\left(1 + 4h^2\right)$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{\frac{1}{100}\left(4 - 5\sqrt{h}\right)}{\frac{\pi}{4}\left(4h^2 + 1\right)}$

$= \frac{4 - 5\sqrt{h}}{25\pi\left(4h^2 + 1\right)}$ **[1 mark]**

VCAA Examination Report **note:**

Most students were able to correctly state $\frac{dV}{dt}$ and find $\frac{dV}{dh}$ and then use this to find $\frac{dh}{dV}$ before proceeding. A few students did not understand the importance of brackets when multiplying the derivative expressions. Many students moved directly from the product of the derivatives to the required expression, without explicitly showing that their product led to the final (given) answer.

ii. $\left.\frac{dh}{dt}\right|_{h=0.25} = \frac{4-5\sqrt{0.25}}{25\pi\left(4(0.25)^2+1\right)}$

$= 0.0153 \quad \text{m/s}$ **[1 mark]**

VCAA Examination Report **note:**

The majority of students were able to use the supplied derivative to find the required rate for the given depth. Some students did not give the answer in the required decimal form.

d. Express the time taken for the depth to reach 0.25 m as a definite integral and evaluate this integral correct to the nearest tenth of a second.

$\frac{dt}{dh} = \frac{25\pi\left(4h^2+1\right)}{4-5\sqrt{h}}$

$t = \int_0^{0.25} \left(\frac{25\pi\left(4h^2+1\right)}{4-5\sqrt{h}}\right) dh$ **[1 mark]**

$= 9.8$ seconds

VCAA Examination Report **note:**

Most students were able to set up a correct definite integral. Transcription errors were occasionally present in the integrand.

e. The water level stabilises when $4 - 5\sqrt{h} = 0$

$\sqrt{h} = \frac{4}{5}$

$h = \frac{16}{25}$ **[1 mark]**

Therefore, the height from the top is $\frac{\sqrt{3}}{2} - \frac{16}{25} = 0.23$ m. **[1 mark]**

VCAA Examination Report **note:**

Most students who attempted this question understood that they need to solve $\frac{dh}{dt} = 0$ to find the limiting water level. Many students who correctly found h did not subtract their value from the height of the top of the fountain.

Question 3

a. $\frac{r}{h} = \frac{0.5}{1} = \frac{1}{2} \Rightarrow r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{12}h^3$

Award **1 mark** for the correct expression.

b. $\frac{dV}{dt} = 0.02\pi - 0.01\pi\sqrt{h}\,r = \frac{\pi}{100}\left(2 - \sqrt{h}\right)$

$\frac{dV}{dh} = \frac{\pi h^2}{4}$

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt} \\ &= \frac{4}{\pi h^2} \times \frac{\pi}{100}\left(2 - \sqrt{h}\right) \\ &= \frac{2 - \sqrt{h}}{25h^2}\end{aligned}$$

$$\begin{aligned}\left.\frac{dh}{dt}\right|_{h=0.25} &= \frac{2 - \sqrt{0.25}}{25(0.25)^2} \\ &= \frac{24}{25} \\ &= 0.96\end{aligned}$$

Award **1 mark** for the correct rates.
Award **1 mark** for the correct chain rule.
Award **1 mark** for the correct rate for h.
Award **1 mark** for the correct final rate.

c. $\frac{dh}{dt} = \frac{2 - \sqrt{h}}{25h^2}$

Invert: $\frac{dt}{dh} = \frac{25^2}{2 - \sqrt{h}}$

$$\begin{aligned}t_{\text{full}} &= \int_0^1 \frac{25h^2}{2 - \sqrt{h}}dh \\ &= 7.3688 \\ &= 7.4 \text{ minutes}\end{aligned}$$

Award **1 mark** for inverting and setting up the correct definite integral.
Award **1 mark** for solving using CAS for the correct time.

d. $\frac{dV}{dt} = 0.05\pi = \frac{\pi}{20}\text{m}^3/\text{min}$

$V = \frac{\pi}{48}\left(x^3 + 6x^2 + 12x\right)$

$$\begin{aligned}\frac{dV}{dx} &= \frac{\pi}{48}\left(3x^2 + 12x + 12\right) \\ &= \frac{\pi}{16}(x + 2)^2\end{aligned}$$

$\frac{dx}{dt} = \frac{dx}{dV} \cdot \frac{dV}{dt}$

$\frac{dx}{dt} = \frac{16}{\pi(x + 2)^2} \times \frac{\pi}{20} = \frac{4}{5(x + 2)^2}$

$\frac{dt}{dx} = \frac{5(x + 2)^2}{4}$

$t = \frac{5}{4}\int (x + 2)^2 dx$

$$t = \frac{5}{12}(x+2)^3 + c$$

$$t = 0,\ x = 0 \Rightarrow\ 0 = \frac{5}{12} \times 8 + c$$

$$c = -\frac{5}{12} \times 8$$

$$t = \frac{5}{12}(x+2)^3 -, \frac{5}{12} \times 8$$

$$t = \frac{5}{12}\left[(x+2)^3 - 8\right] \quad \text{when } x = 1,\ t = \frac{95}{12}$$

$$\frac{12t}{5} = (x+2)^3 - 8$$

$$(x+2)^3 = \frac{12t}{5} + 8$$

$$(x+2)^3 = \frac{4(3t+10)}{5}$$

$$x(t) = \sqrt[3]{\frac{4(3t+10)}{5}} - 2$$

$$= 2\sqrt[3]{(0.3t+1)} - 2 \ \text{ for } 0 \le t \le \frac{95}{12}$$

Award **1 mark** for finding the correct rate.
Award **1 mark** for the correct chain rule.
Award **1 mark** for inverting and solving the differential equation.
Award **1 mark** for the constant of integration.
Award **1 mark** for obtaining the correct expression for x

9.8 Review

Question 1

$$y^2 = x^2 - 2x,\ V = \pi \int_a^b x^2\, dy$$

$$y^2 + 1 = x^2 - 2x + 1 = (x-1)^2$$

$$x = 1 + \sqrt{y^2 + 1}$$

$$V = \pi \int_0^{2\sqrt{2}} \left(1 + \sqrt{y^2+1}\right)^2 dy$$

$$y = \tan(t),\ y^2 + 1 = \tan^2(t) + 1 = \sec^2(t),\ \frac{dy}{dt} = \sec^2(t)$$

$$y = 2\sqrt{2},\ t = \tan^{-1}\left(2\sqrt{2}\right),\ y = 0,\ t = 0$$

$$V = \pi \int_0^{\tan^{-1}\left(2\sqrt{2}\right)} (1 + \sec(t))^2 \sec^2(t)\, dt$$

Award **1 mark** for the correct method.
Award **1 mark** for the correct definite integral.

***VCAA Examination Report* note:**

Very few students answered this question correctly. The most common incorrect answer was an integral in terms of x. Of those that attempted to give an integral in terms of t, most simply replaced dx with dt.

Question 2

a. $f(x) = \dfrac{9}{(x+2)\ (x-4)}$

$f(x) = \dfrac{9}{\left(x^2 - 2x - 8\right)}$

$= 9\left(x^2 - 2x - 8\right)^{-1}$

$f'(x) = \dfrac{-9\,(2x-2)}{\left(x^2 - 2x - 8\right)^2} = 0$

when $x = 1$;

$f(1) = \dfrac{9}{3 \times -3}$

$= -1$

Stationary point, local maximum at $(1, -1)$

Award **1 mark** for solving the gradient equal to zero.

Award **1 mark** for the correct x-value.

Award **1 mark** for the correct coordinates.

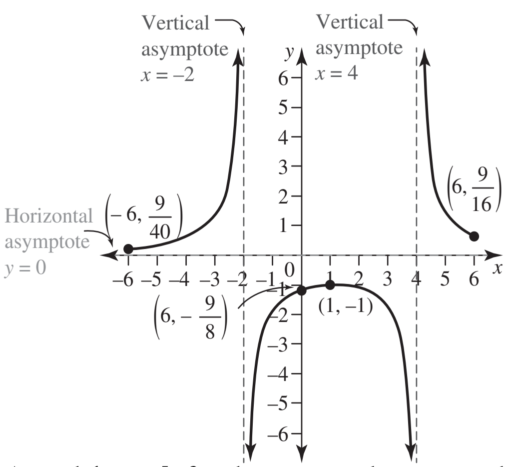
b. Vertical asymptotes $x = 4, x = -2$

Horizontal asymptote $y = 0$

Award **1 mark** for both correct vertical asymptotes.

Award **1 mark** for the horizontal asymptote.

c.



Award **1 mark** for the correct shape over the correct domain.

Award **1 mark** for the correct y-intercept.

Award **1 mark** for correctly approaching asymptotes.

d. i. $V = \pi \displaystyle\int_0^3 \frac{81}{(x\ +\ 2)^2(x\ -\ 4)^2}dx$

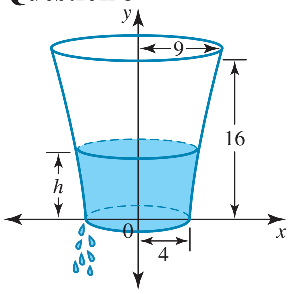
Award **1 mark** for the correct terminals.

Award **1 mark** for the correct definite integral representing the volume.

ii. $V = 12.85$

Award **1 mark** for correctly using CAS to obtain the volume.

Question 3



$$\frac{x^2}{16} - \frac{65y^2}{4096} = 1$$

$$\frac{x^2}{16} = 1 + \frac{65y^2}{4096}$$

$$\frac{x^2}{16} = \frac{4096 + 65y^2}{4096}$$

$$x^2 = \frac{1}{256}\left(65y^2 + 4096\right)$$

$$V = \pi \int_0^h x^2\,dy = \int_0^h \frac{\pi}{256}\left(65y^2 + 4096\right)\,dy$$ **[1 mark]**

$$\frac{dV}{dh} = \frac{\pi}{256}\left(65h^2 + 4096\right), \quad \frac{dV}{dt} = -k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{-k\sqrt{h}}{\frac{\pi}{256}\left(65h^2 + 4096\right)}$$

$$\frac{dt}{dh} = \frac{65h^2 + 4096}{A\sqrt{h}}$$

$$At + B = \int \frac{65h^2 + 4096}{\sqrt{h}}\,dh$$

$$= \int \left(65h^{\frac{3}{2}} + 4096h^{-\frac{1}{2}}\right)\,dh$$

$$= 26h^{\frac{5}{2}} + 8192h^{\frac{1}{2}}$$ **[1 mark]**

When $h = 16 \quad t = 0$

$$B = 26 \times 16^{\frac{5}{2}} + 8192 \times 16^{\frac{1}{2}} = 59392$$

When $h = 9 \quad t = 10$

$$10A + B = 26 \times 9^{\frac{5}{2}} + 8192 \times 9^{\frac{1}{2}} = 30894$$

$$10A = -28498$$

$$A = \frac{-14249}{5} \qquad B = 59392$$

When empty $h = 0$

$$t = -\frac{B}{A} = \frac{59392}{\frac{14249}{5}}$$

$= 20.8$ minutes

So it takes an extra 10.8 minutes **[1 mark]**

Question 4

a. $\int \cos^n(ax)\,dx = C_n$

$= \int \cos(ax)\cos^{n-1}(ax)\,dx$

Let $u = \cos^{n-1}(ax) \quad \frac{dv}{dx} = \cos(ax)$ **[1 mark]**

$\frac{du}{dx} = -(n-1)\,a\cos^{n-2}(ax)\sin(ax) \qquad v = \int \cos(ax)\,dx$

$= \frac{1}{a}\sin(ax)$

$c_n = \frac{1}{a}\sin(ax)\cos^{n-1}(ax) + (n-1)\int \sin^2(ax)\cos^{n-2}(ax)\,dx$

$= \frac{1}{a}\sin(ax)\cos^{n-1}(ax) + (n-1)\int \left(1-\cos^2(ax)\right)\cos^{n-2}(ax)\,dx$

$c_n = \frac{1}{a}\sin(ax)\cos^{n-1}(ax) + (n-1)\int \cos^{n-2}(ax)\,dx - (n-1)\int \cos^n(ax)\,dx$ **[1 mark]**

$\int \cos^n(ax)\,dx = \frac{1}{a}\sin(ax)\quad\cos^{n-1}(ax) + (n-1)\int \cos^{n-2}(ax)\,dx$

$-n\int \cos^n(ax)\,dx + \int \cos^n(ax)\,dx$

$n\int \cos^n(ax)\,dx = \frac{1}{a}\sin(ax)\quad\cos^{n-1}(ax) + (n-1)\int \cos^{n-2}(ax)\,dx$

$\int \cos^n(ax)\,dx = \frac{1}{an}\sin(ax)\quad\cos^{n-1}(ax)$

$+\frac{n-1}{n}\int \cos^{n-2}(ax)\,dx$ **[1 mark]**

b. Let $n = 5, \quad a = 3$

$\int \cos^5(3x)\,dx = \frac{1}{15}\sin(3x)\cos^4(3x) + \frac{4}{5}\int \cos^3(3x)\,dx$ **[1 mark]**

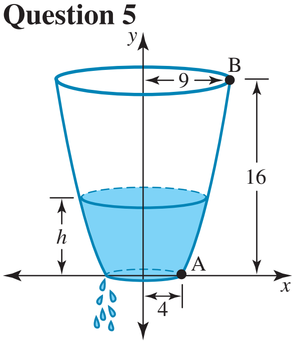
$\int \cos^3(3x)\,dx = \frac{1}{9}\sin(3x)\cos^2(3x) + \frac{2}{3}\int \cos(3x)\,dx$

$= \frac{1}{9}\sin(3x)\cos^2(3x) + \frac{2}{9}\sin(3x)$ **[1 mark]**

$\int \cos^5(3x)\,dx = \frac{1}{15}\sin(3x)\cos^4(3x) + \frac{4}{5}\left[\frac{1}{9}\sin(3x)\cos^2(3x) + \frac{2}{9}\sin(3x)\right] + c$

$= \frac{1}{45}\sin(3x)\left[3\cos^4(3x) + 4\cos^2(3x) + 8\right] + c$ **[1 mark]**

Question 5



$A\ (4, 0) \quad B\ (9, 16)$

$y = ax^2 + b$

(1) $0 = 16a + b$

(2) $16 = 81a + b$

$65a = 16$

$a = \frac{16}{65}\quad b = -\frac{256}{65}$

$y = \frac{16}{65}\left(x^2 - 16\right)$

$\frac{65y}{16} = x^2 - 16$

$x^2 = \frac{65y}{16} + 16 = \frac{65y + 256}{16}$

$V = \pi \int_0^h x^2\, dy = \pi \int_0^h \frac{65y + 256}{16} dy$ **[1 mark]**

$\frac{dV}{dh} = \frac{\pi}{16}(65h + 256),\ \frac{dV}{dt} = -2\sqrt{h}$

$\frac{dt}{dh} = \frac{dt}{dV} \cdot \frac{dV}{dh} = \frac{\pi\ (65h + 256)}{-2\sqrt{h} \times 16}$

$t = \frac{-\pi}{32} \int_{16}^{0} \frac{65h + 256}{\sqrt{h}} dh$ **[1 mark]**

$= \frac{-\pi}{32} \times -\frac{14\,464}{3}$

$= 473.3$ minutes **[1 mark]**

Question 6

a. $x = \sin(t),\ t \in \left[-\frac{\pi}{3}, -\frac{\pi}{3}\right]$

$\dot{x} = \frac{dy}{dx} = \cos(t)$

$y = \frac{1}{2}\sin(t)\tan(t)$

$\dot{y} = \frac{dy}{dt} = \frac{\sin(t)\left(\cos^2(t) + 1\right)}{2\cos^2(t)}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin(t)\left(\cos^2(t) + 1\right)}{2\cos^3(t)} = \frac{1}{2}\tan(t)\left(1 + \sec^2(t)\right)$

Award **1 mark** for the parametric differentiation.

Award **1 mark** for the correct expression for $\frac{dy}{dx}$ in terms of t (other simplifications are possible).

***VCAA Assessment Report* note:**

This question was reasonably well answered. Many students attempted a chain rule relation, but a number of these had x instead of t in what otherwise would have been a correct answer. Some students gave $\frac{dy}{dt}$ as their answer, while others first eliminated t to get y in terms of x, found $\frac{dy}{dx}$, and then expressed their answer in terms of t. The latter method was lengthy and more prone to errors. Some students did unnecessary further working out, attempted to simplify a correct answer and changed it to an incorrect answer.

$\dot{y} = \frac{dy}{dt} = \frac{\sin(t)\left(\cos^2(t) + 1\right)}{2\cos^2(t)}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin(t)\left(\cos^2(t) + 1\right)}{2\cos^3(t)} = \frac{1}{2}\tan(t)\left(1 + \sec^2(t)\right)$

b. $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{\sin\left(\frac{\pi}{6}\right)\left(\cos^2\left(\frac{\pi}{6}\right)+1\right)}{2\cos^3\left(\frac{\pi}{6}\right)} = \frac{7\sqrt{3}}{18}$ **[1mark]**

c. i. $\text{RHS} = \frac{x^2}{2\sqrt{1-x^2}} = \frac{\sin^2(t)}{2\sqrt{1-\sin^2(t)}}$

$= \frac{\sin^2(t)}{2\sqrt{\cos^2(t)}} = \frac{\sin^2(t)}{2\cos(t)}$

$= \frac{1}{2}\sin(t) \times \frac{\sin(t)}{\cos(t)}$

$= \frac{1}{2}\sin(t)\tan(t) = y = \text{LHS}$ **[1 mark]**

***VCAA Assessment Report* note:**

Many responses to this question lacked the necessary steps. Some students assumed what was to be shown and then proceeded to show something else.

ii. $t = \frac{\pi}{3}, x = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, and $t = -\frac{\pi}{3}, x = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Domain: $x \in \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ **[1 mark]**

***VCAA Assessment Report* note:**

This question was answered reasonably well. Common errors included $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$, $(-1, 1)$ and $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$

d. $\text{LHS} = \frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$

$\text{RHS} = \frac{2x^2}{\sqrt{1-x^2}} + \frac{d}{dx}\left(x\sqrt{1-x^2}\right)$

$= \frac{2x^2}{\sqrt{1-x^2}} + x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{d}{dx}(x)$

$= \frac{2x^2}{\sqrt{1-x^2}} + x \times -\frac{x}{\sqrt{1-x^2}} + \sqrt{1-x^2}$

$= \frac{2x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$

$= \frac{2x^2 - x^2 + 1 - x^2}{\sqrt{1-x^2}}$

$= \frac{1}{\sqrt{1-x^2}} = \text{LHS}$

Award **1 mark** for the correct simplification.

Award **1 mark** for the correct verification.

***VCAA Assessment Report* note:**

Students were asked to simplify the right-hand side of the equation in this question. However, some students attempted to work on both sides of the given equation, usually integrating both sides. The second-last step of putting the two square root terms over a common denominator was occasionally omitted.

e. $A = 4\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{2\sqrt{1-x^2}}dx$

$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{2x^2}{\sqrt{1-x^2}} + \frac{d}{dx}\left(x\sqrt{1-x^2}\right)$

then $\int \frac{2x^2}{\sqrt{1-x^2}}dx = \sin^{-1}(x) - x\sqrt{1-x^2}$

$\int \frac{x^2}{2\sqrt{1-x^2}}dx = \frac{1}{4}\left[\sin^{-1}(x) - x\sqrt{1-x^2}\right]$

$A = 4 \times \frac{1}{4}\left[\sin^{-1}(x) - x\sqrt{1-x^2}\right]_0^{\frac{\sqrt{3}}{2}}$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\sqrt{1-\frac{3}{4}}$

$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

$= \frac{4\pi - 3\sqrt{3}}{12}$ units^2

Award **1 mark** for the correct definite integral for the area.
Award **1 mark** for deducing the anti-derivative.
Award **1 mark** for the correct area.

***VCAA Assessment Report* note:**

Only a small number of students seemed to understand this ‘hence’ question. An antiderivative with terminals needed to be written down. The most common answer was an integral for the area, followed by its evaluation using CAS technology.

10 Applications of first-order differential equations

Topic	10	Applications of first-order differential equations
Subtopic	10.2	Growth and decay

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Source: *VCE 2019, Specialist Mathematics Exam 2, Section B, Q3; © VCAA*

Question 1 (9 marks)

a. The growth and decay of a quantity P with respect to time t is modelled by the differential equation

$$\frac{dP}{dt} = kP$$

where $t \geq 0$.

i. Given that $P(a) = r$ and $P(b) = s$, where P is a function of t, show that $k = \frac{1}{a-b}\log_e\left(\frac{r}{s}\right)$. **2 marks**

ii. Specify the condition(s) for which $k > 0$. **2 marks**

b. The growth of another quantity Q with respect to time t is modelled by the differential equation

$$\frac{dQ}{dt} = e^{t-Q}$$

where $t \geq 0$ and $Q = 1$ when $t = 0$.

i. Express this differential equation in the form $\int f(Q)\, dQ = \int h(t)\, dt$. **1 marks**

ii. Hence, show that $Q = \log_e\left(e^t + e - 1\right)$. **2 marks**

iii. Show that the graph of Q as a function of t does not have a point of inflection. **2 marks**

Question 2 (2 marks)

Plutonium-239 is a silvery metal that is used for the production of nuclear weapons. If 3 micrograms of plutonium-239 decomposes to 1 microgram in 38 213 years, determine the half-life of plutonium-239, correct to the nearest year.

Question 3 (1 mark)

The population of Germany in 2010 was approximately 80.827 million and by 2021 it had grown to 83.900 million. Assuming the population of Germany follows the law of natural growth, the year in which the population will reach 100 million is

A. 2062
B. 2063
C. 2066
D. 2072
E. 2073

Source: *VCE 2007, Specialist Mathematics 2, Section 1, Q.14; © VCAA*

Question 4 (1 mark)

The rate at which a type of bird flu spreads throughout a population of 1000 birds in a certain area is proportional to the product of the number N of infected birds and the number of birds still **not** infected after t days. Initially two birds in the population are found to be infected.

A differential equation, the solution of which models the number of infected birds after t days, is

A. $\frac{dN}{dt} = k\frac{(1000 - N)}{1000}$

B. $\frac{dN}{dt} = k(N - 2)(1000 - N)$

C. $\frac{dN}{dt} = kN(1000 - N)$

D. $\frac{dN}{dt} = kN(1000 - (N + 2))$

E. $\frac{dN}{dt} = k(N + 2)(1000 - N)$

Question 5 (5 marks)

The rate at which a type of a hideous infection spreads throughout a population of 5000 wild horses in the badlands of a desert state in the USA is proportional to the product of the number H of infected horses and the number of horses still not infected after t months. The initial study showed 50 horses out of the total 5000 horses to be infected.

Show that $t = \frac{1}{5000p} \ln\left(\frac{99H}{5000 - H}\right)$, where p is a constant.

Question 6 (4 marks)

The population of a town is two million. Five years later the population has grown to three million. Assuming that the population growth is proportional to the current population, show that the population after a **further** ten years is 6.75 million.

Question 7 (1 mark)

The population of a town is one million. Five years later the population has grown to two million. Assuming that the population growth is proportional to the current population, then the population after a further ten years is

A. 3 million
B. 4 million
C. 5 million
D. 6 million
E. 8 million

Question 8 (1 mark)

$N = N(t)$ is the number of bacteria in a culture after a time t hours. Initially there are 200 bacteria present and after two hours the number has increased to 400. If the differential equation describing the bacteria growth is given by $\frac{dN}{dt} = kN$, $N(0) = N_0$, then

A. $k = 2$ and $N_0 = 200$

B. $k = \log_e(2)$ and $N_0 = 200$

C. $k = \log_e\left(\sqrt{2}\right)$ and $N_0 = 200$

D. $k = 2$ and $N_0 = 400$

E. $k = \log_e(2)$ and $N_0 = 400$

Question 9 (1 mark)

A radioactive substance decomposes so that the rate of decay is proportional to the amount of the substance present at that time. The half-life of a certain radioactive substance is 24 days. The percentage of the original amount present after 8 days is closest to

A. 81%

B. 79%

C. 67%

D. 33%

E. 21%

Question 10 (1 mark)

The amount of a substance $Q = Q(t)$ satisfies the differential equation $\frac{dQ}{dt} = -kQ^2$. If the initial amount of the substance present is Q_0 and after a time of T the amount has reduced to $\frac{1}{2}Q_0$, then

A. $kQ_0T = -2$

B. $kQ_0T = -1$

C. $kQ_0T = 1$

D. $\frac{1}{kT}\log_e(2) = -1$

E. $\frac{1}{kT}\log_e(2) = 1$

Question 11 (1 mark)

A research project finds that the rate at which the population of an animal species on an island is declining is proportional to the population, P, at any time, t. If the population of the species at the commencement of the study was 5000 and after five years it was 3500, the rate at which the population is changing can be represented by the differential equation closest to

A. $\frac{dP}{dt} = 5000e^{-.07t}$

B. $\frac{dP}{dt} = -350e^{-.07t}$

C. $\frac{dP}{dt} = -5000e^{-t}$

D. $\frac{dP}{dt} = -3500e^{-5000t}$

E. $\frac{dP}{dt} = \frac{3500}{5000}e^{-.07t}$

Question 12 (1 mark)

An oil spill occurs in a lake that contains 4000 fish. The rate of decline of fish can be modelled by the differential equation $\frac{dF}{dt} = -200e^{-kt}$, where F is the number of fish in the lake t years after the oil spill. If there are 3115 fish in the lake 5 years after the oil spill, then the value of k is closest to

A. 0.2

B. −0.8

C. 0.5

D. −0.05

E. 0.05

Topic	10	Applications of first-order differential equations
Subtopic	10.3	Other applications of first-order differential equations

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Question 1 (2 marks)

The charge, Q units, on a plate conductor t seconds after it starts to discharge is proportional to the charge at that instant. If the charge is 500 units after a half-second and falls to 250 units after 1 second, determine:

a. the original charge. **(1 mark)**

☐ units

b. the time needed for the charge to fall to 125 units. **(1 mark)**

☐ seconds

Question 2 (3 marks)

In an electrical circuit consisting of a resistance of R ohms and a capacitance of C farads, the charge, Q coulombs, at a time t seconds decays according to the differential equation $\frac{dQ}{dt} + \frac{Q}{RC} = 0$. Assuming the initial charge is Q_0, solve the differential equation to determine the charge at any time t after discharging.

Question 3 (3 marks)

When light passes through a medium it loses its intensity as it penetrates depths. The rate of loss of intensity with respect to the depth is proportional to the intensity at that depth. If 5% of light is lost in penetrating 40% of a glass slab, calculate the percentage lost penetrating the whole slab. Give your answer correct to 1 decimal place.

Source: *VCE 2013, Specialist Mathematics, Exam 2, Section 2, Q.3; © VCAA*

Question 4 (13 marks)

The number of mobile phones, N, owned in a certain community after t years, may be modelled by $\log_e(N) = 6 - 3e^{-0.4t}, t \geq 0$.

a. Verify by substitution that $\log_e(N) = 6 - 3e^{-0.4t}$ satisfies the differential equation $\frac{1}{N}\frac{dN}{dt} + 0.4\log_e(N) - 2.4 = 0$. **(2 mark)**

b. Find the initial number of phones owned in the community. Give your answer correct to the nearest integer. **(1 mark)**

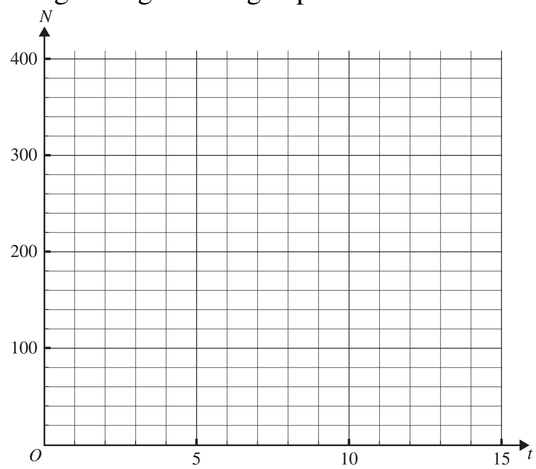
c. Using this mathematical model, find the limiting number of mobile phones that would eventually be owned in the community. **(2 mark)**
Give your answer correct to the nearest integer.

d. The differential equation in part a. can also be written in the form $\frac{dN}{dt} = 0.4N(6 - \log_e(N))$. **(4 mark)**

i. Find $\frac{d^2N}{dt^2}$ in terms of N and $\log_e(N)$. **(2 marks)**

ii. The graph of N as a function of t has a point of inflection. Find the values of the coordinates of this point.
Give the value of t correct to one decimal place and the value of N correct to the nearest integer. **(2 marks)**

e. Sketch the graph of N as a function of t on the axes below for $0 \le t \le 15$. You may wish to upload an image using the image upload tool. **(2 marks)**



Source: *VCE 2010, Specialist Mathematics 2, Section 1, Q.13; © VCAA*

Question 5 (1 marks)

The amount of a drug, x mg, remaining in a patient's bloodstream t hours after taking the drug is given by the differential equation

$$\frac{dx}{dt} = -0.15x.$$

The number of hours needed for the amount x to halve is

A. $2\log_e\left(\frac{20}{3}\right)$

B. $\frac{20}{3}\log_e(2)$

C. $2\log_e(15)$

D. $15\log_e\left(\frac{3}{2}\right)$

E. $\frac{3}{2}\log_e(200)$

Question 6 (1 marks)

The rate at which the price of the shares in a business is declining is proportional to the price, P, at any time, t. If the shares initially cost \$50 and six weeks later they were \$35, then the rate at which the price is changing can be represented by the differential equation

A. $\frac{dP}{dt} = 50e^{-.07t}$

B. $\frac{dP}{dt} = -35e^{-.07t}$

C. $\frac{dP}{dt} = -3.5e^{-.07t}$

D. $\frac{dP}{dt} = -35e^{-50t}$

E. $\frac{dP}{dt} = \frac{35}{50}e^{-.07t}$

Topic	10	Applications of first-order differential equations
Subtopic	10.4	Bounded growth and Newton's law of cooling

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Source: *VCE 2013, Specialist Mathematics Exam 1, Q5; © VCAA*

Question 1 (3 marks)

A container of water is heated to boiling point (100°C) and then placed in a room that has a constant temperature of 20°C. After five minutes the temperature of the water is 80°C.

a. Use Newton's law of cooling $\frac{dT}{dt} = -k(T - 20)$ where T°C is the temperature of the water at time t minutes after the water is placed in the room, to show that $e^{-5k} = \frac{3}{4}$. **(2 marks)**

b. Find the temperature of the water 10 minutes after it is placed in the room. **(1 mark)**

$\square\,°C$

Source: *VCE 2013, Specialist Mathematics Exam 1, Q5; © VCAA*

Question 2 (7 marks)

An RL series circuit consists of a resistance, R ohms, and an inductance, L henries, connected to a voltage source, E volts. The rise of current, i amperes, after a time t seconds satisfies the differential equation

$\frac{di}{dt} + Ri = E$

a. Assuming the initial current is zero, determine the current at any time t. **(5 marks)**

b. Show that the time required for the current to reach half its ultimate value is given by $\frac{L}{R}\log_e\left(\frac{1}{2}\right)$. **(2 marks)**

Question 3 (3 marks)

The temperature of a room is 25°C. A thermometer which was in the room is taken outdoors and in five minutes it reads 15°C. Five minutes later the thermometer reads 10°C. Determine the temperature outdoors.

Question 4 (1 mark)

The temperature in a room is kept at a constant temperature of 5°C. A cup of fluid with a temperature of 75°C is placed on a bench in the room. According to Newton's law of cooling, the time taken for the fluid to cool to a given temperature T can be found using

A. $t = k\log_e \dfrac{(T-5)}{70}$

B. $t = \dfrac{1}{k}\log_e \dfrac{(T-5)}{75}$

C. $t = \dfrac{1}{k}\log_e \dfrac{70}{(T-5)}$

D. $t = \dfrac{1}{k}\log_e \dfrac{(T-5)}{70}$

E. $t = \log_e \dfrac{(T-5)}{75}$

Question 5 (1 mark)

A pot of soup is heated to 100°C and then cools to 90°C in 3 minutes. If the room temperature is 25°C, how much longer will it take until the soup reaches a temperature of 80°C according to Newton's law of cooling?

A. 5 minutes

B. 6.5 minutes

C. 3.5 minutes

D. 6.2 minutes

E. 2.8 minutes

Question 6 (1 mark)

A hot iron at a temperature of 68°C is left to cool in a room where the room temperature is constant at 18°C. The rate at which the temperature of the iron cools is proportional to the excess of its temperature above the room temperature. If k is a positive constant, then the differential equation describing the temperature R of the iron at a time t minutes after it begins to cool is given by

A. $\frac{dR}{dt} + k(R - 68) = 0,\ R(0) = 18$

B. $\frac{dR}{dt} + k(R - 18) = 0,\ R(0) = 68$

C. $\frac{dR}{dt} - k(R - 68) = 0,\ R(0) = 18$

D. $\frac{dR}{dt} - k(R - 18) = 0,\ R(0) = 68$

E. $\frac{dR}{dt} - k(R + 18) = 0,\ R(0) = 68$

Question 7 (1 mark)

A can of beer is taken from a refrigerator and placed in a room. Assuming Newton's law of cooling, the temperature T°C of the can after a time t minutes after it begins to cool is given by $T = 18 - 15e^{-kt}$ where k is a constant. Which of the following is correct?

A. The room temperature is 18°C and the temperature of the refrigerator is −15°C.

B. The room temperature is 18°C and the temperature of the refrigerator is 3°C.

C. The room temperature is 18°C and the temperature of the refrigerator is −3°C.

D. The room temperature is 15°C and the temperature of the refrigerator is 3°C.

E. The room temperature is 15°C and the temperature of the refrigerator is −3°C.

Question 8 (1 mark)

A hot roast 'sicilian' chicken at a temperature of 65°C is left to cool in a room where the room temperature is constant at 21°C. The rate at which the temperature of the chicken cools is proportional to the excess of its temperature above the room temperature. If k is a positive constant, then the differential equation describing the temperature C of the chicken at a time t minutes after it begins to cool is given by

A. $\frac{dC}{dt} - k(R - 21) = 0, \quad R(0) = 65$

B. $\frac{dC}{dt} - k(R + 21) = 0, \quad R(0) = 65$

C. $\frac{dC}{dt} + k(R + 21) = 0, \quad R(0) = 65$

D. $\frac{dC}{dt} + k(R - 21) = 0, \quad R(0) = 65$

E. $\frac{dC}{dt} - k(R - 21) = 0, \quad R(65) = 0$

Topic	10	Applications of first-order differential equations
Subtopic	10.5	Chemical reactions and dilution problems

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Source: *VCE 2018, Specialist Mathematics Exam 1, Q8; © VCAA*

Question 1 (3 marks)

A tank initially holds 16 L of water in which 0.5 kg of salt has been dissolved. Pure water then flows into the tank at a rate of 5 L per minute. The mixture is stirred continuously and flows out of the tank at a rate of 3 L per minute.

a. Show that the differential equation for Q, the number of kilograms of salt in the tank after t minutes, is given by:

$$\frac{dQ}{dt} = -\frac{3Q}{16 + 2t}.$$

(1mark)

b. Solve the differential equation given in part a. to find Q as a function of t. Express your answer in the form $Q = \frac{a}{(16 + 2t)^{\frac{b}{c}}}$, where a, b and c are positive integers. **(1 mark)**

Source: *VCE 2013, Specialist Mathematics Exam 2, Section A, Q13; © VCAA*

Question 2 (3 marks)

Water containing 2 grams of salt per litre flows at the rate of 10 liters per minute into a tank that initially contained 50 liters of pure water. The concentration of salt in the tank is kept uniform by stirring and the mixture flows out of the tank at the rate of 6 liters per minute.

If Q grams is the amount of salt in the tank t minutes after the water begins to flow, the differential equation relating Q to t is

A. $\frac{dQ}{dt} = 20 - \frac{3Q}{25 + 2t}$

B. $\frac{dQ}{dt} = 10 - \frac{3Q}{25 + 2t}$

C. $\frac{dQ}{dt} = 20 - \frac{3Q}{25 - 2t}$

D. $\frac{dQ}{dt} = 10 - \frac{3Q}{25 - 2t}$

E. $\frac{dQ}{dt} = 20 - \frac{3Q}{25}$

Question 3 (3 marks)

A tank initially contains 50 litres of water. A chemical solution is drawn off at a rate of 5 litres per minute, and at the same time a mixture containing the chemical at a concentration of 3 grams per litre is added to the tank at a rate of 4 litres per minute. The contents of the tank are kept well stirred.

a. Set up the differential equation for Q, the amount of the chemical in grams in the tank after t minutes. **(1 mark)**

$\frac{dQ}{dt} = \square$

b. Verify that $Q(t) = 3(50 - t) + C(50 - t)^5$ is a general solution of the differential equation. **(2 marks)**

Topic 10 Subtopic 10.5 Chemical reactions and dilution problems

Source: *VCE 2020, Specialist Mathematics 2, Section A, Q.10; © VCAA*

Question 4 (3 marks)

A tank initially contains 300 grams of salt that is dissolved in 50 L of water. A solution containing 15 grams of salt per litre of water is poured into the tank at a rate of 2 L per minute and the mixture in the tank is kept well stirred. At the same time, 5 L of the mixture flows out of the tank per minute.

A differential equation representing the mass, m grams, of salt in the tank at time t minutes, for a non-zero volume of mixture, is

A. $\frac{dm}{dt} = 0$

B. $\frac{dm}{dt} = -\frac{5m}{50-5t}$

C. $\frac{dm}{dt} = 30 - \frac{m}{10}$

D. $\frac{dm}{dt} = 30 - \frac{5m}{50-3t}$

E. $\frac{dm}{dt} = 30 - \frac{5m}{50-5t}$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.10; © VCAA*

Question 5 (1 mark)

A large tank initially holds 1500 L of water in which 100 kg of salt is dissolved. A solution containing 2 kg of salt per litre flows into the tank at a rate of 8 L per minute. The mixture is stirred continuously and flows out of the tank through a hole at a rate of 10 L per minute.

The differential equation for Q, the number of kilograms of salt in the tank after t minutes, is given by

A. $\frac{dQ}{dt} = 16 - \frac{5Q}{750-t}$

B. $\frac{dQ}{dt} = 16 - \frac{5Q}{750+t}$

C. $\frac{dQ}{dt} = 16 + \frac{5Q}{750-t}$

D. $\frac{dQ}{dt} = \frac{100Q}{750-t}$

E. $\frac{dQ}{dt} = 8 - \frac{Q}{1500-2t}$

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.13; © VCAA*

Question 6 (1 mark)

Water containing 2 grams of salt per litre flows at the rate of 10 litres per minute into a tank that initially contained 50 litres of pure water. The concentration of salt in the tank is kept uniform by stirring and the mixture flows out of the tank at the rate of 6 litres per minute.

If Q grams is the amount of salt in the tank t minutes after the water begins to flow, the differential equation relating Q to t is

A. $\frac{dQ}{dt} = 20 - \frac{3Q}{25 + 2t}$

B. $\frac{dQ}{dt} = 10 - \frac{3Q}{25 + 2t}$

C. $\frac{dQ}{dt} = 20 - \frac{3Q}{25 - 2t}$

D. $\frac{dQ}{dt} = 10 - \frac{3Q}{25 - 2t}$

E. $\frac{dQ}{dt} = 20 - \frac{3Q}{25}$

Question 7 (1 mark)

A tank initially holds 800 L of water when a chlorine solution of concentration 3 g/L is added at a rate of 12 L/min. The mixture is kept uniform by stirring and flows out of the tank at a rate of 8 L/min. If x is the mass of chlorine at any time, t, the volume, V, of the solution can be found using

A. $V = 100 + (12 - 8)\,3t$

B. $V = 100 + 4t$

C. $V = 112 - 8t$

D. $V = 100 - 4t$

E. $V = 100 + \frac{(12 - 8)\,t}{3}$

Question 8 (1 mark)

A tank contains V_0 litres of water in which a grams of salt has been dissolved. A salt solution containing b grams per litre is poured into the tank at a rate of g litres per minute, and the well stirred mixture leaves the tank at a rate of f litres per minute. If the differential equation for the amount of salt Q grams in the tank at a time t minutes is given by $\frac{dQ}{dt} = 12 + \frac{5Q}{t-10}$ $Q(0) = a$, then it is possible that

A. $V_0 = 10, b = 3, g = 4$ and $f = 5$
B. $V_0 = 10, b = 2, g = 6$ and $f = 5$
C. $V_0 = 10, b = 4, g = 3$ and $f = 2$
D. $V_0 = 12, b = 2, g = 6$ and $f = 5$
E. $V_0 = 12, b = 3, g = 4$ and $f = 3$

Question 9 (1 mark)

A tank contains 50 litres of water in which 5 grams of salt has been dissolved. A salt solution containing 3 grams per litre is poured into the tank at a rate of 4 litres per minute, and the well stirred mixture leaves the tank at a rate of 2 litres per minute. If the amount of salt in the tank at a time t minutes is given by Q grams and $Q = Q(t) = 3(50 + 2t) + C(50 + 2t)^n$, where C is an arbitrary constant, then the value of n is

A. -2
B. -1
C. 0
D. 1
E. 2

Topic	10	Applications of first-order differential equations
Subtopic	10.6	The logistic equation

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Question 1 (2 marks)

If a logistic equation has as its solution $N(t) = \dfrac{200}{1 + 99e^{-kt}}$ where $k > 0$, write an expression for $\dfrac{dN}{dt}$.

Question 2 (1 mark)

A logistic equation has the solution $N(t) = \dfrac{500}{1 + 9e^{-kt}}$ where N is the population number of a city in millions at a time t years and k is a positive real constant. Determine the value of the population when the population is increasing most rapidly.

☐ million

Question 3 (7 marks)

Given the general logistic equation $\dfrac{dN}{dt} = kN\left(1 - \dfrac{N}{P}\right)$, $N(0) = N_0$, where k is a positive constant and $P > N_0 > 0$:

a. show, using integration, that $N(t) = \dfrac{PN_0}{N_0 + (P - N_0)^{e^{-kt}}}$. **(2 marks)**

b. show that $\lim\limits_{n \to \infty} N(t) = P$ **(1 mark)**

c. show that $\frac{d^2N}{dt^2} = k^2N\left(1 - \frac{2N}{P}\right)\left(1 - \frac{N}{P}\right)$ and hence that there is a point of inflection at $\left(\frac{1}{k}\log_e\left(\frac{P}{N_o} - 1\right), \frac{P}{2}\right)$ **(2 marks)**

d. verify, using differentiation, that $N(t) = \frac{PN_0}{N_0 + (P - N_0)^{e^{-kt}}}$ is a solution of $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right), N(0) = N_0.$ **(2 marks)**

Topic	10	Applications of first-order differential equations
Subtopic	10.7	Euler's method

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Source: *VCE 2021, Specialist Mathematics Exam 2, Section A, Q8; © VCAA*

Question 1 (1 mark)

Euler's method, with a step size of 0.1, is used to approximated the solution of the differential equation $\frac{dy}{dx} = y\sin(x)$.

Given that $y = 2$ when $x = 1$, the value of y correct to three decimal places, when $x = 1.2$ is

A. 2.168
B. 2.178
C. 2.362
D. 2.370
E. 2.381

Source: *VCE 2020, Specialist Mathematics Exam 2, Section A, Q12; © VCAA*

Question 2 (1 mark)

If $\frac{dy}{dx} = e^{\cos(x)}$ and $y_0 = e$ when $x_0 = 0$ then, using Euler's formula with step size 0.1, y_3 is equal to

A. $e + 0.1\left(1 + e^{\cos(0.1)}\right)$
B. $e + 0.1\left(1 + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$
C. $e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$
D. $e + 0.1\left(e^{\cos(0.1)} + e^{\cos(0.2)} + e^{\cos(0.3)}\right)$
E. $e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)} + e^{\cos(0.3)}\right)$

Source: *VCE 2017, Specialist Mathematics Exam 2, Section A, Q9; © VCAA*

Question 3 (1 mark)

Consider $\frac{dy}{dx} = 2x^2 + x + 1$ where $y(1) = y_0 = 2$.

Using Euler's method with a step size of 0.1 an approximation to $y(0.8) = y_2$ is given by

A. 0.94
B. 1.248
C. 1.6
D. 2.4
E. 2.852

Source: *VCE 2016, Specialist Mathematics 2, Section A, Q.9; © VCAA*

Question 4 (1 mark)

If $f(x) = \frac{dy}{dx} = 2x^2 - x$ where $y_0 = 0 = y(2)$, then y_3 using Euler's formula with step size 0.1 is

A. $0.1f(2)$
B. $0.6 + 0.1f(2.1)$
C. $1.272 + 0.1f(2.2)$
D. $2.02 + 0.1f(2.3)$
E. $2.02 + 0.1f(2.2)$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.11; © VCAA*

Question 5 (1 mark)

Let $\frac{dy}{dx} = x^3 - xy$ and $y = 2$ when $x = 1$.

Using Euler's method with a step size of 0.1, the approximation to y when $x = 1.1$ is

A. 0.9
B. 1.0
C. 1.1
D. 1.9
E. 2.1

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.11; © VCAA*

Question 6 (1 mark)

Consider the differential equation $\frac{dy}{dx} = \frac{1}{3 + 3x + x^2}$, with $y_0 = 1$ when $x_0 = 0$.

Using Euler's method with a step size of 0.1, the value of y_2, correct to three decimal places, is

A. 1.033
B. 1.063
C. 1.064
D. 1.065
E. 1.066

Question 7 (1 mark)

When solving the differential equation $\frac{dy}{dx} = f(x)$ and $y(x_0) = y_0$ using Euler's method, the value of y at $x = x_0 + 2h$ for a small value of h is given by

A. $\int_{x_0}^{x_0+2k} f(x)dx$
B. $\int_{x_0}^{2k} f(x)dx$
C. $y_0 + h\left[f(x_0) + f(x_0 + h)\right]$
D. $(y_0 + hf)(x_0 + 2h)$
E. $(y_0 + 2hf)(x_0 + 2h)$

Question 8 (1 mark)

Euler's method with a step size of 0.2 is used to solve the differential equation $\frac{dy}{dx} = \log_e\left(x^2\right)$, with initial condition $y = 1$ at $x = 2$. When $x = 2.6$ the value obtained for y is closest to

A. 1.59
B. 1.94
C. 1.6
D. 1.4
E. 1.28

Question 9 (1 mark)

Euler's method, with a step size of 0.25, is used to solve the differential equation $\frac{dy}{dx} = \log_e(2x - 3)$, with initial condition $y = 1$ when $x = 3$. The approximation obtained for y when $x = 3$ is given by

A. $\frac{3}{2}\log_e(3) - 1$

B. $1 + \frac{1}{4}\log_e\left(\frac{3}{2}\right)$

C. $1 + \frac{1}{4}\log_e(3)$

D. $1 + \frac{1}{4}\log_e\left(\frac{15}{2}\right)$

E. $3\log_e(3)$

Topic	10	Applications of first-order differential equations
Subtopic	10.8	Slope fields

online only

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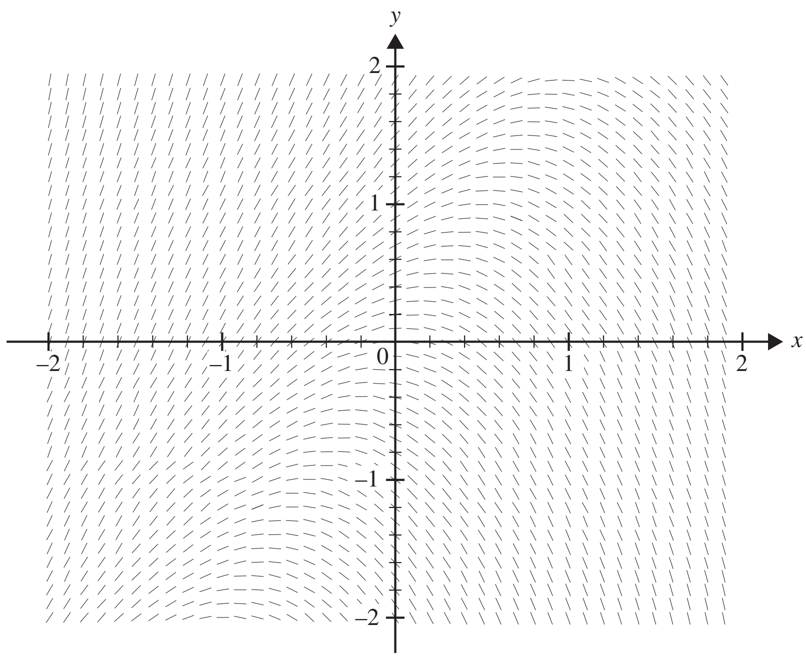
Source: *VCE 2021, Specialist Mathematics Exam 2, Section A, Q.10; © VCAA*

Question 1 (1 mark)



The differential equation that has the diagram above as its direction field is

A. $\dfrac{dy}{dx} = y + 2x$

B. $\dfrac{dy}{dx} = 2x - y$

C. $\dfrac{dy}{dx} = 2y - x$

D. $\dfrac{dy}{dx} = y - 2x$

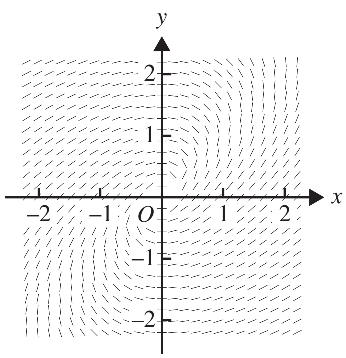
E. $\dfrac{dy}{dx} = x - 2y$

Source: *VCE 2020, Specialist Mathematics Exam 2, Section A, Q9; © VCAA*

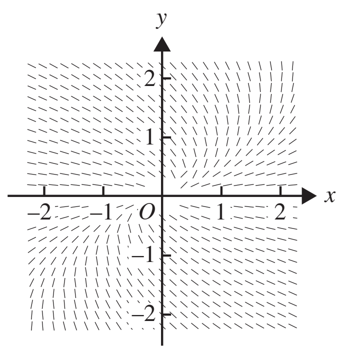
Question 2 (1 mark)

$P(x, y)$ is a point on a curve. The x-intercept of a tangent to point $P(x, y)$ is equal to the y value at P. Which one of the following slope fields best represents this curve?

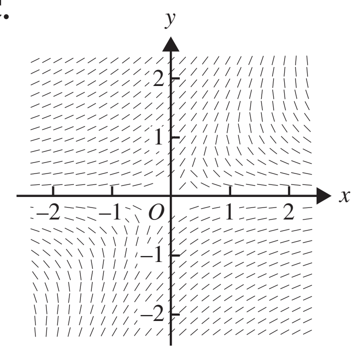
A.



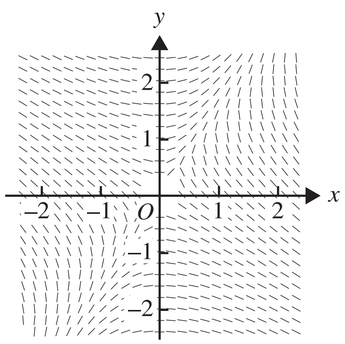
B.



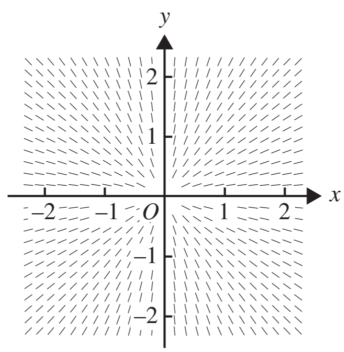
C.



D.

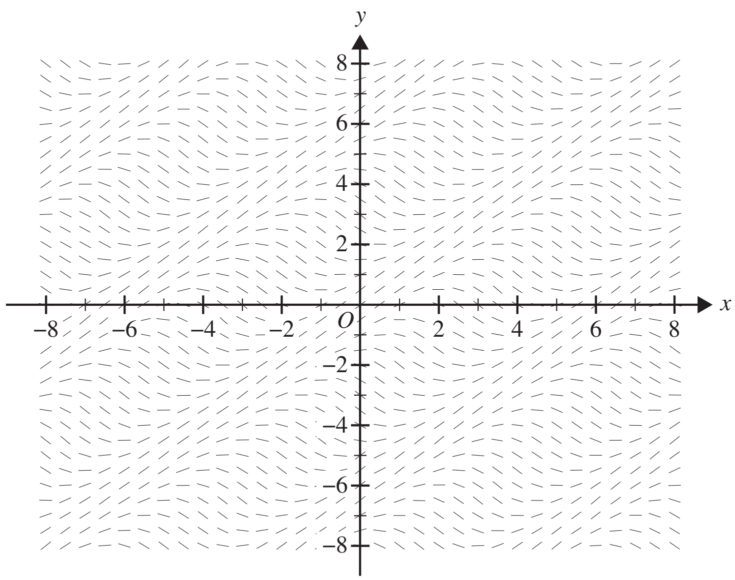


E.



Source: *VCE 2019, Specialist Mathematics Exam 2, Section A, Q9; © VCAA*

Question 3 (1 mark)



The differential equation that has the diagram above as its direction field is

A. $\dfrac{dy}{dx} = \sin(y - x)$

B. $\dfrac{dy}{dx} = \cos(y - x)$

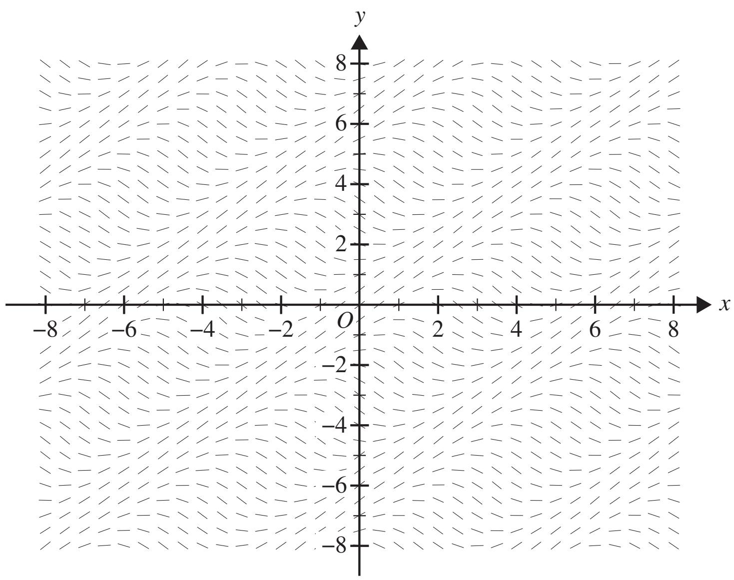
C. $\dfrac{dy}{dx} = \sin(x - y)$

D. $\dfrac{dy}{dx} = \dfrac{1}{\cos(y - x)}$

E. $\dfrac{dy}{dx} = \dfrac{1}{\sin(y - x)}$

Source: *VCE 2019, Specialist Mathematics Exam 2, Section A, Q.9; © VCAA*

Question 4 (1 mark)



The differential equation that has the diagram above as its direction field is

A. $\dfrac{dy}{dx} = \sin(y - x)$

B. $\dfrac{dy}{dx} = \cos(y - x)$

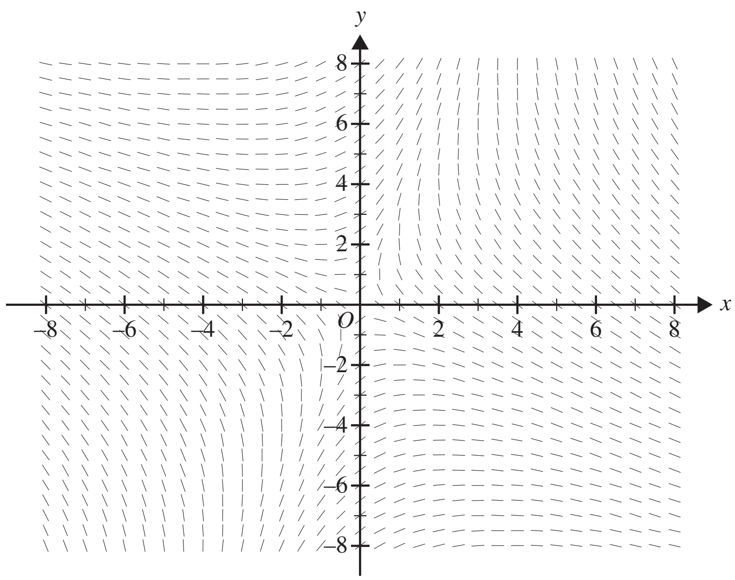
C. $\dfrac{dy}{dx} = \sin(x - y)$

D. $\dfrac{dy}{dx} = \dfrac{1}{\cos(y - x)}$

E. $\dfrac{dy}{dx} = \dfrac{1}{\sin(y - x)}$

Source: *VCE 2018, Specialist Mathematics 2, Section A, Q.10; © VCAA*

Question 5 (1 mark)



The differential equation that best represents the direction field above is

A. $\frac{dy}{dx} = \frac{2x + y}{y - 2x}$

B. $\frac{dy}{dx} = \frac{x + 2y}{2x - y}$

C. $\frac{dy}{dx} = \frac{2x - y}{x + 2y}$

D. $\frac{dy}{dx} = \frac{x - 2y}{y - 2x}$

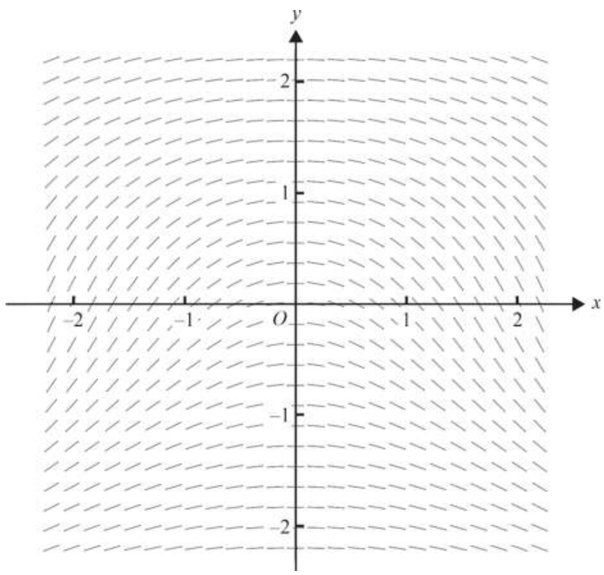
E. $\frac{dy}{dx} = \frac{2x + y}{2y - x}$

Topic 10 | Subtopic 10.8 Slope fields

Source: *VCE 2017, Specialist Mathematics 1, Q.8a; © VCAA*

Question 6 (2 marks)

A slope field representing the differential equation $\frac{dy}{dx} = \frac{-x}{1 + y^2}$ is shown below.

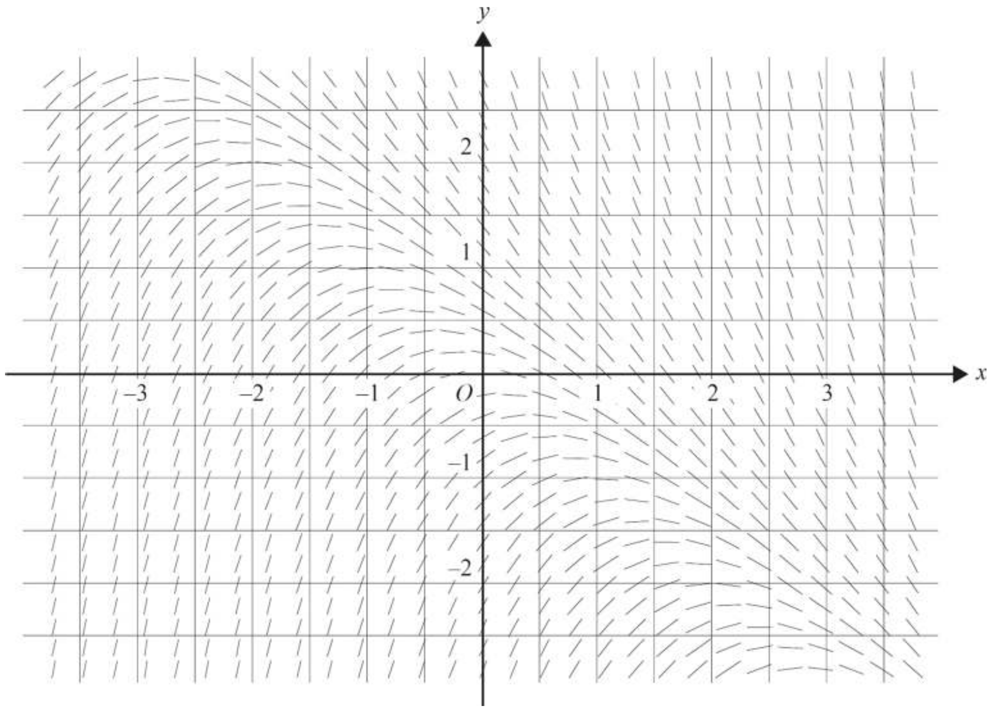


Sketch the solution curve of the differential equation corresponding to the condition $y(-1) = 1$ on the slope field above and, hence, estimate the positive value of x when $y = 0$.
Give your answer correct to one decimal place.

Topic 10 | Subtopic 10.8 Slope fields

Source: *VCE 2016, Specialist Mathematics 2, Section A, Q.10; © VCAA*

Question 7 (1 mark)



The direction field for the differential equation $\frac{dy}{dx} + x + y = 0$ is shown above.

A solution to this differential equation that includes $(0, -1)$ could also include

A. $(3, -1)$

B. $(3.5, -2.5)$

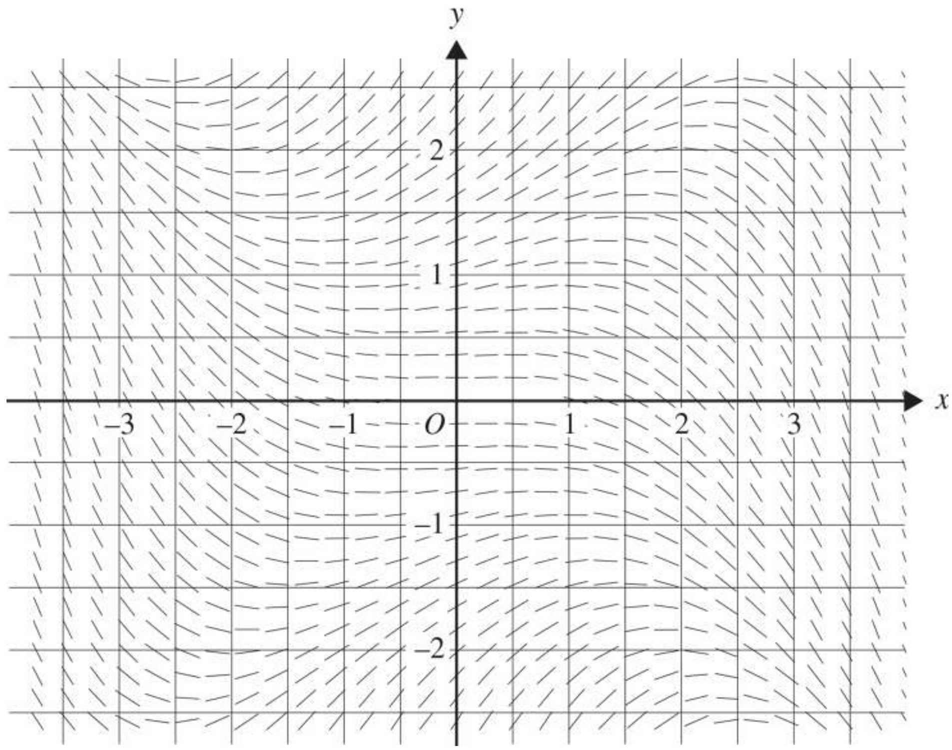
C. $(-1.5, -2)$

D. $(2.5, -1)$

E. $(2.5, 1)$

Source: *VCE 2015, Specialist Mathematics 2, Section 1, Q.13; © VCAA*

Question 8 (1 mark)

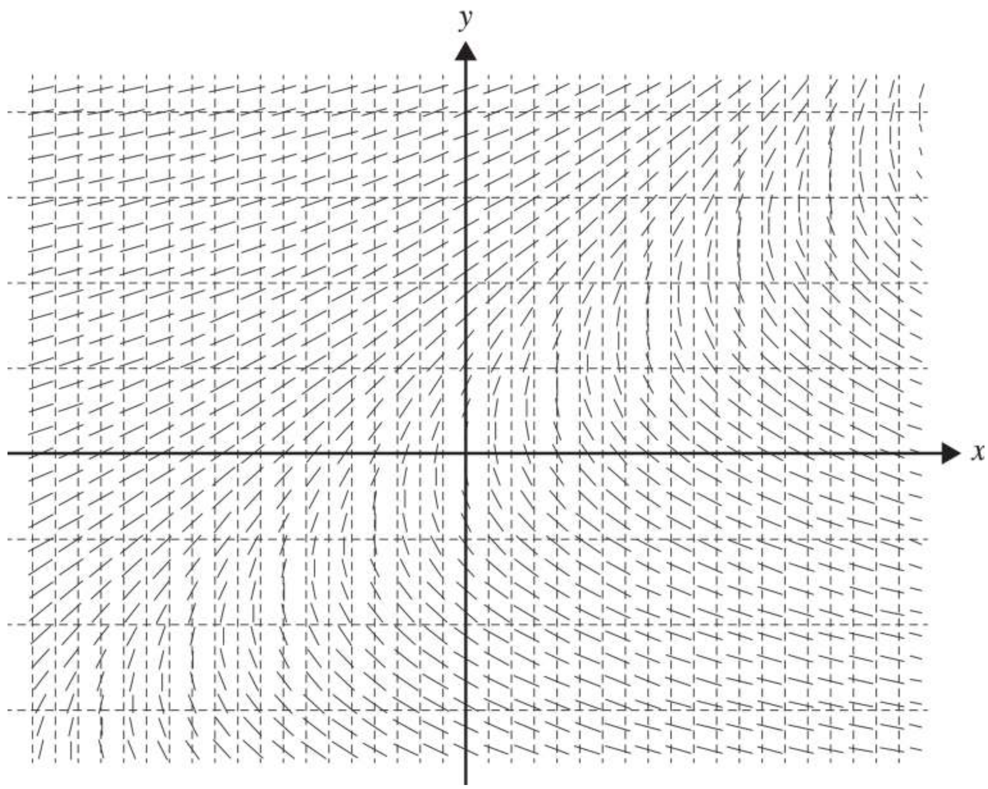


The direction field for a certain differential equation is shown above.
The solution curve to the differential equation that passes through the point $(-2.5, 1.5)$ could also pass through

A. $(0, 2)$
B. $(1, 2)$
C. $(3, 1)$
D. $(3, -0.5)$
E. $(-0.5, 2)$

Source: *VCE 2014, Specialist Mathematics 2, Section 1, Q.14; © VCAA*

Question 9 (1 mark)



The differential equation that is best represented by the above direction field is

A. $\dfrac{dy}{dx} = \dfrac{1}{x - y}$

B. $\dfrac{dy}{dx} = y - x$

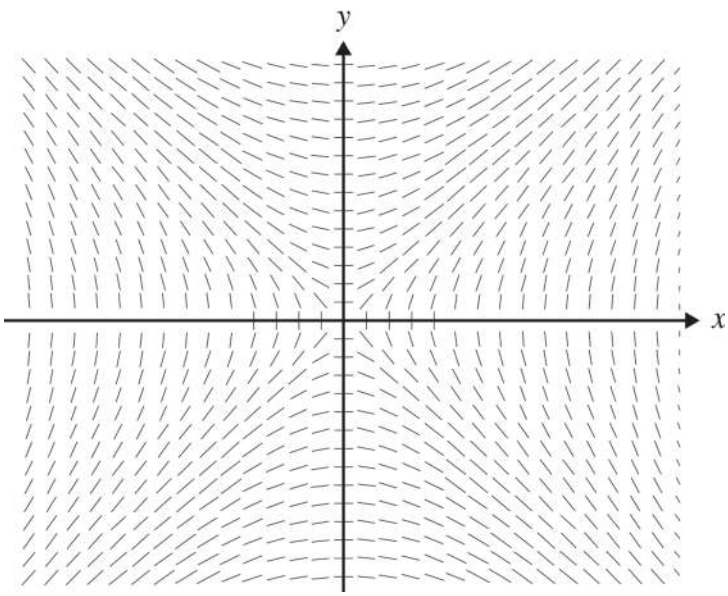
C. $\dfrac{dy}{dx} = \dfrac{1}{y - x}$

D. $\dfrac{dy}{dx} = x - y$

E. $\dfrac{dy}{dx} = \dfrac{1}{y + x}$

Source: *VCE 2013, Specialist Mathematics 2, Section 1, Q.12; © VCAA*

Question 10 (1 mark)



The differential equation that best represents the above direction field is

A. $\frac{dy}{dx} = x^2 - y^2$

B. $\frac{dy}{dx} = y^2 - x^2$

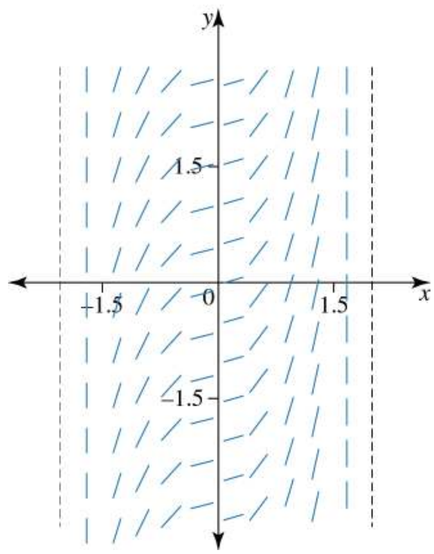
C. $\frac{dy}{dx} = \frac{y}{x}$

D. $\frac{dy}{dx} = -\frac{x}{y}$

E. $\frac{dy}{dx} = \frac{x}{y}$

Topic 10 | Subtopic 10.8 Slope fields

Question 11 (1 mark)
The direction (slope) field for a certain first order differential equation is shown.



The differential equation could be

A. $\dfrac{dy}{dx} = x^3$

B. $\dfrac{dy}{dx} = x^5$

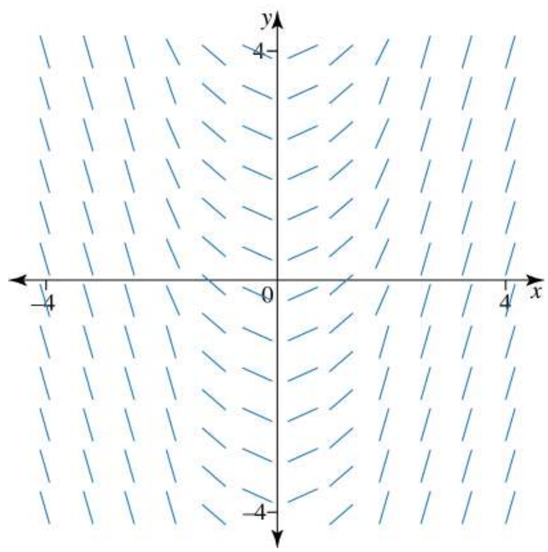
C. $\dfrac{dy}{dx} = 3x^2$

D. $\dfrac{dy}{dx} = x^7$

E. $\dfrac{dy}{dx} = x$

Question 12 (1 mark)

The direction (slope) field for a certain first order differential equation is shown.



The differential equation could be

A. $\frac{dy}{dx} = -x^3$

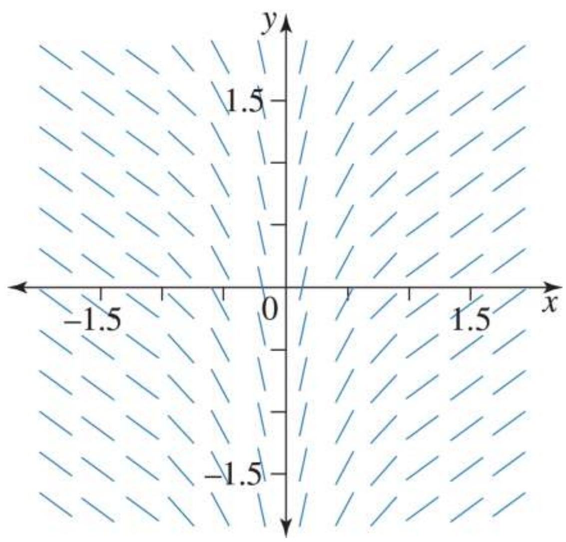
B. $\frac{dy}{dx} = x^4$

C. $\frac{dy}{dx} = x^2$

D. $\frac{dy}{dx} = x^{-1}$

E. $\frac{dy}{dx} = x$

Question 13 (1 mark)
The direction (slope) field for a certain first order differential equation is shown.



The differential equation could be

A. $\frac{dy}{dx} = -x^3$

B. $\frac{dy}{dx} = x^4$

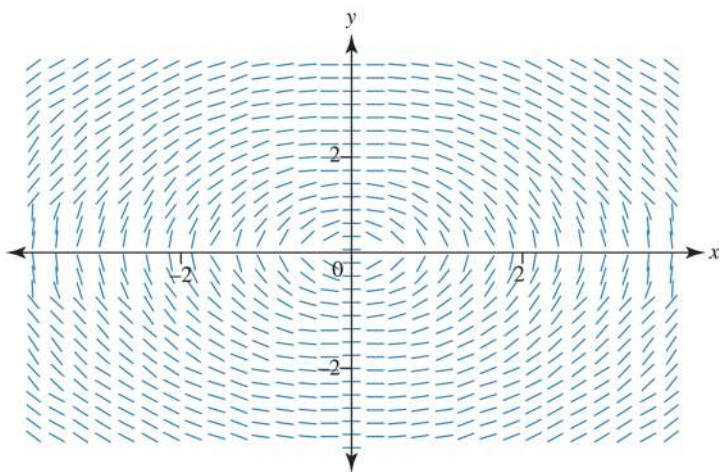
C. $\frac{dy}{dx} = x^2$

D. $\frac{dy}{dx} = \frac{1}{x}$

E. $\frac{dy}{dx} = x$

Question 14 (1 mark)

The direction (slope) field for a certain differential equation is shown. The differential equation could be



A. $\frac{dy}{dx} = -\frac{x}{2y}$

B. $\frac{dy}{dx} = \frac{x}{2y}$

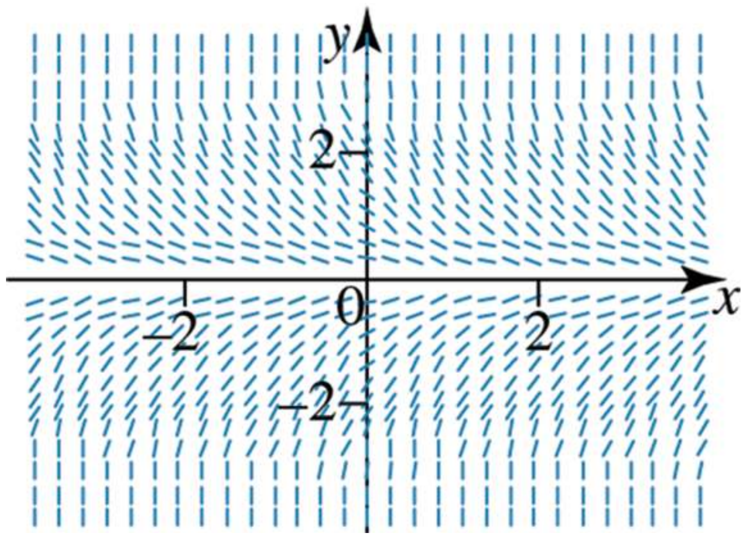
C. $\frac{dy}{dx} = \frac{2y}{x}$

D. $\frac{dy}{dx} = -\frac{2y}{x}$

E. $\frac{dy}{dx} = -\frac{2x}{y}$

Question 15 (1 mark)

The direction (slope) field for a certain differential equation is shown. One solution to the differential equation could be



A. $y = -e^{x}$

B. $y = e^{-x}$

C. $y = e^{x}$

D. $y = \log_{e}(x)$

E. $y = -\frac{1}{x}$

Topic	10	Applications of first-order differential equations
Subtopic	10.9	Review

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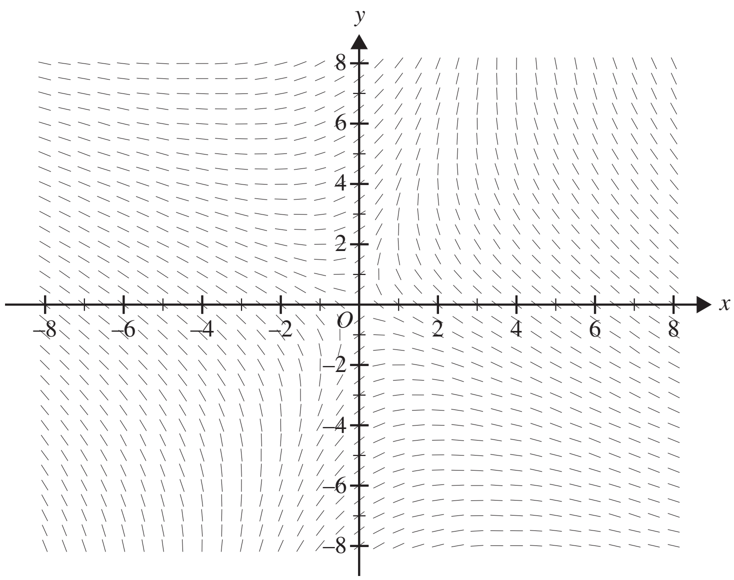
Source: *VCE 2018, Specialist Mathematics Exam 2, Section A, Q 10; © VCAA*

Question 1 (1 mark)

The differential equation that best represents the direction field shown is



A. $\dfrac{dy}{dx} = \dfrac{2x + y}{y - 2x}$

B. $\dfrac{dy}{dx} = \dfrac{x + 2y}{2x - y}$

C. $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 2y}$

D. $\dfrac{dy}{dx} = \dfrac{x - 2y}{y - 2x}$

E. $\dfrac{dy}{dx} = \dfrac{2x + y}{2y - x}$

Source: *VCE 2016, Specialist Mathematics Exam 2, Section A, Q9; © VCAA*

Question 2 (1 mark)

If $f(x) = \dfrac{dy}{dx} = 2x^2 - x$, where $y_0 = 0 = y(2)$, then y_3 using Euler's formula with step size 0.1 is

A. $0.1f(2)$
B. $0.6 + 0.1f(2.1)$
C. $1.272 + 0.1f(2.2)$
D. $2.02 + 0.1f(2.3)$
E. $2.02 + 0.1f(2.2)$

Source: *VCE 2013, Specialist Mathematics Exam 2, Section 2, Q3; © VCAA*

Question 3 (12 marks)

The number of mobile phones, N, owned in a certain community after t years, may be modelled by $\log_e(N) = 6 - 3e^{-0.4t}$, $t \geq 0$.

a. Verify by substitution that $\log_e(N) = 6 - 3e^{-0.4t}$ satisfies the differential equation $\dfrac{1}{N}\dfrac{dN}{dt} + 0.4\log_e(N) - 2.4 = 0$. **(2 marks)**

b. Find the initial number of phones owned in the community. Give your answer correct to the nearest integer. **(2 marks)**

c. Using this mathematical model, find the limiting number of mobile phones that would eventually be owned in the community. Give your answer correct to the nearest integer. **(2 marks)**

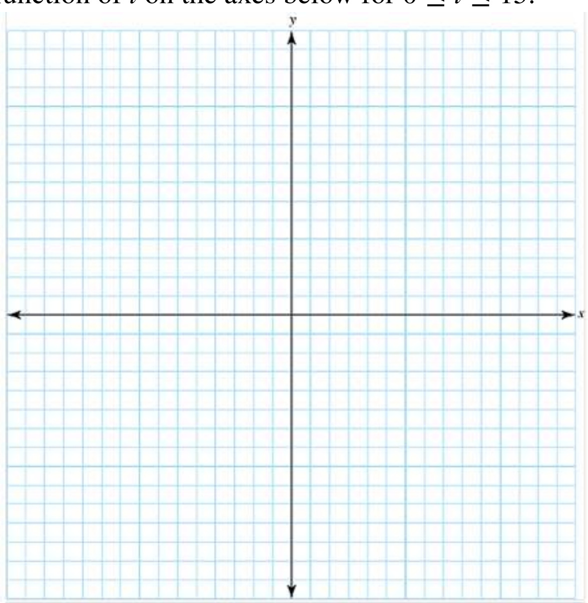
The differential equation in part **a**. can also be written in the form $\frac{dN}{dt} = 0.4N(6 - \log_e(N))$.

d. i. Find $\frac{d^2N}{dt^2}$ in terms of N and $\log_e(N)$. **(2 marks)**

ii. The graph of N as a function of t has a point of inflection.
Find the values of the coordinates of this point.
Give the value of t correct to one decimal place and the value of N correct to the nearest integer. **(2 marks)**

$N = \square$, $t = \square$ years

e. Sketch the graph of N as a function of t on the axes below for $0 \le t \le 15$. **(2 marks)**



Question 4 (4 marks)

A lake can withstand a maximum of 50 ducks. Initially there are 4 ducks in the lake. The number of ducks in the lake grows at a rate proportional to the difference between the maximum number and the current number of ducks in the lake. After 4 months there are 16 ducks in the lake.

a. Write down and solve the differential equation for the number of ducks n in the lake after t months. **(3 marks)**

b. Calculate how many ducks are in the lake after 8 months. **(1 mark)**

☐ ducks

Question 5 (3 marks)

The table shows the population in millions of Canada.

Year	2010	2015	2020
Population	34.148	36.027	37.742

Assuming the population follows a logistic growth rate:

a. determine the maximum population of Canada, correct to the nearest thousand. **(2 marks)**

b. calculate the population of Canada in 2000, correct to the nearest thousand. **(1 mark)**

☐ million

Source: *VCE 2019, Specialist Mathematics 2, Section B, Q.3; © VCAA*

Question 6 (9 marks)

Answer the following.

a. The growth and decay of a quantity P with respect to time t is modelled by the differential equation

$$\frac{dP}{dt} = kP$$

where $t \geq 0$.

i. Given that $P(a) = r$ and $P(b) = s$, where P is a function of t, show that

$$k = \frac{1}{a-b}\log_e\left(\frac{r}{s}\right).$$ **(2 marks)**

ii. Specify the condition(s) for which $k > 0$. **(2 marks)**

b. The growth of another quantity Q with respect to time t is modelled by the differential equation

$$\frac{dQ}{dt} = e^{t-Q}$$

where $t \geq 0$ and $Q = 1$ when $t = 0$.

i. Express this differential equation in the form

$\int f(Q)\,dQ = \int h(t)\,dt.$ **(1 mark)**

ii. Hence, show that $Q = \log_e\left(e^t + e - 1\right)$. **(2 marks)**

iii. Show that the graph of Q as a function of t does not have a point of inflection. **(2 marks)**

Source: *VCE 2016, Specialist Mathematics 2, Section B, Q.3; © VCAA*

Question 7 (11 marks)

A tank initially has 20 kg of salt dissolved in 100 L of water. Pure water flows into the tank at a rate of 10 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 5 L/min.

If x kilograms is the amount of salt in the tank after t minutes, it can be shown that the differential equation relating x and t is

$$\frac{dx}{dt} + \frac{x}{20+t} = 0.$$

a. Solve this differential equation to find x in terms of t. **(3 marks)**

b. A second tank initially has 15 kg of salt dissolved in 100 L of water. A solution of $\frac{1}{60}$ kg of salt per litre flows into the tank at a rate of 20 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 10 L/min.

If y kilograms is the amount of salt in the tank after t minutes, write down an expression for the **concentration**, in kg/L, of salt in the second tank at time t. **(1 mark)**

c. Show that the differential equation relating y and t is $\frac{dy}{dt} + \frac{y}{10+t} = \frac{1}{3}$. **(2 marks)**

d. Verify by differentiation and substitution into the left side that $y = \frac{t^2 + 20t + 900}{6(10+t)}$ satisfies the **differential equation** in **part c.** Verify that the given solution for y also satisfies the **initial condition**. **(3 marks)**

e. Find when the **concentration** of salt in the second tank reaches 0.095 kg/L. Give your answer in minutes, correct to two decimal places. **(2 marks)**

Source: *VCE 2013, Specialist Mathematics 1, Q.6; © VCAA*

Question 8 (4 marks)

Find the value of c, where $c \in R$, such that the curve defined by $y^2 + \frac{3e^{(x-1)}}{x-2} = c$ has a gradient of 2 where $x = 1$.

Source: *VCE 2009, Specialist Mathematics 1, Q.9; © VCAA*

Question 9 (5 marks)

Let $\frac{dy}{dx} = (y+2)^2 + 4$ and $y_0 = y(0) = 0$.

a. Solve the differential equation above, giving y as a function of x. **(3 marks)**

b. Apply Euler's method to find y_1, using a step size of 0.1. **(2 marks)**

Question 10 (3 marks)

If $\frac{dy}{dx} = e^x\sqrt{y},\ y(0) = 1$, find y in terms of x.

Question 11 (3 marks)

If $\frac{dy}{dx} = (e^x)^3(e^y)^2,\ y(0) = 0$, find y in terms of x.

Answers and marking guide

10.2 Growth and decay

Question 1

a. i. $\frac{dP}{dt} = kP$

$\int \frac{1}{P}\,dP = \int k\,dt$

$\log_e(|P|) = kt + c$ but $P > 0$, so the modulus is not needed

$P = P(t) = e^{kt+c} = P_0\,e^{kt}$ where $P_0 = e^c$

(1) $P(a) = P_0\,e^{ka} = r$, (2) $P(b) = P_0\,e^{kb} = s$

$\frac{(1)}{(2)}\quad \frac{r}{s} = \frac{e^{ka}}{e^{kb}} = e^{k(a-b)},\quad k(a-b) = \log_e\left(\frac{r}{s}\right)$

$k = \frac{1}{a-b}\log_e\left(\frac{r}{s}\right)$

Award **1 mark** for correctly integrating and solving.

Award **1 mark** for the correct result.

ii. $k = \frac{1}{a-b}\log_e\left(\frac{r}{s}\right) > 0$, since $P > 0 \Rightarrow \quad r > 0,\ s > 0.$

Case (1) $k > 0,\ \frac{1}{a-b} > 0,\ \log_e\left(\frac{r}{s}\right) > 0 \quad \Rightarrow\ a > b$ and $r > s > 0$ or

Case (2) $k > 0,\ \frac{1}{a-b} < 0,\ \log_e\left(\frac{r}{s}\right) < 0 \quad \Rightarrow\ a < b$ and $s > r > 0$

Award **1 mark** each for the correct cases.

b. i. $\frac{dQ}{dt} = e^{t-Q} = \frac{e^t}{e^Q}$

$\int e^Q\,dQ = \int e^t\,dt$ [1 mark]

ii. $e^Q = e^t + c$ to find c use $Q = 1,\ t = 0$

$e = 1 + c,\quad c = e - 1$

$e^Q = e^t + e - 1$

$Q = \log_e\left(e^t + e - 1\right)$

Award **1 mark** for correctly integrating.

Award **1mark** for the correct proof.

iii. $\frac{d^2Q}{dt^2} = \frac{(e-1)\,e^t}{\left(e^t + e - 1\right)^2} \neq 0$ since $e > 1$ and $e^t > 0$ so there are no inflection points.

Award **1 mark** for the correct second derivative.

Award **1 mark** for the correct argument

Question 2

$m = m_0 e^{-kt},\ m_0 = 3$

when $t = 38213,\ m = 1$

$1 = 3e^{-38213k}$

$e^{-38213k} = \frac{1}{3}$

$-38213k = \log_e\left(\frac{1}{3}\right)$

$$k = \frac{-\log_e\left(\frac{1}{3}\right)}{38213}$$
$$\approx 0.00002875$$

$m = 3e^{-0.00002875t}$ **[1 mark]**

The half-life is the time taken for the mass to reduce to half its initial mass.

$$1.5 = 3e^{-0.00002875t}$$
$$e^{-0.00002875t} = \frac{1}{2}$$
$$\log_e\left(\frac{1}{2}\right) = -0.00002875t$$
$$t = \frac{\log_e\left(\frac{1}{2}\right)}{-0.00002875}$$
$= 24\,109$ years **[1 mark]**

Question 3

$$\frac{dN}{dt} = kN$$
$$N = N_0e^{kt}$$
$$N = 80.827e^{kt}$$

when $t = 11$, $N = 83.900$

$$83.9 = 80.827e^{11k}$$
$$e^{11k} = \frac{83.9}{80.827}$$
$$11k = \log_e\left(\frac{83.9}{80.827}\right)$$
$$11k = 0.0373$$
$$k = 0.00339$$
$$N = 80.827e^{0.00339t}$$
$$100 = 80.827e^{0.00339t}$$
$$e^{0.00339t} = \frac{100}{80.827}$$
$$0.00339t = \log_e\left(\frac{100}{80.827}\right)$$
$$0.00339t = 0.21286$$
$$t = 62.8$$

62.8 years after 2010 is late in 2072.

The correct answer is **D**.

Question 4

$$\frac{dN}{dt} \propto N(1000 - N)$$
$$\frac{dN}{dt} = kN(1000 - N)$$

Answer: **C**

***VCAA Assessment Report* note:**

The number not infected is $1000 - N$. The product is then $N(1000 - N)$; hence, optio **C** is correct.

Question 5

$$\frac{dH}{dt} = pH(5000 - H) \quad \textbf{[1 mark]}$$

$$\frac{dt}{dH} = \frac{1}{pH(5000 - H)}$$

$$\frac{1}{5000p}\left(\frac{1}{H} + \frac{1}{5000 - H}\right) \quad \textbf{[1 mark]}$$

$$t = \frac{1}{5000p}\left(\ln\ H - \ln|5000 - H|\right) + c \ \textbf{[1 mark]}$$

$$t = 0, H = 50$$

$$0 = \frac{1}{5000p}\left(\ln 50 - \ln|5000 - 50|\right) + c$$

$$c = \frac{1}{5000p}\ln 99 \ \textbf{[1 mark]}$$

$$t = \frac{1}{5000p}\ln\left(\frac{99H}{5000 - H}\right) \quad \textbf{[1 mark]}$$

Question 6

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dt}{dN} \propto \frac{1}{N} \Rightarrow \frac{dt}{dN} = \frac{k}{N} \ \textbf{[1 mark]}$$

$$t = k\,\text{In}\ (N) + c$$

$$t = 0,\ \ N = 2, c = -k\,\text{In}\,(2)$$

$$t = k\,\text{In}\ (0.5N) \quad \textbf{[1 mark]}$$

$$t = 5, N = 3$$

$$\Rightarrow k = \frac{5}{\text{In}\,(1.5)} \quad \textbf{[1 mark]}$$

$$t = 5\frac{\text{In}\,(0.5N)}{\text{In}\,(1.5)}$$

$$t = 15,\ \ N = 6.75 \qquad \textbf{[1 mark]}$$

Question 7

$$\frac{dN}{dt} = kN \ \text{ and } \ N(0) = N_0 \Rightarrow N = N(t) = N_0e^{kt}$$

$N(0) = N_0 = 1$ million

N $(5) = 2N_0$

$$2N_0 = N_0e^{5k} \Rightarrow e^{5k} = 2 \quad k = \frac{1}{5}\log_e(2)$$

Now when $t = 15$

$N(15) = N_0e^{3\log_e(2)} = N_0e^{\log_e(8)} = 8N_0 = 8$ million

Question 8

$$\frac{dN}{dt} = kN \text{ and } N(0) = N_0 \Rightarrow N = N(t) = N_0e^{kt}$$

$N(0) = N_0 = 200$

$N(2) = 400$

$$400 = 200e^{2k} \Rightarrow e^{2k} = 2 \quad k = \frac{1}{2}\log_e(2) = \log_e\left(\sqrt{2}\right)$$

Question 9

Q is the amount present at time t

$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ \Rightarrow Q = Q(t) = Q_0 e^{-kt}$ with $k > 0$

$T = \frac{1}{k}\log_e(2) = 24 \Rightarrow k = \frac{1}{24}\log_e(2)$

Now when $t = 8$

$Q(8) = Q_0 e^{-\frac{1}{3}\log_e(2)} = 0.794 Q_0$

So 79% remains.

Question 10

$\frac{dQ}{dt} = -kQ^2$

Inverting: $\frac{dt}{dQ} = -\frac{1}{kQ^2}$

$-k\int 1 dt = \int \frac{1}{Q^2} dQ$

$-kt = \frac{-1}{Q} + C \quad Q(0) = Q_0$

$0 = -\frac{1}{Q_0} + C \Rightarrow C = \frac{1}{Q_0}$

$-kt = -\frac{1}{Q} + \frac{1}{Q_0}$

Now when $t = T, Q = \frac{1}{2}Q_0$

$-kT = -\frac{2}{Q_0} + \frac{1}{Q_0} = -\frac{1}{Q_0}$

$\Rightarrow kQ_0T = 1$

Question 11

$\frac{dP}{dt} = kP$

$$\begin{aligned}
t &= \int \frac{1}{kP} dP + C \\
&= \frac{1}{k}\log_e AP \\
t &= 0, \quad P = 5000 \\
0 &= \frac{1}{k}\log_e(5000A) \\
1 &= 5000A \\
A &= \frac{1}{5000} \\
\therefore t &= \frac{1}{k}\log_e\left(\frac{P}{5000}\right) \\
P &= 5000e^{kt} \\
3500 &= 5000e^{kt} \\
k &= -0.07 \\
P &= 5000e^{-0.07t} \\
\frac{dP}{dt} &= -350e^{-.07t}
\end{aligned}$$

Question 12

$$F = -200\int e^{-kt}dt + C$$
$$= 200\frac{e^{-kt}}{k} + C$$
$$t = 0:$$
$$4000 = \frac{200}{k} + C$$
$$C = 4000 - \frac{200}{k}$$
$$3115 = 200\frac{e^{-5k}}{k} + 4000 - \frac{200}{k}$$
$$-885 = \frac{200\left(e^{-5k} - 1\right)}{k}$$
$$\therefore k = 0.05$$

10.3 Other applications of first-order differential equations

Question 1

a. $\frac{dQ}{dt} = -kQ,\ k > 0,\ Q(0) = Q_0$

$$Q = Q_0 e^{-kt}$$
$$Q(0.5) = 500$$
$$Q(1.0) = 250$$
$$[1]\ :\ 500 = Q_0 e^{-\frac{k}{2}}$$
$$[2]\ :\ 250 = Q_0 e^{-k}$$
$$2 = \frac{e^{-\frac{k}{2}}}{e^{-k}} = e^{\frac{k}{2}}$$
$$\frac{k}{2} = \log_e(2)$$
$$k = 2\log_e(2)$$
$$k = \log_e(4)$$
$$e^k = 4$$
$$Q_0 = 250e^k = 1000\text{ units}$$ **[1 mark]**

b. $Q = 1000e^{-kt}$

$$Q = 125 = 1000e^{-kt}$$
$$\frac{125}{1000} = \frac{1}{8} = e^{-kt}$$
$$-kt = \log_e\left(\frac{1}{8}\right)$$
$$t = -\frac{1}{k}\log_e\left(\frac{1}{8}\right) = \frac{-\log_e\left(\frac{1}{8}\right)}{\log_e(4)}$$
$$= \frac{\log_e(8)}{\log_e(4)} = 1.5$$ **[1 mark]**

Question 2

$$\frac{dQ}{dt}+\frac{Q}{RC}=0,\ Q(0)=Q_0$$

$$\frac{dQ}{dt}=-\frac{Q}{RC}$$

$$\frac{dt}{dQ}=-\frac{RC}{Q}$$

$$t=-RC\int\frac{1}{Q}\,dQ$$ **[1 mark]**

$$=-RC\log_e(Q)+c$$

$$t=0,\qquad Q=Q_0$$

$$0=-RC\log_e(Q_0)+c$$

$$c=RC\log_e(Q_0)$$

$$t=-RC\ln(Q)+RC\ln(Q_0)$$ **[1 mark]**

$$\frac{t}{RC}=\ln\left(\frac{Q_0}{Q}\right)$$

$$\frac{Q_0}{Q}=e^{\frac{t}{RC}}$$

$$\frac{Q}{Q_0}=e^{-\frac{t}{RC}}$$

$$Q=Q(t)=Q_0e^{-\frac{t}{RC}}$$ **[1 mark]**

Question 3

$$\frac{dI}{dx}=-kI$$

$$\frac{dx}{dI}=-\frac{1}{kI}$$

$$x=-\frac{1}{k}\int\frac{1}{I}dI$$

$-kx=\log_e(I)+c$ **[1 mark]**

When $x=0$, $I=I_0$

$$c=-\log_e(I_0)$$

$$-kx=\log_e(I)-\log_e(I_0)$$

$$-kx=\log_e\left(\frac{I}{I_0}\right)$$

$I=I_0e^{-kx}$ **[1 mark]**

When $I=0.95I_0$, $x=0.40$

$$0.95I_0=I_0e^{-0.4k}$$

$$0.95=e^{-0.4k}$$

$$-0.4k=\log_e(0.95)$$

$$k=-\frac{1}{0.4}\log_e(0.95)=0.1282$$

$$I=I_0e^{-0.1282x}$$

$$x=1$$

$$I=I_0e^{-0.1282}=0.880I_0$$

Then the percentage lost is 12.0% **[1 mark]**

Question 4

a. $\log_e(N) = 6 - 3e^{-0.4t}, t \geq 0$

Using implicit differentiation: $\frac{1}{N}\frac{dN}{dt} = 3 \times 0.4e^{-0.4t}$

LHS $\quad \frac{1}{N}\frac{dN}{dt} + 0.4\log_e(N) - 2.4$

$$= 3 \times 0.4e^{-0.4t} + 0.4\left(6 - 3e^{-0.4t}\right) - 2.4$$
$$= 3 \times 0.4e^{-0.4t} + 2.4 - 3 \times 0.4e^{-0.4t} - 2.4 = 0 \qquad \text{(shown)}$$

Award **1 mark** for using implicit differentiation.

Award **1 mark** for correct verification.

***VCAA Assessment Report* note:**

The question was not done well. Many students employed a variety of approaches different to the required method of substitution.

b. When $\quad t = 0, \log_e(N) = 6 - 3e^0 = 3$

$\Rightarrow \qquad N = e^3 \approx 20$

Award **1 mark** for the correct initial value.

***VCAA Assessment Report* note:**

This question was quite well done; however, a number of students left their answer as e^3, instead of giving an answer to the nearest integer.

c. As $\quad t \to \infty, \log_e(N) \to 6 - 3e^{-\infty} = 6$

$\Rightarrow \qquad N = e^6 \approx 403$

Award **1 mark** for the correct limiting value.

***VCAA Assessment Report* note:**

This question was fairly well done, with still some students giving the answer as e^6, rather than the nearest integer, 403. A number of students did not show their working.

d. i. $\frac{d^2N}{dt^2} = \frac{d}{dt}\left(\frac{dN}{dt}\right) = \frac{d}{dt}\left[0.4N\left(6 - \log_e(N)\right)\right]$

Using the product rule:

$$\frac{d^2N}{dt^2} = 0.4\frac{dN}{dt}\left(6 - \log_e(N)\right) + 0.4N \times \frac{-1}{N}\frac{dN}{dt}$$
$$= \frac{dN}{dt}\left[0.4\left(6 - \log_e(N)\right) - 0.4\right]$$
$$= \frac{dN}{dt}\left(2 - 0.4\log_e(N)\right)$$

Substitute for $\frac{dN}{dt} = 0.4N\left(6 - \log_e(N)\right)$

$$= 0.4N\left(6 - \log_e(N)\right)\left(2 - 0.4\log_e(N)\right)$$
$$= \frac{4N}{25}\left(6 - \log_e(N)\right)\left(5 - \log_e(N)\right)$$

Award **1 mark** for correct differentiation using the product rule.

Award **1 mark** for the correct expression for the secondderivative.

***VCAA Assessment Report* note:**

This question required the use of both the chain and product rules. Very few students answered this question correctly. A number of students found $\frac{d^2N}{dt^2}$ in terms of t, while others found $\frac{d^2t}{dN^2}$ and attempted to invert the result, which showed little understanding of the properties of second derivatives. A common error was to find $\frac{d}{dN}\left(\frac{dN}{dt}\right)$.

ii. For inflexion points: $\frac{d^2N}{dt^2} = 0$

Since $\log_e(N) \neq 6 \Rightarrow \log_e(N) = 5 \Rightarrow N = e^5 \approx 148$

$5 = 6 - 3e^{-0.4t} \Rightarrow e^{-0.4t} = \frac{1}{3}, t = \frac{1}{0.4}\log_e(3) = 2.7$ years

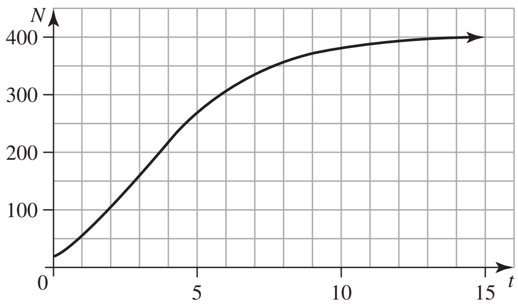
Award **1 mark** for setting the second derivative equal to zero and finding the value of N.

Award **1 mark** for the correct t value.

***VCAA Assessment Report* note:**

The majority of students realised that the second derivative needed to be equated to zero. However, there was a small group who equated the first derivative to zero. Some students obtained the correct result independently of their efforts in part i.

e.



Award **1 mark** for the correct shape on a restricted domain.

Award **1 mark** for drawing a graph passing through the initial point (0, 20) and the inflection point (2.7, 148).

***VCAA Assessment Report* note:**

This question was moderately well done. Major errors included inaccurate position of endpoints, inaccurate placement of the point of inflection and lack of change in concavity.

Question 5

$\frac{dx}{dt} = -0.15x \Rightarrow x = x_0 e^{-0.15t}$

When $x = \frac{1}{2}x_0 \quad t = ?$

$\frac{1}{2} = e^{-0.15t} \Rightarrow \log_e(2) = 0.15t$

$t = \frac{1}{0.15}\log_e(2) = \frac{20}{3}\log_e(2)$

Question 6

$$\frac{dP}{dt} = kP$$

$$t = \int \frac{1}{kP}dt + C$$

$$= \frac{1}{k}\log_e(AP)$$

$$t = 0, P = 50:$$

$$0 = \frac{1}{k}\log_e(50A)$$

$$1 = 50A$$

$$A = \frac{1}{50}$$

$$\therefore t = \frac{1}{k}\log_e\left(\frac{P}{50}\right)$$
$$P = 50e^{kt}$$
$$35 = 50e^{kt} \quad \therefore k = -0.07$$
$$P = 50e^{-0.07t} \quad \therefore \frac{dP}{dt} = -3.5e^{-.07t}$$

10.4 Bounded growth and Newton's law of cooling

Question 1

a. Let $Q = T - 20$

$Q = Q_0 e^{-kt}$

When $t = 0$, $T = 100$, therefore $Q_0 = 80$ **[1 mark]**

When $t = 5$, $T = 80$, therefore $Q(5) = 60$

$$60 = 80e^{-5k}$$
$$e^{-5k} = \frac{60}{80}$$
$$e^{-5k} = \frac{3}{4} \text{ [1 mark]}$$

b. When $t = 10$, $\theta(10) = 80e^{-10k} = 80(e^{-5k})^2 = 80 \times \left(\frac{3}{4}\right)^2$
$$= \frac{80 \times 9}{16} = 45$$

So the temperature of the water is 65°C.

Award **1 mark** for substituting.

Award **1 mark** for the correct indices and finding the temperature difference.

Award **1 mark** for the correct value of the water temperature.

VCAA Assessments Report **note:**

Surprisingly, the result of Question 5a. was rarely used directly, the majority of students expressed k in logarithmic form and often struggled with the algebra. In many cases, their solution to the differential equation had not been expressed with T as the subject but was left with t as the subject. Only a small number of students who did have $T = 20 + 80e^{-kt}$ were able to write in order to solve this part efficiently. Some had difficulties with the exponentials and logarithms. There was some poor arithmetic at the end; for example, $T - 20 = 45$ so $T = 25$ was quit common. Several students multiplied first rather than cancelling and made their working more complicated. A small number of students left their answer in the from $T = \frac{1040}{16}$, $T = \frac{720}{16} + 20$ or $T = \frac{130}{2}$ (unsimplified).

Question 2

a. $L\frac{di}{dt} + Ri = E,\ i(0) = 0$

$$L\frac{di}{dt} = E - Ri$$
$$\frac{1}{L}\frac{dt}{di} = \frac{1}{E - Ri}$$
$$\frac{1}{L}\int dt = \int \frac{1}{E - Ri}\,di \qquad \textbf{[1 mark]}$$
$$\frac{t}{L} = -\frac{1}{R}\log_e(E - Ri) + c$$
$$t = 0, i = 0$$
$$0 = -\frac{1}{R}\log_e(E) + c$$

$$\rightarrow c = \frac{1}{R}\log_e(E) \qquad \textbf{[1 mark]}$$

$$\frac{t}{L} = -\frac{1}{R}\log_e(E - Ri) + \frac{1}{R}\log_e(E) \qquad \textbf{[1 mark]}$$

$$\frac{t}{L} = \frac{1}{R}\left[\log_e(E) - \log_e(E - Ri)\right]$$

$$\frac{Rt}{L} = \log_e\left(\frac{E}{E - Ri}\right)$$

$$e^{\frac{Rt}{L}} = \frac{E}{E - Ri}$$

$$(E - Ri)\,e^{\frac{Rt}{L}} = E \qquad \textbf{[1 mark]}$$

$$E - Ri = Ee^{-\frac{Rt}{L}}$$

$$Ri = E - Ee^{-\frac{Rt}{L}} = E\left(1 - e^{-\frac{Rt}{L}}\right)$$

$$i = i(t) = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \textbf{[1 mark]}$$

b. Max $i(t) = \frac{E}{R}$, $t \rightarrow \infty$, $e^{-\frac{Rt}{L}} \rightarrow 0$

$$i(t) = \frac{E}{2R},$$

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \textbf{[1 mark]}$$

$$e^{-\frac{Rt}{L}} = \frac{1}{2}$$

$$-\frac{Rt}{L} = \log_e\left(\frac{1}{2}\right)$$

$$t = -\frac{L}{R}\log_e\left(\frac{1}{2}\right) \qquad \textbf{[1 mark]}$$

Question 3

Thermometer $T_m = ?$

$t = 0,\ T = 25 \rightarrow \theta_0 = 25 - T_m$

$t = 5,\ T = 15 \rightarrow \theta(5) = 15 - T_m$

$t = 10,\ T = 10 \rightarrow \theta(10) = 10 - T_m$

$$\theta = \theta_0 e^{kt}$$

$[1]:\ \theta_0 = 25 - T_m$

$[2]:\ 15 - T_m = \theta_0 e^{5k}$

$[3]:\ 10 - T_m = \theta_0 e^{10k}$

$$e^{5k} = \frac{15 - T_m}{25 - T_m}$$

$$e^{10k} = \frac{10 - T_m}{25 - T_m}$$

$$e^{10k} = \left(e^{5k}\right)^2 \qquad \textbf{[1 mark]}$$

$$e^{5k} = \frac{15 - T_m}{25 - T_m}$$

$$e^{10k} = \frac{10 - T_m}{25 - T_m}$$

$$e^{10k} = \left(e^{5k}\right)^2 \qquad \textbf{[1 mark]}$$

$$\left(\frac{15-T_m}{25-T_m}\right)^2 = \frac{10-T_m}{25-T_m}$$

$$\frac{(15-T_m)^2}{(25-T_m)^2} = \frac{10-T_m}{25-T_m}$$

$$(15-T_m)^2 = (25-T_m)^2 \times \frac{10-T_m}{25-T_m}$$

$$(15-T_m)^2 = (25-T_m)(10-T_m)$$

$$22 - 30T_m + T_m^2 = 250 - 35T_m + T_m^2$$

$$5T_m = 25$$

$$T_m = 5°\text{C}$$ **[1 mark]**

Question 4

$$\frac{dT}{dt} = k(T-5)$$

$$\frac{dt}{dT} = \frac{1}{k(T-5)}$$

$$t = \frac{1}{k}\int \frac{1}{k(T-5)}dt + \text{C}$$

$$= \frac{1}{k}\log_e A(T-5)$$

$t = 0$:

$$\therefore 0 = \frac{1}{k}\log_e(70A)$$

$$A = \frac{1}{70}$$

$$t = \frac{1}{k}\log_e \frac{(T-5)}{70}$$

Question 5

$$\frac{dT}{dt} = k(T-25)$$

$$\frac{dt}{dT} = \frac{1}{k(T-25)}$$

$$t = \frac{1}{k}\int \frac{1}{k(T-25)}dt + C$$

$$= \frac{1}{k}\log_e A(T-25)$$

$t = 0$:

$$\therefore 0 = \frac{1}{k}\log_e(75A)$$

$$A = \frac{1}{75}$$

$t = 3$:

$$\therefore 3 = \frac{1}{k}\log_e\left(\frac{65}{75}\right)$$

$$k = -0.05$$

$$t = \frac{1}{-0.05}\log_e\left(\frac{55}{75}\right) = 6.5$$

$\therefore$ takes a further 3.5 minutes

Question 6

$\dfrac{dR}{dt} = k(R + 18) \qquad R(0) = 68$ with $k > 0$

The room temperature is 18°C, and the initial temperature of the iron is 68°C.

Question 7

$T = 18 - 15e^{-kt} \Rightarrow 15e^{-kt} = 18 - T$

$\dfrac{dT}{dt} = 15ke^{-kt} = 15k\left(\dfrac{18 - T}{15}\right) = k(18 - T) = -k(T - 18)$ cooling $k > 0$

Also $T(0) = 18 - 15 = 3$°C

The room temperature is 18°C and the temperature of the refrigerator is 3°C.

Question 8

$\dfrac{dC}{dt} = k(R - 21),\ \ R(0) = 65,\ $ with $k > 0$

OR

$\dfrac{dC}{dt} = k(R - 21),\ R(0) = 65, k > 0$

10.5 Chemical reactions and dilution problems

Question 1

a. $\dfrac{dQ}{dt} = \text{inflow} - \text{outflow}$

$V(t) = 16 + (5 - 3)t$

$\dfrac{dQ}{dt} = 5 \times 0 - \dfrac{3Q}{V(t)} = 5 \times 0 - \dfrac{3Q}{16 + (5 - 3)t}$ **[1 mark]**

***VCAA Examination Report* note:**

This problem required students to recognise a difference of rates. The most common error was a failure to explicitly note that the rate in was zero.

b. $\log_e(Q) = -\dfrac{3}{2}\log_e(16 + 2t) + c$ **[1 mark]**

$Q = \dfrac{1}{2},\ t = 0 \Rightarrow \log_e\left(\dfrac{1}{2}\right) + \dfrac{3}{2}\log_e(16) = c$

$\log_e(Q) = -\dfrac{3}{2}\log_e(16 + 2t) + \log_e\left(\dfrac{1}{2}\right) + \dfrac{3}{2}\log_e(16)$ **[1 mark]**

$\log_e(Q) - \log_e\left(\dfrac{1}{2}\right) = -\dfrac{3}{2}\log_e(16 + 2t) + \dfrac{3}{2}\log_e(16)$

$\log_e(2Q) = \log_e\left(\dfrac{16}{16 + 2t}\right)^{\frac{3}{2}}$

$2Q = \left(\dfrac{16}{16 + 2t}\right)^{\frac{3}{2}} = \dfrac{\left(\sqrt{16}\right)^3}{(16 + 2t)^{\frac{3}{2}}} = \dfrac{64}{(16 + 2t)^{\frac{3}{2}}}$

$Q = \dfrac{32}{(16 + 2t)^{\frac{3}{2}}} \qquad a = 32,\ b = 3,\ c = 2$ **[1 mark]**

***VCAA Examination Report* note:**

The majority of students realised that this was a separable differential equation, but many made errors in the subsequent integration with the arbitrary constant of integration frequently missing. Some students made transcription errors that fundamentally changed the problem. Others encountered arithmetic or algebraic issues. Many students took the common factor of 2 from the $16 + 2t$ expression and evaluated

$\frac{1}{2}\int \frac{1}{8+t}dt$ This unnecessary manipulation made subsequent calculations more difficult for these students.

Question 2

$$\frac{dQ}{dt} = \text{inflow} - \text{outflow}$$

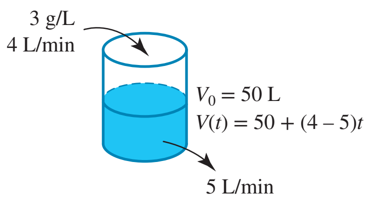
$$\frac{dQ}{dt} = 10 \times 2 - \frac{6Q}{V(t)} \text{ where } V(t) = 50 + (10-6)t$$

$$\frac{dQ}{dt} = 20 - \frac{6Q}{50+4t} = 20 - \frac{3Q}{25+2t}$$

The correct answer is **A**.

Question 3

a.



$$\frac{dQ}{dt} = 3 \times 4 - \frac{5Q}{50 + (4-5)\,t}$$

$$\frac{dQ}{dt} = 12 - \frac{5Q}{50-t} \textbf{ [1 mark]}$$

b. $Q = 3(50-t) + C(50-t)^5$

LHS: $\frac{dQ}{dt} = -3 - 5C(50-t)^4$ **[1 mark]**

RHS: $12 - \frac{5Q}{50-t} = 12 - \frac{55}{50-t}\left[3\,(50-t) + C(50-t)^5\right]$

$= 12 - 15 - 5C(50-t)^4$

$= -3 - 5C(50-t)^4$ **[1 mark]**

Question 4

$$\frac{dm}{dt} = \text{inflow} - \text{outflow}$$

$$= 15 \times 2 - \frac{5m}{50 + (2-5)\,t}$$

$$= 30 - \frac{5m}{50-3t}$$

Question 5

$$\frac{dQ}{dt} = \text{inflow} - \text{outflow}$$

$$\frac{dQ}{dt} = 8 \times 2 - \frac{100}{V(t)}, \quad \text{where } V(t) = 1500 + (8-10)\,t$$

$$\frac{dQ}{dt} = 16 - \frac{10Q}{1500-2t}$$

$$= 16 - \frac{5Q}{750-t}$$

Question 6

$\frac{dQ}{dt} = \text{inflow} - \text{outflow}$

$\frac{dQ}{dt} = 10 \times 2 - \frac{6Q}{V(t)}$ where $V(t) = 50 + (10 - 6)t$

$\frac{dQ}{dt} = 20 - \frac{6Q}{50 + 4t} = 20 - \frac{3Q}{25 + 2t}$

Question 7

$V = 100 + 12t - 8t$

$= 100 + 4t$

Question 8

$\frac{dQ}{dt} = \text{inflow} - \text{outflow} = bg - \frac{fQ}{V}$ and the volume $V = V(t) = V_0 + (g - f)t$

$\frac{dQ}{dt} = bg - \frac{fQ}{V_0 + (g - f)t}$ $Q(0) = a$

$\frac{dQ}{dt} = 12 + \frac{5Q}{t - 10} = 12 - \frac{5Q}{10 - t}$ $Q(0) = a$

$\Rightarrow bg = 12$ $V_0 = 10$ $g - f = -1$

Only possible solution is $f = 5 \Rightarrow g = 4$ and $b = 3$

Question 9

$\frac{dQ}{dt} = 12 - \frac{2Q}{50 + 2t}$

$Q = Q(t) = 3(50 + 2t) + C(50 + 2t)^n$ is the solution

Differentiating:

$\frac{dQ}{dt} = 6 + 2nC(50 + 2t)^{n-1}$

Substituting gives:

$\text{LHS} = 6 + 2nC(50 + 2t)^{n-1}$

$\text{RHS} = 12 - \frac{2Q}{50 + 2t} = 12 - \frac{2}{50 + 2t}\left[3(50 + 2t) + C(50 + 2t)^n\right] = 12 - 6 - 2C(50 + 2t)^{n-1}$

$= 6 - 2nC(50 + 2t)^{n-1}$

$\Rightarrow n = -1$

10.6 The logistic equation

Question 1

$N(t) = \frac{200}{1 + 99e^{-kt}}$

$\frac{dN}{dt} = \frac{0 - 200 \times (-k) \times 99e^{-kt}}{\left(1 + 99e^{-kt}\right)}$ **[1 mark]**

$\frac{dN}{dt} = \frac{200 \times k \times 99e^{-kt}}{\left(1 + 99e^{-kt}\right)^2}$

$= \frac{200k}{1 + 99e^{-kt}} \times \frac{200\left(1 + 99e^{-kt}\right) - 200}{200\left(1 + 99e^{-kt}\right)}$

$$\frac{dN}{dt} = \frac{200k}{1+99e^{-kt}} \times \left(\frac{200\left(1+99e^{-kt}\right)}{200\left(1+99e^{-kt}\right)} - \frac{200}{200\left(1+99e^{-kt}\right)} \right)$$

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{200}\right),\ N(0) = 2$$ **[1 mark]**

Question 2

$$N = \frac{P}{2}$$

$P = 500$

$N = 250$ **[1 mark]**

Question 3

a. $N(0) = N_0,\ P > N_0 > 0$

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right) = kN\left(\frac{P-N}{P}\right) = \frac{k}{P}N(P-N)$$

$$\frac{dt}{dN} = \frac{P}{kN(P-N)}$$

$$kt = P\int \frac{1}{N(P-N)}dN$$

$$\frac{1}{N(P-N)} = \frac{A}{N} + \frac{B}{(P-N)}$$

$$= \frac{A(P-N)+BN}{N(P-N)}$$

$$= \frac{N(B-A)+PA}{N(P-N)}$$

$B - A = 0$

$PA = 1$

$$A = B = \frac{1}{P}$$

$$kt = \int \frac{1}{N} + \frac{1}{P-N}dN$$

$kt = \log_e(N) - \log_e(P-N) + c$ **[1 mark]**

$$0 = \log_e(N_0) - \log_e(P-N_0) + c \rightarrow c = \log_e\left(\frac{P-N_0}{N_0}\right)$$

$$kt = \log_e(N) - \log_e(P-N) + \left(\frac{P-N_0}{N_0}\right)$$

$$= \log_e\left(\frac{N(P-N_0)}{N_0(P-N)}\right)$$

$$e^{kt} = \frac{N(P-N_0)}{N_0(P-N)}$$

$$e^{-kt} = \frac{N_0(P-N)}{N(P-N_0)}$$

$$N(P-N_0)e^{-kt} = N_0(P-N) = N_0P - NN_0$$

$$NN_0 + N(P-N_0)e^{-kt} = N_0P$$

$$N(t) = \frac{N_0P}{N_0 + (P-N_0)e^{-kt}}$$ **[1 mark]**

b. $k > 0,\ \lim_{t\to\infty} N(t) = \dfrac{N_0 P}{N_0 + 0} = P$ **[1 mark]**

c. $\dfrac{dN}{dt} = kN\left(1 - \dfrac{N}{P}\right) = k\left(N - \dfrac{N^2}{P}\right)$

$$\frac{d^2N}{dt^2} = \frac{d}{dt}\left(\frac{dN}{dt}\right) = \frac{d}{dN}\left(\frac{dN}{dt}\right)\cdot\frac{dN}{dt}$$

$$= k\left(1 - \frac{2N}{P}\right)kN\left(1 - \frac{N}{P}\right)$$

$$= k^2N\left(1 - \frac{2N}{P}\right)\left(1 - \frac{N}{P}\right)$$ **[1 mark]**

$$\frac{d^2N}{dt^2} = 0$$

$$1 - \frac{2N}{P} = 0$$

$$N = \frac{P}{2}$$

$$t = ?$$

$$e^{kt} = \frac{N(P - N_0)}{N_0(P - N)}$$

$$= \frac{\frac{P}{2}(P - N_0)}{N_0\left(P - \frac{P}{2}\right)}$$

$$= \frac{P - N_0}{N_0}$$

$$kt = \log_e\left(\frac{P - N_0}{N_0}\right)$$

$$t = \frac{1}{k}\log_e\left(\frac{P}{N_0} - 1\right)$$

∴ Inflection point occurs at

$$\left(\frac{1}{k}\log_e\left(\frac{P}{N_0} - 1\right), \frac{P}{2}\right)$$ **[1 mark]**

d. $N(t) = \dfrac{N_0 P}{N_0 + (P - N_0)e^{-kt}}$

$$N = N_0P\left(N_0 + (P - N_0)e^{-kt}\right)^{-1}$$

$$\frac{dN}{dt} = -PN_0(P - N_0)\times -ke^{-kt}\left(N_0 + (P - N_0)e^{-kt}\right)^{-2}$$

$$= \frac{kPN_0(P - N_0)e^{-kt}}{\left(N_0 + (P - N_0)e^{-kt}\right)^2}$$

$$= \frac{PN_0}{\left(N_0 + (P - N_0)e^{-kt}\right)}\times k(P - N_0)e^{-kt}\times\frac{1}{N_0 + (P - N_0)e^{-kt}}$$

$$\frac{1}{N_0 + (P - N_0)e^{-kt}} = \frac{N}{PN_0}$$ **[1 mark]**

$$N_0 + (P - N_0)e^{-kt} = \frac{PN_0}{N}$$

$$(P-N_0)\,e^{-kt} = \frac{PN_0}{N} - N_0 = \frac{PN_0 - NN_0}{N}$$
$$e^{-kt} = \frac{N_0\,(P-N)}{N\,(P-N_0)}$$
$$\frac{dN}{dt} = N \times \frac{k\,(P-N_0) \times N_0\,(P-N)}{N\,(P-N_0)} \times \frac{N}{PN_0}$$
$$= \frac{kN\,(P-N)}{P}$$
$$= kN\left(1-\frac{N}{P}\right) \quad \textbf{[1 mark]}$$

10.7 Euler's method

Question 1

$\frac{dy}{dx} = y\sin(x) = f(x, y) \quad x_0 = 1, \quad y_0 = 2, \quad h = 0.1$

$y_1 = y_0 + hf(x_0, y_0)$

$y_1 = 2 + 0.1 \times 2\sin(1)$

$y_1 = 2.168$

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = 2.168 + 0.1 \times 2.168\sin(1.1)$

$y_2 = 2.3615$

The correct answer is **C.**

Question 2

$\frac{dy}{dx} = f(x) = e^{\cos(x)}, \; y_0 = e, \; x_0 = 0, h = 0.1$

$y_1 = y_0 + hf(x_0)$

$y_1 = e + 0.1e^{\cos(0)}$

$y_1 = e + 0.1e$

$y_2 = y_1 + hf(x_1)$

$y_2 = e + 0.1e^{\cos(0)} + 0.1e^{\cos(0.1)}$

$y_2 = e + 0.1\left(e + e^{\cos(0.1)}\right)$

$y_3 = y_2 + hf(x_2)$

$y_3 = e + 0.1\left(e + e^{\cos(0.1)}\right) + 0.1e^{\cos(0.2)}$

$y_3 = e + 0.1\left(e + e^{\cos(0.1)} + e^{\cos(0.2)}\right)$

The correct answer is **C.**

Question 3

Using Euler's method, with

$x_0 = 1, \; y_0 = 2, \; h = -0.1, \; \frac{dy}{dx} = f(x) = 2x^2 + x + 1$

So that $x_1 = 0.9$ and $x_2 = 0.8$

$y_1 = y_0 + hf(x_0) = 2 - 0.1\,(4) = 1.6$

$y_2 = y_1 + hf(x_1) = 1.6 - 0.1\left(2 \times 0.9^2 + 0.9 + 1\right)$

$= 1.248$

The correct answer is **B.**

Question 4

$f(x) = \frac{dy}{dx} = 2x^2 - x, \quad y_0 = 0 = y(2), \quad h = 0.1, \quad x_0 = 2, \quad x_1 = 2.1, \quad x_2 = 2.2$

$f(2) = 6, \quad f(2.1) = 6.72, \quad f(2.2) = 7.48$

$$\begin{aligned} y_1 &= y_0 + hf(x_0) \\ &= 0 + 0.1f(2) \\ &= 0.1 \times 6 \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf(x_1) \\ &= 0.6 + 0.1f(2.1) \\ &= 0.6 + 0.1 \times 6.72 \\ &= 1.272 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + hf(x_2) \\ &= 1.272 + 0.1f(2.2) \end{aligned}$$

Question 5

$\frac{dy}{dx} = f(x) = x^3 - xy$

$h = 0.1, \ y_0 = 2, \ x_0 = 1, \ x_1 = 1.1$

$$\begin{aligned} y_1 &= y_0 + hf(x_0) \\ &= 2 + 0.1(1 - 1 \times 2) \\ &= 1.9 \end{aligned}$$

Question 6

$\frac{dy}{dx} = f(x) = \frac{1}{3 + 3x + x^2}$

$h = 0.1, \quad y_0 = 1, \quad x_0 = 0, \quad x_1 = 0.1$

$$\begin{aligned} y_1 &= y_0 + hf(x_0) \\ &= 1 + 0.1\left(\frac{1}{3}\right) = 1.033\,33 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf(x_1) \\ &= 1.033\,33 + 0.1\left(\frac{1}{3 + 3 \times 0.1 + \left(0.1^2\right)}\right) \\ &= 1.064 \end{aligned}$$

Question 7

$y_1 = y_0 + hf(x_0)$

$y_2 = y_1 + hf(x_1) = y_0 + hf(x_0) + hf(x_1) \qquad \text{since} \quad x_1 = x_0 + h$

$y_2 = y_0 + h\left[f(x_0) + f(x_0 + h)\right]$

Question 8

$x_1 = 2.2$

$$\begin{aligned} y_1 &= 1 + 0.2\log_e(2)^2 \\ &= 1.277258872 \end{aligned}$$

$x_2 = 2.4$

$$\begin{aligned} y_2 &= 1.277258872 + 0.2\log_e\left(2.2^2\right) \\ &= 1.5926418 \end{aligned}$$

$x_3 = 2.6$

$$\begin{aligned} y_3 &= 1.5926418 + 0.2\log_e\left(2.4^2\right) \\ &= 1.94 \end{aligned}$$

Question 9

Using Euler's method, with

$x_0 = 2, \quad y_0 = 1, \quad h = \frac{1}{4}, \quad \frac{dy}{dx} = f(x) = \log_e(2x - 3)$

so that $x_1 = \frac{9}{4}, \quad x_2 = \frac{5}{2} \quad$ and $\quad x_3 = \frac{11}{4}$

$y_1 = y_0 + hf(x_0) = 1 + \frac{1}{4}\log_e(1) = 1$

$y_2 = y_1 + hf(x_1) = 1 + \frac{1}{4}\log_e\left(\frac{3}{2}\right)$

$y_3 = y_2 + hf(x_2) = 1 + \frac{1}{4}\log_e\left(\frac{3}{2}\right) + \frac{1}{4}\log_e(2) = 1 + \frac{1}{4}\log_e(3)$

$y_4 = y_3 + hf(x_3) = 1 + \frac{1}{4}\log_e(3) + \frac{1}{4}\log_e\left(\frac{5}{2}\right) = 1 + \frac{1}{4}\log_e\left(\frac{15}{2}\right)$

10.8 Slope fields

Question 1

At the point $x = 1, \; y = 1$, the slope is $m = \tan(135°) = -1$, only $\frac{dy}{dx} = y - 2x$satisfies this.

The correct answer is **D**.

Question 2

$\left.\frac{dy}{dx}\right|_{x=a} = m, \; P(a, b)$

$T: y - b = m(x - a)$

$y = 0 \Rightarrow x = a - \frac{b}{m} = b$

$\frac{b}{m} = a - b, \; m = \frac{b}{a-b}$

$\frac{dy}{dx} = m = \frac{y}{x-y}$ is underfined when $x = y$ and $\frac{dy}{dx} = 0$ when $y = 0$.

The correct answer is **B**.

Question 3

When the gradient $m = 1$, there are no vertical slopes $m = \frac{dy}{dx} = \cos(y - x)$ is the only possible differential equation.

The correct answer is **B**.

Question 4

When $y = x$ the gradient $m = 1$,

there are no vertical slopes $m = \frac{dy}{dx} = \cos(y - x)$ is the only possible differential equation.

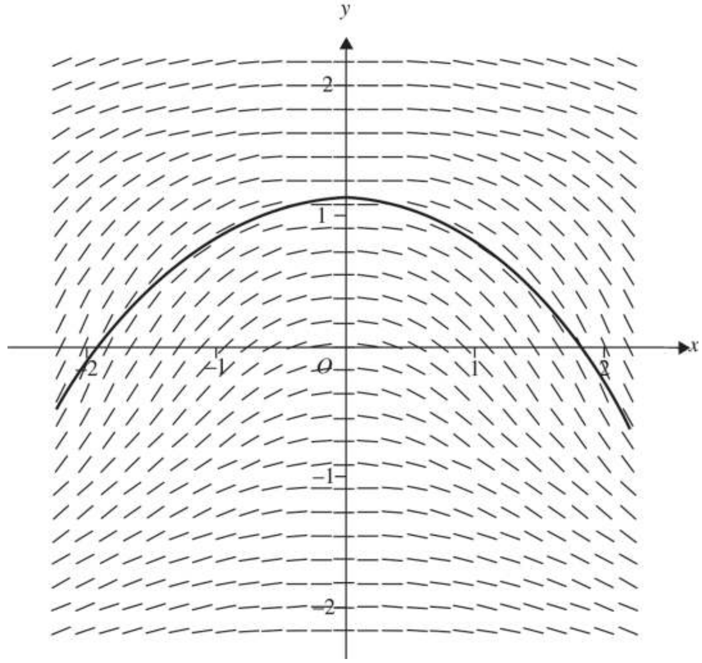
Question 5

When $x = 0$, the gradient $m = 1$; when $y = 0$ the gradient $m = -1$.

These conditions are only satisfied by $m = \frac{dy}{dx} = \frac{2x + y}{y - 2x}$.

Question 6

$\frac{dy}{dx} = \frac{-x}{1+y^2}$, following the slope fields from $(-1,\ 1)$ to $(1.9,\ 0)$, so $x \approx 1.9$



Award **1 mark** for the correct solution curve.
Award **1 mark** for the final correct value.
***VCAA Examination Report* note:**
This question was not answered well. Several curves crossed the slope ticks rather than following them. Errors included:
- the final curve not being symmetrical
- the curve not passing through $(-1,\ 1)$, giving the value for x as around 1.2 (the value of the y intercept)
- finding an approximate value from the solution in part b. even though this was inconsistent with the student's graph (part a. used the word 'hence').

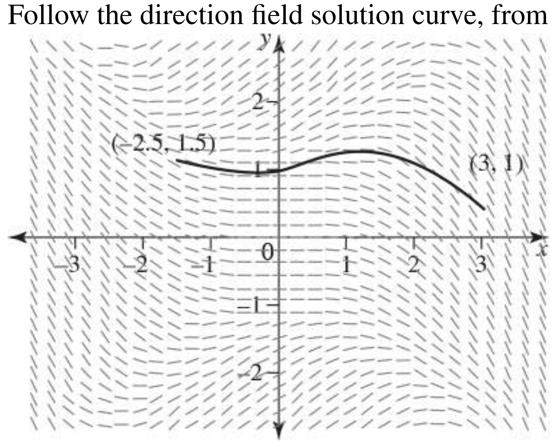
Many graphs were almost flat between $x = -0.5$ and $x = 0.5$, resulting in missing the desired y-intercept. Some drew the graph just to the x-intercepts rather than for the whole domain. Several graphs were not drawn smoothly with sufficient care.

Question 7

Follow the direction field lines from the point $(0, -1)$ to the point $(3.5, -2.5)$.

Question 8

Follow the direction field solution curve, from the point $(-2.5,\ 1.5)$ to the point $(3, 1)$.



Question 9

The slope is $m = \frac{dy}{dx}$.
Along the y-axis, $x = 0$: $y > 0,\ m > 0;\ y < 0,\ m < 0$.
Along the x-axis, $y = 0$: $x > 0,\ m < 0;\ x < 0,\ m > 0$.
When $x = y,\ m = \infty$.
The conditions are only satisfied by $\frac{dy}{dx} = \frac{1}{y - x}$.

Question 10

Slope is $m = \frac{dy}{dx}$ along y-axis. $x = 0$ and $m = 0$ along x-axis.
When $y = 0,\ m = \infty$.
When $x = 1$ and $y = 1,\ m = 1$.
When $x = -1$ and $y = -1,\ m = 1$.
When $x = -1$ and $y = 1,\ m = -1$.
When $x = 1$ and $y = -1,\ m = -1$.
Only satisfied by
$\frac{dy}{dx} = \frac{x}{y}$

Question 11

Slope field shows pattern for $y = x^n$, where n is a positive odd number.
$\frac{dy}{dx} = 3x^2$
$y = \int 3x^2 dx = x^3$ is the only option that satisfies this.

Question 12

Slope field shows pattern for $y = x^n$, where n is a positive even number:
$\frac{dy}{dx} = x$
$y = \int x\, dx = \frac{x^2}{2}$ is the only option that satisfies this.

Question 13

Slope field shows pattern for $y = \log_e |x|$
$\frac{dy}{dx} = \frac{1}{x}$
$y = \int x^{-1}\, dx = \log_e |x|$ is the only option that satisfies this

Question 14

The solution curves are all ellipses of the form $\frac{x^2}{a^2} + y^2 = 1$.
Using implicit differentiation, $\frac{2x}{a^2} + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{a^2 y}$
One solution when $a^2 = 2$ is $\frac{dy}{dx} = -\frac{x}{2y}$. This is the only possible solution.

Question 15

The solution curves are all of the form $y = Ae^{-kx}$ with $k > 0$ and $A < 0$ or $A > 0$.
$\frac{dy}{dx} = -kAe^{-kx} = -ky$. The gradient (slopes) are independent of x; they are the same for each x-value. If $y > 0$, the slopes are always negative, and if $y < 0$, the slopes are always positive. Option B is correct.

10.9 Review

Question 1

When $x = 0$, the gradient $m = 1$; when $y = 0$ the gradient $m = -1$. These conditions are only satified by $m = \frac{dy}{dx} = \frac{2x + y}{y - 2x}$.

The correct answer is **A**.

Question 2

$f(x) = \frac{dy}{dx} = 2x^2 - x,\ y_0 = 0 = y(2),\ h = 0.1$

$x_0 = 2,\ x_1 = 2.1,\ x_2 = 2.2$

$f(2) = 6,\ f(2.1) = 6.72,\ f(2.2) = 7.48$

$$\begin{aligned} y_1 &= y_0 + hf(x_0) \\ &= 0 + 0.1f(2) \\ &= 0.1 \times 6 \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + hf(x_1) \\ &= 0.6 + 0.1f(2.1) \\ &= 0.6 + 0.1 \times 6.72 \\ &= 1.272 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + hf(x_2) \\ &= 1.272 + 0.1f(2.2) \end{aligned}$$

The correct answer is **C**.

Question 3

a. Using implicit differentiation $\frac{1}{N}\frac{dN}{dt} = 3 \times 0.4e^{-0.4t}$

LHS $= \frac{1}{N}\frac{dN}{dt} + 0.4\log_e(N) - 2.4$ **[1 mark]**

$= 3 \times 0.4e^{-0.4t} + 0.4\left(6 - 3e^{-0.4t}\right) - 2.4$

$= 3 \times 0.4e^{-0.4t} + 2.4 - 3 \times 0.4e^{-0.4t}$

$-2.4 = 0$ shown **[1 mark]**

b. When $t = 0$ $\log_e(N) = 6 - 3e^0 = 3$ $\Rightarrow N = e^3 \approx 20$ [1 mark]

$\Rightarrow N = e^3 \approx 20$ **[1 mark]**

c. As $t \to \infty$ $\log_e(N) \to 6 - 3e^{-\infty} = 6$ **[1 mark]**

$N = e^6 \approx 403$ **[1 mark]**

d. i. $\frac{d^2N}{dt^2} = \frac{d}{dt}\left(\frac{dN}{dt}\right) = \frac{d}{dt}\left[0.4N(6 - \log_e(N))\right]$

using product rule

$$\begin{aligned} \frac{d^2N}{dt^2} &= 0.4\frac{dN}{dt}(6 - \log_e(N)) + 0.4N \times \frac{-1}{N}\frac{dN}{dt} \\ &= \frac{dN}{dt}\left[0.4(6 - \log_e(N)) - 0.4\right] \quad \textbf{[1 mark]} \\ &= \frac{dN}{dt}(2 - 0.4\log_e(N)) \end{aligned}$$

substitute for

$$\frac{dN}{dt} = 0.4N(6 - \log_e(N))$$

$$= 0.4N(6 - \log_e(N))(2 - 0.4\log_e(N))$$

$$= \frac{4N}{25}(6 - \log_e(N))(5 - \log_e(N))$$ **[1 mark]**

ii. For inflection points

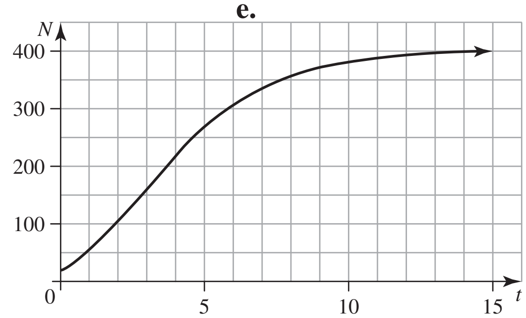
$\frac{d^2N}{dt^2} = 0$ since $\log_e(N) \neq 6 \Rightarrow \log_e(N) = 5$ **[1 mark]**

$\Rightarrow N = e^5 \approx 148$

$5 = 6 - 3e^{-0.4t} \Rightarrow e^{-0.4t} = \frac{1}{3}$ **[1 mark]**

$t = \frac{1}{0.4}\log_e(3) \simeq 2.7$ years

e.



Question 4

a. $\frac{dn}{dt} = k(50 - n)$ **[1 mark]**

$n(0) = 4$

$n(4) = 16$

$$\frac{dt}{dn} = \frac{1}{k(50 - n)}$$

$$kt = \int \frac{1}{(50 - n)}\, dn$$

$kt = -\log_e(|50 - n|) + c$ but $4 \leq n \leq 50$

When $t = 0,\ n = 4$

$0 = -\log_e(46) + c \Rightarrow c = \log_e(46)$

$kt = \log_e(46) - \log_e(50 - \text{n})$

$$kt = \log_e\left(\frac{46}{50 - n}\right)$$

When $t = 4,\ n = 16$

$4k = \log_e\left(\frac{46}{50 - 16}\right) = \log_e\left(\frac{23}{17}\right)$ **[1 mark]**

$$k = \frac{1}{4}\log_e\left(\frac{23}{17}\right)$$

$$e^{kt} = \frac{46}{50 - n}$$

$50 - n = 46e^{-kt}$ **[1 mark]**

$n = n(t) = 50 - 46e^{-kt}$

$n(t) = 2\left(25 - 23e^{-kt}\right)$

b. $t = 8,\ n(8) = 2\left(25 - 23e^{-2\log_e\left(\frac{23}{17}\right)}\right)$

$= 2\left(25 - 23 \times \left(\frac{17}{23}\right)^2\right)$ **[1 mark]**

$= \frac{572}{23}$

≈ 25 ducks

Question 5

a. $N(t) = \dfrac{PN_0}{N_0 + (P - N_0)e^{-kt}}$

In 2010, $t = 0$ and $N(0) = N_0 = 34.148$

In 2015, $t = 5$ and $N(5) = 36.027$

$36.027 = \dfrac{34.148P}{34.148 + (P - 34.148)e^{-5k}}$ (1)

In 2020, $t = 10$ and $N(10) = 37.742$

$37.742 = \dfrac{34.148P}{34.148 + (P - 34.148)e^{-10k}}$ (2) **[1 mark]**

Solving equations (1) and (2) on a CAS calculator:

$P = 48.742,\ k = 0.0383$

The maximum population of Canada will be 48.742 million **[1 mark]**

b. To determine the population in 2000, let $t = -10$.

$N(-10) = \dfrac{48.742 \times 34.148}{34.148 + (48.742 - 34.148)e^{-0.0383\times-10}}$

$= 29.962$

The population of Canada in 2000 was 29.962 million. **[1 mark]**

Question 6

a. i. $\dfrac{dP}{dt} = kP$

$\int \frac{1}{P}dP = \int kdt$

$\log_e(|P|) = kt + c$ but $P > 0$, so the modulus is not needed.

$P = P(t) = e^{kt+c} = P_0e^{kt}$

(1) $P(a) = P_0e^{ka} = r$, (2) $P(b) = P_0e^{kb} = s$

$\dfrac{(1)}{(2)}\ \dfrac{r}{s} = \dfrac{e^{\kappa a}}{e^{kb}} = e^{k(a-b)},\ k(a-b) = \log_e\left(\dfrac{r}{s}\right)$

$k = \dfrac{1}{a-b}\log_e\left(\dfrac{r}{s}\right)$

Award **1 mark** for correctly integrating and solving.

Award **1 mark** for the correct result.

***VCAA Examination Report* note:**

Students solved the differential equation to find the given expression for ***k*** by a variety of correct approaches. Common errors were to neglect a constant of integration or to make mistakes when manipulating logarithmic or exponential terms.

ii. $k = \dfrac{1}{a-b}\log_e\left(\dfrac{r}{s}\right) > 0$, since $P > 0 \Rightarrow r > 0, s > 0$

Case (1) $k > 0, \dfrac{1}{a-b} > 0, \log_e\left(\dfrac{r}{s}\right) > 0 \Rightarrow a > b$

Case (2) $k > 0, \dfrac{1}{a-b} < 0, \log_e\left(\dfrac{r}{s}\right) < 0 \Rightarrow a < b$

Award **1 mark** each for the correct cases.

***VCAA Examination Report* note:**

A significant number of students stated only the first of the above conditions.

b. i. $\dfrac{dQ}{dt} = e^{t-Q} = \dfrac{e^t}{e^Q}$

$\int e^Q dQ = \int e^t dt$

Award **1 mark** for correctly separating the variables.

***VCAA Examination Report* note:**

Most students answered this correctly using the form above or an alternative such as

$\int \dfrac{1}{e^{-Q}} dQ = \int e^t dt.$

ii. $e^Q = e^t + c$ to find c use $Q = 1,\ t = 0$

$e = 1 + c,\ \ c = e - 1$

$e^Q = e^t + e - 1$

$Q = \log_e\left(e^t + e - 1\right)$

Award **1 mark** for correctly integrating.

Award **1 mark** for the correct proof.

iii. $\dfrac{d^2Q}{dt^2} = \dfrac{(e-1)\,e^t}{\left(e^t + e - 1\right)} \neq 0$ since $e > 1$ and $e^t > 0$ so there are no inflexion points.

Award **1 mark** for the correct second derivative.

Award **1 mark** for the correct argument.

***VCAA Examination Report* note:**

Most students supplied a correct second derivative but not all of them went on to reasonably justify why the graph does not have a point of inflection.

Question 7

a. $\dfrac{dx}{dt} + \dfrac{x}{20+t} = 0, x(0) = 20$

$$\frac{dx}{dt} = -\frac{x}{20+t}$$

$$\int \frac{1}{x}dx = \int \frac{-1}{20+t}dt$$

$$\log_e(x) = -\log_e(20+t) + c$$

$$\log_e(20) = -\log_e(20) + c$$

$$\Rightarrow c = 2\log_e(20) = \log_e(400)$$

$$\log_e(x) = \log_e(400) - \log_e(20+t)$$

$$= \log_e\left(\frac{400}{20+t}\right)$$

$$x = x(t) = \frac{400}{20+t}$$

Award **1 mark** for separating the variables.

Award **1 mark** for correct integration.

Award **1 mark** for the correct final expression.

***VCAA Assessment Report* note:**
A range of errors prevented students from achieving full marks for this question. There were many instances where students did not separate variables correctly, offering attempts such as
$\int (20+t)\,dt = \int -x dx \Rightarrow 20t + \frac{t^2}{2} = -\frac{x^2}{2} + c$
Some students did not write down or evaluate the constant. Errors with constants were common among students who added a constant to both sides of the expression before attempting to find its value.
A small number of students used definite integrals from 0 to t and 20 to x on the sides.

b. The volume in this tank is
$V(t) = 100 + (20 - 10)\,t = 100 + 10t$.
Concentration is mass divided by volume, so $c = \frac{y}{100+10t}$ kg/L.
Award **1 mark** for the correct concentration.
***VCAA Assessment Report* note:**
Many students did not demonstrate an understanding of what was required by this question.
Students frequently found an expression for $\frac{dy}{dt}$ rather than the concentration at time t.

c.
$$\begin{aligned}\frac{dy}{dt} &= \text{inflow} - \text{outflow}\\ &= \frac{1}{60} \times 20 - \frac{y}{100+10t} \times 10\\ \frac{dy}{dt} &= \frac{1}{3} - \frac{10y}{100+10t}\end{aligned}$$
$$\frac{dy}{dt} + \frac{y}{10+t} = \frac{1}{3}$$
Award **1 mark** for setting up the differential equation.
Award **1 mark** for the correct differential equation and proof.
***VCAA Assessment Report* note:**
This 'show that' question required students to obtain the expression $\frac{dy}{dt} = \frac{1}{3} - \frac{y}{10+t}$ by logical steps.
Some students incorrectly started with the given expression with no explanation of its origin.
Students frequently did not seem to realise that work done for **part b**. was useful here.

d. $y(t) = \frac{t^2 + 20t + 900}{6(t+10)}$
Differentiating using the quotient rule:
$$\begin{aligned}\frac{dy}{dt} &= \frac{(2t+20) \times 6(10+t) - 6\left(t^2 + 20t + 900\right)}{36\left(10+t^2\right)}\\ &= \frac{12(10+t)^2 - 6\left(t^2 + 20t + 900\right)}{36\left(10+t^2\right)}\\ &= \frac{2(10+t)^2 - \left(t^2 + 20t + 900\right)}{6(10+t)^2}\\ &= \frac{1}{3} - \frac{t^2 + 20t + 900}{6(t+10)^2}\end{aligned}$$

LHS

$$= \frac{dy}{dt} + \frac{y}{10+t}$$

$$= \frac{1}{3} - \frac{t^2+20t+900}{6(t+10)^2} + \frac{t^2+20t+900}{6(t+10)} \times \frac{1}{10+t}$$

$$= \frac{1}{3} - \frac{t^2+20t+900}{6(t+10)^2} + \frac{t^2+20t+900}{6(t+10)^2}$$

$$= \frac{1}{3} = \text{RHS}$$

When $y(0) = \frac{900}{60} = 15$kg, the initial amount of salt dissolved.

Award **1 mark** for correct differentiation using the quotient rule.

Award **1 mark** for the correct proof.

Award **1 mark** for initial conditions.

***VCAA Assessment Report* note:**

Most students were able to find a correct expression for the derivative. It was not always clear how expressions for the left side simplified to the right side. Verification that the given solution satisfied the initial conditions was often absent.

e. Solving $c = \frac{t^2+20t+900}{60(t+10)^2} = 0.095$ using CAS

gives $t = 3.05$ minutes.

Award **1 mark** for the correct equation to be solved.

Award **1 mark** for the correct time.

***VCAA Assessment Report* note:**

Many students did not attempt this question. It was common to see $\frac{dy}{dt} = 0.095$ rather than using the concentration in the equation.

Question 8

$$y^2 + \frac{3e^{(x-1)}}{x-2} = c$$

Using implicit differentitation and the quotientrule

$$2y\frac{dy}{dx} + \frac{3e^{(x-1)}(x-2) - 3e^{(x-1)}}{(x-2)^2} = 0$$

$$2y\frac{dy}{dx} + \frac{3e^{(x-1)}(x-3)}{(x-2)^2} = 0$$

When $x = 1$, $\frac{dy}{dx} = 2$.

$$4y + \frac{3e^0 \times -2}{(-1)^2} = 0 \quad \Rightarrow \quad 4y - 6 = 0 \quad \Rightarrow \quad y = \frac{3}{2}$$

When $x = 1$ and $y = \frac{3}{2}$, $c = \frac{9}{4} + \frac{3e^0}{-1}$

$$c = \frac{9}{4} - 3$$

$$c = -\frac{3}{4}$$

Award **1 mark** for attempting implicit differentiation.

Award **1 mark** for using a quotient rule.

Award **1 mark** for substituting.

Award **1 mark** for the final correct value of c.

VCAA Assessment Report **note:**

This question was reasonably well done. Most students recognised the need for implicit differentiation and so wrote $2y\frac{dy}{dx}$. A reasonable number realised that they needed the quotient rule (or product rule) and the chain rule, although a number had difficulties with algebra. Some students forgot that the derivative of a constant was 0, so a '*c*' remained on the right-hand side after differentiation, meaning that no significant progress was then possible. Some students chose to multiply through by $(x - 2)$ before differentiation. These students were rarely able to make good progress (though a few were able to correctly complete the question this way). Those who attempted to make y the subject often omitted the $\pm$. Typical errors included having a negative sign error in finding y (which nevertheless gave the correct value for c), incorrect differentiation such as $\frac{d}{dx}\left(3e^{x-1}\right) = 3(x-1)e^{x-1}$ and errors in algebra.

Question 9

a. Let $\frac{dy}{dx} = (y+2)^2 + 4$ and $y_0 = y(0) = 0$

$$y = 2\tan\left(2x + \frac{\pi}{4}\right) - 2 \text{ [1 mark]}$$

$$= 2\left(\frac{\tan(2x) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan(2x)\tan\left(\frac{\pi}{4}\right)}\right) - 2$$

$$= \frac{2(\tan(2x) + 1)}{1 - \tan(2x)} - 2 \text{ [1 mark]}$$

$$2x + \frac{\pi}{4} = \tan^{-1}\left(\frac{y+2}{2}\right)$$

$$\frac{y+2}{2} = \tan\left(2x + \frac{\pi}{4}\right)$$

Now make y the subject.

$$0 = \frac{1}{2}\tan^{-1}(1) + C = \frac{\pi}{8} + C \quad \Rightarrow \quad C = -\frac{\pi}{8}$$

$$x = \frac{1}{2}\tan^{-1}\left(\frac{y+2}{2}\right) - \frac{\pi}{8}$$

To find C use $x = 0$ when $y = 0$

$$\frac{dx}{dy} = \frac{1}{4 + (y+2)^2}$$

$$x = \int \frac{1}{4 + (y+2)^2} dy$$

$$x = \frac{1}{2}\tan^{-1}\left(\frac{y+2}{2}\right) + C \text{ [1 mark]}$$

VCAA Assessment Report **note:**

Many students struggled with this question writing incorrectly $\frac{dx}{dy} = \frac{1}{4} + \frac{1}{(y+2)^2}$. Those who had the correct reciprocal were able to get an inverse tan function; however, there were many errors with the constants. Some thought the anti-derivative was a log. Other students correctly found x in terms of y and stopped, while others incorrectly transformed and re-arranged. Other errors included forgetting the constant of integration.

b. $y_0 = 0\; x_0 = 1\; h = 0.1\; f(x, y) = (y+2)^2 + 4$

$y_1 = y_0 + h(f(x_0, y_0))$ **[1 mark]**

$y_1 = 0 + 0.1\left[(0+2)^2 + 4\right]$

$y_1 = 0.8$ **[1 mark]**

***VCAA Assessment Report* note:**
Students tended to either answer this correctly or did not know how to proceed. A large number used $f(2)$ instead of $f'(2)$, while others were careless with brackets and made errors.

Question 10

$$\frac{dy}{\sqrt{y}} = e^x dx$$

$$\int \frac{dy}{\sqrt{y}} = \int e^x dx$$

$$2\sqrt{y} = e^x + \text{c}$$ **[1 mark]**

$$y(0) = 1$$

$$2\sqrt{1} = e^0 + c \Rightarrow c = 1$$ **[1 mark]**

$$2\sqrt{y} = e^x + 1$$

$$y = \frac{(e^x + 1)^2}{4}$$ **[1 mark]**

Question 11

$$\frac{dy}{(e^y)^2} = (e^x)^3 dx$$

$$\int \frac{dy}{(e^y)^2} = \int (e^x)^3 dx$$

$$\int e^{-2y} dy = \int e^{3x} dx$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + c$$ **[1 mark]**

$$y(0) = 0$$

$$-\frac{1}{2} = \frac{1}{3} + c \Rightarrow c = -\frac{5}{6}$$ **[1 mark]**

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} - \frac{5}{6}$$

$$e^{-2y} = -\frac{2}{3}e^{3x} + \frac{5}{3}$$

$$e^y = \left(-\frac{2}{3}e^{3x} + \frac{5}{3}\right)^{-\frac{1}{2}}$$

$$y = \ln\left\{\left(-\frac{2}{3}e^{3x} + \frac{5}{3}\right)^{-\frac{1}{2}}\right\}$$ **[1 mark]**